Revision exercise 5

Find

(a) $\int \frac{1}{9+x^2}dx$ $\left[\frac{1}{3}tan^{-1}\left(\frac{1}{3}x\right)+c\right]$

(b) $\int_{-2}^{2} \frac{1}{4+x^2}dx$ [0.7854]

(c) $\int_{-\sqrt{3}}^{3} \frac{1}{x^2+3}dx$ [1.833]

(d) $\int \frac{1}{4-x^2}dx$ $[sin^{-1}x+c]$

(e) $\int \frac{1}{\sqrt{16-x^2}}dx$ $\left[sin^{-1}\left(\frac{x}{4}\right)+c\right]$

(f) $\int \frac{1}{\sqrt{49-x^2}}dx$ $\left[sin^{-1}\left(\frac{x}{7}\right)+c\right]$

(g) $\int \frac{1}{\sqrt{25-4x^2}}dx$ $\left[\frac{1}{2}sin^{-1}\left(\frac{2}{4}x\right)+c\right]$

(h) $\int \frac{3}{9+x^2}dx$ $\left[tan^{-1}\left(\frac{x}{3}\right)+c\right]$

(i) $\int \frac{1}{25+x^2}dx$ $\left[\frac{1}{5}tan^{-1}\left(\frac{x}{5}\right)+c\right]$

(j) $\int \frac{2}{100+9x^2}dx$ $\left[\frac{1}{15}tan^{-1}\left(\frac{3}{10}x\right)+c\right]$

(k) $\int \frac{1}{3-2x+x^2}dx$ $\left[\frac{1}{\sqrt{2}}tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right)+c\right]$

(l) $\int \frac{1}{x^2\sqrt{4-x^2}}dx$ $\left[-\frac{1}{4}cotsin^{-1}\left(\frac{x}{2}\right)+c\right]$

Integration of exponential and logarithmic functions.

A. From $\frac{d}{dx}e^x = e^x$

- $\int e^x dx = e^x + c$

Example 10

Find

(a) $\int xe^{x^2}dx$

Solution

Let u = $3x^2$ => du = 6xdx i.e. xdx =$\frac{1}{6}du$

$\int xe^{x^2}dx = \frac{1}{6}\int e^u du = \frac{1}{6}e^u + c$

$\therefore \int xe^{x^2}dx = \frac{1}{6}e^{x^2} + c$

(b) $\int secxtanxe^{secx}dx$

Solution

Let u = secx => du=secxtanxdx

$\int secxtanxe^{secx}dx = \int e^u du = e^u + c$

$\therefore \int secxtanxe^{secx}dx = e^{secx} + c$

(c) $\int \frac{e^{cotx}}{sin^2x}dx$

Solution

Let u = cotx

- Du = -cosec^{2}x = $-\frac{1}{sin^2x}$

$\int \frac{e^{cotx}}{sin^2x}dx = -\int e^u du = -e^u + c$

$\therefore \int \frac{e^{cotx}}{sin^2x}dx = -e^{cotx} + c$

(d) $\int \frac{e^{-\frac{1}{x}}}{x^2}dx$

Solution

Let u = $-\frac{1}{x}$ => du = $\frac{1}{x^2}dx$

$\int \frac{e^{-\frac{1}{x}}}{x^2}dx = \int e^u du = -e^u + c$

$\therefore \int \frac{e^{-\frac{1}{x}}}{x^2}dx = e^{-\frac{1}{x}} + c$

B. From $\frac{d}{dx}(Inx) = \frac{1}{x}$

- $\int \frac{1}{x}dx = Inx + c \equiv InAx$

This result shows that

$\int \frac{f\prime(x)}{f(x)}dx = In[f(x)] + c$ **i.e.**

- $\int cot2xdx = \int \frac{cos2x}{sin2x}dx$

$= \frac{1}{2}In(sin2x) + c$

- $\int \frac{a}{b+cx}dx = \frac{a}{c}In(b+cx) + k$

Example 11

Find

(a) $\int \frac{1}{3x+4}dx$

Solution

Let u = 3x+4 => du 3dx i.e. dx=$\frac{1}{3}du$

$\therefore \int \frac{1}{3x+4}dx = \frac{1}{3}\int du = \frac{1}{3}Inu + c$

$= \frac{1}{3}In(3x+4) + c$

(b) $\int \frac{x}{1-5x^2}dx$

Solution

Let u = $1-5x^2$

=> du -10xdx i.e. dx=$-\frac{1}{10x}du$

$\therefore \int \frac{x}{1-5x^2}dx = \frac{1}{10}\int \frac{1}{u}du = \frac{1}{10}Inu + c$

$= \frac{1}{3}In(1-5x^2) + c$

(c) $\int tan^3xdx$

Solution

$\int tan^3xdx = \int tan^2xtanxdx$ (an odd power)

$= \int (sec^2x - 1)tanxdx$

$= \int sec^2xtanxdx - \int tanxdx$

For $\int sec^2xtanxdx$

Let u = tanx, => du = sec^2xdx

$\therefore \int sec^2xtanxdx = \int udu = \frac{1}{2}tan^2x + c$

For $\int tanxdx = \int \frac{sinx}{cosx}dx$

Let u =cosx, => du = -sinxdx

$\therefore \int tanxdx = -\int \frac{1}{u}du = -Inu + c$

= -In(cosx)+c =In(secx)+c

$\therefore \int tan^3xdx = \frac{1}{2}tan^2x + In(secx) + c$

(d) $\int_0^1 \frac{x+1}{3+4x^2}dx$

Solution

$$\int_0^1 \frac{x+1}{3+4x^2}dx = \int_0^1 \frac{1}{3+4x^2}dx + \int_0^1 \frac{1}{3+4x^2}dx$$
$$= \left[\frac{1}{8}In(3+4x^2 + \frac{1}{2\sqrt{3}}tan^{-1}\left(\frac{2}{\sqrt{3}}\right)\right]_0^1$$
$$= \frac{1}{8}In\left(\frac{7}{3}\right) + \frac{1}{2\sqrt{3}}tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$$

(e) $\int_a^{2a} \frac{x^3}{x^4+a^4}dx$

Solution

$$\int_a^{2a} \frac{x^3}{x^4+a^4}dx = \frac{1}{4}[In(x^4+a^4]_a^{2a}$$
$$= \frac{1}{4}(In17a^4 - In2a^4)$$
$$= \frac{1}{4}In\left(\frac{17}{2}\right) = 0.535$$

C. From $\frac{d}{dx}a^x = a^xIna$

⇨ $\int a^xdx = \frac{1}{Ina}a^x + c$

It follows that $\int 2^xdx = \frac{2^x}{In2} + c$

Example 12

Integrate

(a) $\int x^2 2^{3x^2}dx$

Solution

Let u = $3x^3$, => du = $9x^2$ i.e. $x^2dx = \frac{1}{9}du$

$$\int x^2 2^{3x^2}dx = \frac{1}{9}\int 2^u du = \frac{1}{9}\frac{2^u}{In2} + c$$
$$= \frac{1}{9}\frac{2^{3x^2}}{In2} + c$$

(b) $\int cosx.5^{sinx}dx$

Solution

Let u = sinx, => du = cosxdx

$$\int cosx.5^{sinx}dx = \int 5^u du = \frac{5^u}{In5} + c$$
$$= \frac{5^{sinx}}{In5} + c$$

(c) $\int \frac{3^{cotx}}{sin^2x}dx$

Solution

Let u = cotx, => du=-cosec^{2}x

$$\int \frac{3^{cotx}}{sin^2x}dx = -\int 3^{cotx}cosec^2xdx$$
$$= \int 3^u du = \frac{3^u}{In3} + c$$
$$= \frac{3^{cotx}}{In3} + c$$

Revision exercise 6

1. Find the following integrals
   (a) $\int e^x(3+e^x)^2dx$ $\left[\frac{1}{3}(3+e^x)^3 + c\right]$
   (b) $\int 2e^x(e^x-4)^3dx$ $\left[\frac{1}{2}(e^x-4)^4 + c\right]$
   (c) $\int \frac{4e^{-2x}}{(1+e^{-2x})^2}dx$ $\left[\frac{2}{1+e^{-2x}} + c\right]$
   (d) $\int \frac{(e^{-x}+7)^2}{e^x}dx$ $\left[-\frac{1}{3}(e^{-x}+7)^3 + c\right]$
   (e) $\int e^x\sqrt{4+e^x}\,dx$ $\left[\frac{2}{3}\sqrt{(4+e^x)^3} + c\right]$
   (f) $\int e^{5x}\sqrt{e^{5x}+2}\,dx$ $\left[\frac{2}{15}\sqrt{(e^{5x}+2)^3} + c\right]$
   (g) $\int \frac{e^{3x}}{\sqrt{e^{3x}-1}}$ $\left[\frac{2}{3}\sqrt{e^{3x}-1} + c\right]$
   (h) $\int \frac{1}{2e^x\sqrt{1-e^{-x}}}dx$ $\left[\sqrt{1-e^{-x}} + c\right]$
   (i) $\int 5^xdx$ $\left[\frac{5^x}{In5} + c\right]$
   (j) $\int 3^{2x}$ $\left[\frac{3^{2x}}{In9} + c\right]$
   (k) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}$ $\left[2e^{\sqrt{x}} + c\right]$
   (l) $\int x^2e^{x^3}$ $\left[\frac{1}{3}e^{x^3}\right]$
   (m) $\int 4^xdx$ $\left[\frac{4x}{In4}\right]$
   (n) $\int x10^xdx$ $\left[\frac{x10^x}{In10} - \frac{10^x}{(In10)^2} + c\right]$
2. Evaluate
   (a) $\int_1^3 e^xdx$ $[e(e^2-1)]$
   (b) $\int_0^3 e^{-x}dx$ $\left[1 - \frac{1}{e^3}\right]$
   (c) $\int_1^2 2e^{(2x+1)}$ $[e^3(e^2-1)]$
   (d) $\int_{-1}^1 2e^{(1-2x)}$ $\left[e^3 - \frac{1}{e}\right]$
   (e) $\int_0^1 (4xe^{x^2}+1)dx$ $[2e-1]$

Integration involving partial fractions

There are three established types of partial fractions depending on the nature of the denominator.

A. Denominators with linear factors e.g. 3x – 1, x + 2 and 3x – 4.

Each linear factor (ax + b) in the denominator has a corresponding partial fraction of the form $\frac{c}{(ax+b)}$' where a, b and c are constants.

Example 13

(a) Express each of the following in partial fraction. Hence find the integral of each with respect to x.

(i) $\frac{x-1}{(x+1)(x-2)}$

Solution

Let $\frac{x-1}{(x+1)(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x-2)}$

Multiplying by (x+1)(x-2)

⇨ x – 1 = A(x – 2) + B(x+1)

then we find the values of A and B

Putting x = 2: 1 = 3B, => B = $\frac{1}{3}$

Putting x = -1: -2 = -3A, => A = $\frac{2}{3}$

$\therefore \frac{x-1}{(x+1)(x-2)} = \frac{\frac{2}{3}}{(x+1)} + \frac{\frac{2}{3}}{(x-2)}$

$$= \frac{2}{3(x+1)} + \frac{1}{3(x-2)}$$

Hence,

$\int \frac{x-1}{(x+1)(x-2)} dx = \frac{2}{3}\int \frac{1}{(x+1)} dx + \frac{1}{3}\int \frac{1}{(x-2)} dx$

$= \frac{2}{3} In(x+1) + \frac{1}{3} In(x-2) + c$

$= \frac{2}{3} In(x+1)^2(x-2) + c$

(ii) $\frac{1}{x^3-9x}$

Solution

$\frac{1}{x^3-9x} = \frac{1}{x(x^2-9)} = \frac{1}{x(x-3)(x+3)}$

⇨ $\frac{1}{x^3-9x} = \frac{A}{x} + \frac{B}{(x-3)} + \frac{C}{(x+3)}$

Multiplying through with $x(x-3)(x+3)$

1= $A(x^2-9) + B(x^2+3x) + C(x^2-3x)$

Putting x = 0; 1 = -9A => $A = -\frac{1}{9}$

Putting x = 3; 1 = 18B => $B = \frac{1}{18}$

Putting x = -3; 1 = 18C => $C = \frac{1}{18}$

⇨ $\frac{1}{x^3-9x} = -\frac{1}{9x} + \frac{1}{18(x-3)} + \frac{1}{18(x+3)}$

Hence,

$\int \frac{1}{x^3-9x} dx$

$= -\frac{1}{9}\int \frac{1}{x} dx + \frac{1}{18}\int \frac{1}{(x-3)} dx + \frac{1}{18}\int \frac{1}{(x+3)} dx$

$=-\frac{1}{9} Inx + \frac{1}{18} In(x+3) + \frac{1}{18} In(x-3)$+c

$=\frac{1}{18}(In(x+3) + In(x-3) - 2Inx) + c$

$=\frac{1}{18}\left[In\frac{(x+3)(x-3)}{x^2}\right] + c$

(iii) $\frac{2x+1}{(x-1)(3x^2+7x+2)}$

Solution

$\frac{2x+1}{(x-1)(3x^2+7x+2)} = \frac{2x+1}{(x-1)(x+2)(3x+1)}$

$\frac{2x+1}{(x-1)(3x^2+7x+2)} = \frac{A}{(x-1)} + \frac{B}{(x+2)} + \frac{1}{(3x+1)}$

Multiplying by $(x-1)(x+2)(3x+1)$

2x + 1= A(x+2)(3x+1)+B(x-1)(3x+1)+C(x-1)(x+2)

Putting x = 1; 3 = 12A => $A = \frac{1}{4}$

Putting x = -2; -3 = 15B => $B = -\frac{1}{5}$

Putting x = -$\frac{1}{3}$; $\frac{1}{3}$ = $-\frac{20}{9}$C => $C = -\frac{3}{20}$

$\therefore \frac{2x+1}{(x-1)(3x^2+7x+2)} = \frac{1}{4(x-1)} - \frac{1}{5(x+2)} - \frac{3}{20(3x+1)}$

Hence,

$\int \frac{2x+1}{(x-1)(3x^2+7x+2)} dx$

$= \frac{1}{4}\int \frac{1}{(x-1)} dx - \frac{1}{5}\int \frac{1}{(x+2)} dx - \frac{3}{20}\int \frac{1}{(3x+1)} dx$

$= \frac{1}{4} In(x-1) - \frac{1}{5} In(x+2) - \frac{3}{20} In(3x+1)$

$= \frac{1}{20} In\frac{(x-1)^5}{(x+2)^4(3x+1)^3}$

(iv) $\frac{2x^2-x+1}{(x^2-1)(x+2)}$

Solution

$\frac{2x^2-x+1}{(x^2-1)(x+2)} = \frac{2x^2-x+1}{(x+1)(x-1)(x+2)}$

⇨ $\frac{2x^2-x+1}{(x^2-1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{C}{(x+2)}$

Multiplying through by (x+1)(x-1)(x+2)

$2x^2 - x+1= A(x-1)(x+2)+B(x+1)(x+2)+C(x+1)(x-1)$

Putting x = -1; 4 = -2A => $A = -2$

Putting x = 1; 2 = 6B => $B = \frac{1}{3}$

Putting x = -2; 11 = 3C => $C = \frac{11}{3}$

∴ $\frac{2x^2-x+1}{(x^2-1)(x+2)} = \frac{1}{3(x-1)} - \frac{2}{(x+1)} + \frac{11}{3(x+2)}$

Hence,

$\int \frac{2x^2-x+1}{(x^2-1)(x+2)}dx$

$= \frac{1}{3}\int \frac{1}{(x-1)}dx - 2\int \frac{1}{(x+1)}dx + \frac{11}{3}\int \frac{1}{(x+2)}dx$

$= \frac{1}{3}In(x-1) - 2In(x+1) + \frac{11}{3}In(x+2) + c$

(b) Evaluate $\int_1^3 \frac{x^2+1}{x^3+4x^2+3x}dx$

Solution

$\frac{x^2+1}{x^3+4x^2+3x} = \frac{x^2+1}{x(x+1)(x+3)}$

Let $\frac{x^2+1}{x^3+4x^2+3x} = \frac{A}{x} + \frac{B}{(x-3)} + \frac{C}{(x+3)}$

Multiplying with x(x – 3)(x + 3)

$x^2+ 1 = A(x – 3)(x+3)+ B(x)(x+3)+C(x)(x-3)$

Putting x = 0; 1 = 3A => $A = \frac{1}{3}$

Putting x = -1; 2 = -2B => $B = -1$

Putting x = -3; 10 = 6C => $C = \frac{5}{3}$

∴ $\frac{x^2+1}{x^3+4x^2+3x} = \frac{1}{3x} - \frac{1}{(x+1)} + \frac{5}{3(x+3)}$

Hence

$\int_1^3 \frac{x^2+1}{x^3+4x^2+3x}dx$

$= \frac{1}{3}\int_1^3 \frac{1}{x}dx - \int_1^3 \frac{1}{(x+1)}dx + \frac{5}{3}\int_1^3 \frac{1}{(x+3)}dx$

$= \left[\frac{1}{3}Inx - In(x+1) + \frac{5}{3}In(x+3)\right]_1^3$

$= \left\{\frac{1}{3}In3 - In4 + \frac{5}{3}In6\right\} - \left\{\frac{1}{3}In1 - In2 + \frac{5}{3}In4\right\}$

$= 0.3488$

B. Denominators with linear factors
Quadratic factors

Each quadratic factors (ax^2+bx +c) has a corresponding partial fraction of the form $\frac{Ax+B}{(ax^2+bx+c)}$ where a, b, c and A and B are constants.

Example 14

(a) Express $\frac{7x^2+2x-28}{(x-6)(x^2+3x+5)}$ in partial fraction.

Solution

Let $\frac{7x^2+2x-28}{(x-6)(x^2+3x+5)} = \frac{A}{x-6} + \frac{Bx+c}{x^2+3x+5}$

Multiplying through by (x – 6)(x^2+3x+5)

$7x^2$+2x-28= A(x^2+3x+5)+ (Bx +C)(x – 6)

Putting x = 6; 236= 59A, => A = 4

Equating coefficients of x^2

7 = A + B

7 = 4 + B; => B = 3

Equating constants

-28 = 5A – 6C

-28 = 20 – 6C

C = 8

∴ $\frac{7x^2+2x-28}{(x-6)(x^2+3x+5)} = \frac{4}{x-6} + \frac{3x+8}{x^2+3x+5}$

(b) Find the integral of $f(x) = \frac{2x-1}{(x-1)(x^2+1)}$

Solution

Let $\frac{2x-1}{(x-1)(x^2+1)} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+1)}$

Multiplying through by $(x-1)(x^2+1)$

2x – 1 = A(x^2+1)+ (Bx + C)(x-1)

Putting x = 1; 1 = 2A => A =$\frac{1}{2}$

Putting x= 0; -1 = A –C => C = $\frac{3}{2}$

Putting x= -1; 2A + 2B – 2C => B = $-\frac{1}{2}$

∴ $\frac{2x-1}{(x-1)(x^2+1)} = \frac{1}{2(x-1)} + \frac{-\frac{1}{2}x+\frac{3}{2}}{(x^2+1)}$

$\frac{2x-1}{(x-1)(x^2+1)} = \frac{1}{2(x-1)} + \frac{3-x}{2(x^2+1)}$

Note the values of x = 0 and x = -1 are conveniently chosen, but the constants B and C by expansion of the expression and equating constants, i.e.

-1 = A –C => C = $\frac{3}{2}$

2 = C – B

B = $\frac{3}{2} - 2 = -\frac{1}{2}$

Thus,

$$\int \frac{2x-1}{(x-1)(x^2+1)} dx$$

$$= \frac{1}{2}\int \frac{1}{(x-1)} dx + \frac{3}{2}\int \frac{1}{(x^2+1)} dx - \frac{1}{2}\int \frac{x}{(x^2+1)} dx$$

$$= \frac{1}{2} In(x-1) + \frac{3}{2} tan^{-1}x - \frac{1}{4} In(x^2+1) + c$$

(c) Evaluate

(i) $\int_2^3 \frac{3+3x}{x^3-1}$dx

Solution

Note memorize the identities

$$x^3 - 1 = (x-1)(x^2+x+1)$$

$$x^3 + 1 = (x-1)(x^2-x+1)$$

Then

$$\frac{3+3x}{x^3-1} = \frac{3+3x}{(x-1)(x^2+x+1)}$$

$$Let \frac{3+3x}{x^3-1} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+x+1)}$$

Multiplying through by (x – 1)(x^2+x+1)

3+3x = A(x^2+x+1) + (Bx+C)(x – 1)

Putting x = 1, 6 = 3A, => A = 2

By expanding and equating coefficients

x^2: A + B = 0, => B = 0 – 2 = -2

x^0: A – C = 3, => C = 2 – 3 = -1

$$\therefore \frac{3+3x}{x^3-1} = \frac{2}{(x-1)} - \frac{2x+1}{(x^2+x+1)}$$

$$\int_2^3 \frac{3+3x}{x^3-1} dx = 2\int_2^3 \frac{1}{(x-1)} dx - \int_2^3 \frac{2x+1}{(x^2+x+1)}$$

$$= [2In(x-1) - In(x^2+x+1)]_2^3$$

$$= 2In(2) + In\left(\frac{7}{13}\right)$$

$$= 0.7673$$

(ii) $\int_2^3 \frac{x^2}{x^4-1} dx$

Solution

$$\frac{x^2}{x^4-1} = \frac{x^2}{(x-1)(x+1)(x^2+1)}$$

Let $\frac{x^2}{(x-1)(x+1)(x^2+1)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{Cx+D}{(x^2+1)}$

By multiplying through by (x – 1)(x+1)(x^2+1)

x^2= A(x+1)(x^2+1)+ B(x -1)(x^2+1) + (Cx+D)(x^2-1)

By equating coefficients

x^3: A + B + C = 0 …………………… (i)

x^2: A – B + D= 1 ………………………(ii)

x^1: A + B – C = 0 ………………………(iii)

x^0: A – B – D = 0 ………………………(iv)

Eqn. (ii) – Eqn. (iv)

2D = 2 => D = $\frac{1}{2}$

Eqn.(i)+ (iii)

2A+ 2B = 0 ………………………… (v)

Eqn. (ii) + Eqn. (iv)

2A – 2B = 1…………………………(vi)

Eqn. (v)+ Eqn. (vi)

4A = 1 => A = $\frac{1}{4}$

Eqn. (v)

B = $-\frac{1}{4}$

Eqn. (i)

C = 0

$$\therefore \frac{x^2}{x^4-1} = \frac{1}{4(x-1)} - \frac{1}{4(x+1)} + \frac{1}{2(x^2+1)}$$

$$\int \frac{x^2}{x^4-1} dx$$

$$= \frac{1}{4}\int \frac{1}{(x-1)} dx - \frac{1}{4}\int \frac{1}{(x+1)} dx + \frac{1}{2}\int \frac{1}{(x^2+1)} dx$$

$$= \frac{1}{4} In(x-1) - \frac{1}{4} In(x+1) + \frac{1}{2} tan^{-1}x + c$$

$$\int_2^3 \frac{x^2}{x^4-1} dx$$

$= \left[\frac{1}{4}In(x-1) - \frac{1}{4}In(x+1) + \frac{1}{2}tan^{-1}x\right]_2^3$

$= \frac{1}{4}\left[In\frac{x-1}{x+1} + 2tan^{-1}x\right]_2^3$

$= \frac{1}{4}\left\{\left[In\frac{2}{4} + 2tan^{-1}3\right] - \left[In\frac{1}{3} + 2tan^{-1}2\right]\right\}$

$= \frac{1}{4}\left[In\frac{1}{2} - In\frac{1}{3} + 2(tan^{-1}3 - tan^{-1}2)\right]$

$= \frac{1}{4}(0.405 + 0.1\pi)$

$= 0.18$

(iii) $\int_0^1 \frac{x^2+6}{(x^2+4)(x^2+9)}dx$

Solution

Let$\frac{x^2+6}{(x^2+4)(x^2+9)} = \frac{Ax+B}{(x^2+4)} + \frac{Cx+D}{(x^2+9)}$

Multiplying by $(x^2+4)(x^2+9)$

$x^2+6 = (Ax+B)(x^2+9) + (Cx+D)(x^2+4)$

$x^2+6 = (A+C)x^3+(B+D)x^2+(9A+4C)x+9B+4D$

Equating coefficients

x^3: A + C = 0 ……………….(i)

x^2: B+D = 1…………………(ii)

x^1: 9A + 4C = 0 …………..(iii)

x^0: 9B + 4D = 6 ………….(iv)

Solving simultaneously

A= C= 0; B=$\frac{2}{5}$ and D =$\frac{3}{5}$

$\therefore \frac{x^2+6}{(x^2+4)(x^2+9)} = \frac{2}{5(x^2+4)} + \frac{3}{5(x^2+9)}$

$\int_0^1 \frac{x^2+6}{(x^2+4)(x^2+9)}dx$

$= \frac{2}{5}\int_0^1 \frac{1}{(x^2+4)}dx + \frac{3}{5}\int_0^1 \frac{1}{(x^2+9)}dx$

$= \frac{1}{5}\left[tan^{-1}\frac{1}{2}x + tan^{-1}\frac{1}{3}x\right]_0^1$

$= \frac{1}{5}tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}\right) = 0.1571$

C. Repeated factors

Each repeated factor $(ax^2+ b)^n$ in the denominator has corresponding partial fraction of the form:$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$, where a, b, A_i are constants (i = 1, 2, ….n)

Example 15

Express each of the follow in partial fraction and hence find their integrals.

(a) $\frac{4x-9}{(x-3)^2}$

Solution

Let$\frac{4x-9}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2}$

Multiplying through by $(x-3)^2$

4x – 9 = A(x-3) + B = Ax – 3A + B

Equating coefficients

x^1: x = 4

x^0: -3A + B = 4; B = 3

$\therefore \frac{4x-9}{(x-3)^2} = \frac{4}{x-3} + \frac{3}{(x-3)^2}$

Hence

$\int \frac{4x-9}{(x-3)^2}dx = 4\int \frac{1}{x-3}dx + 3\int (x-3)^{-2}dx$

$= 4In(x-3) - \frac{3}{x-3} + c$

(b) $\frac{3x-14}{x^2-8x+16}$

Solution

$\frac{3x-14}{x^2-8x+16} = \frac{3x-14}{(x-4)^2}$

Let$\frac{3x-14}{(x-4)^2} = \frac{A}{x-4} + \frac{B}{(x-4)^2}$

Multiplying through by $(x-4)^2$

3x – 14 = A(x-4) + B = Ax – 4A + B

Equating coefficients

x^1: x = 3

x^0: -4A + B = -14; B = -2

$\therefore \frac{3x-14}{(x-3)^2} = \frac{3}{x-3} - \frac{2}{(x-4)^2}$

Hence

$\int \frac{3x-14}{(x-4)^2}dx = 3\int \frac{1}{x-4}dx - 2\int (x-4)^{-2}dx$

$= 3In(x-4) + \frac{2}{x-4} + c$

(c) $\frac{2x^2-5x+7}{(x-2)(x-1)^2}$

Solution

Let$\frac{2x^2-5x+7}{(x-2)(x-1)^2} = \frac{A}{x-2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$

Multiplying through by $(x-2)(x-1)^2$

$2x^2-5x+7 = A(x-1)^2+B(x-2)(x-1)+C(x-2)$

Putting x = 1: 4 = -C, => C = -4

Putting x = 2: A = 5

Putting x = 0, 7 =A – 2B – 2C; B = -2

$\therefore \frac{2x^2-5x+7}{(x-2)(x-1)^2} = \frac{5}{x-2} - \frac{2}{x-1} - \frac{4}{(x-1)^2}$

Hence

$\int \frac{2x^2-5x+7}{(x-2)(x-1)^2}dx$

$= 5\int \frac{1}{x-2}dx - 2\int \frac{1}{x-1}dx - 4\int (x-1)^{-2}\,dx$

$= 5In(x-2) - 2In(x-1) - \frac{4}{x-1} + c$

(d) $\frac{7x+2}{3x^3+x^2}$

Solution

$\frac{7x+2}{3x^3+x^2} = \frac{7x+2}{x^2(3x+1)}$

Let $\frac{7x+2}{x^2(3x+1)} = \frac{A}{(3x+1)} + \frac{B}{x} + \frac{C}{x^2}$

Multiplying through by $x^2(3x+1)$

$7x+ 2 = Ax^2 +Bx(3x+1)+C(3x+1)$

Putting x = 0; c = 2

Putting x = $-\frac{1}{3}$; $\frac{A}{9} = 2 - \frac{7}{3}$ => A = -3

Putting x= -1; -5 = A + 2B- 2C, => B = 1

$\therefore \frac{7x+2}{x^2(3x+1)} = \frac{-3}{(3x+1)} + \frac{1}{x} + \frac{2}{x^2}$

Hence

$\int \frac{7x+2}{x^2(3x+1)}dx$

$= -\int \frac{3}{(3x+1)}dx + \int \frac{1}{x}dx + 2\int x^{-2}dx$

$= -In(3x+1) + Inx - \frac{2}{x} + c$

$= In\frac{x}{3x+3} - \frac{2}{x} + c$ln

Integration of improper fractions

Improper fractions are those whose index of the numerator is equal to or greater than that of the denominators.

They are first changed to proper fraction by long division or otherwise, before being integrated.

Example 16

(a) Express $\frac{5x^2-71}{(x+5)(x-4)}$ in partial fractions.

Hence find $\int \frac{5x^2-71}{(x+5)(x-5)}dx$

Solution

$\frac{5x^2-71}{(x+5)(x-4)} = \frac{5x^2-71}{x^2+x-20}$

Using long division

$$\begin{array}{r|l} & 5 \\ \hline x^2+x-20 & 5x^2+0x-71 \\ & -\ 5x^2+5x-100 \\ \hline & -5\ +29 \end{array}$$

=> $\frac{5x^2-71}{(x+5)(x-4)} = 5 + \frac{-5x+29}{x^2+x-20}$

Let $\frac{-5x+29}{(x+5)(x-4)} = \frac{A}{x+5} + \frac{B}{x-4}$

Multiplying through by (x+5)(x-4)

-5x + 29 = A(x – 4)+ B(x + 5)

Putting x = 4, B = 1

Putting x = -5; A = -6

$\therefore \frac{-5x+29}{(x+5)(x-4)} = \frac{-6}{x+5} + \frac{1}{x-4}$

Hence

$\int \frac{5x^2-71}{(x+5)(x-4)}dx$

$= 5\int dx - 6\int \frac{1}{x+5}dx + \int \frac{1}{x-4}dx$

$= 5x - 6In(x+5) + In(x-4) + c$

(b) Evaluate $\int_0^1 \frac{3-2x}{1+x}dx$

Solution

$\frac{3-2x}{1+x} = \frac{-2x+3}{x+1}$

Using long division

$$\begin{array}{r|l} & -2 \\ \hline x+1 & -2x+3 \\ & -2x\ -2 \\ \hline & \quad 5 \end{array}$$

$\therefore \frac{3-2x}{1+x} = -2 + \frac{5}{x+1}$

Hence

$\int_0^1 \frac{3-2x}{1+x}dx = -2\int_0^1 dx + 5\int_0^1 \frac{1}{x+1}dx$

$= [-2x + 5In(x+1)]_0^1$

= -2+ 5ln2

= 1.4657

Revision exercise 7

1. Express the following into partial fraction

(a) $\frac{8x}{x^2-4x-12}$ $\left[\frac{6}{x-6}+\frac{2}{x+2}\right]$

(b) $\frac{x^4-x^3+x^2+1}{x^3+x}$ $\left[x-1+\frac{1}{x}+\frac{x-1}{x^2+1}\right]$

(c) $\frac{5x-1}{2x^2+x}-10$ $\left[\frac{3}{2x+5}+\frac{1}{x-2}\right]$

(d) $\frac{2x^2-7x+1}{(2x+1)(2x-1)(x-2)}$

$\left[\frac{1}{2x+1}+\frac{2}{3(2x-1)}-\frac{1}{3(x-2)}\right]$

(e) $\frac{6x+7}{(x^2+2)(x+3)}$ $\left[\frac{x+3}{x^2+2}-\frac{1}{x+3}\right]$

(f) $\frac{5x+7}{(x+1)^2(x+2)}$ $\left[\frac{3}{x+1}+\frac{2}{(x+1)^2}-\frac{3}{x+2}\right]$

(g) $\frac{2x^3+3x^2-x-4}{x^2(x+1)}$ $\left[2+\frac{3}{x}+\frac{4}{x^2}-\frac{2}{x+1}\right]$

2. Find

(a) $\int\frac{x^2}{x^4-1}dx$ $\left[\frac{1}{4}In\left(\frac{x-1}{x+1}\right)+tan^{-1}x+c\right]$

(b) $\int\frac{x^2-4}{(x+1)^2(x-5)}dx$

$\left[\frac{5}{12}In(x+1)-\frac{1}{2(x+1)}+\frac{7}{12}In(x-5)\right]$

(c) $\int\frac{3x^2+x+1}{(x-2)(x+1)^3}dx$

$\left[\frac{5}{9}In(x-2)-\frac{5}{9}In(x+1)-\frac{4}{3(x+1)}+\frac{1}{2(x+1)^2}\right]$

(d) $\int\frac{x^4-x^3+x^2+1}{x^3+x}dx$

$\left[\frac{x^2}{2}-x+Inx+tan^{-1}x-\frac{1}{2}In(1+x^2)+c\right]$

(e) $\int\frac{5x-1}{2x^2+x-10}dx$

$\left[\frac{3}{2}In(2x+5)+In(x-2)+c\right]$

(f) $\int\frac{x^2-9x+2}{(x11)(x-1)(x-2)}dx$

$[2In(x+1)+3In(x-1)-4In(x-2)]+c$

(g) $\int\frac{9x+7}{(2x^2+3)(x+2)}dx$

$\left[\frac{1}{2}In(2x^2+3)+\frac{5}{\sqrt{10}}tan^{-1}\left(\sqrt{\frac{2}{3}}x\right)-In(x+2)+c\right]$

(h) $\int\frac{7+5x-6x^2}{(2x+1)^2(x+2)}dx$

$\left[\frac{3}{2}In(2x+1)-\frac{1}{2x+1}-3In(x+2)+c\right]$

(i) $\int\frac{x^2+7x-14}{(x+5)(x-3)}dx$

$[x+3In(x+5)+2In(x-3)+c]$

3. Evaluate

(a) $\int_0^2\frac{3x^4+7x^3+8x^2+53-186}{(x+4)(x^2+9)}dx$ [-4.5489]

(b) $\int_2^3\frac{x^2}{x^4-1}dx$ [0.18]

(c) $\int_1^3\frac{x^2+1}{x^3+4x^2+3x}$dx [0.3489]

(d) $\int_0^1\frac{x^3}{x^2+1}dx$ [0.1535]

(e) $\int_6^7\frac{x^2-4}{(x+1)^2(x-5)}dx$ [0.4689]

(f) $\int_3^4\frac{3x^2+x+1}{(x-2)(x+1)^3}dx$ [0.3165]

(g) $\int_0^2\frac{8x}{x^2-4x-12}dx$ [1.05]

Integration by parts

This stems from differentiating the product of a function, y = uv,

$$\frac{d}{dx}(uv)=v\frac{du}{dx}+u\frac{dv}{dx}$$

$$u\frac{dv}{dx}=\frac{d}{dx}(uv)-v\frac{du}{dx}$$

$$\int u\frac{dv}{dx}dx=\int\frac{d}{dx}(uv)dx-\int v\frac{du}{dx}dx$$

$$\int u\frac{dv}{dx}dx=uv-\int v\frac{du}{dx}dx$$

Or simply $\int udv=uv-\int vdu$

The function chosen as u should be easily differentiated whereas the other function chosen as v should be easily integrated.

The above expression of the integration by parts can be summarized by using a technique of integration by parts

This is summarized in the table below

Sign	Differentiate	Integrates
+	u_1	$\frac{dv}{dx}$
-	u_2	v_1
+	u_3	v_2
-	u_4	v_3

NB: the signs change as +, -, + etc.

The u function is differentiated until a zero value is obtained otherwise we continue with differentiation.

The integral of the function is equal to the sum of result shown in the table above.

Integration by parts is applied in the following areas:

A. Integration products of polynomials by parts

Example 17

(a) Find

(i) $\int x(x+2)^3 dx$

Solution

Let u = x and $\frac{dv}{dx} = (x+2)^3$

$\frac{du}{dx} = 1$; v = $\frac{1}{4}(x+2)^4$

From $\int udv = uv - \int vdu$

$\int x(x+2)^3 dx$
$= \frac{1}{4}x(x+2)^4 - \int 1.\frac{1}{4}(x+2)^4 dx$
$= \frac{1}{4}x(x+2)^4 - \frac{1}{4}\int (x+2)^4 dx$
$= \frac{1}{4}x(x+2)^4 - \frac{1}{20}(x+2)^5 + c$
$= \frac{1}{20}(x+2)^4(5x - x - 2) + c$
$= \frac{1}{20}(x+2)^4(4x-2) + c$
$= \frac{1}{10}(x+2)^4(x-1) + c$

Or by using basic techniques

Sign	Differentiate	Integrates
+	x	$(x+2)^3$
-	1	$\frac{1}{4}(x+2)^4$
+	0	$\frac{1}{20}(x+2)^5$

$\int x(x+2)^3 dx = \frac{1}{4}x(x+2)^4 - \frac{1}{20}(x+2)^5 + c$
$= \frac{1}{10}(x+2)^4(x-1) + c$

$\therefore \int x(x+2)^3 dx = \frac{1}{10}(x+2)^4(x-1) + c$

(ii) $\int (x+3)(x-4)^5 dx$

Solution

Let u = (x+3) and $\frac{dv}{dx} = (x-4)^5$

$\frac{du}{dx} = 1$; v = $\frac{1}{6}(x-4)^6$

$\int (x+3)(x-4)^5 dx$
$= \frac{1}{6}(x+3)(x-4)^6 - \frac{1}{6}\int 1.(x-4)^6 \text{dx}$
$= \frac{1}{6}(x+3)(x-4)^6 - \frac{1}{6}\int (x-4)^6 dx$
$= \frac{1}{6}(x+3)(x-4)^6 - \frac{1}{42}(x-4)^7 + c$
$= \frac{1}{42}(x-4)^6((7(x+3) - x + 4) + c$
$= \frac{1}{42}(x-4)^6(6x+25) + c$

Sign	Differentiate	Integrates
+	x+3	$(x-4)^5$
-	1	$\frac{1}{6}(x+2)^6$
+	0	$\frac{1}{42}(x+2)^7$

$\int (x+3)(x-4)^5 dx$
$= (x+3)(x-4)^5 - \frac{1}{42}x(x+2)^7 + c$
$\therefore \int (x+3)(x-4)^5 dx$
$= \frac{1}{42}(x-4)^6(6x+25) + c$

(iii) $\int \frac{3x-4}{(x+2)^4} dx$

Solution

$\int \frac{3x-4}{(x+2)^4} dx = \int (3x-4)(x+2)^{-4} \text{dx}$

Let u =(3x – 4) and $\frac{dv}{dx} = (x+2)^{-4}$

$\frac{du}{dx} = 3$; v = $-\frac{1}{3}(x+2)^{-3}$

$\int \frac{3x-4}{(x+2)^4} dx$

$= -\frac{1}{3}(3x-4)(x+2)^{-3} - \int 3.-\frac{1}{3}(x+2)^{-3} dx$

$= -\frac{1}{3}(3x-4)(x+2)^{-3} + \int (x+2)^{-3} dx$

$= -\frac{1}{3}(3x-4)(x+2)^{-3} - \frac{1}{2}(x+2)^{-2} + c$

$= \frac{4-3x}{3(x+2)^3} - \frac{1}{2(x+2)^2} + c$

$= \frac{2(4-3x)-3(x+2)}{6(x+2)^3} + c = \frac{2-9x}{6(x+2)^3} + c$

$\therefore \int \frac{3x-4}{(x+2)^4} dx = \frac{2(4-3x)-3(x+2)}{6(x+2)^3} + c = \frac{2-9x}{6(x+2)^3} + c$

Sign	Differentiate	Integrates
+	3x-4	$(x-4)^{-4}$
-	3	$\frac{1}{3}(x-4)^{-3}$
+	0	$\frac{1}{6}(x-4)^{-2}$

$\int \frac{3x-4}{(x+2)^4} dx$

$= -\frac{1}{3}(3x-4)(x-4)^{-3} - \frac{1}{2}(x-4)^{-2} + c$

$= \frac{2-9x}{6(x+2)^3} + c$

(b) Evaluate

(i) $\int_0^2 x(x-3)^2 dx$

Solution

Sign	Differentiate	Integrates
+	x	$(x-3)^2$
-	1	$\frac{1}{3}(x-3)^3$
+	0	$\frac{1}{12}(x-3)^4$

$\int x(x-3)^2 dx = \frac{1}{3}x(x-3)^3 - \frac{1}{12}(x-3)^4 + c$

$= \frac{1}{12}(x-3)^3(4x - x + 3) + c$

$= \frac{1}{12}(x-3)^3(3x+3) + c$

$= \frac{1}{4}(x-3)^3(x+1) + c$

⇨ $\int_0^2 x(x-3)^2 dx = \left[\frac{1}{4}(x-3)^3(x+1)\right]_0^2$

$= \frac{-3}{4} - \frac{-27}{4} = \frac{24}{4} = 6$

(i) $\int_3^6 \frac{x}{\sqrt{x-2}} dx$

Solution

Sign	Differentiate	Integrates
+	x	$(x-2)^{-\frac{1}{2}}$
-	1	$2(x-2)^{\frac{1}{2}}$
+	0	$\frac{4}{3}(x-2)^{\frac{3}{2}}$

$\int \frac{x}{\sqrt{x-2}} dx = 2x(x-2)^{\frac{1}{2}} - \frac{4}{3}(x-2)^{\frac{3}{2}} + c$

$= \frac{2}{3}(x-2)^{\frac{1}{2}}[3x - 2(x-2)] + c$

$= \frac{2}{3}(x-2)^{\frac{1}{2}}(x+4) + c$

⇨ $\int_3^6 \frac{x}{\sqrt{x-2}} dx = \left[\frac{2}{3}(x-2)^{\frac{1}{2}}(x+4)\right]_3^6$

$= \left[\frac{2}{3}(6-2)^{\frac{1}{2}}(6+4)\right] - \left[\frac{2}{3}(3-2)^{\frac{1}{2}}(3+4)\right]$

$= \frac{2}{3}(20-7) = \frac{26}{3} = 8\frac{2}{3}$

Revision exercise 8

1. Integrate

(a) $\int (x-1)(x+2)^2 dx$

$\left[\frac{1}{4}(x-2)(x+2)^3 + c\right]$

(b) $\int (3x-1)(2x+3)^2 dx$

$\left[\frac{1}{48}(18x-17)(2x+3)^3 + c\right]$

(c) $\int (2-5x)(4-x)^4 dx$

$\left[\frac{1}{30}(25x+8)(4-x)^5\right]$

(d) $\int \frac{x-2}{(2x-3)^2} dx$

$\left[\frac{1}{4}In(2x-3) + \frac{1}{4(2x-3)} + c\right]$

(e) $\int \frac{x+4}{\sqrt{3x-2}} dx$

$\left[\frac{2}{27}(3x+40)\sqrt{3x-2} + c\right]$

(f) $\int \frac{3x+1}{\sqrt{1-2x}} dx$

$In\left(\frac{2-x}{5-x}\right) + \frac{1}{5-x} + c$

2. Evaluate

(a) $\int_{-1}^1 x^2(x+3)^3 dx$ $\left[\frac{108}{5}\right]$

(b) $\int_3^6 \frac{x^2}{\sqrt{1-2x}} dx$ $\left[\frac{586}{15}\right]$

B. Integration products of polynomials and circular/trigonometric functions by parts

Example 18

(a) Find

(i) $\int x\sin x dx$

Solution

Let u = x and $\frac{dv}{dx} = \sin x$

$\frac{du}{dx} = 1$, v = -cosx

$\int x\cos x dx = -x\cos x - \int 1.-\cos x$

$= -x\cos x + \sin x + c$

Or: by using basic technique

Sign	Differentiate	Integrates
+	x	$\sin x$
-	1	$-\cos x$
+	0	$-\sin x$

$\int x\cos x dx = -x\cos x + \sin x + c$

(ii) $\int x^2\cos x dx$

Solution

Let u = x^2 and $\frac{dv}{dx} = \cos x$

$\frac{du}{dx} = 2x$, v = sinx

$\int x^2\cos x dx = x^2\sin x - 2\int x\sin x dx + c$

Let u = x and $\frac{dv}{dx} = \sin x$

$\frac{du}{dx} = 1$, v = -cosx

$\int x^2cosxdx$
$= x^2sinx - 2[-xcosx - \int -cosxdx] + c$
$= x^2sinx - 2[-xcosx + \int cosxdx] + c$
$= x^2sinx - 2[-xcosx + sinx] + c$
$= x^2sinx + 2xcosx - 2sinx + c$

Or using basic technique

Sign	Differentiate	Integrates
+	x^2	$cosx$
-	2x	$sinx$
+	2	$-cosx$
-	0	$-sinx$

$\int x^2cosxdx$ =x^2sins+ 2xcosx – 2sinx + c

(iii) $\int x^2sin^2xdx$

Solution

Let u = x^2 and $\frac{dv}{dx} = sin^2x = \frac{1}{2}(1 - cos2x)$

$\frac{du}{dx} = 2x$, v = $\frac{1}{2}\left(x - \frac{1}{2}sin2x\right)$

$\int x^2sin^2xdx$
$=\frac{1}{2}x^2\left(x - \frac{1}{2}sin2x\right) - \int 2x.\frac{1}{2}\left(x - \frac{1}{2}sin2x\right)dx$
$=\frac{1}{2}x^2\left(x - \frac{1}{2}sin2x\right) - \int x^2dx + \frac{1}{2}\int xsin2xdx$
$=\frac{1}{2}x^2\left(x - \frac{1}{2}sin2x\right) - \frac{1}{3}x^3 + \frac{1}{2}\int xsin2xdx$

Let u = x and $\frac{dv}{dx} = sin2x$

$\frac{du}{dx} = 1$and v = -$\frac{1}{2}$cos2x

$\int xsin2xdx$ =-$\frac{1}{2}$xcos2x+$\frac{1}{4}$sin2x+c

Substituting for$\int xsin2xdx$

$\int x^2sin^2xdx$
$=\frac{1}{2}x^2\left(x - \frac{1}{2}sin2x\right) - \frac{1}{3}x^3 + \frac{1}{2}\left[-\frac{1}{2}\text{xcos2x} + \frac{1}{4}\text{sin2x}\right]$
$=\frac{1}{2}x^2\left(x - \frac{1}{2}sin2x\right) - \frac{1}{3}x^3 - \frac{1}{4}\text{xcos2x} + \frac{1}{8}\text{sin2x} + c$
$= \frac{1}{6}x^3 - \frac{1}{4}x^2sin2x - \frac{1}{4}xcos2x + \frac{1}{8}sin2x + c$

Or by using basic technique

Sign	Differentiate	Integrates
+	x^2	$sin^2x = \frac{1}{2}(1 - cos2x)$
-	2x	$\frac{1}{2}x - \frac{1}{4}sins2x$
+	2	$\frac{1}{4}x^2 + \frac{1}{8}cos2x$
-	0	$\frac{1}{12}x^3 + \frac{1}{16}sin2x$

$\int x^2sin^2xdx$
$= \frac{1}{2}x^2\left(x - \frac{1}{2}sin2x\right) - 2x\left(\frac{1}{4}x^2 + \frac{1}{8}cos2x\right) +$
$2\left(\frac{1}{12}x^3 + \frac{1}{16}sin2x\right)$
$= \frac{1}{6}x^3 - \frac{1}{4}x^2sin2x - \frac{1}{4}xcos2x + \frac{1}{8}sin2x + c$

