A Holistic Approach to A-Level

SUBSIDIARY MATHEMATICS

SECOND EDITION

KAWUMA FAHAD

2 IN 1

WITH SOLUTIONS TO UNEB PAST PAPER QUESTIONS

$$\rho = 1 - \frac{6\sum d^2}{n(n^2-1)} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\sigma = \sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2}$$

$$a.b = |a||b|\cos\theta$$

A Self-Teaching Guide

$$F = ma$$

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GOMBE S.S

A book that guarantees you a point in Sub-Math

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<u>Examination Format</u>

There will be one paper of ***2 hours 40 minutes****. The paper will consist of* ***two sections****: Section A and Section B.*
Section A will comprise short questions on Pure Mathematics, Statistics and Mechanics while Section B will comprise longer questions. Section A will consist of ***eight*** *(8) compulsory questions. Candidates will be required to attempt all questions each carrying 5 marks. Section B will consist of* ***six*** *(6) questions of which candidates will be required to attempt any* ***four*** *(4) each carrying 15 marks.*
In Section B, ***six*** *(6) questions will be set from Pure Mathematics, Mechanics and Probability and Statistics.*

<u>DEDICATION:</u>

Of course, to my beloved parents

Examination tips

- *Read carefully the instructions on the question paper.*
- *If you only have to answer some of the questions, read the questions and choose which to do. Start with the questions you know best.*
- *If the instructions say "Answer all questions", work out steadily through the paper, leaving out any questions you cannot do.*
- *Read each question carefully to be sure what it is you are required to do*
- *Set out all your work carefully and neatly and make your method clear. If the examiners can see what you have done, they will be able to give marks for the correct method even if you have the answer wrong.*
- *If you have to write an explanation as your answer, try to keep it short*
- *Check your answers, especially numerical ones. Look to see if your answers are sensible.*
- *Make sure you know how to use a calculator. They don't all work the same way.*
- *When doing a calculation, keep all figures shown on the calculator, only round off the final answer.*
- *Make sure you take all the equipment you need to the exam that is pens, pencils, ruler, compasses and calculator (make sure its battery is working)*
- *When you have completed the exam, check to see that you have not missed out any questions.*
- *You must show all working. If you give a correct answer without a working, you will receive no marks.*
- *Avoid panic.*

  If you have done your revision, you have no need to panic. If you find the examination difficult, so will everyone else. This means the pass mark will be lower.
- *If you cannot do a question, move on and don't worry about it. Often the answer will come a few minutes later.*
- *If panic occurs, try to find a question you can do. Success will help to calm your nerves.*

CHAPTER 1: SURDS, LOGARITHMS AND INDICES

Expressions such as $\sqrt{4}, \sqrt{25}$ have exact numerical values i.e. $\sqrt{4} = 2, \sqrt{25} = 5$. However expressions such as $\sqrt{2}, \sqrt{3}, \sqrt{5}$ .....can not be written numerically as exact quantities i.e. $\sqrt{2} = 1.414$ *and* $\sqrt{3} = 1.732$. Such numbers are called irrational and it's often more convenient to leave them in the form $\sqrt{2}, \sqrt{3}, \sqrt{5}$.............. hence called **surds**. They are thus irrational numbers, which can be expressed as powers.

Examples:

1. Write the following as the simplest possible surds

(i) $\sqrt{8}$ (ii) $\sqrt{12}$ (iii) $\sqrt{50}$ (iv) $\sqrt{48}$

Solution

(i) $\sqrt{8} = \sqrt{2 \times 4} = \sqrt{4} \times \sqrt{2} = 2\sqrt{2}$

(ii) $\sqrt{12} = \sqrt{4 \times 3} = \sqrt{4} \times \sqrt{3} = 2\sqrt{3}$

(iii) $\sqrt{50} = \sqrt{25 \times 2} = \sqrt{25} \times \sqrt{2} = 5\sqrt{2}$

(iv) $\sqrt{48} = \sqrt{16 \times 3} = \sqrt{16} \times \sqrt{3} = 4\sqrt{3}$

2. Simplify; (i) $\sqrt{75} + \sqrt{108} + \sqrt{27}$

Solution

$$\begin{aligned}\sqrt{75} + \sqrt{108} + \sqrt{27} &= \sqrt{25 \times 3} + \sqrt{36 \times 3} + \sqrt{9 \times 3} \\ &= \sqrt{25} \times \sqrt{3} + \sqrt{36} \times \sqrt{3} + \sqrt{9} \times \sqrt{3} \\ &= 5\sqrt{3} + 6\sqrt{3} + 3\sqrt{3} = 14\sqrt{3}\end{aligned}$$

(ii) $\sqrt{50} + \sqrt{2} - 2\sqrt{18} + \sqrt{8}$

Solution

$$\begin{aligned}\sqrt{50} + \sqrt{2} - 2\sqrt{18} + \sqrt{8} &= \sqrt{25 \times 2} + \sqrt{2} - 2\sqrt{9 \times 2} + \sqrt{4 \times 2} \\ &= \sqrt{25} \times \sqrt{2} + \sqrt{2} - 2\sqrt{9} \times \sqrt{2} + \sqrt{4} \times \sqrt{2} \\ &= 5\sqrt{2} + \sqrt{2} - 2 \times 3\sqrt{2} + 2\sqrt{2} \\ &= 8\sqrt{2} - 6\sqrt{2} = 2\sqrt{2}\end{aligned}$$

3. *Expand and simplify*

(a) $(3 - 3\sqrt{3})(3 + 2\sqrt{3})$

(b) $(5 - 2\sqrt{7})(5 + 2\sqrt{7})$

Solution

(a) $$\begin{aligned}(3 - 3\sqrt{3})(3 + 2\sqrt{3}) &= 6 - 9\sqrt{3} + 4\sqrt{3} - 6(\sqrt{3})^2 \\ &= 6 - 5\sqrt{3} - 6 \times 3 = -12 - 5\sqrt{3}\end{aligned}$$

(b) $$\begin{aligned}(5 - 2\sqrt{7})(5 + 2\sqrt{7}) &= 25 - 10\sqrt{7} + 10\sqrt{7} - 4(\sqrt{7})^2 \\ &= 25 - 4 \times 7 = 25 - 28 = -3\end{aligned}$$

4. Rationalize the denominator of $\frac{3}{\sqrt{2}}$

Solution

Multiply numerator and denominator by $\sqrt{2}$

$$\frac{3}{\sqrt{2}}=\frac{3}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}=\frac{3\sqrt{2}}{2}$$

5. Express $\frac{\sqrt{2}}{2\sqrt{3}}$ *in the form* $\sqrt{\frac{a}{b}}$ *where a and b are real numbers.*

Solution

$$\frac{\sqrt{2}}{2\sqrt{3}}=\frac{\sqrt{2}}{\sqrt{4}\times\sqrt{3}}=\frac{\sqrt{2}}{\sqrt{12}}=\sqrt{\frac{2}{12}}=\sqrt{\frac{1}{6}}$$

Alternatively;

$$\frac{\sqrt{2}}{2\sqrt{3}}=\frac{\sqrt{2}}{2\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}=\frac{\sqrt{6}}{6}=\frac{\sqrt{6}}{\sqrt{36}}=\sqrt{\frac{6}{36}}=\sqrt{\frac{1}{6}}$$

6. Rationalize the denominator of $\frac{3-\sqrt{5}}{1+3\sqrt{5}}$

Solution

Multiply numerator and denominator by the denominator with sign of $3\sqrt{5}$ changed (conjugate of the denominator)

$$\frac{3-\sqrt{5}}{1+3\sqrt{5}}=\frac{3-\sqrt{5}}{1+3\sqrt{5}}\times\frac{1-3\sqrt{5}}{1-3\sqrt{5}}=\frac{(3-\sqrt{5})(1-3\sqrt{5})}{(1+3\sqrt{5})(1-3\sqrt{5})}$$

$$=\frac{3-9\sqrt{5}-\sqrt{5}+3\sqrt{25}}{1^2-(3\sqrt{5})^2}=\frac{3+15-10\sqrt{5}}{1-45}$$

$$=\frac{18-10\sqrt{5}}{-44}=\frac{18}{-44}-\frac{10\sqrt{5}}{-44}$$

$$=\frac{-9}{22}+\frac{5}{22}\sqrt{5}$$

7. Rationalize the denominator of $\frac{1}{3-\sqrt{2}}$

Solution

$$\frac{1}{3-\sqrt{2}}=\frac{1}{3-\sqrt{2}}\times\frac{3+\sqrt{2}}{3+\sqrt{2}}=\frac{3+\sqrt{2}}{3^2-(\sqrt{2})^2}$$

$$=\frac{3+\sqrt{2}}{9-2}=\frac{1}{7}(3+\sqrt{2})$$

8. Express $\frac{2\sqrt{3}+3\sqrt{2}}{2\sqrt{3}-3\sqrt{2}}$ in the form $a+b\sqrt{c}$

Solution

$$\frac{2\sqrt{3}+3\sqrt{2}}{2\sqrt{3}-3\sqrt{2}}=\frac{(2\sqrt{3}+3\sqrt{2})(2\sqrt{3}+3\sqrt{2})}{(2\sqrt{3}-3\sqrt{2})(2\sqrt{3}+3\sqrt{2})}=\frac{2\sqrt{3}\times2\sqrt{3}+2\sqrt{3}\times3\sqrt{2}+3\sqrt{2}\times2\sqrt{3}+3\sqrt{2}\times3\sqrt{2}}{2\sqrt{3}\times2\sqrt{3}+2\sqrt{3}\times3\sqrt{2}-3\sqrt{2}\times2\sqrt{3}-3\sqrt{2}\times3\sqrt{2}}$$

$$=\frac{4\sqrt{9}+6\sqrt{6}+6\sqrt{6}+9\sqrt{4}}{4\sqrt{9}+6\sqrt{6}-6\sqrt{6}-9\sqrt{4}}$$

$$=\frac{12+12\sqrt{6}+18}{12-18}$$

$$=\frac{30+12\sqrt{6}}{-6}=\frac{30}{-6}+\frac{12\sqrt{6}}{-6}=-5-2\sqrt{6}$$

9. Simplify $\frac{1}{1-\sqrt{3}}-\frac{1}{1+\sqrt{3}}$

Solution

$$\frac{1}{1-\sqrt{3}}-\frac{1}{1+\sqrt{3}}=\frac{(1+\sqrt{3})-(1-\sqrt{3})}{(1-\sqrt{3})(1+\sqrt{3})}=\frac{1+\sqrt{3}-1+\sqrt{3}}{1-(\sqrt{3})^2}=\frac{2\sqrt{3}}{1-3}=\frac{2\sqrt{3}}{-2}=-\sqrt{3}$$

Trial questions

1. Simplify (a) $\sqrt{8}+18-2\sqrt{2}$
   (b) $\sqrt{75}+2\sqrt{12}-\sqrt{27}$
   (c) $\sqrt{28}+\sqrt{175}-\sqrt{63}$
   (d) $\sqrt{512}+\sqrt{128}+\sqrt{32}$
   (e) $\sqrt{1000}-\sqrt{40}-\sqrt{90}$
   [Ans: (a)$3\sqrt{2}$ (b) $6\sqrt{3}$ (c) $4\sqrt{7}$ (d) 28 (e) $5\sqrt{10}$]
2. Express $\frac{3\sqrt{2}-2\sqrt{3}}{3\sqrt{2}+2\sqrt{3}}$ in the form $a-b\sqrt{c}$ [Ans: $5-2\sqrt{6}$]
3. Given that $\frac{3\sqrt{5}-2\sqrt{3}}{3\sqrt{5}+2\sqrt{3}}=p+q\sqrt{r}$. Find p ,q and r [$Ans$: $p=\frac{19}{11}, q=\frac{-4}{11}, r=15$]
4. Rationalize the surd $\frac{1}{3\sqrt{5}-\sqrt{3}}+\frac{1}{\sqrt{5}+\sqrt{3}}$ [Ans; $\frac{1}{7}(10+4\sqrt{15})$]
5. Simplify $\frac{\sqrt{3}-2}{2\sqrt{3}+3}$ in the form $p+q\sqrt{3}$ [Ans:$4-\frac{7}{3}\sqrt{3}$]
6. Simplify $\frac{1}{\sqrt{5}-\sqrt{3}}$ [Ans: $\frac{1}{2}(\sqrt{5}+\sqrt{3})$]
7. Rationalize (i) $\frac{1}{3\sqrt{5}-\sqrt{3}}+\frac{1}{\sqrt{5}+3}$ [Ans: $\frac{12\sqrt{5}-10\sqrt{3}}{21}$] (ii) $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$ [Ans: $5-2\sqrt{6}$]
8. Simplify $\frac{1}{3-\sqrt{7}}+\frac{1}{3+\sqrt{7}}$ [Ans: 3]
9. Rationalize the denominator of (a) $\frac{\sqrt{5}+1}{\sqrt{5}-\sqrt{3}}$ (b) $\frac{\sqrt{2}+2\sqrt{5}}{\sqrt{5}-\sqrt{2}}$ (c) $\frac{\sqrt{10}+2\sqrt{5}}{\sqrt{10}+\sqrt{5}}$ (d) $\frac{2\sqrt{2}-\sqrt{3}}{\sqrt{2}+\sqrt{3}}$ (e) $\frac{1}{3\sqrt{2}-2\sqrt{3}}$ (f) $\frac{\sqrt{6}+\sqrt{3}}{\sqrt{6}-\sqrt{3}}$ [Ans: (a) $\frac{1}{2}(5+\sqrt{3}+\sqrt{5}+\sqrt{15})$ (b) $4+\sqrt{10}$ (c) $\sqrt{2}$ (d) $-7+3\sqrt{6}$ (e) $\frac{1}{6}(3\sqrt{2}+2\sqrt{3})$ (f) $\frac{1}{3}(9+2\sqrt{18})$]
10. Express in the form $a+b\sqrt{c}$,
    (a) $(\sqrt{5}+2)^2$ (b) $(1+\sqrt{2})(3-2\sqrt{2})$ (c) $\frac{\sqrt{3}+2}{2\sqrt{3}-1}$ (d) $\sqrt{3}+2+\frac{1}{\sqrt{3}-2}$
    [Ans: (a) $9+4\sqrt{5}$ (b) $-1+\sqrt{2}$ (c) $\frac{8}{11}+\frac{5}{11}\sqrt{3}$ (d) 0]
11. Express $\frac{3\sqrt{7}-2\sqrt{3}}{2\sqrt{7}+\sqrt{3}}$ in the form $\frac{a-b\sqrt{c}}{d}$ where a, b, c and d are integers [Ans: $\frac{48-7\sqrt{21}}{25}$]
12. Given that $\frac{\sqrt{8}-\sqrt{18}}{1-\sqrt{2}}=a+b\sqrt{2}$, determine the values of a and b [Ans: $a=-2, b=-1$]
13. Express $2\sqrt{50}-3\sqrt{2}+\sqrt{800}-2\sqrt{72}$ in the form $a\sqrt{b}$. Given that $\sqrt{2}=1.414$, evaluate the above expression [Ans: $15\sqrt{2}$ or 21.21]
14. Evaluate $\frac{2}{\sqrt{5}-\sqrt{3}}-\frac{1}{\sqrt{5}+\sqrt{3}}$ given that $\sqrt{5}=2.236$ *and* $\sqrt{3}=1.732$ [Ans: 3.716]
15. Given that $t=\frac{1}{2}(\sqrt{5}+1)$. Show that $t^2=1+t$

INDICES:

Index is another word to mean power i.e. for $a^3 = a \times a \times a$, here a is the base and 3 is a power or an index or exponent.

Laws of indices:

1. $a^m \times a^n = a^{m+n}$ i.e. $2^3 \times 2^2 = 2^{3+2} = 2^5 = 32$
2. $a^m \div a^n = a^{m-n}$ ie $3^3 \div 3^2 = 3^{3-2} = 3^1 = 3$
3. $(a^m)^n = a^{mn}$ ie $(4^2)^3 = 4^{2\times 3} = 4^6$
4. $a^0 = 1$ ie $5^0 = 1$, $\left(\frac{2}{7}\right)^0 = 1, 1000^0 = 1$ etc
5. $a^{-n} = \frac{1}{a^n}$ ie $2^{-1} = \frac{1}{2^1}$, $5^{-2} = \frac{1}{5^2} = \frac{1}{25}$
6. $a^{\frac{1}{n}} = \sqrt[n]{a}$ i.e. $4^{\frac{1}{2}} = \sqrt[2]{4} = 2$
7. $a^n \times b^n = (ab)^n$ i.e. $2^2 \times 3^2 = (2\times 3)^2 = 6^2 = 36$
8. $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{(a^m)}$

Examples

1. Simplify (i) $27^{\frac{1}{3}}$ (ii) $4^{\frac{-1}{2}}$ (iii) $100^{1\frac{1}{2}}$ (iv) $(625)^{\frac{-1}{4}}$ (v) $\left(\frac{27}{1000}\right)^{\frac{-1}{3}}$

Solution

(i) $27^{\frac{1}{3}} = (3^3)^{\frac{1}{3}} = 3^{3\times\frac{1}{3}} = 3^1 = 3$

(ii) $4^{\frac{-1}{2}} = (2^2)^{\frac{-1}{2}} = 2^{2\times\frac{-1}{2}} = 2^{-1} = \frac{1}{2}$

(iii) $100^{1\frac{1}{2}} = 100^{\frac{3}{2}} = (10^2)^{\frac{3}{2}} = 10^{2\times\frac{3}{2}} = 10^3 = 1000$

(iv) $(625)^{\frac{-1}{4}} = (5^4)^{\frac{-1}{4}} = 5^{4\times\frac{-1}{4}} = 5^{-1} = \frac{1}{5}$

(v) $\left(\frac{27}{1000}\right)^{\frac{-1}{3}} = \frac{(27)^{\frac{-1}{3}}}{(1000)^{\frac{-1}{3}}} = \frac{(3^3)^{\frac{-1}{3}}}{(10^3)^{\frac{-1}{3}}} = \frac{3^{3\times\frac{-1}{3}}}{10^{3\times\frac{-1}{3}}} = \frac{3^{-1}}{10^{-1}}$

$= 3^{-1} \div 10^{-1} = \frac{1}{3} \div \frac{1}{10} = \frac{1}{3} \times \frac{10}{1} = \frac{10}{3}$

2. Simplify

(i) $\frac{27^{\frac{1}{2}} \times 243^{\frac{1}{2}}}{243^{\frac{4}{5}}}$

Solution

$$\frac{27^{\frac{1}{2}} \times 243^{\frac{1}{2}}}{243^{\frac{4}{5}}} = \frac{(3^3)^{\frac{1}{2}} \times (3^5)^{\frac{1}{2}}}{(3^5)^{\frac{4}{5}}} = \frac{3^{\frac{3}{2}} \times 3^{\frac{5}{2}}}{3^4}$$

$$= \frac{3^{\left(\frac{3}{2}+\frac{5}{2}\right)}}{3^4} = \frac{3^4}{3^4} = 1$$

(ii) $\frac{a^{\frac{1}{n}} + a^{-n}}{a^{\left(\frac{n+1}{n}\right)}}$

Solution

$$\frac{a^{\frac{1}{n}} \div a^{-n}}{a^{\left(\frac{n+1}{n}\right)}} = \frac{a^{\frac{1}{n}} \div \frac{1}{a^{n}}}{a^{\left(\frac{n+1}{n}\right)}} = \frac{a^{\frac{1}{n}} \times a^{n}}{a^{\left(\frac{n+1}{n}\right)}} = \frac{a^{\frac{1}{n}+n}}{a^{\left(\frac{n+1}{n}\right)}} = \frac{a^{\left(\frac{n^2+1}{n}\right)}}{a^{\left(\frac{n+1}{n}\right)}}$$
$$= a^{\left(\frac{n^2+1}{n}\right)-\left(\frac{n+1}{n}\right)} = a^{\left(\frac{n^2-n}{n}\right)} = a^{n-1}$$

(iii) $\frac{8^{\frac{1}{6}}\times 4^{\frac{1}{3}}}{32^{\frac{1}{6}}\times 16^{\frac{1}{12}}}$

Solution

$$\frac{8^{\frac{1}{6}}\times 4^{\frac{1}{3}}}{32^{\frac{1}{6}}\times 16^{\frac{1}{12}}} = \frac{(2^3)^{\frac{1}{6}}\times(2^2)^{\frac{1}{3}}}{(2^5)^{\frac{1}{6}}\times(2^4)^{\frac{1}{12}}} = \frac{2^{\frac{3}{6}}\times 2^{\frac{2}{3}}}{2^{\frac{5}{6}}\times 2^{\frac{4}{12}}} = \frac{2^{\frac{3}{6}+\frac{2}{3}}}{2^{\frac{5}{6}+\frac{4}{12}}} = \frac{2^{\frac{7}{6}}}{2^{\frac{7}{6}}} = 1$$

(iv) $\frac{3(2^{n+1})-4(2^{n-1})}{2^{n+1}-2^{n}}$

Solution

$$\frac{3(2^{n+1})-4(2^{n-1})}{2^{n+1}-2^{n}} = \frac{3(2^n\times 2)-4(2^n\times 2^{-1})}{(2^n\times 2-2^n)} = \frac{6(2^n)-4\left(\frac{2^n}{2}\right)}{2^n(2-1)}$$
$$= \frac{6(2^n)-2(2^n)}{2^n} = \frac{2^n(6-2)}{2^n} = 4$$

(v) $\frac{9^{n+1}\times 6^{n-1}}{3^{2n-1}\times 2^{n}}$

Solution

$$\frac{9^{n+1}\times 6^{n-1}}{3^{2n-1}\times 2^{n}} = \frac{3^{2(n+1)}\times(3\times 2)^{n-1}}{3^{2n-1}\times 2^{n}} = \frac{3^{2n+2}\times 3^{n-1}\times 2^{n-1}}{3^{2n-1}\times 2^{n}}$$
$$= \frac{3^{2n+2+n-1}\times 2^{n-1}}{3^{2n-1}\times 2^{n}} = \frac{3^{3n+1}\times 2^{n-1}}{3^{2n-1}\times 2^{n}}$$
$$= 3^{(3n+1)-(2n-1)}\times 2^{(n-1)-2n}$$
$$= 3^{3n+1-2n+1}\times 2^{n-1-2n}$$
$$= 3^{n+2}\times 2^{-n-1}$$

(vi) $\frac{x^{2n+1}\times x^{\frac{1}{2}}}{\sqrt{x^{3n}}}$

Solution

$$\frac{x^{2n+1}\times x^{\frac{1}{2}}}{\sqrt{x^{3n}}} = \frac{x^{2n+1+\frac{1}{2}}}{x^{\frac{3n}{2}}} = \frac{x^{2n+\frac{3}{2}}}{x^{\frac{3n}{2}}} = \frac{x^{\frac{4n+3}{2}}}{x^{\frac{3n}{2}}} = x^{\frac{4n+3}{2}-\frac{3n}{2}} = x^{\frac{4n+3-3n}{2}} = x^{\frac{n+3}{2}}$$

Equations involving indices:

Solve the following equations;

1. $3^x = 81$

$3^x = 3^4$

$\therefore x = 4$

2. $27^x = \frac{1}{9}$

$(3^3)^x = \frac{1}{3^2}$

$3^{3x} = 3^{-2}$

$3x = -2 \Rightarrow x = \frac{-2}{3}$

3. $4^x = 0.5$

$4^x = \frac{1}{2}$

$(2^2)^x = 2^{-1}$

$2^{2x} = 2^{-1}$

$\therefore 2x = -1, x = \frac{-1}{2}$

3. $2^{2x+3} + 1 = 9(2^x)$

Solution

$2^{2x} \times 2^3 + 1 = 9(2^x)$

$8(2^{2x}) - 9(2^x) + 1 = 0$

$8(2^x)^2 - 9(2^x) + 1 = 0$ *since* $2^{2x} = 2^x \times 2^x = (2^x)^2$

Let $2^x = y$, *then;*

$8y^2 - 9y + 1 = 0$

$8y^2 - 8y - y + 1 = 0$

$(8y - 1)(y - 1) = 0$, either $8y - 1 = 0 \Rightarrow y = \frac{1}{8}$ or $y - 1 = 0 \Rightarrow y = 1$

When $y = \frac{1}{8}, 2^x = \frac{1}{8} \Rightarrow 2^x = \frac{1}{2^3} \Rightarrow 2^x = 2^{-3}, x = -3$

When $y = 1, 2^x = 1 \leftrightarrow 2^x = 2^0, x = 0$

$\therefore x = 0$ *or* $x = -3$

4. $3(3^{2x})+26(3^x) - 9 = 0$

Solution

$3(3^x)^2+26(3^x) - 9 = 0$

Let $y = 3^x$, then $3y^2 + 26y - 9 = 0$

$3y^2 - y + 27y - 9 = 0$

$y(3y - 1) + 9(3y - 1) = 0$

$(3y - 1)(y + 9) = 0$

Either $3y - 1 = 0$, $y = \frac{1}{3}$ or $y + 9 = 0$, $y = -9$

Either $3^x = \frac{1}{3}$, $3^x = 3^{-1}$ $\therefore x = -1$

Or $3^x = -9$ *and value of x does not exist*

5. $2^{2x+1} + 15(2^x) = 8$

Solution

$2^{2x} \times 2^1 + 15(2^x) - 8 = 0$

$2(2^x)^2 + 15(2^x) - 8 = 0$

Let $2^x = y$, *then* $2y^2 + 15y - 8 = 8$

$2y^2 - y + 16y - 8 = 0$

$y(2y - 1) + 8(2y - 1) = 0$

$(2y - 1)(y + 8) = 0$

Either $2y - 1 = 0$ *or* $y + 8 = 0 \Rightarrow y = \frac{1}{2}$ *or* $y = -8$

Either $2^x = \frac{1}{2} = 2^{-1} \Rightarrow x = -1$ or

$2^x = -8$ *and value of x does not exist*

$\therefore x = -1$

6. $4^{(2t+1)} + 4^{(t+3)} = 16\frac{1}{4}$

Solution

$4^{2t} \times 4^1 + 4^t \times 4^3 = \frac{65}{4}$

$4(4^{2t}) + 64(4^t) = \frac{65}{4}$

$4(4^t)^2 + 64(4^t) = \frac{65}{4}$

Let $4^t = y \ \Rightarrow 4y^2 + 64y = \frac{65}{4}$

$16y^2 + 256y - 65 = 0$ on multiplying through by 4

$16 \times -65 = -1040$ *whose factors are* 260 *and* -4 *that add up to* 256

$16y^2 + 260y - 4y - 65 = 0$

$4y(4y + 65) - (4y + 65) = 0$

$(4y + 65)(4y - 1) = 0$

Either $4y + 65 = 0 \Rightarrow 4y = -65$ *which gives* $y = -\frac{65}{4}$

$4^t = -\frac{65}{4}$ *and here value of t does not exist*

or $4y - 1 = 0 \Rightarrow 4y = 1$ *which gives* $y = \frac{1}{4}$

$4^t = \frac{1}{4} \ \Rightarrow 4^t = 4^{-1}$

$\therefore t = -1$

7. $2^{4(x-1)} = (4 \times 8^x)^3$

Solution

$2^{4x-4} = [2^2 \times (2^3)^x]^3$

$2^{4x-4} = [2^2 \times 2^{3x}]^3$

$2^{4x-4} = \left[2^{(2+3x)}\right]^2$

$2^{4x-4} = 2^{2(2+3x)}$

$\Rightarrow 4x - 4 = 4 + 6x$

$4x = 8 + 6x$

$-2x = 8$

$\therefore x = -4$

Trial questions:

1. Simplify

(i) $\frac{8^{\frac{1}{6}} \times 4^{\frac{1}{3}}}{32^{\frac{1}{6}} \times 16^{\frac{1}{12}}}$ $[Ans: 1]$

(ii) $\frac{x^{\frac{-2}{3}} \times x^{\frac{1}{4}}}{x^{\frac{1}{6}}}$ $\left[Ans: x^{\frac{-7}{12}}\right]$

(iii) $\frac{x^{2n+1} \times x^{\frac{1}{2}}}{\sqrt{x^{3n}}}$ $\left[Ans: x^{\left(\frac{n+3}{2}\right)}\right]$

(v) $\frac{12^{\frac{3}{2}} \times 16^{\frac{1}{8}}}{27^{\frac{1}{6}} \times 18^{\frac{1}{2}}}$ [Ans: 8]

(vi) $y^{\frac{3}{2}} = 64$ [Ans: 8]

(vii) $\left(\frac{27}{8}\right)^{\frac{-2}{3}}$ [Ans: $\frac{2}{3}$]

(iv) $\frac{9^{\frac{1}{2}}\times 27^{\frac{1}{3}}}{64^{\frac{1}{3}}\times 16^{\frac{1}{2}}}$ $\left[Ans: \frac{9}{16}\right]$ (viii) $\frac{32^{\frac{3}{4}}\times 16^{0}\times 8^{\frac{5}{4}}}{128^{\frac{3}{2}}}$ [Ans: $\frac{1}{8}$]

(ix) $4^{-\frac{n}{2}}\times 2^{n+3}\times 16^{-\frac{1}{2}}$ [Ans: 2]

2. Solve the following equations

(i) $9^x = \frac{1}{729}$ $[Ans: x = -3]$ (ii) $8^x = 0.25$ $\left[Ans: x = \frac{-2}{3}\right]$

(iii) $32^x = 0.25$ $\left[Ans: x = \frac{-2}{5}\right]$ (iv) $9^x = 27^{\frac{3}{4}}$ [Ans: $x = \frac{9}{8}$]

(v) $2^{2x} - 5(2^x) + 4 = 0$ $[Ans: x = 2\ or\ 0]$

(vi) $2^{2x+2} + 8 = 33(2^x)$ $[Ans: x = -2\ or\ 3]$

(vii) $3^{2x} - 12(3^x) + 27 = 0$ $[Ans: x = 2\ or\ 1]$

(viii) $x^4 - 4x^2 + 3 = 0$ $[Ans: x = \pm 1\ or \pm\sqrt{3}\ , hint\ Let\ y = x^2]$

(ix) $\left(\frac{1}{4}\right)^x \times 2^{x+1} = \frac{1}{8}$ [Ans: $x = 4$]

(x) $3(4^x) - 8(2^x) + 4 = 0$ $[Ans: x = 1\ or -0.585$]

(xi) $3^{x^2+2} = 27^{2x-1}$ [Ans: $x = 1\ or\ 5$]

(xii) $5^{2x-5} = 125^{x^2-2}$ [Ans: $x = 1\ or -\frac{1}{3}$]

(xiii) $5^{2x} - 2(5^x) - 8 = 0$ [Ans: $x = 0.861$]

LOGARITHMS:

Logarithm is another word to mean index or power i.e. if $y = a^x$, then we define x as logarithm of y to base a $(\log_a y)$

$if\ y = a^x$, then $x = \log_a y$

Operating rules for logarithms

1. $\log_a b + \log_a c = \log_a bc$ ie $\log_2 3 + \log_2 5 = \log_2 3\times 5 = \log_2 15$
2. $\log_a b - \log_a c = \log_a\left(\frac{b}{c}\right)$ ie $\log_3 6 - \log_3 8 = \log_3\left(\frac{6}{3}\right) = \log_3 2$
3. $\log_a b^n = n\log_a b$ ie $\log_3 7^2 = 2\log_3 7$
4. $\log_a b = \frac{\log_c b}{\log_c a}$ i.e. $\log_2 3 = \frac{\log_4 3}{\log_4 2}$, $\log_2 3 = \frac{\log_{10} 3}{\log_{10} 2}$. This is known as the change of base rule.
5. $\log_a b = \frac{1}{\log_b a}$
6. $\log_a 1 = 0\ since\ a^0 = 1$
7. $\log_a a = 1\ since\ a^1 = a$

Examples

1. Express the following statements in logarithm notation

(i) $16 = 2^4$ (ii) $27 = 3^3$

Solution

(i) Introducing $\log_2$ on both sides

$\log_2 2^4 = \log_2 16$

$\log_2 16 = \log_2 2^4$

$\log_2 16 = 4$

(ii) Introducing $\log_3$ on both sides

$\log_3 27 = \log_3 3^3$

$\log_3 27 = 3$

2. Express the following in index notation

(i) $\log_2 32 = 5$ (ii) $7 = \log_2 128$

Solution

(i) $2^5 = 32$ (ii) $2^7 = 128$

3. Simplify $\log_4 9 + \log_4 21 - \log_4 7$

Solution

$$\log_4 9 + \log_4 21 - \log_4 7 = \log_4(9 \times 21) - \log_4 7$$
$$= \log_4\left(\frac{9\times21}{7}\right) = \log_4 27 = \log_4 3^3 = 3\log_4 3$$

4. Simplify $\log_5 125 - \log_5 50 + \log_5 2$

Solution

$$\log_5 125 - \log_5 50 + \log_2 2 = \log_5 5^3 - \log_5(2 \times 25) + \log_5 2$$
$$= 3\log_5 5 - [\log_5 2 + \log_5 25] + \log_5 2$$
$$= 3 - \log_5 2 - \log_5 25 + \log_5 2$$
$$= 3 - \log_5 5^2 = 3 - 2\log_5 5 = 3 - 2 = 1$$

5. If $\log_7 2 = 0.356$ *and* $\log_7 3 = 0.566$. *Find the value of* $2\log_7\left(\frac{7}{15}\right) + \log_7\left(\frac{25}{12}\right) - 2\log_7\left(\frac{7}{3}\right)$

Solution

$$= \log_7\left(\frac{7}{15}\right)^2 + \log_7\left(\frac{25}{12}\right) - \log_7\left(\frac{7}{3}\right)^2 = \log_7\left(\frac{49}{225}\right) + \log_7\left(\frac{25}{12}\right) - \log_7\left(\frac{49}{9}\right)$$
$$= \log_7\left(\frac{49}{225} \times \frac{25}{12}\right) - \log_7\left(\frac{49}{9}\right)$$
$$= \log_7\left(\frac{49}{225} \times \frac{25}{12} \div \frac{49}{9}\right) = \log_7\left(\frac{49}{225} \times \frac{25}{12} \times \frac{9}{49}\right)$$
$$= \log_7\left(\frac{1}{12}\right) = \log_7 1 - \log_7 12 = -\log_7 12$$
$$= -\log_7(4 \times 3) = -(\log_7 4 + \log_7 3)$$
$$= -(\log_7 2^2 + \log_7 3) = -(2\log_7 2 + \log_7 3)$$
$$= -(2 \times 0.356 + 0.566)$$
$$= -1.278$$

. Find the value of $\log_7 12$

Solution

Let $x = \log_7 12$, *then* $7^x = 12$

Introducing log_{10} on both sides gives

$$\log_{10} 7^x = \log_{10} 12$$
$$x\log_{10} 7 = \log_{10} 12$$
$$x = \frac{\log_{10} 7}{\log_{10} 12} = \frac{1.0792}{0.8451} = 1.277$$

. Solve for x in $\log_5(4 - x) - \log_5(x + 2) = \log_5 x$

Solution

$\log_5\left(\frac{4-x}{x+2}\right) = \log_5 x$. Since the logarithms are to the same bases on both sides, then;

$\frac{4-x}{x+2} = x$

$4 - x = x(x+2)$

$4 - x = x^2 + 2x$

$x^2 + 3x - 4 = 0$

$x^2 - x + 4x - 4 = 0$

$x(x-1) + 4(x-1) = 0$

$(x-1)(x+4) = 0$ $\therefore$Either $x - 1 = 0 \Rightarrow x = 1$ or $x + 4 = 0 \Rightarrow x = -4$

8. Solve the equation $(0.2)^x = (0.5)^{x+7}$

Solution

Introducing log_{10} on both sides gives;

$\text{Log } 0.2^x = log 0.5^{x+7}$

$xlog\ 0.2 = (x+7)\log 0.5$

$$\frac{x}{x+7} = \frac{\log 0.5}{log\ 0.2} = 0.4307$$

$x = 0.4307(x+7)$

$x = 0.4307x + 3.0147$

$0.5693x = 3.0147$

$\therefore x = 5.2955$

9. If $3 + \log_{10} x = 2\log_{10} y$. $Express\ x\ in\ terms\ of\ y$

Solution

Using $3\log_{10} 10 + \log_{10} x = 2\log_{10} y$ since $\log_{10} 10 = 1$

$\log_{10} 10^3 + \log_{10} x = \log_{10} y^2$

$\log_{10} 1000x = \log_{10} y^2$

Since the logarithms are to the same bases on both sides, then;

$1000x = y^2$

$$\therefore x = \frac{y^2}{1000} \quad or\ x = 0.001y^2$$

10. Solve $(0.1)^x = (0.2)^5$

Solution

$\text{Log } 0.1^x = \log 0.2^5$

$xlog\ 0.1 = 5\log 0.2$

$x = \frac{5\log 0.2}{log\ 0.1} = \frac{5\times -0.69897}{-1} = 3.495$

$\therefore x = 3.495$

11. Express $\log_9 xy$ $in\ terms\ of$ $\log_3 x$ and $\log_3 y$

Solution

Using the change of base rule, $\log_9 xy = \frac{\log_3 xy}{\log_3 9}$

$$= \frac{\log_3 xy}{\log_3 3^2} = \frac{\log_3 xy}{2\log_3 3} = \frac{1}{2}(\log_3 xy)$$

$$= \frac{1}{2}(\log_3 x + \log_3 y)$$

12. Solve for x in the equation ; $\log_5 x + \log_x 5 = 2.5$

Solution

We see that the logarithms are not to the same bases so we make them the same, thus

$\log_5 x + \frac{1}{\log_5 x} = 2.5$ (Since $\log_x 5 = \frac{1}{\log_5 x}$)

Let $\log_5 x = y, then\ y + \frac{1}{y} = 2.5$

$y^2 + 1 = 2.5y$

$10\,y^2 + 10 = 25y$ (multiplying through by 10)

$10\,y^2 - 25y + 10 = 0$

$a = 10, b = -25\ and\ c = 10$

$y = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$= \frac{-(-25)\pm\sqrt{(-25)^2-4\times10\times10}}{2\times10}$

$= \frac{25\pm\sqrt{225}}{20} = \frac{25\pm15}{20}$

$either\ y = \frac{25+15}{20} = \frac{40}{20} = 2\ \ or\ y = \frac{25-15}{20} = \frac{10}{20} = \frac{1}{2}$

$or\ y = \frac{25-15}{20} = \frac{10}{20} = \frac{1}{2}$

Either $\log_5 x = 2 \Rightarrow 5^2 = x, \therefore x = 25$

Or $\log_5 x = \frac{1}{2} \Rightarrow 5^{\frac{1}{2}} = x, \therefore x = \sqrt{5}$

Alternatively, by factorization

$10\,y^2 - 20y - 5y + 10 = 0$

$10y(y-2) - 5(y-2) = 0$

$(y-2)(10y-5) = 0$

$Either\ y - 2 = 0 \Rightarrow y = 2$

Or $10y - 5 = 0 \Rightarrow y = \frac{1}{2}$

13. Simplify $\log_8 4$

Solution

Using the change of base rule i.e. $\log_a b = \frac{\log_c b}{\log_c a}$

We can change the base 8 to the lowest possible base i.e. 2

$$\log_8 2 = \frac{\log_2 4}{\log_2 8} = \frac{\log_2 2^2}{\log_2 2^3} = \frac{2\log_2 2}{3\log_2 2} = \frac{2}{3}$$

14. Show that $\log_8 x = \frac{2}{3}\log_4 x$. Hence without sing tables or calculator evaluate $\log_8 6$ correct to 3 decimal places given that $\log_4 3 = 0.7925$

Solution

By using the change of base rule and changing the base from 8 to 4

$$\log_8 x = \frac{\log_4 x}{\log_4 8}$$

We can now simplify $\log_4 8$ by changing the base to 2

$$\log_4 8 = \frac{\log_2 8}{\log_2 4} = \frac{\log_2 2^3}{\log_2 2^2} = \frac{3\log_2 2}{2\log_2 2} = \frac{3}{2}$$

Thus $\log_8 x = \frac{\log_4 x}{\left(\frac{3}{2}\right)} = (\log_4 x) \times \frac{2}{3} = \frac{2}{3}\log_4 x$

Now $\log_8 6 = \frac{2}{3}\log_4 6\ \ since\ x = 6$

$\frac{2}{3}\log_4 6 = \frac{2}{3}\log_4(2\times3) = \frac{2}{3}[\log_4 2 + \log_4 3]$

$$= \frac{2}{3}\left[\log_4 4^{\frac{1}{2}} + 0.7925\right]$$
$$= \frac{2}{3}\left[\frac{1}{2}\log_4 4 + 0.7925\right]$$
$$= \frac{2}{3}\left[\frac{1}{2} + 0.7925\right] = \frac{2}{3}[0.5 + 0.7925] = \frac{2}{3} \times 1.2925$$
$$= 0.862\ (3\ d.p)$$

15. Solve the equation $3\log_2 P - 6\log_P 2 + 7 = 0$

Solution

Using the laws of logarithms, $\log_P 2 = \frac{1}{\log_2 P}$

$\Rightarrow\ 3\log_2 P - 6\left(\frac{1}{\log_2 P}\right) + 7 = 0$

Let $\log_2 P = y \ \Rightarrow 3y - \frac{6}{y} + 7 = 0$

$3y^2 - 6 + 7y = 0$ on multiplying through by y

$3y^2 + 7y - 6 = 0$

$3y^2 + 9y - 2y - 6 = 0$

$3y(y + 3) - 2(y + 3) = 0$

$(y + 3)(3y - 2) = 0$

$either\ y + 3 = 0 \ \Rightarrow y = -3 \quad or\ 3y - 2 = 0 \Rightarrow y = \frac{2}{3}$

$when\ y = -3,\ \log_2 P = -3 \ \Rightarrow P = 2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

$when\ y = \frac{2}{3}, \log_2 P = \frac{2}{3} \Rightarrow \quad P = 2^{\frac{2}{3}} = 1.587$

16. Solve for x without using tables or a calculator $\log_3 x - 10\log_x 3 = 3$

Solution

$\log_x 3 = \frac{1}{\log_3 x} \Rightarrow\ \log_3 x - \frac{10}{\log_3 x} = 3$

Let $\log_3 x = y$ thus $y - \frac{10}{y} = 3$

$y^2 - 10 = 3y$ on multiplying through by y

$y^2 - 3y - 10 = 0$

$y^2 - 5y + 2y - 10 = 0$

$y(y - 5) + 2(y - 5) = 0$

$(y - 5)(y + 2) = 0$

$Either\ y - 5 = 0 \ \Rightarrow y = 5 \ or\ y + 2 = 0 \ \Rightarrow y = -2$

When $y = 5,\ \log_3 x = 5 \Rightarrow x = 3^5 = 243$

When $y = -2,\ \log_3 x = -2 \Rightarrow x = 3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

Trial questions

1. Simplify $1 + \log_{10}\left(\frac{4}{x^4}\right)^{\frac{-1}{2}} - 2\log_{10} x$ $\quad$ [Ans: $\log_{10} 5$]
2. Solve the simultaneous equations;

   $\log_{10} x - \log_{10} y = \log_{10} 2.5$ and $\log_{10} x + \log_{10} y = 1$ $\quad$ [$Ans: x = 5\ or\ y = 2$]

3. Find $\log_{\sqrt{5}} 30$ correct to 2 decimal places [Ans: 4.23]
4. If $p = \log_7\left(\frac{14}{15}\right), q = \log_7\left(\frac{21}{20}\right), r = \log_7\left(\frac{49}{50}\right)$, *given that* $\log_7 2 = 0.356$ *and* $\log_7 3 = 0.566$, find the values of (a) $p + q - r$ (b) $p + 3q - 2r$

   [Ans: $(a)\ 0$ $(b)\ 0.064$]
5. Solve the equation $(0.2)^x = (0.5)^{x+7}$ [$Ans: x = 5.294$]
6. Solve the equation $\log_3 x - 4\log_x 3 + 3 = 0$ [Ans: $x = \frac{1}{81} or\ 3$]
7. Solve the simultaneous equations $x + y = 20$ *and* $\log_3 x = \log_9 y$

   [Ans: (4,6); (−5,25)]
8. Solve the equation $\log_2 x - \log_x 8 = 2$ [$Ans: x = 8\ or\ \frac{1}{2}$]
9. Solve for x in the equation $log_4(6 - x) = log_2 x$ [Ans: $x = 2$]
10. (i) If $log_a(2 + a) = 2$, *Find a* [Ans: $a = 2$] (ii) $log_a(6 - a) = 2$ [Ans: $a = 2$]
11. Evaluate $\frac{\log 81}{\log 729}$ [Ans: $\frac{2}{3}$]
12. Determine $\log 0.27$ given that $\log 3 = 0.4771$ [Ans: -0.5687]
13. Express as a single logarithm and simplify your answer $log\sqrt{x^2 - 1} + \frac{1}{2}log\left(\frac{x+1}{x-1}\right)$

    [Ans: $\log(x + 1)$]
14. Given that $\log_3 2 = 0.63$. Find the value of x in the equation $3^{2x} = 3^x + 2$ [Ans: $x = 0.63$]
15. Simplify $\log_7 98 - \log_7 30 + \log_7 15$ [Ans: 2]
16. Determine the values of (i) $\log_2 32 - \log_2 128 + \log_2 64$ (ii) $\log_3 90$ [Ans: (i) 4 (ii) 4.096]
17. If $\log_2 8^x = \frac{1}{\sqrt{3}}$, show that $x = \frac{\sqrt{3}}{9}$
18. Solve the simultaneous equations

    $\log_2 x + \log_2 y = 3$ *and* $\log_y x = 2$ [Ans: $x = 4$ *and* $y = 2$]
19. Find the value of x if $1.38^x = 2.628$ [Ans: $x = 3$]
20. Simplify $\log_{10} 160 + 2\log_{10}\left(\frac{5}{2}\right) - 1$ [Ans: 2]
21. Given that $\log_{10} 2 = 0.301, \log_{10} 3 = 0.477$ *and* $\log_{10} 7 = 0.845$, Evaluate $\log_{10}\sqrt{\left(\frac{63}{8}\right)}$

    [Ans: 0.448]
22. Evaluate $\log_{0.4} 50$ [Ans: -4.269]
23. Solve the equation $2\log_3 x + 3\log_x 3 - 5 = 0$ [Ans: $x = 1\ or\ x = 5.2$]
24. Without using tables or calculator, find the value of x in

    $\log\frac{5}{2} + \log\frac{16}{13} - \log\frac{5}{26} = \log(x^2 - 3x)$ [Ans: $x = 5.77\ or\ x = -2.77$]
25. Express $2\log_3 18 + \log_3 3^{-1} - \log_3 6^2 + 1$ *as a single logarithm* $\log_3 Q$ [Ans: $\log_3 9$]
26. Solve for x *in the equation* $\log_4 4x = 2\log_x 4$ [Ans: $x = 4\ or \frac{1}{16}$]

CHAPTER 2: POLYNOMIALS

Polynomials are algebraic expressions that include real numbers and variables. They contain more th… one term. Polynomials are the sums of monomials.
A monomial has one term for example 5y or $-8x^2$ or 3 are monomials.
A sum of two monomials that are not like terms for example; $-3x^2 + 2$, or $9y + 2y^2$ is a special poly… called a binomial. Similarly, a sum of three monomials is a trinomial for example; $-3x^2 + 3x - 2$.

Degree of a polynomial

The degree of the polynomial is the highest exponent of the variable for example; $3x^2$ has a degree of… $2x^5$ has a degree of 5 and 2 has a degree of zero. When the variable does not have an exponent - alw… understand that there is a '1' e.g., $3x$ is the same as $3x^1$.

$x^5 + 3x^4 + 2x^3 + x^2 - 2x + 1$ is a polynomial of degree 5

$4x^3 + 2x^2 - 7$ is a polynomial of degree 3

$2x^2 - 8x + 9$ is a polynomial of degree 2 e.t.c

Note: All quadratic expressions of the form $ax^2 + bx + c$ are polynomials of degree 2

Operations on Polynomials:

Addition of Polynomials:

Below, the steps for addition of polynomials, the first basic operation on polynomials are clearly laid

1. Collect Like terms at one place
2. Add the numerical coefficients of like terms
3. Write the sum in both standard and simplest form

Examples:

1. Add the following polynomials:

(a) $2a + 3b$ and $-4b + 5a$ (b) $6x + 2y - 3z$ and $9z + 3y - 5x$

Solution:

(a) We know what is meant by like terms. They are terms in which literal coefficients are same. So, t… like terms means to add the numerical coefficients of two or more polynomials which have same lite… coefficients.

In 2a + 3b and -4b + 5a:
2a and 5a are like terms and 3b and -4b is another pair of like terms.
So, add them (the like terms):

$$2a + 5a = 7a$$
$$3b - 4b = -b$$

Now, 7a and –b are unlike terms which cannot be added like like terms. So, the two unlike terms 7a … b are written and the symbol '+' is written to indicate the addition operation of polynomials in the gi… question
So, the sum of 2a + 3b and -4b + 5a is 7a – b

(b) In $6x + 2y - 3z$ and $9z + 3y - 5x$ the like terms are $6x$ and $-5x$, $2y$ and $3y$, $-3z$ and $9z$
So, the sum of like terms is

$$6x - 5x = x$$
$$2y + 3y = 5y$$
$$-3z + 9z = 6z$$

Now write these sums connected by the addition sign '+' to indicate the sum of the two polynomials i… question (i.e. the addition operation on polynomials)

$= x + 5y + 6z$

2. Simplify $(2x + 5y) + (3x - 2y)$

Solution

$(2x + 5y) + (3x - 2y) = 2x + 5y + 3x - 2y$
$= 2x + 3x + 5y - 2y$
$= 5x + 3y$

3. Simplify $(3x^3 + 3x^2 - 4x + 5) + (x^3 - 2x^2 + x - 4)$

Solution

$(3x^3 + 3x^2 - 4x + 5) + (x^3 - 2x^2 + x - 4)$
$= 3x^3 + 3x^2 - 4x + 5 + x^3 - 2x^2 + x - 4$
$= 3x^3 + x^3 + 3x^2 - 2x^2 - 4x + x + 5 - 4$
$= 4x^3 + x^2 - 3x + 1$

4. Simplify $(7x^2 - x - 4) + (x^2 - 2x - 3) + (-2x^2 + 3x + 5)$

Solution

It is perfectly okay to have to add three or more polynomials at once. I will just go slowly and do each step thoroughly, and it should work out right.

$(7x^2 - x - 4) + (x^2 - 2x - 3) + (-2x^2 + 3x + 5)$
$= 7x^2 - x - 4 + x^2 - 2x - 3 + -2x^2 + 3x + 5$
$= 7x^2 + 1x^2 - 2x^2 - 1x - 2x + 3x - 4 - 3 + 5$
$= 8x^2 - 2x^2 - 3x + 3x - 7 + 5$
$= 6x^2 - 2$

5. Simplify $(x^3 + 5x^2 - 2x) + (x^3 + 3x - 6) + (-2x^2 + x - 2)$

Solution

$(x^3 + 5x^2 - 2x) + (x^3 + 3x - 6) + (-2x^2 + x - 2)$
$= x^3 + 5x^2 - 2x + x^3 + 3x - 6 + -2x^2 + x - 2$
$= x^3 + x^3 + 5x^2 - 2x^2 - 2x + 3x + x - 6 - 2$
$= 2x^3 + 3x^2 + 2x - 8$

Subtraction of Polynomials:

The steps for subtraction of polynomials, the second basic operation on polynomials are as follows:

1. Subtract similar terms. To do this, change the algebraic sign of what is to be subtracted and add it to the other.
2. To subtract unlike terms, just write the operation sign – before what is to be subtracted

Examples

1. Simplify $(x^3 + 3x^2 + 5x - 4) - (3x^3 - 8x^2 - 5x + 6)$

Solution

$(x^3 + 3x^2 + 5x - 4) - (3x^3 - 8x^2 - 5x + 6)$
$= (x^3 + 3x^2 + 5x - 4) - (3x^3) - (-8x^2) - (-5x) - (6)$
$= x^3 + 3x^2 + 5x - 4 - 3x^3 + 8x^2 + 5x - 6$
$= x^3 - 3x^3 + 3x^2 + 8x^2 + 5x + 5x - 4 - 6$
$= -2x^3 + 11x^2 + 10x - 10$

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Now, 7a and –b are unlike terms which cannot be added like like terms. So, the two unlike terms 7a and –b are written and the symbol '+' is written to indicate the addition operation of polynomials in the given question
So, the sum of 2a + 3b and -4b + 5a is 7a – b

(b) In $6x + 2y - 3z$ and $9z + 3y - 5x$ the like terms are $6x$ and $-5x$, $2y$ and $3y$, $-3z$ and $9z$
So, the sum of like terms is

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$$2y + 3y = 5y$$
$$-3z + 9z = 6z$$

Now write these sums connected by the addition sign '+' to indicate the sum of the two polynomials in the question (i.e. the addition operation on polynomials)

2. Divide $40a^5b^4 + 55a^3b^5 + 35a^3b^4 + 70ab$ by a^2b^2

Solution:

Divide each in the polynomial $40a^5b^4 + 55a^3b^5 + 35a^3b^4 + 70ab$ by a^2b^2

Let us find the quotients separately as follows:

$(40a^5b^4)/(a^2b^2) = 40a^{5-2}.b^{4-2} = 40a^3b^2$

$(55a^3b^5)/(a^2b^2) = 55ab^3$

$(35a^3b^4)/(a^2b^2) = 35ab^2$

$(70ab)/(a^2b^2) = 70/(ab)$

Now write the above four quotients next to each other, separated by the + sign to indicate their addition

$40a^3b^2 + 55ab^3 + 35ab^2 + 70/(ab)$

Factorization of polynomials of degree 2

The polynomials of degree 2 i.e. in the form $ax^2 + bx + c$ can be factorised in the steps in the following example.

Example: Factorize $6x^2 + 13x + 6$

Solution

1. Multiply the *a term* (6 in the example) by the *c* term (also 6 in the example).

$6 \times 6 = 36$

2. Find two numbers that when multiplied equal this number (36) and add up to be the *b* term (13).

$4 \times 9 = 36$ and $4 + 9 = 13$

3. Substitute the two numbers you get into this form as *k* and *h* (order does not matter):

$ax^2 + kx + hx + c$

$6x^2 + 4x + 9x + 6$

4. Factor the polynomial by grouping. Organize the equation so that you can take out the greatest common factor of the first two terms and the last two terms. Both factored groups should be the same. Add the GCF's together and enclose them in brackets next to the factored group.

$6x^2 + 4x + 9x + 6$

$2x(3x + 2) + 3(3x + 2)$

$(2x + 3)(3x + 2)$

Difference of Two Squares

If you see something of the form $a^2 - b^2$, you should remember the formula

$$(a + b)(a - b) = a^2 - b^2$$

Example:

$x^2 - 4 = (x - 2)(x + 2)$

Solving simple polynomials

Example: Solve $3x^3+2x^2-x = 0$

Solution

This is cubic ... but wait, you can factor out "x":

$3x^3+2x^2- x = x(3x^2+2x-1)= 0$

Now we have one root $(x=0)$ and what is left is quadratic, which we can solve exactly by factorizing it.

$either\ x = 0\ \ or\ 3x^2 + 2x - 1 = 0$

Now solving, $3x^2 + 2x - 1 = 0$, we have;

$3x^2 + 3x - x - 1 = 0$

$3x(x + 1) - (x + 1) = 0$

$(x + 1)(3x - 1) = 0$

$either\ x + 1 = 0 \Rightarrow x = -1\ \ or\ 3x - 1 = 0 \Rightarrow x = \frac{1}{3}$

2. Simplify $(6x^3 - 2x^2 + 8x) - (4x^3 - 11x + 10)$

Solution

$(6x^3 - 2x^2 + 8x) - (4x^3 - 11x + 10)$

$= (6x^3 - 2x^2 + 8x) - (4x^3 - 11x + 10)$

$= (6x^3 - 2x^2 + 8x) - (4x^3) - (-11x) - (10)$

$= 6x^3 - 2x^2 + 8x - 4x^3 + 11x - 10$

$= 6x^3 - 4x^3 - 2x^2 + 8x + 11x - 10$

$= 2x^3 - 2x^2 + 19x - 10$

Multiplication of Polynomials:

Multiplication of polynomials is the third important operation on polynomials. Here are the steps to follow when multiplying polynomials;

- First multiply numerical coefficients and literal coefficients separately. Next, multiply these two products
- To multiply two polynomials when each one has more than one term; Multiply each term of one polynomial with each term of the other polynomial and write like terms together.

Examples:

Multiply the following polynomials:

1. 5p and 8q

Solution:

Product of numerical coefficients 5 and 8 is 40 and product of literal coefficients p and q is pq.
Now, write the product of these two as: 40pq

2. $4x^3 + 2$ *and* $2x^2 + 3x$

Solution:

$4x^3 + 2$ *and* $2x^2 + 3x$

Let us apply the 2nd rule

$(4x^3 + 2)(2x^2 + 3x) = 4x^3(2x^2 + 3x) + 2(2x^2 + 3x)$

$= 8x^5 + 12x^4 + 4x^2 + 6x$ *{Apply exponents rule:* $x^m \times x^n = x^{m+n}$ *}*

Division of Polynomials:

1. To divide a monomial by another monomial, divide the numerical coefficients and the literal coefficients separately.
2. To divide a polynomial by a monomial, divide each term in the polynomial by the monomial.

Examples:

Divide the following polynomials

1. $50p^4q^6$ by 5pq

Solution:

Divide the numerical coefficients and write their quotient i.e $\frac{50}{5} = 10$ now divide literal coefficients and write their quotient as

p^4q^6 by pq {recall exponents rule: { $\frac{x^m}{x^n} = x^{m-n}$ }

$$\frac{p^4q^6}{pq} = p^{4-1} \times q^{6-1} = p^3q^5$$

Now, write the coefficients next to each other to denote their product $50p^3q^5$

CHAPTER 3: QUADRATIC EQUATIONS

Any equation of the form $ax^2 + bx + c = 0$ is called a quadratic equation and the values of x, which satisfy the equation, are called roots.

Solution of a quadratic equation that factorizes

Example

1. Find the roots of the equation $x^2 - 5x + 6 = 0$

Solution

$x^2 - 2x - 3x + 6 = 0$

$x(x-2) - 3(x-2) = 0$

$(x-2)(x-3) = 0$

Either $x - 2 = 0, x = 2$ $or\ x - 3 = 0, x = 3$

Solution of a quadratic equation that does not factorize

By completing the square

This method uses the expansion $(x+b)^2 = x^2 + 2bx + b^2$. It is important to note that the last term b^2, is the square of half the coefficient of x, $(2b)$

Note that the coefficient of the highest term x^2 should be 1

Examples

1. Find the roots of the equation $2x^2 - 5x + 1 = 0$

Solution

Dividing through by 2 gives;

$$x^2 - \frac{5}{2}x + \frac{1}{2} = 0$$

$$x^2 - \frac{5}{2}x = -\frac{1}{2}$$

Adding the square of half the coefficient of x to both sides of the equation;

$$x^2 - \frac{5}{2}x + \left(\frac{5}{4}\right)^2 = -\frac{1}{2} + \left(\frac{5}{4}\right)^2$$

$$\left(x - \frac{5}{4}\right)^2 = -\frac{1}{2} + \frac{25}{16}$$

$$\left(x - \frac{5}{4}\right)^2 = \frac{17}{16}$$

$$\sqrt{\left(x - \frac{5}{4}\right)^2} = \sqrt{\frac{17}{16}}$$

$$x - \frac{5}{4} = \frac{\sqrt{17}}{4}$$

$$x = \frac{5 \pm \sqrt{17}}{4} \qquad \therefore x = 2.281 \quad or\ x = 0.219$$

2. Solve $2x^2 - 6x + 4 = 0$

Solution

$2x^2 - 6x + 4 = 0$

$x^2 - 3x + 2 = 0$

$x^2 - 3x = -2$

Adding the square of half the coefficient of x to each side of the equation

$$x^2 - 3x + \left(\frac{3}{2}\right)^2 = -2 + \left(\frac{3}{2}\right)^2$$

$$\left(x - \frac{3}{2}\right)^2 = -2 + \frac{9}{4}$$

$$\left(x - \frac{3}{2}\right)^2 = \frac{1}{4}$$

$$\sqrt{\left(x - \frac{3}{2}\right)^2} = \sqrt{\frac{1}{4}}$$

$$x - \frac{3}{2} = \pm\frac{1}{2}$$

$$x = \frac{3 \pm 1}{2} \qquad \therefore \quad x = 2 \text{ or } x = 1$$

3. Solve $x^2 + 3x - 1 = 0$

Solution

$x^2 + 3x = 1$

Adding the square of half the coefficient of x to each side of the equation gives;

$$x^2 + 3x + \left(\frac{3}{2}\right)^2 = 1 + \left(\frac{3}{2}\right)^2$$

$$\left(x + \frac{3}{2}\right)^2 = 1 + \frac{9}{4}$$

$$\left(x + \frac{3}{2}\right)^2 = \frac{13}{4}$$

$$x + \frac{3}{2} = \pm\frac{\sqrt{13}}{2}$$

$$x = \frac{-3 \pm \sqrt{13}}{2} \quad giving \ x = \frac{-3+\sqrt{13}}{2} = 0.30 \text{ or } x = \frac{-3-\sqrt{13}}{2} = -3.30$$

Maximum and Minimum values

The method of completing the square, used to solve any equation in the form $ax^2 + bx + c = 0$ can be used to find the maximum or minimum value of the expression $ax^2 + bx + c$

For example, consider the expression $x^2 + 3x + 4$

By completing the square;

$$x^2 + 3x + 4 = x^2 + 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 4$$

$$= \left[x^2 + 3x + \left(\frac{3}{2}\right)^2\right] - \frac{9}{4} + 4$$

$$= \left[x + \frac{3}{2}\right]^2 + \frac{7}{4}$$

Now $\left[x + \frac{3}{2}\right]^2$ cannot be negative for any value of x, i.e $\left[x + \frac{3}{2}\right]^2 \geq 0$

Thus $x^2 + 3x + 4$ is always positive and will have a minimum value of $\frac{7}{4}$ when $x + \frac{3}{2} = 0$ i.e. when $x = -\frac{3}{2}$

Example 2

Find the maximum value of $5 - 2x - 4x^2$

Solution

Let us first rewrite $5 - 2x - 4x^2$ as $-4x^2 - 2x + 5$

$$-4x^2 - 2x + 5 = -4\left(x^2 + \frac{1}{2}x\right) + 5$$
$$= -4\left(x^2 + \frac{1}{2}x + \frac{1}{16}\right) + \frac{4}{16} + 5$$
$$= -4\left(x + \frac{1}{4}\right)^2 + \frac{21}{4}$$
$$= \frac{21}{4} - 4\left(x + \frac{1}{4}\right)^2$$

Now $4\left(x + \frac{1}{4}\right)^2 \geq 0$

Thus $5 - 2x - 4x^2$ has a maximum value of $\frac{21}{4}$ when $x = -\frac{1}{4}$

Example 3

Find by completing the square, the greatest value of the function $f(x) = 1 - 6x - x^2$

Solution

$$1 - 6x - x^2 = -x^2 - 6x + 1$$
$$= -[x^2 + 6x] + 1$$
$$= -[x^2 + 6x + 3^2 - 3^2] + 1$$
$$= -[x^2 + 6x + 9 - 9] + 1$$
$$= -[x^2 + 6x + 9] + 9 + 1$$
$$= -(x + 3)^2 + 10 = 10 - (x + 3)^2$$

Since $(x + 3)^2$ is the square of a real number, it cannot be negative, it is zero when $= -3$, otherwise it is positive

$10 - (x + 3)^2$ is therefore always less than or equal to 10

Thus, the greatest value is 10

IN GENERAL

For a quadratic equation $ax^2 + bx + c = 0$, the roots are obtained from the formula;

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example

Solve $x^2 + 3x - 1 = 0$

Solution

Comparing with the general equation $ax^2 + bx + c = 0$; $a = 1,\ b = 3,\ c = -1$

Substituting in the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$x = \frac{-3\pm\sqrt{3^2-4\times1-1}}{2\times1}$

$x = \frac{-3\pm\sqrt{9+4}}{2}$ $\Rightarrow$ $x = \frac{-3+\sqrt{13}}{2}$ Or $x = \frac{-3-\sqrt{13}}{2}$ $\therefore x = 0.30$ or $x = -3.30$

ROOTS OF QUADRATIC EQUATIONS

If the equation $ax^2 + bx + c = 0$ has roots α and β, then its equivalent equation will be:

$(x-\alpha)(x-\beta) = 0$, as it gives $x = \alpha$ or $x = \beta$

$x^2 - \beta x - \alpha x + \alpha\beta = x^2 + \frac{b}{a}x + \frac{c}{a}$

$x^2 - (\alpha+\beta)x + \alpha\beta = x^2 + \frac{b}{a}x + \frac{c}{a}$

By comparing the coefficients on both sides, we obtain

$\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

Hence the equation $ax^2 + bx + c = 0$ can be written in the form:

$x^2 - (Sum\ of\ roots)x + (product\ of\ roots) = 0$

Examples

1.Write down the sum and product of the roots of the following equations:

(i) $3x^2 - 2x - 7 = 0$ (ii) $5x^2 + 11x + 3 = 0$ (iii) $2x^2 + x - 7 = 0$

Solution

(i) $x^2 - \frac{2}{3}x - \frac{7}{3} = 0$;sum of roots $= -\left(-\frac{2}{3}\right) = \frac{2}{3}$ and product of roots $= -\frac{7}{3}$

(ii) $x^2 + \frac{11}{5}x + \frac{3}{5} = 0$; sum of roots $= -\frac{11}{5}$ and product of roots $= \frac{3}{5}$

(iii) $x^2 + \frac{1}{2}x - \frac{7}{2} = 0$; sum of roots $= \frac{1}{2}$ *and product of roots* $= -\frac{7}{2}$

2.Find the equation whose roots are $\frac{3}{4}$ *and* $-\frac{1}{2}$

Solution

Sum of roots $= \frac{3}{4} + \left(-\frac{1}{2}\right) = \frac{1}{4}$ and product of roots $= \frac{3}{4} \times \left(-\frac{1}{2}\right) = -\frac{3}{8}$

The equation is in the form $x^2 - (Sum\ of\ roots)x + (product\ of\ roots) = 0$

$x^2 - \left(\frac{1}{4}\right)x + \left(\frac{-3}{8}\right) = 0$

$8x^2 - 2x - 3 = 0$

3. Find the equation whose roots are $\frac{1}{3}$ *and* $-\frac{1}{4}$

Solution

Sum of roots $= \frac{1}{3} + -\frac{1}{4} = \frac{1}{3} - \frac{1}{4} = \frac{4-3}{12} = \frac{1}{12}$

Product of roots $= \frac{1}{3} \times -\frac{1}{4} = -\frac{1}{12}$

The equation is in the form $x^2 - (Sum\ of\ roots)x + (product\ of\ roots) = 0$

$x^2 - \left(\frac{1}{12}\right) + \left(\frac{-1}{12}\right) = 0$

$12x^2 - x - 1 = 0$

4. The roots of the equation $3x^2 + 4x - 5 = 0$ are α and β, find the values of;

(i) $\frac{1}{\alpha}+\frac{1}{\beta}$ (ii) $\alpha^2+\beta^2$

Solution

$\alpha+\beta = -\frac{4}{3}$ $\alpha\beta = -\frac{5}{3}$

(i) $\frac{1}{\alpha}+\frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta} = \frac{\frac{-4}{3}}{\frac{-5}{3}} = -\frac{4}{3}\times -\frac{3}{5} = \frac{4}{5}$

(ii) From $(\alpha+\beta)^2 = (\alpha+\beta)(\alpha+\beta) = \alpha^2+2\alpha\beta+\beta^2$

$\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$=\left(\frac{-4}{3}\right)^2 - 2\left(\frac{-5}{3}\right)$

$=\frac{16}{9}+\frac{10}{3} = \frac{16+30}{9} = 5\frac{1}{9}$

5.The roots of the equation $2x^2 - 7x + 4 = 0$ are α and β. Find the equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.

Solution

From the given equation, sum of roots, $\alpha+\beta = \frac{7}{2}$ and product of roots $\alpha\beta = 2$

For the new roots, sum $\frac{\alpha}{\beta}+\frac{\beta}{\alpha} = \frac{\alpha^2+\beta^2}{\alpha\beta} = \frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta} = \frac{\left(\frac{7}{2}\right)^2-4}{2} = \frac{\left(\frac{49}{4}\right)-4}{2} = \frac{33}{8}$

Product of new roots, $\frac{\alpha}{\beta}\times\frac{\beta}{\alpha} = 1$

The equation is given by $x^2 - (Sum\ of\ roots)x + (product\ of\ roots) = 0$

$x^2 - \frac{33}{8}x + 1 = 0$

$8x^2 - 33x + 8 = 0$

6.Find the values of k if the roots of the equation $3x^2 + 5x - k = 0$ differ by 2

Solution

Let one root be α, then the other will be $\alpha + 2$

Sum of roots $\alpha + \alpha + 2 = -\frac{5}{3}$, $2\alpha = -\frac{5}{3} - 2 \Rightarrow \alpha = -\frac{11}{6}$

Product of roots $\alpha(\alpha+2) = -\frac{k}{3}$, $\alpha^2 + 2\alpha = -\frac{k}{3}$ .... ***

Substituting for α in equation *** gives;

$\left(\frac{-11}{6}\right)^2 + 2\left(\frac{-11}{6}\right) = -\frac{k}{3}$

$\frac{121}{36} - \frac{22}{6} = -\frac{k}{3}$

$\frac{121-132}{36} = -\frac{k}{3}$, $-\frac{11}{36} = -\frac{k}{3}$ $\therefore k = \frac{11}{12}$

7.If one of the roots of the equation $27x^2 + bx + 8 = 0$ is the square of the other, find b.

Solution

Let one root be α, then the other will be α^2, then;

Sum of roots $\alpha + \alpha^2 = -\frac{b}{27}$ ....(i) and product of roots $\alpha\times\alpha^2 = \frac{8}{27}$ ....(ii)

$\alpha^3 = \left(\frac{2}{3}\right)^3$ $hence\ \alpha = \frac{2}{3}$ Which we substitute in equation (i) to find b;

$\frac{2}{3} + \left(\frac{2}{3}\right)^2 = -\frac{b}{27}$

$\frac{2}{3} + \frac{4}{9} = -\frac{b}{27}$

$\frac{10}{9} = -\frac{b}{27}$ $\therefore b = -30$

8. Given that α and β are the roots of the equation $2x^2 - 5x + 4 = 0$, write down the values of $\alpha + \beta$ and $\alpha\beta$. Determine the equation whose roots are $\frac{1}{2\alpha}$ and $\frac{1}{2\beta}$

Solution

$x^2 - \frac{5}{2}x + 2 = 0$

$\alpha + \beta = \frac{5}{2}$, $\alpha\beta = 2$

Sum of new roots $= \frac{1}{2\alpha} + \frac{1}{2\beta} = \frac{\beta+\alpha}{2\alpha\beta} = \frac{\frac{5}{2}}{2(2)} = \frac{5}{2} \div 4 = \frac{5}{8}$

Product of new roots $= \frac{1}{2\alpha} \times \frac{1}{2\beta} = \frac{1}{4\alpha\beta} = \frac{1}{4(2)} = \frac{1}{8}$

The equation is given by $x^2 - (Sum\ of\ roots)x + (product\ of\ roots) = 0$

$x^2 - \frac{5}{8}x + \frac{1}{8} = 0$

$8x^2 - 5x + 1 = 0$

The discriminant

The value of the expression $(b^2 - 4ac)$ will determine the nature of the roots of the quadratic equation $ax^2 + bx + c = 0$ and it is called discriminant i.e. it discriminates between the roots of the equation.

For;

(i) Two real roots , $b^2 - 4ac > 0$

(ii) Repeated or equal roots $b^2 - 4ac = 0$

(iii) No real roots, $b^2 - 4ac < 0$

Example

Given that the equation $(5a + 1)x^2 - 8ax + 3a = 0$ *has equal roots , find the possible values of a*

Solution

We identify a, b and c from the above equation and then apply the condition for equal roots

a = $(5a + 1)$, b = $-8a$ *and* $c = 3a$

For equal roots, $b^2 - 4ac = 0$

$(-8a)^2 - 4(5a + 1)(3a) = 0$

$64a^2 - 12a(5a + 1) = 0$

$64a^2 - 60a^2 - 12a = 0$

$4a^2 - 12a = 0$

$4a(a - 3) = 0$

Either $4a = 0$ or $a - 3 = 0$ $\therefore a = 0\ or\ a = 3$

Trial questions

1. State (i) the sum (ii) the product of the roots of each of the following equations
(a) $x^2 + 9x + 4 = 0$ (b) $x^2 - 7x + 2 = 0$ (c) $2x^2 - 7x + 1 = 0$ (d) $3x^2 + 10x - 2 = 0$
[Ans: (a) -9, 4 (b) 2, 7 (c) $^7/_2$, $^1/_2$ (d) $\frac{-10}{3}, \frac{-2}{3}$]
2. In each part of this equation, you are given the sum and product of the roots of a quadratic. Find the quadratic equation in the form $ax^2 + bx + c = 0$

	a	b	c	d	e	f	g
Sum	-3	6	7	-2/3	-5/2	-3/4	-1/4
Product	-1	-4	-5	-7/3	-2	-5	-1/3

[Ans: (a) $x^2 + 3x - 1 = 0$ (b) $x^2 - 6x - 4 = 0$ (c) $x^2 - 7x - 5 = 0$ (d) $3x^2 + 2x - 7 = 0$ (e) $x^2 + 5x - 4 = 0$ (f) $2x^2 + 3x - 10 = 0$ (g) $12x^2 + 3x - 4 = 0$]
3. If α, β are the roots of the equation $3x^2 - x - 1 = 0$,form the equations whose roots are;
(i) $2\alpha, 2\beta$ (ii) α^2, β^2 (iii) $\frac{1}{\alpha}, \frac{1}{\beta}$ (iv) $\alpha + 1, \beta + 1$
[Ans: (i) $3x^2 - 2x - 4 = 0$ (ii) $9x^2 - 9x + 1 = 0$ (iii) $x^2 + x - 3 = 0$ (iv) $3x^2 - 7x + 3 = 0$]
4. One of the roots of the equation $ax^2 + bx + c = 0$ is three times the other. Show that $3b^2 - 16ac = 0$
5. If the roots of the equation $2x^2 - 7x + 1 = 0$ *are* α *and* β.find the quadratic equation whose roots are $\frac{1}{\alpha^2}$ *and* $\frac{1}{\beta^2}$ [Ans: $x^2 + 45x + 4 = 0$]
6. Given that α *and* β *are the roots of the quadratic equation* $3x^2 - x - 5 = 0$.Form the equation whose roots are $2\alpha - \frac{1}{\beta}$ *and* $2\beta - \frac{1}{\alpha}$ [Ans: $15x^2 - 13x - 169 = 0$]
7. One root of the equation $2x^2 - x + c = 0$ *is twice the other* .Find the value of c [Ans: c = $\frac{1}{9}$]
8. Find the value of k for which the equation $4(x-1)(x-2) = k$ has roots which differ by 2
[Ans: k = 3]
9. If the roots of the equation $x^2 + px + 7 = 0$ *are equal.* Find the possible values of p
[Ans: p = ± 6]
10. Find the quadratic equation, which has the difference of its roots equal to 2 and the difference of the squares of its roots equal to 5. [Ans: $16x^2 - 40x + 9 = 0$]
11. Each of the following expressions has a maximum or minimum value for all real values Find (i) which it is, maximum or minimum (ii) its value (iii) the value of x
(a) $x^2 + 4x - 3$ [Ans: (i) min (ii) -7 (iii) -2]
(b) $2x^2 + 3x + 1$ [Ans: (i) min (ii) $\frac{-1}{8}$ (iii) $\frac{-3}{4}$]
(c) $x^2 - 6x + 1$ [Ans: (i) min (ii) -8 (iii) 3]
(d) $3 - 2x - x^2$ [Ans: (i) max (ii) 4 (iii) -1]
(e) $5 + 2x - x^2$ [Ans: (i) max (ii) 6 (iii) 1]
12. (i) Express the function $13 + 6x + 3x^2$ *in the form* $a(x+b)^2 + q$ where a, b and q are constants.
(ii) Find the value of x when the function is minimum
(iii) State the minimum value of the function
[Ans: (i) $13 + 6x + 3x^2 = 3(x+1)^2 + 10$ (ii) $x = -1$ (iii) 10]

CHAPTER 4: SERIES

Consider the following sets of numbers

2, 4, 6, 8, 10, ...

1, 2, 4, 8, 16, ...

4, 9, 16, 25, 36, ...

Each set of numbers in the order given has a pattern and there is an obvious rule for obtaining the next number and as many subsequent numbers we wish to find. Such sets are called sequences and each number in the set is a term of a sequence.

Series

When the terms of a sequence are added, a series is formed i.e

$1 + 2 + 4 + 8 + 16 + \ldots\ldots$ is a series

As each term is a power of 2, we can write this series in the form;

$$2^0 + 2^1 + 2^2 + 2^3 + 2^4 + \cdots$$

All the terms of this series are in the form 2^r, so 2^r is the general term

We can thus define the series as the sum of terms of the form 2^r where r takes on integer values i.e if we decide to take the first five terms, r takes on integer values from 0 to 4

Using $\sum$ as the symbol for " the sum of terms such as", we can redefine our series concisely as;

$$\sum_{r=0}^{4} 2^r$$

$\sum_{r=2}^{10} r^3$ *means "the sum of all terms of the form* r^3

where r takes all values from 2 to 10 inclusive

$$\sum_{r=2}^{10} r^3 = 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 + 9^3 + 10^3$$

We shall consider two important series or progressions i.e. arithmetic and geometrical progressions

ARITHMETIC PROGRESSIONS (A.Ps)

The difference between any term (except the first) and its predecessor (the term immediately in front) is constant in this series. This is called the common difference, d, of the A.P. Consider

(a) 5, 8, 11, 14 .... d=3

(b) 1+3+5+7+......+99 d=2

(c) -5, -1, 3, 7, ... d=4

In general, if we denote the first by a, then;

No. of term	1	2	3	4	n
Denoted by	T_1	T_2	T_3	T_4	T_n
Term	a	$a+d$	$a+d+d$ $a+2d$	$a+2d+d$ $a+3d$	$a+(n-1)d$

$\therefore$ *For any A.P,* $T_n = a + (n-1)d$ *whe* T_n *is the nth term*

Examples

1. Find the 35^{th} term of 5, 9, 13,..........

Solution

$a = 5,\ d = 9 - 5 = 4 \quad and\ n = 35$

$T_n = a + (n-1)d$

$T_{35} = 5 + (34)(4) = 141$

2. How many terms are in the A.P $2 + 4 + 6 + \cdots + 46$?

Solution

$a = 2, \quad d = 2 \quad nth\ term = 46$

$T_n = a + (n-1)d$

$46 = 2 + (n-1)2$

$2n = 46 \quad \therefore n = 23\ terms$

3. The 7th term of an A.P is 27 and the 12th term is 47.Find the first term and common difference of the progression.

Solution

$T_7 = a + 6d = 27 \ldots\ldots\ldots\ldots\ldots (i)$

$T_{12} = a + 11d = 47 \ldots\ldots\ldots\ldots\ldots (ii)$

$equation\ (ii) - (i)\ gives$

$5d = 20, \quad d = 4$

Using (i) $a + 6 \times 4 = 27$

$a + 24 = 27 \Rightarrow a = 3$

4. The arithmetic progression A.P is given below

$\log_3(x+2), \log_3(x+3), \log_3(x+5) \ldots\ldots\ldots\ldots\ldots$ Find the value of x.

Solution

In an A.P, there is a constant difference between the respective terms i.e. the common difference and this is what we are going to use to solve this question

$$\log_3(x+5) - \log_3(x+3) = \log_3(x+3) - \log_3(x+2)$$

$$\log_3\left(\frac{x+5}{x+3}\right) = \log_3\left(\frac{x+3}{x+2}\right)$$

Since the logarithm are to the same bases, then;

$$\frac{x+5}{x+3} = \frac{x+3}{x+2}$$

$$(x+5)(x+2) = (x+3)(x+3)$$

$$x^2 + 2x + 5x + 10 = x^2 + 3x + 3x + 9$$

$$x^2 + 7x + 10 = x^2 + 6x + 9$$

$$7x + 10 = 6x + 9$$

$$7x - 6x = 9 - 10$$

$$x = -1$$

Sum of an arithmetic progression

The sum of an AP is obtained using the formula $S_n = \frac{n}{2}(2a + (n-1)d)$

Examples

1.Find the sum of the first 12 terms of the AP 3, 7, 11 ...

Solution

$a = 3,\quad d = 4 \qquad n = 12$

$S_{12} = \frac{12}{2}(2(3) + (12-1)4) = 6(6+44) = 6(50) = 300$

2.Find the sum of the series $11 + 13 + 15 + \cdots + 89$

Solution

$a = 11, d = 2 \qquad nth\ \ term = 89$

$89 = 11 + (n-1) \times 2$

$78 = 2(n-1)$

$39 = n - 1 \qquad \therefore n = 40$

$S_n = \frac{n}{2}(2a + (n-1)d)$

$S_{40} = \frac{40}{2}(2 \times 11 + (40-1)2) = 2000$

3.The fourth term of an AP is 13 and the tenth term is 31 .Find the sum of the first ten terms of the AP

Solution

$T_4 = a + 3d = 13 \ \ldots\ldots\ldots\ldots\ldots (i)$

$T_{10} = a + 9d = 31 \ \ldots\ldots\ldots\ldots\ldots (ii)$

Solving (i) and (ii) simultaneously i.e. (ii)−(i)

6d=18 $\therefore d = 3$

$a + 3 \times 3 = 13$

$a = 4$

For $n = 10,\ S_{10} = \frac{10}{2}(2 \times 4 + (10-1)3) = 5(35) = 175$

$\therefore$ *The sum of the ten terms of the AP is* 175

GEOMETRIC PROGRESSIONS (G.Ps)

The ratio of each term (except the first) to its predecessor is a constant in this series. It is called the common ratio denoted by r i.e.

(a) 2,4,8,16.......... $r = \frac{4}{2} = 2$

(b) -3, 6,-12, 24... $r = \frac{6}{-3} = -2$

(c) 144, 72, 36, 18,....... $r = \frac{72}{144} = \frac{1}{2}$

In general, if we denote the first term by a

T_1	T_2	T_3	T_4	T_5
a	$a \times r$ ar	$ar \times r$ ar^2	$ar^2 \times r$ ar^3	$ar^3 \times r$ ar^4

We can deduce that $T_n = ar^{n-1}$ where T_n *is the nth term*

Examples

1. Find the 20th term of the GP 2, 6, 18...

Solution

Here $r = \frac{6}{2} = 3$ and $a = 2$

$T_n = ar^{n-1}$

$T_{20} = 2 \times 3^{20-1} = 2 \times 3^{19} = 2324522934$

2. Find the third and tenth term of the GP $3 + 6 + \cdots \ldots\ldots + 10$

Solution

$a = 3,\quad common\ ratio(r) = \frac{6}{3} = 2$

3rd term $= ar^2 = 3 \times 2^2 = 12$

10th term $= ar^9 = 3 \times 2^9 = 1536$

3. The second term of a GP is $\frac{8}{9}$ *and the sixth term is* $4\frac{1}{2}$. Find the first term and the first term and common ratio

Solution

$T_2 = ar = \frac{8}{9} \ldots\ldots (i)$

$T_6 = ar^5 = \frac{9}{2} \ldots\ldots\ldots (ii)$

$equation\ (ii) \div (i)$ gives;

$\frac{ar^5}{ar} = \frac{\frac{9}{2}}{\frac{8}{9}} = \frac{81}{16}$

$r^4 = \frac{81}{16} = \frac{3^4}{2^4} = \left(\frac{3}{2}\right)^4$

$\therefore\ r = \frac{3}{2}$

From (i) $ar = \frac{8}{9}$, $a\left(\frac{3}{2}\right) = \frac{8}{9}$

$\therefore\ a = \frac{16}{27}$

4. The second and fourth terms of a G.P are 10 and 40 respectively . Find the possible progressions and the seventh term in each case.

Solution

$2^{nd}\ term;\quad ar = 10 \quad \ldots(i)$

$4^{th}\ term;\quad ar^3 = 40 \ \ldots(ii)$

$\frac{equation\ (ii)}{equation\ (i)}\ gives;\ \frac{ar^3}{ar} = \frac{40}{10}$

$\Rightarrow r^2 = 4$

$r = \sqrt{4} = \pm 2$

Either $r = -2\ or\ r = 2$

When $r = 2,\ a(2) = 10 \Rightarrow a = 5$

Hence the G.P is 5, 10, 20, 40, ...

$7^{th} term\ = ar^6 = 5 \times 2^6 = 320$

$when\ r = -2, a(-2) = 10 \Rightarrow a = -5$

Hence the G.P is -5, 10, -20, 40, ...

$7^{th} term\ = ar^6 = -5 \times 2^6 = -320$

The sum of a GP

The sum of n terms of a GP is given by the formula;

$$s_n = \frac{a(r^n - 1)}{r - 1} \quad if\ r > 1 \quad or \quad s_n = \frac{a(1 - r^n)}{1 - r} \quad if\ r < 1$$

Examples

1.Find the sum of the first ten terms of the series $8 + 4 + 2 + \cdots$.

Solution

Here $a = 8$, $r = \frac{1}{2}$, $n = 10$

$$s_{10} = \frac{8\left(1 - \left(\frac{1}{2}\right)^{10}\right)}{1 - \frac{1}{2}} = \frac{8\left(1 - \frac{1}{2^{10}}\right)}{\frac{1}{2}}$$

$$= 16\left(1 - \frac{1}{1024}\right) = 16\left(\frac{1023}{1024}\right) = \frac{1023}{64} = 15.98$$

$\therefore The\ sum\ of\ the\ first\ ten\ terms\ of\ the\ series\ is\ 15.98$

2.What is the least number of terms of the series $2 + 3 + \frac{9}{2} + \ldots\ldots\ldots$ which must be taken for the sum to exceed 30?

Solution

$a = 2 \quad and \quad r = \frac{3}{2}$

$$s_n = \frac{a(r^n - 1)}{r - 1}$$

$$s_n = \frac{2\left(\left(\frac{3}{2}\right)^n - 1\right)}{\frac{3}{2} - 1} > 30$$

$4(1.5^n - 1) > 30$

$1.5^n - 1 > 7.5$

$1.5^n > 8.5 \quad and\ by\ introducing\ \log_{10}\ on\ both\ sides\ gives;$

$\log 1.5^n > \log 8.5$

$n\log 1.5 > \log 8.5 \quad , \quad n > \frac{\log 8.5}{\log 1.5} = \frac{0.9294}{0.1761} = 5.278$

$n > 5.278$

$since\ n\ must\ be\ an\ integer, we\ must\ take\ 6\ terms\ for\ the\ sum\ to\ exceed\ 30$

$n = 6$

3. Find the eighth term and the sum of the first eight terms of the series $\frac{1}{2}, 1, 2, 4, \ldots\ldots$

Solution

$a = \frac{1}{2},\ r = 1 \div \frac{1}{2} = 2$

$n^{th}\ term = ar^{n-1}$

$8^{th} term = \frac{1}{2}(2)^{8-1} = \frac{1}{2} \times 128 = 64$

$s_n = \frac{a(r^n-1)}{r-1}$

$s_8 = \frac{\frac{1}{2}(2^8-1)}{2-1} = \frac{1}{2}(256-1) = 127.5$

4. The angles of a triangle form a geometrical progression. If the smallest angle is 20^0, determine the largest angle to the nearest degree.

Solution

Here $a = 20$ and the GP is $20, 20r, 20r^2$

And $20 + 20r + 20r^2 = 180$ $(since\ angles\ of\ a\ triangle\ add\ up\ to\ 180^0)$

$20r + 20r^2 = 160$

$r^2 + r - 8 = 0$ (On dividing through by 20 and re arranging)

$a = 1, b = 1, \ c = -8$

Using $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$r = \frac{-(1) \pm \sqrt{(1)^2-4(1)(-8)}}{2(1)} = \frac{-1 \pm \sqrt{33}}{2} = \frac{-1 \pm 5.74}{2}$

$\therefore r = \frac{-1+5.74}{2} = 2.37$

The largest angle $= ar^2 = 20 \times (2.37)^2 = 20 \times 5.6169 = 112.3^0$

$\therefore The\ largest\ angle\ is\ 112^0$

The sum to infinity of a G.P

Consider the general G.P, $a + ar + ar^2 + \cdots$

$S_n = \frac{a(r^n-1)}{r-1} and\ if\ |r| < 1, then\ \lim_{n \to \infty} r^n = 0$

$\Rightarrow \lim_{n \to \infty} S_n = \frac{a}{1-r}$

Therefore, for a G.P the sum to infinity, $S_\infty = \frac{a}{1-r}$

Example 1

Find the sum to infinity of the following G.P 8, 4, 2, 1, ½, ...

Solution

$a = 8, \ r = \frac{4}{8} = \frac{1}{2}$

$S_\infty = \frac{a}{1-r} = \frac{8}{1-\frac{1}{2}} = 8 \div \frac{1}{2} = 16$

Example 2

Find the sum to infinity of the following G.P; $1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \cdots$

Solution

$a = 1, \ r = -\frac{1}{4} \div 1 = -\frac{1}{4}$

$S_\infty = \frac{a}{1-r} = \frac{1}{1-(-\frac{1}{4})} = 1 \div \frac{5}{4} = \frac{4}{5}$

Trial questions

1.Find the number of terms in each of the following A.Ps

(a) $5 + 8 + 11 + 14 + \cdots \ldots . +57$

(b) $1 + 6 + 11 + \cdots \ldots \ldots + 501 + 506$ [Ans: (a) 20 (b) 102]

2.Find the sum to infinity of the series;

(a) $16 + 12 + 9 + \cdots$

(b) $16 + 8 + 4 + 2 + 1 + \cdots$.

