8. (a) Find the locus of a point $P(x,y)$ which moves such that its distance from $(-1,2)$ is twice its distance from the origin. (03 marks)

(b) Derive the condition for the points $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ to be collinear. ~~03~~ (03 marks)

(c) (i) Translate the line $y=3x+1$ by a factor of $(2,3)$ (2 marks)

(ii) Dilate $y=3x-1$ by a factor $(2,5)$ (02 marks)

9. (a) Evaluate the following integrals

(i) $$\int \frac{\ln x}{x\sqrt{1-4\ln x-\ln^2 x}}\,dx$$ (04 marks)

(ii) $$\int \frac{(\sin^{-1}x)^2}{\sqrt{1-x^2}}\,dx$$ (02 marks)

(b) Given two curves represented by $y=2-x^2$ and $y^3=x^2$. ~~(02 marks)~~

(i) sketch to show the region S enclosed by the curves (02 marks)

(ii) Evaluate the area S enclosed between the curves. (02 marks)

10 (a) Find the derivative of the following

(i) $y=\ln(\sec x+\tan x)$ (ii) $y=\sinh^{-1}(\tan x)$ (03 marks)

(b) By using first principle of differentiation, differentiate $y=\frac{1}{x+1}$ (03 marks)

(c) By using Maclaurin's theorem expand $\tan^{-1}x$ in ascending powers of x as far as the term in x^3. Use the resulting expression to evaluate $\tan^{-1}0.1$ correct to four decimal places. (04 marks)

MARKING SCHEME

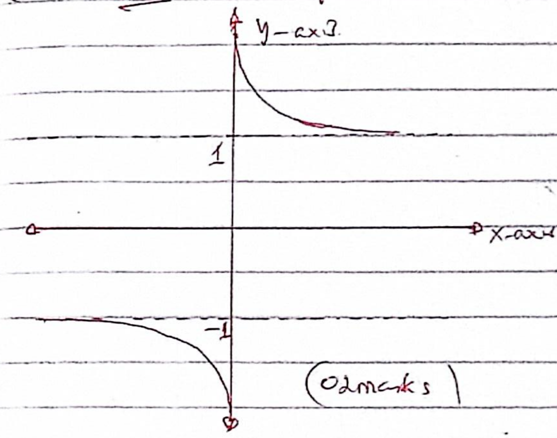
1 (a) (i) 229325.6169 (3 marks)

(ii) 21.6440 (3 marks)

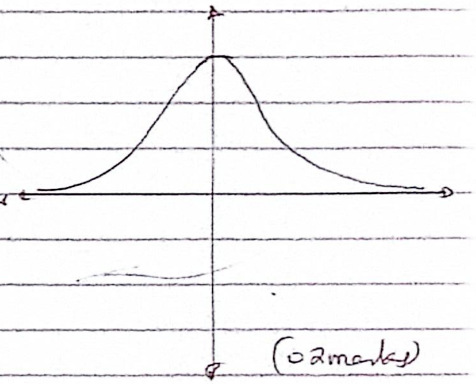
(iii) $|A| = 45$ (02 marks)

$$A^{-1} = \begin{bmatrix} -3/15 & 4/9 & 13/15 \\ 2/15 & -1/9 & 8/45 \\ 1/3 & -1/9 & -2/9 \end{bmatrix}$$ (02 marks)

(02 marks) 2 (a) Graph of $\coth x$ Graph of $\operatorname{sech} x$



(02 marks)



(02 marks)

(b) Given $\tanh^{-1}x/3+\tanh^{-1}x=\tanh^{-1}1/4$

Required value of x

$$\tanh\left[\tanh^{-1}x/3+\tanh^{-1}x\right]=\tanh\left[\tanh^{-1}1/4\right]$$

$$\frac{\tanh(\tanh^{-1}x/3)+\tanh(\tanh^{-1}x)}{1+\tanh(\tanh^{-1}x/3)\cdot\tanh(\tanh^{-1}x)}=1/4$$ (01 mark)

$$\frac{x/3+x}{1+x^2/3}=1/4$$

$$\frac{x+3x}{3+x^2}=1/4$$

$x^2-16x+3=0$ (01 mark)

Solving quadratic $x=15.81$ or 0.1898 (0.5 mark)

The value satisfying the eqn is 0.1898 (0.5 marks)