(iv) $\int xcos^2xdx$

Solution

$\int xcos^2xdx$

Let u = x and $\frac{dv}{dx} = cos^2x = \frac{1}{2} + \frac{1}{2}cos2x$

$\frac{du}{dx} = 1$, v =$\frac{1}{2}\left(x + \frac{1}{2}sin2x\right)$

$\int xcos^2xdx$
$= \frac{1}{2}x\left(x + \frac{1}{2}sin2x\right) - \int 1.\frac{1}{2}\left(x + \frac{1}{2}sin2x\right)$ dx
$= \frac{1}{2}x\left(x + \frac{1}{2}sin2x\right) - \frac{1}{2}\int xdx - \frac{1}{4}\int sin2x\text{dx}$
$= \frac{1}{2}x^2 + \frac{1}{4}sin2x - \frac{1}{4}x^2 + \frac{1}{8}cos2x + c$
$= \frac{1}{4}x^2 + \frac{1}{4}sin2x + \frac{1}{8}cos2x + c$

(b) Evaluate

(i) $\int_0^{\frac{\pi}{2}} x^2cosxdx$

Solution

$\int_0^{\frac{\pi}{2}} x^2cosxdx$

=$[x^2sinx + 2xcosx - 2sinx]_0^{\frac{\pi}{2}}$

=$\left[\frac{\pi^2}{4}sin\frac{\pi}{2} + 2.\frac{\pi}{2}cos\frac{\pi}{2} - 2sin\frac{\pi}{2}\right] - 0$

=$\left(\frac{\pi^2}{4} - 2\right) = 0.4674$

(ii) $\int_0^{\frac{\pi}{4}} xtan^2xdx$

Solution

Let u = x and $\frac{dv}{dx} = tan^2x = sec^2x - 1$

$\frac{du}{dx} = 1$; v = tanx - x

$\int xtan^2xdx$=xtanx - x^2-$\int(tanx - x)dx$
=xtanx - x^2+lncosx+$\frac{1}{2}x^2 + c$

=xtanx +lncosx$-\frac{1}{2}x^2 + c$

Or by using basic technique

Sign	Differentiate	Integrates
+	x	$tan^2x = sec^2x - 1$
-	1	tanx – x
+	0	-Incosx$-\frac{1}{2}x^2 +$

$\int xtan^2xdx$=xtanx - x^2+Incosx+$\frac{1}{2}x^2 + c$

=xtanx +Incosx$-\frac{1}{2}x^2 + c$

Hence;

$$\int_0^{\frac{\pi}{4}} xtan^2xdx = \left[\text{xtanx } + \text{Incosx} - \frac{1}{2}x^2\right]_0^{\frac{\pi}{4}}$$

$$= \left[\frac{\pi}{4}\tan\frac{\pi}{4} + \text{Incos}\frac{\pi}{4} - \frac{1}{2}\left(\frac{\pi}{4}\right)^2\right] - 0$$

$$= 0.1304$$

Revision Exercise 9

1. Integrate each of the following
   (a) $\int xsin2xdx$
   $\left[-\frac{x}{2}cos2x + \frac{1}{4}sin2x + c\right]$
   (b) $\int x^2sinxdx$
   $[-x^2cosx + 2xsinx + 2cosx + c]$
   (c) $\int(x + 1)^2sinxdx$
   $[(1 - 2x - x^2)cosx = 2(x + 1)sinx + c]$
   (d) $\int x^2sinxcosxdx$
   $\left[\frac{1}{8}cos2x(1 - 2x^2) + \frac{1}{4}xsin2x + c\right]$
   (e) $\int x^3 \cos x^2\, dx$
   $\left[\frac{1}{2}x^2 \sin x^2 + \frac{1}{2}\cos x^2 + c\right]$
   (f) $\int(xcosx)^2dx$
   $\left[\frac{1}{6}x^3 + \frac{1}{8}(2x^2 - 1) + \frac{1}{4}xsin2x + c\right]$
2. Evaluate
   (a) $\int_0^{\pi} x^2sinxdx$ [5.8696]
   (b) $\int_0^{\pi} x^2cos2xdx$ [0.0584]
   (c) $\int_0^{\frac{\pi}{4}} xtan^2xdx$ [0.1304]

C. Integration products of polynomials and exponential functions by parts

Examples 19

(a) Find
(i) $\int xe^xdx$

Solution

Let u = x and $\frac{dv}{dx} = e^x$

$\frac{du}{dx} = 1$; v =e^x

$$\int xe^xdx = xe^x - \int 1.e^xdx$$

$$= xe^x - e^x + c$$

Or by using basic technique

Sign	Differentiate	Integrates
+	x	e^x
-	1	e^x
+	0	e^x

$\int xe^xdx = xe^x - e^x + c$

(ii) $\int xe^{-x}dx$

Solution

Let u = x and $\frac{dv}{dx} = e^{-x}$

$\frac{du}{dx} = 1$; v =$-e^{-x}$

$$\int xe^xdx = -xe^{-x} - \int 1.-e^{-x}dx$$
$$= -xe^{-x} + \int e^{-x}dx$$

$$= -xe^x - e^x + c$$

Or by using basic technique

Sign	Differentiate	Integrates
+	x	e^{-x}
-	1	$-e^{-x}$
+	0	e^x

$\int xe^xdx = -xe^x - e^x + c$

(iii) $\int xe^{3x}dx$

Solution

Let u = x and $\frac{dv}{dx} = e^{3x}$

$\frac{du}{dx} = 1$; v =$\frac{1}{3}e^{3x}$

$$\int xe^{3x}dx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + c$$

Or by using basic technique

Sign	Differentiate	Integrates
+	x	e^{3x}
-	1	$\frac{1}{3}e^{3x}$
+	0	$\frac{1}{9}e^{3x}$

$\int xe^xdx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + c$

(b) Find

(i) $\int x.2^x dx$

Solution

Let u = x and $\frac{dv}{dx} = 2^x$

$\frac{du}{dx} = 1$; v = $\frac{2^x}{In2}$

$\int x.2^x dx = \frac{x.2^x}{In2} - \frac{1}{In2}\int 2^x dx$

$= \frac{x.2^x}{In2} - \frac{1}{In2}\left(\frac{2^x}{In2}\right) + c$

$= \frac{2x}{In2}(x-1) + c$

(ii) $\int 3^{\sqrt{(2x-1)}} dx$

Solution

Let p $= \sqrt{(2x-1)}$, $p^2 = 2x - 1$

2pdp = 2dx

pdp = dx

$\Rightarrow \int 3^{\sqrt{(2x-1)}} dx = \int 3^p.pdp$

Let u =p and $\frac{dv}{dp} = 3^p$

$\frac{du}{dp} = 1$, v = $\frac{3^p}{In3}$

$\int 3^p.pdp = \frac{3^p.p}{In3} - \frac{1}{In3}\int 3^p dp$

$= \frac{3^p.p}{In3} - \frac{1}{In3}\left(\frac{3^p}{In3}\right) + c$

$\therefore \int 3^{\sqrt{(2x-1)}} dx$

$= \frac{\sqrt{(2x-1)}3^{\sqrt{(2x-1)}}}{In3} - \frac{1}{In3}\left(\frac{3^{\sqrt{(2x-1)}}}{In3}\right) + c$

$= \frac{3^{\sqrt{(2x-1)}}}{In3}\left(\sqrt{(2x-1)} - \frac{1}{In3}\right) + c$

(c) Evaluate

(i) $\int_0^1 xe^{-x} dx$

Solution

$\int_0^1 xe^{-x} dx = [-xe^{-x} - e^{-x}]_0^1$

$= (-e^{-1} - e^{-1}) - (0 - e^0)$

$= -2e^{-1} + 1$

$= 1 - \frac{2}{e}$

$= 0.2642$

(ii) $\int_0^1 xe^{3x} dx$

Solution

$\int_0^1 xe^{3x} dx = \left[\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x}\right]_0^1$

$= \left[\frac{1}{3}e^3 - \frac{1}{9}e^3\right] - \left[0 - \frac{1}{9}e^0\right]$

$= \frac{2}{9}e^3 + \frac{1}{9} = 4.5746$

Revision exercise 10

1. Integrate each of the following with respect to x
   (a) xe^{3x} $\left[\frac{e^{3x}}{9}(3x-1) + c\right]$
   (b) x^2e^x $[e^x(x^2 - 2x + 2) + c]$
   (c) $x^3e^{x^2}$ $\left[\frac{e^{x^2}}{2}(x^2 - 1) + c\right]$
   (d) x^2e^{-2x} $\left[-\frac{e^{-2}}{4}(2x^2 + 2x + 1) + c\right]$
   (e) $\frac{x^2}{e^{-x^3}}$ $\left[-\frac{1}{3}e^{-x^3} + c\right]$
   (f) $e^x(3 + e^x)^2$ $\left[\frac{1}{3}(3 + e^x)^3 + c\right]$
2. Evaluate each of the following
   (a) $\int_0^1 x^2 e^{2x} dx$ [1.5973]
   (b) $\int_0^1 (x-1)e^x dx$ [2]

D. Integration products of polynomials and inverse trigonometric functions by parts

Example 20

(a) Find

(i) $\int sin^{-1}xdx$

Solution

$\int sin^{-1}xdx = \int 1.sin^{-1}xdx$

Let u=$sin^{-1}x$ and $\frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$; v = x

$\int sin^{-1}xdx = xsin^{-1}x - \int \frac{x}{\sqrt{1-x^2}} dx$

For $\int \frac{x}{\sqrt{1-x^2}} dx$

Let u = $1 - x^2$

Du= -2x

$-\frac{1}{2x}du = dx$

$\int \frac{x}{\sqrt{1-x^2}}dx = \int \frac{x}{u^{\frac{1}{2}}}.-\frac{1}{2x}du$

$= -\frac{1}{2}\int u^{-\frac{1}{2}}du$

$= -\frac{1}{2}\left[2u^{\frac{1}{2}} + c\right] = -u^{\frac{1}{2}} + c$

By substitution

$\int sin^{-1}xdx = xsin^{-1}x + u^{\frac{1}{2}} + c$

$\therefore \int sin^{-1}xdx = xsin^{-1}x + \sqrt{1-x^2} + c$

(ii) $\int cos^{-1}\left(\frac{x}{a}\right)dx$

Solution

$\int cos^{-1}\left(\frac{x}{a}\right)dx = \int 1.cos^{-1}\left(\frac{x}{a}\right)dx$

Let u = $cos^{-1}\left(\frac{x}{a}\right)$ and $\frac{dv}{dx} = 1$

$\frac{du}{dx} = -\frac{1}{\sqrt{a^2-x^2}}$

$\int cos^{-1}\left(\frac{x}{a}\right)dx = xcos^{-1}\left(\frac{x}{a}\right) + \int \frac{1}{\sqrt{a^2-x^2}}$ dx

For $\int \frac{x}{\sqrt{a^2-x^2}}dx$

Let u = $a^2 - x^2$

Du= -2x

$-\frac{1}{2x}du = dx$

$\int \frac{x}{\sqrt{a^2-x^2}}dx = \int \frac{x}{u^{\frac{1}{2}}}.-\frac{1}{2x}du$

$= -\frac{1}{2}\int u^{-\frac{1}{2}}du$

$= -\frac{1}{2}\left[2u^{\frac{1}{2}} + c\right] = -u^{\frac{1}{2}} + c$

By substitution

$\int cos^{-1}\left(\frac{x}{a}\right)dx = xcos^{-1}x + u^{\frac{1}{2}} + c$

$\therefore \int cos^{-1}\left(\frac{x}{a}\right)dx = xcos^{-1}x + \sqrt{a^2-x^2} + c$

(iii) $\int xtan^{-1}x\,dx$

Solution

Let u = $tan^{-1}x$ and $\frac{dv}{dx} = x$

$\frac{du}{dx} = \frac{1}{1+x^2}$; v = $\frac{1}{2}x^2$

$\int xtan^{-1}x\,dx = \frac{1}{2}x^2tan^{-1}x - \frac{1}{2}\int \frac{x^2}{1+x^2}$dx

For $\int \frac{x^2}{1+x^2}$dx $= \int \left(1 - \frac{1}{1+x^2}\right)dx$

$= \int dx - \int \frac{1}{1+x^2}dx$

$= x - tan^{-1}x + c$

By substitution

$\int xtan^{-1}x\,dx$
$= \frac{1}{2}x^2tan^{-1}x - \frac{1}{2}[x - tan^{-1}x] + c$
$= \frac{1}{2}x^2tan^{-1}x - \frac{1}{2}x + \frac{1}{2}tan^{-1}x + c$
$= \frac{1}{2}[(x^2+1)tan^{-1}x - x] + c$

(b) Evaluate $\int_0^1 xsin^{-1}xdx$

Solution

Let u = sin^{-1}x and $\frac{dv}{dx} = x$

$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$; v= $\frac{1}{2}x^2$

$\int xsin^{-1}xdx = \frac{1}{2}x^2sin^{-1}x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}dx$

For $\int \frac{x^2}{\sqrt{1-x^2}}dx$

Let x = sinθ => dx = cosθdθ

$\int \frac{x^2}{\sqrt{1-x^2}}dx = \int \frac{sin^2\theta}{\sqrt{1-sin^2\theta}}.cos\theta d\theta$

$= \int \frac{sin^2\theta}{cos\theta}.cos\theta d\theta$

$= \int sin^2\theta\,d\theta$

$= \frac{1}{2}\int(1 - cos2\theta)\,d\theta$

$= \frac{1}{2}\theta - \frac{1}{4}sin2\theta + c$

$= \frac{1}{2}\theta - \frac{1}{4}(2sin\theta cos\theta) + c$

$= \frac{1}{2}\theta - \frac{1}{2}(sin\theta cos\theta) + c$

$= \frac{1}{2}sin^{-1}x - \frac{1}{2}x\sqrt{1-x^2} + c$

$\therefore \int xsin^{-1}xdx$

$= \frac{1}{2}x^2sin^{-1}x - \frac{1}{4}sin^{-1}x + \frac{1}{4}x\sqrt{1-x^2}$

$\int_0^1 xsin^{-1}xdx$

$= \left[\frac{1}{2}x^2sin^{-1}x - \frac{1}{4}sin^{-1}x + \frac{1}{4}x\sqrt{1-x^2}\right]_0^1$

$= \left[\frac{1}{2}.1.sin^{-1}1 - \frac{1}{4}sin^{-1}(1)\right] - (0)$

$= \frac{\pi}{4} - \frac{\pi}{8} = \frac{\pi}{8}$

Revision exercise 11

1. Find the following integrals

(a) $\int tan^{-1}3xdx$

$\left[xtan^{-1}x - \frac{1}{6}In(1+9x^2) + c\right]$

(b) $\int \frac{xsin^{-1}x}{\sqrt{1-x^2}}dx$

$\left[xsin^{-1}x + \sqrt{1-x^{2+c}}\right]$

(c) $\int sec^{-1}x\,dx$

$\left[xsec^{-1}x - In\left(x+\sqrt{x^2-1}\right) + c\right]$

(d) $\int cot^{-1}xdx$

$\left[xcot^{-1}x + \frac{1}{2}In(1+x^2) + c\right]$

2. Evaluate

(a) $\int_0^1 sin^{-1}xdx$ $\quad \left[\frac{\pi}{2} - 1\right]$

(b) $\int_0^1 cos^{-1}xdx$ $\quad [1]$

E. Integration products of polynomials and logarithmic functions by parts

Example 21

(a) Integrate

(i) $\int Inx^2dx$

Solution

$\int Inx^2dx = \int 1.Inx^2dx$

Let u = Inx^2, $\frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{2x}{x^2} = \frac{2}{x}$; v = x

$\int Inx^2dx = xInx^2 - 2\int x.\frac{1}{x}dx$

$= xInx^2 - 2x + c$

$= 2xInx - 2x + c$

$\therefore \int Inx^2dx = 2xInx - 2x + c$

(ii) $\int xIn(x^2-1)dx$

Solution

Let u = $In(x^2-1)$ and $\frac{dv}{dx} = x$

$\frac{du}{dx} = \frac{2x}{x^2-1}$; $v = \frac{1}{2}x^2$

$\int xIn(x^2-1)dx$

$= \frac{1}{2}x^2In(x^2-1) - \int \frac{1}{2}x^2.\frac{2x}{x^2-1}dx$

$= \frac{1}{2}x^2In(x^2-1) - \int \frac{x^3}{x^2-1}dx$

For $\int \frac{x^3}{x^2-1}dx$

By using long division

$\frac{x^3}{x^2-1} = x + \frac{x}{x^2-1}$

⇨ $\int \frac{x^3}{x^2-1}dx = \int xdx + \int \frac{x}{x^2-1}dx$

$= \frac{1}{2}x^2 + \frac{1}{2}In(x^2-1) + c$

$\therefore \int xIn(x^2-1)dx$

$= \frac{1}{2}x^2In(x^2-1) - \frac{1}{2}x^2 - \frac{1}{2}In(x^2-1) + c$

(iii) $\int x^{-3}Inxdx$

Solution

Let u = Inx and $\frac{dv}{dx} = x^{-3}$

$\frac{du}{dx} = \frac{1}{x}$ and v = $-\frac{1}{2}x^{-2}$

$\int x^{-3}Inxdx = -\frac{1}{2}x^{-2}Inx + \frac{1}{2}\int \frac{1}{x}.x^{-2}$ dx

$= -\frac{1}{2}x^{-2}Inx + \frac{1}{2}\int x^{-3}dx$

$= -\frac{1}{2}x^{-2}Inx - \frac{1}{4}x^{-2} + c$

$= -\frac{1}{4}x^{-2}(Inx+1) + c$

(b) Evaluate $\int_1^{10} x\log_{10}xdx$

Solution

Changing from base 10 to base e

$\log_{10}x = \frac{Ine}{In10}$

$\int_1^{10} x\log_{10}xdx = \frac{1}{In10}\int_1^{10} xInxdx$

Let u= Inx; $\frac{dv}{dx} = x$

$\frac{du}{dx} = \frac{1}{x}$; $v = \frac{1}{2}x^2$

$$\frac{1}{In10}\int_1^{10} xInxdx = \frac{1}{In10}\left[\frac{1}{2}x^2Inx - \frac{1}{4}x^2\right]_1^{10}$$

$= \frac{1}{In10}\left[(50In10 - 25) - \frac{1}{4}\right]$

$= \frac{1}{In10}\left[50In10 - \frac{99}{4}\right] \quad = 50 - \frac{99}{4In10}$

Revision exercise 12

1. Integrate each of the following
   (a) $xInx$ $\left[\frac{x^2}{4}(2Inx-1)+c\right]$
   (b) x^2Inx $\left[\frac{x^2}{9}(3Inx-1)+c\right]$
   (c) $\sqrt{x}Inx$ $\left[\frac{2}{9}\sqrt{x^3}(3Inx-2)+c\right]$
   (d) $(Inx)^2$ $[x(2-2Inx+(Inx)^2)+c]$
   (e) $\frac{Inx}{x^2}$ $\left[-\frac{1}{x}(Inx+1)+c\right]$
   (f) $3^x x$ $\left[\frac{3^x}{(In3)^2}(xIn3-1)+c\right]$
   (g) $x(Inx)^2$
   $\left[\frac{1}{4}x^2(1-2Inx+2(Inx)^2)+c\right]$
2. Evaluate the following
   (a) $\int_2^4 x^3Inxdx$ $[70.9503]$
   (b) $\int_2^4(x-1)In(2x)dx$ $[1.0794]$
   (c) $\int_1^4\frac{Inx}{x^2}dx$ $[0.4034]$

F. Integration of products of exponential and trigonometric functions by parts

Example 22

(a) Find
(i) $\int e^{-x}sinxdx$

Solution

Taking I = $\int e^{-x}sinxdx$

Let u = e^{-x}, $\frac{dv}{dx}=sinx$

$\frac{du}{dx}=-e^{-x}$; v = -cosx

$=> I=-e^{-x}cosx-\int -e^{-x}.-cosxdx$

$I=-e^{-x}cosx-\int e^{-x}.cosxdx$ ...(*)

For $\int e^{-x}.cosxdx$

Let u = e^{-x}, $\frac{dv}{dx}=cosx$

$\frac{du}{dx}=-e^{-x}$; v = sinx

$\int e^{-x}.cosxdx=e^{-x}sinx-\int -e^{-x}sinx$

$=e^{-x}sinx+I$ ....(**)

Substituting for (**) in equation (*)

$I=-e^{-x}cosx-e^{-x}sinx-I$

$2I=-e^{-x}cosx-e^{-x}sinx-I+A$

$I=-\frac{1}{2}e^{-x}(cosx+sinx)+c$

$\therefore\int e^{-x}sinxdx=-\frac{1}{2}e^{-x}(cosx+sinx)+c$

Or by using basic technique

sign	Differentiate	integrate
+	e^{-x}	sinx
-	$-e^{-x}$	-cosx
+	e^{-x}	-sinx

$I=-e^{-x}cosx-e^{-x}sinx-\int e^{-x}sinx$

$I=-e^{-x}cosx-e^{-x}sinx-I$

$2I=-e^{-x}cosx-e^{-x}sinx+A$

$I=-\frac{1}{2}e^{-x}(cosx+sinx)+c$

$\therefore\int e^{-x}sinxdx=-\frac{1}{2}e^{-x}(cosx+sinx)+c$

(ii) $\int e^{2x}cos3xdx$

Solution

Taking I = $\int e^{2x}cos3xdx$

Let u = e^{2x}, $\frac{dv}{dx}=cos3x$

$\frac{du}{dx}=2e^{2x}$; v = $\frac{1}{3}$sin3x

$I=\frac{1}{3}e^{2x}sin3x-\int 2e^{2x}.\frac{1}{3}sin3xdx$

$I=\frac{1}{3}e^{2x}\text{sin3x}-\frac{2}{3}\int e^{2x}\text{sin3xdx}$ ... (*)

For $\int e^{2x}$sin3xdx

Let u = e^{2x}, $\frac{dv}{dx}=sin3x$

$\frac{du}{dx}=2e^{2x}$; v =- $\frac{1}{3}$cos3x

$\int e^{2x}$ sin3xdx

$=-\frac{1}{3}e^{2x}\text{cos3x}-\int 2e^{2x}.-\frac{1}{3}\text{cos3xdx}$

$=-\frac{1}{3}e^{2x}\text{cos3x}+\frac{2}{3}\int e^{2x}\text{cos3xdx}$

$=-\frac{1}{3}e^{2x}\text{cos3x}+\frac{2}{3}I$ ....(**)

Substituting (**) into (*)

$I=\frac{1}{3}e^{2x}\text{sin3x}-\frac{2}{3}\left(-\frac{1}{3}e^{2x}\text{cos3x}+\frac{2}{3}I\right)$

$= \frac{1}{3}e^{2x}\text{sin3x} + \frac{2}{9}e^{2x}\text{cos3x} + \frac{4}{9}I + \text{c}$

$\frac{13}{9}I = \frac{1}{3}e^{2x}\text{sin3x} + \frac{2}{9}e^{2x}\text{cos3x} + \text{A}$

$I = \frac{3}{13}e^{2x}\text{sin3x} + \frac{2}{13}e^{2x}\text{cos3x} + \text{c}$

$I = \frac{1}{13}e^{2x}(3sin3x + 2cos3x) + c$

$\therefore \int e^{2x}cos3xdx = \frac{1}{13}e^{2x}(3sin3x + 2cos3x) + c$

(iii) $\int e^{3x}sin2xdx$

Solution

Taking I = $\int e^{3x}sin2xdx$

Let u = e^{3x}, $\frac{dv}{dx} = sin2x$

$\frac{du}{dx} = 3e^{3x}$; v = $-\frac{1}{2}$cos2x

$I = -\frac{1}{2}e^{3x}cos2x - \int 3e^{3x}.-\frac{1}{2}cos2xdx$

$I = -\frac{1}{2}e^{3x}\text{cos2x} + \frac{3}{2}\int e^{3x}\text{cos2xdx}$ … (*)

For $\int e^{3x}$cos2xdx

Let u = e^{3x}, $\frac{dv}{dx} = cos2x$

$\frac{du}{dx} = 3e^{3x}$; v = $\frac{1}{2}$sin2x

$\int e^{3x}$ cos2xdx

$= \frac{1}{2}e^{3x}\text{sin2x} - \int 3e^{3x}.\frac{1}{2}\text{sin2xdx}$

$= \frac{1}{2}e^{3x}\text{sin2x} - \frac{3}{2}\int e^{3x}\text{sin2xxdx}$

$= \frac{1}{2}e^{3x}\text{sin2x} - \frac{3}{2}I$ ………………….(**)

Substituting (**) into (*)

$I = -\frac{1}{2}e^{3x}cos2x + \frac{3}{2}\left(\frac{1}{2}e^{3x}\text{sin2x} - \frac{3}{2}I\right)$

$= -\frac{1}{2}e^{3x}cos2x + \frac{3}{4}e^{3x}\text{sin2x} - \frac{9}{4}I + \text{c}$

$\frac{13}{4}I = -\frac{1}{2}e^{3x}cos2x + \frac{3}{4}e^{3x}\text{sin2x} + \text{A}$

$I = -\frac{2}{13}e^{3x}\text{cos2x} + \frac{3}{13}e^{3x}\text{sin2x} + \text{c}$

$I = \frac{1}{13}e^{3x}(3sin2x - 2cos2x) + c$

$\therefore \int e^{3x}sin2xdx = \frac{1}{13}e^{2x}(3sin2x - 2cos3x) + c$

Or using basic technique

sign	Differentiate	integrate
+	e^{3x}	sin2x
-	$3e^{3x}$	$-\frac{1}{2}$cos2x
+	$9e^{3x}$	$-\frac{1}{4}$sinx

$I = -\frac{1}{2}e^{3x}cos2x + \frac{3}{4}e^{3x}\text{sin2x} - \frac{9}{4}I + \text{c}$

$\frac{13}{4}I = -\frac{1}{2}e^{3x}cos2x + \frac{3}{4}e^{3x}\text{sin2x} + \text{A}$

$I = -\frac{2}{13}e^{3x}\text{cos2x} + \frac{3}{13}e^{3x}\text{sin2x} + \text{c}$

$I = \frac{1}{13}e^{3x}(3sin2x - 2cos2x) + c$

$\therefore \int e^{3x}sin2xdx = \frac{1}{13}e^{2x}(3sin2x - 2cos3x) + c$

(b) Evaluate $\int_0^\infty e^{-2x}sin3xdx$

Solution

Taking I =$e^{-2x}sin3xdx$

Let u = e^{-2x}, $\frac{dv}{dx} = sin3x$

$\frac{du}{dx} = -2e^{-2x}$; v = $-\frac{1}{3}$cos3x

$I = -\frac{1}{3}e^{-2x}\text{cos3x} - \int 2e^{2x}.\frac{1}{3}\text{sin3xdx}$

$I = \frac{1}{3}e^{3x}\text{sin3x} - \frac{2}{3}\int e^{-2x}\text{cos3xdx}$ …… (*)

For $\int e^{-2x}$cos3xdx

Let u = e^{-2x}, $\frac{dv}{dx} = cosx$

$\frac{du}{dx} = -2e^{2x}$; v = $\frac{1}{3}$sin3x

$\int e^{-2x}$cos3xdx

$= -\frac{1}{3}e^{-2x}\text{sin3x} + \frac{2}{3}\int e^{2x}\text{sins3xdx}$

$= -\frac{1}{3}e^{-2x}\text{sin3x} + \frac{2}{3}I$ ……..(**)

Substituting (**) into (*)

$I = -\frac{1}{3}e^{-2x}\text{cos3x} - \frac{2}{3}\left(\frac{1}{3}e^{-2x}\text{cos3x} + \frac{2}{3}I\right)$

$= -\frac{1}{3}e^{-2x}\text{cos3x} - \frac{2}{9}e^{-2x}\text{sin3x} - \frac{4}{9}I + \text{c}$

$\frac{13}{9}I = -\frac{1}{3}e^{-2x}\text{scos3x} - \frac{2}{9}e^{-2x}\text{sin3x} + \text{c}$

$I = -\frac{1}{13}e^{-2x}(3cos3x + 2sin3x) + c$

$\therefore \int e^{-2x}sin3xdx$

$= -\frac{1}{13}e^{-2x}(3cos3x + 2sin3x) + c$

⇨ $\int_0^{\infty} e^{-2x}sin3xdx$

$=\left[-\frac{1}{13}e^{-2x}(3cos3x + 2sin3x)\right]_0^{\infty}$

$=\frac{3}{13}$ since $e^{\infty} = 0$

Revision exercise 13

Integrate each of the following with respect to x

(a) $e^x cosx$ $\left[\frac{1}{2}e^x(sinx + cosx) + c\right]$

(b) $e^x sinxx$ $\left[\frac{1}{2}e^x(sinx - cosx) + c\right]$

(c) $e^{ax}cosbx$ $\left[\frac{e^{ax}}{b^2+a^2}(acosbx + bsinbx) + c\right]$

(d) $e^{3x}sin2x$ $\left[\frac{1}{13}e^{3x}(3sin2x - 2cos2x) + c\right]$

G. Integration of products of trigonometric functions by parts

A student should take note of the following

(i) $\int tanxdx = In(\sec x)$+ c

Proof

$\frac{d}{dx}In(sex)dx = \frac{secxtanx}{secx} = tanx$

Hence$\int tanxdx = In(\sec x)$+ c

(ii) $\int cosecxdx = -In(cosecx + cotx) + c$

Proof

$$\frac{d}{dx}In(cosecx + cotx)dx = \frac{-cosecxcotx - cosec^2x}{cosecx + cotx}$$
$$= \frac{-cosecx(cotx - cosecx)}{cosecx + cotx}$$
$$= -\text{cosecx}$$

$\therefore \int cosecxdx = -In(cosecx + cotx) + c$

(iii) $\int secxdx = In(secx + tanx) + c$

Proof

$$\frac{d}{dx}In(secx + tanx)dx = \frac{secxtanx + sec^2x}{secx + tanx}$$
$$= \frac{secx(In(secx + tanx))}{secx + tanx}$$
$$= \text{secx}$$

$\therefore \int secxdx = In(secx + tanx) + c$

(iv) $\int cotxdx = In(sinx) + c$

Proof

$\frac{d}{dx}In(sinx) = \frac{cosx}{sinx} = cotx$

$\therefore \int cotxdx = In(sinx) + c$

Example 22

(a) Find

(i) $\int sec^3xdx$

Solution

Taking $I = \int sec^3xdx = \int secxsec^2xdx$

Let u = secx and $\frac{dv}{dx} = sec^2x$

$\frac{du}{dx} = secxtanx$; v = tanx

$I = secxtanx - \int(secxtanx)tanxdx$

$= secxtanx - \int secxtan^2xdx$

$= secxtanx - \int secx(sec^2x - 1)dx$

$= secxtanx - \int sec^3xdx + \int secx\, dx$

$= secxtanx -$ I + In(secx+tanx)+c

2I = $secxtanx$ + In(secx+tanx)+c

$I = \frac{1}{2}[secxtanx + \text{In}(\text{secx} + \text{tanx})] + c$

$\therefore \int sec^3xdx$

$= \frac{1}{2}[secxtanx + \text{In}(\text{secx} + \text{tanx})] + c$

(ii) $\int cosec^3xdx$

Taking $I = \int cosec^3xdx = \int cosecxcosec^2xdx$

Let u = cosecx and $\frac{dv}{dx} = cosec^2x$

$\frac{du}{dx} = cosecxcotx$; v = -cotx

$I = -cotxcosecx - \int(secxcotx)cotxdx$

$= -cosecxcotx - \int secxcot^2xdx$

$= -cosecxcotx - \int cosecx(cosec^2x - 1)dx$

$= -cosecxtanx - \int cosec^3xdx + \int cosecx\, dx$

$= -cotxcosecx - I + \int cosecx\, dx$+A

$2I = -cotxcosex) - In(cosecx + cotx)$+c

$I = -\frac{1}{2}[cotxcosecx + In(cosecx + cotx)] + c$

$\therefore \int cosec^3xdx$

$= -\frac{1}{2}[cotxcosecx + In(cosecx + cotx)] + c$

(b) Show that $\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} sec^3xtanx = \frac{8}{27}(9 - \sqrt{3})$

Solution

$\int sec^3xtanxdx = \int sec^2xsecxtanx$

Let u $= sec^2x$ and $\frac{dv}{dx} = secxtanx$

$\frac{du}{dx} = 2sec^2x; v = secx$

$\int sec^3xtanxdx = sec^3x - 2\int sec^3xtanxdx$

$I - sec^3x - 2I + c$

$3I = sec^3x$

$I = \frac{1}{3}sec^3x + c$

$\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} sec^3xtanx = \left[\frac{1}{3}sec^3x\right]_{-\frac{\pi}{6}}^{\frac{\pi}{3}}$

$= \frac{1}{3}\left[sec^3\left(\frac{\pi}{3}\right) - sec^3\left(-\frac{\pi}{6}\right)\right]$

$= \frac{1}{3}\left[8 - \frac{8}{3\sqrt{3}}\right] = \frac{1}{3}\left[8 - \frac{8\sqrt{3}}{9}\right]$

$= \frac{8}{3}\left[1 - \frac{\sqrt{3}}{9}\right]$

$= \frac{8}{27}\left[9 - \sqrt{3}\right]$

Revision exercise 14

Integrate each of the following with respect to x

1. sec^3x
   $\left[\frac{1}{2}[secxtanx + \text{In}(\text{secx} + \text{tanx})] + c\right]$
2. $cosec^3x$
   $\left[-\frac{1}{2}[cotxcosecx + In(cosecx + cotx)] + c\right]$
3. sec^3xtanx
   $\left[\frac{1}{3}sec^3x\right]$

Integration using t- substitution

Case 1

We know that if t = $tan\frac{1}{2}\theta$, then

$\sin\theta = \frac{2t}{1+t^2}$ and

$cos\theta = \frac{1-t^2}{1+t^2}$

Generally

If t = $tan\frac{1}{2}k\theta$, then

$\sin k\theta = \frac{2t}{1+t^2}$ and

$cosk\theta = \frac{1-t^2}{1+t^2}$

Example 23

Find

(a) $\int cosecxdx$

Solution

Let t = tan$\frac{1}{2}x$

$dt = sec^2\frac{1}{2}xdx$

$2dt = (1 + t^2)dx$

$dx = \frac{2}{1+t^2}dt$

$\int cosecxdx = \int \frac{1}{sinx}dx = \int \frac{1+t^2}{2t}.\frac{2}{1+t^2}dt$
$= \int \frac{1}{t}dt = Int + c$

$\therefore \int cosecxdx = In\left(tan\frac{1}{2}x\right) + c$

(b) $\int secxdx$

Solution

Let t = tan$\frac{1}{2}x$

$dt = sec^2\frac{1}{2}xdx$

$2dt = (1 + t^2)dx$

$dx = \frac{2}{1+t^2}dt$

$\int secxdx = \int \frac{1}{cosx}dx = \int \frac{1+t^2}{1-t^2}.\frac{2}{1+t^2}dt$
$= \int \frac{2}{1-t^2}dt = \int \frac{2}{(1+t)(1-t)}dt$

Let $\frac{2}{(1+t)(1-t)} = \frac{A}{1+t} + \frac{B}{1-t}$

2= A(1 – t) + B(1 + t)

Putting t = 1, B = 1

Putting t = -1; A= 1

=>$\frac{2}{(1+t)(1-t)} = \frac{1}{1+t} + \frac{1}{1-t}$

$\int \frac{2}{1-t^2}dt = \int \frac{1}{1+t}dt + \int \frac{1}{1-t}dt$

$= In(1+t) - In(1-t) + c = In\left(\frac{1+t}{1-t}\right) + c$

$\therefore \int secxdx = In\left(\frac{1+tan\frac{1}{2}x}{1-tan\frac{1}{2}x}\right) + c$

(c) $\int sec3xdx$

Solution

Let t = $tan\frac{1}{2}(3x) = \tan\frac{3}{2}x$

$2dt = 3(1+t^2)dx$

$dx = \frac{2}{3(1+)}dt$

$\int sec3xdx = \int \frac{1}{cos3x}dx = \int \frac{1+t^2}{1-t^2}.\frac{2}{3(1+t^2)}dt$
$= \frac{2}{3}\int \frac{1}{1-t^2}dt = \frac{1}{3}\int \frac{2}{1-t^2}dt$

Let$\frac{2}{(1+t)(1-t)} = \frac{A}{1+t} + \frac{B}{1-t}$

2= A(1 – t) + B(1 + t)

Putting t = 1, B = 1

Putting t = -1; A= 1

=>$\frac{2}{(1+t)(1-t)} = \frac{1}{1+t} + \frac{1}{1-t}$

$\int \frac{2}{1-t^2}dt = \int \frac{1}{1+t}dt + \int \frac{1}{1-t}dt$

$= In(1+t) - In(1-t) + c = In\left(\frac{1+t}{1-t}\right) + c$

$\therefore \int sec3xdx = \frac{1}{3}In\left(\frac{1+tan\frac{1}{2}x}{1-tan\frac{1}{2}x}\right) + c$

(d) $\int \frac{1}{3-2cosx}dx$

Solution

Let t = $tan\frac{1}{2}x$

$dt = sec^2\frac{1}{2}xdx$

$2dt = (1+t^2)dx$

$dx = \frac{2}{1+t^2}dt$

$\int \frac{1}{3-2cosx}dx = \int \frac{1}{3-2\left(\frac{1-t^2}{1+t^2}\right)}.\frac{2}{1+t^2}dt = 2\int \frac{1}{1+5t^2}dt$

$= \frac{2}{\sqrt{5}}tan^{-1}(\sqrt{5}t) + c = \frac{2\sqrt{5}}{5}tan^{-1}(\sqrt{5}t) + c$

$\therefore \int \frac{1}{3-2cosx}dx = \frac{2\sqrt{5}}{5}tan^{-1}\left(\sqrt{5}tan\frac{1}{2}x\right) + c$

(e) $\int \frac{2}{3sin2x+4}dx$

Solution

Let t = $tanx$

$dt = sec^2xdx$

$dt = (1+t^2)dx$

$dx = \frac{1}{1+t^2}dt$

$\int \frac{2}{3sin2x+4}dx = \int \frac{2}{3\left(\frac{2t}{1+t^2}\right)+4}.\frac{1}{1+t^2}dt$

$= \int \frac{1}{2t^2+3t+2}dt = \int \frac{1}{\frac{7}{8}+2\left(t+\frac{3}{4}\right)^2}dt$

$= \frac{1}{\sqrt{2}}.\frac{\sqrt{8}}{\sqrt{7}}tan^{-1}\frac{\sqrt{2}\left(t+\frac{3}{4}\right)}{\sqrt{\left(\frac{7}{8}\right)}} + c$

$= \frac{2}{\sqrt{7}}tan^{-1}\frac{(4t+3)}{\sqrt{7}} + c$

$\therefore \int \frac{2}{3sin2x+4}dx = \frac{2\sqrt{7}}{7}tan^{-1}\left(\frac{4tanx+3}{\sqrt{7}}\right) + c$

(f) $\int \frac{2}{3+5cos\frac{1}{2}x}$

Solution

Let t = $tan\frac{1}{4}x$

$dt = \frac{1}{4}sec^2\frac{1}{4}xdx$

$dt = \frac{1}{4}(1+t^2)dx$

$dx = \frac{4}{1+t^2}dt$

(g) $\int \frac{2}{3+5cos\frac{1}{2}x} = \int \frac{2}{3+5\left(\frac{1-t^2}{1+t^2}\right)}.\frac{4}{1+t^2}dt$
$= \int \frac{2}{4-t^2}dt = \int \frac{2}{(2+t)(2-t)}dt$

Let $\frac{2}{(2+t)(2-t)} = \frac{A}{(2+t)} + \frac{B}{2-t}$

2= A(2 – t)+ B(2 + t)

Putting t = 2; B= $\frac{1}{2}$

Putting t = -2; A= $\frac{1}{2}$

$$\int \frac{2}{3+5cos\frac{1}{2}x}dx = \frac{1}{2}\int \frac{1}{2+t}dt + \frac{1}{2}\int \frac{1}{2-t}dt$$

$$= \frac{1}{2}In(2+t) - \frac{1}{2}In(2-t) + c$$

$$= \frac{1}{2}In\left(\frac{2+t}{2-t}\right) + c$$

$$\therefore \int \frac{2}{3+5cos\frac{1}{2}x}dx = \frac{1}{2}In\left(\frac{2+tan\frac{1}{4}x}{2-tan\frac{1}{4}x}\right) + c$$

Case II

When integrating fractional trigonometric functions containing the square of sinx, cosx, etc.

We use the

t-substitution, t = tanx

For sin^2kx or cos^2kx, we use t = tanx

Example 24

Find the integrals of the following

(a) $\int \frac{1}{4sin^2x-9cos^2x}dx$

Solution

Dividing numerator and denominator by cos^2x

$$\int \frac{1}{4sin^2x-9cos^2x}dx = \int \frac{sec^2x}{4tan^2x-9}dx$$

$$= \int \frac{1+tan^2x}{4tan^2x-9}dx$$

Let t = tanx

$$dt = sec^2xdx = (1+t^2)dx$$

$$dx = \frac{dt}{(1+t^2)}$$

$$\int \frac{1+tan^2x}{4tan^2x-9}dx = \int \frac{1+t^2}{4t^2-9}.\frac{dt}{(1+t^2)}$$

$$= \int \frac{1}{(2t+3)(2t-3)}dt$$

Let $\frac{1}{(2t+3)(2t-3)} = \frac{A}{(2t+3)} + \frac{B}{(2t-3)}$

1 = A(2t – 3) + B(2t + 3)

Putting t = $\frac{3}{2}$; B = $\frac{1}{6}$

Putting t = $-\frac{3}{2}$; A = $-\frac{1}{6}$

$$\Rightarrow \int \frac{1}{(2t+3)(2t-3)}dt$$

$$= \frac{1}{6}\int \frac{B}{(2t-3)}dt + \frac{1}{6}\int \frac{1}{(2t+3)}dt$$

$$= \frac{1}{6}.\frac{1}{2}In(2t-3) - \frac{1}{6}.\frac{1}{2}In(2t+3) + c$$

$$= \frac{1}{12}In\left(\frac{2t-3}{2t+3}\right) + c$$

$$\int \frac{1}{4sin^2x-9cos^2x}dx = \frac{1}{12}In\left(\frac{2tanx-3}{2tanx+3}\right) + c$$

(b) $\int \frac{1}{3+4sin^2 5x}dx$

Solution

Dividing by the numerator and denominator $cos^2 5x$

$$\int \frac{1}{3+4sin^2 5x}dx = \int \frac{sec^2 5x}{3sec^2 5x-4}dx$$

$$= \int \frac{1+tan^2 5x}{3+7tan^2 5x}dx$$

Let t = tan5x

$$dt = sec^2 5xdx = 5(1+t^2)dx$$

$$dx = \frac{dt}{5(1+t^2)}$$

$$\int \frac{1}{3+4sin^2 5x}dx = \int \left(\frac{1+t^2}{3+7t^2}\right).\frac{dt}{5(1+t^2)}$$

$$= \frac{1}{5}\int \frac{1}{3+7t^2}dt$$

$$= \frac{1}{5}.\frac{1}{\sqrt{7}}.\frac{1}{\sqrt{3}}tan^{-1}\left(\frac{\sqrt{7}}{\sqrt{3}}t\right) + c$$

$$= \frac{1}{5\sqrt{21}}tan^{-1}\left(\frac{\sqrt{7}}{\sqrt{3}}tan5x\right) + c$$

$$\therefore \int \frac{1}{3+4sin^2 5x}dx = \frac{\sqrt{21}}{105}tan^{-1}\left(\frac{\sqrt{21}}{3}tan5x\right) + c$$

(c) $\int \frac{sin^2 3x}{1+cos^2 3x}dx$

Solution

Dividing numerator and denominator by $cos^2 3x$

$$\int \frac{sin^2 3x}{1+cos^2 3x}dx = \int \frac{tan^2 3x}{sec^2 3x+1}dx = \int \frac{tan^2 3x}{2+tan^2 3x}dx$$

Let t = tan3x

$$dt = 3sec^2 3xdx = 3(1+t^2)dx$$

$dx = \frac{dt}{3(1+t^2)}$

$\int \frac{tan^2 3x}{2+tan^2 3x} dx = \int \frac{t^2}{2+t^2} \cdot \frac{dt}{3(1+t^2)}$

$= \frac{1}{3} \int \frac{t^2}{(2+t^2)(1+t^2)} dt$

Let $\frac{t^2}{(2+t^2)(1+t^2)} = \frac{Ax+B}{(2+t^2)} + \frac{Cx+D}{(1+t^2)}$

By equating coefficients and solving simultaneously

A = 2, C = -1, B = D = 0

$\int \frac{sin^2 3x}{1+cos^2 3x} dx = \int \frac{2}{(2+t^2)} dt - \int \frac{1}{(1+t^2)} dt$

$= \frac{1}{3}\left[\frac{2}{\sqrt{2}} tan^{-1}\left(\frac{t}{\sqrt{2}}\right) - tan^{-1} t\right] + c$

$= \frac{1}{3}\left[\frac{2}{\sqrt{2}} tan^{-1}\left(\frac{tan3x}{\sqrt{2}}\right) - tan^{-1}(tan3x)\right] + c$

$= \frac{\sqrt{2}}{3} tan^{-1}\left(\frac{tan3x}{\sqrt{2}}\right) - \frac{1}{3} tan^{-1}(tan3x) + c$

(d) $\int \frac{1}{cos2x - 3sin^2 x} dx$

Solution

$\int \frac{1}{cos2x - 3sin^2 x} dx = \int \frac{1}{1 - 5sin^2 x} dx$

Dividing the numerator and denominator by $cos_2 x$

$\int \frac{1}{sec^2 - 5tan^2 x} dx$

Let t = tanx

$dt = sec^2 x dx = (1+t^2) dx$

$dx = \frac{dt}{(1+t^2)}$

$\int \frac{1}{1-5sin^2 x} dx = \int \frac{1}{1-4t^2} \cdot \frac{dt}{(1+t^2)}$

$= \int \frac{1}{1-4t^2} dt = \int \frac{1}{(1+2t)(1-2t)} dt$

Let $\frac{1}{(1+2t)(1-2t} = \frac{A}{1+2t} + \frac{B}{1-2t}$

1 = A(1 – 2t) + B (1 + 2t)

Putting t = $\frac{1}{2}$; B = $\frac{1}{2}$

Putting t = $-\frac{1}{2}$; A = $\frac{1}{2}$

$\int \frac{1}{(1+2t)(1-2t)} dt = \frac{1}{2} \int \frac{dt}{1+2t} + \frac{1}{2} \int \frac{dt}{1-2t}$

$= \frac{1}{2}\left[\frac{1}{2} In(1+2t) - \frac{1}{2} In(1-2t)\right] + c$

$= \frac{1}{4} In\left(\frac{1+2t}{1-2t}\right) + c$

$\therefore \int \frac{1}{cos2x - 3sin^2 x} dx = \frac{1}{4} In\left(\frac{1+2tanx}{1-2tanx}\right) + c$

Revision exercise 14

1. Integrate the following

(a) $\int \frac{4}{3+5sinx} dx$ $\left[\frac{3tan\frac{1}{2}x+1}{tan\frac{1}{2}x+3} + c\right]$

(b) $\int \frac{1}{4+5cosx} dx$ $\left[\frac{1}{3} In\left(\frac{3+tan\frac{1}{2}x}{3-tan\frac{1}{2}x}\right) + c\right]$

(c) $\int \frac{1}{1+5sin2x} dx$ $\left[-\frac{1}{1+tanx} + c\right]$

(d) $\int \frac{4}{5+3cos\frac{1}{2}x} dx$ $\left[tan^{-1}\left(\frac{1}{2} tan\frac{1}{4}x\right) + c\right]$

(e) $\int \frac{4}{2+sin\frac{1}{2}x} dx$

$\left[\frac{8\sqrt{3}}{9} tan^{-1}\left(\frac{2tan\frac{1}{4}x+1}{\sqrt{3}}\right) + c\right]$

