COMPLEX NUMBERS

Complex numbers

Equations $ax^2 + bx + c = 0$ where $a \neq 0$ and $b^2 < 4ac$ can only be solved by introducing $\sqrt{(-1)}$ = i or j; and this is the basis of complex numbers.

Solving equations involving complex numbers

Given $ax^2 + bx + c = 0$

$$x^2 + \left(\frac{b}{a}\right)x = \left(-\frac{c}{a}\right)$$

$$\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b^2}{4a^2} - \frac{c}{a}\right) = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{4a^2}$$

If $b^2 > 4ac$; the equation has two roots which are both real.

If $b^2 = 4ac$; the equation has got two roots which are repeated and real.

If $b^2 < 4ac$; the equation has got two roots which are not real.

In complex numbers equations whose roots are not real are considered

Hence

$$x = -\frac{b}{2a} \pm i\frac{\sqrt{b^2 - 4ac}}{4a^2}$$

We let x = p ± iq

Where $p = -\frac{b}{2a}$ and $q = \frac{\sqrt{b^2-4ac}}{4a^2}$

Example 1

(a) $x^2 + 4 = 0$

Solution

$x^2 = -4$

$x = \sqrt{-4} = \pm 2i$

(b) $x^2 + 2x + 5 = 0$

Solution

From $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-2 \pm \sqrt{4 - 4\,(1)(5)}}{2\,(1)} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2}$$

$x = -1 \pm 2i$

(c) $2x^2 - 3x + 4 = 0$

$$x = \frac{3 \pm \sqrt{9 - 4\,(2)(4)}}{2\,(2)} = \frac{3 \pm \sqrt{-25}}{4} = \frac{3 \pm 5i}{4}$$

$$x = \frac{3}{4} \pm \frac{5}{4}i$$

The above simple examples reveal the following

(a) A complex number Z may be written as Z = x + iy, where x and y are real numbers. The real part of Z is x and the imaginary part is y. i.e. *x = Re(Z)* and *y = im(Z).*

(b) Solution of such equations whose roots are complex occur in pairs, called conjugates. For instance, compare Z = $\pm 2i$, -1 ± 2i and $\frac{3}{4} \pm \frac{5}{4}i$

 (i) If Z = -1 + 2i is one root of an equation, then its conjugate $\bar{Z}$ = -1 – 2i is another root.

 (ii) An equation cannot have odd number of complex roots

(c) If x + iy = 0, then x = 0 and y = 0

This implies that two complex numbers are equal if and only if their corresponding real and imaginary parts are equal

i.e. $Z_1 = x_1 + iy_1$ and $Z_2 = x_2 + iy_2$, then

$Z_1 = Z_2$, only and only if $x_1 = x_2$ and $y_1 = y_2$.

Proof: If Z1 = Z2

$x_1 + iy_1 = x_2 + iy_2$

i.e. $(x_1 – x_2) + (y_1 – y_2) = 0$

=> $x_1 – x_2 = 0$ and $y_1 – y_2 = 0$

$x_1 = x_2$ and $y_1 = y_2$

Algebra of complex numbers

Like in the real plane, complex numbers can be subjected to four basic operations of addition, subtraction, multiplications and division.

Using $Z_1 = x_1 + iy_1$ and $Z_2 = x_2 + iy_2$

Addition of complex number

When adding two or more complex numbers, real parts are added separately and imaginary parts are added separately

$Z_1 + Z_2 = (x_1 + x_2) + i(y_1 + y_2)$

Illustration

If $Z_1 = 2 + 3i$ and $Z_2 = 1 – 2i$

$Z_1 + Z_2 = (2 + 1) + i(3 + -1)$

$= 3 + 2i$

Subtraction of complex numbers

When subtracting two or more complex numbers, real parts are subtracted from real parts and imaginary parts subtracted from imaginary parts.

$Z_1 – Z_2 = (x_1 – x_2) + i(y_1 – y_2)$

Illustration

If $Z_1 = 2 + 3i$ and $Z_2 = 1 – 2i$

$Z_1 + Z_2 = (2 – 1) + i(3 – -1)$

$= 1 + 4i$

Geometrically, the operations of addition and subtraction of complex numbers are diagonals of a parallelogram. (see the Argand diagrams)

Multiplication of complex numbers

When multiplying complex numbers the following properties should be observed

$i^2= \sqrt{-1} x \sqrt{-1} = -1$;

$i^3 = -1 x i = -1$

$i^4 = i^2 x i^2 = -1 x -1 = 1$

$i^5 = i^3 x i^2 =-1 x -1 = 1$

$i^6 = i^3 x i^3 =-1 x -1 = i^2 = -1$

$i^8 = i^4 x i^4 =1 x 1 = 1$

Now

$Z_1Z_2 = (x_1 + iy_1)(x_2 + iy_2)$

$= x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2$

$= (x_1x_2 – y_1y_2) + i(x_1y_2 + x_2y_1)$

Division of complex numbers

When diving a complex number with another, the result is also a complex number. It can be done in two ways

Either:

Let $\frac{Z_1}{Z_s} = p + iq$ for some real numbers p and q.

⇨ $Z_1 = (p + iq)Z_2$

$x_1 + iy_1 = (p + iq)(x_2 + iy_2)$

$= px_2 + ipy_2 + iqx_2 +i2qy_2$

$= px_2 – qy_2 + i(py_2 + qx_2)$

Equating the corresponding real and imaginary parts of the resulting complex number

$x_1 = px_2 – qy_2$ and $x_2= py_2 + qx_2$

Solving these equation simultaneously for p and q p = $\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}$ and q = $\frac{x_1x_2-y_1y_2}{x_2^2+y_2^2}$

Or:

'Realizing' the denominator by multiplying through by it conjugate

$$\frac{Z_1}{Z_s}=\frac{(x_1+iy_1)(x_2-iy_2)}{(x_2+iy_2)(x_2-iy_2)}=\frac{x_1x_2-ix_1y_2-ix_2y_1\,y_1y_2}{x_2^2+y_2^2}$$

$$=\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}+i\,\frac{(x_1x_2-y_1y_2)}{x_2^2+y_2^2}$$

Example 2

Given that Z_1 = 3 – 2i and Z_2= 2 + 3i, by expressing your answer in the form a + bi, find

(a) $Z_1 + Z_2$
Solution
$Z_1 + 3Z_2$ = (3 – 2i) + 3(2 + 3i) = 9 + 7i

(b) $2Z_1$ – Z2
Solution
2(3 – 2i) – (2 + 3i) = 4 – 7i

(c) If Z1 = 2 – 3i and Z2 = 4 + i, find the values of

(i) $Z_1^3Z_2$
Solution

$Z_1^3Z_2 = (2 – 3i)^3(4 + i)$

Using Pascal's triangle

$= [2^3 + 3(2)^2(-3i) + 3(2)(-3i)^2 + (-3i)^3](4+ i)$

=(8 – 36i – 54 + 27i)(4 + i)

=(-46 – 9i)(4 + i)

= -184 – 46i – 36i + 9

= -175 – 82i

(ii) $\frac{Z_2^1}{Z_1}$

Solutions

$$\frac{Z_2^1}{Z_1}=\frac{(4+i)^2}{(2-3i)}=\frac{16+8i-1}{(2-3i)}=\frac{15+8i}{(2-3i)}$$

$$=\frac{(15+8i)(2+3i)}{(2-3i)(2+3i)}=\frac{30+45i+16i-24}{4+9}$$

$$=\frac{6}{13}+i\,\frac{61}{13}$$

Forming quadratic equations given roots

When given roots to a complex number, the quadratic equation say Z can be obtained by finding the sum and the products of the roots

i.e. the equation is given by
Z^2 – (sun of roots)Z + product of roots

Not that if Z is a root of the equation, then its conjugate $\bar{Z}$ is also a root

Example 3

Form a quadratic equation in Z given the following roots

(a) 1 + 5i
Solution
Let Z = 1 + 5i, the Z* = 1 – 5i
Sum of roots (1 + 5i) + (1 - 5i) = 2
Product of roots (1 + 5i)(1 – 5i)= 26
Equation: Z^2 – 2Z + 26 = 0
Or
Let Z = 1 + 5i => Z – 1 – 5i = 0
Z = 1 – 5i => Z – 1 + 5i = 0
⇨ (Z – 1 – 5i)(Z – 1 + 5i) = 0
Z^2 – 2Z + 26 = 0

(b) 4 – 3i
Solution
Let Z = 4 – 3i, the Z* = 4 + 3i
Sum of roots (4 - 3i) + (4 + 3i) = 8
Product of roots (4 - 3i)(4 - 3i)= 41
Equation: Z^2 – 8Z + 41 = 0
Or
Let Z = 4 - 3i => Z – 4 + 3i = 0
Z = 4 + 3i => Z – 4 - 3i = 0
⇨ (Z – 4 + 3i)(Z – 4 - 3i) = 0
Z^2 – 8Z + 41 = 0

Solving cubic equations

It is quite difficult to solve cubic equations, however, we use the approach of inspection, i.e. we substitute the assumed factors into the given equation. If the unction is f(x) = 0 and x – a = 0 is a root, then f(a) = 0.

We then get another equation (quadratic) using long division which can easily be solved to obtain other factors.

Example 4

Solve the following equations

(a) $x^3 - 4x^2 + x + 26 = 0$

Solution

Let $f(x) = x^3 - 4x^2 + x + 26 = 0$

$f(1) = 2 - 4 + 1 + 26 \neq 0$; => $x - 1$ is not a factor

$f(2) = 8 - 8 + 2 + 26 \neq 0$; => $x - 2$ is not a factor

$f(-2) = -8 - 16 - 2 + 26 = 0$; $x + 2$ is a factor

Using long division

$$\begin{array}{r|l} & x^2 - 6x + 13 \\ \hline (x+2) & x^3 - 4x^2 + x + 26 \\ - & \underline{x^3 + 2x^2} \\ & -\ 6x^2 + x + 26 \\ - & \underline{-\ 6x^2 - 12x} \\ & 13x + 26 \\ - & \underline{13x + 26} \\ & 0\ \ +0 \end{array}$$

$\therefore x^3 - 4x^2 + x + 26 = (x+2)(x^2 - 6x + 13) = 0$

Either $x + 2 = 0$; $x = -2$

Or $x^2 - 6x + 13 = 0$

Solving $x^2 - 6x + 13 = 0$

$x = \frac{6 \pm \sqrt{36 - 4(1)(13)}}{2(1)} = \frac{6 \pm \sqrt{16\ x - 1}}{2} = 3 \pm 2i$

The roots of the equation are $x = -2, 3 \pm 2i$

Alternatively

Given the equation $x^2 - 6x + 13 = 0$

Let the roots be $x \pm 2i$

Sum of roots $x + iy + x - iy = 6$

$2x = 6$; $x = 3$

Product of roots $= (x + iy)(x - iy) = 13$

$x^2 + y^2 = 13$

Substituting for x;

$9 + y^2 = 13$; $y = \pm 2$

The roots of the equation are $x = -2, 3 \pm 2i$

(b) $2x^3 - 12x^2 + 25x - 21 = 0$

Solution

Let $f(x) = 2x^3 - 12x^2 + 25x - 21 = 0$

$f(1) = 2 - 12 + 25 - 12 = -6 \neq 0$; $x - 1$ is not a factor

$f(2) = 16 - 48 + 50 - 21 = -3 \neq 0$; $x - 2$ is not a factor

$f(3) = 54 - 108 + 75 - 21 = 0$; $x - 3$ is a factor

Using long division

$$\begin{array}{r|l} & 2x^2 - 6x + 7 \\ \hline (x-3) & 2x^3 - 12x^2 + 25x - 21 \\ - & \underline{2x^2 - 6x^2} \\ & -6x^2 + 25x - 21 \\ - & \underline{-6x^2 + 18x} \\ & 7x - 21 \\ - & \underline{7x - 21} \\ & 0 + 0 \end{array}$$

$\therefore\ 2x^3 - 12x^2 + 25x - 21$

$= (x-3)(2x^2 - 6x + 7) = 0$

Either $x - 3 = 0$; $x = 3$

Or $2x^2 - 6x + 7 = 0$

$x = \frac{6 \pm \sqrt{36 - 4(2)(7)}}{2(2)} = \frac{6 \pm \sqrt{4\ x\ 5\ x - 1}}{4} = \frac{3}{2} \pm \frac{\sqrt{5}}{2}$

