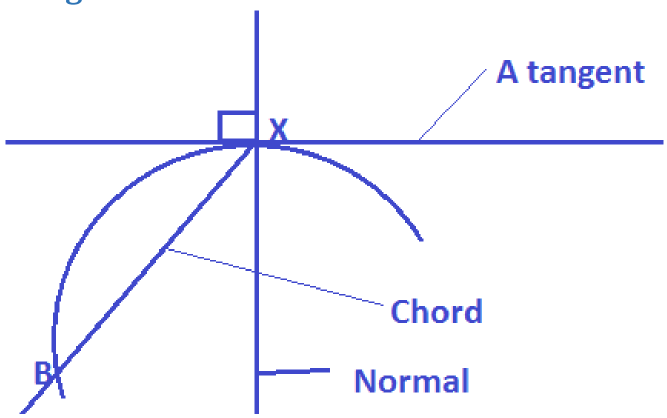
TOPIC 3: DIFFERENTIATION

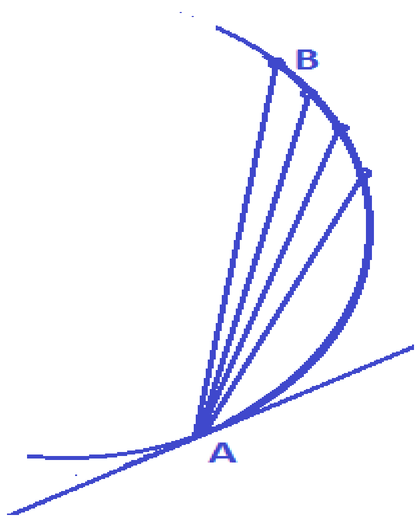
Tangents and Normal



A gradient of a curve at a point

Defined as the gradient of the tangent to the curve at the point

To find a gradient at a point you need to get a point next to the given point and use y-increase over x-increase



If A is the required point then B in the point next to A.

The closer B is to A the better the approximation

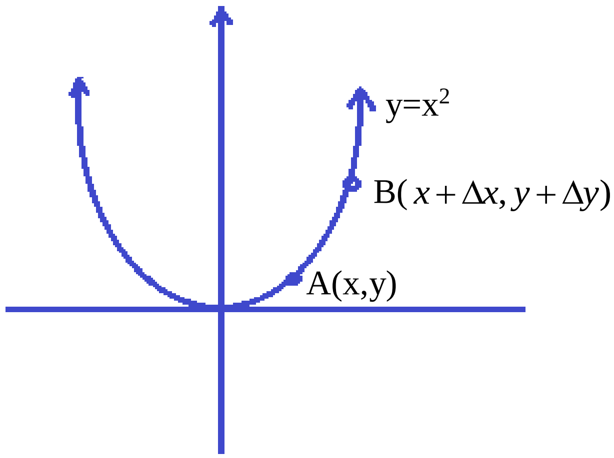
Gradient chord of AB is approximately equal to gradient at A. That is to say the gradient of AB as B tends to A will equal to the gradient at A

Note the Delta prefix

$\Delta x \rightarrow a\, small\ increase in\, x$

$\Delta y \rightarrow Small\ increase\ in\ \ y$

Consider the curve $y=x^2$



$$Grad = \frac{y+\Delta y - y}{x+\Delta x - x} = \frac{\Delta y}{\Delta x}$$

As Δy tends to zero $\frac{\Delta y}{\Delta x} \rightarrow$ grad which $\frac{dy}{dx}$

$$y + \Delta y = (x+\Delta x)^2$$

$$y + \Delta y = x^2 + 2x\Delta x + (\Delta x)^2$$

$$\Delta y = x^2 + 2x\Delta x + (\Delta x)^2 - y$$

$$\Delta y = x^2 + 2x\Delta x + (\Delta x)^2 - x^2$$

$$\Delta y = 2x\Delta x + (\Delta x)^2$$

$$\frac{\Delta y}{\Delta x} = 2x + (\Delta x)$$

$$As \;\; \Delta y \to 0$$

$$\frac{\Delta y}{\Delta x} \to grad$$
$$and \;\; 2x + \Delta x \to 2x$$
$$\therefore grad = 2x$$

$$For \;\; y = x^3$$

$$y + \Delta y = (x + \Delta x)^3$$

$$y + \Delta y = x^3 + 3x^2 \Delta x + 3x(\Delta x)^2 + (\Delta x)^3$$

$$\Delta y = x^3 + 3x^2 \Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3$$

