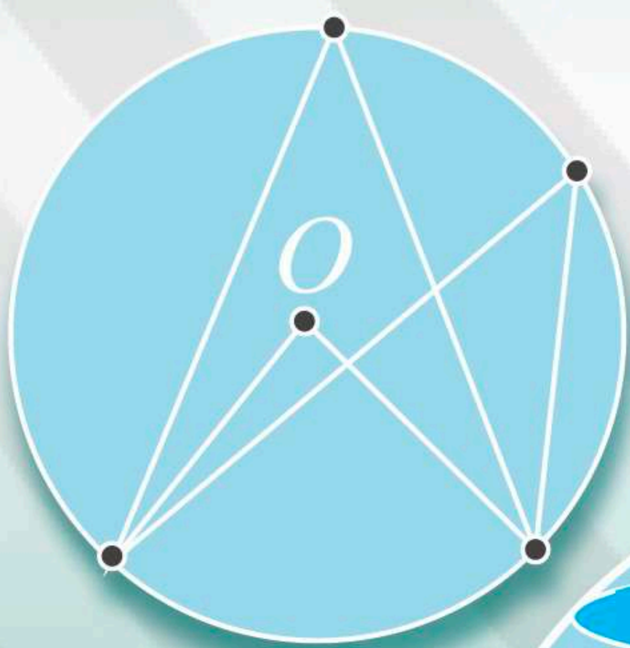
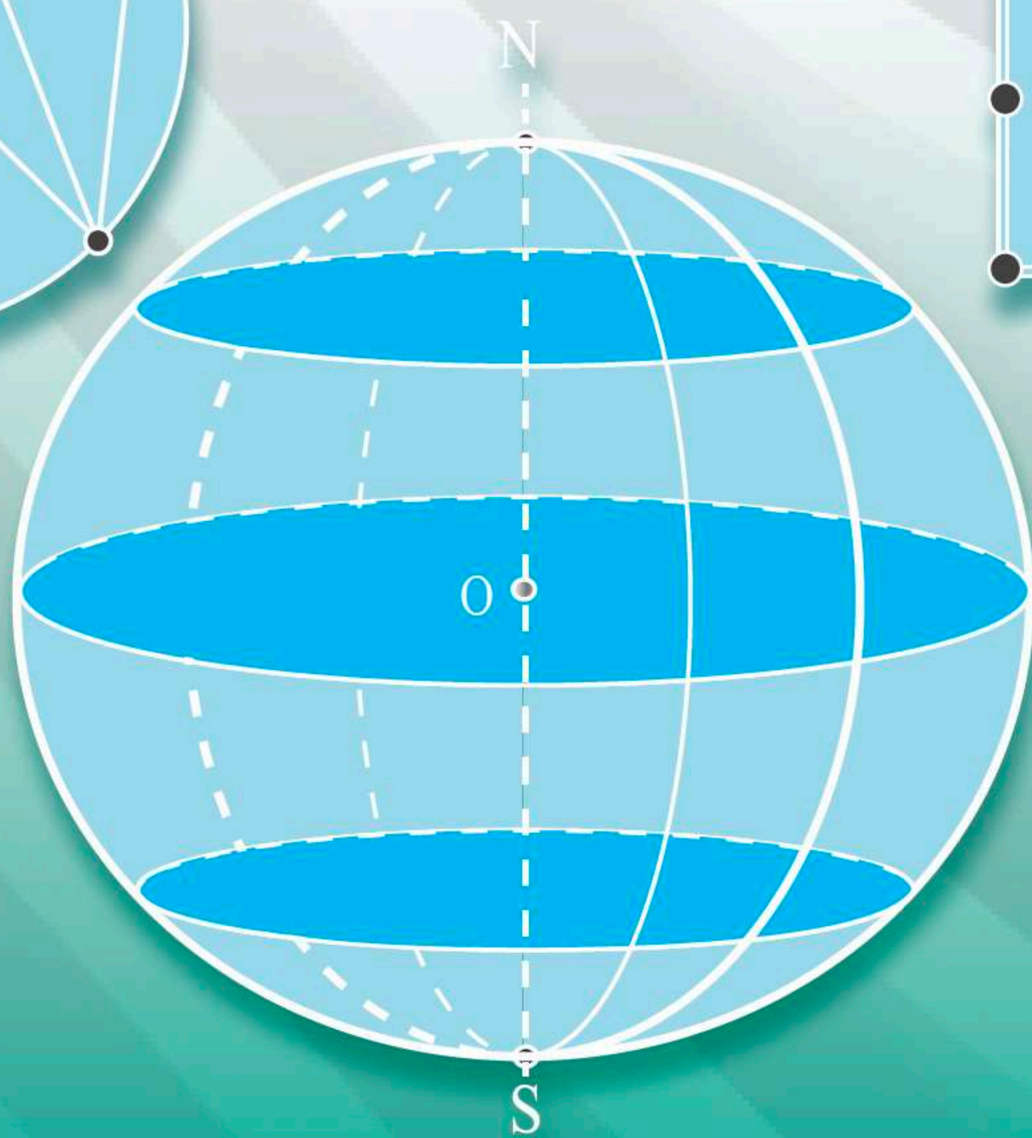
Basic Mathematics

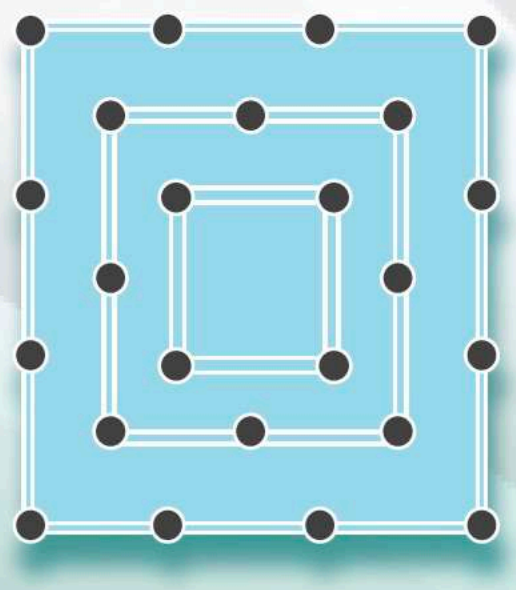
for Secondary Schools

Student's Book

Form Three







Tanzania Institute of Education

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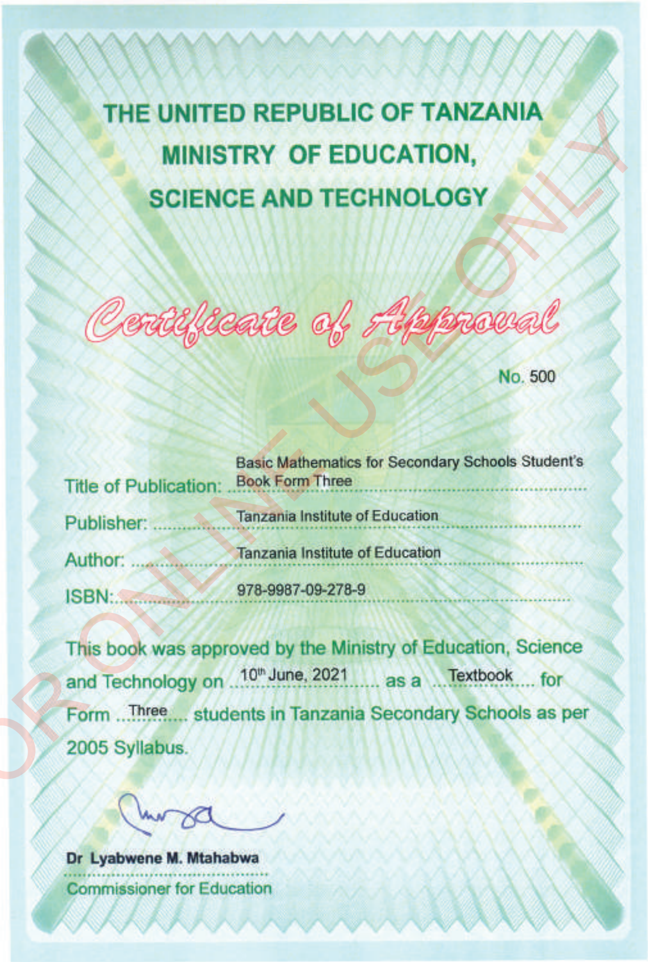
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Dr Lyabwene M. Mtahabwa
Commissioner for Education

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Tanzania Institute of Education
P. O. Box 35094
Dar es Salaam, Tanzania

Mobile numbers: +255 735 041 170
+255 735 041 168
Email: director. general@tie.go.tz
Website: www.tie.go.tz

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Writers: Dr Ratera S. Mayar (TIE), Mr Albert P. Kissima (TIE) & Department of Mathematics, University of Dar es Salaam (UDSM)

Editors: Prof. Ralph P. Massenge (MARUCO), Dr. Maombi J. Mkenyeleye (UDOM), Dr Michael H. Mkwizu (SUA), Dr Aziza J. Iddi (SUA), Dr Augustino I. Msigwa (UDSM), Dr Ali O. Ahmada (ARU), Dr Furaha M. Chuma (DUCE), & Dr Edward K. Ngailo (DUCE)

Designer: Mr Halifa Halifa
Illustrators: Mr Godlove S. Kyando & Mr Fikiri A. Msimbe
Coordinator: Mr Albert P. Kissima

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Dr Aneth A. Komba
Director General
Tanzania Institute of Education

Preface

This textbook, *Basic Mathematics for Secondary Schools* is written specifically for Form Three students in the United Republic of Tanzania. The book is prepared in accordance with the 2005 Basic Mathematics Syllabus for Ordinary Level Secondary Education Form I-IV, issued by the then, the Ministry of Education and Vocational Training (MoEVT).

The book consists of eight chapters, namely Relations, Functions, Statistics, Rates and variations, Sequences and series, Circles, Earth as a sphere, and Accounting. Each chapter comprises of activities, illustrations, and exercises. You are encouraged to do all activities and exercises together with any other assignment provided by your teacher. Doing so, will promote the development of the intended competencies.

Tanzania Institute of Education

Chapter One

Relations

Introduction

In daily life, people and things relate in some ways. For example, in families, there is a unique way on how family members relate. Children relate to their parents by birth. You may relate to a certain school by being a student in that school. In this chapter, you will learn about the relations between two sets. You will also learn how to draw graphs of relations, determine the domain and range of relations, find the inverse of relations, draw graphs of inverse relations, and identify domain and range of inverse relations. The competencies developed in this chapter will help you to analyse properties of things in real life situations such as relationships of family members, fares against travelling distances, changes of temperature with time, students and their performance grades, and many other applications.

Meaning of a relation

A relation is a set of ordered pairs. It associates an element of one set with one or more elements of another set. Usually, a relation is denoted by the letter R. In set notation, the relation R between two sets can be written as

$R = \{(a, b) : a$ is an element of the first set and b is an element of the second set$\}$.

The above relation R is read as R is a set of ordered pair (a, b) such that a is an element of the first set and b is an element of the second set.

Furthermore, if a is an element from set A and b is an element from set B, then the relation R is mathematically written as $R = \{(a,b) : a \in A, b \in B\}$.

The notation, $a \in A, b \in B$ means that a is a member of set A and b is a member of set B. Also, we use $a \mapsto p$ to mean 'a is mapped onto p'. For example, when we write $x \mapsto 3x$, we mean 'x is mapped onto 3 times x'.

Suppose that A and B are two non–empty sets. A relation between A and B consists of connections linking the elements of set A with the elements of set B following a certain rule. The sets could consist of people, things or numbers. For example, consider a family tree in Figure 1.1 which shows three sisters and their children.

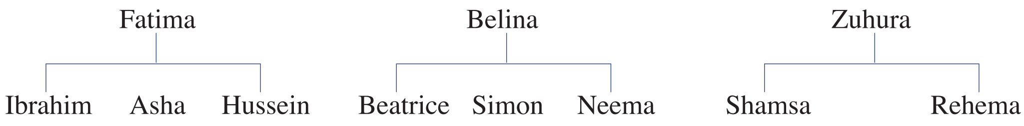


Figure 1.1: *Relationship between sisters and their children*

The relation 'is a nephew of ' can be illustrated as follows. Let A be a set of children and B be a set of sisters. Place set A on the left hand side and set B on the right hand side as shown in Figure 1.2. Arrows from a set of children connect to a set of their aunts. For example, Ibrahim is a nephew of Belina, there is an arrow drawn from Ibrahim to Belina. This type of representation is called an arrow diagram, a mapping diagram or a pictorial diagram.

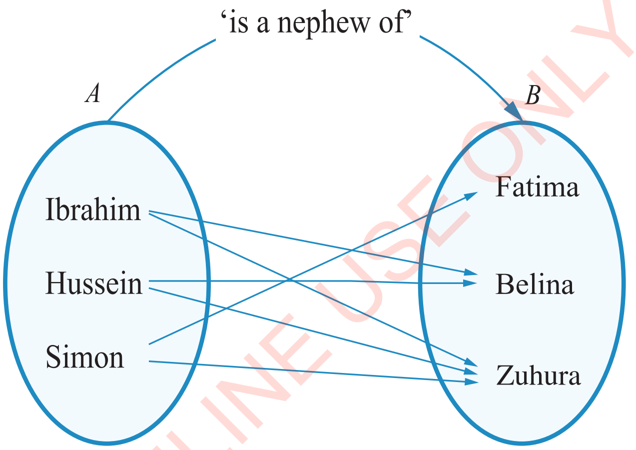


Figure 1.2: *Pictorial representation of a relation 'is a nephew of'*

Sets A and B in Figure 1.2 have no elements in common. However, sets can have elements in common. The sets could even be the same. Figure 1.3 represents the relation 'is a brother of '. In this relation, sets A and B consist of six children who are brothers and sisters of each other born to two sisters as described in Figure 1.1.

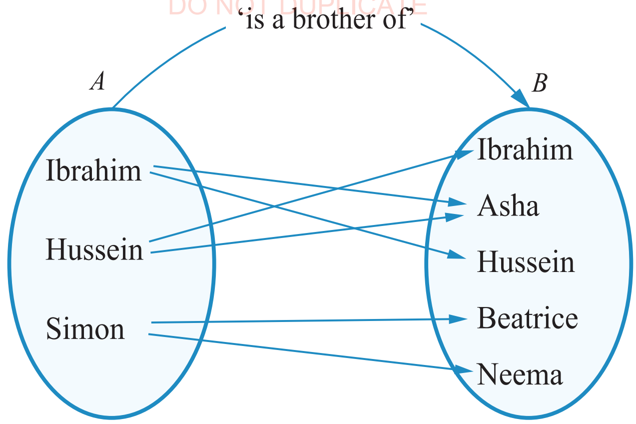


Figure 1.3: *Pictorial representation of the relation 'is a brother of'*

Relations can also exist between sets of numbers. Figure 1.4 represents the relation 'is a factor of '.

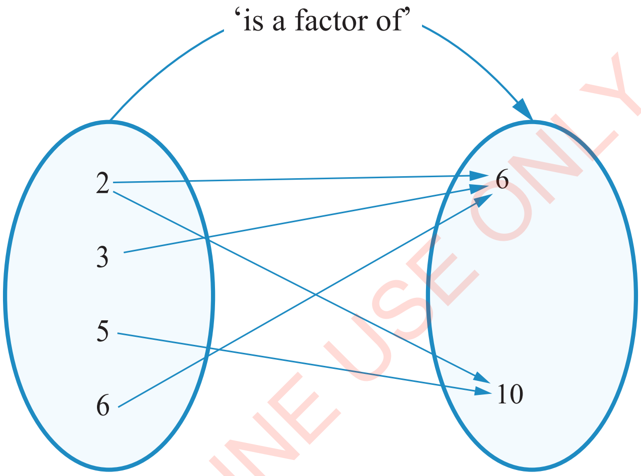


Figure 1.4: *Pictorial representation of the relation 'is a factor of'*

Types of relations

There are four types of relations, namely; one-to-one, many-to-one, one-to-many, and many-to many-relations.

One-to-one relations

A one-to-one relation is a relation in which exactly one element of the first set is mapped onto exactly one element of the second set. For example, the relation $R = \{(1, 2), (2, 4), (3, 6), (4, 8), (5, 10)\}$ is a one-to-one relation. The pictorial diagram of the relation R is shown in Figure 1.5.

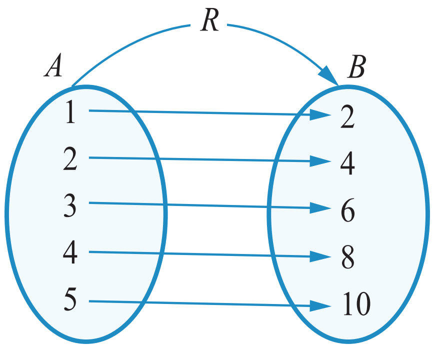


Figure 1.5: *Pictorial diagram of a one-to-one relation*

Many-to-one relations

A many-to-one relation is the relation in which at least two elements from the first set are mapped onto exactly one element of the second set. For example, a relation 'a child of' defined as R = {(Haruna, Khasim), (Khalid, Khasim), (Maimuna, Khasim), (Anes, Joseph), (Juliet, Joseph)} is a many-to-one relation because Haruna, Khalid, and Maimuna are children of Khasim, while Anes and Juliet are children of Joseph. The respective mapping diagram of a relation R, 'is a child of' is shown in Figure 1.6.

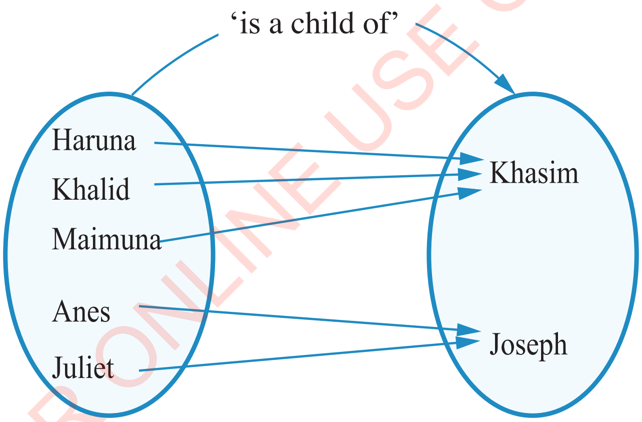


Figure 1.6: *Pictorial diagram of a many-to-one relation*

One-to-many relations

A one-to-many relation is the relation in which at least one element of the first set is mapped onto more than one element of the second set. For example, the relation $R = \{(0,0), (0,-3), (1,-1), (1,1), (4,2), (4,-2), (9,-3), (9,3)\}$ is a one-to-many relation. The pictorial representation of the relation R is shown in Figure 1.7.

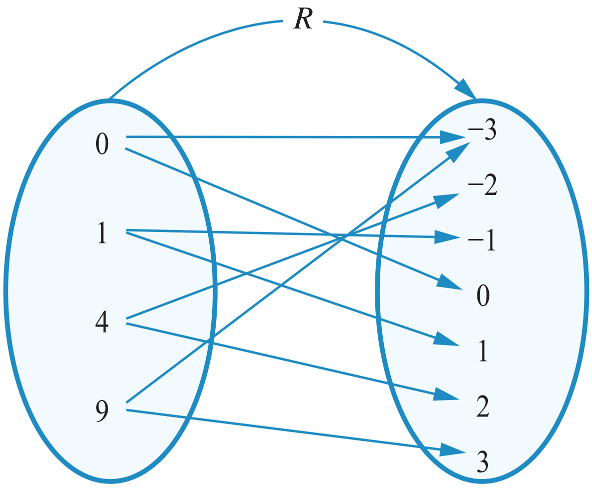


Figure 1.7: *Pictorial diagram of a one-to-many relation*

Many-to-many relations

A many-to-many relation is the relation in which several elements of the first set are mapped onto more than one element of the second set. An example of a many-to-many relation is;

$R = \{(-2, -1), (-2, 0), (-1, -1), (-1, 0), (-1, 2), (0, 1), (1, 0), (1, 2), (2, 1), (2, 2)\}$.

A pictorial representation of the relation R is shown in Figure 1.8.

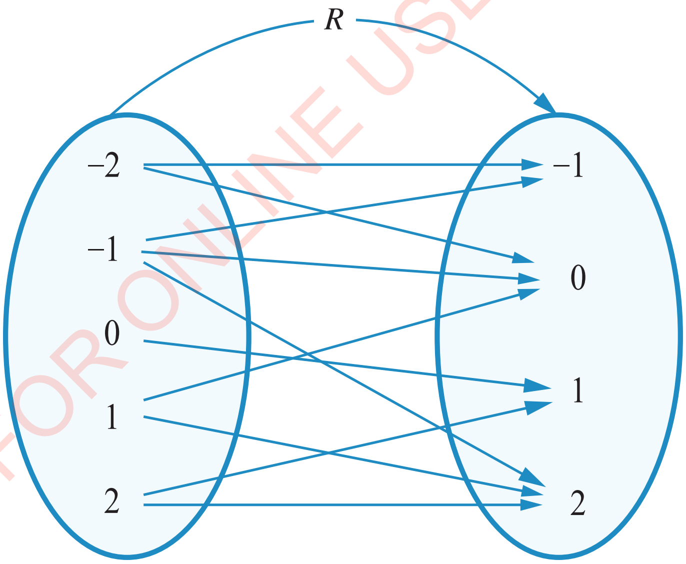


Figure 1.8: *Pictorial diagram of a many-to-many relation*

Activity 1.1: Forming relations among family members

In a group, perform the following tasks:

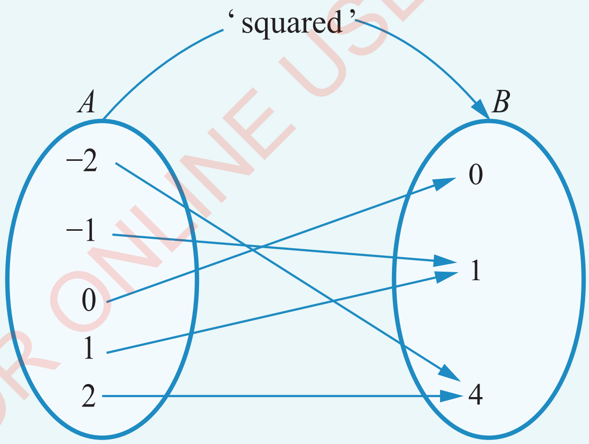
1. Reflect on different ways on how you relate with your families such as parents, children, uncles, aunts, brothers and sisters.
2. Demonstrate that relation in front of the class. Use long sticks or rolled papers as a means of connecting your corresponding family members. Other students should observe and create pictorial diagrams of the relations.
3. After demonstrations, draw a pictorial diagram of the relation in task 2 on a large manila paper.
4. Share your findings through presentation for further inputs.

Example 1.1

Let $A = \{-2, -1, 0, 1, 2\}$ and $B = \{0, 1, 2, 3, 4\}$. Draw an arrow diagram to illustrate the relation which connects each element of set A to its square in set B.

Solution

Put set A on the left hand side and set B on the right hand side. The square of -2 is 4, the square of -1 is 1 and so on. This relation is mathematically represented as $R=\{(a,b): b = a^2\}$. The pictorial diagram is shown as follows.



Example 1.2

Write down the relation in Example 1.1 in set notation. List the elements of the relation R.

Solution

The relation is that elements of set B are the square of the elements of set A.

$R = \{(a, b): a \in A, b \in B, \text{ where } b = a^2\}$.

The elements of relation R consist of the following pairs:

$R = \{(-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4)\}$.

Example 1.3

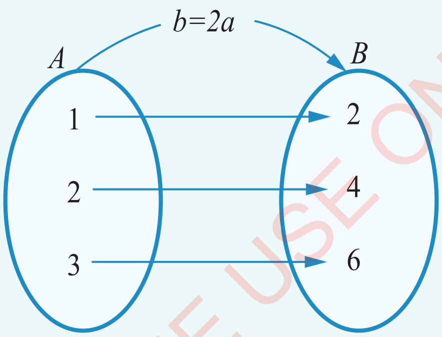
Let $R = \{(a,b): b = 2a, \text{ where } a \in A \text{ and } b \in B\}$. If $A = \{1,2,3\}$, draw a pictorial representation to illustrate the relation R.

Solution

From the relation, $b = 2a$.

If $a = 1,\ b = 2,\ a = 2,\ b = 4,\ a = 3,\ b = 6$,

then, a pictorial representation of the relation R is shown in the following figure:



Exercise 1.1

1. By using Figure 1.1, use pictorial diagrams to illustrate the following relations: (a) 'is a daughter of' (b) 'is a niece of' (c) 'is a mother of'
2. Let C = {mass, length, time} and D = {centimetres, seconds, hours, kilograms, tonnes}. Draw an arrow diagram to illustrate the relation 'can be measured in'.
3. Let $A = \{25°, 132°, 90°, 48°, 308°\}$ and B = {acute angle, right angle, obtuse angle, reflex angle}. Draw a pictorial diagram to illustrate the relation between sets A and B.

4. Let $E = \{1, 2, 3, 4\}$ and $F = \{2, 3, 4, 5\}$. Draw a pictorial arrow diagram to illustrate the relation 'is less than'.
5. Let $G = \{a, e, i, o, u\}$ and $H = \{$Ghana, Gambia, Zambia, Uganda$\}$. Draw an arrow diagram to show an inclusion of vowel in the name of the country.
6. Let $J = \{$pig, cow, horse, lion, tiger$\}$ and $K = \{2, 3, 4, 5\}$. Draw a pictorial diagram to connect the name of each animal with the number of letters in its name.
7. Let $M = \{$Dodoma, Lusaka, Accra, Paris, Freetown$\}$ and $N = \{$France, Zambia, Tanzania, Ghana, Sierra Leone$\}$. Draw an arrow diagram to connect each capital city with its country name.
8. A group of people contains the following members: Pandu Mkoku, Rafael Martin, Aloyce Wambura, and Juma Mkoku. Let P be a set of children, and Q a set of fathers. Draw a pictorial diagram to show the connection between set P and set Q.
9. Let $S = \{$Dar es Salaam, Lindi, Ruvuma, Arusha, Mbeya, Tanga, Tabora$\}$. Draw a pictorial diagram to show the relation 'is north of' between members of set S.
10. Using set S in question 9, draw an arrow diagram to show the relation 'is east of' between members of set S.
11. Let $T = \{$Tanzania, China, Burundi, Nigeria$\}$. Draw a pictorial diagram between members of set T to show the relation 'has a larger population than'.
12. Let $U = \{2, 3, 5\}$ and $V = \{6, 7, 10, 60\}$. Draw a pictorial diagram between set U and set V to illustrate the relation 'is a factor of'.
13. Let $W = \{9, 10, 14, 12\}$ and $X = \{2, 5, 7, 6\}$. Draw a pictorial diagram between set W and set X to illustrate the relation 'is a multiple of'.
14. Let $Y = \{1, 4, -4, 9, 10\}$ and $Z = \{1, -2, 2, 3, -4\}$. Draw a pictorial diagram between set Y and set Z to illustrate the relation 'is a square of'.

Domain and range of a relation

Suppose that a relation between two sets A and B is $R: A \rightarrow B$. It is read as, 'relation R maps set A onto set B'. The domain of this relation is the set of all members of set A which are related to a member of set B, that is, all members of set A which have corresponding members in set B forms the domain of a relation R.