2. Integrate each of the following

(a) $\int \frac{1}{1+2sin^2 x} dx$

$\left[\frac{\sqrt{3}}{3} tan^{-1}(\sqrt{3}tanx) + c\right]$

(b) $\int \frac{1}{1-10sin^2 x} dx$

$\left[\frac{1}{6} In(1+3tanx) - \frac{1}{6} In(1-3tanx)\right] + c$

(c) $\int \frac{\sin^2 x}{1+\cos^2 x} dx$

$\left[\sqrt{2} tan^{-1}\left(\frac{\sqrt{2}}{2} tanx\right) - x + c\right]$

(d) $\int \frac{4}{\cos^2 x + 9\sin^2 x} dx$

$\left[\frac{4}{3} tan^{-1}(3tanx) + c\right]$

(e) $\int \frac{1+\sin x}{\cos^2 x} dx$

$[tanx + secx + c]$

(f) $\int \frac{1}{1+\tan x} dx$

$\left[\frac{1}{2} x + \frac{1}{2} In(cosx + sinx) + c\right]$

3. Evaluate

(a) $\int_0^{\frac{\pi}{2}} \frac{3}{1+sinx} dx$ $[3]$

(b) $\int_0^{\frac{2\pi}{3}} \frac{3}{5+4cosx} dx$ $\left[\frac{1}{3}\pi\right]$

(c) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{3}{4+5cosx} dx$ $[2In2]$

(d) $\int_0^{\frac{\pi}{2}} \frac{5}{3sinx+4cosx} dx$ $[In6]$

Integration of special cases involving splitting the numerator

Case 1

When a fractional integrand with quadratic denominator expressed in the form of $\frac{f(x)}{g(x)}$ is such that g(x) cannot be factorized or written in simple partial fractions, it is normally very useful to express it as a fraction by splitting the numerator.

i.e. Numerator = A(derivative of denominator + B

Example 25

Find the integral of each of the following

(a) $\int \frac{2x-1}{4x^2+3}dx$

Solution

Numerator=$A\left[\frac{d}{dx}(4x^2+3)\right]+B$

2x – 1 =A(8x)+B

Putting x = 0, B= -1

Putting x = 1, A = $\frac{1}{4}$

$\int \frac{2x-1}{4x^2+3}dx = \frac{1}{4}\int \frac{8x}{4x^2+3}dx - \int \frac{1}{4x^2+3}dx$
$= \frac{1}{4}In(4x^2+3) - \frac{\sqrt{3}}{6}tan^{-1}\left(\frac{2\sqrt{3}}{3}x\right) + c$

(b) $\int \frac{2x+3}{x^2+2x+10}dx$

Solution

Numerator=$A\left[\frac{d}{dx}(x^2+2x+10)\right]+B$

2x +3 =A(2x+2)+B

Putting x = -1, B= 1

Putting x = 0, A = 1

$\int \frac{2x+3}{x^2+2x+10}dx$

$= \frac{1}{4}\int \frac{2x+2}{x^2+2x+10}dx + \int \frac{1}{x^2+2x+10}dx$

$= In(x^2+2x+10) + \int \frac{1}{9+(x+1)^2}$dx

$= In(x^2+2x+10) + \frac{1}{3}tan^{-1}\left(\frac{x+1}{3}\right) + c$

(c) $\int \frac{x}{x^2+3x+5}dx$

Solution

Numerator=$A\left[\frac{d}{dx}(x^2+3x+5)\right]+B$

x =A(2x+3)+B

Putting x = $-\frac{3}{2}$, B= $-\frac{3}{2}$

Putting x = 0, A = $-\frac{1}{2}$

$\int \frac{x}{x^2+3x+5}dx$

$= \frac{1}{2}\int \frac{2x+3}{x^2+3x+5}dx - \frac{3}{2}\int \frac{1}{x^2+3x+5}dx$

$= \frac{1}{2}In(x^2+3x+5) - \frac{3}{2}\int \frac{1}{\frac{11}{4}+\left(x+\frac{3}{2}\right)^2}$dx

$= \frac{1}{2}In(x^2+3x+5) - \frac{3}{\sqrt{11}}tan^{-1}\left(\frac{2x+3}{\sqrt{11}}\right) + c$

(d) $\int \frac{1-2x}{9-(x+2)^2}dx$

Solution

$\int \frac{1-2x}{\sqrt{9-(x+2)^2}}dx$
$= \int \frac{1}{\sqrt{9-(x+2)^2}}dx - \int \frac{2x}{\sqrt{9-(x+2)^2}}dx$
$= sin^{-1}\left(\frac{x+2}{3}\right) - \int \frac{2x}{\sqrt{9-(x+2)^2}}dx$
For $\int \frac{2x}{\sqrt{9-(x+2)^2}}dx$
Let sinu = $\frac{x+2}{3}$
3sinu= x +2
3cosudu=dx
$\int \frac{2x}{\sqrt{9-(x+2)^2}}dx = \int \frac{2(3sinu-2)}{\sqrt{9-9sin^2u}}.3cosudu$

$= \int \frac{6sinu-4}{3\sqrt{(1-sin^2u)}}.3cosudu$

$= \int(6sinu-4)du$

$= -6cosu - 4u + c$

$= -6\sqrt{1-\left(\frac{x+2}{3}\right)^2} - 2sin^{-1}\left(\frac{x+2}{3}\right) + c$

Substituting for $\int \frac{2x}{\sqrt{9-(x+2)^2}}dx$

$\int \frac{1-2x}{\sqrt{9-(x+2)^2}}dx$
$= sin^{-1}\left(\frac{x+2}{3}\right) + -6\sqrt{1-\left(\frac{x+2}{3}\right)^2} - 4sin^{-1}\left(\frac{x+2}{3}\right) + c$
$= 5sin^{-1}\left(\frac{x+2}{3}\right) + 6\sqrt{1-\left(\frac{x+2}{3}\right)^2} + c$

Case II

When finding the integral of fractional trigonometric function expressed in the form $\int \frac{acosx+bsinx}{c\ cosx+dsinx}$, a, b, c and d are constants, we split the numerator as:

Numerator = A(derivative of denominator)+ (denominator)

Example 26

1. Find

(a) $\int \frac{2cosx+9sinx}{3cosx+sinx}dx$

Solution

Let 2cosx+ 9sinx =A$\frac{d}{dx}$(3cosx+sinx)+B(3cosx+sinx)

2cosx+ 9sinx =A(-3sinx+cosx)+B(3cosx+sinx)

2cosx+ 9sinx =(A+3B)cosx +(-3A+B)sinx

Equating coefficients:

For cosx: A+3B = 2 ………………. (i)

For sinx: -3A+B = 9……………….(ii)

Solving Eqn. (i) and Eqn. (ii) simultaneously

A= $-\frac{5}{2}$ and B = $\frac{3}{2}$

⇨ $\int \frac{2cosx+9sinx}{3cosx+sinx}dx$

$=-\frac{5}{2}\int \frac{-3sinx+cosx}{3cosx+sinx}dx + \frac{3}{2}\int \frac{3cosx+sinx}{3cosx+sinx}$

$=-\frac{5}{2}$In(3cosx+sinx)+ $\frac{3}{2}x + c$

(b) $\int \frac{3sinx}{4cosx-sinx}dx$

Solution

Let 3sinx = A$\frac{d}{dx}$(4cosx–sinx) + B(4cosx–sinx)

3sinx = A(-4sinx–cosx) + B (4cosx – sinx)

3sinx = (-A+B)cosx +(-4A-B)sinx

Equating coefficients

For cosx: -A + 4B = 0 ………………… (i)

For sin x: -4A- B = 3 ………………… (ii)

Solving Eqn. (i) and Eqn. (ii) simultaneously

A= $-\frac{12}{17}$ and B= $-\frac{3}{17}$

$\int \frac{3sinx}{4cosx-sinx}dx$

$= -\frac{12}{17}\int \frac{-4sinx-cosx}{4cosx-sinx}dx - \frac{3}{17}\int \frac{4cosx-sinx}{4cosx-sinx}dx$

$= -\frac{12}{17}In(4cosx - sinx) - \frac{3}{17}x$+c

2. Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{cosx-sinx}{3cosx+2sinx}dx$

Solution

Let 3sinx = A$\frac{d}{dx}$(3cosx+2sinx) + B(3cosx+2sinx)

cosx-sinx = A(-3sinx+2cosx) + B (3cosx +2sinx)

cosx- sinx = (2A+3B)cosx +(-3A+2B)sinx

Equating coefficients

For cosx: 2A + 3B = 1 ………………… (i)

For sin x: -3A+2B = -1 ………………… (ii)

Solving Eqn. (i) and Eqn. (ii) simultaneously

A= $\frac{5}{13}$ and B= $-\frac{1}{13}$

$\int \frac{cosx-sinx}{3cosx+2sinx}dx$

$=\frac{5}{13}\int \frac{2cosx-3sinx}{3cosx+2sinx}dx + \frac{1}{13}\int \frac{3cosx+2sinx}{3cosx+2sinx}dx$

$=\frac{5}{13}In(3cosx + 2sinx) + \frac{1}{13}x + c$

$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{cosx-sinx}{3cosx+2sinx}dx$

$=\left[\frac{5}{13}In(3cosx + 2sinx) + \frac{1}{13}x + c\right]_{\frac{1}{6}\pi}^{\frac{1}{2}\pi}$

$=\left[\frac{5}{13}In\left(3cos\frac{\pi}{2} + 2sin\frac{\pi}{2}\right) + \frac{1}{13}.\frac{\pi}{2}\right] - \left[\frac{5}{13}In\left(3cos\frac{\pi}{6} + 2sin\frac{\pi}{6}\right) + \frac{1}{13}.\frac{\pi}{6}\right]$

$=\left[\frac{5}{13}In2 + \frac{\pi}{26}\right] - \left[\frac{5}{13}In\frac{2+\sqrt{3}}{2} + \frac{\pi}{78}\right]$

$=\frac{5}{13}In\left(\frac{4}{2+3\sqrt{3}}\right) + \frac{\pi}{39}$

Revision exercise 15

1. Integrate each of the following

(a) $\int \frac{x+2}{x^2+2x+4}dx$

$\left[\frac{1}{2}In(x^2 + 2x + 4) + \frac{\sqrt{3}}{3}tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + c\right]$

(b) $\int \frac{x}{x^2-x+3}dx$

$\left[\frac{1}{2}In(x^2 - x + 3) + \frac{2}{\sqrt{11}}tan^{-1}\left(\frac{2x-1}{\sqrt{11}}\right) + c\right]$

(c) $\int \frac{2(x+1)}{x^2+4x+8}dx$

$\left[In(x^2+4x+8)-tan^{-1}\left(\frac{x+2}{2}\right)+c\right]$

(d) $\int \frac{5x+7}{x^2+4x+8}dx$

$\left[\frac{5}{2}In(x^2+4x+8)-\frac{3}{2}tan^{-1}\left(\frac{x+2}{2}\right)+c\right]$

2. Integrate the following

(a) $\int \frac{cosx-2sinx}{3cosx+4sinx}dx$

$\left[\frac{2}{5}In(4sinx+3cox)-\frac{1}{5}x+c\right]$

(b) $\int \frac{cosx}{2cosx-sinx}dx$

$\left[-\frac{1}{5}In(2cosx-sinx)+\frac{2}{5}x+c\right]$

(c) $\int \frac{cosx}{cosx-2sinx}dx$

$\left[-\frac{2}{5}In(cosx-2sinx)-\frac{1}{5}x+c\right]$

(d) $\int \frac{2cosx+sinx}{4cosx+3sinx}dx$

$\left[-\frac{14}{15}In(4cosx+3sinx)-\frac{11}{5}x+c\right]$

Revision exercise 16: general topical revision questions

1. Find

(a) $\int sinxdx$ $\left[xsin^{-1}x+\sqrt{1-x^2}+c\right]$

(b) $\int xsec^2xdx$ $[xtanx+Incosx+c]$

(c) $\int \frac{x^2}{\sqrt{1-x^2}}dx$ $\left[\sqrt{1-x^2}\left(\frac{-2-x^2}{3}\right)+c\right]$

(d) $\int In(x^2-4)dx$

$\left[xIn(x^2-4)-2x+2\left(In\frac{x+2}{x-2}\right)+c\right]$

(e) $\int \frac{dx}{3-2cosx}dx$ $\left[\frac{2}{\sqrt{5}}tan^{-1}\left(\sqrt{5}tan\frac{x}{2}\right)+c\right]$

(f) $\int 3^{\sqrt{2x-1}}dx$

$\left[\frac{3^{\sqrt{2x-1}}}{In3}\left(\sqrt{2x-1}-\frac{1}{In3}\right)+c\right]$

(g) $\int sin^2xdx$ $\left[\frac{1}{2}\left(x-\frac{1}{2}sin2x\right)+c\right]$

(h) $\int tan^3xdx$ $\left[\frac{1}{2}tan^2x-Incosx+c\right]$

(i) $\int \frac{4x^2}{\sqrt{1-x^6}}dx$ $\left[\frac{4}{3}sin^{-1}(x^3)+c\right]$

(j) $\int \frac{x^2}{x^4-1}dx$ $\left[\frac{1}{4}In\left(\frac{x-1}{x+1}\right)+\frac{1}{2}tan^{-1}x+c\right]$

(k) $\int \frac{2x}{\sqrt{x^2+4}}dx$ $\left[2\sqrt{x^2+4}+c\right]$

(l) $\int xInxdx$ $\left[\frac{x^2}{2}Inx-\frac{x^2}{4}+c\right]$

(m) $\int x^3e^{x^4}dx$ $\left[\frac{1}{4}e^{x^4}+c\right]$

(n) $\int \frac{1}{1+sin^2x}$ $\left[\frac{\sqrt{2}}{2}tan^{-1}(\sqrt{2}tanx)+c\right]$

(o) $\int Inxdx$ $[x(Inx-1)+c]$

(p) $\int x^2sin2xdx$

$\left[-\frac{1}{2}x^2cos2x+\frac{1}{2}xsin2x+\frac{1}{4}cos2x+c\right]$

(q) $\int Inx^2dx$ $[2x(Inx-1)+c]$

(r) $\int \frac{dx}{e^x-1}$ $[In(1-e^{-x})+c]$

(s) $\int \frac{x^2}{(1+x^2)^{\frac{1}{2}}}dx$ $\left[\frac{1}{3}(1+x^2)^{\frac{1}{2}}(x^2-2)+c\right]$

(t) $\int \frac{dx}{1-cosx}$ $\left[-cot\left(\frac{x}{2}\right)+c\right]$

(u) $\int \frac{x^4-x^3+x^2+1}{x^3+x}dx$

$\left[\frac{x^2}{2}-x+Inx+tan^{-1}x-\frac{1}{2}In(1+x^2)+c\right]$

(v) $\int \frac{sin^{-1}2x}{\sqrt{1-4x^2}}dx$ $\left[\left(\frac{sin^{-1}2x}{2}\right)^2+c\right]$

(w) $\int x(1-x^2)^{\frac{1}{2}}dx$ $\left[\frac{1}{3}(1-x^2)^{\frac{3}{2}}+c\right]$

(x) $\int \frac{1+\sqrt{x}}{2\sqrt{x}}dx$ $\left[\sqrt{x}+\frac{x}{2}+c\right]$

(y) $\int x^2e^xdx$ $[x^2e^x-2xe^x+2e^x+c]$

(z) $\int \frac{dx}{x^2\sqrt{(25-x^2)}}dx$ $\left[-\frac{1}{25}\left(\frac{5\sqrt{25-x^2}}{x^2}\right)+C\right]$

2. Evaluate

(a) $\int_0^{\frac{\pi}{2}} xcos^2xdx$ [0.3669]

(b) $\int_1^{\sqrt{3}} (x+tanx)dx$ [1.0003]

(c) $\int_0^{\frac{\pi}{2}} sin2xcosxdx$ $\left[\frac{2}{3}\right]$

(d) $\int_1^3 \frac{x^2+1}{x^3+4x^2+3x}dx$ [0.3489]

(e) $\int_0^1 \frac{x}{\sqrt{1+x}}dx$ [0.3905]

(f) $\int_0^{\frac{\pi}{2}} \frac{cosx}{1+sin^2x}dx$ [0.7854]

(g) $\int_0^{\frac{\pi}{6}} sinxsin3xdx$ [0.1083]

(h) $\int_0^1 \frac{x^3}{x^2+1}dx$ [0.15345]

(i) $\int_0^{\sqrt{\frac{\pi}{2}}} 2xcosx^2dx$ [1]

(j) $\int_0^2 \frac{8x}{x^2-4x-12}dx$ [1.05]

(k) $\int_0^{\frac{\pi}{2}} \frac{dx}{1+sinx+cosx}$ $[In2]$

(l) $\int_0^{\frac{\pi}{2}} sin2xcosxdx$ $\left[\frac{2}{3}\right]$

(m) $\int_4^6 \frac{dx}{x^2-2x-3}$ [0.1905]

(n) $\int_0^{\frac{\pi}{2}} xsin^22xdx$ $\left[\frac{\pi^2}{16}\right]$

(o) $\int_1^{\sqrt{3}} \frac{x^2}{\sqrt{x^4-x^2}} dx$ [2]

(p) $\int_1^3 \frac{3x^2+4x+1}{x^3+2x^2+x} dx$ $[In12]$

(q) $\int_0^{\frac{\pi}{2}} x^2 sinxdx$ $[\pi - 2]$

(r) $\int_0^1 xe^{2x}dx$ $[2.0973]$

(s) $\int_{\frac{\pi}{3}}^{\pi} xsinxdx$ $[2.7992]$

(t) $\int_{\frac{1}{2}}^{1} 10x\sqrt{(1-x^2)}\, dx$ $[2.165]$

(u) $\int_0^{\frac{\pi}{2}} sin5xcos3xdx$ $\left[\frac{1}{2}\right]$

(v) $\int_0^{\frac{\sqrt{3}}{2}} \frac{dx}{9+4x^2} dx$ $\left[\frac{\pi}{36}\right]$

3. Show that

(a) $\int_0^1 \frac{x^2+6}{(x^2+4)(x^2+9)} = \frac{\pi}{20}$

(b) $\int_0^{\frac{\pi}{2}} xtan^2xdx = \frac{1}{32}(8\pi - \pi^2 - 16\log_e 2)$

(c) $\int_2^4 xInxdx = 14In2 - 3$

(d)

4. Given that

$\frac{3x^3+2x^2-6x-2}{(x^2+x-2)(x^2-2)} = \frac{1}{x+2} + \frac{B}{x-1} + \frac{Cx+D}{x^2-2}$

Determine the values of A, B, C, D

Hence evaluate $\int_3^4 \frac{3x^3+2x^2-6x-2}{(x^2+x-2)(x^2-2)}$dx

$[A = B = C = 1, D = 0; 2.4770]$

5. Use the substitution of x = $\frac{1}{u}$ to evaluate

$\int_1^2 \frac{dx}{x\sqrt{x^2-1}}$ $\left[\frac{\pi}{3}\right]$

6. Express $\frac{x^3-3}{(x-2)(x^2+1)}$ as partial fractions

$\left[\frac{x^3-3}{(x-2)(x^2+1)} = 1 + \frac{1}{x-2} + \frac{x+1}{x^2+1}\right]$

Hence find $\int \frac{x^3-3}{(x-2)(x^2+1)} dx$

$\left[x + In(x-2) + \frac{1}{2}In(x^2+1) + tan^{-1}x + c\right]$

7. Express f(x) = $\frac{2x^2-x+14}{(4x^2-1)(x+3)}$ in partial fraction

$\left[\frac{2x^2-x+14}{(4x^2-1)(x+3)} = \frac{-3}{2x+1} + \frac{2}{2x-1} + \frac{1}{x+3}\right]$

Hence evaluate $\int_1^3 f(x)dx$ $[0.7440]$

8. Using the substitution 2x+1 = u, find

$\int_0^1 \frac{xdx}{(2x+1)^2}$ $\left[\frac{1}{18}\right]$

9. Express

(a) f(x) = $\frac{6x}{(x-2)(x+4)^2}$ in partial fraction

$\left[\frac{6x}{(x-2)(x+4)^2} = \frac{1}{3(x-2)} - \frac{1}{3(x+4)} + \frac{4}{(x+4)^2}\right]$

Hence evaluate $\int f(x)dx$

$\left[\frac{1}{3}In\left(\frac{x-2}{x+4}\right) - \frac{4}{(x+4)} + c\right]$

(b) f(x) = $\frac{3x^2+x+1}{(x-2)(x+1)^2}$ in partial fraction

$\frac{3x^2+x+1}{(x-2)(x+1)^2}$

$= \frac{5}{9(x-2)} - \frac{5}{9(x+2)} + \frac{4}{3(x+1)^2} - \frac{1}{(x+1)^3}$

Hence evaluate

$\int_3^4 \frac{3x^2+x+1}{(x-2)(x+1)^2} dx$ $[0.317]$

(c)

10. Using the substitution x = 3sinθ, evaluate

(a) $\int_0^3 \sqrt{\left(\frac{3+x}{3-x}\right)}\, dx$ $[7.7125]$

(b) $\int_0^{\pi} \frac{dx}{3+5cosx}$ $[0.2747]$

(e)

11. Use t= tan$\frac{1}{2}x$ to evaluate

(a) $\int_0^{\frac{\pi}{2}} \frac{dx}{3-cosx}$ $[0.6755]$

(b)

12. Given that $\int_0^a (x^2 + 2x - 6)\, dx = 0$, find the value of a [a=-6]

13. Use the substitution $x^2 = \theta$ to find

$\int \frac{x}{1+\cos x^2} dx$ $\left[\frac{1-3x}{3(x+1)^{\frac{5}{3}}(x-1)^{\frac{4}{5}}}\right]$

14. Resolve $y = \frac{x^3+5x^2-6x+6}{(x-1)^2(x^2+2)}$ into partial fraction

$\left[\frac{x^3+5x^2-6x+6}{(x-1)^2(x^2+2)} \equiv \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{4}{(x^2+2)}\right]$

Hence find $\int ydx$

$\left[In(x-1) + \frac{-2}{(x-1)} + \frac{4}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + c\right]$

15. Express f(x) = $\frac{1}{x^2(x-1)}$ in partial fraction

$\left[\frac{1}{x^2(x-1)} = -\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-1}\right]$

Hence evaluate $\int_2^3 f(x)dx$ $[0.12102\,]$

Application of integration

Like differentiation, integration has a wide spectrum of application, some of which are discussed below

Acceleration, velocity, displacement

Given the acceleration, a, of a particle, its velocity, v and displacement, s can be computed as long as the initial values are known.

Acceleration, a = $\frac{dv}{dt}$ => v = $\int adt$

Also, velocity v = $\frac{ds}{dt}$ => s = $\int vdt$

Example 27

The acceleration of a particle after t seconds is given by a = 5 + t.

If initially, the particle is moving at $1ms^{-1}$, find the velocity after 2s and the distance it would have covered by then

Given $\frac{dv}{dt} = 5 + t$

⇨ $dv = (5 + t)dt$
$v = 5t + \frac{1}{2}t^2 + c$

Whet t = 0, v = 1, =>c = 1

$\therefore v = 5t + \frac{1}{2}t^2 + 1$

When t = 2s

$v = 5(2) + \frac{1}{2}(2)^2 + 1$= $13ms^{-1}$.

And $\frac{ds}{dt} = 5t + \frac{1}{2}t^2 + 1$

$ds = \left(5t + \frac{1}{2}t^2 + 1\right)dt$

s = $\frac{5}{2}t^2 + \frac{1}{6}t^3 + t + c$

when t = 0, s= 0 => c = 0

∴s = $\frac{5}{2}t^2 + \frac{1}{6}t^3 + t$

At t =2s

s = $\frac{5}{2}(2)^2 + \frac{1}{6}(2)^3 + 2$=$13\frac{1}{3}$m

Example 28

A particle with a velocity $(2i+3j)ms^{-1}$ is accelerated uniformly at the rate of $(3ti – 2j)ms^{-1}$ from the origin. Find

(i) The speed reached by the particle at t = 4s.

Solution

Given a= 3ti – 2j

v = $\int adt = \int(3ti - 2j)dt$

$= \frac{3}{2}t^2 i - 2tj + c$

At t= 0, 2i+3j

c =2i+3j

By substitution

$v = \left(\frac{3}{2}t^2 + 2\right)i + (-2t + 3)j$

At t = 4s

$v = \left(\frac{3}{2}(4)^2 + 2\right)i + (-2(4) + 3)j$

= $(26i – 5j)ms^{-1}$

Speed = $|v| = \sqrt{26^2 + (-5)^2}$ =$26.5ms^{-1}$

(ii) The distance travelled by the particle after 2s.

Solution

r = $\int vdt$

r= $\int\left(\left(\frac{3}{2}(4)^2 + 2\right)i + (-2(4) + 3)j\right)dt$

$= \left(\frac{3}{6}t^3 + 2t\right)i + \left(\frac{-2}{2}t^2 + 3t\right)j + c$

At t = 0, r = 0; => c = 0

$\therefore r = \left(\frac{3}{6}t^3 + 2t\right)i + \left(\frac{-2}{2}t^2 + 3t\right)j$

At t = 2

$\therefore r = \left(\frac{8}{2} + 4\right)i + (-4 + 6)j$

$|r| = \sqrt{8^2 + 2^2}$ =8.25m

Hence the distance = 8.25m

Example 29

A particle has initial position of (7i+5j)m. the particle moves with constant velocity of (ai+bi)ms-1 and 3s later its position is (10i – j)m. fins the values of a and b.

Solution

Given v = ai +bj

r =$\int vdt = \int(ai + bi)dt + c$

$= ati + btj + c$

at t = 0; r = c =(7i+5j)m

$\therefore r = (at + 7)i + (bt + 5)j$

After 3s

$10i - j = (3a + 7)i + (3b + 5)j$

Equating corresponding vectors

For i: 10 = 3a+7 => a = 1

For j: -1 = 3b +5 => b = -2

$\therefore a = 1 \text{ and } b = -2$

Example 30

A particle of mass 2kg, initially at rest at (0, 0, 0) is acted on by a force $\begin{pmatrix} 2t \\ 2t \\ 4t \end{pmatrix} N$. Find

(i) its acceleration at time t

from F = Ma

$\begin{pmatrix} 2t \\ 2t \\ 4t \end{pmatrix} = 2a \Rightarrow a = \begin{pmatrix} t \\ t \\ 2t \end{pmatrix}$

(ii) its velocity after 3s

velocity v =$\int adt = \int \begin{pmatrix} t \\ t \\ 2t \end{pmatrix} dt$

v=$\begin{pmatrix} \frac{t^2}{2} \\ \frac{t^2}{2} \\ t^2 \end{pmatrix} + c$

at t = 0, v = 0 => c= 0

∴v=$\begin{pmatrix} \frac{t^2}{2} \\ \frac{t^2}{2} \\ t^2 \end{pmatrix}$

At t = 3s

$v = \frac{9}{2}i + \frac{9}{2}j + 9k$

(iii) the distance of the particle travelled after 3s.

r= $\int vdt = \int \left(\frac{t^2}{2}i + \frac{t^2}{2}j + t^2k\right) dt$

= $\left(\frac{t^3}{6}i + \frac{t^3}{6}j + \frac{1}{3}t^3k\right) + c$

At t = 0, r = 0 => c = 0

$\therefore r = \left(\frac{t^3}{6}i + \frac{t^3}{6}j + \frac{1}{3}t^3k\right)$

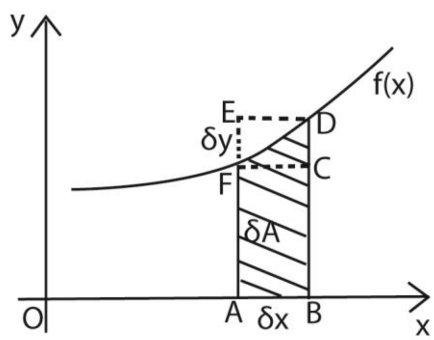
At t = 3

$r = \left(\frac{3^3}{6}i + \frac{3^3}{6}j + \frac{1}{3}.3^3k\right) = \left(\frac{9}{2}i + \frac{9}{2}j + 9k\right)$

$|r| = \sqrt{\left(\frac{9}{2}\right)^2 + \left(\frac{9}{2}\right)^2 + 9^2}$ =11.02m

Area under a curve

If the area under the curve y = f(x) for α≤ x ≤β is required, a small strip can be used for analysis



Suppose the shaded region is δA, the area of the shaded strip lies between areas of the rectangles ABCF and AVDE.

i.e. Area of ABCF ≤ δA ≤ ABDE.

$y\delta x \leq \delta A \leq (y + \delta y)dx$

Dividing by δx

$y \leq \frac{\delta A}{\delta x} \leq (y + \delta y)$

$\lim_{\delta x \to 0} \frac{\delta A}{\delta x} \to \frac{dA}{dx}$ and $\delta y \to 0$

Hence $\frac{dA}{dx} = y$

Integrating both sides with respect to x

$\int \frac{dA}{dx} dx = \int ydx$

Now for the interval $\alpha \leq x \leq \beta$

$A = \int_\alpha^\beta ydx \text{ Or } A = \int_\alpha^\beta f(x)dx$

Note: when finding the area under the curve, it is advisable that you sketch the curve first in order to establish the required region.

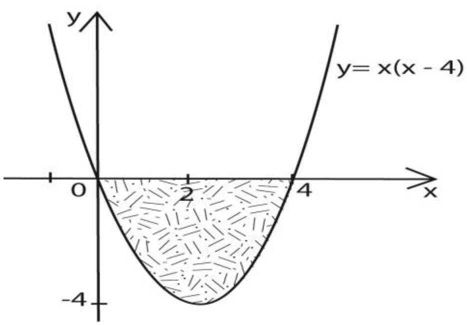
Area between the curve and the x-axis

Example 31

(i) Find the area enclosed by y = x(x – 4) and x-axis

Solution

By sketching the graph y = x(x – 4) with the x-axis we have



Area required = $\int_0^4 x(x-4)dx$

$=\int_0^4 x^2 - 4xdx$

$=\left[\frac{x^3}{3} - 2x^2\right]_0^4$

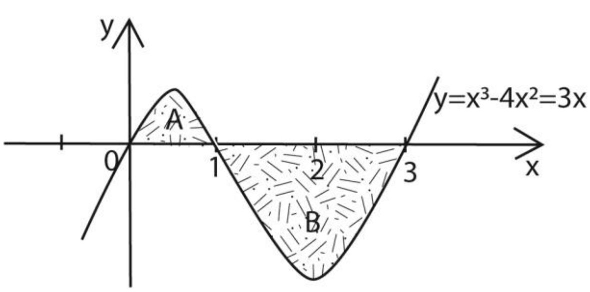
$=\frac{64}{3} - 32 = \frac{-32}{3}$

Hence the area under the curve is $\frac{32}{3}$ $sq.units$ (- sign indicates that the area is below the x-axis).

(ii) Find the area enclosed by the curve y = x^3 – 4x^2 + 3x and the x-axis from x = 0 and x = 3

Solution

By sketching the graph y = x^3 – 4x^2 + 3x with the x-axis we have



Required area = A+ B

Area A $=\int_0^1 (x^3 - 4x^2 + 3x)\,dx$

$=\left[\frac{x^4}{4} - \frac{4x^3}{3} + \frac{3x^2}{2}\right]_0^1$

$=\left(\frac{1}{4} - \frac{4}{3} + \frac{3}{2}\right) - (0) = \frac{5}{12}$

Area B $=\int_1^3 (x^3 - 4x^2 + 3x)\,dx$

$=\left[\frac{x^4}{4} - \frac{4x^3}{3} + \frac{3x^2}{2}\right]_1^3$

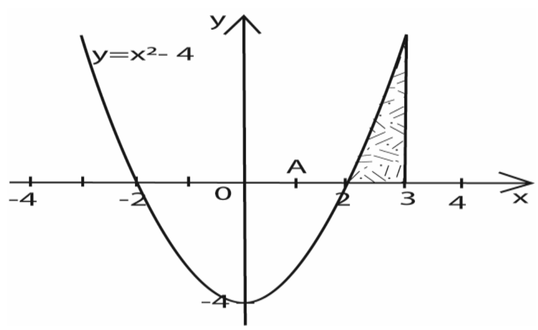
$=\left(\frac{81}{4} - 36 + \frac{27}{2}\right) - \left(\frac{1}{4} - \frac{4}{3} + \frac{3}{2}\right) = -\frac{8}{3}$

Area$=\frac{5}{12} + \frac{8}{3} = \frac{37}{12} sq.units$

(iii) Find the area between y=x^2 – 4, the x-axis and line x = 3.

Solution

By sketching the graph of y=x^2 – 4 with the x-axis, we have



Required $= \int_2^3 (x^2 - 4)\,dx$

$= \frac{1}{3}[x^3 - 4x]_2^3$

$= \frac{7}{3}$ sq. units

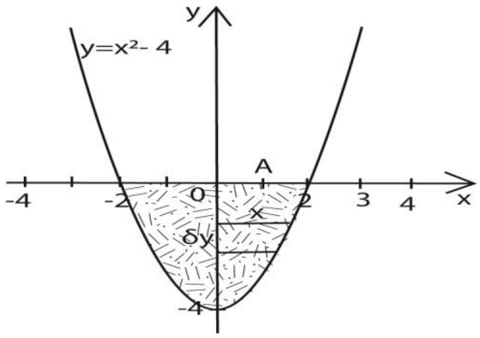
Area between the curve and the y-axis

This involves finding the area under the curve with respect to y or by subtracting the area under the curve with the x-axis from the rectangle (s) formed.

Example 32

Find the area enclosed by the curve y = x^2- 4 and the y = x2 -4 and y-axis between

(i) y = -4 and y = 0
Solution



1st Approach

Integrating with respect to x

Required area $= \int_{-2}^{2}(x^2-4)dx$

$= \frac{1}{3}[x^3-4x]_{-2}^{2}$

$= \left(\frac{8}{3}-8\right)-\left(\frac{-8}{3}+8\right)$

$= \left(\frac{16}{3}-16\right)$sq. units

$= \frac{-32}{3}$

Hence the required area is $\frac{32}{3}\,sq.units$

2nd approach

y = $x^2 - 4$

x = $(y+4)^{\frac{1}{2}}$

Required area = $2\int_{-1}^{0} xdy$

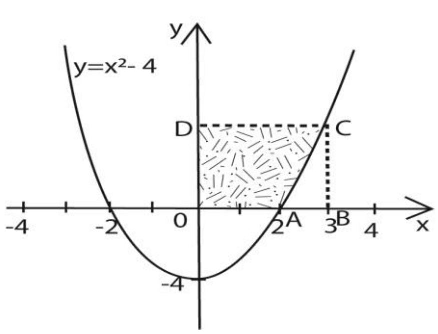
= $2\int_{-1}^{0}(y+4)^{\frac{1}{2}}dy$

= $2\left[\frac{2}{3}(y+4)^{\frac{3}{2}}\right]_{-4}^{0}$

= $2\frac{2}{3}[(8)-(0)]$

= $\frac{32}{3}$ sq. units

(ii) y = 0 and y = 5



1st approach

Required area= 2 x shaded region

Required area = $2\int_{0}^{5} xdy$

= $2\int_{0}^{5}(y+4)^{\frac{1}{2}}dy$

= $2\left[\frac{2}{3}(y+4)^{\frac{3}{2}}\right]_{0}^{5}$

= $2\frac{2}{3}[(2)-(8)]$

= $\frac{76}{3}$ sq. units

2nd approach

Required area = 2 x shaded area

=2[Area of OBCD – area of ABC]

=2[(3 x 5) – $\int_{2}^{3}(x^2-4)dx$]

=2[15– $\frac{7}{3}$]= $\frac{76}{3}$sq.units

Area between two curves

Suppose we want to find the area between two intersecting functions f(x) and g(x), required it to

(i) find the point of intersection of the functions
(ii) sketch the functions f(x) and g(x)

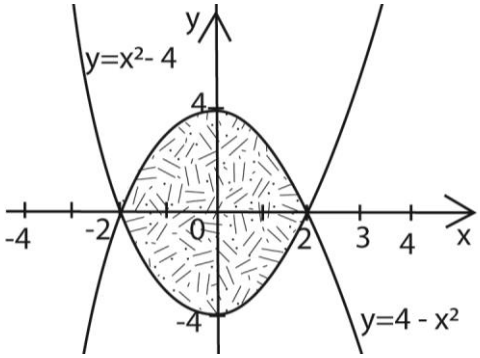
Note if f(x) is above g(x), then the required area

$= \int f(x)dx - \int g(x)dx$

Example 33

Find the area enclosed between the curves

(a) y = $x^2 - 4$ and y = $4 - x^2$
Solution
Finding the points of intersection
$x^2 - 4 = 4 - x^2$
$2x^2 = 8$
x = 2 or x = 2
when x = 2, y = 0
when x = -2, y = 0
The sketch of the functions:



Required area

$= \int_{-2}^{2}[(4 - x^2) - (x^2 - 4)]dx$

$= \int_{-2}^{2}(8 - 2x^2)dx$

$= \left[8x - \frac{2x^3}{3}\right]_{-2}^{2}$

$= \left(16 - \frac{16}{3}\right) - \left(-16 + \frac{16}{3}\right)$

$= \frac{32}{3} + \frac{32}{3} = \frac{64}{3}$ sq. units

(b) y =$2x^2$+7x+3 and y=9+4x-x^2

Solution

Finding the points of intersection

$2x^2 + 7x + 3 = 9 + 4x - x^2$

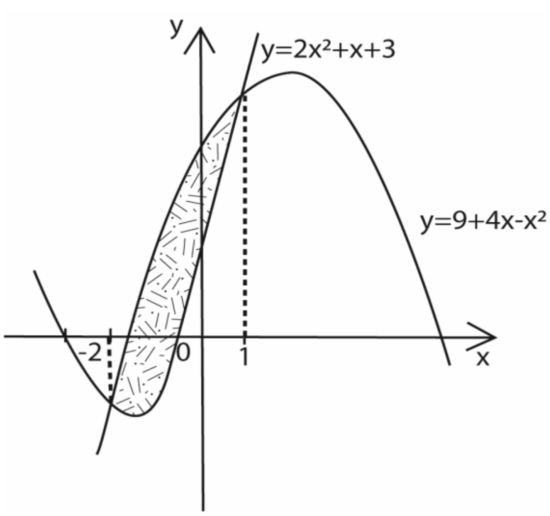
$3x^2 + 3x - 6 = 0$

$(x + 2)(x - 1) = 0$

x = -2 of x = 1

When x = -2, y = -3

When x = 1, y = 12



Required area

$= \int_{-2}^{1}[(9 + 4x - x^2) - (2x^2 + 7x + 3)]\,dx$

$= \int_{-2}^{1}(6 - 3x - 3x^2)\,dx$

$= \left[6x - \frac{3x^2}{2} - x^3\right]_{-2}^{1}$

$= \left(6 - \frac{3}{2} - 1\right) - (-12 - 6 + 8)$

$= 13.5 sq.units$

Example 34

Find the area enclosed between the curve y = x^2 – x – 3 and the line 2x +1

Solution

Finding the points of intersection

x^2 – x – 3 = 2x +1

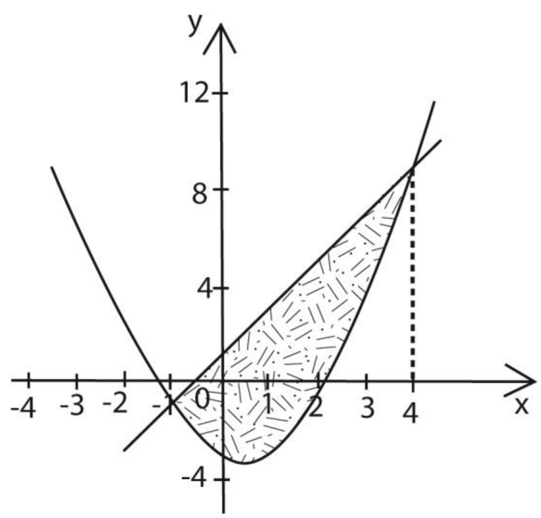
x^2 – 3x – 4 = 0

(x+1)(x-4) = 0

X = -1 or x = 4

When x = -1, y = -1

When x= 4, y = 9



Area required

= $\int_{-1}^{4}[(2x + 1) - (x^2 - x - 3)]\,dx$

=$\int_{-1}^{4}(4 + 3x - x^2)$dx

=$\left[4x + \frac{3}{2}x^2 - \frac{1}{3}x^3\right]_{-1}^{4}$

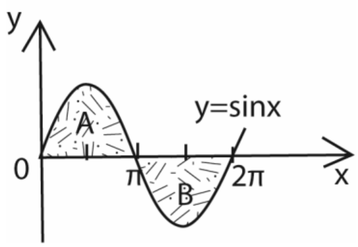
$=\left(16+24-\frac{64}{3}\right)-\left(-4+\frac{3}{3}+\frac{1}{2}\right)$

=20.83 sq.units

Example 35

Find the area enclosed by the curve y = sinx and the x-axis between x = 0 and x = 2π.

Solution



Required area = A + B

$=\int_0^{\pi} sinxdx+\int_{\pi}^{2\pi} sinxdx$

$=[-cosx]_0^{\pi}+[-cosx]_{\pi}^{2\pi}$

$=-(-cos\pi-cos0)-(-cos2\pi-cos\pi)$

=-(-1 – 1)- (-1 – 1)

= 2 +2 = 4sq. units

Volume of a solid of revolution

A solid of revolution is formed when a given area rotates about a fixed axix. Due to the way in which it is formed, it is referred to as solid of revolution.

These bodies have always got axes of symmetry.

The solids formed is subdivided into small cylindrical disks of thickness δx and height y.

Volume of each disk = πy^2dx

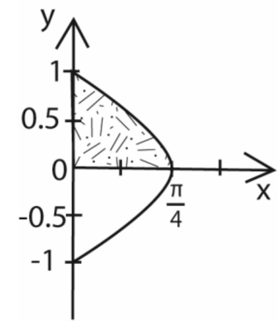
Thereforethe colume of the whole solid of revolution is obtained by rotating through one revolution about the x-axis, the region bounded by the curve y = f(x) and the linex x = a and x =b is given by v = $\int_a^b \pi y^2dx$

If the rotation is about the y-axis, the volume is given by v =$\int_a^b \pi x^2dy$

Example 36

(a) Find the volume of revolution when the portion of the curve y = cos2x for 0≤x≤$\frac{\pi}{4}$ is rotated through four right angles about the x-axis.

Solution



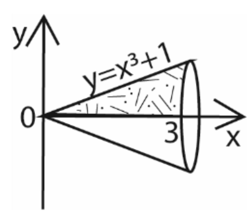
$V=\pi\int_0^{\frac{\pi}{2}} y^2dx=\pi\int_0^{\frac{\pi}{2}} cos^2 2xdx$

$=\pi\int_0^{\frac{\pi}{2}}(1+cos4x)dx$

$=\frac{\pi}{2}\left[x+\frac{1}{4}sin4x\right]_0^{\frac{\pi}{2}}$

$=\frac{1}{8}\pi^2$ cubic units

(b) Find the volume of the area bounded by the curve y = x^3 + 1, the x-axis and limits x = 0 and x = 3 when rotated through four right angles about the x-axis.



$V=\pi\int_0^3 y^2dx=\pi\int_0^3 (x^3+1)^2dx$

$=\pi\int_0^3(x^6+2x^3+1)^2dx$

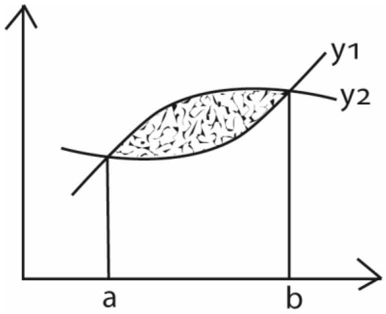
$=\pi\left[\frac{x^7}{7}+\frac{x^4}{2}+x\right]_0^3$

$=\pi\left(\frac{3^7}{7}+\frac{3^4}{2}+3\right)-(0)$

$=1118.25$ cubic units.

Rotation the area enclosed between two curves

If we have two curves y_1 and y_2 that enclose some area between a and b as shown below



Now if we rotate this area about the x-axis the volume of the solid formed is given by

$v = \pi \int_a^b [(y_2)^2 - (y_1)^2]\, dx$

Example 36

(a)A cup is madeby rotating the area between y= x^2 and y = x+1 with x ≥ 0 about the x-axis. Find the volume of the material needed to make the cup.

Solution

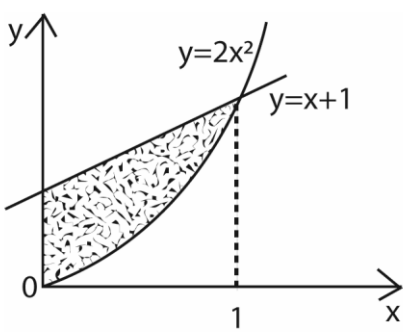
Finding the points of intersection

$2x^2 = x + 1$

$2x^2 – x + 1 = 0$

$(2x+1)(x – 1) = 0$

x =1 since we only need to consider x≥ 0.



V = $\pi \int_0^1 [(y+1)^2 - (2x^2)^2]dx$

$=\pi \int_0^1 (x^2 + 2x + 1 - 4x^4)dx$

$=\pi \left[\frac{x^3}{3} + x^2 + x - \frac{4x^5}{5}\right]_0^1$

$= \pi \left(\frac{1}{3} + 1 + 1 - \frac{4}{5}\right) - 0$

$=\frac{23}{15}\pi$ units cubed

Example 37

Find the volume of revolution when the portion of the area between the curves y = x^2 and x = y^2 is rotated through 360^0 about the x-axis.

Solution

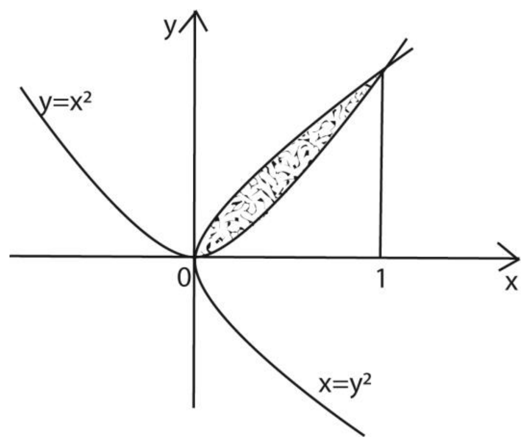
Points of intersection

$x^2 = x^{\frac{1}{2}} => x^4 = x$

$x^4 - x = 0$

$x(x^3 - 1) = 0$

Either x = 0 or x = 1



The volumeof revolution

$= \pi \int_0^1 \left[\left(\sqrt{x}\right)^2 - (x^2)^2\right] dx$

$=\pi \int_0^1 (x - x^4)dx$

$=\pi \left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1$

$=\pi \left[\left(\frac{1}{2} - \frac{1}{5}\right) - 0\right]$

$=\frac{3}{10}\pi$

Example 38

Find the volume generated when the area enclosed by the curve y = 4 – x^2 and the line y = 4 – 2x is rotated through 2π.

Solution

Finding the points of intersection

$4 – 2x = 4 – x^2$

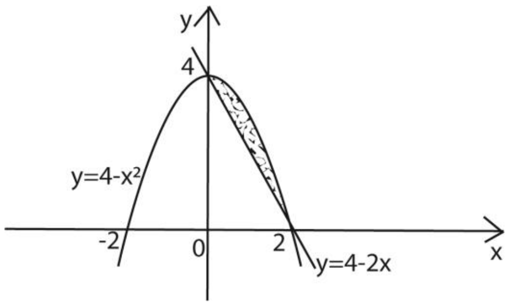
$x^2 - 2x = 0$

$x(x - 2) = 0$

Either x = 0 or x = 2

When x = 0, y = 4

When x = 2, y = 0



Required volume

$= \pi \int_0^2 [(4 - x^2)^2 - (4 - 2x)^2] dx$

$= \pi \int_0^2 [(16 - 8x^2 + x^4) - (16 - 8x - 4x^2)] dx$

$= \pi \int_0^2 (x^4 - 4x^2 + 8x)\, dx$

$= \pi \left[\frac{x^5}{5} - \frac{4x^3}{3} + 4x^2\right]_0^2$

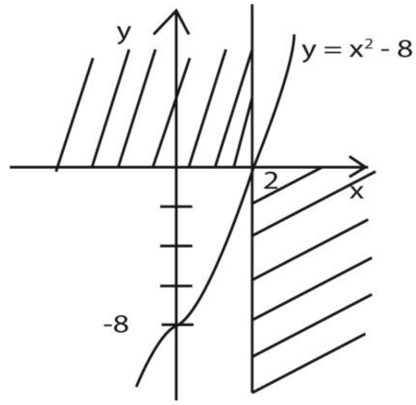
$= \frac{176}{15}\pi = 36.86$cubic units

Example 39

(a) Sketch the curve y = $x^3 - 8$ (08marks)

y = $x^3 - 8$
Intercepts
When x = 0, y = -8
When y = 0, x = 2
(x, y) = (2, 0)
Turning point: $\frac{dy}{dx} = 3x^2$
3x2 = 0
x = 0
$\frac{d^2y}{dx^2} = 6x$
$\frac{d^2y}{dx^2} = 0, x = 0$
Point of reflection= (0, 8)

	x < 2	x > 2
y	-	+



(b) The area enclosed by the curve in (a), the y-axis and x-axis is rotate about the line y = 0 through 360^0. Determine the volume of the solid generated. (04 marks)

V = $\pi \int_0^2 y^2 dx$

$= \pi \int_0^2 (x^3 - 8)^2 dx$

$= \pi \int_0^2 (x^6 - 16x^3 + 64) dx$

$= \pi \left[\frac{x^7}{7} - 4x^4 + 64x\right]_0^2$

$= \pi \left(\frac{128}{7} - 64 + 128\right)$

$= \frac{576\pi}{7}$ =250.5082 units^3

The mean value theorem for integrals

If f(x) is a continuous function on the closed interval [a, b], then there exist a number c in the closed interval such that

Area of the rectangle = f(c).(b-a)

But area under the curve between a and b

$= \int_a^b f(x) dx$

Equating the two

$\int_a^b f(x) dx - f(c).(b - a)$

Dividing both sides by (b – a)

$f(c) = \frac{1}{b-a} \int_a^b f(x) dx$

Where f(c) is the height of the rectangle

This height is the average value of the function over the interval in the question.

