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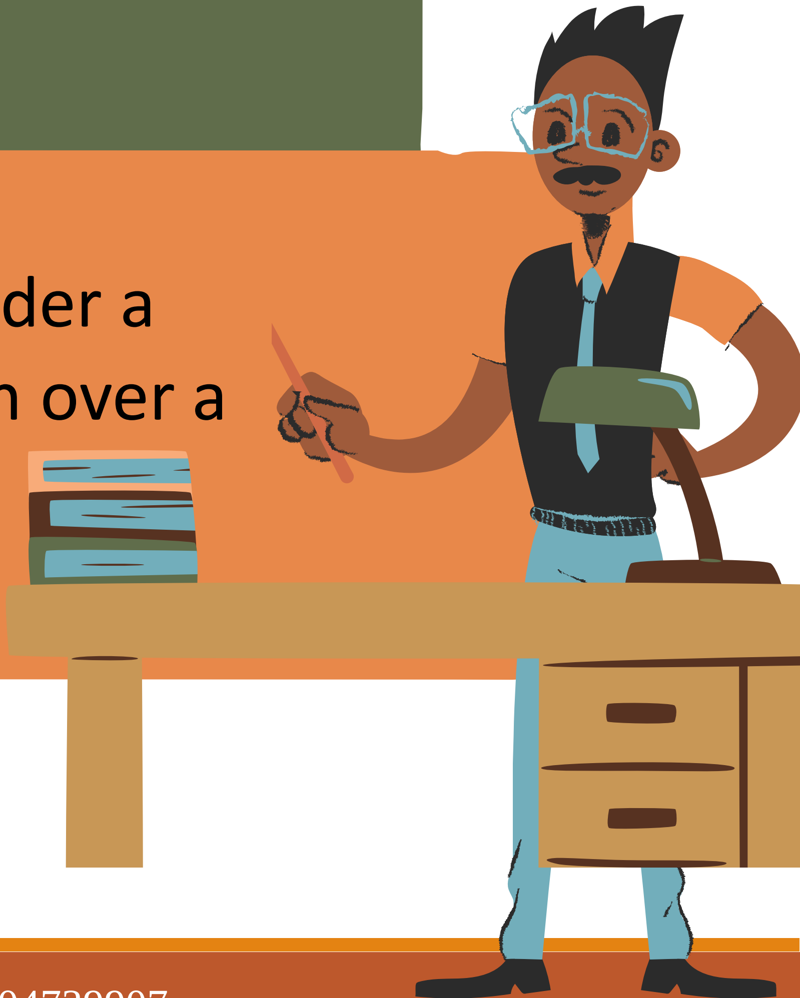
A'LEVEL APPLIED MATHEMATICS