On the other hand, the range of a relation is the set of all elements of set B which are related to at least one element of set A, that is, all members of set B which have corresponding members in set A form the range of a given relation. Range can also be described as the set of values which correspond to a set of values of domain of the relation. In Figure 1.9, the domain is $A = \{2, 3, 5\}$ and the range is $B = \{a, c, e, f, g\}$.

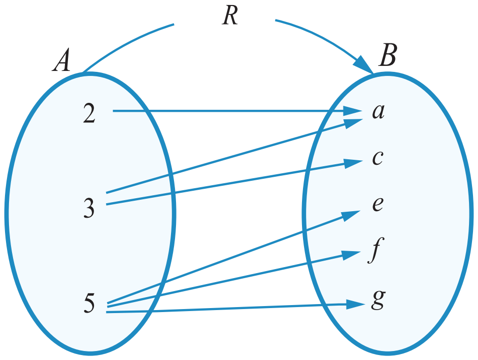


Figure 1.9: *Illustration of domain and range*

Example 1.4

Use the values in Table 1.1 to represent the relation R that exists between x and y in terms of the ordered pairs and write its domain and range.

Table 1.1 : Table of values of the relation R

x	-3	0.5	1	2	5	6
y	-6	1	2	4	10	12

Solution

From Table 1.1, a relation of ordered pairs is

$R = \{(-3, -6), (0.5, 1), (1, 2), (2, 4), (5, 10), (6, 12)\}$.

The domain of R is a set of x values, that is $x = \{-3, 0.5, 1, 2, 5, 6\}$ and the range of R is the set of y values, that is $y = \{-6, 1, 2, 4, 10, 12\}$.

Example 1.5

Given the following ordered pairs:

$(1, 2), (2,1), (-3, 4), (-3, -5), (2, 2), (-8, 0), (-8, -3)$;

(a) Which ordered pairs belong to the relation $R = \{(x,y) : y > x\}$?

(b) State the domain and range of R.

Solution

(a) Choose all ordered pairs (x,y) in the given set such that $y > x$.
The ordered pairs which satisfy this condition are;
$(1, 2), (-3, 4), (-8, 0), (-8, -3)$.

(b) Domain of $R = \{1, -3, -8\}$ and range of $R = \{2, 4, 0, -3\}$.

Example 1.6

Let $R = \{(4, 16), (4, 20), (5, 20), (8, 16), (9, 18)\}$.

(a) Write a relationship between the set of ordered pairs of relation R.

(b) Express R in tabular form.

(c) Find the domain and range of R.

Solution

(a) It can be observed that $R = \{(x,y) : y \text{ is divisible by } x\}$. Thus, the relationship between the ordered pairs of R is that, the second entry is divisible by the first entry without a remainder for all elements of R or the second entry is the multiple of the first entry.

(b) The domain of R is $\{4, 5, 8, 9\}$ and the range of R is $\{16, 18, 20\}$.

(c) The relation R can be expressed in tabular form as shown in Table 1.2.

Table 1.2: Table of values of the relation R

x	4	4	5	8	9
y	16	20	20	16	18

Graphs of relations

In the previous section, you learnt how to represent a relation as a set using pictorial diagrams. In this section, you will learn how to represent relations graphically. Also, you will learn how to draw linear equality, inequality, quadratic, and cubic relations.

Linear equality relations

A linear equality relation R can be represented graphically by taking each pair (x, y) as a point in the xy–plane. If the relation R has an infinite number of elements (pairs), tabulate a few values of x with their corresponding values of y and use them to draw the graph of the relation. In this case, the domain and range of the relation can be obtained from the graph.

Example 1.7

Draw the graph of the relation $R = \{(x,y):y = 2x\}$.

Solution

Tabulate a few values of the relation $R = \{(x,y):y = 2x\}$ as shown in Table 1.3.

Table 1.3: Table of values of the relation $R = \{(x,y):y = 2x\}$

x	–3	–2	–1	0	1	2	3
$y = 2x$	–6	–4	–2	0	2	4	6

The graph of the relation $R = \{(x,y):y = 2x\}$ is obtained by plotting the ordered pairs on the xy–plane as shown in Figure 1.10.

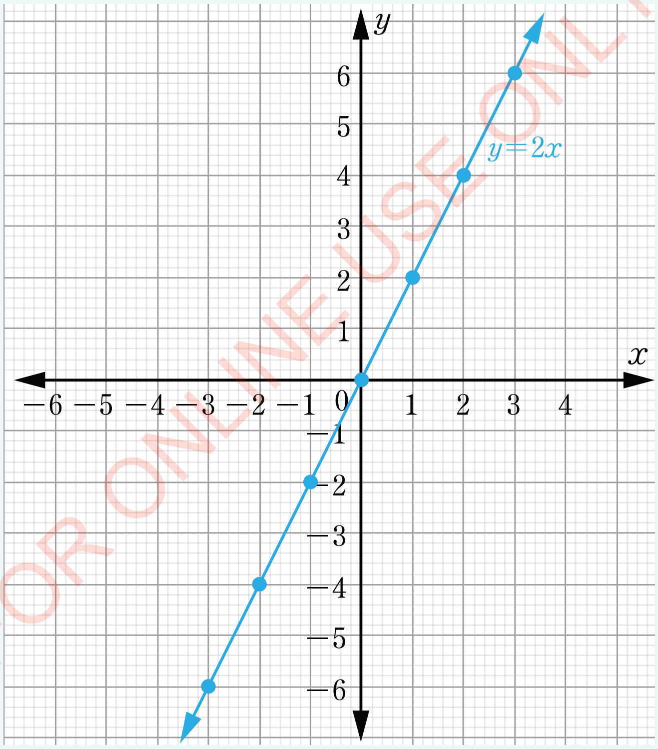


Figure 1.10: *Graph of the relation* $R = \{(x,y):y = 2x\}$

Example 1.8

Draw the graph of the relation $R = \{(x,y): y = 2x + 3\}$.

Solution

Tabulate a few values of a relation $R = \{(x,y): y = 2x + 3\}$ as shown in Table 1.4.

Table 1.4: Table of values of the relation $R = \{(x,y): y = 2x + 3\}$

x	-4	-3	-2	-1	0	1	2	3
y	-5	-3	-1	1	3	5	7	9

The graph of the relation $R = \{(x,y): y = 2x + 3\}$ is given in Figure 1.11.

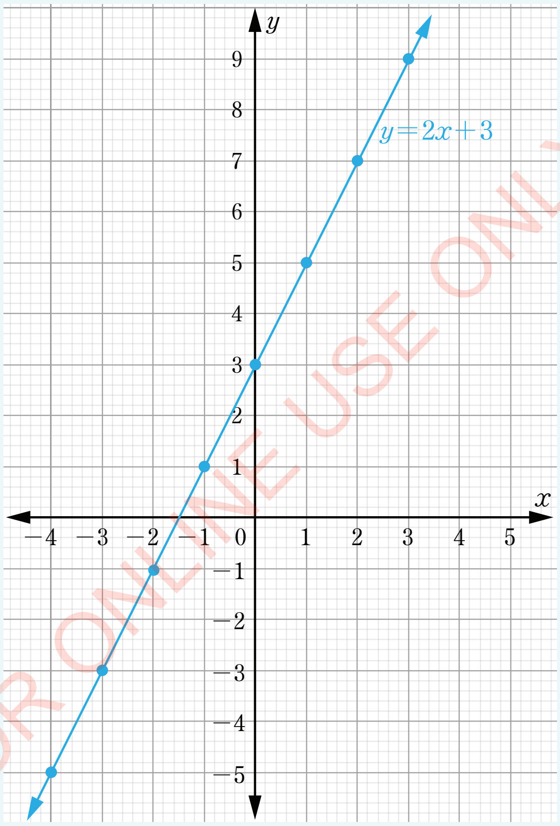


Figure 1.11: *Graph of the relation* $R = \{(x,y): y = 2x + 3\}$

Linear inequality relations

Some linear relations can be represented as inequalities. The graph of an inequality relation is drawn the same way as that of a normal linear relation with some considerations of the inequality signs. This is done by shading the regions containing points that satisfy the given relation. Dotted lines are used to represent inequality relations with less than (<) or greater than (>) signs while solid lines are used for relations which involve inequalities with less than or equal (≤) and greater than or equal (≥) signs.

Example 1.9

Draw a graph of each of the following relations and state its domain and range:
(a) $R = \{(x,y): y \leq x\}$. (b) $R = \{(x,y): y < x\}$. (c) $R = \{(x,y): y >-2x+1\}$.

Solution

(a) The relation $R = \{(x,y): y \leq x\}$ has an inequality sign '≤' which means that all the points on the line $y = x$ are included in the required region. In this case, a solid line is used. Since the line divides the xy–plane into two regions, test few points from each region to determine the region which satisfies the relation. For instance, from the first region, choose (– 1, –4) and (–3, –2) and from the second region choose (2, 1) and (4, 2). Substituting the values of x and y from these ordered pairs gives 1<2 and 2<4, 2<–3 and 4<–1. Hence (2, –1) and (4, 2) satisfy the inequality while (–3, 2) and (–1, 4) do not satisfy the inequality. Shade the region which contain points which satisfy the relation $R = \{(x,y): y \leq x\}$ as shown in Figure 1.12.

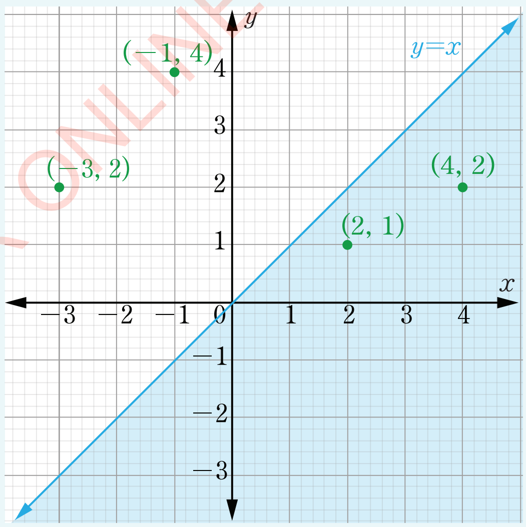


Figure 1.12: *Graph of the relation* $R = \{(x,y): y \leq x\}$

From the graph, the domain of the relation R is $\{x{:}x$ is all real numbers$\}$ and the range of the relation R is $\{y{:}y$ is all real numbers$\}$.

(b) The relation $R = \{(x{:}y){:}y < x\}$ has an inequality sign $<$ which means that all the points on the line $y = x$ are not included in the required region. Therefore, in this case, a dotted line is used. Since the dotted line separates the xy – plane into two regions, test few points from each region. For example, choose (– 4, 4), and (– 2, 0) from the first region and (4, – 4) and (1, – 1) from the second region. Observe that (1, –1) and (4, – 4) satisfy the inequality and (– 4, 4) and (–2, 0) do not satisfy. Shade the region which contain points which satisfy relation R as shown in Figure 1.13.

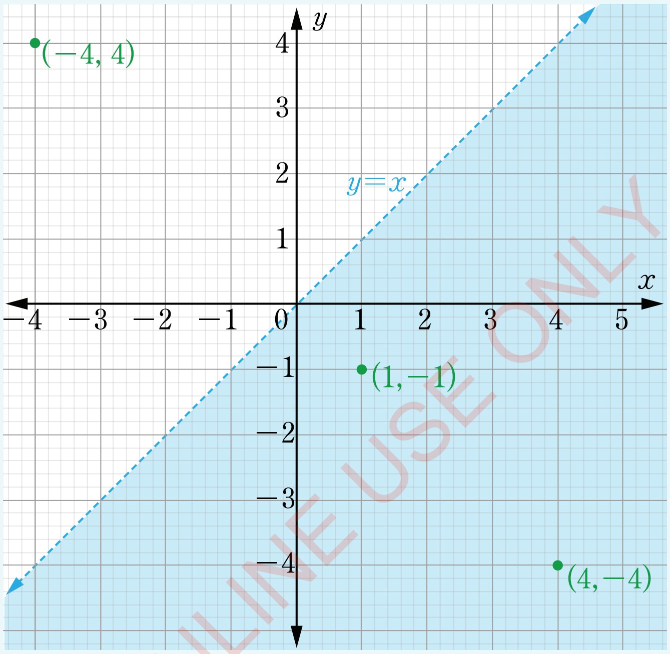


Figure 1.13: *Graph of the relation* $R = \{(x,y){:}y < x\}$

From the graph, the domain and range of the relation R is a set of all real numbers as all values of x and y are involved because the line $y = x$ extends infinitely in both directions.

(c) The relation R $= \{(x, y){:}y > -2x + 1\}$. Test few points which do not lie on the dotted line, for example (–2, 2), (2, –2), (2, –5) and (–2, 6). In this case, the points (–2, 6) and (2, –2) satisfy the relation R, and hence they are in the shaded region, while the points (–2, 2) and (2, –5) do not satisfy the relation R.

Shade the region which contain points which satisfy relation R as shown in Figure 1.14.

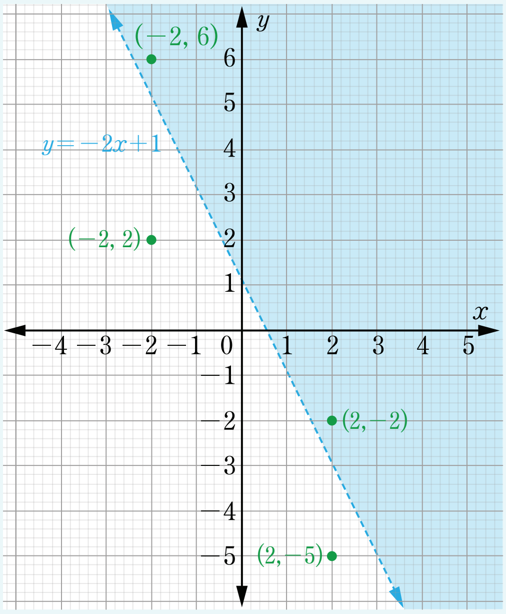


Figure 1.14: *The graph of the relation* $R = \{(x, y): y > -2x + 1\}$

From the graph, as the line extends infinitely in both directions, all values of x and y are involved. Therefore,

Domain $= \{x: x \in \mathbb{R}\}$,

Range $= \{y: y \in \mathbb{R}\}$.

Some linear relations involve more than one inequality. In this case, the solution is the set of all ordered pairs which satisfy all the inequalities in the given relation. This is illustrated in Example 1.10 and Example 1.11.

Example 1.10

Let $R = \{(x, y): x + y \leq 2 \text{ and } y \leq 2\}$ be a relation of real numbers. Draw a graph of R and state its domain and range.

Solution

The relation R has two inequalities. Draw the two lines and then test few points that satisfy both inequalities. Start with one inequality, say $x + y \leq 2$, and choose some points such as $(4, 1)$, $(2, -4)$ and $(-2, -2)$. Test these points by substituting them into the relation and shade the wanted region.

Again, test the second inequality and shade the wanted region. The region where the shading intersect is the wanted region as shown in Figure 1.15.

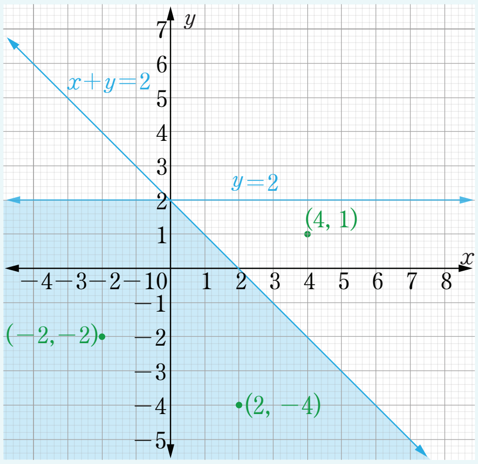
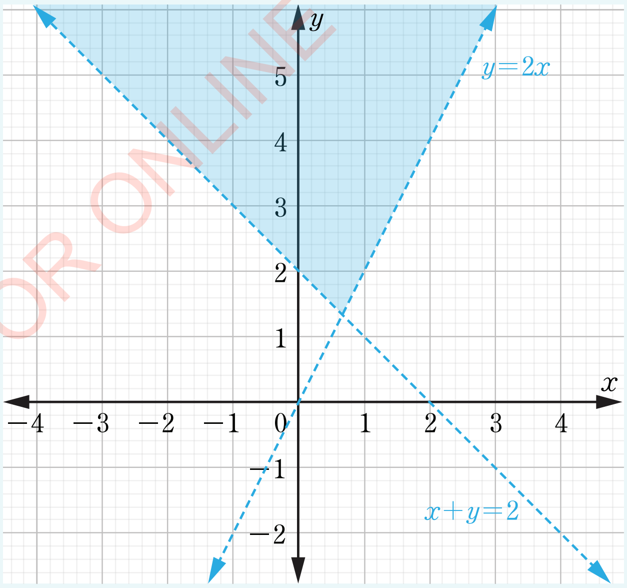


Figure 1.15: *Graph of the relation* $R = \{(x,y): x+y \leq 2 \text{ and } y \leq 2\}$

Therefore, from the graph, the domain of R is $\{x: x \in \mathbb{R}\}$ and the range is $\{y: y \leq 2\}$.

Example 1.11

Find the expression of the relation given by the following graph. State its domain and range.



Solution

Choose any point in the shaded region, say (1, 4). By substituting $x = 1$ and $y = 4$ in the equations $y = 2x$ and $x + y = 2$, we get $5 > 2$ (true) and $4 > 2$ (true). Thus, the shaded region represents the set of all ordered pairs (x,y) of real numbers such that $y > 2x$ and $x + y > 2$. Therefore, the relation R is given by $R = \{(x,y):y > 2x \text{ and } x + y > 2\}$. The domain of the relation $R = \{(x,y):y > 2x \text{ and } x + y > 2\}$ is the set of all real numbers, because as the lines extend to infinity, all values of x are involved. To find the range of R, take all values of y which are in the shaded region. From the graph, the values of y start at a point of intersection $\left(\frac{2}{3}, \frac{4}{3}\right)$ of the two inequalities. Since the lines are dotted, the values at the point of intersection are not included. Therefore, the greater than sign should be used. Hence, the range of R is $\left\{y:y > \frac{4}{3}, \text{ where } y \text{ is a real number}\right\}$.

Quadratic relations

Sometimes relations are represented by quadratic equations or quadratic inequalities. For example, $R = \{(x,y):y = x^2 + 1\}$ and $R = \{(x,y):y \geq x^2 + 1 \text{ and } y < 5\}$ are quadratic relations. Such relations can also be represented graphically.

Example 1.12

Draw the relation $R = \{(x,y):y \geq x^2 + 1 \text{ and } y < 5\}$ and determine its domain and range.

Solution

To sketch the graph of R, first, sketch the quadratic equation $y = x^2 + 1$ using a solid line and a dotted line for $y = 5$. Next, identify the points that satisfy both inequalities. The graph of $y = x^2 + 1$ can easily be drawn by using table of values as shown in Table 1.5.

Table 1.5: Table of values for the relation $y = x^2 + 1$

x	−3	−2	−1	0	1	2	3
$y = x^2 + 1$	10	5	2	1	2	5	10

The graph of R is as shown in Figure 1.16.

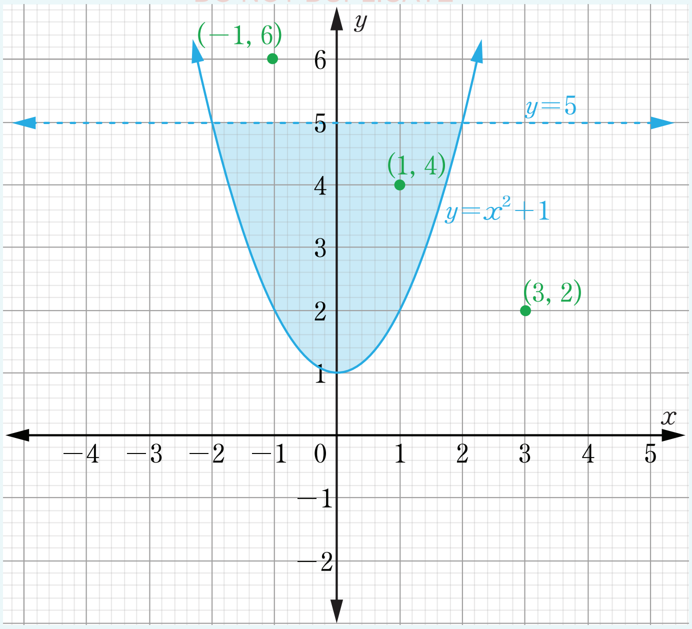


Figure 1.16: *The graph of the relation* $R = \{(x,y):y \geq x^2+1 \text{ and } y < 5\}$

The graph shows that, the domain of R is $\{x: -2 \leq x \leq 2\}$ while its range is $\{y:1 \leq y < 5\}$.

Example 1.13

Let $R = \{(x,y):y \leq -(x-1)^2+1 \text{ and } y \geq x-6\}$ be a relation. Draw the graph of R and determine its domain and range.

Solution

To draw the graph of R, first, sketch the equations $y = -(x-1)^2+1$ and $y = x-6$. Next, identify the points that satisfy both inequalities. A few values for $y = -(x-1)^2+1$ are tabulated in Table 1.6.

Table 1.6: Table of values of the equation $y = -(x-1)^2+1$

x	−2	−1	0	1	2	3	4
$y = -(x-1)^2+1$	−8	−3	0	1	0	−3	−8

The graph of R is as shown in Figure 1.17.

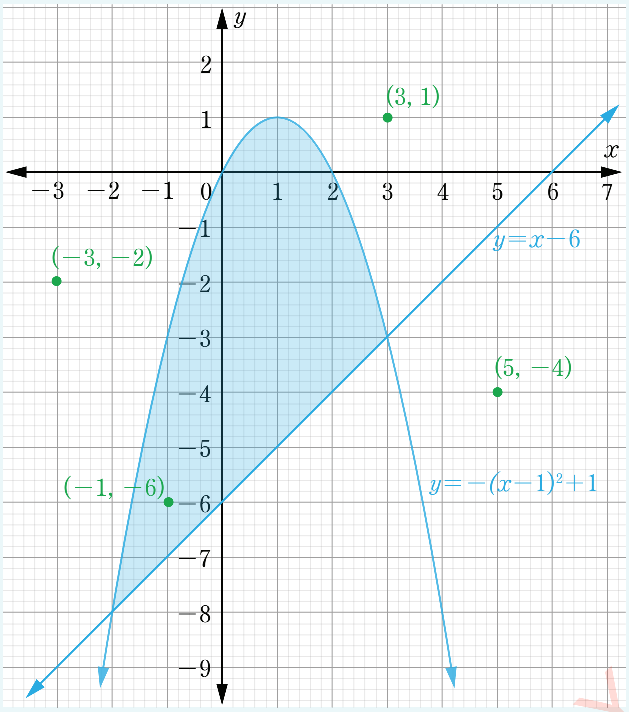


Figure 1.17: *Graph of the relation* $R = \{(x,y): y \leq -(x-1)^2 + 1 \text{ and } y \geq x - 6\}$

The domain of R is $\{x: -2 \leq x \leq 3\}$ and its range is $\{y: -8 \leq y \leq 1\}$.

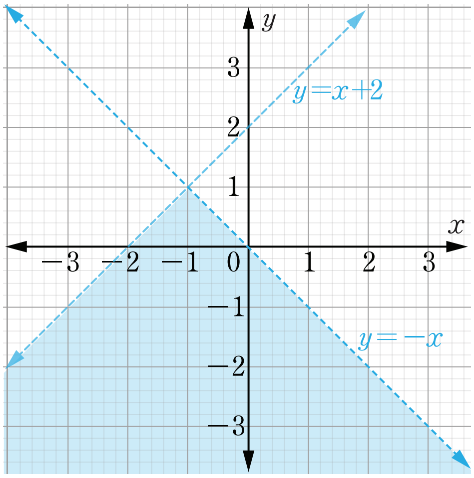
Exercise 1.2

Draw the graphs of the relations in questions 1 to 7 and use the graphs to find the domain and range of each relation.

1. $R = \{(x,y): 2x - y = 10\}$.
2. $R = \{(x,y): x - y < 1 \text{ and } y < 2\}$.
3. $R = \{(x,y): y = x^2\}$.
4. $R = \{(x,y): x + y \geq 1 \text{ and } x \leq y + 2\}$.
5. $R = \{(x,y): y - 2x \geq 0 \text{ and } x \leq -1\}$.
6. $R = \{(x,y): y \geq x - 1\}$.
7. $R = \{(x,y): y \geq x - 1 \text{ and } y < 3\}$.

In questions 8 to 13, find the relations represented by the graphs. Use the graphs to determine the domain and range of each relation.

8.



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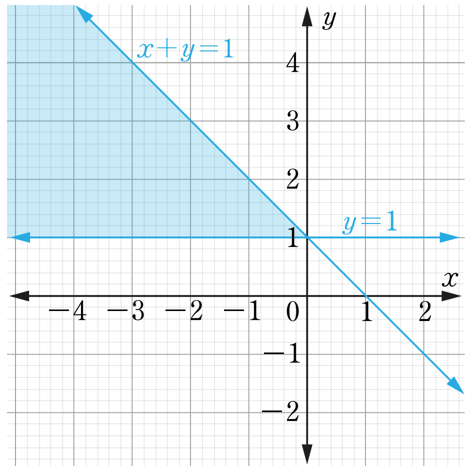

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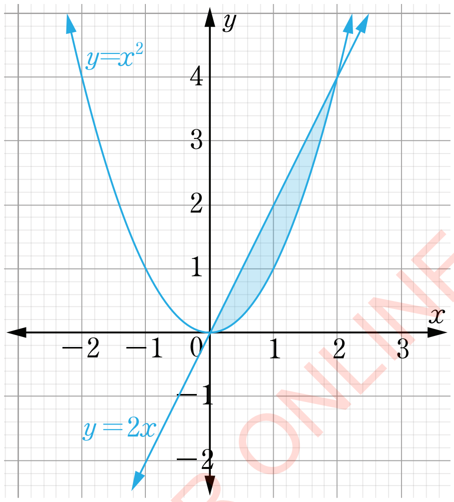

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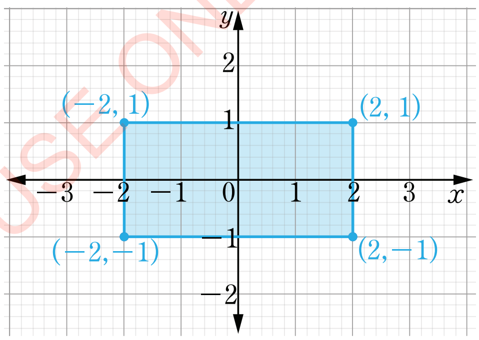

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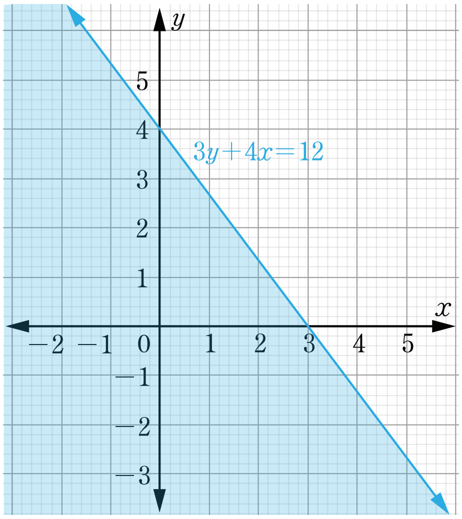

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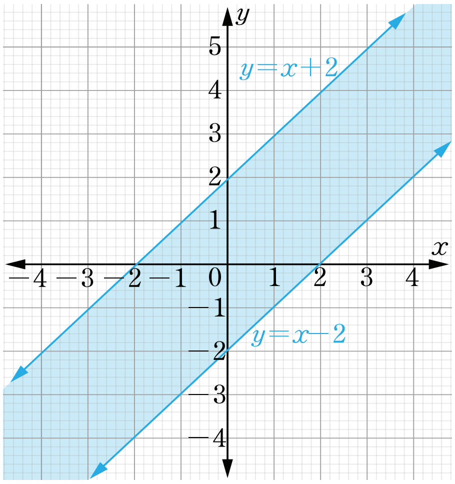


Domain and range of a relation expressed in equation form

When a relation is written in the form of an equation, say $y = 2x + 1$, the domain of the relation is the set of all values of the independent variable x. When these values are substituted into the relation they give values of the dependent variable y. Therefore, the domain of R is $\{x:(x,y)$ belongs to R for some values of $x\}$ and range of R is $\{y:(x,y)$ belongs to R for some values of $y\}$.