Hence the mean value of f(x) over a closed interval (a, b) is given by

M.V $=\frac{1}{b-a}\int_a^b f(x)dx$

Example 40

Find the mean value of y= x^2 + 2 for x = 1 and x = 4.

Solution

M.V $=\frac{1}{4-1}\int_1^4 (x^2+2)dx$

$=\frac{1}{3}\int_1^4 (x^2+2)dx$

$=\frac{1}{3}\left[\frac{x^3}{3}+2x\right]_1^4$

$=\frac{1}{3}\left[\left(\frac{64}{3}+8\right)-\left(\frac{1}{3}+2\right)\right]=9$

Example 41

Find the mean value of

y = $\frac{1}{1+sin^2\theta}$ for 0 ≤ θ≤ $\frac{\pi}{4}$

Solution

M.V $=\frac{1}{\frac{\pi}{4}-0}\int_0^{\frac{\pi}{4}} \frac{1}{1+sin^2\theta}dx$

$=\frac{4}{\pi}\int_0^{\frac{\pi}{4}} \frac{sec^2\theta}{sec^2\theta+tan^2\theta}dx$

$=\frac{4}{\pi}\int_0^{\frac{\pi}{4}} \frac{1+tan^2\theta}{1+2tan^2\theta}dx$

Let t = tan θ => dt = $sec^2\theta d\theta$= $(1+t^2)d\theta$

$d\theta=\frac{dt}{1+t^2}$

Changing limits

When θ = 0, t = 0 and when θ=$\frac{\pi}{4}$, t = 1

$\therefore M.V=\frac{\pi}{4}\int_0^1 \frac{1+t^2}{1+2t^2}.\frac{dt}{1+t^2}$

$=\frac{\pi}{4}\int_0^1 \frac{1}{1+2t^2}dt$

$=\frac{\pi}{4}\left[\frac{1}{\sqrt{2}}tan^{-1}\sqrt{2t}\right]_0^1$

$=\frac{2\sqrt{2}}{\pi}tan^{-1}\sqrt{2}$

= 0.86

Example 42

Find the mean value of y = x(4 – x)in the interval where y ≥ 0.

Solution

Given y ≥ 0 => x(4 –x)≥0 (positive)

The solution is 0 ≤ x ≤ 4

M.V $=\frac{1}{4-0}\int_0^4 x(4-x)dx=\frac{1}{4}\int_0^4(4x-x^2)dx$

$=\frac{1}{4}\left[2x^2-\frac{x^3}{3}\right]_0^4=\frac{8}{3}$

Revision exercise 17

1. Find the volume generate in each case whenthe area enclosed by the curve y = x^2 – 6x + 18 and the line y = 10 is rotated about
   (i) Y = 10 [1541π $units^3$]
   (ii) x –axix, [256π $units^3$]
2. Find the volume generated when the area enclosed by the curve y = x4 from y = 3 and y = 6 is rotated about the y-axis [6.33π $units^3$]
3. The displacement x of a particle at time t is given by x = sint. Find the mean value of its velocity over the interval 0 < t < $\frac{\pi}{2}$
   (i) with respect to t [0.637ms^{-1}]
   (ii) with respect to displacement x [0.785ms^{-1}]
4. (a) Determine the equation of the normal to the curve y = $\frac{1}{x}$ and the point x = 2. Find the coordinates of the other point where the normal meets the curve again.
   [2y – 8x+15 =0; $\left(-\frac{1}{8},-8\right)$
   (b) Find the area of the region bounded by the curve y =$\frac{1}{x(2x+1)}$, the x-axis and the lines x = 1 and x = 2. $\left(In\left(\frac{6}{5}\right)\right)$
5. A shell is formed by rotating the portion of the parabola y2= 4x for which 0 ≤ x ≤ 1 through two right angles about its axis. Find
   (i) the volume of the solid formed [2π]
   (ii) the area of the base of the solid formed [4π $units^2$]

6. Show that the tangents at (-1,3) and (1,5)on the curve y = $2x^2$ + x + 2 passes through the origin. Find the area enclosed between the curve and these two tangents $\left[\frac{4}{3}\right]$
7. Sketch the curve $y = x - \frac{8}{x^2}$ for x > 0, showing any a symptotes. Find the area enclosed by the x-axis, the line x = 4 and the curve $x - \frac{8}{x^2}$. [10 sq. units]
   If this area is now rotated about the x-axis through 3600, determine the volume of the solid generated, correct to 3 significant figures. [42.1 cubic units]
8. Show that the tangents to the curve 4 – 2x – $2x^2$ at points 9-1, 4) and $\left(\frac{1}{2}, 2\frac{1}{2}\right)$respectively passes through the point $\left(-\frac{1}{4}, 5\frac{1}{2}\right)$. Calculate the area of the curve enclosed between the curve and the x-axis. [9sq.units]
9. (i) find the Cartesian equation of the curve given parametrically by
   x = $\frac{1+t}{1-t}$, y = $\frac{2t^2}{1-t}$ $\left[y = \frac{(x-1)^2}{x+1}\right]$
   (ii) sketch the curve
   (iii) find the area enclosed between the curve and the line y = 1 [1.955sq.units]
10. Given the curve y = sin3x, find the
   (a)(i) the value of $\frac{dy}{dx}$ at the point $\left(\frac{\pi}{2}, 0\right)$
   (ii) equation of the tangent to the curve at this point [y= 3x+π)

   (b) (i) sketch the curve y = sin3x

   (ii) Calculate the area bounded by the tangent in (a)(i) above, the curve and y-axis

   [0.9783sq. units]
11. A hemisphericalbowl of internal radius r is fixed with its rim horizontal and contains a liquid to the depth h. show by integration that the volume of the liquid in the bowl is $\frac{1}{3}\pi h^2(3r - h)$
12. Find the volume of the solid of revolution formed by rotating the area enclosed by the curve y = x(1+x), the x-axis, the lines x = 2 and x = 3 through four right anglesabout the x-axis. [31.033π cubic units]

12. Curve sketching

Curve sketching (A-level)

The procedures of curve sketching depend on the nature of the curve to be sketched

Graphs of y = f(x) (Non rational functions)

For any graph of the form y = f(x) where f(x) is not linear, some or all the following steps are followed.

(a) Determine if the curve is symmetrical about either or both axes of coordinates.
- Symmetry about the x-axis occurs if the equation contains only even powers of y. here equation will be uncharged when (-y) is substituted for y. this applies to graphs of the type y^2=f(x)
- Symmetry about the y-axis occurs if the equation contains only even powers of x. Here the equation will be uncharged when (-x) is substituted for x. Here the graph is said to even i.e. f(x) =f(x). For example the graph of y = x^2. **Note** if there are odd powers of x and y then there will be no symmetry.

(b) Determine if there is symmetry about the origin. Here symmetry occurs when a change in the sign of x causes a change in the sign of y without altering its numerical value.

(c) Find the intercepts i.e. the curve cuts the x-axis at a point when y = 0 and cuts the y-axis at the point when x = 0.

(d) The curve passes through the origin if (x, y)= (0, 0)

The behaviour of the neighbour of the curve through the origin is studied by considering the ratio of $\frac{y}{x}$.
- If this ratio is small, the curve keeps close to x-axis near the origin.
- If the ratio is unity, the direction of the curve bisects the angle between the axes.
- If the ratio is large, the curve keeps near the y-axis.

Or:

We consider the behaviour of $\frac{dy}{dx}$ near the origin.

- If $\frac{dy}{dx}$ is very small, then the curve lies near the x-axis.
- If $\frac{dy}{dx}$ is large, then the curve lies near the y-axis.
- If $\frac{dy}{dx}$ is near unity, then the direction of the curve (tangent at origin) bisects the angle between the axes.

(e) Examine the behaviour of the function as x→±∞ and y →±∞ (if any)

(f) Find the turning points and their nature as well as points if inflexion (if any)
Use the second derivative
- For min point, $\frac{d^2y}{dx^2} = +ve$
- For max point, $\frac{d^2y}{dx^2} = -ve$
- Point of inflexion, $\frac{d^2y}{dx^2} = 0$

Example 1

(a) Sketch the graph of y= 5 + 4x – x^2.
Steps taken
- Finding intercepts
  x – intercept; y = 0
  0 = 5 + 4x – x^2.
  5 + 5x –x – x^2 = 0
  5(1 + x) – x(1 + x) =0
  (5 – x)(1 + x)

Either 5 – x = 0; x = 5

Or 1 + x = 0; x = -1

Hence the curve cuts the x-axis at point (-1, 0) and (0, 5)

y – intercept, when x = 0, y = 5

hence the curve cuts the y-axis at point (0, 5)

- As x→+∞, y →-∞ and x→-∞, y →+∞
- Finding turning point

$\frac{dy}{dx} = 4 - 2x$

At turning point $\frac{dy}{dx} = 0$

2x – 4 = 0; x = 2

When x = 2; y= 5 + 4(2) – $(2)^2$= 9

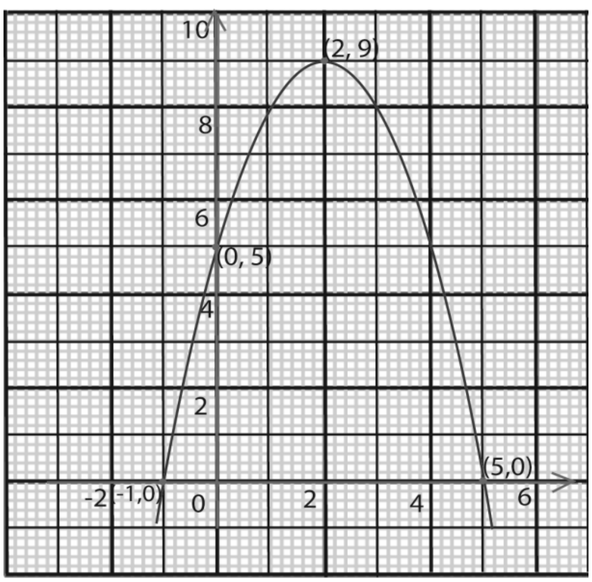
Hence turning point = (2, 9)

Finding the nature of turning point

$\frac{dy}{dx} = 4 - 2x$

$\frac{d^2y}{dx^2} = -2$

Since $\frac{d^2y}{dx^2} < 0$, hence the turning point is maximum.



(b) Sketch the curve y = $x^3 - x^2 - 5x + 6$

Steps taken

For y – intercept; x = 0, y = 0

Hence the y – intercept is (0, 6)

For x – intercept, y = 0

$x^3 - x^2 - 5x + 6 = 0$

error approach is used to find the first factor i.e. (x-2), then other factor is found by long division

$$\begin{array}{r|l} & x^2 + x - 3 \\ (x-2) & x^3 - x^2 - 5x + 6 \\ & -\underline{x^3 - 2x^2} \\ & \quad x^2 - 5x + 6 \\ & \quad -\underline{x^2 - 2x} \\ & \qquad 3x + 6 \\ & \qquad -\underline{3x + 6} \\ & \qquad 0 + 0 \end{array}$$

⇨ $x^3 - x^2 - 5x + 6 = (x - 2)(x^2 + x - 3) = 0$

Solving $x^2 + x - 3 = 0$

$x = \frac{-1\pm\sqrt{1+12}}{2} = \frac{-1\pm3.6}{2}$

x = 1.3 or -2.6

Hence the x- intercepts are (2, 0), (1.3, 0) and (-2.3, 0)

Finding turning points

$y = x^3 - x^2 - 5x + 6$

$\frac{dy}{dx} = 3x^2$-2x-5

At turning point $\frac{dy}{dx} = 0$

$3x^2$-2x-5 = (3x – 5)(x+1) = 0

Either 3x -5 = 0 x = $\frac{5}{3}$

Or x + 1 = 0; x =-1

When x = $\frac{5}{3}$;

$y = \left(\frac{5}{3}\right)^3 - \left(\frac{5}{3}\right)^2 - 5\left(\frac{5}{3}\right) + 6 = \frac{-13}{27}$

When x = -1

$y = (-1)^3 - (-1)^2 - 5(-1) + 6 = 9$

Hence turning points are $\left(\frac{5}{3}; \frac{-13}{27}\right)$ and(-1, 9)

Finding the nature of turning points

$\frac{dy}{dx} = 3x^2$-2x-5

$\frac{d^2y}{dx^2} = 6x - 2$

For$\left(\frac{5}{3}; \frac{-13}{27}\right)$

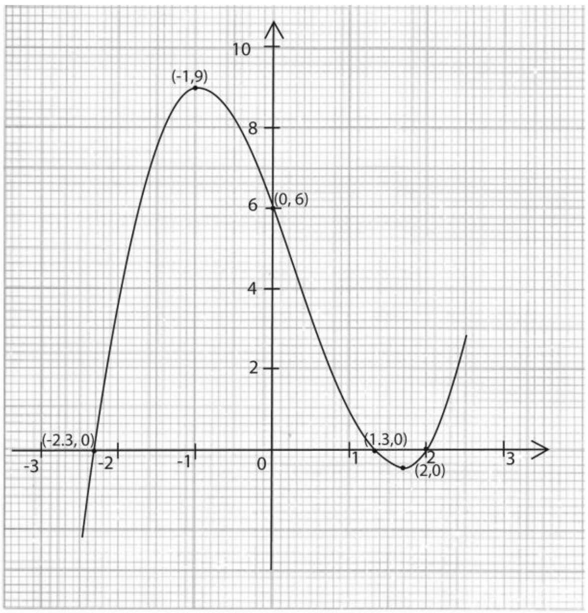
$\frac{d^2y}{dx^2} = 6\left(\frac{5}{3}\right) - 2 = 8\ (> 0)$

$\therefore \left(\frac{5}{3}; \frac{-13}{27}\right)$is minimum

For (-1, 9)

$\frac{d^2y}{dx^2} = 6(-1) - 2 = -8\ (< 0)$

$\therefore (-1; 9\ )$is maximum



(c) sketch the curve y = $x^3 - 8$

y = $x^3 - 8$

Intercepts

When x = 0, y = -8

When y = 0, x = 2

(x, y) = (2, 0)

Turning point: $\frac{dy}{dx} = 3x^2$

$3x^2 = 0$

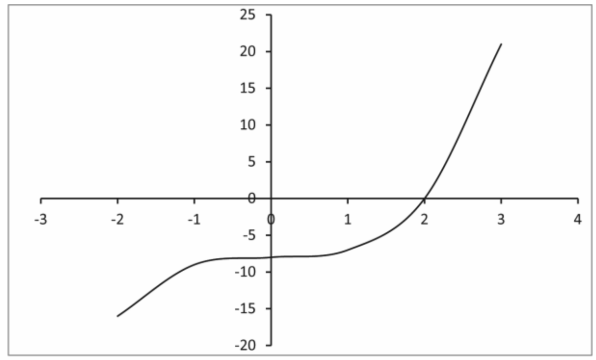
x = 0

$\frac{d^2y}{dx^2} = 6x$

$\frac{d^2y}{dx^2} = 0, x = 0$

Point of reflection= (0, 8)

	x < 2	x > 2
y	-	+



(d) Sketch the curve $y = x^2(x-4)$

Steps taken

- Finding the intercepts

y – intercept, (0,0)

hence y – intercept is (0, 0)

For x – intercept, y= 0

=>$x^2(x-4) = 0$

Either x = 0 or x = 4

Hence x-intercept are (0, 0) and (4, 0)

- As x→+∞, y →+∞ and x→-∞, y →-∞
- Finding turning point(s)

$y = x^2(x-4) = x^3 - 4x^2$

$\frac{dy}{dx} = 3x^2 - 8x$

At turning point, $\frac{dy}{dx} = 0$

⇨ $3x^2 - 8x = x(3x-8) = 0$

Either x = 0

Or x = $\frac{8}{3}$

When x = 0; y=0

When x =$\frac{8}{3}$; =3$\left(\frac{8}{3}\right)^2 - 8\left(\frac{8}{3}\right) = \frac{-256}{27}$

Hence turning points are (0,0) and$\left(\frac{8}{3}, \frac{-256}{27}\right)$

- Finding the nature of turning points

$\frac{dy}{dx} = 3x^2 - 8x$

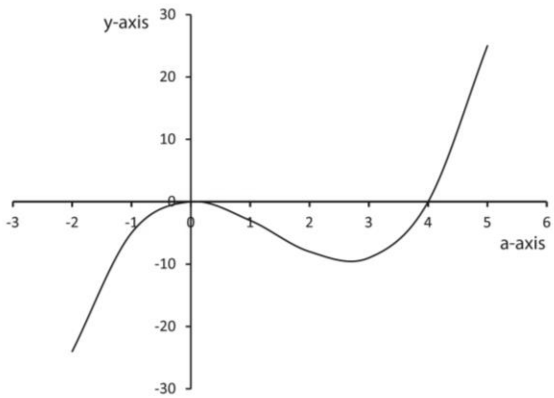
$\frac{d^2y}{dx^2} = 6x - 8$

For (0, 0); $\frac{d^2y}{dx^2} = 6(0) - 8 = -8\ (< 0)$

Hence (0, 0) is maximum

For $\left(\frac{8}{3}, \frac{-256}{27}\right); \frac{d^2y}{dx^2} = 6\left(\frac{8}{3}\right) - 8 = 8\ (>0)$

Hence $\frac{8}{3}$ is minimum



Graphs of rational functions

Rational functions are fractions expressed in the form $y = \frac{f(x)}{g(x)}$.

The basic principles followed when sketching rational curves

(a) Determine if the curve is symmetrical about either or both axes of coordinates.

(b) Find the intercepts on both axes.

(c) Examine the behaviour of the curve as x tends to infinity.

(d) Find the turning points and their nature

(e) Determine the possible asymptotes of the curve

- Vertical asymptote is the value of x which make(s)y tend to infinity. Here we equate the denominator of the function to zero
- Horizontal asymptote is the value of x which make(s)x tend to infinity. Here we divide terms of the numerator and denominators by x with the highest power.

  Alternatively; when finding the horizontal asymptote, we re-arrange the equation and solve for x or make x the subject and then observe the limits, i.e. x→∞, see how y behaves
- Slating asymptotes; this only occurs if horizontal asymptote does not exist and the frictions is improper. Here we divide the terms of the numerator by those of the denominator an y equated to the quotient becomes the asymptote, i.e. asymptote is the y-quotient.

(f) Determine the region where the curve exists/does not exist. This is done by finding a quadratic equation in x such that for real values of x; $b^2>4ac$

Example 2

(a) Sketch the graph of $y = \frac{(x-1)(x+2)}{(x-2)(x+1)}$

Solution

Steps taken

- Finding intercepts

  For y-intercepts; x= 0, $y = \frac{(-1)(2)}{(-2)(+1)} = 1$

  Hence the y-intercept = (0, 1)

  For x-intercept y= 0, $\frac{(x-1)(x+2)}{(x-2)(x+1)} = 0;$

  x=1 or x = -2

  Hence the x-intercept are (1, 0) and (-2, 0)
- Finding turning points

  $y = \frac{(-1)(2)}{(-2)(+1)} = \frac{x^2+x-2}{x^2-x-2}$

  $\frac{dy}{dx} = \frac{(x^2-x-2)(2x+1)-(x^2+x-2)(2x-1)}{(x^2-x-2)^2}$

  At turning point, $\frac{dy}{dx} = 0$

⇨ $\frac{(x^2-x-2)(2x+1)-(x^2+x-2)(2x-1)}{(x^2-x-2)^2} = 0$

  $(2x^3 - x^2 - 5x - 2)$

  $-(2x^3 + x2 - 5x + 2) = 0$

  $2x^2 + 4 = 0$

  $x^2 + 2 = 0$

  There is no real value of x, hence there is no turning points.
- Finding asymptotes;

  Vertical asymptote

  $(x-2)(x+1) = 0$

  Either (x – 2) = 0; x = 2

  Or (x + 1) = 0; x = -1

  Hence the vertical asymptotes are x = 2 and x = 1

  Horizontal asymptotes

  $y = \frac{x^2+x-2}{x^2-x-2}$

  Dividing terms on the LHS by x^2

  $y = \frac{1+\frac{1}{x}-\frac{2}{x^2}}{1-\frac{1}{x}-\frac{2}{x^2}}$

  As x →∞, y→1

Hence the horizontal asymptote y = 1

- Finding the regions where the curve does not exist

$y = \frac{x^2+x-2}{x^2-x-2}$

$y(x^2 - x - 2) = x^2 + x - 2$

$y(x^2 - x - 2) - x^2 - x + 2 = 0$

$(y-1)x^2 + (-y-1)x + (2-2y) = 0$

For real value of x, $b^2 > 4ac$

⇨ $(-y-1)^2 > 8(y-1)(1-y)$

$(y+1)^2 + 8(y-1)^2 > 0$

$9y^2 - 6y + 9 > 0$

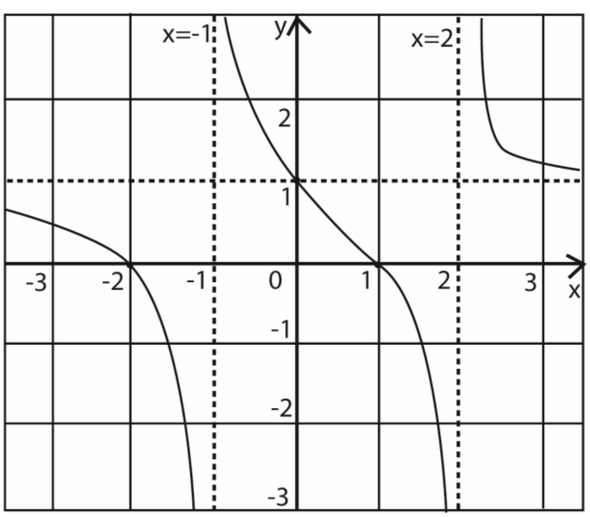
There is no real value of y which means that there is no restriction on y

- Determining the sign of the function throughout its domain. The function will only change sign where the curve cuts the x-axis and vertical asymptotes

The critical value are -, -1, 1, 2

	X<-2	-2<x<-1	-1<x<1	1<x<2	x>2
x-1	-	-	-	+	+
x+2	-	+	+	+	+
x-2	-	-	-	-	+
X+1	-	-	+	+	+
y	+	-	+	-	+

Graph of $y = \frac{(x-1)(x+2)}{(x-2)(x+1)}$



(b) Sketch the graph

Sketch the graph of $y = \frac{x(x-2)}{x+1}$

Steps taken

- Finding the intercepts

For y – intercept; x = 0, and y =0

Hence the y-intercept is (0, 0)

For x- intercept y= 0

⇨ $\frac{x(x-2)}{x+1} = 0$

$x(x-2) = 0$

Either x = 0

Or (x-2) = 0; x = 2

Hence the x-intercepts are(0, 0) and (2, 0)

- Finding turning points

$y = \frac{x^2-2x}{x+1}$

$\frac{dy}{dx} = \frac{x^2-2x(1)-(x+1)(2x-2)}{(x+1)^2} = \frac{x^2+2x-2}{(x+1)^2}$

At turning points, $\frac{dy}{dx} = 0$

⇨ $x^2 + 2x - 2 = 0$

$x = \frac{-2\pm\sqrt{2^2-(4x1x-2)}}{2x1}$

$x = \frac{-2\pm\sqrt{12}}{2} = -1 \pm 1.732$

When x =$-1 + 1.732 = 0.732$

$y = \frac{0.732(0.732-2)}{0.732+1} = -0.54$

When x = $-1 - 1.732 = -2.732$

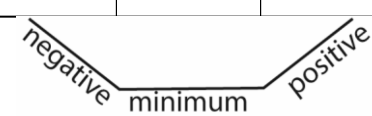
$y = \frac{-2.732(-2.732-2)}{-2.732+1} = -7.46$

Hence the turning points are (0.73, -0.54) and (-2.73, -7.46)

- Finding the nature of the turning points

For (0.73, -0.54)

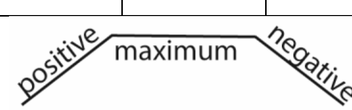
x	O	0.73	3
$\frac{dy}{dx}$	-2	0	0.25



Hence the turning point (0.73, -0.54) is minimum

For (-2.73, -7.46)

x	-3	-2.73	-2
$\frac{dy}{dx}$	0.25	0	-2



Hence the turning point (-2.73, -7.46) is maximum

- Finding asymptotes

For vertical asymptote, the denominator = 0

⇨ x+ 1 = 0; x = -1

since the function is improper fraction, there must be slating asymptote.

Dividing the numerator by denominator;

$$\begin{array}{r|l} & x-3 \\ x+1) & x^2-2x \\ & -\ x^2+x \\ \hline & -\ 3x-3 \\ \hline & 3 \end{array}$$

The slanting asymptote is y = x – 3

X	0	3
y	-3	0

- Finding the region where the curve does not exist.

$y = \frac{x^2-2x}{x+1}$

$y(x+1) = x^2 - 2x$

$x^2 - (2+y)x - y = 0$

For real values of x, $b^2 \geq 4ac$

$(2+y)^2 > 4x\ 1\ x(-y)$

$y^2 + 8y + 4 > 0$

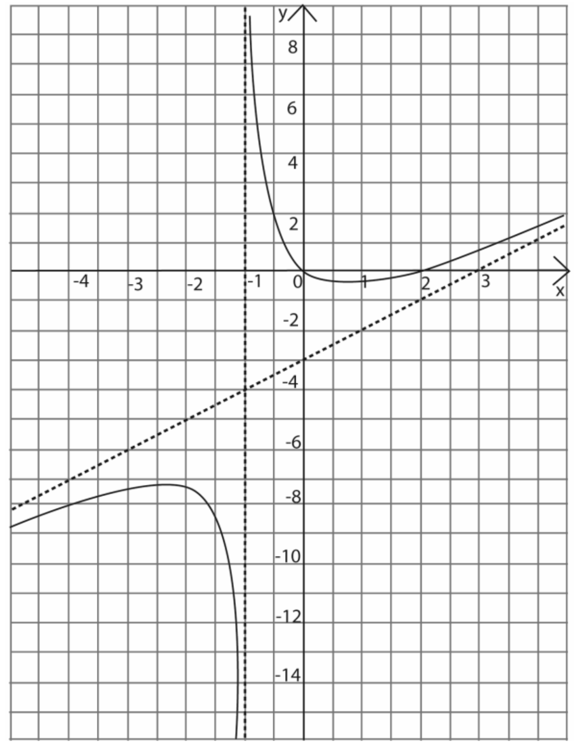
The inequality cannot be factorized therefore we may not proceed further even though there is no real value of y

- Determining the sign of the function through it domain

The critical values are -1, 0, 2

	X<-1	-1<x<0	0<x<2	x>2
x(x-1)	+	+	-	+
x+1	-	+	+	+
y	-	+	-	+

Graph of $y = \frac{x(x-2)}{x+1}$



(c) Given the curve $y = \frac{x(x-1)}{(x-2)(x+1)}$

- Finding intercept

For y-intercept x= 0; y= 0

Hence y-intercept = (0, 0)

For x-intercept y = 0

⇨ x(x-1) = 0

Either x = 0

Or x -1 =0; x = -1

Hence x-intercept as are (0, 0) and (1, 0)

- Finding turning points

$y = \frac{x^2-x}{x^2-x-2}$

$\frac{dy}{dx} = \frac{(x^2-x-2)(2x-1)-(x^2-x)(2x-1)}{(x^2-x-2)^2}$

At turning point $\frac{dy}{dx} = 0$

$(2x-1)\{(x^2-x-2)-(x^2-x)\} = 0$

(2x – 1)(-2) = 0

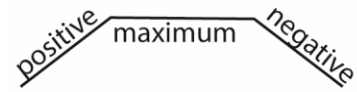
⇨ 2x – 1 = 0; x =$\frac{1}{2}$

When x = $\frac{1}{2}$, $y = \frac{\left(\frac{1}{2}\right)^2-\frac{1}{2}}{\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)-2} = \frac{1}{9}$

Hence turning point is $\left(\frac{1}{2}, \frac{1}{9}\right)$

- Determining nature of turning point

x	0	$\frac{1}{2}$	1
$\frac{dy}{dx}$	2	0	-6



Hence the turning point $\left(\frac{1}{2}, \frac{1}{9}\right)$ is maximum.

- Finding the asymptote(s)

For vertical asymptote

(x – 2)(x + 1) = 0

Either (x – 2) = 0; x = 2

Or (x + 1) =0; x = -1

For horizontal asymptote

Dividing the numerator and denominator on the RHS by x^2.

$y = \frac{1+\frac{1}{x}}{1-\frac{1}{x}-\frac{2}{x^2}}$

As x → ∞, y→1

- Finding the region where x does not exist

$y = \frac{x(x-1)}{(x-2)(x+1)}$

$y(x-2)(x+1) = x(x-1)$

$y(x^2 - x - 2) = x^2 - x$

$yx^2 - yx - 2y - x^2 + x = 0$

$(y-1)x^2 + (1-y)x - 2y = 0$

For real values of x, $b^2 \geq 4ac$

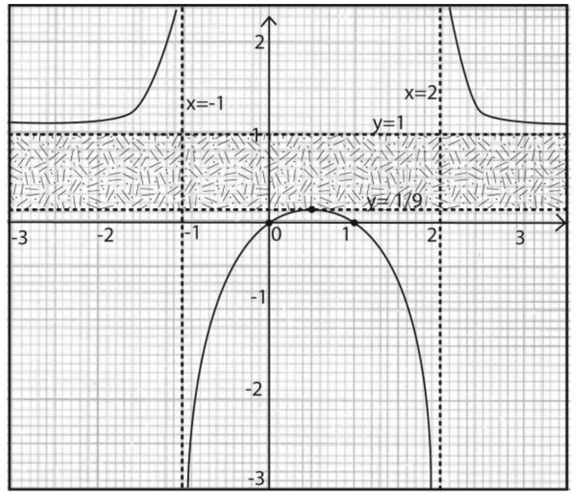
$(1-y)^2 > -8y(y-1)$

$1 - 2y + y^2 + 8y(y-1) > 0$

(9y – 1)(y – 1) = 0

	y<$\frac{1}{9}$	$\frac{1}{9} < y < 1$	Y > 1
9y – 1	-	+	+
y – 1	-	-	+
(9y – 1)(y – 1)	+	-	+

Hence the curve does not lie in the range $\frac{1}{9} < y < 1$



(d) Sketch the curve $y = \frac{x^2+4x+3}{x+2}$

- Finding the range of values over which the curve does not exist

$y = \frac{x^2+4x+3}{x+2}$

$y(x+2) = x^2 + 4x + 3$

$x^2 + (4-y)x + (3-2y) = 0$

For real values of x, $b^2 \geq 4ac$

$(4-y)^2 > 4(3-2y)$

$16 - 8y + y^2 - 12 + 8y \geq 0$

$y^2 + 4 \geq 0$

Since there are no real values of y, this means that there is no restriction on y.

- Finding intercepts

For y intercept, x= 0; $y = \frac{3}{2}$

Hence y- intercept is $\left(0, \frac{3}{2}\right)$

For x – intercepts y = 0

⇨ $\frac{x^2+4x+3}{x+2} = 0$

$x^2 + 4x + 3 = 0$

(x + 3)(x+ 1) = 0

Either (x + 3) = 0; x= -3

Or (x+1) = 0, x = -1

Hence x-intercepts are (-1, 0) and (-3, 0)

- Finding turning points

$y = \frac{x^2+4x+3}{x+2}$

$\frac{dy}{dx} = \frac{(x+2)(2x+4)-(x^2+4x+3)(1)}{(x+2)^2}$

At turning points, $\frac{dy}{dx} = 0$

$x^2 + 4x + 5x = 0$

$x = \frac{-4 \pm \sqrt{4^2 - 4(1)(5)}}{2(1)}$

$x = \frac{-4 \pm \sqrt{-4}}{2}$

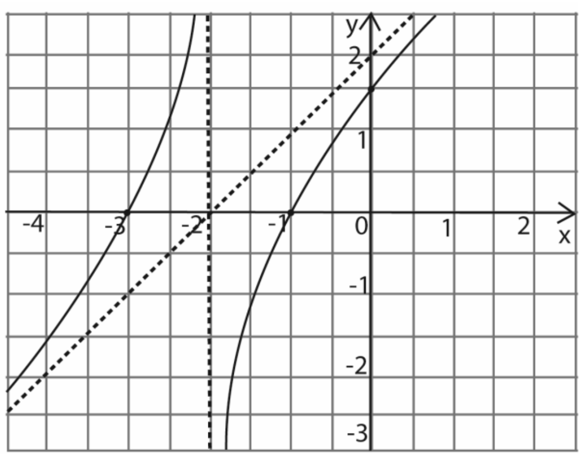
Since there is no real value of x, this means that the curve has no turning point

- Finding vertical asymptote
  (x +2) = 0
  x = -2
  Since the function is improper fraction, there could be a slating asymptote

$$\begin{array}{r|l} & x + 2 \\ (x+2) & x^2 + 4x + 3 \\ & -\ x^2 + 2x \\ & \quad 2x + 3 \\ & \quad -\ 2x + 4 \\ & \quad\quad -1 \end{array}$$

Hence the slanting asymptote is y =x + 2

A curve $y = \frac{x^2+4x+3}{x+2}$



(e) A curve is given by $y = \frac{(x-1)}{(2x-1)(x+1)}$

(i) Show that for real values of x, y cannot take on values in the interval $\left(\frac{1}{9}, 1\right)$

$y = \frac{(x-1)}{(2x-1)(x+1)}$

$y(2x-1)(x+1) = x - 1$

$y(2x^2 + x - y) = x - 1$

$2yx^2 + (y-1)x + (1-y) = 0$

For real values of x, $b^2 \geq 0$

$(y-1)^2 \geq 8y(1-y)$

$(y-1)^2 + 8y(y-1) \geq 0$

(9y – 1)(y – 1) = 0

	$y<\frac{1}{9}$	$\frac{1}{9} < y < 1$	Y > 1
9y – 1	-	+	+
y – 1	-	-	+
(9y – 1)(y – 1)	+	-	+

Hence the curve does not lie in the range $\frac{1}{9} < y < 1$

(ii) Determine the turning points of the curve

$y = \frac{(x-1)}{(2x-1)(x+1)} = \frac{(x-1)}{2x^2+x-1}$

$\frac{dy}{dx} = \frac{(2x^2+x-1)(1)-(x-1)(4x+1)}{(2x^2+x-1)^2}$

$= \frac{-2x^2+4x}{(2x^2+x-1)^2}$

At turning point $\frac{dy}{dx} = 0$

⇨ $-2x^2 + 4x = 0$

−2x(x- 2) = 0

Either 2x = 0; x= 0

Or (x - 2) = 0; x= 2

When x = 0; $y = \frac{-1}{-1} = 1$ => (x, y) =(0, 1)

When x = 2, $y = \frac{1}{3x3} = \frac{1}{9}$=> (x, y) =$\left(2, \frac{1}{9}\right)$

Determining the nature of turning points

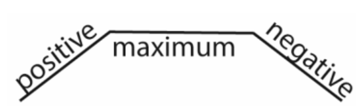
For (0,1)

x	-0.5	0	0.5
$\frac{dy}{dx}$	-2.5	0	1.5

Hence (0, 1) is minimum

For$\left(2, \frac{1}{9}\right)$

x	1	2	3
$\frac{dy}{dx}$	+0.025	0	-0.074



Hence $\left(2, \frac{1}{9}\right)$is maximum

(iii) State with reasons the asymptotes of the curve

For vertical asymptote

(2x – 1)(x + 1) = 0

Either 2x – 1 =0; x=$\frac{1}{2}$

Or (x + 1) = 0; x = -1

For horizontal asymptotes

Dividing the numerator and denominator on the RHS by x

$y = \frac{1-\frac{1}{x}}{2x-1-\frac{1}{x}}$

As x→ ∞, y → 0

Hence horizontal asymptote is y = 0

(iv) Sketch the curve

Finding intercepts

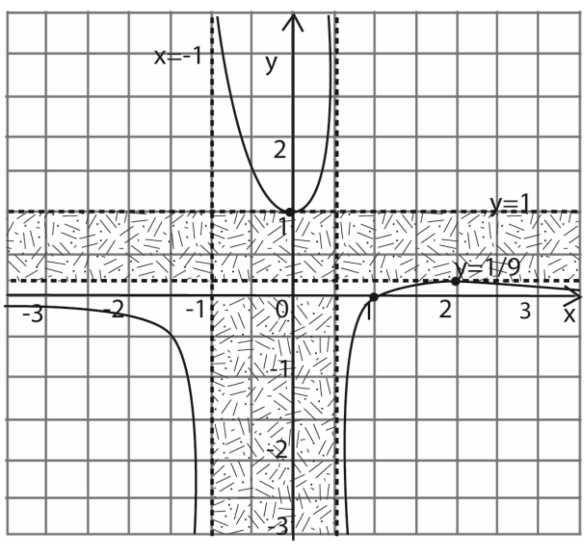
For y-intercept x = 0; y = 1

Hence y-intercept (0,1)

For x-intercept y = 0; x= 1

Hence x – intercept is(1, 0)

A graph of $y = \frac{(x-1)}{(2x-1)(x+1)}$



Sketching graphs of parametric equations

It requires eliminating parameters in the equations given and then following similar steps in above examples.

Example 3

(a) A curve is given by parametric equations

x = t + 2 and y = $\frac{t^2-t}{t+1}$

(a) Find the Cartesian equation of the curve

Solution

x = t – 2; t = x + 2

Substituting t into the equation

$y = \frac{(x-2)^2-(x-2)}{(x-2)+1} = \frac{(x-2)(x-3)}{(x-1)}$

Hence Cartesian equation is $y = \frac{(x-2)(x-3)}{(x-1)}$

(b) Sketch the curve

$y = \frac{(x-2)(x-3)}{(x-1)}$

- Finding intercepts

For y-intercept, x = 0, y = -6

For x-intercept y = 0

⇨ $(x-2)(x-3) = 0$

Either x – 2 = 0; x = 2

Or (x – 3) = 0; x = 3

Hence x- intercepts are (2, 0) and (3, 0)

- Finding the turning points

$y = \frac{x^2-5x+6}{(x-1)}$

$\frac{dy}{dx} = \frac{(x-1)(2x-5)-(x^2-5x+6)(1)}{(x-1)^2} = \frac{x^2-2x-1}{(x-1)^2}$

At turning point $\frac{dy}{dx} = 0$

⇨ $x^2 - 2x - 1 = 0$

$x = \frac{2 \pm \sqrt{(-2)^2-4(1)(-1)}}{2(1)}$

$x = \frac{2\pm\sqrt{8}}{2} = \frac{2\pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$

Either x = 1 $= \sqrt{2}$= 2.4

Or x = 1 $- \sqrt{2}$= -0.4

When x = 2.4

$y = \frac{(2.4-2)(2.4-3)}{(2.4-1)} = -0.17$

When x = -0.4

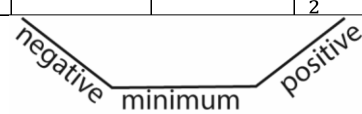
$y = \frac{(-0.4-2)(-0.4-3)}{(-0.4-1)} = -5.83$

Hence the turning points are (2.4, -0.17) and (-0.4, - 5.83)

- Finding the nature of turning points

For (2.4, -0.17)

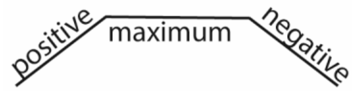
X	2	2.4	3
$\frac{dy}{dx}$	-1	0	$\frac{1}{2}$



Hence the turning point (2.4, -0.17) is minimum.

For (-0.4, -5.83)

X	-1	-0.4	0
$\frac{dy}{dx}$	$\frac{1}{2}$	0	-1



Hence the turning point (-0.4, -5.3) is maximum

- Finding asymptotes

For vertical asymptotes

x – 1 = 0, x= 1

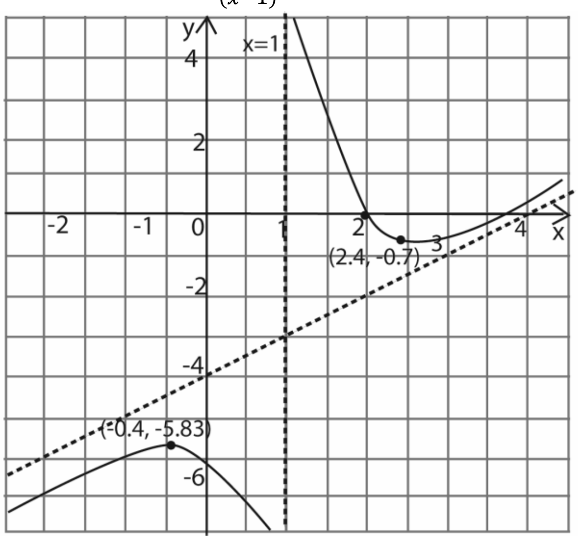
For slanting asymptote

$$\begin{array}{r} x-4 \\ (x-1)\overline{)\,x^2-5x+6} \\ \underline{-\;x^2-x} \\ -4x+6 \\ \underline{-\;-4x+4} \\ 2 \end{array}$$

Hence the slanting asymptote is y = x – 4

x	0	4
y	-4	0

Graph $y = \frac{(x-2)(x-3)}{(x-1)}$



(c) A curve is given by parametric equations

x = cos2θ and y =2sinθ.

(i) Find the equation of the normal to the curve at θ=$\frac{5\pi}{6}$

x = cos2θ

$\frac{dx}{d\theta} = -2sin2\theta$

y =2sinθ.

$\frac{dy}{d\theta} = 2cos\theta$

$\frac{dy}{dx} = \frac{dy}{d\theta}.\frac{d\theta}{dx} = \frac{-2cos\theta}{2sin2\theta} = -\frac{cos\theta}{sin2\theta}$

At θ=$\frac{5\pi}{6}$

$\frac{dy}{dx} = \frac{cos\left(\frac{5\pi}{6}\right)}{sin\left(\frac{5\pi}{3}\right)} = -1$

Gradient of the normal =$\frac{-1}{-1} = 1$

x = cos2θ

$x = cos\frac{5\pi}{3} = \frac{1}{2}$

$y = 2sin\frac{5\pi}{6} = 1$

Let a point (x, y) lie on the normal

⇨ $\frac{y-1}{x-\frac{1}{2}} = 1$

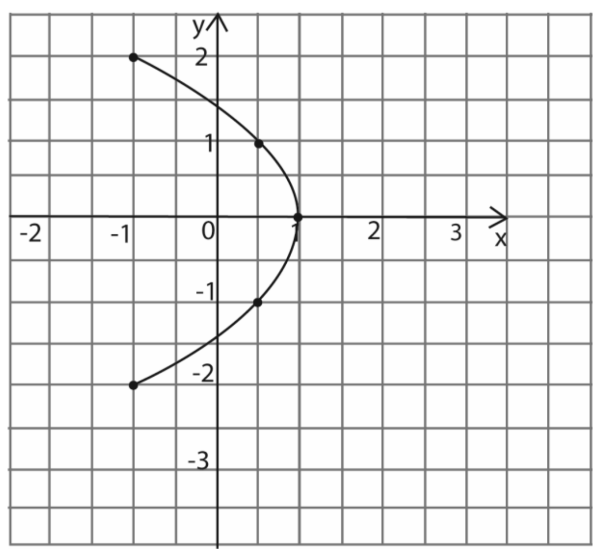
$y = x + \frac{1}{2}$

Hence the equation of the normal to the curve at $\theta = \frac{5\pi}{6}$ is $y = x + \frac{1}{2}$

(ii) Sketch the curve for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
x=cos2θ	-1	-0.5	0.5	1	0.5	-0.5	-1
y=2sinθ	-2	-1.73	-1	0	1.73	1.73	2

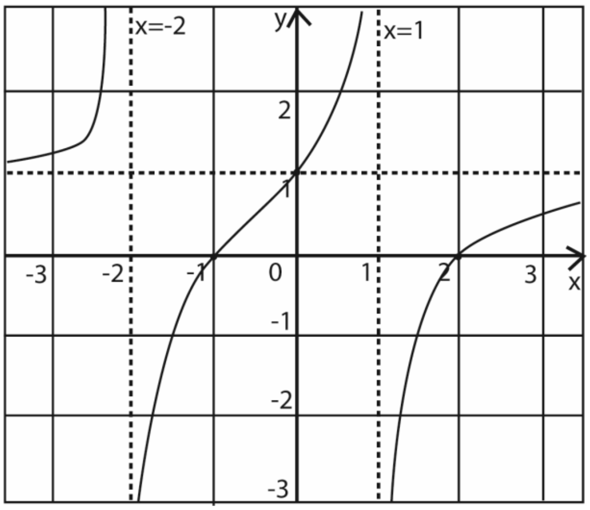
A graph of x = 1 $-\frac{y^2}{2}$



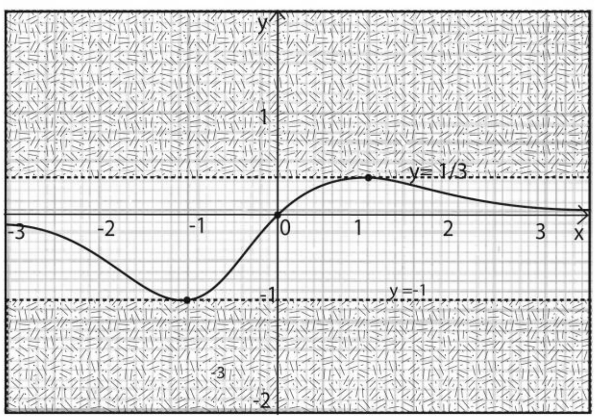
Revision question 1

1. Sketch the graph $y=\frac{(x-2)(x+1)}{(x-1)(x+2)}$

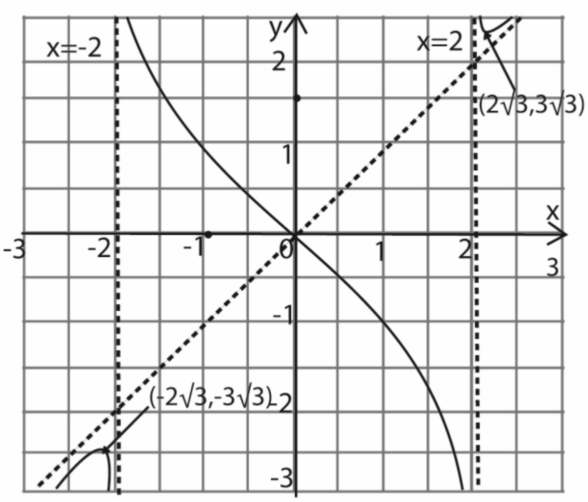
A graph $y=\frac{(x-2)(x+1)}{(x-1)(x+2)}$



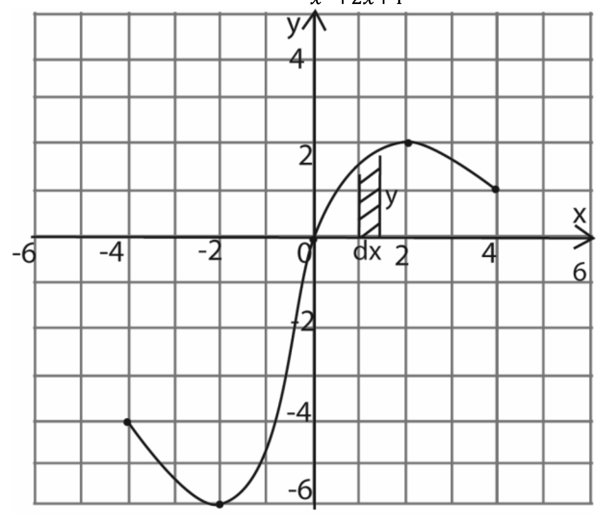
2. Sketch the graph $y=\frac{x}{x^2+x+1}$



3. Sketch the curve$y=\frac{x^2}{x^2-4}$



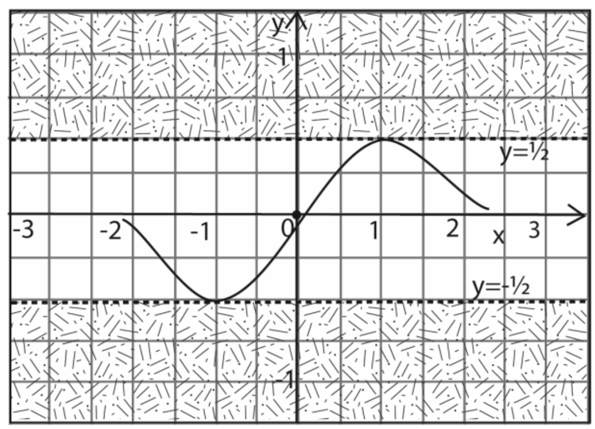
4. (a) Sketch the curve $y=\frac{12}{x^2+2x+4}$



(b) Find the area enclosed by the curve, x-axis and 0≤ x ≤4 [0.259 to3dp)

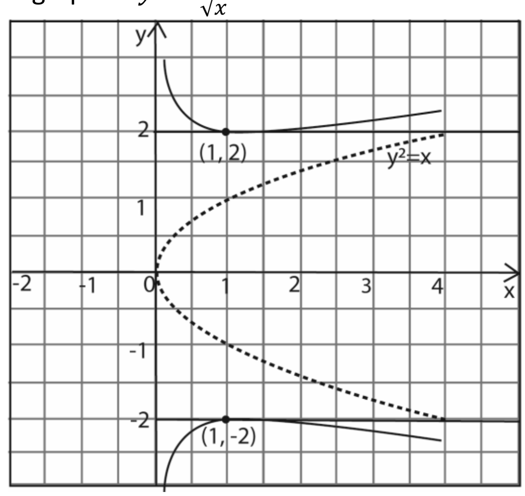
5. Determine the stationary points (including points of inflexion) of the curve $y=\frac{x}{x^2+1}$.