TRAPEZIUM RULE

SUITABLE FOR S.5 AND S.6

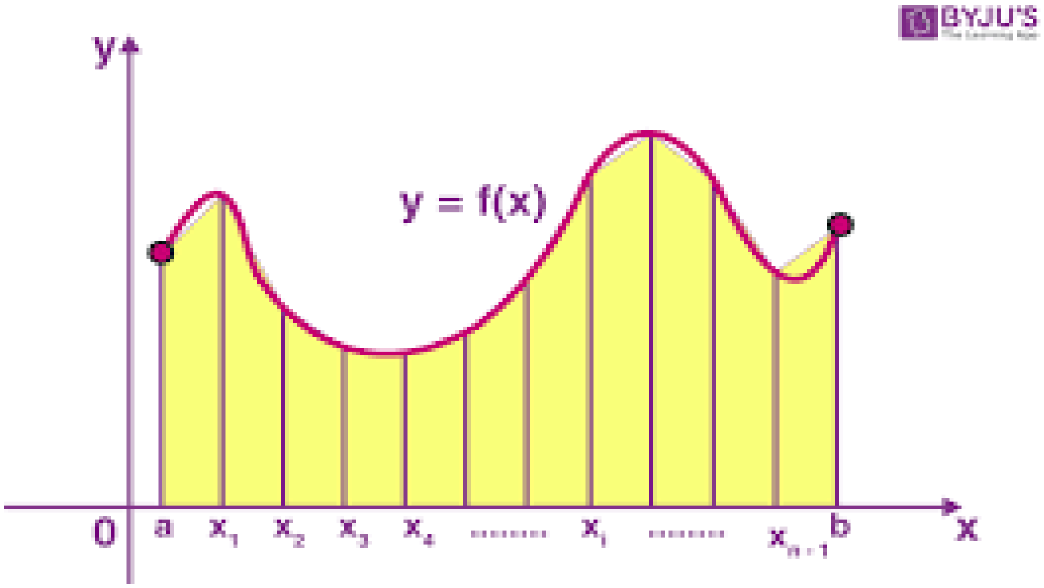
TRAPEZIUM RULE

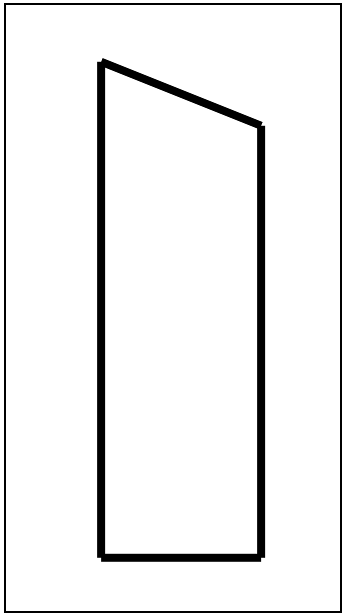
The trapezium rule is used for estimating an integral area under a curve of a continuous function over a given interval
[a, b]



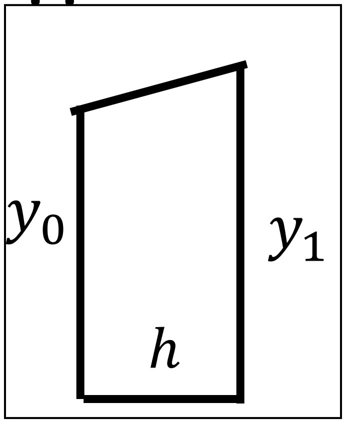
For example if y = f(x) , then $A = \int_a^b y\,dx$

If we are to approximate that area using the trapezium rule, then the area under the curve is divided into a number of strips of equal width. The top edge of each strip is replaced by a straight line so the strips become trapezia.





The total area of the trapezia gives an approximation to the area under the curve.



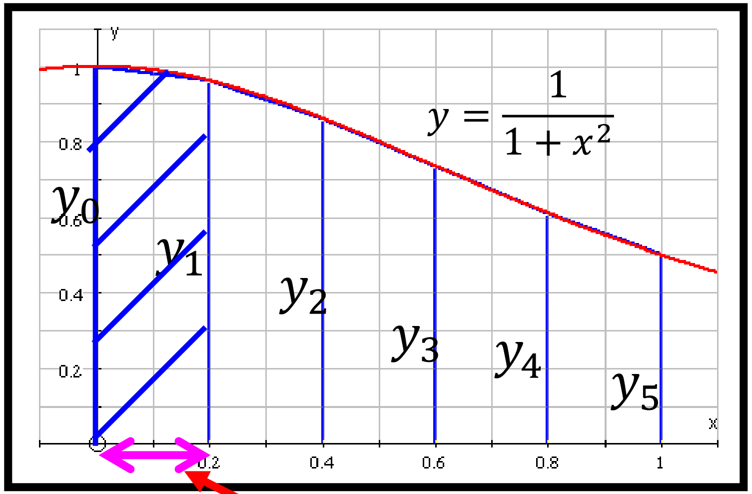
The formula for the area of the trapezium above is:

$$A = \frac{1}{2}(y_0 + y_1) \times h$$

h seems a strange letter to use for width but it is always used in the trapezium rule

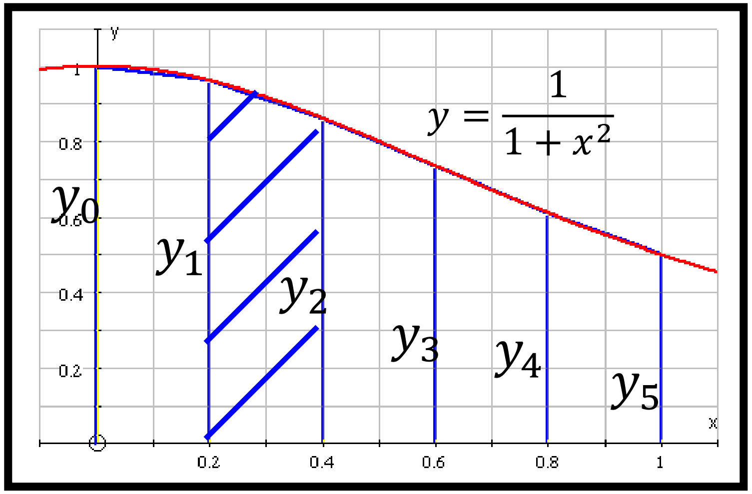
Example 1 **Use the trapezium rule with 5strips to evaluate;**

$$\int_0^1 \frac{1}{1+x^2}\,dx$$



The parallel sides of the trapezia are the y-values of the function.