$$\frac{\Delta y}{\Delta x} = 3x^2 \Delta x + 3x\Delta x + (\Delta x)^2$$

$$\text{As} \;\; \Delta x \to 0, \;\; 3x\Delta x, \Delta x)^2 = 0$$

$$\therefore grad = 3x^2$$

This implies that for the curve y= x^n

Its grad = nx^{n-1}

EXAMPLE 1

Finding the grad function of the curve y = 2x(x- 4) = $2x^2$-8x

$$y + \Delta y = (x + \Delta x)^2 \text{ -} 8(x + \Delta x)$$

$$y + \Delta y = 2(x^2 + 2x\Delta x + (\Delta x)^2 \text{ -} 8x - 8\Delta x$$

$$= 2x^2 + 4x\Delta x + (2\Delta x)^2 \text{ -} 8x - 8\Delta x - 2x^2 + 8x$$

$$= 4x\Delta x + (2\Delta x)^2 - 8\Delta x$$

$$\frac{\Delta y}{\Delta x} = 4x + 2\Delta x - 8$$

$$As \quad \Delta x \to 0$$

$$grad = 4x - 8$$

This is what is referred to as differentiating from first principles

In the same examples one should use the formula nx^{n-1} to find the gradient

Grad of curve $y=2x^2-8x$

Grad = $2(2x^{2-1}) -8(1x^{1-1})$

=4x-8

The first principle approach can be deduced to a formula

$$\underset{\Delta x \to 0}{Limit} \left(\frac{\Delta y}{\Delta x} \right) = \underset{\Delta x \to 0}{Lim} + \frac{(x+\Delta x) - f(x)}{\Delta x}$$

EXAMPLE 1

Find the gradient of the curve $y=x^2-x+3$ at x=1

$$\frac{dy}{dx} = 2x - 1$$

At x=1

Grad =2(1)-1

=1

Equation of Tangents and Normal

Example 1

Find the tangent to the curve $y = \frac{1}{3}x^3 + 3x^2 + 7x + 1 \ \ at \ \ x = -1$

Grad of tangent $=\frac{1}{3}(x^{3-1}+3(2x^{2-1})+7(1x^{1-1})+1 \ \ at \ \ x=-1$

Grad $=x^2+6x+7$

At x =-1

Grad = $(-1)^2$+6(-1)+

=1-6+7 =2

Get the *y* value of the point when x=-1

$$y=\frac{1}{3}(-1)^3+3(-1)^2+7(-1)+1$$

$$=-\frac{1}{3}+3-7+1$$

$$=-\frac{1}{3}-3 \quad =-\frac{10}{3}$$

The point $\left(-1,-\frac{10}{3}\right)$

Equation of tangent

$$\frac{\left(y+\frac{10}{3}\right)}{(x+1)}=2$$

$$y+\frac{10}{3}=2(x+1)$$

$$y=2x+2-\frac{10}{3}$$

$$y = 2x - \frac{4}{3}$$
$$3y = 6x - 4$$

Example

Find the equation of the normal at the point where the curve $y=x^2 +3x-$ cuts the y-axis