Example 1.14

Find the domain and range of the relation $y = 2x + 1$.

Solution

Given the relation $y = 2x + 1$, any real value of x gives a definite real value of y. Therefore, the domain of this relation is the set of all real numbers. The range is obtained by writing x in terms of y and determine the possible real values of y for which x is defined, that is, $y = 2x + 1$, which implies that

$$x = \frac{y-1}{2}.$$

From $x = \frac{y-1}{2}$, it is observed that any real value of y gives a definite real value of x. Hence, range of $y = 2x + 1$ is also a set of all real numbers.

Example 1.15

Find the domain and range of the relation $y = 3x^2 + 2$.

Solution

All values of x give definite values of y. Thus, the domain of $y = 3x^2 + 2$ is the set of all real numbers, that is, domain $= \{x : x \in \mathbb{R}\}$.

For the range, first write x in terms of y, that is, from

$$y = 3x^2 + 2,$$

Making x the subject gives

$$x = \pm\sqrt{\frac{y-2}{3}}.$$

Note that, the values in the square root are defined only for the non-negative real numbers. So, we have

$$\frac{y-2}{3} \geq 0,$$
$$y \geq 2.$$

Thus, range $= \{y : y \geq 2\}$.

Therefore, domain $= \{x : x \in \mathbb{R}\}$ and range $= \{y : y \geq 2\}$.

Example 1.16

Find the domain and range of the relation $R = \{(x, y) : y$ is divisible by x, where x and y are integers$\}$.

Solution

If x and y are integers and y is divisible by x, then $\frac{y}{x} = k$, where k is an integer. Since the fraction is not defined when the denominator is zero, the domain of the relation R is a set of all integers except zero, that is,

$$\text{domain} = \{x : x \in \mathbb{Z}, x \neq 0\}.$$

To find the range of the relation R, note that y as a numerator takes all integer values. Thus, the range of R is the set of all integers, that is,

$$\text{range} = \{y : y \in \mathbb{Z}\}.$$

Therefore, domain $= \{x : x \in \mathbb{Z}, x \neq 0\}$ and range $= \{y : y \in \mathbb{Z}\}$.

Example 1.17

Let R be a relation on real numbers given by $y = \frac{1}{x+3}$. Find the domain and range of R.

Solution

Given the relation $y = \frac{1}{x+3}$, the denominator needs to be a non-zero number. So, $x \neq -3$. All other values of x satisfy the equation. Therefore,

$$\text{Domain} = \{x : x \in \mathbb{R}\ x \neq -3\}.$$

To find the range of the relation R, we write x in terms of y for the given equation.

Given $y = \frac{1}{x+3}$, which implies that $x = \frac{1}{y} - 3$.

Note that for the equation $x = \frac{1}{y} - 3$ to be defined, y must not be equal to zero. All other values of y satisfy the equation $x = \frac{1}{y} - 3$. Thus,

$$\text{range} = \{y : y \in \mathbb{R},\ y \neq 0\}.$$

Therefore, domain $= \{x : x \in \mathbb{R} \text{ except } -3\}$ and range $= \{y : y \in \mathbb{R},\ y \neq 0\}$.

Exercise 1.3

1. Find the domain and range of each of the following relations:
   (a) $R = \{(x,y) : x$ and y are real numbers and $y = \sqrt{x}\}$.
   (b) $R = \{(x,y) : x$ and y are natural numbers and y is a multiple of $x\}$.
2. Let $R = \{(x,y) : y = 3x + 1\}$ be a relation on real numbers. Find:
   (a) The set of ordered pairs which belong to R from the set
   $$\left\{(1,\ 2),\ (0,\ -1),\ \left(\tfrac{1}{2},\ \tfrac{5}{2}\right),\ (-2,\ -5),\ (3,\ 10)\right\}.$$
   (b) The domain of the set obtained in (a).
   (c) The range of the set obtained in (a).
3. Let $R = \{(x,y) : x^2 + y^2 = 1\}$ be a relation on the set of real numbers.
   (a) From the following set of pairs of numbers, write a subset whose members belong to R.

$$\left\{(0,\ -1),\ (1,\ 1),\ (2,\ -\sqrt{3}),\ (0,\ 1),\ \left(\frac{1}{2},\ \frac{\sqrt{3}}{2}\right),\ \left(\sqrt{2},\ \frac{\sqrt{2}}{2}\right),\ \left(\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)\right\}.$$

(b) Give the domain of the set obtained in (a).
(c) Give the range of the set obtained in (a).

4. What is the range of the following mappings if the domain of each relation is the set of integers?
   (a) $R{:}x \mapsto 2x - 1$
   (b) $R{:}x \mapsto x^2$
   (c) $R{:}x \mapsto x^3$
5. Find the domain and range of the following relations:
   (a) $y = x^2 + 3$.
   (b) $y^2 = 4x$.

Inverse of a relation

Activity 1.2: Converting currencies

Currencies of different countries relate linearly depending on a given exchange rate between the currencies. Perform the following tasks individually or in a group:

1. Choose a currency of any country and find its current exchange rate against the Tanzanian currency.
2. Express mathematically the relationships between quantities in task 1 by using an equation. Write the relation in such a way that Tanzanian currency forms the domain and foreign curreny forms the range.
3. Prepare a table of values with 10 values of the domain and range depending on your choice of foreign currency.
4. Prepare a new table and exchange the values of the columns of the table prepared in task 3.
5. Use the values obtained in task 4 to form a new equation.
6. Compare the new equation with the one formed in in task 2 and note down any similarity and differences.
7. Choose some values of the range from the table in task 2 and substitute into the new equation. What values do you get and how are they related to those in the table prepared in task 2? Comment on the observations.
8. Without using a table of values, use the equation formed in task 2 to obtain the equation formed in task 5.
9. Based on the observations in task 7, what is the new equation referred to as with reference to the equation in task 2?
10. Share your findings to the rest of the class.

Given a relation R, one can find another relation from R by reversing the order in all the ordered pairs belonging to R. This type of a relation is called the inverse relation of R and is denoted by R^{-1}. Here '-1' is not an exponent but a symbol to indicate the inverse. The symbol R^{-1} is read as 'R inverse' or 'the inverse of R'.

Let R be a relation where elements of set A are mapped onto members of set B. Then, R^{-1} is the relation mapping elements in set B onto elements in set A. This relation is defined by $R^{-1} = \{(b,a):(a,b) \text{ belongs to } R\}$.

Example 1.18

Let R be a relation given by $R = \{(a,b),(c,d),(e,f),(t,s)\}$. Find:

(a) The domain of R.
(b) The range of R.
(c) The inverse of R.
(d) The domain of R^{-1}.
(e) The range of R^{-1}.

Solution

(a) The domain of R is $\{a, c, e, t\}$.
(b) The range of R is $\{b, d, f, s\}$.
(c) To find R^{-1}, interchange the first and the second coordinates in each pair. Hence,

$$R^{-1} = \{(b, a), (d,c), (f,e), (s,t)\}.$$

(d) The domain of R^{-1} is $\{b, d, f, s\}$.
(e) The range of R^{-1} is $\{a, c, e, t\}$.

Remarks: It is observed from Example 1.18 that, the domain of R^{-1} is the range of R and the range of R^{-1} is the domain of R. These statements are generally true for any relation.

Pictorial representation of the inverse of a relation

The pictorial representation of inverse of a relation R is obtained by reversing the direction of all arrows in R. For example, the pictorial representations of the relation R and its inverse are given in Figure 1.18 and Figure 1.19, respectively.

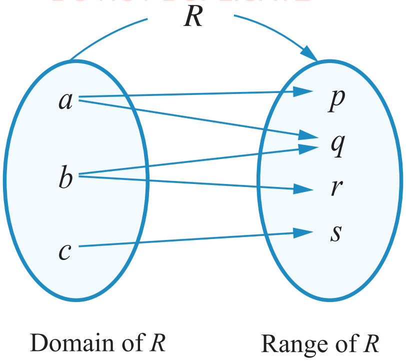


Figure 1.18: *Pictorial representation of the relation R*

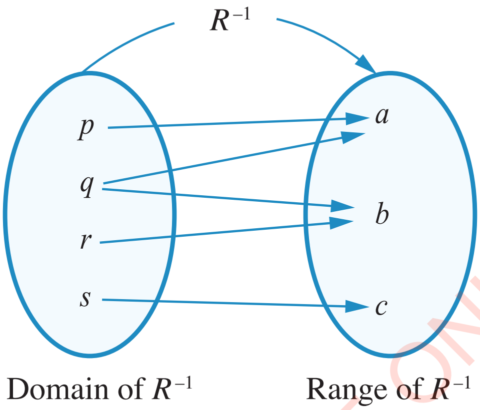


Figure 1.19: *Pictorial representation of the relation* R^{-1}

Finding the inverse of a relation algebraically

The inverse of a relation R which is in form of an equation or an inequality is obtained using the following steps:

Step 1: Interchange the variables x and y.

Step 2: Write y in terms of x.

Step 3: The resulting equation represents the inverse of the relation.

Example 1.19

Find the inverse of the relation $R = \{(x,y):y = 5x\}$.

Solution

Interchange the variables x and y, and then write y in terms of x. The two steps can be summarized as follows:

Step 1: Interchange the variables. That is $y = 5x$ becomes $x = 5y$.

Step 2: Write y in terms of x:
From $x = 5y$, we obtain $y = \frac{1}{5}x$.

Therefore, $R^{-1} = \left\{(x,y) : y = \frac{1}{5}x\right\}$.

Remark: The ordered pairs belonging to R^{-1} are denoted by (x,y) and not by (y,x). This is because a coordinate point is always written by starting with the x coordinate followed by the y coordinate.

Example 1.20

Find the inverse of the relation $R = \{(x, y)$: student y is taller than student $x\}$.

Solution

The inverse relation (R^{-1}) of the relation R is obtained by interchanging the variables x and y. Therefore, $R^{-1} = \{(x,y)$:student x is taller than student $y\}$.

Example 1.21

Given the relation $R = \{(x,y):y = 4x^2\}$, find:

(a) The inverse of R.

(b) The domain and range of R^{-1}.

Solution

(a) Given that $R = \{(x,y):y = 4x^2\}$, we interchange the variables x and y to get $x = 4y^2$. Rewrite y in terms of x such that

$$y^2 = \frac{x}{4} \Leftrightarrow y = \pm\frac{\sqrt{x}}{2}.$$

Hence, the inverse relation of R is given by

$$R^{-1} = \left\{(x, y): y = \pm\frac{\sqrt{x}}{2}\right\}.$$

(b) The domain of $R^{-1} = \{x : x \geq 0\}$ and the range of $R^{-1} = \{y : y$ is a real number$\}$.

Example 1.22

Given the relation $R = \{(x,y) : y < x + 4\}$, find:

(a) The inverse of R.

(b) The domain and range of R^{-1}.

Solution

(a) From the inequality $y < x + 4$, interchange the variables x and y to obtain $x < y + 4$. This is equivalent to $x - 4 < y$ which can be written as $y > x - 4$.

Hence, $R^{-1} = \{(x,y) : y > x - 4\}$.

(b) The domain of $R^{-1} = \{x : x \in \mathbb{R}\}$ and the range of $R^{-1} = \{y : y \in \mathbb{R}\}$

Example 1.23

Given the relation $R = \left\{(x,y) : y = \frac{1}{x-1}\right\}$, find the:

(a) Domain of R.

(b) Range of R.

(c) Inverse of R.

(d) Domain of R^{-1}.

(e) Range of R^{-1}.

Solution

(a) Since the denominator of a fraction cannot be equal to zero, then in the equation $y = \frac{1}{x-1}$, $x \neq 1$. Therefore, the domain of $R=\{x{:}x \in \mathbb{R},\ x \neq 1\}$.

(b) To find the range of R, first express x in terms of y, that is, from $y = \frac{1}{x-1}$,

we get $x = \frac{1}{y} + 1$.

Observe that $y \neq 0$ for $x = \frac{1}{y} + 1$ to be defined. Hence, the range of $R = \{y{:}y \in R,\ y \neq 0\}$.

(c) For the inverse of R, given that $y = \frac{1}{x-1}$, upon interchanging x and y, it becomes $x = \frac{1}{y-1}$. Making y the subject of the equation we obtain $y = \frac{1}{x} + 1$. Therefore, $R^{-1} = \left\{(x, y) : y = \frac{1}{x} + 1\right\}$.

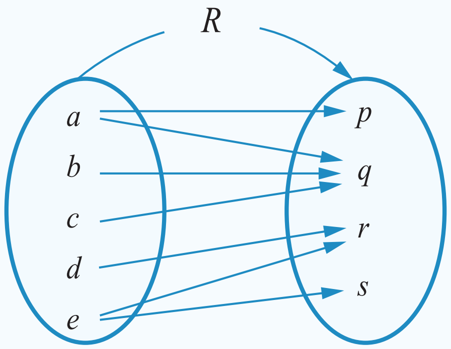
(d) From the equation for R^{-1}, that is , $y = \frac{1}{x} + 1$, observe that x should never be zero. Thus, the domain of $R^{-1} = \{x{:}\ x \in \mathbb{R},\ x \neq 0\}$. Notice that, one could also use the fact that, the range of R is the domain of R^{-1} to obtain the same result. However, remember to replace x by y and vice versa.

(e) Using the fact that the domain of R is the range of R^{-1}, the range of $R^{-1} = \{y{:}y \in \mathbb{R},\ y \neq 1\}$ or $\{y{:}y$ is all real numbers except $1\}$.

Exercise 1.4

In questions 1 to 3, find the inverse of each of the given relations.

1. R is the set of all pairs of coordinate points in the xy–plane.
2. R is the set of all ordered pairs (x, y) of real numbers where y is the cube root of x.
3. R is the set of all pairs of coordinate points in the xy–plane which lie below the line $y = 4$.
4. Study the relation R shown in the following figure and answer the questions that follow.



(a) Make a drawing to show the inverse relation R^{-1}.

(b) Make a list of all ordered pairs belonging to R^{-1}.

(c) Which set is the domain of R^{-1}?

(d) Which set is the range of R^{-1}?

In questions 5 to 12, find:

(a) The inverse of a relation.

(b) The domain and range of the inverse of a relation.

5. $R = \{(x,y): y = 2x+1\}$.
6. $R = \{(x,y): 2y-1 = 4x\}$.
7. $R = \{(x,y): y = x^2\}$.
8. $R = \{(x,y): y < 0\}$.
9. $R = \{(x,y): 3x > y\}$.
10. $R = \{(x,y): 2x+5 < y\}$.
11. $R = \left\{(x,y): y = \dfrac{2}{x+5}\right\}$.
12. $R = \{(x,y): y = 3x^2 - 1\}$.

Graphs of inverse of a relation

Consider the relation $R = \{(x,y): y > x\}$. Its inverse is given by $R^{-1} = \{(x,y): y < x\}$. In this case, R is the relation "greater than" for all real numbers and R^{-1} is the relation "less than" for all real numbers.

The graphs of $R = \{(x,y): y > x\}$ and $R^{-1} = \{(x,y): y < x\}$ are shown in the shaded regions in Figure 1.20 and Figure 1.21, respectively.

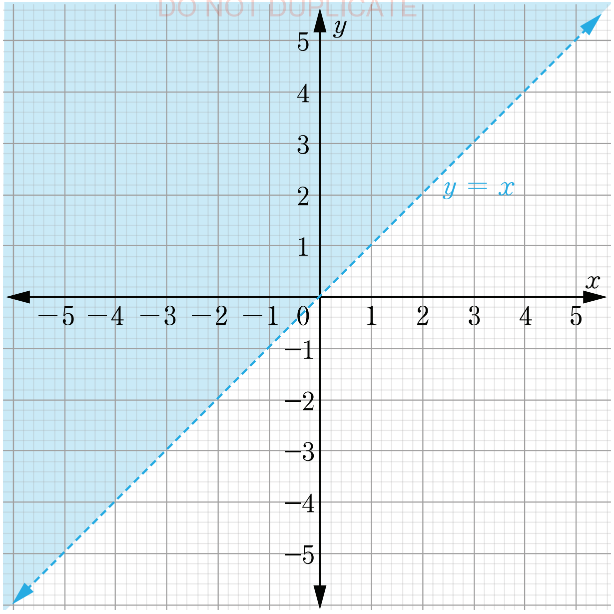


Figure 1.20: *Graph of the relation* $R = \{(x, y) : y > x\}$

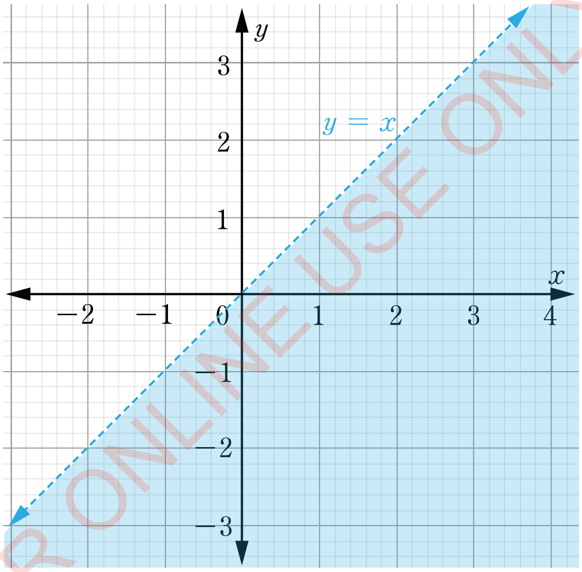


Figure 1.21: *Graph of the relation* $R^{-1} = \{(x,y):y < x\}$

One way of understanding graphically the relationship between a relation R and its inverse R^{-1} is to open a book and align the binding with the line $y = x$ as shown in Figure 1.22.

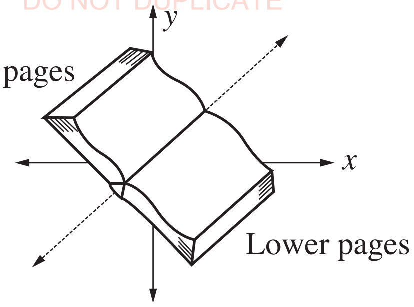


Figure 1.22: *A demonstration of inverse of a relation using an opened book*

In this case, if the upper pages represents the graph of the relation R, then the lower pages represents the graph of its inverse. For example, if the pairs $(0,1)$, $(1,2)$, and (x, y) belong to the relation R (upper pages), then the points $(1,0)$, $(2,1)$, and (y, x) belong to R^{-1} (lower pages).

So, reversing the coordinates of any point in the plane is the same as reflecting it in the line $y = x$. Here, the line $y = x$ acts as a mirror.

Example 1.24

Consider the relation given by $R = \{(x,y):y > 2x\}$. Its inverse is $R^{-1} = \left\{(x,y):y < \frac{x}{2}\right\}$. The graphs of R and R^{-1} are shown in Figure 1.23 on the same pair of axes.

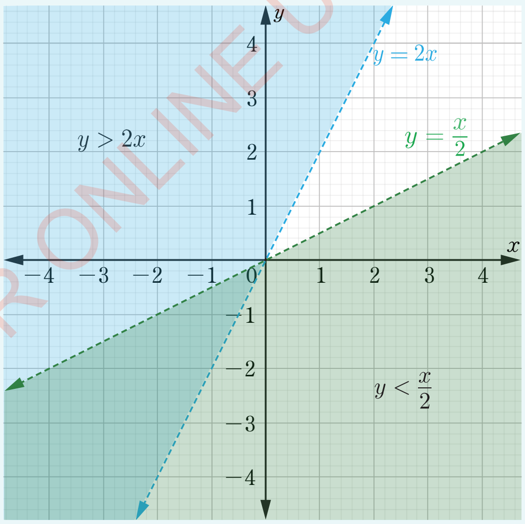


Figure 1.23: *Graph of relations* $R = \{(x,y):y > 2x\}$ and $R^{-1} = \left\{(x,y):y < \frac{x}{2}\right\}$

Therefore, given any relation R, the graph of R^{-1} can be obtained by reflecting the graph of R about the line $y = x$.

Example 1.25

Draw the graph of the inverse of the relation $R = \{(x,y): y = 3x\}$.

Solution

From the equation $y = 3x$, interchange the variables x and y to obtain $x = 3y$. Writing y in terms of x, we have $y = \frac{x}{3}$. Hence, $R^{-1} = \left\{(x,y): y = \frac{x}{3}\right\}$ and its graph is shown in Figure 1.24.

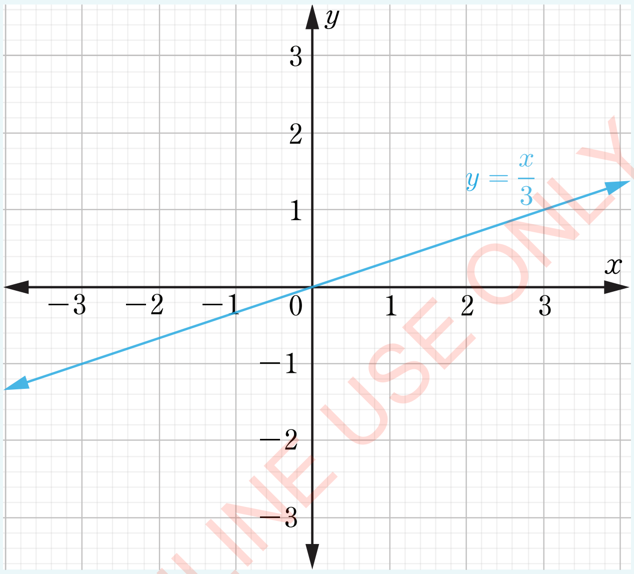


Figure 1.24: *Graph of* $R^{-1} = \left\{(x,y): y = \frac{x}{3}\right\}$

Example 1.26

Draw the graph of the inverse of the relation $R = \{(x,y): y \le 0 \text{ and } y \ge x\}$, and then find the domain and range of R^{-1}.

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Solution

Note that, after interchanging x and y, we have that $R^{-1} = \{(x,y):x \leq 0$ and $y \leq x\}$.

Therefore, the graph of R^{-1} is as shown in Figure 1.25.

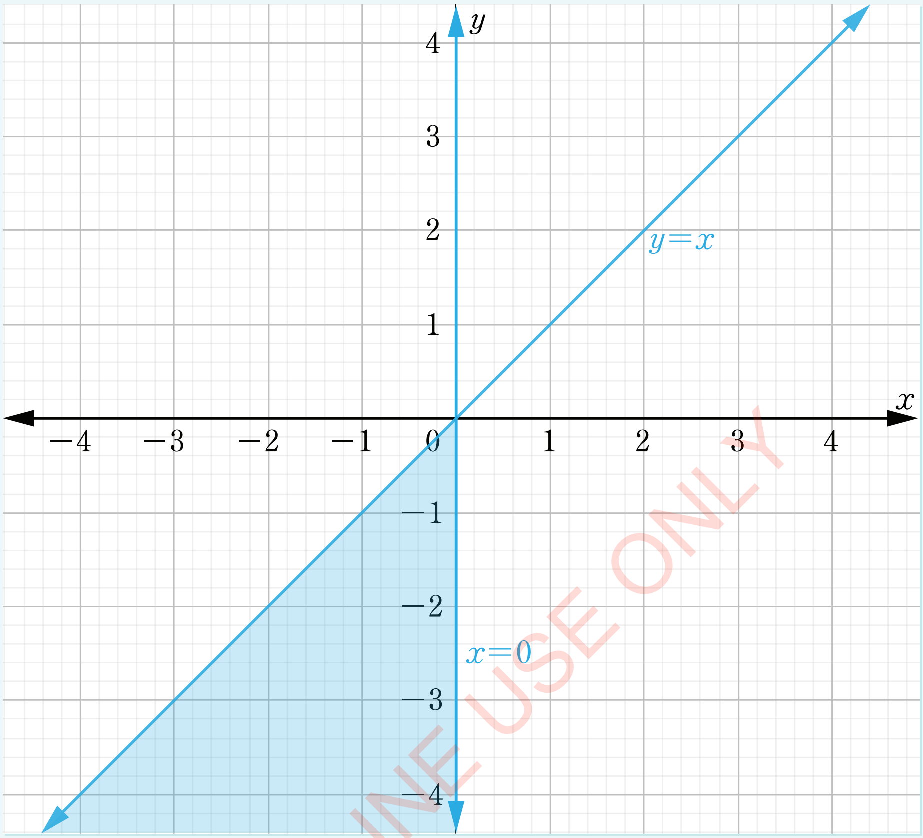
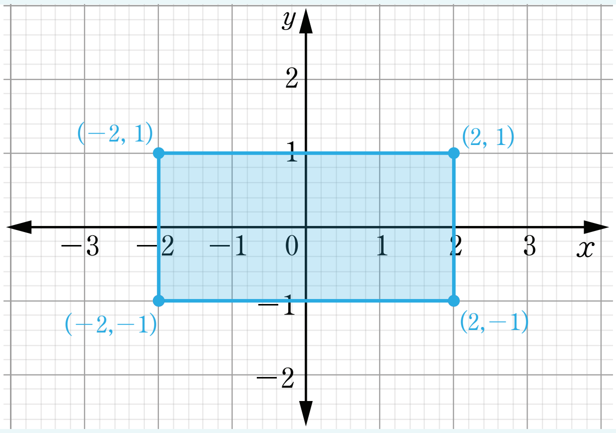


Figure 1.25: *Graph of* $R^{-1} = \{(x,y):x \leq 0$ and $y \leq x\}$

From the graph, it is observed that the domain of $R^{-1} = \{x:x \leq 0\}$ and the range of $R^{-1} = \{y:y \leq 0\}$.

Example 1.27

Draw the graph of the inverse of the relation R shown in the following figure. Use the graph to find the domain and range of the inverse relation.