(c) Required to show

$$\frac{d}{dx}\left\{\coth^{-1}(\sin x)\right\} = \sec x$$

let $y = \coth^{-1}(\sin x)$

$\coth y = \sin x$ (01mark)

$-\operatorname{cosech}^2 y \frac{dy}{dx} = \cos x$ (01mark)

$\frac{dy}{dx} = \frac{\cos x}{-\operatorname{cosech}^2 y} = -\frac{\cos x}{\coth^2 y - 1}$ (01 mark)

$\frac{dy}{dx} = -\frac{\cos x}{\sin^2 x - 1} = \frac{\cos x}{1 - \sin^2 x}$ (01mark)

$= \frac{\cos x}{\cos^2 x} = \frac{1}{\cos x}$ (01mark)

$= \sec x$

$\therefore \frac{d}{dx}\left[\coth^{-1}(\sin x)\right] = \sec x$

3. let x be the number of belts of type A manufactured (0.5mark)

y be the number of belts of type B manufactured (0.5mark)

Objective function $f(x,y) = 0.4x + 0.3y$ (0.5mark)

Linear constraints

$x + y \leq 800$ (0.5mark)

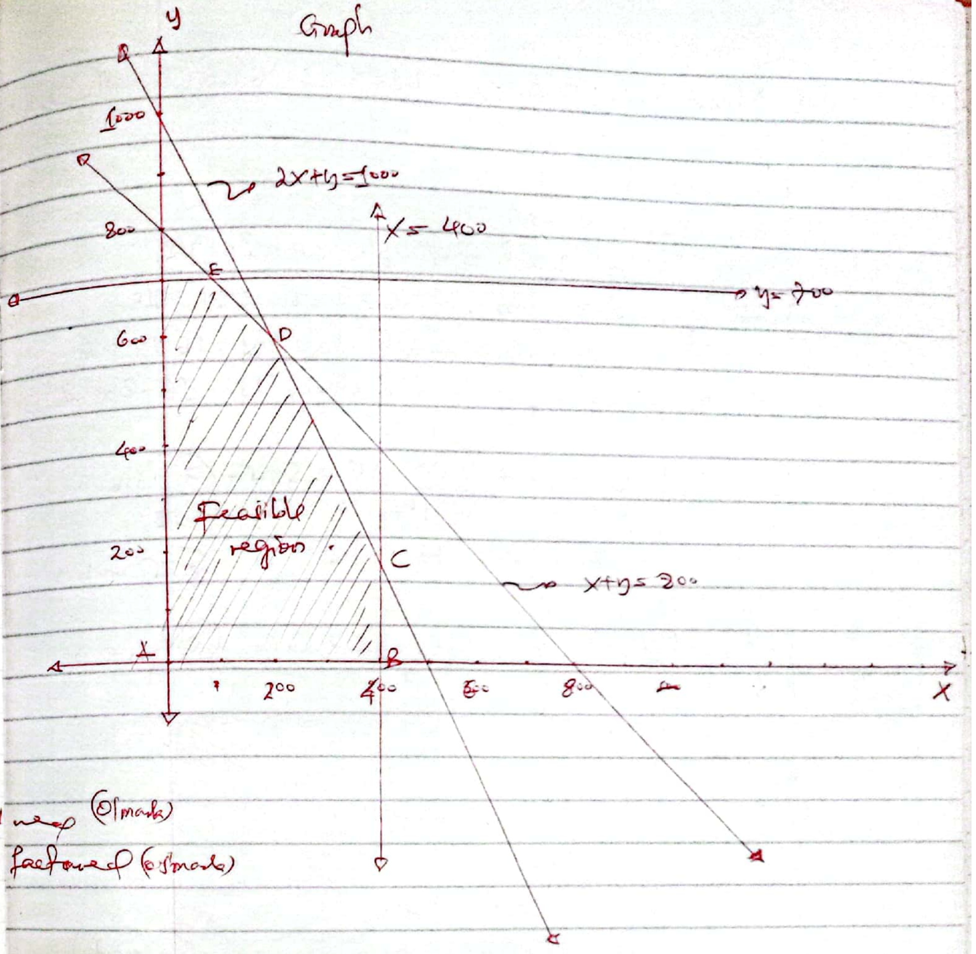
$2x + y \leq 800$ (0.5mark)

$x \leq 400$ (0.5mark)

$y \leq 700$ (0.5mark)