(c) $84 - 42 + 21 - 10\frac{1}{2} + \cdots \ldots$ [Ans: (a) 64 (b) 32 (c) 56]

3.Find the sum of each of the following A.Ps

(a) $2 + 4 + 6 + 8 + 10 + \cdots . +146$

(b) $100 + 95 + 90 + 85 + \cdots \ldots \ldots \ldots . -20$

(c) $4 + 10 + 16 + 22 + 28 + \cdots \ldots . +334$

[Ans: (a) 5402 (b) 1000 (c) 9464]

4.The 5th term of an arithmetic progression is 12 and the sum of the first 5 terms is 80. Determine the 1st term and common difference. [*Ans*: $a = 20$, $d = -2$]

5.What is the number of terms of a geometric progression 5, 10, 20....... that can give a sum greater than 500,000? [Ans: 17]

6. The 10th term of an arithmetic progression is 20 and the 15th term is 44.Find the value of the first term and the common difference ,hence find the sum of the first 60 terms [Ans:5430]

7.The 8th term of an AP is twice the third term and the sum of the first eight terms is 39. Find the sum of the of the first 21 terms of the AP [Ans: 204.75]

8.Five numbers are in a geometric progression , the first being 8 and the last 648.Find the common ratio [Ans; 3]

9. How many terms of the GP $3 + 5 + 8\frac{1}{3} + \cdots$ must be taken for the sum to exceed 100?

[Ans; n = 7]

10. In an AP, the 18th term is twice the 9th term. Find the ratio of the sum of 18 terms to the sum of 9 terms of this AP. [Ans: 19:5]

11. An AP has 37 terms of which 9 is the fourth and $58\frac{1}{2}$ is the last. Find the sum of the AP. [Ans: $1165\frac{1}{2}$]

12. If $3\frac{5}{9}$ and $40\frac{1}{2}$ are the first and last terms of a GP respectively and that there are seven terms altogether , find the second term [Ans: $\frac{16}{3}$]

13. The sum of the first twenty terms of an AP is 800. Given that the sum of the first twenty six terms is 1352, determine (i) the first term and common difference (ii) the 23rd term [Ans: (i) $a = 2, d = 4$ (ii) 90]

14. The first and last terms of an AP are -3 and 58 respectively. The sum of all the terms of the progression is 5060. Find the number of terms and the common difference. [Ans : $n = 184, d = \frac{1}{3}$]

15. The first term of an AP is 15 and the fourth term is 6. Find the 10th term and the sum of the first 10 terms [Ans: -12, 15]

16. The series of an AP $a_1, a_2 and\ a_3$ sum up to 12. The sixth term a_6 is 16. Determine the sum of the first 10 terms of the progression. [Ans; $S_{10} = 145$]

17. The first two terms of an AP and a GP are alike. The first term of each progression is 20. The sum of the first five terms of the AP is 80. Find the common difference of the AP and the common ratio of the GP, hence find the difference between the five terms
[Ans: $d = -2, r = \frac{9}{10}$, 1.122]

18. The sum of the first 12 terms of an AP is 72. The eighth term is four times the sum of the fourth and fifth terms. Determine (i) the first term and the common difference of the AP (ii) the sum of the first 20 terms
[Ans: (i) $a = -7.2, d = 2.4$ (ii) 312]

19. The sum of the second and third terms of a GP is 48. The sum of the fifth and sixth terms is 1296. Find (i) the common ratio and the first term of the GP (ii) the sum of the first twelve terms of the GP
[Ans : (i) $r = 3$, $a = 4$ (ii) 1062880]

20. The first, third and eleventh terms of an AP are also the first second and third terms of a GP. Given that the first term of the AP is 2, find the (i) common ratio, r and common difference, d (ii) the sum of the first 10 terms of the AP (iii) number of terms of a GP that give a total of 699050
[Ans: (i) $d = 3, r = 4$ (ii) 155 (iii) $n = 10$]

21. The first two terms of an AP are 4 and -8. Find the number of terms whose sum is -156.
[Ans: n = 78]

22. Given that 4 and -8 are the first two terms of a GP, find the fifth term and the sum of the first five terms
[Ans: 48, 44]

CHAPTER 5: DIFFERENTIATION

Differentiation is the process of obtaining the gradient of the curve. The gradient function $\frac{dy}{dx}$ (pronounced as " dee y dee x ") or the differential coefficient of y with respect to x.

The form ax^n

If $y = ax^n, then \frac{dy}{dx} = nax^{n-1}$ where a is a constant

(Multiply by the power and then decrease the power by one)

Example

Differentiate the following with respect to x

$(i)\ x^8$ $\quad (ii)\ 3x^5 \quad (iii)\ \frac{3}{x^2} \quad (iv)\ \sqrt{x}$

Solution

(i) $let\ y = x^8 \Rightarrow \frac{dy}{dx} = 8 \times x^{8-1} = 8x^7$

(ii) $let\ y = 3x^5 \Rightarrow \frac{dy}{dx} = 5 \times 3x^{5-1} = 15x^4$

(iii) $let\ y = \frac{3}{x^2} = 3x^{-2} \Rightarrow \frac{dy}{dx} = -2 \times 3x^{-2-1} = -6x^{-3} = -\frac{6}{x^3}$

(iv) $let\ y = \sqrt{x} = x^{\frac{1}{2}} \Rightarrow \frac{dy}{dx} = \frac{1}{2} \times x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$

Differentiating a constant (k)

$If\ y = k$ and k is a constant, $\frac{dy}{dx} = 0$

$\Rightarrow$ The derivative of a constant is zero

$If\ y = 6, then\ \frac{dy}{dx} = 0$

Differentiating a sum or difference

When differentiating a sum or difference, we differentiate separately i.e.

$If\ y = f(x) + g(x) - h(x)$

$\frac{dy}{dx} = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x)) - \frac{d}{dx}(h(x))$

$\frac{dy}{dx} = f'(x) + g'(x) - h'(x)$

Example 1

If $y = ax^2 + bx + c$. Find $\frac{dy}{dx}$ if a, b and c are constants

Solution

$\frac{dy}{dx} = 2ax + b$

Example 2

$If\ y = 3x^2 - 6x + \frac{2}{x^2}$. find $\frac{dy}{dx}$

Solution

$y = 3x^2 - 6x + \frac{2}{x^2} = 3x^2 - 6x + 2x^{-2}$

$\frac{dy}{dx} = 6x - 6 + (-2 \times x^{-2-1})$

$= 6x - 6 - 4x^{-3} = 6x - 6 - \frac{4}{x^3}$

Example 3

Find $\frac{dy}{dx}$ $if\ y = x^3 + 7x^2 - 2x + 12$

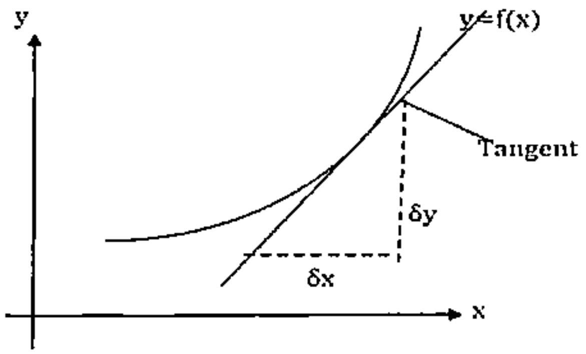
Solution

$\frac{dy}{dx} = 3x^2 + 14x - 2$

Gradient of a function / curve

The gradient of a curve is not a constant therefore the gradient of a curve is determined at a particular point. The gradient of the curve at any point is defined as the gradient of the tangent to the curve at that point and measures the rate of increase of y with respect to x

The gradient of the curve at a point is equal to the gradient of the tangent at that same point



Gradient of tangent = $\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$

Therefore the gradient of the tangent to the curve $y = f(x) is\ given\ by\ \frac{dy}{dx}\ or\ f'(x)$

Example 1

Find the gradient of the curve $y = x^3 + x^2 - 2x$

Solution

$\frac{dy}{dx} = 3x^2 + 2x - 2$

$\therefore$ *the gradient* of the curve is $3x^2 + 2x - 2$

Example 2

Find the gradient of the curve $y = 2x^2 - 5$ at a point at the point (1, -2)

Solution

$\frac{dy}{dx} = 4x$

At a point (1, -2) , $x = 1$

$\frac{dy}{dx} = 4 \times 1 = 4$

$\therefore$ The gradient of the curve is 4 at the point (1, -2)

Example 3

The curve $y = ax^3 - 2x^2 - x + 7$ has a gradient of 3 at the point where $x = 2$. Determine the value of a.

Solution

Gradient of the curve $= \frac{dy}{dx} = 3$ $at\ x = 2$

If $y = ax^3 - 2x^2 - x + 7$, $\frac{dy}{dx} = 3ax^2 - 4x - 1$

$at\ x = 2, \frac{dy}{dx} = 3a(2)^2 - 4(2) - 1 = 12a - 9$

$\Rightarrow 12a - 9 = 3$

$12a = 12$, $\therefore a = 1$

Equation of the tangent to the curve at a point

When given a point on a certain curve, the equation of the tangent at that point can be obtained. The tangent to curve at a point is the line touches the curve at that point.

Example 1

Find the equation of the tangent to the curve $y = x^3 - 3x + 2$ at the point where $x = 2$

Solution

Gradient of the tangent $= \frac{dy}{dx} = 3x^2 - 3$

When $x = 2, \frac{dy}{dx} = 3(2)^2 - 3 = 9$

It is important that you get the y coordinate by substituting the x-value in the equation of the curve

i.e. $y = 2^3 - 3(2) + 2 = 4$

The point is (2, 4)

The equation of the tangent can be obtained from;

$\frac{y - 4}{x - 2} = 9$

$y - 4 = 9x - 18$

$y = 9x - 14$ is the equation of the tangent to the curve

Example 2

Find the equation of the tangent to the curve $4y = x^2$ at a point (2, 1)

Solution

$y = \frac{x^2}{4}$

$\frac{dy}{dx} = \frac{x}{2}$ At (2, 1) ,$x = 2$ *hence* $\frac{dy}{dx} = \frac{2}{2} = 1$

Gradient of tangent =1

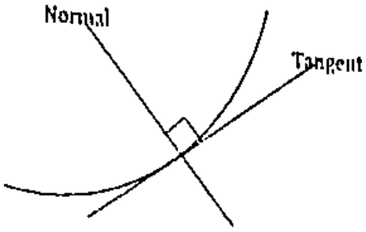
$\frac{y-1}{x-2} = 1$

$y - 1 = x - 2$

$y = x - 1$ is the equation of the tangent to the curve

Equation of normal at a given point

A line perpendicular to a tangent at the given point of contact on the curve is called the normal at that point.



If the gradient of the tangent is m, the product of the normal is $\frac{-1}{m}$ (from gradient of ⊥ lines)
In other words, the product of the gradients of perpendicular lines is -1.

Gradient of tangent × gradient of normal = -1

Gradient of tangent = $\frac{dy}{dx}$, $Gradient\ of\ normal = -1\Big/\frac{dy}{dx} = \frac{-1}{gradient\ of\ tangent}$

Example 1

Find the gradient of the normal to the curve $y = 4x - x^2$ at the point where $x = 1$

Solution

$\frac{dy}{dx} = 4 - 2x$

At $x = 1$, $\frac{dy}{dx} = 4 - 2(1) = 2$

$when\ x = 1,\ y = 4(1) - 1^2 = 3$

Gradient of tangent = 2, Gradient of normal = $\frac{-1}{2}$ at a point (1, 3)

The equation of the normal is thus given by;

$\frac{y-3}{x-1} = \frac{-1}{2}$

$2(y - 3) = -1(x - 1)$

$2y - 6 = -x + 1$

$\therefore\ 2y + x = 7$ *is the equation of normal to the curve*

Example 2

Find the equation of the tangent and normal to the curve $y = x^2 - 4x + 1$ at the point (-2, 13)

Solution

$y = x^2 - 4x + 1, \frac{dy}{dx} = 2x - 4$

At (-2, 13), $\frac{dy}{dx} = 2(-2) - 4 = -8$

Thus gradient of the tangent is -8

Equation of tangent is given by; $\frac{y-13}{x+2} = -8 \Rightarrow y - 13 = -8(x + 2)$

$y - 13 = -8x - 16$

$y = -8x + 3$

Gradient of the normal = $\frac{-1}{-8} = \frac{1}{8}$

Example

Find the maximum and minimum values of the function $y = 2x^3 - 13x^2 - 12x$

Solution

$\frac{dy}{dx} = 6x^2 - 6x - 12 = 6(x^2 - x - 2)$

For maximum and minimum values, $\frac{dy}{dx} = 0$

$$\Rightarrow 6(x^2 - x - 2) = 0$$
$$x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0 \ \Rightarrow \ x = -1 \ or\, x = 2$$

Therefore, the turning points are where $x = -1$ *and* $x = 2$.

The sign of $\frac{dy}{dx}$ is now tested on each side of the turning values of x

x	L	2	R	L	-1	R
sign of $\frac{dy}{dx}$	−	0	+	+	0	−
	\	—	/	/	—	\
		Minimum			Maximum	

Hence at $x = -1$, *the function has a maximum value of* 7 *and when* $x = 2$, *the function has* a minimum value of -20

Example 2

Find the turning points of $y = x^3 - 6x^2 - 15x + 3$ and distinguish between them

Solution

$y = x^3 - 6x^2 - 15x + 3$

$\frac{dy}{dx} = 3x^2 - 12x - 15$

For turning points; $\frac{dy}{dx} = 0 \Rightarrow 3x^2 - 12x - 15 = 0$

$$3x^2 - 12x - 15 = 0$$
$$x^2 - 4x - 5 = 0$$
$$x^2 - 5x + x - 5 = 0$$
$$x(x-5) + (x-5) = 0$$
$$(x-5)(x+1) = 0$$
$$x = 5 \ or\, x = -1$$

When $x = 5$, $y = (5)^3 - 6(5)^2 - 15(5) + 3 = 125 - 150 - 75 + 3 = -97$

when $x = -1, y = (-1)^3 - 6(-1)^2 - 15(-1) + 3 = -1 - 6 + 15 + 3 = 11$

The sign of $\frac{dy}{dx}$ is now tested on each side of the turning values of x to identify the nature of the turning points

Applications of differentiation

Stationary points

A point on a curve at which $\frac{dy}{dx} = 0$ is called a stationary point. At such point, the tangent to the curve is parallel to the x-axis

Example

Find the stationary points of the curve $y = 4x^3 + 15x^2 - 18x + 7$

Solution

$\frac{dy}{dx} = 12x^2 + 30x - 18$

For stationary values $\frac{dy}{dx} = 0 \quad \Rightarrow 12x^2 + 30x - 18 = 0$

$$2x^2 + 5x - 3 = 0 \quad \text{(on dividing through by 6)}$$

$$2x^2 + 2x + 3x - 3 = 0$$

$$(2x - 1)(x + 3) = 0$$

Either $x = -3$ or $x = \frac{1}{2}$

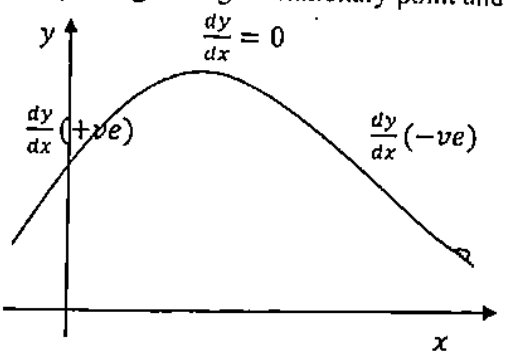
When $x = -3, y = 4(-3)^3 + 15(-3)^2 - 18(-3) + 7 = 88$

When $x = \frac{1}{2}, y = 4(\frac{1}{2})^3 + 15(\frac{1}{2})^2 - 18\left(\frac{1}{2}\right) + 7 = \frac{9}{4}$

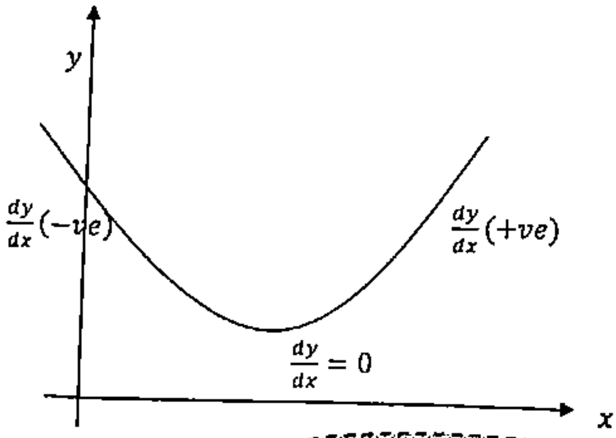
Therefore the stationary points are $(3, 88)$ and $(\frac{1}{2}, \frac{9}{4})$

Maximum and minimum turning points

Consider a curve passing through a stationary point and reaching a maximum value at that point



If it reaches a minimum value at a stationery point



Example

Find the maximum and minimum values of the function $y = 2x^3 - 13x^2 - 12x$

Solution

$\frac{dy}{dx} = 6x^2 - 6x - 12 = 6(x^2 - x - 2)$

For maximum and minimum values, $\frac{dy}{dx} = 0$

$\Rightarrow 6(x^2 - x - 2) = 0$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0 \;\; \Rightarrow \;\; x = -1 \;\; or\, x = 2$

Therefore, the turning points are where $x = -1$ *and* $x = 2$.

The sign of $\frac{dy}{dx}$ is now tested on each side of the turning values of x

x	L	2	R	L	-1	R
$sign\ of\ \frac{dy}{dx}$	$-$	0	+	+	0	$-$
	\	—	/	/	—	\

Minimum Maximum

Hence at $x = -1$, *the function has a maximum value of* 7 *and when* $x = 2$, *the function has* a minimum value of -20

Example 2

Find the turning points of $y = x^3 - 6x^2 - 15x + 3$ and distinguish between them

Solution

$y = x^3 - 6x^2 - 15x + 3$

$\frac{dy}{dx} = 3x^2 - 12x - 15$

For turning points; $\frac{dy}{dx} = 0 \Rightarrow 3x^2 - 12x - 15 = 0$

$3x^2 - 12x - 15 = 0$

$x^2 - 4x - 5 = 0$

$x^2 - 5x + x - 5 = 0$

$x(x-5) + (x-5) = 0$

$(x-5)(x+1) = 0$

$x = 5 \; or \; x = -1$

When $x = 5$, $y = (5)^3 - 6(5)^2 - 15(5) + 3 = 125 - 150 - 75 + 3 = -97$

when $x = -1, y = (-1)^3 - 6(-1)^2 - 15(-1) + 3 = -1 - 6 + 15 + 3 = 11$

The sign of $\frac{dy}{dx}$ is now tested on each side of the turning values of x to identify the nature of the turning points

x	L	5	R	L	-1	R
sign of $\frac{dy}{dx}$	−	0	+	+	0	−
	\	—	/	/	—	\

Minimum Maximum

Therefore (5, -97) is a minimum turning point and (-1, 11) is a maximum turning point

The Second derivative test

The nature of the turning points can be found by using the second derivative test in such a way that if the curve has a maximum turning point, then $\frac{d^2y}{dx^2} < 0$ (negative) and the curve has a minimum turning point if $\frac{d^2y}{dx^2} > 0$ (positive)

Example

Determine the nature of the turning points of the following curves

(a) $y = 15 - 2x^2$ (b) $y = x^2 - 3x + 2$

Solution

(a) $y = 15 - 2x^2$

$\frac{dy}{dx} = -4x$

$\frac{d^2y}{dx^2} = -4$ (*which is less than* 0 / *negative*)

The curve therefore has a maximum turning point

(b) $y = x^2 - 3x + 2$

$\frac{dy}{dx} = 2x - 3$

$\frac{d^2y}{dx^2} = 2$ (*which is greater than zero*/ positive)

The curve therefore has a minimum turning point

Curve sketching

For the function $y = f(x)$, we can plot values of x against the corresponding values of y and obtain an accurate graph of the function. A less accurate representation, which we call a sketch is adequate for many purposes provided that the sketch still shows the salient and noteworthy features of the function. We shall now learn to sketch curves and we shall mainly concentrate on quadratic curves (in the form $y = ax^2 + bx + c$). The other curves are beyond the scope of the syllabus.

The main guidelines are:-

- Determining the intercepts (where the curve cuts the axes) of the curve i.e. where x = 0 and y = 0
- The position and nature of the turning point i.e. maximum or minimum

Examples

1. Sketch the curve $y = 2x^2 - 6x + 4$

Solution

Intercepts

$when\ y = 0, 2x^2 - 6x + 4 = 0$

$$2x^2 - 2x - 4x + 4 = 0$$
$$2x(x-1) - 4(x-1) = 0$$
$$(x-1)(2x-4) = 0$$
$$x = 1 \quad or\ x = 2$$

(1, 0) and (2, 0) are the intercepts on the x-axis

$when\ x = 0, y = 4\ ,\ \ hence\ (0, 4)\ \ is\ the\ intercept$ on the y-axis

Turning point

$\frac{dy}{dx} = 4x - 6$

For turning points, $\frac{dy}{dx} = 0 \quad \Rightarrow 4x - 6 = 0$

$$x = \frac{3}{2}$$

We need to find the corresponding y-value of the turning point

$$when\ x = \frac{3}{2} \quad ; \ y = 2(\tfrac{3}{2})^2 - 6\left(\tfrac{3}{2}\right) + 4 = -\tfrac{1}{2}$$

$\therefore (\frac{3}{2}, -\frac{1}{2})$ is the turning point

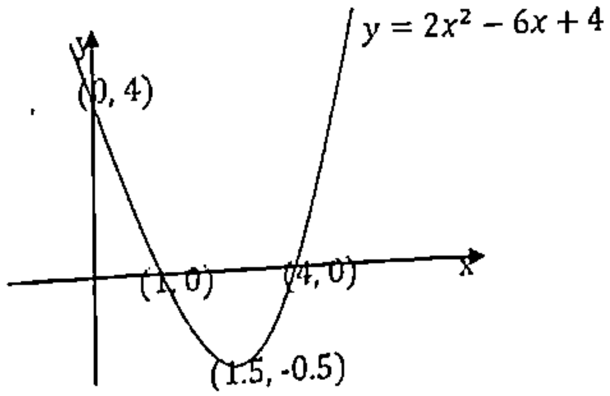
We now investigate for the nature of the turning point of the curve by using the values on the immediate left and right of the turning point.

Value of x	L	1.5	R
Sign of $\frac{dy}{dx}$	−	0	+

We now come to realize that (1.5, -0.5) is a minimum turning point

Alternatively; using the second derivative method, $\frac{d^2y}{dx^2} = 4\ which\ is\ greater\ than\ 0$

Implying that the curve has a maximum turning point.



2. Sketch the curve $y = 4x - x^2$

Solution

Intercepts

$when\ y = 0, 4x - x^2 = 0$

$x(4 - x) = 0$

Either $x = 0, or\ x = 4, \Rightarrow (0,0)\ and\ (4,0)\ are\ the\ x - intercepts$

Turning point

$y = 4x - x^2$

$\frac{dy}{dx} = 4 - 2x$

$4 - 2x = 0 \Rightarrow x = 2$

$when\ x = 2, y = 4(2) - 2^2 = 4$

(2, 4) is a turning point

We now investigate for the nature of the turning point of the curve

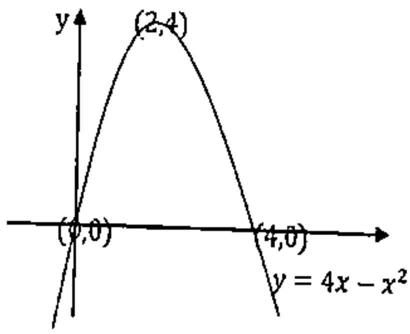
Value of x	L	2	R
Sign of $\frac{dy}{dx}$	+	0	−
	/	—	\

We observe that (2, 4) is a maximum turning point

Alternatively; if we would wish to investigate the nature of the turning point using the second derivative, we find out that $\frac{d^2y}{dx^2} = -2$ $which\ is\ less\ than$ 0 $(\frac{d^2y}{dx^2} < 0)$

Hence, the curve has a maximum turning point

We can now sketch the curve



3. Sketch the graph of the function $y = 5 + 4x - x^2$

Solution

Intercepts

$when\ y = 0, 5 + 4x - x^2 = 0$

$5 + 5x - x - x^2 = 0$

$5(1 + x) - x(1 + x) = 0$

$(1 + x)(5 - x) = 0$

$either\ x = 5\ or\ x = -1$

The curve cuts the x-axis at (5, 0) and (-1, 0)

$when\ x = 0, y = 5 + 4(0) - 0^2 = 5$

The curve cuts the y-axis at (0, 5)

Turning point

$y = 5 + 4x - x^2$

$\frac{dy}{dx} = 4 - 2x$

For turning points, $\frac{dy}{dx} = 0 \quad \Rightarrow 4 - 2x = 0\ which\ gives\ x = 2$

Now we need to find the y - value corresponding to the x-value obtained above

$when\ x = 2, y = 5 + 4(2) - (2)^2 = 5 + 8 - 4 = 9$

Thus (2, 9) is the turning point

Nature of the turning point

	L	2	R
Sign of $\frac{dy}{dx}$	+ /	0 —	− \

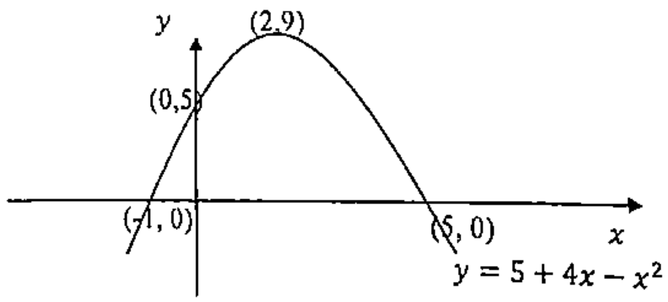
Thus (2, 9) is a maximum turning point

Alternatively, we can find the nature of the turning point using the second derivative

From $\frac{dy}{dx} = 4 - 2x$

$\frac{d^2y}{dx^2} = -2$

Since $\frac{d^2y}{dx^2} < 0, it\ is\ a\ maximum\ turning\ point$



4. Sketch the curve $y = x^2 + 2x - 3$

Solution

Intercepts

$when\ y = 0, x^2 + 2x - 3 = 0$

$x^2 + 3x - x - 3 = 0$

$x(x + 3) - (x + 3) = 0$

$(x + 3)(x - 1) = 0$

$either\ x = -3\ or\ x = 1$

The curve cuts the x-axis at (1, 0) and (-3, 0)

$when\ x = 0, y = (0)^2 + 2(0) - 3 = -3$

The curve cuts the y-axis at (0, -3)

Turning point

$y = x^2 + 2x - 3$

$\frac{dy}{dx} = 2x + 2$

For turning points, $\frac{dy}{dx} = 0 \quad \Rightarrow 2x + 2 = 0\ which\ gives\ x = -1$

Now we need to find the y - value corresponding to the x-value obtained above

$when\ x = 2, y = (-1)^2 + 2(-1) - 3 = -4$

Thus (-1, -4) is the turning point

Nature of the turning point

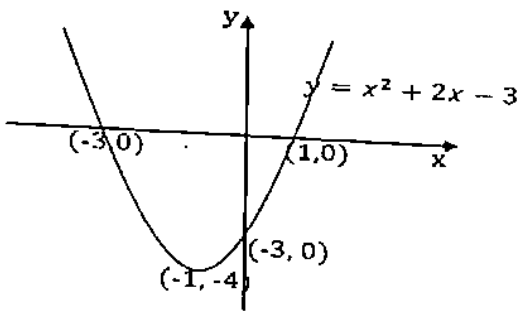
	L	-1	R
Sign of $\frac{dy}{dx}$	−	0	+

Thus (2, 9) is a minimum turning point

Alternatively, we can find the nature of the turning point using the second derivative

From $\frac{dy}{dx} = 2x + 2$

$\frac{d^2y}{dx^2} = 2\ which\ is\ positive\ (> 0)\ hence\ mininmum\ turning\ point$



Note:

From all the graphs of the functions, it follows that the curve $y = ax^2 + bx + c$ has a maximum turning point when $a < 0$ and a minimum turning point when $a > 0$

Velocity and acceleration

$If\ y = f(x), \frac{dy}{dx}\ is\ the\ rate\ of\ change\ of\ y\ with\ respect\ to\ x$

Similarly $if\ u = f(v),\ then\ \frac{du}{dv}\ is\ the\ rate\ of\ change\ of\ change\ of\ u\ with\ respect\ to\ v$

Now the velocity v of a body is defined as the rate of displacement s of a body from some fixed origin, with respect to time i.e. $v = \frac{ds}{dt}$

The acceleration a of a body is defined as the rate of the velocity of a body with respect to time i.e. $a = \frac{dv}{dt}$

So displacement, velocity and acceleration are linked up with the process of differentiation with respect to time.

Examples

1. The displacement s metres of a body from an origin O at a time t seconds is given by $s = 2t^2 - 3t + 6$. Find (a) the displacement (b) the velocity (c) the acceleration of the body when t = 1

Solution

Given $s = 2t^2 - 3t + 6$

(a) When $t = 1, s = 2(1)^2 - 3(1) + 6 = 5\ m$

(b) Since $v = \frac{ds}{dt}$

$v = \frac{ds}{dt} = 4t - 3$

$when\ t = 1, v = 4(1) - 3 = 1\ m/s$

(c) From $a = \frac{dv}{dt}$

$v = 4t - 3,\ a = \frac{dv}{dt} = 4\ ms^{-2}$

2. If $v = t^2 - 4t + 3$, find (a) the values of t when the body is at rest (b) the acceleration when t = 5

Solution

(a) At rest, v = 0

$t^2 - 4t + 3 = 0$

$t^2 - t - 3t + 3 = 0$

$t(t - 1) - 3(t - 1) = 0$

$(t - 1)(t - 3) = 0$

$t = 1\ or\ t = 3$

(b) Using $a = \frac{dv}{dt}$

$v = t^2 - 4t + 3 \Rightarrow \frac{dv}{dt} = 2t - 4$

$a = 2t - 4$

$when\ t = 5, a = 2(5) - 4 = 6m/s^2$

3. If $s = 5t^3 - t$, find the expressions of v and a in terms of t

Solution

$v = \frac{ds}{dt} = 15t^2 - 1$

$a = \frac{dv}{dt} = 30t$

Differentiation of trigonometric functions

Derivatives of $\sin x$ and $\cos x$

When we differentiate $\sin x$ with respect to x, we get $\cos x$ and if we differentiate $\cos x$ with respect to x, we get $-\sin x$. The learner should be in position to recall the above statement.

$\frac{dy}{dx}\sin x = \cos x$

$\frac{dy}{dx}\cos x = -\sin x$

Derivatives of $\sin kx$ and $\cos kx$

If we let $y = \sin kx$ and we let $u = kx$ where k is a constant

$$\text{then } y = \sin u \Rightarrow \frac{dy}{du} = \cos u \text{ and } \frac{du}{dx} = k$$

Using the chain rule; $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = k \times \cos u = k\cos kx$

$$\therefore \frac{d}{dx}\sin kx = k\cos kx$$

Similarly;

If we let $y = \cos kx$ and we let $u = kx$ where k is a constant

$$\text{then } y = \cos u \Rightarrow \frac{dy}{du} = -\sin u \text{ and } \frac{du}{dx} = k$$

Using the chain rule; $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = k \times -\sin u = -k\sin kx$

$$\therefore \frac{d}{dx}\cos kx = -k\sin kx$$

Examples

Find $\frac{dy}{dx}$ if (a) $y = \sin 4x$ (b) $y = \cos 7x$

Solution

(a) let $y = \sin 4x$ and we let $u = 4x$

$$\text{then } y = \sin u \Rightarrow \frac{dy}{du} = \cos u \text{ and } \frac{du}{dx} = 4$$

Using the chain rule; $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 4 \times \cos u = 4\cos 4x$

$$\therefore \frac{dy}{dx} = 4\cos 4x$$

(b) let $y = \cos 7x$ and we let $u = 7x$

$$\text{then } y = \cos u \Rightarrow \frac{dy}{du} = -\sin u \text{ and } \frac{du}{dx} = 7$$

Using the chain rule; $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 7 \times -\sin u = -7\sin 7x$

$$\therefore \frac{dy}{dx} = -7\sin 7x$$

With practice, the reader will soon be able to differentiate such functions directly e.g

$$\text{if } y = \cos 4x, \frac{dy}{dx} = (-\sin 4x) \times \frac{d}{dx}(4x) = -4\sin 4x$$

$$\text{if } y = \sin 7x, \frac{dy}{dx} = (\cos 7x) \times \frac{d}{dx}(7x) = 7\sin 7x$$

Derivatives of $a\sin kx$, $a\cos kx$, $a\sin kx + c$ and $a\cos kx + c$

Where a, k and c are constants

$$\frac{d}{dx}a\sin kx = a\frac{d}{dx}\sin kx = ak\cos kx$$

$$\frac{d}{dx}a\cos kx = a\frac{d}{dx}\cos kx = -ak\sin kx$$

$$\frac{d}{dx}[a\sin kx + c] = \frac{d}{dx}a\sin kx + \frac{d}{dx}c = ak\cos kx$$

$$\frac{d}{dx}[a\cos kx + c] = \frac{d}{dx}a\cos kx + \frac{d}{dx}c = -ak\sin kx$$

Examples

Find $\frac{dy}{dx}$ if (a) $y = 5\sin 9x$ (b) $y = 3\cos 8x$ (c) $y = 6\sin 3x + 4$ (d) $y = 3\cos 2x + 6$

Solution

(a) $\frac{dy}{dx} = 5\frac{d}{dx}\sin 9x = 5 \times 9\cos 9x = 45\cos 9x$

(b) $\frac{dy}{dx} = 3\frac{d}{dx}\cos 8x = 3 \times -8\sin 8x = -24\sin 8x$

(c) $\frac{dy}{dx} = \frac{d}{dx}6\sin 3x + \frac{d}{dx}(4) = 18\cos 3x$

(d) $\frac{dy}{dx} = \frac{d}{dx}3\cos 2x + \frac{d}{dx}(6) = -6\sin 2x$

Derivatives of $a\sin(kx+c)$ and $a\cos(kx+c)$

If we let $y = a\sin(kx+c)$ and we let $u = kx + c$

Then $y = a\sin u \Rightarrow \frac{dy}{du} = a\cos u$ and $\frac{du}{dx} = k$

Using the chain rule; $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = k \times a\cos u = ak\cos(kx+c)$

$\therefore \frac{d}{dx}a\sin(kx+c) = ak\cos(kx+c)$

Similarly;

If we let $y = a\cos(kx+c)$ and we let $u = kx + c$

then $y = a\cos u \Rightarrow \frac{dy}{du} = -a\sin u$ and $\frac{du}{dx} = k$

Using the chain rule; $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = k \times -a\sin u = -ak\sin(kx+c)$

$\therefore \frac{d}{dx}a\cos(kx+c) = -ak\sin(kx+c)$

Examples:

Find $\frac{dy}{dx}$ if (a) $y = 5\sin(4x+3)$ (b) $y = 3\cos(5x+10)$

Solution

(a) $\frac{dy}{dx} = 5 \times 4\cos(4x+3) = 20\cos(4x+3)$

(b) $\frac{dy}{dx} = -3 \times 5\sin(5x+10) = -15\sin(5x+10)$

Trial questions

1. Differentiate the following with respect to x

   (i) $9x^2 + 3x + 5$ (ii) $(x+2)^2$ (iii) $\frac{x^5 - x^2 + 1}{x^2}$

   [Ans: (i)]

2. Find the equation of the tangent to the curve $y = 3x^2 + 7x - 2$ at the point, P where $x = -1$ [Ans:]

3. Find the gradient of the curve $y = x^2 + 6x - 4$ at the point where the curve cuts the y-axis
[Ans: 6]
4. Find the equations of the tangent and normal to the following curves at the points indicated
(a) $y = x^2$ at $(3,9)$ [Ans: $y = 6x - 9$, $6y + x = 57$]
(b) $y = 5 - 2x^2$ at $(-1, 3)$ [Ans: $y = 4x + 7$, $4y + x = 11$]
(c) $y = 4 + x - 2x^2$ $at\ a\ point\ where\ x = 1$ [Ans: $y + 3x = 6$, $3y = x + 8$]
5. Find the coordinates of any stationary points on the given curves and distinguish between them.
   (a) $y = 2x^2 - 8x$ [Ans: min at $(2, -8)$]
   (b) $y = 18x - 20 - 3x^2$ [Ans: Max at $(3,7)$]
   (c) $y = x^3 + 3x^2 - 9x - 5$ [Ans: max at $(-3, 22)$, min at $(1, -10)$]
6. Sketch the following curves, clearly indicating on your sketch the coordinates of any turning points and any points where the curve cuts the axes.
   (a) $y = (1 - x)(x - 5)$
   (b) $y = x^2 - 8x - 20$
   (c) $y = x^2 + 2x - 3$
   (d) $y = x^2 + 10x + 10$
   (e) $y = 3x - x^2$
7. Differentiate the following trigonometric functions with respect to x
   (a) $7\sin(8x + 3)$ [Ans: $56\cos(8x + 3)$]
   (b) $15\cos(3x + 4)$ [Ans: $-45\sin(3x + 4)$]
   (c) $24\cos 2x + 3$ [Ans: $-48\sin 2x$]
   (d) $13\sin 4x + 3$ [Ans: $52\cos 4x$]

CHAPTER 6: MATRICES

A matrix is a rectangular array of numbers called elements or entries. Information can conveniently be presented as an array of rows and columns.

Order of a matrix

The order of a matrix gives the format of how a matrix should be written. It is always in the form $m \times n$ where m is the number of rows and n is the number of columns in the matrix. For example

(i) $A\ 2 \times 2\ matrix$

In thus matrix, the number of rows is 2 and the number of columns are also 2 i.e.

$\begin{pmatrix} 8 & 1 \\ -3 & 4 \end{pmatrix}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

(ii) $A\ 3 \times 3\ matrix$

In thus matrix, the number of rows is 3 and the number of columns are also 3 i.e.

$\begin{pmatrix} 3 & 1 & 0 \\ 4 & 0 & 1 \\ 1 & 9 & 2 \end{pmatrix}$; $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

(iii) $A\ 2 \times 3\ matrix$

$\begin{pmatrix} 2 & 4 & 0 \\ 1 & 6 & 9 \end{pmatrix}$

(iv) $A\ 3 \times 2\ matrix$

$\begin{pmatrix} 1 & 3 \\ 5 & 2 \\ 7 & 4 \end{pmatrix}$

Note: Other matrices of different order are possible i.e. $1 \times 2, 2 \times 1, 1 \times 3, 3 \times 1, e.t.c$

Operations on matrices

Addition and Subtraction

Two or more matrices can be added if they have the same order i.e. the number of rows and columns in the first matrix must be equal to the number of rows and columns in the second matrix.

Examples

1. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$
2. $\begin{pmatrix} -2 & 0 \\ 3 & 2 \end{pmatrix} + \begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} -2+-1 & 0+3 \\ 3+0 & 2+2 \end{pmatrix} = \begin{pmatrix} -3 & 3 \\ 3 & 4 \end{pmatrix}$
3. $\begin{pmatrix} 1 & 0 & 1 \\ 3 & -1 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 2 & 1 \\ 2 & 0 & -3 \end{pmatrix} = \begin{pmatrix} 1+3 & 0+2 & 1+1 \\ 3+2 & -1+0 & 2 \pm 3 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 2 \\ 5 & -1 & -1 \end{pmatrix}$
4. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a-e & b-f \\ c-g & d-h \end{pmatrix}$
5. $\begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix} - \begin{pmatrix} -1 & -3 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 3--1 & 1--3 \\ -2-0 & 0-2 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ -2 & -2 \end{pmatrix}$
6. $\begin{pmatrix} 6 & 3 \\ 1 & 2 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 8 & 1 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 6-0 & 3--1 \\ 1-8 & 2-1 \\ 1-3 & 0-0 \end{pmatrix} = \begin{pmatrix} 6 & 4 \\ -7 & 3 \\ -2 & 0 \end{pmatrix}$

Multiplication of matrices

Scalar multiplication

This is the type of multiplication where we multiply a given matrix with a constant, which is taken as a scalar.

Examples

1. Expand $a\begin{pmatrix} b & c \\ e & f \end{pmatrix}$

Solution

$$a\begin{pmatrix} b & c \\ e & f \end{pmatrix} = \begin{pmatrix} a \times b & a \times c \\ a \times e & a \times f \end{pmatrix} = \begin{pmatrix} ab & ac \\ ae & af \end{pmatrix}$$

2. Given the matrix $A = \begin{pmatrix} 3 & 0 \\ 1 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 3 \\ -2 & 8 \end{pmatrix}$. Find (i) $2A$ (ii) $4B - A$ (iii) $3(A + B)$

Solution

(i) $2A = 2\begin{pmatrix} 3 & 0 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 2 \times 3 & 2 \times 0 \\ 2 \times 1 & 2 \times -2 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 2 & -4 \end{pmatrix}$

(ii) $4B = 4\begin{pmatrix} 0 & 3 \\ -2 & 8 \end{pmatrix} = \begin{pmatrix} 4 \times 0 & 4 \times 3 \\ 4 \times -2 & 4 \times 8 \end{pmatrix} = \begin{pmatrix} 0 & 12 \\ -8 & 32 \end{pmatrix}$

$4B - A = \begin{pmatrix} 0 & 12 \\ -8 & 32 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} -3 & 12 \\ -9 & 34 \end{pmatrix}$

(iii) $A + B = \begin{pmatrix} 3 & 0 \\ 1 & -2 \end{pmatrix} + \begin{pmatrix} 0 & 3 \\ -2 & 8 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ -1 & 6 \end{pmatrix}$

$3(A + B) = 3\begin{pmatrix} 3 & 3 \\ -1 & 6 \end{pmatrix} = \begin{pmatrix} 9 & 9 \\ -3 & 18 \end{pmatrix}$

3. Given the matrix $A = \begin{pmatrix} 4 & 1 \\ 5 & 2 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 1 \\ 2 & 3 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 3 \\ 2 & -1 \end{pmatrix}$. Find $\det(2A + 3B - C)$

Solution

$$2A + 3B - C = 2\begin{pmatrix} 4 & 1 \\ 5 & 2 \end{pmatrix} + 3\begin{pmatrix} -1 & 1 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 0 & 3 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 2 \\ 10 & 4 \end{pmatrix} + \begin{pmatrix} -3 & 3 \\ 6 & 9 \end{pmatrix} - \begin{pmatrix} 0 & 3 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 16 & 13 \end{pmatrix} - \begin{pmatrix} 0 & 3 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 14 & 14 \end{pmatrix}$$

$$\det(2A + 3B - C) = 5 \times 14 - 14 \times 2 = 70 - 28 = 42$$

General Multiplication of matrices

We can multiply two or more matrices if and only if the number of columns in the first matrix is equal to the number of rows in the second matrix.

Examples

Expand

1. $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}$

Solution

When we are expanding, we multiply row by column

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a \times e + b \times g & a \times f + b \times h \\ c \times e + d \times g & c \times f + d \times h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

2. $\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 3 & 1 \end{pmatrix}$

Solution

$$\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 3 \times 0 + 1 \times 3 & 3 \times 1 + 1 \times 1 \\ 2 \times 0 + 1 \times 3 & 2 \times 1 + 1 \times 1 \end{pmatrix} = \begin{pmatrix} 0+3 & 3+1 \\ 0+3 & 2+1 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 3 & 3 \end{pmatrix}$$

3. $\begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix}^2$

Solution

$$\begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix}^2 = \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 3 \times 3 + 4 \times 2 & 3 \times 4 + 4 \times 5 \\ 2 \times 3 + 5 \times 2 & 2 \times 4 + 5 \times 5 \end{pmatrix}$$
$$= \begin{pmatrix} 9+8 & 12+20 \\ 6+10 & 8+25 \end{pmatrix} = \begin{pmatrix} 17 & 32 \\ 16 & 33 \end{pmatrix}$$

4. $\begin{pmatrix} 8 & 9 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} -2 & 3 \\ 4 & 0 \end{pmatrix}$

Solution

$$\begin{pmatrix} 8 & 9 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} -2 & 3 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} 8 \times -2 + 9 \times 4 & 8 \times 3 + 9 \times 0 \\ 5 \times -2 + -1 \times 4 & 5 \times 3 + -1 \times 0 \end{pmatrix}$$
$$= \begin{pmatrix} -16+36 & 24+0 \\ -10-4 & 15+0 \end{pmatrix} = \begin{pmatrix} 20 & 24 \\ -14 & 15 \end{pmatrix}$$

5. Given the matrices $= \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} x \\ y \end{pmatrix}$. Find AB

Solution

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

However, the product BA is not defined because it is impossible

6. Given the matrices $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ *and* $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. *Find the matrix products* AB and BA.

What conclusion can you draw from your solution.

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 \times a + 2 \times c & 1 \times b + 2 \times d \\ 3 \times a + 4 \times c & 3 \times b + 4 \times d \end{pmatrix} = \begin{pmatrix} a + 2c & b + 2d \\ 3a + 4c & 3b + 4d \end{pmatrix}$$

$$BA = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} a \times 1 + b \times 3 & a \times 2 + b \times 4 \\ c \times 1 + d \times 3 & c \times 2 + d \times 4 \end{pmatrix} = \begin{pmatrix} a + 3b & 2a + 4b \\ c + 3d & 2c + 4d \end{pmatrix}$$

We can observe that $AB \neq BA$

Note: in general, when multiplying two matrices, the commutative law does not hold i.e. $AB \neq BA$ as we have seen above.

The determinant of a matrix

Consider a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the determinant is denoted by det A or $|A|$ defined as the difference between the product of the elements in the major/leading diagonal and the product of the elements in the minor diagonal i.e.

$|A| = \det A = ad - bc$

Examples

1. If $M = \begin{pmatrix} 4 & 1 \\ 3 & -1 \end{pmatrix}$, find det M

Solution

$\det M = (4 \times -1) - (1 \times 3) = -4 - 3 = -7$

2. $If\ A = \begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix}$, find $|A|$

Solution

$|A| = (1 \times 0) - (3 \times 1) = 0 - 3 = -3$

3. Given that $A = \begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$, *Find* (*i*) $\det(3A + B)$ (*ii*) $\det(2A - B)$

Solution

(i) $3A + B = 3\begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 9 \\ 3 & 0 \end{pmatrix} + \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 11 \\ 4 & 1 \end{pmatrix}$

$\det(3A + B) = (6 \times 1) - (4 \times 11) = 6 - 44 = -38$

(ii) $2A - B = 2\begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 2 & 0 \end{pmatrix} - \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ 1 & -1 \end{pmatrix}$

$\det(2A - B) = (-1 \times -1) - (4 \times 1) = 1 - 4 = -3$

Singular matrix

A matrix whose determinant is zero is called a singular matrix.

Examples

1. Given that $A = \begin{pmatrix} 4 & 6 \\ 1 & 1 \end{pmatrix}$ *and* $B = \begin{pmatrix} -1 & -4 \\ 2 & 1 \end{pmatrix}$. Show that $A + B$ is a singular matrix.

Solution

$A + B = \begin{pmatrix} 4 & 6 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} -1 & -4 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix}$

$\det(A + B) = 3 \times 2 - 2 \times 3 = 6 - 6 = 0$

Since the det (A + B) = 0, A + B is a singular matrix

2. Given that matrix $A = \begin{pmatrix} 4 & 2 \\ a & 3 \end{pmatrix}$ is a singular matrix, find the value of a.

Solution

$\det A = 4 \times 3 - 2 \times a = 12 - 2a$

For a singular matrix, $\det A = 0$

$\Rightarrow 12 - 2a = 0$

$2a = 12\ thus\ a = 6$

3. Given that $M = \begin{pmatrix} 3a & a-6 \\ -6 & a+2 \end{pmatrix}$, *find the values of a for which the matrix M is singular*

Solution

$\det M = 3a(a+2) - (-6)(a-6) = (3a^2 + 6a) - (-6a + 36)$
$= 3a^2 + 6a + 6a - 36$
$= 3a^2 + 12a - 36$

Since matrix M is singular, then det M = 0

$\Rightarrow \quad 3a^2 + 12a - 36 = 0$
$a^2 + 4a - 12 = 0$
$a^2 - 2a + 6a - 12 = 0$
$a(a-2) + 6(a-2) = 0$
$(a-2)(a+6) = 0$
$\therefore a = 2 \text{ or } a = -6$

Inverse of a matrix

The inverse of a matrix A is given by $\frac{1}{\det A} \times$ *the adjoint matrix*. The inverse of a matrix A is denoted by A^{-1}. To get the adjoint, we interchange the entries of the major diagonal and multiply the entries of the minor diagonal by -1 i.e.

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $Adjoint\ A = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$\det A = ad - bc$

$A^{-1} = \frac{1}{\det A} \times Adjoint\ A = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Note: The inverse of a singular matrix does not exist because we end up with a division by zero, which is undefined.

Examples

1. If $A = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix}$, *find* (i) A^{-1} (ii) B^{-1} (iii) $(A+B)^{-1}$

Solution

(i) $\det A = (3 \times 1) - (1 \times 0) = 3 - 0 = 3$

Adjoint $A = \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}$

$A^{-1} = \frac{1}{3}\begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}$

(ii) $\det B = (-1 \times 3) - (2 \times 1) = -3 - 2 = -5$

$Adjoint\ B = \begin{pmatrix} 3 & -2 \\ -1 & -1 \end{pmatrix}$

$B^{-1} = \frac{1}{-5}\begin{pmatrix} 3 & -2 \\ -1 & -1 \end{pmatrix}$

(iii) $A + B = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$

$\det(A+B) = (2 \times 4) - (3 \times 1) = 8 - 3 = 5$

Adjoint $(A+B) = \begin{pmatrix} 4 & -3 \\ -1 & 2 \end{pmatrix}$

$(A+B)^{-1} = \frac{1}{5}\begin{pmatrix} 4 & -3 \\ -1 & 2 \end{pmatrix}$

2. Given the matrix $A = \begin{pmatrix} 4 & 1 \\ 5 & 2 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 1 \\ 2 & 3 \end{pmatrix}$, $find\ (AB)^{-1}$

Solution

$AB = \begin{pmatrix} 4 & 1 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 4\times -1 + 1\times 2 & 4\times 1 + 1\times 3 \\ 5\times -1 + 2\times 2 & 5\times 1 + 2\times 3 \end{pmatrix}$

$= \begin{pmatrix} -4+2 & 4+3 \\ -5+4 & 5+6 \end{pmatrix} = \begin{pmatrix} -2 & 7 \\ -1 & 11 \end{pmatrix}$

$\det(AB) = (-2\times 11) - (-1\times 7) = -22 + 7 = -15$

$Adjoint\ AB = \begin{pmatrix} 11 & -7 \\ 1 & -2 \end{pmatrix}$

$(AB)^{-1} = \frac{1}{-15}\begin{pmatrix} 11 & -7 \\ 1 & -2 \end{pmatrix}$

Identity matrix

An identity matrix is a matrix which has the entries in the major diagonal equal to one and the entries in the minor diagonal all equal to zero e.g. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $is\ a\ 2\times 2$ idententiy matrix, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $is\ a\ 3\times 3$ identity matrix. The identity Matrix is denoted by I.

Examples

1. Given that $A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$, find (i) A^{-1} (ii) AA^{-1}

Solution

(i) $\det A = (1\times 1) - (1\times 2) = 1 - 2 = -1$

$Adj\ A = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}$

$A^{-1} = \frac{1}{-1}\begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$

(ii) $AA^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1\times -1 + 2\times 1 & 1\times 2 + 2\times -1 \\ 1\times -1 + 1\times 1 & 1\times 2 + 1\times -1 \end{pmatrix}$

$= \begin{pmatrix} -1+2 & 2-2 \\ -1+1 & 2-1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Note: We can observe that the above product gives an identity matrix. Thus for all matrices, the product $AA^{-1} = I$ where I is an identity matrix.

2. Given that A is the matrix $\begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix}$. Determine the scalars $x\ and\ y$ such that $A^2 + xA + yI = 0$ where $I\ is\ a\ 2\times 2$ identity matrix.

Solution

$A^2 = \begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 2\times 2 + 4\times -1 & 2\times 4 + 4\times 3 \\ -1\times 2 + 3\times -1 & -1\times 4 + 3\times 3 \end{pmatrix} = \begin{pmatrix} 0 & 20 \\ -5 & 5 \end{pmatrix}$

$xA = x\begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 2x & 4x \\ -x & 3x \end{pmatrix}$

$yI = y\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}y & 0\\0 & y\end{pmatrix}$

Given $A^2 + xA + yI = 0$

$\begin{pmatrix}0 & 20\\-5 & 5\end{pmatrix} + \begin{pmatrix}2x & 4x\\-x & 3x\end{pmatrix} + \begin{pmatrix}y & 0\\0 & y\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$

$\begin{pmatrix}2x+y & 20+4x\\-5-x & 5+3x+y\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$

$-5 - x = 0 \ldots\ldots(i)$

$x = -5$

$2x + y = 0 \ldots..(ii)$

$y = -2x$

$y = -2(-5) = 10$

3. Given the matrix $P = \begin{pmatrix}-2 & -4\\3 & 5\end{pmatrix}$, determine the matrix Q such that $QP = I$ where I is a 2×2 identity matrix

Solution

$let\ Q = \begin{pmatrix}a & b\\c & d\end{pmatrix}$

$\Rightarrow \begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}-2 & -4\\3 & 5\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$

$\begin{pmatrix}-2a+3b & -4a+5b\\-2c+3d & -4c+5d\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$

$-2a + 3b = 1 \ldots..(i)$

$-4a + 5b = 0 \ldots..(ii)$

We can solve the simultaneous equations (i) and (ii) using elimination or substitution method.

Using elimination

$2(i);\quad -4a + 6b = 2$

(ii) $\underline{-\quad -4a + 5b = 0}$

$b = 2$

Substitute for b in eqn (ii)

$-4a + 5 \times 2 = 0$

$-4a = -10 \Rightarrow a = \frac{10}{4} = \frac{5}{2} = 2.5$

Similarly $-2c + 3d = 0 \ldots..(iii)$

$-4c + 5d = 1 \ldots\ldots..(iv)$

Solving equation (iii) and (iv) simultaneously gives;

$2(iii)\quad -4c + 6d = 0$

$\underline{-\quad -4c + 5d = 1}$

$d = -1$

Substitute for d in (iv)

$-4c + 5(-1) = 1$

$-4c = 6 \Rightarrow c = -\frac{6}{4} = -1.5$

Therefore the matrix $Q = \begin{pmatrix}2.5 & 2\\-1.5 & -1\end{pmatrix}$

Solving simultaneous equations using matrices

One of the most important applications of matrices is to find the solution of linear simultaneous equations. It is a requirement to first re-arrange the given simultaneous equations into matrix format.

Example 1

Consider the simultaneous equations

$$x + 2y = 4$$
$$3x - 5y = 1$$

Provided you understand how matrices are multiplied together, you will realize these can be written in matrix form as;

$$\begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

Writing $A = \begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ $\quad and\ B = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

$we\ have\ AX = B$

This is the **matrix form** of the simultaneous equations. Here the unknown matrix is X, since A and B are already known. A is called the **matrix of coefficients.**

Now given $AX = B$, we can multiply both sides by the inverse of A, provided it exists to obtain;

$A^{-1}AX = A^{-1}B$

But $AA^{-1} = I, the\ identity\ matrix$

Further more $IX = X$ since multiplying any matrix by an identity matrix of the appropriate order leaves the matrix unaltered. Therefore $X = A^{-1}B$

$Thus\ if\ AX = B, then\ X = A^{-1}B$

This result gives us a method for solving simultaneous equations. All we need to do is to write them in matrix form, calculate the inverse of the matrix of coefficients, and finally perform and finally perform a matrix multiplication.

Solution

$$\begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

We need to calculate the inverse of $A = \begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix}$

$$detA = (1 \times -5) - (2 \times 3) = -11$$

$$A^{-1} = -\frac{1}{11}\begin{pmatrix} -5 & -2 \\ -3 & 1 \end{pmatrix}$$

$$X = A^{-1}B = -\frac{1}{11}\begin{pmatrix} -5 & -2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = -\frac{1}{11}\begin{pmatrix} -5 \times 4 + -2 \times 1 \\ -3 \times 4 + 1 \times 1 \end{pmatrix} = -\frac{1}{11}\begin{pmatrix} -22 \\ -11 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \Rightarrow x = 2 \quad and\ y = 1$$

Example 2

Using matrices, calculate the values of x and y for the following simultaneous equations

$$2x - 2y - 3 = 0$$
$$8y = 7x + 2$$

Solution

Step 1: write the equations in the form $ax + by = c$

$2x - 2y = 3$
$7x - 8y = -2$

Step 2: write the equations in matrix form

$$\begin{array}{r} \text{coefficients of first equation} \rightarrow \\ \text{coefficients of second equation} \rightarrow \end{array} \begin{pmatrix} 2 & -2 \\ 7 & -8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \begin{array}{l} \leftarrow \text{constant of first equation} \\ \leftarrow \text{constant of second equation} \end{array}$$

Step 3: Find the inverse of a 2×2 matrix

$$determinant = (2 \times -8) - (-2 \times 7) = -16 + 14 = -2$$

$$inverse = -\frac{1}{2}\begin{pmatrix} -8 & 2 \\ -7 & 2 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 3.5 & -1 \end{pmatrix}$$

Step 4: Multiply both sides of the matrix equation by the inverse

$$\begin{pmatrix} 4 & -1 \\ 3.5 & -1 \end{pmatrix}\begin{pmatrix} 2 & -2 \\ 7 & -8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 3.5 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \times 3 + -1 \times -2 \\ 3.5 \times 3 + -1 \times -2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 14 \\ 12.5 \end{pmatrix}$$

Therefore $x = 14 \quad and\ y = 12.5$

Example 3

Solve the simultaneous equations below using the matrix method.

$x + 2y = 4$
$x + y = 3$

Solution

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$let\ A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix},\ B = \begin{pmatrix} x \\ y \end{pmatrix}\ and\ C = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

$Now\ AB = C \quad \Rightarrow B = \frac{C}{A}$ which gives $B = A^{-1}C$

$$B = A^{-1}C$$

$$\det A = (1 \times 1) - (2 \times 1) = 1 - 2 = -1$$

$$A^{-1} = \frac{1}{-1}\begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$$

But $B = A^{-1}C \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \times 4 + 2 \times 3 \\ 1 \times 4 + -1 \times 3 \end{pmatrix} = \begin{pmatrix} -4 + 6 \\ 4 + -3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

From the equality of matrices, $x = 2\ and\ y = 1$

Example 4

Solve the simultaneous equations using the matrix method

$2x + y = 3$
$4x - 2y = 10$

Solution

$$\begin{pmatrix}2 & 1\\4 & -2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}3\\10\end{pmatrix}$$

$$let\ A=\begin{pmatrix}2 & 1\\4 & -2\end{pmatrix},\ B=\begin{pmatrix}x\\y\end{pmatrix}\ and\ C=\begin{pmatrix}3\\10\end{pmatrix}$$

$$\det A=(2\times -2)-(4\times 1)=-4-4=-8$$

$$A^{-1}=-\tfrac{1}{8}\begin{pmatrix}-2 & -1\\-4 & 2\end{pmatrix}$$

$$\text{But } B=A^{-1}C\Rightarrow\begin{pmatrix}x\\y\end{pmatrix}=-\tfrac{1}{8}\begin{pmatrix}-2 & -1\\-4 & 2\end{pmatrix}\begin{pmatrix}3\\10\end{pmatrix}$$

$$\begin{pmatrix}x\\y\end{pmatrix}=-\tfrac{1}{8}\begin{pmatrix}-2\times 3+-1\times 10\\-4\times 3+2\times 10\end{pmatrix}=-\tfrac{1}{8}\begin{pmatrix}-16\\8\end{pmatrix}=\begin{pmatrix}2\\-1\end{pmatrix}$$

$$x=2\ and\ y=-1$$

Solving mathematical problems using the matrix method

In day-to-day life, we are obligated to solving problems which require the ideas of finding total expenditures, the would be amount of profit got after transacting a business and others. These mathematical problems can be solved using the matrix approach

Example 1

A grocery sells two kinds of meat products A and B. Athieno bought 4 kg of A and 6 kg of B paying a total of B paying a total of shs 5280/=, Namusisi bought 5kg of A and 3 kg of B at a total cost of 4440/=

(i) Write down two equations to describe Athieno's and Namusisi's purchase
(ii) By combining the two equations in matrix form, determine the cost of 1 kg of each meat product
(iii) How much would Mugisha pay for 6 kg of A and 5 kg of B

Solution

Let $x = cost\ of\ grade\ A$ per kg

$y = cost\ of\ grade\ B$ per kg

Athieno's purchase; $4x+6y=5280$

Namusisi's purchase; $5x+3y=4440$

(i) The equations are;

$$4x+6y=5280$$
$$5x+3y=4440$$

(ii) $\begin{pmatrix}4 & 6\\5 & 3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}5280\\4440\end{pmatrix}$

Determinant of the matrix of coefficients $=(4\times 3)-(5\times 6)=-18$

The inverse matrix $=-\frac{1}{18}\begin{pmatrix}3 & -6\\-5 & 4\end{pmatrix}$

Thus $\begin{pmatrix}x\\y\end{pmatrix}=-\frac{1}{18}\begin{pmatrix}3 & -6\\-5 & 4\end{pmatrix}\begin{pmatrix}5280\\4440\end{pmatrix}$

$$\begin{pmatrix}x\\y\end{pmatrix}=-\tfrac{1}{18}\begin{pmatrix}15840-26240\\-26400+17760\end{pmatrix}$$

$$\begin{pmatrix}x\\y\end{pmatrix}=-\tfrac{1}{18}\begin{pmatrix}-10800\\8640\end{pmatrix}$$

$$x=-\tfrac{1}{18}\times -10800=600$$

$$y = -\frac{1}{18} \times -8640 = 480$$

Hence the cost of 1 kg of A = sh 600

And the cost of 1 kg of B = sh 480

(iii) Mugisha paid

$$(6 \times 600) + (5 \times 480) = sh\ 6000$$

Example 2

At Jenga-mwili supermarket, Mercy bought 5 trays of eggs and 7 kg of irish potatoes at shs 11800. Moses bought 6 trays of eggs and 8 kg of irish potatoes at shs. 14000. If shs t and shs p are the prices of a tray of eggs and a kg of potatoes respectively

(i) Write two equations to describe the purchase of the two men

(ii) By combining the two equations to a matrix form, determine the cost of purchasing each item

(iii) How much would Hanifa pay for 2 trays of eggs and 2 kilograms of irish potatoes?

Solution

(i) $5t + 7p = 11800 \ldots\ldots\ldots\ldots (i)$

$6t + 8p = 14000 \ldots\ldots\ldots\ldots (ii)$

(ii) $\begin{pmatrix} 5 & 7 \\ 6 & 8 \end{pmatrix}\begin{pmatrix} t \\ p \end{pmatrix} = \begin{pmatrix} 11800 \\ 14000 \end{pmatrix}$

Determinant of the matrix of coefficients = $(5 \times 8) - (7 \times 6) = -2$

The inverse matrix = $-\frac{1}{2}\begin{pmatrix} 8 & -7 \\ -6 & 5 \end{pmatrix}$

Thus $\begin{pmatrix} t \\ p \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 8 & -7 \\ -6 & 5 \end{pmatrix}\begin{pmatrix} 11800 \\ 14000 \end{pmatrix}$

$$\begin{pmatrix} t \\ p \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 94400 - 98000 \\ -70800 + 70000 \end{pmatrix}$$

$$\begin{pmatrix} t \\ p \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} -3600 \\ -800 \end{pmatrix}$$

$$t = -\frac{1}{2} \times -3600 = 1800/=$$

$$p = -\frac{1}{2} \times -800 = 400/=$$

(iii) For 2 trays and 2 kg of irish potatoes, Hanifa pays

$$2 \times 1800 + 2 \times 400 = 4400$$

Example 3

Four students; Kale, Linda, Musa and Nana went to a stationery shop.

Kale bought 4pens, 6 counter books and 1 graph book

Linda bought 10 pens and 5 counter books

Musa bought 3 pens and 3 graph books

Nana bought 5 pens, 2 counter books and 8 graph books

The costs of a pen, a counter book and a graph were shs 400, shs 1200 and shhs 1000 respectively.

(a) (i) Write a 4×3 matrix for the items bought by the four students

(ii) Write a 3×1 matrix for the costs of each item

(b) Use the matrices in (a) to calculate the amount of money spent by each student

(c) If each student was to buy 4 pens, 10 counter books and 6 graph books, how much money would be spent by all the four students

Solution

(a) Let P = pen , C = counter books and G = graph books

(i)

$$\begin{array}{cc} & \begin{matrix} P & C & G \end{matrix} \\ \begin{matrix} Kale \\ Linda \\ Musa \\ Nana \end{matrix} & \begin{pmatrix} 4 & 6 & 1 \\ 10 & 5 & 0 \\ 3 & 0 & 3 \\ 5 & 2 & 8 \end{pmatrix} \end{array}$$

Hence the 4×3 matrix for the items bought is;

$$\begin{pmatrix} 4 & 6 & 1 \\ 10 & 5 & 0 \\ 3 & 0 & 3 \\ 5 & 2 & 8 \end{pmatrix}$$

(ii)

$$\begin{array}{cc} & \text{Cost} \\ \begin{matrix} P \\ C \\ G \end{matrix} & \begin{pmatrix} 400 \\ 1200 \\ 1000 \end{pmatrix} \end{array}$$

Hence the 3×1 matrix for the cost of each item is;

$$\begin{pmatrix} 400 \\ 1200 \\ 1000 \end{pmatrix}$$

(b) By multiplying the two matrices

$$\begin{array}{cc} & \begin{matrix} P & C & G \end{matrix} \\ \begin{matrix} Kale \\ Linda \\ Musa \\ Nana \end{matrix} & \begin{pmatrix} 4 & 6 & 1 \\ 10 & 5 & 0 \\ 3 & 0 & 3 \\ 5 & 2 & 8 \end{pmatrix} \end{array} \begin{pmatrix} 400 \\ 1200 \\ 1000 \end{pmatrix} = \begin{matrix} Kale \\ Linda \\ Musa \\ Nana \end{matrix} \begin{pmatrix} 4 \times 400 + 6 \times 1200 + 1 \times 1000 \\ 10 \times 400 + 5 \times 1200 + 0 \times 1000 \\ 3 \times 400 + 0 \times 1200 + 3 \times 1000 \\ 5 \times 400 + 2 \times 1200 + 8 \times 1000 \end{pmatrix}$$

$$= \begin{matrix} Kale \\ Linda \\ Musa \\ Nana \end{matrix} \begin{pmatrix} 1600 + 7200 + 1000 \\ 4000 + 6000 + 0 \\ 1200 + 0 + 3000 \\ 2000 + 2400 + 8000 \end{pmatrix} = \begin{matrix} Kale \\ Linda \\ Musa \\ Nana \end{matrix} \begin{pmatrix} 9800 \\ 10000 \\ 4200 \\ 12400 \end{pmatrix}$$

Hence kale spent shs 9800, Linda shs 10000, Musa shs 4200 and Nana shs 12400

(c)

$$\begin{array}{cc} & \begin{matrix} P & C & G \end{matrix} \\ \begin{matrix} Kale \\ Linda \\ Musa \\ Nana \end{matrix} & \begin{pmatrix} 4 & 10 & 6 \\ 4 & 10 & 6 \\ 4 & 10 & 6 \\ 4 & 10 & 6 \end{pmatrix} \end{array} \begin{pmatrix} 400 \\ 1200 \\ 1000 \end{pmatrix}$$

Cost by Kale = $4 \times 400 + 10 \times 1200 + 6 \times 1000 = 1600 + 12000 + 6000 = 19600$

Total cost by all the four students = $4 \times 19600 = 78{,}400$

Example 4

A retail trader ordered for shirts from a Kampala wholesale shop as follows

Colour	Size			
	Small	Medium	Large	Extra large
Blue	0	40	20	0
Green	30	0	25	0
Yellow	0	20	0	10

Given below is the cost for each size of shirt

	Size			
	Small	Medium	Large	Extra large
Cost (U shs)	3000	3600	4200	4800

(a) Write down a
   (i) 4×3 matrix for the order of the shirts made
   (ii) A 4×1 cost matrix
(b) Given that the trader had to pay 17% tax of the cost of the shirts purchased, find his expenditure on the order.

Solution

(i)

$$\begin{array}{r c} & \begin{matrix} B & G & Y \end{matrix} \\ \begin{matrix} small \\ Medium \\ Large \\ Extra-large \end{matrix} & \begin{pmatrix} 0 & 30 & 0 \\ 40 & 0 & 20 \\ 20 & 25 & 0 \\ 0 & 0 & 10 \end{pmatrix} \end{array}$$

(ii)

$$\begin{array}{r c} \begin{matrix} small \\ Medium \\ Large \\ Extra-large \end{matrix} & \begin{pmatrix} 3000 \\ 3600 \\ 4200 \\ 4800 \end{pmatrix} \end{array}$$

(b)

$$\text{Cost incurred} = \begin{pmatrix} 0 & 40 & 20 & 0 \\ 30 & 0 & 25 & 0 \\ 0 & 20 & 0 & 10 \end{pmatrix}\begin{pmatrix} 3000 \\ 3600 \\ 4200 \\ 4800 \end{pmatrix}$$

$$= \begin{pmatrix} 0 + 40 \times 3600 + 20 \times 4200 + 0 \\ 30 \times 3000 + 0 + 25 \times 4200 + 0 \\ 0 + 20 \times 3600 + 0 + 10 \times 4800 \end{pmatrix}$$

$$= \begin{pmatrix} 144000 + 84000 \\ 90000 + 105000 \\ 72000 + 48000 \end{pmatrix} = \begin{pmatrix} 228000 \\ 195000 \\ 120000 \end{pmatrix}$$

$$= 228000 + 195000 + 120000 = 543000$$

Tax paid $= \frac{17}{100} \times 543000 = 92310$

Total expenditure $= 543000 + 92310 = 635310/=$

Trial questions

1. Solve the following sets of simultaneous equations using the inverse matrix method
   (a) $5x + y = 13 \quad and \quad 3x + 2y = 5$
   (b) $3x + 2y = -2 \quad and \; x + 4y = 6$
   (c) $4x + 2y = 6 \; and \; 3x + 5y = 5$
   (d) $7x + 4 = 5y \quad and \; 4 - 2x + y = 0$
   (e) $2y - 4x + 2 = 0 \quad and \; 3x - 2y = 5$
   (f) $3x + y = 5 \quad and \; 7x = 2y + 2$
   (g) $3x + 2y = 5 \quad and \; 5x + 4y = 11$
   [Ans: (a) $x = 3, y = -2$ (b)$x = -2, y = 2$ (c) $x = \frac{10}{7}, y = \frac{1}{7}$ (d) $x = 8, y = 12$
   (e) $x = -3, y = -7$ (f) $x = \frac{12}{13}, y = \frac{29}{13}$ (g)$x = -1, y = 4$]
2. Given the matrices $A = \begin{pmatrix} 1 & 0 \\ 4 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 6 & -1 \\ -2 & -3 \end{pmatrix}$ find
   (i) $Matrix\ C\ which\ is\ equal\ to\ 2A - 3B$
   (ii) AB
   (iii) $Show\ that\ \det(AB) = (\det A)(\det B)$ [Ans: (i) $\begin{pmatrix} -16 & 3 \\ 14 & 19 \end{pmatrix}$ (ii) $\begin{pmatrix} 6 & -1 \\ 14 & -19 \end{pmatrix}$]
3. Given that $A = \begin{pmatrix} 3 & 0 \\ 1 & -2 \end{pmatrix}$ $and\ B = \begin{pmatrix} 1 & 2 \\ 5 & 0 \end{pmatrix}$, determine (i) $A + B$ (ii) $(AB)^2$
   [Ans: (i) $\begin{pmatrix} 4 & 2 \\ 6 & -2 \end{pmatrix}$ (ii) $\begin{pmatrix} -45 & 30 \\ -45 & -50 \end{pmatrix}$]
4. Given that $\begin{pmatrix} 3-a & 3 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} -3 \\ x \end{pmatrix} = \begin{pmatrix} -3 \\ x \end{pmatrix}$ $find\ the\ values\ of\ a\ and\ x$
   [$Ans: a = 1, x = 1$]
5. Given that $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$ and $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ $find\ AB - BA$ [Ans: $\begin{pmatrix} -4 & 0 \\ 0 & 4 \end{pmatrix}$]
6. A and B are two matrices such that $A = \begin{pmatrix} 1 & 3 \\ 4 & 11 \end{pmatrix}$ $and\ B = \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix}$; find
   (i) matrix $P = AB$ (ii) P^{-1} [Ans: (i) $\begin{pmatrix} 2 & 11 \\ 7 & 41 \end{pmatrix}$ (ii) $-\frac{1}{5}\begin{pmatrix} 41 & -11 \\ -7 & 2 \end{pmatrix}$]
7. Given the matrices $P = \begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix}, Q = \begin{pmatrix} 3 & 2 \\ 4 & -1 \end{pmatrix}, R = \begin{pmatrix} 5 & -4 \\ -1 & 2 \end{pmatrix}$; determine
   (i)$P.Q + R$.(ii) $the\ determinant\ of\ (P.Q + R)$ [Ans: (i) $\begin{pmatrix} 3 & 2 \\ 3 & 1 \end{pmatrix}$ (ii) -3]
8. Find the inverse of $A = \begin{pmatrix} 4 & -1 \\ 2 & 3 \end{pmatrix}$, [$Ans$: $\frac{1}{14}\begin{pmatrix} 3 & 1 \\ -2 & 4 \end{pmatrix}$]
9. Given that $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ $show\ that\ A^2 - 4A = I\ where\ I\ is\ a\ 2 \times 2\ identity\ matrix$
10. Given that matrix $A = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$, $find\ the\ values\ of\ the\ scalar\ \lambda\ for\ which\ A - \lambda I$ is singular where I is a 2×2 identity matrix. [Ans: $\lambda = 4\ or -1$]
11. Given that $A = \begin{pmatrix} 2 & -1 \\ 0 & -1 \end{pmatrix}$ $and\ B = \begin{pmatrix} 6 & 8 \\ 10 & -12 \end{pmatrix}$; $Find\ (AB)^{-1}$ [Ans: $\frac{1}{304}\begin{pmatrix} 12 & -28 \\ 10 & 2 \end{pmatrix}$]
12. Given that $P = \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}, Q = \begin{pmatrix} -1 & 1 \\ 4 & 5 \end{pmatrix}$ $and\ R = \begin{pmatrix} 4 & 6 \\ 10 & 15 \end{pmatrix}$, $find\ the\ matrix\ T$ such that $T = P^2 + 3Q - R$ [Ans: $\begin{pmatrix} 2 & 9 \\ 26 & 33 \end{pmatrix}$]

13. The matrix $\begin{pmatrix}0 & 4\\3 & -1\end{pmatrix}$ is pre multiplied by the column matrix $\begin{pmatrix}x\\y\end{pmatrix}$ to give $\begin{pmatrix}-8\\x\end{pmatrix}$, find the values of x and y. [Ans: $x = -1, y = -2,$]
14. Given the matrix $P = \begin{pmatrix}2 & 3\\1 & 2\end{pmatrix}$ $and\ Q = \begin{pmatrix}2 & -3\\-1 & 2\end{pmatrix}$, find (i) PQ (ii) a 2×2 matrix R such that QR = P [Ans: (i) $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ $(ii)\begin{pmatrix}7 & 12\\4 & 7\end{pmatrix}$]
15. Given that $P = \begin{pmatrix}2 & -1\\3 & -2\end{pmatrix}, Q = \begin{pmatrix}1 & 5\\2 & -3\end{pmatrix}$ $and\ R = \begin{pmatrix}4 & 3\\1 & -2\end{pmatrix}$, find (i) $QR - P$ (ii) determinant of QR – P [Ans: (i) $\begin{pmatrix}7 & -6\\2 & 14\end{pmatrix}$ (ii) 110]
16. Given the matrix $A = \begin{pmatrix}2 & 3\\5 & 7\end{pmatrix}$, $find\ a\ matrix\ B\ such\ that\ A + B = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$
[Ans: $\begin{pmatrix}-1 & -3\\-5 & -6\end{pmatrix}$]
17. If $\begin{pmatrix}4 & 1\\x & -1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}4\\8\end{pmatrix}$, $determine\ the\ values\ of\ x\ and\ y$ [Ans: $x = 2\ or - 6, y = -4\ or\ 28$]
18. Given that $A = \begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}$ $and\ B = \begin{pmatrix}-1 & -2\\0 & 1\end{pmatrix}$, $Evaluate\ (A + B)^2$ [Ans: $\begin{pmatrix}0 & 0\\15 & 25\end{pmatrix}$]
19. If $\begin{pmatrix}2 & 4\\-3 & 3\end{pmatrix} + k\begin{pmatrix}3 & 1\\0 & n\end{pmatrix} = \begin{pmatrix}8 & 6\\-3 & -1\end{pmatrix}$, $find\ the\ values\ of\ k\ and\ n$ [Ans: k = 2, n = -2]
20. Find the values of a and b such that $\begin{pmatrix}3 & b\\4 & a\end{pmatrix}\begin{pmatrix}7a\\2\end{pmatrix} = \begin{pmatrix}43\\30\end{pmatrix}$ [Ans: $a = 1, b = 11$]
21. Given that $A = \begin{pmatrix}3 & -2\\-4 & 5\end{pmatrix}$, $find\ a\ matrix\ B\ such\ that\ AB = \begin{pmatrix}7 & 0\\0 & 7\end{pmatrix}$. Hence or otherwise find the inverse of matrix A. [Ans: $B = \begin{pmatrix}5 & 20\\4 & 3\end{pmatrix}$; $A^{-1} = \frac{1}{7}\begin{pmatrix}5 & 2\\4 & 3\end{pmatrix}$]
22. Given the matrices $A = \begin{pmatrix}4.5 & 1\\0 & 7\end{pmatrix}$ $and\ B = \begin{pmatrix}2 & -1\\3 & 1\end{pmatrix}$, find a matrix M such that $3M - 2I = 2A - B$ $where\ I\ is\ a\ 2 \times 2\ identity\ matrix$ [Ans: $M = \begin{pmatrix}3 & 1\\-1 & 5\end{pmatrix}$]
23. If $A = \begin{pmatrix}1 & -1\\2 & -1\end{pmatrix}$ $and\ B = \begin{pmatrix}1 & 1\\4 & -1\end{pmatrix}$. $Show\ that\ (A + B)^2 = A^2 + B^2$
24. Given that $D = \begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}$ $and\ I$ $is\ a$ 2×2 identity matrix, obtain the values of p and q such that $D^2 = pD + qI$. [Ans: p = 2, q = -1]
25. Given the matrices $A = \begin{pmatrix}2 & 1\\2 & 3\end{pmatrix}$; find λ $such\ that$ $|A - \lambda I| = 0$, where I is a 2×2 identity matrix. [Ans: $\lambda = 1\ or\ 4$]
26. Find the possible values x can take on given $A = \begin{pmatrix}x^2 & 3\\1 & 3x\end{pmatrix}$ $and\ B = \begin{pmatrix}3 & 6\\2 & x\end{pmatrix}$ and AB = BA. [Ans: $x = 3\ or\ x = -\frac{1}{2}$]
27. Given the matrices $A = \begin{pmatrix}1 & 0\\2 & -1\end{pmatrix}$ $and\ B = \begin{pmatrix}-2 & 1\\3 & 0\end{pmatrix}$, find $(A + B)^2$[Ans: $\begin{pmatrix}11 & -4\\-10 & 11\end{pmatrix}$]
28. Given the matrix $C = \begin{pmatrix}1 & -1\\2 & 4\end{pmatrix}$, find the values of b such that $C^2 - 5C + bI = 0$
[Ans: b = 6]
29. Sarah found out that she could buy 12 pencils and 10 books for shs. 2100. Alternatively she could buy 20 pencils and 4 books for shs. 1600 at the same price per item. Find the cost of each item [Ans: pencil = 50/= : book = 150/=]

30. Two ladies Sarah and Fiona went for shopping. Sarah bought 1 kg of sugar, 500 gm of tea, 2 loaves of bread while Fiona bought 2 kg of sugar, 1 loaf of bread and 3 litres of milk. The cost price of the items were; sugar shs 2200 per kg, tea shs 500 per kg, shs 3500 per loaf and shs 1200 per litre of milk.

(a) Write down a matrix for the

(i) Purchases

(ii) Cost price

(b) Use matrix multiplication to determine the difference in expenditure of the two ladies.

[Ans: (a) (i) $\begin{array}{c} \\ Sarah \\ Fiona \end{array} \begin{pmatrix} S & T & B & M \\ 1 & \frac{1}{2} & 2 & 0 \\ 2 & 0 & 1 & 3 \end{pmatrix}$ (ii) $\begin{array}{c} S \\ T \\ L \\ M \end{array} \begin{pmatrix} 2200 \\ 500 \\ 3500 \\ 1200 \end{pmatrix}$ (iii) 2050/=]

CHAPTER 7: INTEGRATION

Integration is the process of obtaining an original function from a given gradient function; hence, it is the reverse of differentiation.

$$If\ \frac{dy}{dx} = x^n\ , then\ y = \frac{x^{n+1}}{n+1}\quad where\ n \neq -1$$

$$\Rightarrow \int ax^n\, dx = \frac{x^{n+1}}{n+1} + C\ \ where\ C\ is\ an\ arbitrary\ constant$$

The general rule when integrating a power of x is that we add one onto the exponent/power and then divide by the new exponent/power. It clear (hopefully) that we will need to avoid $n = -1$ in this formula because we will end up with division by zero, which is undefined.

Indefinite integrals

We call $\int f(x)\, dx$ an indefinite integral because it does not give a definite answer and we add an arbitrary constant after integrating.

We know that $y = x^3, y = x^3 + 5\ , y = x^3 - 6$ all satisfy $\frac{dy}{dx} = 3x^2$, for this reason

When we integrate $3x^2, we\ write\ y = x^3 + C$ because we are not certain whether the original function had a constant or not as when we differentiate a constant we get zero.

Note: We always integrate with respect to a certain variable i.e. $\int f(x)\, dx$ means integrating the function with respect to x and $\int f(t)\, dt$ means integrating the function with respect to t.

Example 1

Integrate the following with respect to x

(a) 5

Solution

$$\int 5\, dx = \int 5x^0\, dx = \frac{5x^{0+1}}{0+1} + C = 5x + C$$

Hence $\int k\, dx = kx + C, where\ k\ and\ C\ are\ constants$

(b) x^3

Solution

$$\int x^3\, dx = \frac{x^{3+1}}{3+1} + C = \frac{x^4}{4} + C$$

(c) $x^{\frac{3}{2}}$

Solution

$$\int x^{\frac{3}{2}}\, dx = \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + C = \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + C = \frac{2x^{\frac{5}{2}}}{5} + C$$

(d) $4\sqrt{x}$

Solution

$$\int 4\sqrt{x}\, dx = \int 4x^{1/2}\, dx = 4\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{4x^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{8x^{\frac{3}{2}}}{3} + C$$

(e) $7x^5$

Solution

$$\int 7x^5\,dx = \frac{7x^{5+1}}{6} + C = \frac{7x^6}{6} + C$$

(f) $\frac{1}{\sqrt{x}}$

Solution

$$\int \frac{1}{\sqrt{x}}dx = \int \frac{1}{x^{\frac{1}{2}}}dx = \int x^{-\frac{1}{2}}\,dx = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C = 2x^{\frac{1}{2}} + C = 2\sqrt{x} + C$$

Integrating a sum or difference

When integrating a sum or difference, just like differentiating, we integrate the terms separately.

Example 2

Integrate $3x^3 - 4x^2 + 5x - 1$ with respect to x

Solution

$$\int (3x^3 - 4x^2 + 5x - 1)\,dx = \int 3x^3\,dx - \int 4x^2\,dx + \int 5x\,dx - \int 1\,dx$$

$$= \frac{3x^4}{4} - \frac{4x^3}{3} + \frac{5x^2}{2} - x + C$$

Note: with practice, the reader can do the above integral at once without even separating the terms as in the next example.

Example 3

Integrate $5t^3 - 10t^{-6} + 4$ *with respect to* t

Solution

$$\int (5t^3 - 10t^{-6} + 4)\,dt = 5\left(\frac{1}{4}\right)t^4 - 10\left(\frac{1}{-5}\right)t^{-5} + 4t + C$$

$$= \frac{5}{4}t^4 + 2t^{-5} + 4t + C$$

Techniques of integrating

Some integrals require simplification before you can do the integrating. The methods of simplifying can be expanding if given a product or dividing to simplest form as in the following examples.

Example 4

Find $\int (x-1)(x^3+2)\,dx$

Solution

We need to first expand the product $(x-1)(x^3+2)$

$$(x-1)(x^3+2) = x^4 + 2x - x^3 - 2 = x^4 - x^3 + 2x - 2$$

We can integrate the result of expansion, thus

$$\int (x-1)(x^3+2)\,dx = \int (x^4 - x^3 + 2x - 2)\,dx$$

$$= \frac{x^5}{5} - \frac{x^4}{4} + \frac{2x^2}{2} - 2x + C$$

$$= \frac{x^5}{5} - \frac{x^4}{4} + x^2 - 2x + C$$

Example 5

Find $\int \frac{x^5 - x^2 + 1}{x^2} dx$

Solution

We need to first simplify $\frac{x^5 - x^2 + 1}{x^2}$ by dividing the terms separately

$$\frac{x^5 - x^2 + 1}{x^2} = \frac{x^5}{x^2} - \frac{x^2}{x^2} + \frac{1}{x^2} = x^3 - 1 + x^{-2}$$

We can integrate the result of division, thus

$$\int \frac{x^5 - x^2 + 1}{x^2} dx = \int (x^3 - 1 + x^{-2})\, dx = \frac{x^4}{4} - x + \frac{x^{-1}}{-1} + C$$

$$= \frac{x^4}{4} - x - \frac{1}{x} + C$$

Definite integrals

A definite integral is one that gives a definite answer since it has limits i.e.

$\int_a^b f(x)\, dx$ is a definite integral where a is the lower limit and b, the upper limit.

Suppose $\int f(x)\, dx = F(x) + c$

$$\int_{x=a}^{x=b} f(x)\, dx = [F(b) + c] - [F(a) + c]$$

$$= F(b) - F(a)$$

We usually write this as $\int_{x=a}^{x=b} f(x)\, dx = [F(x)]_a^b$

Note: The constants of integration cancel out in case of a definite integral thus there is no need to add an arbitrary constant to the final answer.

Examples

Evaluate the following integrals

1. $\int_1^4 x^{\frac{1}{2}}\, dx$

Solution

$$\int_1^4 x^{\frac{1}{2}}\, dx = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4 = \left[\frac{2}{3}x^{\frac{3}{2}}\right]_1^4 = \frac{2}{3}(4)^{\frac{3}{2}} - \frac{2}{3}(1)^{\frac{3}{2}}$$

$$= \frac{16}{3} - \frac{2}{3} = \frac{14}{3}$$

2. $\int_0^2 x^2 + 1\, dx$

Solution

$$\int_0^2 x^2 + 1\, dx = \left[\frac{x^3}{3} + x\right]_0^2 = \left(\frac{2^3}{3} + 2\right) - \left(\frac{0^3}{3} + 0\right)$$

$\frac{8}{3}+2=\frac{14}{3}$

3. $\int_{-3}^{1} 6x^2-5x+2\,dx$

Solution

$$\int_{-3}^{1} 6x^2-5x+2\,dx=\left[2x^3-\frac{5}{2}x^2+2x\right]_{-3}^{1}$$
$$=\left(2(1)^3-\tfrac{5}{2}(1)^2+2(1)\right)-\left(2(-3)^3-\tfrac{5}{2}(-3)^2+2(-3)\right)$$
$$=\left(2-\tfrac{5}{2}+2\right)-\left(-54-\tfrac{45}{2}-6\right)$$
$$=\tfrac{3}{2}-\left(-\tfrac{165}{2}\right)=\tfrac{168}{2}=84$$

4. $\int_{1}^{3}(x^2-4x+1)\,dx$

Solution

$$\int_{1}^{3}(x^2-4x+1)\,dx=\left[\frac{x^3}{3}-2x^2+x\right]_{1}^{3}$$
$$=\left(\tfrac{(3)^3}{3}-2(3)^2+3\right)-\left(\tfrac{(1)^3}{3}-2(1)^2+1\right)$$
$$=(9-18+3)-\left(\tfrac{1}{3}-2+1\right)$$
$$=(-6)-\left(-\tfrac{2}{3}\right)=-\tfrac{16}{3}$$

5. $\int_{1}^{2} y^2+y^{-2}\,dy$

Solution

$$\int_{1}^{2} y^2+y^{-2}\,dy=\left[\frac{y^3}{3}+\frac{y^{-1}}{-1}\right]_{1}^{2}=\left[\frac{y^3}{3}-\frac{1}{y}\right]_{1}^{2}$$
$$=\left(\tfrac{(2)^3}{3}-\tfrac{1}{2}\right)-\left(\tfrac{(1)^3}{3}-\tfrac{1}{1}\right)$$
$$=\tfrac{8}{3}-\tfrac{1}{2}-\tfrac{1}{3}+1=\tfrac{17}{6}$$

6. $\int_{0}^{4}\sqrt{t}(t-2)\,dt$

Solution

We need to first multiply the integral before integrating.

$$\sqrt{t}(t-2)=t^{\frac{1}{2}}(t-2)=t^{\frac{3}{2}}-2t^{\frac{1}{2}}$$

Thus $\int_{0}^{4}\sqrt{t}(t-2)\,dt=\int_{0}^{4}t^{\frac{3}{2}}-2t^{\frac{1}{2}}\,dt=\left[\frac{t^{\frac{5}{2}}}{\frac{5}{2}}-2\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]_{0}^{4}=\left[\frac{2}{5}t^{\frac{5}{2}}-\frac{4}{3}t^{\frac{3}{2}}\right]_{0}^{4}$

$$=\left(\tfrac{2}{5}(4)^{\frac{5}{2}}-\tfrac{4}{3}(4)^{\frac{3}{2}}\right)-(0)$$
$$=\tfrac{64}{5}-\tfrac{32}{3}=\tfrac{32}{15}$$

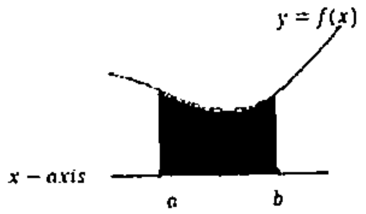
In the evaluation process, recall that;

$$(4)^{\frac{5}{2}}=\left((4)^{\frac{1}{2}}\right)^5=2^5=32$$

$$(4)^{\frac{3}{2}} = \left((4)^{\frac{1}{2}}\right)^3 = 2^3 = 8$$

Area under the curve

The area between the graph of $y = f(x)$ and the x-axis is given by the definite integral of the function of the curve.



$$Area = \int_a^b f(x)\,dx$$

This formula gives a positive result for the area above the x-axis and a negative result for the area below the x-axis.

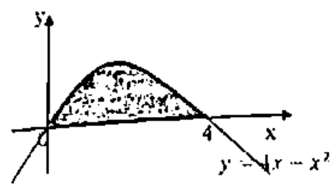
Note: If asked to find the area under the curve, it is a requirement to first sketch the curve i.e. by getting the intercepts and knowing the nature of the turning point.
The nature of the turning point can be known by mere looking at the equation of the curve i.e. the coefficients of the x^2 [This is very important]
If first asked to sketch the curve, then go smoothly through the processes of curve sketching.

Examples

1. Find the area enclosed by the curve $y = 4x - x^2$

Solution

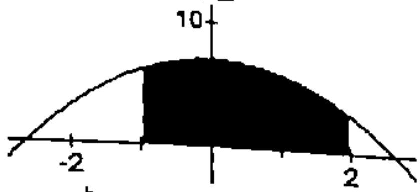
The x-intercepts are when y = 0, i.e $4x - x^2 = 0$, $x(4 - x) = 0$; $x = 0$ and $x = 4$
The curve has a maximum turning point [Recall concept]
We can now sketch the curve as follows;



$$A = \int_a^b y\,dx = \int_0^4 (4x - x^2)\,dx$$
$$= \left[2x^2 - \frac{x^3}{3}\right]_0^4 = \left(2(4)^2 - \frac{(4)^3}{3}\right) - 0$$
$$= \left(32 - \frac{64}{3}\right) - 0 = \frac{32}{3}\ sq.units$$

2. Find the area between $y = 7 - x^2$ and the x-axis between the values $x = -1$ and $x = 2$

Solution



$$A = \int_a^b y\,dx = \int_{-1}^{2} (7 - x^2)\,dx$$

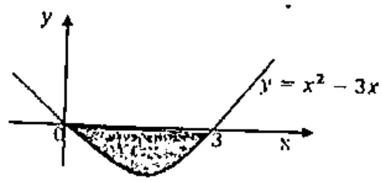
$$= \left[7x - \frac{x^3}{3}\right]_{-1}^{2} = \left(7(2) - \frac{(2)^3}{3}\right) - \left(7(-1) - \frac{(-1)^3}{3}\right)$$

$$= \left(14 - \frac{8}{3}\right) - \left(-7 + \frac{1}{3}\right) = 14 - \frac{8}{3} + 7 - \frac{1}{3} = 21 - \frac{9}{3} = 18\ sq.units$$

3. Find the area enclosed by the curve $y = x^2 - 3x$ and the x-axis

Solution

First make a sketch of the curve to where your area lies



$$A = \int_a^b y\,dx = \int_0^3 (x^2 - 3x)\,dx$$

$$= \left[\frac{x^3}{3} - \frac{3x^2}{2}\right]_0^3 = \left(\frac{(3)^3}{3} - \frac{3(3)^2}{2}\right) - 0$$

$$= \left(9 - \frac{27}{2}\right) = -\frac{9}{2}\ sq.units$$

The area has a negative value because it lies below the x-axis but we shall always take the positive value. Therefore $A = \frac{9}{2}\ sq\ units$

Displacement, Velocity and Acceleration

We earlier saw that displacement, velocity and acceleration are linked up process of differentiation. Similarly; acceleration (a) , velocity (v) and displacement (s) in the reverse order are linked up together by a process of integration.