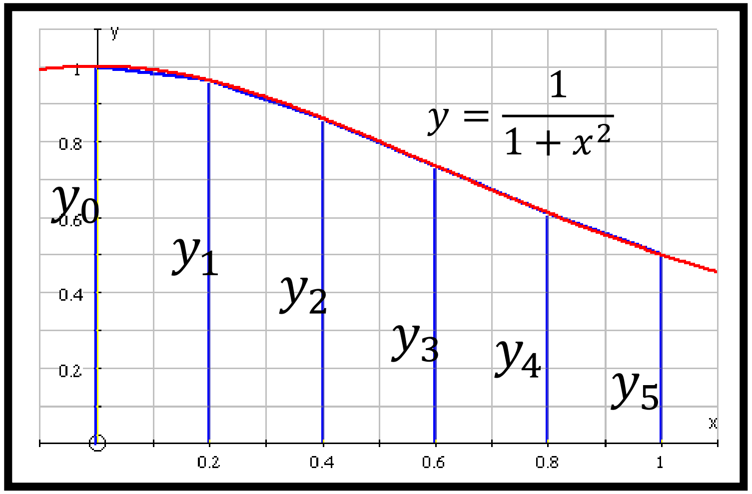
The area of the 1st trapezium $= \frac{1}{2}(y_0 + y_1) \times h$



The area of the 1st trapezium $= \frac{1}{2}(y_0 + y_1) \times h$

The area of the 2nd trapezium $= \frac{1}{2}(y_1 + y_2) \times h$

etc.



The total area, A

$$\mathbf{A} \approx \frac{1}{2}(y_0 + y_1) \times h + \frac{1}{2}(y_1 + y_2) \times h + \ldots + \frac{1}{2}(y_4 + y_5) \times h$$

$$\approx \frac{1}{2} \times h(y_0 + y_1 + y_1 + y_2 + y_2 + \ldots + y_5)$$

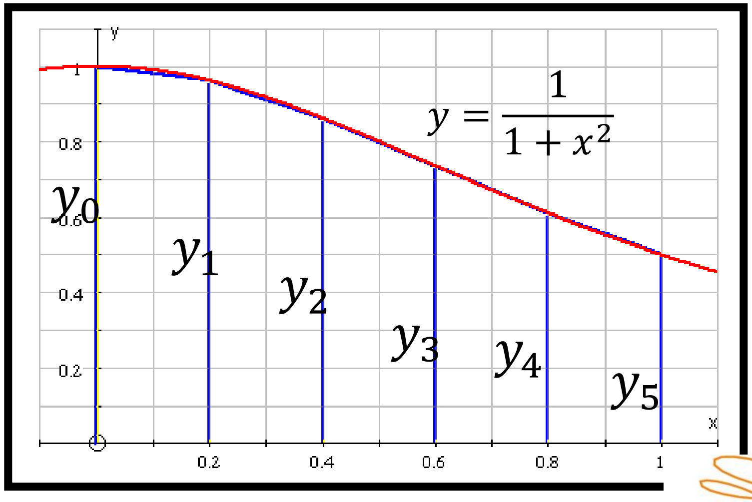
Since y_1 is the side of 2 trapezia, it occurs twice in the formula. The same is true for y_2 to y_4 .

So, $\int_0^1 \frac{1}{1+x^2}\,dx \approx \frac{1}{2} \times h(y_0 + 2(y_1 + y_2 + y_3 + y_4) + y_5)$

$$\int_0^1 \frac{1}{1+x^2}dx = \frac{h}{2}(y_0 + y_5 + 2(y_1 + y_2 + y_3 + y_4))$$

$$h = \frac{b-a}{n}, \qquad h = \frac{1-0}{5}, \ h = 0.2$$

n	x_n	y_o, y_5	$y_1,\ldots,y_4$
0	0	1	
1	0.2		0.9615
2	0.4		0.8621
3	0.6		0.7353
4	0.8		0.6098
5	1.0	0.5	
Total		1.5	3.1687



So, $$\int_0^1 \frac{1}{1+x^2}\,dx \approx \frac{1}{2} \times 0\cdot 2(y_0 + y_5 + 2(y_1 + y_2 + y_3 + y_4))$$

$$\approx 0\cdot 1(1.5 + 2(3.1687))$$

$$\approx 0\cdot 784 \rightleftarrows (\text{ 3.s.f. })$$

The general formula for the trapezium rule is

$$\int_a^b y\,dx = \frac{h}{2}(y_0 + 2(y_1 + y_2 + y_3 + \ldots + y_{n-1}) + y_n)$$

where n is the number of strips.

The y-values are called ordinates.

There is always 1 more ordinate than the number of strips.

The width, h, of each strip is given by

$$h = \frac{b-a}{n}$$

To improve the accuracy we just need to use more strips.

e.g.2 Use the trapezium rule to evaluate $\int_0^{\pi} \sin x \, dx$ using 6 strips and giving the answer to 2 d. p.