$x, y \geq 0$ (0.5 marks)

Graph



4. The distribution table

C. Interval	x	f	c.f	fx
30-39	34.5	6	6	207
40-49	44.5	12	18	534
50-59	54.5	14	32	763
60-69	64.5	16	48	1032
70-79	74.5	8	56	596
80-89	84.5	6	62	507
90-99	94.5	6	68	567
Total.		68		4206

Column for x, $f(x)$ @ 0.5mark, Total = 1mark

(i) Mean $\bar{x} = \frac{\sum fx}{N} = \frac{4206}{68} = 61.85$ (1 mark)

For mode

Modal class = 60–69

Lower limit of modal class, $L = 60 - 0.5 = 59.5$

Width of modal class, $C = 69.5 - 59.5 = 10$

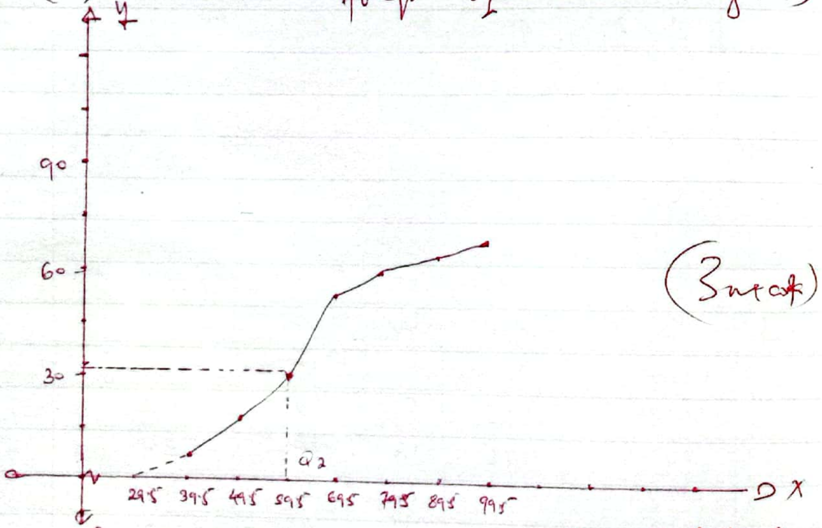
Excess frequency before modal class, $t_1 = 16 - 14 = 2$

Excess frequency after modal class, $t_2 = 16 - 8 = 8$

$\therefore$ Mode $= L + \left(\frac{t_1}{t_1 + t_2}\right)C = 59.5 + \left(\frac{2}{2+8}\right)10$

$= \cancel{46}\ 61.5$ (0.5 marks)

(ii) Cumulative frequency Curve (ogive)



(3 mark)

Upper Real limits

from the ogive, the estimated median is $59.5 + 1.5 = 61$ (1 mark)

(iii) Quartiles are given by

$$Q_i = L + \left(\frac{\frac{iN}{4} - f_c}{f_Q}\right)C \quad \text{(0.5 marks)}$$

Lower Quartile

Position of lower quartile $= N/4 = 68/4 = 17$

Lower quartile class = 40–49, f

$f_c = 6$

$f_Q = 12$

$c = 10$

$L = 39.5$

(0.5 mark)

$Q_1 = 39.5 + \left(\frac{17 - 48}{12}\right)10 = 48.67$ (0.5 marks)

Upper Quartile

Position of upper quartile $= \frac{3N}{4} = \frac{3 \times 68}{4} = 57$

Upper quartile class = 70–79

$L = 69.5$

$f_c = 48$

$f_Q = 8$

$C = 10$

$Q_3 = 69.5 + \left(\frac{51 - 48}{8}\right)10 = 73.25$ (1 mark)

Semi-interquartile range $= \frac{1}{2}$ (upper quartile − lower quartile)

$= \frac{1}{2}[73.25 - 48.67]$

$= 12.29$ (0.5 marks)

5 (a)(i) $[A\cap(A'\cup B)]\cup[B\cap(A'\cup B')]$

$= [(A\cap A')\cup(A\cap B)]\cup[(B\cap A')\cup(B\cap B')]$ Distributive law

$= [\varnothing\cup(A\cap B)]\cup[(B\cap A')\cup\varnothing]$ Compliment law

$= (A\cap B)\cup(B\cap A')$ Identity law

$= (A\cap B)\cup(A'\cap B)$ Commutative law

$= (A\cup A')\cap B$ Distributive law

$= U\cap B$ Compliment law

$= \underline{B}$ Identity law (02 marks)

