P425/1
Pure Mathematics
Paper 1
June 2024
3 hours

UGANDA ADVANCED CERTIFICATE OF EDUCATION

Pure Mathematics

Paper 1

3 hours

INSTRUCTIONS TO CANDIDATES

- *Attempt all the questions in* **Section A** *and only five questions from* **Section B**.
- *All working must be shown clearly.*
- *Begin each answer on a fresh sheet of paper.*
- *Mathematical tables with a list of formulae and squared papers are provided.*
- *Silent, non-programmable scientific calculators may be used.*
- *State the degree of accuracy at the end of the answer to each question attempted using a calculator or tables and indicate* **"Cal"** *for calculator or* **"Tab"** *for Mathematical tables.*

SECTION A (40 marks)

Attempt **ALL** *the questions*

1. Solve the simultaneous equations:

$$3y + x - 3z = -4, \quad 3x - y + 2z = 1, \quad -2x + y + z = 7$$ *(05 marks)*

2. Prove that $\tan^2\left(\frac{\pi}{4} + \theta\right) = \left(\frac{1+\sin 2\theta}{1-\sin 2\theta}\right)$. *(05 marks)*

3. The first term of an Arithmetic progression and Geometric progression are each $\frac{2}{3}$. Their common difference and common ratio are each to x and the sum of their first three terms are also equal. Find the two possible values of x. *(05 marks)*

4. Find the point of intersection between the lines $\mathbf{r} = (-2\mathbf{i} + 5\mathbf{j} - 11\mathbf{k}) + \lambda(3\mathbf{i} + \mathbf{j} + 3\mathbf{k})$, and $\mathbf{r} = (8\mathbf{i} + 9\mathbf{j}) + t(4\mathbf{i} + 2\mathbf{j} + 5\mathbf{k})$. *(05 marks)*

5. Evaluate $\int_1^4 \left(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{2}}\right) dx$, giving your answer in simplified surd. *(05 marks)*

6. $A(-3,\ 0)$ and $B(3,\ 0)$ are fixed points. Show that the locus of a point $P(x,\ y)$ which moves such that $PB = 2PA$ is a circle and find its centre and radius. *(05 marks)*

7. Air is pumped into a spherical balloon at a rate of $256\pi\ cm^3 s^{-1}$. When the radius of the balloon is 15 cm, find the rate at which its radius is increasing. *(05 marks)*

8. Find the equation of the tangent to the curve $x^2y - xy^2 = 12$ at the point where $(4,\ 3)$. *(05 marks)*

SECTION B

Attempt **FIVE** *the questions from this section*

9. (a) Solve the equation: $\sqrt{(y+6)} - \sqrt{(y+3)} = \sqrt{(2y+5)}$. Verify your answers. *(06 marks)*

(b) Solve for x and y: $\begin{array}{l}\log(x+y) = 1 \\ 2\log y - \log(30 - x) = 0\end{array}$. *(06 marks)*

10. (a) Show that $1 + 2i$ is a root of the equation $2z^3 - z^2 + 4z + 15 = 0$, hence find the other roots. *(06 marks)*

(b) If $z = 1 + 2i$ is a root of the equation $z^3 + az + b = 0$ where a and b are real, find the values of a and b. *(06 marks)*

11. (a) Differentiate from first principles $y=\frac{1}{\sqrt{x}}$. *(05 marks)*

(b) A hemispherical bowl is being filled with water at a uniform rate. When the height of the water is $h\,cm$ the volume is $\pi\left(rh^2-\frac{1}{3}h^3\right)cm^3$, $r\,cm$ being the radius of the hemisphere. Find the rate at which the water level is rising when it is half way to the top, given that $r=6\,cm$ and the bowl fills in $1\min$. *(07 marks)*

12. (a) Prove that $\tan\frac{1}{2}(A-B)+\tan\frac{1}{2}(A+\mathbf{B})=\frac{2\sin A}{\cos A+\cos B}$. *(05 marks)*