The roots of the equation are $x = 3, \frac{3}{2} \pm \frac{\sqrt{5}}{2}$

Alternatively

Given the equation $2x^2 - 6x + 7 = 0$

Or $x^2 - 3 + \frac{7}{2} = 0$

Let the roots be $x \pm 2i$

Sum of roots $x + iy + x - iy = 3$

$2x = 3$; $x = \frac{3}{2}$

Product of roots = $(x + iy)(x - iy) = \frac{7}{2}$

$x^2 + y^2 = \frac{7}{2}$

Substituting for x;

$\frac{9}{4} + y^2 = \frac{7}{2}$; $y = \frac{\pm\sqrt{5}}{2}$

The roots of the equation are $x = 3, \frac{3}{2} \pm \frac{\sqrt{5}}{2}$

Finding the square root of a complex number

Suppose that the square root of Z is a + bi

Then $a + bi = Z^{\frac{1}{2}} \Rightarrow (a + bi)^2 = Z$

Example 5

Find the square roots of

(a) -5 + 12i

Solution

Approach 1

Let $\sqrt{(-5+12i)} = x + iy$

By squaring both sides

$-5 + 12i = x^2 - y^2 + i2xy$

Equating corresponding parts

$x^2 - y^2 = -5$ .... (i)

$2xy = 12$

$x = \frac{6}{y}$.................. (ii)

Substituting eqn.(ii) into eqn. (i)

$\left(\frac{6}{y}\right)^2 - y^2 = -5$

$36 - y^4 = -5y^2$

Let k = y2

$k^2 - 5k - 36 = 0$

$(k - 9)(k + 4) = 0$ i.e. k = 9 or k = -4

When k = -4, $y^2 = -4$ which is inadmissible since y must be real.

When k = 9; $y^2 = 9 \Rightarrow y = \pm 3$

⇨ $x = \frac{6}{\pm 3} = \pm 2$

$\therefore \sqrt{(-5+12i)} = 2 + 3i$ or $-2 - 3i \equiv \pm(2+3i)$

Approach 2

Let $\sqrt{(-5+12i)} = x + iy$

$x^2 - y^2 + i2xy = -5 + 12i$

Equating corresponding parts

$x^2 - y^2 = -5$ .... (i)

$2xy = 12$............... (ii)

Eqn. $(i)^2$ + eqn. $(ii)^2$

$$\begin{array}{rr} & x^4 - 2x^2y^2 + y^4 = 25 \\ + & 4x^2y^2 = 144 \\ \hline & x^4 + 2x^2y^2 + y^4 = 169 \end{array}$$

$(x^2 + y^2)^2 = 169$

$x^2 + y^2 = 13$ ....(iii)

Eqn.(i) + eqn. (iii)

$2x^2 = 8$; $x = \pm 2$

Substituting for x in eqn. (iii)

$4 + y^2 = 13$; $y = \pm 3$

$\therefore \sqrt{(-5+12i)} = 2 + 3i$ or $-2 - 3i \equiv \pm(2+3i)$

(b) 3 + 4i

Let $\sqrt{(3+4i)} = x + iy$

$x^2 - y^2 + i2xy = 3 + 4i$

Equating corresponding parts

$x^2 - y^2 = 3$ .... (i) and $2xy = 4$............... (ii)

Eqn. $(i)^2$ + eqn. $(ii)^2$

$$\begin{array}{rr} & x^4 - 2x^2y^2 + y^4 = 9 \\ + & 4x^2y^2 = 16 \\ \hline & x^4 + 2x^2y^2 + y^4 = 25 \end{array}$$

$(x^2 + y^2)^2 = 25$

$x^2 + y^2 = 5$ ……………(iii)

Eqn.(i) + eqn. (iii)

$2x^2 = 8$; $x = \pm 2$

Substituting for x in eqn. (iii)

$4 + y^2 = 5$; $y = \pm 1$

$\therefore \sqrt{(3+4i)} = 2 + i$ or $-2 - i \equiv \pm(2+i)$

Other equations

Example 6

Solve the simultaneous equations

$Z_1 + Z_2 = 8$

$4Z_1 - 3iZ_2 = 26 + 8i$

Solution

$Z_1 + Z_2 = 8$

$Z_1 = 8 - Z_2$ ……………………… (i)

$4Z_1 - 3iZ_2 = 26 + 8i$………… (ii)

Substituting eqn. (i) into eqn. (ii)

$4(8 - Z_2) + 3iZ_2 = 26 + 8i$

$32 - 4Z_2 - 3iZ_2 = 26 + 8i$

$(4 + 3i)Z_2 = 6 - 8i$

$z_2 = \frac{(6-8i)(4-3i)}{(4+3i)(4-3i)} = \frac{-50i}{25} = -2i$

Substituting for Z_2 into (i)

$Z_1 = 8 + 2i$

$\therefore Z_1 = 8 + 2i$ and $Z_2 = -2i$

Example 7

Solve the equation 6x2 – 2(1+ 2i)x – 1 = 0

Solution

Using $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$x = \frac{2(1+2i) \pm \sqrt{4(1+2i)^2 - 4(6)(-1)}}{2(6)}$$

$= \frac{1+2i\pm\sqrt{(3+4i)}}{6}$

But $\sqrt{(3+4i)} = \pm(2+i)$

⇨ $x = \frac{1+2i\pm(2+i)}{6}$

Either $x = \frac{1}{2} + \frac{1}{2}i$ or $x = -\frac{1}{6} + \frac{1}{6}i$

Example 8

Solve the equation $\left(\frac{Z-1}{Z+1}\right)^2 = i$, If Z is a complex number in form a + bi

Solution

$(Z - 1)^2 = i(Z + 1)^2$

$Z^2 - 2Z + 1 = iZ^2 + 2iZ + i$

$(i - 1)Z^2 + 2(i + 1)Z + i = 0$

⇨ $Z^2 - 2iZ + 1 = 0$

$Z = \frac{2i \pm\sqrt{-4-4}}{2} = \frac{2i\pm 2i\sqrt{2}}{2} = i \pm \sqrt{2}$

$\therefore Z = (1 \pm \sqrt{2})i$

Revision exercise 1

1. Simplify each of the following and express your answer in the form a + bi
   (a) (3 + 4i) + (2 + 3i) [5 + 7i]
   (b) (3 + 4i) – (2 – 3i) [1+ 7i]
   (c) $(2i)^2$ [-4]
   (d) (2 + 3i)(2 – 3i) [13]
   (e) (3 + 4i)(1 – 2i) [11 – 2i]
2. If Z is a complex number in form (a + bi) solve
   (a) $\left(\frac{Z-1}{Z+1}\right)^2 = i$ [$Z = (1 \pm \sqrt{2})i$]
   (b) $\frac{Z}{Z+1} = 1 + 2i$ $\left[-1 + \frac{1}{2}i\right]$

(c) Express each of the following complex numbers in the form a + bi
   (a) $Z_1 = (1 - i)(1 + 2i)$ [3 + i]
   (b) $Z_2 = \frac{2+6i}{3-i}$ [2i]
   (c) $Z_3 = \frac{-4i}{1-i}$ [2 – 2i]

(d) Find the square root of
   (a) 12i – 5 [2 +3i or -2 – 3i]
   (b) 5 + 2i [±(3 +2i)]
   (c) 15 + 8i [±(4 + i)]

(d) $7 - 24i$ [$\pm(4 - 3i)$]

(e) Given that Z = 3 + 4i, find the value of the expression $Z + \frac{25}{Z}$ [6]

(f) Given that the complex number Z* and its complex conjugate satisfy the equation $ZZ^* + 2iZ = 12 + 6i$, find the possible values of Z [3 + 3i; 3 – i]

(g) Solve the simultaneous equation

$Z_1 + Z_2 = 8$

$4Z_1 - 3iZ_2 = 26 + 8i$

[$Z_1 = 8 + 2i$ and $Z_2 = -2i$]

(h) Express each of the following in the form a + bi

(a) $\frac{2+3i}{1-i}\left[-\frac{1}{2}+\frac{5}{2}i\right]$

(b) $\frac{3-i}{1+2i}\left[\frac{1}{5}-\frac{7}{5}i\right]$

(c) $\frac{4}{1+i}$ [-1 + i]

(d) $\frac{3+4i}{3-4i}-\frac{3-4i}{4+4i}\left[\frac{48i}{25}\right]$

(i) Solve the following equations

(a) $2x^2 +32 = 0$ [±4i]

(b) $4x^2 + 9 = 0$ $\left[\pm\frac{3}{2}i\right]$

(c) $x^2 + 2x + 5 = 0$ [-1 ± 2i]

(d) $x^2 - 4x + 5 = 0$ [2 ± i]

(e) $2x^2 + x + 1 = 0$ $\left[\frac{-1\pm\sqrt{71}}{4}\right]$

(j) Form quadratic equations having roots

(a) 3i, -3i [$x^2 + 9 = 0$]

(b) 1 + 2i, 1- 2i [$x^2 - 2x + 5 = 0$]

(c) 2 + I, 2 – i[$x^2 - 4x + 5 = 0$]

(d) 3+4i, 3 – 4i [$x^2 - 6x + 25 = 0$]

(k) Solve the following equations if each has at least one real root

(a) $x^3 - 7x^2 + 19x - 13 = 0$ [1, 3 ± 2i]

(b) $2x^3 - 2x^2 - 3x - 2 = 0$ $\left[2, -\frac{1}{2}\pm\frac{1}{2}i\right]$

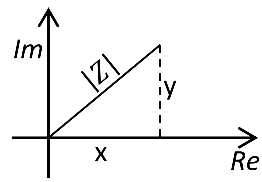
(c) $x^3 + 3x^2 + 5x + 3 = 0$ [-1, $\pm\sqrt{2}i$]

The Argand diagram

Complex numbers can be represented graphically just as coordinates are. However, instead of the x – axis and the y – axes, thereal (*Re*) and the imaginary (*Im*) exes are used respectively instead. This representation was first suggested by a mathematician named R. Argand, hence the Argand diagram. Each complex number is represented by a line of certain length in a particular direction. Thus each complex number is shown as a vector on Argand diagram.

The sum and the difference of the two complex numbers can be shown on an Argand diagram in the same way as we show vectors which are normally added or subtracted.

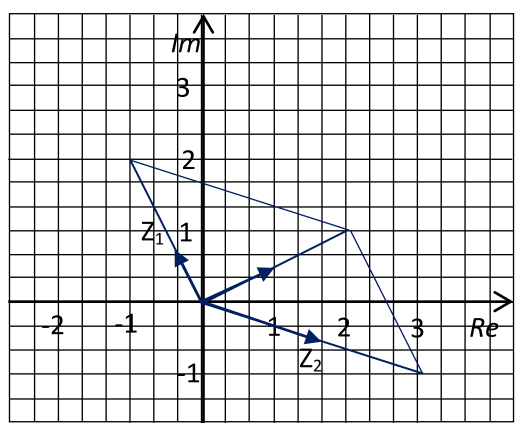
The complex number x = x + iy is represented as shown below



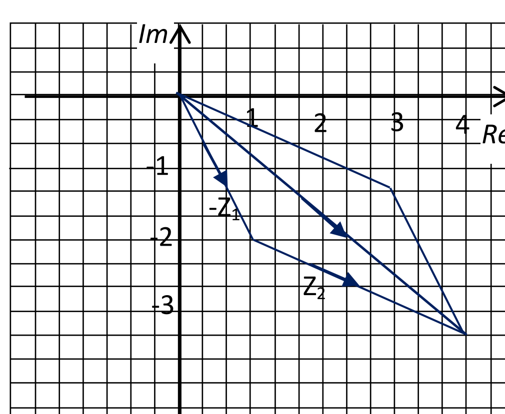
Example 8

Given $Z_1 = -1 + 2i$ and $Z_2 = 3 - i$ show $Z_1 + Z_2$ and $Z_2 - Z_1$ using Argand diagram

(a) $Z_1 + Z_2 = -1 + 2i + 3 - i = 2 + i$



(b) $Z_2 - Z_1 = (3 - i) - (-1 + 2i) = 4 - 3i$



The modulus and argument of a complex number

From the above diagram

$r^2 = |Z|^2 = x^2 + y^2$

i.e. r = $|Z| = \sqrt{x^2 + y^2}$ which is modulus Z = mod(Z) and tanθ = $\frac{x}{y}$, i.e. $\theta = \tan^{-1}\frac{y}{x}$, which is the argument, arg(Z) or amplitude, amp(Z).