Sketch the curveSketch

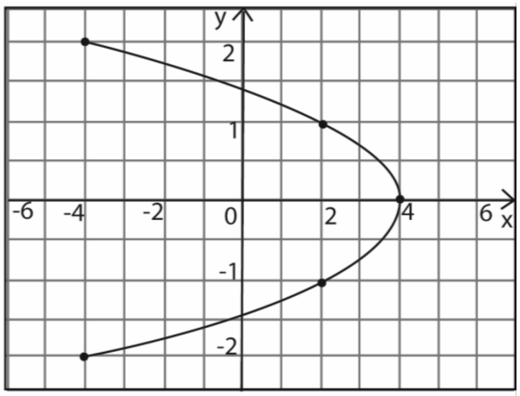


6. Sketch the curve give by the following parametric equations x= t^2 and $y=t+\frac{1}{t}$.

A graph of $y=\frac{x+1}{\sqrt{x}}$

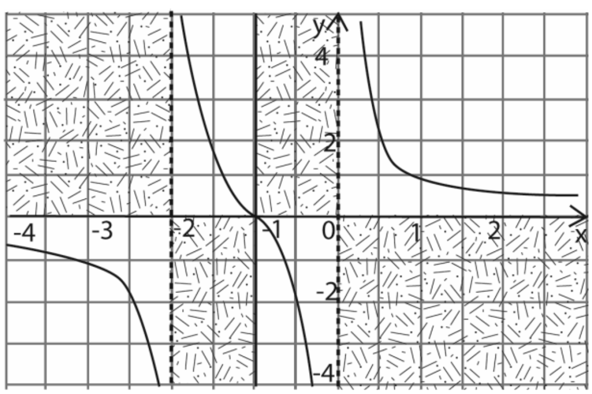


7. A curve is given by the parametric equations x= 4cos2t and y = 2sint
   (i) Find the equation of the normal to the curve at t = $\frac{5\pi}{6}$ [y = 4x – 7]
   (ii) Sketch the curve for $-\frac{\pi}{2} < t < \frac{\pi}{2}$

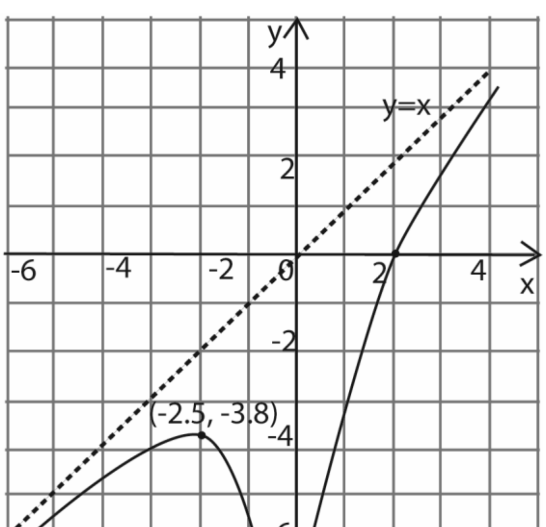


   (iii) Find the area enclosed by the curve and the y-axis [7.543 units (3d,p)]

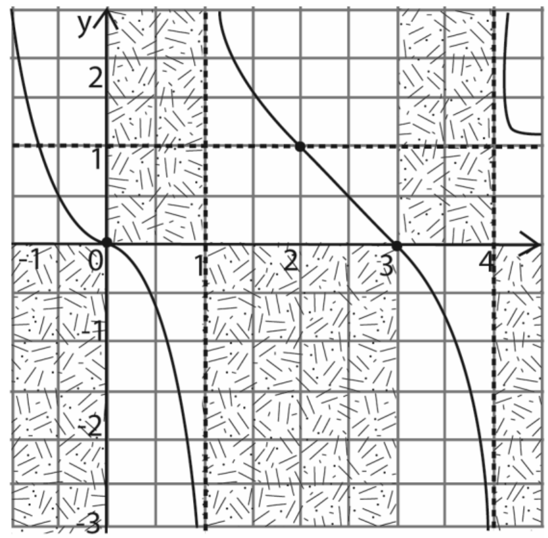
8. Sketch the curve $y = \frac{x+1}{x^2+2x}$



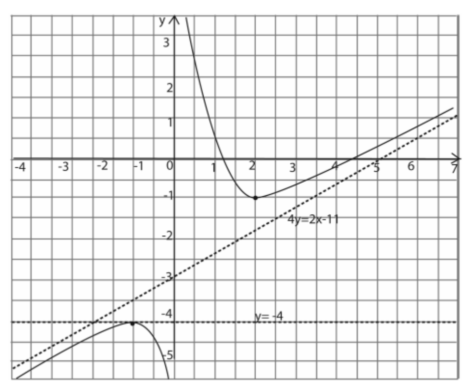
9. Sketch the curve =$y = x - \frac{8}{x^2}$



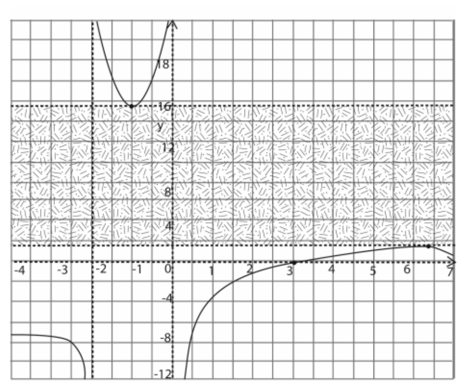
10. Given the curve $y = \frac{x(x-3)}{(x-1)(x-4)}$
   (i) Show that the curve does not have turning points $\left[\frac{dy}{dx} = 0; has\ no\ roots)\right]$
   (ii) Find the equations of asymptotes. Hence sketch the graph



11. Determining the nature of the turning points of the curve $y = \frac{x^2-6x+5}{2x+1}$, sketch the graph of the curve for x= -2 to x=7. Show any asymptotes.



12. Sketch the curve $y = \frac{4(x-3)}{x(x+2)}$



13. Approximations

Approximations

The Maclaurin's Theorem

The polynomial of Maclaurin's series of any infinitely differentiable function, f(x)m whose value and all values of all its derivatives, exist at x = 0

$$f(x) = f(0)x + \frac{f'(0)}{2!}x^2 + \frac{f''(0)}{3!}x^3 + \cdots$$

Maclaurin's series of sinx

Let $f(x) = sinx \;=>\; f(0) = \sin(0) = 0$

$f'(x) = cosx \;=>\; f'(0) = cos(0) = 1$

$f''(x) = -sinx \;=>\; f''(0) - sin(0) = 0$

$f'''(x) = -cosx => f'''(0) = -cos(0) = -1$

$f^{iv}(x) = sinx => f'''(0) = sin(0) = -1$

Note that the fourth derivative takes us back to the starting point. So these values repeat in a cycle of four as 0, 1, 0, -1; 0, 1, 0, -1; etc.

By substitution, we have

$$sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

The Maclaurin's series of sin x is valid for all values of x.

Maclaurin series of cos x

$$cosx = \frac{d}{dx}(sinx)$$

$$= \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{3!} - \frac{x^6}{6!} + \cdots$$

The Maclaurin's series of cos x is valid for all values of x.

Maclaurin's series of e^x

Let f(x) = e^x => f(0) = e^0=1

f'(x) = e^x => f'(0) = e^0=1

f''(x) = e^x => f''(0) = e^0=1

f'''(x) = e^x => f'''(0) = e^0=1 etc.

Here we see that the function and all its derivatives are the same, so these values repeat themselves indefinitely at 1, 1, 1,1, etc. by substitution we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots$$

The Maclaurin's series of e^x is valid for all values of x

Maclaurin series of Inx

Let f(x) = Inx => f(0) =In(0) = ?

f'(x) =$\frac{1}{x}$ => f'(0) = $\frac{1}{0}$=?

f''(x) = $-\frac{1}{x^2}$ => f''(0) = $\frac{1}{0^2}$=?

Here we notice that neither the function nor any of the derivatives exist as x= 0, so there is no polynomial Maclaurin's expansion of natural logarithm, Inx.

Maclaurin series of In(1+x)

Let f(x) = In(1+x) => f(0) =In(1+0) = 0

f'(x) =$-\frac{1}{1+x}$ => f'(0) = $-\frac{1}{1+0}$= 1

f''(x) = $\frac{-1}{(1+x)^2}$ => f''(0) = $\frac{1}{(1+0)^2}$ =-1

f'''(x) = $\frac{2}{(1+x)^3}$ => f'''(0) = $\frac{2}{(1+0)^3}$ =2

f^{iv}(x) = $-\frac{3x2}{(1+x)^4}$ => f^{iv} (0) = $\frac{-3x2}{(1+0)^4}$ =-3x2 etc by substitution we have

$$In(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$$

The Maclaurin's series of In(1+x) is valid for -1< x ≤ 1

Note the validity of **Maclaurin series** is arrived at by using ratio test theorem whose derivation is outside the scope of our coverage

Summary

f(x)	Expansion	Validity
e^x	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$	for all x
e^{-x}	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$	for all x
sinx	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	for all x
cosx	$1 - \frac{x^2}{3!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	for all x
$tan^{-1}x$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	for -1< x ≤ 1
In(1+x)	$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	for -1< x ≤ 1
$(1+x)^k$	$1 + kx - \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$	for -1< x ≤ 1
$\frac{1}{1-x}$	$1 + x + x^2 + x^3 + \ldots\ldots$	for -1< x ≤ 1
$\frac{1}{(1-x)^2}$	$1 + 2x + 3x^2 + 4x^3 + \ldots\ldots$	for -1< x ≤ 1

Answering questions

The questions usually require to produce Maclaurin's series of a function to a specifies nth term and then its application.

Examples

1. Find the Maclaurin series for $(1+x)^{-1}$ as far as x^4.
   Deduce the Maclaurin series for
   (i) In(1+x)
   (ii) In(1-x)
   (iii) $In\left(\frac{1-x}{1+x}\right)$

   Solution

   Maclaurin expansion series is given by
   $$f(x) = f(0) + f'^{(0)}x + \frac{f''(0)}{2!} + \frac{f'''(0)}{3!} + \ldots$$

   Let $f(x) = (1+x)^{-1} \Rightarrow f(0) = (1+0)^{-1} = 0$

   $f'(x) = -1(1+x)^{-2} \Rightarrow f'(0) = -1(1+0)^{-2} = -1$

   $f''(x) = 2(1+x)^{-3} \Rightarrow f'(0) = 2(1+0)^{-3} = 2$

   $f'''(x) = -6(1+x)^{-4} \Rightarrow f'''(0) = -6(1+0)^{-4} = -6$

   $f^{iv}(x) = 14(1+x)^{-5} \Rightarrow f^{iv}(0) = 24(1+0)^{-5} = 24$

   By substitution we have

   $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4$

   (i) We know the $\int \frac{dx}{1+x}$=In(1+x)
   ⇨ In(1+x)=$\int(1 - x + x^2 - x^3 + x^4)dx$
   $= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$

   This valid for-1 < x ≤ 1

   (ii) Replacing x by –x in (i)

   In(1-x) $= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5}$

   This valid for-1 < x ≤ 1

   (iii) Subtracting (ii) from (i)

   In(1+x) – In(1-x)= $2x + \frac{2x^3}{3} + \frac{2x^5}{5}$

2. Find the Maclaurin series for (1+x)-1 as far as x^4.
   Deduce the Maclaurin series for

(i) $\frac{1}{1+x^2}$ as far as x^6.

(ii) $tan^{-1}x$ as far as x^7.

Show that $tan^{-1}\left(\frac{1}{2}\right) + tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{4}$

Solution

Maclaurin expansion series is given by

$f(x) = f(0) + f'^{(0)}x + \frac{f''(0)}{2!} + \frac{f'''(0)}{3!}+...$

Let f(x) = $(1+x)^{-1}$ => f(0)= $(1+0)^{-1}$ = 0

f'(x) = $-1(1+x)^{-2}$ => f'(0) = $-1(1+0)^{-2}$=-1

f''(x) = $2(1+x)^{-3}$ => f'(0) = $2(1+0)^{-3}$= 2

f'''(x) = $-6(1+x)^{-4}$ => f'''(0) = $-6(1+0)^{-4}$= -6

f^{iv}(x) = $14(1+x)^{-5}$ => f^{iv}(0) = $24(1+0)^{-5}$= 24

By substitution we have

$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4$

(i) Replacing x by x^2 gives

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6$$

(ii) We know that $\int \frac{dx}{1+x^2} = tan^{-1}x$

⇨ $tan^{-1}x = \int(1 - x^2 + x^4 - x^6)\,dx$

$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$

We also know that

$tan^{-1}A + tan^{-1}B = \frac{A+B}{1-A.B}$

⇨ $tan^{-1}\left(\frac{1}{2}\right) + tan^{-1}\left(\frac{1}{3}\right) = \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}$

$= tan^{-1}(1) = \frac{\pi}{4}$

3. Use Maclaurin theorem to expand e^x up to the term x^4, use your expansion to evaluate e correct to 4 decimal places.

Let f(x) = e^x => f(0) = e^0=1

f'(x) = e^x => f'(0) = e^0=1

f''(x) = e^x => f''(0) = e^0=1

f'''(x) = e^x => f'''(0) = e^0=1 etc.

by substitution we have

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

Evaluating e

e =e^1, substituting for x = 1

$e^1 = 1 + (1) + \frac{(1)^2}{2!} + \frac{(1)^3}{3!} + \frac{(1)^4}{4!}$

$= 2 + \frac{1}{6} + \frac{1}{24} = \frac{65}{24} = 2.7083\ (4d.p)$

4. Expand $\sqrt{\left(\frac{1+2x}{1-x}\right)}$ up to the term x^2. Hence find the value of $\sqrt{\left(\frac{1.04}{0.98}\right)}$ to four significant figures. (12marks)

$\sqrt{\left(\frac{1+2x}{1-x}\right)} = (1+2x)^{\frac{1}{2}}(1-x)^{\frac{-1}{2}}$

Using $(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \cdots$

$\sqrt{\left(\frac{1+2x}{1-x}\right)} = \left(1 + x - \frac{1}{2}x^2\right)\left(1 + \frac{1}{2}x + \frac{3}{8}x^2\right)$

$= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + x + \frac{1}{2}x^2 - \frac{1}{2}x^2$

$= 1 + \frac{3}{2}x + \frac{3}{8}x^2$

$\therefore \sqrt{\left(\frac{1+2x}{1-x}\right)} \approx 1 + \frac{3}{2}x + \frac{3}{8}x^2$

Substituting for x = 0.02

$\sqrt{\left(\frac{1.04}{0.98}\right)} = \sqrt{\frac{1+2(0.02)}{1-0.02}}$

$= 1 + \frac{3}{2}(0.02) + \frac{3}{8}(0.02)^2$

$= 1.030$

5. Obtain the first two non-zero terms of Maclaurin's series for sec x

$f(x) = f(0) + f'^{(0)}x + \frac{f''(0)}{2!} + \frac{f'''(0)}{3!}+...$

$f(x) = \sec x$ => $f(0) = sec0 = 1$

f'(x) =secxtanx => f'(0)= sec0tan0 =0

f''(x) =secxsec^2x + tanxsecxtanx

⇨ f''(0)=sec0$sec^2$0+tan0sec0tan0= 1+0 = 1

Hence the first two non-zero terms of Maclaurin series of $sec\ x = 1 + \frac{x^2}{2}$

Revision exercise

1. Use Maclaurin theorem to expand the following up to
   (i) $In\left(\frac{1+x}{1-x}\right)$ up to x^3. Hence, find the approximation of In2 correct to 3 significant figure

   $\left[In\left(\frac{1+x}{1-x}\right) = 2x + \frac{2}{3}x^3; 0.691\right]$

   (ii) $e^{-x}sinx$
   $\left[x - x^2 + \frac{1}{3}x^3\right]$

   (iii) $In\sqrt{\left(\frac{1+sinx}{1-sinx}\right)}$ $\left[2x + \frac{x^3}{6}\right]$

   (iv) $In(1 + sinx)$ $\left[x - \frac{x^2}{2} + \frac{x^3}{6}\right]$

   (v) $In(1 + x)^2$ $\left[2x - x^2 + \frac{2x^3}{3} - \frac{x^4}{2}\right]$

   $(vi) \frac{1}{\sqrt{(1+x)}}$ $\left[1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16}\right]$

2. Given $y = tan^{-1}\sqrt{1-x}$ show that
   (i) $(2-x)\frac{dy}{dx} + \frac{1}{2\sqrt{(1-x)}} = 0$
   (ii) $(2-x)\frac{d^2y}{dx^2} - \frac{dy}{dx} + \frac{1}{4}(1-x)^{-\frac{1}{2}} = 0$

3. Use Maclaurin theorem to show that
   (i) $\frac{cosx}{1-x^2} = 1 + \frac{1}{2}x^2 + \frac{11}{24}x^4$
   (ii) $e^{-x}sinx = \frac{x}{3}(x^2 - 3x + 3)$. Hence evaluate$e^{-x}sin\frac{\pi}{3}$ to 4d.p [0.3334]

4. Given that $y = e^{tan^{-1}x}$, show that $(1 + x^2)\frac{d^2y}{dx^2} + (2x - 1)\frac{dy}{dx} = 0$. Hence or otherwise, determine the first four non-zero terms of the Maclaurin expansion of y
   $\left[1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right]$

5. Given that y = $In\left\{e^x\left(\frac{x-2}{x+2}\right)^{\frac{3}{4}}\right\}$, show that
   $\frac{dy}{dx} = \frac{x^2-1}{x^2-4}$

6. Use Maclaurin's theorem to express In(sinx+cosx) as a power series up to the term x^2.[$x - x^2$]

Trapezium rule

It is used for estimating an integral area under a curve of continuous function over a given interval [a, b]

if y = f(x)

A = $\int_a^b ydx$

Using several strips between x = a and x = b of equal width, trapezium rule can be used to determine the area.

A $\approx \frac{1}{2}h[(first + last\ ordinates) + 2(sum\ of\ the\ middle\ ordinates)]$

where h = $\frac{b-a}{subintervals}$

Note

(i) sub-intervals, subdivision and strips are the same
(ii) subinterval = ordinates
(iii) when dealing with a trigonometric function, calculators must be in radian mode
(iv) when the final answer is required to a specific number of d.p'c, the working's should be done at least a d.p higher but the final answer rounded to the required d.p's

Example 1

Use the trapezium rule with four-intervals to estimate $\int_{0.2}^{1.0}\left(\frac{2x+1}{x^2+x}\right)dx$. Correct to two decimal places.

Let y = $\left(\frac{2x+1}{x^2+x}\right)$

h = $\frac{1.0-0.2}{4}$ = 0.2

x	y = $\frac{2x+1}{x^2+x}$	
0.2	5.8333	
0.4		3.2143
0.6		2.2917
0.8		1.8056
1.0	1.5000	
Sum	7.3333	7.3116

$\int_{0.2}^{1.0}\left(\frac{2x+1}{x^2+x}\right)dx = \frac{1}{2}\ x\ 0.2(7.3333 + 7.3116\ )$
= 2.1955
= 2.20 (2D)

Example 2

Use the trapezium rule with seven coordinates to estimate

$\int_0^3[(1.2)^x - 1]^{\frac{1}{2}}$dx correct to 2 decimal places (05marks)

Solution

For 7 ordinates, there are 6 subintervals

Width, h = $\frac{b-a}{subinterval} = \frac{3-0}{6} = 0.5$

Let y = $\sqrt{(1.2)^x - 1}$

x	y	
0	0	
0.5		0.309
1		0.447
1.5		0.561
2		0.663
2.5		0.760
3	0.853	
Sum	0.853	2.74

Using the trapezium rule

$\int_0^3 [(1.2)^x - 1]^{\frac{1}{2}}$dx = $\frac{0.5}{2}[0.853 + 2(2.74)]$ = 1.58

Example 3

(a) Use the trapezium rule with 6-ordinated to estimate the value of $\int_0^{\frac{1}{2}} (x + sinx)dx$, correct to three decimal places.

$h = \frac{\frac{1}{2}-0}{5} = \frac{\pi}{10}$

x	y	
0	0	
$\frac{\pi}{10}$		0.6232
$\frac{2\pi}{10}$		1.2161
$\frac{3\pi}{10}$		1.7515
$\frac{4\pi}{10}$		2.2077
$\frac{\pi}{2}$	2.5708	
Sum	2.5708	5.7985

$\int_0^{\frac{1}{2}} (x + sinx)dx = \frac{1}{2}\ x\ \frac{\pi}{10}(2.5708 + 2\ x\ 5.7985)$

= 2.225

(b)(i) Evaluate $\int_0^{\frac{1}{2}} (x + sinx)dx$, correct to three decimal places

$\int_0^{\frac{1}{2}} (x + sinx)dx = \left|\frac{x^2}{2} - \cos x\right|_0^{\frac{1}{2}}$

$= \frac{1}{2}\left(\frac{\pi^2}{4} - 0\right) - (\cos\frac{\pi}{2} - \cos 0)$

$= \frac{\pi^2}{8} + 1$

= 2.234

(ii) Calculate the error in your estimation in (a) above

Error = $|2.234 - 1.225|$ = 0.009

(iii) Suggest how the error may be reduced (06marks)

Increasing on number of strips or subintervals

Example 4

A student used the trapezium rule with five sub-intervals to estimate

$\int_2^3 \frac{x}{(x^2-3)} dx$ correct to **three** decimal places

Determine;

(a) The value the student obtained (06marks)

h = $\frac{3-2}{5}$ =0.2

X	y_1, y_6	y_2 ……y_5
2.0	2.0	
2.2		1.1956
2.4		0.8696
2.6		0.6915
2.8		0.5785
3	0.5	
Sum	2.5	3.3352

$\int_2^3 \frac{x}{(x^2-3)} dx = \frac{1}{2} x\ 0.2[2.5 + 2(3.3352)]$

= 0.91704 = 0.917 (3D)

(b) The actual value of the integral (03marks)

$\int_2^3 \frac{x}{(x^2-3)} dx = \left[\frac{1}{2} Inx^2 - 3\right]_2^3$

$=\frac{1}{2}(In\ 6 - In\ 1)$

= 0.896

(c) (i) the error the student made in the estimate

Error = $|0.896 - 0.917|$ = 0.021

(ii) how the student can reduce the error(03marks)

Increasing on the number of sub-intervals or ordinates or reducing the width of h

Example 5

Use trapezium rule with 4 subintervals to estimate to 3 decimal places $\int_0^{\frac{\pi}{2}} \cos x dx$

Solution

h = $\frac{\frac{\pi}{2}-0}{4} = \frac{\pi}{8}$

x	f(x) = cos x	
0	1.0000	
$\frac{\pi}{8}$		0.9239
$\frac{2\pi}{8}$		0.7071
$\frac{3\pi}{8}$		0.3827
$\frac{4\pi}{8}$	0.0000	
sum	1.0000	2.0137

$\int_0^{\frac{\pi}{2}} \cos x dx = \frac{1}{2} x \ \frac{\pi}{8}[1 + 2\ x\ 2.0137]$

=0.987

Example 6

Use trapezium rule with 7 ordinates to estimate $\int_0^3 \frac{1}{1+x} dx$ correct to 3dp

Solution

h = $\frac{3-0}{7-1} = 0.5$

x	f(x) = cos x	
0	1.0000	
0.5		0.6667
1.0		0.5000
1.5		0.4000
2.0		0.3333
2.5		0.2757
3.0	0.2500	
sum	1.2500	2.157

$\int_0^3 \frac{1}{1+x} dx = \frac{1}{2} x\ 0.5[1.25 + 2\ x\ 2.1857]$

= 1.405

Example 7

(a) Use the trapezium rule to estimate the integral value of $\int_2^3 \frac{x}{1+x^2} dx$ using five subinterval and correct to 3d.p.

(b) (i) find the exact value of $\int_2^3 \frac{x}{1+x^2} dx$

(ii) suggest how the error may be reduced.

(a) h = $\frac{3-2}{5} = 0.2$

x	f(x) = $\frac{x}{1+x^2}$	
2.0	0.40000	
2.2		0.37671
2.4		0.35503
2.6		0.33505
2.8		0.31674
3.0	0.30000	
sum	0.70000	1.3353

$\int_2^3 \frac{x}{1+x^2} dx = \frac{1}{2} x\ 0.2[0.7 + 2\ x\ 1.38353]$

= 0.3467

(b)(i) $\int_2^3 \frac{x}{1+x^2}dx = \left[\frac{1}{2}In(1+x^2)\right]_2^3 = \frac{1}{2}(In10\ In5)$ = 0.3466

(ii) error = $|exact\ value - approximate\ value| = |0.3466 - 0.3467|$ = 0.0001

(iii) the error can be reduced by reducing h or increasing the number of sub-intervals.

Example 8

(a) Use trapezium rule to estimate the integral value of $\int_0^1 x^2e^x\,dx$

(b) (i) find exact value of $\int_0^1 x^2e^x\,dx$

(ii) determine the percentage error in your estimation

(a) h = $\frac{1-0}{5} = 0.2$

x	f(x) = x^2e^x	
0	0	
0.2		0.0489
0.4		0.2387
0.6.		0.6560
0.8		1.4243
1.0	2.7183	
sum	2.7183	2.3679

$\int_0^1 x^2e^x\,dx = \frac{1}{2}x\ 0.2[2.713 + 2\ x\ 2.3679]$

=0.74541≈ 0.745

(b)(i) $\int_0^1 x^2e^x\,dx = [x^2e^x - 2xe^x + 2e^x]_0^1$

= 0.718

(ii) error = $|0.718 - 0.745|$= 0.027

Percentage error = $\frac{error}{exact\ value}\ x\ 100\%$

= $\frac{0.027}{0.718}\ x\ 100$ = 3.8%

Revision Exercise

1. (a) Use trapezium rule with six strips to estimate $\int_0^{\pi} xsinxdx$ [3.069]

   (b) Determine the percentage error in your determination. [2.3%]

2. Use the trapezium rule to estimate the approximate value of $\int_0^1 \frac{1}{1+x^2dx}$ using 6 ordinates and correct to 3 decimal places. [0.784]

3. (a) Use trapezium rule with six strips to estimate $\int_2^4 \frac{10}{2x+1}dx$ correct 4dp. [2.9418]

   (b) Determine the percentage error in your estimation and suggest how this error may be reduce. [0.098%]

4. (a) Use trapezium rule to estimate the area of y = 3x between x-axis, x = 1 and x = 2, using five subintervals. Give your answer correct to four significant figures. [5.483]

   (b) Find the exact value of $\int_1^2 3^x dx$ [5.461]

   (c) Find the exact percentage error in calculations (a) and (b) above. [0.4028%]

5. Use trapezium rule with 7 ordinates to estimate $\int_0^3 \frac{1}{1+x}dx$, correct to 3 decimal places [1.405]

6. Use the trapezium rule with 6 ordinates to evaluate $\int_0^1 e^{-x^2}$ correct to 2 decimal place. [0.74]

7. Use the trapezium rule with 6 ordinates to estimate $\int_1^2 \frac{Inx}{x}dx$.Give your answer correct to 3 decimal places [0.237]

8. Find the approximate value to one decimal place of $\int_0^1 \frac{dx}{1+x}$, using the trapezium rule with five strips. [0.7]

9. (a) Use trapezium rule with five subintervals to estimate $\int_0^{\frac{\pi}{3}} \tan x\,dx$ correct to 3dp. [0.704]

(b) (i) Find the exact value of $\int_0^{\frac{\pi}{3}} \tan x \, dx$ to 3 d.p. [0.693]

(ii) Calculate the percentage error in your estimation in (a) above [1.587%]

(iii) Suggest how the percentage error in (b)(ii) may be reduced.

10. Use the trapezium rule with four subdivisions to estimate $\int_0^{\frac{\pi}{2}} \frac{1}{1+sinx} dx$. Give your answer correct to three decimal places. [1.013]
11. Find the approximate value of $\int_0^2 \frac{1}{1+x^2} dx$ using trapezium rule with 6 ordinates. Give your answer to 3 decimal places (05marks)[1.105]
12. Use the trapezium rule with five subintervals to estimate $\int_2^4 \frac{5}{(x+1)} dx$. Give your answer correct to 3 decimal places (05marks)[2.559]
13. A student used the trapezium rule with five sub-intervals to estimate $\int_2^3 \frac{x}{(x^2-3)} dx$ correct to **three** decimal places

    Determine;

    (a) The value the student obtained (06marks) [0.917]

    (b) The actual value of the integral (03marks) [0.896]

    (c) (i) the error the student made in the estimate [0.021]

    (ii) how the student can reduce the error(03marks)

14. (a) Use the trapezium rule with 6-ordinated to estimate the value of $\int_0^{\frac{1}{2}} (x + sinx)dx$, correct to three decimal places, [2.225]

    (b)(i) Evaluate $\int_0^{\frac{1}{2}} (x + sinx)dx$, correct to three decimal places [2.234]

    (ii) Calculate the error in your estimation in (a) above [0.009]

    (iii) suggest how the error may be reduced (06marks)

14.Coordinate geometry 1

Coordinate geometry 1

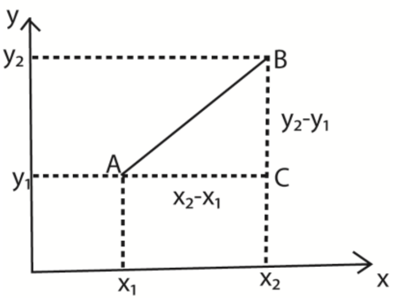
This is the area of mathematics where geometrical relationships are described algebraically by reference to the coordinates

The length of the line segment

Given two points A(x1, y1) and B(x2, y2) in x-y plane, the distance between A and B, denoted by $\overline{AB}$ is $\overline{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)}$

Proof

Geometrical approach



Using Pythagoras theorem

$$\overline{AB} \ = \overline{AC} \ + \overline{BC}$$

$$\overline{AB} \ = (x_2 - x_1)^2 + (y \ - y_1)$$

$$\overline{AB} = \sqrt{(x_2 - x_1)^2 + (y \ - y_1)}$$

Vector method approach

AB = OB – OA

$$= \begin{pmatrix} x \\ y_2 \\ z \end{pmatrix} - \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

$$= \begin{pmatrix} x & - x_1 \\ y & - y_1 \\ z & - z_1 \end{pmatrix}$$

Using Pythagoras theorem

$$\overline{AB} = \sqrt{(x \ - x_1) \ + (y \ - y_1)}$$

Example 1

(a) Find the between the points

(i) A(1, 3) and B(7, 11)

Solution

Using $\overline{AB} = \sqrt{(x \ - x_1) \ + (y \ - y_1)}$

$\overline{AB} = \sqrt{(7 - 1) \ + (8 - 3)}$

$= \sqrt{6 \ + 8}$

$= \sqrt{100}$ = 10

(ii) P(-1, 2) and Q(3, 7)

Solution

$\overline{PQ} = \sqrt{(-1 - 3) \ + (2 - 7)}$

$= \sqrt{(-4) \ + (-5)}$

$= \sqrt{25}$ = 5

(b) The points A, B and C have coordinates A(-3, 2), B(-1, -2) and C(0, n) where n is a constant. Given that $\overline{BC} = \frac{1}{5}AC$, find the possible values of n.

Solution

Using $\overline{AB} = \sqrt{(x \ - x_1) \ + (y \ - y_1)}$

$\overline{BC} = \sqrt{(0 - (-1)) \ + (n - (-2))}$

$= \sqrt{1 \ - (n + 2)}$

=$\sqrt{n \ + 4n + 5}$

$\overline{AC} = \sqrt{(0 - (-3)) \ + (n - 2)}$

$= \sqrt{3 \ - (n - 2)}$

=$\sqrt{n \ - 4n + 13}$

Bu t$\overline{BC} = \frac{1}{5}AC$

⇨ $\sqrt{n \ + 4n + 5}$=$\frac{1}{5}\sqrt{n \ - 4n + 13}$

$5\sqrt{n \ + 4n + 5} = \sqrt{n \ - 4n + 13}$

Squaring both sides

$25(n \ + 4n + 5)$= $n \ - 4n + 13$

$24n^2 + 104n + 112 = 0$

$3n^2 + 13n + 14 = 0$

$(3n + 7)(n + 2) = 0$

Either n = — – or n = -2

Hence the values of n are— – and -2

To show that given points are vertices of a right-angle triangle.

Suppose that the points A, B and C are vertices of a triangle ABC, to show that ABC is a right angled triangle, then by applying the Pythagoras theorem

Either $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$

or $\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2$

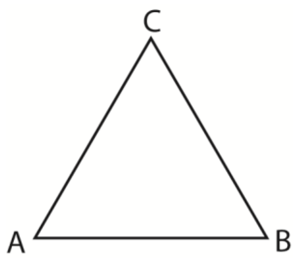
or $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$

Example 2

Prove that the following points are vertices of a right-angled triangle

(a) A(2, 3), B(5, 6) and C(8, 3)

Solution



$\overline{AC}^2 = (8-2) \ + (3-3) \ = 36$

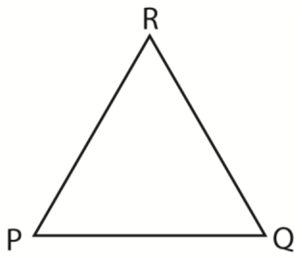
$\overline{AB}^2 = (5-2) \ + (6-3) \ = 18$

$\overline{BC}^2 = (8-5) \ + (3-6) \ = 18$

Since $\overline{AB}^2$ $+\overline{BC}^2$ $=\overline{AC}^2$ = 36, the triangle ABC is a right angled triangle

(b) P(2, 1), Q(5, -1) and R(9, 5)

Solution



$\overline{P2} = (5-2) \ + (-1-1) \ = 13$

$\overline{PR} = (9-2) \ + (5-1) \ = 65$

$\overline{QR} = (9-5) \ + (5-(-1)) \ = 52$

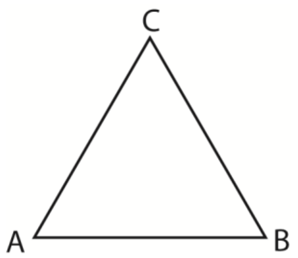
Since $\overline{PQ}$ + $\overline{QR}$ = $\overline{PR}$ = 65, the triangle PQR is a right angled triangle

Note: if the triangle is **isosceles**, then two of the sides must be equal and for **equilateral** triangle all the sides must be equal

Example 3

Prove that the following points A(1, 2), B(3, 7) and C(6, 14) are vertices of an isosceles triangle.

Solution



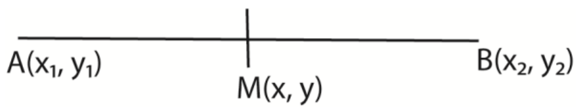
$\overline{AC} = (6-1) \ + (14-2) \ = 169$

$\overline{AB} = (13-1) \ + (7-2) \ = 169$

$\overline{BC} = (6-13) \ + (14-7) \ = 98$

Since $\overline{AC}$ = $\overline{AB}$ = 169, hence, the triangle ABC, is isosceles triangle.

The mid-point of a line segment



The mid-point , M of a line segment AB with $A(x_1, y_1)$ and $B(x_2, y_2)$ is given as

$M\left(\frac{x_1+x_2}{},\frac{y_1+y_2}{}\right)$

Example 4

(a) Find the coordinates of the midpoint of the line joining each of the following pairs of points

(i) A(8, 4) and B(2, -4)

Solution

The midpoint of AB = $M\left(\frac{x_1+x_2}{},\frac{y_1+y_2}{}\right)$

= $M\left(\frac{8+}{},\frac{4-4}{}\right)$

= M(5, 0)

(ii) P(-6, -2) and Q(-4, -5)

Solution

The midpoint of PQ = $M\left(\frac{x_1+x_2}{},\frac{y_1+y_2}{}\right)$

= $M\left(\frac{-6-4}{},\frac{-2-5}{}\right)$

= M(-5, -3.5)

(b) Find the coordinated of point S given that M(3, -2) is the midpoint of the straight line joining S to T(9, -2)

Solution

The midpoint of PQ = $M\left(\frac{x_1+x_2}{},\frac{y_1+y_2}{}\right)$

⇨ $\frac{x+9}{} = 3$

$x = -3$

Also $\frac{y-2}{} = -2$

y = -2

Hence S(-3, -2)

Exercise 1

1. Find the distance between each of the following pairs of points
   (a) (-7, 3) and (-2, 5) [$\sqrt{29}$]
   (b) (2, -3) and (7, 7) [$5\sqrt{5}$]
   (c) (4, -1) and (-2, 1) [$2\sqrt{10}$]
2. Prove that a triangle with vertices (1, 2), (13, 7) and (6, 14) is isosceles.
3. Find the midpoint of the following point
   (a) (2, 1) and (4, 5) [3, 3]
   (b) (-1, 4) and (3, 1) [1, 3]
   (c) (-2, 6) and (0, 2) [-1, 4]
4. Prove that the points A(-2, 0), B(0, $2\sqrt{5}$) and C(2, 0) are vertices of an equilateral triangle.
5. The points L, M and N have coordinates (3, 1), ((2, 6) and (x, 5) respectively. Given that the distance LM is equal to the distance MN, calculate the possible values of x. [-3 or 7]
6. Given that the distance between P(r, 4) and Q(2, 3) is equal to the distance between R(3, -1) and S(-2, 4). Calculate the possible value of r. [-5 or 7]
7. A triangle has vertices A(6, 2), B(x, 6) and C(-2, 6). Given that the triangle is isosceles with AB = BC, Calculate the value of x. [3]
8. F(5, 1), G(x, 7) and H(8, 2) are vertices of a triangle. Given that the length of the side FG is twice the length of side FH, find the value of x. [3 or 7]
9. Given that the distance from A(13, 10) to B(1, y) is three times the distance from B to C(-3, -2), find the value of y. [1 or -8]
10. M(6,5) is the midpoint of a straight line joining the point A to point B, find the coordinates of B [10, 7]

Gradient of a straight line

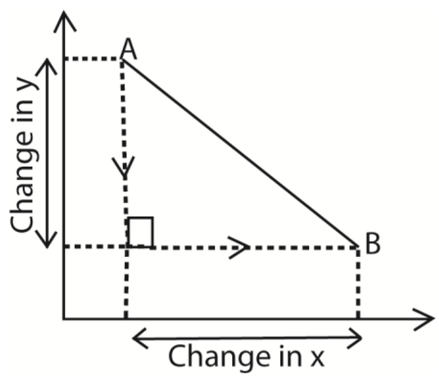
The gradient of a line joining points A(x1, y1) and B(x2, y2) is the measure of steepness of the line AB and it is a ratio of the change in y-coordinate to the change in x-coordinate.

i.e., gradient $=\frac{change\ in\ y}{change\ in\ x} = \frac{\Delta y}{\Delta x} = \frac{y_2-y_1}{x_2-x_1}$

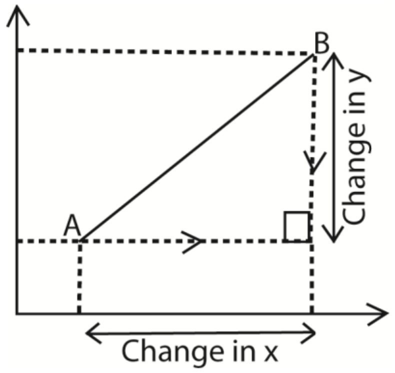
The gradient is usually denoted by m which may be positive or negative

Note:

(i) If the line slopes downwards from left to right, the gradient

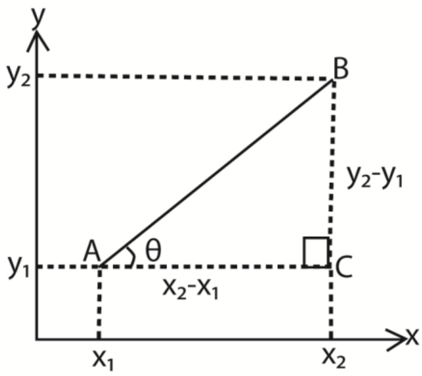


(ii) On the other hand, if the line slopes upwards from left to right, the gradient is positive.



Angle of a straight line to the horizontal

Suppose that the line in the second illustration makes θ is the horizontal as shown below



From trigonometry, $\tan\theta = \frac{BC}{AC}$

$= \frac{y_2-y_1}{x_2-x_1}$

$$\theta = \tan^{-1}\left(\frac{y_2-y_1}{x_2-x_1}\right)$$

This means that if m = $\frac{y_2-y_1}{x_2-x_1}$

$$\theta = \tan^{-1} m$$

Example 5

(a) Find the gradient of the straight line joining each of the following pairs of points.

(i) A(7, 4) and B(-1, -2)

Solution

m = $\frac{y_2-y_1}{x_2-x_1} = \frac{-2-4}{-1-7} = \frac{-6}{-8} = \frac{}{4}$

(ii) A(-3, -2) and B(-4, -5)

Solution

m = $\frac{y_2-y_1}{x_2-x_1} = \frac{-5-(-2)}{-4-(-3)} = \frac{-3}{-1} = 3$

(b) Find the angle which the straight line joining each of the following pairs of points makes with the horizontal

(i) A(5, 4) and B(6, 8)

Solution

Angle $\theta = \tan^{-1}\left(\frac{y_2-y_1}{x_2-x_1}\right)$

$= \tan^{-1}\left(\frac{8-4}{6-5}\right)$

$= 76^0$

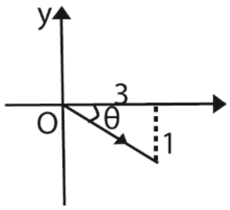
(ii) A(-3, -5) and (-4, -2)

Angle $\theta = \tan^{-1}\left(\frac{y_2-y_1}{x_2-x_1}\right)$

$= \tan^{-1}\left(\frac{-2-(-5)}{-4-(-2)}\right)$

$= -71.6^0$

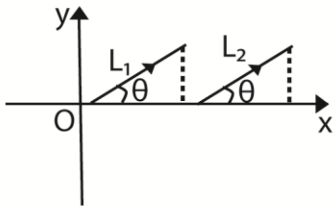
i.e.



Hence the angle which AB makes with the horizontal is 71.6^0 with positive x-axis downwards as shown in the diagram.

Gradient of parallel lines

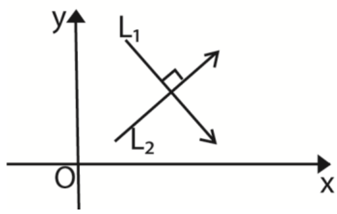
If two are parallel, their gradients are equal.



In the diagram above L_1 is parallel to L_2 and both L_1 and L_2 have the same gradient.

Gradient of perpendicular lines

If two lines are perpendicular, the product of their gradients is -1.



If m_1 and m_2 are the gradients of L_1 and L_2 respectively, then $m_1 \times m_2 = -1$

Example 6

(a) Given the points A(2, 3), B(5, 5), C(7,2) and D(4, 0)

(i) Prove that AB is parallel to DC

Gradient of AB, $m_1 = \frac{5-3}{5-} = \frac{}{3}$

Gradient of DC, $m_2 = \frac{-0}{7-4} = \frac{}{3}$

Since the gradient of AB and DC are equal, the lines AB and DC are parallel

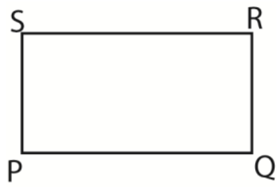
(ii) Prove that AC is perpendicular to BD

Gradient of AC, $m_1 = \frac{-3}{7-} = \frac{-1}{5}$

Gradient of BD, $m_2 = \frac{0-5}{4-5} = 5$

Since $m_1 \times m_2 = -1$, AC is perpendicular BD

(b) Prove that the point P(1, 3), Q(3, 4), R(5, 0) and S(3, -1) form a parallelogram.



Gradient of PQ, $m_1 = \frac{4-3}{3-1} = \frac{1}{}$

Gradient of SR, $m_2 = \frac{0-(-1)}{5-3} = \frac{1}{}$

∴ PQ and SR are parallel

$$\overline{PQ} = \sqrt{(3-1) \ + (4-3)} \ = \sqrt{5}$$

$$\overline{SR} = \sqrt{(5-3)^2 + (0-(-1))} \ = \sqrt{5}$$

∴ PQ and SR are equal

Gradient of PS, $m_1 = \frac{-1-}{3-1} = -2$

Gradient of QR, $m_2 = \frac{0-4)}{5-3} = -2$

∴ PS and QR are parallel

So the figure PQRS could either be a rectangle of parallelogram

For a rectangle

PQ and PS are perpendicular, thus the product of their gradients = -1

Since the gradient of PQ and x gradient of PS

$= \frac{1}{}x - 2 = -1$

Hence the figure PQRS is a rectangle not a parallelogram.

(c) The quadrilateral ABCD has vertices A(-2, -3), B(1, -1), C(7, -10) and D(2, -9).