From $a = \frac{dv}{dt}$, $it\ follows\ that\ v = \int a\,dt$

Similarly from $v = \frac{ds}{dt}$, $it\ follows\ that\ s = \int v\,dt$

Examples

1. If $v = 3t^2 - 8t$ $and\ s = 3\ when\ t = 0, find\ the\ expression\ for\ s\ in\ terms\ of\ t$

Solution

$s = \int v\,dt = \int (3t^2 - 8t)\,dt$

$s = t^3 - 4t^2 + C$

But s = 3 when t = 0, [This helps you calculate the value of the constant C]

$3 = C$

$\therefore s = t^3 - 4t^2 + 3$

2. If $v = t^2 - 4t + 3$ and $s = 4$ when $t = 3$, find the displacement when t = 1

Solution

$s = \int v\,dt = \int (t^2 - 4t + 3)\,dt$

$s = \frac{t^3}{3} - 2t^2 + 3t + C$

but $s = 4$ when $t = 3$

Substituting; $4 = 9 - 18 + 9 + C$

$C = 4$

$s = \frac{t^3}{3} - 2t^2 + 3t + 4$

Displacement when t = 1, $s = \frac{(1)^3}{3} - 2(1)^2 + 3(1) + 4$

$s = \frac{16}{3}m$

3. If $a = 1 - t$ and when $t = 2, v = 1$ and $s = \frac{13}{3}$, find the expressions for v and s in terms of t.

Solution

$v = \int a\,dt = \int (1 - t)\,dt$

$v = t - \frac{t^2}{2} + C$

But when t = 2, v = 1

$1 = 2 - \frac{2^2}{2} + C$

$C = 1$

Therefore $v = t - \frac{t^2}{2} + 1$

Similarly $s = \int v\,dt = \int \left(t - \frac{t^2}{2} + 1\right) dt$

$s = \frac{t^2}{2} - \frac{t^3}{6} + 2t + C$

But when t = 1, $s = \frac{13}{3}$

$\frac{13}{3} = \frac{1}{2} - \frac{1}{6} + 2 + C$

$\frac{13}{3} = \frac{1}{3} + 2 + C$

$C = 2$

Thus $s = \frac{t^2}{2} - \frac{t^3}{6} + 2t + 2$

4. A body moves in a straight line. At time t seconds, its acceleration is given by $a = 6t + 1$. When $t = 0$, the velocity of the body is 2 m/s and its displacement is 1m. Find the expressions of v and s in terms of t.

Solution

$a = \frac{dv}{dt} = 6t + 1$

$v = \int a\,dt = \int 6t + 1\,dt$

$v = 3t^2 + t + C$

But when t = 0, v = 2, thus $2 = 0 + C \Rightarrow C = 2$

Substituting for C; $v = 3t^2 + t + 2$

Now using, $v = \frac{ds}{dt} = v = 3t^2 + t + 2$

$s = \int v\,dt = \int (3t^2 + t + 2)\,dt$

$s = t^3 + \frac{t^2}{2} + 2t + C$

But when t = 0, s = 1,

$\Rightarrow 1 = 0 + C$ *thus* $C = 1$

Substituting; $s = t^3 + \frac{t^2}{2} + 2t + 1$

Trial questions

1. Integrate the following with respect to x

(i) x^5 (ii) $\frac{1}{x^5}$ (iii) $\sqrt[4]{x}$ (iv) x^{-3} (v) $\frac{1}{x^{\frac{5}{2}}}$ (vi) $x^{-\frac{1}{2}}$ (vii) x (viii) $\frac{1}{\sqrt[3]{x}}$

[Ans: (i) $\frac{1}{6}x^6 + c$ (ii) $-\frac{1}{4}x^{-4} + c$ (iii) $\frac{4}{5}x^{\frac{5}{4}} + c$ (iv) $-\frac{1}{2}x^{-2} + c$

(v) $-\frac{2}{3}x^{-\frac{3}{2}} + c$ (vi) $2x^{\frac{1}{2}} + c$ (vii) $\frac{1}{2}x^2 + c$ (viii) $\frac{3}{2}x^{\frac{2}{3}} + c$]

2. Evaluate each of the following definite integrals

(i) $\int_0^2 x^3\,dx$ (ii) $\int_1^2 x^5\,dx$ (iii) $\int_2^4 (x^2 + 4)\,dx$ (iv) $\int_0^3 (x^2 + 2x - 1)\,dx$

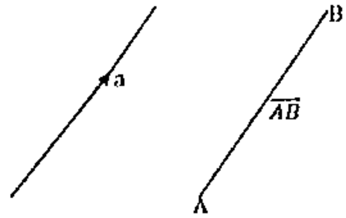
(v) $\int_0^2 (x^3 - 3x)\,dx$ (vi) $\int_1^2 (x^3 - 3x^2 + 2x)\,dx$ (vii) $\int_{-1}^1 2x - 3\,dx$ (viii) $\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{1}{x^3}\,dx$

[Ans: (i) 4 (ii) $\frac{63}{6}$ (iii) $26\frac{2}{3}$ (iv) 15 (v) -2 (vi) $-\frac{1}{4}$ (vii) -6 (viii) 6]

3. Find the area enclosed by the curve $y = x^2 - 1$ *and the* $x-$ *axis* [Ans: $\frac{4}{3}$ sq units]
4. Find the area enclosed by the curve $y = x^2 - 6x$ *and the* $x-$ *axis* [Ans: 36sq units]
5. Find the area enclosed by the curve $y = 4 + 3x - x^2$ *and* $x-$ *axis* [Ans: $20\frac{5}{6}$]
6. Find the area enclosed by the curve $y = x^2 - 4x - 5$ *and the* $x-$ *axis* [Ans: 36]
7. If $a = 6t - 12$, *and when* $t = 0, v = 9$ *and* $s = 6$, *find the expressions* for the velocity and displacement. [Ans: $v = 3t^2 - 12t + 9,\ s = t^3 - 6t^2 + 9t + 6$]

CHAPTER 8: VECTORS

A vector quantity is one that has magnitude and it is related to a definite direction in space i.e.



Equal vectors

For any two vectors to be be equal, they must have the same magnitude and direction

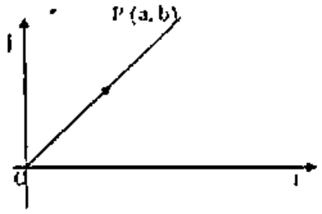
Parallel vectors

Two vectors a and b are parallel if one is a scalar multiple of the other i.e.

$a = \lambda b$

Position vectors

A position vector whose distance and direction from the origin is specific. Consider a vector $ai + bj$



The position vector $\overline{OP}$ is given by $\overline{OP} = ai + bj$

Addition and subtraction of vectors

Example

If $a = 3i + 4j$ and $b = 2i + 8j$. Find (a) a+b (b) a-2b

Solution

(a) $a + b = 3i + 4j + 2i + 8j = 5i + 12j$

(b) $a - 2b = 3i + 4j - 2(2i + 8j)$

$= 3i + 4j - 4i - 16j$

$= -i - 12j$

Modulus of a vector

The modulus of a vector a is the magnitude of a i.e. the length of the line representing a. The modulus of a vector a is denoted by $|a|$

$$|ai + bj| = \sqrt{a^2 + b^2}$$

Note: the vector $ai + bj$ can be denoted by $\begin{pmatrix} a \\ b \end{pmatrix}$ which is a column vector.

Example

Given that $a = 3i + 4j$ and $b = 2i + 8j$. Find (a) $|a|$ (b) $|b|$ (c) $|a + b|$

Solution

(a) $|a| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

(b) $|b| = \sqrt{2^2 + 8^2} = \sqrt{68} = 2\sqrt{17}$

(c) $a + b = 3i + 4j + 2i + 8j$

$|a + b| = \sqrt{5^2 + 12^2} = \sqrt{169} = 13$

Unit vectors

A unit vector is a vector whose magnitude or length is one. It is usually written as $\tilde{a}$. The unit vector of a is given by $\tilde{a} = \frac{a}{|a|}$

Example 1

Find the unit vector of $2i - j$

Solution

$|2i - j| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$ Units

The unit vector will be $\frac{1}{\sqrt{5}}(2i - j)$

Example2

Find the unit vector of a if $a = 3i + 2j$

$|a| = \sqrt{3^2 + 2^2} = \sqrt{13}$

$\tilde{a} = \frac{a}{|a|}$

$\tilde{a} = \frac{1}{\sqrt{13}}(3i + 2j)$

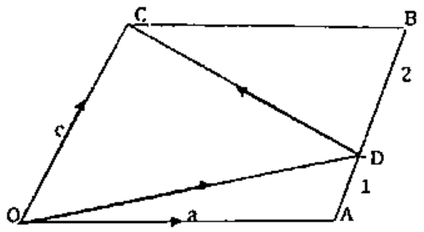
Further examples

1. In a parallelogram OABC, $\overrightarrow{OA} = a$ and $\overrightarrow{OC} = c$. The point D lies on AB such that AD: DB = 1:2. Express the following vectors in terms of a and c.

(a) $\overrightarrow{CB}$ (b) $\overrightarrow{BC}$ (c) $\overrightarrow{AB}$ (d) $\overrightarrow{AD}$ (e) $\overrightarrow{OD}$ (f) $\overrightarrow{DC}$

Solution

Let us draw the parallelogram



(a) $\overrightarrow{CB}$ is the same length as $\overrightarrow{OA}$ *and it is in the same direction* $\Rightarrow \overrightarrow{CB} = \overrightarrow{OA}$

$\therefore \overrightarrow{CB} = a$

(b) $\overrightarrow{BC}$ is the same length as $\overrightarrow{CB}$ but it is in the same direction $\Rightarrow \overrightarrow{BC} = -\overrightarrow{CB}$

$\therefore \overrightarrow{BC} = -a$

(c) $\overrightarrow{AB}$ is the same length as $\overrightarrow{OC}$ and is in the same direction $\Rightarrow \overrightarrow{AB} = \overrightarrow{OC}$

$\therefore \overrightarrow{OC} = c$

(d) AD:DB= 1: 2

$\overrightarrow{AD} = \frac{1}{3}\overrightarrow{AB}$

$\overrightarrow{AD} = \frac{1}{3}c$

(e) $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD}$

$= a + \frac{1}{3}c$

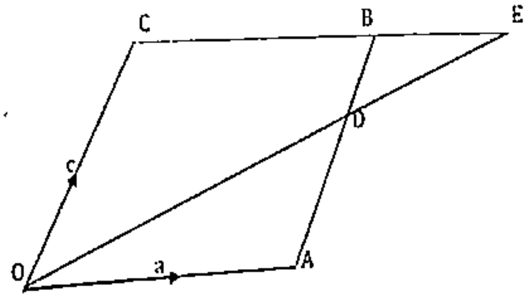
$= \frac{1}{3}(3a + c)$

(f) $\overrightarrow{DC} = \overrightarrow{DB} + \overrightarrow{BC}$

$= \frac{2}{3}c + (-a)$

$= \frac{2}{3}(2c - 3a)$

2.The diagram below shows a parallelogram OABC with $\overrightarrow{OA} = a$ and $\overrightarrow{OC} = c$.D is a point on AB such that AD: DB=2:1. OD produced meets CB produced at E. $\overrightarrow{DE} = h\overrightarrow{OD}$ and $\overrightarrow{BE} = k\overrightarrow{CB}$ .Find



(a) $\overrightarrow{BE}$ in terms of a and k

(b) $\overrightarrow{DE}$ in terms of h, a and c

Solution

(a) $\overrightarrow{BE} = k\overrightarrow{CB}$

But $\overrightarrow{CB} = \overrightarrow{OA} = a$

$\therefore \overrightarrow{BE} = ka$

(b) $\overrightarrow{DE} = h\overrightarrow{OD}$

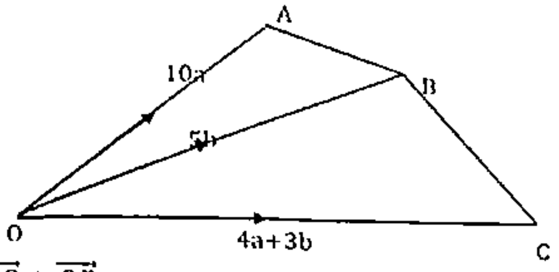
But $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD} = a + \frac{2}{3}\overrightarrow{AB}$

$= a + \frac{2}{3}c = \frac{1}{3}(3a + 2c)$

$\therefore \overrightarrow{DE} = h \times \frac{1}{3}(3a + 2c) = \frac{h}{3}(3a + 2c)$

3.If O, A, B, C are four points such that $\overrightarrow{OA} = 10a$, $\overrightarrow{OB} = 5b$, $\overrightarrow{OC} = 4a + 3b$. Show that A,B and C are collinear

Solution



$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$= -10a + 5b$

$= 5b - 10a$

$= 5(b - 2a)$

$\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC}$

$= -5b + (4a + 3b)$

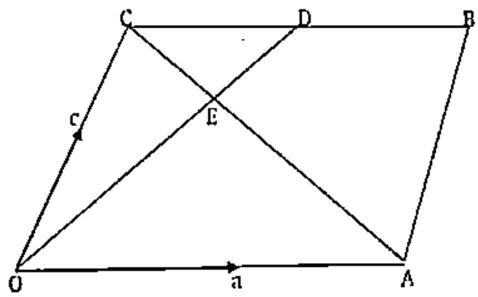
$= 4a - 2b$

$= -2(b - 2a)$

$\therefore \overrightarrow{BC} = -\frac{2}{5}\overrightarrow{AB}$

$\overrightarrow{AB}$ *and* $\overrightarrow{BC}$ are in opposite direction and since B is a common point, A, B and C are collinear

4.OABC is a parallelogram with OA=a and OC=c, D is a midpoint of $\overrightarrow{BC}$ *and* $\overrightarrow{OD}$ meets $\overrightarrow{AC}$ at E



Given that OE=hOD and AE=kAC

Find in terms of vectors a and c , the vectors OE, AE and CE

Solution

From OE=hOD

OD = OC + CD

CD = $\frac{1}{2}CB = \frac{1}{2}OA = \frac{1}{2}a$

OD = c + $\frac{1}{2}a = \frac{1}{2}(2c + a)$

$OE = \frac{h}{2}(2c + a)$

$AE = AO + OE$

$= -a + \frac{h}{2}(2c + a)$

$= \frac{2hc+ha-2a}{2} = \frac{1}{2}(2hc + (h-2)a)$

Alternatively;

From AE=kAC

$AC = AO + OC$

$AC = OC - OA = c - a$

$AE = k(c - a)$

$CE = CO + OE$

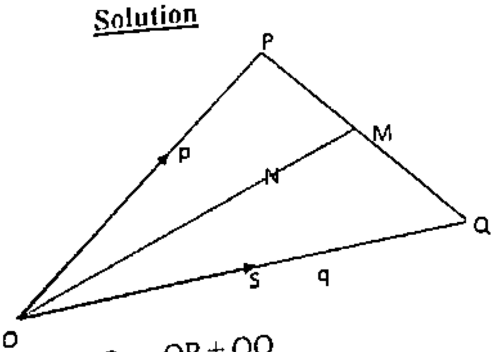
$= OE - OC$

$= \frac{h}{2}(2c + a) - c$

$= \frac{2hc+ha-2c}{2} = \frac{1}{2}(2c(h-1) + ha)$

5. M is the mid-point of $\overline{PQ}$ in the triangle OPQ. If OP = p and OQ = q, find in terms of the vectors p and q, the vectors PQ, PM and OM. N is a point on $\overline{OM}$ such that ON: NM = 2: 1. Express ON and PN in terms of p and q. Given that S is a mid-point of $\overline{OQ}$, use vector methods to show that N lies on $\overline{PS}$ and hence determine the ratio $\overline{PN}$: $\overline{SN}$

Solution



$PQ = PO + OQ = -OP + OQ$

$PQ = OQ - OP$

$PQ = q - p$

$PM = \frac{1}{2}PQ$, since M is the mid-point of $\overline{PQ}$

$PM = \frac{1}{2}(q - p)$

$OM = OP + PM$

$= p + \frac{1}{2}q - \frac{1}{2}p$

$= \frac{1}{2}(p + q)$

ON : NM=2:1 (total ratio = 2 + 1=3)

$\Rightarrow ON = \frac{2}{3}OM = \frac{2}{3} \times \frac{1}{2}(p + q)$

$\therefore ON = \frac{1}{3}(p + q)$

PN = PO + ON (see diagram)

= -OP + ON = ON − OP = $\frac{1}{3}(p+q) - p = \frac{1}{3}q - \frac{2}{3}p$

$\therefore PN = \frac{1}{3}(q - 2p)$

Now NS = NO + OS = OS − ON

But OS = $\frac{1}{2}OQ$ since S is the mid-point of OQ

$\Rightarrow OS = \frac{1}{2}q$

Also $ON = \frac{1}{3}(p+q)$ from above

$\Rightarrow NS = \frac{1}{2}q - \frac{1}{3}(p+q) = \frac{1}{2}q - \frac{1}{3}p - \frac{1}{3}q = \frac{3q-2p-2q}{6} = \frac{q-2p}{6}$

NS = $\frac{1}{6}(q-2p)$

But PN = $\frac{1}{3}(q-2p)$ so $\frac{PN}{NS} = \frac{\frac{1}{3}(q-2p)}{\frac{1}{6}(q-2p)} = \frac{1}{3} \times \frac{6}{1} = 2$

$\frac{PN}{NS} = 2 \Rightarrow PN = 2NS$

Since PN is a scalar multiple of NS, then PN is parallel to NS.

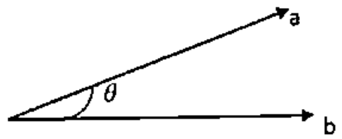
But since both vectors contain a common point N, N and S are collinear (lie on a straight line) and so N lies on PS as required.

Now $\frac{PN}{NS} = \frac{2}{1}$ so PN : NS = 2 : 1

Hence $\overline{PN} : \overline{SN} = 2 : 1$

The scalar product

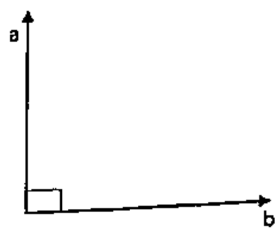
The scalar dot product of two vectors is defined as the product of the magnitude of the two vectors and the cosine of the angle between the two vectors



$a.b = |a||b|\cos\theta$

Properties of the scalar product

1. From definition $a.b = |a||b|\cos\theta$, it follows that two perpendicular vectors have a scalar product of zero



$a.b = |a||b|\cos 90$

$\Rightarrow a.b = 0$

2. $a.b = b.a$
3. $a.(b + c) = a.b + a.c$
4. $\lambda(a.b) = a.(\lambda b) = (\lambda a).b = \lambda|a||b|\cos\theta$

Consider the vectors $a = x_1i + y_1j$ and $b = x_2i + y_2j$

Now $a.b = (x_1i + y_1j).(x_2i + y_2j)$

$= x_1x_2i.i + x_1y_2i.j + y_1x_2j.i + y_1y_2j.j$

But $i.i = j.j = 1$ and $i.j = j.i = 0$

$\therefore a.b = x_1x_2 + y_1y_2 = |a||b|\cos\theta$ where θ is the acute angle between a and b.

Example 1

Find the angle between the vectors a and b given that $a = 3i + 4j$ and $b = 5i - 12j$

Solution

Let θ be the required angle

$a.b = |a||b|\cos\theta$

$|a| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

$|b| = \sqrt{5^2 + (-12)^2} = \sqrt{169} = 13$

$a.b = \begin{pmatrix}3\\4\end{pmatrix}.\begin{pmatrix}5\\-12\end{pmatrix} = (3 \times 5) + (4 \times -12) = 15 - 48 = -33$

$-33 = 5 \times 13\cos\theta$

$-\frac{33}{65} = \cos\theta$

$\cos\theta = -0.50769$

$\theta = \cos^{-1}(-0.50769) = 120.51^0$

The angle between the vectors is 120.51^0

Example 2

The points A, B, C and D have position vectors $-2i + 3j,\ 3i + 8j, 7i + 6j$ and $7i - 4j$ respectively. Show that AC is perpendicular to BD.

Solution

To show that two vectors are perpendicular, we must get their dot product and it should be equal to zero.

$\overrightarrow{OA} = \begin{pmatrix}-2\\3\end{pmatrix},\ \overrightarrow{OB} = \begin{pmatrix}3\\8\end{pmatrix}, \overrightarrow{OC} = \begin{pmatrix}7\\6\end{pmatrix}, \overrightarrow{OD} = \begin{pmatrix}7\\-4\end{pmatrix}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix}7\\6\end{pmatrix} - \begin{pmatrix}-2\\3\end{pmatrix} = \begin{pmatrix}9\\3\end{pmatrix}$

$\overrightarrow{BD} = \overrightarrow{OD} - \overrightarrow{OB} = \begin{pmatrix}7\\-4\end{pmatrix} - \begin{pmatrix}3\\8\end{pmatrix} = \begin{pmatrix}4\\-12\end{pmatrix}$

$AC.BD = \begin{pmatrix}9\\3\end{pmatrix}.\begin{pmatrix}4\\-12\end{pmatrix} = (9 \times 4) + (3 \times -12) = 36 - 36 = 0$

Therefore since $AC.BD = 0, AC$ *is perpendicular to* BD

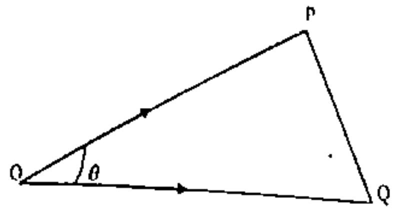
Example 3

The position vectors $OP = \begin{pmatrix}2\\3\end{pmatrix}$ and $OQ = \begin{pmatrix}4\\-1\end{pmatrix}$ are joined to form triangle OPQ. Determine (i) the lengths of triangle OPQ

(ii) the angle between OP and OQ

(iii) the area of the triangle OPQ

Solution



(i) $\overline{OP} = |OP| = \sqrt{2^3 + 3^2} = \sqrt{13} units$

$\overline{OQ} = |OQ| = \sqrt{4^3 + (-1)^2} = \sqrt{17} units$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix}4\\-1\end{pmatrix} - \begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}2\\-4\end{pmatrix}$

$\overline{PQ} = |PQ| = \sqrt{2^2 + (-4)^2} = \sqrt{20}\ units$

(ii) $OP.OQ = \begin{pmatrix}2\\3\end{pmatrix}.\begin{pmatrix}4\\-1\end{pmatrix} = (2\times 4) - (3\times -1) = 8 - 3 = 5$

Using $OP.OQ = |OP||OQ|\cos\theta$

$5 = \sqrt{13}\times\sqrt{17}\cos\theta$

$\cos\theta = \frac{5}{\sqrt{221}} = 0.336$

$\theta = 70.37^0$

(iii) $A = \frac{1}{2}|OP||OQ|\sin\theta$

$= \frac{1}{2}\times\sqrt{13}\times\sqrt{17}\sin 70.37^0$

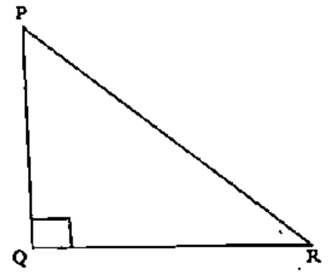
$= 7\ sq\ units$

Example 4

A triangle PQR has vertices P(1, -1), Q(6, 4) and R(3, 7). Using vectors, show that triangle PQR has a right angle at Q. Hence, find the area of the triangle PQR.

Solution

Let us first assume the right-angled triangle



If PQR is right angled, then it must satisfy the Pythagoras theorem i.e. $a^2 + b^2 = c^2$

In this case $\overline{PQ}^2 + \overline{QR}^2 = \overline{PR}^2$

$PQ = OQ - OP = \begin{pmatrix}6\\4\end{pmatrix} - \begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}5\\5\end{pmatrix}$

$|PQ| = \sqrt{5^2 + 5^2} = \sqrt{50}$

$QR = OR - OQ = \begin{pmatrix}3\\7\end{pmatrix} - \begin{pmatrix}6\\4\end{pmatrix} = \begin{pmatrix}-3\\3\end{pmatrix}$

$|QR| = \sqrt{(-3)^2 + 3^2} = \sqrt{18}$

$PR = OR - OP = \begin{pmatrix}3\\7\end{pmatrix} - \begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}2\\8\end{pmatrix}$

$|PR| = \sqrt{2^2 + 8^2} = \sqrt{68}$

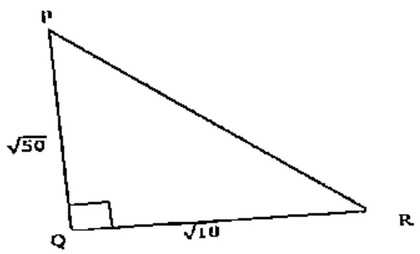
Now $\overline{PQ}^2 = (\sqrt{50})^2 = 50$

$\overline{QR}^2 = (\sqrt{18})^2 = 18$

$\overline{PR}^2 = (\sqrt{68})^2 = 68$

From above; $\overline{PQ}^2 + \overline{QR}^2 = 50 + 18 = 68 = \overline{PR}^2$

Therefore, the triangle PQR is right-angled at Q



$A = \frac{1}{2} \times b \times h = \frac{1}{2} \times \sqrt{18} \times \sqrt{50} = 15\ sq.units$

Trial questions

1. If the point P has position vector $7i - 3j$ *and point* Q has position vector $5i + 5j$. Find (a) $\overrightarrow{PQ}$ (b) $\overrightarrow{QP}$ [*Ans:* (a) $-2i + 8j$ (b) $2i - 8j$]
2. The point P has position vector $3i - 2j$ and Q is a point such that $\overrightarrow{QP} = 2i - 3j$. Find the position vector of Q. [Ans: $i + j$]
3. Given that $a = 3i - j$ and $b = 2i + j$, find (a) $|a|$ (b) $|b|$ (c) $a + b$ (d) $|a + b|$ [Ans: (a) $\sqrt{10}$ (b) $\sqrt{5}$ (c) $5i$ (d) 5]
4. Given that $a = \begin{pmatrix}2\\3\end{pmatrix}$ and $b = \begin{pmatrix}1\\-3\end{pmatrix}$;find (a) $a + 2b$ (b) $|a + 2b|$ (c) $2a + 3b$ (d) $|2a + 3b|$ [Ans: (a) $\begin{pmatrix}4\\-3\end{pmatrix}$ (b) 5 (c) $\begin{pmatrix}7\\-3\end{pmatrix}$ (d) $\sqrt{58}$]
5. The three points A, B, and C have position vectors a, b and c respectively. If c = 3b – 2a . Show that A, B and C are collinear
6. The three points A, B and C have position vectors $i - j, 5i - 3j$ and $11i - 6j$ respectively. Show that A, B and C are collinear.
7. Find the angle between each of the following pairs of vectors
   (a) $a = 3i + 4j$ and $b = 5i + 12j$ [*Ans:* 14^0]
   (b) $c = 5i - j$ and $d = 2i + 3j$ [*Ans:* 68^0]
8. The points A, B, C and D have position vectors $5i + j, -3i + 2j, -3i - 3j$ and $i - 6j$ respectively. Show that AC is perpendicular to BD
9. The points E, F and G have position vectors $2i + 2j, i + 6j$ and $-7i + 4j$. Show that the triangle EFG is right angled at F.

10. If $a = \binom{3}{-4}$ find a unit vector parallel to a [Ans: $\begin{pmatrix} \frac{3}{5} \\ \frac{-4}{5} \end{pmatrix}$

11. The points A, B and C have position vectors $4i - j, i + 3j$ and $-5i + 2j$ respectively. Find (a) $\overrightarrow{AB}$ (b)$\overrightarrow{BC}$ (c) $\overrightarrow{CA}$ (d)the angles of a triangle ABC

[Ans: (a) $-3i + 4j$ (b) $-6i - j$ (c)$9i - 3j$ (d)$35^0, 117^0, 28^0$]

12. E is the centre of the rectangle ABCD and $\overrightarrow{AB} = a, \overrightarrow{BC} = b$, Express in terms of a and b the vectors (i)$\overrightarrow{AC}$ (ii)$\overrightarrow{CD}$ (iii) $\overrightarrow{BD}$ (iv) $\overrightarrow{EB}$ (v) $\overrightarrow{EA}$

[Ans: (i) $a + b$ (ii) $-a$ (iii)$b - a$ (iv)$\frac{1}{2}(a - b)$ (v) $-\frac{1}{2}(a + b)$]

13. The position vectors of three points A, B and C relative to the origin O are p, 3q – p and 9q – 5p respectively. Show that the points A, B and C lie on the same straight line, state the ratio AB : BC Given that OBCD is a parallelogram and that E is the point such that DB = $\frac{1}{3}DE$, find the position vectors of D and E relative O [Ans: 1 : 2, 6q – 4p , 5p – 3q]

14. The position vectors of the points A and B with respect to the origin O are $2i + 3j$, $-i + 5j$ respectively. Find the position vector C such that $\overrightarrow{AC} = 2\overrightarrow{AB}$. Calculate the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{OB}$ [Ans: $-4i + 7j$, 45^0]

15. The position vectors of points A, B and C relative to the origin O are $\binom{1}{3}, \binom{2}{6}$ and $\binom{5}{5}$ respectively. Write down the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{BC}$. Use the vector methods to calculate (i) angle BAC (ii) angle ABC. State the special property of triangle ABC and deduce its area.

[Ans: $\binom{1}{3}$, $\binom{4}{2}$, $\binom{3}{-1}$, $45^0, 90^0$, *right angled isosceles* , 5 *sq units*]

16. Given the vectors $a = 2i - 4j$ and $b = 3i + 5j$. Find the
    (a) Modulus of the vector $5a + 2b$
    (b) Angle between the vectors a and b [Ans: (a) 18.87 (b) 122.47]

17. If $a = 2i + j$ and $b = i - 2j$. Express in terms of i and j
    (i) $2a + b$ (ii) $-3a + 4b$, hence find the angle between the vectors $2a + b$ and $-3a + 4b$
    [Ans: (i) 5i (ii) -2i – 11j ; 100.30^0]

18. The position vectors of the points P, Q and R are $-2i + 4j, 2i + 4j$ and $2i + 8j$ respectively. Show that QP and QR are perpendicular

19. Given that $p = 7i + 4j$, $q = 3i - 5j$, determine (i) the angle between p and q (ii) $|3p - 4q|$
[Ans: (i) 88.87^0 (ii) 33.24]

20. Given that $a = 3i - 4j$ and $b = -5i + 12j$, find (i) $(3a + b).b$ (ii) the angle between a and b
[Ans: (i) -20 (ii) 165.75^0]

21. The points A, B and C have position vectors $5i + 2j, 4i + 6j$ and $-2i + 6j$ respectively Determine (i) $|BC|$ (ii)$|AC|$ (iii)the angle between BC and AC

[Ans: (i) 6 (ii) 8.06 (iii) 29.72^0]

22. The points P, Q, R and S have position vectors $p = \binom{5}{1}, q = \binom{-3}{2}, r = \binom{-3}{-3}$ and $s = \binom{1}{-6}$ respectively. Show that PR is perpendicular to QS.

PR = a

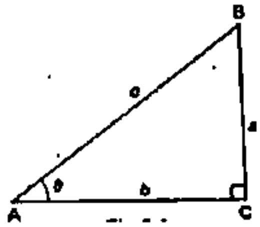
CHAPTER 9: TRIGONOMETRY

Trigonometry is a branch of mathematics that studies the relationship between the three sides and the three angles of a right angled triangle in terms of ratios and representing them as trigonometric ratios; sine, cosine and tangent.

Trigonometric ratios for the general angle

The trigonometric ratios include the main three mentioned above and the others include secant, cosecant and cotangent abbreviated as sec, cosec and cot respectively.

If we consider a right angled triangle ABC



then $\sin\theta = \frac{a}{c}$, $\cos\theta = \frac{b}{c}$

Also $\tan\theta = \frac{a}{b} = \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{\sin\theta}{\cos\theta}$

It can also be obsceved that the remaining angle, $B = 90 - \theta$

And that $\sin(90-\theta) = \frac{b}{c} = \cos\theta$ *and* $\cos(90-\theta) = \frac{a}{c} = \sin\theta$

Therefore $\sin\theta = \cos(90-\theta)$ *and* $\cos\theta = \sin(90-\theta)$

Now using the Pythagoras theorem i.e. $a^2 + b^2 = c^2$

Dividing through by c^2 *gives*

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1$$

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$

$$\Rightarrow \boldsymbol{sin^2\theta + cos^2\theta = 1}$$

Though we have derived these relationships using an acute angle, they are identities i.e. true for any angle and should be memorized.

The three remaining trigonometric ratios are reciprocals of sine, cosine and tangent.

They are; $secant = \frac{1}{cosine}$; $cosecant = \frac{1}{sine}$; $cotangent = \frac{1}{tangent} = \frac{cosine}{sine}$

Further trigonometric identities

Using $sin^2\theta + cos^2\theta = 1$

Dividing through by $cos^2\theta$

$$\frac{sin^2\theta}{cos^2\theta} + \frac{cos^2\theta}{cos^2\theta} = \frac{1}{cos^2\theta}$$

But $\frac{\sin\theta}{\cos\theta} = \tan\theta$ *and* $\frac{1}{\cos\theta} = \sec\theta$

Therefore; $\boldsymbol{tan^2\theta + 1 = sec^2\theta}$

Now if we divide the original identity by $sin^2\theta$, we obtain;

$$\frac{\sin^2\theta}{\sin^2\theta}+\frac{\cos^2\theta}{\sin^2\theta}=\frac{1}{\sin^2\theta}$$

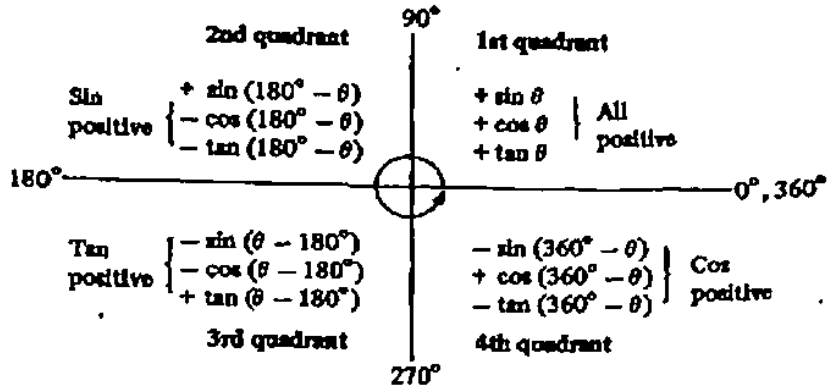
But $\frac{\cos\theta}{\sin\theta}=\cot\theta$ and $\frac{1}{\sin\theta}=cosec\,\theta$

Therefore $\mathbf{1+cot^2\theta = cosec^2\theta}$

These three identities will be found useful later when solving equations

Trigonometric ratios for general angles

The relationship between the ratios of the general angles and the corresponding acute angles depends on which quadrant the basic angle lies in. The angles can lie in four quadrants following in the anti-clockwise direction. In the 1st quadrant, all are positive, in the 2nd quadrant only sine is positive, in the 3rd quadrant only tan is positive and in the 4th quadrant only cosine is positive. Note that the positive angle are measured in the anticlockwise direction and the negative angles are measured in the clockwise direction. The relationships can be summarized as shown below.



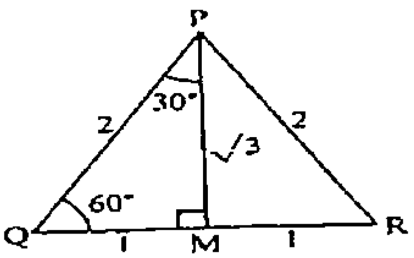
It can also be remembered by a student saying "All Scientist Take Chemistry" in the anticlockwise direction.

Trigonometric ratios for special angles

The trigonometric ratios of the angles 0^0, 30^0, 45^0, 60^0 and 90^0 are used often in mechanics and other branches of mathematics and so it is useful to have their values in surd form.

30^0 and 60^0

Suppose ΔPQR *is* equilateral, with sides 2 units and that PM is the perpendicular bisector of QR



Using the Pythagoras theorem, $MP^2+MQ^2=PQ^2$

$$MP=\sqrt{2^2-1^2}=\sqrt{3}$$

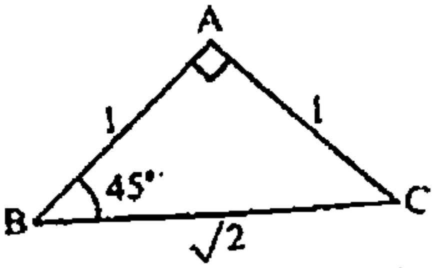
Since ΔPQR *is* equilateral, $PQM=60^0$ *and* $QPM=30^0$

From ΔPQM; $\sin 30^0=\frac{1}{2}$; $\cos 30^0=\frac{\sqrt{3}}{2}$; $\tan 30^0=\frac{1}{\sqrt{3}}$ *or* $\frac{\sqrt{3}}{3}$

And $\sin 60^0=\frac{\sqrt{3}}{2}$; $\cos 60^0=\frac{1}{2}$; $\tan 60^0=\frac{\sqrt{3}}{1}=\sqrt{3}$

45^0

Consider a right-angled triangle which is isosceles and in which the equal sides are 1 unit in length. The equal sides will each be 45^0



Using the Pythagoras theorem; $BC^2 = 1^2 + 2^2 \quad or\ BC = \sqrt{2}$

Hence $\sin 45^0 = \frac{1}{\sqrt{2}}\ or \frac{\sqrt{2}}{2}$; $\cos 45^0 = \frac{\sqrt{2}}{2}$; $\tan 45^0 = \frac{1}{1} = 1$

0^0 *and* 90^0

$\sin 0^0 = 0\,, \cos 0^0 = 1 \quad and \tan 0^0 = 0$

$\sin\ 90^0 = 1,\ \cos 90^0 = 0 \quad and \tan 90^0 = \frac{\sin 90^0}{\cos 90^0} = \frac{1}{0} = \infty\ (undefined)$

The results can be summarized in the table below

Angle	sin	cos	tan
0°	0	1	0
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	1	0	∞

Examples

1. *Show that* $\cos^2 30^0 + \cos 60^0 \sin 30^0 = 1$

Solution

The left hand side is $\cos^2 30^0 + \cos 60^0 \sin 30^0$

$L.H.S = (\cos 30^0)(\cos 30^0) + \cos 60^0 \sin 30^0$

$= \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{3}{4} + \frac{1}{4} = 1 = R.H.S$ as required

Obtuse angles

Trigonometric ratios of obtuse angles cannot be defined by means of a right angled triangle.
The sine, cosine or tangent of an obtuse angle is the sine, cosine or tangent of a supplement angle, with the appropriate sign

If θ is an obtuse angle;

$\sin\theta = +\sin(180^0 - \theta)$; $\cos\theta = -\cos(180^0 - \theta)$; $\tan\theta = -\tan(180^0 - \theta)$

2. Write each of the following as trigonometric ratios of an acute angle
(a) $\sin 155^0$ (b) $\cos 140^0$ (c) $\tan 130^0$

Solution

(a) $\sin 155^0 = +\sin(180^0 - \theta) = \sin 25^0$
(b) $\cos 140^0 = -\cos(180^0 - 140^0) = -\cos 40^0$
(c) $\tan 1300 = -\tan(180^0 - 130^0) = -\tan 50^0$

3. $If \sin 35^0 = 0.5736, find\ the\ values\ of\ (a) \sin 145^0 \quad (b) \cos 125^0$

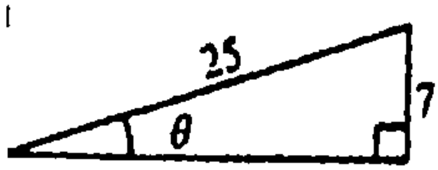
Solution

(a) $\sin 145^0 = +\sin(180^0 - 145^0) = \sin 35^0 = 0.5736$
(b) $\cos 125^0 = -\cos(180^0 - 125^0) = -\cos 55^0 = -\cos(90^0 - 35^0)$
$= -\sin 35^0 = -0.5736$

4. Given that $\sin\theta = \frac{7}{25}$ *and that* θ *is an acute angle, find* $(a) \cos\theta$ $(b) \tan\theta$

Solution

First sketch a right angled triangle containing an angle θ and with two sides of the length 7 and 25 such that $\sin\theta = \frac{7}{25}$

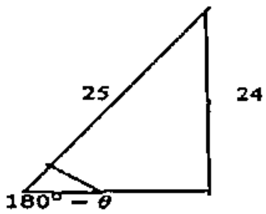


Using the Pythagoras theorem, the third side of the triangle $= \sqrt{25^2 - 7^2} = 24$
Since θ is acute, the trigonometric ratios of θ will be positive, hence
(a) $\cos\theta = \frac{24}{25}$ (b) $\tan\theta = \frac{7}{24}$

5. Given that $\sin\theta = \frac{24}{25}$ *and that* θ *is an obtuse angle, find* $(a) \cos\theta$ $(b) \tan\theta$

Solution

As θ is obtuse, sketch a right-angled triangle containing an angle $(180^0 - \theta)$ and with two sides of length 24 and 25 units.



As $\sin\theta$ *and* $\sin(180^0 - \theta)$ *are numerically equal,* $\sin(180^0 - \theta) = \frac{24}{25}$
Using the Pythagoras theorem, the third side of the triangle $= \sqrt{25^2 - 7^2} = 24$
$(a) \cos\theta = -\cos(180^0 - \theta) = -\frac{7}{25}$
$(b) \tan\theta = -\tan(180^0 - \theta) = -\frac{24}{7}$

Maximum and minimum values of sine and cosine

The trigonometric ratios of all angles differ from the trigonometric ratios of acute angles only is sign.

From the definition of sine and cosine

The maximum value of $\sin\theta$ $is + 1$ $(when\ \theta = 90^0, 450^0 \ldots\ldots\ldots)$

And the minimum value of $\sin\theta$ $is - 1$ $(when\ \theta = 270^0, 630^0 \ldots\ldots\ldots)$

The maximum value of $\cos\theta$ $is + 1(when\ \theta = 0^0, 360^0, \ldots\ldots\ldots)$

And the minimum value of $\cos\theta$ $is - 1(when\ \theta = 180^0, 540^0, \ldots\ldots)$

Example

Write down the maximum and minimum values of each of the following and state the smallest value of θ, from 0^0 $to\ 360^0$ $, for\ which\ these\ occur$

(a) $1 - 2\cos\theta$ (b) $3\sin\theta - 1$

Solution

(a) $\cos\theta$ $varies$ -1 to +1, hence $2\cos\theta$ $varies - 2\ to + 2$

Thus the maximum value of $1 - 2\cos\theta$ is $1 - (-2) = 3$ and occurs when $\cos\theta = -1$ $i.e\ when\ \theta = 180^0$

The minimum value of $1 - 2\cos\theta$ is $1 - 2 = -1$ and occurs when $\cos\theta = 1$ i.e. when $\theta = 0^0$

(b) $\sin\theta$ $varies$ -1 to +1, hence $3\sin\theta$ $varies - 3\ to + 3$

Thus the maximum value of $3\sin\theta - 1$ is $3 - 1 = 2$ and occurs when $\sin\theta = 1$ $i.e\ when\ \theta = 90^0$

The minimum value of $3\sin\theta - 1$ is $-3 - 1 = -4$ and occurs when $\sin\theta = -1$ i.e. when $\theta = 270^0$

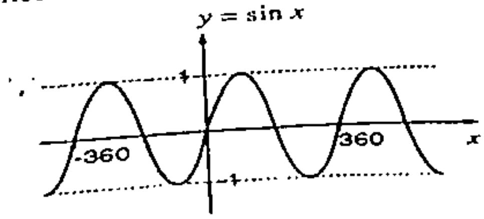
Graphs of trigonometric functions

$y = \sin x$

The graph is continuous

It ranges from -1 to 1 $(-1 \le \sin x \le 1)$

It is periodic i.e. it repeats itself every 360^0



$y = \cos x$

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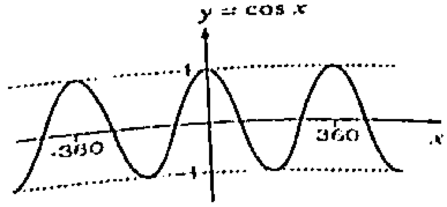
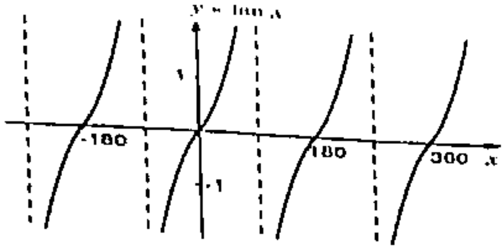


$y = \tan x$

The graph is not continuous being undefined when $\theta = 90^0, 270^0, 450^0$ $e.t.c$

Ranges from $-\infty$ to ∞ $(-\infty \leq \tan x \leq \infty)$

$\tan 0^0 = \tan 180^0 = \tan 360^0$



Solving trigonometric equations

Example 1

Solve the equation $\cos x = \frac{\sqrt{3}}{2}$ $for\ values\ of\ x\ such\ that$ $0^0 \leq x \leq 360^0$

Solution

The acute angle with a cosine of $\frac{\sqrt{3}}{2}$ is 30^0 , $so\ solutions$ will make an angle of 30^0 with the x-axis. The fact that the cosine is positive indicates that there are solutions in the 1st and 4th quadrants.

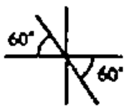
$For\ the\ range$ $0^0 \leq x \leq 360^0$, $x = 30^0$ or 330^0

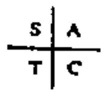
Example 2

Solve the equation $\tan x = -\sqrt{3}$ $for\ values\ of\ x\ such\ that$ $-180^0 \leq x \leq 180^0$

Solution

We first ignore the minus sign and we find $\tan^{-1}\sqrt{3} = 60^0$ and so the solutions will make an angle of 60^0 with the x-axis. Using the fact the tangent is negative; there are solutions in the 2nd and 4th quadrants.





For the range $-180^0 \leq x \leq 180^0, x = -60^0$ or 120^0

Example 3

Solve the following equations for $0^0 \leq x \leq 360^0$

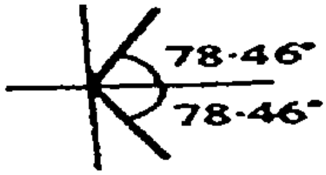
(a) $\cos x = 0.2$ $\quad (b)$ $\sin x = -0.2$

Solution

(a) We require the acute angle with cosine of 0.2. This is found by either using the inverse cosine function (written as $\cos^{-1}$ $or\ arcos$) on a calculator or by using cosine tables.

$\cos^{-1} 0.2 = 78.46^0$ i.e. solutions will make an angle of 78.46^0 with the x-axis

Because the cosine is positive, solutions are found in the 1st and 4th quadrants. Thus the solutions can be sketched as shown below.

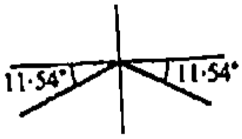


For the range $0^0 \leq x \leq 360^0$, $x = 78.46^0$ *or* 281.54^0

(b) We first ignore the minus sign in $\sin x = -0.2$

From a calculator or tables, $\sin^{-1} 0.2 = 11.54^0$

But sine is negative, solutions are found in the 3rd and 4th quadrants thus solutions can be sketched as below;



For the range $0^0 \leq x \leq 360^0, x = 191.54^0$ *or* 348.46^0

Example 4

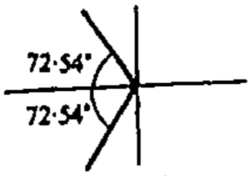
Solve $\cos x = -0.3 \ for -180^0 \leq x \leq 180^0$

Solution

First ignore the minus sign

$\cos^{-1} 0.3 = 72.54^0$

Because the cosine is negative, the solutions are in the 2nd and 3rd quadrants. Thus, solutions can be sketched as shown.



For a range $-180^0 \leq x \leq 180^0, x = 107.46^0$ *or* -107.46^0

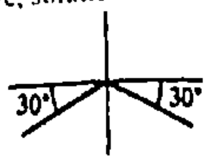
Example 5

Solve $\sin(x + 10^0) = -0.5$ for $0^0 \leq x \leq 360^0$

Solution

$\sin^{-1} 0.5 = 30^0$ *i.e* solutions will make an angle 30^0 with the x-axis

Because sine is negative, solutions will be in the 3rd and 4th quadrants



Thus $x + 10^0 = 210^0$ *or* $x + 10^0 = 330^0$

$x = 200^0$ *or* 320^0

Example 6

Solve $3(\tan x + 1) = 2\ for - 180^0 \leq x \leq 180^0$

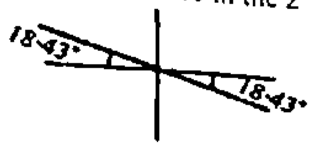
Solution

Expanding; $3\tan x + 3 = 2$

$\tan x = -\frac{1}{3}$

$\tan^{-1}\frac{1}{3} = 18.43^0$ i.e. solutions will make an angle of 18.43^0 with the x-axis

Since tan is negative, solutions are in the 2nd and 4th quadrants



For $-180^0 \leq x \leq 180^0, x = -18.43^0\ \ or\ 161.57^0$

Example 7

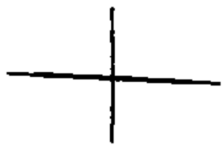
Solve $sin^2x + \sin x \cos x = 0$ for $0^0 \leq x \leq 360^0$

Solution

Factorizing $\sin x\,(\sin x + \cos x) = 0$

Thus $Either\ \sin x = 0\ \ or\ \sin x + \cos x = 0$

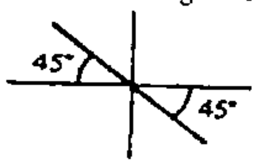
$\sin^{-1} 0 = 0\ \ and\ solutions\ acn\ be\ sketched$



$x = 0^0, 180^0, 360^0$

$\sin x = -\cos x \Rightarrow \frac{\sin x}{\cos x} = -1\ thus \tan x = -1$

$\tan^{-1} 1 = 45^0$, and because the tangent is negative, solutions will occur in the 2nd and 4th quadrants



$x = 135^0\ \ or\ 315^0$

Thus 0 for $0^0 \leq x \leq 360^0, x = 0^0, 135^0, 180^0, 315^0, 360^0$

Note: It is important that we factorize $sin^2x + \sin x \cos x$ in the above example and do not attempt to cancel by $\sin x\,.Cancelling\ will\ lead\ to\ sinx = -\cos x$ and so solutions arising from $\sin x = 0$ would be lost.

Example 8

Solve the equation $4\sin\theta = \tan\theta\ \ \ for\ 0^0 \leq x \leq 360^0$

Solution

$4\sin\theta = \frac{\sin\theta}{\cos\theta}$

$4\sin\theta\cos\theta = \sin\theta$

$4\sin\theta\cos\theta - \sin\theta = 0$

$\sin\theta\,(4\cos\theta - 1) = 0$

Either $\sin\theta = 0,\ \ \theta = 0^0, 180^0, 360^0$

Or $4\cos\theta - 1 = 0$

$\cos\theta = \frac{1}{4}$

$\cos^{-1}\frac{1}{4} = 75.52^0$

Cosine is positive so the angles will lie in the 1st and 4th quadrants

$\theta = 75.52^0 \quad or\ 248.48^0$

$for\ 0^0 \le x \le 360^0, \theta = 0^0, 75.52^0, 180^0, 284.48^0, 360^0$

Example 9

Solve $6cos^2x - \cos x - 1 = 0 \ \ for\ 0^0 \le x \le 360^0$

Solution

We need to first factorize the expression

$6cos^2x - 3\cos x + 2\cos x - 1 = 0$

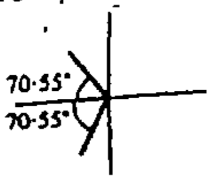
$3\cos x\,(2\cos x - 1) + (2\cos x - 1) = 0$

$(2\cos x - 1)(3\cos x + 1) = 0$

$either 2\cos x - 1 = 0$

$\cos x = \frac{1}{2}$

Now $\cos^{-1}\frac{1}{2} = 60^0$ and because cos is positive, solutions lie in the 2nd and 3rd quadrants

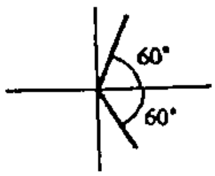


$x = 109.47^0, 250.53^0$

$or \quad 3\cos x + 1 = 0$

$\cos x = -\frac{1}{3}$

$\cos^{-1}\frac{1}{3} = 70.53^0$ and because cos is negative, solutions lie in the 1st and 4th quadrants



$x = 60^0, 300^0$

Thus for the range $0^0 \le x \le 360^0; x = 60^0, 109.47^0, 250.53^0, 300^0$

Note: In some cases, factorizing can give a bracket that does not lead to any solutions. For example if we had to solve

$(2\cos x - 1)(\cos x - 2) = 0\ for\ 0^0 \le x \le 360^0$

Either $2\cos x - 1 = 0 \quad or \quad \cos x - 2 = 0$

$\cos x = \frac{1}{2}$ $\qquad \cos x = 2$

$\cos^{-1}\frac{1}{2} = 60^0$ $\qquad \cos^{-1} 2 = no\ solutions$

Therefore for $0^0 \le x \le 360^0,\ x = 60^0, 300^0$

Example 10

Solve the equation $4\cos x - 3\sec x = 2\tan x \quad for - 180^0 \le x \le 180^0$

Solution

$$4\cos x - 3\sec x = 2\tan x$$

$$4\cos x - \frac{3}{\cos x} = \frac{2\sin x}{\cos x}$$

Multiplying through by $\cos x$ *gives;*

$$4cos^2x - 3 = 2\sin x$$

$$But\ cos^2x = 1 - sin^2x$$

$$4(1 - sin^2x) - 3 = 2\sin x$$

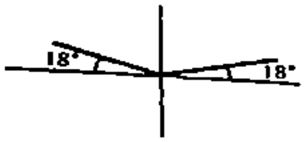
$$4 - 4sin^2x - 3 = 2\sin x$$

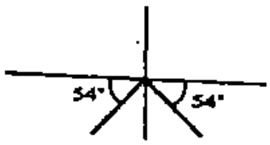
$$4sin^2x + 2\sin x - 1 = 0$$

$$\sin x = \frac{-2 \pm \sqrt{(4+6)}}{8} = \frac{-2 \pm \sqrt{10}}{8}$$

Hence $\sin x = 0.3090 \quad or \quad -0.8090$

$Now\ \sin^{-1} 0.3090 = 18^0 \qquad and\ \sin^{-1} 0.8090 = 54^0$





Thus for the range $-180^0 \le x \le 180^0, x = -126^0, -54^0, 18^0, 162^0$

The compound angle formulae

The compound angle formula gives the relationship between compound angles i.e. the sum and difference between two angles. Their proofs are not required at this level. They are as follows;

$$\sin(A + B) = \sin A\cos B + \sin B\cos A$$

$$\sin(A - B) = \sin A\cos B - \sin B\cos A$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\cos(A - B) = \cos A\cos B + \sin A\sin B$$

Example 1

Evaluate the following without using tables or a calculator (a) $\cos 75^0$ $(b)\sin 75^0$ $(c)\cos 15^0$ $(d)\sin 15^0$ $(e)\sin 330^0$ $(f)\ \cos 240^0$

Solution

(a) $\cos 75^0 = \cos(45^0 + 30^0) = \cos 45^0\cos 30^0 - \sin 45^0\sin 30^0$

$$= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{3} - \frac{\sqrt{2}}{2} \times \frac{1}{2} = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$$

$$\cos 75^0 = \frac{\sqrt{6}-\sqrt{2}}{4}$$

(b) $\sin 75^0 = \sin(45^0 + 30^0) = \sin 45^0\cos 30^0 + \sin 30^0\cos 45^0$

$$= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \times \frac{1}{2} = \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$

$$\sin 75^0 = \frac{\sqrt{6}+\sqrt{2}}{4}$$

(c) $\cos 15^0 = \cos(45^0 - 30^0) = \cos 45^0 \cos 30^0 + \sin 45^0 \sin 30^0$

$= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \times \frac{1}{2} = \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$

$\cos 15^0 = \frac{\sqrt{6}+\sqrt{2}}{4}$

(d) $\sin 15^0 = \sin(45^0 - 30^0) = \sin 45^0 \cos 30^0 - \sin 30^0 \cos 45^0$

$= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \times \frac{1}{2} = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$

$\sin 15^0 = \frac{\sqrt{6}-\sqrt{2}}{4}$

(e) $\sin 330^0 = \sin(360^0 - 30^0) = \sin 360^0 \cos 30^0 - \cos 360^0 \sin 30^0$

$= 0 \times \cos 30^0 - (1) \times \sin 30^0 = -\sin 30^0 = -\frac{1}{2}$

(f) $\cos 240^0 = \cos(180^0 + 60^0) = \cos 180^0 \cos 60^0 - \sin 180^0 \sin 60^0$

$= (-1) \times \cos 60^0 - 0 \times \sin 60^0$

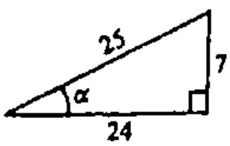
$= -\cos 60^0 = -\frac{1}{2}$

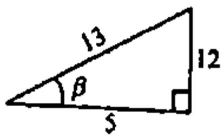
Example 2

Given that α and β are acute angles with $\sin\alpha = \frac{7}{25}$ and $\cos\beta = \frac{5}{13}$, find without using tables or calculator the values of (a) $\sin(\alpha+\beta)$ (b) $\cos(\alpha+\beta)$

Solution

We first sketch the two triangles to represent the given situations and get the remaining sides where necessary





since $\sin\alpha = \frac{7}{25}$; $\cos\alpha = \frac{24}{25}$ since $\cos\beta = \frac{5}{13}$, $\sin\beta = \frac{12}{13}$

(a) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$

$= \frac{7}{25} \times \frac{5}{13} + \frac{12}{13} \times \frac{24}{25} = \frac{323}{325}$

(b) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= \frac{24}{25} \times \frac{5}{13} - \frac{7}{25} \times \frac{12}{13} = \frac{120}{325} + \frac{84}{325} = \frac{204}{325}$

Example 3

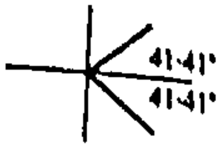
Solve the equation $\cos\theta\cos 20^0 + \sin\theta\sin 20^0 = 0.75$ $for\ 0^0 \le x \le 360^0$

Solution

$\cos\theta\cos 20^0 + \sin\theta\sin 20^0 = 0.75$

$\cos(\theta - 20^0) = 0.75$

$\cos^{-1}(0.75) = 41.41^0$ *and because the cosine is positive*; solutions are in the 1st and 4th quadrants.



$\Rightarrow \theta - 20^0 = 41.41^0$ or 318.59^0

$\therefore \theta = 61.41^0$ or 338.59^0

Example 4

Prove the following identities (a) $\sin(90^0 - \theta) = \cos\theta$ (b) $\cos(180^0 - \theta) = -\cos\theta$

Solution

(a) $\sin(90^0 - \theta) = \sin 90^0 \cos\theta - \sin\theta \cos 90^0$

$= (1)\cos\theta - (\sin\theta)(0) = \cos\theta$

(b) $\cos(180^0 - \theta) = \cos 180^0 \cos\theta + \sin 180^0 \sin\theta$

$= (1)\cos\theta + (0)\sin\theta = \cos\theta$

Example 5

Prove that $\cos(A+B)\cos(A-B) \equiv \cos^2 A - \sin^2 B$

Solution

$L.H.S = \cos(A+B)\cos(A-B)$

$= (\cos A\cos B - \sin A\ \sin B)(\cos A\cos B - \sin A\ \sin B)$

$= \cos^2 A\cos^2 B + \cos A\cos B\sin A\sin B - \sin A\sin B\cos A\cos B - \sin^2 A\sin^2 B$

$= \cos^2 A\cos^2 B - \sin^2 A\sin^2 B$

$= \cos^2 A(1 - \sin^2 B) - (1 - \cos^2 A)\sin^2 B$

$= \cos^2 A - \cos^2 A\sin^2 B - \sin^2 B + \cos^2 A\sin^2 B$

$= \cos^2 A - \sin^2 B = R.H.S$

The solution of triangles

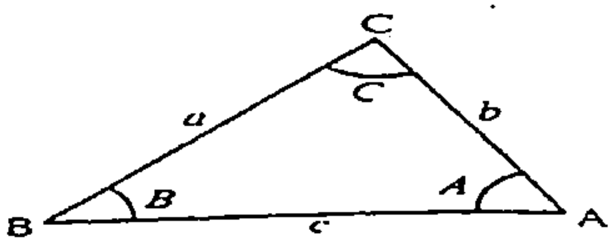
A triangle possesses six elements i.e. the three sides and the three angles. If any three elements (other than three angles.) are given, the remaining three elements can be found .This is called solving the triangle,

In solving the triangle, two geometrical facts are useful i.e.

1. In any triangle the sun of the angles is 180^0
2. In any triangle, the largest side opposite the greater angle and the shortest side is opposite the angle

The sine rule

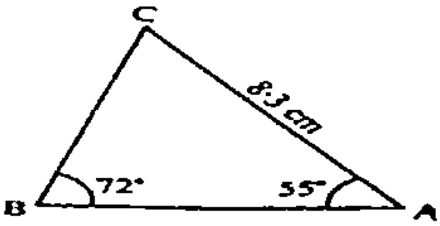
Consider a triangle ABC with sides a, b and c



$Then \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Example 1

Find the length of the side BC in the given triangle



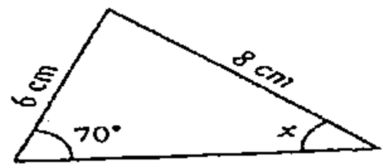
Solution

Using the sine rule; $\frac{BC}{\sin 55} = \frac{8.3}{\sin 72}$

$$BC = \frac{8.3 \sin 55}{\sin 72} = 7.15\ cm$$

Example 2

Find the angle x in the given triangle



Solution

Using the sine rule; $\frac{8}{\sin 70} = \frac{6}{\sin x}$

$$\sin x = \frac{6 \sin 70}{8} = 0.7048$$

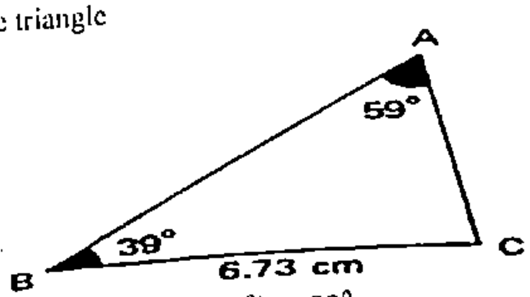
$$x = \sin^{-1}(0.7048) = 44.81^0$$

Example 3

In the triangle ABC, $A = 59^0, B = 39^0$ and $a = 6.73cm$. Find the length of the smallest side and the length of the remaining side.

Solution

First sketch the triangle



$$C = 180^0 - (39^0 + 59^0) = 82^0$$

The smallest side corresponds to the smallest angle, which is 39^0

Using the sine rule, $\frac{b}{\sin 39} = \frac{6.73}{\sin 59}$

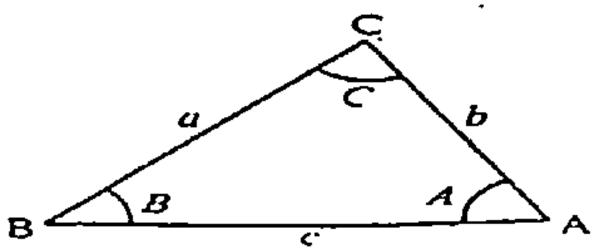
$$b = \frac{6.73 \sin 39}{\sin 59} = 4.94\ cm$$

$\frac{c}{\sin C} = \frac{6.73}{\sin 59}$ but $C = 82^0$

$c = \frac{6.73 \sin 82}{\sin 59} = 7.78\ cm$

The cosine rule

Consider the triangle ABC shown below with the sides a, b and c



Then; $a^2 = b^2 + c^2 - 2bc\cos A$

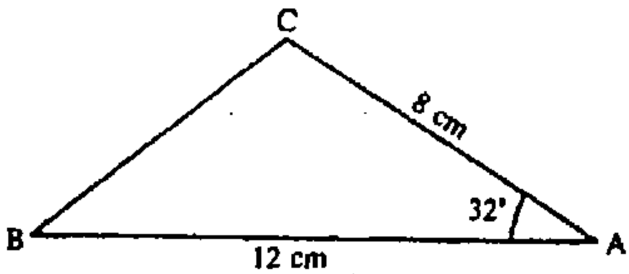
$b^2 = a^2 + c^2 - 2ac\cos B$

$c^2 = b^2 + a^2 - 2ab\cos C$

Example 1

Find the length of side BC in the following triangle

Solution



Solution

By using the cosine rule; $BC^2 = 12^2 + 8^2 - 2(12)(8)\cos 140^0$

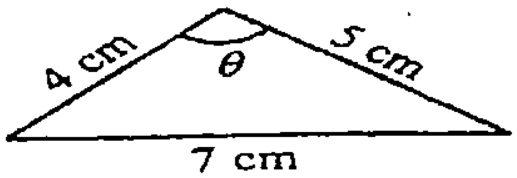
$= 144 + 64 - 192(-0.776)$

$= 208 + 147.1 = 355.1$

$BC = \sqrt{355.1} = 18.8\ cm$

Example 2

Find the angle θ in the triangle below



Using the cosine rule; $7^2 = 5^2 + 4^2 - 2(4)(5)\cos\theta$

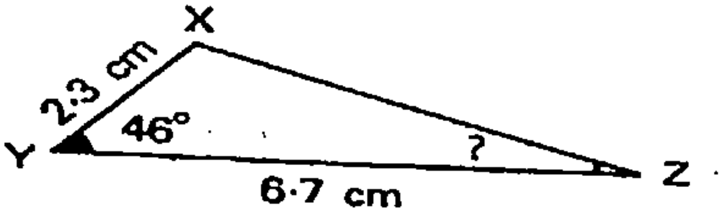
$\cos\theta = \frac{25+16-49}{40} = -0.2$

$\theta = \cos^{-1}(-0.2) = 101.54^0$

Example 3

In the triangle XYZ, YZ = 6.7 cm, XY = 2.3 cm and angle XYZ = 46.53^0. Calculate angle XYZ.

Solution



From the cosine rule; $XZ^2 = YZ^2 + XY^2 - 2(YZ)(XY)\cos(XYZ)$

$$= 6.7^2 + 2.3^2 - 2(6.7)(2.3)\cos 46.53^0$$

$$= 44.89^0 + 5.29^0 - 21.20^0 = 28.98$$

$$XZ = \sqrt{28.98} = 5.383\ cm$$

Now by using the sine rule;

$$\frac{XY}{\sin XZY} = \frac{XZ}{\sin XYZ}$$

$$\sin XZY = \frac{XY\sin XYZ}{XZ} = \frac{2.3\sin 46.53}{5.383} = 0.3101$$

$$XZY = \sin^{-1}(0.3101) = 18.06^0$$

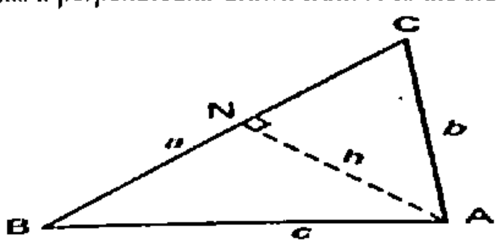
The area of a triangle

Consider the triangles shown below;

The area of the triangle is obtained from the formula;

$$Area = \frac{1}{2} \times base \times Altitude = \frac{1}{2} \times b \times h$$

A triangle ABC has a perpendicular drawn from A to the side BC



If h is the length of the perpendicular, then $h = b\sin C$ or $c\sin B$ *from the* triangles ACN and ABN

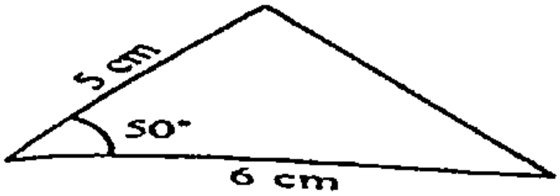
$$\text{New area of triangle} = \frac{1}{2}ah$$

$$= \frac{1}{2}ab\sin C$$

$$= \frac{1}{2}ac\sin B$$

Similarly it can be shown that the area equals to $\frac{1}{2}bc\sin A$.

Example 1
Find the area of the triangle shown below



$Area = \frac{1}{2} \times 5 \times 6 \times \sin 50^0 = 11.5\ cm^2$

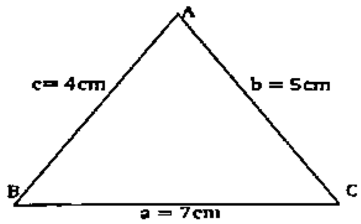
Example 2
In the triangle ABC, AB = 5 cm, BC = 6 cm and angle ABC = 60^0. Find the area of the triangle ABC.

Solution

$Area = \frac{1}{2}ac\sin B = \frac{1}{2} \times 6 \times 5\sin 60 = 15 \times 0.866 = 12.99\ cm^2$

Example 3
Given triangle ABC in which AB = 4 cm, a = 7 cm and AC = 5 cm. Find the
(i) Angle ABC (ii) Area of triangle ABC

Solution



(i) using the cosine rule; $b^2 = a^2 + c^2 - 2ac\cos B$

$$\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{7^2+4^2-5^2}{2\times7\times4} = \frac{40}{56} = 0.71432$$

$$B = \cos^{-1}(0.71432) = 44.4^0$$

(ii) Area of ABC $= \frac{1}{2}ac\sin B = \frac{1}{2} \times 7 \times 4 \times \sin 44.4^0$

Note: since we are given all the three sides of the triangle, the area of this triangle can be calculated using Hero's formula.

Hero's formula

Using Hero's formula, the area of a triangle can be found from the three sides in a triangle ABC as follows;

$$Area = \sqrt{s(s-a)(s-b)(s-c)}$$

Where the sides are a, b and c and s is the semi-perimeter gotten from $s = \frac{1}{2}(a+b+c)$

Example 1
The sides of a triangle are a = 12.7 cm, b = 13.9 cm and c = 8.6 cm. Calculate the area of the triangle.

Solution

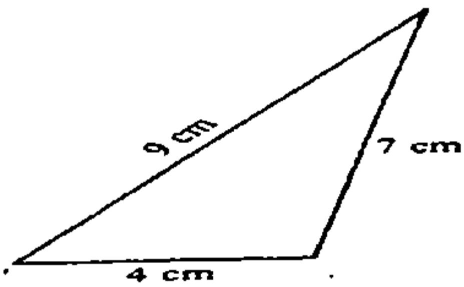
Semi-perimeter $= \frac{1}{2}(12.7 + 13.9 + 8.6) = 17.6$

$$s - a = 17.6 - 12.7 = 4.9$$
$$s - b = 17.6 - 13.9 = 3.7$$
$$s - c = 17.6 - 8.6 = 9.0$$

Area $= \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{17.6 \times 4.9 \times 3.7 \times 9} = 53.59\ cm^2$

Example 2

Find the area of the following triangle



Solution

$s = \frac{4+7+9}{2} = \frac{20}{2} = 10$

$Area = \sqrt{10(10-4)(10-7)(10-9)} = \sqrt{180} = 13.4\ cm^2$

Trial questions

1. Solve the following equations for the values of x *between* 0^0 *and* 360^0
   (a) $5\cos x = \cot x$ $[Ans: 11.53^0, 90^0, 168.47^0, 270^0]$
   (b) $3\tan x = \sec x$ $[Ans: 41.82^0, 138.18^0]$
   (c) $\sin^2 x = \frac{1}{4}$ $[Ans: 30^0, 150^0, 210^0, 330^0]$
   (d) $\tan^2 x = \frac{1}{3}$ $[Ans: 30^0, 150^0, 210^0, 330^0]$
   (e) $2\cos^2 x + 3\cos x + 1 = 0$ $[Ans: 120^0, 180^0, 240^0]$
   (f) $\cos^2 x + \sin x + 1 = 0$ $[Ans: 270^0]$
   (g) $2\cot^2 x + \tan x - 3 = 0$ $[Ans: 30^0, 41.82^0, 138.18^0, 150^0]$
   (h) $\sec^2 x - 3\tan x + 1 = 0$ $[Ans: 45.63^0, 43^0, 225^0, 243.43^0]$
   (i) $2\cot x + \tan x - 3 = 0$ $[Ans: 45^0, 63.43^0, 225^0, 243.43^0]$
   (j) $2\sin^2 x + 3\sin x = 2$ $[Ans: 30^0, 150^0]$
2. Show that $\cos(x + 120^0) + \cos(x + 240^0) = 0$
3. Prove the following identities
   (a) $\frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} = 2cosec\,\theta$
   (b) $\frac{cosec\,\theta}{\cot\theta + \tan\theta} = \cos\theta$
   (c) $\sec^4\theta - \sec^2\theta = \tan^4\theta + \tan^2\theta$

4. A and B are acute angles such that $\sin A = \frac{12}{13}$ $and \cos B = \frac{4}{5}$. Without using tables or calculator, find the values of $(a) \sin(A - B)$ $(b) \cos(A - B)$ [Ans: (a) $\frac{33}{65}$ $(b) \frac{56}{65}$]
5. C and D are both obtuse angles such that $\sin C = \frac{3}{5}$ $and \sin D = \frac{5}{13}$. Without the use of tables or a calculator, find the values of $(a) \sin(C + D)$ $(b) \cos(C - D)$
   [Ans: (a) $-\frac{56}{65}$ $(b) \frac{63}{65}$]
6. In the triangle ABC, AB =15 cm, BC = 6 cm and angle ABC = 60^0, find (i) AC (ii) the remaining angles (iii) area of the triangle.
   [Ans: (i) AC = 5.57 cm (ii) $51.1^0, 68.9^0$ $(iii) 12.99\ cm^2$]
7. Prove the following identities
   (a) $\cos(90^0 + \theta) = -\sin\theta$
   (b) $\sin(A + B) + \sin(A - B) = 2\sin A \sin B$
   (c) $\sin(90^0 + \theta) = \cos\theta$
   (d) $\cos(A + B) - \cos(A - B) = -2\sin A \sin B$
   (e) $\sin\theta\tan\theta + \cos\theta = \sec\theta$
   (f) $cosec\,\theta - \sin\theta = \cot\theta\cos\theta$
   (g) $(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2 = 2$
8. If A and B are acute angles such that $\sin A = 0.28$ $and \cos B = 0.8$. Without using tables or a calculator, find the values of;
   (a) $\sin(A + B)$ $(b) \cos(A - B)$ [Ans: $(a) 0.8$ $(b) 0.936$]
9. Solve the following equations for $0^0 \leq \theta \leq 360^0$
   (a) $\cos 40^0 \cos\theta - \sin 40^0 \sin\theta = 0.4$ [Ans: $26.4^0, 253.6^0$]
   (b) $\sin(\theta + 45^0) = \sqrt{2}\cos\theta$ [Ans: 45^0, 225^0]
10. Solve the triangle ABC given that $A = 66^0, C = 44^0$ $and\ a = 7cm$
    [Ans: $B = 70^0, c = 5.32\ cm, b = 7.2\ cm$]
11. Solve the triangle ABC given that $A = 45^0, c = 5\ cm\ and\ b = 6\ cm$
    [Ans: $a = 4.31\ cm, C = 55.1^0, B = 79.9^0$]
12. Solve the triangle ABC given that C = $50^0, c = 8cm, and\ a = 10\ cm$
    [Ans: $A = 73.2^0, B = 56.8^0, b = 8.73\ cm$ $or\ A = 106.8^0, B = 23.2^0, b = 4.12\ cm$]

CHAPTER 10: DIFFERENTIAL EQUATIONS

A differential equation is an equation that contains a differential coefficient e.g.

$$\frac{dy}{dx} = 3x$$

$$\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} = 6$$

Order of a differential equation

The order of a differential equation is the highest derivative, which appears in it for instance;

The equation $\frac{dy}{dx} - 4x = 3$ is a first order differential equation because it contains only a first differential coefficient i.e. $\frac{dy}{dx}$

The equation $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} = 9$ is a second order differential equation because it contains a second differential coefficient, $\frac{d^2y}{dx^2}$

Note: Any differential equation represents a relationship between two variables say x and y and the same relationship can often be expressed in a form that does not contain a differential coefficient e.g. $y = x^2 + C$ and $\frac{dy}{dx} = 2x$

Solution to a differential equation

The solution of a differential equation is an equation relating the variables involved but containing no differential coefficient like $\frac{dy}{dx}$. There are two types of solutions i.e.

(i) The general solution

This contains an arbitrary constant

(ii) Particular solution

It may be obtained if the " x-value" and the corresponding "y-value" are given. These are called initial conditions are used to calculate the value of the constant.

Consider $\frac{dy}{dx} = 3x^2$; *this is a first order differential equation*

By separating the variables

$$dy = 3x^2dx$$

Integrating on both sides

$$\int dy = \int 3x^2\,dx$$

$y = x^3 + C$; This is a general solution

$If\ x = 1, when\ y = 2$ (*these are initial conditions*)

$$2 = 1 + C$$

$$\Rightarrow C = 1$$

$\therefore y = x^3 + 1$ (This is a particular solution)

Separable differential equations

A separable differential equation is any differential equation that we can write in the following form

$N(y)\frac{dy}{dx} = M(x)$

Note that in order for a differential equation to be separable, all the y's in the differential equation must be multiplied by the derivative and the x's in the differential equation must be on the other side of the equal sign.

Solving separable differential equations is fairly easy. We first rewrite the differential equation as;

$$N(y)dy = M(x)dx$$

Then you integrate on either sides

$$\int N(y)\,dy = \int M(x)\,dx$$

Examples

1. Find the general solutions to the following differential equations

(a) $3y\frac{dy}{dx} = 5x^2$

Solution

By separating variables i.e. by separating $dy\ from\ dx$ and collecting on one side all terms involving y together with dy, while all the x terms with dx, it gives;

$$3ydy = 5x^2dx$$

We now integrate both sides of the equation

$$\int 3ydy = \int 5x^2dx$$

$$\frac{3y^2}{2} = \frac{5x^3}{3} + C$$

(b) $u\frac{du}{dv} = v + 2$

Solution

$$u\,du = (v + 2)dv$$

Integrating on both sides gives;

$$\int u\,du = \int(v + 2)dv$$

$$\frac{u^2}{2} = \frac{v^2}{2} + 2v + C$$

(c) $\frac{dy}{dx} = y^2$

Solution

$$\frac{1}{y^2}dy = dx$$

$$\int y^{-2}\,dy = \int dx$$

$$-y^{-1} = x + c$$

$$-\frac{1}{y} = x + C$$

(d) $\frac{1}{x}\frac{dy}{dx} = \frac{1}{y^2-2}$

Solution

By separating the variables;

$(y^2 - 2)\, dy = x\, dx$

Integrating on both sides;

$\int (y^2 - 2)\, dy = \int x\, dx$

$\frac{y^3}{3} - 2y = \frac{x^2}{2} + C$

(c) $\frac{dy}{dx} = x^2 - 2x$

Solution

$dy = (x^2 - 2x)dx$

$\int dy = \int (x^2 - 2x)dx$

$y = \frac{x^3}{3} - x^2 + C$

2. Find the particular solutions of the following differential equations

(a) $y^2 \frac{dy}{dx} = x^2 + 1 \;\; if\ y = 1\ when\ x = 2$

Solution

$y^2 dy = (x^2 + 1\ )dx$

$\int y^2 dy = \int (x^2 + 1\ )dx$

$\therefore\ \frac{y^3}{3} = \frac{x^3}{3} + x + C$

This is a general solution but since the initial condition is given, we can find the value of C

$\frac{(1)^3}{3} = \frac{(2)^3}{3} + 2 + C$

$C = -\frac{13}{3}$

$therefore\ \frac{y^3}{3} = \frac{x^3}{3} + x - \frac{13}{3}$ is the particular solution

(b) $\frac{dy}{dx} = 6y^2x\ if\ y = \frac{1}{25}\ \ when\ x = 1$

Solution

$y^{-2}dy = 6x\, dx$

$\int y^{-2}dy = \int 6x\, dx$

$\frac{y^{-1}}{-1} = \frac{6x^2}{2} + C$

$-\frac{1}{y} = 3x^2 + C$

So, we now have the general solution. Let us apply the initial condition and find the value of C

$\frac{1}{\frac{1}{25}} = 3(1)^2 + C\ \ which\ gives\ C = -28$

Substitute this value in the general solution to get the particular solution

$-\frac{1}{y} = 3x^2 - 28$

(c) $\frac{dy}{dx} = \frac{3x^2+4x-4}{2y-4}$ $\quad if\ y = 3\ when\ x = 1$

Solution

This differential equation is clearly separable, so let us put it in the proper form and then integrate both sides

$$(2y-4)dy = (3x^2+4x-4)dx$$
$$\int(2y-4)dy = \int(3x^2+4x-4)dx$$
$$y^2 - 4y = x^3 + 2x^2 - 4x + C$$

When $x = 1, y = 3$

$$(3)^2 - 4(3) = (1)^3 + 2(1)^2 - 4(1) + C$$
$$C = -2$$

Therefore $y^2 - 4y = x^3 + 2x^2 - 4x - 2$

Natural occurrences of differential equations

Differential equations often arise when a physical situation is interpreted mathematically (i.e. when a mathematical model is made of the physical situation). Differential equations are used to solve applied problems such as those involving carbon dating and radioactive decay; the amount of drug in an organ: mixtures; supply and demand; logistic growth and marginal productivity.

Example 1

A body moves with a velocity v, which is inversely proportional to its displacement s from a fixed point. Form a differential equation to represent the information

Solution

Velocity is the rate of change of displacement with respect to time

$$v \propto \frac{1}{s}$$
$$k = \frac{k}{s} \quad where\ k\ is\ a\ constant$$

But $v = \frac{ds}{dt}$

$$\Rightarrow \frac{ds}{dt} = \frac{k}{s}$$
$$s\frac{ds}{dt} = k \quad is\ the\ differential\ equation$$

Example 2

A particle moves in a straight line with an acceleration that is inversely proportional to its velocity (acceleration is the rate of change of velocity)

(a) Form the differential equation to represent this data

(b) Given that the acceleration is 2 m/s^2 when the velocity is 5 m/s, solve the differential equation

Solution

(a) Using $\frac{dv}{dt}$ $for\ acceleration, we\ have;$

$$\frac{dv}{dt} \propto \frac{1}{v} \quad \Rightarrow \frac{dv}{dt} = \frac{k}{v}$$

(b) $If\ v = 5, \frac{dv}{dt} = 2$

Then $2 = \frac{k}{5} \Rightarrow k = 10$

$$\frac{dv}{dt} = \frac{10}{v}$$

$v\,dv = 10\,dt$

$\int v\,dv = \int 10\,dt$

$\frac{v^2}{2} = 10t + c$

Example 3

The rate of change of the price with respect to time is inversely proportional to the current price, P. Form a differential equation to represent the above information and solve it.

Solution

$\frac{dP}{dt} \propto \frac{1}{P}$

$\frac{dP}{dt} = \frac{k}{P}$

$P\,dP = k\,dt$

$\int P\,dP = \int k\,dt$

$\frac{P^2}{2} = kt + C$

Trial questions

1.Solve the differential equation $\frac{dy}{dx} = 3x^2y^2$ *given that* $y = 1$ *when* $x = 0$

[Ans: $x^3y = y - 1$]

2. Find the general solution to the differential equation $6t\frac{dt}{ds} + 1 = 0$, and the particular solution given by the conditions s = 0 when t = -2 [Ans: $s = 12 - 3t^2$]

3. Find the general solutions of the following differential equations

(a) $\frac{dy}{dx} = 3x$ (b) $2y\frac{dy}{dx} = 3$ (c) $\frac{dy}{dx} = \frac{x-4}{4y^3}$

(d) $\frac{dy}{dx} = -\frac{x}{y}$ (e) $\frac{dy}{dx} = y^{\frac{4}{5}}$

[Ans: (a) $y = \frac{3x^2}{2} + C$ (b) $y^2 = 3x + C$ (c) $y^4 = \frac{x^2}{2} - 4x + C$ (d) $\frac{y^2}{2} = \frac{-x^2}{2} + C$

(e) $5y^{\frac{1}{5}} = x + C$]

4. Find the particular solution of the differential equation $\frac{dx}{dt} = t$, *where* $x = 3$ *when* $t = 1$.

[Ans: $x = \frac{t^2}{2} + \frac{5}{2}$]

5. The rate of change of y with respect to x is proportional to the square of x. Write a differential equation that models this statement. [Ans: $\frac{dy}{dx} = kx^2$]

6. Given that $\frac{dy}{dx} = x^2 + kx$ where k is a constant. If y has a turning point at the point (3, -2), calculate the value of (i) k (ii) y when $x = 4$ [Ans: (i) $k = -3$ (ii) $y = -\frac{1}{6}$]

CHAPTER 11: DESCRIPTIVE STATISTICS

This is the branch of mathematics dealing with collection, interpretation, presentation and analysis of data where data refers to the facts in the day-to-day life.

Statistical methods are used in research to collect, analyze and formulate research findings in every field at higher institutions of learning.

Statistical data can be categorized into two .i.e. Qualitative and Quantitative.

Qualitative data measures attributes such as sex, colour, and so on while Quantitative data can be represented by numerical quantity

Quantitative data is of two forms. i.e. Continuous or discrete.

Discrete data is the information collected by counting and usually takes on integral values e.g. number of students in a class, school etc.

Continuous data can take on any value i.e. weight, height, mass, etc.

The quantity, which is counted or measured, is called the variable.

CRUDE/RAW/UNGROUPED DATA

These are individual values of a variable that have been arranged in order and grouped in small number of classes.

GROUPED / CLASSIFIED DATA

These are individual values of a variable that have been arranged in order and grouped in small number of classes.

POPULATION AND SAMPLES

A population is a total set of Items under consideration and it's defined by some characteristics of these items.

A sample is a finite subset of a population.

PRESENTATION OF DATA

The ways of presenting data include:

- Bar graphs
- Histogram
- Frequency Polygon
- The Ogive
- Pie chart

BAR GRAPH

A bar graph or bar chart is a graph where the class frequencies are plotted against class limits.

HISTOGRAM

A histogram is a graph where the class frequencies are plotted versus class boundaries.

Example 1

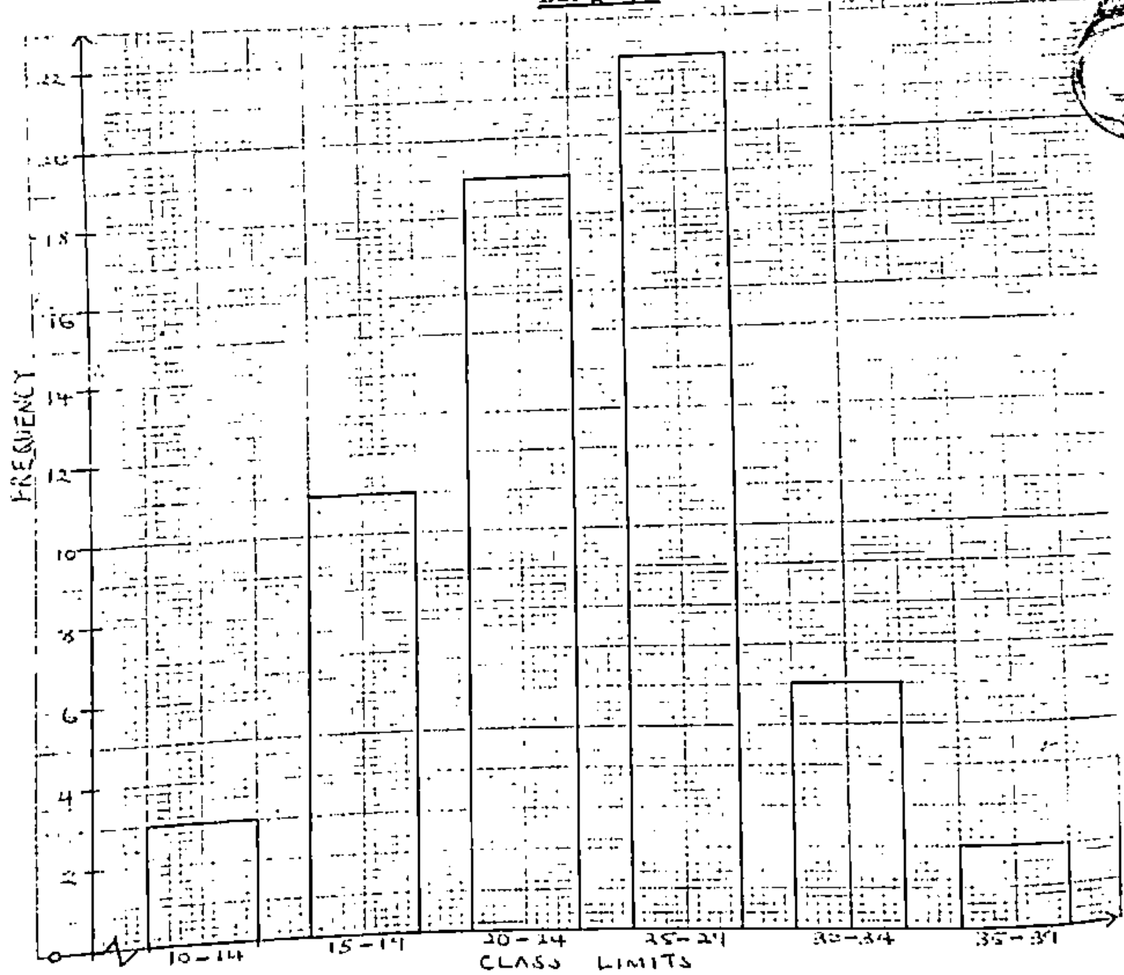
The times taken by rats to pass through a maze are recorded in the table below. Use the data given to plot a bar graph and histogram.