Note: for principal values, $-\pi \le Arg(Z) \le \pi$

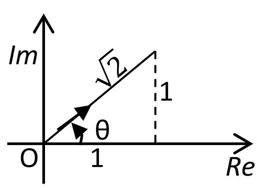
Example 9

Find the modulus and argument of the following

(a) $Z_1 = 1 + i$

Solution

$|Z_1| = \sqrt{(1^2 + 1^2)} = \sqrt{2}$

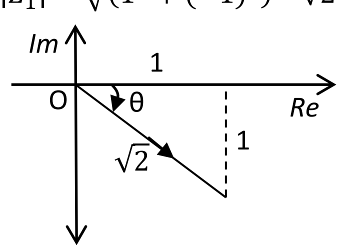


Arg (Z_1) = $\tan^{-1}\left(\frac{1}{1}\right) = 45^0 \; or \; \frac{\pi}{4}$

(b) $Z_2 = 1 - i$

Solution

$|Z_1| = \sqrt{(1^2 + (-1)^2)} = \sqrt{2}$

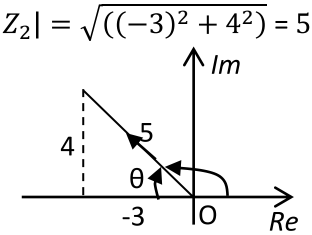


Arg(Z_2) = $\tan^{-1}(-1) = 45^0 \; or \; -\frac{\pi}{4}$

(c) $Z_3 = -3 + 4i$

Solution

$|Z_2| = \sqrt{((-3)^2 + 4^2)} = 5$

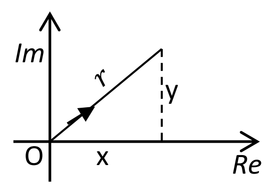


Arg(Z_3) = $\tan^{-1}\left(\frac{4}{3}\right)$ = $180^0 - 53.1^0 = 126.9^0$

Note: Due care must be taken in obtaining the argument of complex numbers as above example shows. Although the moduli for Z_1 and Z_2 are the same, the arguments differ because the fall in different quadrants. A sketch may be thus may be a necessary pre-requisite.

The polar (modulus – argument) form of a complex number.

Given Z = x + iy (the Cartesian form)



x = rcosθ and y = rsinθ

∴Z= x + iy = rcosθ + rsinθ

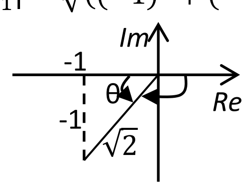
i.e. Z = rcosθ + rsinθ or simply Z = rcis. This form is referred to as polar form of Z.

Example 10

Express $Z_1 = -1 - i$ and $Z_2 = -\sqrt{3} + 3i$

Solution

$(a)|Z_1| = \sqrt{((-1)^2 + (-1)^2)} = \sqrt{2}$

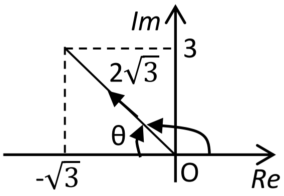


Arg(Z_1) = $\tan^{-1}\left(\frac{-1}{-1}\right) = 225^0 \;\; = -135^0 or -\frac{\pi}{4}$

$Z_1 = \sqrt{2}(\cos(-135^0) + isin135^0)$

$= \sqrt{2}\left(cos\frac{3\pi}{4} - isin\frac{3\pi}{4}\right)$

$(b)|Z_2| = \sqrt{((-\sqrt{3})^2 + 3^2)} = 2\sqrt{3}$



Arg(Z_2) = $\tan^{-1}\left(\frac{3}{-\sqrt{3}}\right)$= 120^0 = $\frac{2\pi}{3}$

Z_2 = $2\sqrt{3}$(cos120^0 + isin120^0)

Z_2 = $2\sqrt{3}$(cos$\frac{2\pi}{3}$ + isin$\frac{2\pi}{3}$)

Multiplication and division of polar form

The two operations offer valuable results in complex plane

Let $|Z_1| = r_1$ and arg(Z_1) = θ_1 i.e.

$|Z_1| = r_1(cos\theta_1 + isin\theta_1)$

Let $|Z_2| = r_2$ and arg(Z_2) = θ_2 i.e.

$|Z_2| = r_2(cos\theta_2 + isin\theta_2)$

(a) Multiplication of polar form

Z_1Z_2 =$r_1r_2(cos\theta_1 + isin\theta_1)(cos\theta_2 + isin\theta_2)$

=r_1r_2(cos$\theta_1$cosθ_2 + i cos$\theta_1$sinθ_2 + isin$\theta_1$cosθ_2 – sin$\theta_1$cosθ_2)

= r_1r_2{(cos$\theta_1$cosθ_2 - sin$\theta_1$cosθ_2)+ i(cos$\theta_1$sinθ_2 + sin$\theta_1$cosθ_2)}

= r_1r_2{cos(θ_1 + θ_2) + isin(θ_1 + θ_2)}

Deductions:

(i) $|Z_1Z_2| = r_1r_2 = |Z_1||Z_2|$
(ii) $arg(Z_1Z_2) = \theta_1 + \theta_1$
$= \arg(Z_1) + \arg(Z_2)$
(iii) $|Z^2| = |Z||Z| = |Z|^2$
(iv) $arg(Z^2) = \arg(Z) + \arg(Z) = 2\arg(Z)$

In general

(i) $|Z_1Z_2Z_3 \ldots Z_n| = |Z_1||Z_2||Z_2| \ldots .. |Z_n|$
(ii) $|Z^n| = |Z||Z||Z| \ldots |Z| = |Z|^n$
(iii) $arg(Z_1Z_2Z_3 \ldots Z_n)$

$= \arg(Z_1) + \arg(Z_2) + \arg(Z_3) \ldots \arg(Z_n)$

(iv) $arg(Z^n) = n\arg(Z)$

Example 11

Given that Z_1 = 3 + 4i and Z_2 = 1 – i, find

(i) $|Z_1Z_2|$
(ii) $Arg(Z_1Z_2)$

Solution

$|Z_1| = \sqrt{3^2 + 4^2} = 5$

arg(Z_1) =$\tan^{-1}\left(\frac{4}{3}\right) = 53.1^0$

$|Z_2| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

arg(Z_2) =$\tan^{-1}\left(-\frac{1}{1}\right) = -45^0$

(i) $|Z_1Z_2| = |Z_1||Z_2| = 5\sqrt{2}$
(ii) $arg(Z_1Z_2) = \theta_1 + \theta_1$
$= 53.1^0 + (-45^0) = 8.1^0$

Example 12

Given that Z_1= 3(cos$\frac{\pi}{3}$ + isin$\frac{\pi}{3}$)

And Z_2 = 5(cos$\frac{\pi}{4}$ + isin$\frac{\pi}{4}$) find

(i) $|Z_1Z_2|$
(ii) $Arg(Z_1Z_2)$

Solution

For

Z_1= 3(cos$\frac{\pi}{3}$ + isin$\frac{\pi}{3}$); $|Z_1| = 3\ and\ \arg(Z_1) = \frac{\pi}{3}$

For

Z_2 = 5(cos$\frac{\pi}{4}$ + isin$\frac{\pi}{4}$); $|Z_2| = 5\ and\ \arg(Z_2) = \frac{\pi}{4}$

(i) $|Z_1Z_2| = |Z_1||Z_2| = 3x5 = 15$
(ii) $arg(Z_1Z_2) = \theta_1 + \theta_1$ =$\frac{\pi}{3} + \frac{\pi}{4}$ = $\frac{7\pi}{12}$

(b) Division of polar form

$\frac{Z_1}{Z_2}$
$= \frac{r_1}{r_2}\left\{\frac{cos\theta_1cos\theta_2 - icos\theta_1sin\theta_2 + isincos\theta_2 - sin\theta_1sin\theta\right.}{(cos^2\theta_2 + sin^2\theta_2)}$

=$\frac{r_1}{r_2}\{cos\theta_1cos\theta_2 - icos\theta_1sin\theta_2 + isincos\theta_2 - sin\theta_1sin\theta_2\}$

=$\frac{r_1}{r_2}$$\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$

Deduction

(a) $\frac{Z_1}{Z_2} = \frac{r_1}{r_2} = \frac{|Z_1|}{|Z_2|}$

(b) $arg\left(\frac{Z_1}{Z_2}\right) = \theta_1 - \theta_2$

$= \arg(Z_1) - \arg(Z_2)$

Example 13

Given that $Z_1 = 3 + 4i$ and $Z_2 = 1 - i$, find

(i) $\left|\frac{Z_1}{Z_2}\right|$

(ii) $Arg\left(\frac{Z_1}{Z_2}\right)$

Solution

$|Z_1| = \sqrt{3^2 + 4^2} = 5$

$\arg(Z_1) = \tan^{-1}\left(\frac{4}{3}\right) = 53.1^0$

$|Z_2| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$\arg(Z_2) = \tan^{-1}\left(-\frac{1}{1}\right) = -45^0$

(i) $\left|\frac{Z_1}{Z_2}\right| = \frac{|Z_1|}{|Z_2|} = \frac{5}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$

(ii) $arg\left(\frac{Z_1}{Z_2}\right) = \theta_1 - \theta_1$

$= 53.1^0 - (-45^0) = 98.1^0$

Example 14

Three complex numbers are given as

$Z_1 = -1 - i$; $Z_2 = 3 - i\sqrt{3}$; $Z_3 = -1 + i\sqrt{3}$

Find

(a) $\left|\frac{Z_1}{Z_2}\right|$

(b) $arg\left(\frac{Z_2}{z_1 Z_3}\right)$

(c) $\left|\frac{{Z_1}^2}{{Z_2}^3 Z_3}\right|$

(d) $arg\left(\frac{{Z_3}^3 {Z_2}^2}{Z_1}\right)$

Solution

$|Z_1| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

$\arg(Z_1) = \tan^{-1}\left(\frac{-1}{-1}\right) = -135^0 = -\frac{3\pi}{4}$

$|Z_2| = \sqrt{3^2 + (\sqrt{3})^2} = 2\sqrt{3}$

$\arg(Z_2) = \tan^{-1}\left(-\frac{\sqrt{3}}{3}\right) = -30^0 = -\frac{\pi}{6}$

$|Z_3| = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$

$\arg(Z_1) = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right) = 120^0 = -\frac{2\pi}{3}$

(a) $\left|\frac{Z_1}{Z_2}\right| = \frac{|Z_1|}{|Z_2|} = \frac{\sqrt{2}}{2}$

(b) $arg\left(\frac{Z_2}{z_1 Z_3}\right) = argZ_2 - arg(z_1 Z_3)$

$= argZ_2 - argZ_1 - argZ_3$

$= -30^0 + 135^0 - 120^0 = -15 = -\frac{\pi}{12}$

(c) $\left|\frac{{Z_1}^2}{{Z_2}^3 Z_3}\right| = \frac{|Z_1|^2}{|Z_2|^3|Z_3|} = \frac{2}{24\sqrt{3}(2)} = \frac{\sqrt{3}}{72}$

(d) $arg\left(\frac{{Z_3}^3 {Z_2}^2}{Z_1}\right)$

$= arg\left({Z_3}^2\right) + \arg\left({Z_2}^2\right) - \arg(Z_1)$

$= 3\arg(Z_3) + 2\arg(Z_2) - \arg(Z_1)$

$= 3(120^0) - 2(30^0) + 135^0 = 75^0 = \frac{5\pi}{12}$

Note: the reader should correlate the computations of arguments Z_1Z_2 and $\frac{Z_1}{Z_2}$ with the laws of logarithms