(i) Prove that Ad is parallel to BC

Gradient of AD, $m_1 = \frac{-9-(-3)}{-(-2)} = \frac{-3}{}$

Gradient of BC, $m_2 = \frac{-10-(-1)}{7-1} = \frac{-3}{}$

Since the gradient of AD and BC are equal, the lines AD and BC are parallel

(ii) Prove that AD is perpendicular to BC

Gradient of AD, $m_1 = \frac{-1-(-3)}{1-(-2)} = \frac{}{3}$

Gradient of BC, $m_2 = \frac{-10-(-1)}{7-1} = \frac{-3}{}$

Since $m_1 \times m_2 = -1$, AD is perpendicular BC

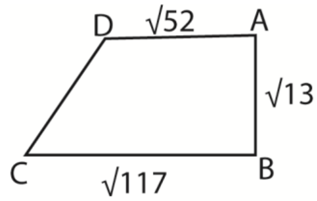
(iii) Prove that the area of the quadrilateral ABCD is 32½ sq. units

$$\overline{AB} = \sqrt{(1-(-2)) \ + (-1-(-3))}$$
$$= \sqrt{13}$$

$$\overline{BC} = \sqrt{(7-1) \ + (-10-(-1))}$$
$$= \sqrt{117}$$

$$\overline{AD} = \sqrt{(2-(-2)) \ + (-9-(-3))}$$
$$= \sqrt{52}$$

Since all the sides of a quadrilateral are different, the figure is a trapezium



$$Area = \frac{1}{2}\sqrt{13}(\sqrt{52} + \sqrt{117})$$

$$= 32\frac{1}{2} \ sq.units$$

(d) A quadrilateral ABCD has vertices A(-2, 1), B(0, 4), C(3, 2) and D(1, -1)

(i) Prove that all sides of the quadrilateral have the same length.

Solution

$$\overline{AB} = \sqrt{(0-(-2)) \ + (4-1)}$$
$$= \sqrt{13}$$

$$\overline{BC} = \sqrt{(3-0)\ \ + (2-4)}$$
$$= \sqrt{13}$$
$$\overline{CD} = \sqrt{(1-3)\ \ + (-1-2)}$$
$$= \sqrt{13}$$

$$\overline{AD} = \sqrt{(-2-1)\ \ + \left(1-(-2)\right)}$$
$$= \sqrt{13}$$

Hence all the sides of the quadrilateral are equal in length

(ii) Prove that AB is parallel to AD

Gradient of AB, $m_1 = \frac{4-1)}{0-(-2)} = \frac{3}{}$

Gradient of DC, $m_2 = \frac{-(-1)}{3-1} = \frac{3}{}$

Since the gradient of AB and DC are equal, the lines AB and DC are parallel

(iii) Prove that AD is parallel to BC

Gradient of AD, $m_1 = \frac{-1-1)}{1-(-2)} = \frac{-2}{3}$

Gradient of BC, $m_2 = \frac{-4}{3-0} = \frac{-2}{3}$

Since the gradients of AD and BC are equal, the lines AD and BCC are parallel

(iv) What is the name of the quadrilateral ABCD?

It could be a square or a rhombus

For square, gradient of AB x gradient of BC $= \frac{-2}{3} x \frac{3}{} =$ -1

Hence the quadrilateral is a square.

Exercise 2

1. Find the gradient of the straight line of each of the following pairs of points.
   (a) (-2, 5) and (5, -3) $\left[\frac{-8}{7}\right]$
   (b) (3, 7) and (7, -4) $\left[\frac{-11}{4}\right]$
   (c) (6, 3) and (7, 4) [1]
2. Find the angle between a line joining the following points with the horizontal
   (a) (2, 5) and (-3, -2) [54.46^0]
   (b) (3, 7) and (-6, 11) [-23.96^0]
   (c) (5, -3) and (5, 2) [90^0]
3. A triangle has vertices A(3, -2), B(2, -14) and C(-2, -4).Find the gradients of the straight lines AB, BC and CA. Hence prove that the triangle is right-angled
   $\left[12, \frac{-5}{}, \frac{}{5}; hence\ BC\ and\ CA\ are\ perpendicular\right]$
4. The straight line joining the points P(6, 5) to Q(q, 2) is perpendicular to the straight line joining point Q to R(9, -1). Find the value of q. [3, 11]
5. Prove that the quadrilateral PQRS with vertices P(-1, 3), Q(2, 4), R(4, -2) and S(1, -3) is a rectangle and calculate the area [20 sq. units]
6. The four points A(5, 4), B(6,2), C(12, 5) and D(11, 7) are vertices of a quadrilateral. Prove that the quadrilateral is a rectangle and calculate its area. [15 sq. units]
7. Prove that the points A(2, 3), B(4, 8), C(8, 9) and D(4, -1)for a trapezium.
8. The quadrilateral ABCD has vertices S(1, 1), T(4, 5), U(12, -1) and V(1, -1) are vertices of a quadrilateral STUV.
   (a) Prove that ST is perpendicular to TU, and that SV is perpendicular to UV.
   (b) Calculate the length of each of the sides ST, TU, UV, and VS. [5, 10, 11, 2]
   (c) Prove that the area of the quadrilateral STUV is 36 square units.
9. The quadrilateral CDEF has vertices, C4, 0), D(8, 4), E(2, -8) and F(0, 2). The points P, Q, R and S are midpoints of the sides CD, DE, EF and FC respectively. Prove that the Quadrilateral PQRS is a rhombus and show that its area is 15 sq. units.

Equation of a straight line

The general equation of a straight line is given by y = mx + c, where m = gradient of the line i.e. m $= \frac{y_2-y_1}{x_2-x_1}$ and c = y-intercept. For lines passing through the origin, c = 0; hence y = mx.

Example 7

(a) Find the equation of with gradient = 5 and passes through the point

(i) A(2, 5)

Solution

Method 1

The general equation of a line is y = mx + c

Substituting for m = 5 and points of A

5 = 5(2) + c

c = -5

Hence the line is y = 5x – 5

Method 2

Let B(x, y) lie on the line

From m $=\frac{y_2-y_1}{x_2-x_1}$

5 $=\frac{y-5}{x-2}$

y – 5 = 5(x – 2)

y = 5x – 5

(ii) P(2, 7)

Using y = mx + c

Substituting for m = 5 and points of P

7 = 5(2) + c

c = 17

Hence the line is y = 5x +17

(b) Find the equation of a straight line joining the following points

(i) A(0, 5) and B(3, 4)

Solution

Gradient = $\frac{4-5}{3-0} = \frac{-1}{3}$

Substituting coordinates for A in general equation

5 $=\frac{-1}{3}(0) + c$ => C = 5

Hence equation of the line is y = $\frac{-1}{3}x + 5$

(ii) P(-2, -5) and Q(3, -7)

Solution

Gradient = $\frac{-7-(-5)}{3-(-2)} = \frac{-2}{5}$

Substituting coordinates for P in general equation

-5 $=\frac{-2}{5}(-2) + c$ => C = -5 - $\frac{4}{5}$ = -$\frac{29}{5}$

Hence equation of the line is y = $\frac{-2}{5}x - \frac{29}{5}$

(c) Find the equation of a perpendicular bisector of the line joining the following points

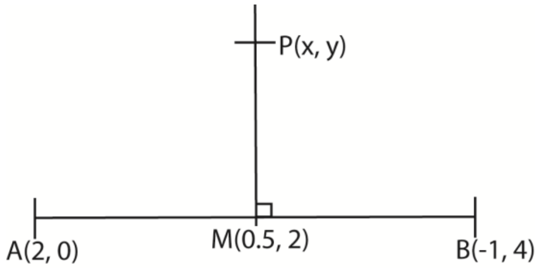
(i) (2, 0) and (-1, 4)

Solution

Let M be the midpoint of AB

⇨ M($\frac{-1}{}$, $\frac{0+4}{}$ =M(0.5, 2)

Let P(x, y) lie on the perpendicular bisector



Gradient of AB, $m_1 = \frac{4-0}{-1-} = \frac{4}{-3}$

Gradient of MP, $m_2 = \frac{y-2}{x-0.5}$

But m_1 x m_2 = -1

$\frac{4}{-3}\left(\frac{y-2}{x-0.5}\right) = -1$

4(y – 2) = 3(x – $\frac{1}{}$)

8y -16 = 6x – 3

8y – 6x = 13

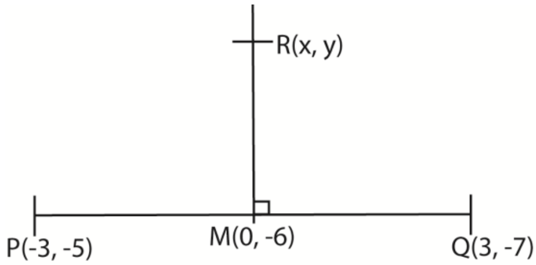
(ii) P(-3, -5) and Q(3, -7)

Solution

Let M be the midpoint of AB

⇨ M($\frac{-3+3}{}$, $\frac{-5-7}{}$ =M$(0, -6)$

Let R(x, y) lie on the perpendicular bisector



Gradient of PQ, $m_1 = \frac{8-(-7)}{-3-3} = \frac{15}{-6} = \frac{5}{-2}$

Gradient of MR, $m_2 = \frac{y-(-6)}{x-0} = \frac{y+6}{x-0}$

But m_1 x m_2 = -1

$\frac{5}{-2}\left(\frac{y+6}{x-0}\right) = -1$

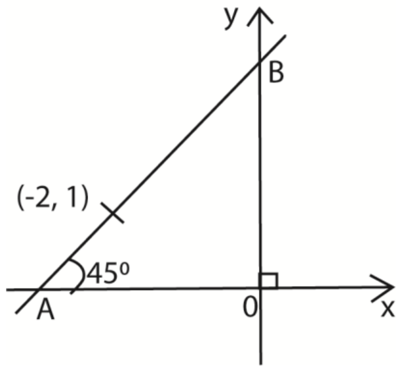
5(y+6) =2x

5y – 2x + 30 = 0

(d) A straight line, L passes through point (-2, 1) and makes an angle of 45^0 with the horizontal.

(i) Find the equation of the line L.

Solution



Gradient of the line L = tan 45^0 = 1

⇨ m = 2

Substituting the coordinates of the point in the general equation, y = mx + c

1 = 1(-2) + c => that c = 3

Hence the equation of the line L is y = x + 3

(ii) Given that L intersects the x-axis at A and the y-axis at B find the distance AB

Solution

At A, y = 0; x = -3

Hence coordinates of A(-3, 0)

At B x = 0 y = 3

Hence coordinates of B(0, 3)

$\overline{AB} = \sqrt{(0-(-3))^2 + (3-0)}$

$= 3\sqrt{2}$ units

Perpendicular distance of a point to a line

Just like in vectors, the shortest distance of a point from a given line is the perpendicular distance of a point from the line

Suppose that the equation of the line is in the form ax + by + c = 0

The perpendicular distance of a point

P(x_1, y_1)=d $=\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$

Example 8

Find the perpendicular distance of the following points from the given lines

(a) (-2, 6) from x + y + 4 = 0

Solution

a =1, b = 1 and c = 4

Substituting for (x, y) = (-2, 6)

d $=\left|\frac{1(-2)+1(6)+4}{\sqrt{1^2+1^2}}\right| = \frac{8}{\sqrt{2}}$=$4\sqrt{2}$units

(b) (7, -4) from 3x – 5y = 7

Solution

Rearranging the equation

3x – 5y – 7 = 0

a =3, b = -5 and c = -7

Substituting for (x, y) = (7, -4)

d $=\left|\frac{3(7)-5(-4)-7}{\sqrt{3^2+(-5)^2}}\right| = \frac{34}{\sqrt{34}}$=$\sqrt{34}$units

(c) (8, -5) from y = 3x -1

Solution

Rearranging the equation

3x – y – 1 = 0

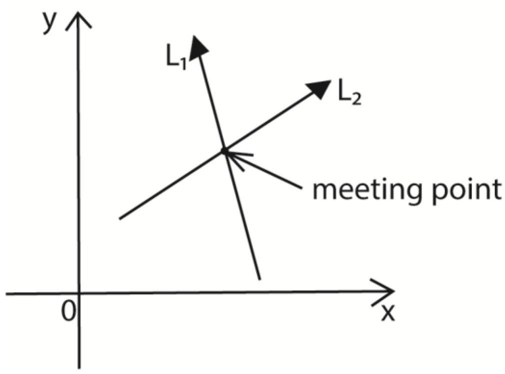
a =3, b = **-1** and c = -1

Substituting for (x, y) = (8, -5)

d $=\left|\frac{3(8)-1(-5)-1}{\sqrt{3^2+(-1)^2}}\right| = \frac{28}{\sqrt{10}}$=8.85units

Intersection of two lines

The point of intersection of two or more lines is obtained by solving the equations of the lines simultaneously.



Example 9

(a) Find the coordinates of the point of intersection of each of the following pairs of straight lines

(i) y = x + 3 and y = 4x + 6

Solution

Let y = x + 3 ……………………(1)

And y = 4x + 6 …………………(2)

Equating (1) and (2)

x + 3 = 4x + 6; x = -1

substituting x I (1)

y = 4(-1) + 6; y = 2

Hence the point of intersection is (-1, 2)

(ii) 2x - 3y = 7 and 3x – 7y = 13

Let 2x – 3y = 7 ……………………(1)
And 3x – 7y = 13 …………………(2)
Equating (1) and (2)
x + 3 = 4x + 6; x = -1
3eqn,(1) – 2eqn. (2)
5y = -5 => y = -1
Substituting for y into eqn. (1)
2x – 3(-1) = 7; x = 2
Hence the point of intersection (2, -1)

(b) (i) Find the equation of the straight line L, which passes through the point P(2, 4) and perpendicular to the line 5y + x = 7

Solution

Let Q(x, Y) lie on the same line

Gradient of PQ, $m_1 = \frac{y-4}{x-2}$

From5y + x = 7; $y = -\frac{1}{5}x + \frac{7}{5}$

Gradient, $m_2 = -\frac{1}{5}$

But for perpendicular lines, $m_1 \times m_2 = -1$

⇨ $-\frac{1}{5}\left(\frac{y-4}{x-2}\right) = -1$

$y - 4 = 5(x - 2)$

$y = 5x - 6$

(ii) Given that the line L meets line y = x + 6 at point S. find the coordinates of S.

Solution

At S the two equations are equal

⇨ $5x - 6 = x + 6$; x = 3

Substituting x in y = x + 6

y = 3 + 6 = 9

Hence coordinates of S are (3, 9)

(c) The line L has equation 2x – y – 1 = 0. The line M passes through point A(0, 4) and is perpendicular to the line L. The line N passes through point B(3, 0) and is parallel to M.

(i) Find an equation of M and show that he line L and M intersect at the point A(0, 4).

Solution

Let K(x, y) lie on line M

Gradient of AK, $m_1 = \frac{y-4}{x-0} = \frac{y-4}{x}$

From 2x – y = 1; y = 2x -1

Gradient, $m_2 = 2$

For perpendicular line $m_1 \times m_2 = -1$

⇨ $2\left(\frac{y-4}{x}\right) = -1$

2(y – 4) = -x

2y + x = 8

At the point, P of intersection, the two lines are equal

2x – 1 = y ……………………….(1)
2y + x = 8 ………………………. (2)
Substituting for y in eqn. (1) into equation (2)
2(2x-1) + x = 8
x = 2
Substituting for x in eqn. (1)
y = 2(2) – 1 = 3
Hence point of intersection P (2, 3)

(ii) Find an equation of N and hence find the coordinates of point Q where the Line L and Line N intersect

Let D (x, y) lie on N

Gradient BD, m1 = $\frac{y-0}{x-3} = \frac{y}{x-3}$

From line 2y + x = 8

$y = -\frac{1}{-} + 4$

Gradient $m_2 = -\frac{1}{-}$

For parallel lines $m_1 = m_2$.

⇨ $-\frac{1}{-} = \frac{y}{x-3}$

$2y = 3 - x$

2y + 3x = 3

At point Q, Line L = line N

2x – 1 = y ………………….(1)
2y + x = 3 ……………………(2)
Substituting for y in eqn. (2)
2(2x -1) + x = 3
5x – 2 = 3
x = 1
Substituting for x in eqn. (1)
y = 2(1) -1 = 1
hence Q(1, 1)

(iii) Prove that $\overline{AP} = \overline{BQ} = \overline{PQ}$

$\overline{AP} = \sqrt{(2-0) \ \ + (3-4)\ } = \sqrt{5}$

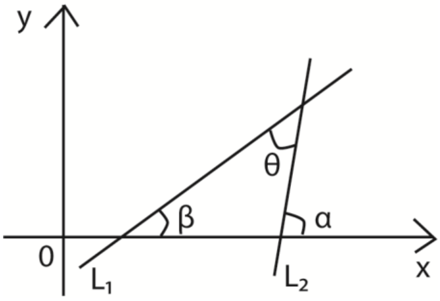
$\overline{BQ} = \sqrt{(1-3) \ \ + (1-0)\ } = \sqrt{5}$

$\overline{PQ} = \sqrt{(1-2) \ \ + (1-3)\ } = \sqrt{5}$

Hence $\overline{AP} = \overline{BQ} = \overline{PQ}$

Angle between two lines

Suppose that two lines y = m1x + c1 and y = m2x + c2 are inclined at angle α and β respectively as shown below



The sum of two interior angles = opposite exterior angle

⇨ θ + β = α

θ = α – β

tan θ = tan (α – β)

$= \frac{tan\alpha - tan\beta}{1+tan\alpha tan\beta}$

$= \frac{m_1 - m_2}{1 - m_1 m_2}$

θ = $\tan^{-1}\left(\frac{m_1-m_2}{1+m_1m_2}\right)$

Hence the acute angle between two intersecting lines is $\tan^{-1}\left(\frac{m_1-m_2}{1+m_1m_2}\right)$

Note that the angle between parallel lines is zero while that between perpendicular line is 90^0.

Example 10

(a) Find the acute angle between the following pairs of lines

(i) 4y – 3x = 6 and 2y + x = 3

Solution

For 4y – 3x = 6

y = $\frac{3}{4}x + \frac{6}{4}$ => $m_1 = \frac{3}{4}$

For 2y + x = 3

y = $-\frac{1}{}x + \frac{3}{}$ => $m_2 = -\frac{1}{}$

From θ = $\tan^{-1}\left(\frac{m_1-m_2}{1+m_1m_2}\right)$

θ = $\tan^{-1}\left(\frac{\frac{3}{4}-\left(-\frac{1}{2}\right)}{1+\frac{3}{4}\cdot\left(-\frac{1}{2}\right)}\right) = \tan^{-1}\left(\frac{\frac{3}{4}+\frac{1}{2}}{1-\frac{3}{8}}\right)$

$=\tan^{-1} 2 = 63.4^0$

(ii) 2x – 5y = 15 and 3y + 2x = 6

Solution

For 2x – 5y = 15

y = $-x + 3$ => $m_1 = \frac{}{5}$

For 3y + 2x = 6

y = $-\frac{}{3}x + 2$ => $m_2 = -\frac{}{3}$

From θ = $\tan^{-1}\left(\frac{m_1-m_2}{1+m_1m_2}\right)$

θ = $\tan^{-1}\left(\frac{\frac{2}{5}-\left(-\frac{2}{3}\right)}{1+\frac{2}{5}\cdot\left(-\frac{2}{3}\right)}\right) = \tan^{-1}\left(\frac{\frac{2}{5}+\frac{2}{3}}{1-\frac{4}{15}}\right)$

$=\tan^{-1}\left(\frac{16}{11}\right) = 46.8^0$

(iii) y = 4 and 3y + 2x – 6 = 0

Solution

For y = 4, => $m_1 = 0$

For 3y + 2x – 6 = 0

y = $-\frac{}{3}x - 2$ => $m_2 = -\frac{}{3}$

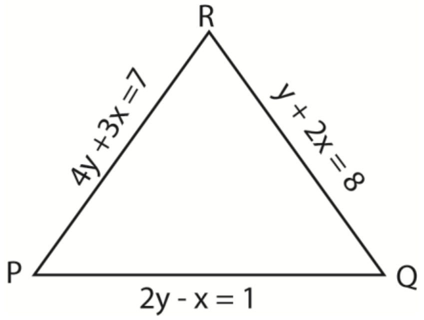
From θ = $\tan^{-1}\left(\frac{m_1-m_2}{1+m_1m_2}\right)$

θ = $\tan^{-1}\left(\frac{0-\left(-\frac{2}{3}\right)}{1+0.\left(-\frac{2}{3}\right)}\right) = \tan^{-1}\left(\frac{\frac{2}{3}}{1}\right)$

$=\tan^{-1}\left(\frac{2}{3}\right) = 33.7^0$

(b) Calculate the area of the triangle which has sides given by the equations 2y – x = 1, y + 2x = 8 and 4y + 3x = 7

Solution



At point P

4y + 3x = 7 ....(1)

2y – x = 1 ....(2)

Eqn. (1) – 2eqn.(2)

10y = 10; y = 1

Substituting for y in eqn. (2)

x = 2(1) – 1 = 1

Hence P(1, 1)

At point Q

y + 2x = 8 ....(1)

2y – x = 1 ....(2)

Eqn. (1) + 2eqn.(2)

5y = 10; y = 2

Substituting for y in eqn. (2)

x = 2(2) – 1 = 3

Hence Q(3, 2)

At point R

y + 2x = 8 ………………………………(1)

4y + 3x = 7 ………………………………(2)

3Eqn. (1) - 2eqn.(2)

-5y = 10; y = -2

Substituting for y in eqn. (1)

2x = 8 + 2 =10 => x = 5

Hence r(-2, 5)

Finding dimensions

$\overline{PR} = \sqrt{(5-1)^2 + (-2-1)} = 5$

$\overline{PQ} = \sqrt{(3-1)^2 + (2-1)} = \sqrt{5}$

$\overline{QR} = \sqrt{(5-3) + (-2-2)} = 2\sqrt{5}$

Finding <QPR

For 4y + 3x = 7

y = $-\frac{3}{4}x + \frac{7}{4}$ => $m_1 = -\frac{3}{4}$

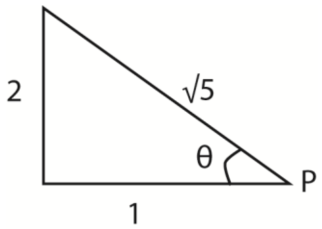
For 2y – x = 1

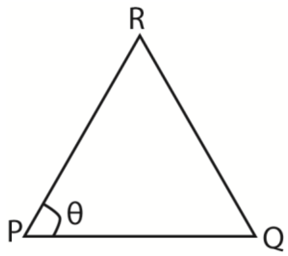
y = $\frac{1}{}x + \frac{1}{}$ => $m_2 = \frac{1}{}$

From θ = $\tan^{-1}\left(\frac{m_1-m_2}{1+m_1m_2}\right)$

<QPR = $\tan^{-1}\left(\frac{\frac{1}{2}-\left(-\frac{3}{4}\right)}{1+\frac{1}{2}.\left(-\frac{3}{4}\right)}\right) = \tan^{-1}\left(\frac{\frac{1}{2}+\frac{3}{4}}{1-\frac{3}{8}}\right)$

=$\tan^{-1}(2)$





Method 1 using sine rule

Area of PQR = $\frac{1}{}xPQ\ x\ PC\ sin\theta$

=$\frac{1}{}x\sqrt{5}\ x\ 5x\ \frac{}{\sqrt{5}} = 5$sq. units

Method 2 using Heron's formula

Area of PQR = $\sqrt{s(s-a)(s-b)(s-c)}$

Where a, b, and c are sides of a triangle and

s = $\frac{1}{}(a+b+c)$

s = $\frac{1}{}(5+\sqrt{5}+2\sqrt{5} = \frac{5+3\sqrt{5}}{}$

Area = $\sqrt{\left(\frac{5+3\sqrt{5}}{}\right)\left(\frac{3\sqrt{5}-5}{}\right)\left(\frac{5+\sqrt{5}}{}\right)\left(\frac{5-\sqrt{5}}{}\right)}$

= 5sq. units

Exercise 3

1. Find the gradient of each of the following straight lines
   (a) y = 4x -2 [4]
   (b) y = 2x + 3 [3]
   (c) y = 2 – 5x [-5]
   (d) $\frac{y}{} - \frac{x}{5} = 4$ $\left[\frac{}{5}\right]$
2. Find the equation of the straight line that has the following properties
   (a) Gradient 1 and passes through (2, 4) [y = x + 2]
   (b) gradient$\frac{1}{4}$ and passes through (2, 5) $\left[y = \frac{1}{4}x + \frac{9}{}\right]$
3. Find the equation of a straight line that has the following properties
   (a) Passes through (-2, 3) and parallel to y = 5x + 4 $[y = 5x + 13]$
   (b) Passes through (6, -2 and is perpendicular to y = -3x + 4 $\left[y = \frac{1}{3}x - 4\right]$
   (c) Passes through $\left(-\frac{1}{3}, -\frac{}{3}\right)$ and is perpendicular to 3y + 10x – 8 = 0 $[10y = 3x - 3]$
4. Find the equation of a straight line joining the following pairs of points
   (a) (2, 4) and (-1, 0) [3y = 4x +4]
   (b) (-4, 1) and(6, 2) [10y = x + 14]
   (c) (3, 4) and (-1, 4) [y =4]
5. Find the equation of the perpendicular bisector of the straight lines joining each of the following pairs of point
   (a) (5, 5) and (2, -2) [4y+ 2x = 11]
   (b) (-1, 4)and (3, 3) [2y – 8x + 1 = 0]
   (c) (3, 2) and (-4, 1) [y = -7x – 2]
6. Find the equation of a straight line:
   (a) L_1 which is perpendicular bisector of points A(-2, 3) and B(1, -5) [16y – 6x + 13 = 0]
   (b) L_2 which is a perpendicular bisector of the points B(1, -5) and (17, 1) [3y + 8x – 60 = 0]
   (c) Show that L_1 is perpendicular to L_2.

7. The perpendicular bisector of a straight line joining the points (3, 2) and (5, 6) meet the x-axis at A and they-axis at B. Prove that the distance AB is equal to $\sqrt{5}$.
8. A is a point (1, 2) and B is a point (7, 4). The straight line L_1 passes through B and is perpendicular to AB; the straight line L_2 passes through A and is also perpendicular to AB. The line L_1 meets the x-axis at P and the y-axis at Q. Line L_2 meets the x-axis at R and the y-axis at S.
   (a) Find the equations of each of L_1 and L_2. [y = -3x + 25, y = -3x + 5]
   (b) Calculate the area of the triangle OPQ. $\left[104\frac{1}{6}\right]$
   (c) Calculate the area of the triangle ORS. $\left[4\frac{1}{6}\right]$
   (d) Find the area of the trapezium PQSR. [100 sq. units]
9. P is the point with coordinates (2, 1) and L is the straight line which is perpendicular to OP and which passes through P.
   (a) Find the equation of L. [y = -2x + 5]
   (b) Given that line L meets the x-axis at A and y-axis at B. calculate
      (i) the area of the triangle OAP. [1.25]
      (ii) The area of the triangle OBP [5]
      (iii) Find the ratio of the area OAP to that of OBP [1:4]
10. Find the shortest distance between each of the following
    (a) The point (2, 4) and the line 3x – 4y + 8 = 0 $\left[\frac{}{5}\right]$
    (b) The point (5, -1) and the line 12x + 5y – 3 = 0 [4]
    (c) The point (9, -3) and the line y = x. $[6\sqrt{2}]$
11. Find the coordinates of the point of intersection of each of the following pairs of straight lines
    (a) y = 2x + 3 and y = 4x + 1 [1, 5]
    (b) y = x + 3 and y = 4x + 6 [-1, 2]
    (c) 2x – 3y = 7 and 3x – 7y = 13 [2, -1]
    (d) x + 3y – 2 = 0 and 3x + 5y – 8 = 0 $\left[\frac{7}{}, -\frac{1}{}\right]$
12. Find the equation of the straight line L, which passes through the point (2, 4) and perpendicular to the line 5y + x = 7 [y = 5x – 6]
    (b) Given that the line L meets the line y = x + 6 at point S, find the coordinates of point S.[3, 9]
13. Calculate the area of the triangle which has sides given by the equations 2y – x = 1, y + 2x = 8 and 4y + 3x = 7 [5sq. units]
14. The point A has coordinates (2, -5). The straight line 3x + 4y – 36 = 0 cuts the x-axis at B and the y – axis at C. Find
    (a) The equation of the line through A which is perpendicular to the line BC. [4x – 3y = 23]
    (b) The perpendicular distance from the line BC. [10]
    (c) The area of the triangle ABC. [75 sq. units]

Locus

A locus is the set of all points in a plane that satisfy some condition. For example the locus of points equidistant from two given points say A and B is a perpendicular bisector of AB.

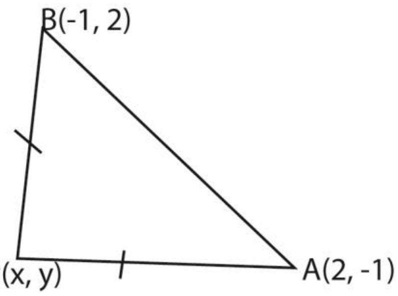
Locus can be expressed in terms of the Cartesian coordinates (x, y) of the form (r, θ)

Example 11

(a) Find the locus of point P(x, y) that is equidistant from the point A(2, -1) and B(-1, 2)

Solution

Method 1

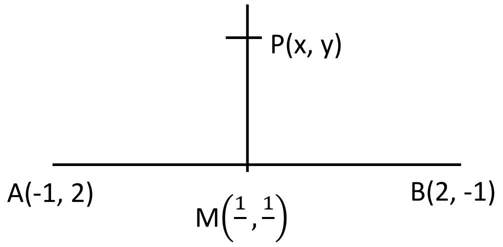


$\overline{AP} = \overline{PB}$ i.e. $\overline{AP} = \overline{PB}$

$(x-2) + (y+1) = (x+1) + (y-2)$

y = x

Method 2



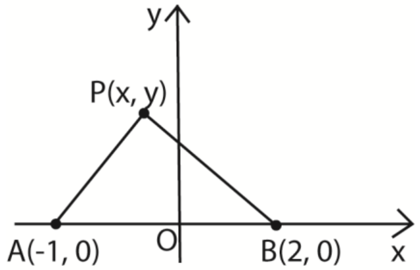
Gradient of AB = $\frac{+1}{-1-} = -1$

Since AB and MP are perpendicular, the product of their gradients = -1

Gradient of MP=$\frac{x-\frac{1}{2}}{y-\frac{1}{2}} = 1$

y = x

(b) Find the locus of a point which moves so that of the squares of its distance from points A(-2, 0) and B(2, 0) is 25 units.



$\overline{AP} \;\; + \overline{BP} \;\; = 25$

$(x+1) \;\; + y \;\; + (x-2) \;\; + y \;\; = 25$

$x \;\; + y \;\; = 25$

(c) The locus of P(x, y) is such that the distance OP is half the distance PR, where O is the origin and R is the point (-3, 6)

(i) Show that the locus of P describes a circle in the x-y plane

Solution

Given OP = $\frac{1}{}PR$ i.e. $4OP^2 = PR^2$

$4(x^2+ y^2) = (x+ 3)^2 + (y - 6)^2$

$4x^2 + 4y^2 = x^2 + 6x + 9 + y^2 - 12y + 36$

$x^2 + y^2 - 2x + 4y - 15 = 0$ (a circle in x – y plane)

(ii) Determine the radius of the circle and the centre of the circle.

Solution

$x^2 + y^2 - 2x + 4y = 15$

After completing squares we have

$(x - 1)^2 + (y + 2)^2 = 20$

The centre of the circle is (1, -2) and the radius= $\sqrt{20} = 2\sqrt{5}$ units

(iii) Where does P cut the line x = 3?

Solution

Substituting x = 3 in the equation

$(x - 1)^2 + (y + 2)^2 = 20$

$((3 - 1)^2 + (y + 2)^2 = 20$

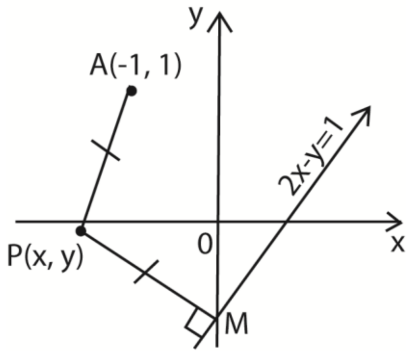
$(y + 2)^2 = 16$ i.e. $y + 2 = \pm 4$

$y = -2 \pm 4$ i.e. y = -6 or y = 2

∴P cuts the line x = 3 at the point (3, 2) and (3, -3)

(d) Find the locus of P(x, y) if its distance from A(-1, 1) is equal to the distance form the line 2x – y = 1

Solution



$\sqrt{(x+1) \;\; + (y-1)^2} = \frac{|2x-y-1|}{\sqrt{(2^2+(-1)^2)}}$

⇨ $5[(x+1)^2 + (y-1)^2] = (2x - y - 1)$

$5x^2+10x + 5 + 5y^2 - 10y + 5 = 4x(y+1)+ (y+1)^2$

$x^2 + 4y^2 + 4xy + 14x - 12y + 9 = 0$ is the locus

(e) A point R moves so that its distance from point (2, 0) is twice its distance from (0, -1). Show that the locus of R is a circle and determine its radius and its centre.

Solution

Let P (2, 0), Q(0, 1) and R (x, y)

Given PR = 2QR => $PR^2 = 4QR^2$

$(x - 2)^2 + y^2 = 4[x^2 + (y+ 1)^2]$

$x^2 - 4x + 4 +y^2 = 4x2 + 4y^2 + 8y + 4$

$3x^2 + 3y^2 + 4x + 8y = 0$ hence a circle

⇨ $x \;\; + \frac{4}{3}x + y \;\; + \frac{8}{3}y = 0$

$\left(x + \frac{}{3}\right) \;\; + \left(y + \frac{4}{3}\right) \;\; = \frac{20}{9}$

The centre is at $\left(-\frac{}{3}, \frac{4}{3}\right)$ and the radius

$= \sqrt{\frac{20}{9}} = \frac{2\sqrt{5}}{3}$

(f) Find the locus of a point P(x, y) whose distance from the point A(3, -2) is always 5units.

Solution

AP = 5 => AP2 = 25

i.e. $(x – 3)^2 + (y + 5)^2 = 25$

$x^2 + y^2 – 6x + 4y – 12 = 0$

The locus is a circle with centre (3, -2) and radius 5 units.

(g) A is a point on the x-axis and C is a point (2, 3). The perpendicular to AC through C meets the y-axis at B. Find the locus of the midpoint of AB.

Solution

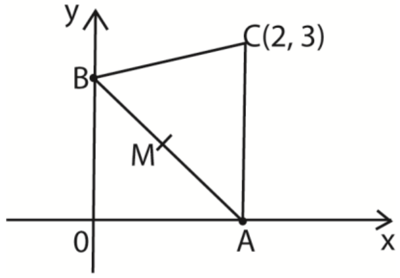
Let A(a, 0), B(0, b) and M(x, y) is the midpoint of AB

⇨ M$\left(\frac{a+0}{}, \frac{0+b}{}\right)$ i.e. M$\left(\frac{a}{}, \frac{b}{}\right)$

Since AC is perpendicular BC

⇨ $\frac{0-3}{a-2} \cdot \frac{b-3}{0-2} = -1$

$3(b -3)$=2(2-a) ……………………(i)



Now at M , x = $\frac{a}{}$and y = $\frac{b}{}$

⇨ a = 2x and b = 2y

Substituting into eqn. (i)

3(2y - 3) =2(2 – 2x)

i.e. 4x + 6y= 13

(h) P is a variable point given by parametric equations

$x = \frac{1+t}{1-t}$ and $y = \frac{t}{1-t}$

Show that the locus of P is a straight line

Solution

By eliminating t

From $x = \frac{1+t}{1-t}$; t = $\frac{x-1}{x+1}$

From $y = \frac{t}{1-t}$; t = $\frac{y}{+y}$.

Equating t

$\frac{x-1}{x+1} = \frac{y}{+y}$

(x -1)(2 +y) =y(x – 1)

y – x = -1 (which is a line)

(i) A point P moves such that its distance from two points A(-2, 0) and B (8,6) are in ratio AP: PB = 3:2. Show that the locus of P s a circle. (05marks)

$\frac{\overline{AP}}{\overline{PB}} = \frac{3}{}$ => $2\overline{AP} = 3\overline{PB}$

$2\sqrt{(x+2)\ + y}\ =$

$3\sqrt{(x-8)^2 + (y-6)}$

Squaring both sides

$4(x^2 + 4x + 4 + y^2)$

$= 9(x^2 -16x + 64 + y^2 -12y + 36)$

$4x^2 + 16x + 16 + 4y^2$

$= 9x^2 – 144x + 900 + 9y^2 – 108y$

$5x^2 + 5y^2- 160x -108y + 884 = 0$

Exercise 4

1. Find the equation to the locus of a point which moves so that its distance from the point (3, 4) is always 5 units.
   [$x^2 + y^2 – 6x – 8y = 0$]
2. A and B are points (1, 0) and (7, 8) respectively. A point P moves so that the angle APB is a right angle. Find the equation of locus of P.

   [$x^2 + y^2$ – xx – 8y + 7 = 0]

3. A variable point P is given by parametric equation x = ct, y = $\frac{c}{t}$. Show that the locus of P is xy = c^2.
4. Find the locus of point P which is equidistant from the line x = 2and the circle $x^2 + y^2 = 1$.
   [$x^2 + 6x – 9 = 0$]
5. A Point P is twice as far from the line x + y = 5 and from the point (3, 0). Find the locus of P.

   [$7x^2 + 7y^2 – 38x + 10y – 2xy + 47 = 0$]

6. Find the Cartesian equation of a curve given parametrically by $x = \frac{1+t}{1-t}$ and $y = \frac{t}{1-t}$
   [y – x + 1= 0]
7. P(3, 2) and Q(5, 6) are two fixed points and R moves so that angle PRQ is right angle. Show that the locus of R is given by the equation $x^2 + y^2 – 9x – 6y + 26 = 0$

8. Show that the locus of P is $\frac{1}{4}x^2 + \frac{1}{3}y^2 = 1$, given that PA + PB = 4 where A and B are the points (1, 0) and (-1, 0) respectively

Exercise 5 (Topical question)

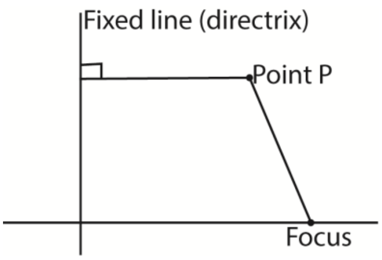
1. ABCD is a quadrilateral with A(2, -2), B(5, -1), C(6, 2) and D(3, 1). Show that the quadrilateral is a rhombus.
2. PQRS is a quadrilateral with vertices P(2, -1), Q(4, -1) and S(2, 1). Show that the quadrilateral is a rhombus
3. The Locus of P is such that the distance OP is half the distance PR, where O is the origin and R id the point (-3, 6).
   (a) Show that the locus of P describes a circle in x – y plane
   [$x^2 + y^2 - 2x + 4y - 15 = 0$ (a circle in x – y plane)]
   (b) Determine the centre and radius of the circle.
   [The centre of the circle is (1, -2) and the radius= $\sqrt{20}$ = $2\sqrt{5}$ units]
   (c) Where does P cuts the line x = 3
   [(3, 2) and (3, -3)]
4. A Point P is twice as far from the line x + y = 5 as from the point (3, 0). Find the locus of P.
   [$7x^2 + 7y^2 -38x + 10y + 47 = 0$]
5. Find the locus of point P which is equidistant from the line x = 2and the circle $x^2 + y^2 = 1$.
   [$x^2 + 6x - 9 = 0$]
6. The points R(2, 0) and P (3. 0)lie on the x-axis and Q(0, -y) lie on the y-axis. The perpendicular from the origin to RQ meets PQ at S(X, -Y). Determine the locus Of S in terms of X and Y. [$2X^2+ 3Y^2 - 6X = 0$]
7. The point A(2, 1), P(α, β) and point B(1, 2) lie in the same plane. PA meets the x-axis at point (h, 0) and PB meets the y-axis at point (0, k). Find h and k in terms of α and β.
   $\left[h = \frac{\beta-\alpha}{\beta-1}; k = \frac{\alpha-\beta}{\alpha-1}\right]$
8. A is a point (1, 3) and B is a point (4, 6). P is a variable point which moves in such a way that $\overline{AP} \;\; + \overline{PB} \;\; = 34$. Show that the locus of P describes a circle. Find the centre and radius of the circle
   $\left[centre\ is\left(\frac{5}{-},\frac{9}{-}\right), radius = \frac{2\sqrt{5}}{}units\right]$
9. Find the locus of the point P(x, y) which moves such that its distance from the point S(-3, 0) is equal to its distance from a fixed line x = 3 [$y^2 + 12x = 0$]
10. Given the vector a = i – 3j + 3k and b = -i – 3j + 2k. find
    (j) acute angle between vectors a and b
    [30.86^0]
    (ii) equation of the plane containing a and b
    [-3x+ 5y +6z = 0]
11. The points A and B lie on the positive sides of the x-axis and y-axis respectively. If the length of AB is 5 units and angle OAB is θ, where O is the origin, find the equation of the line AB (leave θ in your answer)
    [y = -xtanθ + 5sinθ]
12. (a) Find the equation of the locus of a point which moves such that its distance from D(4, 5) is twice its distance from the origin.
    [$8x^2 + 8y^2 + 8x + 10y - 41 = 0$]
    (b) the line y = mx intersects the curve 2x2 – x at point A and B. Find the equation of the locus of point P which divides AB in the ratio 2:5. [$y = 7x^2 - x$]

15. Coordinate geometry 2

Conic section

The section is the circle, parabola, ellipse and hyperbola. They are called conic sections because they are the shapes one sees when he slices a double cone at various angles.

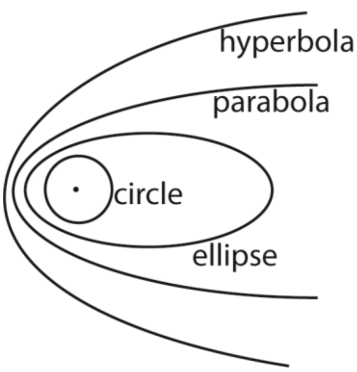
Mathematically a conic is a locus of points that move such that its distance from a fixed point (focus) bears a constant ratio (eccentricity, e) to its distance from a fixed line (directrix).



There are four different conics depending on the magnitude of the eccentricity, e:

When

- $e = 0$, it is a circle
- $e < 1$, it is an ellipse
- $e = 1$, it is a parabola
- $e > 1$ it is hyperbola



In a Cartesian coordinate system, a conic is a curve that has an equation of the second degree given in the form

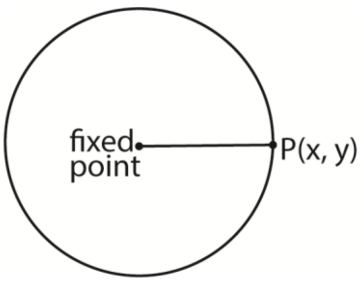
$ax^2 + 2hxy + by^2 + 2gx + 2gy + C = 0$

Now when

- $a = b$ and $h = 0$, it is a circle
- $h^2 < ab$, it is an ellipse
- $h^2 = ab$, it is a parabola
- $h^2 > ab$, it is a hyperbola
- $a = -b$, it is a rectangular hyperbola

The circle

A circle is a locus of point that moves so that its from a fixed point is constant



The fixed point is the **centre** of the circle and the constant distance is the **radius**.

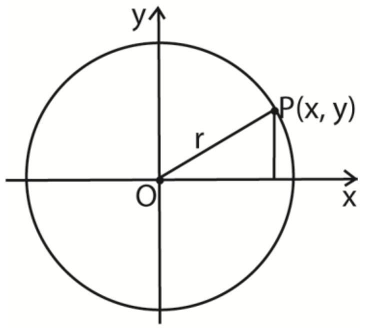
Equation of a circle

There are several ways of obtaining the equation of the circle.

I. Given the radius and the centre.

(a) The centre at the origin

Let P(x, Y) be a point on the circumference of the circle

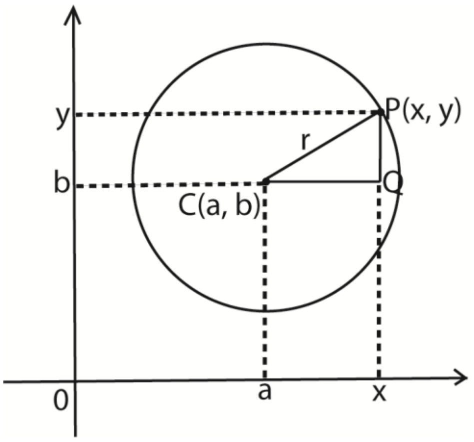


From Pythagoras theorem

$(x - 0)^2 + (y - 0)^2 = r^2$

$x^2 + y^2 = r^2$

(b) The centre at any point C(a, b)



In the triangle CPQ

$(x - a)^2 + (y - b)^2 = r^2$ ……………………………….(i)

$x^2 - 2ax + a^2 + y^2 - 2by + b^2 = r^2$

$x^2 + y^2 - 2ax - 2by + a^2 + b^2 - r^2 = 0$

$x^2 + y^2 + 2gx + 2fy + c = 0$ where $g = -a$, $f = -b$ and $c = a^2 + b^2 - r^2$ (a constant)

Equation (i) show a the equation of a circle is obtained given its centre and radius.

Equation (ii) represents the general equation of a circle with centre (-g, -f) and radius,

$r = \sqrt{g^2 + f^2 - c}$

Example 1

(a) Find the equation of the circle with centre at

(i) origin and radius 5 units

Solution

Using general equation

$(x - 0)^2 + (y - 0)^2 = 5^2$

$x^2 + y^2 = 25$

(ii) (-1, 2) and radius 4 units

Solution

$(x + 1)^2 + (y - 2)^2 = 4^2$

$x^2 + y^2 + 2x - 4y = 11$

(iii) $\left(2, -\frac{3}{5}\right)$ and radius $\sqrt{2}$

Solution

$(x - 2)^2 + (y + \frac{3}{5})^2 = (\sqrt{2})^2$

$x^2 + y^2 - 4x + \frac{6y}{5} - \frac{59}{25} = 0$

$25x^2 + 25y^2 - 100x + 30y = 59$

Note the three above equation of the circle show that

- The coefficient of x^2 and y^2 are always equal
- There is no term in xy
- The equation of a circle is of second degree equation in x and y.

The method of completing square can enable one to deduce the centre and the radius of the circle given its equation

(b) Find the centre and the radius of the following circles:

(i) $x^2 + y^2 - 4x + 6y + 1 = 0$

solution

$(x - 2)^2 + (y + 3)^2 = -1 + 4 + 9$

$(x - 2)^2 + (y + 3)^2 = 12$

∴the centre is at (2, -3) and the radius $= \sqrt{12} = 4\sqrt{3}$ units

(ii) $3x^2 + 3y^2 + 6x - 2y - 9 = 0$

Solution

$x^2 + y^2 + 2x - \frac{2}{3}y - 3 = 0$

$(x + 1)^2 + (y - \frac{1}{3})^2 = 3 + 1 + \frac{1}{9} = \frac{37}{9}$

∴the centre is at $(-1, \frac{1}{3})$ and the radius

$= \sqrt{\frac{37}{9}} = 2.028$ units

In general given the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$

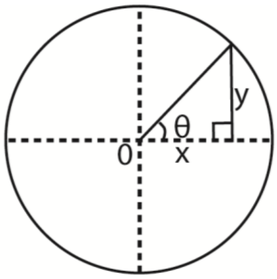
$(x + g)^2 + (y + f)^2 = -c + g^2 + f^2$

∴the centre is at (-g, -f) and the radius $= \sqrt{(-c + g^2 + f^2)}$ units

II. Given the polar equation

Suppose the equation of the circle in polar for is r = acosθ, required is to find its equation in Cartesian form

From the polar coordinates, we have



From the figure above

$x^2 + y^2 = r^2$ .... (i)

x = rcosθ

y = rsinθ

But r = acosθ

⇨ $r = a\left(\frac{x}{r}\right)$

$r^2 = ax$

Substituting for r^2 into eqn. (i)

$x^2 + y^2 = ax$

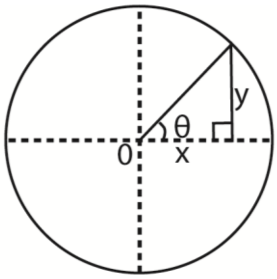
$x^2 + y^2 – ax = 0$

Example 2

Find in Cartesian form the equation of a circle given

(i) r = 2cosθ

Solution



$x^2 + y^2 = r^2$ .... (i)

x = rcosθ

y = rsinθ

But r = 2cosθ

⇨ $r = 2\left(\frac{x}{r}\right)$

$r^2 = 2x$

Substituting for r^2 into eqn. (i)

$x^2 + y^2 = 2x$

$x^2 + y^2 – 2x = 0$

(ii) r = 3cosθ

Solution

$x^2 + y^2 = r^2$ .... (i)

x = rcosθ

y = rsinθ

But r = 3cosθ

⇨ $r = 3\left(\frac{x}{r}\right)$

$r^2 = 3x$

Substituting for r^2 into eqn. (i)

$x^2 + y^2 = 3x$

$x^2 + y^2 – 3x = 0$

III. Given three non-collinear points on the circumference of the circle

(a circle circumscribing a triangle)

Method I

The general equation of the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ has three unknown: g, f, and c.