Time(seconds)	10-14	15-19	20-24	25-29	30-34	35-39
Frequency	3	11	19	22	6	2

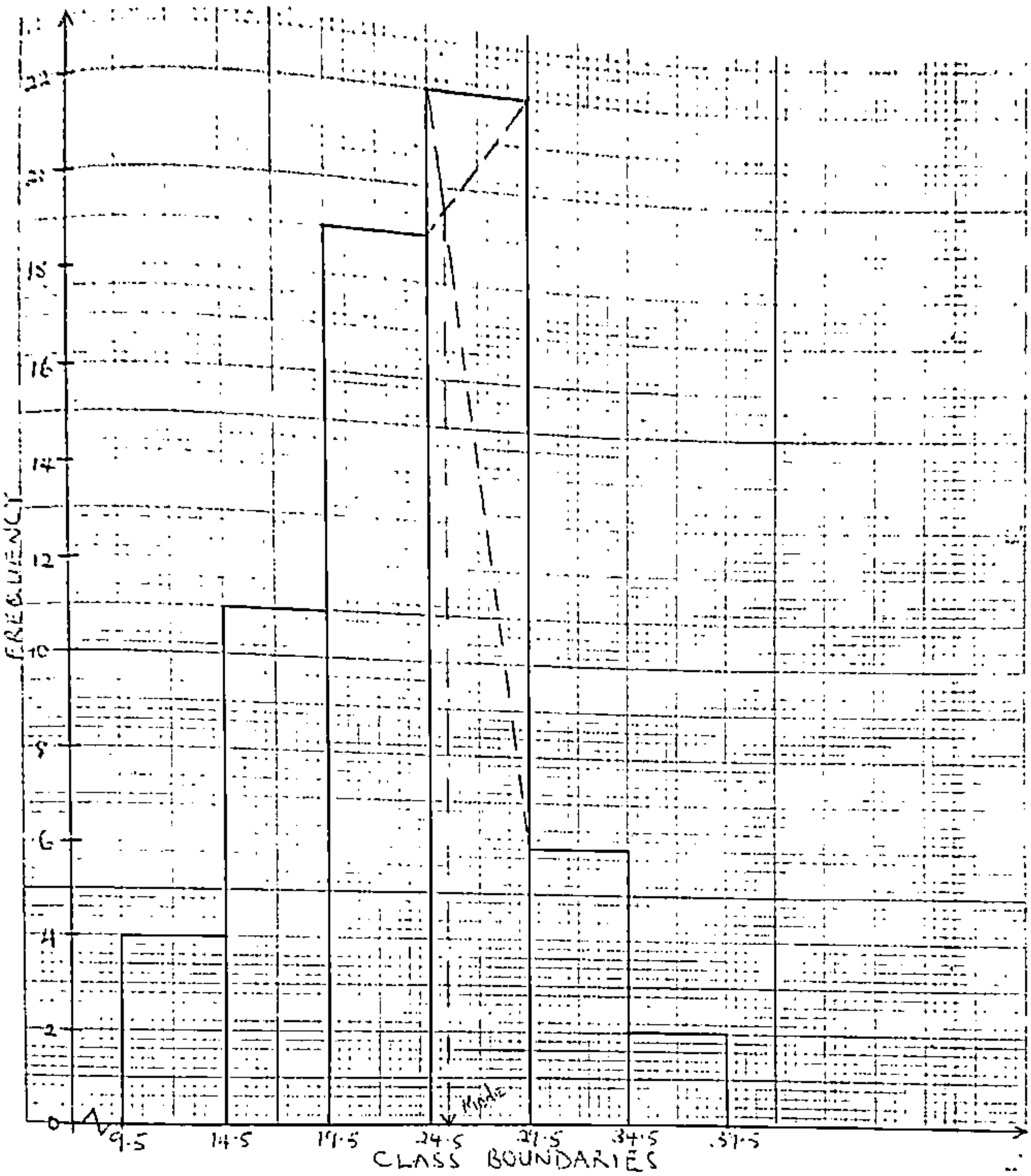
Solution

Class limits	Class boundaries	Frequency
10 - 14	9.5 - 14.5	3
15 - 19	14.5 - 9.5	11
20 - 24	19.5 - 24.5	19
25 - 29	24.5 - 29.5	22
30 - 34	29.5 - 34.5	6
35 - 39	34.5 - 39.5	2

Bar graph



Histogram



Note: The mode can be obtained from histogram as shown above.

Estimated mode from histogram = 24.5 + 1 = 25.5

The reader should also note that these are spaces between the bars for a bar graph while there are no spaces for a histogram.

Shading of the histogram is not important and if it is used, let it be uniform

Example 2

The table below shows the population of Kampala in millions for different age groups

Age group	Population in millions
Below 10	2
10 and under 20	8
20 and under 30	10
30 and under 40	14
40 and under 50	5
50 and under 60	1

Draw a histogram to represent the above data

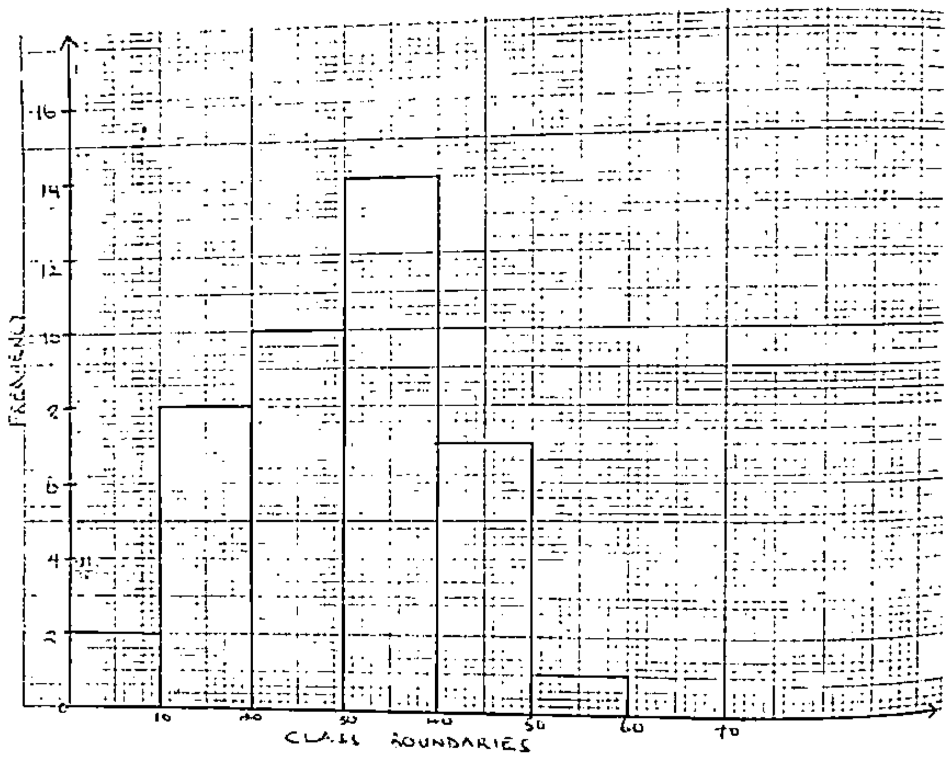
Solution

Class	Frequency
0 - < 10	2
10 - < 20	8
20 - < 30	10
30 - < 40	14
40 - < 50	5
50 - < 60	1

In this case,

The class boundaries are given i.e. 0 - < 10

Histogram



FREQUENCY POLYGON

The frequency polygon is obtained by plotting class frequencies versus class marks. Then the consecutive points are joined using a straight line.

Class mark/ mid interval value (x) = $\frac{1}{2}$ (Lower class limit + upper class)

i.e. for the class 10-14, class mark(x) = $\frac{1}{2}(10 + 14) = 12$

The class mark is also known as the mid mark

Example 3

The age distribution of a group of people is given in the table below.

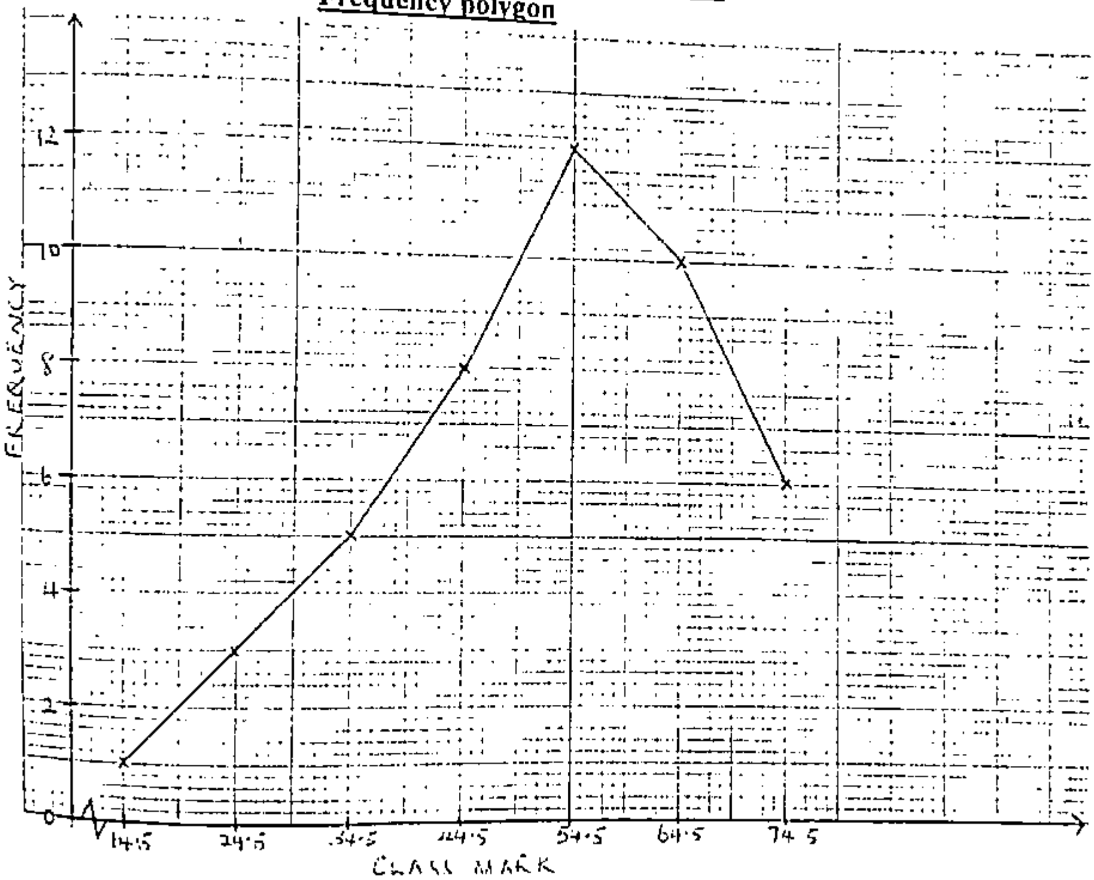
Marks	10 - 19	20 - 29	30 - 39	40 - 49	50 – 59	60 - 69	70 - 79
Frequency	1	3	5	8	12	10	6

Construct a frequency polygon for the data above

Solution

Class Limits	Class mark	Frequency
10-19	14.5	1
20-29	24.5	3
30-39	34.5	5
40-49	44.5	8
50-59	54.5	12
60-69	64.5	10
70-79	74.5	6

Frequency polygon



MEASURES OF CENTRAL TENDENCY

The measures of central tendency include mean, mode and median. They are called so because they are centered about the same value.

MEAN

This is the sum of the data values divided by the number of values in the data. It is denoted by $\bar{X}$

Mean, $\bar{X} = \frac{\sum x}{n}$ $\quad$ *where $\sum$ means summation*

The mean for grouped data can also be calculated from;

(i) $\bar{X} = \frac{\sum fx}{\sum f}$ where x is the class mark and f is the frequency

(ii) $\bar{X} = A + \frac{\sum fd}{\sum f}$ where A is the Assumed mean / Working mean and d is the deviation where $d = x - A$

Examples

1. The measured weight for a child over eight-year period gave the following results (in kgs) 32, 33, 35, 38, 43, 53, 63, 65. Calculate the mean weight of the child.

Solution

$$\text{Mean} = \frac{\sum x}{n} = \frac{32+33+35+38+43+53+63+65}{8}$$
$$= 45.25\text{kg}$$

2. The information below gives the age in years of 49 students. Determine the mean age.

Age	14	15	16	17	18	21
Frequency	2	6	14	10	9	8

Solution

Age(x)	Frequency(f)	fx
14	2	28
15	6	90
16	14	224
17	10	170
18	9	162
21	8	168
$\sum$	49	842

$$\text{Mean}, \bar{X} = \frac{\sum fx}{\sum f} = \frac{842}{8} = 17.184 \text{ years}$$

3. The data below shows the weights in kg of an S.5 class in a certain school.

Weight(kg)	10 – 14	15 – 19	20 – 24	25 – 29	30 – 34	35 – 39	40 – 44	45 – 49
Frequency	5	9	12	18	25	15	10	6

Calculate the mean weight of the class

Solution

Class	Class mark (x)	Frequency(f)	fx
10-14	12	5	60
15-19	17	9	153
20-24	22	12	264
25-29	27	18	486
30-34	32	25	800
35-39	37	15	555
40-44	42	10	420
45-49	47	6	282
$\sum$		100	3020

Mean, $\bar{X} = \frac{\Sigma fx}{\Sigma f} = \frac{3020}{100} = 30.20$

4. The table below shows the marks obtained by students in a sub-maths test marked out of 40

Marks	10 – 14	15 - 19	20 – 24	25 – 29	30 – 34	40 – 49
Cumulative frequency	2	8	17	38	45	50

Calculate the mean mark of the students

Solution

In this case, the cumulative frequencies are given instead of the frequencies; therefore we have to find the frequencies. The first value of the cumulative frequency is the first value of frequency. We obtain the next values of frequency by subtracting the consecutive cumulative frequencies.

Marks	Cumulative frequency(F)	Frequency(f)	Class mark (x)	fx
10 – 14	2	2	12	24
15 – 19	8	6	17	102
20 – 24	17	9	22	198
25 – 29	38	21	27	567
30 – 34	45	7	32	224
35 – 39	50	5	37	185
$\sum$		50		1300

Mean, $\bar{X} = \frac{\Sigma fx}{\Sigma f} = \frac{1300}{50} = 26$

Mean from assumed/Working mean

We can calculate the mean when give the assumed mean which is also known as the working mean using the formula;

$\bar{X} = A + \frac{\Sigma fd}{\Sigma f}$ *where A is the Assumed mean / Working mean* and d is the deviation where $d = x - A$

5. The height to the nearest class of 30 pupils is shown in the table below. Using 152cm as the assumed mean, calculate the mean height.

Height(x cm)	148	149	150	151	152	153	154	155	156
No. of Pupil	1	2	2	3	6	7	4	3	2

Solution

Assumed mean, A =152

Height(x)	Frequency(f)	Deviation, $d = x - A$	fd
148	1	-4	-4
149	2	-3	-6
150	2	-2	-4
141	3	-1	-3
152	6	0	0
153	7	1	7
154	4	2	8
155	3	3	9
156	2	4	8
$\sum$	30		15

Mean, $\bar{X} = A + \frac{\sum fd}{\sum f} = 152 + \frac{15}{30} = 152 + 0.5 = 152.5\ cm$

6. The number of accidents that took place at black spot on a certain road in 2008 were recorded as follows:

No. of accidents	0 – 4	5 – 7	8 – 10	11 - 13	14 – 18
No. of days	2	5	10	8	5

Using 9 as the working mean, calculate the mean number of accidents per day.

Solution

A = 9

Class	Mid value(x)	Freq(f)	Deviation($d = x - A$)	fd
0 – 4	2	2	-7	-14
5 – 7	6	5	-3	-15
8 – 10	9	10	0	0
11 – 13	12	8	3	24
14 – 18	16	5	7	35
$\sum$		30		30

$Mean, \bar{X} = A + \frac{\sum fd}{\sum f} = 9 + \frac{30}{30} = 9 + 1 = 10$

MEDIAN

The median of a group of numbers is the number in the middle when the numbers are in order of magnitude.

Determine the median for the following observations

(i) 4,1,6,2,6,7,8

Solution

1,2.4,6,6,7,8

The median is 6

(ii) 3,3,3,7,6,4,7,6,4,8

Solution

3,3,3,4,4,6,6,7,7,8

The median $= \frac{4+6}{2} = \frac{10}{2} = 5$

For grouped data, we can use the following formula to calculate the median i.e

$$Median = L_1 + \left(\frac{\frac{N}{2} - F_b}{f_m}\right) \times C$$

Where:

L_1 = lower class boundary of the median class

N = Total number of observations or the total frequency

F_b = Cumulative frequency before median class

f_m = Frequency of the median class

C = Class width

Class width

This is the difference between the lower and upper class boundaries i.e. for the class 40 – 44, the class width is 44.5 – 39.5 = 5.

Note that it depends on the degree of accuracy i.e. for the class 7.0 – 7.4, the class width will be 7.45 – 6.95 = 0.5

Advantages of the median

It is easy to understand and calculate

It is not affected by extreme values

Disadvantage of the median

It is only one or two values to decide the median

THE MODE

This is the number in a set of numbers that occurs the most i.e. the modal value of 5, 6, 3, 4, 5, 2, 5, 4 and 3 is 5 because there are more 5s than any other number.

For grouped data, the mode is calculated from;

$$Mode = L_1 + \left(\frac{\Delta_1}{\Delta_1 + \Delta_2}\right) \times C$$

Where;

L_1 = lower class boundary of the modal class

Δ_1 = difference between the modal frequency and the value before it

Δ_2 = difference between the modal frequency and the value after it

C = class width

Note: The modal class is identified as the class with the highest frequency and the mode can as well be estimated from the histogram as we have already seen.

Example 1

The following were the heights of people in a certain town of Uganda.

Height(cm)	101 – 120	121 – 130	131 – 140	141 – 150	151 – 160	161 – 170	171 – 190
No. of p'ple	1	3	5	7	4	2	1

Calculate the mean, mode, and median for the data.

Solution

Class	f	Class mark(x)	fx	F	Class boundaries
101 – 120	1	110.5	110.5	1	100.5 – 120.5
121 – 130	3	125.5	376.5	4	120.5 – 130.5
131 – 140	5	135.5	677.5	9	130.5 – 140.5
141 – 150	7	145.5	1018.5	16	140.5 – 150.5
151 – 160	4	155.5	622	20	150.5 – 160.5
161 – 170	2	165.5	331	22	160.5 – 170.5
171 – 190	1	180.5	180.5	23	170.5 – 190.5
Σ	23		3316.5		

Mean, $\bar{X} = \frac{\Sigma fx}{\Sigma f} = \frac{3.5316}{23} = 144\ cm$

Median class is 141 – 150

$$Median \doteq L_1 + \left(\frac{\frac{N}{2} - F_b}{f_m}\right) \times C$$

$$L_1 = 140.5, F_b = 9\ , f_m = 7, \frac{N}{2} = \frac{23}{2} = 11.5, C = 10$$

$$Median = 140.5 + \left(\frac{11.5-9}{7}\right) \times 10 = 140.5 + \frac{2.5}{7} \times 10$$

$$= 140.5 + 3.57 = 144.1 \text{ cm}$$

The modal class is 141 – 150

$$Mode = L_1 + \left(\frac{\Delta_1}{\Delta_1 + \Delta_2}\right) \times C$$

$$L_1 = 140.5,\ \Delta_1 = 7 - 5 = 2, \qquad \Delta_2 = 7 - 4 = 3$$

$$\text{Mode} = 140.5 + \left(\frac{2}{2+3}\right) \times 10 = 140.5 + 4 = 144.5 \text{ cm}$$

Note: Do you realize that the mean, mode and median have closely the same values? Hence they are the measures of central tendency.

Example 2

1. The data below shows the weights in kg of an S.5 class in a certain school.

Weight(kg)	10 – 14	15 – 19	20 – 24	25 – 29	30 – 34	35 – 39	40 – 44	45 – 49
Frequency	5	9	12	18	25	15	10	6

Calculate the median and modal weight of the class

Solution

Class	freq(f)	F	Class boundaries
10 – 14	5	5	9.5 – 14.5
15 – 19	9	14	14.5 – 19.5
20 – 24	12	26	19.5 – 24.5
25 – 29	18	44	24.5 – 29.5
30 – 34	25	69	29.5 – 34.5
35 – 39	15	84	34.5 – 39.5
40 – 44	10	94	39.5 – 44.5
45 – 49	6	100	44.5 – 49.5

$$Median = L_1 + \left(\frac{\frac{N}{2} - F_b}{f_m}\right) \times C$$

The median class is 30 – 34, $L_1 = 29.5, f_m = 25\ , F_b = 44,\ \frac{N}{2} = \frac{100}{2} = 50$

$$median = 29.5 + \left(\frac{50-44}{25}\right) \times 5 = 29.5 + 1.2 = 30.7 \text{ kg}$$

$$\text{Mode} = L_1 + \left(\frac{\Delta_1}{\Delta_1+\Delta_2}\right) \times C$$

The modal class is 30 – 34, $\Delta_1 = 25 - 18 = 7$, $\Delta_2 = 25 - 15 = 10$

$$mode = 29.5 + \left(\frac{7}{7+10}\right) \times 5 = 29.5 + 2.06 = 31.56 \text{ kg}$$

THE OGIVE

The Ogive is also known as the cumulative frequency curve where by cumulative frequency curve is plotted against the upper class boundaries and the consecutive points are joined into a smooth curve using free hand.

Examples

1. The frequency distributed table shows the weights of 100 children measured to the nearest kg.

Weight	10 – 14	15 – 19	20 – 24	25 – 29	30 – 34	35 – 39	40 – 44	45 – 39
No. of Children	5	9	12	18	25	15	10	6

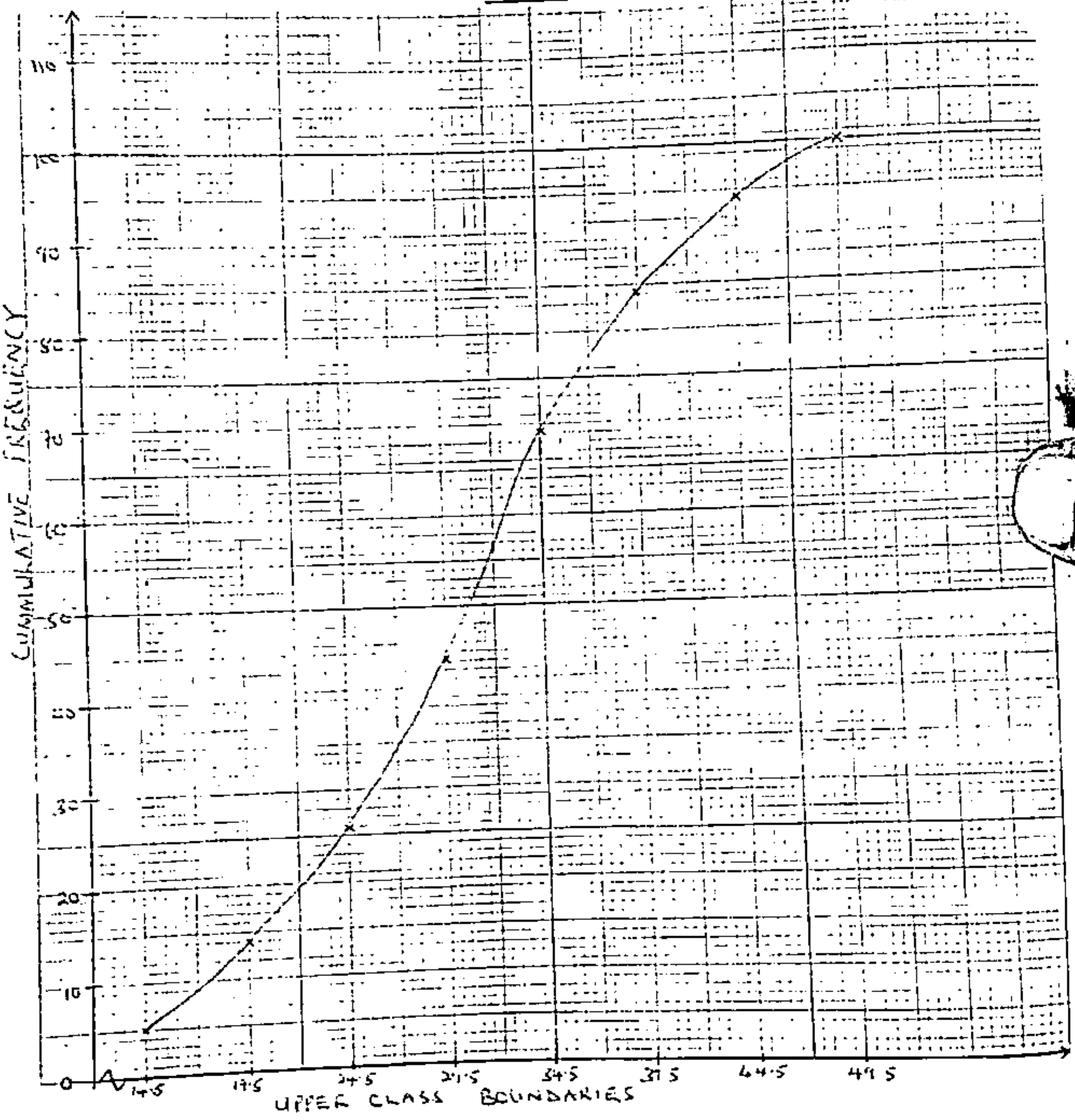
Draw a cumulative frequency curve for the data.

Solution

We need to draw a distribution table with the cumulative frequencies and class boundaries

Class	$freq(f)$	$Cumulative\ frequency\ (F)$	Class boundaries
10 – 14	5	5	9.5 – 14.5
15 – 19	9	14	14.5 – 19.5
20 – 24	12	26	19.5 – 24.5
25 – 29	18	44	24.5 – 29.5
30 – 34	25	69	29.5 – 34.5
35 – 39	15	84	34.5 – 39.5
40 – 44	10	94	39.5 – 44.5
45 – 49	6	100	44.5 – 49.5

OGIVE



Estimating median and quartiles using the Ogive

2. The marks obtained by 40 pupils in a mathematics examination were as follows:

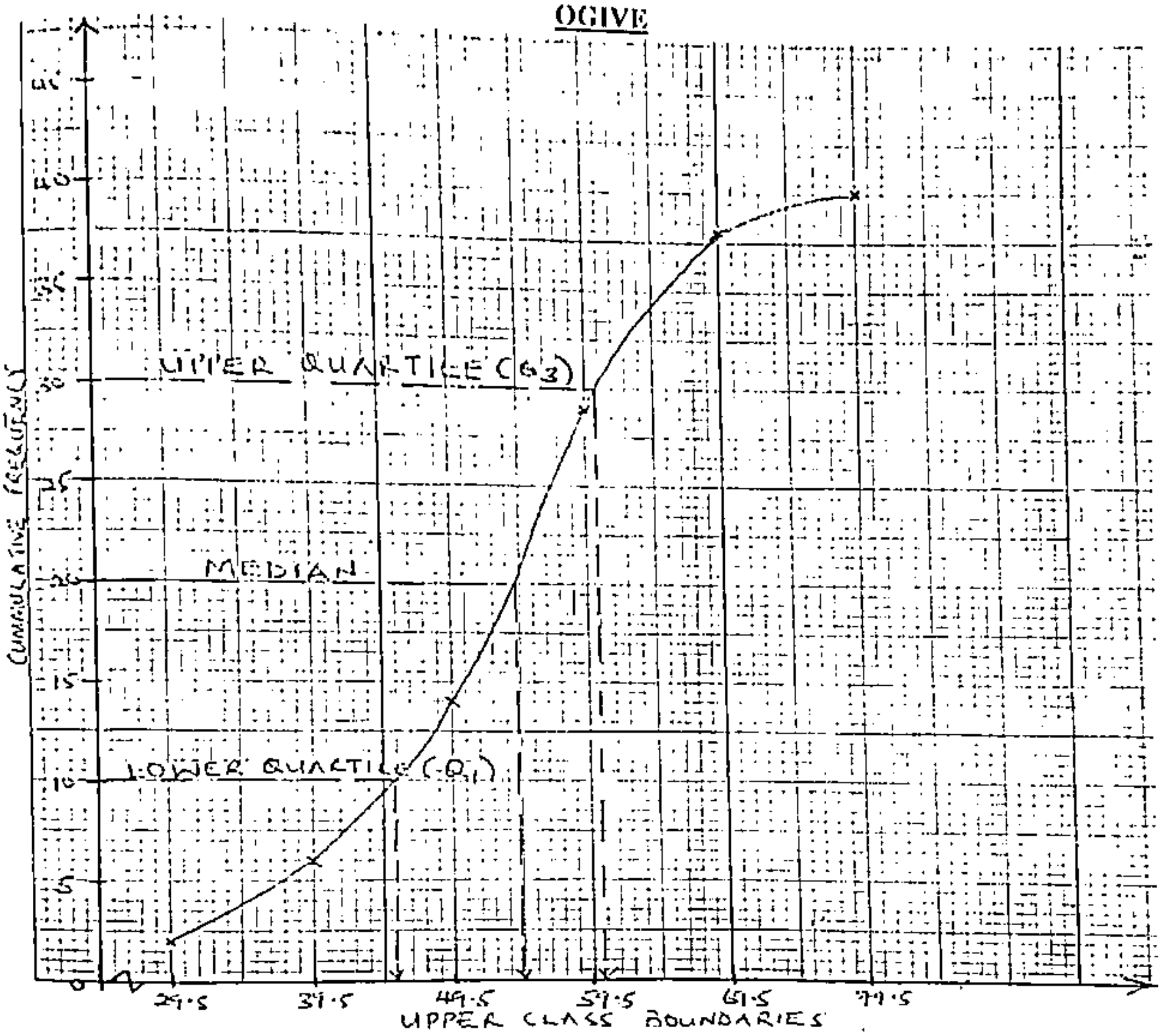
Marks	20 – 29	30 – 39	40 – 49	50 – 59	60 – 69	70 – 79
No. of pupils	2	4	8	15	9	2

Plot a cumulative frequency curve and use it to estimate the median mark, upper quartile, lower quartile and the inter quartile range

Solution

Class	Freq(f)	F	Upper class boundaries
20 – 29	2	2	29.5
30 – 39	4	6	39.5
40 – 49	8	14	49.5
50 – 59	15	29	59.5
60 – 69	9	38	69.5
70 – 79	2	40	79.5

OGIVE



$$Median = \left(\frac{1}{2}N\right)^{th} measure = \frac{1}{2} \times 40 = 20^{th} measure$$

Draw a dotted line across the graph from F = 20 to meet the curve and drop a vertical dotted line to meet the horizontal axis. This gives the estimated median

Hence the median = 54 marks

Quartiles

The quartiles divide a distribution into four equal parts.
The lower quartile(Q_1)is the value 25% way through the distribution and the value 75% way through the distribution is called the upper quartile(Q_3).

Lower quartile$(Q_1) = \left(\frac{1}{4}N\right)^{th}$ $measure = \frac{1}{4} \times 40 = 10^{th} measure$

From the graph, lower quartile = 45.5

Upper quartile$(Q_3) = \left(\frac{3}{4}N\right)^{th}$ $measure = \frac{3}{4} \times 40 = 30^{th} measure$

From the graph, upper quartile = 60

The difference between the upper quartile and lower quartile is called the Interquartile range.

The Interquartile range = $Q_3 - Q_1$

For the given graph, interquartile range = 60 – 45.5 = 14.5

The semi-interquartile range or quartile deviation = $\frac{1}{2}(Q_3 - Q_1)$

For the given graph, interquartile range = $\frac{1}{2} \times 14.5 = 7.25$

Percentiles

The percentiles divide a distribution into one hundred equal parts.
The lower quartile, Q_1 is the 25th percentile P25, the median is the 50th percentile P50 and the upper quartile Q_3 is the 75th percentile P75.

Example

The data shows the marks obtained by 80 form IV pupils in a certain school.

Mark	1-10	11-20	21-30	31-40	41-50	51-60	61-70	71-80	81-90	91-100
Frequency	3	5	5	9	11	15	14	8	6	4

Draw a cumulative frequency and use your graph to estimate

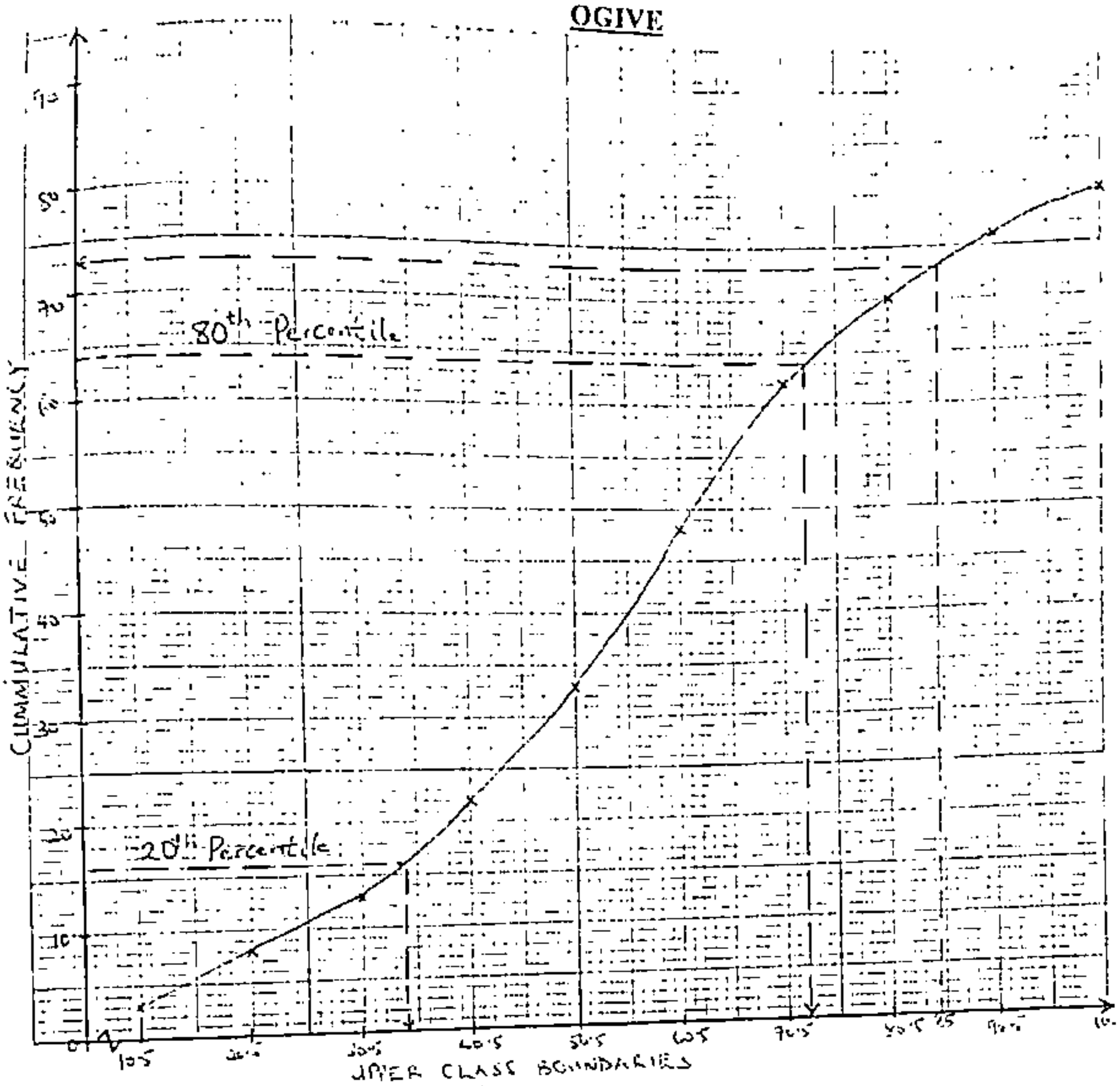
(i) the 20th and 80th percentile mark

(ii) the number of pupils who scored a distinction given that the mark for a distinction was 85

Solution

Marks	$Freq(f)$	F	Upper class boundaries
1 – 10	3	3	10.5
11 – 20	5	8	20.5
21 – 30	5	13	30.5
31 – 40	9	22	40.5
41 – 50	11	33	50.5
51 – 60	15	48	60.5
61 – 70	14	62	70.5
71 – 80	8	70	80.5
81 – 90	6	76	90.5
91 – 100	4	80	100.5

OGIVE



(i) 20th percentile mark $= \frac{20}{100} \times 80 = 16^{th} measure$

From the graph, 20th percentile mark = 32.5

80th percentile mark$= \frac{80}{100} \times 80 = 64^{th} measure$

From the graph, 80th percentile mark = 71.5

(ii) To obtain the number of pupils who scored a distinction, we draw a dotted vertical line from 85 on the horizontal axis to meet the curve at a certain point. From that point, draw a horizontal dotted line to meet the cumulative frequency axis

From the graph, we read off 73 but this is not the number of pupils that scored above 85 marks.
To obtained the required number, we subtract this value from the total number of pupils i.e
Number of pupils that scored above 85 $= 80 - 73 = 7\ pupils$

MEASURES OF DISPERSION

The spread of observations in relation to a measure of central tendency of the given data is known as dispersion. In order to compare data, the measure of dispersion is taken into account along with the measure of central tendency.

The range

This is the difference between the largest and the smallest values of the data.
i.e. for the data about lengths of leaves in garden tree, 5,6,7,7,4,5,3,2,9,8,8,6,5,3

Range = 9 – 2 = 7

Standard deviation

This is the positive square root of variance. Standard deviation is denoted by σ

Standard deviation $(\sigma) = \sqrt{Variance}$

The following expressions can be used to calculate the standard deviation;

$$\sigma = \sqrt{\frac{\Sigma f x^2}{\Sigma f} - \left(\frac{\Sigma f x}{\Sigma f}\right)^2}$$

When using the assumed/ working mean, A

$$\sigma = \sqrt{\frac{\Sigma f d^2}{\Sigma f} - \left(\frac{\Sigma f d}{\Sigma f}\right)^2}$$

Note: the expression under the square root is the variance

Examples

1. Calculate the standard deviation for the distribution of marks in the table below.

Marks	5	6	7	8	9
Frequency	3	8	9	6	4

Solution

Marks(x)	Frequency(f)	fx	fx^2
5	3	15	75
6	8	48	288
7	9	63	441
8	6	48	384
9	4	36	324
Σ	30	210	1512

$$\sigma = \sqrt{\frac{\Sigma f x^2}{\Sigma f} - \left(\frac{\Sigma f x}{\Sigma f}\right)^2}$$

$$\sigma = \sqrt{\frac{1512}{30} - \left(\frac{210}{30}\right)^2} = \sqrt{50.4 - 49} = \sqrt{1.4} = 1.183$$

2. The table below shows the weights to the nearest kg of 150 patients who visited a certain health unit during a certain week

Weight(kg)	10 – 19	20 – 29	30 – 39	40 – 49	50 – 59	60 – 69	70 – 79
No. of patients	30	16	24	32	28	12	8

Calculate the standard deviation of the weights of the patients.

Solution

Class	Freq(f)	x	fx	fx^2
10-19	30	14.5	435	6307.5
20-29	16	24.5	392	9604
30-39	24	34.5	828	28566
40-49	32	44.5	1424	63368
50-59	28	54.5	1526	83167
60-69	12	64.5	774	49923
70-79	8	74.5	596	44402
$\sum$	150		5975	285337.5

$$\sigma = \sqrt{\frac{\Sigma fx^2}{\Sigma f} - \left(\frac{\Sigma fx}{\Sigma f}\right)^2}$$

$$\sigma = \sqrt{\frac{285337.5}{150} - \left(\frac{5975}{150}\right)^2} = \sqrt{1902.25 - 1586.69} = \sqrt{315.56} = 17.76$$

3. The table below gives the points scored by a team in various events. Find the mean and standard deviation using working mean A = 4

Points	0	1	2	3	4	5	6	7
No. of events	1	3	4	7	5	5	2	3

Solution

Points	Frequency	$d = x - A$	fd	fd^2
0	1	-4	-4	16
1	3	-3	-9	27
2	4	-2	-8	16
3	7	-1	-7	7
4	5	0	0	0
5	5	1	5	5
6	2	2	4	8
7	3	3	9	27
$\sum$	30		-10	100

$$Mean, \bar{X} = A + \frac{\Sigma fd}{\Sigma f} = 4 + \frac{-10}{30} = 4 - 0.33 = 3.67\ points$$

Standard deviation, $\sigma = \sqrt{\frac{\Sigma fd^2}{\Sigma f} - \left(\frac{\Sigma fd}{\Sigma f}\right)^2}$

$$\sigma = \sqrt{\frac{106}{30} - \left(\frac{-10}{30}\right)^2} = \sqrt{3.533 - 0.111} = \sqrt{3.422} = 1.85 \text{ points}$$

4. The table below shows the weight in kg of 100 boys in a certain school

Weight(kg)	60 – 62	63 – 65	66 – 68	69 – 71	72 – 74
Frequency	8	10	45	30	7

Using the assumed mean of 67, calculate the mean and standard deviation

Solution

Weight	Freq(f)	Mid value (x)	$d = x - A$	fd	fd^2
60-62	8	61	-6	-48	288
63-65	10	64	-3	-30	90
66-68	45	67	0	0	0
69-71	30	70	3	3	270
72-74	7	73	6	6	252
Σ	100			54	900

$$Mean, \bar{X} = A + \frac{\Sigma fd}{\Sigma f} = 67 + \frac{54}{100} = 67 + 0.54 = 67.54\ points$$

$$\text{Standard deviation, } \sigma = \sqrt{\frac{\Sigma fd^2}{\Sigma f} - \left(\frac{\Sigma fd}{\Sigma f}\right)^2}$$

$$\sigma = \sqrt{\frac{900}{100} - \left(\frac{54}{100}\right)^2} = \sqrt{9 - 0.2916} = \sqrt{8.7084} = 2.951$$

Trial questions

1. The table below shows the weekly wages of a number of workers at a small factory.

Weekly wages	75-84	85-94	95-104	105-114	115-124	125-134	135-144	145-154
Frequency	2	3	7	11	10	8	4	1

Calculate the modal, median and the mean wage

2. Below are heights, measured to the nearest cm of 50 pupils

157	167	165	162	160	157	160	152	157	162
157	165	152	162	155	160	157	160	162	160
157	152	167	157	160	160	162	165	157	160
157	157	157	160	157	162	155	157	160	157
150	162	152	160	157	157	165	160	162	150

a) Make a frequency distribution table by dividing them into class intervals of 5 starting with the class 148-152

b) Draw a cumulative frequency curve and use it to estimate

(i) The median (ii) Interquartile range

3 The table below shows marks obtained by students of mathematics in a certain school.

Marks	30-<40	40-<50	50-<60	60-<70	70-<80
No. of students	2	15	10	11	27

(i) Calculate the mean, median and standard deviation for the above data

(ii) . Draw an Ogive for the above data

2. Below are heights, measured to the nearest cm of 50 pupils.

157	167	165	162	160	157	160	152	157	162
157	165	152	162	155	160	157	160	162	160
157	152	167	157	160	160	162	165	157	160
157	157	157	160	157	162	155	157	160	157
150	162	152	160	157	157	165	160	162	150

a) Make a frequency distribution table by dividing them into class intervals of 5 starting with the class 148-152

b) Draw a cumulative frequency curve and use it to estimate

(i) The median (ii) Interquartile range

3. The table below shows marks obtained by students of mathematics in a certain school

Marks	30-<40	40-<50	50-<60	60-<70	70-<80
No. of students	2	15	10	11	27

(i) Calculate the mean, median and standard deviation for the above data

(ii) Draw an Ogive for the above data

4. Sixty pupils were asked to draw a free hand line of length 20cm. The lengths of the lines were measured to nearest cm, and were recorded as shown in the table.

Length(cm)	11-13	13-15	15-17	17-19	19-21	21-23	23-25
Frequency	3	6	11	15	13	10	2

a) Calculate the mean length

b) Draw a cumulative frequency graph and estimate the median, the upper and the lower quartiles.

5. Below are the heights to the nearest cm of 40 students

150	170	152	155	169	167	157	158	157
167	164	165	164	163	162	163	158	158
160	160	159	161	161	161	160	160	160
159	162	160	159	160	161	161	156	150

a) Make a frequency distribution table starting with class interval 150-152

b) Draw an Ogive and use it to estimate the median, Interquartile range and the 20th percentile height.

6. Calculate the mean and the standard deviation of the following distribution of scores

Scores	1-5	6-10	11-15	16-20	21-25	26-30	31-35
Frequency	3	19	38	69	45	21	5

7. The numbers of the eggs collected from a poultry farm for 40 consecutive days were as follows.

138	145	145	157	150	142	154	140
146	135	128	149	164	147	152	138
168	142	135	125	158	135	148	176
146	150	165	144	126	153	136	163
161	156	144	132	176	140	147	130

a) Construct a frequency distribution table with classes of equal interval width 5, starting from 125-129.

b) Draw a cumulative frequency curve (Ogive) and use it to estimate the (i) Interquartile range

(iii) Median number of eggs

8. The following marks were obtained by 85 students in an English examination;

96 81 23 62 44 18 62 70 72 40 81 70 30 28 23 02

60 20 48 50 19 33 32 58 71 62 19 12 83 53 81 73

52 25 71 61 46 64 35 59 82 82 42 63 43 17 35 72

37 54 47 76 18 44 65 45 70 38 63 89 31 37 93 03

63 25 52 53 38 57 53 71 70 63 89 31 37 93 58 58

a) Using class intervals of 10 marks, and starting with a class of 0-9; construct a frequency distribution table.

b) Using your table to find the (i) Median mark

(ii) Mean mark

(iii) Standard deviation

9. The marks obtained by 50 students in a test were:

76 17 57 63 12 96 38 46 82 48

61 93 44 19 70 60 71 18 40 54
50 27 62 42 63 52 53 38 62 25
62 23 32 81 31 63 64 18 70 27
52 81 35 63 38 37 44 19 70 32

a) Construct a grouped frequency distribution table with equal class intervals of 10 marks, starting with the 10 – 19 class group.
b) Draw a histogram and use it to estimate the modal mark.
c) Calculate the mean and standard deviation of the mark.

10. The times taken by a group of students to solve a mathematical problem are given below.

Time(min)	5-9	10-14	15-19	20-24	25-29	30-34
No. of students	5	14	30	17	11	3

(a) Draw a histogram for the data. Use it to estimate the modal time for solving a problem.
(b) Calculate the mean time and standard deviation of solving a problem.

11. The table below shows the weights (in kg) of 150 patients who visited a certain health unit during a certain week.

Weight (kg)	0-19	20-29	30-39	40-49	50-59	60-69	70-79
No. of patients	30	16	24	32	28	12	8

a) Calculate the appropriate mean and modal weights of the patients.
b) Plot an Ogive for the above data. Use the Ogive to estimate the median and semi interquartile for the weights of patients.

12. In agricultural experiment, the gains in mass (in kg) of 100 cows during a certain period were recorded as follows;

Gain in mass (kg)	5-9	10-14	15-19	20-24	25-29	30-34
Frequency	2	29	37	16	14	2

Calculate the (i) mean mass gained
(ii)Standard deviation (iii)Median

13. The table below shows the marks of 36 candidates in oral examination.

30 31 55 49 56 47
36 41 39 45 39 50
42 43 44 39 46 56
30 48 53 38 50 63
40 54 61 46 56 44
53 60 56 50 62 52

(i) Construct a frequency distribution table having an interval of 6marks starting with the 30-35 class group.
(ii) Draw a cumulative frequency curve and use it to estimate the median mark.
(iii) Calculate the mean mark.

14. Construct a frequency distribution of the following data on the length 5of time (in minutes), it took 50 persons to complete a certain application form.

29 22 38 28 34 32 23 19 21 31
16 28 19 18 12 27 15 21 25 16
30 17 22 29 18 29 25 20 16 11
17 12 15 24 25 21 22 17 18 15
21 20 23 18 17 15 16 26 23 22

Using class intervals of length 5minutes starting with the interval 10-14. Calculate the (i) Mean (ii)Standard deviation using Assumed mean A= 22

15. The ages of students in an Institution were as follows.

Age	18-<19	19-<20	20-<21	21-<22	23-<24	24-<25
No. of students	12	35	38	24	8	3

(i) Draw a histogram of the data and use it to estimate the modal age.

(ii) Use the data to estimate the median, upper and lower quartile ages.

16. The table below shows the masses of 40 students to the nearest kg

Mass(kg)	20 – 29	30 – 39	40 – 49	50 – 59	60 – 69	70 – 79
Number of students	2	m	9	4m	8	1

(a) Find the value of m

(b) Using 50 as the working mean, calculate the mean mass

(c) Calculate the standard deviation

17. The table below gives the frequency distribution of the marks obtained by 130 students in a physics test.

Marks	21 – 30	31 – 40	41 – 50	51 – 60	61 – 70	71 – 80	81 – 90	91 – 100
Frequency	1	3	6	24	30	31	22	13

(a) Calculate the median mark

(b) (i) draw a histogram to represent the data and use the histogram to estimate the mode

18. The table below shows the marks obtained by 76 candidates in a biology examination

Marks	50 – 54	55 – 59	60 – 64	65 – 69	70 – 74	75 – 79	80 – 84
Number of candidates	3	7	15	22	19	8	2

(a) Calculate (i) mean mark (ii) median mark

(b) Draw a histogram and use it to estimate the mode

19. The table below shows the number of students of a certain school who took breakfast in a certain month of 30 days

170	145	168	158	135	124
182	152	171	159	164	192
165	190	158	173	194	132
177	151	179	154	131	160
215	167	143	122	203	130

(a) Construct a frequency distribution table of equal class interval, starting with 120 – 129

(b) Plot a cumulative frequency curve and use it to estimate the

(i) Median (ii) interquartile range

20. The table below shows the masses(in kg) of 30 girls selected at random from a certain school

59	60	68	68	52	62
55	65	43	50	58	45
60	70	52	49	54	59
62	42	70	60	46	64
54	45	73	58	60	45

(i) Make a frequency distribution table with classes having an interval of 5 kg beginning with 40 – 44 class

(ii) Calculate the mean mass for the sample

(iii) Plot a cumulative frequency curve and use it to estimate the median

(iv) What is the probability that a girl in the school chosen at random weighs between 50.5 kg and 63 kg

21. The table below shows the frequency distribution of marks of 800 candidates who sat a national examination.

Marks(%)	1 – 10	11 – 20	21 – 30	31 – 40	41 – 50	51 – 60	61 – 70	71 – 80	81 – 90	91 – 100
Frequency	30	50	100	150	150	130	90	60	30	10

(a) (i) construct a cumulative frequency distribution for the data

(ii) draw a cumulative frequency curve for the distribution

(b) Use your graph to estimate;

(i) median mark

(ii) percentage number of candidates that failed if the pass mark was 50%

(iii) inter quartile range

(c) calculate the mean mark

22. The table below shows the results of candidates who sat a mathematics examination marked out of 60

Marks	0 – 9	10 – 19	20 – 29	30 – 39	40 – 49	50 – 59
Number of candidates	7	26	40	46	28	13

(i) Draw a cumulative frequency curve and use it to estimate the range of marks of the middle 50% of the candidates

(ii) Find the mean mark for the examination

23. The table below shows the weights in kilograms of 200 cows

Weight (kg)	Frequency
44 – 47	15
48 – 51	3
52 – 55	45
56 – 59	7
60 – 63	46
64 – 67	20
68 – 71	61
72 – 75	3

(a) Find the mean and standard deviation

(b) Calculate the modal weight

(c) Draw an Ogive and use it to estimate.

(i) Semi-interquartile range (ii) The percentage of cows weighing below 65 kg

24. The table below shows the marks obtained by 100 students in a sub maths test

Marks	1 – 10	11 – 20	21 – 30	31 – 40	41 – 50	51 – 60	61 – 70	71 – 80
Cumulative frequency	5	13	25	40	64	82	92	100

(a) Draw a cumulative frequency curve and use it to estimate the

(i) Median mark

(ii) Mark at which a distinction was awarded if 20 students obtained a distinction

(b) Calculate the mean mark using the working mean of 45.5

CHAPTER 12: INDEX NUMBERS

An index number is a statistical measure, which represents the change in a variable or group of variables with respect to time, environment or other characteristics.

Base year
This is the year against which all the other years are compared. The price in the base year is normally denoted as P_0

Current year
This is the year (period) for which the index is to be calculated. The price in the current year is normally denoted by P_1

Basic characteristics of index numbers

- The index for the base period is which is standard practice The statement "2012 = 100 " is used to identify the base
- The change in the value of the index from the base period to any given period is simply a measure of percentage change from the base period for two periods
- The change in the value of an index does not indicate percentage change unless one time period is the base period
- Index numbers measure relative changes. They measure the relative change in the value of a variable or a group of related variables over a period of time or between places.

Types of index numbers

- Simple index numbers
- Aggregate index numbers
- Composite index numbers
- Value index numbers

SIMPLE INDEX NUMBERS

A simple price index measures the relative change from the base period for a single measurement .This includes price index, quantity index, wage index etc.
Simple price index is often known as a price relative and it is given by

$$\text{Simple index number} = \frac{\text{price in the current year}}{\text{price in the base year}} \times 100 = \frac{P_1}{P_0} \times 100$$

Wage index

This measures the changes in wages of workers

$$\text{Simple wage index} = \frac{W_1}{W_0} \times 100$$

Where W_1 is the wage of workers in the current year
W_0 is the wage of workers in the base year

Quantity index

This type of index number pertains to measuring changes in volumes of commodities like goods produced or goods consumed.

$$\text{Quantity index} = \frac{q_1}{q_0} \times 100$$

Example 1

An article cost shs 500 in 1990 and shs 800 in 1994. Taking 1990 as the base year, find the price relative in 1994.

Solution

Price relative $= \frac{P_1}{P_0} \times 100 = \frac{800}{500} \times 100 = 160$

This indicates that the price of the article has gone up by 60%

Example 2

The wage of nurses in Uganda in 1995 was shs 20,000. The wage of the same nurses in 1997 was increased by 25000. Using 1995 as the base year, calculate the nurses' wage index for 1997.

Solution

$W_1 = 20,000 + 25,000 = 45,000$

$W_0 = 20,000$

Wage index $= \frac{W_1}{W_0} \times 100 = \frac{45000}{20000} \times 100 = 225$

Therefore the nurses wage increased by 125% in 1995

Note: The percentage sign is always omitted in the final answer.

Construction of index numbers

The construction of index numbers can be divided into two types i.e.

(a) Unweighted indices

(b) Weighted indices

UNWEIGHTED INDEX NUMBERS

The following are the methods of constructing unweighted index numbers

(i) Simple aggregative method

(ii) Simple average of price relative method

Simple aggregative method

This is a simple method for constructing index numbers. In this, the total of current year prices for various commodities is divided by the corresponding base year total and multiplying the result by 100

Simple aggregate price index $= \frac{\sum P_1}{\sum P_0} \times 100$

Where $\sum P_1 = \textit{the total of commodity prices in the current year}$

$\sum P_0 = \textit{the total of same commodity prices in the base year}$

Example 1

Calculate the price index number for 2003, taking the year 2000 as the base year

Commodity	Price in the year 2000	Price in the year 2003
A	600	800
B	500	600
C	700	1000
D	1200	1600
E	1000	1500

Solution

Simple aggregate price index $= \frac{\sum P_1}{\sum P_0} \times 100 = \frac{800+600+1000+1600+1500}{600+500+700+1200+1000} \times 100$

$$= \frac{5500}{4000} \times 100 = 137.5$$

This shows that there is an increase of 37.5% in the prices of the commodities

Example 2

An average family in Kampala spent the following amounts on the items shown in the years 1997 and 1998

ITEM	Amount in shs (1997)	Amount in shs (1998)
Housing	80,000	90,000
Clothing	20,000	20,000
Electricity	20,000	25,000
Water	25,000	25,000
Food	140,000	160,000
Transport	30,000	36,000
Medical	30,000	35,000
Miscellaneous	30,000	40,000

Using 1997 as the base year, calculate the simple aggregate expenditure index for 1998

Solution

Total expenditure in 1997

= 80,000+20,000+20,000+25,000+140,000+30,000+30,000+30,000 = 375,000/=

Total expenditure in 1998

=90,000+20,000+25,000+25,000+160,000+36,000+35,000+40,000 = 431,000/=

Simple aggregate expenditure $= \frac{431000}{375000} \times 100 = 114.93$

Example 3

Data chip manufactures and sells three computer chip models; the basic, financial and scientific. The respective retail prices are 950, 3500 and 7000 in 1994; 150, 1800 and 2500 in 1998; 80, 600 and 1250 in 2002. Calculate the simple aggregate price index for 1998 and 2002 taking 1994 as the base year.

Solution

Chip Model	Retail price		
	1994	1998	2002
Basic	950	150	80
Financial	3500	1800	600
Scientific	7000	2500	1250
$\sum$	11450	4450	1930

Simple aggregate price index for 1998 $= \frac{4450}{11450} \times 100 = 38.86$

Simple aggregate price index for 2002 $= \frac{1930}{11450} \times 100 = 16.86$

Conclusion:

Since the index in the base period, 1994 is 100, the difference in the indices for 1994 and 1998 indicates that the average price of the three models is declined by;

$$100 - 38.86 = 61.14\%$$

The decline in price from 1994 to 2002 is $100 - 16.86 = 83.14\%$

Simple average of price relatives method

In this method, the price relatives for all commodities is calculated and then their average is taken to calculate the index number

Simple average of price indices $= \frac{\sum \frac{P_1}{P_0} \times 100}{n}$ where n is the number of items

Solution

(i)

Year	Retail Price	Price index$\left(\frac{P_1}{P_0} \times 100\right)$
1983	110	100
1984	120	109.09
1985	130	118.18
1986	150	136.36
1987	165	150
1988	185	168.18

Price index in 1988 = 168.18

Implying the price of sugar increased by 168.18 – 100 = 68.18%

Hence the family's consumption reduced by 68.18%

(ii) Base value P_0 = 150

Year	Retail Price	Price index$\left(\frac{P_1}{P_0} \times 100\right)$
1983	110	73.33
1984	120	80
1985	130	86.78
1986	150	100
1987	165	110
1988	185	123.33

(iii) Average retail price from 1983 – 1985 = $\frac{110+120+130}{3} = 120$

$$\frac{P_{1999}}{120} \times 100 = 160$$

$$P_{1999} = \frac{160 \times 120}{100} = 192$$

WEIGHTED PRICE INDICES

Index numbers at times are needed where there is more than just one item i.e an index number that compares the cost of living depends on food, housing, clothing, entertainment, c.t.c. They can be calculated in terms of the weight using the weighted price index. The weighted price index can also be referred to as the cost of living index.

Construction of the weighted price indices

Case1:

When prices of the items and weights attached are given, here we consider three situations

(i) When the units of the items under consideration are uniform, here the aggregate weighted price index is used

Weighted aggregate price index = $\frac{\sum P_1 W}{\sum P_0 W} \times 100$

Note: Weighted aggregate price index can also be referred to as composite index

(ii) When the units of the items are not uniform, here we use the simple average of price relatives

i.e $\frac{\sum \frac{P_1}{P_0} \times 100}{n}$

Weighted average price index $= \frac{\sum \frac{P_1}{P_0} \times W}{\sum W} \times 100$

(iii) When the price relatives and weights of the items are given, here the weighted price index is given by;

Weighted price index $= \frac{\Sigma PW}{\Sigma W}$ where P is the price relative

Examples

1. Find the cost of living index based on the following data

Item	Price index	Weight
Food	120	172
Clothing	124	160
Housing	125	170
Transport	135	210
Others	104	140

Solution

Item	Price index(P)	Weight	PW
Food	120	172	20640
Clothing	124	160	19840
Housing	125	170	21250
Transport	135	210	28350
Others	104	140	14560
Σ		852	104640

Cost of living index = weighted price index $= \frac{\Sigma PW}{\Sigma W} = \frac{104640}{852} = 122.82$

2. Nsambya Hillside High School bought three types of chicken feeds in 1998 and 2000. The weights of the corresponding price (in shs) are in the table below.

Chicken feed	Weight	Price 1998	Price 2000
A	120	500	600
B	60	300	360
C	50	250	400

Using 1998 as the base year, calculate the weighted average price index

Solution

Commodity	Weight	Price (P_0)	Price(P_1)	$\frac{P_1}{P_0}$	$\frac{P_1}{P_0}W$
A	120	500	600	1.2	144
B	60	300	360	1.2	72
C	50	250	400	1.6	80
Σ	230				296

Weighted aggregate price index $= \frac{296}{230} \times 100 = 128.7$

Conclusion: the price of chicken feeds increased by 28.7%

3. The following items are used in the assembly of a TV set; 8 transistors, 22 resistors, 9 capacitors, 2 diodes and a circuit board. Due to inflation, the price of each component has increased as shown below

Item	Transistors	Resistors	Capacitors	Diodes	Circuit
Price in 1980	120	165	150	160	200
Price in 1986	180	210	170	180	250

Calculate the composite index number of the assembled TV set in 1986 using 1980 as the base year.

Solution

Item	Weight (W)	Price (P_0)	WP_0	Price (P_1)	WP_1
Transistor	8	120	960	180	1440
Resistor	22	165	3630	210	4620
Capacitor	9	150	1350	170	1530
Diode	2	160	320	180	360
Circuit	1	200	200	250	250
$\sum$			6460		8200

Composite index = Weighted aggregate price index $= \frac{\sum P_1 W}{\sum P_0 W} \times 100$

$= \frac{8200}{6460} \times 100 = 127$

4. The prices of unit values of four commodities A, B, C and D in the years 1994 and 1996 were as below.

Commodities	Price 1994	Price 1996	Weights
A	400	500	7
B	900	1100	2
C	600	700	3
D	600	800	6

(a) Taking 1994 as the base year, Calculate the
 (i) Simple price index for 1996
 (ii) Weighted average price index

(b) Suppose that the actual quantities consumed per week by a family in 1994 were 5kg, 2kg, 3 litres and $1\frac{1}{2}$ kg of A, B, C and D respectively. Determine the average percentage increase in the price of commodities in 1996.

Solution

(a)

Commodity	Weight(W)	Price(P_0)	Price(P_1)	$\frac{P_1}{P_0} \times 100$	PW
A	7	400	500	125	875
B	2	900	1100	122.22	244.44
C	3	600	700	116.67	350
D	6	600	800	133.33	800
$\sum$	18			497.22	2269.44

(i) Simple price index $= \frac{\sum \frac{P_1}{P_0} \times 100}{n} = \frac{497.22}{4} = 124.31$

(ii) Weighted average price index $= \frac{\sum PW}{\sum W} \times 100 = \frac{2269.44}{18} = 126.1$

(b) We can find the percentage increase in the prices using the weighted aggregate price index since the weights have been given

Commodity	Weight(W)	Price (P_0)	P_0W	Price (P_1)	P_1W
A	5	400	2000	500	2500
B	2	900	1800	1100	2200
C	3	600	1800	700	2100
D	1.5	600	900	800	1200
$\sum$			6500		8000

Weighted aggregate price index $= \frac{\sum P_1 W}{\sum P_0 W} \times 100 = \frac{8000}{6500} \times 100 = 123.1$

Percentage increase in the price of commodities = 123.1 – 100 = 23.1%

5. The cost of making a cake is calculated from the cost of baking flour, sugar, milk and eggs. The following table gives the cost of these items in 1990 and 1996

Item	Price 1990	Price 1996	Weight
Flour per kg	600	780	12
Sugar per kg	500	400	5
Milk per litre	250	300	2
Eggs per egg	100	150	1

Using 1990 as the base year,

(i) Calculate the price relative for each item. Hence find the simple price index for the cost of making a cake.

(ii) Find the weighted aggregate price index for the cost of a cake.

Solution

Item	1990(P_0)	1996(P_1)	Weight	Price relatives ($\frac{P_1}{P_0} \times 100$)	P_0W	P_1W
Flour/kg	600	780	12	130	7200	9360
Sugar/kg	500	400	5	80	2500	2000
Milk/ltr	250	300	2	120	500	600
Eggs/egg	100	150	1	150	100	150
$\sum$				480	10300	12110

(i) The price relatives are indicated by $\frac{P_1}{P_0} \times 100$ in the table

Simple price index $= \frac{480}{4} = 120$

(ii) Weighted aggregate price index $= \frac{\sum P_1W}{\sum P_0W} \times 100 = \frac{12110}{10300} \times 100 = 117$

Case 2:

When the prices and quantities of items are given, here two approaches are used

(i) Laspeyres' method

(ii) Paasche method

These two methods differ only in the period used for weighting. The Laspeyres' method uses base-period weights, that is the original prices and quantities of the items bought are used to find the percentage change over a period of time in either price or quantity consumed depending on the problem. The Paasche method uses current year weights.

Laspeyres' Price index $= \frac{\sum p_1q_0}{\sum p_0q_0} \times 100$

Where P_1 is the current year price

P_0 is the base year price

q_0 is the quantity used in the base period

Paasche's price index $= \frac{\sum p_1q_1}{\sum p_0q_1} \times 100$

Where q_1 is the quantity used in the base period.

Examples
1. The table below shows the prices for six items and the number of units of each consumed by a typical family in 1995 and 2005.

Item	1995		2005	
	Price	Quantity	Price	Quantity
Bread (loaf)	770	50	1980	55
Eggs(dozen)	1850	26	2980	20
Milk (litre)	880	102	1980	130
Apples (500g)	1460	30	1750	40
Juice (300ml)	1580	40	1700	41
Coffee (400g)	4400	12	4750	12

Determine the weighted price index and interpret the result

Solution

Using the Laspeyres' method

Item	P_0	Q_0	P_1	Q_1	P_1Q_0	P_0Q_0
Bread (loaf)	770	50	1980	55	99000	38500
Eggs(dozen)	1850	26	2980	20	77480	48100
Milk (litre)	880	102	1980	130	201960	89760
Apples (500g)	1460	30	1750	40	52500	43800
Juice (300ml)	1580	40	1700	41	68000	63200
Coffee (400g)	4400	12	4750	12	57000	52800
$\sum$					555940	336160

Weighted Price index $= \frac{\sum P_1q_0}{\sum P_0q_0} \times 100$

$= \frac{555940}{336160} \times 100 = 165.4$

This result indicates that there has been an increase of 65.4% in the prices of the items between 1995 and 2005

Alternatively by using the Paasche method

Item	P_0	Q_0	P_1	Q_1	P_1Q_1	P_0Q_1
Bread (loaf)	770	50	1980	55	108900	42350
Eggs(dozen)	1850	26	2980	20	59600	37000
Milk (litre)	880	102	1980	130	257400	114400
Apples (500g)	1460	30	1750	40	70000	58400
Juice (300ml)	1580	40	1700	41	69700	64780
Coffee (400g)	4400	12	4750	12	57000	52800
$\sum$					622600	369730

Weighted price index $= \frac{\sum P_1q_1}{\sum P_0q_1} \times 100$

$= \frac{622600}{369730} \times 100 = 168.4$

This result indicates that there has been an increase of 68.4% in the prices of the items between 1995 and 2005

Note:
For examinations, the student is free to use any of the above methods at his own discretion but not both.
The consumer price index (C.P.I) is an example of the Laspeyres' index

VALUE INDEX

A value index measures changes in both the price and quantities involved. A value index, such as the index of department store sells, needs the original base year prices, the original base year quantities, the present year prices and the present year quantities for its construction. It is given by;

$$\text{Value index} = \frac{\sum p_1 q_1}{\sum p_0 q_0} \times 100$$

Example 1

The prices and quantities sold at Waleska Department Store for various items for May 2000 and May 2005 are given in the table below.

Item	Price 2000 (dollars)	Quantity sold in 2000	Price 2005 in dollars	Quantity sold in 2005
Ties (each)	10	1000	12	900
Suits (each)	300	100	400	120
Shoes (pair)	100	500	120	500

What is the value index for May 2005 using May 2000 as the base period?

Solution

Item	P_0	Q_0	P_1	Q_1	P_0Q_0	P_1Q_1
Ties (each)	10	1000	12	900	10000	10800
Suits (each)	300	100	400	120	30000	48000
Shoes (pair)	100	500	120	500	50000	60000
$\sum$					90000	118800

$$\text{Value index} = \frac{\sum p_1 q_1}{\sum p_0 q_0} \times 100 = \frac{118800}{90000} \times 100 = 132$$

This implies that the sales increased by 32% from May 2000 to May 2005

Example 2

The table below represents the changes in the domestic consumption of the indicated food items

Commodity	Unit	Price in shillings		Quantity	
		2009	2010	2009	2010
Matooke	kg	180	150	1500	2500
Bread	loaf	500	700	80	100
Milk	litre	400	700	60	60
Vegetables	kg	1000	800	45	60
Fruits	kg	700	600	120	100

Using 2009 as the base year, calculate the

(i) Price index for each food item for 2010

(ii) Simple aggregate price index for 2010

(iii) The value index for 2010

Solution

(i) Price index for matooke $= \frac{150}{180} \times 100 = 83.33$

For bread $= \frac{700}{500} \times 100 = 140$

For milk $= \frac{700}{400} \times 100 = 175$

For vegetables $\frac{800}{1000} \times 100 = 80$

For fruits $= \frac{600}{700} \times 100 = 85.71$

(ii) Simple aggregate price index $= \frac{\sum P_1}{\sum P_0} \times 100$

$= \frac{150+700+700+800+600}{180+500+400+1000+700} \times 100 = \frac{2950}{2780} \times 100 = 106.2$

(iii)

Item	P_0	Q_0	P_1	Q_1	$P_0 Q_0$	$P_1 Q_1$
Matooke	180	1500	150	2500	270000	375000
Bread	500	80	700	100	40000	70000
Milk	400	60	700	60	24000	42000
Vegetables	1000	60	800	60	45000	48000
Fruits	700	100	600	100	84000	60000
$\sum$					463000	595000

Value index $= \frac{\sum P_1 q_1}{\sum P_0 q_0} \times 100 = \frac{595000}{463000} \times 100 = 128.51$

Trial questions

1. The average prices for bananas, milk and wheat were as follows for 1997 and 1998

Item	Prices in shillings	
	1997	1998
Bananas	3000 per bunch	5000 per bunch
Milk	700 per litre	800 per litre
Meat	2500 per kg	2000 per kg

(a) Calculate the price relatives for these commodities for 1998 taking 1997 as the base year hence the simple price index

(b) Given that meat, bananas and milk are given weights of 2,1 and 3 respectively, taking 1997 as the base, calculate the index number for the total costs of the commodities for a family in 1998 [Ans: (a) 120.32 (b) 112.87]

2. The cost of making a cake is calculated from the cost of baking flour, sugar, milk and eggs. The following table gives the cost of these items in 1985 and 1986

Item	Price 1985	Price 1986	Weight
Flour per kg	60	78	12
Sugar per kg	50	40	5
Milk per litre	25	30	2
Egg per egg	10	15	1

Using 1985 as the base year,

(i) Calculate the price index for each item hence find the simple price index of making a cake

(ii) Find the weighted aggregate price index for the cost of a cake

(iii) If the cost of making a cake in 1986 was shs 30, find the cost in 1985 using the two indices in (i) and (ii) [Ans: (i) 120 (ii) 117.57 (iii) 26/=]

3. The table below represents the weights and index for five items

Item	Food	Tobacco	Housing	Transport	Medical
Weight	304	129	331	120	116
Index	124	126	127	119	128

Determine the weighted index number for all items [Ans: 125.1 2]

4. The following table shows the prices and quantities of some four commodities A, B, C and D for the years 2006 and 2007

Item	Price per unit		Quantities	
	2006	2007	2006	2007
A	100	120	36	42
B	110	100	96	88
C	50	65	10	12
D	80	85	11	10

Using 2006 as the base year, Calculate

(i) Price index number for 2007

(ii) Simple aggregate price index number

(iii) Weighted price index number

(iv) Value index number

[Ans: (i) 111.79 (ii) 108.82 (iii) 107 (iv) 105.21]

5. The table below shows the prices of four commodities and their weights in 2006 and 2007

Commodity	Price (U shs)		Weight
	2006	2007	
Banana(1 bunch)	3000	8000	4
Meat(1kg)	2500	3000	3
Milk (1 litre)	300	400	1
Sugar (1kg)	1500	1800	2

Taking 2006 as the base year, find for 2006, the;

(a) (i) price relative for each commodity

(ii) simple aggregate price index

(b) Weighted price index for all the commodities [Ans:(a) (ii) 180.82 (b) 197.37]

6. The prices per unit (in U shs) of four food stuffs A, B,C and D in December 2004 and December 2005 are shown in the following table

Food stuff	Price (U shs) in December	
	2004	2005
A	635	887.5
B	720	815
C	730	1045
D	362	503

The weights of the food stuffs A, B, C and D are 6, 4, 3 and 7 respectively. Taking 2004 as the base year, calculate for 2005 the

(a) price relative for each food stuff hence the simple price index

(b) (i) weighted price index

(ii) price of food stuff costing shs 500 in December 2004 using the weighted aggregate index.

[Ans: (a) (ii) 133.7 (b) (i) 133.53 (ii) 667.64/=]

7. The price (in shillings) per litre of fuel in months of January and May of a certain year are given in the table below.

	January	May
Petrol	1470	1420
Diesel	1270	1220
Kerosene	1140	1160

Find the simple price index for May taking January as the base [Ans: 98.15]

8. The following table shows the quantities and prices for the years 1998 and 2005 for Sam's student Centre'

Item	1998		2005	
	Price (U shs)	Quantity	Price (U shs)	Quantity
Pens (dozen)	900	5	1100	6
Pencils(dozen)	650	5	800	6
Erasers(each)	450	25	550	28
Paper (pkg)	890	50	1090	75
Printer drum (each)	6000	30	5000	45
Printer cartridges	16000	15	20000	20

Taking 1998 as the base year, calculate the
- (i) Simple price index for 2005
- (ii) Weighted price index
- (iii) Value index [Ans: (i) 116.4 (ii) 109.05 (iii) 151.72]

9. The table below shows the cost per kg of some items commonly used by a certain family

Item	Sugar	Posho	Salt	Rice	Millet
Cost per kg	2200	2000	500	2600	3000

(a) Using posho as the base, calculate the price relatives of each item hence determine the cost of living index

(b) Comment on the results in (a) above [Ans: (a) 103 (b) increase by 3%]

10. The table shows the prices of items in uganda shillings and their weights in 2010 and 2013

Items	2010	2013	Weights
Rice	2400	2800	3
Meat	5000	7000	1
Posho	1200	1600	2
Beans	2000	2500	4

Calculate the aggregate weighted index for the items taking 2010 as the base year [Ans:126.55]

11. The price relatives for five commodities A, B, C, D and E are shown in the table below with their respective weights.

Commodity	A	B	C	D	E
Price relative	145	125	130	x	120
Weight	2	3	4	5	1

Find the value of x if the weighted price index is 127. [Ans: 91]

12. The expenses of a house hold (in thousands of Uganda shillings) for the first month of three consecutive years (2000 – 2002) were as follows

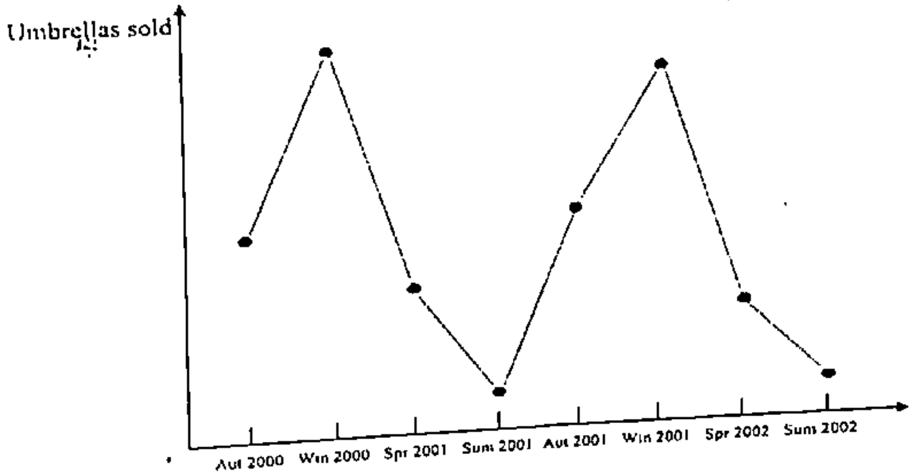
Item	Year		
	2000	2001	2002
Food	240	300	320
Fuel	40	50	56
Transport	80	120	120
Others	120	150	160

- (i) Taking 2000 as the base year, find the price relatives for the years2001 and 2002
- (ii) Using the weights of 4, 1, 2 and 3 for food, fuel, transport and others respectively, calculate the weighted aggregate index for 2001 and 2002.

[Ans: (i) 127.63 (ii) 135.26]

CHAPTER 13: MOVING AVERAGES

Many sets of data display trends, which depend upon time of year or the particular month or even the time of the day etc.
For example, we would expect the sales of umbrellas against the seasons of the year (autumn, winter, summer, spring etc.). We would expect some wildly fluctuating graph as follows;



This is called a **time series graph**
In attempting to glean meaningful information from such graphs, we really need to isolate the different seasons, each of which exerts its own seasonal influence.
One way of doing this is to use what are termed moving averages, which are designed to level out the large fluctuations which can occur in a set of data that varies over time.

Example 1
Suppose you have measured the weight of a child over an eight-year period and have the following figures (in kg) 32, 33, 35, 38, 43, 53, 63, 65
We can take the average of each 3 years period. These are 3 year / 3 point moving averages
We can take the average of each 3 years period. These are 3 year / 3 point moving averages
Let M_n denote moving averages i.e M_1 will denote the first, M_2 the second and so on

$$M_1 = \frac{32+33+35}{3} = \frac{100}{3} = 33.3$$
$$M_2 = \frac{33+35+38}{3} = \frac{106}{3} = 35.3$$
$$M_3 = \frac{35+38+43}{3} = \frac{116}{3} = 38.7$$
$$M_4 = \frac{38+43+53}{3} = \frac{134}{3} = 44.7$$
$$M_5 = \frac{43+53+63}{3} = \frac{159}{3} = 53.0$$
$$M_6 = \frac{53+63+65}{3} = \frac{181}{3} = 60.3 \quad (\textit{this is the last})$$

If we are to calculate the 4 year/ 4point moving averages
$$M_1 = \frac{30+33+35+38}{4} = \frac{136}{4} = 34$$

$M_2 = \frac{33+35+38+43}{4} = \frac{149}{4} = 37.3$

$M_3 = \frac{35+38+43+53}{4} = \frac{169}{4} = 42.3$

$M_4 = \frac{38+43+53+63}{4} = \frac{197}{4} = 49.3$

$M_5 = \frac{43+53+63+65}{4} = \frac{224}{4} = 56$ (*this is the last*)

Example 2

A college records the number of people who sign up for adult education classes each term. The table shows the numbers from December 2000 to June 2002

Term	December 2000	March 2001	June 2001	December 2001	March 2002	June 2002
Number of people	520	300	380	640	540	599

(a) calculate the three point moving averages for the data

(b) Plot the three point moving averages with the original data together on the same axis.

(c) (i) comment on the trend of the number of people who sign up in the period

(ii) use your graph to the three point moving average that will be plotted at june 2002

Solution

(a) The three point moving averages are;

$M_1 = \frac{520+300+380}{3} = \frac{1200}{3} = 400$

$M_2 = \frac{300+380+640}{3} = \frac{1320}{3} = 440$

$M_3 = \frac{380+640+540}{3} = \frac{1560}{3} = 520$

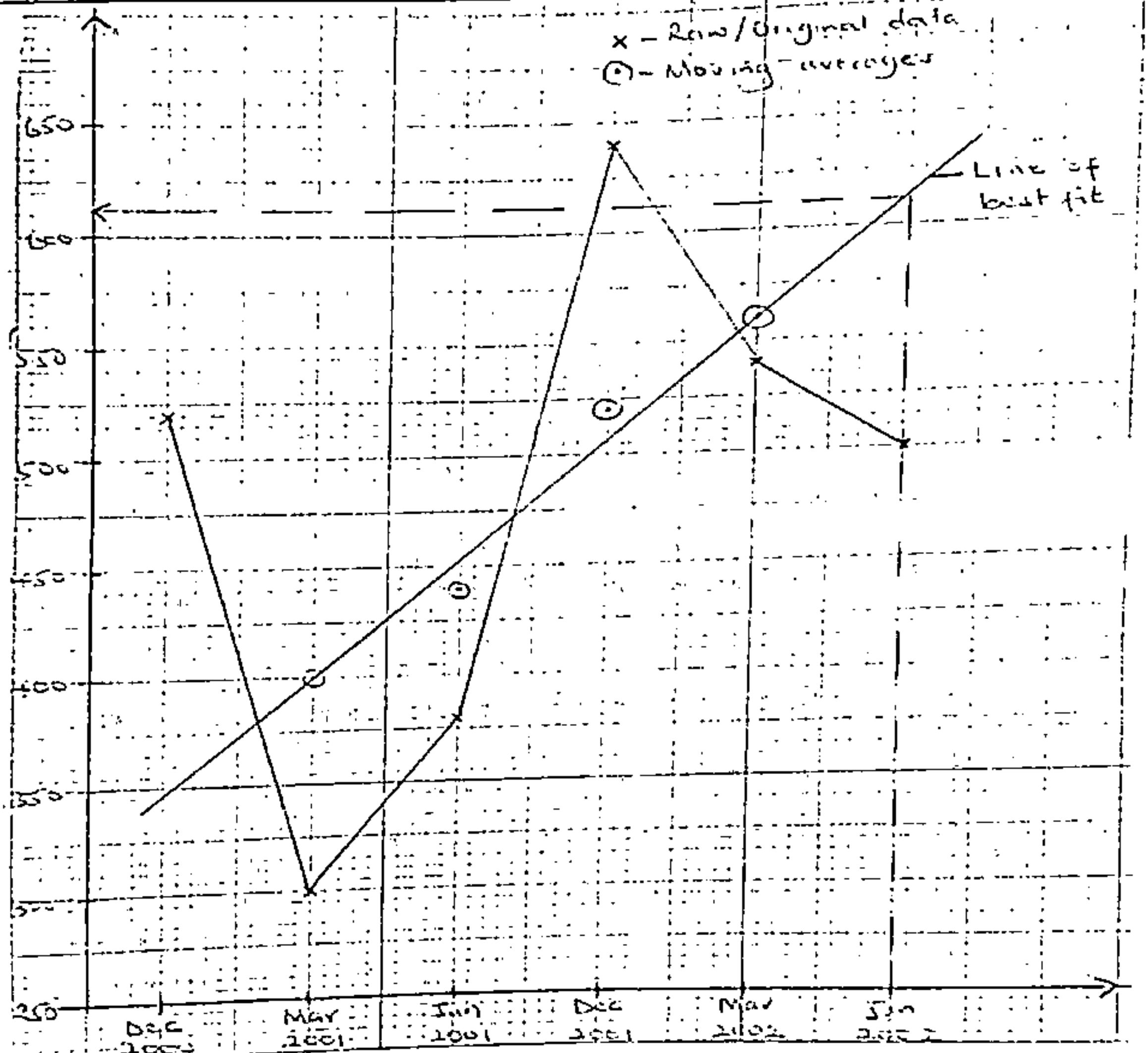
$M_4 = \frac{640+540+500}{3} = \frac{1680}{3} = 560$

Alternatively, we can decide to calculate the moving averages in a table as below

Term	Dec 2000	March 2001	June 2001	Dec 2001	March 2002	June 2002
Number of people	520	300	380	640	540	599
Moving totals (M.T)		1200	1320	1560	1680	
Moving averages(M.A)		$\frac{1200}{3} = 400$	$\frac{1320}{3} = 440$	$\frac{1560}{3} = 520$	$\frac{1680}{3} = 560$	

(b) We always plot the moving averages in the **middle** of the respective grouping

A graph showing the number of people and the three point moving averages



(c) (i) The nature of the line of best fit gives us the trend and we can identify that the number of people who sign up for adult education generally increases over the given period.

(ii) From the line of best fit, we can estimate the moving average plotted at June to be 610

Example 3

The table below shows the amounts of Jenny's gas bills from September 2001 to December 2002 in dollars.

Date	September 2001	December 2001	March 2002	June 2002	September 2002	December 2002
Amount of bill	28.70	32.40	29.10	7.80	30.30	38.60

(a) Calculate the four point moving averages for the data

(b) Graph the raw data and the moving averages

(c) Comment on the trend of the bills over the given period

Solution

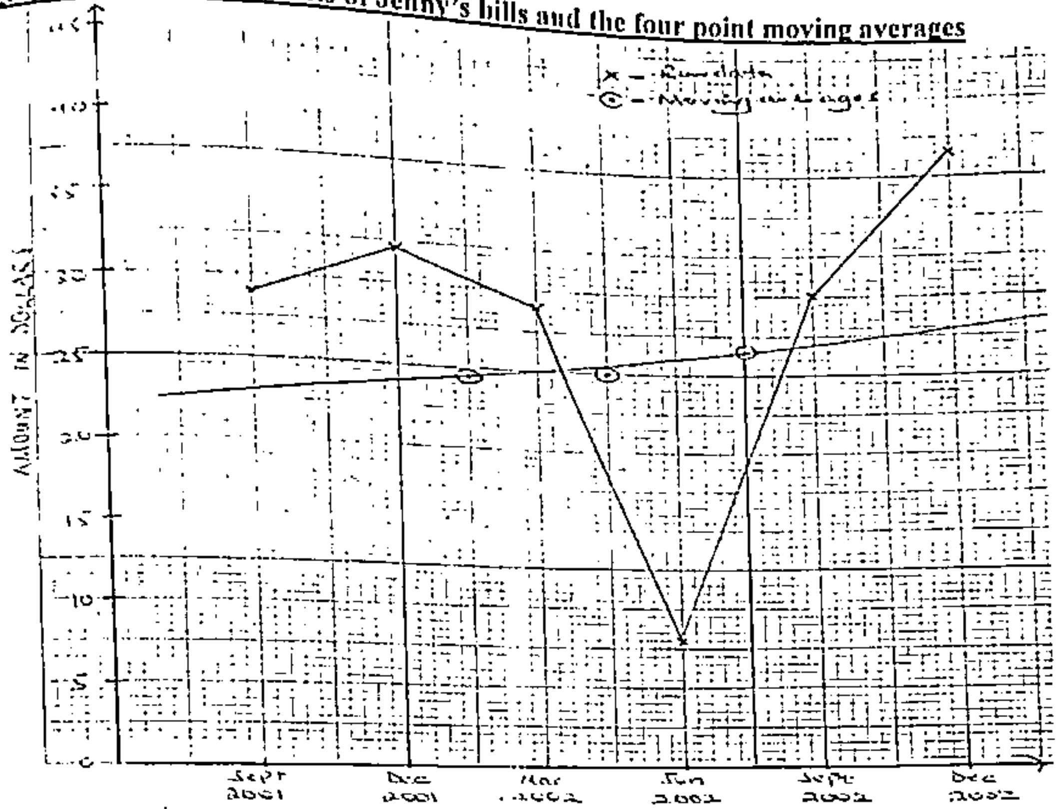
(a) $$M_1 = \frac{28.70+32.40+29.10+7.80}{4} = \frac{98}{4} = 24.5$$

$$M_2 = \frac{32.40+29.10+7.80+30.30}{4} = \frac{99.6}{4} = 24.9$$

$$M_3 = \frac{29.10+7.00+30.30+38.90}{4} = \frac{106.1}{4} = 26.53$$

(b)

A graph showing the amounts of Jenny's bills and the four point moving averages



(c) There is a general increase in Jenny's gas bills

Example 4

The amount of water used every after 6 months over a period of 4 years is shown in the following table.

Year	2008		2009		2010		2011	
Month	Mar	Oct	Mar	Oct	Mar	Oct	Mar	Oct
Water used (m^3)	36	45	29	43	38	45	52	46

(a) Calculate the three point moving averages for the data

(b) On the same axis, plot the original data and the moving averages

(c) (i) comment on the amount of water used in the period

(ii) Using your graph, estimate the amount of water that was used in March 2012

Solution

(a) $M_1 = \frac{36+45+29}{3} = \frac{110}{3} = 36.7$

$M_2 = \frac{45+29+43}{3} = \frac{117}{3} = 39$

$M_3 = \frac{29+43+38}{3} = \frac{110}{3} = 36.7$

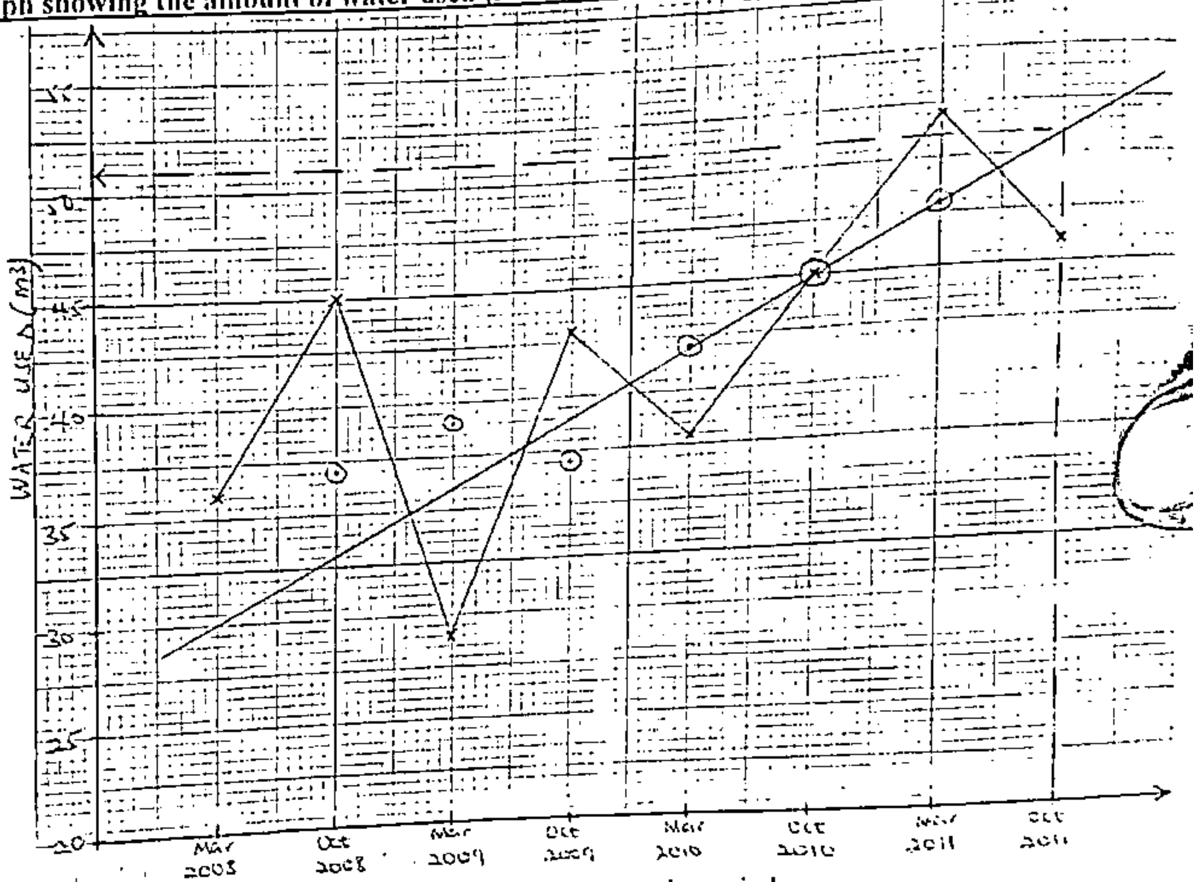
$M_4 = \frac{43+38+45}{3} = \frac{126}{3} = 42$

$M_5 = \frac{38+45+52}{3} = \frac{135}{3} = 45$

$M_6 = \frac{45+52+46}{3} = \frac{143}{3} = 47.7$

(b)

A graph showing the amount of water used (in m³) and the three point moving averages



(c)(i) the amount of water used generally increases over the period

(ii)

Let the amount in March 2012 *be x*

Moving average plotted at October 2011 = 51

Hence $\frac{52+46+x}{3} = 51$

$98 + x = 153$

$x = 55m^3$

Example 5

The table below shows the annual production of copper in millions of kilograms in a certain country for the period 1960 – 1970

Year	1960	'61	'62	'63	'64	'65	'66	'67	'68	'69	1970
Annual production	196	146	172	178	155	152	130	154	166	164	135

(a) Construct the 5 year moving averages

(b) On the same axis, plot the original data and the moving averages

(c) Comment on the trend of production over the 11 year period.