Example 15

(a) Given that $Z_1 = 3 + 4i$, $Z_2 = 12 + 5i$ find

(i) $|Z_1 Z_2|$

(ii) $\arg(Z_1 Z_2)$

(iii) $\left|\frac{Z_1}{Z_2}\right|$

(iv) $\arg(Z_1 Z_2)$

Solution

$|Z_1| = \sqrt{(3)^2 + (4)^2} = 5$

$Arg(Z_1) = \tan^{-1}\left(\frac{4}{3}\right) = 53.1^0$

$|Z_3| = \sqrt{(12)^2 + (3)^2} = 13$

$Arg(Z_1) = \tan^{-1}\left(\frac{5}{12}\right) = 22.6^0$

(i) $|Z_1 Z_2| = |Z_1||Z_2| = 5\ x\ 13 = .65$

(ii) $\arg(Z_1 Z_2) = \arg(Z_1) + \arg(Z_2)$

$= 53.1^0 + 22.6^0 = 75.7^0$

(iii) $\left|\frac{Z_1}{Z_2}\right| = \frac{|Z_1|}{|Z_2|} = \frac{5}{13}$

(iv) $\arg\left(\frac{Z_1}{Z_2}\right) = \arg(Z_1) - \arg(Z_2)$

$= 53.1^0 - 22.6^0 = 30.5^0$

(b) Express Z = -1 -i$\sqrt{3}$ in modulus-urgument form. Hence find $\frac{1}{Z}$ in form a + bi where a and b are real numbers

Solution

$$|Z| = \sqrt{(-1)^2 + (-\sqrt{3})^2} = 2$$

Arg(Z) =$\tan^{-1}\left(\frac{-\sqrt{3}}{-1}\right) = -120^0 = -\frac{2}{3}\pi$

$\therefore$Z =$2\left(\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right)$

Hence, $\frac{1}{Z} = Z^{-1} = 2^{-1}\left(\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right)^{-1}$

$$.= \frac{1}{2}\left(\cos\left(-\frac{2\pi}{3}\right) - i\sin\left(-\frac{2\pi}{3}\right)\right)$$

$$= \frac{1}{2}\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$$

$$= -\frac{1}{4} + i\frac{\sqrt{3}}{4}$$

Solving equations of higher order

When one or more roots of a complex number is(are) given, their conjugates are also the roots of the same equation

To prove that a certain root is a factor of a complex function, we either use long division to show that the quotient of division is zero or we substitute the factor given into the equation to note the remainder or we get the product of the already existing roots and then carry on long division to observe the remainder or we use the synthetic approach

Example 16

Given that (1+ 3i) and (2 – i) are roots of the equation

$aZ^4 + bZ^3 + cZ^2 + dZ + e = 0$, find

(i) The other two roots
(ii) The sum of the four roots
(iii) The product of the four roots

Solution

(i) Since 1 + 3i and 2 – I are roots of the equation, then their conjugates are also roots of the same equations. Hence the other two roots are 1 – 3i and 2 + i.

(ii) Sum of roots = (1 + 3i) + (1 – 3i) + (2 – i) + (2 + i) = 6

(iii) Products of roots
= (1 + 3i)(1 – 3i)(2 – i (2 + i) = 6
= $(1 - 9i^2)(4 - i^2)$
=(1 + 9)(4 + 1)= 50

Example 17

Show that Z_1 = 2 + 3i is a root of the equation $Z^4 - 5Z^3 + 18Z^2 - 17Z + 13 = 0$

Hence find the other remaining roots

Solution

Approach 1

Given 2 + 3i is a root, its conjugate 2 - 3i is also a root of the equation

The equation of these two roots

Sum of roots 2 + 3i + 2 – 3i = 4

Product of roots = (2 + 3i)(2 – 3i) = 13

The equation is $Z^2 - 4Z + 13 = 0$

Using long division

Let $f(Z) = Z^4 - 5Z^3 + 18Z^2 - 17Z + 13$

$$\begin{array}{r|l}
 & Z^2 - Z + 1 \\
\hline
Z^2 - 4Z + 13 & Z^4 - 5Z^3 + 18Z^2 - 17Z + 13 \\
 & \underline{Z^4 - 4Z^3 + 13Z^2} \\
 & \quad -Z^3 + 5Z^2 - 17Z + 13 \\
 & \quad \underline{-Z^3 + 4Z^2 - 13Z} \\
 & \qquad Z^2 - 4Z + 13 \\
 & \qquad \underline{Z^2 - 4Z + 13} \\
 & \qquad 0 + 0 + 0
\end{array}$$

Since the remainder is zero, $Z^2 - Z + 1$is also a factor of f(Z)

⇨ $f(Z) = (Z^2 - 4Z + 13)(Z^2 - Z + 1)$

Either $Z^2 - 4Z + 13 = 0$ or $Z^2 - Z + 1 = 0$

Factors of $Z^2 - Z + 1$

$$Z = \frac{1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = \frac{1 \pm i\sqrt{3}}{2}$$

The roots are $2 \pm 3i$ and $\frac{1 \pm i\sqrt{3}}{2}$

Approach 2

Taking Z = 2 + 3i and substituting in the given equation

LHS = $(2 + 3i)^4 - 5(2 + 3i)^3 + 18(2 + 3i)^2 - 17(2+3i) + 13 = 0$ (RHS)

Z1 = 2 + 3i is a root

This means $Z_2 = 2 - 3i$ is also a root of the equation as it is a conjugate of the given root.

Given $f(Z) = Z^4 - 5Z^3 + 18Z^2 - 17Z + 13 = 0$

Sum of roots = 5 and

Product of roots = 13

Let the other roots be α and β

Sum of the roots = 2 + 3i + 2 – 3i + α + β = 5

4 + α + β = 5

α + β = 1

α = (1 – β) .... (i)

Product of roots = (2 + 3i)(2 – 3i)αβ = 13

13αβ = 13

αβ = 1 .... (ii)

Substituting (i) into (ii)

(1 – β)β = 1 i.e. β2 – β + 1 = 0

$$\beta = \frac{1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = \frac{1 \pm i\sqrt{3}}{2}$$

∴ The roots are $2 \pm 3i$ and $\frac{1 \pm i\sqrt{3}}{2}$

Approach 3: using synthetic method

2 + 3i	1	-5	18	-17	13
	0	2+3i	-15-3i	15+3i	-13
	1	-3 + 3i	3 -3i	-2+3i	0

Since the last value in the table is zero, therefore 2 + 3i is a root of the equation. Other roots can be obtained as shown above.

Illustration of the synthetic method

Procedure

(a) Write down the root and coefficients of the expression in the first row (as shown above)
(b) Write zero immediately below the first coefficient (this is the only entry in the second row that is simply written, the other are to be obtained by multiplication)
(c) At the first coefficient to zero to get the first entry in the third row
(d) Obtain the second entry in the second row by multiplying 1 by 2 + 3i, then add to -5 to get the second entry in the third row i.e. -3 + 3i
(e) Repeat (d) by multiplying -3 + 3i to get the third entry in the second row, this entry is added to 18 to get to get the third row i.e. 3 – 3i. continue with this trend up to the last entry. If the last entry in the table is zero, then the value being tested is a root.

Example 18

Show that Z1 = 1 + I is a root of the equation $Z^4 - 6Z^3 + 25Z^2 - 34Z + 26 = 0$

Solution

Using synthetic division

1 + i	1	-6	25	-34	26
	0	2+i	-6-4i	21+13i	-26
	1	-5 + i	17 - 4i	-3+ 13i	0

Since the last entry in the table is zero, 1 + I is a root.

The conjugate 1 – i is also a root

Let the other roots be α and β

Sum of roots = 1 + i + 1 – i + α + β = 6

2 + α + β = 6

α + β = 4

α = 4 – β ……………… (i)

Product of roots (1 + i)(1 – i)αβ = 26

2αβ = 26

Aβ = 13 ……………… (ii)

Substituting eqn.(i) into eqn. (ii)

(4 – β)β = 13

β^2 - 4β + 13 = 0

$\beta = \frac{4\pm\sqrt{16-4(1)(13)}}{2(1)} = \frac{4\pm\sqrt{-36}}{2} = \frac{4\pm 6i}{2}$

= 2 ± 3i

Hence the other roots are 2 + 3i and 2 – 3i

Revision exercise 2

1. Given Z_1 = 7 – 4i and Z_2 = -1 + 3i, express in the modulus-argument form the following
   (a) Z_1Z_2
   {r = 25.45 [cos(0.437π) + isin(0.437π)]}
   (b) $\left|\frac{Z_1}{Z_2}\right|$
   $\{p = 2.55[cos(1.233\pi) + isin(1.233\pi)]$
2. Find the modulus and the principal arguments of
   (a) $\frac{1-i}{1+i}\left[1, \frac{-\pi}{3}\right]$
   (b) $\frac{-1-7i}{4+3i}\left[\sqrt{2}, \frac{-3\pi}{4}\right]$
   (c) $\frac{1+i}{2-i}\left[\frac{\sqrt{10}}{5}, 125\right]$
   (d) $\frac{(3+i)^2}{1-i}\left[2\sqrt{5}, 143\right]$
3. If z_1 and z_2 are complex numbers, solve the simultaneous equation, giving your answer in the form a + bi.
   $4z_1 + 3z_2$= 23
   $z_1 + iz_2$ = 6 [2 +3i, 5 – 4i]
4. Given that 2 + I is a root of the equation z^3 – 11z + 20 = 0. Find the remaining roots [2 – i, -4]
5. Show that 2+ i is a root of the equation $2z^3 – 9z^2$ + 14z – 5 = 0. Hence find other roots [2- i, ½]
6. Find equation whose roots are -1 ± i, where i = $\sqrt{-1}$ [z^2 +2z + 2 = 0]
7. Show that z = 1 is a root of the equation $z^3 -5z^2$ + 9z – 5 = 0 [2 – i, 2 + i]
8. The complex number satisfies $\frac{z}{z+2} = 2 - i$ find the real and imaginary parts of z, and the modulus and argument of z.
   [*Re*(z) = -3,*Im*(z) = -1, $\sqrt{10}$, -2.82rads]
9. One root of the equation z2 + az + b = 0 where a and b are real constants is 2 + 3i. Find the values of a and b [a = -4, b = 13]
10. If z_1 = $2\left(cos\frac{2\pi}{3} + isin\frac{2\pi}{3}\right)$ and
    z_2 =$6\left(cos\left(\frac{-3\pi}{4}\right) + isin\left(\frac{-3\pi}{4}\right)\right)$ find
    (a) $\left|\frac{z_1}{z_2}\right| \left[\frac{1}{3}\right]$
    (b) $arg\left(\frac{z_1}{z_2}\right)\left[\frac{17\pi}{12}\right]$
    (c) $\left|\frac{z_2}{z_1}\right|$[3]
    (d) $arg\left(\frac{z_2}{z_1}\right)\left[\frac{-17\pi}{12}\right]$
11. If z1 and z2 are two complex numbers such that $|z_1 - z_2| = |z_1 + z_2|$. Show that the difference of their arguments is $\frac{\pi}{2}$ or $\frac{3\pi}{2}$.
12. Find the modulus and argument of $\frac{(2-i)^2(3i-1}{i+3}$ [5, 0.6435 rad]
13. If $z_1 = \frac{1+7i}{1-i}$ and $z_2 = \frac{17-7i}{2+2i}$ find the moduli of
    (a) z_1 [5]
    (b) z_2 [6.5]
    (c) $z_1 + z_2$ [2.061]
    (d) z_1z_2 [32.5]

Demoivre's theorem

This states that for all rational integers

r(cosθ + isinθ)^n = r(cos*n*θ + isin*n*θ)

The theorem can be proved by induction approach or otherwise

By induction

Given that (cosθ+ isinθ)^n = cos*n*θ + isin*n*θ

Suppose n = 1

cosθ+ isinθ = cos*1*θ + isin*1*θ

⇨ cosθ+ isinθ = cosθ+ isinθ i.e. the proof holds for n = 1

Suppose n = k

$(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

For n = k + 1

$(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)^1$

$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$

$= \cos k\theta\cos\theta + i\cos k\theta\sin\theta + i\sin k\theta\cos\theta + i^2\sin k\theta\sin\theta$

$= \cos k\theta\cos\theta - \sin k\theta\sin\theta + i(\cos k\theta\sin\theta + \sin k\theta\cos\theta)$