Solution:

$$\int_a^b y dx = \frac{h}{2}(y_0 + 2(y_1 + y_2 + y_3 + y_4 + y_5) + y_6)$$

$$h = \frac{b-a}{n}$$

$$\Rightarrow h = \frac{\pi - 0}{6} = \frac{\pi}{6}$$

$$\int_0^{\pi} \sin x \, dx = \frac{h}{2}(y_0 + y_6 + 2(y_1 + y_2 + y_3 + y_4 + y_5))$$

Radians!

To make sure that we have the required accuracy we must use at least 1 more d. p. than the answer requires. It is better still to store the values in the calculator's memories.

$$h = \frac{\pi}{6}$$

$$\int_0^{\pi} \sin x \, dx = \frac{h}{2}(y_0 + y_6 + 2(y_1 + y_2 + y_3 + y_4 + y_5))$$

n	x_n	y_0, y_5	$y_1, \cdots, y_4$
0	0	0	
1	$\frac{\pi}{6}$		0.5
2	$\frac{2\pi}{6}$		0.8660
3	$\frac{3\pi}{6}$		1
4	$\frac{4\pi}{6}$		0.8660
5	$\frac{5\pi}{6}$		0.5
6	π	0	
Total		0	3.232

$$\int_0^{\pi} \sin x \, dx \approx \frac{h}{2}(y_0 + y_6 + 2(y_1 + y_2 + y_3 + y_4 + y_5))$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sin x \, dx \approx \frac{1}{2} x \frac{\pi}{6}(0 + 2(3.732))$$

$$\approx 1 \cdot 95(\text{ 2 d. p. })$$

The exact value of $\int_0^{\pi} \sin x \, dx$ is 2.

The percentage error in our answer is found as follows:

$$\text{Percentage error} = \frac{\text{error}}{\text{exact value}} \times 100$$

(where, error = |exact value – the approximate value|).

$$\approx \frac{|2 - 1 \cdot 95|}{2} \times 100$$

$$= 2 \cdot 5 \ \%$$

SUMMARY

- The trapezium law for estimating an area is

$$\int_a^b y\,dx = \frac{h}{2}(y_0 + 2(y_1 + y_2 + y_3 + \ldots + y_{n-1}) + y_n)$$

where n is the number of strips.

- The width, h, of each strip is given by $h = \dfrac{b-a}{n}$

- The number of ordinates is 1 more than the number of strips.
- The trapezium law underestimates the area if the tops of the trapezia lie under the curve and overestimates it if the tops lie above the curve.

- The accuracy can be improved by increasing n.

Exercises

Use the trapezium rule to estimate the areas given by the integrals, giving the answers to 3 s. f.

1. $\int_0^1 \sqrt{(1+x)}\,dx$ **using 4 strips**

How can your answer be improved?

2. $\int_0^{\frac{\pi}{2}} \cos x \, dx$ **using 4 ordinates**

Find the approximate percentage error in your answer, given that the exact value is 1**.**

Solutions 1.

$$\int_0^1 \sqrt{(1+x)}\,dx = \frac{h}{2}(y_0 + y_4 + 2(y_1 + y_2 + y_3))$$

n	x_n	y_o, y_4	$y_1,\cdots,y_3$
0	0	1	
1	0.25		1.118
2	0.5		1.225
3	0.75		1.323
4	1	1.414	
Total		2.414	3.666

Solutions 1.

$n = 4,$

$h = 0 \cdot 25$

$$A \approx \frac{h}{2}(y_0 + y_4 + 2(y_1 + y_2 + y_3))$$

$$A \approx \frac{h}{2}(2.414 + 2(3.666))$$

$$\approx 1 \cdot 22(\text{ 3 s. f. })$$

The answer can be improved by using more strips.

Solutions

2. $$\int_0^{\frac{\pi}{2}} \cos x\, dx = \frac{h}{2}(y_0 + y_3 + 2(y_1 + y_2))$$

n	x_n	y_o, y_3	$y_1,\ldots,y_2$
0	0	1	
1	$\frac{\pi}{6}$		0.8660
2	$\frac{2\pi}{6}$		0.5
3	$\frac{\pi}{2}$	0	
Total		1	1.3660

Solutions

2. $\int_0^{\frac{\pi}{2}} \cos x \, dx$ $\quad n = 3,$ $\quad h = \frac{\pi}{6}$

using 4 ordinates

$$A \approx \frac{h}{2}(y_0 + y_3 + 2(y_1 + y_2))$$

$$A \approx \frac{h}{2}(1 + 2(1.3660))$$

$$\approx 0 \cdot 977(\text{ 3 s. f. })$$

The exact value is 1, so the percentage error

$$\approx \frac{1 - 0 \cdot 977}{1} \times 100 \quad = 2 \cdot 3 \ \%$$