At y-axis   x=0   y = -2

(0,-2) is the point

Grad of tangent = $2x^{2-1} +3(1x^{1-1})$ to grad =2x+3

At x = 0

Grad of tangent = 3

But grad of tangent multiplied with grad of normal = -1

Grad of normal x3 =-1

Grad of normal $= -\frac{1}{3}$

Equation of normal

$$\frac{y+2}{x-0} = \frac{-1}{3}$$

$$3y + 6 = -x$$
$$3y + x + 6 = 0$$

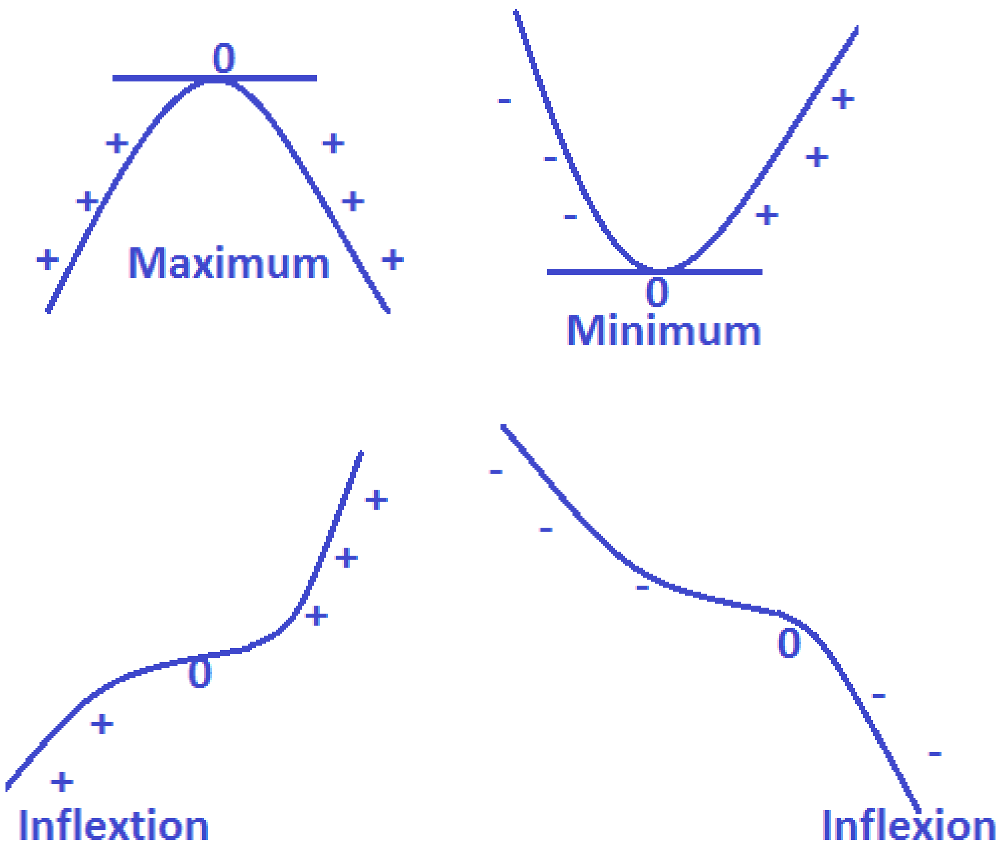
Finding turning points

Note:

At turning points gradient function is zero and the nature is determined by considering change of grad in the neighborhood of the turning point. If is changes from positive to *zero to negative the curve is maximum and if changes from negative to zero to positive then the nature of the curve will be minimum if the signs do not change then it is referred to as point of inflexion*

Grad +0 -   - 0 +,   -O- or + 0 +

Max   min   Point of inflexion



Finding The Turning Points Of Curve And Nature

There are two methods of finding the turning point and nature.

1. Differentiating the function and equating it to zero . Find the value or values of the turning point and draw a table to find the nature of the turning point as stated above.

2. Differentiating as above to get values of turning points and differentiate it for the second time and use the values above to find whether the second derivative gives a positive for minimum or negative for maximum.

Example 1

Find the turning points of the curve y = $x^4 + 4x^3 - 6$ and their nature

Step 1

Differentiate

$$\frac{dy}{dx} = 4x^3 + 12x^2$$

At turning point

$$\frac{dy}{dx} = 0$$

$$4x^3 + 12x^2 = 0$$
$$4x^2(x+3) = 0$$

x =0 0r x=-3

These are the values of x where the curve turns

.Find the corresponding values of x substitution in the equation of the curve above

x=0

y= 0+0-6=-6

(0,6) is a point

And when x =-3

y =$(-3)^4+4(-3)^3$ -6

= 81-108-6

=81-114 =-33

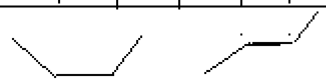
(-3, -33) is also a turning point

The Nature Of Turning Point

Method 1

Using a table

	L		R	L		R
X	-4	-3	-1	-1	0	1
$\frac{dy}{dx}$	-	0	+	+	0	+



NOTE

When inserting them in the table begin with the smallest and end with the bigger one choose a small figure next to the value of turning point on both left and right hand side for both note the sign it gives when you substitute in the gradient function

For x=-3 left hand side is negative and right hand side positive leading to minimum turning and for x=0 it is a point of inflexion of positive to positive

Method 2

After getting the values of x and y for the turning points

Differentiate the second time

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(4x^3 + 12x^2)$$

$$= 12x^3 + 24x$$

When x=-3

$\frac{d^2y}{dx^2} = 12(-3)^3 + 24(-3) = Positive$ There is a minimum turning point