$= \cos(k+1)\theta + i\sin(k+1)\theta$

which is true when n = k + 1

Therefore the proof holds for all values of n

Using otherwise

Given z = cosθ + isinθ

$Z^2 = (\cos\theta + i\sin\theta)^2$

$= \cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta$

$= \cos^2\theta - \sin^2\theta + i2\sin\theta\cos\theta$

$= \cos2\theta + i\sin2\theta$

$Z^3 = (\cos2\theta + i\sin2\theta)^2(\cos\theta + i\sin\theta)$

$= \cos2\theta\cos\theta - \sin2\theta\sin\theta + i(\sin2\theta\cos\theta + \cos2\theta\sin\theta)$

$= \cos3\theta + i\sin3\theta$

⇨ $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ hold for all positive values of n

Also, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ holds for fractional and negative indices

(a) $z^{-1} = \frac{1}{Z} = \frac{1}{\cos\theta + i\sin\theta}$

$= \left(\frac{1}{\cos\theta + i\sin\theta}\right)\left(\frac{(\cos\theta - i\sin\theta)}{(\cos\theta - i\sin\theta)}\right)$

$= \frac{(\cos\theta - i\sin\theta)}{(\cos^2\theta - i^1\sin^2\theta)} = \cos\theta - i\sin\theta$

⇨ $z^{-1} = \cos(-1\theta) + i\sin(-1\theta)$

(b) suppose n = -m

$z^{-m} = \frac{1}{z^m} = \frac{1}{(\cos\theta + i\sin\theta)^m}$

$= \left(\frac{1}{\cos m\theta + i\sin m\theta}\right)\left(\frac{(\cos m\theta - i\sin m\theta)}{(\cos m\theta - i\sin m\theta)}\right)$

⇨ $(\cos\theta + i\sin\theta)^{-m}$

$= \cos(-m\theta) + i\sin(-m\theta)$

Hence the theorem holds for all values of n.

Applications of Demoivre's theorem

It is mainly employed in proving trigonometric functions and finding roots of complex function.

Proving identities

Example 19

By using Demoivre's theorem, show that

(a) $\tan5x = \frac{5\tan x - 10\tan^3x + \tan^5x}{1 - 10\tan^2x + \tan4x}$

Solution

cos5x + isin5x = (cosx + sinx)5

$= \cos^5x + 5i\sin x\cos^4x - 10\sin^2x\cos^3x - 10i\sin^3x\cos^2x + \sin^4x\cos x + i\sin^5x$

Equating real parts

$\text{Cos}5x = \cos^5x - 10\cos^3x\sin^2x + \cos x\sin^4x$

Equating imaginary parts

$\text{Sin}5x = 5\cos^4x\sin x - 10\cos^2x\sin^3x + \sin^5x$

$\tan5x = \frac{\sin5x}{\cos5x} = \frac{5\cos^4x\sin x - 10\cos^2x\sin^2x + \sin^5x}{\cos^5x - 10\cos^3x\sin^2x + \cos x\sin^4x}$

Dividing terms of the R.H.S by $\cos^5x$

$\tan5x = \frac{5\tan x - 10\tan^3x + \tan^5x}{1 - 10\tan^2x + \tan4x}$

(b) $\frac{\cos3x + i\sin3x}{\cos5x + i\sin5x} = \cos8x + i\sin8x$

Solution

$\frac{\cos3x + i\sin3x}{\cos5x + i\sin5x} = \frac{(\cos x + i\sin x)^3}{(\cos x + i\sin x)^5}$

$= (\cos x + i\sin x)^{3-(-5)}$

$= (\cos x + i\sin x)^8$

$= \cos8x + i\sin8x$

(c) $\frac{(\cos x + i\sin x)(\cos2x + i\sin2x)}{\cos\frac{x}{2} + i\sin\frac{x}{2}} = \cos\frac{5x}{2} + i\sin\frac{5x}{2}$

Solution

$\frac{(\cos x + i\sin x)(\cos2x + i\sin2x)}{\cos\frac{x}{2} + i\sin\frac{x}{2}}$

$= \frac{(\cos x + i\sin x)^1(\cos x + i\sin x)^2}{(\cos x + i\sin x)^{\frac{1}{2}}}$

$=(cosx + isinx)^{1+2-\frac{1}{2}}$

$=(cosx + isinx)^{\frac{5}{2}} = cos\frac{5x}{2} + isin\frac{5x}{2}$

(d) 16sin⁵x = sin5x -5sin3x + 10sinx

Given Z = cosx + isinx

$\bar{Z}$ = cosx – isinx

(Z – $\bar{Z}$) = 2isinx

This means that $(Z - \bar{Z})^n$ = 2isin$n$x

$(Z – Z^*)^5$ = 2isin5x = 32isin^{5}x

$(Z - \bar{Z})^5$

= Z^5 -5$Z^4\bar{Z}$ + 10$Z^3\bar{Z}^2$ + 10$Z^2(-\bar{Z})^3$+ 5Z$\bar{Z}^4$ - $\bar{Z}^5$

= Z^5 -5Z^3 + 10Z - 10($\bar{Z}$)+ 5$\bar{Z}^3$ - $\bar{Z}^5$

$=[Z^5 - \bar{Z}^5] - 5[Z^3 - \bar{Z}^3] + 10[Z - \bar{Z}]$

= 2isin5x – 10isin3x + 20isinx

⇨ 32isin⁵x = 2isin5x – 10isin3x + 20isinx

∴16sin⁵x = sin5x -5sin3x + 10sinx

Finding roots

Suppose that z = r(cosθ + isinθ)

Then $z^n = r^n$(cos$n$θ + isin$n$θ)

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(cos\frac{1}{n}\theta + isin\frac{1}{n}\theta\right)$$

The general expression for finding the roots of a complex function is given by

$z = r[cos(\theta + 2\pi k) + isin(\theta + 2\pi k)]$ where k = 0, 1, 2, …k and θ is usually in radians.

Now for nth root, we have

$z^n = r^n[cosn(\theta + 2\pi k) + isin\, n(\theta + 2\pi k)]$

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(cos\frac{1}{n}(\theta + 2\pi k) + isin\frac{1}{n}(\theta + 2\pi k)\right)$$

Example 20

Use Demoivre's theorem to find the square root of

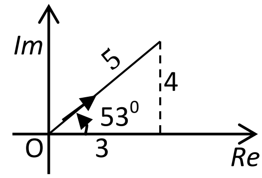
(a) 3 + 4i

Solution

$|z| = \sqrt{(9 + 16} = 5$

$\theta = \tan^{-1}\frac{4}{3} = 53.1^0 = 0.295\pi$

The principal angle lies in the first quadrant



z = $5[cos(0.295\pi + 2\pi k) + isin(0.295\pi + 2\pi k)]$

$z^{\frac{1}{2}}$

$= 5^{\frac{1}{2}}\left[cos\left(\frac{0.295\pi+2\pi k}{2}\right) + isin\left(\frac{0.295\pi+2\pi k}{2}\right)\right]$

Substituting for k = 0

$z_1^{\frac{1}{2}} = 5^{\frac{1}{2}}\left[cos\left(\frac{0.295\pi}{2}\right) + isin\left(\frac{0.295\pi}{2}\right)\right]$= -2 - i

$\therefore \sqrt{3 + 4i} =$ 2±i

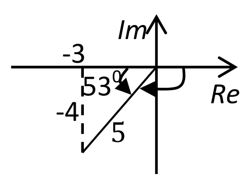
(b) -3 – 4i

Solution

$|z| = \sqrt{(9 + 16} = 5$

$\theta = \tan^{-1}\frac{-4}{-3} = -127^0 = -0.7\pi$

The principal angle lies in the third quadrant



z = $5[cos(0.295\pi + 2\pi k) + isin(0.295\pi + 2\pi k)]$

$z^{\frac{1}{2}} = 5^{\frac{1}{2}}\left[cos\left(\frac{0.295\pi+2\pi k}{2}\right) + isin\left(\frac{0.295\pi+2\pi k}{2}\right)\right]$

Substituting for k = 0

$z_1^{\frac{1}{2}} = 5^{\frac{1}{2}}\left[cos\left(\frac{0.295\pi}{2}\right) + isin\left(\frac{0.295\pi}{2}\right)\right]$= -2 - i

Substituting for k = 1

$z_2 = 5^{\frac{1}{2}}\left[cos\left(\frac{0.295\pi+2\pi}{2}\right) + isin\left(\frac{0.295\pi+2\pi}{2}\right)\right]$

= ±(1-2i)

$\therefore \sqrt{-3 - 4i} =$ ±(1 – 2i)

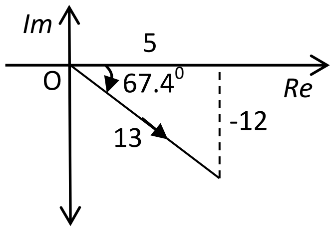
(c) 5 – 12i

Solution

$|z| = \sqrt{(25 + 144} = 13$

$\theta = \tan^{-1}\frac{-12}{5} = -67.4^0 = -0.374\pi$

The principal angle lies in the third quadrant



z = $13[cos(-0.374\pi + 2\pi k) + isin(-0.374\pi + 2\pi k)]$

$z^{\frac{1}{2}} = 13^{\frac{1}{2}}\left[cos\left(\frac{-0.374\pi+2\pi k}{2}\right) + isin\left(\frac{-0.374\pi+2\pi k}{2}\right)\right]$

Substituting for k = 0

$z_1 = 13^{\frac{1}{2}}\left[cos\left(\frac{-0.374\pi}{2}\right) + isin\left(\frac{-0.374\pi}{2}\right)\right]$

= 3 - 2i

Substituting for k = 1

$z_2 =$
$13^{\frac{1}{2}}\left[cos\left(\frac{-0.374\pi+2\pi}{2}\right) + isin\left(\frac{-0.374\pi+2\pi}{2}\right)\right]$

= -3 + 2i

$\therefore \sqrt{5 - 12i} = \pm(3 - 2i)$

Example 21

(a) Use roots 1, α and α^2 of unity. Hence show that $1 + \alpha + \alpha^2 = 0$

Solution

Let $\sqrt[1]{1} = z$

z = $1^{\frac{1}{3}} = (cos0^0 + isin0^0)$ (In mod-Arg form

=$[cos(0 + 2\pi k) + isin\ (0 + 2\pi k)]^{\frac{1}{3}}$, k = 0,1,2

= $cos(\frac{2\pi k}{3}) + isin\ (\frac{2\pi k}{3})$

When k = 0

$z_1 = cos0^0 + isin\ 0^0 = 1$

When k = 1

$z_2 = cos(\frac{2\pi}{3}) + isin\ (\frac{2\pi}{3}) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

When k = 2

$z_3 = cos(\frac{4\pi}{3}) + isin\ (\frac{4\pi}{3}) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

∴the cube roots are 1, $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

Using otherwise

Taking

$z = 1^{\frac{1}{3}}$, then $z^3 - 1 = 0$

Let $f(z) = z^3 - 1$

$f(1) = 1^3 - 1 = 0$

∴ z – 1 is a factor.