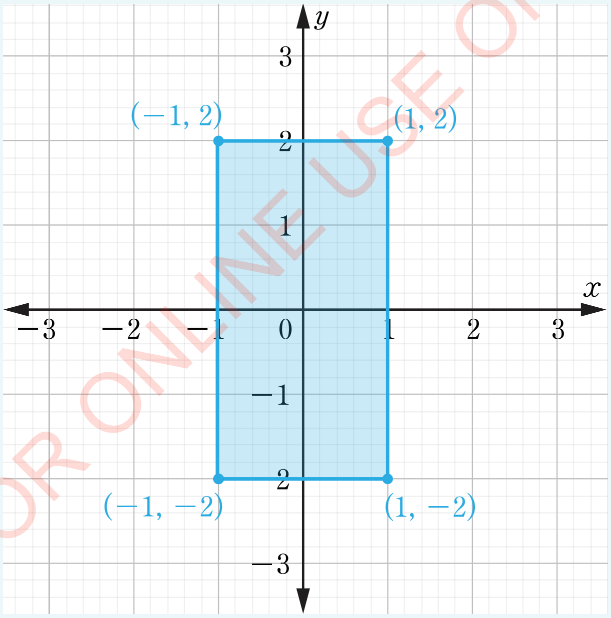


Solution

Using the coordinates of some of the points on the boundary of the relation R, we have $R = \{(-2, 1), (-2, -1), (0, -1), (2, 0), (-2, 0), (1, 1), (-1, -1)\}$.

Interchanging the coordinates gives

$R^{-1} = \{(1, -2), (-1, -2), (-1, 0), (0, 2), (0, -2), (1, 1), (-1, -1)\}$. The graph of R^{-1} is as shown in the following figure.



From the graph, domain of $R^{-1} = \{x : -1 \leq x \leq 1\}$ and range of $R^{-1} = \{y : -2 \leq y \leq 2\}$.

Basic Mathematics Form Three

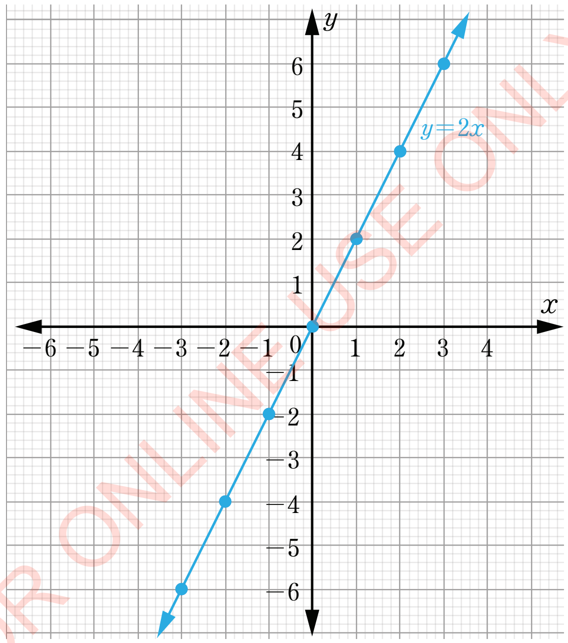
Exercise 1.5

In questions 1 to 6, draw the graph of the inverse of each of the given relations, hence obtain the domain and range of each inverse relation.

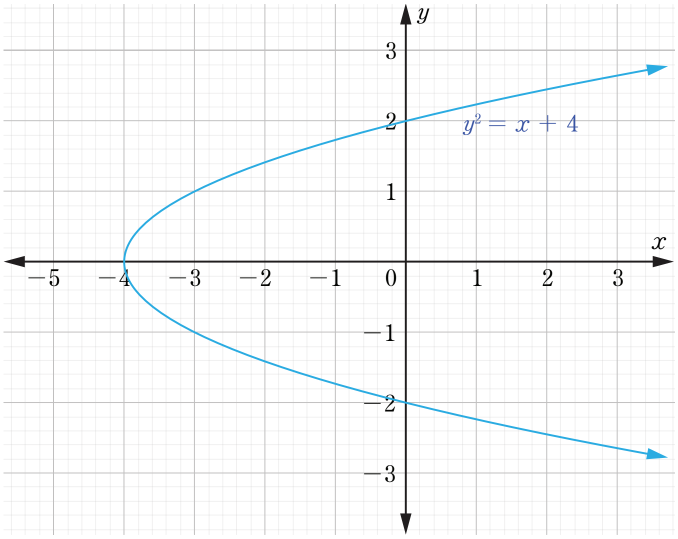
1. $R = \{(x,y): y = 2x - 2\}$.
2. $R = \{(x,y): y = x^2\}$.
3. $R = \{(x,y): 2y - x = 4\}$.
4. $R = \{(x,y): x > 0\}$.
5. $R = \{(x,y): 2y > x\}$.
6. $R = \{(x,y): x + y < 1 \text{ and } y < x - 1\}$.

In questions 7 to 12, draw the graph of the inverse of each of the given relations, hence obtain the domain and range of the inverse relation.

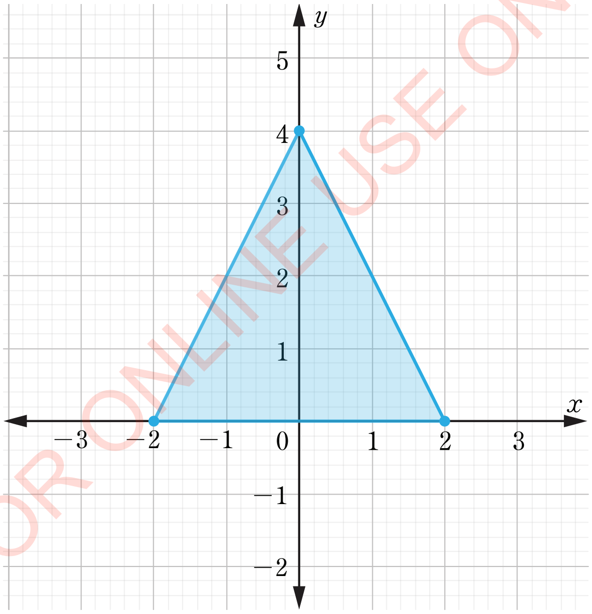
7.



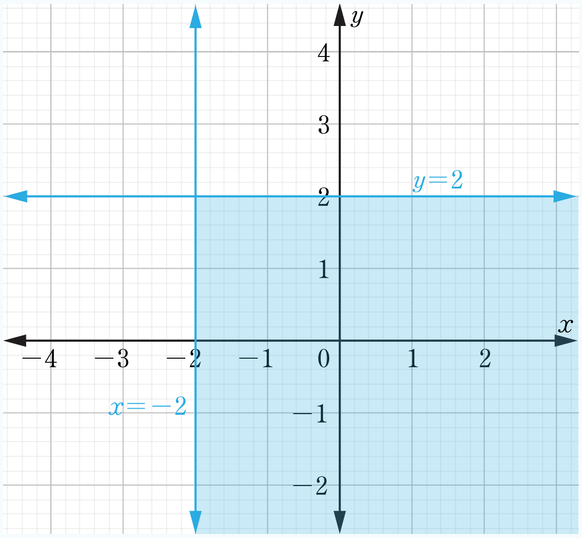
8.



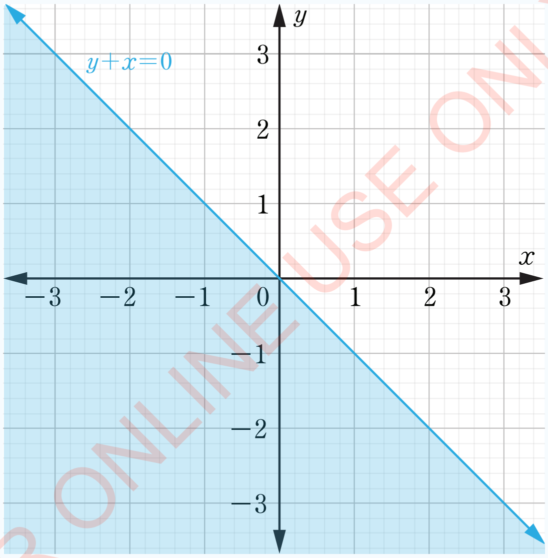
9.



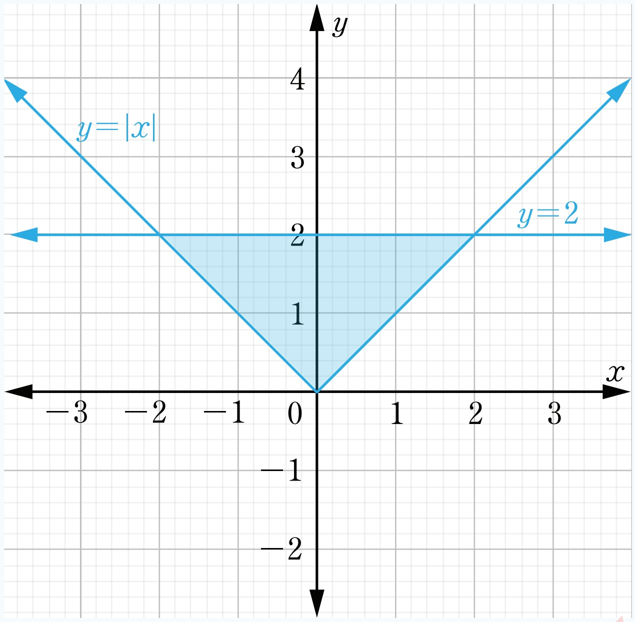
10.



11.



12.



Activity 1.3: Identifying real life relations

Individually or in a group, perform the following tasks:

1. Prepare a list of five real life examples of relations.
2. Describe the domain and range of each relation.
3. Represent the relations pictorially and in a form of equations, if possible.
4. Represent the inverse of each relation pictorially and as an equation, if possible.
5. Prepare a poster (or any other means of your choice) of the summary of your work, hence share the results to the rest of the class.

Chapter summary

1. A relation between two sets A and B is a set of ordered pairs (x, y), with $x \in A$ and $y \in B$. A relation can be represented by a pictorial diagram.
2. The domain of a relation is the set of values of set A which occur in the relation. The range is the set of values of set B which correspond to values of set A in the relation.

3. A relation between sets of numbers can be represented on a graph.
4. An equation in x and y is a relation. It can be represented on a graph.
5. An inequality in x and y is a relation. It can be represented by a shaded region.
6. The inverse of a relation of sets A and B is found by interchanging the elements in sets A and B. On a graph, it is shown by interchanging the x and y coordinates.

Revision exercise 1

In questions 1 to 4, use $\{1,2,3,4\}$ as a set of the domain of the relation R to give the range of each of the given mappings:

1. $R:x \mapsto 2x$.
2. $R:x \mapsto 3x - 7$.
3. $R:x \mapsto 4x + 5$.
4. $R:x \mapsto x^2$.
5. If $X = \{4, 3, 8\}$ and $Y = \{a, 1\}$, list the ordered pairs of the following relations:
   (a) $X \rightarrow Y$.
   (b) $Y \rightarrow X$.
   (c) $X \rightarrow X$.
   (d) $Y \rightarrow Y$.

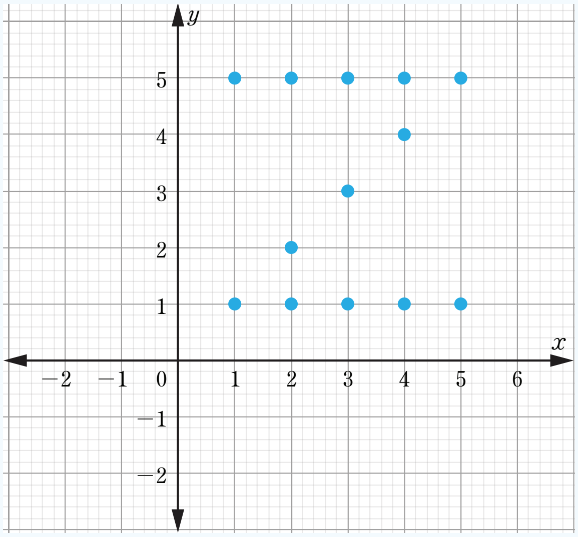
6. Represent the following relations on the xy-plane:
   (a) $R = \{(3, 2), (2, 2)\}$.
   (b) $R = \{(1, 2), (1, 5), (1,7), (3, 5), (2,2), (2,7)\}$.
   (c) $R = \{(2,4), (2,3), (7,1), (5,4)\}$.

7. Represent each of the following relations by using arrow diagrams:
   (a) $R = \{(3, 2), (2, 2)\}$.
   (b) $R = \{(1, 2), (1, 5), (1, 7), (3, 5), (2, 2), (2, 7)\}$.
   (c) $R = \{(2, 4), (2, 3), (7, 1), (5, 4)\}$.

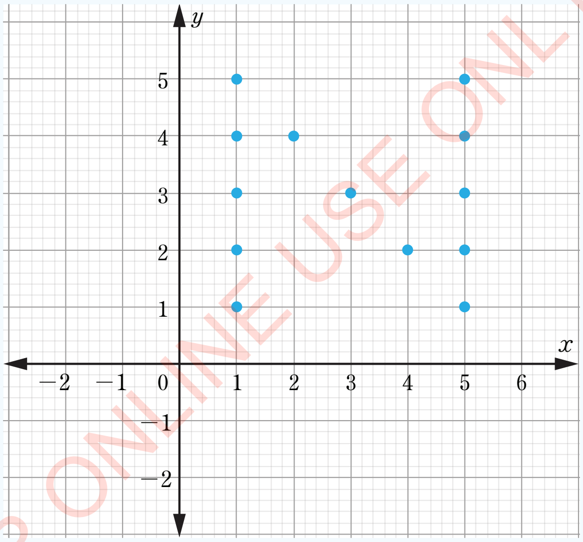
8. Write the inverses of the following relations:
   (a) $R = \{(a, b), (a, c), (a, d)\}$.
   (b) $R = \{(a, e), (b, f), (c, g), (d, h)\}$.

9. Draw the graphs of the inverses of the following relations:

(a)



(b)



10. For each of the given relations, write down its inverse and the range of the inverse relation:

(a) $x \mapsto x+4$, domain: $\{6, 7, 8, 9\}$.

(b) $x \mapsto x^2$, domain: $\left\{-1, \frac{-1}{2}, 0, \frac{1}{2}, 1\right\}$.

Chapter Two

Functions

Introduction

A function provides a way of representing real world situations mathematically, where a certain input is required to give a certain output. It can be compared to a machine that produces a unique output based on one or more inputs. In this chapter, you will learn the concept of a function, types of functions, representing functions pictorially, identifying functions among relations, drawing graphs of functions, stating domain and range of functions, and finding inverse functions. The competencies developed in this chapter will enable you to perform real life activities related to functions, such as finding circumferences and areas of large objects, recording quantities of things in related measuring units, analysing the supply and demand in terms of prices of goods, among many other applications.

Concept of a function

Some relations have special properties. Let us consider two sets A and B, where A is a set of children and B is a set of mothers as shown in Figure 2.1 and Figure 2.2.

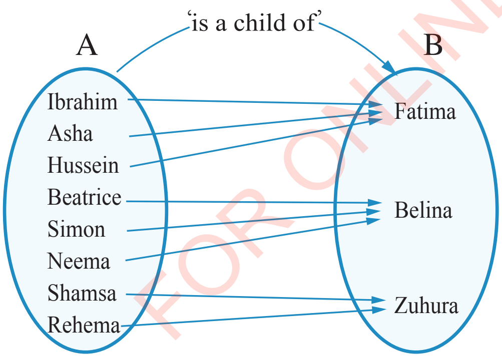


Figure 2.1: *Pictorial representation of relation "is a child of"*

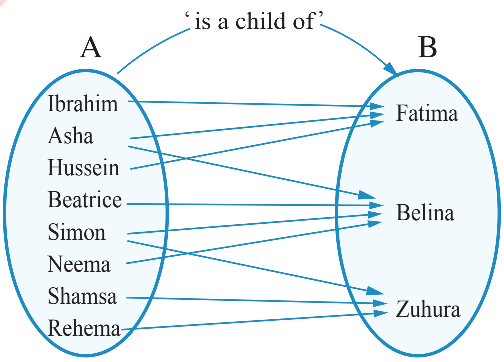


Figure 2.2: *Pictorial representation of relation "is a child of"*

In Figure 2.1, every child in the left-hand side set is connected to exactly one mother in their right-hand side set. This implies that, every child is born to exactly

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one mother. This relation is true because, in real life, a child is born to only one mother. The fact that this relation is true in real life, therefore, it is a function.

In Figure 2.2, some children in the left-hand side set are connected to more than one mother in the right-hand side set. This implies that, a child is born to more than one mother, which is not true in real life. The fact that this relation is not true in real life, therefore, it is not a function.

Based on the observations in Figure 2.1 and Figure 2.2, a relation is therefore a function if and only if;

(a) each element in the left-hand side set has an arrow leaving from it.
(b) no element on the left-hand side has more than one arrow leaving from it.

Activity 2.1: Identifying functions among relations

Individually or in a group, perform the following tasks

1. Go through the types of relations discussed in Chapter One.
2. Use the knowledge of function to identify types of relations which are functions and which are not.
3. Illustrate through pictorial diagrams the difference between types of relations which are functions and those which are not functions.
4. Discuss the observed differences with the rest of the class.

Generally, a function can be defined as a relation that associates each element x of a set X, called the domain of the function with a single element y of another set Y, called the range of the function. Simply, a function is a special type of a relation with a property that each input element has a unique output. Examples of functions are shown in Figure 2.3 to Figure 2.5.

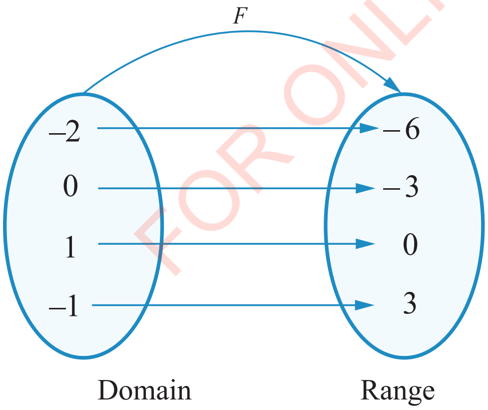


Figure 2.3: *Pictorial representation of a function F*

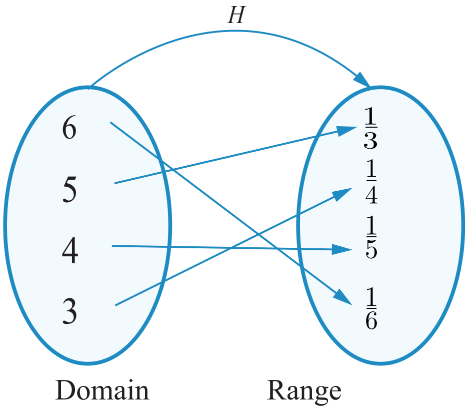


Figure 2.4: *Pictorial representation of a function H*

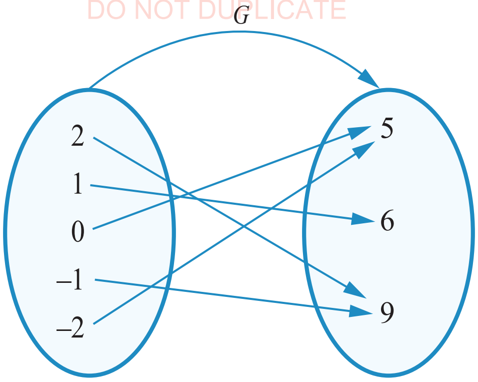


Figure 2.5: *Pictorial representation of a function G*

The pictorial diagrams in Figure 2.3, Figure 2.4, and Figure 2.5 can be expressed as a set of ordered pairs as follows:

Figure 2.3 : $F = \{(-2, -6), (0, -3), (1, 0), (-1, 3)\}$

Figure 2.4 : $H = \left\{(3, \frac{1}{4}), (4, \frac{1}{5}), (5, \frac{1}{3}), (6, \frac{1}{6})\right\}$

Figure 2.5 : $G = \{(2, 9), (1, 6), (0, 5), (-1, 9), (-2, 5)\}$

Example 2.1:

Plot the coordinates of function F = $\{(4,2), (3,7), (1,4), (4,4)\}$ in the xy – plane:

Solution

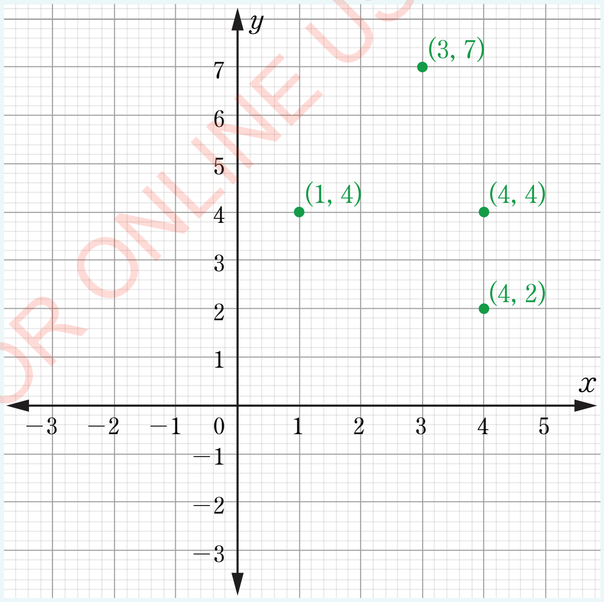


Figure 2.6 *Coordinates represented in the xy–plane*

Example 2.2:

Draw a pictorial diagram of the function F with coordinates (1, 2), (3, 4), (4, 6), (5, 7).

Solution

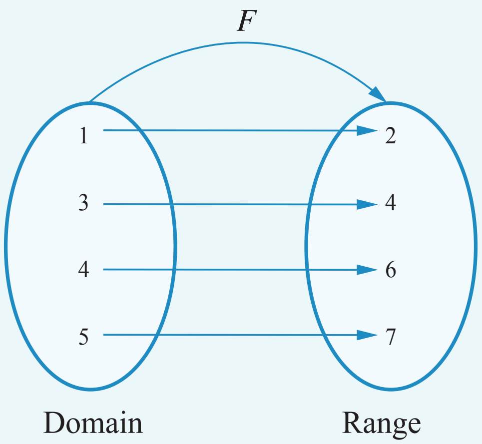
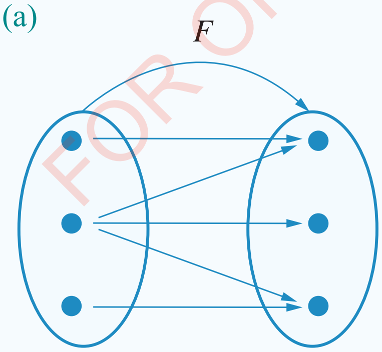


Figure 2.7: *Pictorial diagram of the function F*

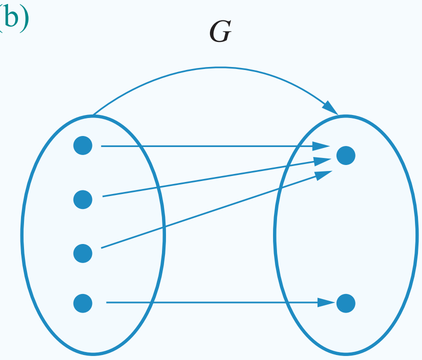
Exercise 2.1

1. Represent the following set of ordered pairs in a pictorial diagram: $\{(1,-3),(2,2),(-1,3),(-2,4)\}$
2. Represent the following coordinates in the *xy*-plane: (0, 0), (1, 1), (2, 2), (3, 3), (4, 4)
3. Which of the following relations are functions?

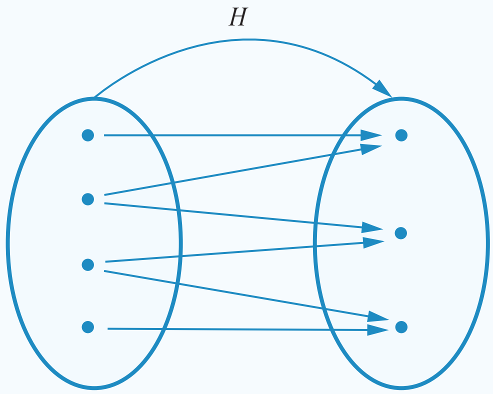
(a)



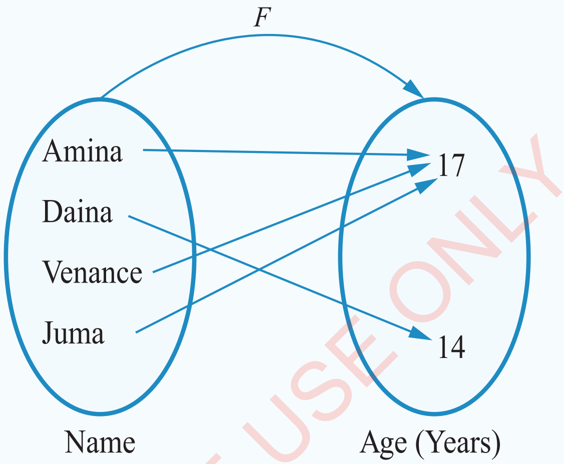
(b)



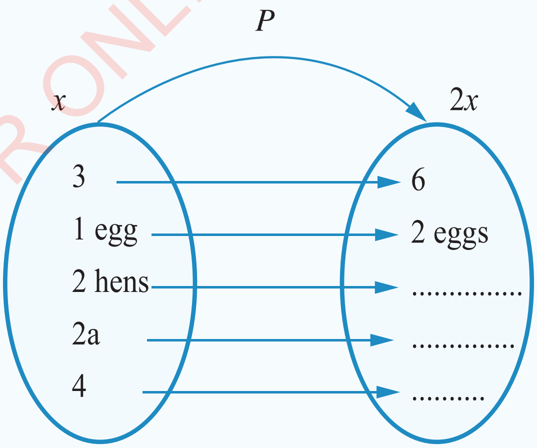
(c)



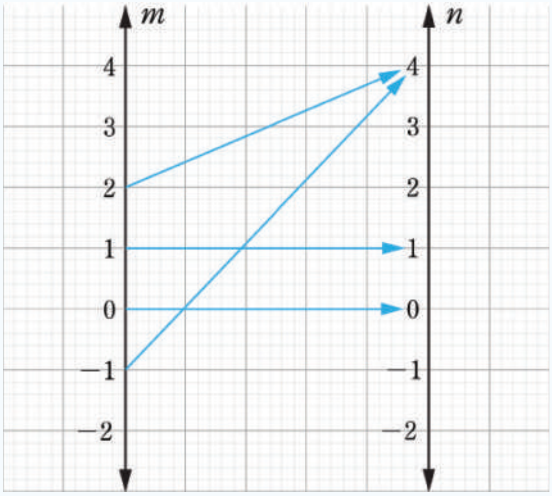
4. List the coordinates of the mapping in the following figure:



5. The function P doubles whatever is fed into it. Complete the missing outputs in the following figure.

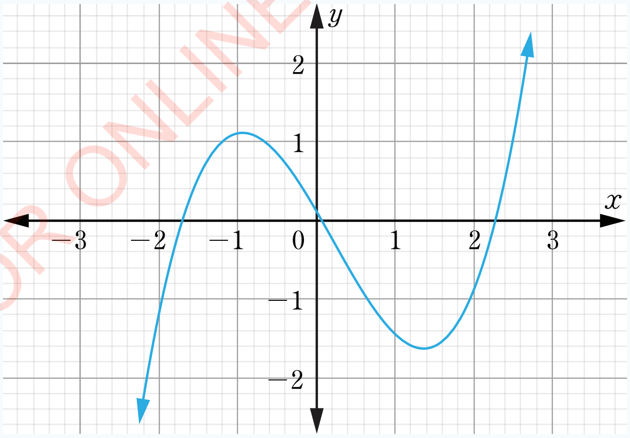


6. Write down the ordered pairs (x, y) from the given mapping in the following figure.

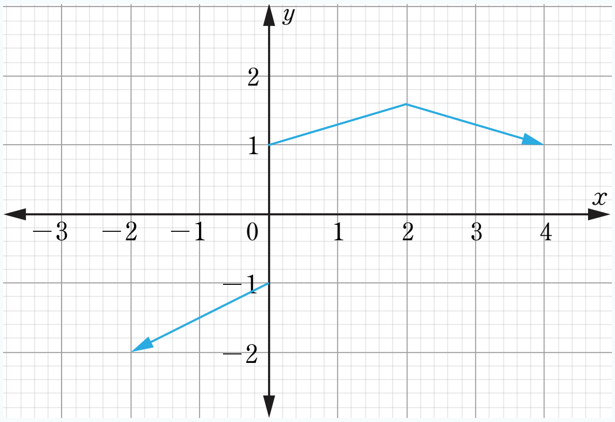


7. Which of the following relations are functions?
   (a) (positive even number) ⟶ (its largest prime factor)
   (b) (word) ⟶ (number of letters in that word)
   (c) (name) ⟶ (person)
   (d) (student) ⟶ (month of his/her birthday)
   (e) (person) ⟶ (name)

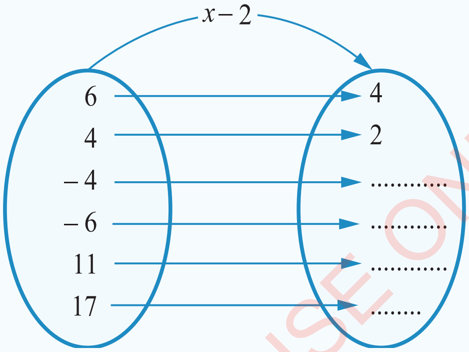
8. Which of the following graphs represent a function?
   (a)



(b)



9. The following figure represents the function 'subtract 2 from x'. Copy and complete the pictorial diagram:



Function notation

Consider a function f which maps x onto $3x+1$. In mathematical language, it is denoted by $f:x \rightarrow 3x+1$. This notation is read as 'f is a function that maps x onto $3x+1$'. Each value of f is obtained by multiplying each value of x by 3 and adding 1. The above function notation can also be written as $f(x) = 3x+1$ and read as 'f of x equals $3x$ plus 1'. The value of $f(x)$ can be found by substituting the values of x. For example, if $x = 11$, in the function $f(x) = 3x+1$, then $f(11) = 3 \times 11 + 1 = 34$. Likewise, if $f(x) = 16$, the value of x can be determined by solving the equation $3x+1 = 16$ giving $x = 5$.