(ii) $(A\cap B')\cup(A'\cup B')$

$= [A\cup(A'\cup B')]\cap[B'\cup(A'\cup B')]$ Distributive law

$= [(A\cup A')\cup B']\cap[A'\cup(B'\cup B')]$ Associative law

$= (U\cup B')\cap(A'\cup B')$ Compliment and Identity law

$= B'\cap(A'\cup B')$ Identity law

$= (\varnothing\cup B')\cap(A'\cup B')$ Distributive law

$= \cancel{\varnothing\cup B'}$

$= (\varnothing\cap A')\cup B'$ Identity law

$= B'$ Identity law

(b) Given $A = \{x: x\in R;\ 3<x\leq 6\}$

$B = \{x: x\in R,\ -4<x\leq 4\}$

$A' = \{x: x\in R;\ x\leq 3 \text{ and } x>6\}$

$B' = \{x: x\in R;\ x\leq -4 \text{ and } x>4\}$

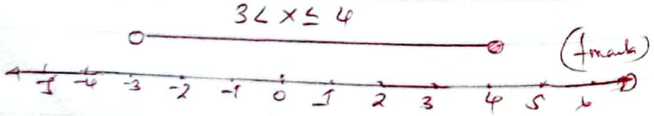
$A\cup B = \{x: x\in R;\ -4<x\leq 6\}$

$A\cap B = \{x: x\in R;\ 3<x\leq 4\}$ (3 marks)

$A'\cap(A\cup B) = \{x: x\in R;\ -4<x\leq 3\}$

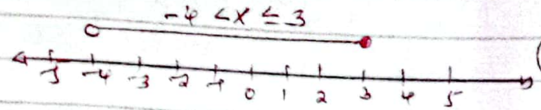
$A-B = \{x: x\in R;\ 4<x\leq 6\}$

(i) $A\cap B$



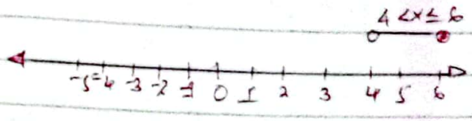
(1 mark)

(ii) $A'\cap(A\cup B)$



(1 mark)

(iii) $A-B$



(1 mark)

6. (a)(i) Given $f(x) = \sqrt{2x-4}$, $x\geq 2$ and

$g(x) = \frac{x^2+4}{2}$, $x\geq 0$

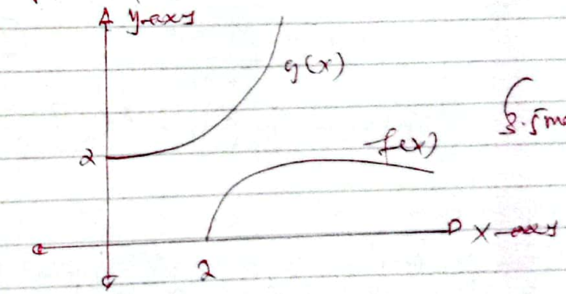
For inverse function $f\circ g(x) = g\circ f(x) = x$

$f\circ g(x) = \sqrt{\frac{2(x^2+4)}{2} - 4} = x$ (1 mark)

$g\circ f(x) = \frac{(\sqrt{2x-4})^2+4}{2} = x$ (1 mark)

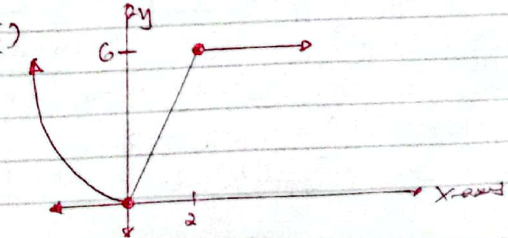
Hence $f(x)$ and $g(x)$ are inverse function of one another. (0.5mark)

(ii) Graphs of $f(x)$ and $f^{-1}(x)$



(3.5 marks)

(b) (i)



(02 marks)

(ii) Domain = $\{x : x \in R\}$ (0.5 mark)

Range = $\{y : y \in R;\ y \geq 0\}$ (0.5 mark)

(iii) $f \circ f \circ f(-1) = f \circ f(1) = f(3) = 6$ (1 mark)