Using long division

$$\begin{array}{r} z^2 + z + 1 \\ z-1\overline{)\,z^3 - 1\quad} \\ \underline{z^3 - z^2} \\ z^2 - 1 \\ \underline{z^2 - 1} \end{array}$$

$z^3 - 1 = (z - 1)(z^2 + z + 1)$

Either z – 1 = 0, z = 1

Or $z^2 + z + 1 = 0$

$z = \frac{-1\pm\sqrt{1^2-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

∴the cube roots are 1, $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

Hence

Letting α = $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$, the

$\alpha^2 = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

Letting α = $-\frac{1}{2} - i\frac{\sqrt{3}}{2}$, the

$\alpha^2 = \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

Hence the roots can be written as 1, $\alpha + \alpha^2$

Adding these roots

$1 - \frac{1}{2} - i\frac{\sqrt{3}}{2} - \frac{1}{2} + i\frac{\sqrt{3}}{2} = 0$

(b) Evaluate $(-16)^{\frac{1}{4}}$

Solution

Let z = $(-16)^{\frac{1}{4}} = 16^{\frac{1}{4}}(-1)^{\frac{1}{4}}$ = $2(-1)^{\frac{1}{4}}$

= $[cos180^0 + isin180^0]^{\frac{1}{4}}$

=$2\left[cos\left(\frac{\pi+2\pi k}{4}\right) + isin\left(\frac{\pi+2\pi k}{4}\right)\right]$, k = 0, 1, 2, 3

When k = 0

z_1 = $[cos45^0 + isin45^0]$=$\sqrt{2} + i\sqrt{2}$

When k = 1

Z_2 = $[cos135^0 + isin135^0]$=-$\sqrt{2} + i\sqrt{2}$

When k = 2

Z_3 = $[cos225^0 + isin225^0]$=$\sqrt{2} - i\sqrt{2}$

When k = 3

Z_3 = $[cos315^0 + isin315^0]$=-$\sqrt{2} - i\sqrt{2}$

$\therefore\ (-16)^{\frac{1}{4}} = \pm(\sqrt{2} \pm i\sqrt{2})$

Example 23

Evaluate $(1 + \sqrt{3}i)^{\frac{2}{3}}$

Solution

$(1 + \sqrt{3}i)^{\frac{2}{3}} = \left[(1 + \sqrt{3}i)^2\right]^{\frac{1}{3}}$

$= [(1 + \sqrt{3}i)(1 + \sqrt{3}i)]^{\frac{1}{3}}$

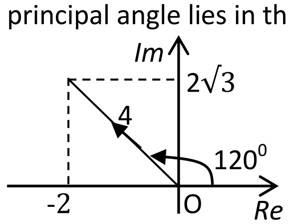
$= [-2 + 2\sqrt{3}i]^{\frac{1}{3}}$

Let z = -2 + 2$\sqrt{3}$i

$|z| = \sqrt{(-2)^2 + (2\sqrt{3})^2} = 4$

Arg(z) = $\tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = -60^0 = 120^0 = \frac{2\pi}{3}$

The principal angle lies in the second quadrant



$z = 4\left[cos\left(\frac{2\pi}{3} + 2\pi k\right) + isin\left(\frac{2\pi}{3} + 2\pi k\right)\right]$

$z^{\frac{1}{3}} = 4^{\frac{1}{3}}\left[cos\left(\frac{2\pi+6\pi k}{9}\right) + isin\left(\frac{2\pi+6\pi k}{9}\right)\right]$

When k = 0

$z_1 = 4^{\frac{1}{3}}\left[cos\left(\frac{2\pi}{9}\right) + isin\left(\frac{2\pi}{9}\right)\right]$

=1.2160 +1.0204i

When k = 1

$z_2 = 4^{\frac{1}{3}}\left[cos\left(\frac{8\pi}{9}\right) + isin\left(\frac{8\pi}{9}\right)\right]$

= -1.4917 + 0.5429i

When k = 2

$z_3 = 4^{\frac{1}{3}}\left[cos\left(\frac{14\pi}{9}\right) + isin\left(\frac{14\pi}{9}\right)\right]$

= 0.2756 – 1.5633i

$\therefore\ (1 + \sqrt{3}i)^{\frac{2}{3}}$ =1.2160 +1.0204i, -1.4917 + 0.5429i, 0.2756 – 1.5633i

Example 24

Find the root of z^4 + 4 = 0 using Demoivre's theorem

z^4 = -4 = 4(-1 + 0i)

arg(z) =$\tan^{-1}\left(\frac{0}{-1}\right) = 180^0 = \pi$

⇨ z^4 = $4[cos180^0 + sin180^0]$

$z = 4^{\frac{1}{4}}\left[cos\left(\frac{\pi+2\pi k}{4}\right) + sin\left(\frac{\pi+2\pi k}{4}\right)\right]$

When k = 0

$z_1 = 4^{\frac{1}{4}}\left[cos\left(\frac{\pi}{4}\right) + sin\left(\frac{\pi}{4}\right)\right]$

$= \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = 1 + i$

When k = 1

$z_2 = 4^{\frac{1}{4}}\left[cos\left(\frac{3\pi}{4}\right) + sin\left(\frac{3\pi}{4}\right)\right]$

$= \sqrt{2}\left(\frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = -1 + i$

When k = 2

$z_5 = 4^{\frac{1}{4}}\left[cos\left(\frac{5\pi}{4}\right) + sin\left(\frac{5\pi}{4}\right)\right]$

$= \sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = -1 - i$

When k = 3

$z_4 = 4^{\frac{1}{4}}\left[cos\left(\frac{7\pi}{4}\right) + sin\left(\frac{7\pi}{4}\right)\right]$

$= \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = 1 - i$

∴ the roots of $z^4 + 4 = 0$ are 1 ± i and -1 ± i

The locus of a complex number

This is an equation of a set of point representing a variable complex number or a path traced. It may be in form of an equation of a circle or a straight line.

Example 25

Determine the locus of z if

(a) $|z - 4| = 3$
(b) Arg(z+2) = $\frac{\pi}{6}$

Solution

Let z = x + iy

(a) $|x + iy - 4| = 3$
$|(x - 4) + iy| = 3$
$(x - 4)^2 + y^2 = 9$
$x^2 + y^2 - 8x + 7 = 0$
The locus of z is a circle with centre (4, 0)

(b) Arg(x + iy +2) = $\frac{\pi}{6}$
Arg((x+2) + iy) = $\frac{\pi}{6}$
$\tan^{-1}\left(\frac{y}{x+2}\right) = \frac{\pi}{6}$
$\frac{y}{x+2} = tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$
$\frac{y}{x+2} = \frac{\sqrt{3}}{3}$
$y = \frac{\sqrt{3}}{3}(x + 2)$

The locus of z is a straight line

Example 26

Determine the locus of z if:

(a) $|z| = 2$ and sketch it
(b) $\arg(z - 1) = \frac{\pi}{4}$ and sketch it
(c) $2|z - 1| = |z + i|$
(d) $\arg\left(\frac{z-i}{z+1}\right) = \frac{\pi}{3}$

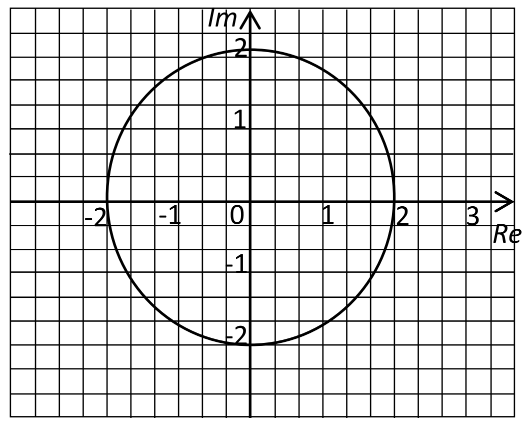
Solution

Let z = x + iy

(a) $|x + iy| = 2$ i.e. $\sqrt{x^2 + y^2} = 2$

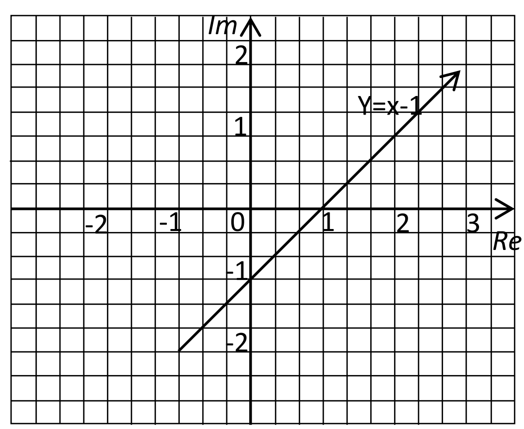
$x^2 + y^2 = 4 \; or \; (x - 0)^2 + (y - 0)^2 = 2^2$

The locus of z is a circle with centre (0, 0) and radius 2units



(b) Arg[(x – 1) + iy] = $\frac{\pi}{4}$
$\tan^{-1}\left(\frac{y}{x-1}\right) = \frac{\pi}{4}$
$\frac{y}{x-1} = tan\frac{\pi}{4} = 1$
$y = x - 1$

The locus is a straight line



(c) $2|(x - 1) + iy| = |x + i(y + 1)|$

$\left[2\sqrt{(x-1)^2 + y^2}\right]^2 = \left[\sqrt{x^2 + (y+1)^2}\right]^2$

$3x^2 + 3y^2 - 8x - 2y + 3 = 0$

$x^2 + y^2 - \frac{8}{3}x - \frac{2}{3}y + 1 = 0$

$= \left(x - \frac{4}{3}\right)^2 + \left(y - \frac{1}{3}\right)^2 = -1 + \frac{16}{9} + \frac{1}{9}$

$= \left(x - \frac{4}{3}\right)^2 + \left(y - \frac{1}{3}\right)^2 = \frac{8}{9}$

The locus of Z is a circle with the centre $\left(\frac{4}{3}, \frac{1}{3}\right)$

and radius $\sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$

(d) $\arg\left(\frac{x+iy-i}{x+iy+1}\right) = \frac{\pi}{3}$

$\arg\left(\frac{x+i(y-1)}{(x+1)+iy}\right) = \frac{\pi}{3}$

$\arg[x + i(y-1)] - \arg[x+1) + iy] = \frac{\pi}{3}$

$\tan^{-1}\left(\frac{y-1}{x}\right) - \tan^{-1}\left(\frac{y}{x+1}\right) = \frac{\pi}{3}$

$\frac{\left(\frac{y-1}{x}\right) - \left(\frac{y}{x+1}\right)}{1 + \left(\frac{y-1}{x}\right)\left(\frac{y}{x+1}\right)} = \sqrt{3}$

$\frac{y-x-1}{x^2+y^2+x-y} = \sqrt{3}$

$\sqrt{3}x^2 + \sqrt{3}y^2 + (\sqrt{3}+1)x - (\sqrt{3}+1)y + \frac{\sqrt{3}}{3} = 0$

$x^2 + y^2 + \left(\frac{3+\sqrt{3}}{3}\right)x - \left(\frac{3+\sqrt{3}}{3}\right)y + \frac{\sqrt{3}}{3} = 0$

The locus is a circle with centre at

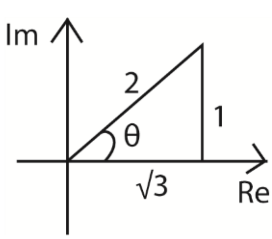
$\left[-\frac{1}{6}(3+\sqrt{3}) - \frac{1}{6}(3+\sqrt{3})\right]$ and radius $\sqrt{\frac{2}{3}}$

Example 27

The complex number z = $\sqrt{3} + i$.
$\bar{Z}$ *is the conjugate of* Z.

(a) Express Z in the modulus argument form

$|z| = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{3+1} = 2$



tanθ = $\frac{1}{\sqrt{3}}$

θ = Arg (z) = $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

Hence z = $2\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$

or z = $2[\cos 30^0 + i\sin 30^0]$

(i) On the same Argand diagram plot $\bar{Z}$ and $2\bar{Z} + 3i$

$\bar{z} = \sqrt{3} - i$

$|z| = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{3+1} = 2$

$2\bar{Z} + 3i = 2(\sqrt{3} - i) + 3i = 2\sqrt{3} + i$

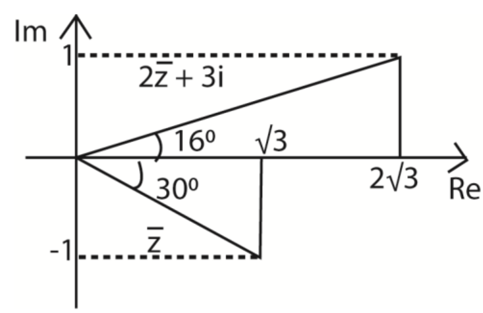
$|2\bar{Z} + 3i| = \sqrt{(2\sqrt{3})^2 + 1^2} = \sqrt{13}$

Finding arg($\bar{z}$):

Arg($\bar{z}$) = $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -30^0$

Finding arg($2\bar{Z} + 3i$):

Arg($2\bar{Z} + 3i$) = $\tan^{-1}\left(\frac{1}{2\sqrt{3}}\right) = 16.2^0$



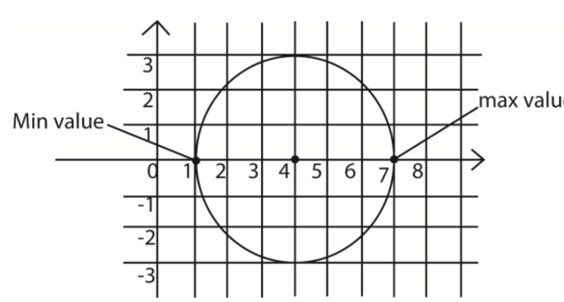
(b) What are the greatest and least values of $|Z|$ *if* $|Z - 4| \leq 3$?