A function can be represented graphically by setting $y = f(x)$. The function $f(x) = 3x+1$ can therefore be written as $y = 3x+1$. The equation tells us that the values of y depend on the values of x. Thus, the values of y can be calculated by substituting the corresponding values of x.

The graph of a function $f(x)$ is the same as the graph of the equation $y = f(x)$. If point (a, b) satisfies the equation, then a represents the value of x and b represents the value of y.

The graph of $y = 3x + 1$ can be drawn on the xy-plane as shown in Figure 2.8.

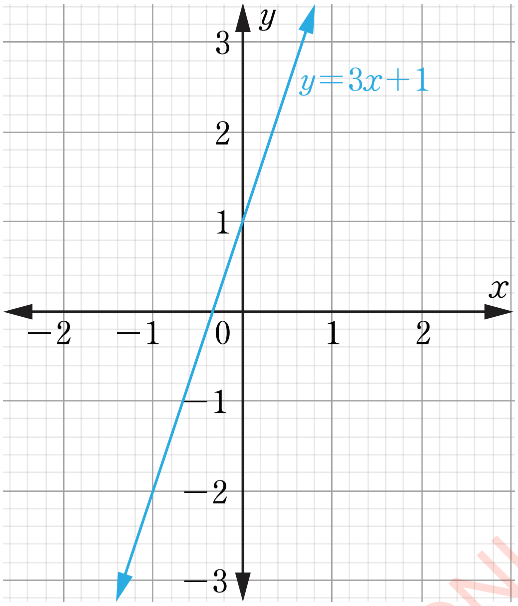


Figure 2.8: *Graph of* $f(x) = 3x + 1$

Identifying a function from a graph

You have learnt that all functions are relations but not all relations are functions. You have also learnt that all one-to-one and many-to-one relations are functions. When given a relation pictorially, you can easily identify whether or not the relation is a function. What if you are given a graph? How can you identify whether it represents a function? In this section, you will learn a vertical line test for identifying a function from a graph and a horizontal line test to identify if a function is one-to-one.

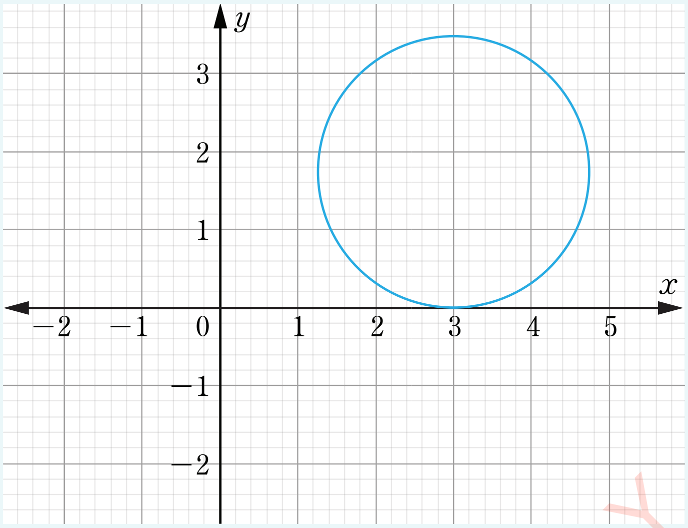
Vertical line test for identifying functions

Graphically, to test if a relation is a function, draw a parallel line to the y–axis. If the line crosses the graph at only one point, then the relation is a function. It implies that, if the vertical line cross the graph at only one point, then one value of x corresponds to only one value of y which satisfies the definition of a function.

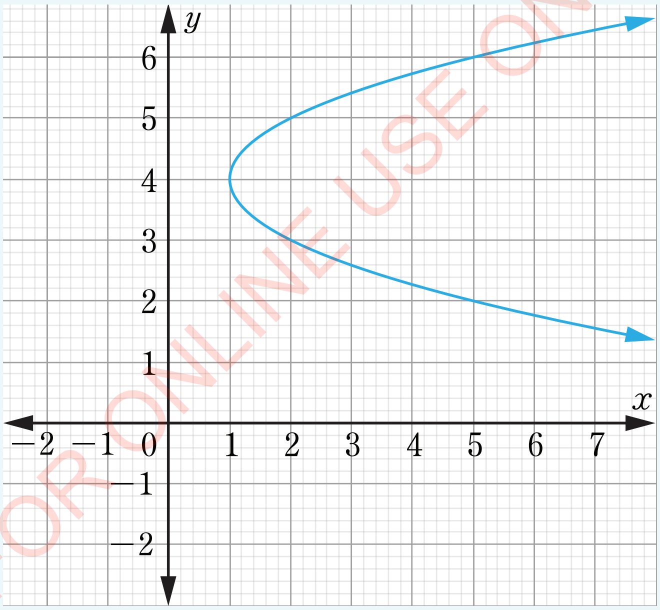
Example 2.3

Study the following figures in (a) up to (c), then determine the graphs which represent functions.

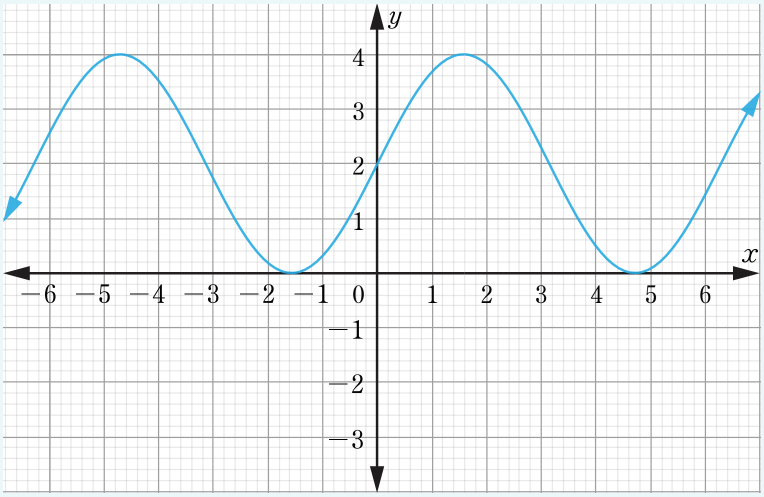
(a)



(b)



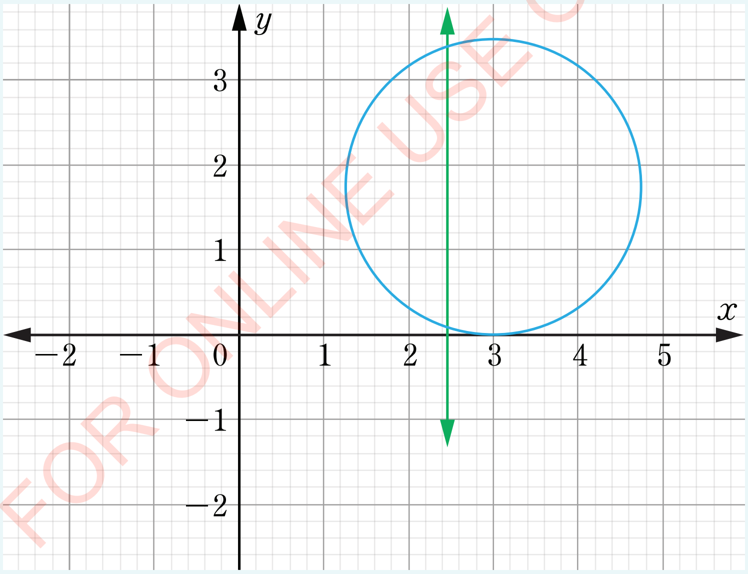
(c)



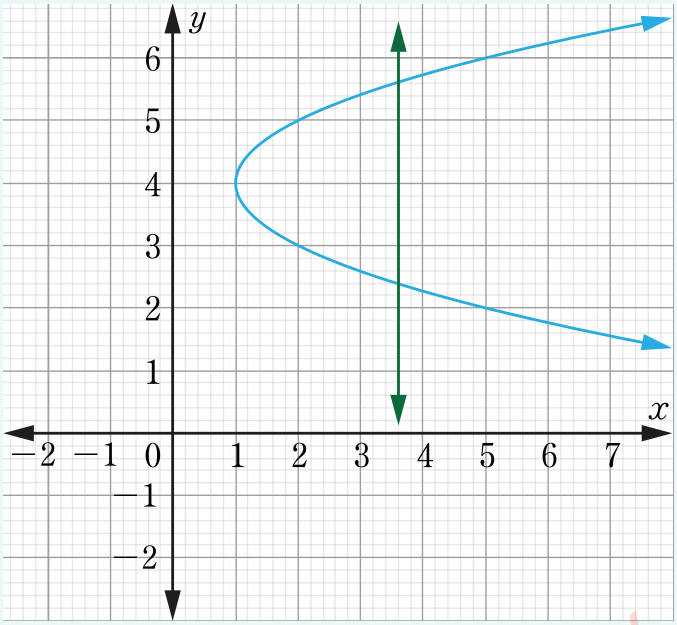
Solution

Draw a straight line parallel to the y – axis in each of the given graphs as shown in the following figures:

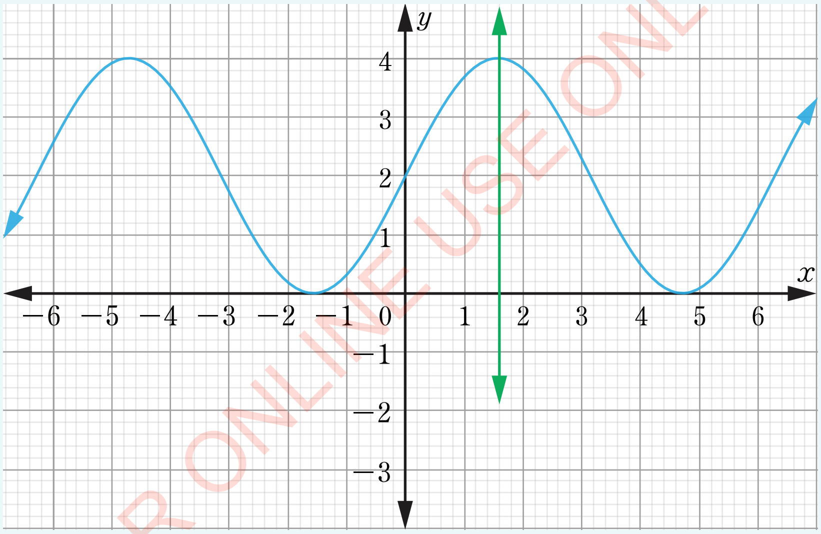
(a)



(b)



(c)



From the figures, it can be observed that the vertical lines drawn in (a) and (b) cross the graphs more than once, therefore the graphs in (a) nd (b) are not functions, while the vertical line drawn in (c) crosses the graph once, hence (c) represents a function.

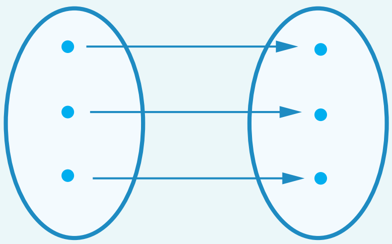
Types of functions

The following activity will help you to identify types of functions.

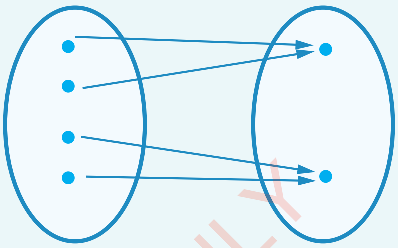
Activity 2.2: Identifying types of functions

1. Study the figures in (a) and (b) then identify their main differences in terms of how the sets of ordered pairs are related.
2. What is your conclusion based on how the sets of ordered pairs are related in each pictorial figure?
3. What is the name of each type of function represented by each of the pictorial figure?

(a)



(b)



Generally, there are functions in which each value from the domain has only one value in the range. These are called one-to-one functions. There are other functions in which several values of the domain are related to one value of the range. These functions are called many-to-one functions.

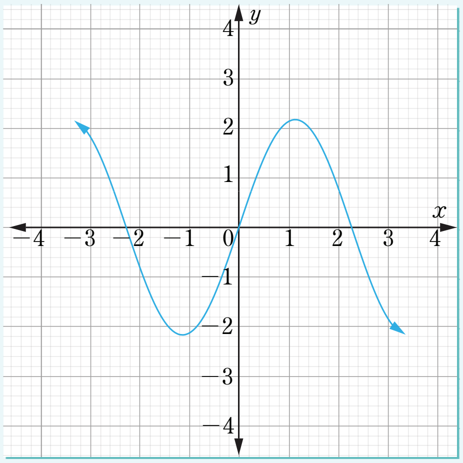
Horizontal line test for identifying one-to-one functions

A function whose graph is such that any line drawn parallel to the x–axis crosses the graph at only one point is called a one-to-one function.

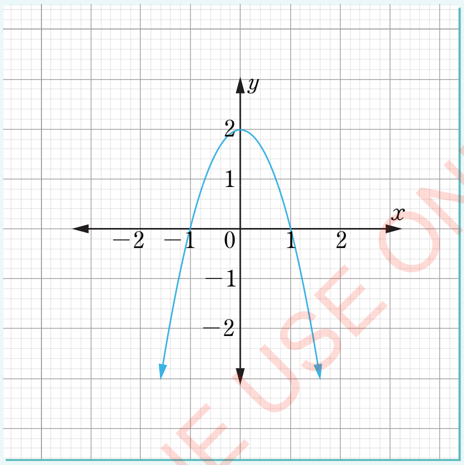
Example 2.4

Study the following figures and identify the functions which are one-to-one.

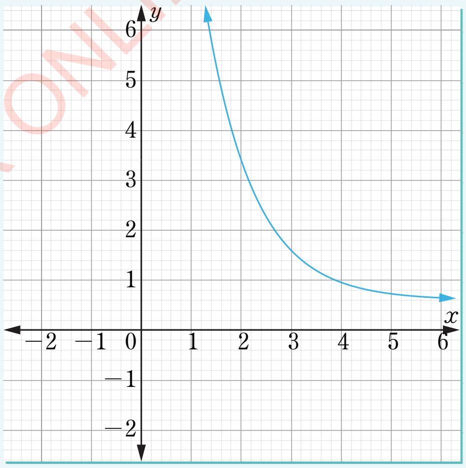
(a)



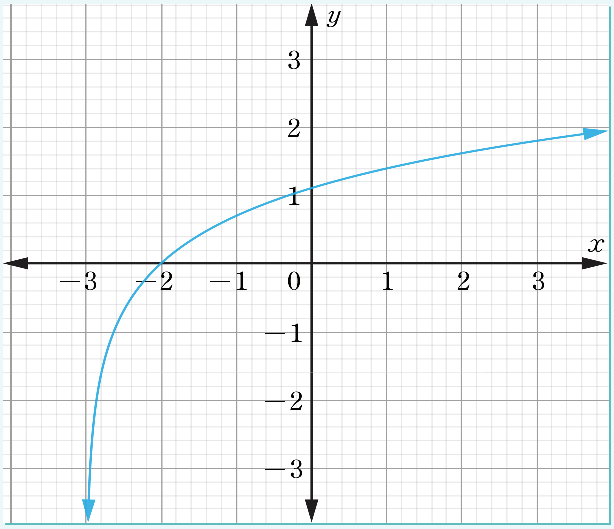
(b)



(c)



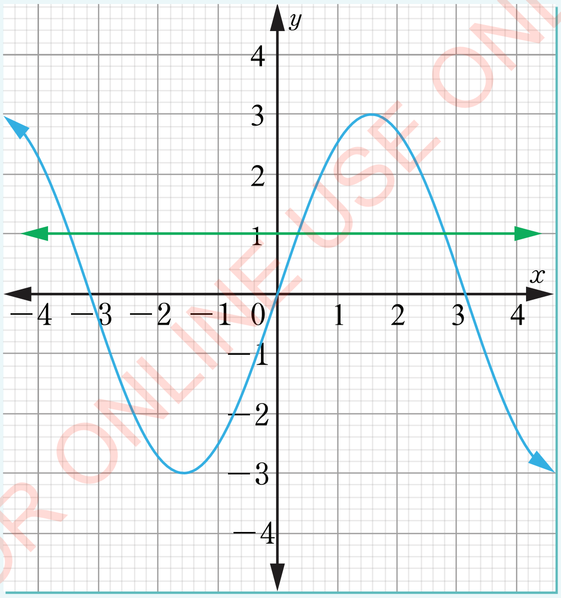
(d)



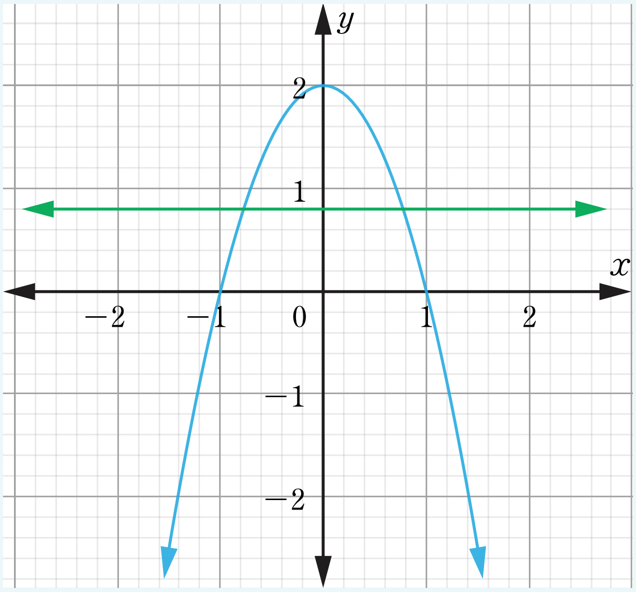
Solution

In order to test whether or not a function is one-to-one, draw a line parallel to the x – axis as shown in the following figures.

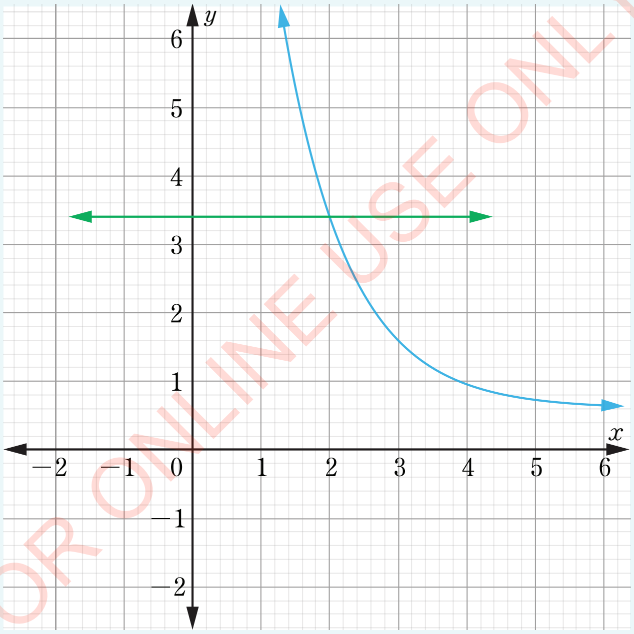
(a)



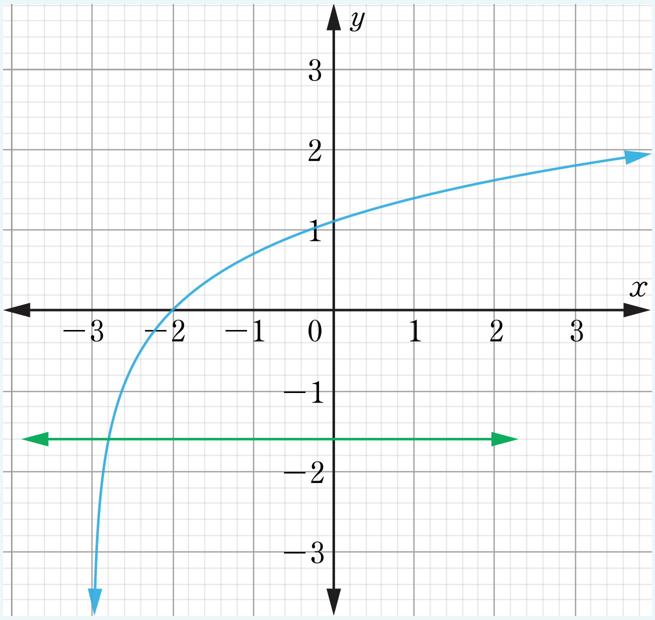
(b)



(c)



(d)



From the figures, it can be observed that the horizontal lines drawn in (a) and (b) crosses the graph more than once, therefore the functions are not one-to-one, while the horizontal lines drawn in (c) and (d) crosses the graphs once, hence the functions are one-to-one.

Example 2.5

Write the function 'a number is squared and then multiplied by 2' in the form $f:x \rightarrow f(x)$, and then find:
(a) $f:2$. (b) $f(3)$.

Solution

(a) $f:x \rightarrow 2x^2$

$f:2 \rightarrow 2(2)^2$

$f:2 \rightarrow 8$

Therefore, $f:2 \rightarrow 8$.

(b) $f(x) = 2x^2$

$f(3) = 2(3)^2$

$= 2(9)$

$= 18$

Therefore, $f(3) = 18$.

Example 2.6

Consider the function defined by $g(x) = 3x + 3$. Find $g(-2)$.

Solution

$$g(x) = 3x + 3$$
$$g(-2) = 3(-2) + 3$$
$$= -6 + 3$$
$$= -3$$

Therefore, $g(-2) = -3$.

Example 2.7

Draw the graph of the function defined by $f(x) = 2x + 1$ and identify if it is a one-to-one function.

Solution

Choose few values of x near the origin and then use them to calculate the corresponding values of $f(x)$ as shown in Table 2.1.

Table 2.1: Table of values of the function $f(x) = 2x + 1$

x	-2	-1	0	1	2	3
$y = f(x)$	-3	-1	1	3	5	7

The ordered pairs are plotted on the xy-plane and joined by a straight line as shown in Figure 2.9.

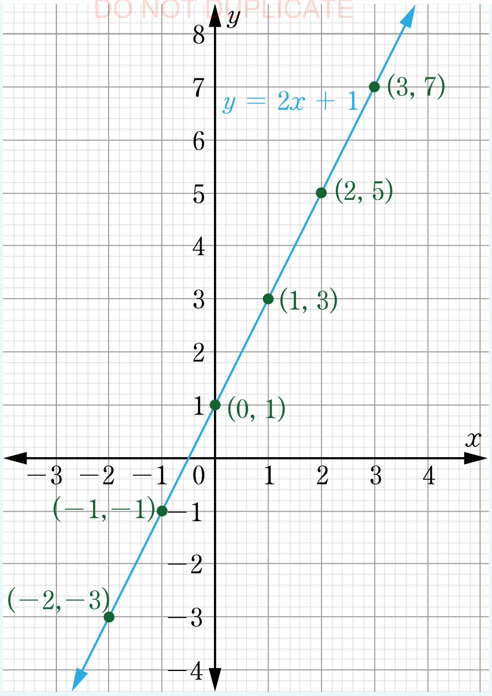


Figure 2.9: *The graph of* $f(x) = 2x + 1$

In this case, any horizontal line drawn crosses the graph at only one point which indicates that the function represented by a graph is a one-to-one function.

Example 2.8

Draw the graph of the function defined by $f(x) = x^2$ and determine if it is a one-to-one function.

Solution

Select few points on the graph of the function $f(x) = x^2$ as shown in Table 2.2.