Substituting the three points given in this equation will give us three equations in three unknowns which are solved simultaneous

Method II

The perpendicular bisectors of two or more chords pass through the centre of the circle.

Example 3

(a) Find the equation of the circle passing through the points

(i) A(1, 1), B(2, 0)and C(3, 1)

Method 1

The general equation is

$x^2 + y^2 + 2gx + 2fy + c = 0$

At A(1,1): 1+ 1+ 2g + 2f + c = 0

i.e. 2g + 2f + c = -2 ....(i)

At B(2, 0): 4 + 4g + c = 0

i.e. 4g + c =-4 ....(ii)

At C(3, 1): 9 + 1 + 6g + 2f + c = 0

i.e. 6g + 2f + c = -10 .... (iii)

(i) – (iii) : -4g = 8; g = -2

Substituting for g in eqn. (ii)

-8+ c = -4: c = 4

Substituting for g and c into eqn. (i)

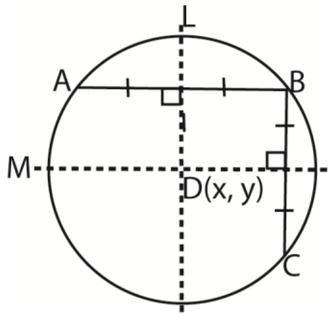
-4 + 2f + 4 = -2; f = -1

By substitution, the equation of the circle is

$x^2 + y^2 - 4x - 2y + 4 = 0$

Method 2

Let L and M be the perpendicular bisector of chords AB and BC



Equation of L

$(x - 1)^2 + (y - 1)^2 = (x - 2)^2 + (y - 0)^2$

$2x - 2y - 2 = 0$ i.e. $x - y = 1$ ………. (i)

Equation of M:

$(x - 2)^2 + (y - 0)^2 = (x - 3)^2 + (y - 1)^2$

$2x + 2y = 6$ i.e. $x + y = 3$ ……………..(ii)

Eqn. (i) + Eqn. (ii): $2x = 4 \Rightarrow x = 2$

Substituting for into eqn. (i), y = 1

∴The centre of the circle is at D(2,1) with radius AD(or BD or CD)

Let point P(x, y) lie on the circle.

Considering AD as the centre,

Radius, $r = \sqrt{(2-1)^2 + (1-1)^2} = 1$

The equation of the circle is

(x – 2)2 + (y – 1)2 = (2 – 1)2 + (1 – 1)2

$x^2 - 4x + 4 + y^2 + 2y + 1 = 1$

$x^2 + y^2 - 4x - 2y + 4 = 0$

(ii) P(-2, 2), Q(2, 4) and R(5, -5)

Method 1

The general equation is

$x^2 + y^2 + 2gx + 2fy + c = 0$

At P(-2, 2): 4+ 4 – 4g + 4f + c = 0

i.e. 4g – 4f + c = 8 …………………(i)

At Q(4, 4): 4 + 16+ 4g + 8f+ c = 0

i.e. 4g + 8f + c =-20 ……………………….(ii)

At R(5, -5): 25 + 25 + 10g – 10f + c = 0

i.e. 10g – 10f + c = -50 ………….. (iii)

(i) + (iii) : 8g + 4f = -12

2g + f = -3 ……………………………(iv)

Eqn. (ii) – eqn. (iii)

-6g+ 18f = 30

-g + 3f = 5 ……………………………(v)

2eqn. (iv) + eqn. (v)

7f = 7, f = 1

Substitution for f into eqn. (iv)

g = -2

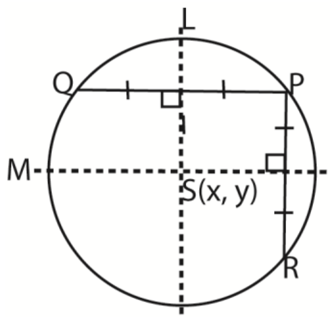
Substitution for g and c into eqn. (i)

c =-20

By substitution, the equation of the circle is

$x^2 + y^2 - 4x + 2y - 20 = 0$

Method 2



Equation of the perpendicular bisector, M of chord PQ.

Equation of M

$(x + 2)^2 + (y - 2)^2 = (x - 2)^2 + (y - 4)^2$

$2x + y = 3$ ………………………………………. (i)

Equation of L:

$(x + 2)^2 + (y - 2)^2 = (x - 5)^2 + (y + 5)^2$

$x - y = 3$ ……………………………………………(ii)

Eqn. (i) + Eqn. (ii): 3x = 6; x = 2

Substituting for x into eqn. (i): y = -1

∴ the centre is at S(2, -1) with radius

$SP = \sqrt{(2+2)^2 + (-1-2)^2}$

The equation is
$(x - 2)^2 + (y + 1)^2 = (2 + 2)^2 + (-1 - 2)^2$

$x^2 + y^2 - 4x + 2y - 20 = 0$

(b) Find the centre and radius of a circle circumscribing triangle ABC with vertices A(3, -2), B(1, 5) and C(-1, -1)

The general equation is

$x^2 + y^2 + 2gx + 2fy + c = 0$

At A(3, -2): $9 + 4 + 6g - 4f + c = 0$

i.e. $6g - 4f + c = -13$ ……………………….(i)

At B(1, 5): $1 + 25 + 2g + 10f + c = 0$

i.e. $2g + 10f + c = -26$ ……………………..(ii)

At C(-1, -1): $1 - 2g - 2f + c = 0$

i.e. $-2g - 2f + c = -2$ …………………….. (iii)

Solving equation (i), (ii) and (iii) simultaneously, we obtain
$g = \frac{-19}{26}, f = \frac{-37}{26}, c = \frac{-108}{13}$
The equation of the circle

$x^2 + y^2 - \frac{19}{13}x - \frac{37}{13}y - \frac{108}{13} = 0$

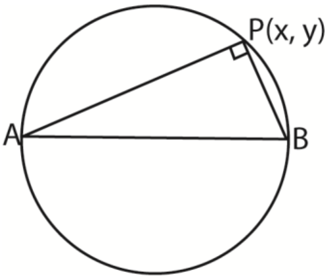
$(x - \frac{19}{26})^2 + (y - \frac{37}{26})^2 = \frac{108}{13} + \left(\frac{19}{26}\right)^2 + \left(\frac{37}{26}\right)^2$

$\therefore$ The centre is $\left(\frac{19}{26}, \frac{37}{26}\right)$ and

$r = \sqrt{\frac{108}{13} + \left(\frac{19}{26}\right)^2 + \left(\frac{37}{26}\right)^2}$

IV. Given the end points of the diameter of a circle.

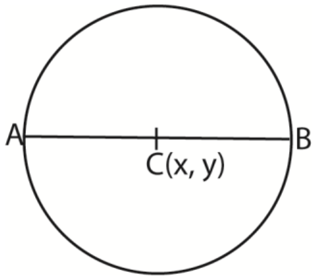
Method 1



Let P(x, y) be a general point on the circumference.

Since AP is perpendicular to BP (Angle is a semicircle)
⇨ (Gradient of AP) x (Gradient BP) = -1

Method 2



Let C(x, y) be the centre of the circle.

Since AC and BC are the radii of the circle,

AC = CB

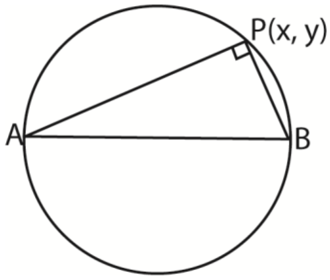
Example 4

(a) Find the equation of a circle in each case given the end points of the diameter

(i) A(1, 2) and B(-3, 4)

Solution

Either: let P(x, y) lie on the circle



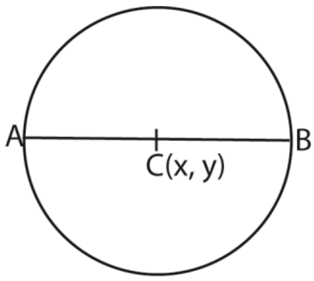
(Gradient of AP) x (Gradient BP) = -1

$\frac{y-2}{x-1} \cdot \frac{y-4}{x+3} = -1$

$(y - 2)(y - 4) + (x - 1)(x + 3) = 0$

$x^2 + y^2 + 2x - 6y + 5 = 0$

Or:



Taking C(x, y) as the centre of the circle,

x $=\frac{1}{2}(1-3)=-1$ and y $=\frac{1}{2}(2+4)=3$

∴ the centre of the circle is at (-1, 3) and radius, r = $\sqrt{AC^2}=\sqrt{CB^2}$

$$=\sqrt{(-1-1)^2+(3-2)^2}=\sqrt{5}$$

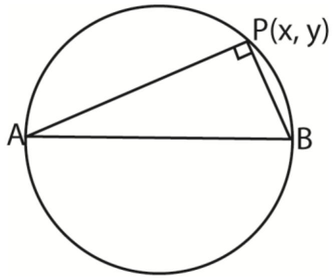
$$Equation\ of\ the\ circle:$$

$(x + 1)^2 + (y – 3)^2 = 5$

i.e. $x^2 + y^2 + 2x – 6y + 5 = 0$

(ii) P(5, -2) and Q(-1, 3)

Let R(x, y) lie on the circle



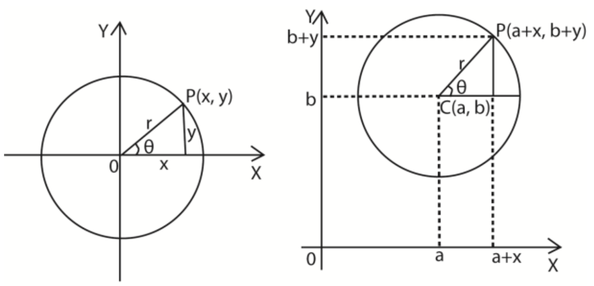
(Gradient of PR) x (Gradient RQ) = -1

$\frac{y+2}{x-5}\cdot\frac{y-3}{x+1}=-1$

(y + 2)(y – 3)+(x -5)(x+1) = 0

$x^2 + y^2 – 4x – y – 11 = 0$

Parametric equation of a circle



(i) centre at (0, 0) (ii) centre at (a, b)

In diagram (i), any point P(x, y) has parametric coordinates, x = rcosθ and y = rsinθ and the circle has centre (0, 0)

In diagram (ii) any point P(a+x, b+y) has parametric coordinates, x = a + rcosθ and y = b + rsinθ and the circle has centre (a, b)

Example 5

Show that the parametric equations (3+2cosθ, -1+sinθ) represent a circle. Determine the centre and the radius.

Solution

Given x = 3+2cosθ and y= -1+2sinθ

Cosθ=$\frac{1}{2}(x-3)$ and sinθ = $\frac{1}{2}(y+1)$

⇨ $\frac{1}{4}(x-3)^2+\frac{1}{4}(y+1)^2$ since $\cos^2θ+\sin^2θ = 1$

$(x-3)^2+(y+1)^2 = 2^2$

The locus is a circle with centre (3, -1) and radius, r = 2 units

Equation of the tangent and normal to the circle

The gradient of the tangent to the a circle may be got in two ways

Using gradient of radius or by differentiation of the function implicitly:

Example 6

(a) Find the equation of the tangent and normal to the circle

(i) $x^2 + y^2 + 2x - 8y + 4 = 0$ at point (2, 2)

Solution

Method 1

$x^2 + y^2 + 2x - 8y + 4 = 0$

$(x + 1)^2 + (y – 4)^2 = 13$

Centre (-1, 4)

Gradient of the radius = $\frac{4-2}{-1-3}=\frac{-2}{3}$

Gradient of the tangent = $\frac{3}{2}$

Equation of tangent

$y – 2 = \frac{3}{2}(x – 2)$

2y – 3x + 2 = 0

Equation of the normal

$y – 2 = -\frac{3}{2}(x – 2)$

2x + 3y = 0

Method 2

By differentiation of the function implicitly.

$2x + 2y\frac{dy}{dx} + 2 - 8\frac{dy}{dx} = 0$

$(2y-8)\frac{dy}{dx} = -2x-2$

$\frac{dy}{dx} = \frac{1+x}{4-y}$

At (2, 2), $\frac{dy}{dx} = \frac{1+2}{4-2} = \frac{3}{2}$

Equation of tangent

$y-2 = \frac{3}{2}(x-2)$

$2y-3x+2=0$

Equation of the normal

$y-2 = -\frac{3}{2}(x-2)$

$2x+3y=0$

(ii) $(x-3)^2 + (y+2)^2 = 4$ at point (1, 0)

Solution

Either

Centre at (3, -2)

Gradient of the radius = $\frac{-2-0}{3-1} = -1$

Gradient of the tangent = 1

Equation of tangent

$y-0 = (x-1)$

$y-x+1=0$

Equation of the normal

$y-0 = -1(x-1)$

$y+x-1=0$

Or:

$2(x-3)+2(y+2)\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{3-x}{2+y}$

At (1, 0); $\frac{dy}{dx} = \frac{3-1}{2+0} = 1$

Gradient of the tangent = 1

Gradient of the normal = -1

Equation of tangent

$y-0 = (x-1)$

$y-x+1=0$

Equation of the normal

$y-0 = -1(x-1)$

$y+x-1=0$

(b) Fine the equation of the tangents to the circle $x^2+y^2-8x-6y+9=0$ which are parallel to the straight line $4x-3y+2=0$

Solution

Let the tangent be of the form $y = mx + c$

From $4x-3y+2=0$

$y = \frac{4}{3}x + c$

Gradient = $\frac{4}{3}$

Equation of the tangent

⇨ $y = \frac{4}{3}x + c$ or $4x-3y+c=0$

Finding the radius

Method 1

Given $x^2+y^2-8x-6y+9=0$ i.e.

$(x-4)^2 + (y-3)^2 = 42$

The centre is at (4, 3) and the radius is, r = 4

Note

The perpendicular distance, d, from the point (α, β) to the line ax + by + c = 0 is $d = \frac{|a\alpha+b\beta+c|}{\sqrt{a^2+b^2}}$ units.

Now the distance from the centre to the tangent is the radius=$\frac{|444-3x3+c|}{\sqrt{4^2+(-3)^2}} = 4$;

=> c = -27 or c = 13

Hence the equation of the two tangents become

$4x-3y+13=0$ **or** $4x-3y-27=0$

Method 2

From $4x-3y+c=0$ => $y = \frac{1}{3}(4x+c)$

Substituting y into the equation of the circle

$x^2 + \frac{1}{9}(4x+c)^2 - 8x - 2(4x+c) + 9 = 0$

If the line is a tangent then, $b^2 = 4ac$

$64(c-18) = 4(25)(c^2-18c+81)$

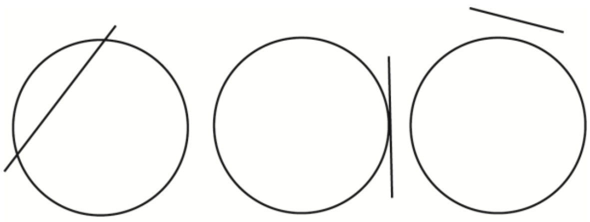
$c^2 + 14c - 3159 = 0$

c = 13 or c = -27

Hence the equation of the two tangents become

$4x-3y+13=0$ **or** $4x-3y-27=0$

Intersection of the line and the circle



(i) $b^2 > 4ac$ (chord)

(ii) $b^2 = 4ac$ (tangent)

(iii) $b^2 < 4ac$

Given the line y = mx + c and the circle

$x^2 + y^2 = r^2$

Substituting for y into the equation of a circle:

$x^2 + (mx+c)^2 = r^2$

$(1 + m^2)x^2 + 2mcx + c^2 - r^2 = 0$
$4m^2c^2 = 4(1+ m^2)(c^2 - r^2)$ i.e. $(b^2 = 4ac)$
$m^2c^2 = c^2 - r^2 + m^2c^2 - m^2r^2$
$c^2 = r^2(1+m^2)$
$c = \pm r\sqrt{(1+m^2)}$
The line $y = mx = \pm r\sqrt{(1+m^2)}$ is a tangent to the circle $x^2 + y^2 = r^2$ for all values of m.

Alternatively

The circle $x^2 + y^2 = r^2$ has a centre (0, 0) and radius, r.
The distance, d from O to the line y = mx + c must be the radius
Writing y = mx +c as mx – y + c = 0
The distance, r= $\left|\frac{m(0)-0+c}{\sqrt{(m^2+(-1)^2)}}\right|$
i.e. $c = \pm r\sqrt{(1+m^2)}$, implying, the equation of the tangent is
$y = mx = \pm r\sqrt{(1+m^2)}$

Example 7

(a) Find the value of the constant k so that the line 4x + 3y + k = 0 is a tangent to the circle $x^2 + y^2 = 36$

Solution

Either 4x + 3y + k = 0
$m = \frac{dy}{dx} = -\frac{4}{3}$
The tangent is $y = mx \pm 6\sqrt{(1+m^2)}$
Substituting form into the equation of the line: $y = -\frac{4}{3}x \pm 6\sqrt{\left(1+\frac{16}{9}\right)}$
$y = -\frac{4}{3}x \pm 10$
$3y + 4x \pm 30$
Hence $k = \pm 30$

Or:

Given x2 + y2 = 36
Centre (0, 0) and radius r = 6
The distance from O to the line is 6
$6 = \left|\frac{4(0)-3(0)+k}{\sqrt{(4^2+3^2)}}\right|$
Hence $k = \pm 30$

(b) Show that the line 2x – 3y + 26 = 0 is a tangent to the circle $x^2 + y^2 - 4x + 6y - 104 = 0$ and find the coordinates of the point of contact.

Solution

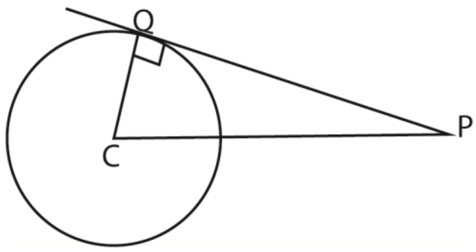
Method 1

From 2x – 3y + 26 = 0 => $x = \frac{1}{2}(3y - 26)$
Substituting for x into the equation of the circle;
$\frac{1}{4}(3y - 26)^2 + y^2 - 4.\frac{1}{2}(3y - 26) + 6y - 104 = 0$
$13y^2 - 156y + 468 = 0$ ……………….(i)
For equal roots: $156^2 = 4 \times 13 \times 468$ which is true, hence a tangent.
Solving eqn. (i)
$y = \frac{156}{2 \times 13} = 6$ and $x = \frac{1}{2}(3x6 - 26) = -4$
Thus, the coordinates of the point of contact are (-4, 6)

Method 2

If the line is a tangent, then its perpendicular distance from the centre of the circle must is the radius
$x^2 + y^2 - 4x + 6y - 104 = 0$
i.e. $(x - 2)^2 + (y + 3)^2 = 117$
the centre is (2, -3) and radius $r = 3\sqrt{13}$
Now, $\left|\frac{2(2)-3(-3)+26}{\sqrt{(2^2+(-3)^2)}}\right| = \frac{39}{\sqrt{13}} = \frac{39\sqrt{13}}{13} = 3\sqrt{13}$
Since the distance from the centre to the line is the radius, then the line is a tangent.

The length of a tangent from a given point to a circle



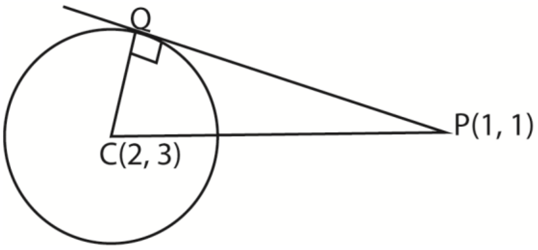
Since the radius is perpendicular to the tangent,
$QC^2 + PQ^2 = PC^2$ i.e. $PQ = \sqrt{(PC^2 - QC^2)}$

Example 8

(a) Find the value of a if the length of the tangent from the point (1, 1) the circle $x^2 + y^2 - 4x - 6y + 3a = 0$ is 2 units

Solution

Equation of the circle
$x^2 + y^2 - 4x - 6y + 3a = 0$ i.e.
$(x - 2)^2 + (y - 3)^2 = 13 - 3a$
Centre is (2, 30 and r2 = 13 – 3a.



PQ =$\sqrt{(PC^2 - QC^2)}$

$PC^2 = (2 – 1)^2 + (3 – 1)^2 = 5$

$CQ^2 = r^2 = 13 – 3a$

$\therefore PQ^2 = 5-(13 -3a)= 2^2=4$

$3a = -5 + 13 + 4 = 12$

a = 4

(b) Find the equation of the tangent from the point (2, 11) to the circle $x^2 + y^2 = 25$

Solution

Method 1

Let the equation of the tangents be

y = mx + c which passes through (2, 11)

⇨ 11 = 3m + c i.e. c = 11 – 2m

The equation of the tangent is

mx - y + 11 – 2m = 0

As it is a tangent to the circle

$(x – 0)^2 + (y – 0)^2 = 5^2$

⇨ 5= $\left|\frac{m(0)+11-2m}{\sqrt{(m^2+(-1)^2)}}\right|$

11 – 2m =$5\sqrt{1+m^2}$

121 – 44m + 4m2 = 25 + 25m2

$21m^2 + 72m - 96 = 0$

m = $\frac{4}{3}$ or m = $\frac{-24}{7}$

When m = $\frac{4}{3}$, the equation of the tangent is

$\frac{4}{3}x - y + 11 - 2\left(\frac{4}{3}\right) = 0$

$4x - 3y + 25 = 0$

When m = $\frac{-24}{7}$, the equation of the tangent is

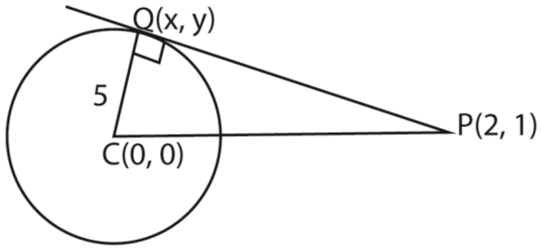
$\frac{-24}{7}x - y - 2\left(\frac{-24}{7}\right) = 0$

$4x - 3y - 125 = 0$

Method 2

The circle: $(x – 0)^2 + (y -0)^2 = 5^2$

i.e. centre C(0,0) and radius r = 5



$PQ^2 + 5^2 = CP^2$

$PQ^2 = (2 – 0)^2 +(11 – 0)^2 – 52 = 100$

PQ = 10 units

Thus $CQ^2 = x^2 + y2 =5^2$

$x^2 + y2 =5^2$.................................(i)

and $PQ^2 = (x-2)^2 +(y-11)^2 = 100$

$x^2 + y^2 – 4x – 22y = -25$ ....(ii)

Eqn. (i) – eqn. (ii)

4x + 22y = 50

$y = \frac{1}{11}(25 - 2x)$

Substituting for y into eqn. (i)

$x^2 + \frac{1}{121}(25 - 2x)^2 = 25$

$125x^2 – 100x – 2400 = 0$

$5x^2 – 4x – 96 =0$

(5x – 24)(x + 4) = 0

x = -4 or x = $\frac{24}{5}$

When x = -4; $y = \frac{1}{11}(25 - 2(-4))$ = 3 and

when x =$\frac{24}{5}$; $y = \frac{1}{11}\left(25 - 2\left(\frac{24}{5}\right)\right) = \frac{7}{5}$

The possible coordinates of Q(x, y) are (-4, 3) and $\left(\frac{24}{5}, \frac{7}{5}\right)$

Taking P(2, 1) and Q(-4, 3)

Equation of PQ: $\frac{y-3}{x+4} = \frac{3-11}{-4-2}$

$\frac{y-3}{x+4} = \frac{4}{3}$

4x – 3y + 25 = 0

Taking P(2, 11) and $Q\left(\frac{24}{5}, \frac{7}{5}\right)$

Equation of PQ: $\frac{y-11}{x-2} = \frac{11-\frac{7}{5}}{2-\frac{24}{5}}$

$\frac{y-11}{x-2} = -\frac{24}{7}$

24x + 7y – 125 = 0

The equations of the tangent from the point (1, 1) to the circle $x^2 + y^2 = 25$ are

4x – 3y + 25 = 0 and 24x + 7y – 125 = 0

(c) Find the length of the tangent from the origin to the circle $x^2 + y^2 - 10x + 2y + 13 = 0$.

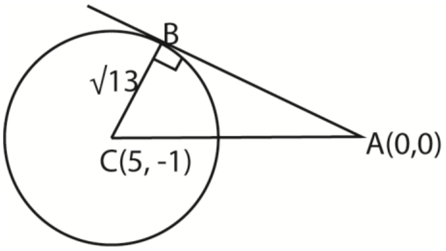
Solution

$x^2 + y^2 - 10x + 2y + 13 = 0$

$(x-5)^2 + (y+1)^2 = -13 + 15 + 1$

$(x-5)^2 + (y+1)^2 = 13$

Centre, C(5, -1) and radius, r = $\sqrt{13}$



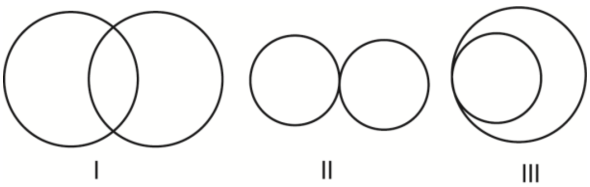
$AB^2 = AC^2 - BC^2$

$AB^2 = (5-0)2 + (-1-0)2 = 13$

AB = $\sqrt{13}$

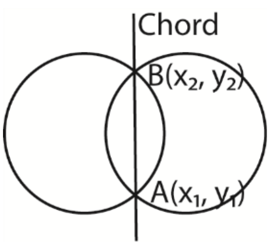
Intersection of two circles

Two circles may intersect at two distinct points or merely just touch each other at particular point.



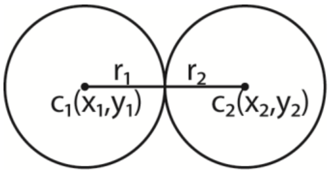
I. Shows intersection of two circles at two distinct points.
II. Shows touching of two circles externally
III. Shows touching of two circles internally

When two circles intersect at two distinct points, they do so on a common chord



If circles touch each other, they may do so either internally or externally

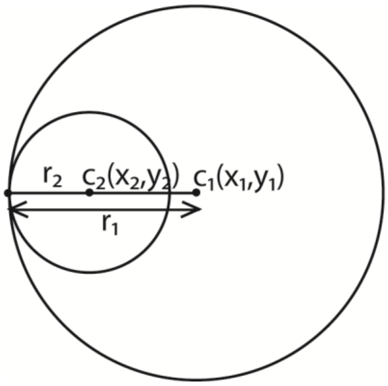
External touching



We now have $C_1C_2 = r_1 + r_2$

Where $r_1 + r_2 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Internal touching



We now have $C_1C_2 = r_1 - r_2$

Where $r_1 - r_2 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Concentric circles have common centre but different radii.

Example 9

(a) Show that the circles $x^2 + y^2 - 2x - 6y + 1 = 0$ and $x^2 + y^2 - 8x - 8y + 31 = 0$ intersect in two distinct points and hence find the length of the common chord.

Solution

Let $x^2 + y^2 - 2x - 6y + 1 = 0$ …………..(i)

and $x^2 + y^2 - 8x - 8y + 31 = 0$ …………(ii)

Eqn. (i) – eqn. (ii)

$6x + 2y - 30 = 0$

$y = 3(5 - x)$

This is the equation of the common chord.

Substituting y into eqn. (i)

$x^2 + 9(5-x)^2 - 2x - 18(5-x) + 1 = 0$

$10x^2 - 74x + 136 = 0$

$5x^2 - 37x + 68 = 0$

$(5x - 17)(x - 4) = 0$

$x = 4$ or $x = \frac{17}{5}$

When $x = 4$, $y = 3(5-4) = 3$ and

when $x = \frac{17}{5}$, $y = 3(5 - \frac{17}{5}) = \frac{24}{5}$

The circle intersect in two distinct point $(4, 3)$ and $\left(\frac{17}{5}, \frac{24}{5}\right)$

The length of the chord is the distance d between these points

$= \sqrt{\left(4-\frac{17}{5}\right)^2+\left(3-\frac{24}{5}\right)^2} = \frac{3\sqrt{10}}{3}$

(b) Show that the circles $x^2 + y^2 - 4x + 2y - 8 = 0$ and $x^2 + y^2 + 6x - 13y + 22 = 0$ touch each other.

Solution

Method 1

Let $x^2 + y^2 - 4x + 2y - 8 = 0$ ....(i)

and $x^2 + y^2 + 6x - 13y + 22 = 0$.............(ii)

Eqn. (ii) – eqn. (i)

$10x - 15y + 30 = 0$

$y = \frac{2}{3}(x+3)$

Substituting for y into eqn. (i)

$x^2 + \frac{4}{9}(x+3)^2 - 4x + \frac{4}{3}(x+3) - 8 = 0$

$x = 0$

when $x = 0$, $y = \frac{2}{3}(0+3) = 2$

∴The circles intersect at only one point (0, 2), hence touching each other.

Method 2

If the circles touch, the distance between their centres C_1 and C2 is the sum of their radii r_1 and r_2, i.e. $r_1 + r_2 = C_1C_2$.

Also $(r_1 + r_2)^2 = (C_1C_2)^2$

For $x^2 + y^2 - 4x + 2y - 8 = 0$

$(x-2)^2 + (y+1)^2 = 8 + 4 + 1$

$(x-2)^2 + (y+1)^2 = 13$

∴ $C_1(2, -1)$ and $r_1 = \sqrt{13}$

For $x^2 + y^2 + 6x - 13y + 22 = 0$

$(x+3)^2 + (y-6.5)^2 = -22 + 9 + 42.25$

$(x+3)^2 + (y-6.5)^2 = 29.25$

∴ $C_2(-3, 6.5)$ and $r_1 = \sqrt{29.25}$

$C_1C_2 = \sqrt{(2-(-3))^2+(-1-6.5)^2}=9.014$

$r_1 + r_2 = \sqrt{13}+\sqrt{29.25}=9.014$

∴the circles touch each other externally since $C_1C_2 = r_1 + r_2$

(c) Show that the circles $x^2 + y^2 - 6x - 2y + 1 = 0$ and $x^2 + y^2 + 2x - 8y + 13 = 0$ touch each other and find the equation of the tangent at the point of contact.

Solution

For $x^2 + y^2 - 6x - 2y + 1 = 0$

$x^2 + y^2 - 6x - 2y = -1$

$(x-3)^2 + (y-1)^2 = -1 + 9 + 1$

$(x-3)^2 + (y-1)^2 = 9$

∴$C_1(3, 1)$ and radius $= \sqrt{9} = 3$

For $x^2 + y^2 + 2x - 8y + 13 = 0$

$x^2 + y^2 + 2x - 8y = -13$

$(x+1)^2 + (y-4)^2 = -13 + 1 + 16$

$(x+1)^2 + (y-4)^2 = 4$

$C_1C_2 = \sqrt{(3-(-1))^2+(1-4)^2}=5$

$r_1 + r_2 = 3 + 2 = 5$

∴the circles touch each other externally since $C_1C_2 = r_1 + r_2$

Finding the equation of the tangent;

Let $x^2 + y^2 - 6x - 2y + 1 = 0$ ....(i)

And $x^2 + y^2 + 2x - 8y + 13 = 0$.............(ii)

Eqn. (i) – eqn. (ii)

$8x - 6y + 12 = 0$

$3y = 4x + 6$

$y = \frac{4}{3}x + 2$

(d) A circle A passes through the point (t+2, 3t) and has the centre at (t, 3t). Circle B has radius 2 and its centre at (t+2), 3t).

(i) Determine the equations of the circles A and Bin terms of t.

Solution

Radius of circle A:

$r^2 = (t+2-t)^2 + (3t-3t)^2 = 2^2$

Equation of circle A

$(x-t)^2 + (y-3t)^2 = 22$

$x^2 + y^2 - 2tx - 6ty + 10t^2 - 4 = 0$

Equation of B

$(x-(t+2))^2 + (y-3t)^2 = 2^2$

$x^2 + y^2 - 2(t+2)x - 6ty + 10t^2 - 4t = 0$

(ii) If t = 1, show that circles A and B intersect at $(2, 3\pm\sqrt{3})$

Solution

When t = 1

Equation of A

$x^2 + y^2 - 2x - 6y + 6 = 0$ ....(i)

Equation of B

$x^2 + y^2 - 6x - 6y + 14 = 0$ ....(ii)

Eqn. (i) – eqn. (ii)

$4x - 8 = 0 \Rightarrow x = 2$

Substituting x into eqn. (i)

$4 + y^2 - -4 - 6y$

$y^2 - 6y + 6 = 0$

$y = \frac{6 \pm \sqrt{36-4(1)(6)}}{2(1)}$

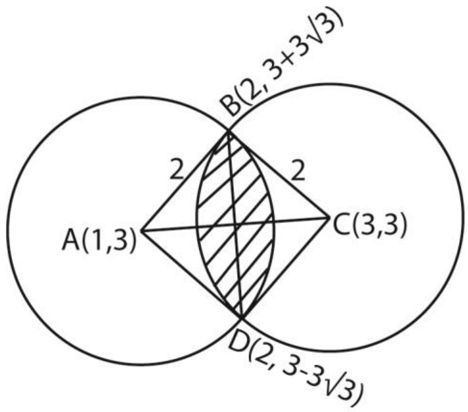
$= \frac{6 \pm \sqrt{9-6)}}{3}$

$= 3 \pm \sqrt{2}$

Hence the point of intersection is $(2, 3\pm\sqrt{3})$

(iii) Show that the area of the region of intersection of the two circles A and B is$8\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$.

Solution



$BD^2 = (2 - 2)^2 + \left(3 + \sqrt{3} - (3 - \sqrt{3})\right)^3$

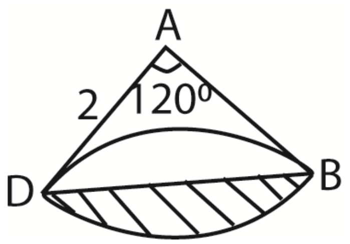
$= (2\sqrt{3})^2$

$BD = 2\sqrt{3}$

In triangle ABD: cosB = cos D =$\frac{\sqrt{3}}{2}$

Angle B = angle D = 30^0

Angle A = 120^0 (angle sum of triangle)



Total area of the figure = $\frac{120}{360}\pi x 2 x 2 = \frac{4\pi}{3}$

Area of triangle ABD = $\frac{1}{2} x 2 x 2 sin 120$

$= 2x\frac{\sqrt{3}}{2} = \sqrt{3}$

Area of the shaded region = $\frac{4\pi}{3} - \sqrt{3}$

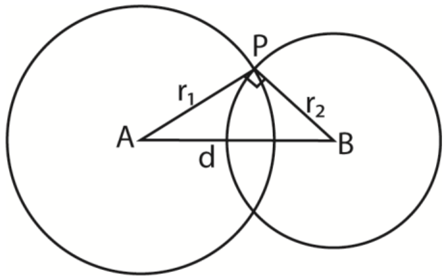
Area of the region of intersection

$=2\left(\frac{4\pi}{3} - \sqrt{3}\right)$

$=8\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$

Orthogonal circles

Two circles cut orthogonally if the tangent at the point of intersection is at right angle.



From the circles above

$PA^2 + PB^2 = AB^2$

$r_1^2 + r_2^2 = d^2$ (condition for orthogonality)

Alternatively

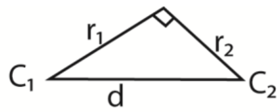
Given two circles $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

By completing squares, we have

$(x+g_1)^2 + (y+f_1)^2 = -c_1 + g_1^2 + f_1^2$

Centre, $C_1(-g_1, -f_1)$ radius, $r_1^2 = -c_1 + g_1^2 + f_1^2$

Similarly, $C_1(-g_2, -f_2)$ radius, $r_2^2 = -c_2 + g_2^2 + f_2^2$



$(C_1C_2)^2 = r_1^2 + r_2^2$

$(g_2 - g_1)^2 + (f_2 - f_1)^2 = -c_1 + g_1^2 + f_1^2 - c_2 + g_2^2 + f_2^2$

$g_2^2 - 2g_2g_1 + g_1^2 + f_2^2 - 2f_2f_1 + f_1^2$

$= -c_1 + g_1^2 + f_1^2 - c_2 + g_2^2 + f_2^2$

$2g_2g_1 + 2f_2f_1 = c_1 + c_2$ (condition for orthogonality)

Example 10

(a) Show that the circle $x^2 + y^2 + 10x - 4y - 3 = 0$ and $x^2 + y^2 - 2x - 6y + 5 = 0$ are orthogonal.

Solution

Method 1

Completing squares

For $x^2 + y^2 + 10x - 4y - 3 = 0$

$(x + 5)^2 + (y - 2)^2 = 3 + 25 + 4 = 32$

$C_1 = (-5, 2)$ and $r_1^2 = 32$

For $x^2 + y^2 - 2x - 6y + 5 = 0$

$(x - 1)^2 + (y - 3)^2 = -5 + 1 + 9 = 5$

$C_2 = (1, 3)$ and $r_1^2 = 5$

Let d = distance between the centres of the two circles.

$d^2 = (1+5)^2 + (3 - 1)^2 = 37$

and $r_1^2 + r_2^2 = 32 + 5 = 37$ hence orthogonal.

Method 2

Using $2g_2g_1 + 2f_2f_1 = c_1 + c_2$

$2(-5)(1) + 2(2)(3) = -3 + 5$

$-10 + 12 = 2$

$2 = 2$ (hence orthogonal)

(b) Find the equation of the circle which passes through points (1, 1), (1, -1) and is orthogonal to $x^2 + y^2 = 4$.

Solution

The general equation is

$x^2 + y^2 + 2gx + 2fy + c = 0$ ………………… (i)

Centre C(-g, -f) and $r^2 = g^2 + f^2 - c$

For $x^2 + y^2 = 4$.

C(0, 0) and $r^2 = 4$

But $(C_1C_2)^2 = r_1^2 + r_2^2$

$g^2 + f^2 - c = g^2 + f^2 - c + 4$

Through (1, 1) => $1 + 1 + 2g + 2f + 4 = 0$

$2g + 2f = -6$ ……………………………………(ii)

Through (1, -1) => $1 + 1 + 2g - 2f + 4 = 0$

$2g - 2f = -6$ ……………………………………(iii)

Eqn. (ii) + eqn. (iii)

$4g = -12$; $g = -2$

Substituting for g in equation (ii)

$2(-3) + 2f = -6$; $f = 0$

Substituting for c, g and f into eqn. (i)

The equation of the circle

$x^2 + y^2 - 6x + 4 = 0$

(c) Find the equation of the circle which passes through the origin and cuts the two circles $x^2 + y^2 - 6x + 8 = 0$ and $x^2 + y^2 - 2x - 2y - 7 = 0$ orthogonally

Solution

The general equation is

$X^2 + y^2 + 2gx + 2fy + c = 0$

when it passes through the origin c = 0, hence it becomes

$x^2 + y^2 + 2gx + 2fy = 0$ ………………… (i)

Centre C(-g, -f) and $r^2 = g^2 + f^2$

Given $x^2 + y^2 - 6x + 8 = 0$

$(x - 3)^2 + (y - 0)_2 = -8 + 9 = 1$

Centre $C_1(3, 0)$ and $r_1^2 = 1$

If they cut orthogonally

$(C_1C_2)^2 = r_1^2 + r_2^2$

$(3 + g)^2 + (0 + f)^2 = g^2 + f^2 + 1$

$9 + 6g + g^2 + f^2 = g^2 + f^2 + 1$

$6g = -8$

$g = -\frac{4}{3}$

Similarly, given $x^2 + y^2 - 2x - 2y - 7 = 0$

$(x - 1)^2 + (y - 1)^2 = 7 + 1 + 1 = 9$

Centre $C_1(1, 1)$ and $r_1^2 = 9$

If it cuts orthogonally,

$(C_1C_2)^2 = r_1^2 + r_2^2$

$(1 + g)^2 + (1 + f)^2 = g^2 + f^2 + 9$

$1 + 2g + g^2 + 1 + 2f + f^2 = g^2 + f^2 + 1$

$2g + 2f = 7$

Substituting for g

$2\left(-\frac{4}{3}\right) + 2f = 7, f = \frac{29}{6}$

Substituting for g and f into eqn. (i), the equation of the circle is

$x^2 + y^2 + 2\left(-\frac{4}{3}\right)x + 2\left(\frac{29}{6}\right)y = 0$

$x^2 + y^2 - \frac{8}{3}x + \frac{29}{3}y = 0$

$3x^2 + 3y^2 - 8x + 29y = 0$

Example 11

1. A circle whose centre is in the first quadrant touches the x – and y –axes and the line $8x - 15y = 120$. Find the

   (a) equation of the circle (10marks)

   Solution

   Radius a = $\frac{|8a - 15a - 120|}{\sqrt{8^2 + (-15)^2}}$

   $= \frac{|-7a + 120|}{17}$

   $17a = 7a + 120$

   $10a = 120$

   $a = 12$

   Equation of the circle

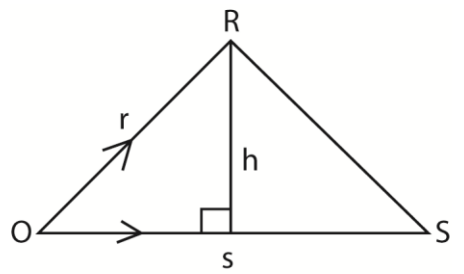
   $(x - 12)^2 + (y - 12)^2 = 12^2$

   $X^2 + y^2 - 24x - 24y + 144 = 0$

(b) point at which the circle touches the x-axis. (02marks)
$y = 0$
$(x - 12)^2 = 0$
$x = 12$
the point (12, 0)

Example 12

The position vectors of the vertices of a triangle are O, r and s, where O is the origin. Show that its area (A) is given by $4A^2 = |r|^2|s|^2 - (r.s)^2$. (06marks)



$r.s = |r||s| \cos O$
$(r.s)^2 = |r|^2|s|^2 cos^2\ O$
$sin^2\ O = 1 - \frac{(r.s)^2}{|r|^2|s|^2} = \frac{|r|^2|s|^2 - (r.s)^2}{|r|^2|s|^2}$
$A = \frac{1}{2}|r||s|sinO$
$2A = |r||s|sinO$
$4A^2 = |r|^2|s|^2 sin^2\ O$
$4A^2 = |r|^2|s|^2 . \frac{|r|^2|s|^2 - (r.s)^2}{|r|^2|s|^2}$
$4A^2 = |r|^2|s|^2 - (r.s)^2$

Hence, find the area of a triangle when $r = \begin{pmatrix}2\\3\end{pmatrix}$ and $s = \begin{pmatrix}1\\4\end{pmatrix}$ (06marks)

$|r|^2 = 2^2 + 3^2 = 13$
$|s|^2 = 1^2 + 4^2 = 17$
$r.s = \begin{pmatrix}2\\3\end{pmatrix}.\begin{pmatrix}1\\4\end{pmatrix} = (2\ x\ 1) + (3\ x\ 4) = 14$
$\therefore 4A^2 = 13 \times 17 - 14^2 = 25$
$A = \sqrt{\frac{25}{4}} = 2.5$units

Exercise 1

1. A point P is such that its distance from the origin is five times its distance from the point (12, 0). Show that the locus of P is a circle and find its radius. $[(5, -6);\ \sqrt{61}]$
2. If $P(x_1, y_1)$ is a point outside the circle, $x^2 + y^2 + 2gx + 2fy + c = 0$, show that the length of the tangent PT from P to the circle is given by $PT^2 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$
   Two circles have centres A(1, 3) and B(6, 8) and intersect at C(2, 6) and D, find the equation of each of the circles and that of line CD. The tangents to the circles from a point P are of equal length. Verify that lie on CD.
   $$\begin{bmatrix} x^2 + y^2 - 10x = 0 \\ and \\ x^2 + y^2 - 11x - 7y + 30 = 0 \end{bmatrix}$$
3. (a) find the equation of the tangent and the normal to the circle
   $3x^2 + 3y^2 + 6x - 4y - 15 = 0$ at the point (-2, 3)
   $[7y - 3x + 15 = 0, 3y + 7y + 5 = 0]$
   (b) Show that the circles
   $3x^2 + 3y^2 - 2x - 2y + 1 = 0$ and
   $3x^2 + 3y^2 - 6x - 4y + 9 = 0$ cut orthogonally.
   (c) Find the equation of the circle which passes through points (1, 1), (1, -1) and is orthogonal to $x^2 + y^2 = 4$. $[x^2 + y^2 - 6x + 4 = 0]$
4. A circle A passes through the point (t+2, 3t) and has the centre at (t, 3t). Circle B has radius 2 and its centre at (t+2), 3t).
   (i) Determine the equations of the circles A and Bin terms of t.
   $[x^2 + y^2 - 2(t+2)x - 6ty + 10t^2 - 4t = 0]$
   (ii) If t = 1, show that circles A and B intersect at $(2, 3\pm\sqrt{3})$
   (iii) Show that the area of the region of intersection of the two circles A and B is$8\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$.
5. The equation of circle, centre O is given by $x^2 + y^2 + Ax + By + C = 0$ where A, B and C are constant. Given that 4A = 3B, 3A = 2c and C = 9
   (a) Determine
   (i) The coordinates of the centre of the circle [(-3, -4)]
   (ii) The radius of the circle [4 units]

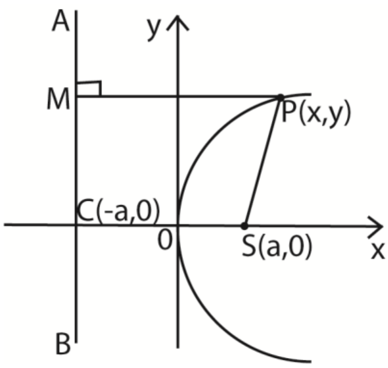
(b) A tangent is drawn from the point Q(3, 2) to the circle. Find
   (i) the coordinates of P, the point where the tangent meets the circle [(-4.16, -0.17) or (0.83, -5.16)]
   (ii) the area of the triangle QPO. [14.96]

6. find the orthocentre (point of intersection of the altitude) of the triangle with vertices A(-2, 1), B(3, -4) and C(-6, -1) [(-2, -4)
7. (a) Find the equation of the circle circumscribing the triangle whose vertices are A(1, 3), B(4, -5) and C(9, -1). Find also the radius of the circle.
$\left[x^2+y^2-\frac{113}{13}x+\frac{8}{13}y-\frac{41}{13}=0; r=4.71\right]$
   (b) If the tangent to the circle at A(1, 3) meets the x-axis at P(h, 0) and the y-axis at Q(10, k), find the values of h and k. [h= -2, k = 2]
8. Find the equation of a circle which passes through the points (5, 7), (1, 3) and 2, 2).
$[x^2+y^2-7x-9y+24=0]$
9. (a) If x = 0 and y = 0 are tangents to the circle $x^2+y^2+2gx+2fy+c=0$; show that $c = g^2 = f^2$.
   (b) Given that the line 3x – 4y + 6 = 0 is also tangent to the circle in (a) above; determine the equation of the circle lying in the first quadrant.
   $[x^2+y^2-2x-2y+1=0]$
10. Form the equation of a circle that passes through points A(-1, 4), B(2, 5) and C(0, 1)
$[x^2+y^2-2x-6y+5=0]$
11. The line x + y = c is a tangent to the circle $x^2 + y^2 - 4y + 2 = 0$. Find the coordinates of the point of contact of the tangent for each value of c. [(1, 3)]
12. ABCD is a square inscribed in a circle $x^2 + y^2 - 4x - 3y = 36$. Find the length of the diagonal and the area of the square.
[13, 84.5]

Parabola

A parabola is a locus of a point say P which moves so that its distance from a fixed point (the focus) is always equal to its perpendicular distance from a fixed straight line (the directrix)

The general shape is as follows.