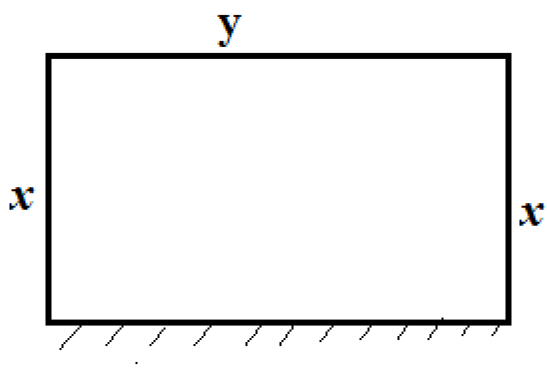
And for x=0

$$\frac{d^2y}{dx^2} = 0$$

No meaning can be derived from this. In this case use the table to find out.

Example

A farmer has an adjustable electric fence to enclose a rectangular grazing are on three sides , the tourism side being a fixed hedge. Find the maximum area he can enclose given that it is 100m long



$$A = xy = x(100 - 2x) = 100x - 2x^2$$

$$\frac{dA}{dx} = 100 - 4x$$

At ,maximum or minimum $\frac{dA}{dx} = 0$

100-4x=0

x = 25

Maximum area =25(100-50)=1250m^2

If they were two values test using second derivative method or the table

While differentiating the function which in product from and quotient form , simplify before differentiating

Example

Differentiate the function

y=(x+3)(2x+1)

Expand first

y= $2x^2+7x+3$

$$\frac{dy}{dx} = 4x + 7$$

Example

Differentiate the function $y = \frac{10x^5 + 3x^4}{2x^2}$

$$y = \frac{10x^5}{2x^2} + \frac{3x^4}{2x^2} \text{(Split the numerator)}$$

$$y = 5x^3 + \frac{3}{2}x^2$$

$$\frac{dy}{dx} = 5(3x^2) + \frac{3}{2}(2x)$$

$$= 15x^2 + 3x$$

Sketching The Curve

Steps

1. Finding the turning point
2. The nature of points
3. Cut points
4. Sketch the curve

Example

Sketch the curve $y = 4x^3 - 3x^4$

The turning points

$$\frac{dy}{dx} = 12x^2 - 12x^3$$

$$\frac{dy}{dx} = 12x^2(1-x)$$

At turning point $\frac{dy}{dx} = 0$

$12x^2(1-x) = o$

$x=0$ or $x=1$

L R L R

X	-1	0	½	½	1	2
$\frac{dy}{dx}$	+	0	+	+	0	1

(0,0) (inflexion) (1,1) Maximum

x=0

y =0

x=1

y =4-3

y =1

Cut points

x=0

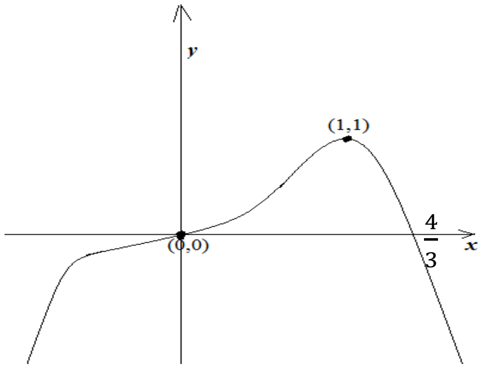
y=0 (0,0)

y=0)

$0=x^3(4-3x)$

x=0 , x= 4/3

$(\frac{4}{3}, 0)$



Example

Sketch the curve y = (x+2)(x+1)(3-x)

$y=(x+2)(3x-x^2+3-x)$

$y=(x+2)(2x-x^2+3)$

$y = 2x^2 - x^3 + 3x + 4x - 2x^2 + 6$

$y = -x^3 + 7x + 6$

$$\frac{dy}{dx} = -3x^2 + 7$$

$3x^2 = 7$

$$x^2 = \frac{7}{3}$$

$$x = \pm\sqrt{\frac{7}{3}}$$

Nature

$$\frac{d^2y}{dx^2} = -6x$$

For $x = \sqrt{\frac{7}{3}}$

The point is minimum

For $x = -\sqrt{\frac{7}{3}}$

The point is maximum

(1.5, 13.1)   (1.5,-1.1)

Min   max

Cut points

When x =0

Y =(2)(1)(3)   (0,6)

=6

When y =0

0=(x+2)(x+1)(3-x)

X=-2, x=-1, x=3

(-2,0), (-1,0) and (3,0)