Table 2.2: Table of values of the function $f(x) = x^2$

x	−3	−2	−1	0	1	2	3
$y = f(x)$	9	4	1	0	1	4	9

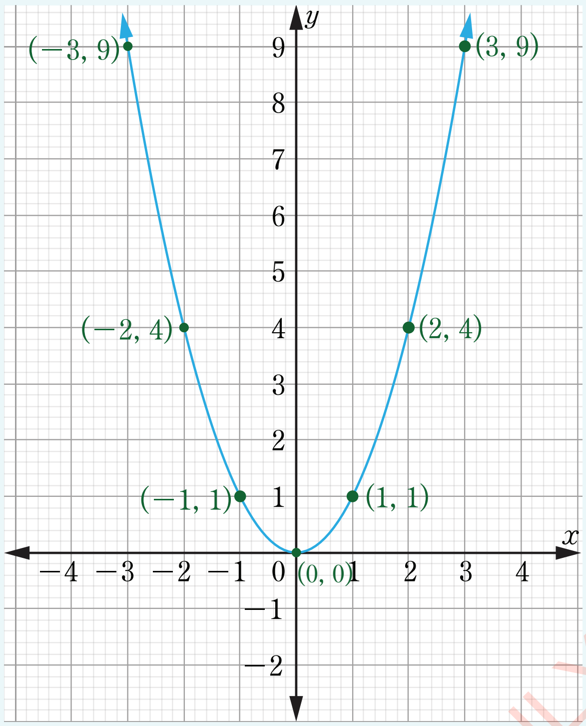


Figure 2.10: *Graph of* $f(x) = x^2$

Any horizontal line drawn in Figure 2.10 crosses the graph at more than one point. This indicates that the function represented by the graph is not a one-to-one function, but rather a many-to-one function.

Exercise 2.2

Write each of the following functions in the form $f{:}x \rightarrow f(x)$.

1. Divide a number by 5 and add 2.
2. Add a number to 2 and divide the sum by 5.
3. Square a number and subtract 7.
4. Subtract 7 from a number and square.
5. Subtract a number from 8.
6. Cube a number and double.

Find the values of the following functions for each of the given values of x:

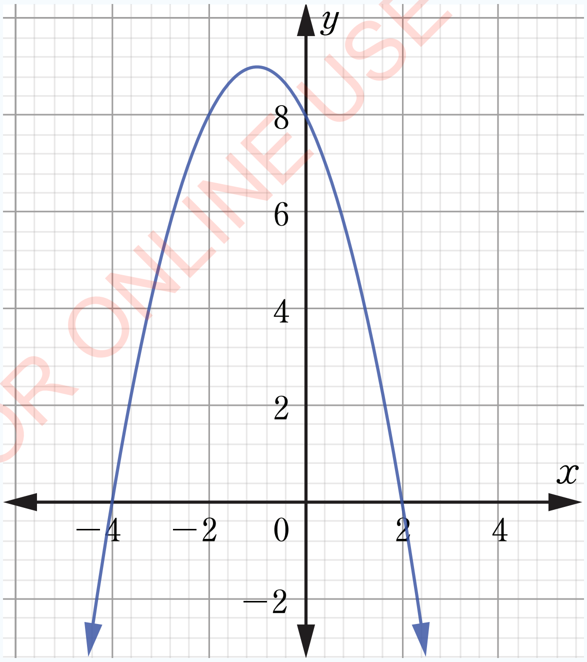
7. $f(x) = 2x + 3$; $x = 1$, $x = -2$, $x = a$
8. $C(x) = x^3$; $x = 3$, $x = 7$, $x = 2$
9. $S(x) = x^2$; $x = -1$, $x = 5$, $x = 4$
10. $K(x) = 3$; $x = 1$, $x = 7$

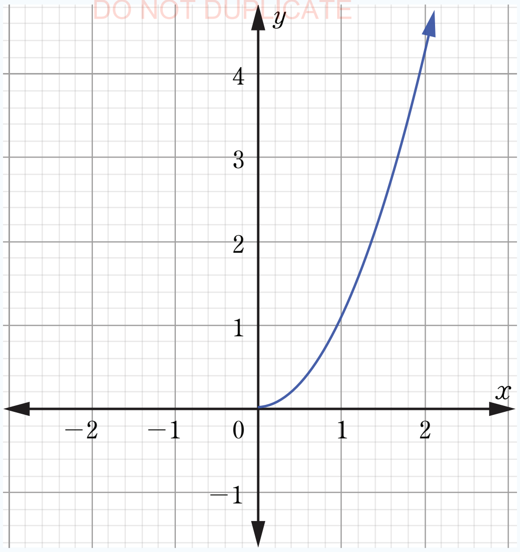
Draw the graph of each of the following functions for some appropriate values of x and determine if it is a one-to-one function:

11. $f(x) = 2x + 3$
12. $f(x) = x - 1$
13. $f(x) = -2$
14. $f(x) = x^3$
15. $f(x) = -x - 1$
16. $f(x) = -x^2$
17. $f(x) = x^2 - x + 1$
18. $f(x) = 3 - x$

Determine which of the following functions are one-to-one:

19. $f = \{(x, y)$: x is a pupil in your school and y is her name$\}$
20. $f = \{(x, y)$: y is the father of $x\}$
21. $f = \{(x, y): y = x + 4\}$
22.



23. 

Domain and range of a function

A function is a special type of a relation, hence it has a domain and range. The domain of a function f is the set of all values for which the function is defined. The range of a function f is the set of all values that the function takes. So, if the function is of the form $y = f(x)$, the domain is the set of all possible values of x, and the range is the set of all possible values of y.

Activity 2.3: Determining the domain and range of functions

In a group or individually, perform the following tasks:

1. Study carefully the following relation formed by pairs of children and mothers: (Jane, Grace), (Asha, Jamila), (Samson, Alice), (Mohamed, Jamila), (Moses, Grace), (Jane, Lilian), (Peter, Alice), (Hussein, Jamila).
2. Represent the given set of ordered pairs using pictorial diagrams.
3. Write a statement which connects the set of ordered pairs.
4. Study the pictorial diagram in task 2 and state the extent to which the relation is realistic.
5. Discuss whether or not the relation is a function.

6. If the relation is not realistic, make an appropriate rearrangement to make the relation realistic and state its domain and range.
7. Use your final work to discuss with the rest of the class on how you have reached at your conclusion.

Consider the function defined by mapping set A onto set B, that is, $f{:}A \rightarrow B$. Set A is called the domain and set B is called the range of the function. For a given domain of a function, there is a corresponding set of values of the range.

Example 2.9

Find the domain and range of the function $f(x) = 5x - 1$.

Solution

Domain: The set of all possible values of x which can give real values of $f(x)$. In this case, it is a set of all real numbers. Thus, domain = $\{x{:}x \in \mathbb{R}\}$.

To find the range of $f(x)$, first, write x in terms of y, that is $x = \frac{1}{5}y + \frac{1}{5}$. It is clear that any value of y from the set of real numbers will give a unique value of x. Thus, the range $= \{y{:}y \in \mathbb{R}\}$.

Example 2.10

Find the domain and range of the function $f(x) = x^2$, for $-4 \leq x \leq 4$.

Solution

Domain = $\{x{:} -4 \leq x \leq 4\}$.

To get the range of the function, let $y = f(x)$, then write x in terms of y, that is, $x = \pm\sqrt{y}$. Since there is no square root of a negative number, we see that for the given domain, the range = $\{y{:}0 \leq y \leq 16\}$.

Therefore, domain = $\{x{:} -4 \leq x \leq 4\}$ and range = $\{y{:}0 \leq y \leq 16\}$.

Example 2.11

Find the domain and range of $f(x) = x^2 + 1$.

Solution

Domain = $\{x : x \in \mathbb{R}\}$

Range: $f(x) = x^2 + 1$. Set $f(x) = y$ which implies that $x^2 = y - 1$, which means $x = \pm\sqrt{y-1}$. Since there is no square root of a negative number, then $y - 1 \geq 0$, giving $y \geq 1$. Hence, the range is $\{y : y \geq 1\}$.

Therefore, domain = $\{x : x \in \mathbb{R}\}$ and range = $\{y : y \geq 1\}$.

Example 2.12

Find the domain and range of $f(x) = x^3$.

Solution

Domain = $\{x : x \in \mathbb{R}\}$. Since $y = x^3$, we get $x = \sqrt[3]{y}$. Hence, range = $\{y : y \in \mathbb{R}\}$.

Exercise 2.3

Determine the domain and range of the following functions defined by:

1. $f(x) = 4x + 7, \; -10 \leq x \leq 10$.
2. $f(x) = \sqrt{x}, \; 0 \leq x \leq 5$.
3. $f(x) = \frac{1}{x}, \; 1 \leq x \leq 2$.
4. $f(x) = \sqrt{x}$
5. $g(x) = x - 1$
6. $h(t) = t^2 + 2t + 3$
7. $f(t) = 2 - t^2$
8. $g(x) = 3 + 2x - x^2$
9. $f(x) = x^3 + 1$
10. $g(x) = -(1 - x^2)$
11. $f(x) = 4$

Graphs of polynomial functions

In Chapter One, you learnt how to draw graphs of relations. The same procedures are used to draw graphs of functions. In this section, you will learn how to draw graphs of linear, quadratic, cubic, and step functions.

Polynomial functions

A polynomial function in x is a function whose expression contains powers of x where the power is a whole number $n \geq 0$. The highest power of x in a polynomial is called the degree of the polynomial. The following are examples of polynomial functions of different degrees:

(a) $f(x) = 4x^3 - 6x^2 + 8x + 2$ is of degree 3.

(b) $f(x) = x^2 + 1$ is of degree 2.

(c) $f(x) = 4$ is of degree 0.

(d) $f(x) = -x - 3$ is of degree 1.

Exercise 2.4

In questions 1 to 11, state the functions which are polynomials.

1. $f(x) = \frac{3}{x^3} + \frac{3}{x} + 4$
2. $f(x) = \frac{3}{2}$
3. $f(x) = x^{-4}$
4. $f(x) = x$
5. $f(y) = \frac{3}{y} + \frac{3}{y^2}$
6. $f(y) = 6 + y$
7. $f(a) = a + 2a^2 + 5a^3$
8. $f(x) = \frac{x}{5}$
9. $f(y) = 0$
10. $f(a) = 0.5 + 3a + a^2$
11. $f(x) = x^3 + 2x^{-2} - 1$

In questions 12 to 16, which expressions are polynomials?

12. $6x^2 + 5x + 2$
13. $\frac{(x-5)(x-1)}{2}$
14. $\frac{(x-3)(x-1)}{(x-1)}$
15. $\frac{(x+5)^2}{x-5}$
16. $\sqrt{a^2 + 3}$

What is the degree of each of the polynomials in questions 17 to 19?

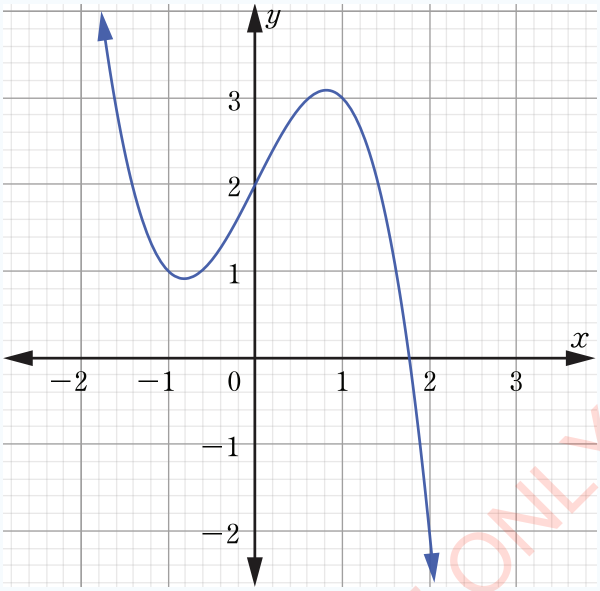
17. $1+5-4x^2+2x^3$

18. $4+2x^2-x^3$

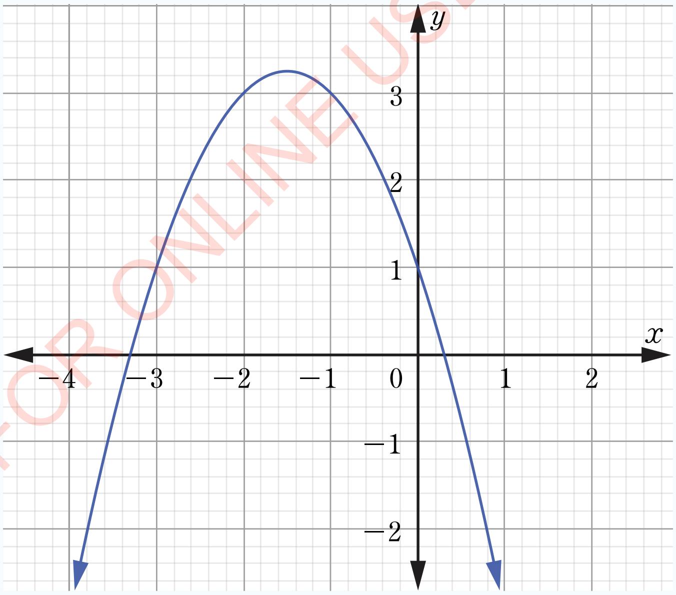
19. $1-9x+5x^2$

What is the degree of each of the polynomials represented by the following graphs?

20.



21.



Linear functions

A linear function f is a function of the form $f(x) = mx + c$, where m and c are real numbers. In a linear function, m represents the gradient or slope and c the y –intercept.

When the slope m is zero, the function simplifies to $f(x) = c$, which is called a constant function and its line graph is horizontal or parallel to the x-axis. For example, $f(x) = 3$ and $f(x) = -5$ represent the lines $y = 3$ and $y = -5$, respectively. When m is positive or negative number, the line graph is inclined.

The graph of a linear function is a straight line $y = mx + c$ as shown in Figure 2.11 for positive m (line 2), negative m (line 1) and m=0 (line 3).

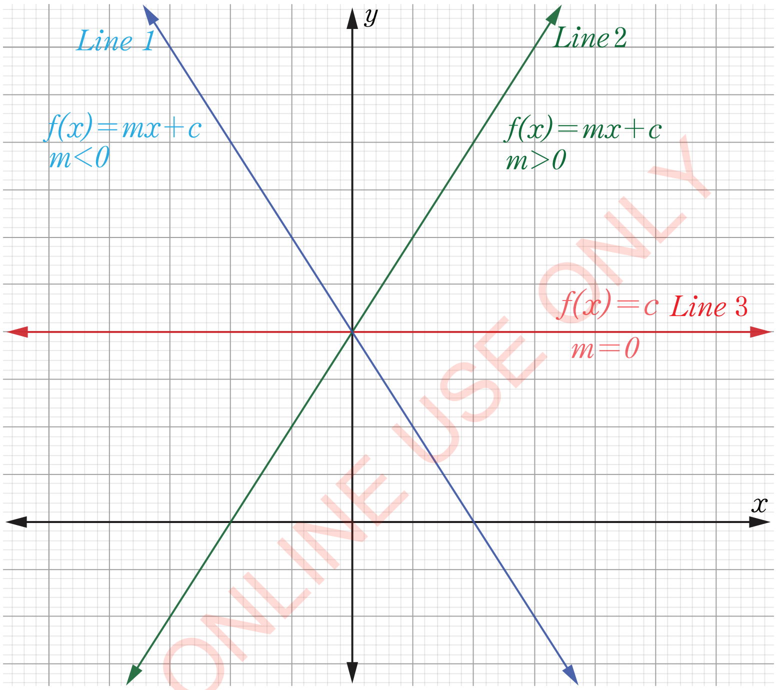


Figure 2.11: *Graphs of $f(x) = mx + c$ with gradient $m < 0$, $m = 0$, and $m > 0$*

Example 2.13

Find a linear function $y = f(x)$ with a gradient –2 and $f(1) = 3$, and then, draw its graph.

Solution

From the given information, draw a graph with gradient $m = -2$ which passes through the point (1, 3). The equation of a line passing through a point (x_1, y_1) with a gradient m can be found by using the point and its gradient as follows:

gradient $(m) = \dfrac{\text{chage in } y}{\text{change in } x} = \dfrac{y - y_1}{x - x_1}$, multiplying both sides by $x - x_1$ gives

$$y - y_1 = m(x - x_1) \text{ or } y = m(x - x_1) + y_1.$$

Setting $m = -2$, $x_1 = 1$ and $y_1 = 3$, we have $y = -2(x-1)+3$ which gives $y = -2x + 5$. Since $y = f(x)$, it follows that $f(x) = -2x + 5$. The graph of this function is shown in Figure 2.12.

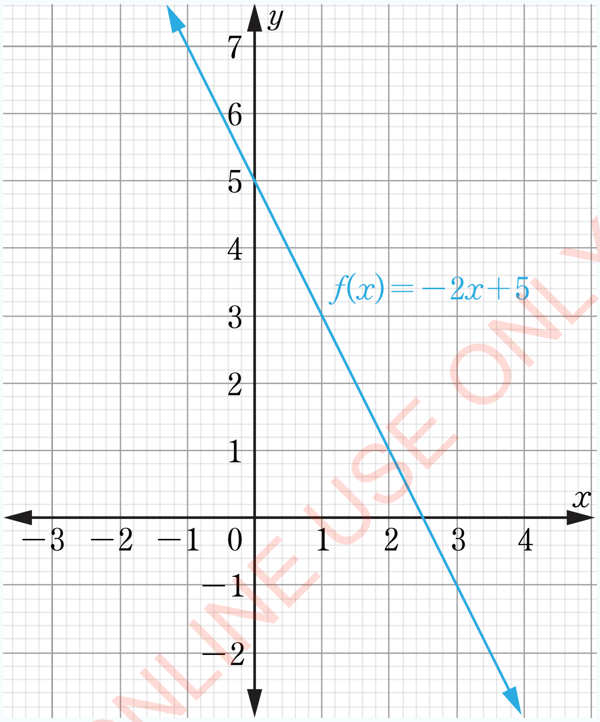


Figure 2.12: *Graph of* $f(x) = -2x + 5$

In general, the equation of a linear function with gradient m passing through the point (x_0, y_0) is given by the equation $f(x) = m(x - x_0) + y_0$.

Example 2.14

Find a linear function with gradient -1 which passes through the point (1, 2).

Solution

$f(x) = m(x - x_0) + y_0$, where $x_0 = 1$, $y_0 = 2$ and $m = -1$

$f(x) = -1(x - 1) + 2$

$= -x + 3$

$f(x) = -x + 3$

Remark: The graph of a straight line can be drawn if any two points are known. When an equation is given, the two points on a graph can be identified by finding the x and y–intercepts.

Example 2.15

Draw the graph of the function $f(x) = 2x + 1$.

Solution

The equation has the form $f(x) = mx + c$ with gradient $m = 2$ and 1 as y–intercept, $c = f(0) = 1$. Therefore, the graph passes through the point (0, 1). The x-intercept is obtained from setting $f(x) = 0$, that is $2x + 1 = 0$, which when solved gives $x = -\frac{1}{2}$. Thus, the graph also passes through the point $\left(-\frac{1}{2}, 0\right)$ as shown in Figure 2.13.

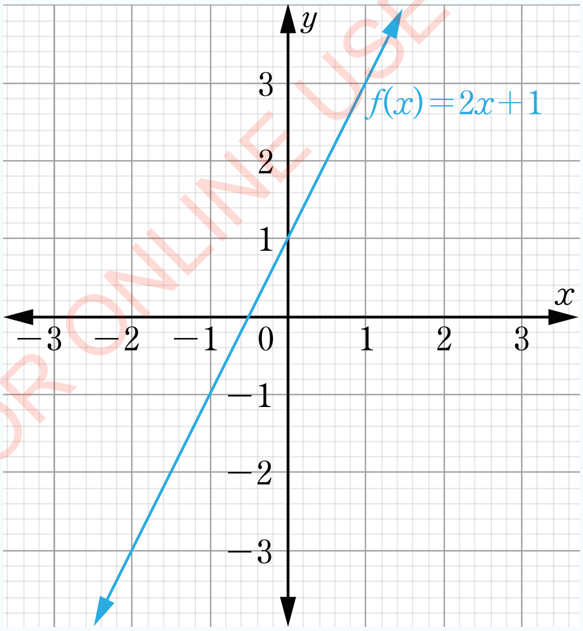


Figure 2.13: *Graph of* $f(x) = 2x + 1$

Example 2.16

Draw the graph of $f(x) = x + 3$ and find its domain and range.

Solution

Linear equations are easily drawn by using x and y-intercepts as shown in Table 2.3.

Table 2.3: x and y-intercepts of $f(x) = x + 3$

x	0	−3
y	3	0

The graph of the function $f(x) = x + 3$ is shown in Figure 2.14.

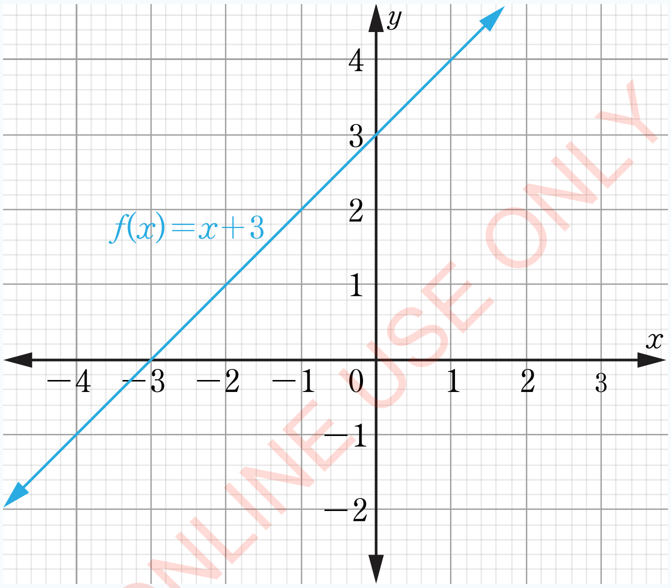


Figure 2.14: *The graph of* $f(x) = x + 3$

From the graph in Figure 2.14, it can be observed that the graph contains all the values of x and y as it extends in both sides. Hence, the domain and range is the set of all real numbers.

Exercise 2.5

In questions 1 to 6, find the equation of a linear function $f(x)$ which satisfies the given values. In each case, m denotes the gradient:

1. $m = -3,\ f(1) = 3$
2. $m = 2,\ f(0) = 5$
3. $m = -2,\ f(0) = 0$
4. $f(1) = 2,\ f(-2) = 3$
5. $m = 4,\ f(0) = 8$
6. $m = 0$, y - intercept is 2

In questions 7 to 11, draw the graph of each of the given functions without using table of values:

7. $f(x) = \frac{7}{4} - x$
8. $f(x) = 5 - x$
9. $f(x) = \frac{1}{2}x$
10. $f(x) = 5 + 2x$
11. $f(x) = 4$

Sketching quadratic functions

A quadratic function is a function of the form $f(x) = ax^2 + bx + c$, where $a \neq 0$. Some examples of quadratic functions are $f(x) = 3x^2 + 2x + 1$, $f(x) = x^2 + 5$, and $f(x) = x^2$. A sketch of a graph of a function represents a rough shape of a graph which includes only the key features while a plot of a graph of a function represents an accurate shape of a graph.

The graph of a quadratic function must look similar to one of the graphs in Figure 2.15 and Figure 2.16. Notice that, a graph of a quadratic function has either a minimum or maximum point called the vertex. We can find this point by the method of completing the square.

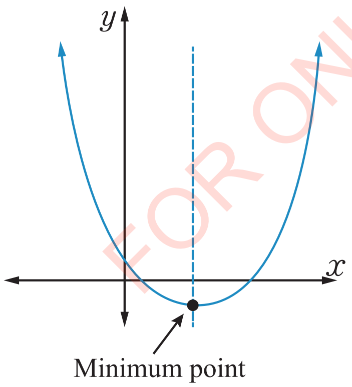


Figure 2.15: *Graph of a quadratic function opening upwards*

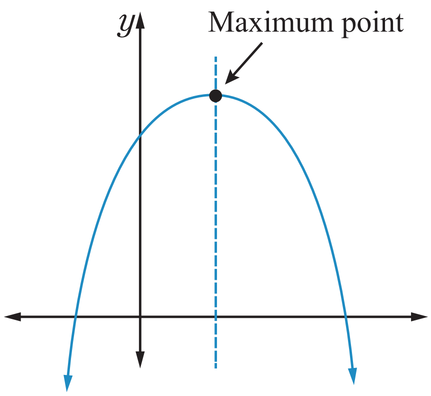


Figure 2.16: *Graph of a quadratic function opening downwards*

Note also that the graphs of quadratic functions are symmetric about the vertical line through the minimum or maximum point. This is the line of symmetry of the graph. The curve of a graph of any quadratic function is called a parabola.

Properties of quadratic functions

Consider the function $f(x) = ax^2 + bx + c$, where a, b and c are constants. By completing the square, the function can be written as:

$$y = ax^2 + bx + c,$$

$$\frac{y}{a} = x^2 + \frac{b}{a}x + \frac{c}{a}$$

$$\frac{y}{a} - \frac{c}{a} = x^2 + \frac{b}{a}x$$

$$\frac{y}{a} - \frac{c}{a} + \left(\frac{b}{2a}\right)^2 = x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2$$

$$\frac{y}{a} - \frac{c}{a} + \frac{b^2}{4a^2} = \left(x + \frac{b}{2a}\right)^2$$

$$y - c + \frac{b^2}{4a} = a\left(x + \frac{b}{2a}\right)^2$$

$$y = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

$$= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

From which we get $y - \left(\frac{4ac - b^2}{4a}\right) = a\left(x + \frac{b}{2a}\right)^2$

Thus if $a > 0$, then $a\left(x + \frac{b}{2a}\right)^2 \geq 0$, hence $y \geq \frac{4ac - b^2}{4a}$. This means that, the function has a minimum value of $y = \frac{4ac - b^2}{4a}$, which is attained when $x + \frac{b}{2a} = 0$, that is, at $x = -\frac{b}{2a}$, in this case, the graph opens upwards.

Similarly, if $a < 0$, then $a\left(x + \frac{b}{2a}\right)^2 \leq 0$, hence $y \leq \frac{4ac - b^2}{4a}$. This means that, the function has a maximum value of $y = \frac{4ac - b^2}{4a}$, which is attained when $x + \frac{b}{2a} = 0$, that is, at $x = -\frac{b}{2a}$, in this case, the graph opens downwards.

The line given by $x = -\frac{b}{2a}$ is the axis of symmetry of the graph of a quadratic function. The point at which the quadratic function attains its maximum or minimum value is called the turning point of the function and given by;

$$\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$$

where $x = -\frac{b}{2a}$ is the axis of symmetry and $y = \frac{4ac - b^2}{4a}$ is either the maximum or the minimum value of the quadratic function.

Remarks: The axis of symmetry, maximum and minimum values, and the x and y–intercepts are important in sketching graphs of quadratic functions.

Example 2.17

Find the turning point of $f(x) = x^2 - 2x - 3$ and sketch its graph. Hence, on the graph indicate its turning point and the axis of symmetry.

Solution

Coordinates of turning point $= \left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$

From $f(x) = x^2 - 2x - 3$, $a = 1$, $b = -2$ and $c = -3$

$$\text{Turning point} = \left(-\frac{-2}{2(1)}, \frac{4(1)(-3) - (-2)^2}{4(1)}\right)$$
$$= \left(1, \frac{-12 - 4}{4}\right)$$
$$= (1, -4)$$

Therefore, the turning point is $(1, -4)$ which is a minimum point, since $a = 1$, which implies that $a > 0$. Hence the graph of $f(x) = x^2 - 2x - 3$ opens upwards.