- S(a,0)is the focus (fixed point)
- P(x, y) is the variable point
- x = -a is the equation of the directrix (fixed line or line AB.
- O is the vertex of the above parabola(line of symmetry)
- OS is the focal length
- PM is the perpendicular distance from the curve at P to the directrix.
- The focus , S lies on the x-axis and has coordinates S(a, 0) where a is a constant.

Equation of a parabola

By the above definition

$$\frac{PS}{PM}=1$$

$$PS^2=PM^2$$

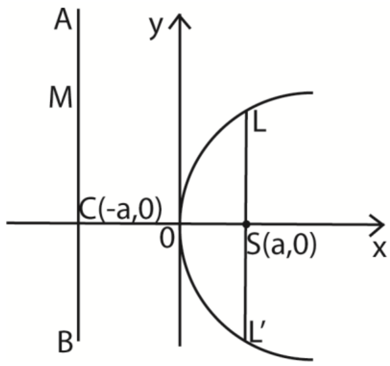
$$(x-a)^2+y^2=(x+a)^2$$

$$x^2-2ax+a^2+y^2=x^2+2ax+a^2$$

$$y^2=4ax$$

Length of the lactus rectum of a parabola

A lactus rectum is a line perpendicular to the axis of the parabola and passing through the focus S(a, 0)



In the diagram above, LL' is the length of **latus rectum**.

Its length can be derived as follows

The x-coordinates of L is a.

The corresponding y-coordinate is obtained by substituting the value of x in the equation of the parabola $y^2 = 4ax$

$y^2 = 4a^2$ or $y = \pm 2a$

The coordinates of L and L' are L(a, 2a) and L'(a, -2a)

Length LL' = 2a + 2a = 4a

∴ the length of the latus rectum is 4a units.

Parametric equation of a parabola

A typical point on the parabola can be represented by the equation

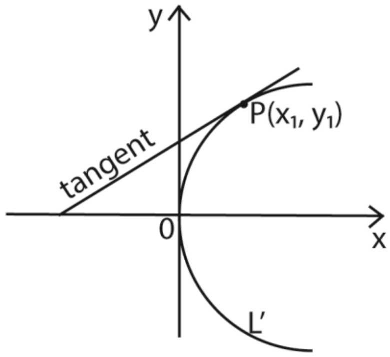
$x = at^2$ and $y = 2at$, where t is the parameter. These express the coordinates of a point on the curve in terms the parameter, t. such a point can be referred to as 't'.

However, other parameters may be used like p and q, hence $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ etc. Using the above reference, these are points 'p' and 'q' respectively.

To show the point $P(at^2, 2at)$ lies on the parabola y2 = 4ax

This is done in two ways

(i) When the point on the parabola is (x_1, y_1). The tangent to the parabola at this point is established by differentiating the $y^2 = 4ax$ with respect to x.



Using the equation $y^2 = 4ax$

$2y\frac{dy}{dx} = 4a$

$\frac{dy}{dx} = \frac{2a}{y}$

At the point $P(x_1, y_1)$

$\frac{dy}{dx} = \frac{2a}{y_1}$

⇨ $\frac{y-y_1}{x-x_1} = \frac{2a}{y_1}$

$yy_1 - {y_1}^2 = 2ax - 2ax_1$

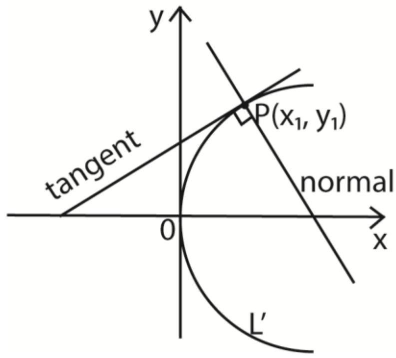
Since the point $P(x_1, y_1)$ lies on the parabola, we replace ${y_1}^2$ and $4ax_1$

⇨ $yy_1 - 4ax = 2ax - 2ax_1$

$yy_1 = 2ax + 2ax_1$

$yy_1 = 2a(x + x_1)$

Finding the equation of the normal at $P(x_1, y_1)$



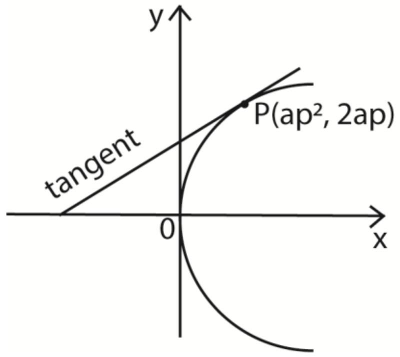
Now the gradient of the normal at $P(x_1, y_1) = -\frac{y_1}{2a}$

$\frac{y-y_1}{x-x_1} = -\frac{y_1}{2a}$

$2ay - 2ay_1 = y_1(x - x_1)$

$y - y_1 = \frac{y_1}{2a}(x - x_1)$

(ii) Finding the equation of the normal given parametric equation



At P(ap^2, 2ap), x = ap^2 and y = 2ap

$\frac{dx}{dp} = 2ap \; and \; \frac{dy}{dp} = 2a$

The gradient $\frac{dy}{dx} = \frac{dy}{dp} x \frac{dp}{dx} = \frac{2a}{2ap} = \frac{1}{p}$

Gradient of the normal = -p

The equation of the normal at P(ap^2, 2aP)

$\frac{y-2ap}{x-ap^2} = -p$

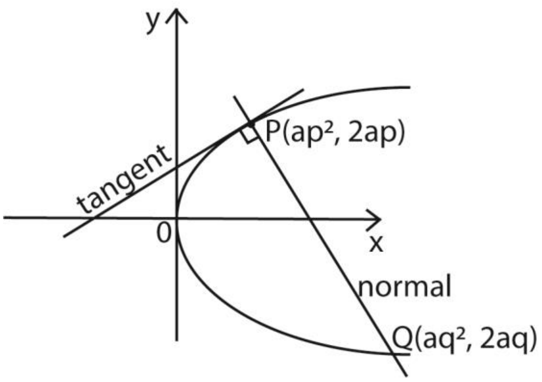
$y - 2ap = -p(x - ap^3)$

$y + px - ap(2 - p^2) = 0$

The point where the normal meets the parabola again

The coordinates of the point where the normal meets the parabola again can be computed as follows

Let the normal at P(ap^2, 2ap) meets the parabola again at Q(aq^2, 2aq)



Either

The gradient of the normal, PQ = -p

⇨ $\frac{2ap-2aq}{ap^2-aq^2} = -p$

$\frac{2(p-q)}{(p-q)(p+q)} = -p$

$\frac{2}{(p+q)} = -p$

$q = \left(p + \frac{2}{p}\right)$

Substituting for the parameter q in the point of Q(aq^2, 2aq) the coordinates are

$Q\left(a\left(p + \frac{2}{p}\right)^2, 2a\left(p + \frac{2}{p}\right)\right)$

Or

At the point S of intersection of the normal and parabola:

Equation of the normal:

$y + px = ap^2 + ap^3$ ……………………….(i)

Equation of parabola: y^2 = 4ax …………(ii)

From eqn. (i) y = $ap(2 - p^2) - px$

Substituting in eqn. (ii): $[ap(2 + p^2]^2$=4ax

p^2x^2 -2a[2+p^2(2+p^2)]x + $a^2p^2(2+p^2)^2$ = 0

Solving the above quadratic equation of gives

x = ap^2 or x =$a\left(p + \frac{2}{p}\right)^2$

the 1st value is the x-coordinate of P and the second value is the required x- coordinate of Q.

Or

The equation of the normal at P is

$y + px = ap^2 + ap^3$

If this normal passes through Q, then the coordinates of Q satisfy the equation

Substituting for x = aq^2 and y = 2aq

p(aq^2)+ 2aq = 2ap + ap^3

2a(q – p) = ap(p^2 – q^2)

-2a(p – q) = ap(q2 – p2)

-2a(p – q) = ap(q – p)(q + p)

-2 = p(q +p)

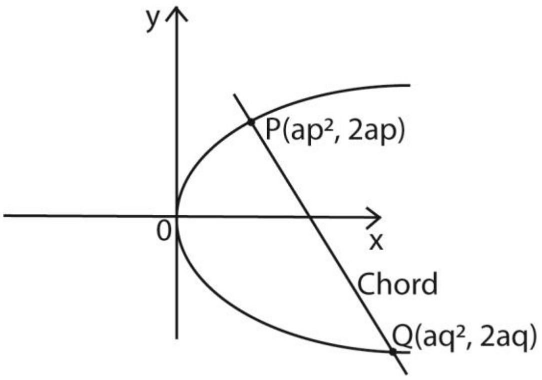
$q = \left(p + \frac{2}{p}\right)$

Substituting for the parameter q in the point of Q(aq^2, 2aq) the coordinates are

$Q\left(a\left(p + \frac{2}{p}\right)^2, 2a\left(p + \frac{2}{p}\right)\right)$

Equation of the chord

The chord of a parabola is the straight line meeting the curve at two distinct points.



The gradient of the chord= $\frac{2ap-2aq}{ap^2-aq^2} = \frac{2}{p+q}$

The equation of the chord PQ

$\frac{y-2ap}{x-ap^2} = \frac{2}{p+q}$

(p + q)y = 2x + 2apq

Deducing the equation of the tangent from the equation of the chord

For the tangent at P, visualize the chord PQ as it turns about P, i.e. q→ p.

i.e. Equation of the tangent at P arises when p = q

From (p + q)y = 2x + 2apq

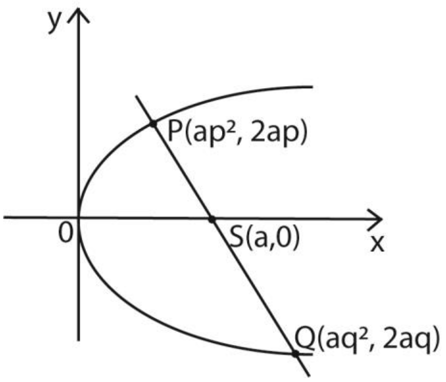
When q = p: 2py = 2x + 2ap^2 or py = x + ap^2

Similarly, the equation of the tangent at Q arises as p→ q; then

qy =x + aq^2

Equation of the focal chord

Focal chord is a chord passing through the focus.



The coordinates of the focus, S(a, 0) satisfy the equation of the chord as it passes through it.

From the chord: (p + q)y = 2x + 2apq

Substituting for x=a and y = 0

pq + 1 = 0

This relationship between the parameters p and q is the condition for the chord to pass through the focus.

It can also be derived as follows

The gradient of PS = gradient of SQ

$\frac{2ap-0}{ap^2-1} = \frac{0-2aq}{a-aq^2}$

$pq + 1 = 0$

The midpoint of the chord

Given two points P(ap2, 2ap) and Q(aq2, 2aq), the midpoint is $M\left(\frac{1}{2}a(p^2+q^2), a(p+q)\right)$

The locus of the midpoint of the focal chord

Eliminating the given parameters from the given equation gives the locus of any curve.

From x = $\frac{1}{2}a(p^2+q^2)$, $p^2+q^2 = \frac{2x}{a}$ ....(i)

And y = $a(p+q)$=> p + q =$\frac{y}{a}$ ....(ii)

Squaring eqn. (ii); p^2 + q^2 + 2pq = $\frac{y^2}{a^2}$ ....(iii)

Substituting eqn. (i) into eqn. (ii)

$\frac{2x}{a} + 2pq = \frac{y^2}{a^2}$

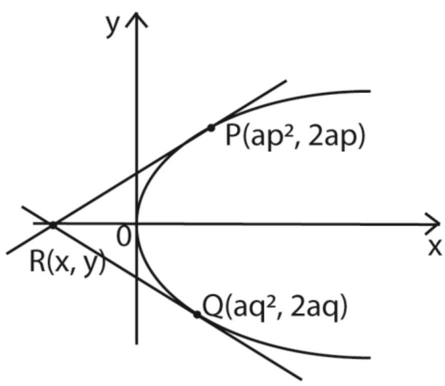
Now for a focal chord, pq = -1

$\frac{2x}{a} + 2(-1) = \frac{y^2}{a^2}$

y^2 = 2a(x – a)

Intersection of two tangents

Let the tangents at P and Q intersect at R(x, y)



Equation of the tangent at P: py = x + ap^2 .....(i)

Equation of the tangent at Q: qy = x + aq^2 .....(ii)

Eqn. (i) – eqn. (ii)

$(p - q)y = a(p^2 - q^2)$ or $y = a(p + q)$

Substituting y into equation (i)

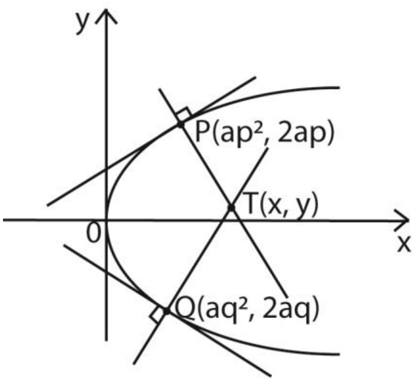
$ap(p+q) = x + ap^2$ i.e. $x = apq$

$R(x, y) = (apq, a(p+q))$

If PQ is a focal chord, the x-coordinate of R becomes a(-1) = -a; x = -a

∴ The tangent at P and Q meets on the directrix if PQ is a focal chord.

Intersection of two normal



Eqn. of the normal at P:
$px + y = 2ap + ap^3$ ....(i)
Eqn. of the normal at Q:
$qx + y = 2aq + aq^3$ ....(ii)
eqn. (i) – eqn. (ii)
$(p - q)x = 2a(p - q) + a(p - q)(p^2 + pq + q^2)$

$x = 2a + a(p^2 + pq + q^2)$
Substituting for x into eqn. (i)
$p[2a + a(p^2 + pq + q^2)] + y = 2ap + ap3$
$y = -apq(p+ q)$
$\therefore T(2a + a(p^2 + pq + q^2), -apq(p+ q))$

The condition that the line y = mx + c is a tangent to the parabola.

At the point of tangency, the roots must be equal.

Equation of the line: $y = mx + c$ ....(i)

Equation of parabola: $y^2 = 4ax$ ....(ii)

Solving eqn. (i) and eqn. (ii) simultaneously, substitute for y into eqn. (ii)

$(mx + c)^2 = 4ax$

$m^2x^2 + 2mcx + c^2 = 4ax$

$m^2x^2 + 2(mc - 2a)x + c^2 = 4ax$

This line is a tangent when $b^2 = 4ac$

$4(mc - 2a)^2 = 4m^2c^2$

$m^2c^2 - 4amc + 4a^2 = m^2c^2$

$mc = a$ i.e. $c = \frac{a}{m}$

∴The line y = mx + c is a tangent to the curve $y = 4ax$ when $c = \frac{a}{m}$

Finding the point of contact between the tangent and the parabola

From $= m^2x^2 + 2(mc - 2a)x + c^2 = 0$

$m^2x^2 + 2(m.\frac{a}{m} - 2a)x + c^2 = 0$

$m^2x^2 - 2ax + c^2 = 0$

$x^2 - \frac{2ax}{m^2} + \frac{c^2}{m^2} = 0$

If the roots of this equation are equal, they are given by half the sum of roots

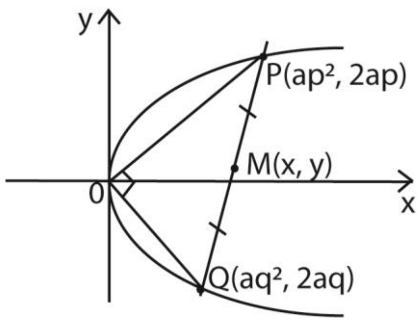
Now sum of roots $= \frac{2a}{m^2}$

⇨ $x = \frac{1}{2}.\frac{2a}{m^2} = \frac{a}{m^2}$ and $y = m.\frac{a}{m^2} + \frac{a}{m} = \frac{2a}{m}$

Hence the point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m;p\ xz}\right)$

Example 13

(a) A chord of the parabola $y^2 = 4ax$ subtends a right angle at the vertex. Show that the locus of the midpoint of the chord is
$y^2 = 2a(x - 4a)$
Solution



Given two points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ the midpoint,(x, y) of PQ is

$x = \frac{1}{2}(ap^2 + aq^2) = \frac{1}{2}a(p^2 + q^2)$ and

y $=\frac{1}{2}(2ap + 2aq) = \frac{1}{2}a(p + q)$

The coordinates of

M(x, y) $=\left(\frac{1}{2}a(p^2 + q^2), \frac{1}{2}a(p + q)\right)$

Eliminating parameters, p and q from the given equations

$x = \frac{1}{2}a(p^2 + q^2) => p^2 + q^2 = \frac{2x}{a}$ .....(i)

$and \;\; y = \frac{1}{2}a(p + q) => p + q = \frac{y}{a}$ .....(ii)

Squaring eqn. (ii) $p^2 + q^2 + 2pq = \frac{y^2}{a^2}$ ....(iii)

Substituting eqn. (i) into eqn. (iii)

$\frac{2x}{a} + 2pq = \frac{y^2}{a^2}$ ....(iv)

Since OP and OQ are perpendicular, the gradient of OP x gradient of OQ = -1

$\frac{2ap-0}{ap^2-0} \cdot \frac{aq^2-0}{2aq-0} = -1$

$\frac{2}{p} \cdot \frac{2}{q} = -1$

$pq = -4$

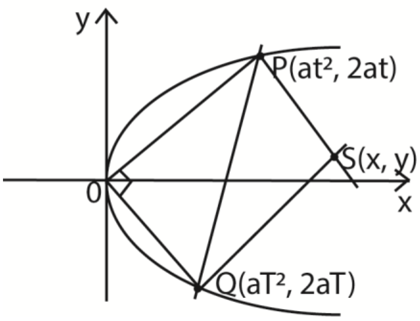
Substituting for pq into eqn. (iv)

$\frac{2x}{a} - 4 = \frac{y^2}{a^2}$

$y^2 = 2ax - 8a^2$

$y^2 = 2a(x - 4a)$; As required

(b) The chord of parabola $y^2 = 4ax$ subtends a right angle at the vertex. Given that the coordinates of P and Q are $(at^2, 2at)$ and $(aT^2, 2aT)$ respectively. Prove that $tT + 4 = 0$ and that the locus of their point of intersection S of the normal at P and Q is given by $y^2 = 16x (x - 6a)$



Since OP and OQ are perpendicular, the gradient of OP x gradient of OQ = -1

$\frac{2at-0}{at^2-0} \cdot \frac{aT^2-0}{2aT-0} = -1$

$\frac{2}{t} \cdot \frac{2}{T} = -1$

$tT + 4 = 0$ As required

Eqn. of the normal at P:

$tx + y = 2at + at^3$ ....(i)

Eqn. of the normal at Q:

$Tx + y = 2aT + aT^3$ .... (ii)

Eqn. (i) – eqn.(ii)

$(t - T)x = 2a(t - T) + a(t^3 - T^3)$

$(t - T)x = 2a(t - T) + a(t - T)(t^2 + tT + T^2)$

$x = 2a + a(t^2 + tT + T^2)$ ....(iv)

Substituting x into eqn. (i)

$2at + at(t^2 + tT + T^2) + y = 2at + at^3$

$atT(t + T) + y = 0$

$y = atT(t + T)$ ....(iv)

Substituting for tT = -4 into eqn. (iv)

$y = 4a(t + T)$

⇨ $t + T = \frac{y}{4a}$

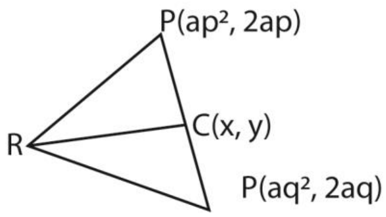
But $(t + T)^2 = t^2 + T^2 + 2tT$

i.e. $\left(\frac{y}{4a}\right)^2 = \frac{x+2a}{a} + 2(-4)$

$y^2 = 16a(x + 2a) - 16a^2(-8)$

$y^2 = 16a(x - 6a)$ as required

(c) C is the midpoint of a variable chord PQ of the parabola. The tangents at P and Q meet at R. Prove that RC is parallel to the axis of the parabola.



This means

$x = \frac{1}{2}(ap^2 + aq^2) = \frac{1}{2}a(p^2 + q^2)$ and

y $=\frac{1}{2}(2ap + 2aq) = \frac{1}{2}a(p + q)$

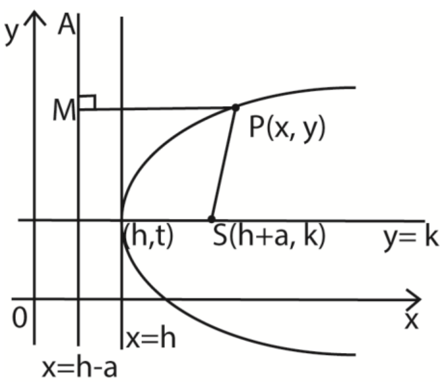
$\therefore C\left[\frac{1}{2}a(p^2 + q^2), \; \frac{1}{2}a(p + q)\right]$

The equations of the tangents at P and Q are

$py = x + ap^2$ and $qy = x + aq^2$ respectively
solving these simultaneously
$x = apq$ and $y = a(p+q)$
∴ R(apq, a(p + q) for both.
∴ RC is parallel to the axis of the parabola
$y = 0$

The parabola with vertex at the point (h, k)

Suppose that the focus of the parabola is shifted by h horizontally and by k vertically, we have;



Using the definition of a parabola: $PS^2 = PM^2$

$[x - (h + a)^3] + (y - k)^2 = [x - (h - a)]^2$

$x^2 - 2(h+a)x + (h+a)^2 + (y - k)^2 = x^2 - 2(h-a)x + (h-a)^2$

$(y - k)^2 = 4a(x - h)$

The above parabola has the following properties

- Vertex is at (h, k)
- Focus is at (h+a, k)
- Equation of the directrix, $x = h - a$
- Length of the latus rectum 2(2a + k)

Example 14

(a) Determine the coordinates of the vertex and the focus and the equation of the directrix of parabola $y^2 - 12x + 2y + 25 = 0$. Hence find the length of latus rectum

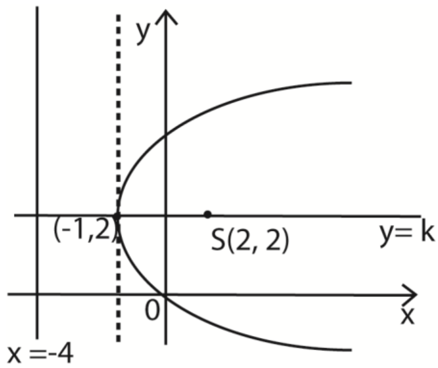
Solution

⇨ $y^2 + 2y = 12x - 25$
$(y + 1)^2 = 12x - 25 + 1$
$(y + 1)^2 = 12(x - 2)$
i.e. $(y + 1)^2 = 4.3(x - 2)$in the form
$(y - k)^2 = 4a(x - 2)$
⇨ a = 3, k = -1, and h = 2
The equation of the directress is x = -1
⇨ x = 2 – 3 = -1

Hence the equation of directrix is x= -1
The length of the latus rectum is
2(2a +k) = 2(6 -1) = 10

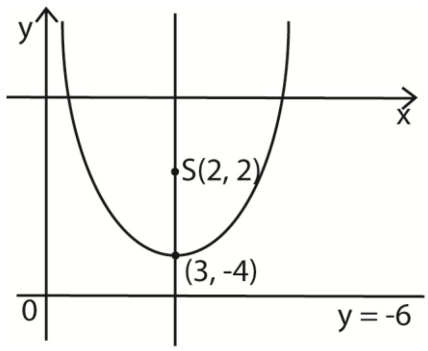
(b) Determine the equation of the parabola with the following respective vertexes and foci

(i) (-1, 2) and (2, 2)
Solution
The required equations can be obtained from first principles using the definition of a parabola or using the general derived rule of $(y - k)2 = 4a(x - h)$



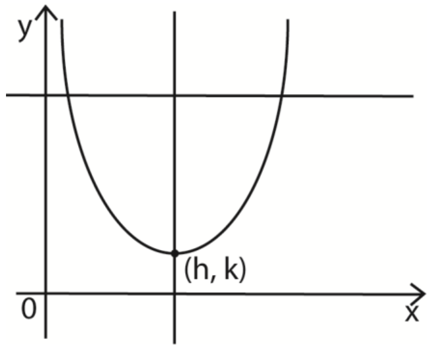
Using
$(y - k)^2 = 4a(x - h)$
For vertex = (-1, 2), h = -1 and k = 2
For focus S(2, 2), h + a = -1+ a = 2
⇨ a = 3
by substitution,
$(y - 2)^2 = 4 \times 2(x - {}^{-}1)$
$y^2 - 4y + 4 = 12(x + 1)$
$y^2 - 4y - 12x - 8 = 0$

(ii) (3, 4) and (3, -2)



The distance from the vertex to the focus is 2 units. i.e. a = 2
$(x - 3)^2 = 4(2)(y + 4)$
⇨ $x^2 - 8y - 6x - 23 = 0$

Note: the parabola of the form $(x - h)^2 = 4a(y - 4)$ has the vertex at (h, k) but its axis is parallel to the y-axis.



(c) Find the equation of a parabola with a horizontal axis passing through (1, 4) and vertex (-2, 2). Hence find its focus and directrix.

Solution

Using $(y - k)^2 = 4a(x - h)$

For vertex, (h, k) = -2, 2), h = -2 and k =2

By substitution

$(y - 2)^2 = 4a(x - -2)$

Since the curve passes through (1, 4),

Substitute for x =1and y = 4.

$(4 - 2)^2 = 4a(1 + 2)$

4 = 12a

$a = \frac{4}{3}$

Substituting for h, k and a, the equation becomes $(y - 2)^2 = 4.\frac{4}{3}(x + 2)$

Or $y^2 + 12y - 4x + 6 = 0$

Equation of directrix is x = h – a

$\therefore x = -2 - \frac{4}{3} = -\frac{10}{3}$

(d) Find the focus and directrix of the parabola $y^2 = 8(x - 12)$

Solution

Comparing y2 =8(x – 12) with

$(y - k)^2 = 4a(x - h)$

⇨ (y – 0)2 = 4x 2(x – 12)

k = 0, a = 2 and h = 12

Focus is S(h+a, k) = S(14, 0)

Hence S(14, 0)

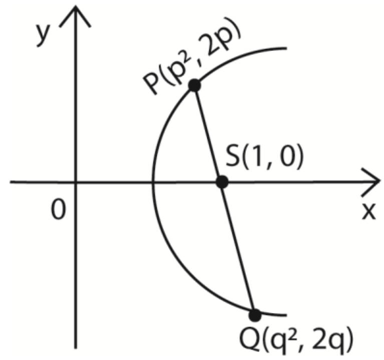
Directrix is h – a = 12 – 2 = 10

Hence directrix is 10

(e) A focal chord PQ, to the parabola $y^2 = 4x$, has a gradient m = 1. Find the coordinates of the mid-point of PQ. (05marks)

Solution

Method 1



Gradient of PS = gradient of PQ = 1

$\frac{2p-0}{p^2-1} = 1$

$p^2 - 2p - 1 = 0$

$p = \frac{2\pm\sqrt{2^2-4(1)(-1)}}{2(1)} = \frac{2\pm\sqrt{4+4}}{2} = 1\pm\sqrt{2}$

$\Rightarrow P\left[\left(1+\sqrt{2}\right)^2, \left(2+2\sqrt{2}\right)\right]$,

$Q\left[\left(1-\sqrt{2}\right)^2, \left(2-2\sqrt{2}\right)\right]$,

Let M(x, y) be the coordinates of the mid-point of PQ.

$x = \frac{1}{2}\left[\left(1+\sqrt{2}\right)^2, \left(2+2\sqrt{2}\right)\right] = 3$

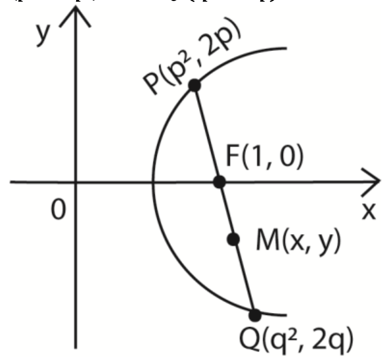
$y = \frac{1}{2}\left[\left(1-\sqrt{2}\right)^2, \left(2-2\sqrt{2}\right)\right] = 2$

∴M(3, 2)

Method 2

From $y^2 = 4x$,; a =1

$\therefore P(p^2, 2p) and\ Q(q^2, 2q)$



From focal chord, pq = 1

Gradient = $\frac{2q-2p}{q^2-p^2} = 1$

$= \frac{2(q-p)}{(q+p)(q-p)} = 1$

⇨ q + p = 2

$M(\frac{p^2+q^2}{2}, p + q)$

$x = \frac{p^2+q^2}{2}$

$= \frac{1}{2}[(p+q)^2 - 2pq]$

$= \frac{1}{2}[(-2)^2 - 2(1)]$

$= \frac{1}{2}(4+2) = 3$

y = p + q = 2

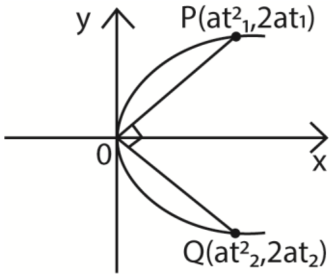
∴M(3, 2)

The point P($at_1^2, 2at_1$) and Q($at_2^2, 2at_2$) are on parabola $y^2 = 4ax$. OP is perpendicular to OQ, where O is the origin.

Show that $t_1t_2 + 4 = 0$.

Method 1

OP.OQ = 0



Gradient of OP, $= m_1 = \frac{2at_1}{at_1^2} = \frac{2}{t_1}$

Gradient of OQ $= m_2 = \frac{2at_2}{at_2^2} = \frac{2}{t_2}$

But $m_1m_2 = -1$

$\frac{2}{t_1} \cdot \frac{2}{t_2} = -1$

$t_1t_2 + 4 = 0$

Method 2

OP.OQ = 0

$\begin{pmatrix} at_1^2 \\ at_1 \end{pmatrix}\begin{pmatrix} at_2^2 \\ at_2 \end{pmatrix} = 0$

$at_1^2 . at_2^2 + 2at_1 . 2at_2 = 0$

$aa^2t_1t_2(t_1t_2 + 4) = 0$

$t_1t_2 + 4 = 0$

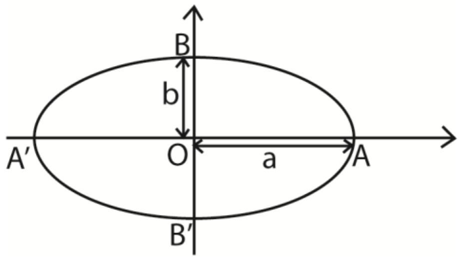
Exercise 4

1. Find the focus and directrix of the following parabolas
2. Prove that the equation of the tangent at (at^2, 2at) on the parabola y^2 =4ax is is $x – ty + at^2 = 0$.A and B meet at P. the line PM is parallel to the axis of parabola and meets the line AB at M. Prove that M is the midpoint of AB. If the parameters of the point A and B are t and 2t respectively and the tangents meet at P, find the coordinates of P and show that it always lies on the parabola $2y^2 = 9ax$.
3. Show that if the normal at P(at^2, 2at) to the parabola $y^2 = 4ax$ meets the parabola again at Q(aT^2, 2aT), $t^2 + tT + 2 = 0$. Hence show that the locus of the point of the tangents at P and Q is described by the curve $(x + 2a)y^2 + 4a^2 = 0$
4. Show that O is the line y = mx + d is a tangent to the parabola $y^2 = Ax + By + C$ if $d = \frac{1}{4mA}(A + Bm)^2 + \frac{cm}{A}$.
5. Given that O is the vertex of the parabola $y^2 = 4ax$ and P is a point on the parabola, find the coordinates of the point of intersection of the normal and the perpendicular bisector of the line OP
6. Prove that the equation of the tangent to the parabola $y^2 = 4ax$ at the point (at^2, 2at) is $ty = x + at^2$. The tangent at P meets the y-axis at a point such that FQPR is a parallelogram, find
   (a) the coordinates of R($h+at^2$, at)
   (b) the equation of the locus of R
   [$y^2 = a(x – h)$, where F(h,0) is the focus.]
7. (a)Show that the locus of the midpoint of the line joining the parabola $y^2 = 8x$ and the point (8, 0) is a parabola.
   (b) Determine the point at which lines from the new focus are perpendicular to the parabola $y^2 = 8x$
   $[(1, 2\sqrt{2}) and (1, -2\sqrt{2})]$
   (c) Find the y-coordinates of the point at which the tangent at one of the points meets the y-axis. $[\sqrt{2}]$
8. (a) Tangent from the point T(t^2, 2t) touches the curve $y^2 = 4x$. Find
   (i) the equation of the tangent
   [$ty – x – t^2 = 0$]
   (ii) the equation of the line L parallel to the normal at (t^2, 2t) and passing through (1, 0). [y + tx – t = 0]
   (iii) the point of intersection X of the line L and the tangent. [X(0, t)
   (b) A point P(x, y) is equidistant from X and T. Show that the locus of P is
   $t^3 + 3t – 2(xt + y) = 0$

9. Prove that the tangents to the parabola $y^2 = 4ax$ at points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ meet at the point $T(apq, a(p+q))$
   (a) If M is the midpoint of PQ, prove that TM is bisected by the parabola
   (b) If P and Q vary on the parabola in such a manner that PQ is always parallel to the fixed line y = mx, show that T always lies on the fixed line parallel to x-axis
   $$\left[\begin{array}{c} show\ that\ y = \frac{2a}{m};\ since\ a\ and\ m\ are \\ constant, then\ T\ lies\ a\ fixed\ line \\ that\ is\ parallel\ to\ x-axis \end{array}\right]$$
10. $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ are two points on the parabola, $y^2 = 4ax$, PQ is a focal chord. Prove that pq = -1 and hence that if the tangents at P and Q intersect at T, the locus of T is given by x + a = 0. PM and QN are perpendiculars onto x + a = 0, s = (a,0). Prove that <MSN = <PTQ = 90^0.
11. (a) (i) Find the equation of the chord through the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ of the parabola $y^2 = 4at$.
   $[(t_1 + t_2)y - 2at_1t_2 = 0]$
   (ii) Show that the chord cuts the directric when $y = \frac{2a(t_2t_1-1)}{t_1+t_2}$
   (b) Find the equation of the normal to the parabola $y^2 = 4ax$ at $(at^2, 2at)$ and determine the point of intersection with the directrix.$[at(t^2 + 3), (a, at(t^2+3)$
12. (a) Find the equation of the tangent to the parabola $y^2 = \frac{x}{16}$ at the point $\left(t^2, \frac{t}{4}\right)$.
   (b) If the tangents to the parabola in (a) above at the point $\left(p^2, \frac{p}{4}\right)$ and $\left(q^2, \frac{q}{4}\right)$ meet on the line y = 2.
      (i) show that p + q = 16
      (ii) deduce that the midpoint of PQ lies on the line y = 2.

The Ellipse

An ellipse takes the form of an oval shape whose length is 2a and width 2b.

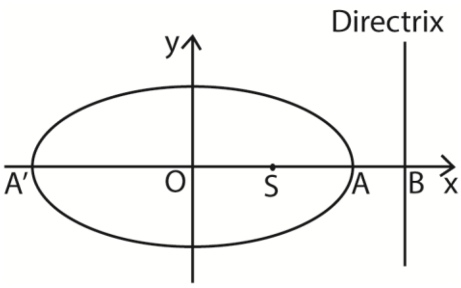


In conics, an ellipse is defined as a locus of all points of P such that the distance from P to a fixed point (focus) bears a constant ratio e (< 1) to the distance from P to a fixed line (the directrix).

Note

- O is the centre of the ellipse
- A'OA is the major axis of which OA = OA' is the semi-major axis.
- B'OB is the minor axis in which OB = OB' is the semi-minor axis.

Finding directrix and foci of an ellipse



Let OA = OA' = a

Using the definition of a conic

At A: AS = eAB

OS – OA = e(OB – OA)

a – s = e(b – a) ……………………………….(i)

At A: A,S = eA'B

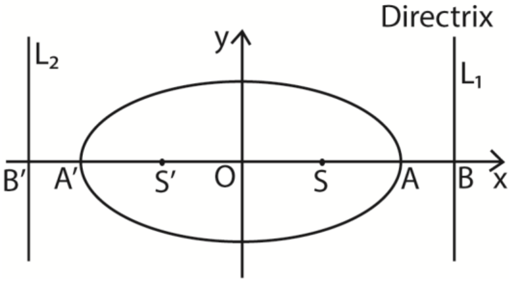
A'O – OS = e(A'O + OB)

a + s = e(b + a) ……………………………….(ii)

eqn. (i) + eqn. (ii)

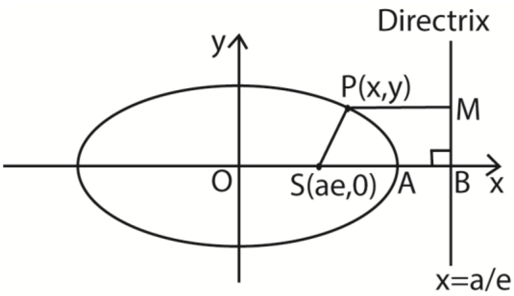
2s = 2ae; s = ae

Now, for an ellipse above, there are other point S' and B' such that the same locus would be obtained with these as focus and point of directrix (L_2)



Hence the equation of the directrix are given as $x = \pm\frac{a}{e}$ and the coordinates of the foci are $(\pm ae, 0)$ where $0 < e < 1$

Equation of an ellipse



By definition

$PS^2 = e^2PM^2$

$(x - ae)^2 + y^2 = e^2\left(\frac{a}{e} - x\right)^2$

$(1 - e^2)x^2 + y^2 = a^2(1 - e^2)$

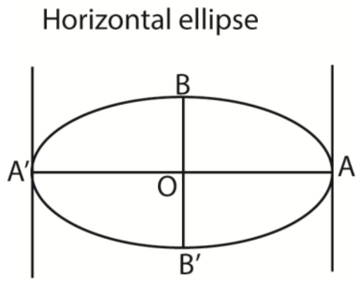
Dividing through by $a^2(1 - e^2)$

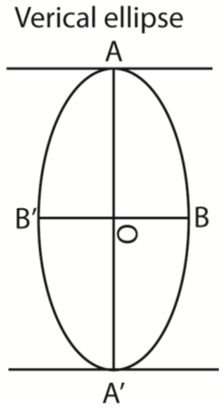
$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

This is an equation of the ellipse with centre at the origin.

Bear in mind that an ellipse may be horizontal or vertical. However, the properties remain the same.

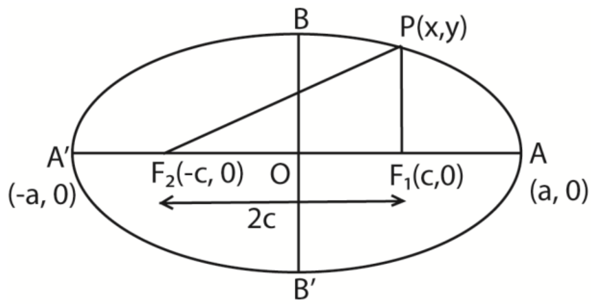




If a > b, the ellipse is horizontal and if a < b, the ellipse is vertical

Properties of an ellipse

(i) If P is any point of the ellipse with foci F_1 and F_2 and length of the major axis 2a.



Then $|F_1P| + |F_2P| = 2a$

This property can be used to derive the equation of an ellipse as follows:

$|F_1P| + |F_2P| = 2a$

$\sqrt{(x+c)^2 + (y-0)^2} + \sqrt{(x-c)^2 + (y-0)^2} = 2a$

$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$

$\sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2}$

Squaring both sides and simplifying

$cx - a^2 = -a\sqrt{(x-c)^2 + y^2}$

squaring both sides again

$(cx - a^2)^2 = a^2[(x-c)^2 + y^2]$

$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$

Dividing through by $a^2(a^2 - c^2)$

$\frac{x^2}{a^2} + \frac{y^2}{a^2-c^2} = 1$

Taking $b^2 = a^2 - c^2$

$c^2 = a^2 - b^2$

$c = \pm\sqrt{a^2 - b^2}$

where c = ae

Note

By equating the two:

$ae \pm \sqrt{a^2 - b^2}$

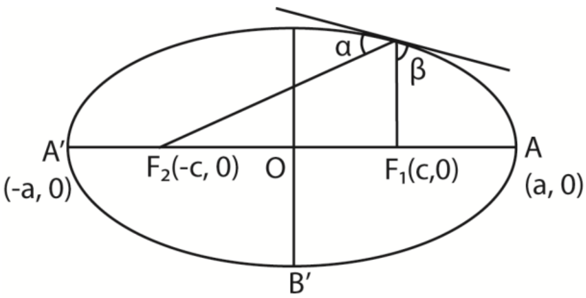
$a^2e^2 = a^2 - b^2$

$b^2 = a^2 - a^2e^2$

$b^2 = a^2(1 - e^2)$

Hence this property is noteworthy using for derivation of the equation of an ellipse.

(ii) The angles formed between the tangents to an ellipse and the lines through the fociare equal.



i.e. α = β

Parametric equation of an ellipse

The equation x = acosθ and y = bsinθ for a parameterθ, are used to represent a point on an ellipse. Where 0≤θ≤ 2π. Other parameters like α and β represent the points; i.e. (acosα, bsinα) and (acosβ, bsinβ) respectively.

Gradient of the tangent can also be obtained eitherparametrically or from the equations of given ellipse.

From the parameters:

X = a cosθ and y = bsinθ

$\frac{dx}{d\theta} = -asin\theta, \frac{dy}{d\theta} = bcos\theta$

$\frac{dy}{dx} = \frac{-bcos\theta}{asin\theta}$

From the equation:

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

⇨ $b^2x^2 + a^2y^2 = a^2b^2$

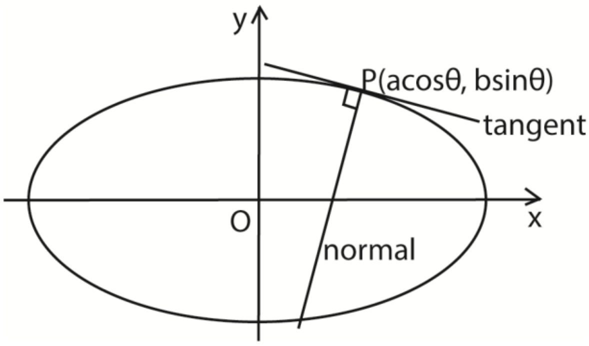
$2b^2x + 2a^2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-b^2x}{a^2y}$

At P(acosθ, bsinθ)

$\frac{dy}{dx} = \frac{-bcos\theta}{asin\theta}$

Equation of the tangent and normal



(a) Equation of the tangent at P(acosθ, bsinθ)

$y - bsin\theta = \frac{-bcos\theta}{asin\theta}(x - acos\theta)$

$aysin\theta - absin^2\theta = -bxcos\theta + abcos^2\theta$

$bxcos\theta + aysin\theta = ab$

Or $\frac{x}{a}cos\theta + \frac{y}{b}sin\theta = 1$

(b) Gradient of the normal at P(acosθ, bsinθ)

$= \frac{asin\theta}{bcos\theta}$

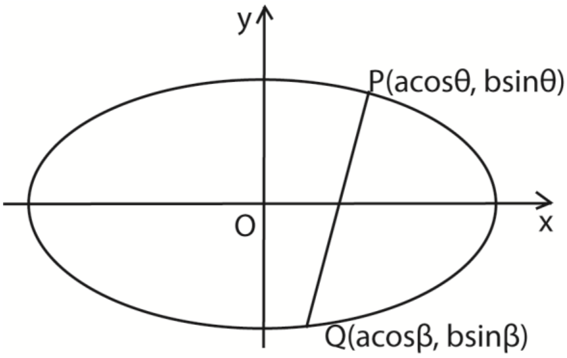
Equation of the normal at P(acosθ, bsinθ)

$y - bsin\theta = \frac{asin\theta}{bcos\theta}(x - acos\theta)$

$bysin\theta - b^2sin\theta cos\theta = axsin\theta - a^2sin\theta cos\theta$

$axsin\theta - bysin\theta = (a^2 - b^2)\, sin\theta cos\theta$

Equation of the chord



The equation of the chord joining the points P($a\cos\theta$, $b\sin\theta$) and Q($a\cos\beta$, $b\sin\beta$) can be obtained as follows

Gradient of PQ = $\frac{b(\sin\beta-\sin\theta)}{a(\cos\beta-\cos\theta)}$

For any point (x, y) on the chord

$\frac{y-b\sin\theta}{x-a\cos\theta}=\frac{b(\sin\beta-\sin\theta)}{a(\cos\beta-\cos\theta)}$

Simplifying

$bx\cos\frac{1}{2}(\beta+\theta) + ay\sin\frac{1}{2}(\beta+\theta)= ab\cos\frac{1}{2}(\beta-\theta)$

A line y = mx + c as a tangent to the ellipse

Equation of the line $y = mx + c$ ....(i)

Equation of ellipse $b^2x^2 + a^2y^2 = a^2b^2$ ....(ii)

Substituting eqn. (i) into eqn. (ii)

$b^2x^2 + a^2(m^2x^2 + 2mcx + c^2) = a^2b^2$

$(b^2 + a^2m^2)x^2+ 2a^2mcx + a^2(c^2 – b^2)= 0$

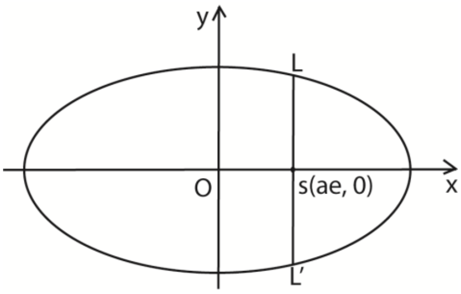
For equal roots

$2a^2m^2c^2 – 4(b^2 + a^2m^2)a^2(c^2 – b^2)$

$b^2c^2 = b^2(b^2 + a^2m^2)$ i.e. $c^2 = b^2 + a^2m^2$

∴ the line $y = mx + c$ is a tangent to ellipse $b^2x^2 + a^2y^2 = a^2b^2$ when $c^2 = b^2 + a^2m^2$ and the line becomes $y = mx \pm\sqrt{b^2 + a^2m^2}$

The length of the latus rectum



At L, $x = \pm ae$

Substituting for x into $b^2x^2 + a^2y^2 = a^2b^2$

$b^2a^2e^2 + a^2y^2 = a^2b^2$

$y^2 = b^2(1 – e^2)$

But for an ellipse, $b^2 = a^2(1- e^2)$

⇨ $y^2 = b^2\left(\frac{b^2}{a^2}\right)= \frac{b^4}{a^2}$

$y = \frac{b^2}{a}$

The length of the latus rectum is

LL' = 2y = $\frac{2b^2}{a}$

Equation of the director circle

If two perpendicular tangent are drawn from a point P(x, y) to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, thelocus of P as the points of contact vary is called a director circle.

The line of the tangent is

$y = mx \pm\sqrt{b^2 + a^2m^2}$

Squaring both sides:

$y^2 – 2myx + m^2x^2 = b^2 + a^2m^2$

$(a^2 – x^2)m^2 + 2xym + b^2 – y^2 = 0$; a quadratic in m.

If m_1 and m_2 are gradients of two tangents, then

Sum of roots = $m_1 + m_2 = \frac{-2xy}{a^2-x^2}$ and product of roots $m_1m_2= \frac{b^2-y^2}{a^2-x^2}$

Since the tangents are perpendicular,

$m_1m_2 = -1$

⇨ $\frac{b^2-y^2}{a^2-x^2} = -1$

$b^2 – y^2 = x^2 – a^2$ or $x^2 + y^2 = a^2 + b^2$

This is the equation of the director circle.

Note: The centre of the director circle is the centre of the ellipse, O(0, 0) and its radius is $\sqrt{(a^2 + b^2)}$ units.

Example 15

(a) Show that the equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at P(x_1, y_1) is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

Solution

$b^2x^2 + a^2y^2 = a^2b^2$

Differentiating with respect to x