$|z - 2| \leq 3$

$|x - 4 + iy| \leq 3$

$(x-4)^2 + y^2 \leq 3$

This is an equation of the circle with centre (4, 0) and radius 3.



Greatest value of $|z| = 4 + 3 = 7$

Lowest value of $|z| = 4 - 3 = 1$

Example 28

(a) Given that the complex number Z and its conjugate $\bar{Z}$ satisfy the equation

$$Z\bar{Z} + 2iZ = 12 + 6i. \text{ Find Z.}$$

Let z = x + iy then $\bar{z} = x - iy$
(x + iy)(x – iy) +2i(x + iy) = 12 + 6i
$x^2 + y^2 - 2y + 2xi = 12 + 6i$
Equating imaginary parts
2xi = 6i
x = 3
Equating real parts
$x^2 + y^2 - 2y = 12$
By substituting for x
$9 + y^2 - 2y = 12$
$y^2 - 2y - 3 = 0$
(y – 3)(y+1) = 0
y = 3 or y = -1
when y = 3; z = 3 + 3i
when y = -1; z = 3 - i
∴ the possible values of z are 3 +3i and 3- i.

(b) One root of the equation $Z^3 - 3Z^2 + 9Z + 13 = 0$ is $2 + 3i$. Determine other roots

Solution

Given root 2 + 3i; its conjugate is also a root of the equation

The equation of these two is

z^2 – (sum of roots)z + product of roots = 0

Sum = 2 + 3i + 2 – 3i = 4

Product of roots =(2 + 3i)(2 – 3i) = 2 + 9 = 13

$$z^2 - 4z + 13 = 0$$

$$\begin{array}{r|l} & z + 1 \\ z^2 - 4z + 13 & z^3 - 3z^2 - 9z + 13 \\ & -z^3 - 4z^2 - 13z \\ \hline & \quad z^2 - 4z + 13 \\ & \quad -z^2 - 4z + 13 \\ \hline \end{array}$$

$$(z^2 - 4z + 13)(z + 1) = 0$$

z + 1= 0
z = -1

So the roots are 2±3i and -1

Example 29

(a) If $z_1 = \frac{2i}{1+3i}$ and $z_2 = \frac{3+2i}{5}$, find $|z_1 - z_2|$

Solution

$$z_1 - z_2 = \frac{2i}{1+3i} - \frac{3+2i}{5},$$

$$= \frac{10i-(1+3i)(3+2i)}{5(1+3i)} = \frac{10i-[3+2i+9i-6]}{5(1+3i)}$$

$$= \frac{10i-11i+3}{5(1+3i)} = \frac{3-i}{5(1+3i)}$$

$$= \frac{(3-i)(3-i)}{5(1+3i)(3-i)} = \frac{3-9i-i-3}{5(1+9)} = \frac{-i}{5}$$

$$|z_1 - z_2| = \sqrt{0^2 - \left(-\frac{1}{5}\right)^2} = \frac{1}{5}$$

Alterative 2

$$z_1 = \frac{2i}{1+3i} = \frac{2i\,(1-3i)}{(1+3i)(1-3i)} = \frac{2i+6}{1+9}$$

$$= \frac{2i+6}{10} = \frac{3+2i}{5}$$

$$z_1 - z_2 = \frac{3+2i}{5} - \frac{3+2i}{5} = \frac{-i}{5}$$

$$|z_1 - z_2| = \sqrt{0^2 - \left(-\frac{1}{5}\right)^2} = \frac{1}{5}$$

(b) Given the complex number z = x + iy

(i) Find $\frac{z+i}{z+2}$

$$\frac{z+i}{z+2} = \frac{x+i(1+y)}{(x+2)+iy} = \frac{[x+i(1+y)][(x+2)-iy]}{[(x+2)+iy][(x+2)-iy]}$$

$$= \frac{x[(x+2)-iy]-i(1+y)[(x+2)-iy]}{(x+2)^2+y^2}$$

$$= \frac{x^2+2x-ixy+i(x+2+xy+2y)+y+y^2}{(x+2)^2+y^2}$$

$$= \frac{x^2+2x+y^2+y+i(2+x+2y)}{(x+2)^2+y^2}$$

(ii) Show that the locus of $\frac{z+i}{z+2}$ is a straight line when its imaginary part is zero. State the gradient of the line.

If imaginary part is zero

$(2 + x + 2y) = 0$

$2y = -x - 2$

y = $-\frac{1}{2}x + 1$

Comparing with y = mx + c

The gradient = $-\frac{1}{2}$

Example 30

(a) Given that a complex number Z* and its conjugate satisfy the equation

ZZ* + 2iZ = 12 + 6i, find the possible values of Z.

Solution

Let z* = x + iy, then z = x –iy

(x + iy)(x – iy) + 2i(x – iy) + 6i = 12 + 6i

$x^2 + y^2 + 2y + 2xi = 12 + 6i$

Equating real parts

$x^2 + y^2 + 2y = 12$

Equating imaginary parts

2x = 6 => x = 3

$9 + y^2 + 2y = 12$

$y^2 + 2y - 3 = 0$

(y – 1)(y + 3) = 0 => y = 1 or y = -3

When y = -3; z = 3 + 3i and

When y = 1, z = 3 –i

The possible values of z are 3 + 3i and 3 - i

(b) Find the Cartesian equation, in its simplest form of the curve described by $|z - 3 + 6i| = 2|z|$ where z is the complex number x + iy.

Hence sketch an Argand diagram the region satisfying $|z - 3 + 6i| \leq 2|z|$

Solution

Substituting z = x = iy

$|(x - 3) + i(y - 6)| = 2|x + iy|$

$(x - 3)^2 + (y + 6)^2 = 4(x^2 + y^2)$

$x^2 + y^2 - 6x + 12y + 45 = 4x^2 + 4y^2$

$x^2 + y^2 + 2x - 4y - 15 = 0$

Hence sketching the region satisfying $|z - 3 + 6i| \leq 2|z|$

i.e. $x^2 + y^2 + 2x - 4y - 15 \geq 0$

substituting (0, 0)

LHS = - 15 ; RHS = 0

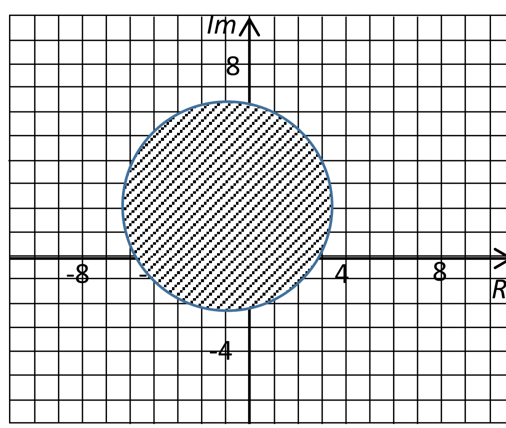
Since LHS< RHS, the point (0, 0) does not lie in the region$|z - 3 + 6i| \leq 2|z|$ therefore it lies in unwanted region.

Now $x^2 + y^2 + 2x - 4y - 15 \geq 0$

= $(x + 1)^2(y - 2)^2 \geq 20$

The locus is a circle centre (-1, 2) and radius $\sqrt{20}$

The plot of $|z - 3 + 6i| \leq 2|z|$



Example 31

If n is a variable and z = 4n + 3i(1 – n), show that the locus of Z is a straight line. Determine the minimum value of $|z|$

Solution

Let z = x = iy

⇨ x + iy = 4n +3i(1 – n)

Equating real parts

x = 4n => n = $\frac{x}{4}$

Equating imaginary parts

y = 3(1 – n)

⇨ y = $3\left(1-\frac{x}{4}\right)$

y =$-\frac{3x}{4}+3$ which is a straight line with gradient$-\frac{3}{4}$ and intercept 3

Given z = 4n + 3i(1 – n)
$|z|^2=16n^2+9(1-n)^2$

$|z|^2=25n^2-18n+9$
The minimum value can be obtained by completing squares or differentiation
By completing squares
$|z|^2=25(n^2-\frac{18}{25}n+\frac{9}{25})$
$=25\left[\left(n-\frac{9}{25}\right)^2+\frac{9}{25}-\frac{81}{625}\right]$
$=25\left[\left(n-\frac{9}{25}\right)^2+\frac{144}{625}\right]$
$=25\left(n-\frac{9}{25}\right)^2+\frac{144}{25}$
$|z|^2$ is minimum when $n=\frac{9}{25}$
$|z|^2{}_{min}$ = $\frac{144}{25}$
⇨ $|z|_{min}=\frac{12}{5}$

By differentiation

$2|z|\left|\frac{dz}{dn}\right|=50n-18=0$

n = $\frac{18}{50}=\frac{9}{25}$

$|z_{min}|^2=25\left(\frac{9}{25}\right)^2-18\left(\frac{9}{25}\right)+9=\frac{3600}{625}$

⇨ $|z|_{min}=\frac{60}{25}=\frac{12}{5}$

Example 32

Show that the locus of z if $\frac{z-3}{z+i}$ is wholly imaginary is a circle with centre at $\frac{3}{2}-\frac{1}{2}i$ and radius $\frac{1}{2}\sqrt{10}$

Solution

Let z = x + iy

$\frac{z-3}{z+i}=\frac{(x-3+iy)}{(x+i(y+1))}\cdot\frac{(x-i(y+1))}{(x-i(y+1))}$

= $\frac{x(x-3)+y(y+1)+i[(xy-(x-3)(y+1)}{x^2+(y+1)^2}$

If the above expression is wholly imaginary, then the real part must be zero

i.e. $\frac{x(x-3)+y(y+1)}{x^2+(y+1)^2}=0$

=> $x^2 + y^2 - 3x + y = 0$

$\left(x-\frac{3}{2}\right)^2+\left(y+\frac{1}{2}\right)^2=\frac{9}{4}+\frac{1}{4}=\frac{10}{4}$

The locus is a circle with centre with $\frac{3}{2}-\frac{1}{2}i$ and radius $\frac{1}{2}\sqrt{10}$

Revision exercise 4

1. Use Demoivre's theorem to show that
(a) $\cos6x = \cos^6x - 15\cos^4x\sin x + 15\cos x\sin^4x - \sin^6x$
(b) tan3x = $\frac{3tanx-tan^3x}{1-3tan^2x}$
2. (a) Evaluate $\left(1+i\sqrt{3}\right)^{\frac{2}{3}}$
[1.2160 + 1.0204i, -1.4917 +0.5429i]
(b) Find the roots of $z^4 + 4 = 0$ using Demoivre's theorem
[1± i, -1± i]
3. (a) Given that the complex number Z and its conjugate $\overline{Z}$ satisfy the equation
$Z\overline{Z}-2Z+2\overline{Z}=5-4i$. Find possible values of Z. (06marks)
[Z = $\pm2+1$]
(b) Prove that if $\frac{Z-6i}{Z+8}$ is real, then the locus of the point representing the complex number Z is a straight line. (06marks) [y = $\frac{3}{4}x$ + 6]
4. (a)Express 5 + 12i in polar form
[5 + 12i = 13(cos 67.38^0 + isin67.38^0)]
Hence, evaluate $\sqrt[3]{5+12i}$, giving your answer in the form a + ib where a and b are corrected to two decimal places. (12marks)

[z = 2.17 + 0.90i, -1.86 + 1.43i, -0.31 – 2.33i]

5. (a) The total impedance z in an electric circuit with two branches z_1 and z_2 is given by $\frac{1}{z}=\frac{1}{z_1}+\frac{1}{z_2}$. Given that z_1= 3 + 4i and z_2

= 5 + 5i where i =$\sqrt{-1}$, calculate the total impedance z in form a + bi
$\left[z = \frac{275}{145} + i\frac{325}{145}\right]$