Solution

(a) $M_1 = \frac{196+146+172+178+155}{5} = 169.4$ $\quad M_5 = \frac{155+152+130+154+166}{5} = 151.4$

$M_2 = \frac{146+172+178+155+152}{5} = 160.6$ $\qquad M_6 = \frac{152+130+154+166+164}{5} = 153.2$

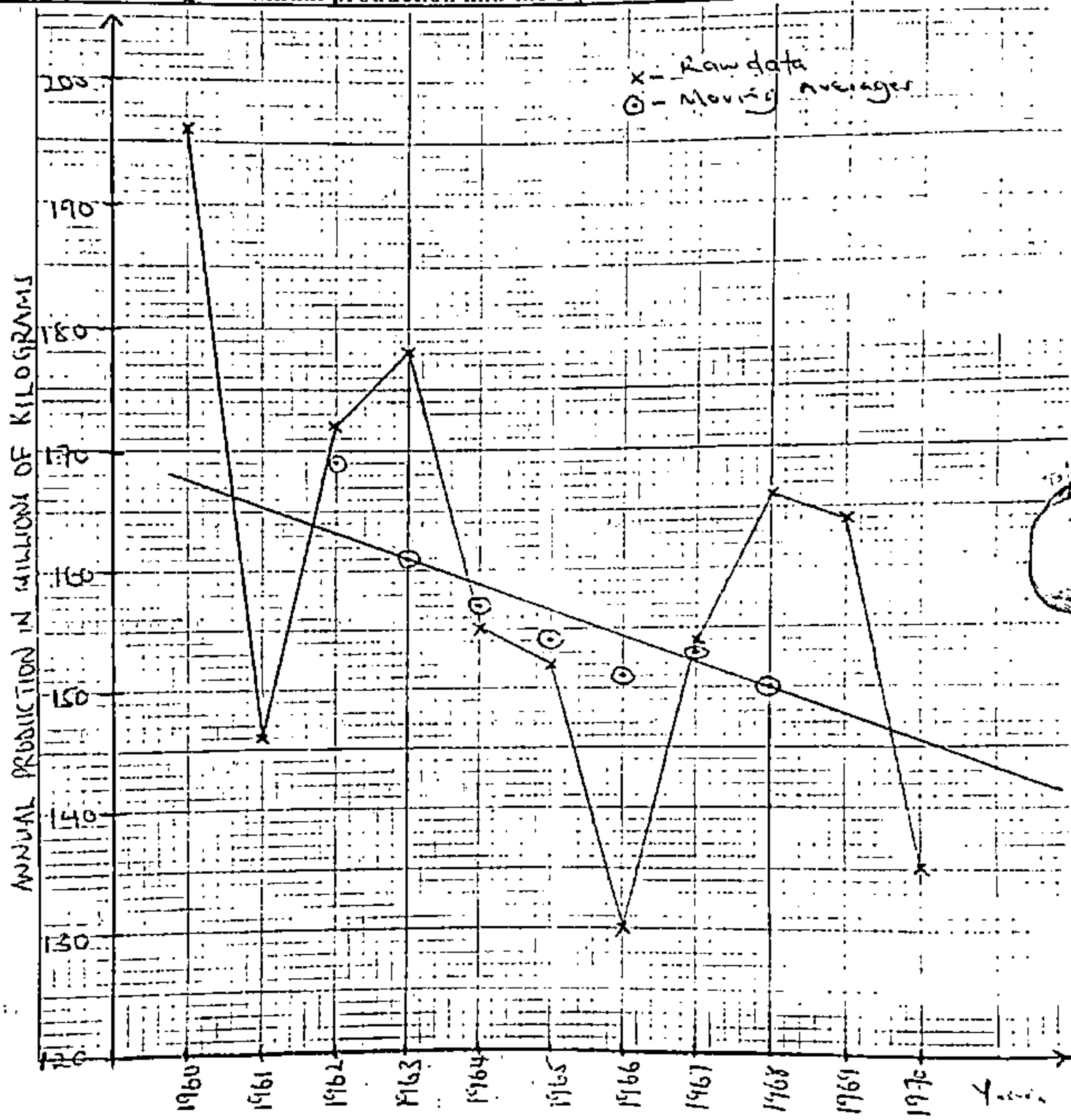
$M_3 = \frac{172+178+155+152+130}{5} = 157.4$ $\qquad M_7 = \frac{130+154+166+164+135}{5} = 149.8$

$M_4 = \frac{178+155+152+130+154}{5} = 153.8$

Alternatively, if you used the table to calculate the five year moving averages, you would have the following;

Year	1960	'61	'62	'63	'64	'65	'66	'67	'68	'69	1970
Annual production	196	146	172	178	155	152	130	154	166	164	135
Moving totals	X	X	847	803	787	769	757	766	749	X	X
Moving averages	X	X	169.4	160.6	157.4	153.8	151.4	153.2	149.8	X	X

(b) **A graph showing the annual production and the 5 year moving averages**



(c) The production generally decreases over the period

Example 6

The table below shows the amount of milk (in thousands of litres) produced by a certain exotic farm in yearly quarters for the 1986 – 1989 period.

Year	QUARTER			
	1st	2nd	3rd	4th
1986	19.5	30.0	32.5	25.0
1987	30.5	37.0	38.5	26.5
1988	36.5	44.5	46.6	35.0
1989	45.5	50.5	52.5	42.5

(a) Calculate the four point moving averages for the data

(b) On the same axes, plot the four point moving averages and the original data

(c) (i) comment on the trend of milk production over the period of 4 years

(ii) Use your graph to estimate the amount of milk that will be plotted in the 1st quarter of 1990.

Solution

(a) $M_1 = \frac{19.5+30.0+32.5+25.0}{4} = 26.75$

$M_2 = \frac{30.0+32.5+25.0+30.5}{4} = 29.5$

$M_3 = \frac{32.5+25.0+30.5+37.0}{4} = 31.25$

$M_4 = \frac{25.0+30.5+37.0+38.5}{4} = 32.75$

$M_5 = \frac{30.5+37.0+38.5+26.5}{4} = 33.13$

$M_6 = \frac{37.0+38.5+26.5+36.5}{4} = 34.63$

$M_7 = \frac{38.5+26.5+36.5+44.5}{4} = 36.5$

$M_8 = \frac{26.5+36.5+44.5+46.5}{4} = 38.5$

$M_9 = \frac{36.5+44.5+46.5+35.0}{4} = 40.63$

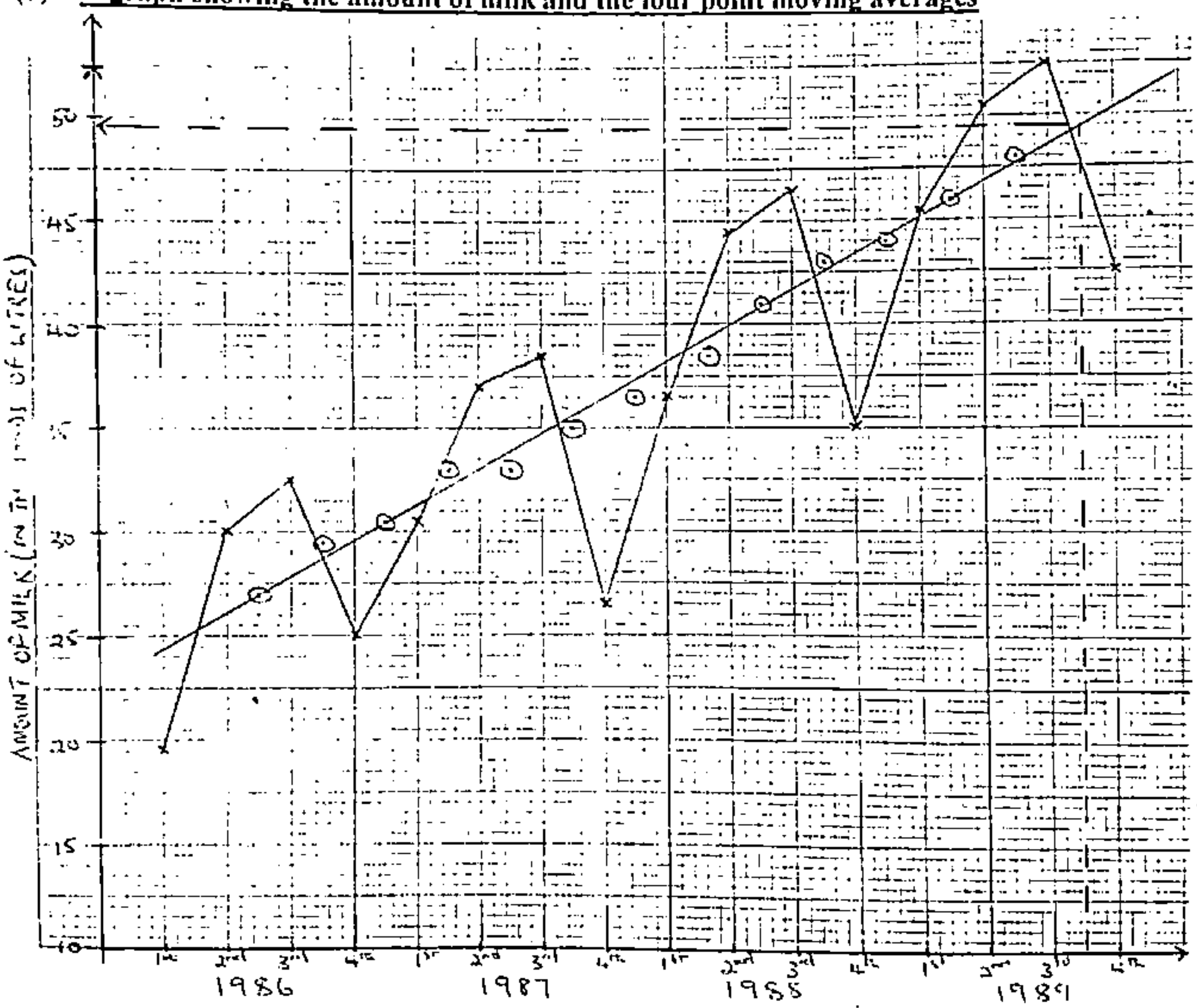
$M_{10} = \frac{44.5+46.5+35.0+45.5}{4} = 42.88$

$M_{11} = \frac{46.5+35.0+45.5+50.0}{4} = 44.25$

$M_{12} = \frac{35.0+45.5+50.0+52.5}{4} = 45.75$

$M_{13} = \frac{45.5+50.0+52.5+42.5}{4} = 47.63$

(b) **A graph showing the amount of milk and the four point moving averages**



(c) (i) The production generally increases over the given period

(ii) Let the amount of milk produced in the 1st quarter of 1990 be x

From the graph, the 14th moving average, $M_{14} = 49.5$

$\Rightarrow \frac{50.0+52.5+42.5+x}{4} = 49.5$

$145 + x = 198 \Rightarrow x = 53$

Example 7

The table below indicates the quarterly variation in a certain school earnings in millions of shillings from 1950 – 1952.

Year	QUARTER			
	1st	2nd	3rd	4th
1950	11.0	10.0	10.0	9.4
1951	10.5	9.7	9.4	9.3
1952	9.9	9.3	9.0	8.6

(a) Calculate the quarterly moving averages for the data

(b) On the same axes, plot the four point moving averages and the original data

(c) (i) comment on the trend of the school earnings over the period

(ii) Estimate the earning that will be recorded in the first quarter of 1953

Solution

(a) $M_1 = \frac{11.0+10.0+10.0+9.4}{4} = \frac{40.4}{4} = 10.1$

$M_2 = \frac{10.0+10.0+9.4+10.5}{4} = \frac{39.9}{4} = 9.97$

$M_3 = \frac{10.0+9.4+10.5+9.7}{4} = \frac{39.6}{4} = 9.9$

$M_4 = \frac{9.4+10.5+9.7+9.4}{4} = \frac{39}{4} = 9.75$

$M_5 = \frac{10.5+9.7+9.4+9.3}{4} = \frac{38.9}{4} = 9.58$

$M_6 = \frac{9.7+9.4+9.3+9.9}{4} = \frac{38.3}{4} = 9.58$

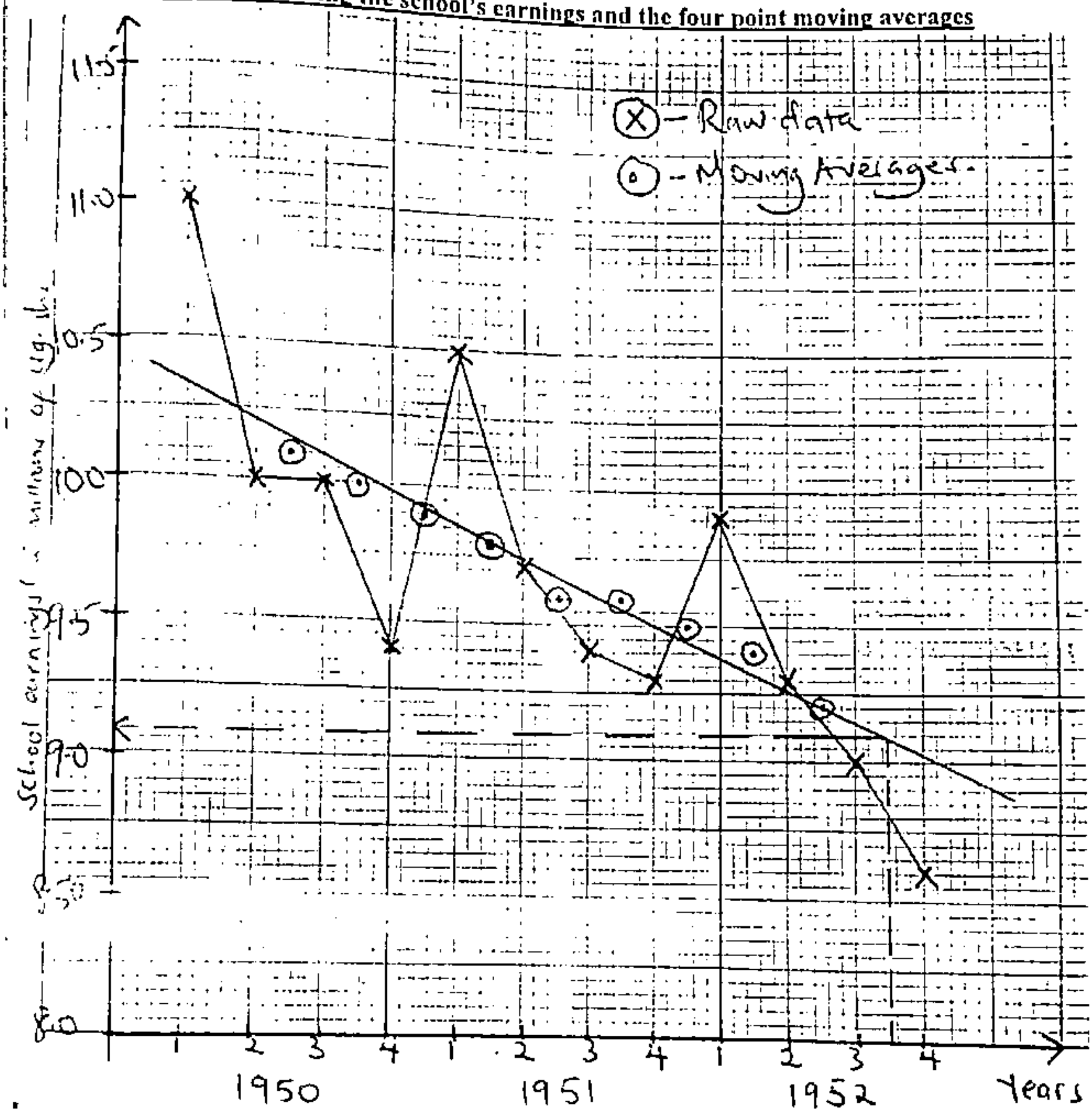
$M_7 = \frac{9.4+9.3+9.9+9.3}{4} = \frac{37.9}{4} = 9.48$

$M_8 = \frac{9.3+9.9+9.3+9.0}{4} = \frac{37.5}{4} = 9.38$

$M_9 = \frac{9.9+9.3+9.0+8.6}{4} = \frac{36.8}{4} = 9.2$

(b)

A graph showing the school's earnings and the four point moving averages



(c) (i) the school's earnings generally decrease over the period

(ii) let the school's earnings in the first quarter of 1953 be x

The 10^{th} moving average from the graph $M_{10} = 9.1$

$$\frac{9.3+9.0+8.6+x}{4} = 9.1$$

$$26.9 + x = 4 \times 9.1$$

$$x = 36.4 - 26.9 = 9.5$$

Therefore, it is estimated that the school's earnings in the 1^{st} quarter of 1953 would be 9.5 million Ug .shs.

Trial questions

1. The sales (in thousands of shillings) of a computer accessories company for the period 2002 to 2004 are given in the table below.

Year	QUARTER			
	1st	2nd	3rd	4th
2002	1235	1242	1410	1400
2003	1275	1270	1450	1480
2004	1302	1280	1510	1500

   (a) Calculate the four point moving averages
   (b) On the same axes, plot graphs of the sales and the moving averages against time. Comment on the general trend of the sales for the three years period
   (c) Use your graph to estimate the sales of computer accessories in the first quarter of 2005.

2. The table below shows the quarterly cost (in 1000's Uganda shillings) of electricity for a house hold over a period of 3years 1992 – 1994

Year	QUARTER			
	1st	2nd	3rd	4th
1992	68	60	59	65
1993	82	80	80	92
1994	94	78	90	105

   (a) Calculate the four point moving averages
   (b) On the same axes, plot both the raw data and the moving averages
   (c) Comment on the trend of electricity over the period of 3 years

3. The table below shows the electricity supplied (in million kilowatt hours) to a company on a quarterly basis between 1988 and 1991

Year	QUARTER			
	1st	2nd	3rd	4th
1988	8.9	7.1	6.7	9.3
1989	10.1	7.5	7.1	10.5
1990	11.7	7.5	8.3	16.9
1991	12.5	8.3	9.5	17.7

   (a) Calculate the quarterly moving averages
   (b) On the same axes, represent the data above and the quarterly moving averages
   (c) Comment on the trend of power supply to the company over the four years period

4. The average prices of a bunch of matooke in each third of a year over a period of $3\frac{1}{3}$ years are given in Uganda shillings in the table below.

Year	1st third	2nd third	3rd third
1998	4500	5000	5200
1999	5500	5700	6000
2000	6200	6500	6800
2001	7000	X	

   (a) Calculate the three point moving averages
   (b) On the same graph, plot the raw data and the moving averages
   (c) (i) comment on the trend of prices of matooke for this period
   (iii) Use your graph to estimate the value of X in the table

5. The table below shows the number of first grades scored by a candidate class of a certain school on termly basis for the years 2008 to 2011

Year	FIRST GRADES		
	1st term	2nd term	3rd term
2008	72	80	84
2009	76	92	80
2010	96	100	98
2011	102	108	x

(i) Calculate a three point moving averages for the data

(ii) Plot the first grades and the 3-point moving averages on the same graph

(iii) From the graph , predict the number of first grades x obtained in 3rd term of 2011

(iv) Comment on the trend of performance

6. The table below shows the number of bags sold by a certain shop, over the period of 12 weeks.

Week	1	2	3	4	5	6	7	8	9	10	11	12
No. of bags sold	422	318	349	252	386	230	256	141	264	168	272	260

(a) Calculate the 3 weekly moving averages

(b) On the same axes, show the weekly sales and the 3- weekly moving averages

(c) Comment on the trend of sales of the bags over the 12 weeks period

7. The table below shows the students' enrollment in school over a period of 12 years

Year	1992	1993	1994	1995	1996	1997	1998	1999	2000	2001	2002	2003
students	700	765	810	840	900	925	975	1034	1059	1110	1150	1188

(a) Calculate the four point moving averages

(b) (i) plot the four point moving averages on the graph

(ii) Use the graph to predict the students' enrollment for the year 2004.

8. The table below shows the average school fees per student of a certain school in thousands of Uganda shillings from 2001 to 2004

Year	School term fees in thousands of Ug shs		
	1st term	2nd term	3rd term
2001	180	200	210
2002	190	230	200
2003	240	250	245
2004	255	270	

(i) Calculate the 3-point moving averages for the data

(ii) Plot the term fees and the 3-point moving averages on the same graph

(iii) From the graph, predict the fees per student for the third term of 2004.

9. The expenditure on school fees (in thousands of Ug shs) by a family for three years is shown in the following table.

Year	QUARTER			
	1st	2nd	3rd	4th
2001	84	64	61	82
2002	92	70	70	85
2003	100	81	81	96

(i) Calculate the four point moving averages

(ii) Plot the expenditure on fees and the four point moving averages on the same graph

(iii) Comment on the trend of the school fees over the given period.

10. The table below shows the monthly sales of a certain product (in kg) in the year 1995

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Sales	672	636	680	704	744	700	756	784	828	800	840	880

(a) Calculate the 6-point moving averages

(b) Plot on the same axes the actual sales and the moving averages. Comment on the trend of sales during the year.

11. The table below shows the annual production of oil in millions of litres by a company for a period 1998 – 2006

Year	1998	1999	2000	2001	2002	2003	2004	2005	2006
Prodn	195	145	170	177	154	152	132	155	167

(a) Construct a 5-year moving averages for the oil production

(b) On the same axes, plot the graphs of annual production and moving averages

(c) Comment on the general trend of oil production over the year period

CHAPTER 14: SCATTER GRAPHS AND CORRELATION

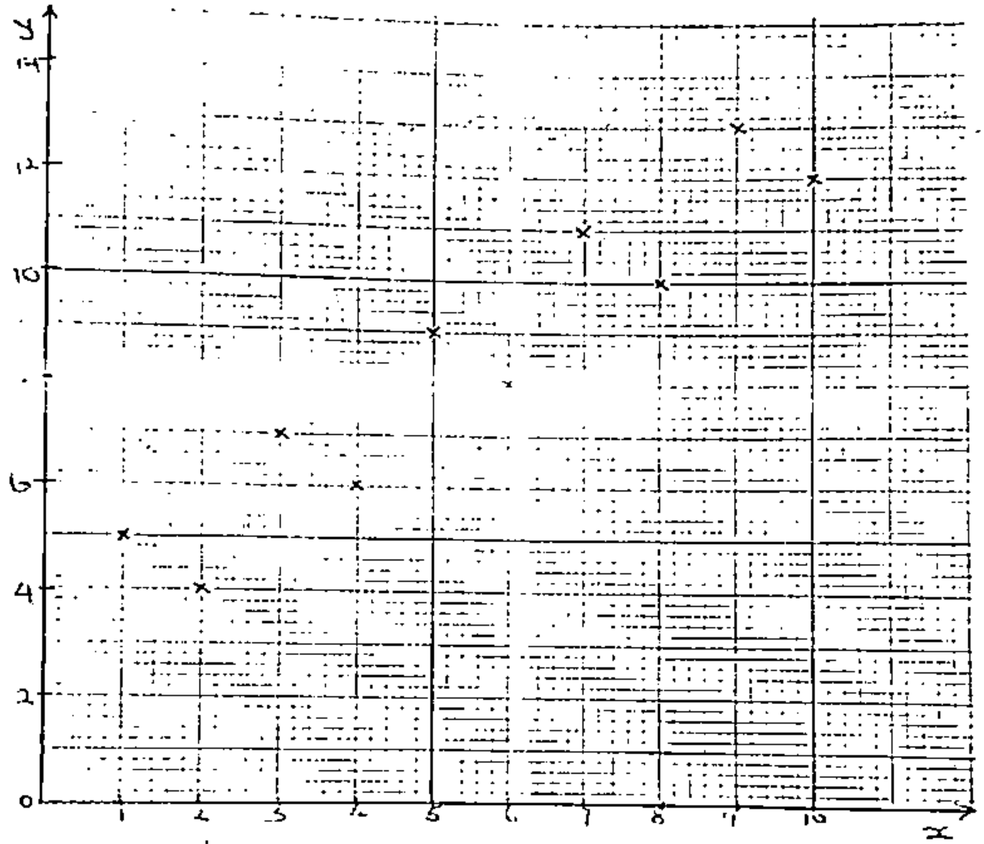
The relation between two variables can more appreciably be shown on diagrams or graphs known as scatter graphs. A scatter graph is obtained by representing the scores of one variable on the vertical axis and the other corresponding values on the horizontal axis.

Example

Draw a scatter diagram for the following data

X	1	2	3	4	5	6	7	8	9	10
Y	5	4	7	6	9	8	11	10	13	12

Solution



REGRESSION LINE

When a scatter graph is plotted, a line of best fit can be drawn through the points. This line is called the regression line.

The regression line passes through the point $(\bar{X}, \bar{Y})$ where $\bar{X} = \frac{\Sigma X}{n}$ and $\bar{Y} = \frac{\Sigma Y}{n}$

The regression line passes through the above point such that there are equal number of points both below and above the line.

Example 1

In two different athletics competition C_1 and C_2, ten schools A, B, C, D, E, F, G, H, I, J participated and their performances in points are given below.

Competition	A	B	C	D	E	F	G	H	I	J
C_1	556	473	502	514	435	499	507	510	560	540
C_2	590	496	578	608	524	504	600	560	597	572

(i) Plot the points on a scatter diagram C_2 against C_1

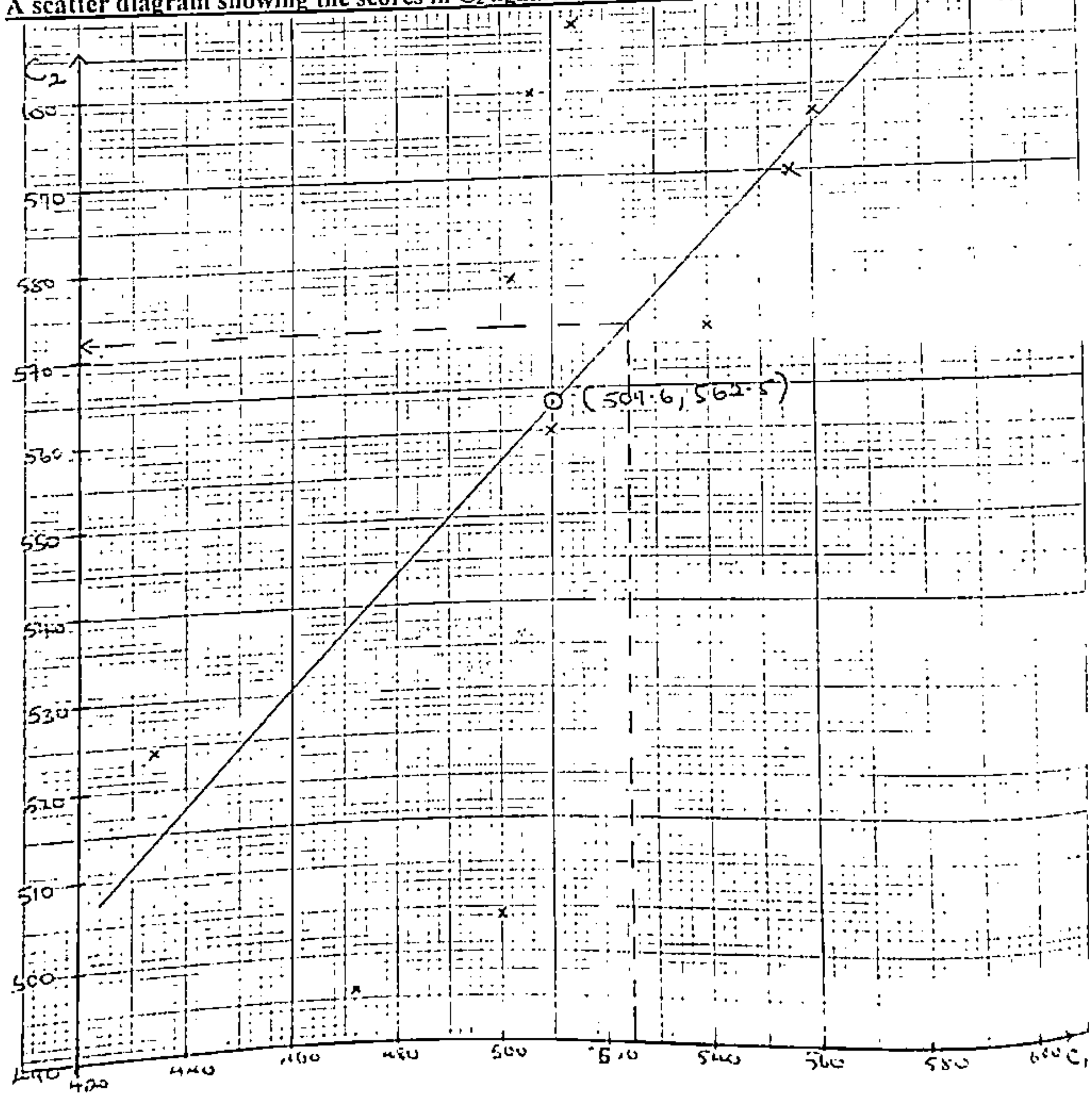
(ii) Draw a line of best fit through the plotted points on your scatter diagram

(iii) Estimate how many points a school would have scored in competition C_2 if it had scored 525 points in the competition C_1

Solution

(i)

A scatter diagram showing the scores in C_2 against those in C_1



(i) $\bar{X} = \frac{\Sigma C_1}{10} = \frac{5096}{10} = 509.6$

and $\bar{Y} = \frac{\Sigma C_2}{10} = \frac{5625}{10} = 562.5$

The line of best fit passes through the point (509.6, 562.5)

(ii) From the graph, the school would have scored 572 points in C_2 if it had scored 525 points in C_1.

Example 2

The table below shows the marks scored in mathematics and fine art of 10 students in a certain school.

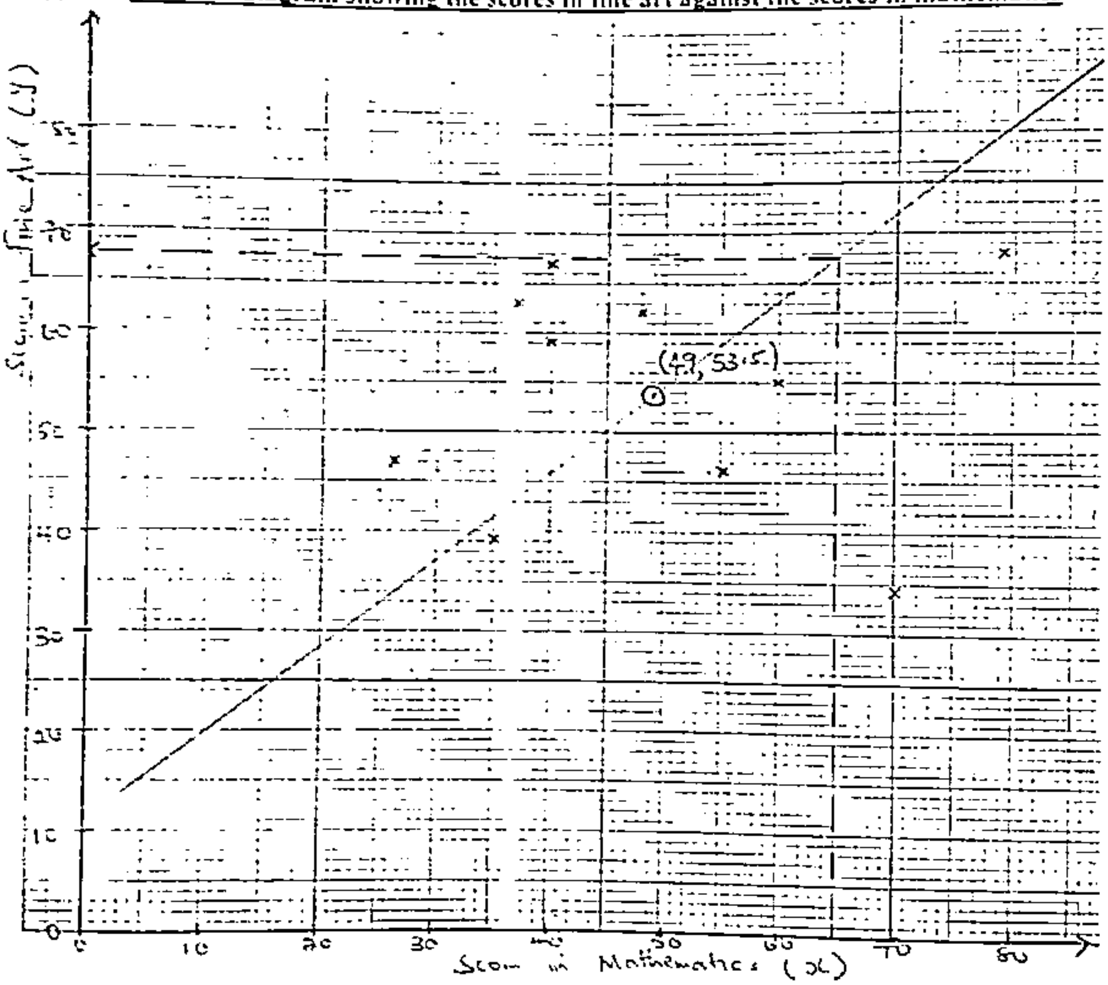
Mathematics	40	48	79	26	55	35	37	70	60	40
Fine Art	59	62	68	47	46	39	63	29	55	67

(i) Draw a scatter diagram and comment on your result

(ii) Plot a line of best fit on your scatter diagram and estimate the score in fine art if 65 marks were scored by a student in Mathematics

Solution

(i) A scatter diagram showing the scores in fine art against the scores in mathematics



(ii) $\bar{X} = \frac{\Sigma X}{10} = \frac{490}{10} = 49$ and $\bar{Y} = \frac{\Sigma Y}{10} = \frac{535}{10} = 53.5$

From the graph, a student who scored 65 marks in mathematics would have scored 68 marks in fine art.

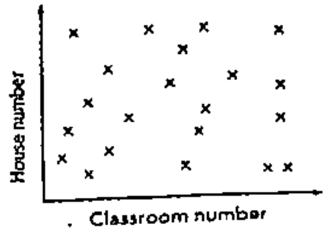
CORRELATION

Correlation is a method used to determine the relation between two or more variables. Correlation coefficient is the index used to measure the degree of correlation

Types of correlation

This diagram shows a **positive correlation**

This diagram shows a **negative correlation**



This diagram shows **no correlation**

Interpretation of the magnitude of correlation coefficient

Correlation coefficient	Interpretation
0 – 0.19	Chance correlation
0.2 – 0.39	Slight correlation
0.4 – 0.59	Moderate correlation
0.6 – 0.79	Substantial correlation
0.8 – 1.0	High correlation

Note: the sign associated with the correlation coefficient will be the one responsible for the type of coefficient i.e. -0.85 would indicate a high negative correlation, 0.24 would indicate a slight positive correlation and so on.

Rank correlation

The degree of relationship can be calculated using the spearman's correlation coefficient (ρ) as indicated below;

Spearman's rank correlation coefficient, $\rho = 1 - \frac{6\sum d^2}{n(n^2-1)}$

Where d is the difference between the rankings of a given scores and n is the number of pairs.

Examples

1. Two examiners X and Y each marked the scripts of 10 candidates who sat a mathematics examination. The table below shows the examiner's rankings of the candidates

Examiner	A	B	C	D	E	F	G	H	I	J
X	5	3	6	1	4	7	2	10	8	9
Y	6	3	7	2	5	4	1	10	9	8

Solution

Candidate	R_x	R_y	$d = R_x - R_y$	d^2
A	5	6	-1	1
B	3	3	0	0
C	6	7	-1	1
D	1	2	-1	1
E	4	5	-1	1

F	7	4	3	9
G	2	1	1	1
H	10	10	0	0
I	8	9	-1	1
J	9	8	1	1
				$\sum d^2 = 16$

$\rho = 1 - \frac{6\sum d^2}{n(n^2-1)} = 1 - \frac{6\times16}{10(100-1)} = 1 - 0.097 = 0.903$

Comment: there is a very high positive correlation between the two examiners X and Y

2. The table below shows the marks of eight students in physics and mathematics. Rank the results and find the value of the rank correlation. Comment on the result.

Physics (X)	65	65	70	75	75	80	85	85
Mathematics(Y)	50	55	58	55	65	58	61	65

Solution

X	Y	Rx	RY	d = Rx− RY	d^2
65	50	7.5	8	0.5	0.25
65	55	7.5	6.5	1	1
70	58	6	4.5	-1.5	2.25
75	55	4.5	6.5	-2	4
75	65	4.5	1.5	3	9
80	58	3	4.5	-1.5	2.25
85	61	1.5	3	-1.5	2.25
85	65	1.5	1.5	0	0
					$\sum d^2 = 21$

$\rho = 1 - \frac{6\sum d^2}{n(n^2-1)} = 1 - \frac{6\times21}{8(64-1)} = 1 - 0.25 = 0.75$

There is a positive substantial correlation between the two subjects

3. Eight students of a certain school participated in the 1989 and 1990 national mathematics contests. Their scores were as follows.

Participant	A	B	C	D	E	F	G	H
1989	72	60	56	76	68	52	80	64
1990	56	44	60	74	66	38	68	52

(a) Calculate the mean scores for the participants each year
(b) Compute a rank correlation coefficient for the performance of the participants in the two years.
(c) Use the values obtained in (a) and (b) to comment on
  (i) The level of difficulty of the two contests
  (ii) Whether the two contests examined had the same mathematic aptitude

Solution

(a) For 1989, Mean = $\frac{72+60+56+76+68+52+80+64}{8} = \frac{528}{8} = 66$

For 1990, Mean = $\frac{56+44+60+74+66+38+68+52}{8} = \frac{458}{8} = 57.25$

(b)

Participant	1989(X)	1990(Y)	Rx	Ry	d	d^2
A	72	56	3	5	-2	4
B	60	44	6	7	-1	1
C	56	60	7	4	3	9
D	76	74	2	1	1	1
E	68	66	4	3	1	1
F	52	38	8	8	0	0
G	80	68	1	2	-1	1
H	64	52	5	6	-1	1
						$\sum d^2 = 18$

$\rho = 1 - \frac{6\sum d^2}{n(n^2-1)} = 1 - \frac{6\times 18}{8(64-1)} = 1 - 0.2143 = 0.7857$

(c) (i) the contest in 1990 was generally harder than that of 1989

(ii) the two contests had the same mathematical aptitude

4. The following table gives marks (X) obtained by 12 students in an examination in statistics at the end of one term together with the marks (Y) obtained at the end of the following term.

Students	A	B	C	D	E	F	G	H	I	J	K	L
Marks(X)	53	74	48	71	68	60	47	72	48	65	80	40
Marks(Y)	41	50	44	38	41	48	45	57	36	46	50	47

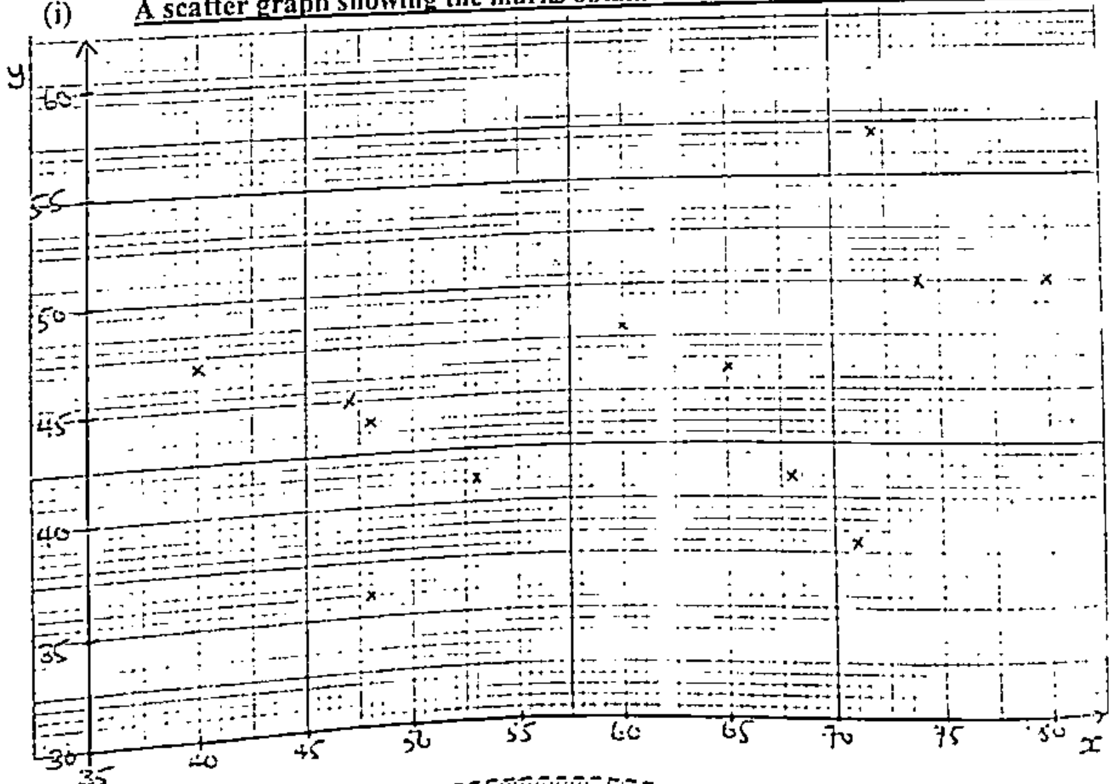
(i) Plot a scatter graph for the above data

(ii) Calculate the rank correlation coefficient for the data

(iii) What conclusions can one draw from your result in (ii) above

Solution

(i) **A scatter graph showing the marks obtained by 12 students in statistics examinations**



(ii)

Student	X	Y	Rx	Ry	d = Rx − Ry	d²
A	53	41	8	9.5	-1.5	2.25
B	74	50	2	2.5	-0.5	0.25
C	48	44	9.5	8	1.5	2.25
D	71	38	4	11	-7	49
E	66	41	5	9.5	-4.5	20.25
F	60	48	7	4	3	9
G	47	45	11	7	4	16
H	72	57	3	1	2	4
I	48	36	9.5	12	-2.5	6.25
J	65	46	6	6	0	0
K	80	50	1	2.5	-1.5	2.25
L	40	47	12	5	7	49
						$\sum d^2 = 160.5$

$\rho = 1 - \frac{6\sum d^2}{n(n^2-1)} = 1 - \frac{6\times160.5}{12(144-1)} = 1 - 0.5612 = 0.4388$

(iii) Conclusion: there is a moderate positive correlation between the marks X and Y

5. The following table gives the marks obtained in calculus, physics and statistics by seven students;

Calculus	72	50	60	55	35	48	82
Physics	61	55	70	50	30	50	73
Statistics	50	40	62	70	40	40	60

Draw scatter diagrams and determine the rank correlation coefficients between the performances of the students in

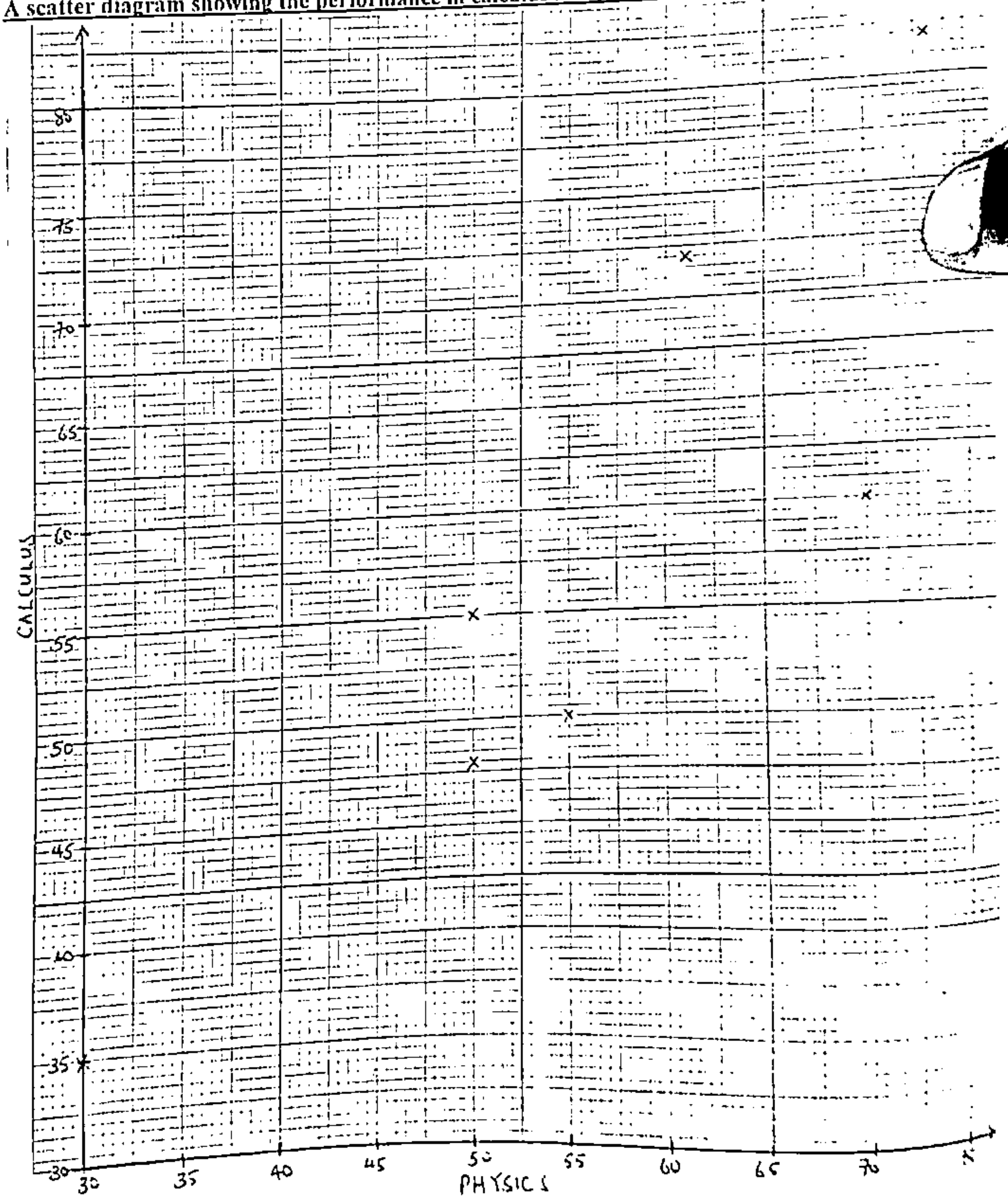
(i) Calculus and Physics

(ii) Calculus and statistics

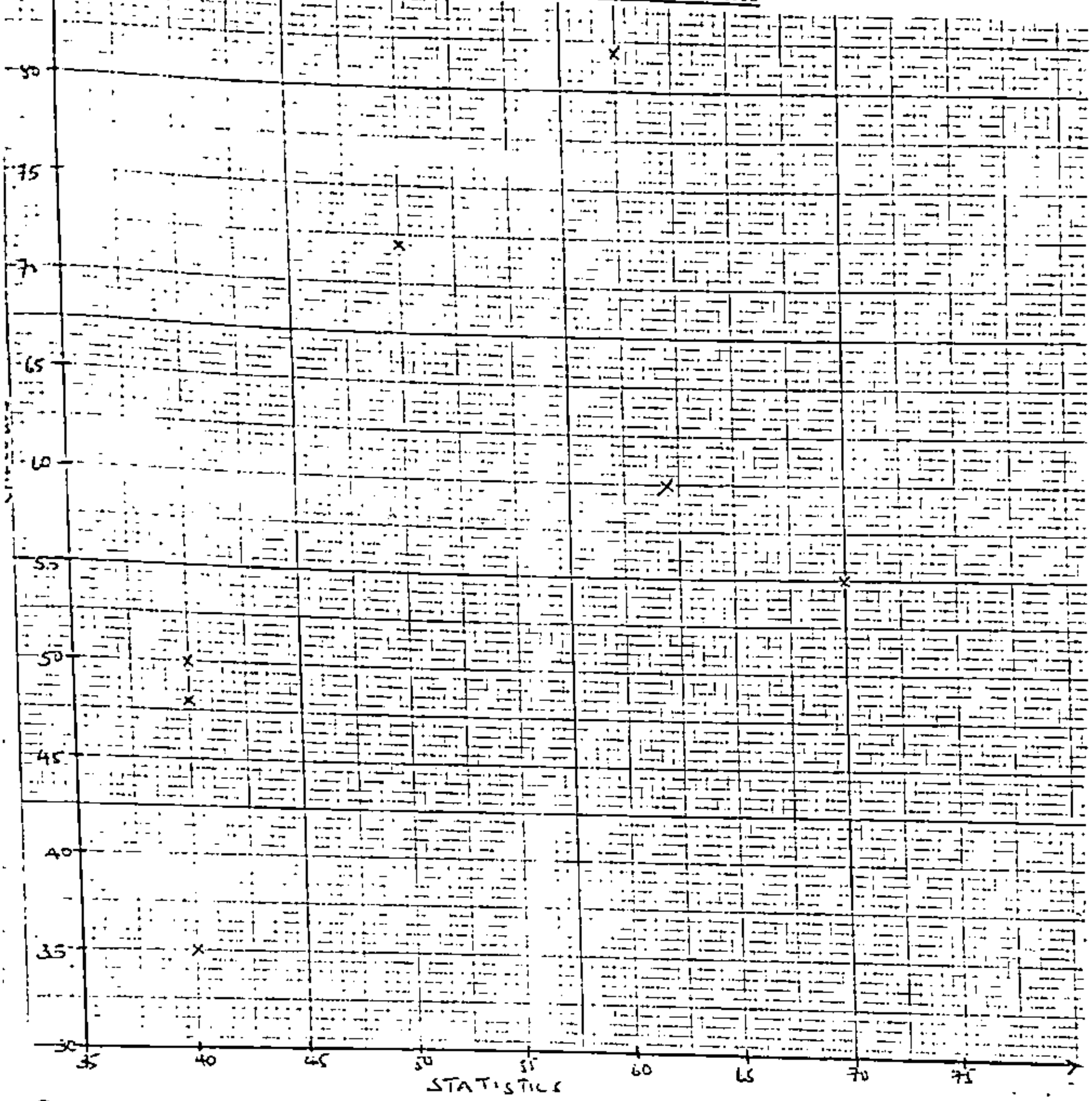
Give interpretations to your results

Solution

A scatter diagram showing the performance in calculus and physics



scatter diagram showing the performance in calculus and statistics



Let C = calculus, P = Physics, S = Statistics

C	P	S	R_C	R_P	R_S	$d = R_C - R_P$	d^2	$d = R_C - R_S$	d^2
72	61	50	2	3	4	-1	1	-2	4
50	55	40	5	4	6	1	1	-1	1
60	70	62	3	2	2	1	1	1	1
55	50	70	4	5.5	1	-1.5	2.25	3	9
35	30	40	7	7	6	0	0	1	1
48	50	40	6	5.5	6	0.5	0.25	0	0
82	73	60	1	1	3	0	0	-2	4
$\sum d^2$							5.5		20

(i) Calculus and physics

$$\rho = 1 - \frac{6\sum d^2}{n(n^2-1)} = 1 - \frac{6\times5.5}{7(49-1)} = 1 - \frac{33}{336} = 0.9018$$

There is a high positive correlation

(ii) Calculus and statistics

$$\rho = 1 - \frac{6\sum d^2}{n(n^2-1)} = 1 - \frac{6\times 20}{7(49-1)} = 1 - \frac{120}{336} = 0.6429$$

There is a substantial positive correlation

6. The table below represents the scores obtained in biology and geography by 10 students

Biology (X)	51	63	43	60	61	32	71	40	48	59
Geography(Y)	61	41	36	45	49	53	70	38	60	52

Assuming that the highest mark represents the first rank and so on

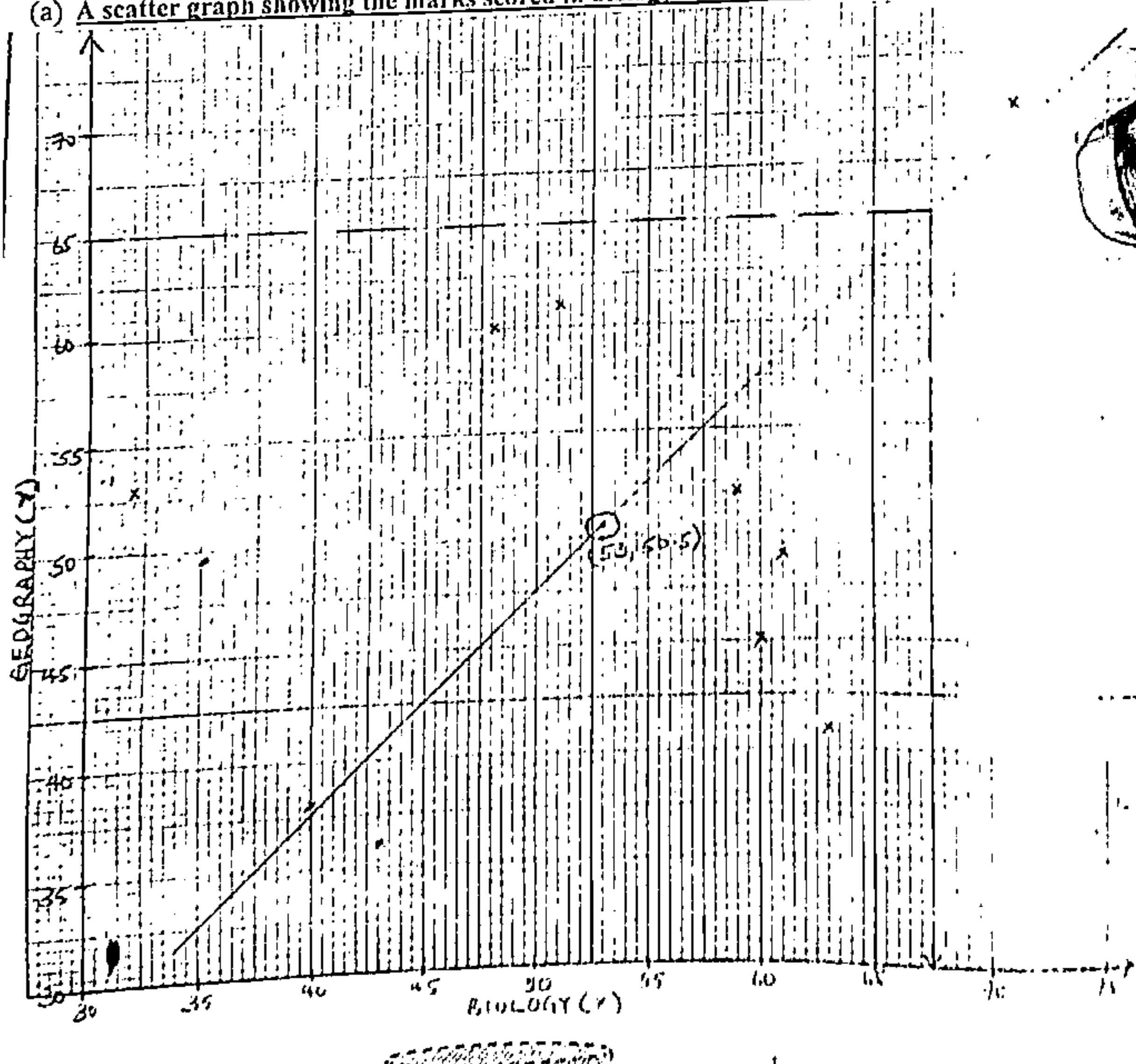
(a) Construct a scatter diagram and line of best fit

(b) Use your line of best fit to estimate the score obtained in biology by a student who scores 65 in geography

(c) Calculate the spearman's rank correlation coefficient and comment on your result

Solution

(a) **A scatter graph showing the marks scored in biology and geography**



(b) Let R_B = rank of biology, R_G = rank of geography

R_B	R_G	$d = R_B - R_G$	d^2
6	3	3	9
2	8	-6	36
8	10	-2	4
4	7	-3	9
3	6	-3	9
10	5	5	25
1	1	0	0
9	9	0	0
7	4	3	9
5	2	3	9
$\sum d^2$			110

Spearman's rank correlation coefficient, $\rho = 1 - \frac{6\sum d^2}{n(n^2-1)} = 1 - \frac{6\times110}{10(100-1)}$

$= 1 - \frac{660}{990} = 0.333$

There is a slight positive correlation

Trial questions

1. The table below shows the marks scored by eight students A, B, C, D, E, F, G, H in three tests of English during the first term of the school calendar.

Student	A	B	C	D	E	F	G	H
TEST I	80	70	60	74	65	65	48	58
TEST II	70	70	65	70	64	60	58	50
TEST III	75	80	86	82	70	64	60	64

(a) Calculate the rank correlation coefficient for the performance between

(i) Test I and Test II

(ii) Test I and Test III [Ans: (a)(i) 0.875 (ii) 0.744]

(b) Comment on the relationship of the performance in the three terms

2. The following are final examination scores which 12 students obtained in Psychology, X and economics, Y

Psychology(X)	35	56	65	78	49	82	22	90	77	35	52	93
Economics (Y)	57	72	63	76	53	100	38	82	82	19	43	79

(i) Draw a scatter diagram for the data

(ii) Find the rank correlation coefficient for the performance of the students. [Ans: 0.874]

3. The marks obtained in two tests X and Y were as follows

X	51	62	64	47	54	44	68	61	56
Y	45	54	58	46	49	43	59	56	53

(i) Plot a scatter diagram and comment on your graph

(ii) calculate the rank correlation coefficient. comment on your result [Ans: 0.97]

4. The marks scored by 12 students in an English and mathematics examination were

English	74	52	43	65	39	56	48	37	52	68	45	68
Mathematics	46	56	38	42	48	51	59	54	45	51	35	61

(i) Draw a scatter diagram and comment on the performance of the students in the two subjects.

(ii) Calculate the rank correlation coefficient and comment on your result [Ans: 0.157]

5. The table below shows the percentage preference of nine most popular holiday destinations as sampled by a tour company for two years 1996 and 1997

Holiday destination	F	G	S	I	A	Y	C	H	B
1996	90	80	78	78	50	40	30	20	10
1997	79	90	80	60	60	35	50	60	22

(i) Plot a scatter diagram for the data and comment on the correlation between the figures for the two years

(ii) Calculate the spearman's rank correlation coefficient and comment on your result. [Ans: 0.813]

6. The table below shows the height of each boy (X cm) and the distance (Y cm) to which he can throw the ball.

Boy	A	B	C	D	E	F	G	H	I	J
X (cm)	122	124	133	138	144	156	158	161	164	168
Y (cm)	41	28	52	56	29	34	59	61	63	67

(i) Draw a scatter diagram for this data

(ii) Comment on the relationship between the boys' heights and the distances they throw the ball

(iii) Draw a line of best fit. Use the line to estimate the distance the ball can be thrown by a boy of height 175cm

(iv) Calculate the rank correlation coefficient between X and Y [Ans: 0.818]

7. The marks obtained by 8 students in English (X) and French (Y) are given below;

English (X)	55	42	37	59	38	48	56	48
French (Y)	60	48	41	63	35	39	51	55

(i) Plot a scatter graph for the performance of the 8 students in the two subjects. Comment on your graph

(ii) Calculate the rank correlation coefficient of the performance of the students in the two subjects. Comment on your result [Ans: 0.768]

8. The marks obtained in physics and chemistry by 10 students in end of year examinations were;

Chemistry (X)	54	58	60	60	70	65	71	68	73	66
Physics(Y)	57	61	63	64	74	68	70	73	75	78

(i) Draw a scatter diagram and comment on it

(ii) Calculate the rank correlation coefficient of the performance [Ans: 0.839]

9. Three examiners X, Y and Z each marked the scripts of 10 candidates who sat a mathematics examination. The table below shows the examiners' ranking of the candidates.

	CANDIDATES									
EXAMINERS	A	B	C	D	E	F	G	H	I	J
X	8	5	9	2	10	1	7	6	3	4
Y	5	3	6	1	4	7	2	2	8	9
Z	6	3	7	2	5	4	1	10	9	8

Calculate the rank correlation coefficient of rankings between

(i) X and Y

(ii) Y and Z [Ans: (i) -0.217 (ii) 0.515]

10. The following table shows the marks scored by 13 students in biology and chemistry tests.

Biology	40	28	28	20	21	31	22	36	29	24	30	25	27
Chemistry	40	30	28	20	22	45	25	35	27	23	31	27	26

(a) Draw a scatter diagram to represent the performance of the students in the two subjects. Comment on the relationship between the performance in biology and chemistry

(b) Calculate the spearman's rank correlation coefficient between the marks of the two subjects [Ans: 0.945]

11. The table below shows the performance of 10 students in their inter-house music competition and their performance in their end of term mathematics test.

Student	A	B	C	D	E	F	G	H	I	J
Scores in music	280	270	276	232	250	228	182	205	220	150
Scores in mathematics	70	64	72	68	52	55	50	48	61	40

(a) Represent the performances on the same scatter graph. Comment on the graph

(b) Calculate the rank correlation coefficient between the students' performance in music competition and their performance in end of term mathematics. Comment on this result.

12. The table below shows the marks of 10 students in three papers [Ans: 0.867]

Paper I	81	42	55	67	36	46	59	78	30	67
Paper II	64	50	54	70	48	32	49	54	46	58
Paper III	59	47	78	43	60	54	31	52	68	62

(a) Calculate the rank correlation coefficient between

(i) Paper I and paper II

(ii) Paper II and Paper III

(iii) Comment on the relationship between the performance in Paper I and the other two papers II and III [Ans: (i) 0.788 (ii) -0.124]

13. Eight applicants for a certain job obtained the following marks in aptitude and written tests.

Applicant	A	B	C	D	E	F	G	H
Aptitude test	33	45	15	42	45	35	40	48
Written test	57	60	40	75	58	48	54	68

Calculate the rank correlation coefficient of the applicant's performance in the two tests. Comment on your result. [Ans: 0.744]

14. The table below shows the percentage of sand Y in the soil at different depths X(in cm)

Soil depth(X)	35	65	55	25	45	75	20	90	51	60
Percentage of sand	86	70	84	92	79	68	96	58	86	77

(a) (i) Plot a scatter diagram for the data. Comment on the relationship between the depth of the soil and the percentage of sand in the soil

(ii) Draw a line of best fit through the points of the scatter diagram. Use it to estimate the

- Percentage of sand in the soil at the depth of 31 cm
- Depth of soil with 54% sand

(b) Calculate the rank correlation coefficient between the percentage of sand in the soil and the depth of soil. [Ans: -0.948]

15. Given the variables X and Y below

X	80	75	86	60	75	92	86	50	64	75
Y	62	58	60	45	68	68	81	48	50	70

Obtain the rank correlation coefficient between the variables X and Y. Comment on your result.

[Ans: 0.715]

16. The table below shows the marks scored by 10 candidates in two subjects x and y

Candidate	A	B	C	D	E	F	G	H	I	J
Subject x	34	21	27	28	29	32	39	24	32	36
Subject y	52	46	50	48	50	51	55	47	49	51

(i) Plot a scatter diagram for the data. What does the diagram show?

(ii) Draw a line of best fit and find x when y = 45

(iii) Calculate a rank correlation coefficient and comment on your results [Ans: 0.891]

17. Below are the marks scored by 8 students A, B, C, D, E, F, G and H in statistics and mechanics test in a given term

Student	A	B	C	D	E	F	G	H
Mechanics	35	40	60	54	63	40	55	72
Statistics	52	75	41	60	81	31	65	52

(a) (i) Plot a scatter diagram for the data. Comment on the relationship between mechanics and statistics performance

(ii) Draw a line of best fit through the points of the scatter diagram. Use your result to estimate the marks in statistics for a student who got 47 in mechanics

(b) Calculate the rank correlation coefficient for the two tests. Comment on your result [Ans: 0.190]

18. The table below shows the scores of eight houses in a music competition for two consecutive years

Houses	A	B	C	D	E	F	G	H
1st year scores	70	84	65	90	75	70	76	92
2nd year scores	76	80	70	84	75	72	75	82

(i) Draw a scatter diagram for the data above and comment on your diagram

(ii) Calculate a rank correlation coefficient between the 1st and 2nd year scores. Comment on your result. [Ans: 0.869]

19. The table below shows the scores of nine employees in interview (x) and job performance (y)

x	57	35	56	57	66	79	81	84	52
y	66	51	63	34	47	70	84	84	53

(a) (i) Draw a scatter diagram for the data.

(ii) Comment on the relationship between the interview and job performance

(b) (i) Calculate the rank correlation coefficient between x and y .

(ii) Comment on your result [Ans: 0.642]

20. Eight athletes A, B, C, D, E, F, G and H scored the following points in two events long jump and high jump.

Athletes	A	B	C	D	E	F	G	H
Long jump	15	14	12	10	8	7	4	1
High jump	1	3	3	5	7	7	9	12

(a) Plot a scatter diagram for the data and comment on the relationship between the two events

(b) Calculate a rank correlation coefficient and comment on the value obtained.

[Ans: -0.964]

CHAPTER 15: THE PROBABILITY THEORY

Probability theory is a branch of mathematics concerned with prediction or uncertainty.
The term probability arose from the games of chance and gambling i.e. tossing a coin, rolling a die, playing cards etc.
The probability of an event is the measure of the likelihood that it will occur and it is given on a numerical scale from 0 to 1.The numbers representing probabilities can be written as percentages, fractions or decimals

- A probability of zero implies that the event is impossible
- A probability of one (100%) indicates that the event is certain to occur.
- All other events have a probability between 0 and 1

The closer the probability of an event to 1, the more likely it is to happen and the closer the probability of an event to 0, the less likely it is to happen
We talk about a 'fair' coin or a 'fair' dice meaning that all outcomes are equally likely

For a fair coin, $P(H) = P(T) = \frac{1}{2}$

The alternative is that the coin or dice is biased

Sample space and generation of the sample space

Sample space (S) is the set of all possible outcomes of an experiment. Outcomes are events that can occur after an experiment.

Each possible outcome is called a sample point

Example

Rolling a die; S = (1, 2, 3, 4, 5, 6)
Tossing a coin; S = (H, T)
Note: 1, 2, 3, 4, 5, 6 and H, T are sample points

Generation of the sample space

The ways of generating a sample space include the following;
(i) Table of out comes
(ii) Permutations
(iii) Tree diagram

Terms used in the set theory

An event is a subset of a sample space

Intersection of events

Consider A and B as two events of a sample space S, the intersection of these two events is given by A ∩ B i.e. containing sample points common to both A and B

Union of events

This is a set of all sample points in either A or B or both It is denoted A ∪B

Complement of an event

If A is an event of a sample space S, the compliment of A is given by the set containing all sample points in S that are not in A It is denoted A '

The Or situation

If A and B are two events, the probability that either event A or B or even both occurs is denoted by $P(A \cup B)$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Probability function

The probability function of an event A is denoted by P(A) and is the sum of the probabilities of the sample points in A. (i) P(A)$\leq$ 1 (ii) P(A)$\geq$ 0 (iii) P(S) = 1

$\text{P (A)} = \frac{n(A)}{n(S)}$

The sum of the probabilities if all outcomes to an experiment must be 1

Example 1

When you toss a coin, what is the probability that it lands heads up?

Solution

When you toss a coin, there are two possibilities that it lands heads or tails up

$$P(heads) + P(tails) = 1$$

But both are equally likely so

$$P(heads) = P(tails) = \frac{1}{2}$$

Example 2

The probability that it rains tomorrow is $\frac{2}{3}$. What is the probability that it doenot rain tomorrow?

Solution

Tomorrow it must either rain or not rain so

$$P(rain) + P(no\ rain) = 1$$

$$\frac{2}{3} + P(no\ rain) = 1$$

$$P(no\ rain) = 1 - \frac{2}{3} = \frac{1}{3}$$

Example 3

Find the probability of choosing a defective pen in a lot of 12 out of which 4 are defective, if a single draw is made

Solution

No. of ways the event can happen = 4

Total no. of possibilities = 12

Hence probability = $\frac{4}{12} = \frac{1}{3}$

Example 4

What is the probability of throwing a number greater than 4 for a die whose faces are numbered from 1 to 6?

Solution

$S = \{1,2,3,4,5,6\}$ $\quad n(S) = 6$

$A = \{5,6\}$ $\quad n(A) = 2$

$\text{P (A)} = \frac{2}{6} = \frac{1}{3}$

The outcome of two events

When dealing with probabilities for two events, it is important to be able to identify all the possible outcomes. We can use 2-way tables or tree diagrams

Example 1

A six-sided die and a coin are tossed. List all the possible outcomes.

Solution

The coin can land heads (denoted by H) or tails (T)

While the die can show 1, 2, 3, 4, 5 or 6

So for heads on coin, the possible outcomes are;

H1, H2, H3, H4, H5 and H6

While for tails, they are;

T1, T2, T3, T4, T5 and T6

This list can conveniently be summarized in a 2-way table i.e

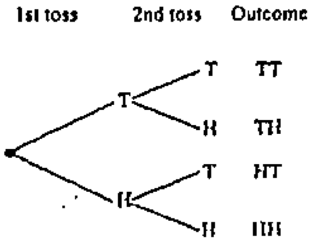
	Die					
	1	2	3	4	5	6
Coin H	H1	H2	H3	H4	H5	H6
Coin T	T1	T2	T3	T4	T5	T6

Example 2

A coin is tossed twice. List all the possible outcomes

Solution

You can use a tree diagram to represent this solution



Note that 'TH' is not the same as 'HT'

This is an excellent method but it can lead to problems when you have too many branches

Determining probabilities

When the outcomes of an event are all equally likely, then the probabilities can be found by considering all the possible outcomes

The probability of an outcome is given by;

$$\frac{\text{number of ways of obtaining outcome}}{\text{number of possible outcomes}}$$

Example

In a class of 30 children, 16 are girls, 4 wear glasses and 3 are left handed. A child is selected at random from the class. What is the probability that this child is;

(a) a girl (b) right handed (c) wearing glasses

Solution

(a)In a class, there are 16 girls so

$$P(girl) = \frac{16}{30} = \frac{8}{15}$$

(b) There are three left handed children and so the other 27 must be right handed so

$P(right\ handed) = \frac{27}{30} = \frac{9}{10}$

(c) There are 4 children wearing glasses so

$P(wears\ glasses) = \frac{4}{30} = \frac{2}{15}$

Probability of two events

When two events take place and every outcome is equally likely to happen, the probability of a particular outcome can be readily found from the formula

$$probability = \frac{number\ of\ successful\ outcomes}{total\ number\ of\ outcomes}$$

Example 1

Two dice are thrown together. Find the probability that the total score is 9

Solution

The table shows all the possible outcomes and total scores

Second die

First die	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

There are 36 possible outcomes and each one is equally likely to occur.

The outcomes that give a total of 9 have been circled and there are 4 such outcomes

Now the probability can be found

$P(9) = \frac{4}{36} = \frac{1}{9}$

Example 2

A spinner which forms part of a children's game can point to one of the four regions A, B, C or D. What is the probability that when the two children spin the spinner, it points to the same letters

Solution

The table shows all the possible outcomes

Second child

First child	A	B	C	D
A	AA	AB	AC	AD
B	BA	BB	BC	BD
C	CA	CB	CC	CD
D	DA	DB	DC	DD

There are 16 possible outcomes. Each is equally likely to occur. The outcomes that are the same for the children have been circled. There are 4 outcomes of this type
The probability that both have the same letter will be given by;

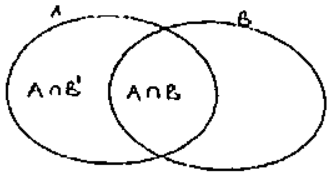
$$P(same\ letter) = \frac{4}{16} = \frac{1}{4}$$

Interaction with the set theory

The following results can be deduced from the set theory

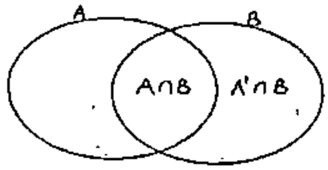
Result 1

For any two events A and B;
$P(A) = P(A \cap B) + P(A \cap B')$



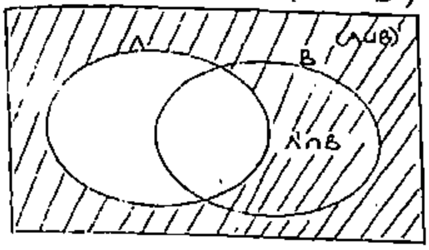
Result 2

For any two events A and B;
$P(B) = P(A \cap B) + P(A' \cap B)$



Result 3

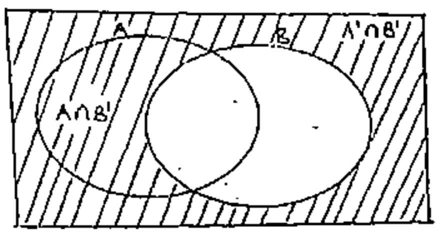
$P(A') = P(A' \cap B) + P(A' \cap B')$



Note that $A' \cap B' = (A \cup B)'$

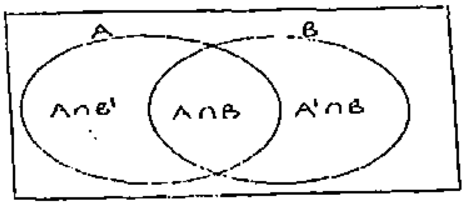
Result 4

$P(B') = P(A \cap B') + P(A' \cap B')$



Result 5

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$



Result 6

For any two events; (i) $P(A \cup B)' = P(A' \cap B')$
(ii) $P(A' \cup B') = P(A \cap B)'$

The contingency table

The alternative way of recalling the first four results is by using the contingency table

1	P(A)	P(A')
P(B)	P(A∩ B)	P(A'∩ B)
P(B')	P(A∩ B')	P(A' ∩ B')

$P(A) + P(A') = 1$ and $P(B) + P(B') = 1$

Examples

1. Events A and B are such that $P(A)=\frac{19}{30}$, $P(B)=\frac{2}{5}$ and $P(A\cup B)=\frac{4}{5}$. Find $P(A\cap B)$.

Solution

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$
$$\Rightarrow P(A\cap B)=P(A)+P(B)-P(A\cup B)$$
$$=\frac{19}{30}+\frac{2}{5}-\frac{4}{5}=\frac{7}{30}$$

2. The probability that a student passes mathematics is $\frac{2}{3}$ and the probability that he passes physics is $\frac{4}{9}$. If the probability that he passes at least one of them is $\frac{4}{5}$, find the probability that he passes both papers.

Solution

Let M denote event passing mathematics and P denote passing physics

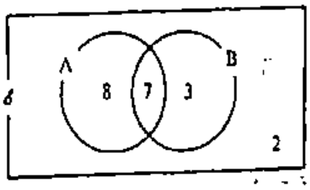
$P(M)=\frac{2}{3}$ $\quad P(P)=\frac{4}{9}$ $\quad P(M\cup P)=\frac{4}{5}$

$$P(M\cup P)=P(M)+P(P)-P(M\cap P)$$
$$\Rightarrow P(M\cap P)=P(M)+P(P)-P(M\cup P)$$
$$=\frac{2}{3}+\frac{4}{9}-\frac{4}{5}=\frac{14}{45}$$

3. One element is randomly selected from a universal set of 20 elements. Sets A and B are subsets of the universal set and n(A) = 15, n(B) = 10 and $n(A\cap B)=7$. If P(A) is the probability of the selected element belonging to Set A, find
(i) P(A) (ii) $P(A\cap B)$ (iii) P(A') (iv) $P(A\cup B)$

Solution

Using the venn diagram, we can find the required probabilities



(i) $P(A)=\frac{15}{20}=\frac{3}{4}$

(ii) $P(A\cap B)=\frac{7}{20}$

(iii) $P(A')=\frac{5}{20}=\frac{1}{4}$

(iv) $P(A\cup B)=\frac{18}{20}=\frac{9}{10}$

4. If A and B are two events such that $P(A)=\frac{1}{4}$, $P(B)=\frac{1}{2}$ and $P(A\cap B)=\frac{1}{8}$, find
(a) $P(A\cup B)$ (b) $P(A\cup B)'$

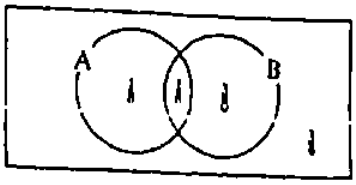
Solution

Method I

(a)
$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$
$$=\frac{1}{4}+\frac{1}{2}-\frac{1}{8}=\frac{5}{8}$$

(b) $P(A \cup B)' = 1 - P(A \cup B) = 1 - \frac{5}{8} = \frac{3}{8}$

Method II



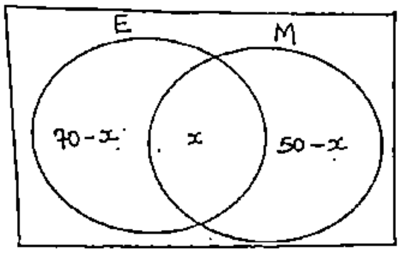
(a) $P(A \cup B) = \frac{1}{8} + \frac{1}{8} + \frac{3}{8} = \frac{5}{8}$

(b) $P(A \cup B)' = \frac{3}{8}$

5. In a class of 100 students, 70 offer economics while 50 students offer mathematics. Each student offers at least one of the subjects. Determine the probability for the number of students who offer both subjects.

Solution

Let the number of students who offer both subjects be x



$$70 - x + x + 50 - x = 100$$
$$120 - x = 100$$
$$\Rightarrow x = 20$$

Thus $P(E \cap M) = \frac{20}{100} = 0.2$

6. Two dice are thrown, what is the probability of scoring a double or a sum greater than 8?

Solution

A table of outcomes can be used to generate the sample space.

	First die					
Second die	1,1	1,2	1,3	1,4	1,5	1,6
	2,1	2,2	2,3	2,4	2,5	2,6
	3,1	3,2	3,3	3,4	3,5	3,6
	4,1	4,2	4,3	4,4	4,5	4,6
	5,1	5,2	5,3	5,4	5,5	5,6
	6,1	6,2	6,3	6,4	6,5	6,6

Table of sums

2	3	4	5	6	7
3	4	5	6	7	8
4	5	6	7	8	9
5	6	7	8	9	10
6	7	8	9	10	11
7	8	9	10	11	12

From the table of sums we can see that the total number of outcomes is 36

Let event A denote scoring a double

$A = \{(1,1)\ (2,2)\ (3,3)\ (4,4)\ (5,5)\ (6,6)\}$

$n(A) = 6 \Rightarrow P(A) = \frac{6}{36}$

Let event B denote scoring a sum greater than 8

$$B = \{9, 9, 9, 9, 10, 10, 10, 11, 11, 12\}$$

$$n(B) = 10 \Rightarrow P(B) = \frac{10}{36}$$

$$A \cap B = \{10,\ 12\} \Rightarrow n(A \cap B) = 2 \text{ thus } P(A \cap B) = \frac{2}{36}$$

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{6}{36} + \frac{10}{36} - \frac{2}{36} = \frac{14}{36} = \frac{7}{18}$$

Mutually exclusive events

If two events A and B have no sample points in common i.e. if $A \cap B = \{\ \}$ or they cannot occur at the same time, then we say that A and B are mutually exclusive. $P(A \cap B) = 0$

Examples

1. Given that A and B are mutually exclusive events such that P(A) = 0.5, P(B) = 0.9, find (i) $P(A' \cup B)$ (ii) $P(A' \cap B')$

Solution

(i) For mutually exclusive events, $P(A \cap B) = 0$

$$P(A \cup B) = P(A) + P(B) \Rightarrow P(B) = P(A \cup B) - P(A)$$

$$= 0.9 - 0.5 = 0.4$$

$$P(A' \cup B) = P(A') + P(B) - P(A' \cap B)$$

From the contingency table,

$$P(A' \cap B) = P(B) - P(A \cap B)$$

$$= 0.4 - 0 = 0.4$$

$$P(A') = 1 - P(A) = 1 - 0.5 = 0.5$$

Therefore $P(A' \cup B) = 0.5 + 0.4 - 0.4 = 0.5$

(ii)

$$P(A' \cap B') = P(A \cup B)' = 1 - P(A \cup B)$$

$$= 1 - 0.9 = 0.1$$

2. A and B are mutually exclusive events such that P(A)= 0.3, P(B) =0.5 . Find (i) $P(A \cup B)$ (ii) P(A') (iii) $P(A' \cap B')$

Solution

(i) $P(A \cap B) = 0$

$$P(A \cup B) = P(A) + P(B)$$

$$= 0.3 + 0.5 = 0.8$$

(ii) $P(A') = 1 - P(A) = 1 - 0.3 = 0.7$

(iii) $P(B) = P(A \cap B) + P(A' \cap B)$

$$\Rightarrow P(A' \cap B) = P(B) - P(A \cap B)$$

$$= 0.5 - 0 = 0.5$$

From $P(A') = P(A' \cap B') + P(A' \cap B)$

$$P(A' \cap B) = P(A') - P(A' \cap B) = 0.7 - 0.5 = 0.2$$

3. Given that A and B are mutually exclusive events and that $P(A) = \frac{2}{5}$ and $P(B) = \frac{1}{2}$. Find (i) $P(A \cup B)$ (ii) $P(A \cap B')$ (iii) $P(A' \cap B')$

Solution

(i) $P(A \cup B) = P(A) + P(B)$

$= \frac{2}{5} + \frac{1}{2} = \frac{9}{10}$

(ii)

From $P(A) = P(A \cap B') + P(A \cap B)$

$\Rightarrow P(A \cap B') = P(A) - P(A \cap B)$

$= \frac{2}{5} - 0 = \frac{2}{5}$

(iii)

From $P(B') = P(A \cap B') + P(A' \cap B')$

$P(A' \cap B') = P(B') - P(A \cap B')$

But $P(B') = 1 - P(B) = 1 - \frac{1}{2} = \frac{1}{2}$

$\Rightarrow P(A' \cap B') = \frac{1}{2} - \frac{2}{5} = \frac{1}{10}$

Independent events

Independent events are events such that the occurrence of one does not affect/influence the occurrence of the other. If A and B are independent events, then

$$P(A \cap B) = P(A) \times P(B)$$

Examples

1. A die is rolled twice. If event A is the first throw shows a six and event B is the second will show a six
   (a) Are the events A and B independent?
   (b) find P(A and B)

Solution

(a) The events are independent as the number obtained on the first throw does not affect the number obtained on the second throw

(b) $P(A) = \frac{1}{6}$ and $P(B) = \frac{1}{6}$

$P(A \cap B) = P(A) \times P(B)$

$= \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

2. Two events A and B are independent such that P(B) = 0.6 and $P(A \cup B) = 0.94$. Find,
   (i) P(A) (ii) $P(A \cap B)$

Solution

(i) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

But $P(A \cap B) = P(A) \times P(B) = 0.6P(A)$

Thus $0.94 = P(A) + 0.6 - 0.6P(A)$

$0.94 - 0.6 = P(A) - 0.6P(A)$

$0.34 = 0.4\,P(A)$

$\therefore P(A) = \frac{0.34}{0.4} = 0.85$

(ii)

$P(A \cap B) = P(A) \times P(B)$

$= 0.85 \times 0.6 = 0.51$

3. Two events A and B are independent events such that P(A) =0.40, P(B)= a and $P(A \cup B) = 0.70$. Find (i) $P(A \cup B)'$ (ii) the value of a (iii) $P(A \cap B)$ (iv) $P(A \cap B')$

Solution

(i) $P(A \cup B)' = 1 - P(A \cup B)$

$= 1 - 0.70 = 0.3$

(ii)

From $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Since A and B are independent, $P(A \cap B) = P(A) \times P(B) = 0.40 \times a = 0.4a$

$\Rightarrow 0.7 = 0.4 + a - 0.4a$

$0.7 - 0.4 = 0.6a$

$0.6a = 0.3 \Rightarrow a = \frac{0.3}{0.6} = 0.5$

(iii)

$P(A \cap B) = 0.4a = 0.4 \times 0.5 = 0.2$

(iv) Using the contingency table

$P(A) = P(A \cap B) + P(A \cap B')$

$\Rightarrow P(A \cap B') = P(A) - P(A \cap B)$

$= 0.4 - 0.2 = 0.2$

4. Given that A and B are events such that $P(A) = \frac{2}{3}$ and $P(B) = \frac{1}{5}$ .Find

(i) $P(A \cup B)$ (ii) $P(A \cap B')$ (iii) $P(A' \cap B')$ if A and B are independent events.

Solution

(i) From $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cap B) = P(A) \times P(B) = \frac{2}{3} \times \frac{1}{5} = \frac{2}{15}$

$P(A \cup B) = \frac{2}{3} + \frac{1}{5} - \frac{2}{15} = \frac{11}{15}$

(ii) from the contingency table

$P(A) = P(A \cap B) + P(A \cap B')$

$\Rightarrow P(A \cap B') = P(A) - P(A \cap B)$

$= \frac{2}{3} - \frac{2}{15} = \frac{8}{15}$

(iii) from $P(B') = P(A \cap B') + P(A' \cap B')$

$P(A' \cap B') = P(B') - P(A \cap B')$

$= \frac{4}{5} - \frac{8}{15} = \frac{4}{15}$

Conditional Probability

The conditional probability of an event B in relation to an event A is the probability that event B occurs after or given that A has already occurred. If A and B are events, then the conditional probability of A given B denoted as P(A/B) is given by;

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \quad provided\ P(B) \neq 0$$

Example 1

The probability that a regular scheduled plane departs on time is 0.83 and the probability that it arrives on time is 0.92. The probability that it departs on time and arrives on time is 0.78. Find the probability that the plane;

(i) arrives on time given that it departs on time

(ii) departs on time given that it arrives on time

Solution

Let D denote event plane departs on time $\Rightarrow P(D) = 0.83$

Let A denote event plane arrives on time $\Rightarrow P(A) = 0.92$

$$P(A \cap D) = P(D \cap A) = 0.78$$

(i) $P(^A/_D) = \frac{P(A \cap D)}{P(D)} = \frac{0.78}{0.83} = 0.94$

(ii) $P(^D/_A) = \frac{P(D \cap A)}{P(A)} = \frac{0.78}{0.92} = 0.85$

Example 2

Events A and B are such that $P(A) = \frac{1}{2}$ $P(B) = \frac{3}{8}$ $P(^A/_B) = \frac{7}{12}$. Find

(i) $P(A \cap B)$ (ii) $P(B/A')$]

Solution

(i) $P(^A/_B) = \frac{P(A \cap B)}{P(B)} = \frac{7}{12}$

$$P(A \cap B) = \frac{7}{12}P(B) = \frac{7}{12} \times \frac{3}{8} = \frac{7}{32}$$

(ii) $P(B/A') = \frac{P(B \cap A')}{P(A')}$

$P(A') = 1 - P(A) = 1 - \frac{1}{2} = \frac{1}{2}$

But $P(B \cap A') = P(A' \cap B) = P(B) - P(A \cap B)$

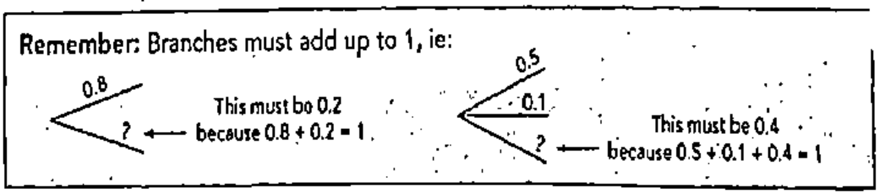
$$= \frac{3}{8} - \frac{7}{32} = \frac{5}{32}$$

$$P(B/A') = \frac{^5/_{32}}{^1/_2} = \frac{5}{32} \times \frac{2}{1} = \frac{5}{16}$$

Probability tree diagrams

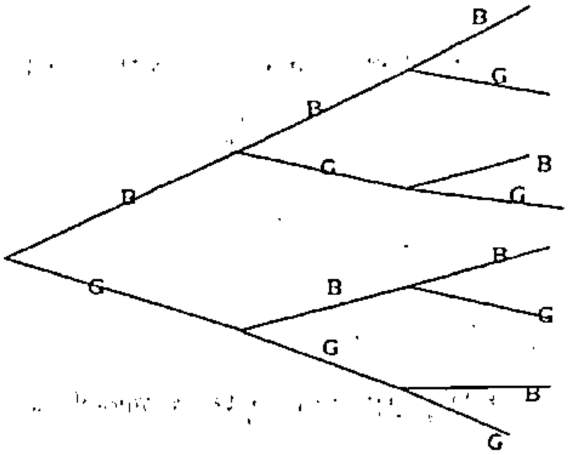
Tree diagrams can be used to obtain the possible outcomes of an experiment when the outcomes are not necessarily equally likely or generate a sample space.

When using tree diagrams, you always multiply along the branches to determine the probability of combined events.



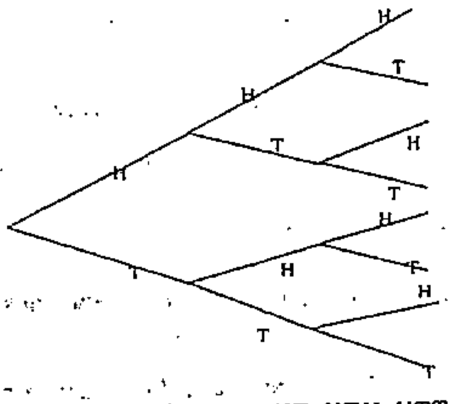
The tree diagram can be used to list the possible outcomes.

For example; if a family plans to have three children



The possible outcomes are $\{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG\}$

Now if a coin is tossed three times



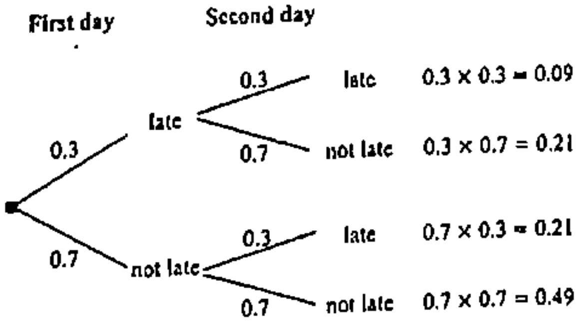
The possible outcomes are $\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

Examples

1. The probability that Jenny is late for school is 0.3. Find the probability that on two consecutive days, she is
(a) never late (b) late only once

Solution

The probability of being late is $1 - 0.3 = 0.7$



The probabilities in each set of branches are multiplied together to give the probability of that outcome

(a) The probability that Jenny is never late is given by the bottom set of branches and has a probability 0.49

(b) The probability that she is late race is given by the two middle sets of branches which both have a probability 0.21

So the probability that she is late once is given by

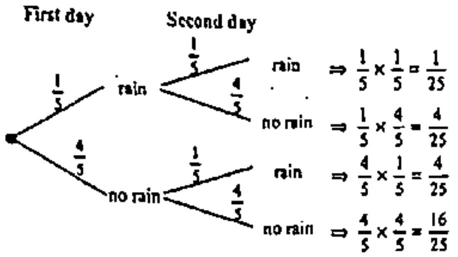
$$0.21 + 0.21 = 0.42$$

2. If the probability that it rains on any day is $\frac{1}{5}$, find the probability that
   (a) It rains on two consecutive days
   (b) It rains on only one of the two consecutive days

Solution

The tree diagram shows all the possible outcomes. The probability of each event can be placed on the appropriate branch of tree.