(b) If $\tan\alpha=p$, $\tan\beta=q$, $\tan\gamma=r$, prove that $\tan(\alpha+\beta+\gamma)=\frac{p+q+r-pqr}{1-pr-rq-pq}$,

hence, show that $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\frac{\pi}{4}$. *(07 marks)*

13. (a) The profit y generated from the sale is given by the function $y=72x+3x^2-2x^3$. Calculate how many terms should be sold to receive maximum profit and determine the maximum profit. *(05 marks)*

(b) Find the area enclosed by the curve $y=x(8-x)$ and the line $y=12$. *(07 marks)*

14. Determine the turning points and asymptotes of the curve $y=\frac{4x^2-10x+7}{(x-1)(x-2)}$ hence, sketch the curve. *(12 marks)*

15. (a) If the position vectors of points A and B are $2\mathbf{i}+4\mathbf{j}+6\mathbf{k}$ and $-3\mathbf{i}+2\mathbf{j}+8\mathbf{k}$ respectively, find the position vector of the point P which divides **AB** externally in the ratio $5:3$. *(06 marks)*

(b) Find the angle between the lines $\mathbf{r}=\begin{pmatrix}3\\3\\-1\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\-2\end{pmatrix}$ and $\mathbf{r}=\begin{pmatrix}4\\-5\\-8\end{pmatrix}+\mu\begin{pmatrix}3\\7\\-6\end{pmatrix}$. *(06 marks)*

16. Given that the line $y=mx+c$ is a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, show that $c^2=a^2m^2+b^2$. Hence, determine the equations of the tangents at the point $(-3,\ 3)$ to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$. *(12 marks)*