7.(a)(i) Required to show that

$$x_{n+1} = \frac{1}{2}\left[x_n + \frac{A}{x_n}\right]$$

for square root of A

let $x = \sqrt{A}$ so that $x^2 = A$

$x^2 - A = 0$ (0.5 mark)

$f(x_n) = x_n^2 - A$ (0.5 mark)

$f'(x) = 2x_n$ (0.5 mark)

Then from $x_{n+1} = x_n - \dfrac{f(x)}{f'(x)}$ (0.5 mark)

$x_{n+1} = x_n - \dfrac{[x_n^2 - A]}{2x_n}$ (0.5 mark)

$x_{n+1} = \dfrac{x_n^2 + A}{2x_n}$ (~~1 mark~~)

$x_{n+1} = \frac{1}{2}[x_n + A/x_n]$ (1 mark)

(ii) Given $x^3 - 3x - 20$

for existence of a root in the interval $(3, 4)$, $f(3)\,f(4) < 0$

$f(x) = x^3 - 3x - 20$ (0.5 mark)

$f(3) = 3^3 - 3(3) - 20 = -2$ (0.5 mark)

~~Now, $f(3) \times f(4) = -32 \times -2$~~

$f(4) = 4^3 - 3(4) - 20 = +32$ (0.5 mark)

Now $f(3) \times f(4) = 32 \times -2 = -64$ (0.5 mark)

Hence there is a root in the interval $(3, 4)$

Root approximation by secant formula.

$$x_{n+2} = x_{n+1} - \left[\frac{x_{n+1} - x_n}{f(x_{n+1}) - f(x_n)}\right] f(x_{n+1})$$ (0.5 mark)

first iteration; $n = 0$, $x_0 = 3$ and $x_1 = 4$

$f(x_0) = 3^3 - (3 \times 3) - 20 = -2$ and $f(x_1) = 32$ (0.5 mark)

$x_2 = 4 - 32\left(\frac{1}{34}\right) = 3.0588$

Second iteration $n = 1$, $f(x_2) = -0.5575$

$x_3 = 3.0588 - \left[\dfrac{3.0588 - 4}{-0.5575 - 32}\right] \times (-0.5575) = 3.0749$ (0.5 mark)

Third iteration $n = 2$, $f(x_3) = -0.15149$

$x_4 = 3.0749 - \left[\dfrac{3.0749 - 3.0588}{-0.15149 + 0.5575}\right] \times (-0.15149)$

$= 3.0809$ (0.5 mark)

Fourth iteration $n = 3$, $f(x_4) = 0.00103$

$x_5 = 3.0809 - \left[\dfrac{3.0809 - 3.0749}{0.00103 + 0.15149}\right](0.00103)$

$= 3.08086$ (0.5 mark)

Hence the approximate root is 3.0809 to four decimal places.

(b) Using Simpson's rule to find $I = \int_0^1 \frac{1}{x+1}\,dx$

Using $n=4$, $a=0$, and $b=1$

width of intervals, $h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$ (0.5 marks)

Table of values.

n	x	y_n	y_0+y_4	odd ordinates	even ordinates
0	0.00	1.000	1.000		
1	0.25	0.8000		0.8000	
2	0.50	0.6667			0.6667
3	0.75	0.5714		0.5714	
4	1.00	0.5000	0.5000		
Total			1.5	1.3714	0.6667

(1.5 marks)

$I = \frac{h}{3}\left[(y_0+y_4) + 4\sum \text{odd ordinates} + 2\sum \text{even ordinates}\right]$ (0.5 marks)

$I = \frac{0.25}{3}\left[1.5 + 4(1.3714) + 2(0.6667)\right] = 0.693$ (0.5 marks)

8. (a) Given distance of $P(x,y)$ from $(-1,2)$ = 2 distance of $P(x,y)$ from origin

By distance formula.