The probability of no rain is $1 - \frac{1}{5} = \frac{4}{5}$



(a) The probability that it rains on two consecutive days is given by the top set of branches and is $\frac{1}{25}$

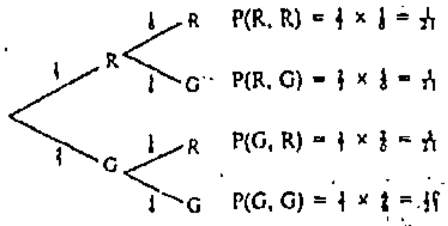
(b) There are two outcomes where there is rain on only one of the two days. These are rain – no rain, with a probability of $\frac{4}{25}$ and no rain – rain with a probability of $\frac{4}{25}$

The probability of rain on only one day is found by adding these two probabilities together;

$$\frac{4}{25} + \frac{4}{25} = \frac{8}{25}$$

3. A bag contains 7 discs, 2 of which are red and 5 are green. Two discs are removed at random without replacement and their colours are noted. Find the probability that the discs will be;
   (a) both red (b) different colours (c) the same colour

Solution



(a) P(both red) = $\frac{1}{21}$

(b) P(different) = P(R, G) + P(G,R)

$$= \frac{5}{21} + \frac{5}{21} = \frac{10}{21}$$

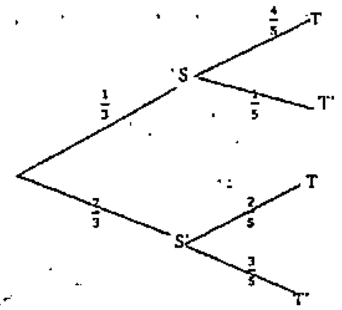
(c) P(same) = P(R,R) + P (G,G)

$= \frac{1}{21} + \frac{10}{21} = \frac{11}{21}$

4. The probability that it will be sunny tomorrow is $\frac{1}{3}$. If it is sunny, the probability that Vivianne plays tennis is $\frac{4}{5}$. If it is not sunny, the probability that she plays tennis is $\frac{2}{5}$. Find the probability that Vivianne plays tennis tomorrow.

Solution

Let S denote event sunny and T denote event Vivianne playing tennis

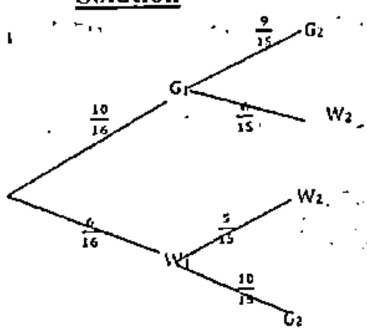


$P(T) = P(S \cap T) + P(S' \cap T)$

$= \frac{1}{3} \times \frac{4}{5} + \frac{2}{3} \times \frac{2}{5} = \frac{4}{15} + \frac{4}{15} = \frac{8}{15}$

5. A box contains 10 green and 6 white marbles. A marble is chosen at random, its colour noted and it is not replaced. This is repeated once more. What is the probability that the marbles chosen at random are of the same colour?

Solution



P(marble of the same colour) = P($W_1 \cap W_2$) + P ($G_1 \cap G_2$)

$= \frac{6}{16} \times \frac{5}{15} + \frac{10}{16} \times \frac{9}{15} = \frac{1}{8} + \frac{3}{8} = \frac{4}{8} = \frac{1}{2}$

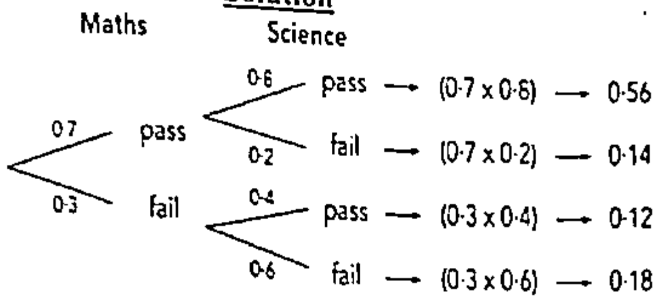
6. The probability of a person passing maths is 0.7. The probability of a person who passed maths, passing science is 0.8. The probability of a person who has failed maths, passing science is 0.4. Find the probability of a person;

(a) passing maths and science

(b) failing maths and science

(c) passing one subject and failing the other

Solution



(a) P(passing maths and science) = 0.56

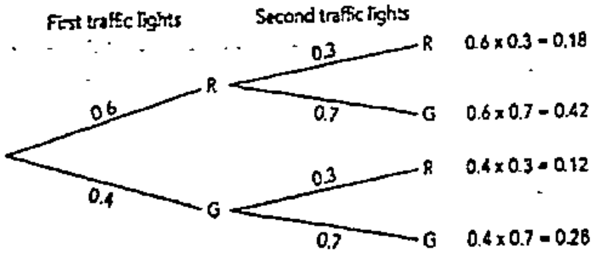
(b) P(failing maths and science) = 0.18

(c) P(passing one subject and failing the other)
= P(pass math and fail science) + P(fail math and pass science)
$= 0.7 \times 0.2 + 0.3 \times 0.4 = 0.14 + 0.12 = 0.26$

7. A car driver passes through two sets of traffic lights on his way to work. The lights can either be red or green. The probability of red at the first lights is 0.6. The probability of red at the second lights is 0.3. Find the probability that;
   (a) both lights are red
   (b) both lights are green
   (c) one set of lights is and one is green
   (d) at least one set of lights is red

Solution

This problem is independent probability i.e. the colour of the second traffic lights is not affected by the colour of the first set.



Solution

(a) $P(R, R) = 0.6 \times 0.3 = 0.18$

(b) $P(G, G) = 0.4 \times 0.7 = 0.28$

(c) Red and green or green and red

$P(R \text{ and } G) + P(G \text{ and } R) = 0.6 \times 0.7 + 0.4 \times 0.3 = 0.42 + 0.12 = 0.54$

(d) Red and red or red and green or green and red

$P(R \text{ and } R) + P(R \text{ and } G) + P(G \text{ and } R) = 0.18 + 0.42 + 0.12 = 0.72$

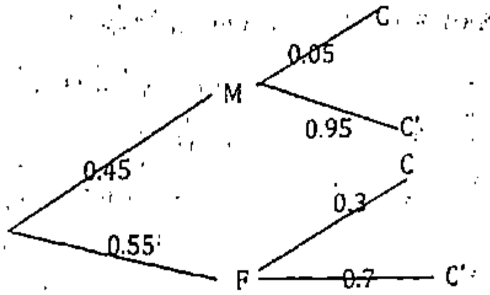
Alternatively;

P(at least red) = 1 − P(green and green) = 1 − 0.28 = 0.72

8. The proportion of female students at Makerere University is 55%. If 30% of the male students and 5% of the female students study computer. What is the probability that a computer student chosen at random is a female?

Solution

Let F denote Female student and Male student and let C event student studying computer.



$P(F/C) = \frac{P(F \cap C)}{P(C)}$

$P(C) = P(F \cap C) + P(M \cap C)$

$= 0.55 \times 0.05 + 0.45 \times 0.3 = 0.0275 + 0.135 = 0.1625$

$P(F/C) = \frac{0.0275}{0.1625} = 0.1692$

Trial questions

1. If $P(A) = \frac{3}{5}$ and $P(B) = \frac{1}{3}$, find (a) $P(A \cup B)$ if A and B are mutually exclusive events
   (b) $P(A \cap B)$ if A and B are independent events [Ans: (a) $\frac{14}{15}$ (b) $\frac{1}{5}$]
2. Given that $P(A) = \frac{2}{3}$, $P(B) = \frac{1}{2}$, $P(A \cap B) = \frac{1}{3}$ and $P(A \cup B) = \frac{5}{6}$. State whether each of the following statements is true or false
   (a) A and B are mutually exclusive events
   (b) A and B are independent events [Ans: (a) False (b) True]
3. If A and B are independent events such that $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{4}$, find
   (a) $P(A \cap B)$ (b) $P(A \cup B)$ [Ans: (a) $\frac{1}{8}$ (b) $\frac{5}{8}$]
4. A bag contains 9 discs, 2 of which are green and 7 yellow. Two discs are removed at random in succession, without replacement. Find the probability that the discs will
   (a) Both be green (b) be of the same colour (c) be of different colours
   [Ans: (a) $\frac{1}{36}$ (b) $\frac{11}{18}$ (c) $\frac{7}{18}$]
5. If A and B are two events such that $P(A) = \frac{5}{8}$ and $P(B/A) = \frac{3}{7}$, find $P(A \cap B)$
   [Ans: $\frac{15}{56}$]
6. If A and B are two events such that $P(A) = \frac{2}{5}$ and $P(A \cap B) = \frac{3}{10}$, find $P(B/A)$
   [Ans: $\frac{3}{4}$]
7. Two events A and B are such that $P(A) = 0.2$, $P(A' \cap B) = 0.22$, $P(A \cap B) = 0.18$.
Evaluate (i) $P(A \cap B')$ (ii) $P(A/B)$ [Ans: (i) 0.02 (ii) 0.45]

8. Two events A and B are such that $P(A) = \frac{1}{2}$, $P\left(\frac{A}{B'}\right) = \frac{2}{3}$, $P\left(\frac{A}{B}\right) = \frac{3}{7}$, *where* B' is the event B does not occur. Find (i) $P(A \cap B)$ (ii)$P(A \cup B)$ (iii)$P(B)$ (iv)$P(B/A)$.
[Ans: (i)$\frac{3}{10}$ (ii) $\frac{9}{10}$ (iii)$\frac{7}{10}$ (iv)$\frac{3}{5}$]
9. A bag contains 4 white balls, 3 black balls and 1 red ball. Two balls are picked at random in succession without replacement. Find the probability that
   (i) both are of the same colour
   (ii) at least one black ball is picked [Ans: (i)$\frac{9}{28}$ (ii)$\frac{9}{14}$]
10. A box contains 7 red balls and 6 blue balls. Two balls are selected at random without replacement. Find the probability that;
   (i) they are of the same colour
   (ii) at least one is blue [Ans: (i)$\frac{6}{13}$ (ii)$\frac{19}{26}$]
11. Two boxes P and Q contain white and brown cards. P contains 6 white cards and 4 brown cards. Q contains 2 white cards and 3 brown cards. A box is selected at random and a card selected. Find the probability that;
   (i) a brown card selected
   (ii) box Q is selected given that the card is white [Ans: (i)$\frac{2}{5}$ (ii) $\frac{1}{2}$]
12. Bag A contains 3 green and 2 red balls. Bag B contains 4 green and 3 red balls. If a ball is picked at random from a bag chosen at random, find the probability that a red ball is (i) picked (ii) not picked [Ans: (i)$\frac{29}{70}$ (ii) $\frac{41}{70}$]
13. Two independent events A and B are such that P(A) = 0.40, P(B) = a, P(A∪ B) = 0.70. Find (i) $P(A \cup B)'$ (ii)*the value of a* (iii)$P(A \cap B)$ (iv)$P(A \cap B')$
[Ans: (i) 0.3 (ii) 0.5 (iii) 0.2 (iv) 0.2]
14. Given that A and B are two events such that P(A) = 0.5, P(B) =0.7 and $P(A \cup B) = 0.8$. Find (i) $P(A \cap B)$ (ii)$P(A \cap B')$ [*Ans*: (i)0.4 (ii) 0.1]
15. Two events A and B are independent such that P(A) =0.2 and $P(A \cup B) = 0.8$. Find
   (i) P(B) (ii) $P(A' \cup B')$ [Ans: (i)$\frac{3}{4}$ (ii) $\frac{17}{20}$]
16. In a school canteen, the probability that a child has chips with their meal is 0.9 and the probability that they have baked beans is 0.6. Find the probability that a child;
   (i) has both chips and beans
   (ii) has chips but not beans
   (iii) neither chips nor beans [Ans: (i) 0.54 (ii) 0.36 (iii) 0.04]
17. Paul travels to London on an early train. The probability that he arrives late is $\frac{1}{10}$. He catches the train on two consecutive days. What is the probability that he arrives;
   (a) On time on both days (b) on time at least one day (c) late on both days
[Ans: (a)$\frac{81}{100}$ (b) $\frac{99}{100}$ (c)$\frac{1}{100}$]
18. When Jackie's phone rings, the probability that the call is for her is $\frac{3}{4}$
   (a) What is the probability that the call is not for Jackie?
   (b) Find the probabilities that;

(i) Both calls are for Jackie (ii) only one call is for Jackie (iii) neither call is for Jackie.

[Ans: (a) $\frac{1}{4}$ (b) (i) $\frac{9}{16}$ (ii) $\frac{3}{8}$ (iii) $\frac{1}{16}$]

19. John has 8 red socks and six white socks all mixed up in his sock drawer. He takes two socks at random in succession from the drawer without replacement.
 (a) If the first sock that John takes is red, what is the probability that the second sock will also be red?
 (b) What is the probability that John will take two socks of the same colour?

 [Ans: (a) $\frac{7}{13}$ (b) $\frac{43}{91}$]

20. A game contains two tetrahedral dice which have faces numbered 1 to 4 . The two dice are thrown and the total score is noted. Find the probability;
 (i) that a score of 3 is obtained
 (ii) getting a score greater than 4
 (iii) which score is most likely? [Ans: (i) $\frac{1}{8}$ (ii) $\frac{5}{8}$ (iii) 5]

21. In a certain city suburb 30% of the residents read New Vision paper only, 55% read both New Vision and Monitor. If 10% do not read any paper, find the probability that a person picked at random reads;
 (i) Monitor
 (ii) Monitor or New Vision but not both [Ans: (i) 0.6 (ii) 0.9]

22. On a route to school, a bus must pass through two sets of traffic lights. The probability that the bus has to stop at a set of lights is 0.6. What is the probability that the bus;
 (i) does not have to stop at a set of traffic lights?
 (ii) gets to school without having to stop at a traffic light?
 (iii) stops at both sets of traffic lights?
 (iv) stops at one set of traffic lights? [Ans: (i) 0.4 (ii) 0.16 (iii) 0.36 (iv) 0.48]

23. On average, Maurice comes to tea on 2 days out of every 5. If comes to tea , the probability that we have jam tarts is 0.7. If he does not come for tea, the probability that we have jam tarts is 0.4. What is the probability that we have jam tarts for tea tomorrow?

 [Ans: 0.52]

24. A die is thrown twice. Find the probability that;
 (a) two odd numbers are obtained
 (b) the same two numbers are obtained [Ans: (a) $\frac{1}{4}$ (b) $\frac{1}{6}$]

25. Given that A and B are mutually exclusive events such that P(A) = 0.4, $P(A \cup B) = 0.7$ .Find (i) $P(A' \cap B')$ (ii) $P(A' \cup B)$ [Ans: (i) 0.3 (ii) 0.6]

CHAPTER 16: PERMUTATIONS AND COMBINATIONS

Permutations

A permutation is an ordered arrangement of a number of items

For example suppose a photographer must arrange three girls Anne (A), Banks (B) and Catherine (C) in a row for a photograph. He can do this in six possible ways

ABC, ACB, BAC, BCA, CAB, CBA

Each arrangement is a possible permutation of the girls A, B and C and so there are six permutations altogether.

Now if there are four different books on a shelf. In how many ways could they be arranged in order?

If we label the books A, B, C and D for convenience, writing the arrangement in which A comes first;

A B C D   A C D B
A B D C   A D B C
A C B D   A D C B   i.e. 6 arrangements

If we take book B first, there will be six arrangements as well (try it out); the same applies to book C and D coming first. Here there is a total of 24 arrangements of the four books

Alternatively, if we have four boxes into each of which one book can be put

Box 1	Box 2	Box 3	Box 4
Any one of 4	Any one of 3	Any one of 2	No choice

There are four ways of filling the first box and three ways of filling the second since three books are left after filling the first and so on

There are $4 \times 3 \times 2$ ways of filling the first three boxes and for the fourth, it is only one way since one book is left

Altogether, they become $4 \times 3 \times 2 \times 1 = 24$ ways

Example 1

In how many ways can 3 books be arranged in order if 7 different books are available?

Solution

Box 1	Box 2	Box 3
7 ways	6 ways	5 ways

Here the number of arrangements $= 7 \times 6 \times 5 = 210$

Example 2

In how many ways can the 1st, 2nd and 3rd prizes be awarded in a race if there are 10 competitors?

Solution

The 1st prize can be awarded in 10 ways, the 2nd in 9 ways and the third in 8 ways

Total number of ways $= 10 \times 9 \times 8 = 720$ ways

Factorial notation

Let n be an integer, then the continued product of the 1st n natural numbers is called n factorial denoted by n! It is very important to note that $0! = 1$

Hence $n! = n(n-1)(n-2)(n-3)\ldots\ldots 3 \times 2 \times 1$

i.e $5! = 5 \times 4 \times 3 \times 2 \times 1$, $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

Thus the number of ways of arranging n unlike objects in a row is given by $n!$

Example 1

Find the number of ways of arranging the letters of the word **THURSDAY**

Solution

THURSDAY has 8 different/unlike letters

Number of ways of arranging the letters = 8! = 40320 ways

Example 2

Now consider the already looked at example in the introduction of this topic, we can see that the four books can be arranged in $4! = 4 \times 3 \times 2 \times 1 = 24\ ways$

Example 3

Evaluate (a) $\frac{6!}{2\times4!}$ (b) $\frac{7!}{4!\times2!}$

Solution

(a) $\frac{6!}{2\times4!} = \frac{6\times5\times4\times3\times2\times1}{2\times(4\times3\times2\times1)} = 15$

(b) $\frac{7!}{4!\times2!} = \frac{7\times6\times5\times4\times3\times2\times1}{4\times3\times2\times1\times(2\times1)} = 105$

Permutations of objects selected from a group

Suppose we wish to arrange r objects from n unlike objects, we usually say that the number of permutations of r objects selected from n unlike objects is ^nP_r

$$^nP_r = \frac{n!}{(n-r)!}$$

Example 1

In how many ways can the letters of the word **MEASURING** be arranged or permuted?

Solution

Number of letters = 9.

So we are arranging 9 letters out of 9

$$^9P_9 = \frac{9!}{(9-9)!} = \frac{9!}{0!} = 9! = 362{,}880\ ways$$

Example 2

In how many ways can the 1st, 2nd and 3rd prizes be awarded in a race if there are 10 competitors?

Solution

We are arranging 3 competitors out of 10 thus

$^{10}P_3 = \frac{10!}{(10-3)!} = \frac{10!}{7!} = 720\ ways$ (compare with the method used earlier)

Example 3

Find the number of arrangements using any of the three letters of the word **CHEMISTRY**?

Solution

CHEMISTRY has 9 letters so arranging 3 letters out of 9 gives;

$^9P_3 = \frac{9!}{(9-3)!} = \frac{9!}{6!} = 504\ ways$

Example 4

Find the number of arrangements using four letters of the word **SPHERICAL?**

Solution

SPHERICAL has 9 letters so arranging 4 letters out of 9 gives;

$^9P_4 = \frac{9!}{(9-4)!} = \frac{9!}{5!} = 3024\ ways$

Arrangement of n objects selected from a group with like objects

If we wish to arrange n objects with p like objects, q like objects and r like objects, we can obtain it as follows;

$$\text{Number of arrangements} = \frac{n!}{p!q!r!}$$

Example 1

Find the number of ways of arranging letters of the word **BIOLOGY**

Solution

BIOLOGY has 7 letters but two letters are the same i.e. 2O's

Number of ways $= \frac{7!}{2!} = 2520$

Example 2

Find the number of ways of arranging the letters of the word **MESSAGE**

Solution

MESSAGE has 7 letters with 2S's and 2E's

Number of ways $= \frac{7!}{2!\times 2!} = 1260$

Example 3

In how many ways can the letters of the word **MATHEMATICS** be arranged in a row?

Solution

MATHEMATICS has 11 letters with 2M's, 2A's and 2T's

Number of ways $= \frac{11!}{2!2!2!} = 6652800$

Example 4

How many words can be performed from the letters of the word **DAUGHTER** so that

(i) The vowels always come together

(ii) The vowels are never together

Solution

(i) The given word contains 8 different letters. When the vowels AUE are always, they can be treated as an entity i.e DGHTR (AUE)

There are six letters which can be arranged = 6! =720

But the three vowels can also be arranged in 3! = 6

Total number of ways $= 720 \times 6 = 4320$

(ii) The total number of ways of arranging the word DAUGHTER = 8! = 40320

$$\begin{pmatrix}\text{number of ways when the}\\ \text{vowels are never together}\end{pmatrix} = \begin{pmatrix}\text{total number}\\ \text{of ways}\end{pmatrix} - \begin{pmatrix}\text{number of ways when the}\\ \text{vowels are always together}\end{pmatrix}$$

$$= 40320 - 4320 = 36000$$

Example 5

Find the number of ways in which the letters of the word **SHALLOW** can be arranged (a) if the two L s must not come together (b) if the two L s must always be together

Solution

Leaving out the two L s, the letter **SHAOW** can be arranged in 5! Ways

$$\uparrow\ S\ \uparrow\ H\ \uparrow\ A\ \uparrow\ O\ \uparrow\ W\ \uparrow$$

(a) With each of these ways the first L can be inserted in any one of the places. When this is done, there are then 5 possible places for the second L not next to the first. Hence the total number of arrangements with the two L s separated is $5! \times 6 \times 5$ provided the L s can be distinguished. They cannot and so the number of arrangements is

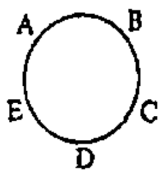
$\frac{5!\times6\times5}{2} = 5! \times 15 = 1800$

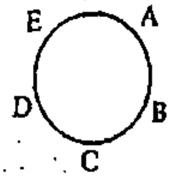
(b) In this case take the two L s (LL) as one object. There are then six places for it in each of the 5! arrangements of the letters SHAOW. Hence the number of arrangements is

$6 \times 5! = 6! = 720$

Circular arrangements

With circular permutations, it is the relative positions of the items being arranged which is important. For example A B C D E is a different arrangement from E A B C D in a row,

but 

is not a different arrangement from 

When arranging in a circle, we always arrange relative to one object i.e we fix one object and arrange the remaining objects relative to it.

Therefore the number of arrangements of n unlike things in a circle will be $(n - 1)!$

In the cases where clockwise and anticlockwise arrangements are not considered to be different, this reduces to $\frac{1}{2}(n-1)!$

Examples

1. Five girls Vivianne, Pearl, Praise, Sonia and Joan are to be seated at a circular table. In how many ways can this be done?

Solution

If one girl is fixed, 4 girls remain to be arranged

Therefore, number of ways of arranging the five girls = $4! = 24\ ways$

2. Find the number of ways in which ten boys can be arranged on a table

Solution

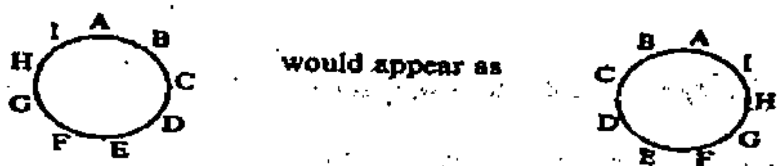
One of the boys must be fixed and then we arrange the remaining nine

Number of arrangements = $9! = 362880$

3. Nine beads, all of different colours are to be arranged on a circular wire. Two arrangements are not considered to be different if they appear the same when the ring is turned over. How many different arrangements are possible?

Solution

When viewed from one side, these arrangements are only different in that one is a clockwise arrangement and the other is anticlockwise. For example the arrangement below;



If one bead is fixed, there are $(9-1)!$ ways of arranging the remaining beads relative to the fixed one i.e. $8!$ ways. But half of these arrangements will appear the same as the other half when the ring is turned over, because for every clockwise arrangement there is a similar anticlockwise arrangement.

Hence;

Number of arrangements $= \frac{1}{2}(8!) = 20160\ ways$

4. In how many ways can five people, Smith, James, Clark, Brown and White be arranged around a circular table if
   (a) Smith must sit next to brown
   (b) Smith must not sit next to brown

Solution

(a) Since Smith and Brown must sit next to each other, Consider these two bonded together as one person.
There are now 4 people to sit
Fixing one of them, the remaining 3 can be sited in $3 \times 2 \times 1 = 6$ ways relative to the one that was fixed.
In each of these arrangements, Brown and smith are seated together in a particular way.
Brown and smith could now change their seats giving another 6 ways of arranging the 5 people
Total number of arrangements $= 2 \times 6 = 12$ ways

(b) If Smith is not to sit next to brown, then this situation is mutually exclusive with situation in (a) above

Hence $\begin{pmatrix} number\ of\ ways\ when\ Smith \\ does\ not\ sit\ next\ to\ Brown \end{pmatrix} = \begin{pmatrix} total\ number \\ of\ ways \end{pmatrix} - \begin{pmatrix} number\ of\ ways\ in\ which \\ Smith\ sits\ next\ to\ Brown \end{pmatrix}$

Total number of arrangements of 5 people on a circular table $= (5-1)! = 4! = 24$

Required number of arrangements $= 24 - 12 = 12$

Thus the number of arrangements in which Smith does not sit next to Brown is 12

COMBINATIONS

A combination is the number of ways of selecting a group of objects from a given set of objects e.g. an A-Level subject combinations such as HEG, PCB, PCM, MEG, etc. In making a selection from a number of items, only the contents of the group selected are important, not the order in which the items are selected i.e GEH, GHE, HGE, EGH, EHG are all the same as HEG

The number of possible combinations of n different objects, taken r at a time, is given by ^nC_r also written as $\binom{n}{r}$ where

$$^nC_r = \frac{n!}{(n-r)!r!}$$

Examples

1. How many selections of 6 letters can be made from the 9 letters A, B, C, D, E, F, G, H, I?

Solution

The number of selections is ^nC_r where r is the number of things selected from a group of n

Hence for this case n = 9 and r = 6

$$^9C_6 = \frac{9!}{(9-6)!6!} = \frac{9!}{3!6!} = 84$$

There are 84 selections of 6 letters which can be made from the 9 letters

2. In how many ways can 4 boys be chosen from 6?

Solution

The number of selections = $^6C_4 = \frac{6!}{(6-4)!4!} = \frac{6!}{2!4!} = 15$

3. A committee of 2 men and 3 women is to be chosen from 5 men and 4 women. How many different committees can be formed?

Solution

The two men can be selected in $^5C_2 = 10$ ways

The three women can be chosen in $^4C_3 = 4$ ways

The possible committees are $10 \times 4 = 40$

Note: 10 is multiplied by 4 since the choice of the men and the choice of the women are independent operations.

4. How many different committees, each consisting of 3 boys and 2 girls can be chosen from 7 boys and 2 girls?

Solution

Number of ways of choosing 3 boys from 7 = $^7C_3 = 35$

Number of ways of choosing 2 girls from 5 = $^5C_2 = 10$

Number of committees which can be chosen = $35 \times 10 = 350$

5. A group consists of 4 boys and 7 girls. In how many ways can a team of five be selected if it is to contain

(a) no boys (b) 2 boys and 3 girls (c) at least 3 boys ?

Solution

(a) No boys are selected, so the team is chosen from the 7 girls
number of ways of choosing 5 girls from 7 = ${}^7C_5 = 21$

(b) 2 boys can be chosen from 4 in ${}^4C_2 = 6$ ways
3 girls can be chosen from 7 in ${}^7C_3 = 35$ ways
Number of teams = $6 \times 35 = 210$

(c) If the team is to have at least 3 boys, then there must be either 3 or 4 boys
Number of teams with 3 boys and 2 girls = ${}^4C_3 \times {}^7C_2 = 84$
Number of teams with 4 boys and 1 girl = ${}^4C_4 \times {}^7C_1 = 7$
These are mutually exclusive events,
so number of teams with at least 3 boys = 84 + 7 = 91

6. A group consists of 6 men and 5 women. If a committee of five members is to be formed, in how many ways can this be done if it must contain
(a) At least one woman (b) not more than three men ?

Solution

(a) If the committee is to have at least one woman, then it can have 1, 2, 3, 4 or 5 women.
With 1 woman and 4 men, number of ways = ${}^5C_1 \times {}^6C_4 = 75$
With 2 women and 3 men, number of ways = ${}^5C_2 \times {}^6C_3 = 200$
With 3 women and 2 men, number of ways = ${}^5C_3 \times {}^6C_2 = 150$
With 4 women and 1 man, number of ways = ${}^5C_4 \times {}^6C_1 = 30$
With 5 women and no man, number of ways = ${}^5C_5 \times {}^6C_0 = 1$
Total number of ways = 75 + 200 + 150 + 30 + 1 = 456 ways

(b) If the committee is not to have more than 3 men, then it can have 3, 2, 1 or no man
With 3 men and 2 women, number of ways = ${}^6C_3 \times {}^5C_2 = 200$
With 2 men and 3 women, number of ways = ${}^6C_2 \times {}^5C_3 = 150$
With 1 man and 4 women, number of ways = ${}^6C_1 \times {}^5C_4 = 30$
With no man and 5 women, number of ways = ${}^6C_0 \times {}^5C_5 = 1$
Total number of ways = 200 + 150 + 30 + 1 = 381

Trial questions

1. Evaluate without using a calculator
(a) $\frac{8!}{6!}$ (b) $\frac{9!}{3\times5!}$ (c) $\frac{5!\times4!}{6!}$ [Ans: (a) 56 (b) 1008 (c) 4]
2. In how many ways can a group of ten children be arranged in a line? [Ans: 10!]
3. Find the number of permutations of two different letters taken from the letters A, B, C, D, E, F [Ans: 30]
4. In how many ways can six books be arranged on a shelf when the books are selected from ten different books? [Ans: 151200]
5. How many code words each consisting of five different letters, can be formed from the letters A, B, C, D, E, F, G and H ? [Ans: 6720]

6. In how many ways can the letters of the word **MEDIAN** be arranged? [Ans: 720]

7. How many different teams of 7 players can be chosen from 10 girls? [Ans: 120]

8. Three students are to be promoted from a particular class. If five students are under consideration for promotion, in how many ways can the group to be promoted be chosen? [Ans: 10]

9. A librarian has to make a selection of 5 newspapers and 7 magazines from the 8 newspapers and 9 magazines which are available. In how many ways can she make her selection? [Ans: 2016]

10. Find the number of different selections of 3 letters from the word **METHOD**

[Ans: 20]

11. A group consists of 5 boys and 8 girls. In how many ways can a team of four be chosen, if it contains (a) no girls (b) not more than one girl (c) at least two boys? [Ans: (a) 5 (b) 85 (c) 365]

12. In how many ways can a committee of five people be selected from 7 men and 3 women if it must contain (a) 3 men and 2 women (b) 3 women and 2 men (c) at least 1 woman? [Ans: (a) 105 (b) 21 (c) 231]

13. In how many ways can a committee of 7 people be selected from 4 men and 6 women if the committee must have at least 4 women on it? [Ans : 100]

14. A group consists of 5 boys and 8 girls. In how many ways can a team of five be chosen if it is to contain (a) no girls (b) no boys (c) at least one boy ?

[Ans: (a) 1 (b) 56 (c) 1231]

15. A tennis club has to select two mixed double pairs from a given group of 5 men and 4 women. In how many ways can this be done? [Ans: 120]

16. A circular ring has ten different beads. In how many ways can the beads be arranged along the ring? [Ans: 181440]

17. Find the number of arrangements of the letters of the word **COMMITTEE**

[Ans: 60480]

18. A combination of five vehicles is to be chosen from six saloon cars and seven vans. If at least three saloon cars must be chosen. In how many ways can the combination be done?

[Ans: 531]

19. Determine the number of different arrangements of the letters in the word **ARRANGE**?

[Ans: 1260]

20. In how many different ways can the letters of the word **REVERSES** be arranged?

[Ans: 1680]

21. There are 6 women and 4 men wedding preparation meeting. 5 people are chosen at random to constitute an ushering committee. Find the number of committees that can be formed containing at least two women. [Ans: 246]

22. A committee of five people is to be selected from 7 women and 4 men. In how many ways can the committee be chosen if there has to be at least a man on the committee?

[Ans: 441]

23. Evaluate the following with out using a calculator

(i) ${}^{7}C_{4}$ (ii) ${}^{7}P_{4}$

[Ans: (i) 35 (ii) 840]

24. How many possible committees of 5 members can be formed from 6 boys and 5 girls, if there must be at least a boy and a girl on each committee formed [Ans: 455]

CHAPTER 17: RANDOM AND CONTINUOUS VARIABLES

When carrying out an experiment, variables are used to describe the event. A variable in this case can be defined as a characteristic that can assume different values. Letters of the alphabet such as X, Y, or Z can be used to represent variables. Since the variables are associated with probability, they are called random variables. Random variables may be either discrete or continuous. A discrete random variable is the variable that has values that can be counted while a continuous is one that has uncountable domain.

DISCRETE RANDOM VARIABLES

When a variable is discrete, it is possible to specify or describe all its possible numerical values, for example;

- The number of females in a group of four students; the possible values are 0, 1, 2, 3, 4
- The number of heads obtained when a coin is thrown two times; the possible values are 0, 1, 2.
- The number of boys possible if a family plans to have three children is 0, 1, 2, 3, or 4.

Consider this situation:

By mistake, three faulty fuses are put into a box containing two good fuses are put into a box containing two good fuses. The faulty and good fuses become mixed up making a total of five fuses and indistinguishable by sight. You choose to take two fuses from the box. What is the probability that you take

(a) No faulty fuses

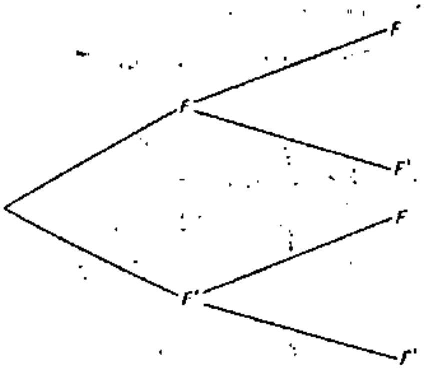
(b) One faulty fuse

(c) Two faulty fuses ?

Solution

It is possible to show the outcomes and probabilities on a tree diagram

Let event F denote faulty fuse and F ' denote not faulty



	Probability	*Outcome*
	$P(F,F) = \frac{3}{5} \times \frac{2}{4} = 0.3$	2 faulty fuses
	$P(F,F') = \frac{3}{5} \times \frac{2}{4} = 0.3$	1 faulty fuse
	$P(F',F) = \frac{2}{5} \times \frac{3}{4} = 0.3$	1 faulty fuse
	$P(F',F') = \frac{2}{5} \times \frac{1}{4} = 0.1$	0 faulty fuses

(a) P(no faulty fuses) = 0.1

(b) P(one faulty fuse) = 0.3 + 0.3 = 0.6

(c) P(two faulty fuses) = 0.3

The variable being considered here is "the number of faulty fuses" and is denoted by X

The values that X can take are 0, 1 or 2

The probability that there are no faulty fuses, i.e. the probability that the variable X takes on the value 0, can be written as P(X=0), so $P(X = 0) = 0.1$

Similarly P(X =1) = 0.6 and P(X = 2) = 0.3

When defining variables, the variable is usually denoted by a capital letter (X, Y, R, etc.) and a particular value that variable takes by a small letter (x, y , r etc.), so that P(X = x) means "the probability that the variable X takes the value x "

The probability distribution for x can be summarized in the table below

x	0	1	2
$P(X = x)$	0.1	0.6	0.3

If the sum of the probabilities is 1, the variable is said to be random

In this example; $P(X = 0) + P(X = 1) + P(X = 2) = 0.1 + 0.6 + 0.3 = 1$

So X is a discrete random variable.

For a discrete random variable, the sum of the probabilities is 1,

i.e. $\sum_{all\,x} P(X = x) = 1$

also $P(X = x) \geq 0 \; for\; all\; values\; of\; x$

The function responsible for allocating probabilities, $P(X = x)$is known as the probability density function of X, sometimes abbreviated as p.d.f of X. The probability density function can either list the probabilities individually or summarize them in a formula

Examples

1. The discrete random variable X has the following probability distribution

x	1	2	3	4	5
$P(X = x)$	0.2	0.25	0.4	a	0.05

(a) Find the value of a

(b) Find (i) P(1≤ X ≤ 3) (ii) P (X > 2) (iii) P (2 < X < 5) (iv) the mode

Solution

(a) Using the property $\sum_{all\,x} P(X = x) = 1$

$0.2 + 0.25 + 0.4 + a + 0.05 = 1$

$0.9 + a = 1$

$a = 0.1$

(b) (i) $P(1 \leq X \leq 3) = P(X = 1) + P(X = 2) + P(X = 3)$

= 0.2 + 0.25 + 0.4 = 0.85

(ii) $P(X > 2) = P(X = 3) + P(X = 4) + P(X = 5)$

= 0.4 + a + 0.05 = 0.4 + 0.1 + 0.05 = 0.55

(iii) $P(2 < X < 5) = P(X = 3) + P(X = 4)$

= 0.4 + a = 0.4 + 0.1 = 0.5

(iv) The mode is the value of x with the highest probability. The highest probability in this case is 0.4 hence the mode is 3

2. The p.d.f of a discrete random variable X is given by $P(X = x) = kx^2 \; for\; x = 0, 1, 2, 3, 4$. Given that k is a constant, find the value of k.

Solution

By drawing the table, it would help us write out the probability distribution of X

x	0	1	2	3	4
$P(X = x)$	0	k	4k	9k	16k

Since X is a discrete random variable, $\sum_{all\,x} P(X = x) = 1$

So $0 + k + 4k + 9k + 16k = 1$

$30k = 1 \Rightarrow k = \frac{1}{30}$

3. Suppose that a coin is tossed twice so that the sample space S = {HH, HT, TH, TT}. If X represents the number of heads that come up, find the probability function corresponding to the random variable X

Solution

$P(HH) = \frac{1}{4}$, $P(HT) = \frac{1}{4}$, $(TH) = \frac{1}{4}, P(TT) = \frac{1}{4}$

$P(X = 0) = P(TT) = \frac{1}{4}$

$P(X = 1) = (HT) + P(TH) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$P(X = 2) = P(HH) = \frac{1}{4}$

The probability function is thus given in the table below

x	0	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Expectation of X, E(X)

E(X) is read as 'E of X' and it gives an average or typical value of X, known as the expected value or expectation of X. This is comparable with the mean in descriptive statistics.

The expectation of X (expected value or mean), written as E(X) is given by;

$$E(X) = \sum_{all\,x} xP(X = x)$$

The symbol μ, pronounced 'mew' is often used for the expectation, where $\mu = E(X)$

Examples

1. A random variable X has the following probability distribution

x	-2	-1	0	1	2
$P(X = x)$	0.3	0.1	0.15	0.4	0.05

Find the expectation E(X)

Solution

x	-2	-1	0	1	2
$P(X = x)$	0.3	0.1	0.15	0.4	0.05
$xP(X = x)$	-0.6	-0.1	0	0.4	0.1

$$E(X) = \sum_{all\,x} xP(X = x)$$

$= -0.6 + -0.1 + 0 + 0.4 + 0.1 = -0.2$

2. X is the number of heads obtained when two coins are tossed. Find the expected number of heads.

Solution

$P(HH) = \frac{1}{4}$, $P(HT) = \frac{1}{4}$, $(TH) = \frac{1}{4}, P(TT) = \frac{1}{4}$

$P(X = 0) = P(TT) = \frac{1}{4}$

$P(X=1) = (HT) + P(TH) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$P(X=2) = P(HH) = \frac{1}{4}$

x	0	1	2
$P(X=x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
$xP(X=x)$	0	$\frac{1}{2}$	$\frac{1}{2}$

$E(X) = 0 + \frac{1}{2} + \frac{1}{2} = 1$

3. X is the random variable 'the number of likely boys obtained' for the family that plans to have three children. Find E(X)

Solution

The possible outcomes are {BBB, BBG, BGB, BGG,GBB, GBG, GGB, GGG}

x	0	1	2	3
$P(X=x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$
$xP(X=x)$	0	$\frac{3}{8}$	$\frac{6}{8}$	$\frac{3}{8}$

$E(X) = \frac{3}{8} + \frac{6}{8} + \frac{3}{8} = \frac{12}{8} = 1.5$

4. Two fair tetrahedral dice whose faces are numbered 1, 2, 3 and 4 are thrown at the same time. The score is the sum of the numbers which show up on the faces of the dice.
   (i) Construct a probability distribution table for the scores
   (ii) Calculate the expected score for the throw

Solution

Table of outcomes

die 1	die 2: 1	2	3	4
1	1, 1	1, 2	1, 3	1, 4
2	2, 1	2, 2	2, 3	2, 4
3	3, 1	3, 2	3, 3	3, 4
4	4, 1	4, 2	4, 3	4, 4

table for sum

2	3	4	5
3	4	5	6
4	5	6	7
5	6	7	8

Let X be the random variable sum of the scores

The p.d.f of X is as shown in the table below;

x	2	3	4	5	6	7	8
$P(X=x)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

(ii)

x	2	3	4	5	6	7	8	$\sum$
$P(X=x)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$	1
$xP(X=x)$	$\frac{2}{16}$	$\frac{6}{16}$	$\frac{12}{16}$	$\frac{20}{16}$	$\frac{18}{16}$	$\frac{14}{16}$	$\frac{8}{16}$	$\frac{80}{16}$

$E(X) = \frac{80}{16} = 5$

Variance of X, Var (X)

For a discrete random variable, with $E(X) = \mu$, the variance is defined as follows;
The variance of X written as Var (X) = $E(X-\mu)^2$
Alternatively, $Var(X) = E(X-\mu)^2$

$$= E(X^2 - 2\mu X + \mu^2)$$
$$= E(X^2) - 2\mu E(X) + E(\mu^2)$$
$$= E(X^2) - 2\mu^2 + \mu^2$$
$$= E(X^2) - \mu^2$$ [This format is easier to work with]

Note: $\mu = E(X)$ and $\mu^2 = [E(X)]^2$

Therefore Var (X) = $E(X^2) - [E(X)]^2$ $\quad where\ E(X^2) = \sum_{all\,x} x^2 P(X = x)$

Var (X) is sometimes written as the square of the standard deviation i.e σ^2

Thus standard deviation of X, $\sigma = \sqrt{Var(X)}$

Examples

1. The random variable X has a probability distribution as shown in the table

x	1	2	3	4	5
$P(X=x)$	0.1	0.3	0.2	0.3	0.1

Find (a) $E(X)$ (b) $E(X^2)$ (c) $Var\ (X)$ (d) $the\ standard\ deviation\ of\ X$

Solution

x	1	2	3	4	5
$P(X=x)$	0.1	0.3	0.2	0.3	0.1
$xP(X=x)$	0.1	0.6	0.6	1.2	0.5
x^2	1	4	9	16	25
$x^2P(X=x)$	0.1	1.2	1.8	4.8	2.5

(a) $E(X) = 0.1 + 0.6 + 0.6 + 1.2 + 0.5 = 3$

(b) $E(X^2) = 0.1 + 1.2 + 1.8 + 4.8 + 2.5 = 10.4$

(c) Var (X) = $E(X^2) - [E(X)]^2$
$= 10.4 - 3^2 = 10.4 - 9 = 1.4$

(d) standard deviation of X, $\sigma = \sqrt{Var(X)} = \sqrt{1.4} = 1.18\ (2\ d.p)$

2. The discrete random variable X has p.d.f $P(X = x)$ $for\ x = 1,2,3$

x	1	2	3
$P(X=x)$	0.2	0.3	0.5

Find (a) $E(X)$ (b) $E(X^2)$ (c) $Var\ (X)$ (d) $the\ standard\ deviation\ of\ X$

Solution

x	1	2	3
$P(X=x)$	0.2	0.3	0.5
$xP(X=x)$	0.2	0.6	1.5
x^2	1	4	9
$x^2P(X=x)$	0.2	1.2	4.5

(a) $E(X) = 0.2 + 0.6 + 1.5 = 2.3$

(b) $E(X^2) = 0.2 + 1.2 + 4.5 = 5.9$

(c) $\text{Var}(X) = E(X^2) - [E(X)]^2$

$= 5.9 - (2.3)^2 = 5.9 - 5.29 = 0.61$

(d) standard deviation of X, $\sigma = \sqrt{Var(X)} = \sqrt{0.61} = 0.781\ (3\ d.p)$

Trial questions

1. A random variable X has the probability distribution as shown in the table

x	1	2	3	4	5
$P(X = x)$	0.1	0.3	a	0.2	0.05

Find (a) the value of a (b) $P(X \geq 4)$ (c) $P(X < 1)$ (d) $P(2 \leq X \leq 4)$

[Ans: (a) 0.35 (b) 0.25 (c) 0 (d) 0.65]

2. The probability distribution of a random variable X is as shown in the table below

x	1	2	3	4	5
$P(X = x)$	0.1	0.3	y	0.2	0.1

Find (a) the value of y (b) E(X) [Ans: (a) 0.3 (b) 2.9]

3. Find the expected number of heads when two fair coins are tossed [Ans: 1]
4. The discrete random variable X has p.d.f ; P(X=0) = 0.05, P(X=1) = 0.45, P(X=2) = 0.5. Find (a) E(X) (b) E(X^2) [Ans: (a) 1.45 (b) 2.45]
5. The discrete random variable X has a p.d.f $P(X = x) = kx\ for\ x = 1,2,3,4,5$ where k is a costant. Find E(X) [Ans: $\frac{11}{3}$]
6. Find Var (X) for each of the following probability distributions

(a)

x	-3	-2	0	2	3
$P(X = x)$	0.3	0.3	0.2	0.1	0.1

(b)

x	1	3	5	7	9
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{6}$

(c)

x	0	2	5	6
$P(X = x)$	0.11	0.35	0.46	0.08

[Ans: (a) 4.2 (b) $\frac{22}{3}$ (c) 3.67]

7. X is a random variable 'the number on the biased die' and the p.d.f is as shown

x	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{5}$	y	$\frac{1}{5}$	$\frac{1}{6}$

Find (a) the value of y (b) E(X) (c) E(X^2) (d) Var (X)

[Ans: (a) 0.1 (b) 3.5 (c) 15.23 (d) 0.933]

8. A discrete random variable X can take on the values 0, 1, 2 or 3 and its probability distribution is given by P(X = 0) = k, P(X = 1) = 3k, P(X = 2) = 4k, P(X =3) = 5k, where k is a constant. Find (a) the value of k (b) the mean and variance of X [Ans: (a) $\frac{1}{13}$ (b) 2 , $\frac{12}{13}$]
9. a discrete random variable X represents the number of heads obtained when three coins are tossed. (a) Construct the probability distribution table for X

(b) Calculate the expected number of heads
(c) Find the variance of X [Ans: (b) 1.5 (c) 0.75]

10. The table below shows a random variable X with the following probability distribution

x	1	2	3	4	5
$P(X = x)$	$\frac{1}{16}$	2k	$\frac{1}{4}$	K	$\frac{3}{8}$

Find the (i) value of k (ii) variance of X [Ans: (i) k = 5/48 (ii) 1.8]

11. A random variable X has the p.d.f given by P(X = 0) = 0.1, P(X = 1) = 0.3, P(X = 2) =0.4 and P(X = 3) = 0.2. Find the
   (i) Expectation of X (ii) variance of X [Ans: (i) 1.7 (ii) 0.81]

12. A discrete random variable X takes on the values 0, 1, 2, 3 and 4. Its probability distribution is;
$P(X = 0) = c, P(X = 1) = 2c, P(X = 2) = 3c , P(X = 3) = 4c$ *and* $P(X = 4) = 5c$
Find the (i) the value of the constant c
   (ii) standard deviation of the distribution [Ans: (i) 1/15 (ii) 1.24]

13. A random variable X has the probability distribution;
$P(X = 0) = P(X = 1) = 0.1$
$P(X = 2) = 0.2$
$P(X = 3) = P(X = 4) = 0.3$
Find the mean and variance of X [Ans: 2.6 ; 1.64]

14. The random variable X has the distribution shown in the table below

x	-1	0	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{20}$	$\frac{1}{20}$	m	$\frac{3}{20}$	$\frac{6}{20}$	$\frac{2}{20}$	$\frac{3}{20}$	$\frac{1}{20}$

Find the (i) value of m
   (ii) expectation and variance of X [Ans: (i) 3/20 (ii) 2.75 ; 3.0875]

15. A random variable X has a p.d.f f(x) given as;

x	-1	0	1
$P(X = x)$	a	$\frac{1}{2}$	b

Where a and b are the probabilities of P(X = -1) and P(X = 1) respectively
Given that E(X) = $\frac{1}{6}$,
   (i) determine the values of a and b
   (ii) calculate the variance and standard deviation of X
   (iii) find P(X > -1) [Ans: (i) a = 1/6 , b = 1/3 (ii) 0.472 ; 0.687 (iii) 5/6]

16. A discrete random variable X is represented by the p.d.f

$$P(X = x) = \begin{cases} \frac{1+i}{ik} & ; i = 1,2,3 \dots\dots 6 \\ 0 & elsewhere \end{cases}$$

Find the (i) value of k (ii) expectation of X [Ans: (i) k = 20/169 (ii) 3.195]

17. A discrete random variable x has the probability distribution given below
$P(x = 0) = P(x = 4) = k, P(x = 1) = P(x = 3) = 2k$ *and* $P(x = 2) = 4k$ where k is a constant.
   (i) Find the value of k
   (ii) State the mode
   (iii) Calculate the mean [Ans: (i) 0.1 (ii) 2 (ii) 2]

CONTINOUS RANDOM VARIABLES

A continuous random variable is one which has a continuous or countable domain

For example

- The mass, in grams, of a bag of sugar packed by a particular machine
- The time taken in minutes, to perform a given task
- The height, in metres, of a five year old girl
- The life time in hours of a 100-watt bulb
- The amounts of rainfall in a certain city

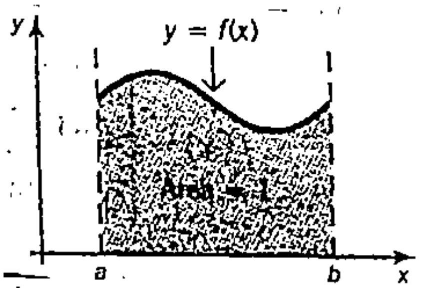
Probability density function (p.d.f)

A continuous random variable X is given by its probability distribution function (p.d.f), which is specified for the range of values for which x is valid. The probabilities are given by the area under the curve. it is denoted by f(x)

Properties of the probability density function

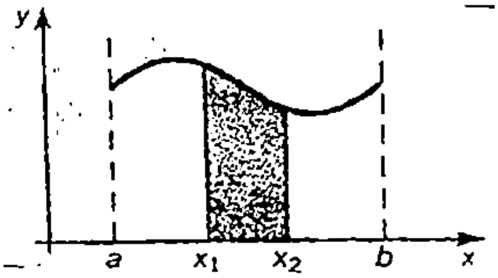
1. It is non negative i.e $f(x) \geq 0$ $for\ all\ x$
2. For a continuous random variable X, with a p.d.f $f(x)$ valid over the range $a \leq x \leq b$

(a)



$\int_a^b f(x)\,dx = 1$

(b)



$P(x_1 \leq X \leq x_2) = \int_{x_1}^{x_2} f(x)\,dx$

Examples

1. A continuous random variable X has a p.d.f $f(x) = kx^2$ $for\ 0 \leq x \leq 4, find$ the
   (a) value of constant k (b) $P(1 \leq X \leq 3)$

Solution

(a) $\int_a^b f(x)\,dx = 1$

$\int_0^4 kx^2\,dx = 1$

$k\int_0^4 x^2\,dx = 1$

$k\left[\frac{x^3}{3}\right]_0^4 = 1$

$k\left[\frac{4^3}{3} - 0\right] = 1$

$\frac{64k}{3} = 1 \Rightarrow k = \frac{3}{64}$

$\therefore f(x) = \frac{3}{64}x^2$ $for\ 0 \leq x \leq 4$

(b) $P(x_1 \leq X \leq x_2) = \int_{x_1}^{x_2} f(x)\,dx$

$P(1 \leq X \leq 3) = \int_1^3 \frac{3}{64}x^2\,dx$

$= \frac{3}{64}\int_1^3 x^2\,dx$

$= \frac{3}{64}\left[\frac{x^3}{3}\right]_1^3$

$= \frac{3}{64}\left[\frac{4^3}{3} - \frac{1^3}{3}\right]$

$= \frac{3}{64}\left[\frac{27}{3} - \frac{1}{3}\right]$

$= \frac{3}{64} \times \frac{26}{3} = 0.40625 = 0.41(2d.p)$

2. X is a continuous random variable, the mass, in kilograms of a substance produced per minute in an industrial process where;

$$f(x) = \begin{cases} \frac{1}{36}x(6-x) & 0 \le x \le 6 \\ 0 & otherwise \end{cases}$$

Find the probability that the mass is more than 5kg

Solution

We are required to find P(X > 5)

$P(X > 5) = \int_5^6 \frac{1}{36}x(6-x)\,dx = \frac{1}{36}\int_5^6 (6x - x^2)\,dx$

$= \frac{1}{36}\left[3x^2 - \frac{x^3}{3}\right]_5^6$

$= \frac{1}{36}\left[\left(3(6^2) - \frac{6^3}{3}\right) - \left(3(5^2) - \frac{5^3}{3}\right)\right]$

$= \frac{1}{36}\left[(108 - 72) - \left(75 - \frac{125}{3}\right)\right]$

$= \frac{1}{36}\left[36 - \frac{100}{3}\right] = \frac{1}{36} \times \frac{8}{3} = \frac{8}{108} = 0.074$

$6x^2$ $\quad \frac{6x^{1+1}}{2}$ $\quad 3x^2$

Example 3

The continuous random variable X has p.d.f $f(x)$ where;

$$f(x) = \begin{cases} k & 0 \le x < 2 \\ k(2x-3) & 2 \le x \le 3 \\ 0 & otherwise \end{cases}$$

Find (a) the value of the constant k (b) P $(X \le 1)$ (c) $P(X \ge 2.5)$ (d) $P(1 \le X \le 2.3)$

Solution

(a) $\int_{all\,x} f(x)\; dx = 1$

$\int_0^2 k\,dx + \int_2^3 k(2x-3)\,dx = 1$

$k\int_0^2 dx + k\int_2^3 (2x-3)\,dx = 1$

$k[x]_0^2 + k[x^2 - 3x]_2^3 = 1$

$k(2-0) + k([3^2 - 3(3)] - [2^2 - 3(2)]) = 1$

$2k + k(0 - (-2)) = 1$

$2k + 2k = 1, thus\ 4k = 1 \Rightarrow k = \frac{1}{4}$

$$f(x) = \begin{cases} \frac{1}{4} & 0 \le x < 2 \\ \frac{1}{4}(2x-3) & 2 \le x \le 3 \\ 0 & otherwise \end{cases}$$

(b) $P(X \le 1) = \int_0^1 \frac{1}{4}dx = \left[\frac{x}{4}\right]_0^1 = \left(\frac{1}{4} - 0\right) = \frac{1}{4}$

(c) $P(X \ge 2.5) = \int_{2.5}^3 \frac{1}{4}(2x-3)\,dx = \frac{1}{4}\int_{2.5}^3 (2x-3)\,dx$

$= \frac{1}{4}[x^2 - 3x]_{2.5}^3$

$=\frac{1}{4}([3^2-3(3)]-[2.5^2-3(2.5)])$
$=\frac{1}{4}(0-(-1.25))$
$=\frac{1}{4}\times 1.25=0.3125$

(d) $P(1\le X\le 3)=\int_1^2\frac{1}{4}dx+\int_2^{2.3}\frac{1}{4}(2x-3)\,dx$
$=\left[\frac{x}{4}\right]_1^2+\frac{1}{4}\left[x^2-3x\right]_2^{2.3}$
$=\left(\frac{2}{4}-\frac{1}{4}\right)+\frac{1}{4}([2.3^2-3(2.3)]-[2^2-3(2)])$
$=\frac{1}{4}+\frac{1}{4}(-1.61+2)$
$=0.25+0.25(0.39)=0.25+0.0975=0.3475$

Expectation of X, E(X)

For a continuous random variable with a p.d.f $f(x)$

$$E(X)=\int_{all\,x}xf(x)\,dx$$

E(X) is referred to as the mean or expectation of X and is often denoted by μ

Examples

1. The p.d.f of a continuous random variable X is given below

$$f(x)=\begin{cases}\frac{1}{9}x^2 & 0\le x\le 3\\ 0 & otherwise\end{cases}$$

Find (a) μ, *the mean of X* (b) $P(X<\mu)$

Solution

(a) $\mu=E(X)$

$E(X)=\int_{all\,x}xf(x)\,dx$
$=\int_0^3 x\times\frac{1}{9}x^2dx=\frac{1}{9}\int_0^3x^3\,dx$
$=\frac{1}{9}\left[\frac{x^4}{4}\right]_0^3=\frac{1}{9}\left(\frac{3^4}{4}-0\right)=\frac{1}{9}\times\frac{81}{4}=\frac{81}{36}=2.25$

(b) $P(X<\mu)=P(X<2.25)$
$P(X<2.25)=\int_0^{2.25}\frac{1}{9}x^2\,dx=\frac{1}{9}\left[\frac{x^3}{3}\right]_0^{2.25}=\frac{1}{9}\left(\frac{2.25^3}{3}-0\right)=0.42$

2. A teacher of young children is thinking of asking her class to guess her height in metres. The teacher considers that the height guessed by a randomly selected child can be modelled by the random variable X with the probability density function;

$$f(x)=\begin{cases}\frac{3}{16}(4x-x^2) & 0\le x\le 2\\ 0 & otherwise\end{cases}$$

Using this model, (a) find P(X < 1) (b) show that E(X) = 1.25

Solution

(a) $P(X < 1) = \int_0^1 \frac{3}{16}(4x - x^2)\,dx = \frac{3}{16}\int_0^1 (4x - x^2)\,dx$

$= \frac{3}{16}\left[2x^3 - \frac{x^3}{3}\right]_0^1$

$= \frac{3}{16} \times \left(\left[2(1) - \frac{1}{3}\right] - 0\right)$

$= \frac{3}{16} \times \frac{5}{3} = \frac{5}{16} = 0.3125$

(b) $E(X) = \int_{all\,x} xf(x)\,dx$

$= \frac{3}{16}\int_0^2 x(4x - x^2)\,dx = \frac{3}{16}\int_0^2 (4x^2 - x^3)\,dx$

$= \frac{3}{16}\left[\frac{4x^3}{3} - \frac{x^4}{4}\right]_0^2 = \frac{3}{16}\left(\left(\frac{4(2)^3}{3} - \frac{2^4}{4}\right) - (0)\right)$

$= \frac{3}{16}\left(\frac{32}{3} - \frac{16}{4}\right) = \frac{3}{16} \times \frac{20}{3} = \frac{20}{16} = 1.25$

3. A continuous random variable X has a p.d.f $f(x)$ where

$$f(x) = \begin{cases} 0.25x & 0 \le x < 2 \\ 1 - 0.25x & 2 \le x \le 4 \\ 0 & otherwise \end{cases}$$

Find E(X)

Solution

$E(X) = \int_{all\,x} xf(x)\,dx$

$= \int_0^2 x \times 0.25x\,dx + \int_2^4 x(1 - 0.25x)\,dx$

$= 0.25\int_0^2 x^2 dx + \int_2^4 (x - 0.25x^2)\,dx$

$= 0.25\left[\frac{x^3}{3}\right]_0^2 + \left[\frac{x^2}{2} - 0.25\frac{x^3}{3}\right]_2^4$

$= 0.25\left(\frac{2^3}{3} - 0\right) + \left(\left(\frac{4^2}{2} - 0.25\frac{(4)^3}{3}\right) - \left(\frac{2^2}{2} - 0.25\frac{(2)^3}{3}\right)\right)$

$= 0.25 \times \frac{8}{3} + \left(\left(8 - \frac{16}{3}\right) - \left(2 - \frac{2}{3}\right)\right)$

$= \frac{2}{3} + \left(\frac{8}{3} - \frac{4}{3}\right) = \frac{2}{3} + \frac{4}{3} = \frac{6}{3} = 2$

4. A continuous random variable X has a p.d.f defined by;

$$f(x) = \begin{cases} \frac{kx}{4} & 0 \le x < 4 \\ k & 4 \le x \le 6 \\ 0 & otherwise \end{cases} \quad where\ k\ is\ a\ constant$$

Find the (a) value of k (b) E(X) (c) value of b for which $P(x \le b) = 0.2$

Solution

(a) $\int_{all\,x} f(x)\,dx = 1$

$\int_0^4 \frac{kx}{4}\,dx + \int_4^6 k\,dx = 1$

$$\frac{k}{4}\int_0^4 x\,dx + k\int_4^6 dx = 1$$
$$\frac{k}{4}\left[\frac{x^2}{4}\right]_0^4 + k[x]_4^6 = 1$$
$$\frac{k}{4}\left(\frac{4^2}{2} - 0\right) + k(6-4) = 1$$
$$\frac{k}{4}\times 8 + 2k = 1$$
$$2k + 2k = 1, \quad 4k = 1 \quad \Rightarrow k = \frac{1}{4}$$

(b) $E(X) = \int_{all\,x} xf(x)\,dx$

It is important that we first rewrite the p.d.f replacing k with its value

$$f(x) = \begin{cases} \frac{x}{16} & 0 \le x < 4 \\ \frac{1}{4} & 4 \le x \le 6 \\ 0 & otherwise \end{cases}$$

$$E(X) = \int_0^4 x \times \frac{x}{16}dx + \int_4^6 x \times \frac{1}{4}dx$$
$$= \int_0^4 \frac{x^2}{16}dx + \int_4^6 \frac{x}{4}dx = \frac{1}{16}\int_0^2 x^2\,dx + \frac{1}{4}\int_4^6 x\,dx$$
$$= \frac{1}{16}\left[\frac{x^3}{3}\right]_0^4 + \frac{1}{4}\left[\frac{x^2}{2}\right]_4^6$$
$$= \frac{1}{16}\left(\frac{4^3}{3} - 0\right) + \frac{1}{4}\left(\frac{6^2}{2} - \frac{4^2}{2}\right)$$
$$= \frac{1}{16}\times\frac{64}{3} + \frac{1}{4}(18-8)$$
$$= \frac{4}{3} + \frac{1}{4}\times 10 = \frac{4}{3} + \frac{5}{2} = \frac{23}{6} = 3.833$$
$$\therefore E(X) = 3.833$$

(c) $P(x \le b) = 0.2$

Since 0.2 is less than 0.5, then we expect b to lie in the range $0 \le x \le 4$

$$P(x \le b) = \int_0^b \frac{x}{16}dx = 0.2$$
$$\frac{1}{16}\int_0^b x\,dx = 0.2$$
$$\frac{1}{16}\left[\frac{x^2}{2}\right]_0^b = 0.2$$
$$\frac{1}{16}\left(\frac{b^2}{2} - 0\right) = 0.2$$
$$\frac{b^2}{32} = 0.2$$

$$b^2 = 32 \times 0.2 = 6.4$$
$$b = \sqrt{6.4} = 2.53$$

Expectation of any function of X

If g(x) is any function of the continuous random variable X having a p.d.f f(x), then

$$E(g(x)) = \int_{all\,x} g(x)f(x)\,dx$$

In particular $E(X^2) = \int_{all\,x} x^2 f(x)\,dx$

Example

The continuous random variable X has a p.d.f $f(x)$ where

$$f(x) = \begin{cases} \frac{1}{20}(x+3) & 0 \le x \le 4 \\ 0 & otherwise \end{cases}$$

Find (a) E(X) (b) $E(X^2)$

Solution

(a) $E(X) = \int_{all\,x} xf(x)\,dx = \int_0^4 \frac{1}{20}x(x+3)\,dx$

$$= \frac{1}{20}\int_0^4 (x^2+3x)\,dx$$

$$= \frac{1}{20}\left[\frac{x^3}{3}+\frac{3x^2}{2}\right]_0^4 = \frac{1}{20}\left(\left(\frac{4^3}{3}+\frac{3(4)^2}{2}\right)-(0)\right)$$

$$= \frac{1}{20}\times\frac{136}{3} = \frac{34}{15} = 2.267$$

(b) $E(X^2) = \int_{all\,x} x^2 f(x)\,dx$

$$= \int_0^4 \frac{1}{20}x^2(x+3)\,dx$$

$$= \frac{1}{20}\int_0^4 (x^3+3x^2)\,dx$$

$$= \frac{1}{20}\left[\frac{x^4}{4}+x^3\right]_0^4 = \frac{1}{20}\left(\left(\frac{4^4}{4}+4^3\right)-(0)\right)$$

$$= \frac{1}{20}\times 128 = 6.4$$

Note: $E(X^2)$ is an important value which is needed when calculating the variance of X

Variance of X, Var (X)

If X is a continuous random variable with p.d.f $f(x)$, then ;

$$Var(X) = \int_{all\,x} x^2 f(x)\,dx - \mu^2$$ [Recall: Var (X) = $E(X^2) - [E(X)]^2$]

where $\mu = E(X) = \int_{all\,x} xf(x)\,dx$

The standard deviation of X is often written as σ, where $\sigma = \sqrt{Var(X)}$

Examples

1. The continuous random variable X has p.d.f $f(x)$ where $f(x) = \frac{1}{8}x\,;0 \le x \le 4$

Find (a) E(X) (b) $E(X^2)$ (c) Var (X) (d) the standard deviation, σ of X

Solution

(a) $E(X) = \int_{all\,x} xf(x)\,dx$

$$= \int_0^4 x\times\frac{1}{8}x\,dx = \frac{1}{8}\int_0^4 x^2\,dx = \frac{1}{8}\left[\frac{x^3}{3}\right]_0^4 = \frac{1}{8}\left(\frac{4^3}{3}-0\right) = \frac{1}{8}\times\frac{64}{3} = 2.7$$

(b) $E(X^2) = \int_{all\,x} x^2 f(x)\,dx$

$$= \int_0^4 x^2\times\frac{1}{8}x\,dx = \frac{1}{8}\int_0^4 x^3\,dx = \frac{1}{8}\left[\frac{x^4}{4}\right]_0^4 = \frac{1}{8}\left(\frac{4^4}{4}-0\right) = \frac{1}{8}\times\frac{64}{3} = 8$$

(c) Var (X) = $E(X^2) - [E(X)]^2$

$$= 8-(2.7)^2 = 8-7.29 = 0.71\ (2\ d.p)$$

(d) Standard deviation = $\sqrt{Var(X)} = \sqrt{0.71} = 0.8439$

2. As an experiment, a temporary roundabout is installed at cross roads. The time X in minutes which vehicles have to wait before entering the roundabout has a probability density function $f(x) = \begin{cases} 0.8 - 0.32x & 0 \le x \le 2.5 \\ 0 & otherwise \end{cases}$

Find the mean and standard deviation of X

Solution

$E(X) = \int_{all\,x} xf(x)\,dx$

$= \int_0^{2.5} x(0.8 - 0.32x)\,dx = \int_0^{2.5}(0.8x - 0.32x^2)\,dx$

$= \left[0.8\frac{x^2}{2} - 0.32\frac{x^3}{3}\right]_0^{2.5}$

$= \left(\left(0.8\frac{(2.5)^2}{2} - 0.32\frac{(2.5)^3}{3}\right) - (0)\right)$

$= 0.833$ minutes

$E(X^2) = \int_{all\,x} x^2 f(x)\,dx = \int_0^{2.5}(0.8x^2 - 0.32x^3)\,dx$

$= \left[0.8\frac{x^3}{3} - 0.32\frac{x^4}{4}\right]_0^{2.5}$

$= \left(\left(0.8\frac{(2.5)^3}{3} - 0.32\frac{(2.5)^4}{4}\right) - (0)\right)$

$= 4.167 - 3.125$

$= 1.042$

Var (X) = $E(X^2) - [E(X)]^2 = 1.042 - 0.833^2 = 0.348$

Standard deviation of X = $\sqrt{0.348} = 0.59$

Trial questions

1. The continuous random variable X has a p.d.f $f(x)$ where $f(x) = kx^2\ for\ 0 \le x \le 2$
   (a) Find the value of the constant k
   (b) Find $P(X \ge 1)$
   (c) Find $P(0.5 \le x \le 1.5)$ [Ans: (a) $\frac{3}{8}$ (b) $\frac{7}{8}$ (c) $\frac{13}{32}$]
2. A continuous random variable X has a p.d.f $f(x)$ $where\ f(x) = kx\ ; 0 \le x \le 4$
   (a) Find the value of the constant k (b) find $P(1 \le x \le 2.5)$
   [Ans: (a) 0.125 (b) 0.328]
3. Find E(X) for each of the following continuous random variables
   (a) $f(x) = \frac{3}{4}(x^2 + 1)\ ;\ 0 \le x \le 1$
   (b) $f(x) = \frac{3}{4}x(2 - x);\ 0 \le x \le 2$
   (c) $f(x) = kx^3\ ;\ 0 \le x \le 2$
   (d) $f(x) = \begin{cases} \frac{3}{8} & \frac{2}{3} \le x \le 2 \\ \frac{3}{32}x(4 - x) & 2 \le x \le 4 \\ 0 & otherwise \end{cases}$

   [Ans: (a) $\frac{9}{16}$ (b) 1 (c)1.6 (d) 2.042]

4. A random variable X has a probability density function f(x) given by

$$f(x) = \begin{cases} kx(5-x) & 0 \le x \le 5 \\ 0 & otherwise \end{cases}$$

Show that $k = \frac{6}{125}$ and find the mean of X [Ans: 2.5]

5. For each of the questions (a) to (d), find
(i) E(X) (ii) $E(X^2)$ (iii) Var (X) (iv) standard deviaton of x

(a) $f(x) = \frac{3}{8}x^2$; $0 \le x \le 2$

(b) $f(x) = \frac{1}{4}(4-x)$; $1 \le x \le 3$

(c) $f(x) = 4x^3$; $0 \le x \le 1$

(d) $f(x) = \begin{cases} \frac{1}{4} & 0 \le x \le 2 \\ \frac{1}{4}(2x-3) & 2 \le x \le 3 \\ 0 & otherwise \end{cases}$

[Ans: (a) (i) 1.5 (ii) 2.4 (iii) 1.5 (iv) 0.387

(b) (i) $\frac{11}{6}$ (ii) $\frac{11}{3}$ (iii) $\frac{11}{36}$ (iv) 0.553

(c) (i) $\frac{4}{5}$ (ii) $\frac{2}{3}$ (iii) $\frac{2}{75}$ (iv) 0.163

(d) (i) $1\frac{19}{24}$ (ii) $2\frac{1}{24}$ (iii) $\frac{479}{576}$ (iv) 0.912]

6. A continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} ax & ; \quad 0 \le x \le 2 \\ a(4-x); & 2 < x \le 4 \quad \text{where } a \text{ is a constant} \\ 0 & ; \quad elsewhere \end{cases}$$

(a) Find the value of a

(b) Calculate the expectation; E(X) [Ans: (a) ¼ (b) 2]

7. A random variable X has a probability density function (p.d.f)

$$f(x) = \begin{cases} ax & 0 \le x \le 1 \\ \frac{1}{2}(4-x) & 1 < x \le 3 \\ 0 & elsewhere \end{cases}$$

(a) Find the value of a

(b) The expectation of X, E(X)

(c) $P(2 \le X \le 2.5)$ [Ans: (a) -2 (b) 3 (c) 0.4375]

7. A random variable X has the probability density function

$$f(x) = \begin{cases} kx ; & 0 < x < 1 \\ \left(\frac{k}{2}\right)x ; & 1 \le x \le 2 \\ 0 ; & elsewhere \end{cases}$$

Find (i) the value of k
(ii) E(X)
(iii) Standard deviation of X [Ans: (i) 4/5 (ii) 1.2 (iii) 0.51]

CHAPTER 18: BINOMIAL DISTRIBUTION

There are some probability situations that may result into only two outcomes, or even be reduced to only two. Such situations may include:

i) When a baby is born, it may be either male or female

ii) In a final football match, a team either wins or loses.

Other situations that are reduced to only two possible outcomes may include:

i) A person taking a Pioneer bus may arrive either on time or not on time.

ii) A company producing items that are either defective or not defective

iii) A drug administered to a patient may be either effective or ineffective.

All the above mentioned situations are called binomial or Bernoulli experiments and the outcomes of a binomial experiment are classified as successes or failures.

For a situation to be described using a binomial model,

- a finite number, n, trials are carried out .
- the trials are independent
- the outcome of each trial is deemed either a success or a failure
- the probability, p, of a successful outcome is the same for each trial

The discrete random variable, X, is **the number of successful outcomes in n trials.**
If the above conditions are satisfied, X is said to follow a binomial distribution. This is written

$$X \sim B(n,p)$$

Note: the number of trials, n, and the probability of success, p, are both needed to describe the distribution completely. They are known as the parameters of the binomial distribution.

Writing P(failure) as $q \ where \ q = 1 - p$

If $X \sim B(n,p)$, the probability of obtaining r successes in n trials is P (X = r) where;

$$P(X = r) = \ {}^{n}C_{r}p^{r}q^{n-r} \ \ for\ r = 0,1,2\ 3, \dots\dots n$$

<u>Examples</u>

1. a coin is tossed three times. Find the probability of getting exactly three heads

<u>Solution 1</u>

The problem can be obtained by looking at the sample space, there are three ways of getting 2 heads out of 8 i.e $\{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$

The answer is $\frac{3}{8} = 0.375$

<u>Solution 2</u>

Looking at the problem above from the stand point of a binomial experiment, one can show that it meets the four requirements i.e.

1. There are only two outcomes for each trial, head or tail
2. There is a fixed number of trials, three
3. The outcomes are independent of each other (the outcome of one toss in no way affects the outcome of another toss)
4. The probability of success(heads) is ½ in each case.

In this case; n = 3, X = 2, p = ½ , q = ½

Hence substituting in this formula gives;

$$P(X = r) = \ {}^{n}C_{r}p^{r}q^{n-r}$$

$$P(2\ heads) = P(X = 2) = {}^3C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^1 = 3 \times \frac{1}{4} \times \frac{1}{2} = \frac{3}{8}$$

which is the same answer obtained by using the sample space

2. In a particular population, 10% of the people have blood type B. If three people are selected at random from the population, what is the probability that exactly two of them have blood type B?

Solution

Let X be the random variable people with blood type B

When three people are selected, n = 3, p = 0.1, q = 1 − 0.1 = 0.9

X is the number of outcomes in 3 trials, so $X \sim B(3, 0.1)$

$$P(X = r) = {}^nC_r p^r q^{n-r}$$

$$P(X = 2) = {}^3C_2(0.1)^2(0.9)^1 = 0.072$$

3. A biased coin is tossed three times. The probability of heads on any toss is 0.3. Let X denote the number of heads that come up. Calculate

(i) P(X = 2) (ii) P(X = 3) (iii) P(1 < X ≤ 5)

Solution

If we call heads a success, then X has a binomial distribution with parameters n = 6 and p = 0.3

(i) $P(X = 2) = {}^6C_2(0.3)^2(0.7)^4 = 0.324135 \approx 0.324\ (3\ d.p)$

(ii) $P(X = 3) = {}^6C_3(0.3)^3(0.7)^3 = 0.18522 \approx 0.185\ (3\ d.p)$

(iii) $P(1 < X \le 5) = P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5)$

$= 0.324 + 0.185 + {}^6C_4(0.3)^4(0.7)^2 + {}^6C_5(0.3)^5(0.7)^1$

$= 0.324 + 0.185 + 0.059 + 0.01 = 0.578$

4. A die is tossed three times. If X is the number of fives obtained. Find the probability that

(i) no fives turn up (ii) 1 five turns up (iii) 3 fives turn up

Solution

$X \sim B(n, p)$

n = 3 , p = $\frac{1}{6}$, $q = 1 - \frac{1}{6} = \frac{5}{6}$

(i) $P(X = 0) = {}^3C_0\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^3 = \frac{125}{216} = 0.5787$

(ii) $P(X = 1) = {}^3C_1\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^2 = \frac{75}{216} = 0.34722$

(iii) $P(X = 3) = {}^3C_3\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^0 = \frac{1}{216} = 0.00463$

5. Hospital records show that of the patients suffering from a certain disease, 75% die of it. What is the probability that of 6 randomly selected patients, 4 will recover?

Solution

Let X = number who will recover; percentage that recovers = 100 − 75 = 25%

$X \sim B(n, p)$ $n = 6, p = 0.25\ \ and\ q = 0.75\ (\ failure\ if\ they\ die)$

$P(X = 4) = {}^6C_4(0.25)^4(0.75)^2$
$= 15 \times 0.0021973 = 0.0329595$

6. In the old days, there was a probability of 0.8 of success in any attempt to make a telephone call (this depended on the importance of the person making the call or the operator's curiosity) Calculate the probability of 7 successes in 10 attempts.

Solution

Probability of success, p = 0.8, so q = 0.2
Let X = success in getting through
$P(X = r) = {}^nC_r p^r q^{n-r}$
$P(X = 7) = {}^{10}C_7(0.8)^7(0.2)^3$
$= 0.20133$

7. A blind folded marks man finds that on average, he hits the target 4 times out of 5. If he fires 4 shots. What is the probability of;
   (a) more than two hits
   (b) at least three misses

Solution

(a) $n = 4, p = \frac{4}{5} = 0.8, q = 1 - 0.8 = 0.2$
Let X = number of hits
$P(X > 2) = P(X = 3) + P(X = 4)$
$= {}^4C_3(0.8)^3(0.2)^1 + {}^4C_4(0.8)^4(0.2)^0$
$= 0.8192$

(b) At least 3 misses means 3 misses or more i.e 3 or 4 misses
3 misses mean 1 hit and 4 misses mean 0 hit
P(at least 3 misses) $= P(X = 1) + P(X = 0)$
$= {}^4C_1(0.8)^1(0.2)^3 + {}^4C_0(0.8)^0(0.2)^4$
$= 0.0272$

8. A manufacturer of metal pistons finds that on average, 12% of his pistons are rejected because they are oversize or undersize. What is the probability that a batch of 10 pistons will contain;
   (a) not more than two rejects
   (b) at least two rejects

Solution

Let X = number of rejects, in this case 'success' means rejection
$n = 10, p = 0.12 \ and \ q = 0.88$
(a) not more than two rejects means two rejects or less i.e. $X \leq 2$
$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$
$= {}^{10}C_0(0.12)^0(0.88)^{10} + {}^{10}C_1(0.12)^1(0.88)^9 + {}^{10}C_2(0.12)^2(0.88)^8$
$= 0.2785 + 0.37977 + 0.23304$
$= 0.89131$

(b) at least two rejects means two rejects or more i.e $X \geq 2$

We would work out all the cases for X = 2, 3, 4 … 10. But it would be hectic. It is much easier using the tables or proceed as follows;

Probability of at least 2 rejects = P(X = 2) + P(X = 3) + ……+ P(X = 10)

$$= 1 - P(X \le 1)$$
$$= 1 - [P(X = 0) + P(X = 1)]$$
$$= 1 - [0.2785 + 0.37977]$$
$$= 0.34173$$

9. If a student randomly guesses at five multiple choice questions, find the probability that the student gets exactly three correct answers if each question has five possible choices.

Solution

$n = 5, X = 3$ *and* $p = \frac{1}{5} = 0.2,$ *since there is one chance in five of* guessing a correct answer. q = 0.8

$$P(X = 3) = {}^5C_3(0.2)^3(0.8)^2 = 0.05$$

10. A certain survey found out that 30% of teenage consumers receive their spending money from part-time jobs. If five teenagers are selected at random, find the probability that at least three of them will have part-time jobs.

Solution

Let X = number having part-time jobs

$n = 5, p = 0.3, q = 0.7$

$$P(X \ge 3) = P(X = 3) + P(X = 4) + P(X = 5)$$
$$= {}^5C_3(0.3)^3(0.7)^2 + {}^5C_4(0.3)^4(0.7)^1 + {}^5C_5(0.3)^5(0.7)^0$$
$$= 0.132 + 0.028 + 0.002$$
$$= 0.162$$

11. The probability that a pen drawn at random from a box of pens is 0.1. If a sample of 6 pens is taken, find the probability that it will contain

(i) No defective pens

(ii) 5 or 6 defective pens

(iii) Less than 3 defective pens

Solution

$n = 6, p = 0.1\,,\ q = 0.9$

Let X = number of defective pens

(i) $P(X = 0) = {}^6C_0(0.1)^0(0.9)^6 = 0.5314$

(ii) $P(5\ or\ 6) = P(X = 5) + P(X = 6)$
$$= {}^6C_5(0.1)^5(0.9)^1 + {}^6C_6(0.1)^6(0.9)^0$$
$$= 0.000054 + 0.000001 = 0.000055$$

(iii) $P(X < 3) = P(X = 0) + P(X = 1) + P(X = 2)$
$$= {}^6C_0(0.1)^0(0.9)^6 + {}^6C_1(0.1)^1(0.9)^5 + {}^6C_2(0.1)^2(0.9)^4$$

$= 0.5314 + 0.3543 + 0.0984$

$= 0.9841$

Note that most of the above probabilities can be obtained using the tables. The value of n is located from the table and its corresponding p (probability of success noted). The probability can be read off from the table for r = 0, 1, 2, 3, ...n

Now let's use the above example to obtain P(X = 0)

Solution

n = 6, p = 0.1

Table

B(n, p) individual terms

n	r	Probability of success 0.01 0.05 0.1	0.15 0.2 0.25 0.3 0.35 ....0.5
6	0	0.5314	
	1	0.3543	
	2	0.0984	
	3	0.0146	
	4	0.0012	
	5	0.0001	
	6		

From the table, P(X = 0) = 0.5314 obtained by reading n = 6, r = 0 and p = 0.1

Then $P(X = 1) = 0.3543$

$P(X = 2) = 0.0984$

12. A multiple choice question paper has 15 questions, each with 4 possible answers of which only one is the correct answer. Determine the probability that by mere guessing, one gets;
    (i) Exactly five correct answers (ii) five incorrect answers

Solution

$n = 15, p = \frac{1}{4} = 0.25$

Let X = number of correct answers

(i) $P(X = 5)$

From the table when n = 15, r = 5, p = 0.25

P(X = 5) = 0.1651

(ii) P(five incorrect answers) = P(ten correct answers)

$= P(X = 10)$

Using the tables, n = 15, r = 10, p = 0.25

$P(X = 10) = 0.0007$

P(five incorrect answers) = 0.0007

Mean, Variance and Standard deviation of a binomial distribution

The mean, variance and standard deviation of a variable that has the binomial distribution can be found by using the formulas

Mean, $\mu = np$

$Variance, \sigma^2 = npq$

Standard deviation, $\sigma = \sqrt{Var(X)} = \sqrt{npq}$

Examples

1. A coin is tossed four times. Find the mean, variance and standard deviation of the number of heads that will be obtained.

Solution

$n = 4,\ p = \frac{1}{2}\ \ , q = \frac{1}{2}$

$mean = np = 4 \times \frac{1}{2} = 2$

$variance = npq = 4 \times \frac{1}{2} \times \frac{1}{2} = 1$

Standard deviation = $\sqrt{Var(X)} = \sqrt{1} = 1$

2. In Makerere University, it is known that $\frac{1}{3}$ of the students play volleyball. In a sample of 12 students, what is the expected value and the standard deviation of the number of volley ballers?

Solution

Let X = number of volley ballers

$X \sim B(n, p)$

$n = 12,\ p = \frac{1}{3}\ \ , q = 1 - \frac{1}{3} = \frac{2}{3}$

$mean = np = 12 \times \frac{1}{3} = 4$

Standard deviation = $\sqrt{12 \times \frac{1}{3} \times \frac{2}{3}} = \sqrt{\frac{8}{3}} = 1.633$

Trial questions

1. 30% of students in a school travel to school by bus. From a sample of ten students chosen at random, find the probability that;
   (a) only three travel by bus
   (b) less than half travel by bus [Ans: (a) 0.267 (b) 0.850]
2. in a survey on washing powder, it is found that the probability that a shopper chooses Omo is 0.25. Find the probability that in a random sample of nine shoppers
   (a) exactly three choose Omo
   (b) more than seven chose Omo [Ans: (a) 0.234 (b) 0.000107]
3. The random variable X is B (6, 0.42). Find (a) P (X = 6) (b) P (X = 4)
   (c) $P(X \leq 2)$ [Ans: (a) 0.00549 (b) 0.157 (c) 0.503]
4. An unbiased die is thrown seven times. Find the probability of throwing at least 5 sixes
   [Ans: 0.002]
5. A fair coin is tossed six times. Find the probability of throwing at least four heads [Ans: 0.344]

6. In a test, there are ten multiple choice questions. For each, there is a choice of four answers, only one of which is correct. A student guesses each of the answers
   (a) Find the probability that he gets more than seven correct
   (b) If he needs to obtain over half marks to pass and each question carries equal weight, find the probability that he passes the test
   [Ans: (a) 0.000416 (b) 0.0197]
7. The probability that it will be a fine day is 0.4. Find the
   (a) Expected number of fine days in a week
   (b) The standard deviation in the week [Ans: (a) 2.8 (b) 1.3]
8. The probability of a football team winning a match is 0.75. If the team has five matches to play, find the probability that it will win at least three of these matches
   [Ans: 0.8965]
9. Of 1000 patients who visited a health center, 250 of them were diagnosed of malaria. If a sample of 5 was drawn at random from the patients, what is the probability that
   (i) 2 of the patients had malaria
   (ii) 4 of the patients did not have malaria [Ans: (i) 0.2637 (ii) 0.3955]
10. In a large city, one person in five is left handed. Find the probability that in a random sample of 10 people;
   (a) exactly three will be left handed
   (b) more than a half are left handed [Ans: (i)0.2013 (ii) 0.0064]
11. A man's chance of hitting a target with each of his shots is $\frac{1}{5}$.
   (a) If he has to fire five shots, calculate the probability that;
      (i) Exactly 3 shots hit the target
      (ii) At least two shots hit the target
   (b) Given that he has 20 shots to fire, determine the mean number and variance of his shots at the target. [Ans: (a) (i) 0.0512 (ii) 0.2627 (b) 4, 3.2]
12. The probability that a student guesses the answer correctly to a multiple choice question is $\frac{1}{4}$. If a quiz has 15 multiple choice questions, determine the probability that a student guesses correctly the answers to
   (i) Exactly six questions
   (ii) At most three questions
   (iii) Between three and eight questions [Ans: (i) 0.0917 (ii) 0.4613 (iii) 0.5213]
13. It was found out that 20% of a sample of chicken recovered from a rare disease after treatment. In a random sample of 5 of such treated chicken, find the probability that;
   (i) There is more than one that recovered
   (ii) Either 3 or 4 recovered
   (iii) Less than 4 recovered [Ans: (i) 0.2627 (ii) 0.0567 (iii) 0.9933]

CHAPTER 19: NORMAL DISTRIBUTION

The normal distribution is one of the most important distributions in statistics. Many measured quantities in natural sciences follow a normal distribution and under certain circumstances, it is also a useful approximation to the binomial distribution

The normal variable X is continuous and its probability density function $f(x)$ depends on its mean μ and the standard deviation, where

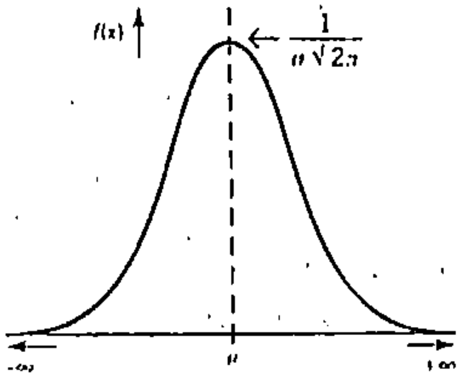
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}; \quad -\infty \leq x \leq \infty$$

This is very complicated and has been included just for reference. You would not be expected to memorize it

To describe the distribution, write $X \sim N(\mu, \sigma^2)$

Note that the description gives the variance σ^2, *rather than the standard deviation*, σ

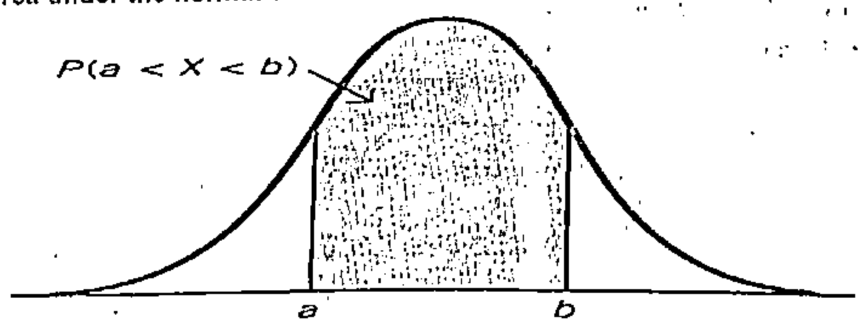
The normal distribution curve has the following features;



- It is bell shaped
- It is symmetrical about the mean, μ
- It extends from $-\infty$ to ∞
- The total area under the curve is 1
- The maximum value of $f(x)$ is $\frac{1}{\sigma\sqrt{2\pi}}$

Finding probabilities

The probability that X lies between a and b is written P(a < X < b). To find the probabilities, you need to find the area under the normal curve between a and b



One way of finding areas is to integrate but since the normal function is complicated and very difficult to integrate, tables are used instead.

The standard normal tables

In order to use the same set of tables for all possible values of μ and σ^2, the variable X is standardised so that the mean is 0 and the standard deviation is 1. Note that the variance is the square of the standard deviation hence the variance is also 1. This standard normal variable is called Z and $Z \sim N(0, 1)$

Using the standard normal tables for any random variable

Standardize X, where $X \sim N(\mu, \sigma^2)$

- Subtract the mean μ
- Then divide by the standard deviation, σ, to obtain

$Z = \frac{X-\mu}{\sigma}$ where $Z \sim N(0, 1)$

Examples

1. The lengths of metal strips produced by a machine are normally distributed with mean length of 150 cm and standard deviation of 10 cm. Find the probability that the length of a randomly selected strip is shorter than 165 cm.