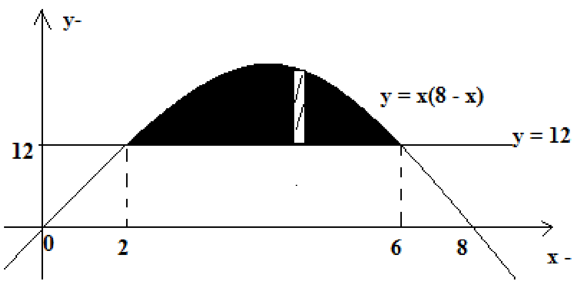
END

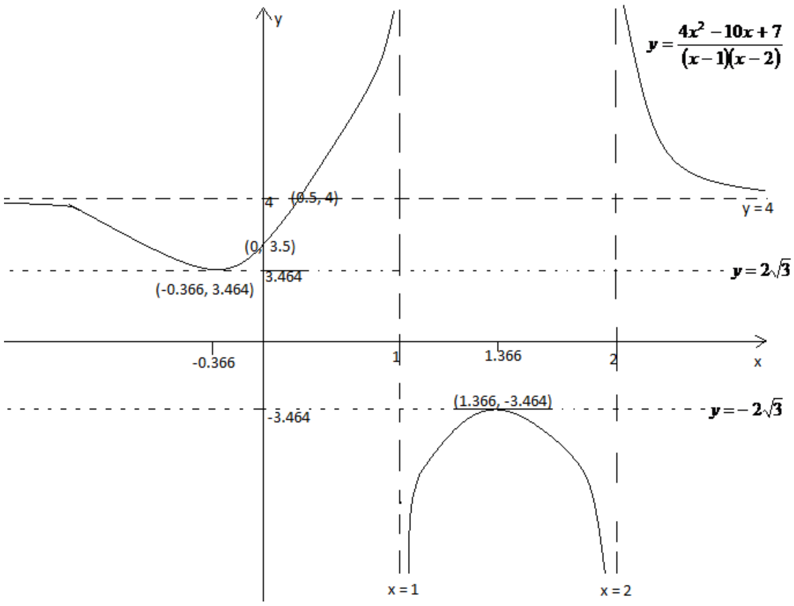
S.6 INTERNAL MOCK MARKING GUIDE

1.	$x+3y-3z=-4, (i)\quad 3x-y+2z=1, (ii)\quad -2x+y+z=7.\ (iii)$ 3 eqn (i) – eqn (ii), $10y-11z=-13\ldots$(iv) 2 eqn (i) + eqn (iii), $7y-5z=-1\ldots$(v) 7 eqn (iv) - 10eqn (v), $-27z=-81$, $z=3$, $y=2$, $x=-1$.	
2	$\tan^2\left(\frac{\pi}{4}+\theta\right)=\left(\frac{\tan\frac{\pi}{4}+\tan\theta}{1-\tan\frac{\pi}{4}\tan\theta}\right)^2 =\left(\frac{1+\tan\theta}{1-\tan\theta}\right)^2 =\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)^2$ $=\left(\frac{\cos^2\theta+2\cos\theta\sin\theta+\sin^2\theta}{\cos^2\theta-2\sin\theta\cos\theta+\sin^2\theta}\right) =\left(\frac{1+\sin 2\theta}{1-\sin 2\theta}\right)$	
3	$a+a+x+a+2x=a+ax+ax^2$ $3.\tfrac{2}{3}+3x=\tfrac{2}{3}\left(1+x+x^2\right)$ $2x^2-7x-4=0, \quad \Leftrightarrow (x-4)(2x+1)=0$ $x=4,\quad x=-\frac{1}{2}$	
4	$\begin{pmatrix}-2\\5\\-11\end{pmatrix}+\begin{pmatrix}3\lambda\\\lambda\\3\lambda\end{pmatrix}=\begin{pmatrix}8\\9\\0\end{pmatrix}+\begin{pmatrix}4t\\2t\\5t\end{pmatrix}$ $\Rightarrow -2+3\lambda=8+4t\ \ldots$(i) $5+\lambda=9+2t\ \ldots$(ii) $-11+3\lambda=5t\ \ldots$(iii) Eqn(i) – eqn(ii)x3 $\begin{array}{rcl}-2+3\lambda & = & 8+4t\\ -(15+3\lambda) & = & -(27+6t)\end{array}$ to get $-17=-19-2t$ $\Rightarrow t=-1$ then from eqn(i) $\lambda=2$ Substitute $t\ \&\ \lambda\ in\ (i)$, LHS. $-11+6=-5=$RHS $=4\mathbf{i}+7\mathbf{j}-5\mathbf{k}$	
5	$\int_1^4\left(\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{2}}\right)dx=\int_1^4\left(x^{-\frac{1}{2}}+\frac{1}{\sqrt{2}}\right)dx$ $=\left[2x^{\frac{1}{2}}+\frac{x}{\sqrt{2}}\right]_1^4$ $=\left(2\left(\sqrt{4}+\frac{4}{\sqrt{2}}\right)\right)-\left(2\sqrt{1}+\frac{1}{\sqrt{2}}\right)$ $=4+2\sqrt{2}-\left(\frac{4+\sqrt{2}}{2}\right)$ $=2+\frac{3}{2}\sqrt{2}$	