$\sqrt{(x+1)^2 + (y-2)^2} = 2\sqrt{x^2+y^2}$

Squaring both sides, we get

$(x+1)^2 + (y-2)^2 = 4(x^2+y^2)$ (1 mark)

$x^2 + 2x + 1 + y^2 - 4y + 4 = 4x^2 + 4y^2$ (1 mark)

$3x^2 + 3y^2 - 2x + 4y - 5 = 0$ (1 mark)

(b) Required, To derive the condition for the points, $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ to be collinear.

for collinear points, slope of line AB = slope of line AC

$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_1}{x_3 - x_1}$ (1 mark)

$x_3y_2 - x_1y_2 - x_3y_1 + \cancel{x_1y_1} = x_2y_3 - x_2y_1 - x_1y_3 + \cancel{x_1y_1}$

$x_1(y_3 - y_1) - x_2(y_1 - y_3) + x_3(y_2 - y_1) = 0$ (0.5 marks)

$x_1\begin{vmatrix} y_3 & 1 \\ y_2 & 1 \end{vmatrix} - x_2\begin{vmatrix} y_1 & 1 \\ y_3 & 1 \end{vmatrix} + x_3\begin{vmatrix} y_2 & 1 \\ y_1 & 1 \end{vmatrix} = 0$ (1 mark)

$x_3\begin{vmatrix} y_2 & 1 \\ y_1 & 1 \end{vmatrix} - x_2\begin{vmatrix} y_1 & 1 \\ y_3 & 1 \end{vmatrix} + x_1\begin{vmatrix} y_3 & 1 \\ y_2 & 1 \end{vmatrix} = 0$ (0.5 marks)

$\therefore \begin{vmatrix} x_3 & y_3 & 1 \\ x_2 & y_2 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} = 0$ (0.5 marks)

(c) (i) Given $y = 3x+1$

Required to translate by factor (2,3)

$(y-3) = 3(x-2)+1$

$y - 3 = 3x - 6 + 1$

$y = 3x - 2$ (2 marks)

(ii) Given $y = 3x - 1$

Required to dilate by a factor (2,5) then

$\frac{y}{5} = 3\left(\frac{x}{2}\right) - 1$

$2y = 15x - 10$

$2y - 15x + 10 = 0$ (2 marks)

9. (c) Evaluating the integral

$$I = \int \frac{\ln x}{x\sqrt{1-4\ln x - \ln^2 x}}\,dx$$

$$= \int \frac{\ln x}{\sqrt{1-4\ln x-(\ln x)^2}}\,\frac{1}{x}dx$$

let $t = \ln x$, so that $dt = \frac{1}{x}dx$ (0.5 mark)

$$= \int \frac{t}{\sqrt{1-4t-t^2}}\,dt$$

$$= -\frac{1}{2}\int \frac{(-2t-4)+4}{\sqrt{1-4t-t^2}}\,dt - \int \frac{4}{\sqrt{1-4t-t^2}}\,dt$$ (0.5 mark)

$$= -\frac{1}{2}\left(2\sqrt{1-4t-t^2}\right) - 2\int \frac{1}{\sqrt{5-(t+2)^2}}\,dt$$ (1 mark)

$$= -\sqrt{1-4t-t^2} - 2\sin^{-1}\left(\frac{t+2}{\sqrt{5}}\right) + C$$ (1 mark)

$$\therefore \int \frac{\ln x}{x\sqrt{1-4\ln x-\ln^2 x}}\,dx = -\sqrt{1-4\ln x-\ln^2 x} - 2\sin^{-1}\left(\frac{2+\ln x}{\sqrt{5}}\right) + C$$ (1 mark)

(ii) Evaluating integral

$$I = \int \frac{(\sin^{-1}x)^2}{\sqrt{1-x^2}}\,dx$$

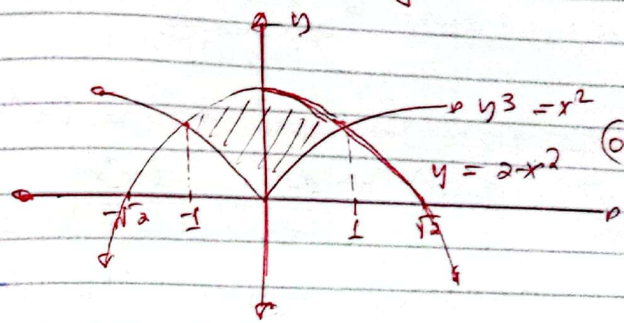
let $t = \sin^{-1}x$ so that $dt = \frac{1}{\sqrt{1-x^2}}dx$ (1/2 mark)

$$= \int t^2\,dt$$ (0.5 mark)

$$= \frac{t^3}{3} + C$$ (0.5 mark)

$$= \frac{1}{3}(\sin^{-1}x)^3 + C$$ (1 mark)