Solution

Let X be the length in centimeters of a metal strip

$\mu = 150,\ \sigma = 10,\ X \sim N(150, 10^2)$

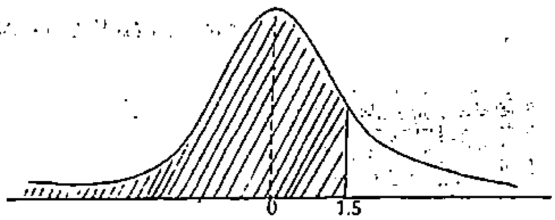
You need to find the probability that the length of is shorter than 165 cm i.e. $P(X < 165)$

To be able to use the standard normal tables, standardize the X- variable by subtracting the mean, 150 and then dividing by the standard deviation, 10

X becomes $\frac{X-150}{10} = Z$

$P(X < 165) = P(Z < \frac{165-150}{10})$

$= P(Z < 1.5)$



$P(Z < 1.5) = 0.5 + P(0 < Z < 1.5)$

$= 0.5 + 0.4332 = 0.9332$

Therefore the probability that the length is shorter than 165 cm is 0.9332

2. The time taken by the milk man to deliver to Kampala Market Street is normally distributed with a mean of 12 minutes and standard deviation of 2 minutes. He delivers milk every day. Estimate the number of days during the year when he takes
   (a) Longer than 17 minutes
   (b) Less than 10 minutes
   (c) Between 9 and 13 minutes

Solution

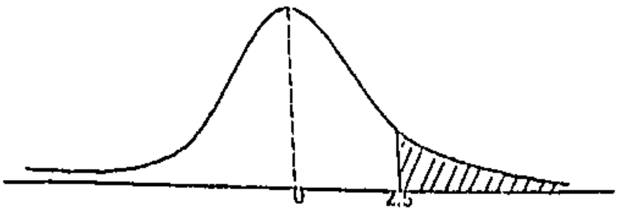
Let X be the time in minutes, taken to deliver milk to Market Street

$X \sim N(12, 2^2)$

Standardizing X using $Z = \frac{X-\mu}{\sigma}$ i.e $\frac{X-12}{2}$

(a) $P(X > 17) = P(Z > \frac{17-12}{2})$

$= P(Z > 2.5)$



$P(Z > 2.5) = 0.5 - P(0 < Z < 2.5)$

$= 0.5 - 0.4938 = 0.0062$

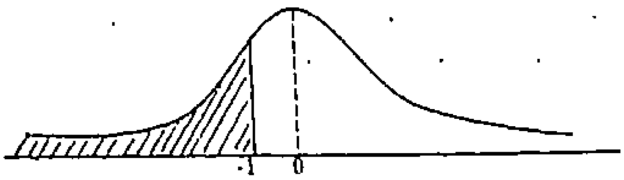
To find the number of days, multiply by 365

$365 \times 0.0062 = 2.263 \approx 2$

On 2 days in the year, he takes longer than 17 minutes

(b) $P(X < 10) = P(Z < \frac{10-12}{2})$

$= P(Z < -1)$



$P(Z < -1) = 0.5 - P(0 < Z < -1)$

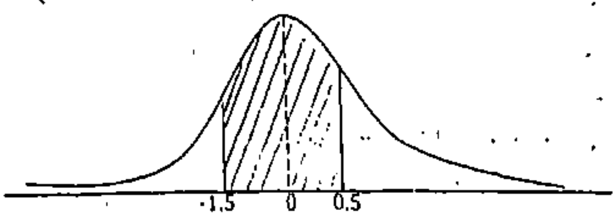
$= 0.5 - 0.3413 = 0.1587$

Now $365 \times 0.1587 = 57.92 \approx 58$

On 58 days in the year, he takes less than 10 minutes

(c) $P(9 < X < 13) = P(\frac{9-12}{2} < Z < \frac{13-12}{2})$

$= P(-1.5 < Z < 0.5)$



$P(-1.5 < Z < 0.5) = P(-1.5 < Z < 0) + P(0 < Z < 0.5)$

$= 0.1915 + 0.4332 = 0.6247$

Now $365 \times 0.6247 = 228.01 \approx 228$

On 228 days in the year, he takes between 9 and 13 minutes

3. A product sold in packets whose masses are normally distributed with a mean of 1.42 kg and a standard deviation of 0.025 kg

(a) Find the probability that the mass of a packet selected at random lies between 1.37 kg and 1.45 kg

(b) Estimate the number of packets in an output of 5000, whose mass is less than 1.35 kg

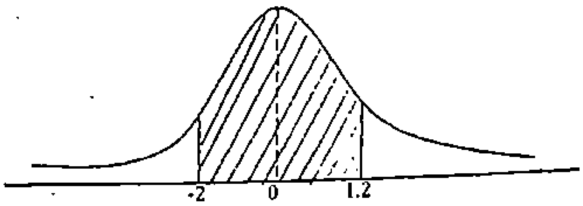
Solution

Let X be the mass in kilograms of a packet

$X \sim N(1.42, 0.025^2)$

(a) $P(1.37 < X < 1.42) = P(\frac{1.37-1.42}{0.025} < Z < \frac{1.45-1.42}{0.025})$

$= P(-2 < X < 1.2)$

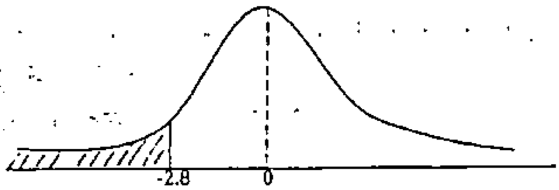


$P(-2 < X < 1.2) = P(0 < Z < -2) + P(0 < Z < 1.2)$

$= 0.4772 + 0.3849 = 0.8621$

(b) $P(X < 1.35) = P(Z < \frac{1.35-1.42}{0.025})$

$= P(Z < -2.8)$



$P(Z < -2.8) = 0.5 - P(0 < Z < -2.8)$

$= 0.5 - 0.4974 = 0.0026$

Since there are 5000 packets, multiply the probability by 5000

$5000 \times 0.0026 = 13$

Therefore 13 packets have a mass less than 1.35 kg

4. A machine used for filling bags with ground coffee, can be set to dispense any required mean weight of coffee in the bag can be modeled by a normal distribution with a mean of 128g and standard deviation of 1.95g per bag. Calculate the percentage of bags that contain less than 125g.

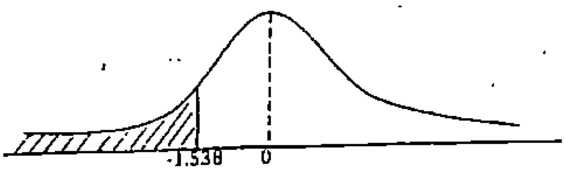
Solution

X is the weight in grams of coffee in the bag from the machine

$X \sim N(128, 1.95^2)$

$P(X < 125) = P(Z < \frac{125-128}{1.95})$

$= P(Z < -1.538)$



$P(Z < -1.538) = 0.5 - P(0 < Z < -1.538)$
$= 0.5 - 0.4938 = 0.062$

Percentage of bags that contain less than 125g = 0.062 × 100 = 6.2%

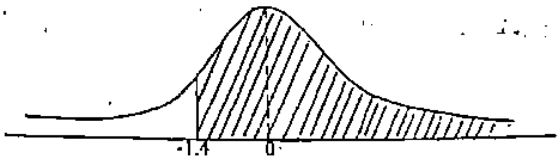
5. The distribution of the masses of adult husky dogs may be modeled by the normal distribution with mean 37 kg and standard deviation of 5 kg. Calculate the probability that an adult husky dog has a mass greater than 30 kg.

Solution

Let X be the mass in kg of a husky dog

$X \sim N(37, 5^2)$

$P(X > 30) = P(Z > \frac{30-37}{5})$
$= P(Z > -1.4)$



$P(Z > -1.4) = 0.5 + P(0 < Z < -1.4)$
$= 0.5 + 0.4192$
$= 0.9192$

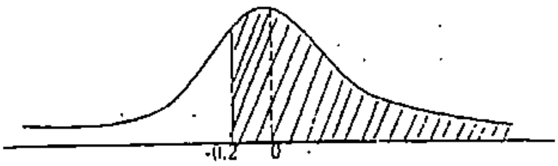
6. The marks of 500 candidates in an examination are normally distributed with a mean of 45 marks and standard deviation of 20 marks. Given that the pass mark is 41, estimate the number of candidates who passed the examination.

Solution

Let X be the examination mark scored by a candidate

$X \sim N(45, 20^2)$

$P(X > 41) = P(Z > \frac{41-45}{20})$
$= P(Z > -0.2)$



$P(Z > -0.2) = 0.5 + P(0 < Z < -0.2)$
$= 0.5 + 0.0793 = 0.5793$

Since there are 500 candidates, to find the number of candidates who pass; multiply the probability by 500

$500 \times 0.5793 = 289.65 \approx 290$

Therefore 290 candidates passed the examination

7. A continuous random variable X is denoted by N(15, 16). Find the probability that
   (i) X is less than 10
   (ii) X lies between 14 and 18

Solution

In this case, we need to find the mean and standard deviation.

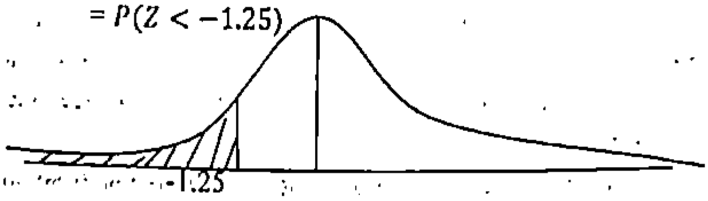
Comparing with $N(\mu, \sigma^2)$

$\mu = 15 \quad and\ \sigma^2 = 16 \Rightarrow \sigma = \sqrt{16} = 4$

We can now standardize using $Z = \frac{X-\mu}{\sigma}$

(i) $P(X < 10) = P(Z < \frac{10-15}{4})$

$= P(Z < -1.25)$



$P(Z < -1.25 = 0.5 - P(-1.25 < Z < 0)$

$= 0.5 - 0.3944 = 0.1056$

(ii) $P(14 < X < 18) = P(\frac{14-15}{4} < Z < \frac{18-15}{4})$

$= P(-0.2 < Z < 0.75)$

$P(-0.2 < Z < 0.75) = P(-0.2 < Z < 0) + P(0 < Z < 0.75)$

$= 0.0793 + 0.2734 = 0.3527$

Trial questions

1. The masses of packages from a particular machine are normally distributed with a mean of 200g and standard deviation 2g. Find the probability that a randomly selected package from the machine weighs
   (a) Less than 197g
   (b) More than 200.5g
   (c) Between 198.5g and 199.5g [Ans: (a) 0.0668 (b) 0.4013 (c) 0.1747]
2. The heights of boys at a particular age follow a normal distribution with mean 150.3 and variance 25 cm. Find the probability that a boy chosen at random from his age group has a height
   (a) Less than 153 cm
   (b) More than 158 cm
   (c) Between 150 cm and 158 cm [Ans: (a) 0.7054 (b) 0.0618 (c) 0.4621]
3. The random variable X is distributed normally such that $X \sim N(50, 20)$. Find
   P (X > 60.3) (b) P(X < 59.8) [Ans: (a) 0.0106 (b) 0.9857]
4. The masses of a certain type of cabbage are normally distributed with a mean of 1000g and standard deviation of 150g. In a batch of 800 cabbages, estimate how many have a mass between 750g and 1290g [Ans: 740]

5. The lifetime of a certain make of electric bulbs is known to be normally distributed with a mean life of 2000 hours and standard deviation of 120 hours. Estimate the probability that the life of such a bulb will be:
   (a) greater than 2150 hours
   (b) greater than 1910 hours
   (c) between 1850 hours and 2090 hours [Ans: (a) 0.1056 (b) 0.7734 (c) 0.6678]
6. The manufacturers of a new model of a car state that, when travelling at 56 miles per hour, the petrol consumption has a mean value of 32.4 miles per gallon with standard deviation of 1.4 miles per gallon. Assuming a normal distribution, calculate the probability that a randomly chosen car of that model will have petrol consumption greater than 30 miles per gallon when travelling at 56 miles per hour. [Ans : 0.957]
7. The processing time of a newly manufactured product is normally distributed with mean 110.5 minutes and standard deviation 12 minutes. Find the probability that the product is processed between 108 and 119 minutes [Ans: 0.3429]
8. In an orange plantation, the weights of oranges are normally distributed with a mean of 210g and variance 30g. Find the percentage of oranges that;
   (i) weigh between 201g and 221g
   (ii) weigh 197g and below [Ans: (i) 0.9276 (ii) 0.0088]
9. The time taken by Sam to pray is normally distributed with a mean of 24 minutes and a standard deviation of 4 minutes
   (a) If he prays every day, find the probability that his prayers take
      (i) more than 34 minutes
      (ii) at most 20 minutes
   (b) in 1000 days, estimate the number of days in which he prays between 34 and 36 minutes.
   [Ans: (a)(i) 0.0062 (ii) 0.1587 (b) 5]
10. The marks obtained in an aptitude test are normally distributed with mean 54 and standard deviation 14.2. Determine the probability that an examinee scored
    (i) between 60 and 70 marks
    (ii) at least 40 marks [Ans: (i) 0.2063 (ii) 0.3379]
11. The marks obtained by UACE candidates were found to be normally distributed with mean 50 and standard deviation 10
    (i) Determine the percentage of candidates who obtained more than 70 marks
    (ii) What percentage of the candidates obtained between 40 and 60 marks?
    (iii) What is the probability that a candidate selected at random from those who scored well above the average, scored more than 65?

    [Ans: (i) 2.28% (ii) 68.3% (iii) 0.0688]
12. The mean lifetime of a certain make of dry cells is 150 days and standard deviation 32 days. Their duration is normally distributed.
    (a) Find the probability that the cells will last between 125 and 210 days
    (b) If there are 300 dry cells, calculate how many will need replacement after 225 days.
    [Ans: (a) 0.7522 (b) 297]

13. A certain type of sweet potatoes has a mass which is normally distributed with mean 1.0 kg and standard deviation 0.15 kg. In a lorry load of 800 of these potatoes, estimate how many will have a mass between 0.85 and 1.5 kg [Ans: 673]
14. For a normal distribution with mean 5 and standard deviation 3, find
    (i) $P(X \geq 5.18)$ (ii) $P(X \leq 4.6)$ [Ans: (i) 0.476 (ii) 0.447]
15. The weights of army recruits form a normal distribution with mean 67.6 kg and standard deviation 6.2 kg. At most how many can be expected to weigh more than 79 kg if there are 1000 recruits? [Ans: 350]
16. The heights of college students are normally distributed with mean 164 cm and standard deviation 7.2 cm. Calculate the probability that the mean height of the students will
    (i) exceed 168 cm
    (ii) lie between 162 and 166 cm [Ans: (i) 0.2891 (ii) 0.2188]
17. The life time of a certain make of battery is normally distributed with an average life of 30 months and a standard deviation of 6 months. What percentage of these batteries can be expected to last from 24 months to 36 months? [Ans: 68.3%]
18. A random variable X is normally distributed with mean 40 and standard deviation 5. Determine the probability that X lies between 43 and 54 [Ans: 0.2716]
19. The masses of women in a certain town are normally distributed with mean 69.8 kg and standard deviation 6.2 kg. If 1000 women are selected, how many will be expected to weigh more than 80 kg? [Ans: 50]
20. The weights of 10000 cattle on a commercial farm are normally distributed with mean of 115 kg and standard deviation of 3kg.
    (i) If one of the cattle was selected at random from the farm, find the probability that its weight would lie between 115kg and 118kg
    (ii) Find how many cattle would weigh between 109 kg and 121 kg.
    [Ans: (i) 0.3413 (ii) 954]

CHAPTER 20: LINEAR MOTION

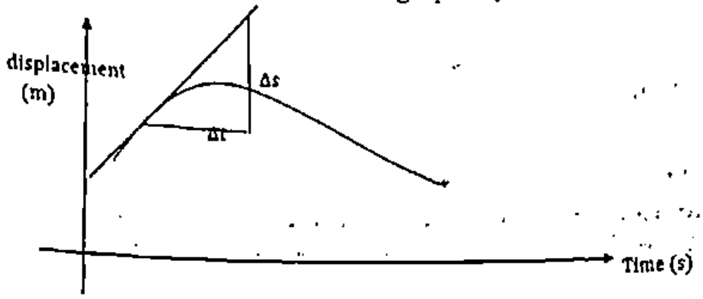
Linear motion refers to motion in a straight line

Displacement: this is the distance covered in a particular direction. It is a vector quantity.

Distance travelled = speed × time

$$S = V \times t$$

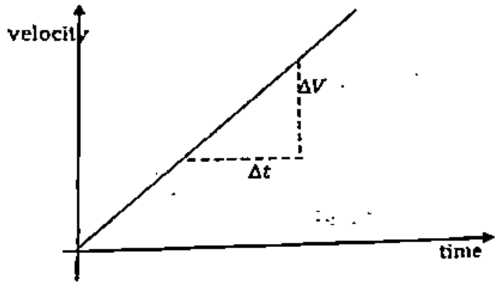
The gradient of the displacement time graph represents velocity



$$Gradient = \frac{\Delta S}{\Delta T} = Velocity\,(ms^{-1})$$

Velocity

This is the measure of the speed at which the body travels in a given direction. The area under the velocity time graph represents displacement and the gradient of the velocity time graph represents acceleration.



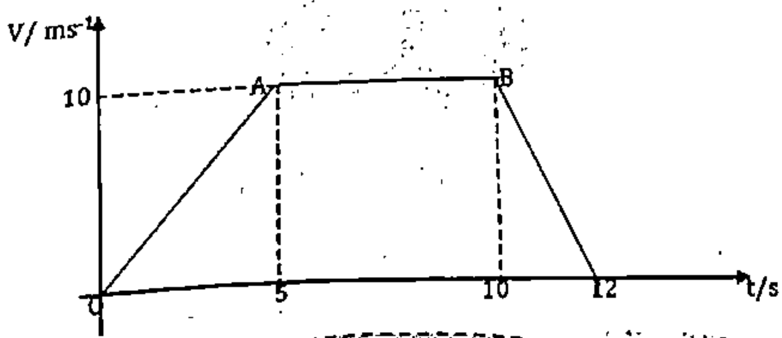
$$Gradient = \frac{\Delta V}{\Delta T}$$

But $acceleration = \frac{change\ in\ velocity}{time\ taken\ to\ change} = \frac{\Delta V}{\Delta T}$

$$a = \frac{\Delta V}{\Delta T}$$

Note: Acceleration is the rate of change of velocity and negative acceleration means retardation or deceleration

Example of a velocity time graph



- The body starts at $t = 0$, from rest (i.e. with zero velocity). From O to A, the velocity increases steadily until it reaches 10m/s at time $t = 5$s. Since OA is a straight line, the acceleration is uniform or constant and equal to $\frac{10}{5} = 2ms^{-2}$. At A, acceleration ceases.
- From A to B, the body travels with uniform/constant velocity of $10ms^{-1}$
- From B to C, the velocity decreases steadily and the body comes to rest again and it has a uniform retardation of $\frac{10}{5} = 2ms^{-2}$
- Average speed $= \frac{total\ distance\ covered}{total\ time\ taken}$

The equations of linear motion

There are three equations of linear motions which are expressed in terms of initial velocity, final velocity, displacement/ distance, time and acceleration. The students are only required to memorize these equations and apply them to solve problems related to linear motion. Their derivation is not important at this stage.

Equation 1: $V = U + at$

Equation 2: $S = Ut + \frac{1}{2}at^2$

Equation 3: $V^2 = U^2 + 2aS$

Example 1

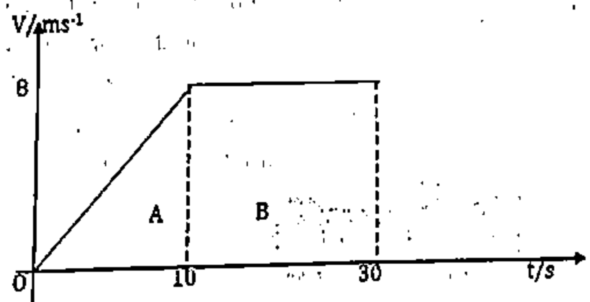
A car starts from rest, accelerates at $0.8ms^{-2}\ for\ 10s$ and then continues at a steady speed for a further 20s. Draw a velocity time graph and find the total distance travelled.

Solution

$U = 0$, $t = 10s$, $a = 0.8ms^{-2}, v =?$, $S =?$

From $V = U + at$

$V = 0 + 0.8 \times 10 = 8ms^{-1}$



Total distance covered = Area under the graph = Area A + Area B

$= \frac{1}{2} \times 10 \times 8 + 20 \times 8$

$= 40 + 160 = 200\ m$

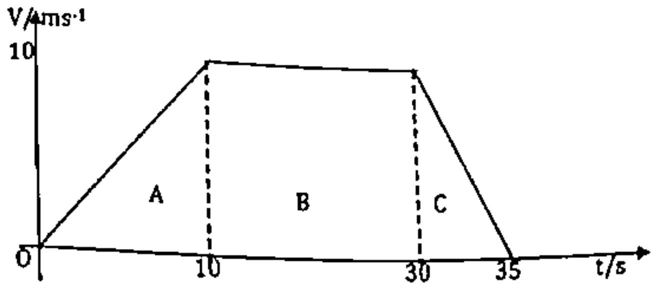
Example 2

A car starts from rest, accelerating at $1ms^{-1}$ for 10s. It then continues at a steady speed for a further 20s and decelerates to rest in 5s. Find (i) the distance travelled in metres (ii) average speed in ms^{-1} (iii) time taken to cover half the distance.

Solution

(i) $U = 0, a = 1ms^{-2}, t = 10s, V =?$

$V = 0 + 10 \times 1 = 10ms^{-1}$



Total distance covered = $area\ under\ the\ graph$

$= Area\ A + Area\ B + Area\ C$

$= \frac{1}{2} \times 10 \times 10 + 20 \times 10 + \frac{1}{2} \times 5 \times 10$

$= 50 + 200 + 25 = 275\,m$

(ii) Average speed $= \frac{total\ distance\ covered}{total\ yime\ taken} = \frac{275}{35} = 7.857ms^{-1}$

(iii) Half the distance $= \frac{1}{2} \times 275 = 137.5$

Time taken $= \frac{distance}{average\ speed} = \frac{137.5}{7.857} = 17.5s$

Example 3

A body decelerating at $0.8ms^{-2}$ passes a certain point with a speed of $30ms^{-1}$. Find its velocity after 10s and the distance covered in that time.

Solution

$a = -0.8ms^{-2}, \quad U = 30ms^{-1}, t = 10s,\ V = ?\ , S = ?$

Using the first equation of motion;

$V = U + at$

$V = 30 + (-0.8)(10) = 30 - 8 = 22ms^{-1}$

Now using the third equation of motion

$V^2 = U^2 + 2aS$

$22^2 = 30^2 + 2 \times (-0.8S)$

$484 = 900 - 1.6S$

$1.6S = 416$

$S = \frac{416}{1.6} = 260\,m$

Example 4

A particle moving along a straight line with uniform acceleration covers the first two consecutive distances of 100m and 140m in the time intervals of 20s and 40s respectively.

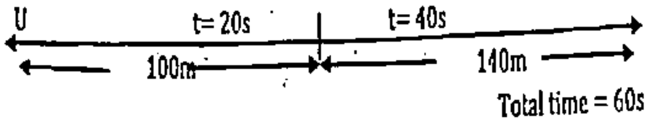
Calculate the;

(i) Acceleration and initial velocity of the particle

(ii) Total time taken before it comes to rest

Solution

Let the initial velocity and acceleration be U and a respectively



Total distance = 240m

Using the 2nd equation of motion; $S = Ut + \frac{1}{2}at^2$

$$100 = 20U + \frac{1}{2}a(20)^2$$

$$100 = 20U + 200a \text{ ………… } (i)$$

Similarly;

$$240 = 60U + \frac{1}{2}a(60)^2$$

$$240 = 60U + 1800a \text{ ………… } (ii)$$

Solving eqn (i) and eqn (ii) gives;

$$9(i) - (ii); \quad 660 = 120U$$

$$U = \frac{660}{120} = 5.5ms^{-1}$$

Now substituting for U in eqn (i) gives;

$$100 = 110 + 200a$$

$$-10 = 200a$$

$$a = -\frac{10}{200} = -0.05ms^{-2}$$

Note: the negative sign indicates deceleration

Example 5

A particle is moving in a straight line with a constant velocity of $60ms^{-1}$ for $3s$. Then there is a constant acceleration of $-3ms^{-2}$ for $5s$.

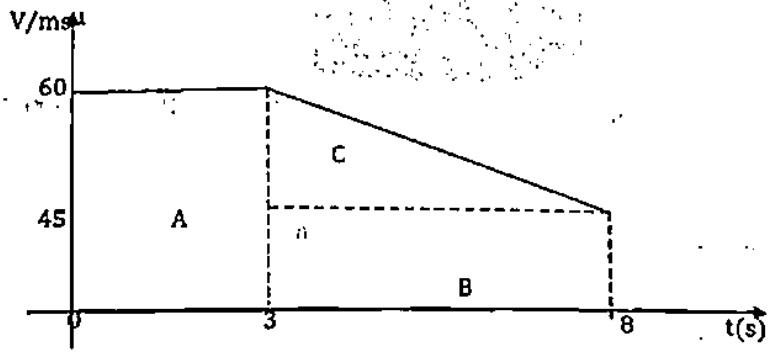
(a) Draw a velocity time graph for its motion

(b) Find the distance it has travelled.

Solution

(a) For $U = 60ms^{-1}, t = 5a, a = -3ms^{-2}$

$$V = U + at = 60 + (-3)(5) = 45ms^{-1}$$



(b) Total distance covered $= area\ under\ the\ graph$

$$= Area\ A + Area\ B + Area\ C$$

$$= (3 \times 60) + (5 \times 45) + \frac{1}{2} \times 5 \times 15$$

$$= 180 + 225 + 37.5 = 442.5\ m$$

Example 6

A particle starts from rest, moving with a constant acceleration of $1.5ms^{-2}$ $for\ 12\ seconds$. For the next 48s, the acceleration is $\frac{1}{8}ms^{-2}$ and for the last 10s, it decelerates uniformly to rest.

(i) Sketch a velocity time graph for the particle's motion
(ii) Find the distance travelled by the particle
(iii) Calculate the average velocity of its motion.

Solution

(i) First acceleration

$a = 1.5ms^{-2}$

$U = 0$

$V = U + at$

$V = 0 + 1.5 \times 12 = 90$

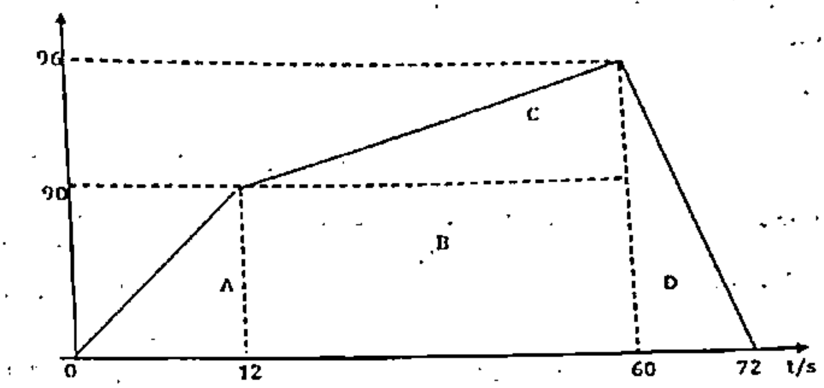
second acceleration

$U = 90ms^{-1},\ V = ?$

$a = \frac{1}{8}$

$t = 48s$

$V = 90 + \frac{1}{8} \times 48 = 96$



(ii) Total distance covered = $area\ under\ the\ graph$

$= Area\ A + Area\ B + Area\ C + Area\ D$

$= \frac{1}{2} \times 12 \times 90 + 48 \times 90 + \frac{1}{2} \times 48 \times 6 + \frac{1}{2} \times 10 \times 96$

$= 540 + 4320 + 144 + 480 = 5484\ m$

(iii) Average speed = $\frac{total\ distance\ covered}{total\ time\ taken} = \frac{5484}{70} = 78.34ms^{-1}$

Example 7

A particle starts form rest, moving with a constant acceleration of $1.5ms^{-2}$ for 30s. For the next 60s, the acceleration is $0.3ms^{-2}$. For the rest 25s, it decelerates uniformly to rest.

(i) Sketch the velocity time graph for the motion of the particle
(ii) Find the acceleration of the particle during the rest period of the journey.
(iii) Determine the total distance travelled by the particle
(iv) The a average speed for the whole journey

Solution

(i) First acceleration

$a = 1.5ms^{-2}$

$t = 30s$

$U = 0$

$V = 0 + 1.5 \times 30 = 45\ ms^{-1}$

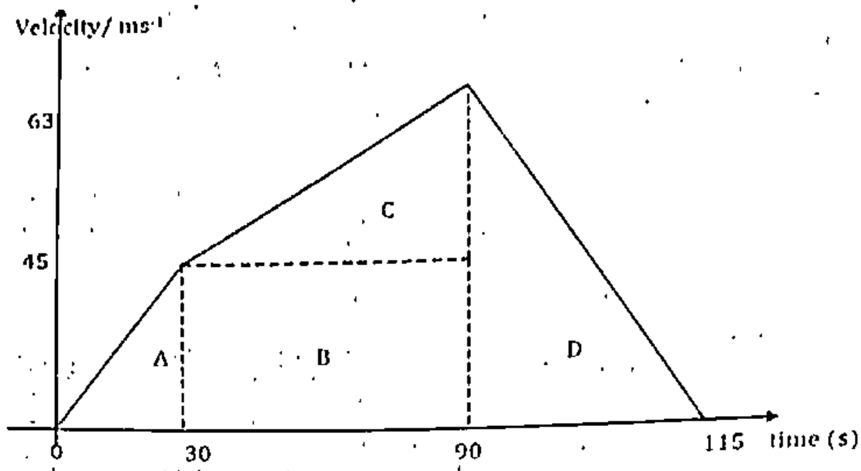
second acceleration

$a = 0.3ms^{-2}$

$t = 60s$

$U = 45ms^{-1}$

$V = 45 + 0.3 \times 60 = 63ms^{-1}$



(ii) $U = 63ms^{-1}, V = 0,\ t = 25s$

$a = \frac{v-u}{t} = \frac{0-63}{25} = -2.52ms^{-2}$

(iii) Total distance covered = $area\ under\ the\ graph$

$= Area\ A + Area\ B + Area\ C + Area\ D$

$= \frac{1}{2} \times 30 \times 45 + 60 \times 45 + \frac{1}{2} \times 60 \times 18 + \frac{1}{2} \times 25 \times 63$

$= 675 + 2700 + 540 + 787.5 = 4702.5\ m$

(iv) Average speed = $\frac{total\ distance\ covered}{total\ time\ taken} = \frac{4702.5}{115} = 40.89\ ms^{-1}$

Example 8

The table below shows the velocity of a particle during the course of its motion

t(s)	0	5	10	15	20	25	30
V(ms^{-1})	0	4	8	8	8	7	6

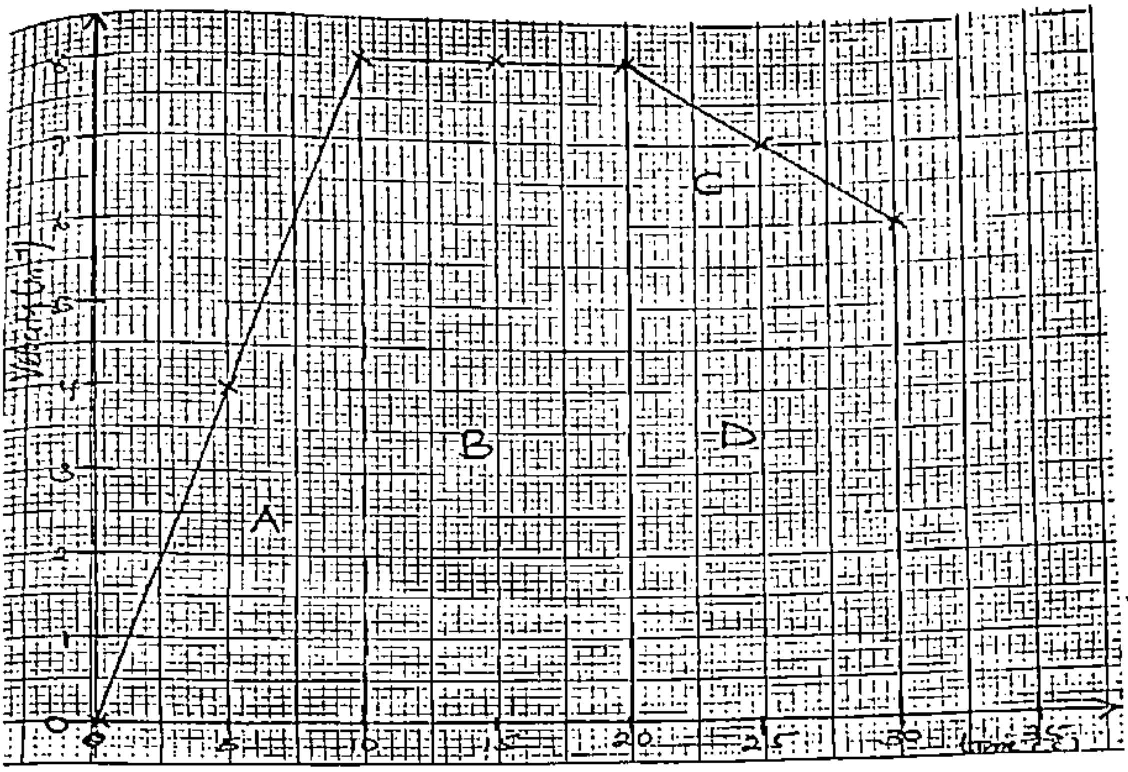
Plot a graph of velocity against time and use it to find

(i) The retardation of the body during the last 10s

(ii) The total distance travelled by the body

(iii) Describe the condition of the particle during the period 10s to 20s

Solution



(i) Retardation $= \frac{6-7}{30-25} = -\frac{1}{5} = -0.2ms^{-2}$

(ii) Total distance covered $=$ *area under the graph*

$= Area\ A + Area\ B + Area\ C + Area\ D$

$= \frac{1}{2} \times 10 \times 8 + 10 \times 8 + \frac{1}{2} \times 10 \times 2 + 10 \times 6$

$= 40 + 80 + 10 + 60 = 190\ m$

(iii) The particle travels with a constant velocity of $8ms^{-1}$

Trial questions

1. A particle travelling with an acceleration of 0.7 ms^{-2} passes a point O with speed 5 ms^{-1}. How long will it take to cover a distance of 250 m from? What will its speed be at this time?
(Ans: 20s, 20ms^{-1})

2. If particle passes a certain point with speed 5ms^{-1} and is accelerating at 3 ms^{-2}. How far will it travel in the next 25s? How long will it take to travel 44 m from the start?
(Ans; 16m, 4s)

3. A car starting from rest moves with a constant acceleration of x ms^{-2} for 10s and travels with constant velocity for a further 10s and then retards at 2x ms^{-2} to come to rest 300m from its starting point. Find the value of x (Ans: $x = \frac{12}{7}$ hint: sketch a v-t graph)

4. A motorist starting a car from rest accelerates uniformly to a speed of Vms^{-1} in 10s. He maintains this speed for another 50s and then applies brakes and decelerates uniformly to rest. His deceleration is numerically equal to twice the acceleration

(i) Sketch a velocity time graph

(ii) Calculate the time during which the deceleration takes place

(iii) Given that the total distance covered is 575m, calculate the value of V

(iv) Calculate the initial acceleration [Ans: (ii) 5 s (iii) 10 ms^{-1} (iv) 1 ms^{-2}]

5. Four points A, B, C and D lie on a straight road such that BC and CD are 448 cm and 576 cm respectively. A cat moving along this road covers each of these distances from A to D at 8s intervals with a constant acceleration

Find (i) the constant acceleration

(ii) Its speed at A to D

(iii) The distance AB.

6. A motorist accelerates uniformly from rest at a rate of a ms^{-2} for 10s and then travels at a constant speed for 20s and slows down to rest at a constant retardation of 2a ms^{-2} . If the total distance is 550 m.

(i) Sketch the velocity time graph for the motion of the motorist

(ii) Find the value of a [Ans: 2 ms^{-2}]

(iii) Find the maximum speed attained by the motorist [Ans: 20 ms^{-1}]

7. A body starts from rest and accelerates at 3ms^{-2} for 4s. It then travels with a maximum velocity for a further 3s and it finally to rest with a uniform retardation after another 5s. By sketching a velocity time graph. Find the average velocity for the whole journey. [Ans: 7.5 ms^{-1}]

8. A car travels along a straight road between two trading centres P and Q. The car starts from rest at P and accelerates at 2.5ms^{-2} until it reaches a speed of 40ms^{-1}. It then travels at this steady speed for distance of 3,120m and then decelerates at 4ms^{-2} to come to rest at Q.

(a) Sketch a velocity time graph for the motion of the car

(b) Determine the

(i) Total time taken for the car to move from P to Q

(ii) Distance from P to Q

(iii) Average speed of the car [Ans: 104 s, 3640 m, 35 ms^{-1}]

9. A body moves a long a straight line uniformly increasing its velocity from 2ms^{-1} to 18ms^{-1} in a time interval of 10s. Find the acceleration of the body during this time and the distance travelled. (Ans: 1.6ms^{-2}, 100m)

10. The table below shows the velocity of a particle during the course of its motion

Time(s)	0	5	10	20	30	60
Velocity (ms^{-1})	0	10	20	20	20	0

Plot a a graph of velocity against time and use it to find the

(i) Acceleration in the first 10s

(ii) Total distance covered

(iii) Describe the motion of the particle during the period $t = 10s$, $t = 30s$

[Ans: 2ms^{-2}, 800 m, constant]

11. A cyclist starting from rest accelerates uniformly until he reaches his maximum speed of 12ms^{-2} .He continues at this steady speed for next 2.4 km. He then applies brakes and decelerates to rest at a rate numerically equal to four times that of his acceleration. Sketch a velocity time graph. Given that the total distance travelled by the cyclist is 2.52km, calculate.

(i) The time during which the acceleration takes place (Ans: 4 s)

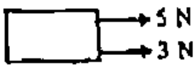
(ii) The distance over which deceleration takes place (24 m)

(iii)The total time for which the cyclist is in motion (240 s)

CHAPTER 21: RESULTANT AND COMPONENTS OF FORCES

Resultant of two forces

The resultant R of two forces P and Q is that single force which could completely take the place of the two forces. The resultant R must have the same effect as the forces P and Q. When only parallel forces are involved. It is easy to find the resultant. For example

the forces 

could be replaced by 

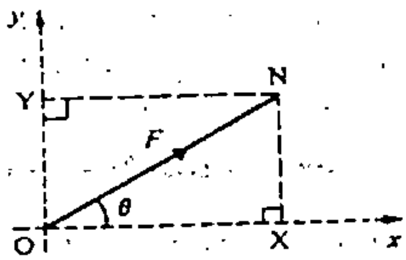
the forces 

could be replaced by 

Resolving a force; components

The component of the force F in any given direction is the measure of the effect of the force F in that direction.

Consider s force F acting at an angle θ to the x-axis as shown below. Let AB represent the force F and the angle BXO = 90^0, AX and AY represent the horizontal and vertical components of F, along the x and y axes respectively.



$\frac{AX}{AB} = \cos\theta$ and $\frac{AY}{AB} = \cos BAY$

$AX = AB\cos\theta$ $AY = AB\cos(90^0 - \theta)$

$AX = F\cos\theta$ $AY = F\sin\theta$

Hence the components are $F\cos\theta$ *and* $F\sin\theta$ along the x and y axes respectively

Note: The rule for finding components may be stated as;

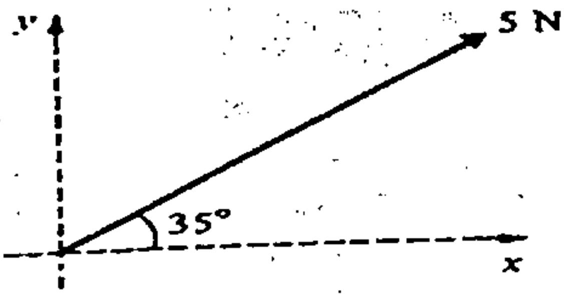
The component of force in any direction is the product of the magnitude of the force and the cosine of the angle between the force and the required direction i.e. $F\cos\theta$ *and* $F\cos(90^0 - \theta)$

Example 1

Find the components of the given forces in the direction of

(i) The x-axis (ii) the y-axis

(a)

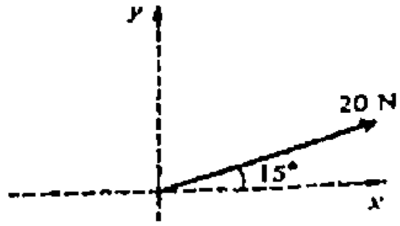


Solution

(i) Component along the x-axis = $5\cos 35^0 = 4.10\,N$

(ii) Component along the y-axis = $5\cos(90^0 - 35^0) = 5\sin 35^0 = 2.87N$

(b)



Solution

(i) Component along the x-axis = $20\cos 15^0 = 19.3\,N$

(ii) Component along the y-axis = $20\sin 15^0 = 5.18\,N$

(c)

(i) Component along the x-axis = $-10\cos 42^0 = -7.43\,N$

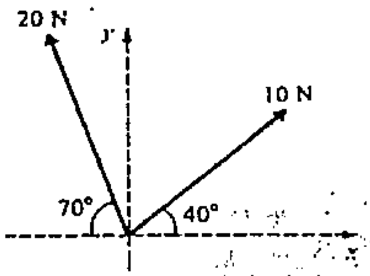
(ii) Component along the y-axis = $10\sin 42^0 = 6.69\,N$

Example 2

Find the sum of the components of the given factors in the direction of

(i) The x-axis (ii) the y-axis

(a)



Solution

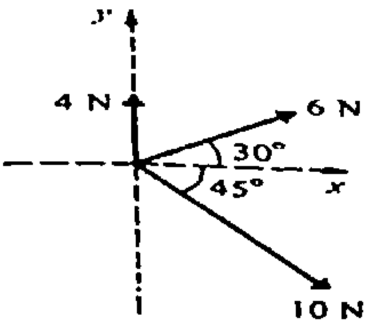
(i) Resolving along the x-axis

$10\cos 40^0 - 20\cos 70^0 = 7.66 - 6.84 = 0.82\,N$

(ii) Resolving along the y-axis

$10\sin 40^0 + 20\sin 70^0 = 6.43 + 18.79 = 25.22\,N$

(b)



Solution

(i) Resolving along the x-axis

$6\cos 30^0 + 10\cos 45^0 = 5.20 + 7.07 = 12.27\ N$

(ii) Resolving along the y-axis

$4 + 6\sin 30^0 - 10\sin 45^0 = 4 + 3 - 7.07 = -0.07N.$

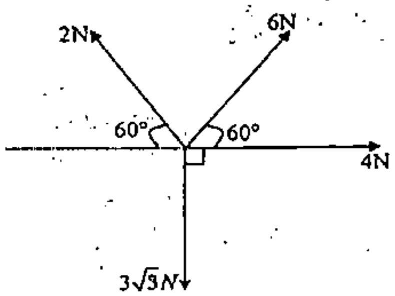
Note: when summing up components of forces, due regard should be given to the directions of the components.

Again if x and y are the magnitude of the perpendicular components of force AB, the magnitude of AB, known as the resultant force is given by;

$$F = \sqrt{x^2 + y^2}. \quad \textit{and the direction is given by } \tan\theta = \frac{\Sigma x}{\Sigma y}$$

Example 3

Find the resultant of the forces shown in the figure below



Solution

Resolving,

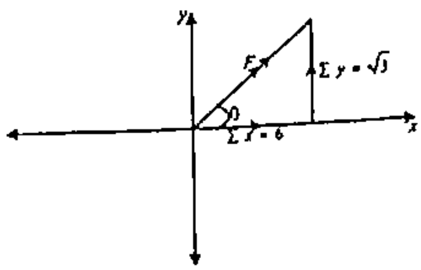
Force, F	Horizontally(→)	vertically (↑)
4	$4\cos 0 = 4$	$4\sin 0 = 0$
6	$6\cos 60 = 3$	$6\sin 60 = 3\sqrt{3}$
2	$2\cos 60 = 1$	$2\sin 60 = \sqrt{3}$
3	$3\sqrt{3}\cos 90 = 0$	$-3\sqrt{3}\sin 90 = -3\sqrt{3}$
	$\Sigma_x = 6$	$\Sigma_y = \sqrt{3}$

$\sum x$ *and* $\sum y$ are the summations of the horizontal and vertical components of the resultant force.

If F is the resultant force, then,

$$R = \sqrt{x^2 + y^2} = \sqrt{6^2 + \left(\sqrt{3}\right)^2} = 6.24\text{N}$$

Since force is a vector quantity, we also have to find its direction



$$\tan\theta = \frac{\Sigma x}{\Sigma y} = \frac{\sqrt{3}}{6} \Rightarrow \theta = 16.1^0$$

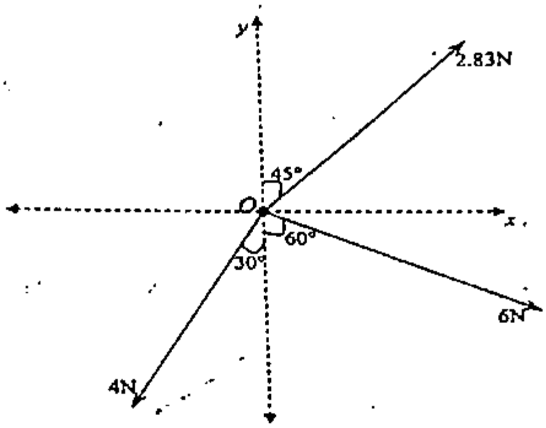
Therefore, the resultant force is 6.24N and makes an angle of 16.1^0 to the horizontal

Example 4

In the figure forces of 4N, 6N and 2.83N act on a particle O.

(i) Find the resultant force,

(ii) Find the acceleration of the particle if it has a mass 2kg

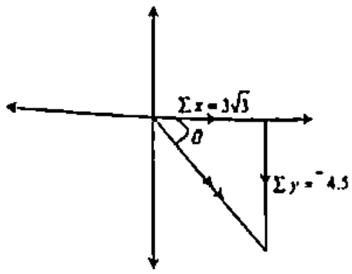


Solution

(i) Resolving,

Force, F	*Horizontally(→)*	*vertically (↑)*
4	- 4 cos 60 = -2	- 4 sin 60 = - 3.5
2.83	2.83 cos 45 = 2	2.83 sin 45 = 2
6	6 cos 30 = $3\sqrt{3}$	3 sin 30 = -3
	$\Sigma_x = 3\sqrt{3}$	$\Sigma_y = -4.5$

$$\text{Resultant} = \sqrt{\sum x^2 + \sum y^2} = \sqrt{(3\sqrt{3})^2 + (-4.5)^2} = 6.88 \text{ N}$$



$\tan\theta = \frac{\Sigma x}{\Sigma y} = \frac{4.5}{3\sqrt{3}} \Rightarrow \theta = 40.9^0$

Therefore the resultant force is 6.88N and is 40.9^0 below the horizontal

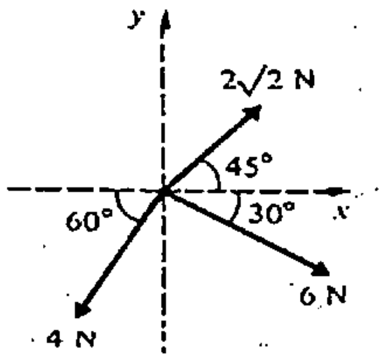
(ii) Force $= mass \times acceleration$

$\Rightarrow 6.88 = 2a$

$\therefore a = 3.44ms^{-2}$

Example 5

Find the magnitude and the direction of the resultant force



Solution

Resultant force horizontally $= 2\sqrt{2}\cos 45^0 - 6\cos 30^0 - 4\cos 60^0$

$= 2 - 5.20 - 2 = -5.20\ N$

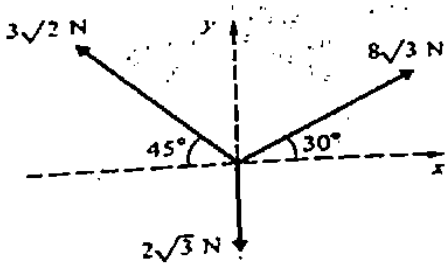
Resultant force vertically $= 2\sqrt{2}\sin 45^0 - 6\sin 30^0 - 4\sin 60^0$

$= 2 - 3 - 3.464 = -4.464$

Resultant force, $R = \sqrt{(-5.20)^2 + (-4.464)^2} = \sqrt{46.97} = 6.85\ N$

Example 6

Find the magnitude and the direction of the resultant force of the given forces below



Solution

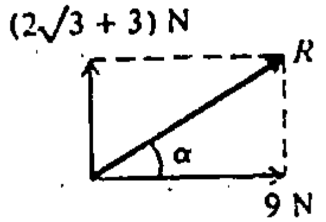
Resultant force horizontally $= 8\sqrt{3}\cos 30^0 - 3\sqrt{2}\cos 45^0 = 8\sqrt{3}\times\frac{\sqrt{3}}{2} - 3\sqrt{2}\times\frac{\sqrt{2}}{2}$

$= 12 - 3 = 9\,N$

Resultant force vertically $= 8\sqrt{3}\sin 30^0 + 3\sqrt{2}\sin 45^0 - 2\sqrt{3}$

$= 8\sqrt{3}\times\frac{1}{2} + 3\sqrt{2}\times\frac{\sqrt{2}}{2} - 2\sqrt{3}$

$= (2\sqrt{3}+3)N$



$R^2 = 9^2 + (2\sqrt{3}+3)^2 = 81 + 41.78 = 122.78$

$R = \sqrt{122.78} = 11.1\,N$ and the direction is at an angle α to x-axis, where

$\tan\alpha = \frac{(2\sqrt{3}+3)}{9} = 0.718$

$\alpha = \tan^{-1} 0.718 = 35.69^0$

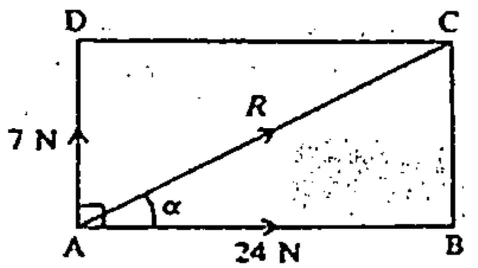
Therefore the resultant force is 11.1 N at an angle of 35.69^0 above the x-axis

Example 7

Two forces of 7 N and 24 N act away from the point A and make an angle of 90^0 with each other. Find the magnitude and direction of their resultant.

Solution

First make a sketch



$R^2 = 7^2 + 24^2$

$R = \sqrt{625} = 25\,N$

$\tan\alpha = \frac{7}{24}$

$\alpha = \tan^{-1}(\frac{7}{24}) = 16.26^0$

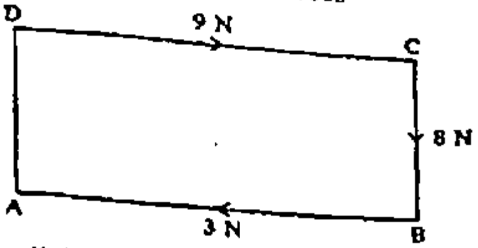
The resultant force is 25 N making an angle of 16.26^0 with the 24 N force

Example 8

ABCD is a rectangle. Forces of 9 N, 8 N and 3 N act along the lines DC, CB and BA respectively, in the directions as indicated by the order of the letters. Find the magnitude of the resultant and the angle it makes with DC.

Solution

First draw a diagram showing the forces

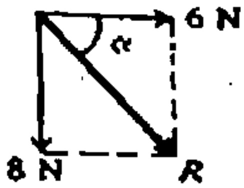


Resolving parallel to DC gives horizontal component = 9 – 3 = 6 N
Resolving parallel to BC gives vertical component = 8 N

$$R^2 = 6^2 + 8^2 = 36 + 64 = 100$$

$$R = \sqrt{100} = 10\ N$$

Draw a diagram to show the two components



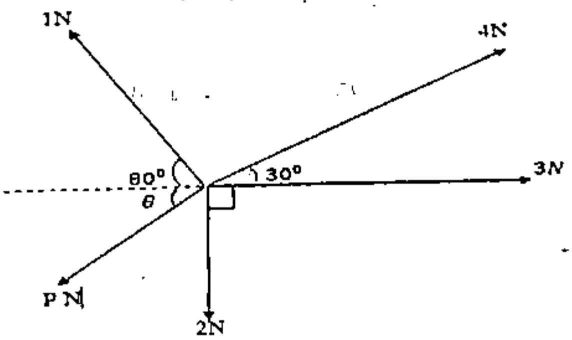
The direction is given by; $\tan\alpha = \frac{8}{6}$

$$\alpha = \tan^{-1}\left(\frac{8}{6}\right) = 53.13^0$$

The resultant is 10 N making an angle of 53.13^0 with DC

Example 9

Forces of magnitudes 1N, 4N, 3N, 2N and PN act on a particle as shown below



If the particle is in equilibrium, find the values of P and θ

Solution

At equilibrium; vertically

Resultant upward force = resultant downward force

$$4\sin 30^0 + \sin 80^0 = 2 + P\sin\theta$$

$$P\sin\theta = 4\sin 30^0 + \sin 80^0 - 2$$

$$P\sin\theta = 2 + 0.985 - 2$$

$$P\sin\theta = 0.985 \quad \ldots\ldots\ldots\ldots\ldots(i)$$

Similarly, horizontally,

Components to the R.H.S = components to the L.H.S

$$3 + 4\cos 30^0 = \cos 80^0 + P\cos\theta$$
$$P\cos\theta = 3 + 4\cos 30^0 - \cos 80^0$$
$$P\cos\theta = 3 + 3.464 - 0.174$$
$$P\cos\theta = 6.29 \;\ldots\ldots\ldots\ldots (ii)$$

Now dividing eqn (i) by eqn (ii) gives;

$$\frac{eqn\ (i)}{eqn\ (ii)};\quad \frac{P\sin\theta}{P\cos\theta} = \frac{0.985}{6.29}$$
$$\tan\theta = 0.157$$
$$\theta = \tan^{-1} 0.157 = 8.9^0$$

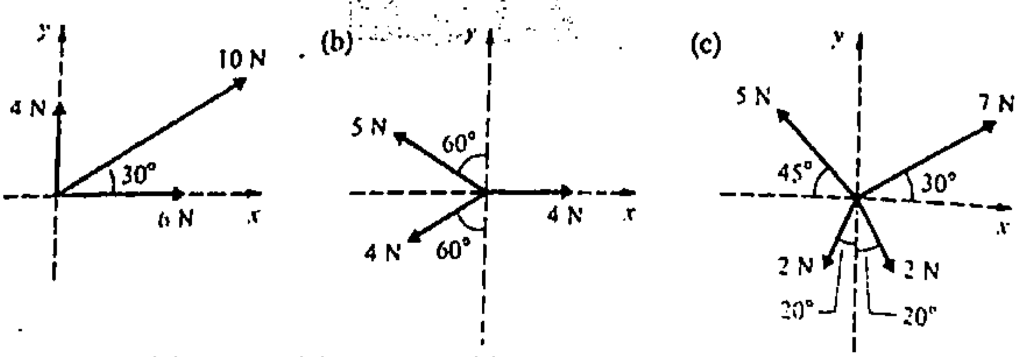
Substitution for θ in eqn (i) ; $P\sin\theta = 0.985$

$$P = \frac{0.985}{\sin 8.9^0} = 6.4\,N$$

Note: Alternatively, the reader can use the fact that the resultant of the components of the forces in any direction is zero at equilibrium i.e. $\sum x = 0 \quad and\ \sum y = 0$

Trial questions

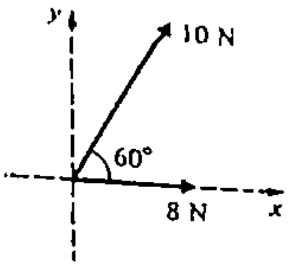
1. ABCD is a rectangle. Forces $6\sqrt{3}\,N, 2\,N\ and\ 4\sqrt{3}$ act along AB, CB and CD respectively, in the direction indicated by the order of the letters. Find the magnitude of the resultant and the angle it makes with AB [Ans: 4 N, 30^0]
2. ABCD is a rectangle. Forces of 8N, 4N, 10N and 2N act along AB, CB, CD and AD respectively in the directions indicated by the order of the letters. Find the magnitude and direction of the resultant force [Ans: 2.83N, 45^0 with BA]
3. ABCD is a rectangle. Forces of 3N, 4N and 1N act along AB, BC and DC respectively in the directions indicated by the order of the letters. Find the magnitude of the resultant and the angle it makes with AB. [Ans: 5.66 N, 45^0]
4. Four forces of magnitude 2N, 4N, 3N and 4N act at a point in the directions whose bearings are 000^0, 060^0, 180^0, 270^0 respectively. Calculate the magnitude of the resultant force [Ans: 1.49N]
5. Each of the following diagrams shows a number of forces. Calculate the magnitude of their resultant.



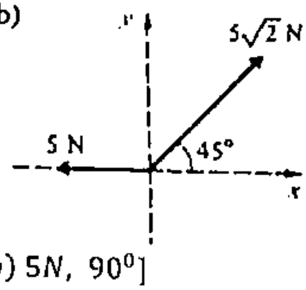
[Ans: $(a)17.2\,N$ (b) $3.8N$ (c) $4.1\,N$]

6. Find the magnitude and direction of the forces given in the diagrams below

(a)



(b)



[Ans: $(a) 15.6\,N, 33.7^0$ $(b)\, 5N,\ 90^0$]

7. ABCD is a square. Forces of 4N, 3N, 2N and 5N act along the sides AB, BC, CD and AD respectively in the directions indicated by the letters. Calculate the magnitude of the resultant and the angle it makes with AB [Ans: 8.2N, 76^0]

CHAPTER 22: FRICTION

The frictional force is the force that acts to oppose the relative motion of two bodies in contact i.e. if one pushes the block on the table with a small force P, the frictional force F comes in to operation and opposes the possible movement.

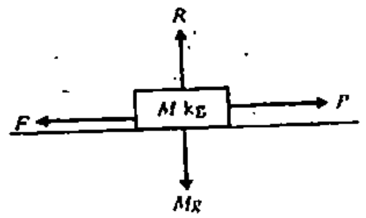
Coefficient of friction

The magnitude of maximum frictional force is proportional to the normal reaction R .The friction constant is called the coefficient of friction, μ, for two surfaces in contact.

$$F_{max} = \mu R$$

For a perfectly smooth, $\mu = 0$

Consider a block of mass M kg resting on a horizontal table and a horizontal force P N is applied.



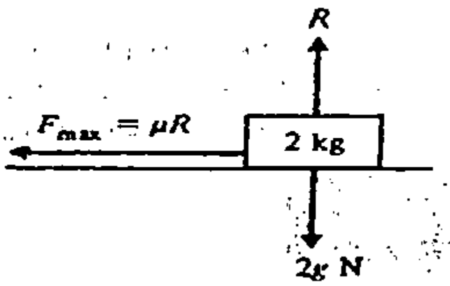
$$R = Mg$$

$$F = \mu Mg \quad where\ g = acceleration\ due\ to\ gravity$$

Example 1

Calculate the maximum frictional force which can act when a block of mass 2 kg rests on a rough horizontal surface if the coefficient of friction is (a) 0.7 (b) 0.2

Solution



(a) There is no motion perpendicular to the plane

Resolving vertically; R = 2g

$$R = 2 \times 9.8 = 19.6N$$

$$F_{max} = \mu R = 0.7 \times 19.6 = 13.72N$$

(b) Similarly, as before;

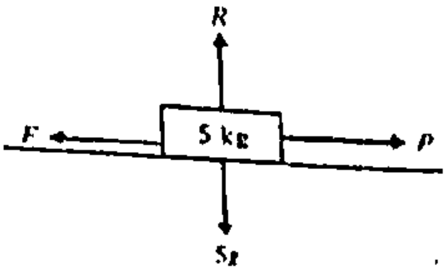
$$F_{max} = \mu R = 0.2 \times 19.6 = 3.92N$$

Example 2

A block of mass 5 kg rests on a rough horizontal plane, the coefficient of friction between the block and plane being 0.6. Calculate the frictional force acting on the block when a horizontal force P is applied to the block and the magnitude of P is

(a) 12N (b) 28N (c) 36 N. Also calculate the magnitude of any acceleration which may occur.

Solution



There is no motion perpendicular to the plane

Resolve vertically; $R = 5g = 5 \times 9.8 = 49\ N$

The frictional force will act in the direction opposite to that in which the force P acts. The maximum value of frictional force is μR

$$\mu R = 0.6 \times 49 = 29.4\ N$$

(a) If P = 12N, then P is less than μR, *so there is no motion*

Friction force, F = P

F = 12 N

(b) If P = 28N, then again P is less than μR and there is no motion

Friction force, F = P = 28N

(c) If P = 36N, then P is greater than the maximum value of frictional force, which is 29.4 N.

Frictional force acting = 29.4N, which does not prevent motion.

The block will move and the maximum value of μR will be maintained.

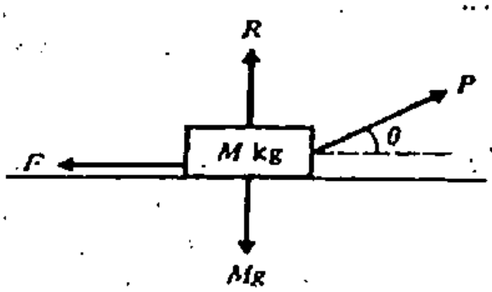
Using F = ma, the equation of motion is;

$$P - \mu R = ma$$

$$36 - 29.4 = 5a$$

$$a = 1.32 ms^{-2}$$

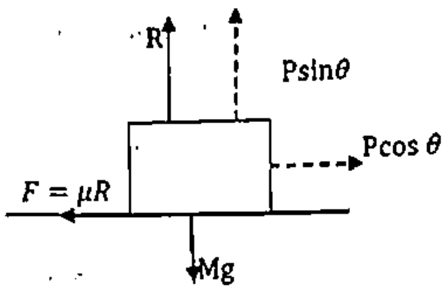
Applied force not horizontal



When a force on the block of mass M kg is inclined at an angle θ above the horizontal, this has two effects;

(i) The component of P in the vertical direction decreases the magnitude of the normal reaction R

(ii) Only the component of P in a horizontal direction is tending to move the block



Resolving vertically; $R + P\sin\theta = Mg$

$R = Mg - P\sin\theta \quad \ldots\ldots(i)$

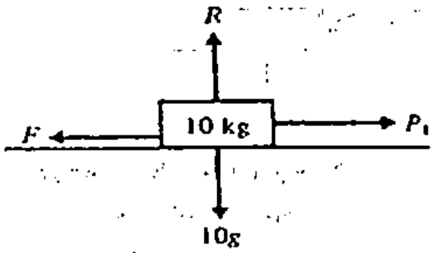
Resolving horizontally; $P\cos\theta = F = \mu R \quad \ldots\ldots(ii)$

Example 1

A 10kg truck lies on a horizontal floor. The coefficient of friction between the truck and the floor is $\frac{\sqrt{3}}{4}$. Calculate the magnitude of force P which is necessary pull the truck horizontally if P is applied (i) horizontally (ii) at 30^0 above the horizontal

Solution

(a)



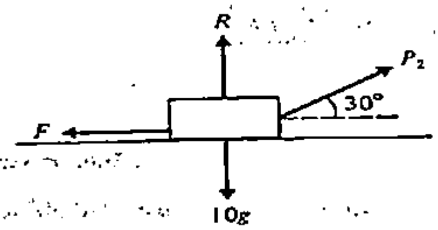
In the position of limiting equilibrium, $P_1 = F$

$P_1 = \mu R$

$P_1 = \frac{\sqrt{3}}{4} \times 10g = \frac{\sqrt{3}}{4} \times 10 \times 9.8 = 42.43\ N$

For motion to take place, the applied force must exceed 42.43 N

(b)



Resolving vertically; $R + P\sin 30^0 = 10g$

$R = 98 - P\sin 30^0$

$R = 98 - \frac{P}{2}$

In the position of limiting equilibrium;

$P\cos 30^0 = \mu R$

$P \times \frac{\sqrt{3}}{2} = \mu\left(98 - \frac{P}{2}\right)$

$P \times \frac{\sqrt{3}}{2} = 98\mu - \frac{\sqrt{3}}{4} \times \frac{P}{2}$

$\frac{\sqrt{3}}{2}P + \frac{\sqrt{3}}{8}P = \frac{\sqrt{3}}{2} \times 98$

$P(5\sqrt{3}) = 2\sqrt{3} \times 98$

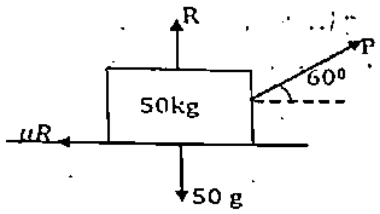
$P = 39.2\ N$

Example 2

A box of mass 50 kg is to be pushed along a rough floor by a force acting at the centre of its top surface. The force is at an angle of 60^0 to the horizontal. If the coefficient of friction is 0.25, calculate the least force which will move the box

Solution

Let the force be P



Resolving vertically;

$R + P\sin 60^0 = 50\text{g}$

$R = 50\text{g} - P\sin 60^0$

Resolving horizontally;

$P\cos 60^0 = \mu R$

$P\cos 60^0 = 0.25(50\text{g} - \text{P}\sin 60^0)$

$\frac{P}{2} = 12.5\text{g} - \text{P} \times \frac{\sqrt{3}}{2} \times \frac{1}{4}$

$\frac{P}{2} + \frac{\sqrt{3}}{8}P = 12.5 \times 9.8$

$P\left(\frac{1}{2} + \frac{\sqrt{3}}{8}\right) = 122.5$

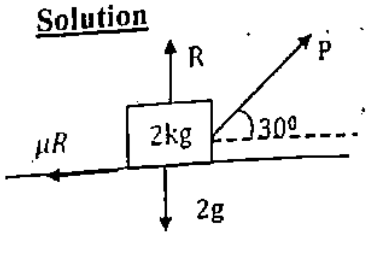
$P(4 + \sqrt{3}) = 8 \times 122.5$

$P = \frac{980}{4+\sqrt{3}} = 170.97\ N$

Example 3

A particle of mass 2 kg rests on a rough horizontal ground. The coefficient of friction between the particle and the ground is $\frac{1}{2}$. Find the magnitude of a force P acting upwards on the particle at 30^0 to the horizontal which will just move the particle.

Solution



Resolving vertically;

$$R + P\sin 30^0 = 2g$$

$$R = 2g - P\sin 30^0$$

Resolving horizontally;

$$P\cos 30^0 = \mu R$$

$$P\cos 30^0 = \tfrac{1}{2}(2g - P\sin 30^0)$$

$$2P \times \tfrac{\sqrt{3}}{2} = 2g - P \times \tfrac{1}{2}$$

$$P\sqrt{3} + 0.5P = 19.6$$

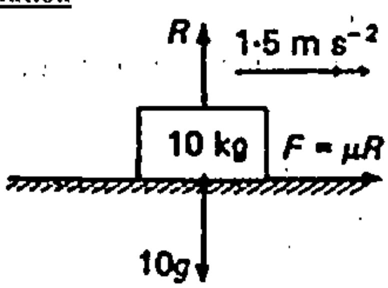
$$2.232P = 19.6$$

$$P = \frac{19.6}{2.232} = 8.78\ N$$

Example 4

A parcel of mass 10 kg rests on a lorry. When the lorry is accelerating at 1.5 ms^{-1}, the parcel is on the point of sliding backwards. What is the coefficient of friction between the parcel and the lorry?

Solution



Vertically; $R = 10g$

From; $F = 10 \times 1.5 = 15$

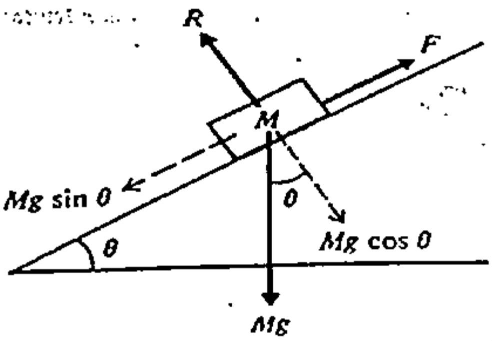
$$\mu R = 15$$

$$\mu \times 10g = 15$$

$$\mu = \frac{15}{10 \times 9.8} = 0.153$$

Rough inclined plane

Consider a body of mass M kg resting on a plane which is inclined at an angle θ to the horizontal.



Resolving at right angles to the plane;

$$R = Mg\cos\theta$$

The component $Mg\sin\theta$ accting down the plane will cause motion unless the frictional force, F acting up the plane balances it.

At equillibrium; $F = Mg\sin\theta$

$$F_{max} = \mu R$$

$$F = \mu Mg\cos\theta = Mg\sin\theta$$

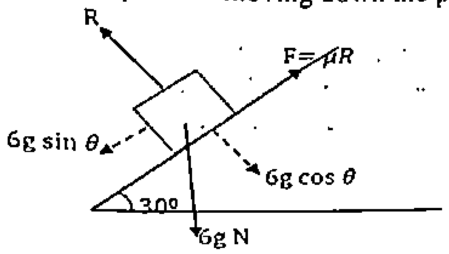
$$\mu = \tan\theta$$

Example 1

A body of mass 6 kg rests in limiting equillibrium on a rough plane inclined at 30^0 to the horizontal. Find the coefficient of friction between the body and the plane.

Solution

Since the body is on the point of moving down the plane, the friction force acts up the plane



Resolving at right angles to the plane;

$$R = 6g\cos 30^0$$

Resolving parallel to the surface of the plane;

$$6g\sin 30^0 = \mu R$$

$$6g\sin 30^0 = \mu \times 6g\cos 30^0$$

$$\mu = \frac{6g\sin 30^0}{6g\cos 30^0} = 0.577$$

Example 2

A mass of 10 kg is placed on a plane inclined at an angle of 30^0 to the horizontal. What force parallel to the plane is required to

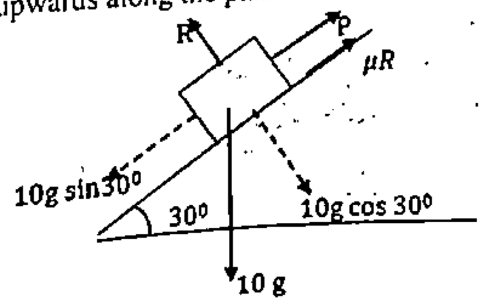
(i) hold the mass at rest

(ii) make the mass move steadily up the plane with an acceleration of $2ms^{-1}$

coefficient of friction between the mass and the plane is 0.4

Solution

(a) As the block is just held at rest, it is on the verge of slipping down, hence friction force $F = \mu R$ acts upwards along the plane.



Resolving at right angles to the plane;

$R = 10g\cos 30^0$

Resolving parallel to the plane;

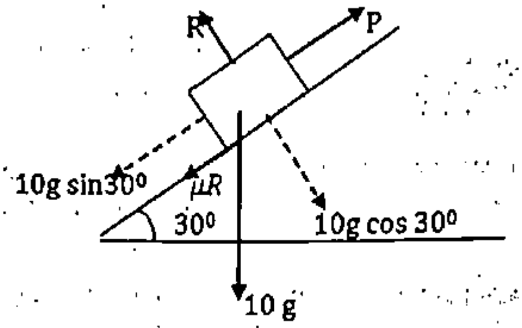
$P + \mu R = 10g\sin 30^0$

$P + 0.4(10g\cos 30^0) = 10g\sin 30^0$

$P = 10g\sin 30^0 - 4g\cos 30^0$

$P = 49 - 33.95 = 15.5\ N$

(b) If the mass is moving steadily up the plane, the friction force, $F = \mu R$ acts down the plane



Resolving at right angles to the plane;

$R = 10g\cos 30^0$

Resolving parallel to the plane;

Resultant force = ma

$P - (10g\sin 30^0 + \mu R) = 10 \times 2$

$P - (10g\sin 30^0 + 0.4 \times 10g\cos 30^0) = 20$

$P - (49 + 33.25) = 20$

$P = 20 + 82.35 = 102.35\ N$

Example 3

A parcel of mass 2 kg is placed on a rough plane which is inclined at 45^0 to the horizontal. The coefficient of friction between the parcel and the plane is 0.25. Find the force that must be applied in the direction parallel to the plane so that;

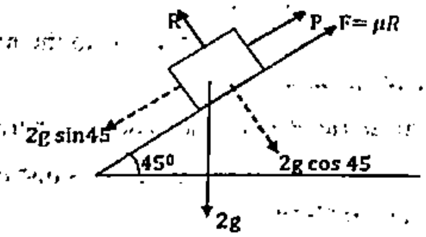
(i) The parcel is just prevented from sliding down the plane

(ii) The parcel moves up the plane with an acceleration of $1.5ms^{-2}$.

(iii) If it is prevented from sliding down the plane, find the frictional force that acts up the plane.

Solution

(i) Let the force be P



$R = 2g\cos 45^0$

Parallel to the plane;

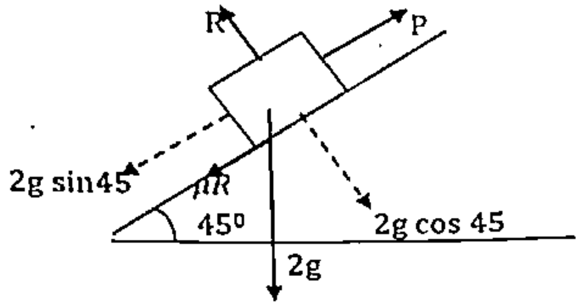
$P + \mu R = 2g\sin 45^0$

$P + 0.25(2g\cos 45^0) = 29\sin 45^0$

$P = 2g\sin 45^0 - 0.5g\cos 45^0$

$P = 10.39\ N$

(ii) On the point of moving up the plane, the friction force acrs down the plane



Parallel to the plane; $R = 2g\cos 45^0$

Along the plane; $P = 2g\sin 45^0 + \mu R$

$P = 2g\sin 45^0 + 0.25(2g\cos 45^0)$

$P = 2 \times 9.81 \times \sin 45^0 + 0.5 \times 9.81 \times \cos 45^0$

$P = 13.86 + 3.46 = 17.32\ N$

(iii) If the parcel moves up the plane;

Then Resultant force, F = mass × acceleration

Resultant force up the plane = $P - (2g\sin 45^0 + \mu R)$

$P - (2g\sin 45^0 + 0.25 \times 2g\cos 45^0) = 2 \times 1.5$

$P - (13.86 + 3.46) = 3$

$P = 3 + 17.32 = 20.32\ N$

Trial questions

1. If a force of 10N is just sufficient to move a mass of 2kg resting on a rough horizontal table, find the coefficient of friction [Ans: 0.51]
2. A block of mass 10kg is placed on an inclined plane at an angle of 30^0 to the horizontal where the coefficient of friction between the plane and the surface is 0.5. Find
   (a) The force required to make the block move up the plane
   (b) Keep the block at rest
   (c) Acceleration of the block down the plane
   [Ans: (a) 91.4 N (b) 6.6 N (c) 0.66 ms^{-2}]
3. A block of mass 5 kg placed on an inclined plane of angle 60^0 to the horizontal is just at rest. Find the force parallel to the plane required to push the block up the plane [Ans: 98 N]
4. A box of mass 20kg starts from rest and slides down a slope inclined at 30^0 to the horizontal. If the coefficient of friction is 0.4, find the acceleration of the box. [Ans:]
5. A particle is placed on a rough plane inclined at an angle, θ to the horizontal, where $\tan\theta = \frac{3}{4}$. The coefficient of friction between the plane and the particle is 0.5. The particle is rest under the action of a force F applied in an upward direction parallel to the plane.
   (a) Calculate the value of F when the particle is about to move down the plane
   (b) Calculate the value of F when the particle is about to move up the plane.
   [Ans: (a) 2 N (b) 10 N]

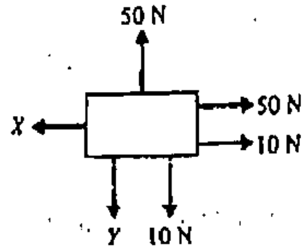
CHAPTER 23: NEWTON'S LAWS OF MOTION

First law:
A change in the state of motion of a body is caused by a force acting on the body otherwise.

Body at rest

If forces act on a body and does not move, the forces must balance. Hence if a number of forces act on a body and it remains at rest, the resultant force in any direction must be zero.

Example 1
A body is at rest when subjected to the forces shown below. Find x and y



Solution

The horizontal forces must balance

$$X = 50 + 10 = 60\ N$$

The vertical forces must also balance

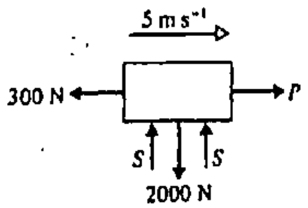
$$Y + 10 = 50$$

$$Y = 40\ N$$

A body in motion

A body can only change its velocity i.e increase its speed, slow down or change direction if a resultant force acts upon it. Thus if a body is moving with a constant velocity, there can be no resultant force acting on it.

Example 2
A body moves horizontally at a constant speed of $5ms^{-1}$ and is subjected to the forces shown. Find P and S.



$$S + S = 2000\ N$$

$$2S = 2000 \Rightarrow S = 1000\ N$$

Since the horizontal velocity is contant;

$$P = 300\ N$$

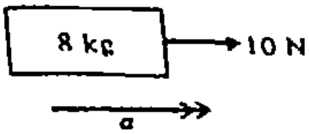
Second law:

Newton's second law can be summarised by the equation F = ma, where F is the resultant force on the body, m is the mass of the body and a is the accleration of the body produced in the direction of the applied force or resultant force.

Example 3

A body of mass 8 kg is acted upon by a force of 10N. Find its acceleration

Solution

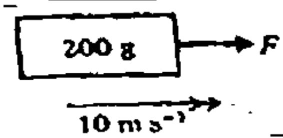


$$F = ma$$
$$10 = 8a$$
$$a = \frac{10}{8} = 1.25ms^{-2}$$

Example 4

Find the resultant force that would give a body of mass 200 g an acceleration of $10ms^{-1}$.

Solution



$m = 200g = \frac{200}{1000} = 0.2\ kg$

Using $F = ma$

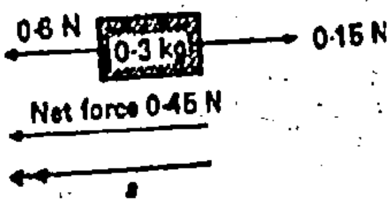
$F = 0.2 \times 10 = 2N$

The force is 2N

Example 5

A horizontal force of 0.6N acts on a body of mass 0.3 kg. There is a resistance of 0.15 N opposing the first force. What acceleration will be produced?

Solution



$Net\ force = 0.6 - 0.15 = 0.45\ N$

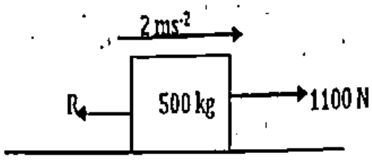
The acceleration will take place in the direction of the resultant force

$$F = ma$$
$$0.45 = 0.3a$$
$$a = \frac{0.45}{0.3} = 1.5\ ms^{-2}$$

Example 6

A car of mass 500 kg moves along a level road with an acceleration of $2ms^{-2}$. If it is exerting a forward force of 1100 N, what resistance is the car experiencing?

Solution



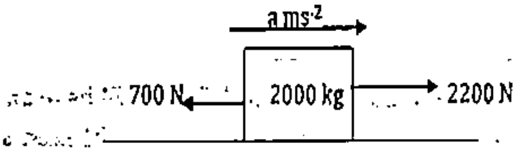
$$Net\ force = ma$$
$$Net\ force = 1100 - R$$
$$1100 - R = 500 \times 2$$
$$1100 - R = 1000$$
$$R = 1100 - 1000 = 100\ N$$

Example 7

A van of mass 2 tonnes moves along a level road against a resistance of 700N. If its engine is exerting a forward force of 2200N, find the acceleration of the van.

Solution

Mass = 2×1000 =2000kg

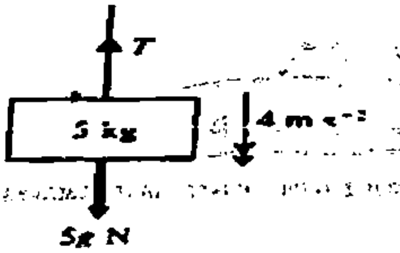


$$Net\ force = Ma$$
$$Net\ force = 2200 - 700 = 1500\ N$$
$$1500 = 2000a$$
$$a = \frac{1500}{2000} = 0.75\ ms^{-2}$$

Example 8

A box of mass 5 kg is lowered vertically by a rope. Find the force in the rope when the box is lowered with an acceleration of $4ms^{-2}$.

Solution



$$Mass\ of\ box = 5\ kg$$
$$weight\ of\ box = 5g\ N$$

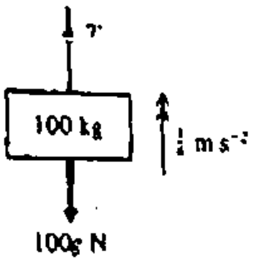
Resultant vertical force = (5g – T) *down wards*

$$5g - T = 5 \times 4$$
$$5 \times 9.8 - T = 20$$
$$T = 49 - 20 = 29$$

Example 9

A pack of bricks of mass 100 kg is hoisted up the side of the house. Find the force in the lifting rope when the bricks are lifted with an accelertaion of 0.25 ms⁻².

Solution



Mass of bicks $= 100\text{kg}$

Weight of bricks $= 100g$ N

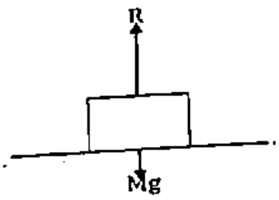
$\mathit{The\ resultant\ upward\ force} = (T - 100g)$ [since motion is upward]

$$T - 100g = 100 \times 0.25$$
$$T = 100g + 25 = 100 \times 9.8 + 25 = 1005 \text{ N}$$

Newton's third law

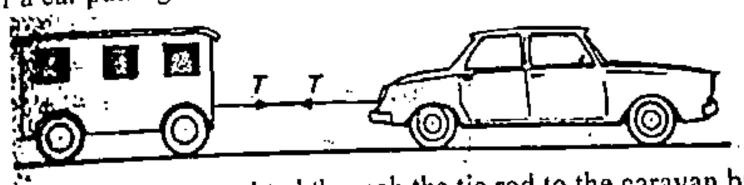
If two bodies are in contact, actually touching or connected by a string, rope or rod. They will have an effect on each other. Newton's third law states that if two bodies A and B are in contact, A will exert a force on B and B will exert an equal but opposite force on A i.e. equal in magnitude but directed in the opposite sense along the same line.

1. Consider a box of mass M kg resting on a horizontal floor



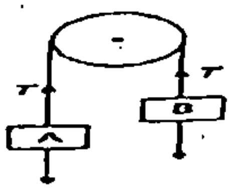
The box exerts a force on the floor and the floor reacts by exerting an equal but opposite force R

2. Consider a car pulling a caravan



The pull of the car is transmitted through the tie rod to the caravan but the caravan equally pulls the car backwards.

3. Consider two masses suspended by a string over a frictionless (smooth) pulley

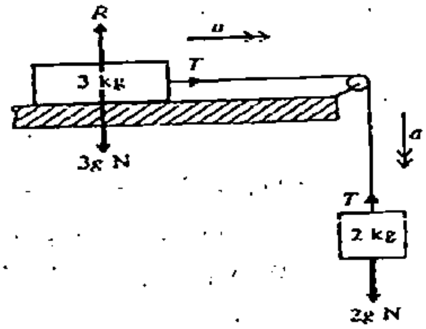


The string transmits a tension which pulls A upwards when considering A but pulls B upwards when considering B

CONNECTED PARTICLES

Case 1

Consider a body of mass 3 kg at rest on a smooth horizontal table. This body is connected by a light string which passes over a smooth pulley at the edge of the table to another body of mass 2 kg hanging freely.



The 3 kg mass will not move in a vertical direction, so the vertical forces acting on it must balance.

R = 3g

The horizontal force acting on the 3 kg mass is T,

Using F = ma gives the equation of motion as;

$T = 3a$ ………………….(i)

The 2 kg mass moves vertically downwards

Using F = ma gives the equation of motion as;

$2g - T = 2a$ ……………………..(ii)

Solving the two equations (i) and (ii) simultaneously i.e (i) + (ii)

$2g = 3a + 2a$

$a = \frac{2}{5}g = \frac{2}{5} \times 9.8 = 3.92\ ms^{-2}$

Substituting for a into equation (i)

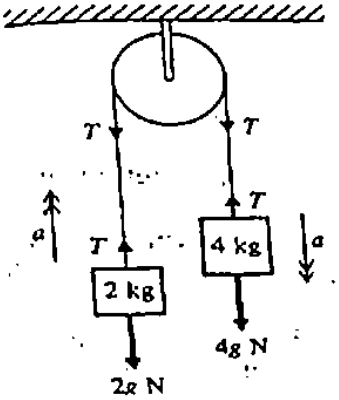
$T = 3 \times 3.92 = 11.76\ N$

Case 2

Particles of mass 4kg and 2 kg are connected by a light string over a smooth fixed pulley. The particles hang freely and are released from rest. Find the acceleration of the two particles and the tension in the string.

Solution

Let the acceleration be a and the tension in the string be T



Using F = ma

For 2kg : $T - 2g = 2a$ ………………….(i)

For 4kg: $4g - T = 4a$ …………………….(ii)

Adding equations (i) and (ii) gives;

$$2g = 6a$$

$$a = \frac{1}{3}g = \frac{1}{3} \times 9.8 = 3.27ms^{-2}$$

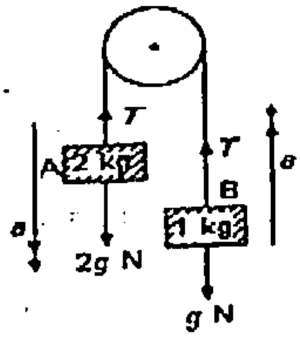
Substituting for a in equation (i);

$$T = 2a + 2g$$

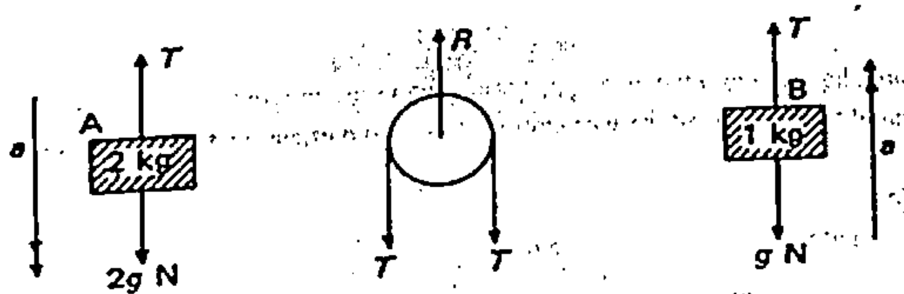
$$T = 2(3.27) + 2(9.8) = 26.13\ N$$

Example 2

A light inextensible string is placed over a smooth pulley. To the ends of the string are attached masses of 2 kg (A) and 1 kg (B) and both parts of the string are vertical. With what acceleration does the system move? What is the reaction at the axle of the pulley?



The 1 kg mass accelerates upwards as the 2kg mass accelerates downwards



For mass A, since acceleration is downwards; $2g - T = 2a$ ……….(i)

For mass B, acceleration is upwards; $T - g = a$ ……………….(ii)

For the pulley, since it has no acceleration; R = 2T …………….(iii)

Now solving equations (i) and (ii) for a and T

$Eqn\ (i) + (ii)$

$g = 3a$

$a = \frac{1}{3}g = \frac{1}{3} \times 9.8 = 3.27ms^{-2}$

From equation (ii); $T = a + g$

$= 3.27 + 9.8 = 13.1\ N$

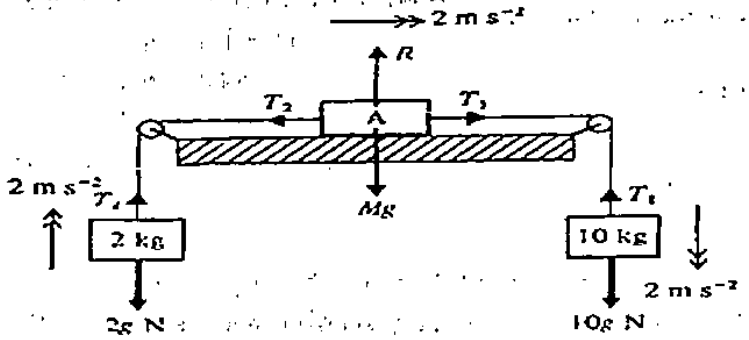
From (iii) ; R = 2T

$R = 2 \times 13.1 = 26.2$ N

Therefore the reaction at the axle of the pulley is 26.2 N

Case 3:

A body A rests on a smooth horizontal table. Two bodies of mass 2 kg and 10 kg, hanging freely are attached to A by strings which pass over smooth pulleys at the edges of the table. The strings are taut. When the system is released from rest, it accelerates at 2 ms^{-2}. Find the mass of A .



Let the mass of A be M kg. The tensions in the two strings will be different; Let them be $T_1 and\ T_2$.

Using F = ma gives;

For 2 kg mass: $T_2 - 2g = 4$ …………..(i)

For A: $T_1 - T_2 = 2M$ ………………(ii)

For 10 kg mass; $10g - T_1 = 20$ …………(iii)

Adding equations (i) , (ii) and (iii) gives;

$8g = 2M + 24$

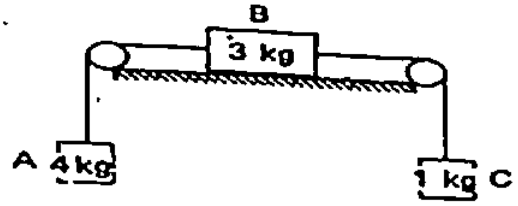
$2M = 54.4$

$M = 27.2$

The mass of the body A is 27.2 kg

Case 4

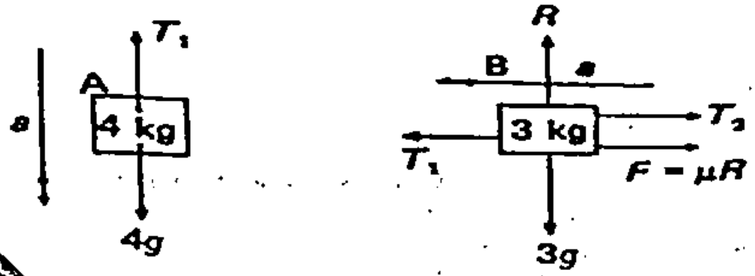
Consider three masses A, B and C connected by light inextensible strings as shown in the figure below where B is held on a rough horizontal plane whose coefficient of friction is 0.6. The pulleys are smooth.



When B is released, what will be the acceleration of the masses?

Solution

The forces that act on the masses can be shown in the diagrams below



$4g - T_1 = 4a$ ………(i)

$F = \mu R$ (as B moves) and acts against the motion

$R = 3g$

$\Rightarrow F = 0.6 \times 3g = 1.8g$

$T_1 - (T_2 + F) = 3a$ ………………(ii)

$T_2 - g = a$ ……………(iii)

From (i) $T_1 = 4g - 4a$

From (iii) $T_2 = g + a$

Substituting for T_1 and T_2 gives;

$4g - 4a - (g + a + 1.8g) = 3a$

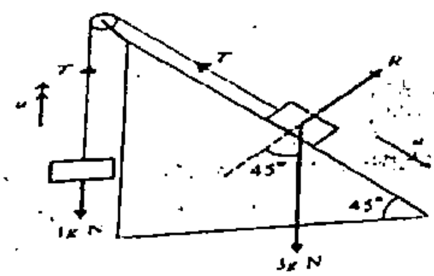
Which simplifies to ; $1.2\ g = 8a$

$a = \frac{1.2}{8}g = \frac{1.2}{8} \times 9.8 = 1.47\ ms^{-2}$

Note: it is necessary to decide in certain problems in which direction the friction is going to act.

Case 5

The bodies shown are connected by a light string which passes over a smooth pulley. The 1 kg mass moves upwards while the 3 kg mass moves downwards. Calculate the tension T, the normal reaction, R and the acceleration, a.



Solution

Applying F = ma in the vertical direction for the 1 kg mass gives;

$T - g = a$ ……………… (i)

Applying F = ma down the plane for the 3 kg mass gives;

$3g \sin 45^0 - T = 3a$

$\frac{3\sqrt{2}}{2}g - T = 3a$ ……………… (ii)

Adding equations (i) and (ii)

$\frac{3\sqrt{2}}{2}g - g = 4a$

$1.1213g = 4a$

$a = \frac{1.1213}{4} \times 9.8 = 2.747ms^{-2}$

Substituting for a in equation (i);

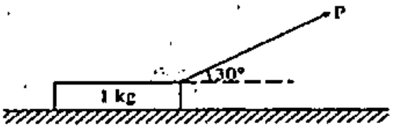
$T = g + 2.747 = 9.8 + 2.747 = 12.547$ N

To get the normal reaction, we resolve at right angles to the surface of the plane, for the 3kg mass (note that in this direction, there is no acceleration)

$R = 3g \cos 45^0 = 20.79\ N$

Trial questions

1. A block of mass 1 kg rests in equilibrium on a rough horizontal table under the action of a force P which acts at an angle of 30^0 to the horizontal as shown in the diagram below.