6	$\sqrt{(x-3)^2+(y-0)^2}=2\sqrt{(x+3)^2+(y-0)^2}$ $x^2-6x+9+y^2=4(x^2+6x+9+y^2)$ $3x^2+3y^2+30x+27=0$ $x^2+y^2+10x+9=0$, $(x+5)^2+(y-0)^2=16$ Which is an equation of a circle with centre $(-5,\ 0)$ and $r=4$	
7	Vol of a sphere is $V=\frac{4}{3}\pi r^3$ and the rate $\frac{dV}{dt}=256\pi\ cm^3s^{-1}$ Vol of hemisphere is $\frac{dV}{dr}=4\pi r^2$, so $\frac{dr}{dt}=\frac{dr}{dV}\times\frac{dV}{dt}$ $\frac{dr}{dt}=\frac{1}{4\pi r^2}\times 256\pi$ for $r=16$, $\frac{dr}{dt}=\frac{1}{4\pi(15)^2}\times 256\pi=\frac{64}{225}\ cm\ s^{-1}$	
8	$2xy+x^2\frac{dy}{dx}-y^2-2xy\frac{dy}{dx}=0$, $\frac{dy}{dx}=\frac{y^2-2xy}{x^2-2xy}$ $(4,\ 3)$, $\frac{dy}{dx}=\frac{9-24}{16-24}=\frac{-15}{-8}=\frac{15}{8}$ Equation: $\frac{y-3}{x-4}=\frac{15}{8}$, $8y=15x-36$	
9.	$\sqrt{y+6}-\sqrt{y+3}=\sqrt{2y+5}$, square both sides $y+6+y+3-2\sqrt{(y+6)(y+3)}=2y+5$, $\sqrt{y^2+9y+18}=2$ $y^2+9y+18=4$, $y^2+9y+14=0$, $(y+7)(y+2)=0$, $y=-7$, $y=-2$ Check: $\sqrt{(y+6)}-\sqrt{(y+3)}=\sqrt{(2y+5)}$ $y=-7$, L.H.S $=\sqrt{-1}-\sqrt{-4}\neq\sqrt{-9}$ $\neq$ R.H.S $x=-2$, L.H.S $=\sqrt{4}-\sqrt{1}=\sqrt{1}$ = R.H.S Therefore, $x=-2$, is the only correct solution.	
b)	$x+y=10$ ……….(i) $y=10-x$ $\frac{y^2}{30-x}=1$, $y^2=30-x$ ……..(ii) $(10-x)^2=30-x$ $x^2-19x+70=0$, $(x-5)(x-14)=0$ $x=5,\ x=14$ corresponding values are $y=5,\ y=-4$ respectively.	
10 a)	If $z=1+2i$ is a root, then $\bar{z}=1-2i$ the conjugate root is the other root. Sum of roots is $1+2i+1-2i=2$ and product is $(1+2i)(1-2i)=5$ thus the equation is $z^2-2z+5=0$.	

	$$\begin{array}{r} 2z+3 \\ z^2-2z+5\overline{)2z^3-z^2+4z+15} \\ 2z^3-4z^2+10z \\ \hline 3z^2-6z+15 \\ 3z^2-6z+15 \end{array}$$ By long division Thus $2z+3=0$ gives us $z=-\frac{3}{2}$ Other roots are $2+i,\ -\frac{3}{2}$	
b)	$(1+2i)^3+a(1+2i)+b=0$ $-11-2i+a+2ai+b=0,\ (-11+a+b)+(2a-2)i=0$ Thus, $2a-2=0,\ a=1$ $(-11+1+b)=0,\ b=10$	
11 a)	$y+\partial y=\frac{1}{\sqrt{x+\partial x}},\ \partial y=\frac{1}{\left(\sqrt{x+\partial x}\right)}-\frac{1}{\sqrt{x}}$ $\partial y=\frac{\sqrt{x}-\sqrt{x+\partial x}}{\sqrt{x}\left(\sqrt{x+\partial x}\right)},\ \partial y=\frac{\sqrt{x}-\sqrt{x+\partial x}}{\sqrt{x}\left(\sqrt{x+\partial x}\right)}\times\frac{\left(\sqrt{x}+\sqrt{x+\partial x}\right)}{\left(\sqrt{x}+\sqrt{x+\partial x}\right)}$ $\partial y=\frac{x-x-\partial x}{\sqrt{x}\left(\sqrt{x+\partial x}\right)\left(\sqrt{x}+\sqrt{x+\partial x}\right)},\ \frac{\partial y}{\partial x}=\frac{-1}{\sqrt{x}\left(\sqrt{x+\partial x}\right)\left(\sqrt{x}+\sqrt{x+\partial x}\right)}$ As $\partial x\to 0, \frac{\partial y}{\partial x}\to\frac{dy}{dx}$, so $\frac{dy}{dx}=\frac{-1}{2x^{3/2}}$	
b)	Vol of hemisphere is $V=\frac{2}{3}\pi r^3$ and the rate $\frac{dV}{dt}=\frac{\pi r^3}{90}cm^3s^{-1}$ $V=\pi\left(rh^2-\frac{1}{3}h^3\right),\ \frac{dV}{dh}=\pi\left(2rh-h^2\right)$ So, $\frac{dh}{dt}=\frac{dh}{dV}\times\frac{dV}{dt},\ \frac{dh}{dt}=\frac{1}{\pi\left(2rh-h^2\right)}\times\frac{\pi r^3}{90}$ for $r=6,\ h=3$ $\frac{dh}{dt}=\frac{216}{90\times 27}=\frac{4}{45}cm\,s^{-1}$	
12 a)	From the L.H.S $\tan\frac{1}{2}(A-B)+\tan\frac{1}{2}(A+\mathbf{B})=\frac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A-B)}+\frac{\sin\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A+B)}$ $=\frac{\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)+\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}$ $=\frac{\frac{1}{2}(\sin A-\sin B)+\frac{1}{2}(\sin A+\sin B)}{\frac{1}{2}(\cos A+\cos B)}$ $=\frac{2\sin A}{\cos A+\cos B}$ as the R.H.S	