The values of x and y–intercepts are obtained by setting $x = 0$ and $y = 0$, respectively. The values of x and y–intercepts are shown in Table 2.4. Since this is a quadratic function, there are two values of the x – intercept.

Table 2.4: The x and y – intrcepts of $f(x) = x^2 - 2x - 3$

x	-1	0	3
y	0	-3	0

Using the turning point, the intercepts, and the fact that the graph opens upward the sketch of the graph of $f(x) = x^2 - 2x - 3$ is as shown in Figure 2.17.

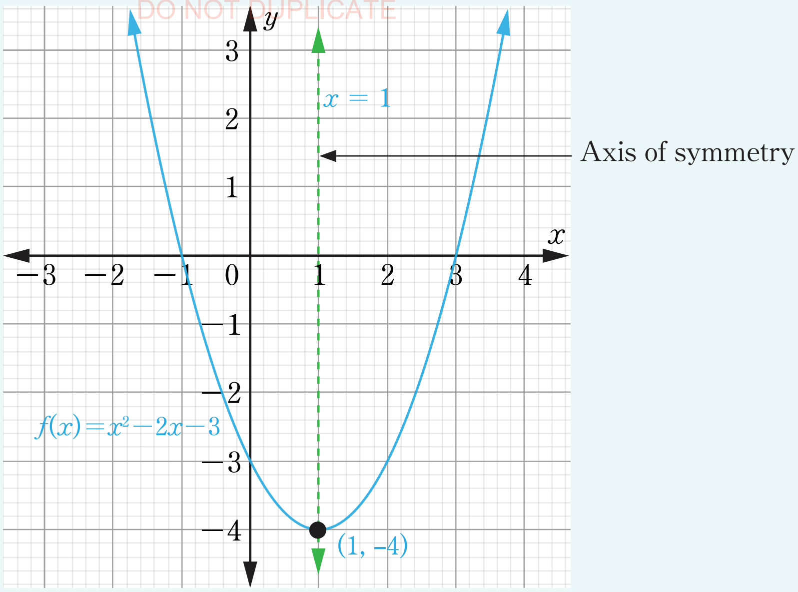


Figure 2.17: *Graph of* $f(x) = x^2 - 2x - 3$

Therefore, it can be concluded that, a quadratic function in the form $f(x) = ax^2 + bx + c$ has a minimum value if $a > 0$ and a maximum value if $a < 0$. In other words, the graph opens upwards if $a > 0$ and downwards if $a < 0$.

Example 2.18

Find the domain and range of the function $f(x) = x^2 - 4x + 5$.

Solution

Domain = {x: x is all real numbers} since any real value of x when substituted in the quadratic equation gives real value of y.

In order to find the range of this function, it is important to understand whether the function has a maximum or a minimum value. Since the value of a is 1, which is greater than 0, then the graph has a minimum value. Hence,

$$\text{Range} = \left\{y : y \geq \frac{4ac - b^2}{4a}\right\}$$

$$= \left\{y : y \geq \frac{4(1)(5) - (-4)^2}{4(1)}\right\}$$

$$= \{y : y \geq 1\}.$$

Alternatively, the range is found by completing the square as follows:

$$y = x^2 - 4x + 5$$
$$y = x^2 - 4x + 4 + 5 - 4$$
$$y = (x-2)^2 + 5 - 4$$
$$y = (x-2)^2 + 1$$

Thus the domain is a set of all real numbers. Since the curve opens upwards, the range is $\{y : y \geq 1\}$.

Example 2.19

Without using a table of values, draw the graph of $f(x) = -x^2 + 4x - 5$ and use it to solve the equation $-x^2 + 4x - 5 = -10$.

Solution

$$y = -(x^2 - 4x) - 5$$
$$= -(x^2 - 4x + 4) - 5 + 4$$
$$\therefore y = -(x-2)^2 - 1$$

Thus, the maximum point is (2,−1) and the axis of symmetry of the quadratic function is $x = 2$. In order to find the point(s) at which the graph crosses the y-axis, set $x = 0$ in the function to get $y = -5$. Since the value of a is negative, the graph opens downwards. Also, since the turning point is below the $x-$axis, the graph does not cross the x-axis as shown in Figure 2.18.

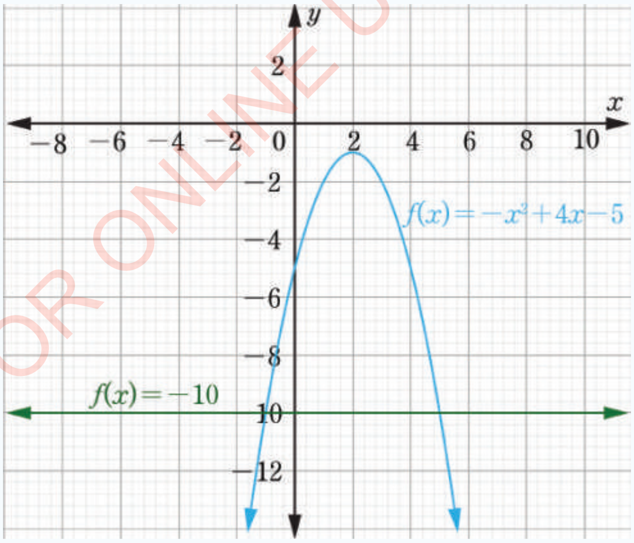


Figure 2.18: *Graph of* $f(x) = -x^2 + 4x - 5$

To solve $-x^2 + 4x - 5 = -10$, draw the line represented by $y = -10$ on the same graph as shown in Figure 2.18. Read the x-coordinates which represent the intersection of the graph and the straight line. The solutions to the equation $-x^2 + 4x - 5 = -10$ are therefore $x = -1$ and $x = 5$.

Example 2.20

Find the maximum or minimum value of the function $f(x) = 4 - 3x - 2x^2$.

Solution

Here $a = -2$, $b = -3$, and $c = 4$. Since the coefficient of x^2 is negative, the function has a maximum value at the turning point. Thus the maximum value is given by

$$y = \frac{4ac - b^2}{4a}$$

$$y = \frac{4(-2)(4) - (-3)^2}{4(-2)} = \frac{41}{8}$$

An alternative way of finding the maximum value is by completing the square:

$$y = -2x^2 - 3x + 4$$

$$x \text{ and } y = -2\left(x^2 + \frac{3}{2}x\right) + 4$$

$$y = -2\left(x + \frac{3}{4}\right)^2 + 4 + \frac{9}{8}$$

$$y = -2\left(x + \frac{3}{4}\right) + \frac{41}{8}$$

Therefore, the maximum value is $\frac{41}{8}$.

Exercise 2.6

1. Draw the graph of the function $f(x) = x^2 - 6x + 5$. Find the minimum value of this function and the corresponding value of x.
2. Draw the graph of the function $y = x^2 - 4x + 2$. Find the minimum value of the function and the corresponding value of x. Use the curve to solve the following equations:
   (a) $x^2 - 4x - 2 = 0$
   (b) $x^2 - 4x + 3 = 0$

In questions 3 to 5, write the functions in the form of $y = a(x+b)^2 + c$, where a, b, and c are constants.

3. $y = x^2 + 8x + 5$
4. $y = 3x^2 + 8x - 1$
5. $y = 5 - 6x - 9x^2$

In questions 6 to 9, find the maximum or minimum value and the axis of symmetry:

6. $y = x^2 - 8x + 18$
7. $y = 1 + 6x - 3x^2$
8. $y = 2x^2 + 3x + 1$
9. $y = 2 - x - x^2$
10. Draw the graph of the function $f(x) = 2x - x^2$, then find its maximum or minimum value and the axis of symmetry.

Sketching cubic functions

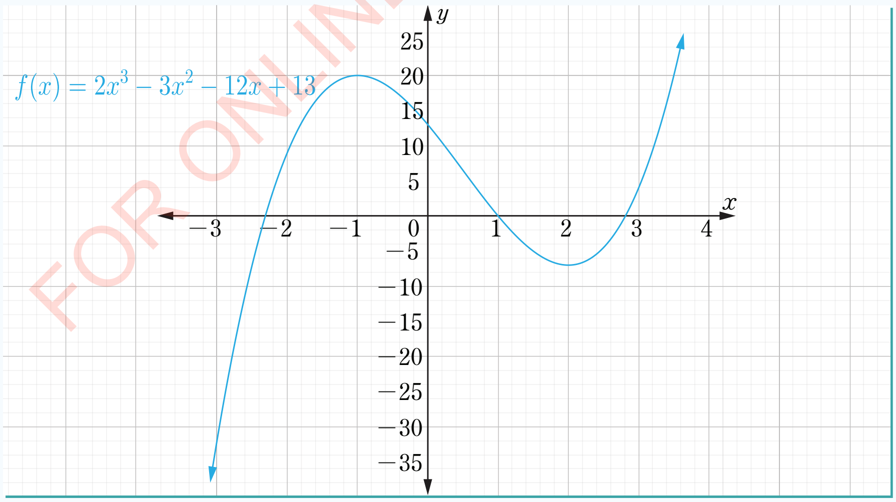
A third degree polynomial function $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$, is called a cubic function. Just like in quadratic functions, sketching a graph of a cubic equation requires an understanding of its properties.

A graph of a function normally provides useful information about the function as compared to an equation. For example, through the graph, one can study its basic properties as you will learn in Activity 2.4.

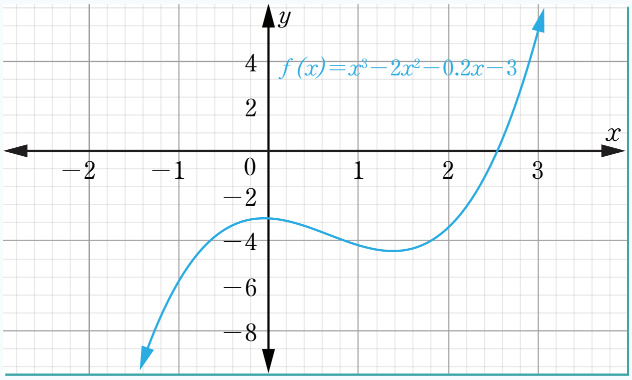
Activity 2.4: Identifying the features of a cubic function

Study carefully each of the following graphs of cubic functions and peform the tasks that follow:

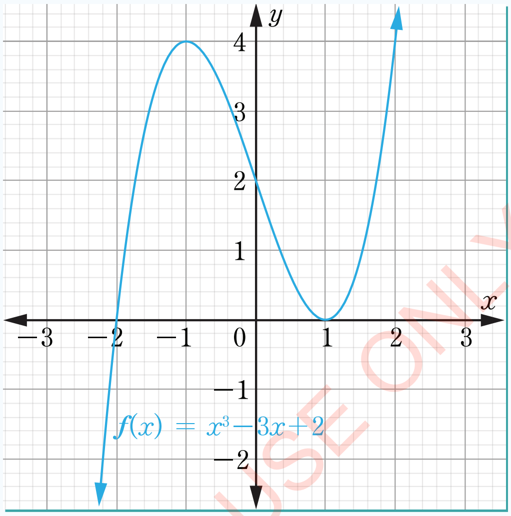
(a)



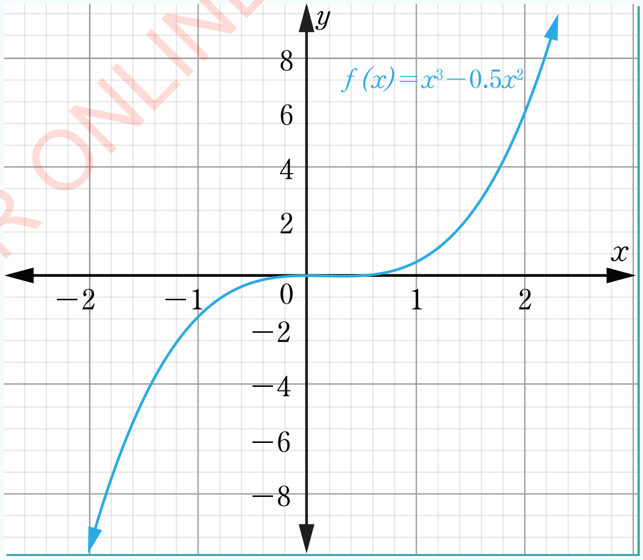
(b)


y
x
$f(x)=x^3-2x^2-0.2x-3$
4
2
0
−2
−4
−6
−8
−2
−1
1
2
3

(c)


y
x
$f(x) = x^3-3x+2$
4
3
2
1
0
−1
−2
−3
−2
−1
1
2
3

(d)


y
x
$f(x)=x^3-0.5x^2$
8
6
4
2
0
−2
−4
−6
−8
−2
−1
1
2

1. List down as many features as possible that you observe from the graphs.
2. Among the features, summarize the features which are common to all the graphs. If some graphs have unique features, indicate them.
3. Write down any cubic equation and use the features to sketch its graph.
4. Prepare and present a summary of the key features in the class. Display your sketch of the graph in the class using manila sheets.
5. Use the feedback from other students to improve your work.

Properties of cubic functions

In Activity 2.4, you may have discovered that the graph of a polynomial function crosses the y-axis at only one point. This point can be determined by setting $x = 0$ in the polynomial. You may also have discovered that the graph of a cubic fuction crosses the x–axis at least once and at most three times.

In general, a graph of a cubic function has two turning points. One turning point opens upwards and another opens downwards. This implies that, one end of the graph goes upward and the other end goes downwards.

Therefore, in sketching cubic functions, the following basic features may be considered:

1. Finding a few points on the graph to determine the position of the turning points as well as the direction of the graph.
2. Finding the y–intercept.
3. Finding the values of x–intercept(s) by solving $f(x) = 0$. This is easily determined if the cubic function can be factorized.
4. In general, a polynomial of degree n cannot cross the x-axis more than n times. In some cases, the graph of a cubic function crosses the x-axis less than 3 times.

Example 2.21

Which of the following functions are polynomial functions?

(a) $f(x) = 2x^3 + x^2 - 7x - 9$

(b) $f(x) = \dfrac{1}{6+x}$

Solution

(a) $f(x) = 2x^3 + x^2 - 7x - 9$ is a polynomial function of degree three with coefficients $a_3 = 2$, $a_2 = 1$, $a_1 = -7$ and $a_0 = -9$.

(b) $f(x) = \dfrac{1}{6+x}$ is not a polynomial function because it is a rational function.

Example 2.22

Sketch the graph of the function $f(x) = x^3 - 6x^2 + 11x - 6$.

Solution

Let $y = f(x)$

$y = x^3 - 6x^2 + 11x - 6$

Find the x and y–intercepts:

To find the y– intercept, set $x = 0$, this gives y– intercept $= -6$.

The x–intercepts are obtained by setting $y = 0$.

Thus, $y = x^3 - 6x^2 + 11x - 6$ gives $x^3 - 6x^2 + 11x - 6 = 0$.

The values of x can be obtained by using the factors of a constant term, in this case, the constant term is 6. Hence, the factors are ± 1, ± 2 ± 3 and ± 6.

Substitute each of the factors into the equation to determine the factors which satisfy the equation $f(x) = 0$. It can be observed that the factors 1, 2, and 3 satisfy the equation. Therefore, the graph will cross the x–axis at points (1, 0), (2, 0), and (3, 0).

Then, choose some values which are on the left and right of the x–intercepts to determine the direction of the graph. Some of these points are shown in Table 2.5.

Table 2.5: Selected values of the function $f(x) = x^3 - 6x^2 + 11x - 6$

x	-1	1.5	2.5	4
y	-24	0.4	-0.4	6

From Table 2.5, you can notice that the point $(-1, -24)$ indicates that the graph extends down in the third quadrant, (the region between negative x – axis and negative y–axis. The point $(1.5, 0.4)$ indicates that part of the graph should be in the first quadrant (the region between positive x–axis and positive y–axis), the point $(2.5, -0.4)$ is in the fourth quadrant (the region between negative y–axis and positive x–axis), and (4,6) is in the first quadrant. So, the graph starts from the first quadrant, turns into the fourth quadrant and goes to the first quadrant again and turns so that it crosses the y –axis at -6 and extends to negative infinity. Thus, the sketch of the graph is as shown in Figure 2.19.

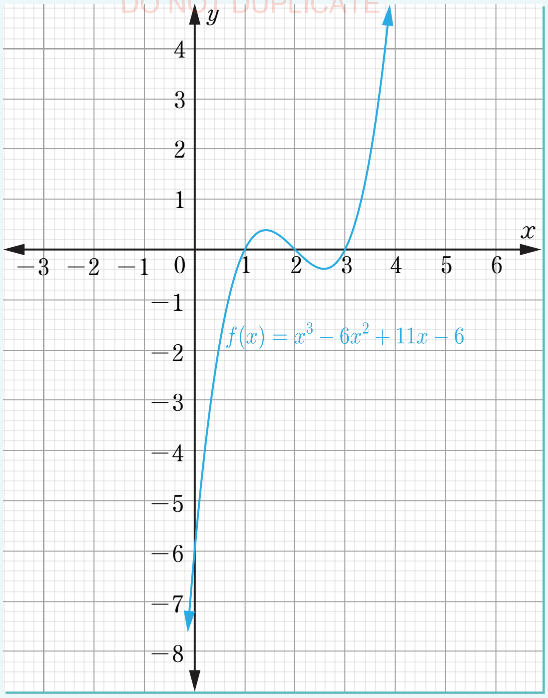


Figure 2.19: *The sketch of the cubic function* $f(x) = x^3 - 6x^2 + 11x - 6$

Example 2.23

Draw the graph of the polynomial function $f(x) = 2x^3 - 3x^2 - 12x + 13$

Solution

Let $y = 2x^3 - 3x^2 - 12x + 13$.

Find the x and y–intercepts:

To find y–intercept, set $x = 0$ to get $y = 13$. To obtain the x–intecepts, set $y = 0$ and solve for x. List all the factors of 13 which are ± 1 and ± 13. The only factor which gives $y = 0$ is 1.

Choose some values which are on the right handside and left handside of 1 to determine the direction of the graph.

Table 2.6: Table of values of $f(x) = 2x^3 - 3x^2 - 12x + 13$

x	−3	−2	−1	2	3	3.5
y	−32	9	20	−7	4	20

The point $(-3, -32)$ shows that the graph passes through the third quadrant. The points $(-2, 9)$ and $(-1, 20)$ are in the second quadrant. This shows that the turning point is above the x–axis. The points $(2, -7)$ and $(3, 4)$ show that the second turning point is below the x–axis in the fourth quadrant. Thus, the sketch of the graph is as shown in Figure 2.20.

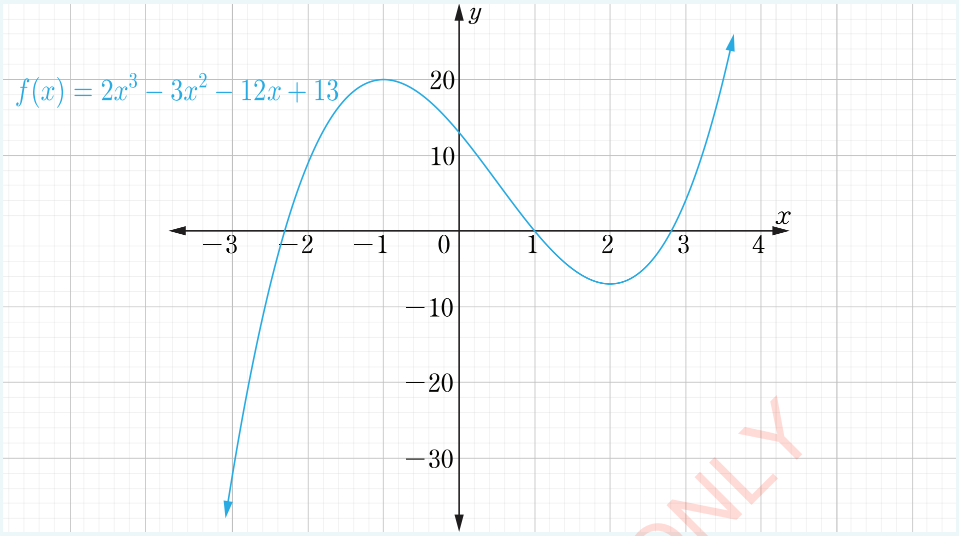


Figure 2.20: *Graph of* $f(x) = 2x^3 - 3x^2 - 12x + 13$

Exercise 2.7

Sketch the graph of each of the following polynomials:

1. $p(x) = -4 - x^3$
2. $p(x) = 4x^3 - 12x^2 + 3$
3. $p(x) = x^3 - x^2 + 2x$
4. $p(x) = x^3 - 2x^2 - 2$
5. $p(x) = 1 + x + x^3$
6. $p(x) = -x^3 - 8$

Step functions

It is not necessary that x and y be related in a linear or polynomial form in order for a functional relationship to exist between them. For example, if y is the cost in Tanzanian shillings of posting an air mailing letter and x is the mass of the letter in grams, then y is a function of x (see Table 2.7).

Table 2.7: The range of mass of letters against the cost of posting the letters

x (mass in grams)	$0 < x \leq 1$	$1 < x \leq 2$	$2 < x \leq 3$	$3 < x \leq 4$	$4 < x \leq 5$
y (cost in Tsh)	150	200	250	300	350

The graph of this relationship is illustrated in Figure 2.21.

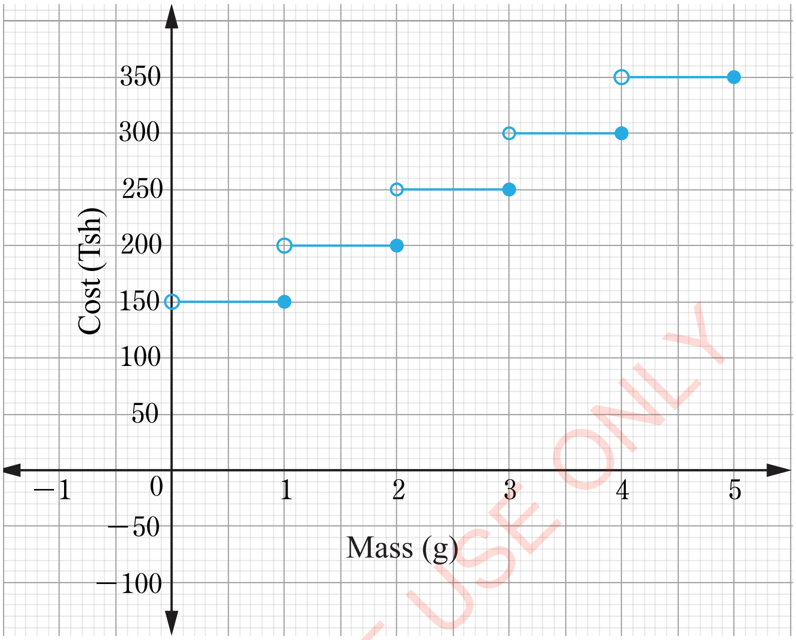


Figure 2.21: *Relationship between mass of letters and cost of postage*

The property of this function is that it is subjected to jumps. Such a function is called a step function. We use a small open circle to show that the end points are not included and a small solid dot to show that they are included. In this example, the domain is a set of positive real numbers such that $0 < x \leq 5$. The range is {150, 200, 250, 300, 350}.

Example 2.24

Let $f(x)$ be defined as follows:

$$f(x) = \begin{cases} -2 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 2 & \text{if } x > 0 \end{cases}$$

(a) Find:

(i) $f(-8)$

(ii) $f(7)$

(b) Sketch the graph of $f(x)$.

(c) Find the range of $f(x)$.

Solution

(a) (i) When $x = -8$, then $x < 0$, hence, the first part of the definition of the function applies, so that $f(-8) = -2$.

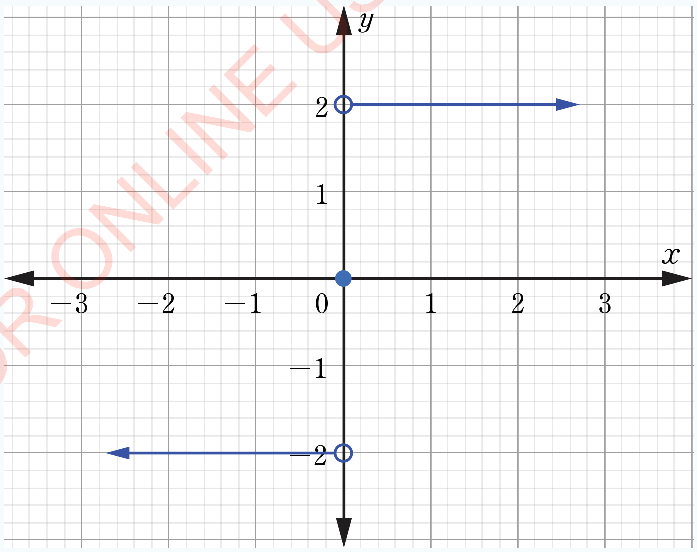
(ii) When $x = 7$, then $x > 0$, hence the third part of the definition of the function applies, so that $f(7) = 2$.

Therefore, $f(-8) = -2$ and $f(7) = 2$.

(b) (i) When $x < 0$, $f(x) = -2$. So, draw a horizontal line for all the negative values of x at $y = -2$ starting at $x = 0$ with an open circle.

(ii) $f(0) = 0$. So, put a solid dot at (0, 0).

(iii) When $x > 0$, $f(x) = 2$, draw a horizontal line for all the positive values of x at $y = 2$ starting at $x = 0$ with open circle. The result is as shown in the following figure.



(c) The function takes the values –2, 0 and 2. Therefore, the range is {–2, 0, 2}.

Example 2.25

Given a function f defined as

$$f(x) = \begin{cases} -3 & \text{if} \quad x \leq 1 \\ 1 & \text{if} \quad 1 < x \leq 2 \\ 4 & \text{if} \quad 2 < x \end{cases}$$

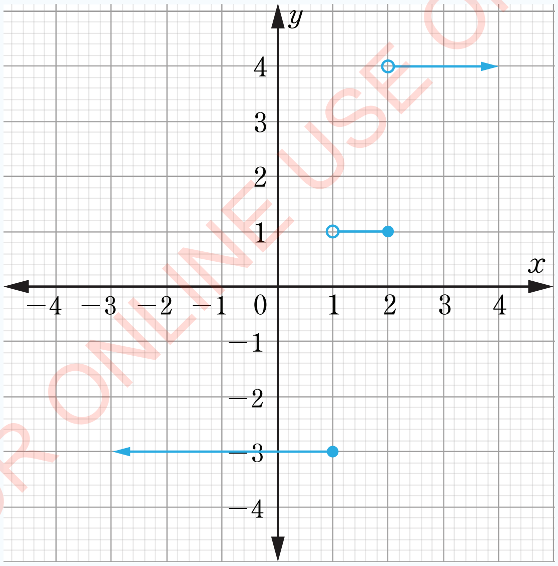
Draw the graph of f and hence state its domain and range.

Solution

Let $y = f(x)$. For $y = -3$, draw a horizontal arrow with a solid dot starting from 1 towards the negative x values (since $x \leq 1$).