Given that the magnitude of P is 2.53 N, calculate

(i) The normal exerted by the table on the block

(ii) The friction force on the block

Given that the block is about to slip, calculate the coefficient of friction

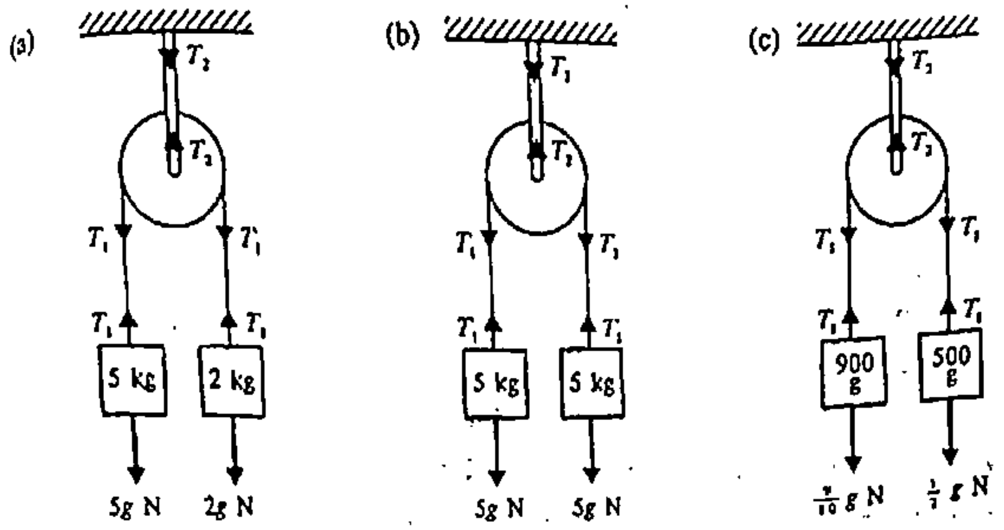
[Ans: (i) 8.54 N (ii) 2.19 N ; 0.26]

2. A light inextensible string passes over a smooth fixed pulley and carries freely hanging masses of 800g and 600g at the ends. Find the acceleration of the system and the force on the pulley. [Ans: 1.4 ms^{-2} , 13.44N]
3. A car of mass 900kg tows a caravan of mass 700kg along a level road. The engine of the car exerts a forward force of 2.4 kN and there is no resistance to motion [Ans: 1.5 ms^{-2} , 1050N]
4. A car of mass 900kg tows a trailer of mass 600kg by means of a rigid tow bar. The car experiences a resistance of 200N and the trailer a resistance of 300N. If the car engine exerts a forward force of 3 kN, find the tension in the tow bar and the acceleration of the system [Ans: 1300 N , 1.67 ms^{-2}]
5. Each of the following diagram shows two freely hanging masses connected by a light inextensible sting passing over a smooth fixed pulley. For each system, find the

(i) Acceleration of the masses.

(ii) Magnitude of the tension T_1

(iii) Magnitude of the reaction T_2



[Ans: (a) (i) 4.2 ms^{-2} (ii) 28 N (iii) 56 N (b) (i) 0 (ii) 49N (iii) 98N
(c) (i) 2.8 ms^{-2} (ii) 6.3 N (iii) 12.6 N]

6. A body of mass 65g lies on a smooth horizontal table. A light inextensible string runs from this body, over a smooth fixed pulley at the edge of the table to a body of mass 5g hanging freely, with the string taut, the system is released from rest. Find

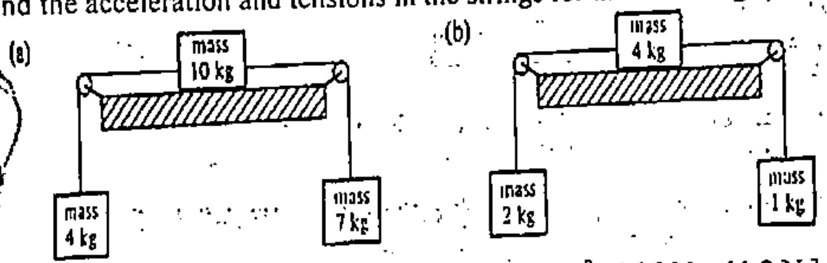
(a) the acceleration of the system

(b) The tension in the string

(c) the distance moved by the 5g mass in the first 2 seconds of motion

[Ans: (a) 0.7 ms^{-2} (b) 0.0455 N (c) 1.4 m]

7. Find the acceleration and tensions in the strings for the following systems

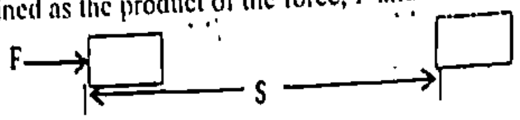


[Ans: (a) 1.4 ms^{-2}, 44.8 N, 58.8 N, (b) 1.4 ms^{-2}, 16.8 N, 11.2 N]

CHAPTER 24: WORK, ENERGY AND POWER

WORK

Work is defined as the product of the force, F and the distance, S moved in the direction of force.

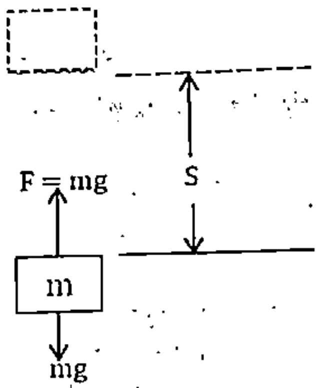


$$W = F \times S$$

The unit of work is Joules (J)

Work done against gravity

In order to raise a mass m kg vertically at a constant speed, a force mg N must be applied vertically upwards to the mass.



In raising the mass a distance, S metres, the work done against gravity will be given by;

Work done against gravity = mgS where g is the acceleration due to gravity

Example

Find the work done against gravity when an object of mass 3.5kg is raised through a vertical distance of 6m

Solution

Vertical force required, F = 3.5g = $3.5 \times 9.8 = 34.3\,N$

and S = 6m

$work\ done = F \times S = 34.3 \times 6 = 205.8\,J$

The work done against gravity is 205.8 J

General motion at constant speed

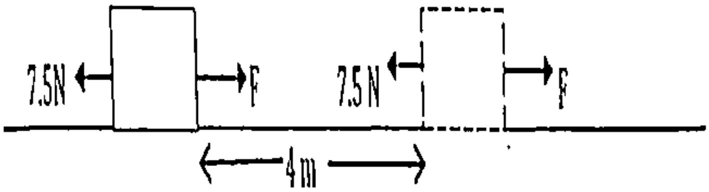
In order to move a body at a constant speed, a force equal in magnitude to the forces of resistance acting on the body has to be applied to the body.

Example 1

A block of wood is pulled a distance of 4m across a horizontal surface against resistances totaling to 7.5 N. If the block moves at a constant velocity, find the work done against the resistances.

Solution

Let the pulling force be F



Resolving horizontally; F = 7.5N

Work done against resistances = $force \times horizontal\ distance\ moved$

$= 7.5 \times 4 = 30\,J$

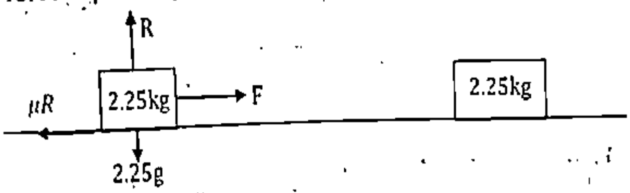
The work done against the resistances is 30J

Example 2

A horizontal force pulls a body of 2.25 kg a distance of 8m across a rough horizontal surface, coefficient of friction is $\frac{1}{3}$. The body moves with a constant velocity and the only resisting force is that due to friction. Find the work done against friction.

Solution

The friction force is μR



Resolve vertically; R = 2.25g = 2.25 × 9.8 = 18

Work done against friction = $\mu R \times distance\ moved$

$= \frac{1}{3} \times 18 \times 8 = 58.8\,J$

The work done against friction is 58.8 J

Work done against gravity and friction

When a body is pulled at a uniform speed up the surface of a rough inclined plane, work is done both against gravity and against the frictional force which is acting on the body due to the contact with the rough surface of the plane.

Example

A rough surface is inclined at $\tan^{-1}(\frac{7}{24})$ to the horizontal. A body of mass 5kg lies on the surface and is pulled at a uniform speed, a distance of 0.75 m up the surface by a force acting along the line of greatest slope. The coefficient of friction between the body and the surface is $\frac{5}{12}$. Find ;

(i) the work done against gravity

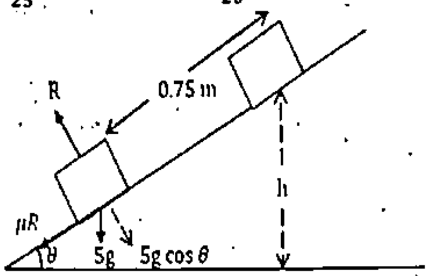
(ii) the work done against friction

Solution

$\theta = \tan^{-1}(\frac{7}{24}) \Rightarrow \tan\theta = \frac{7}{24}$

Using the Pythagoras theorem and the right angled triangle

$\sin\theta = \frac{7}{25}$ and $\cos\theta = \frac{24}{25}$



(i) Work done against gravity = $force \times vertical\ distance\ moved$

Since $\sin\theta = \frac{h}{0.75}$

Vertical distance moved, $h = 0.75\sin\theta = 0.75 \times \frac{7}{25} = 0.21\ m$

Work done against gravity = $5g \times 0.21 = 5 \times 9.8 \times 0.21 = 10.29J$

The work done against gravity is 10.29J

(ii) Resolving perpendicular to the plane gives;

$R = 5g\cos\theta = 5 \times 9.8 \times \frac{24}{25} = 47.04$

But frictional force $= \mu R = \frac{5}{12} \times 47.04 = 19.6\ N$

Work done against friction = $19.6 \times 0.75 = 14.7$ J

Trial questions

1. Find the work done against gravity when a body of mass 5 kg is raised through a vertical distance of 2m [Ans: 98 J]
2. Find the work done against gravity when a body of mass 1 kg is raised through a vertical distance of 3m [Ans: 29.4]
3. A body of mass 10 kg is pulled a distance of 20 m across a horizontal surface against resistances totaling to 40 N. If the body moves with uniform velocity, find the work done against the resistances [Ans: 800J]
4. A surface is inclined at $\tan^{-1}(\frac{3}{4})$ to the horizontal. A body of mass 50 kg lies on the surface and is pulled at a uniform speed, a distance of 5m up a line of greatest slope against resistances totaling to 50 N. Find the;
   (i) Work done against gravity
   (ii) Work done against the resistances [Ans: (i) 1960 J (ii) 250J]

ENERGY

The energy of a body is a measure of the capacity which the body has to do work. When a force does work on a body, it changes the energy of a body. Energy can exist in a number of different forms, but we shall consider two main types; kinetic energy and potential energy

Kinetic energy

The kinetic energy of a body is that energy which it possesses by virtue of its motion. When a force does work on a body so as to increase its speed, then work done is the measure of the increase in the kinetic energy of the body.

The quantity $\frac{mv^2}{2}$ is defined as the kinetic energy of a mass m moving with a velocity v. A body at rest therefore has zero kinetic energy.

Example 1

Find the kinetic energy of a particle of mass 0.25kg moving with a speed of $6ms^{-1}$

Solution

Kinetic energy $= \frac{1}{2}mv^2$

$= \frac{1}{2} \times 0.25 \times 6^2 = 4.5$ J

The kinetic energy of the particle is 4.5 J

Example 2

A body of mass 4kg decreases its kinetic energy by 32 J. If it initially had a speed of $5ms^{-1}$, find its final speed.

Solution

$initial\ kinetic\ energy = \frac{1}{2}mv^2 = \frac{1}{2} \times 4 \times 5^2 = 50J$

$Final\ kinetic\ energy = 50 - 32 = 18J$

Let the final speed be V,

Then $\frac{1}{2}(4)V^2 = 18$

$V^2 = 9$

$V = 3ms^{-1}$

The final speed of the body is $3ms^{-1}$

Example 3

A cricket ball of mass 400g moving at $3ms^{-1}$ and a golf ball of mass 100 g have equal kinetic energies. Find the speed at which the golf ball is moving

Solution

Kinetic energy of cricket ball $= \frac{1}{2}mv^2 = \frac{1}{2} \times 0.4 \times 3^2 = 1.8J$

Kinetic energy of golf ball $= \frac{1}{2} \times 0.1V^2$

But kinetic energy of golf ball = kinetic energy of cricket ball

$\frac{1}{2} \times 0.1V^2 = 1.8$

$0.1V^2 = 1.8 \times 2$

$V^2 = \frac{3.6}{0.1} = 36$

$V = \sqrt{36} = 6ms^{-1}$

The golf ball is moving at $6ms^{-1}$

Potential energy

The potential energy of a body is that energy it possesses by virtue of its position. When a body of mass m kg is raised vertically through a distance of h metres, the work done against gravity is mgh joules. The work done against gravity is the measure of the increase in the potential energy of the body i.e. the capacity of the body to do work is increased.

Example 1

Find the potential energy of a child of mass 48 kg when ascending a vertical distance of 2 m.

Solution

$Potential\ energy = mgh$

$= 48 \times 9.8 \times 2 = 94.08J$

The potential energy is 94.08 J

Example 2

Find the potential energy gained by a ball of mass 0.075 kg at a distance of 32 m above the ground.

Solution

$Potential\ energy = mgh$

$= 0.075 \times 9.8 \times 32 = 23.52J$

The potential energy is 23.52 J

Trial questions

Find the potential energy given by;

(i) A body of mass 5 kg raised through a vertical distance of 10m

(ii) A man of mass 60 kg ascending a vertical distance of 5m

(iii) A body of mass 20 kg above a vertical distance of 2m from the ground.

[Ans: (i) 490 J (ii) 2940 J (iii) 392 J]

The principle of conservation of energy

Suppose we have a situation involving a moving body in which

(a) There is no work done against friction, and

(b) Gravity is the only external force which does work on the body (or against which the body has to do work)

The total mechanical energy possessed by the body will then be the total of its kinetic energy and its potential energy and by the principle of conservation of energy, this will be constant i.e.

Total energy = kinetic energy (K.E) + potential energy (P.E)

Or total energy in the initial state = total energy in the final state

Example 1

The point A is vertically below the point B. A particle of mass 0.1 kg is projected from point A vertically upwards with a speed of 21 ms^{-1} and passes point B with a speed of 7 ms^{-1}

Find the distance from A to B.

Solution

We shall choose to measure the P.E from the level of A. Let the distance from A to B be h metres.

At A;

$K.E = \frac{1}{2}mv^2 = \frac{1}{2} \times 0.1 \times 21^2 = 22.05\,J$

P.E = 0 J since h = 0 at A

Total energy = K.E + P.E = 22.05 J

At B; $K.E = \frac{1}{2}mv^2 = \frac{1}{2} \times 0.1 \times 7^2 = 2.45\,J$

$P.E = mgh = 0.1 \times 9.8 \times h = 0.98h\,J$

Total energy $= K.E + P.E = (2.45 + 0.98h)\,J$

But from the principle of conservation of energy, total energy at A = total energy at B

$\Rightarrow 2.45 + 0.98h = 22.05$

$0.98h = 22.05 - 2.45$

$0.98h = 19.6$

$h = \frac{19.6}{0.98} = 20\,m$

The distance from A to B is 20m

Example 2

The point A is 4 metres vertically above the point B. A body of mass 0.2 kg is projected from A vertically downwards with a speed of 3 ms^{-1}. Find the speed of the body when it reaches B.

Solution

At A;

$K.E = \frac{1}{2}mv^2 = \frac{1}{2} \times 0.2 \times 3^2 = 0.9\,J$

P.E = mgh = $0.2 \times 9.8 \times 4 = 7.84\,J$

Total energy at A = $0.9 + 7.84 = 8.74\,J$

At B;

$K.E = \frac{1}{2}mv^2 = \frac{1}{2} \times 0.2V^2 = 0.1V^2$

P.E = 0

Total energy = $0.1V^2$

Using the principle of conservation of energy; $0.1V^2 = 8.74$

$V^2 = 87.4$

$V = \sqrt{87.4} = 9.35\,ms^{-1}$

The speed of the body is 9.35 ms^{-1} when it reaches B

Trial questions

1. A body of mass 6kg is released from rest and it falls freely under gravity. Find the distance it has fallen when its speed is 7 ms^{-1}.
2. A body of mass 20 kg is projected vertically downwards from a point A with speed 4 ms^{-1}. The body passes through a point B, 5m below A. Find the speed of the body at B.

3. A body of mass 5 kg is released from rest and falls freely under gravity. Find its speed when it has fallen a distance of 10m
   [Ans: 1. 2.5 m 2. 10.68 ms^{-1} 3. 14 ms^{-1}]

POWER

Power is a measure of the rate at which work is being done. If 1 joule of work is done in 1 second, the rate of working is 1 Watt (W). Thus the unit of power is Watts (W)

$$Power = \frac{work\ done}{time\ taken} = \frac{Force\ x distance}{time\ taken} = force\ \times \frac{distance}{time\ taken} = force\ \times velocity$$

Example 1

Find the work done by a force of 6N in moving a body from A to B where AB = 10 m and also the average rate at which the force is working if it takes 5 seconds to move the body from A to B.

Solution

Work done $= force\ \times distance$

$= 6 \times 10 = 60\,J$

Rate of working/ power $= \frac{work\ done}{time\ taken} = \frac{60}{5} = 12W$

The force does 60J of work and its average rate of working is 12 Watts

Example 2

Find the rate at which work is being done when a mass of 20kg is lifted vertically at a constant speed of $5ms^{-1}$.

Solution

Work done $= force\ \times distance$

But for body vertically above the ground, F = weight = mg

$Force = 20 \times 9.8 = 196\,N$

Rate of doing work $= force\ \times velocity = 196 \times 5 = 980\,W$

The rate at which work is being done is 980 W

Trial questions

1. What is the rate at which work must be done in lifting a mass of 500 kg vertically at a constant speed of $3ms^{-1}$?
2. What is the average rate at which work must be done in lifting a mass of 100kg a vertical distance of 5m in 7 seconds?
   [Ans: 1. 14700W 2. 700W]

SOLUTIONS TO UNEB 2013

SECTION A (40 MARKS)

All questions are supposed to be attempted

1. Given that $p = \log_a(a^3y^{-2})$ and $\log_a(ay^2)$, find the value of p + q (05 marks)

Solution

$p + q = \log_a(a^3y^{-2}) + \log_a(ay^2)$

$= \log_a[a^3y^{-2} \times ay^2]$ ......refer to the laws of logarithms

$= \log_a[(a^3 \times a) \times (y^{-2} \times y^2)]$ ....collecting the like terms

$= \log_a[a^{(3+1)} \times y^{(-2+2)}]$ ......using the law of indices $(a^m \times a^n = a^{m+n})$

$= \log_a[a^4 \times y^0]$ but $y^0 = 1$ since any number power zero = 1

$= \log_a a^4 = 4\log_a a$ refer to the laws $\log_a a^n = n\log_a a$ and $\log_a a = 1$.

$\therefore p + q = 4 \times 1 = 4$

2. The table below shows the age in years of mothers at the time they had their first child.

Age in years	15 −	20 −	25 −	30 −	35 −	40 − 45
Number of numbers	2	14	29	43	33	9

Calculate the modal age of the mothers (05 marks)

Solution

The class boundaries are provided in this case and not the class limits

$Mode = L_1 + \left(\frac{\Delta_1}{\Delta_1+\Delta_2}\right)C$

$\Delta_1 = 43 - 29 = 14$ and $\Delta_2 = 43 - 33 = 10$, class width = 5

$Mode = 30 + \left(\frac{14}{14+10}\right) \times 5 = 30 + 2.917 = 32.917$

The modal age is 32.917 years

3. Find the sum of the first ten terms of the geometric progression (GP)

$8 + 4 + 2 + \cdots$ (05 marks)

Solution

Identify the first term a and the common ration r since it is a G.P

$a = 8,\ r = \frac{4}{8} = \frac{1}{2}$

We can see that $r < 1$ *so we use the formula* $S_n = \frac{a(1-r^n)}{1-r}$ to find the sum

$S_n = \frac{8\left(1-\left(\frac{1}{2}\right)^{10}\right)}{1-\frac{1}{2}} = \frac{8\left(1-\frac{1}{1024}\right)}{\frac{1}{2}} = \frac{8\left(\frac{1024-1}{1024}\right)}{\frac{1}{2}} = 8\left(\frac{1023}{1024}\right) \times \frac{2}{1}$

$= 16\left(\frac{1023}{1024}\right) = 15.98$

4. The table below shows the prices of items and their corresponding weights in years 2000 and 2004.

Items	Price(Ushs)		Weight
	2000	2004	
Food	55,000	60,000	4
Housing	48,000	52,000	2
Transport	16,000	20,000	1

Using 2000 as the base year, calculate the weighted price index for the items in 2004.

Solution

Weighted aggregate price index $= \frac{\sum P_1W}{\sum P_0W} \times 100$

$$= \frac{60000\times4+52000\times2+20000\times1}{55000\times4+48000\times2+16000\times1} \times 100$$

$$= \frac{364000}{332000} \times 100 = 109.64$$

Alternatively;

Weighted average price index $= \frac{\sum\frac{P_1}{P_0}\times W}{\sum W} \times 100$

$$= \frac{\frac{60000}{55000}\times4+\frac{52000}{48000}\times2+\frac{20000}{16000}\times1}{4+2+1} \times 100$$

$$= \frac{4.36+2.17+1.25}{7} \times 100 = \frac{7.78}{7} \times 100 = 111.11$$

5. Solve the differential equation $8y\frac{dy}{dx} = 9x^2$

Hence find the equation given that $y = 2$ and $x = 1$. (05 marks)

Solution

$8y\frac{dy}{dx} = 9x^2$

By separating the variables

$8y\,dy = 9x^2\,dx$

$\int 8y\,dy = \int 9x^2\,dx$ ... Integrating on both sides

$\frac{8y^2}{2} = \frac{9x^3}{3} + C$

$4y^2 = 3x^3 + C$

This is a general solution and since the initial conditions are given, we can find the value of the constant C

$when\ x = 1,\ y = 2$

$4(2)^2 = 3(1)^3 + C$

$16 = 3 + C \Rightarrow C = 13$

$\therefore 4y^2 = 3x^3 + 13$ is the particular solution

6. Solve the equation $\sec^2\theta - \tan\theta = 1$ $for\ 0^0 \leq \theta \leq 90^0$ (05 marks)

Solution

$sec^2\theta - \tan\theta = 1$

using the identity $sec^2\theta = 1 + tan^2\theta$

$(1 + tan^2\theta) - \tan\theta = 1$

$tan^2\theta - \tan\theta + 1 - 1 = 0$

$tan^2\theta - \tan\theta = 0$

$\tan\theta(\tan\theta - 1) = 0$ (It should be noted that we factorize not to cancel because we might not get all the required angles)

Either $\tan\theta = 0 \Rightarrow \theta = \tan^{-1}0 = 0^0$

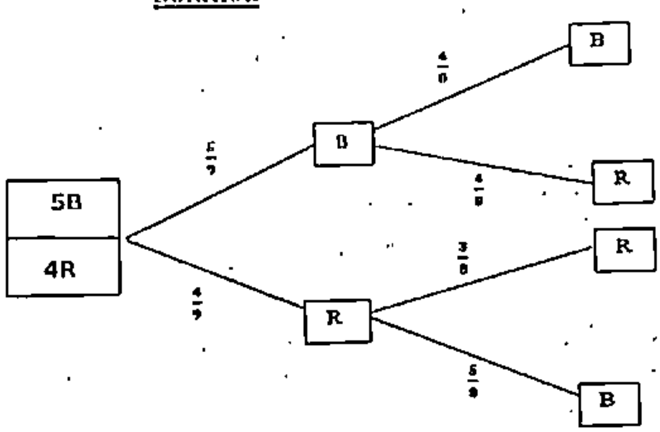
Or $\tan\theta - 1 = 0 \Rightarrow \tan\theta = 1$ *thus* $\theta = \tan^{-1}1 = 45^0$

Since the range of angles required was $0^0 \leq \theta \leq 90^0$ i.e in the first quadrant, then

For $0^0 \leq \theta \leq 90^0, \theta = \{0^0, 45^0\}$

7. A bag contains 5 black pens (B) and 4 red pens (R). Two pens are picked at random, one after the other without replacement. Find the probability that both pens are of the same colour. (05 marks)

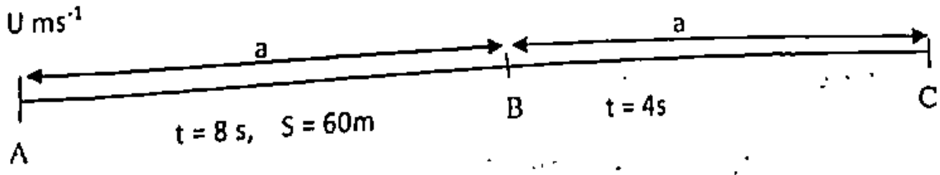
Solution



$$P(\text{Same color}) = P(B_1 \cap B_2) + P(R_1 \cap R_2)$$
$$= \frac{5}{9} \times \frac{4}{8} + \frac{4}{9} \times \frac{3}{8}$$
$$= \frac{20}{72} + \frac{12}{72} = \frac{32}{72} = \frac{4}{9}$$

8. A powered trolley in factory is moving in a straight line with a constant acceleration. It passes point A with a velocity of Ums^{-1}. It takes 8 seconds to travel 60m from a point A to point B. Finally it takes 4 seconds to travel from point B to point C. Find the value of U. (05 marks)

Solution



Let the acceleration be a

Considering the motion from A to B
Using the 2nd equation
$S = Ut + \frac{1}{2}at^2$
$60 = U(8) + \frac{1}{2}(a)(8)^2$
$60 = 8U + 32a$ ....(i)

Considering the motion from A to C ...
total time $= 8 + 4 = 12s$
acceleration $= a$, initial velocity $= U$
$S = Ut + \frac{1}{2}at^2$
$S = U(12) + \frac{1}{2}(a)(12)^2$
$S = 12U + 72a$ ....(ii)

But S = distance AC is not given thus it becomes complicated to obtain the value of U

SECTION B (60 MARKS)

Four questions are supposed to be attempted in this section

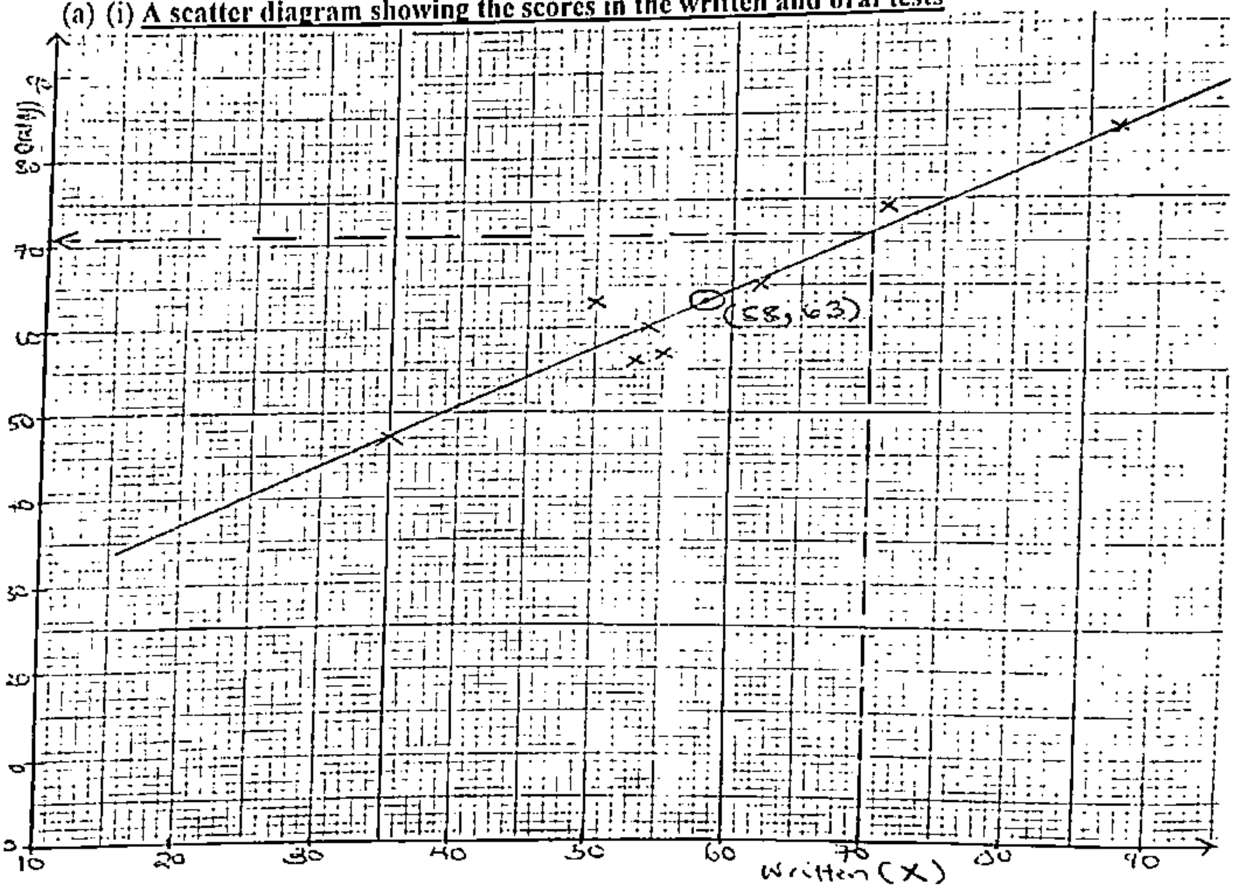
9. Eight candidates seeking admission to a university course sat for written and oral tests. The scores were as shown in the table below:

Written(X)	55	54	35	62	87	53	71	50
Oral(Y)	57	60	47	65	83	56	74	63

(a) (i) Draw a scatter diagram for the data
(ii) Draw a line of best fit on your scatter diagram
(iii) Use the line of best fit to find the value of Y when X = 70 (08 marks)
(b) Calculate Spearman's rank correlation co-efficient. Comment on your result. (07 marks)

Solution

(a) (i) **A scatter diagram showing the scores in the written and oral tests**



(ii) The line of best fit should pass through the point $(\bar{X}, \bar{Y})$ where $\bar{X} = \frac{\sum X}{n}$ and $\bar{Y} = \frac{\sum Y}{n}$

$\bar{X} = \frac{467}{8} = 58.4$ and $Y = \frac{505}{8} = 63.1$

The line of best fit passes through (58.4, 63.1) leaving equal number of points below and above the line

(i) From the graph, it is estimated that when X = 70 , Y = 71

(b)

Written(X)	Oral (Y)	R_X	R_Y	$d = R_X - R_Y$	d^2
55	57	4	6	-2	4
54	60	5	5	0	0
35	47	8	8	0	0
62	65	3	3	0	0
87	83	1	1	0	0
53	56	6	7	-1	1
71	74	2	2	0	0
50	63	7	4	3	9
					$\sum d^2 = 14$

Spear man's rank correlation coefficient, $\rho = 1 - \frac{6\sum d^2}{n(n^2-1)}$

$= 1 - \frac{6\times14}{8(8^2-1)} = 1 - \frac{84}{8(63)} = 1 - \frac{84}{504} = 1 - 0.167 = 0.833$

Comment: There is a high positive correlation between the two tests

10.(a) Sketch the curve $y = 5 + 4x - x^2$ (10 marks)

(b) Find the area enclosed between the curve and the x-axis from $x = -1$ to $x = 5$ (05 marks)

Solution

(a) The requirements to sketch a curve are; find the intercepts i.e. where the curve cuts the axes and the turning point and its nature i.e. where $\frac{dy}{dx} = 0$

Intercepts

$when\ x = 0, y = 5 + 4(0) - (0)^2 = 5$

$\Rightarrow (0,5)\ is\ the\ y - intercept$

$when\ y = 0, 5 + 4x - x^2 = 0$ [This is a quadratic equation which needs to be factorized]

Multiplying $5 \times -1 = -5, factors\ of\ -5\ which\ give\ us\ a\ sum\ of\ 4\ are\ 5\ and\ -1$

Thus $5 + 5x - x - x^2 = 0$

$5(1 + x) - x(1 + x) = 0$

$(1 + x)(5 - x) = 0$

Either $1 + x = 0 \Rightarrow x = -1$ or $5 - x = 0 \Rightarrow x = 5$

Thus (-1, 0) and (5, 0) are the x – intercepts

Turning point

$y = 5 + 4x - x^2$

$\frac{dy}{dx} = 4 - 2x$

For turning points, $\frac{dy}{dx} = 0 \Rightarrow 4 - 2x = 0\ which\ gives\ x = 2$

Now we need to find the y- value corresponding to the x-value obtained above

$when\ x = 2, y = 5 + 4(2) - (2)^2 = 5 + 8 - 4 = 9$

Thus (2, 9) is the turning point

Nature of the turning point

	L	2	R
Sign of $\frac{dy}{dx}$	+	0	−
	/	—	\

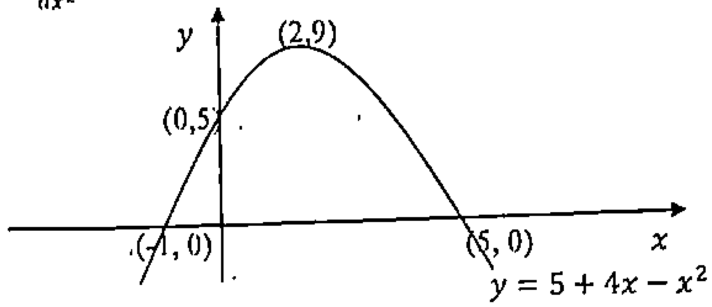
Thus (2, 9) is a maximum turning point

Alternatively we can find the nature of the turning point using the second derivative

$$\text{From } \frac{dy}{dx} = 4 - 2x$$

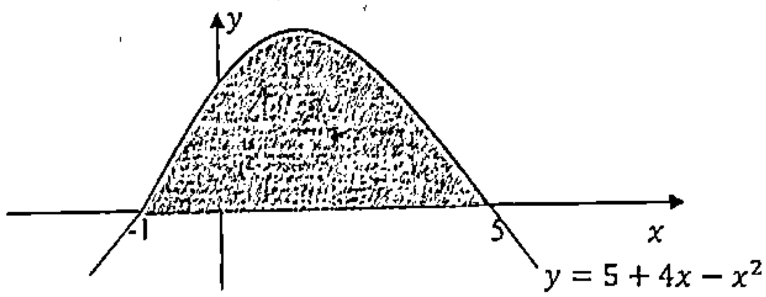
$$\frac{d^2y}{dx^2} = -2$$

Since $\frac{d^2y}{dx^2} < 0$, *it is a maximum turning point*



Note that a student is required to use one of the two methods not both in an exam

(b)



$$A = \int_a^b y\,dx$$

$$= \int_{-1}^{5} (5 + 4x - x^2)dx$$

$$= \left[5x + \frac{4x^2}{2} - \frac{x^3}{3}\right]_{-1}^{5}$$

$$= \left[5x + 2x^2 - \frac{x^3}{3}\right]_{-1}^{5}$$

$$= \left(5(5) + 2(5)^2 - \frac{5^3}{3}\right) - \left(5(-1) + 2(-1)^2 - \frac{(-1)^3}{3}\right)$$

$$= \left(25 + 50 - \frac{125}{3}\right) - \left(-5 + 2 + \frac{1}{3}\right)$$

$$= \left(75 - \frac{125}{3}\right) - \left(-3 + \frac{1}{3}\right)$$

$$= \frac{100}{3} + \frac{8}{3} = \frac{108}{3} = 36\ sq.units$$

11. The table below shows the number of bags of sugar sold by a certain wholesale shop from the year 2009 to 2012.

Year	QUARTER			
	1st	2nd	3rd	4th
2009	192	280	320	260
2010	300	360	380	270
2011	342	420	430	320
2012	424	480	510	412

(a) Calculate the four-point moving averages for the data (06 marks)

(b) (i) on the same axes, plot the original data and the four-point moving averages (05 marks)

(ii) Comment on the trend of the number of bags of sugar sold over the four-year period (01 mark)

(iii) Use your graph to estimate the number of bags to be sold in the first quarter of 2013. (03 marks)

Solution

(a) $M_1 = \frac{192+280+320+260}{4} = 263$

$M_2 = \frac{280+320+260+300}{4} = 290$

$M_3 = \frac{320+260+300+360}{4} = 310$

$M_4 = \frac{260+300+360+380}{4} = 325$

$M_5 = \frac{300+360+380+270}{4} = 327.5$

$M_6 = \frac{360+380+270+342}{4} = 338$

$M_7 = \frac{380+270+342+420}{4} = 353$

$M_8 = \frac{270+342+420+430}{4} = 365.5$

$M_9 = \frac{342+420+430+320}{4} = 378$

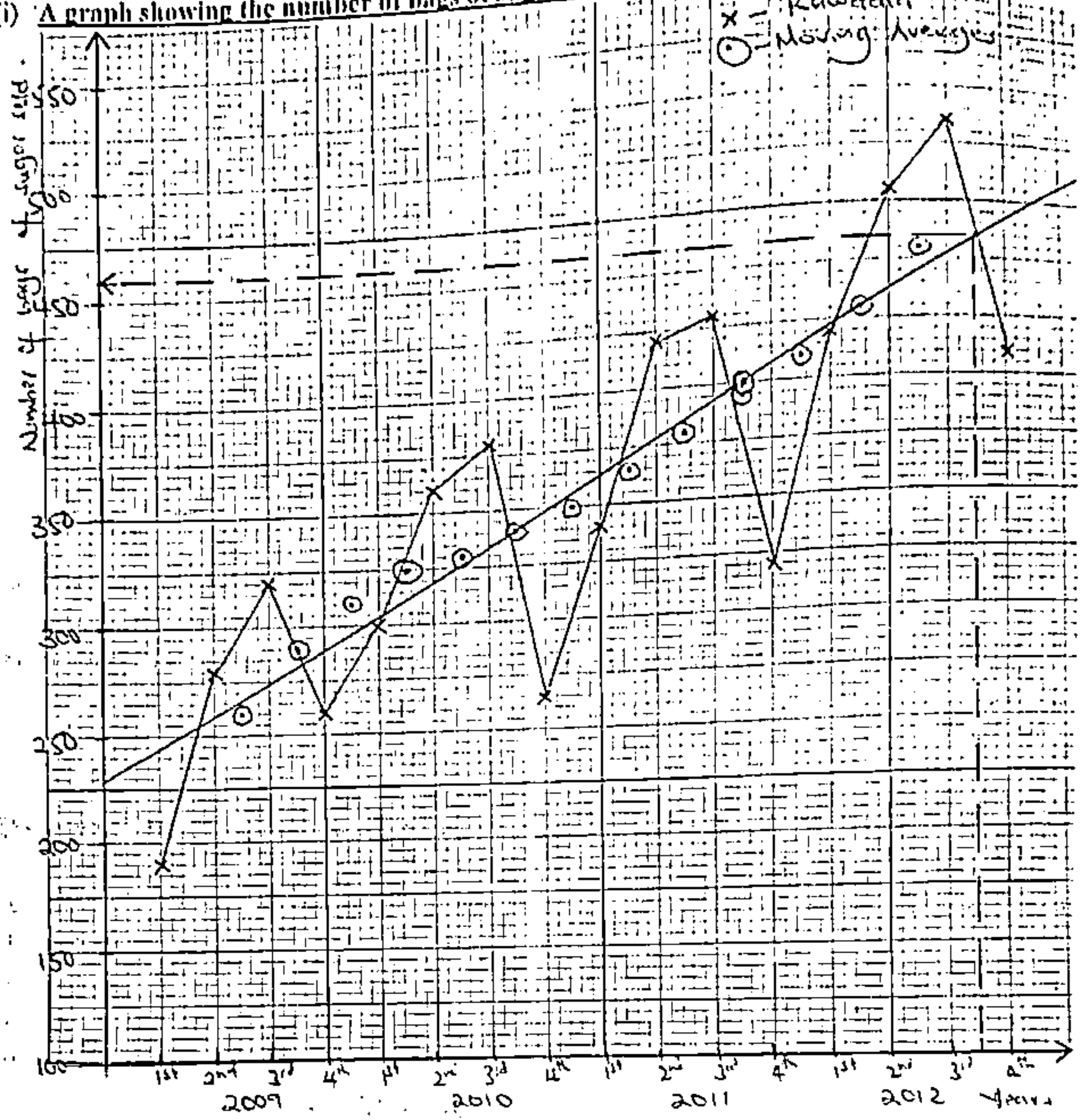
$M_{10} = \frac{420+430+320+424}{4} = 398.5$

$M_{11} = \frac{430+320+424+480}{4} = 413.5$

$M_{12} = \frac{320+424+480+510}{4} = 433.5$

$M_{13} = \frac{424+480+510+412}{4} = 456.5$

(b) (i) A graph showing the number of bags of sugar sold and the four-point moving averages.



(ii) There is a general increase in the number of bags of sugar sold over the given period

(iii) Let the number of bags to be sold in the first quarter of 2013 be x

From the graph, we can estimate the 14th moving average $M_{14} = 460$

$$\frac{480+510+412+x}{4} = 460$$

$$1402 + x = 4 \times 460$$

$$x = 1840 - 1402 = 438$$

The number of bags to be sold would be 438

12. The points P and Q have position vectors $OP = -2i - 5j$ and $OQ = i - 2j$ respectively. R is a point such that $OR = OP + \lambda PQ$

(a) Find the ;

(i) value of OP. OQ

(ii) angle between the two vectors OP and OQ (07 marks)

(b) Determine
(i) vector PQ
(ii) vector OR in terms of λ
(iii) value of λ for which OR is perpendicular to PQ (08 marks)

Solution

$$OP = -2i - 5j = \begin{pmatrix}-2\\-5\end{pmatrix}$$

$$OQ = i - 2j = \begin{pmatrix}1\\-2\end{pmatrix}$$

(a) (i) $OP.OQ = \begin{pmatrix}-2\\-5\end{pmatrix}.\begin{pmatrix}1\\-2\end{pmatrix} = (-2\times 1) + (-5\times -2) = -2 - 10 = -12$

(ii) Let the angle be θ

$$OP.OQ = |OP||OQ|\cos\theta$$

$$|OP| = \sqrt{(-2)^2 + (-5)^2} = \sqrt{4+25} = \sqrt{29}$$

$$|OQ| = \sqrt{(1)^2 + (-2)^2} = \sqrt{1+4} = \sqrt{5}$$

$$-12 = \sqrt{29}\times\sqrt{5}\cos\theta$$

$$\cos\theta = \frac{-12}{\sqrt{29}\times\sqrt{5}} = -0.9965$$

$$\theta = \cos^{-1}(-0.9965) = 175.24^0$$

The angle between the two vectors OP and OQ is 175.24^0

(b) (i) $PQ = OQ - OP = \begin{pmatrix}1\\-2\end{pmatrix} - \begin{pmatrix}-2\\-5\end{pmatrix} = \begin{pmatrix}3\\3\end{pmatrix}$

(ii) $OR = OP + \lambda PQ = \begin{pmatrix}-2\\-5\end{pmatrix} + \lambda\begin{pmatrix}3\\3\end{pmatrix} = \begin{pmatrix}-2+3\lambda\\-5+3\lambda\end{pmatrix}$

(iii) The dot product of perpendicular vectors is zero i.e $OR.OQ = 0$

$$\begin{pmatrix}-2+3\lambda\\-5+3\lambda\end{pmatrix}.\begin{pmatrix}3\\3\end{pmatrix} = 0$$

$$3(-2+3\lambda) + 3(-5+3\lambda) = 0$$

$$-6 + 9\lambda - 15 + 9\lambda = 0$$

$$18\lambda - 21 = 0$$

$$\lambda = \frac{21}{18} = \frac{7}{6}$$

13. A bakery produces loaves of bread whose weight is normally distributed with mean 1000 g and standard deviation 40 g

(a) Find the probability that a randomly selected loaf has a weight of utmost 1020 g. (07 marks)

(b) Assuming that the bakery makes 10500 loaves, find the approximate number of loaves with a weight greater than 950 g. (08 marks)

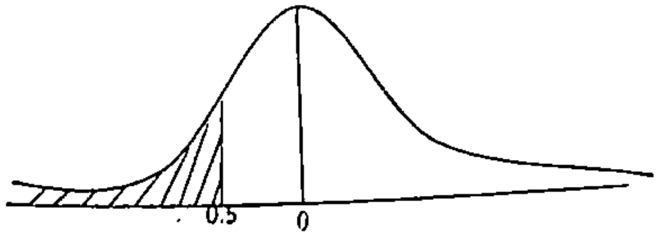
Solution

Let X be the random variable weight of loaves of bread

$$X \sim N(1000, 40^2)$$

(a) Utmost 1020 g means a value that does not exceed 1020 i.e. it must be below or equal to.

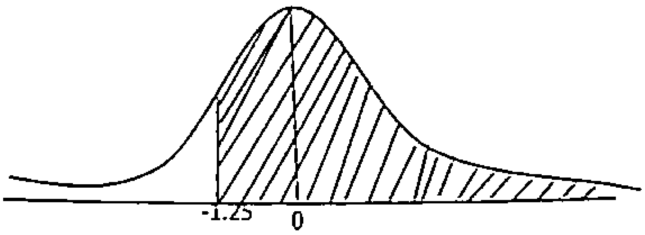
$$P(X \le 1020) = P\left(Z \le \frac{1020-1000}{40}\right) = P\left(Z \le \frac{20}{40}\right) = P(Z \le 0.5)$$



$$P(Z \leq 0.5) = 0.5 - P(0.5 \leq Z \leq 0)$$
$$= 0.5 - 0.1915 = 0.3085$$

The probability that a loaf has a weight of utmost 1020 g is 0.3085.

(b) $P(X > 950) = P\left(Z > \frac{950-1000}{40}\right) = P\left(Z > -\frac{50}{40}\right) = P(Z > -1.25)$

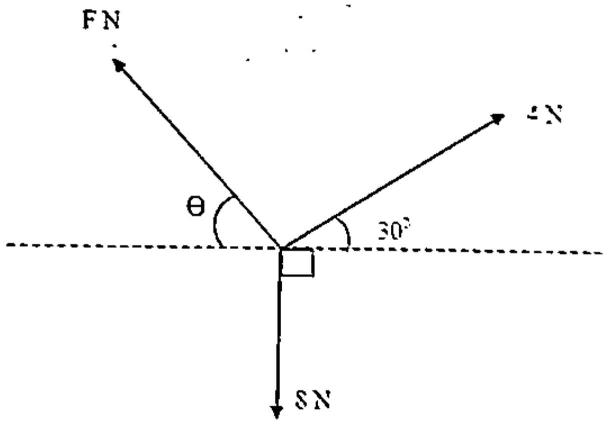


$P(Z > -1.25) = 0.5 + P(-1.25 < Z < 0) = 0.5 + 0.3944 = 0.8944$

Approximate number of loaves $= 10500 \times 0.8944 = 9391.2 \approx 9391$

The number of loaves with a weight greater than 950 g is 9391

14.(a) The diagram below shows three forces FN, 4N and 8N acting on a particle



If the forces are in equilibrium, find the value of

(i) θ (ii) F (05 marks)

(b) In the rectangle ABCD, AB = 4m and BC = 3m. Forces of magnitudes 3N, 10N, 4N, 6N and 5N act in the directions of the letters AB, BC, CD, DA and AC respectively. Taking AB as horizontal find the magnitude of the resultant force. (09 marks)

(a) **Solution**

Force	Horizontally	Vertically
F	$F\cos\theta$ ←	$F\sin\theta$ ↑
4	$4\cos 30$ →	$4\sin 30$ ↑
8	0	8 ↓

For forces in equilibrium, sum of upward forces = sum of downward forces

$$F\sin\theta + 4\sin 30^0 = 8$$

$$F\sin\theta = 8 - 4\sin 30^0$$

$$F\sin\theta = 6 \quad \dots\dots\dots\dots(i)$$

Similarly; horizontally;

$$F\cos\theta = 4\cos 30^0$$

$$F\cos\theta = 3.464 \quad \dots\dots\dots(ii)$$

Dividing eqn (i) by eqn (ii)

$$\frac{F\sin\theta}{F\cos\theta} = \frac{6}{3.464}$$

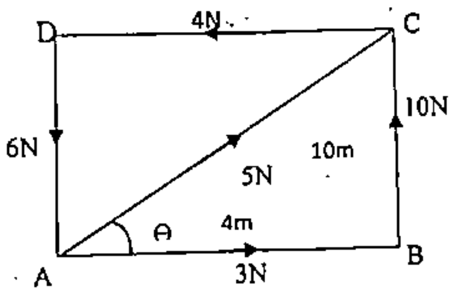
$$\tan\theta = 1.732$$

$$\theta = \tan^{-1} 1.732 = 60^0$$

(ii) Using $F\cos\theta = 3.464$

$$F = \frac{3.464}{\cos 60^0} = 6.928\ N$$

(b)



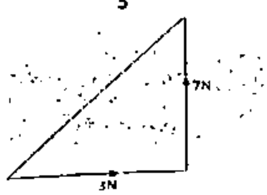
$AC = \sqrt{4^2 + 3^2} = \sqrt{25} = 5\ cm$

Resultant horizontal force, $F_x = 3 + 5\cos\theta - 4$

$$= 3 + 5 \times \frac{4}{5} - 4 = 3 + 4 - 4 = 3N \rightarrow$$

Resultant vertical force $= 10 + 5\sin\theta - 6$

$$= 10 + 5 \times \frac{3}{5} - 6 = 10 + 3 - 6 = 7N \uparrow$$



Resultant horizontal force, $R = \sqrt{F_x{}^2 + F_y{}^2} = \sqrt{3^2 + 7^2} = \sqrt{58} = 7.62\ N$

SOLUTIONS TO UNEB 2014

SECTION A (40 MARKS)

1. The roots of the equation $2x^2 + 4x - 1 = 0$ are α and β. Find the value of $\alpha^2 + \beta^2$

Solution

$2x^2 + 4x - 1 = 0$

$x^2 + 2x - \frac{1}{2} = 0$ on dividing throughout by 2

Sum of roots $\alpha + \beta = -2$ and product of roots $\alpha\beta = -\frac{1}{2}$

From $(\alpha + \beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2$

It follows that $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= (-2)^2 - 2\left(-\frac{1}{2}\right) = 4 + 1 = 5$

2. The ninth term of an arithmetic progression (A.P) is greater than the fifth term by 6. The sum of the first twelve terms is 123. Find the:
(a) common difference of the A.P
(b) first term of the A.P

Solution

$n^{th} term = a + (n - 1)d$

$9^{th} term = a + 8d$; $5^{th} term = a + 4d$

(a) $9^{th} term - 5^{th} term = 6$

$(a + 8d) - (a + 4d) = 6$

$a + 8d - a - 4d = 6$

$4d = 6 \Rightarrow d = \frac{6}{4} = 1.5$

(b) $S_{12} = 123$

From $S_n = \frac{n}{2}[2a + (n - 1)d]$

$\frac{12}{2}[2a + 11 \times 1.5] = 123$

$6[2a + 16.5] = 123$

$2a + 16.5 = 20.5$

$2a = 4 \Rightarrow a = 2$

3. (a) How many arrangements can be made using the letters in the word "TROTTING"
(b) In how many of these arrangements are the letters N and G next to each other

Solution

(a) TROTTING has 8 letters with 3T's

Number of arrangements $= \frac{8!}{3!} = 6720$

(b) we can take N and G as one such that they are always together

TROTTI(NG) has 7 letters with 3T's

Number of arrangements $= \frac{7!}{3!} = 840$

Number of arrangements of N and G =2! =2

Total number of arrangements = $840 \times 2 = 1680$

4. Solve the differential equation $\frac{dy}{dx} = 2x + 5$, given that $y = -1$ and $x = 3$

Solution

$$\frac{dy}{dx} = 2x + 5$$

$$dy = (2x + 5)dx$$

$$\int dy = \int (2x + 5)dx$$ on integrating either sides

$$y = \frac{2x^2}{2} + 5x + C$$

$$y = x^2 + 5x + C$$

$when$ $x = 3, y = -1$

$$(-1) = (3)^2 + 5(3) + C$$

$$C = -25$$

$$\therefore\ y = x^2 + 5x - 25$$

5. A class of n students sat for a mathematics test. Given that $\sum fx = 400, \sum fx^2 = 6500$ and the mean $\bar{x} = 16$, $where$ x is the mark and f is the frequency; determine the value of

(a) n

(b) the standard deviation

Solution

(a) Mean, $\bar{x} = \frac{\sum fx}{n}$

$$16 = \frac{400}{n}$$

$$16n = 400 \Rightarrow n = \frac{400}{16} = 25$$

(b) Standard deviation $= \sqrt{\frac{\sum fx^2}{n} - \left(\frac{\sum fx}{n}\right)^2}$

$$= \sqrt{\frac{6500}{25} - 16^2} = \sqrt{260 - 256} = \sqrt{4} = 2$$

6. Show that $sec^2\theta + cosec^2\theta = sec^2\theta cosec^2\theta$

Solution

From $\sec\theta = \frac{1}{\cos\theta}$ and $cosec\ \theta = \frac{1}{\sin\theta}$

$$L.H.S = sec^2\theta + cosec^2\theta = \frac{1}{cos^2\theta} + \frac{1}{sin^2\theta} = \frac{sin^2\theta + cos^2\theta}{cos^2\theta sin^2\theta}$$

$$= \frac{1}{cos^2\theta sin^2\theta}$$ $since$ $sin^2\theta + cos^2\theta = 1$

$$= \frac{1}{cos^2\theta} \times \frac{1}{sin^2\theta} = sec^2\theta cosec^2\theta = R.H.S$$ as required

7. In a bimodal experiment, the probability of a success for n trials is 0.6. If the mean is 7.2, find the (a) value of n

(b) probability of obtaining 7 successes

Solution

(a) $Mean = np$

$7.2 = 0.6n$

$n = \frac{7.2}{0.6} = 12$

(b) $P(X = r) = {}^{n}C_r p^r q^{n-r}$

$P(X = 7) = {}^{12}C_7 (0.6)^7 (0.4)^5$

$= 0.227$

8. A cyclist rides along a straight road from shop P to shop Q. He passes shop P with a velocity of 2 ms^{-1} and accelerates uniformly at 1.25 ms^{-2} until he attains a velocity of 12 ms^{-1} at shop Q. Find the : (a) time taken by the cyclist to reach Q

(b) distance PQ

Solution

P ———————————————— Q

U = 2ms^{-1}, a =1.25ms^{-1} V = 12ms^{-1}

(a) using $V = U + at$

$12 = 2 + 1.25t$

$1.25t = 10 \Rightarrow t = \frac{10}{1.25} = 8s$

(b)

Using $V^2 = U^2 + 2aS$

$12^2 = 2^2 + 2 \times 1.25S$

$144 = 4 + 2.5S$

$2.5S = 140 \Rightarrow S = 56m$

Alternatively; $S = ut + \frac{1}{2}at^2$

$S = 2 \times 8 + \frac{1}{2} \times 1.25 \times 8^2$

$S = 16 + 40 = 56m$

SECTION B

9. The table below shows the marks of 8 students in the mid-term test and end of term test in economics.

Mid-term tests (x)	99	71	50	67	77	81	96	72
End of term test (y)	99	55	35	60	75	70	99	50

(a) (i) draw a scatter diagram for the data

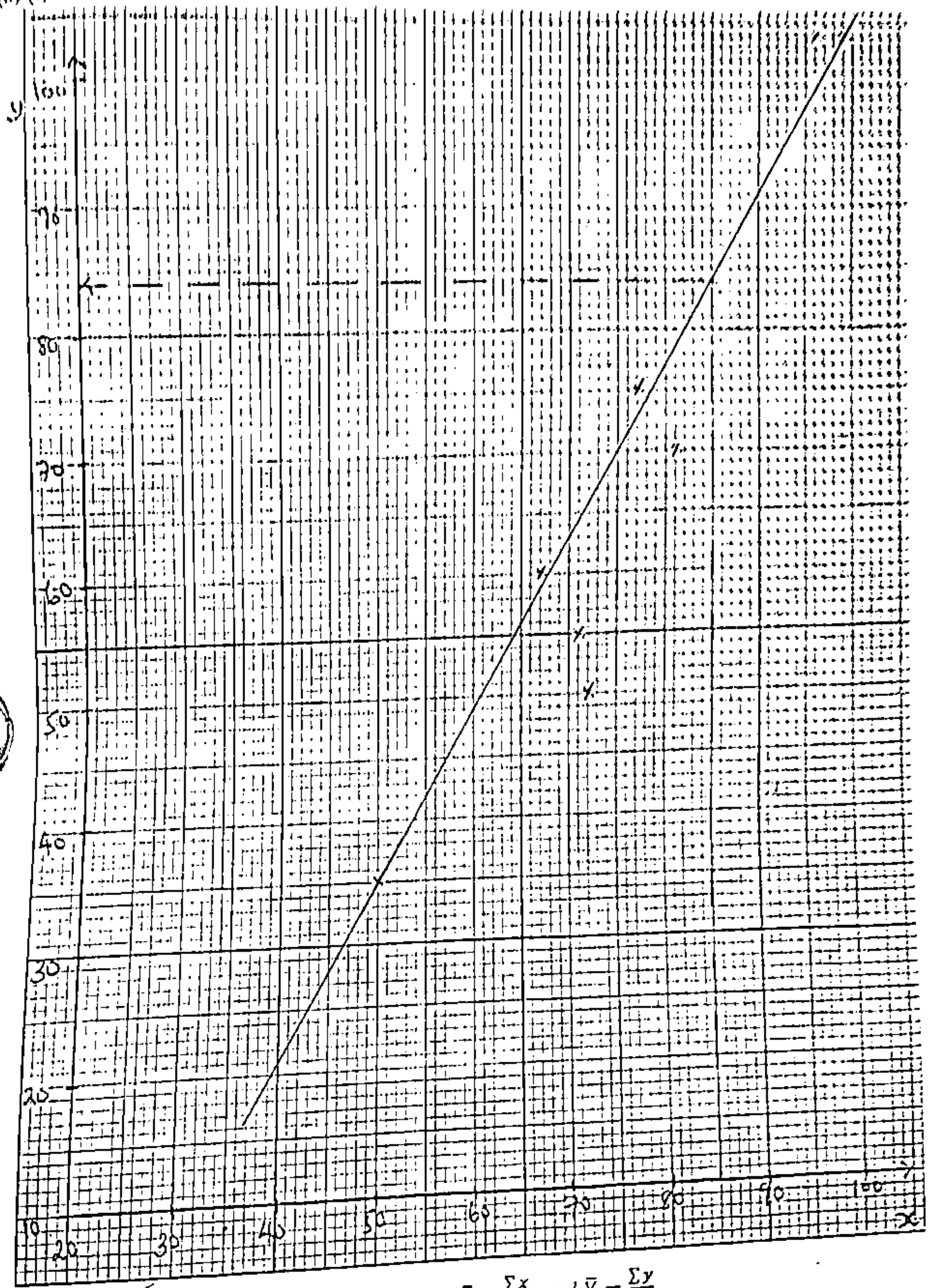
(ii) on the same diagram draw a line of best fit

(iii) Use the line of best fit to find the value of y when $x = 85$

(b) Calculate the spearman's rank correlation coefficient. Comment on your result

Solution

(a) (i)



(ii) Line of best fit passes through $(\bar{X}, \bar{Y})$ where $\bar{X} = \frac{\Sigma x}{n}$ and $\bar{Y} = \frac{\Sigma y}{n}$

$$\bar{X} = \frac{613}{8} = 76.625 \approx 77 \text{ and } \bar{Y} = \frac{543}{8} = 67.875 \approx 68$$

Line of best fit passes through (77, 68)

(iii) when $x = 85, y = 84$

(b)

Mid-term tests (x)	99	71	50	67	77	81	96	72	
End of term test (y)	99	55	35	60	75	70	99	50	
R_x	1	6	8	7	4	3	2	5	
R_y	1.5	6	8	5	3	4	1.5	7	
$d = R_x - R_y$	-0.5	0	0	2	1	-1	0.5	-2	$\sum d^2 = 10.5$
d^2	0.25	0	0	4	1	1	0.25	4	

$$\rho = 1 - \frac{6\sum d^2}{n(n^2-1)} = 1 - \frac{6 \times 10.5}{8(63)} = 1 - 0.125 = 0.875$$

Comment: There is a high positive correlation between the marks of the students for the two tests

10. (a) Given that $A = \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 1 \\ 0 & -2 \end{pmatrix}$; find

(i) AB (ii) BA

Comment on your result

(b) A family bought the following items for three successive days. The first day it bought three bunches of matooke, two kilograms of rice, five kilograms of meat and two kilograms of sugar. The second day it bought only one kilogram of sugar. The third day the family bought a bunch of matooke and two kilograms of rice. A bunch of matooke costs Shs15,000. A kilogram of rice, meat and sugar cost Shs3,300, Shs8,000 and Shs3,000 respectively.

(i) represent the family's requirements in a 3 x 4 matrix

(ii) write down the cost of each item as a column matrix

(iii)Use the matrices in b(i) and (ii) to find the family's total expenditure for the three days

Solution

(a) (i) $AB = \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 2\times 4 + -3\times 0 & 2\times 1 + -3\times -2 \\ 1\times 4 + 1\times 0 & 1\times 1 + 1\times -2 \end{pmatrix} = \begin{pmatrix} 8 & 8 \\ 4 & -1 \end{pmatrix}$

(ii) $BA = \begin{pmatrix} 4 & 1 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 4\times 2 + 1\times 1 & 4\times -3 + 1\times 1 \\ 0\times 2 + -2\times 1 & 0\times -3 + -2\times 1 \end{pmatrix} = \begin{pmatrix} 9 & -11 \\ -2 & -2 \end{pmatrix}$

Comment: $AB \neq BA$ hence the commutative law doesn't apply for matrices

(b) (i) let B – Bunch of Matooke, R – kilogram of rice , M – kilogram of meat, S – Kilogram of sugar

$$\begin{array}{c c} & \begin{array}{cccc} B & R & M & S \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 3 & 2 & 5 & 2 \\ 0 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \end{pmatrix} \end{array}$$

(ii)

$$\begin{array}{c c} \begin{array}{c} B \\ R \\ M \\ S \end{array} & \begin{pmatrix} 15000 \\ 3300 \\ 8000 \\ 3000 \end{pmatrix} \end{array}$$

(iii) Using matrix multiplication

$$\begin{pmatrix} 3 & 2 & 5 & 2 \\ 0 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \end{pmatrix}\begin{pmatrix} 15000 \\ 3300 \\ 8000 \\ 3000 \end{pmatrix} = \begin{pmatrix} 3\times 15000 + 2\times 3300 + 5\times 8000 + 2\times 3000 \\ 0\times 15000 + 0\times 3300 + 0\times 8000 + 1\times 3000 \\ 1\times 15000 + 2\times 3300 + 0\times 8000 + 0\times 3000 \end{pmatrix}$$

$$= \begin{pmatrix} 45000 + 6600 + 6000 \\ 3000 \\ 15000 + 6600 \end{pmatrix} = \begin{pmatrix} 57600 \\ 3000 \\ 21600 \end{pmatrix}$$

Expenditure on the first day = 57600/=

Expenditure on the second day = 3000/=

Expenditure on the third day = 21600/=

Total expenditure = 57600 + 3000 + 21600 = 82200/=

11 (a) the table below shows the price (U Shs) of flour and eggs in the years of 2000 and 2010

COMMODITY	PRICE (UShs)	
	2000	2010
Flour (kg)	3000	5000
Eggs (1 tray)	5000	7000

Taking 2000 as the base year, calculate the:

(i) price relative of each commodity

(ii) simple aggregate price index

Comment on your result

(b) the data below shows items with their corresponding price relatives and weights

ITEM	PRICE RELATIVE	WEIGHT
Food	120	172
Clothing	124	160
Housing	125	170
Transport	135	210
Others	104	140

(i) Find the cost of living index

(ii) Comment on your result

Solution

(a) (i) Price relative for flour $= \frac{5000}{3000} \times 100 = 166.67$

Price relative for eggs $= \frac{7000}{5000} \times 100 = 140$

(ii) Simple aggregate price index $= \frac{\sum P_1}{\sum P_0} \times 100$

Total price in 2010 i.e $\sum P_1 = 5000 + 7000 = 12000$

Total price in 2000 i.e. $\sum P_0 = 3000 + 5000 = 8000$

Simple aggregate price index $= \frac{12000}{8000} \times 100 = 150$

Comment: The prices of the commodities increased by 50% from 2000 to 2010

(b) (i)

Item	Price index(P)	Weight(W)	PW
Food	120	172	20640
Clothing	124	160	19840
Housing	125	170	21250
Transport	135	210	28350
Others	104	140	14560
Σ		852	104640

Cost of living index = weighted price index = $\frac{\sum PW}{\sum W} = \frac{104640}{852} = 122.82$

(ii) Comment: The cost of living increased by 22.82%

12. Given the curve $y = 3x^3 - 4x^2 - x$

(a) find the turning points of the curve

(b) distinguish between the nature of the turning points

Solution

(a) For turning points; $\frac{dy}{dx} = 0$

$$y = 3x^3 - 4x^2 - x$$

$$\frac{dy}{dx} = 9x^2 - 8x - 1$$

$$9x^2 - 8x - 1 = 0$$

$$9x^2 - 9x + x - 1 = 0$$

$$9x(x-1) + (x-1) = 0$$

$$(x-1)(9x+1) = 0$$

$$Either\ x - 1 = 0 \Rightarrow x = 1\ \ or\ 9x + 1 = 0 \Rightarrow x = -\frac{1}{9}$$

When $x = 1, y = 3(1)^3 - 4(1)^2 - (1) = -2$

$(1, -2)$ is a turning point

When $x = -\frac{1}{9}, y = 3\left(-\frac{1}{9}\right)^3 - 4\left(-\frac{1}{9}\right)^2 - \left(-\frac{1}{9}\right) = \frac{14}{243}$

$\left(-\frac{1}{9}, \frac{14}{243}\right)$ is a turning point

(b) Nature of the turning points

x	L	1	R	L	$-\frac{1}{9}$	R
sign of $\frac{dy}{dx}$	$-$	0	+	+	0	$-$
	\	—	/	/	—	\
		Minimum			Maximum	

Therefore $(1, -2)$ is a minimum turning point and $\left(-\frac{1}{9}, \frac{14}{243}\right)$ is a maximum turning point

13. The table below shows the probability distribution of the number of Compact Discs (CDs) sold.

Number of CD's sold	0	1	2	3	4
Probability, P(X= x)	0.05	0.28	c	0.22	0.99

Determine the:

(a) value of c

(b) probability that at least 2 CD's are sold

(c) expectation E(X)

(d) standard deviation

Solution

(a) $\sum_{all\,x} P(X = x) = 1$

$$0.05 + 0.28 + c + 0.22 + 0.09 = 1$$

$$c + 0.64 = 1 \Rightarrow c = 0.36$$

(b) $P(X \geq 2) = P(X = 2) + P(X = 3) + P(X = 4)$

$$= c + 0.22 + 0.09$$

$$= 0.36 + 0.22 + 0.09 = 0.67$$

(c)

x	$P(X = x)$	$xP(X = x)$	$x^2P(X = x)$
0	0.05	0	0
1	0.28	0.28	0.28
2	0.36	0.72	1.44
3	0.22	0.66	1.98
4	0.09	0.36	1.44
$\sum$	1	2.02	5.14

Expectation E(X) = $\sum xP(X = x) = 2.02$

(d) standard deviation = $\sqrt{Var\ (X)}$

$$Var(X) = E(X^2) - [E(X)]^2 = 5.14 - (2.02)^2 = 1.0596$$

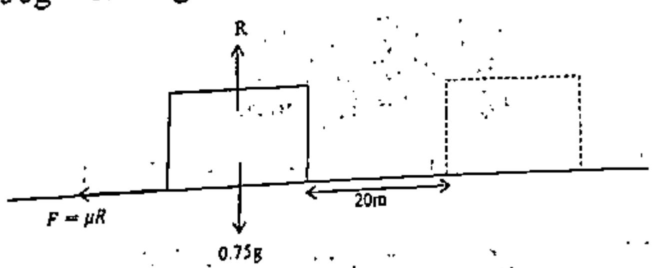
Standard deviation = $\sqrt{1.0596} = 1.0294$

14. (a) A brick of mass 750 g is dragged by a horizontal force at a uniform speed along a rough horizontal surface, through a distance of 20m. The work done against friction is 49.8J. Calculate the coefficient of friction between the brick and the surface

(b) A truck of mass 8 tonnes has a maximum speed of 20 ms^{-1} up an incline of arcsin$\frac{1}{50}$ when the engine is working against resistances of 30,000 N. Calculate the maximum power of the engine.

Solution

(a) M = 750g = 0.75 kg



Work done against friction = Friction force × distance

$$49.8 = F \times 20$$

$F = \frac{49.8}{20} = 2.49\ N$

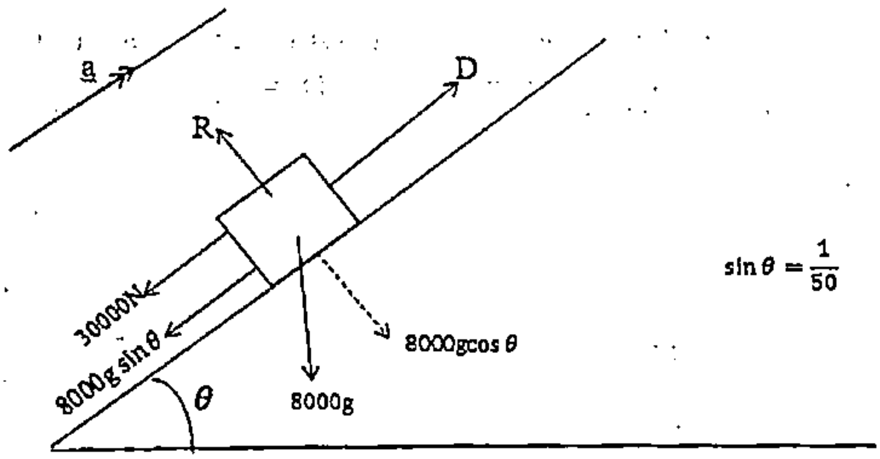
But friction force , $F = \mu R$

$R = 0.75g = 0.75 \times 9.8 = 7.35\ N$

$\Rightarrow 2.49 = 7.35\mu$

$\mu = \frac{2.49}{7.35} = 0.339$

(b)



At maximum speed, $a = 0$

$\Rightarrow D - [30000 + 8000g \sin\theta] = 0$

$D = 30000 + 8000 \times 9.8 \times \frac{1}{50}$

$\therefore D = 30000 + 1568 = 31568\ N$

Maximum power of the engine = driving force (D) × Velocity (V)

$= 31568 \times 20 = 631360\ W$

SOLUTIONS TO UNEB 2015

SECTION A (40 MARKS)

1. Evaluate $\frac{\log_6 216+\log_2 64}{\log_3 243-\log_{10} 0.1}$

Solution

$$\frac{\log_6 216+\log_2 64}{\log_3 243-\log_{10} 0.1}=\frac{\log_6 6^3+\log_2 2^6}{\log_3 3^5-\log_{10} 10^{-1}}=\frac{3\log_6 6+6\log_2 2}{5\log_3 3+\log_{10} 10}=\frac{3+6}{5+1}=\frac{9}{6}=\frac{3}{2}$$

2. The table below shows the ranks of marks awarded by Judge 1 (R_X) and Judge 2 (R_Y) to 7 choir groups A to G.

Choir	A	B	C	D	E	F	G
Rank Judge 1 (R_X)	2	4	6	1	5	3	7
Rank Judge 2 (R_Y)	2	3	5	1	6	4	7

Calculate Spearman's rank correlation coefficient between the marks awarded by the two judges.

Comment on your result.

Solution

Choir	A	B	C	D	E	F	G	
R_X	2	4	6	1	5	3	7	
R_Y	2	3	5	1	6	4	7	
$d=R_X-R_Y$	0	1	1	0	-1	-1	0	
d^2	0	1	1	0	1	1	0	$\sum d^2=4$

$$\rho=1-\frac{6\sum d^2}{n(n^2-1)}=1-\frac{6\times 4}{7(48)}=0.9286$$

Comment: There is a very high positive correlation between the judges

3. Solve the equation $3\sin^2\theta+\cos\theta+1=0$ for values of θ from 0^0 to 180^0 inclusive.

Solution

From $\sin^2\theta+\cos^2\theta=1, \Rightarrow \sin^2\theta=1-\cos^2\theta$

$$3(1-\cos^2\theta)+\cos\theta+1=0$$
$$3-3\cos^2\theta+\cos\theta+1=0$$
$$4-3\cos^2\theta+\cos\theta=0$$
$$3\cos^2\theta-\cos\theta-4=0$$
$$3\cos^2\theta+3\cos\theta-4\cos\theta-4=0$$
$$3\cos\theta(\cos\theta+1)-4(\cos\theta+1)=0$$

$(\cos\theta + 1)(3\cos\theta - 4) = 0$

Either $\cos\theta + 1 = 0 \Rightarrow \cos\theta = -1$

$\theta = \cos^{-1}(1) = 0^0 \Rightarrow \theta = \{180^0\}$

Or $3\cos\theta - 4 = 0 \Rightarrow \cos\theta = \frac{4}{3}$

$\cos\theta = \cos^{-1}\left(\frac{4}{3}\right)$ *(values do not exist)*

$\therefore$ *for* $0^0 \le \theta \le 180^0 = \{180^0\}$

4. A committee of 5 people is to be formed from a group of 6 men and 7 women.

(a) Find the number of possible committees

(b) What is the probability that there are only 2 women on the committee?

Solution

(a)

Men/6	Women/7	Number of committees
0	5	${}^6C_0 \times {}^7C_5 = 21$
1	4	${}^6C_1 \times {}^7C_4 = 210$
2	3	${}^6C_2 \times {}^7C_3 = 525$
3	2	${}^6C_3 \times {}^7C_2 = 420$
4	1	${}^6C_4 \times {}^7C_1 = 105$
5	0	${}^6C_5 \times {}^7C_0 = 6$
Total		1287

Number of possible committees = 1287

(b) Number of committees with two women = 525

$$P(2 \text{ women}) = \frac{525}{1287} = 0.408$$

5. Find the gradient of the curve $y = 4x^2(3x + 2)$ at the point $(1, 20)$

Solution

$$y = 4x^2 \times 3x + 4x^2 \times 2 = 12x^3 + 8x^2$$

$$\frac{dy}{dx} = 36x^2 + 16x$$

At $(1, 20)$, $x = 1$

$$\frac{dy}{dx} = 36(1)^2 + 16(1) = 52$$

The gradient of the curve is 52

6. Three events A, B and C are such that P(A) = 0.6, P(B) = 0.8, $P(B/A) = 0.45$ and $P(B \cap C) = 0.28$. Find

(a) $P(A \cap B)$

(b) $P(C/B)$

Solution

(a) From $P(B/A) = \frac{P(B \cap A)}{P(A)}$

$$P(A \cap B) = P(A) \times P(B/A)$$

$$= 0.6 \times 0.45 = 0.27$$

(b) $P(C/B) = \frac{P(C \cap B)}{P(B)} = \frac{0.28}{0.8} = 0.35$

7. The matrix $A = \begin{pmatrix} 2 & 1 \\ -3 & 0 \end{pmatrix}$ and I is a 2× 2 identity matrix. Determine the matrix B such that $A^2 + \frac{1}{2}B = I$

Solution

$A = \begin{pmatrix} 2 & 1 \\ -3 & 0 \end{pmatrix}$ $\qquad$ $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Let the matrix, $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$A^2 = \begin{pmatrix} 2 & 1 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -3 & 0 \end{pmatrix} = \begin{pmatrix} 2 \times 2 + 1 \times -3 & 2 \times 1 + 1 \times 0 \\ -3 \times 2 + 0 \times -3 & -3 \times 1 + 0 \times 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -6 & -3 \end{pmatrix}$$

$$A^2 + \frac{1}{2}B = I$$

$$\begin{pmatrix} 1 & 2 \\ -6 & -3 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 + \frac{a}{2} & 2 + \frac{b}{2} \\ -6 + \frac{c}{2} & -3 + \frac{d}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$1 + \frac{a}{2} = 1 \Rightarrow a = 0$

$2 + \frac{b}{2} = 0 \Rightarrow b = -4$

$-6 + \frac{c}{2} = 0 \Rightarrow c = 12$

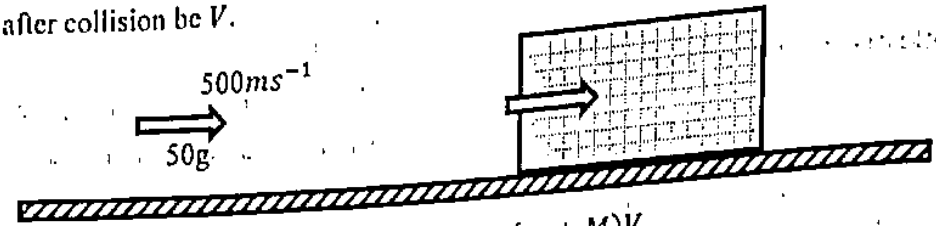
$-3 + \frac{d}{2} = 1 \Rightarrow d = -6$

$$\therefore B = \begin{pmatrix} 0 & -4 \\ 12 & -6 \end{pmatrix}$$

8. A bullet of mass 50g is fired towards a stationary wooden block and enters the block when travelling horizontally with a speed of $500ms^{-1}$. The wooden block provides a constant resistance of 36,000 N. Find how far into the block the bullet will penetrate.

Solution

Let the mass of the bullet be m with the velocity u and the mass of the wooden block be M. Let their velocity after collision be V.



$$mu + 0 = (m + M)V$$

$$V = \frac{mu}{m+M} = \frac{50\times10^{-3}\times500}{50\times10^{-3}+M} = \frac{25}{0.05+M}ms^{-1}$$

$0 - f = ma$ where f is the resistance force

$$a = -\frac{36{,}000}{50\times10^{-3}} = -720{,}000ms^{-2}$$

From $v^2 = u^2 + 2aS$

$$0 = V^2 + 2\times -720{,}000 \times S$$

$$S = \frac{V^2}{1{,}440{,}000}\ m$$

Note: In this question, the mass of the wooden block was not given and the final velocity of the wooden block and the bullet would not be calculated.

SECTION B (60 MARKS)

9. The table below shows the number of students and the marks scored in a test

MARKS	NUMBER OF STUDENTS
0 – 4	10
5 – 9	7
10 – 14	5
15 – 19	3
20 – 24	7
25 – 29	11
30 – 34	37
35 – 39	20

(a) (i) Draw a cumulative frequency curve (Ogive) for the data

(ii) Use the Ogive to estimate the median mark

(b) Calculate the;

(i) mean mark

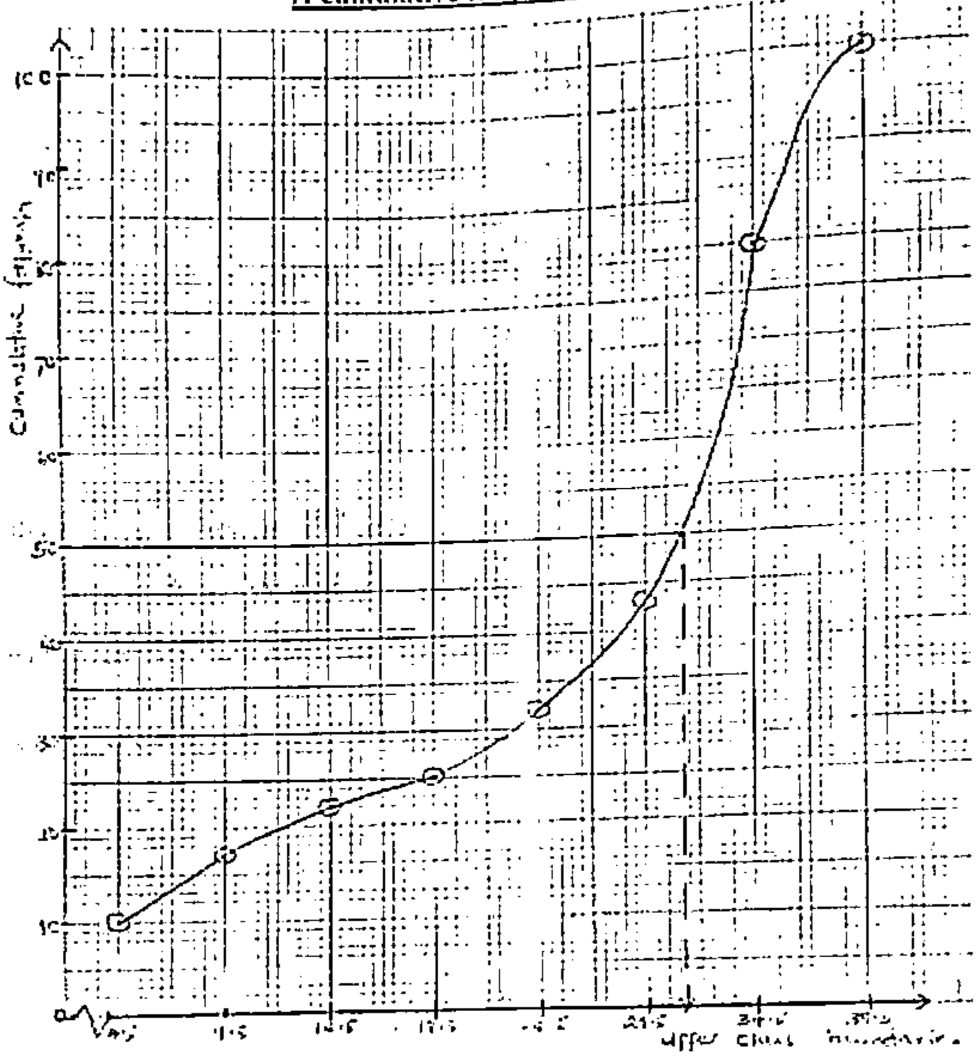
(ii) standard deviation

Solution

Class	f	x	fx	fx^2	F	Upper class boundaries
$0-4$	10	2	20	40	10	4.5
$5-9$	7	7	49	343	17	9.5
$10-14$	5	12	60	720	22	14.5
$15-19$	3	17	51	867	25	19.5
$20-24$	7	22	154	3388	32	24.5
$25-29$	11	27	297	8019	43	29.5
$30-34$	37	32	1184	37888	80	34.5
$35-39$	20	37	740	27380	100	39.5
$\sum$	100		2555	78645		

(a)(i)

A cumulative frequency curve



(ii) Median $= \left(\frac{N}{2}\right)^{th}$ $value = 50^{th} value$

From the graph, median = 31.5

(b)(i) Mean mark $= \frac{\Sigma fx}{\Sigma f} = \frac{2555}{100} = 25.55$

(ii) Standard deviation $= \sqrt{\frac{\Sigma fx^2}{\Sigma f} - \left(\frac{\Sigma fx}{\Sigma f}\right)^2} = \sqrt{\frac{78645}{100} - (25.55)^2} = \sqrt{133.6475} = 11.56$

10. The rate of decay of a radioactive material is proportional to the amount x grams of the material present at any time t. Initially there was 60 grams of the material. After 8 years the material had reduced to 15 grams.

(a) Form a differential equation for the rate of decay of the material

(b) Solve the differential equation formed in (a) above

(c) Find the time taken for the material to reduce to 10 grams.

Solution

a) $\frac{dx}{dt} \propto x$

$\frac{dx}{dt} = -kx$

(b) $\frac{dx}{x} = -kdt$

$\int \frac{dx}{x} = \int -k\,dt$

$\ln x = -kt + C$

$at\ t = 0, x = x_0 \Rightarrow \ln x_0 = C$

$\ln x = -kt + \ln x_0$

$\ln x - \ln x_0 = -kt$

$\ln\left(\frac{x}{x_0}\right) = -kt$

$x_0 = 60,\ at\ t = 8, x = 15$

$\ln\left(\frac{15}{60}\right) = -k(8) \Rightarrow k = \frac{1}{8}\ln 4$

$\ln\left(\frac{x}{x_0}\right) = -\left(\frac{1}{8}\ln 4\right)t$

(c) $at\ t = ?,\quad x = 10$

$\ln\left(\frac{10}{60}\right) = -\left(\frac{1}{8}\ln 4\right)t$

$t = 10.34\ years$

11. The table below shows the prices (in Ug Shs) of some food items in January, June and December together with the corresponding weights.

Item	Price (in Ug Shs)			Weight
	January	June	December	
Matooke (1 bunch)	15,000	13,000	18,000	4
Meat (1 kg)	6,500	6,000	7,150	1
Posho (1kg)	2,000	1,800	1,600	3
Beans(1kg)	2,200	2,000	2,860	2

Taking January as the base month, calculate the;

(a) simple aggregate price index for June. Comment on your result.

(b) weighted aggregate index price index for December. Comment on your result.

Solution

(a) Simple aggregate price index $= \frac{\sum P_1}{\sum P_0} \times 100$

For June, S.A.P.I $= \frac{13000+6000+1800+2000}{15000+6500+2000+2200} \times 100 = \frac{22800}{25700} \times 100 = 88.72$

Comment: The prices of the commodities reduced by 11.28% in June

(b) Weighted aggregate price index $== \frac{\sum P_1 W}{\sum P_0 W} \times 100$

For December, W.A.P.I $= \frac{(18000\times4)+(7150\times1)+(1600\times3)+(2860\times2)}{(15000\times4)+(6500\times1)+(2000\times3)+(2200\times2)} \times 100$

$= \frac{89670}{76900} \times 100 = 116.61$

Comment: The prices of the commodities increase by 16.61% in December.

12. The roots of the equation $2x^2 - 6x + 7 = 0$ are α and β. Determine the

(a) values of $(\alpha - \beta)^2$ and $\frac{1}{\alpha^2\beta} + \frac{1}{\alpha\beta^2}$

(b) quadratic equation with integral coefficient whose roots are $(\alpha - \beta)^2$ and $\frac{1}{\alpha^2\beta} + \frac{1}{\alpha\beta^2}$

Solution

$$2x^2 - 6x + 7 = 0$$

$$x^2 - 3x + \frac{7}{2} = 0$$

$$\alpha + \beta = 3, \quad \alpha\beta = \frac{7}{2}$$

(a) $(\alpha - \beta)^2 = (\alpha - \beta)(\alpha - \beta) = \alpha^2 - \alpha\beta - \alpha\beta + \beta^2 = \alpha^2 + \beta^2 - 2\alpha\beta$

But $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$\Rightarrow (\alpha - \beta)^2 = (\alpha + \beta)^2 - 2\alpha\beta - 2\alpha\beta = (\alpha + \beta)^2 - 4\alpha\beta$

$= 3^2 - 4\left(\frac{7}{2}\right) = 9 - 14 = -5$

$\frac{1}{\alpha^2\beta} + \frac{1}{\alpha\beta^2} = \frac{\beta+\alpha}{\alpha^2\beta^2} = \frac{\alpha+\beta}{(\alpha\beta)^2} = \frac{3}{\left(\frac{7}{2}\right)^2} = 3 \div \frac{49}{4} = 3 \times \frac{4}{49} = \frac{12}{49}$

(b) Sum of roots $= -5 + \frac{12}{49} = \frac{12}{49} - 5 = -\frac{233}{49}$

Product of roots $= -5 \times \frac{12}{49} = -\frac{60}{49}$

$$x^2 - (sum\ of\ roots)x + product\ of\ roots = 0$$

$$x^2 - \left(-\frac{233}{49}\right)x - \frac{60}{49} = 0 \quad or\ 49x^2 + 233x - 60 = 0$$

13. A continuous random variable X has a probability density function given by,

$$f(x) = \begin{cases} \frac{kx}{6}, & 1 \le x \le 2, \\ 0 & otherwise, \end{cases}$$

where k is a constant

(a) Find

(i) the value of k

(ii) $P(X \ge 1.5)$

(iii) the mean of X, E(X)

(b) Sketch the graph of $f(x)$

Solution

(a) (i) $\int_{all\,x} f(x)\,dx = 1$

$$\int_1^2 \frac{kx}{6}\,dx = 1$$

$$\frac{k}{6}\int_1^2 x\,dx = 1$$

$$\frac{k}{6}\left[\frac{x^2}{2}\right]_1^2 = 1$$

$$\frac{k}{6}\left(2 - \frac{1}{2}\right) = 1$$

$$k = 4$$

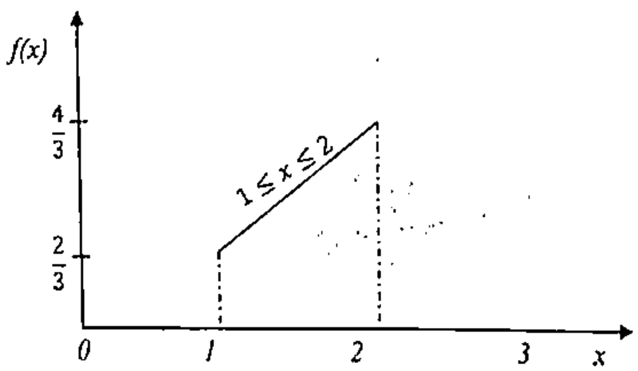
(ii) $f(x) = \begin{cases} \frac{2x}{3}, & 1 \le x \le 2, \\ 0 & otherwise, \end{cases}$

$$P(X \ge 1.5) = \int_{1.5}^2 \frac{2x}{3}\,dx = \left[\frac{x^2}{3}\right]_{1.5}^2 = \left(\frac{2^2}{3} - \frac{1.5^2}{3}\right) = \frac{4}{3} - \frac{2.25}{3} = 0.583$$

(iii) Mean, E(X) $= \int_{all\,x} x f(x)\,dx$

$$= \int_1^2 \frac{2x^2}{3}\,dx = \left[\frac{2x^3}{9}\right]_1^2 = \left(\frac{16}{9} - \frac{2}{9}\right) = \frac{14}{9} = 1.56$$

(b)



14. A motorist moving at $90kmh^{-1}$ decelerates uniformly to a velocity $V\ ms^{-1}$ in 10 seconds. He maintains this speed for 30 seconds and then decelerates uniformly to rest in 20 seconds.

(a) Sketch a velocity-time graph for the motion of the motorist

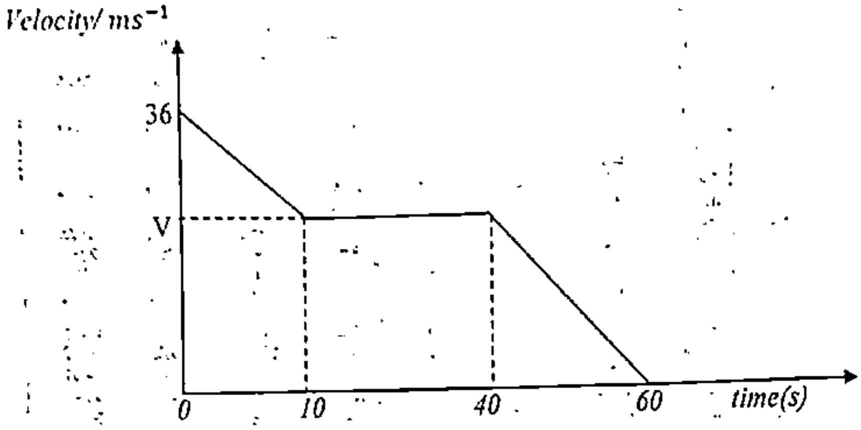
(b) Given that the total distance travelled is 800m, use your graph to calculate the value of V.

(c) Determine the two decelerations.

Solution

$$90\frac{km}{hr} = \frac{9000}{3600}ms^{-1} = 36ms^{-1}$$

(a)



(b) Total distance covered = Area under graph

$$800 = \frac{1}{2}(V+36)10 + (30\times V) + \frac{1}{2}\times 20\times V$$

$$800 = 5(V+36) + 30V + 10V$$

$$800 = 5V + 90 + 30V + 10V$$

$$710 = 45V$$

$$V = \frac{710}{45} = 15.78ms^{-1}$$

(c) First deceleration $= \frac{15.78-36}{10} = -2.02ms^{-2}$

Second deceleration$= \frac{0-15.78}{20} = -0.789ms^{-2}$