b)	From the L.H.S, $\tan(\alpha+\beta+\gamma)=\tan((\alpha+\beta)+\gamma)$ $=\dfrac{\tan(\alpha+\beta)+\tan\gamma}{1-\tan(\alpha+\beta)\tan\gamma}$ $=\dfrac{\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}+\tan\gamma}{1-\left[\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\right]\tan\gamma}$ $=\dfrac{\tan\alpha+\tan\beta+\tan\gamma-\tan\alpha\tan\beta\tan\gamma}{1-\tan\alpha\tan\gamma-\tan\alpha\tan\beta-\tan\beta\tan\gamma}$ $\tan(\alpha+\beta+\gamma)=\dfrac{p+q+r-pqr}{1-pr-rq-pq}$ $\tan^{-1}\dfrac{1}{3}+\tan^{-1}\dfrac{1}{4}+\tan^{-1}\dfrac{2}{9}=\dfrac{\pi}{4}$ $\alpha+\beta+\gamma=\tan^{-1}\left(\dfrac{p+q+r-pqr}{1-pr-rq-pq}\right)$, $=\tan^{-1}\left(\dfrac{\frac{1}{3}+\frac{1}{4}+\frac{2}{9}-\frac{1}{3}\times\frac{1}{4}\times\frac{2}{9}}{1-\frac{1}{3}\times\frac{1}{4}-\frac{1}{3}\times\frac{2}{9}-\frac{2}{9}\times\frac{1}{4}}\right)=\tan^{-1}\dfrac{\frac{85}{108}}{\frac{85}{108}}$ $=\tan^{-1}1=\dfrac{\pi}{4}$	
13 a)	$y=72x+3x^2-2x^3$ $\dfrac{dy}{dx}=72+6x-6x^2$, for max $\dfrac{dy}{dx}=0$ So, $72+6x-6x^2=0$, thus $x^2-x-12=0$ $(x-4)(x+3)=0$ so $x=4,\ x\neq-3$ For $x=4$, $y=288+48-128=208$	
b)	For points of integration to get the limits of integration, $12=x(8-x)$, $x^2-8x+12=0$ $(x-6)(x-2)=0$ so $x=6,\ x=2$, points are $(2,\ 12)$, & $(6,\ 12)$  $A=\int_2^6\left(8x-x^2-12\right)dx$ $=\left[4x^2-\dfrac{x^3}{3}-12x\right]_2^6$	