(b) If n is a variable and z = 4n + 3i(1 – n), show that the locus of Z is a straight line. Determine the minimum value of $|z|$ $\left[|z|_{min} = \frac{12}{5}\right]$

6. Given that z_1 = 3 + 4i and z_2 = -1 + 2i
   (a) Express z_1, z_2, $z_1 + z_2$ and $z_1 - z_2$ in the form r(cosθ + isinθ)
   [z_1= 5(scos53.1^0+ isin53.1^0),
   z_2 =$\sqrt{5}$(cos116.6^0 + isin116.6^0),
   $z_1 + z_2$ = 2$\sqrt{10}$(cos71.6^0 + isin71.6^0)
   $z_1 - z_2$ = 2$\sqrt{5}$(cos-153.4^0 + isin-153.4^0)]
   (b) Determine the angle between $z_1 + z_2$ and z_1- z_2 [225^0]
7. Solve the simultaneous equation
   $z_1 + z_2 = 8$
   $4z_1 - 3iz_2 = 26 + 8i$
   Using the values of z_1 and z_2, find the modulus and argument of $z_1 + z_2 - z_1z_2$.
   [z_1 = 8 +2i, z_2 = -2i; 6.49, 75.96^0]
8. (a) Given that z = 3 + 4i, find the value of expression z + $\frac{25}{z}$ [6]
   Given that $\left|\frac{z-1}{z+1}\right| = 2$, show that the locus of the complex number is
   $x^2 + y^2 + \frac{10x}{3} + 1 = 0$. Sketch the locus
   [The centre of the circle is $\left(\frac{-5}{3}, 0\right)$ it passes through -3 but does not touch the y-axis]
9. (a)(i) Express each of the following complex numbers in the form a + bi
   z1 = (1 –i)(1 + 2i) [3 + i]
   z2 = $\frac{2+6i}{3-i}$ [2i]
   z3 = $\frac{-4i}{1-i}$ [2 – 2i]
   (b) Find the modulus and argument of $z_1z_2z_3$ [17.889, 63.4349^0]
   (c) Find the square root of 12i – 5
   [1 + 3i or -2 – 3i]
10. (a) given that z = $\sqrt{3}$+ i, find the modulus and argument
   (i) z^2 $\left[4, \frac{\pi}{3}\right]$
   (ii) $\frac{1}{z}$ $\left[\frac{1}{2}, -\frac{\pi}{6}\right]$
   (iii) Show in an Argand diagram the points representing complex numbers z, z^2 and $\frac{1}{z}$.
   (b) In an Argand diagram, P represents a complex number z such that $2|z-2| = |z-6i|$ show that P lies on a circle, find
   (i) The radius of this circle[4.2164]
   (ii) The complex number represented by it centre
   $\left[z = \frac{8}{3} - 2i\right]$
11. (a) given the complex number z_1 = 1 – i, z_2 = 7 + I represent z_1z_2 and $z_1 - z_2$ on the Argand diagram.
   Determine the modulus and argument of $\frac{z_1-z_2}{z_1z_2}$ [0.6325, -124.7^0]
   (b) If z is a complex number of form a + bi, solve $\left(\frac{z-1}{z+1}\right)^2 = i$ [z = 1 ±i$\sqrt{2}$]xz
12. (a) show the region represented by $|z-2+i| < 1$ on an Argand diagram
   [it is a circle of centre(2, -1) and radius 1 unit, the circle is drawn with dotted line since the equation is less than, and the unwanted region is the area outside the circle]
   (b) Express the complex number z = 1 - i$\sqrt{3}$ in modulus argument form and hence find z^2 and $\frac{1}{z}$ in the form a + bi
   $\left[\begin{array}{c} z = 2\left(cos\left(-\frac{\pi}{3}\right) + isin\left(-\frac{\pi}{3}\right)\right), \\ z^2 = -2 - i2\sqrt{3}, \frac{1}{z} = \frac{1}{4} + i\frac{\sqrt{3}}{4} \end{array}\right]$
13. (a)Given that z_1 = -i + 1, z_2 = 2 + i, and z_3 = 1 + 5i, represent z_2z_3, z_2-z_1 $\frac{1}{z_1}$and $\frac{z_2z_3}{z_2-z_1} + \frac{1}{z_1}$ o, Argand diagram
   (b) Prove that for positive integers $(cos\theta + sin\theta)^n = cosn\theta + sinn\theta$.
   Deduce that this formula is also true for negative values of n.
14. If z is a complex number, describe and illustrate on the Argand diagram the locus given by each of the following

(a) $\left|\frac{z+i}{z-2}\right| = 3$

[it is a circle centre $\left(\frac{9}{4}, \frac{1}{8}\right)$ radius 0. 8385]

(ii) Arg (z + 3) = $\frac{\pi}{6}$ [is it a straight line represented by equation y = $\frac{\sqrt{3}}{3}x + 3$]

15. (a) Use Demoivre's theorem or otherwise to simplify $\frac{(cos\theta+isin\theta)(cos2\theta+isin2\theta}{cos\frac{\theta}{2}+isin\frac{\theta}{2}}$

$\left[cos\left(\frac{5\theta}{2}\right) + isin\left(\frac{5\theta}{2}\right)\right]$

(b) Express $\frac{i}{4+6i}$ in modulus-argument form.

{z = 0.1387[cos(0.187π)+ isin(0.187π)]}

(c) Solve (z + zz*)z = 5 + 2z where z* is the complex conjugate of z.

[z = 1 + 2i, z* = 1 -2i]

16. Show that 2 + I is a root of the equation $2z^3 - 9z^2 + 14 - 5 = 0$. Hence find the other roots 2 – i and ½

17. (a) find the equation whose root are -1 ± i where i = $\sqrt{-1}$ [$z^2 + 2z + 2 = 0$]

(b) Find the sum of the first 10 terms of the series 1 + 2i+ -4 – 8i + 16 + ….

[205 + 410i]

(c) Prove by induction that

$(cos\theta + sin\theta)^n = cosn\theta + sinn\theta.$

18. Show that z = 1 is a root of the equation $z^3 - 5z2 + 9z - 5 = 0$.

Hence solve the equation for other roots

[2 ± i]

19. (a) Use Demoivre's theorem to express tan5θ in terms of tanθ.

(b) Solve the equation $z^3 + 1 = 0$

$\left[-1, \frac{1}{2} + i\frac{\sqrt{3}}{2}, \frac{1}{2} - i\frac{\sqrt{3}}{2}\right]$

20. (a) Without using tables or calculators, simplify $\frac{\left(cos\frac{\pi}{17}+isin\frac{\pi}{17}\right)^8}{\left(cos\frac{\pi}{17}-isin\frac{\pi}{17}\right)^9}$ [-1]

(b) Given that x and y are real, find the values of x and y which satisfy the equation: $\frac{2y+4i}{2x+y} - \frac{y}{x-i} = 0$ [x = -1 when y = -2 and x = 1 when y = 2]

21. Given the complex number z =$\frac{(3i+1)(i-2)^2}{i-3}$

(i) Determine z in form a + bi where a and b are constant [-4,- 3i]

(ii) arg(z) [-143.13^0]

22. (a) Express the complex numbers z_1 =4i and z_2 = 2 – 2i in trigonometric form r(cosθ + isinθ).

[z_1 =4(cos90^0 + isin90^0),

z_2 = 2$\sqrt{2}$(cos-45^0 + isin-45^0)]

Hence or otherwise evaluate $\frac{z_1}{z_2}$ $\left[\frac{1}{2}\right]$

(b) Find the values of x and y in $\frac{x}{3+3i} - \frac{y}{(2-3i)} = \frac{(6+2i)}{(1+8i)}$ [x = 2.8, y =0.4]

23. Find the fourth root of 4 + 3i

[±(1.4760 + 0.23971i), ±(0.2397 – 1.4760i)]

24. (a) Given that $\frac{ix}{1+iy} = \frac{3x+4i}{x+3y}$; find the values of x and y

[x = 2, y= 1.5 or x = -2 and y = -1.5]

(b) If z = x +iy, find the equation of the locus $\left|\frac{z+3}{z-1}\right| = 4$

$\left[x^2 + y^2 - \frac{38}{15}x + \frac{7}{15} = 0\right]$

25. (a) given that the complex number z and its conjugate Z*, satisfy the equation zz*+ 3z* = 34 – 12i, find the value of z.

[z = 3 - 4i and z = -6 + 4i]

(b) Find the Cartesian equation of the locus of a point P represented by equation$\left|\frac{z+3}{z+2-4i}\right| = i$

[the locus P is an equation of line 8y + 2x = 11]

26. (a) form a quadratic equation having -3 + 4i as one of its roots. [$z^2 + 6z + 25$]

(b) Given z_1 = -1 + i$\sqrt{3}$ and z_2 = -1 - i$\sqrt{3}$

(i) Express $\frac{z_1}{z_2}$ in form a + i$\sqrt{b}$, where a and b are real number

$\left[\frac{-1}{2} - i\frac{\sqrt{3}}{2}\right]$

(ii) Represent $\frac{z_1}{z_2}$ on an Argand diagram

(iii) find$\left|\frac{z_1}{z_2}\right|$ [1]

27. If z = $\frac{(2-i)(5+12i)}{(1+2i)^2}$

(a) Find the

(i) Modulus of z [5.814]
(ii) Argument z [-86.055^0]

(b) Represent z on a complex plane

(c) Write z in the polar form
[z = 5.814(cos(-86.055^0) + isin(-86.055^0) or z = 5.814(cos (0.47π) + isin(0.47π)]

28. (a) the complex number z = $\sqrt{3} + i$.
$\bar{Z}$ *is the conjugate of* Z.
(i) Express Z in the modulus argument form
[z = $2\left[\cos\left(\frac{\pi}{6}\right) + isin\left(\frac{\pi}{6}\right)\right]$ or z = $2[\cos 30^0 + isin30^0]$]
(ii) On the same Argand diagram plot $\bar{Z}$ and $2\bar{Z} + 3i$

(b) What are the greatest and least values of $|Z|\ if\ |Z - 4| \leq 3$?

[Greatest value of $|z| = 7$, Lowest value of $|z| =$ = 1]

29. (a) Given that the complex number Z and its conjugate $\bar{Z}$ satisfy the equation
$Z\bar{Z} + 2iZ = 12 + 6i$. Find Z.
[possible values of z are 3 +3i and 3- i]
(b)One root of the equation $Z^3 - 3Z^2 + 9Z + 13 = 0\ is\ 2 + 3i$. Determine other roots
[other roots are 2-3i and -1]

30. (a) If $z_1 = \frac{2i}{1+3i}$ and $z_2 = \frac{3+2i}{5}$, find $|z_1 - z_2|$
$\left[\frac{1}{5}\right]$

(b) Given the complex number z = x + iy
(i) Find $\frac{z+i}{z+2}$ $\left[\frac{x^2+2x+y^2+y+i(2+x+2y)}{(x+2)^2+y^2}\right]$
(ii) Show that the locus of $\frac{z+i}{z+2}$ is a straight line when its imaginary part is zero. State the gradient of the line. [y = $-\frac{1}{2}x + 1$]

31. (a) Given that the complex number Z and its conjugate $\overline{Z}$ satisfy the equation
$Z\overline{Z} - 2Z + 2\overline{Z} = 5 - 4i$. Find possible values of Z. [Z = $\pm 2 + 1$]

(b) Prove that if $\frac{Z-6i}{Z+8}$ is real, then the locus of the point representing the complex number Z is a straight line. [y = $\frac{3}{4}x + 6$]

32. (a) Express 5 + 12i in polar form
Hence, evaluate $\sqrt[3]{5 + 12i}$, giving your answer in the form a + ib where a and b are corrected to two decimal places. (12marks)

[$\sqrt[3]{5 + 12i} = \sqrt[3]{13}\left[\cos\left(\frac{67.38+2\pi k}{3}\right) + isin\left(\frac{67.38+2\pi k}{3}\right)\right]$ *taking* $k = 0, 1, 2$;
2.17 + 0.90i; -1.86 + 1.43i; -0.31 – 2.33i]

33. Show that the modulus of $\frac{(1-i)^6}{(1+i)}$=$4\sqrt{2}$