(b) (i) A sketch of the region S



(02 marks)

(ii) Area of region S

$$A = \int_{-1}^{1}\left((2-x^2) - x^{2/3}\right)dx$$ 1 mark

$$= 2x - \frac{x^3}{3} - \frac{3}{5}x^{5/3}\Big|_{-1}^{1}$$

$$= \left(2-\frac{1}{3}-\frac{3}{5}\right) - \left(-2+\frac{1}{3}+\frac{3}{5}\right)$$

$$= 32/15 \text{ square units}$$ (1 mark)

10. (a)(i) $y = \ln(\sec x + \tan x)$

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(\sec x + \tan x)}{(\sec x + \tan x)} = \frac{\sec x\tan x + \sec^2 x}{\sec x + \tan x}$$

$$= \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x} = \sec x$$ (1.5 marks)

$$\therefore \frac{d}{dx}\left(\ln(\sec x + \tan x)\right) = \sec x.$$

(i) $y = \sinh^{-1}(\tan x)$

$$d(\sinh^{-1} f(x)) = \frac{f'(x)}{\sqrt{1+(f(x))^2}}$$ (0.5marks)

Where $f(x) = \tan x$ and $f'(x) = \sec^2 x$

$$\frac{d}{dx}\left(\sinh^{-1}(\tan x)\right) = \frac{\sec^2 x}{\sqrt{1+\tan^2 x}} = \frac{\sec^2 x}{\sqrt{\sec^2 x}} = \frac{\sec^2 x}{\sec x}$$

$$\frac{d}{dx}\left(\sinh^{-1}(\tan x)\right) = \sec x$$ (1 mark)

(b) $y = \frac{1}{1+x} = f(x)$

$$f(x+h) = \frac{1}{x+1+h}$$

$$f'(x) = \lim_{h\to 0}\left[\frac{f(x+h) - f(x)}{h}\right]$$ (0.5mark)

$$f'(x) = \lim_{h\to 0}\left[\frac{1}{h}\left[\frac{1}{x+1+h} - \frac{1}{x+h}\right]\right] =$$

$$= \lim_{h\to 0}\left[\frac{1}{h}\left[\frac{-h}{(x+1+h)(x+1)}\right]\right]$$ (1 mark)

$$= \lim_{h\to 0}\left[\frac{-1}{(x+1+h)(x+1)}\right] = \frac{-1}{(x+1+0)(x+1)}$$ (1 mark)

$$= \frac{-1}{(x+1)^2}$$

(c) Required to expand $\tan^{-1}x$ by maclaurin series

$f(x) = \tan^{-1}x$, $f(0) = \tan^{-1} 0 = 0$

$$f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1}$$

$f'(x) = 1 - x^2 + x^4 - x^6 + x^8 \ldots$ $f'(0) = 1$ (Binomial theorem)

$f''(x) = -2x + 4x^3 - 6x^5 + 8x^7 \ldots$ $f''(0) = 0$

$f'''(x) = -2 + 12x^2 - 30x^4 + 56x^6 \ldots$ $f'''(0) = -2$

$f^{iv}(x) = 24x - 120x^3 + 336x^5 \ldots$ $f^{iv}(0) = 0$

$f^{v}(x) = 24 - 360x^2 + 1680x^4 \ldots$ $f^{v}(0) = 24$

$f^{vi}(x) = -720x + 6720x^3 \ldots$ $f^{vi}(0) = 0$

$f^{vii}(x) = -720x + 20160x^3 \ldots$ $f^{vii}(0) = -720$ (1.5mark)

By maclaurins series

$$f(x) = \frac{f(0)}{0!} + \frac{x f'(0)}{1!} + \frac{x^2 f''(0)}{2!} + \frac{x^3 f'''(0)}{3!} + \frac{x^4 f^{iv}(0)}{4!} \ldots$$ (0.5mark)

$$f(x) = 0 + \frac{x}{1} - \frac{2x^3}{3!} + \frac{24x^5}{5!} - \frac{720x^7}{7!} \ldots$$

$$\therefore \tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$$

$$\tan^{-1} 0.1 = 0.1 - \frac{(0.1)^3}{3} + \frac{(0.5)^5}{5} - \frac{(0.1)^7}{7}$$ (1mark)

$= 0.0996686652$

≈ 0.997 to 4 decimal places (1mark)