	$=(144-72-72)-\left(16-\frac{8}{3}-24\right)=\frac{32}{3}\,sq.units$	
14	Turning points, $\frac{dy}{dx}=\frac{(x^2-3x+2)(8x-10)-(4x^2-10x+7)(2x-3)}{(x^2-3x+2)^2}$ For $\frac{dy}{dx}=0$, we have, $8x^3-10x^2-24x^2+30x+16x-20-8x^3+12x^2+20x^2-30x-14x+21=0$ To get, $-2x^2+2x+1=0$ $x=\frac{-2\pm\sqrt{2^2-(-4\times2\times1)}}{-4}=\frac{1}{2}\pm\frac{\sqrt{3}}{2}$ so $x=-0.366,\ x=1.366$ Thus, $(1.366,\ -3.464)\max$, $(-0.366,\ 3.464)\min$ Intercepts: $x=0,\ y=\frac{7}{2}$ so $(0,\ 3.5)$ $y=0,\ 4x^2-10x+7=0$ has no real roots since $(-10)^2-(16\times7)<0$ Vertical asymptotes: $x=1,\ x=2$ Horizontal asymptote, $y=4$, $4x^2-12x+8=4x^2-10x+7$, $x=\frac{1}{2}$, thus curve crosses horizontal asymptote at $(0.5,\ 4)$ 	
15 a)	$\mathbf{AP}:\mathbf{PB}=-5:3$ *or* $5:-3$ $\Rightarrow +3\,\mathbf{AP}=-5\mathbf{PB}$, $3(\mathbf{AO}+\mathbf{OP})=-5(\mathbf{PO}+\mathbf{OB})$ $3\mathbf{OP}-5\mathbf{OP}=-5\mathbf{OB}+3\mathbf{OA}$ $-2\mathbf{OP}=-5\begin{pmatrix}-3\\2\\8\end{pmatrix}+3\begin{pmatrix}2\\4\\6\end{pmatrix}$, $\mathbf{OP}=-\frac{1}{2}\left[\begin{pmatrix}15\\-10\\-40\end{pmatrix}+\begin{pmatrix}6\\12\\18\end{pmatrix}\right]$	

	$\mathbf{OP}=\begin{pmatrix}-21/2\\-1\\11\end{pmatrix}$ so $\mathbf{OP}=\frac{-21}{2}\mathbf{i}-\mathbf{j}+11\mathbf{k}$.	
b)	Let $\mathbf{u}=\begin{pmatrix}1\\-1\\-2\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}3\\7\\-6\end{pmatrix}$ and let $\mathbf{a}=\begin{pmatrix}x\\y\\z\end{pmatrix}$ be the normal to $\mathbf{u}$ & $\mathbf{v}$. $\Rightarrow x-y-2z=0\ \ldots$(i) and $3x+7y-6z=0\ \ldots$(ii) Using (i) and (ii) $7x-7y-14z=0$, $3x+7y-6z=0$, to get $x=2z$, thus from (i) $y=0$ So $\mathbf{a}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}=2z\mathbf{i}+0\mathbf{j}+z\mathbf{k}=z(2\mathbf{i}+\mathbf{k})$ $\Rightarrow$ normal vector is $2\mathbf{i}+\mathbf{k}$ But $ax+by+cz=ax_o+by_o+cz_o$ $\Rightarrow\ 2x+0+z=0+0+0$ $\therefore\ 2x+z=0$.	
16.	$\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1$ $(a^2m^2+b^2)x^2+2a^2cmx+a^2c^2-a^2b^2=0$ For a tangent, $B^2-4AC=0$, so $4a^4c^2m^2-4a^2(a^2m^2+b^2)(c^2-b^2)=0$ for $4a^2b^2c^2\neq 0$, then $a^2c^2m^2-a^2c^2m^2+\ a^2m^2b^2-b^2c^2+b^4=0$ $c^2=a^2m^2+b^2$ For $\frac{x^2}{16}+\frac{y^2}{9}=1,\ \ a^2=16,\ \ b^2=9$ $c^2=16m^2+9$, and for $(-3,\ 3)$, $-3=3m+c$, so, $c=-3-3m$ $16m^2+9=9+\mathbf{18m}+\mathbf{9m^2}$, $m(7m-18)=0$ thus $m=0,\ \ \& \ m=\frac{18}{7}$, similarly, $c=-3,\ \ \& \ c=-\frac{75}{7}$ Equations are $y=-3$ and $7y=18x-75$	

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