For $y = 1$, draw a horizontal line segment at $y = 1$ which starts from 1 to 2. At $x = 1$, the line segment should have an open circle since $x = 1$ is not included $(1 < x)$ and the other end at $x = 2$, should have a solid dot since 2 is included $(x \leq 2)$.

For $y = 4$, draw a horizontal arrow at $y = 4$ starting at $x = 2$ towards the right since $x > 2$ with an open circle at $x = 2$. The graph of the function is as shown in the following figure.



Domain = $\{x : x \in \mathbb{R}\}$

Range = $\{-3, 1, 4\}$

Greatest and least integer function

Greatest and least integer function is a form of step function. The symbol $\lfloor x \rfloor$ is used to denote the greatest integer which is less than or equal to x. The function f defined by $f(x) = \lfloor x \rfloor$ is called the greatest integer function. It is also termed as the floor function.

The symbol $\lceil x \rceil$ is used to denote the least integer which is greater than or equal to x. The function f defined by $f(x) = \lceil x \rceil$ is called the least integer function. It is also known as the ceiling function. Table 2.8 shows examples of outputs of different values substituted to the greatest and least integer functions.

Table 2.8: Solutions of the greatest and least integer functions

x	$f(x) = \lfloor x \rfloor$	$f(x) = \lceil x \rceil$
1	1	1
1.3	1	2
0.5	0	1
4.2	4	5
9.8	9	10

Example 2.26

Suppose that a function F is defined by $F(x) = n$, where $n \leq x < n+1$ and n is an integer such that $-3 \leq x < 2$. Sketch the graph of F and hence state its domain and range.

Solution

Table 2.9: Table of values of $F(x) = n$

x	$F(x)$
$-3 \leq x < -2$	-3
$-2 \leq x < -1$	-2
$-1 \leq x < 0$	-1
$0 \leq x < 1$	0
$1 \leq x < 2$	1

The graph of the function $F(x) = n$ is shown in Figure 2.21.

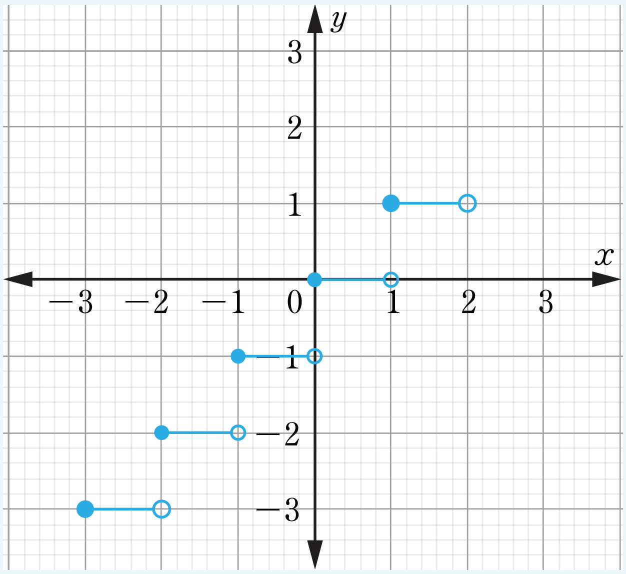


Figure 2.21: *Graph of* $F(x) = n$

Domain $= \{x: -3 \le x < 2\}$

Range $= \{-3, -2, -1, 0, 1\}$.

Example 2.27

If G is a function defined by $G(x) = \lfloor x \rfloor - x$, where $n \le x < n+1$, draw the graph of G and then state its domain and range.

Solution

If $-3 \le x < -2$, then $\lfloor x \rfloor = -3$; thus $G(x) = -3 - x$

If $-2 \le x < -1$, then $\lfloor x \rfloor = -2$; thus $G(x) = -2 - x$

If $-1 \le x < 0$, then $\lfloor x \rfloor = -1$; thus $G(x) = -1 - x$

If $0 \le x < 1$, then $\lfloor x \rfloor = 0$; thus $G(x) = -x$

If $1 \le x < 2$, then $\lfloor x \rfloor = 1$; thus $G(x) = 1 - x$

If $2 \le x < 3$, then $\lfloor x \rfloor = 2$; thus $G(x) = 2 - x$

The output of $G(x) = \lfloor x \rfloor - x$ is summarized in Table 2.10 and its graph is as shown in Figure 2.23.

Table 2.10: Table of values of $G(x) = \lfloor x \rfloor - x$

x	$G(x)$
$-3 \leq x < -2$	$-3-x$
$-2 \leq x < -1$	$-2-x$
$-1 \leq x < 0$	$-1-x$
$0 \leq x < 1$	$-x$
$1 \leq x < 2$	$1-x$
$2 \leq x < 3$	$2-x$

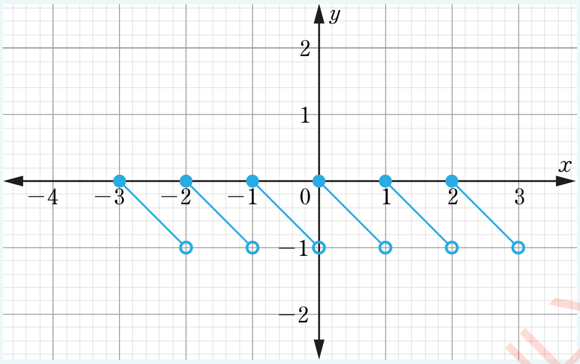


Figure 2.23: *The graph of* $G(x) = \lfloor x \rfloor - x$

Domain = $\{x : x \in \mathbb{R}\}$ and Range = $\{y : -1 < y \leq 0\}$.

Absolute value functions

Before defining the absolute value function, let us review the definition of the absolute value of a number. If x is any real number, its absolute value is written as $|x|$ which means that it is x if $x \geq 0$ and $-x$ if $x < 0$.

For instance, the absolute value of 6, written as $|6|$ is 6 because 6 is greater than zero. The absolute value of -8, written as $|-8|$ is 8 since $|-8| = -(-8)$ from the definition of an absolute value of a negative number, that is $|-x| = -(-x)$.

The absolute value function applies the same concept as the absolute value of a number. It is written as $y = |x|$ and is defined as

$$y = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

From the definition of the absolute value function, its graph can be plotted as shown in Figure 2.24.

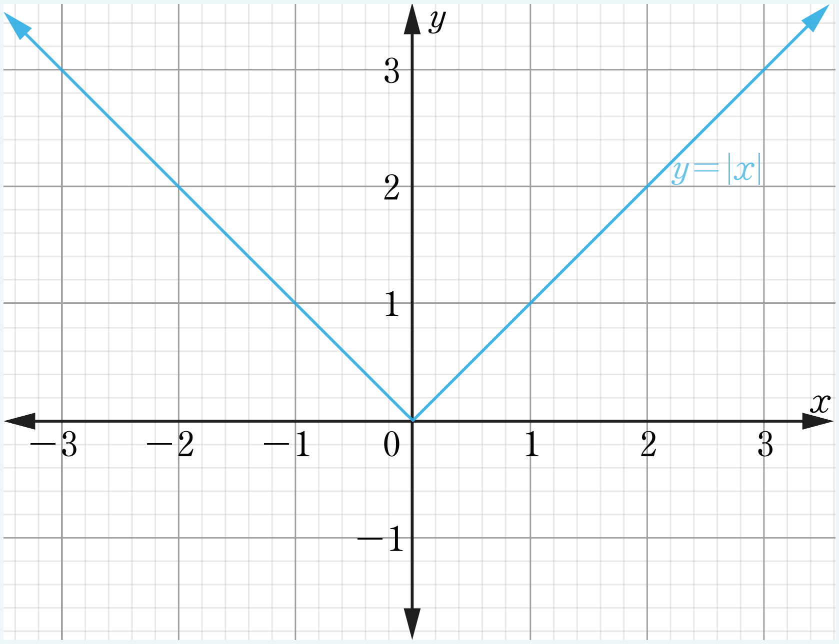


Figure 2.24: *Graph of* $y = |x|$

Example 2.28

Sketch the graph of $y = |x - 3|$ and give its domain and range.

Solution

Table 2.11: Table of values for $y = |x - 3|$

x	−2	−1	0	1	2	3	4	5	6
y	5	4	3	2	1	0	1	2	3

The graph of $y = |x - 3|$ is shown in Figure 2.25.

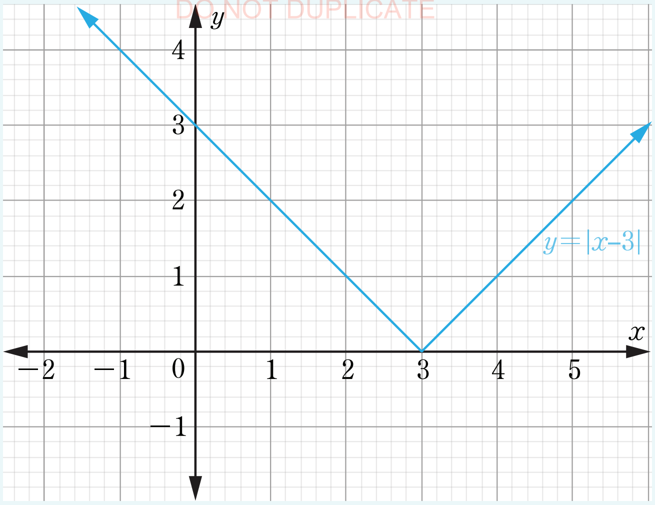


Figure 2.25: *The graph of* $y = |x - 3|$

Domain= $\{x: x \in \mathbb{R}\}$ and range = $\{y: y \geq 0\}$.

Example 2.29

Sketch the graph of $y = |x| + 1$ and give its domain and range.

Solution

Table 2.12: Table of values of $y = |x| + 1$

x	−3	−2	−1	0	1	2	3
y	4	3	2	1	2	3	4

The graph of $y = |x| + 1$ is presented in Figure 2.26.

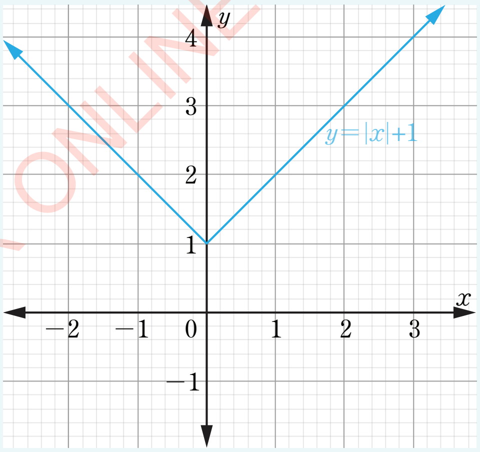


Figure 2.26: *The graph of* $y = |x| + 1$

Domain= $\{x: x \in \mathbb{R}\}$ and range = $\{y: y \geq 1\}$

Exercise 2.8

1. Find the value of :

   (b) $\left\lfloor \frac{23}{4} \right\rfloor$ (b) $\lceil 0.1 \rceil$ (c) $\lceil -1.01 \rceil$ (d) $\lfloor 102 \rfloor$

In questions 2 to 8, sketch graphs of the functions:

2. $f(x) = 2n$ for $n-1 \leq x < n$, n is a positive integer.
3. $f(x) = 3n$ for $n-1 \leq x < n$, n is a positive integer.
4. $f(x) = \begin{cases} -1 \text{ if } x < 0 \\ 0 \text{ if } x = 0 \\ 1 \text{ if } x > 0 \end{cases}$
5. $g(x) = x\lfloor x \rfloor$ for $n \leq x < n+1$, where n is an integer.
6. $u(x) = \begin{cases} 0 \text{ if } x < 0 \\ 1 \text{ if } x \geq 0 \\ -1 \text{ if } x < 0 \end{cases}$
7. (a) $g(x) = \lceil x \rceil$ for $n < x \leq n+1$ where n is an integer.

   (b) $h(x) = \lfloor x \rfloor$ for $n \leq x < n+1$ where n is an integer.
8. $y = \left\lfloor x + \frac{1}{2} \right\rfloor$ for $n \leq x < n+1$, where n is an integer.
9. The function $f(x)$ is defined as follows;

   $$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

   Find: (a) $f(3)$ and $f(-2)$ (b) Sketch the graph of $f(x)$

   (c) Find the range of $f(x)$
10. Sketch a graph of $y = 3\lceil x \rceil$ for $n-1 < x \leq n$ where n is an integer.
11. Sketch the graph of $g(x) = 2 - |x|$ and state its domain and range.
12. Sketch the graph of each of the following functions and state its domain and range.

    (a) $f(x) = |x| - 1$

    (b) $g(x) = 1 - |x-2|$

    (c) $h(x) = |x+2| + 4$

Inverse of a function

In Chapter One, you learnt about the inverse of a relation. The inverse of a function is also a function provided that the function is one–to–one. Study Figure 2.27 and Figure 2.28 and learn how to differentiate a one-to-one function from its inverse.

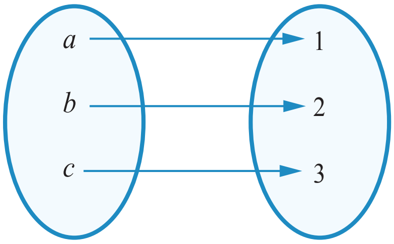


Figure 2.27: *Pictorial representation of a one-to-one function*

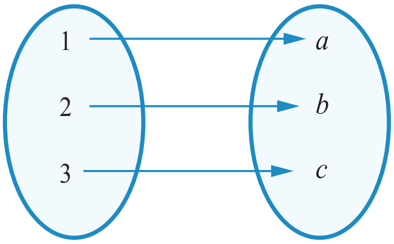


Figure 2.28: *Pictorial representation of the inverse of a one-to-one function*

The inverse function reverses the direction of the arrows. Figure 2.27 shows a one–to–one function. Figure 2.28 is the inverse of the function in Figure 2.27. You can see that the inverse takes each domain value to only one range value, hence it is a function.

Figure 2.29 represents a many–to–one function. In Figure 2.30, we can see that the inverse takes one domain value to more than one range values, hence it is not a function.

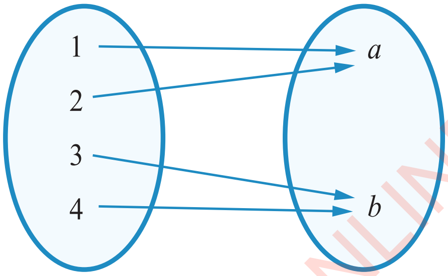


Figure 2.29: *The pictorial representation of many-to-one function*

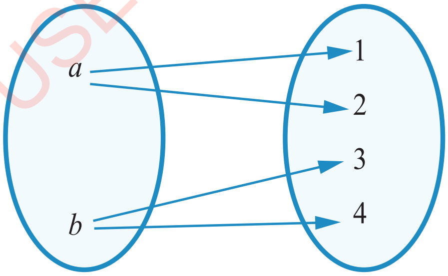


Figure 2.30: *The pictorial represention of the inverse of many-to-one function*

Activity 2.5: Creating units conversion tool

In Chapter One, you learnt about the inverse of a relation. Functions are special relations. The concept of inverse of a function applies only to functions which are one-to-one. A rich collection of one-to-one functions are found in everyday life when converting various measuring units such as lengths, weights, volumes or even in exchange of currencies.

In a group or individually, perform the following tasks which reflect one of these applications.

1. List down 10 items whose units are in inches. Such items are like nails and pipes.
2. Choose any common metric unit of length such as metres, centimetres or millimetres.
3. Generate a tool which will simplify conversion of the inches into the common metric units of your choice.
4. Prepare a table with two columns, one is for items in inches and another is for any unit of your choice.
5. Use the created conversion tool to create another tool which will convert back the units into inches.
6. Create another table and fill the values based on the new tool.
7. Present the data in both tables and pictorial diagrams, hence compare the results.
8. State how the two tools created are related to one other.
9. Share the findings to the rest of the class through a presentation.

In Activity 2.5, you may have noticed that, there exist a function which is a reverse of the other. This function is called an inverse function. Given a function $f(x)$ its inverse is denoted by $f^{-1}(x)$. The inverse of a one–to–one function is found using the following steps.

1. Interchange the variables x and y.
2. Make y the subject of the formula.

Example 2.30

Find the inverse of the function $f(x) = 3x + 7$ and draw the graphs of both functions on the same xy–plane.

Solution

Let $y = f(x)$, such that $y = 3x + 7$. To find the inverse, interchange y and x to obtain $x = 3y + 7$. Making y the subject gives $y = \frac{x}{3} - \frac{7}{3}$. Hence, the inverse of the function is $f^{-1}(x) = \frac{x}{3} - \frac{7}{3}$. The graphs of $f(x)$ and $f^{-1}(x)$ are shown in Figure 2.31.

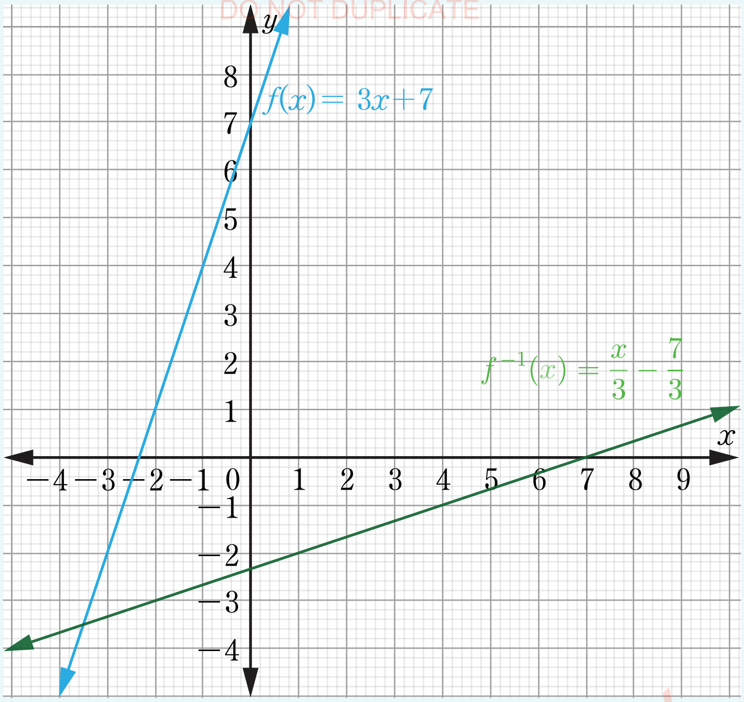


Figure 2.31: Graph of $y = 3x + 7$ and its inverse $f^{-1}(x) = \frac{x}{3} - \frac{7}{3}$

Example 2.31

Find the inverse of the function $y = x^2$, then draw its graph and its inverse on the same *xy*–plane. State whether the inverse is a function.

Solution

To find the inverse, interchange x and y to obtain $y^2 = x$. Making y the subjects gives $y = \pm\sqrt{x}$. Therefore, $y^{-1} = \pm\sqrt{x}$. The graph of $y = x^2$ and its inverse is shown in Figure 2.32.

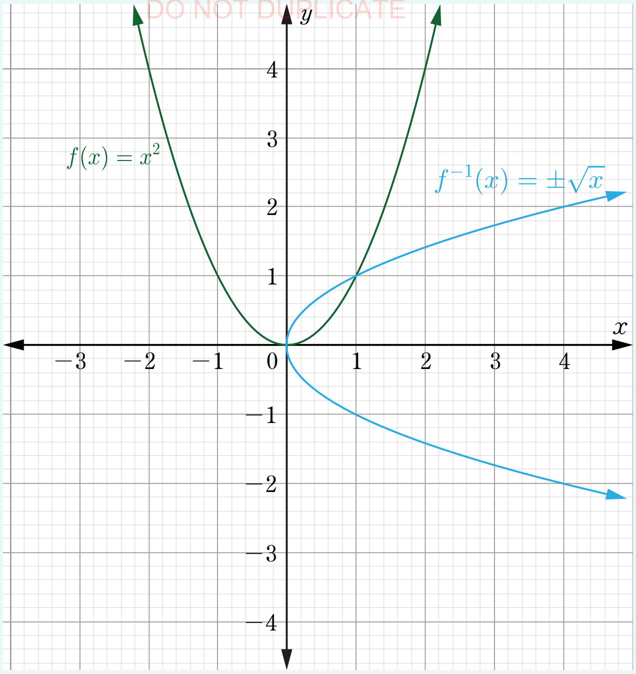


Figure 2.32: *Graph of* $y = x^2$ *and* $y^{-1} = \pm\sqrt{x}$

Figure 2.32 shows that the inverse of $y = x^2$ is not a function because each value of x, has two values of y^{-1}. The domain of the inverse of this relation is $x \geq 0$ and the range is the set of all real numbers.

Remarks: Inverses of some functions are not functions unless the domains are restricted. However, the inverses of all one-to-one functions are functions.

Example 2.32

Find the inverse of the function $f(x) = \frac{4x-2}{3}$.

Solution

Let $y = \frac{4x-2}{3}$, interchange y and x to get $x = \frac{4y-2}{3}$.

Making y the subject gives $y = \frac{3x+2}{4}$.

Therefore, $f^{-1}(x) = \frac{3x+2}{4}$.

Example 2.33

Find the inverse of the function $f(x) = \sqrt{x-3}$, for $x \geq 3$.

Solution

Let $y = \sqrt{x-3}$, upon interchanging the variables y and x we get $x = \sqrt{y-3}$. Making y the subject of the equation gives $y = x^2+3$. Hence, $f^{-1}(x) = x^2+3$, for $x \geq 0$.

Example 2.34

Find the inverse of the function $\mathrm{F} = \left\{(1,2),(3,6),(5,10),(-2,-4),\left(\frac{1}{2},1\right)\right\}$.

Solution

The inverse of F is obtained by interchanging the x and y coordinates in each point.

Therefore $\mathrm{F}^{-1} = \left\{(2,1),(6,3),(10,5),(-4,-2),\left(1,\frac{1}{2}\right)\right\}$.

Example 2.35

Find the inverse of the function $f(x) = \sqrt{1-x^2}$ for $-1 \leq x \leq 1$.

Solution

Let $y = \sqrt{1-x^2}$, interchanging the variables x and y gives $x = \sqrt{1-y^2}$ and hence $x^2 = 1-y^2$ or $y^2 = 1-x^2$ or $y = \pm\sqrt{1-x^2}$ for $0 \leq x \leq 1$. Therefore, $f^{-1}(x) = \pm\sqrt{1-x^2}$ for $0 \leq x \leq 1$. Note that, in this example $f^{-1}(x)$ is not a function.

Exercise 2.9

In questions 1 to 3, (a) find the inverses of the given functions (b) state whether or not the inverses are functions.

1. $f(x) = x^2+2x+1$ 2. $f(x) = x+2$ 3. $f(x) = 3x-2$
4. A function is defined by $f(x) = x-3$. Find;
   (a) its inverse.
   (b) the value of $f^{-1}(1)$.
   (c) the value of $f^{-1}(2)$.

In questions 5 to 10, find the inverses of each of the given functions:

5. $f(x) = \sqrt{x}$ for $x \geq 0$
6. $f(x) = |x - 2|$
7. $F = \{(2, 3), (3, 4), (4, 5), (7, 6)\}$
8. $f(x) = x$
9. $f(x) = x^2 - 2$
10. $f(x) = |x + 1|$

Activity 2.6: Applying functions in real life situations

Individually or in a group, perform the following tasks:

1. Identify one real life problem which can be represented as a function.
2. Design a function model for the problem you have identified. The model may be in terms of a figure, picture, drawing, writing or any other type of a model of your choice.
3. Use the concept of a function to describe the usefulness of the functions in a real life situation.
4. Share your final work to the rest of the class.

Chapter summary

1. A function from set A to set B is a relation between set A and set B. Every element of A is assigned to exactly one element of set B.
2. In a one–to–one function, each element of set A is assigned to only one element of set B. A function that is not one–to–one is many–to–one function.
3. The graph of a function is drawn by setting up a table of values.
4. The inverse of a function $f(x)$ is written as $f^{-1}(x)$. It is obtained by interchanging x and y, and then making y the subject of the equation.
5. Some functions such as step functions are defined by more than one equation.
6. The absolute value function is always non-negative.
7. The step function $\lfloor x \rfloor$ is the greatest integer function which is less than or equal to x, and $\lceil x \rceil$ is the least integer function which is greater than or equal to x.
8. If $f^{-1}(x)$ is the inverse of $f(x)$, then the domain and range of $f(x)$ are the range and domain of $f^{-1}(x)$, respectively.

Revision exercise 2

In questions 1 to 3, state the degree of the given polynomials

1. $p(x) = 2 + 6x + 21x^2 + 23x^3$
2. $p(x) = -11x^3 - 13x^2 + 10x$
3. $p(x) = a_0 + a_1x + a_2x^2$
4. Let F be a function that takes a whole number to twice of that number.
   (a) Write a formula for this function.
   (b) Make a sketch of the function.
5. Which of the following expressions are polynomials?
   (a) $\sqrt{a^2+3}$
   (b) $a + a + a$
   (c) $a + \sqrt{3}$
   (d) $\dfrac{x^2+6x+5}{2}$
   (e) $\dfrac{x^2+4}{x^3+4}$
6. Find the point at which the graph of each of the following polynomials crosses the y–axis:
   (a) $f(x) = x^3 + 6x^2 - 8$ (b) $f(x) = 9 - x^2$
7. Sketch the graph of the inverse of $y = \sqrt{1-x^2}$, for $0 \le x \le 1$.
8. Determine which of the following functions are one-to-one.
   (a) $f(x) = \{(x,y): y = x^2 - 4,\ x \le 0\}$
   (b) $f(x) = \{(x,y): y = x^2 - 3x + 2,\ x \ge 0\}$
   (c) $f(x) = \{(x,y): y = x^3\}$
   (d) $f(x) = \{(x,y): y = 3x + 5\}$
9. If $g(x) = \begin{cases} x+2, & x < 0 \\ 2, & 0 \le x \le 2 \end{cases}$
   (a) find (i) $g(-1)$ (ii) $g(-4)$ (iii) $g(1.6)$
   (b) sketch the graph of $g(x)$
   (c) find the domain and range of g.

10. If $g(x) = \begin{cases} x, & \text{if } x < 0 \\ 2x-1, & \text{if } 1 \leq x < 2 \\ x-2, & \text{if } x \geq 2 \end{cases}$

(a) find (i) $g(1)$ (ii) $g(1.5)$ (iii) $g(4)$ (iv) $g\left(\frac{21}{2}\right)$

(b) sketch the graph of $g(x)$

(c) find the domain and range of $g(x)$

11. Find the coordinates of the turning points of the following functions:

(a) $y = 2x^2 - 7x + 6$

(b) $y = 4x^2 + x + 1$

(c) $y = 2x^2 + 3x + 4$

12. Find the inverse of each of the following functions and state its domain and range:

(a) $y = \sqrt{x^2 - 1}$

(b) $f(x) = -\frac{1}{2}x + 3$

13. Which of the following functions are polynomial functions? Why?

(a) $f(x) = 3x^4 + 4x^2 + 7x - 81$

(b) $f(x) = \frac{1}{5+x}$

(c) $f(x) = \frac{2}{x^2} + \frac{5}{x} + 8$

Chapter Three

Statistics

Introduction

Decision makers use analysed data to draw conclusions related to various daily activities. The branch of mathematics dealing with data collection, analysis, and presentation is called statistics. Statistics is concerned with scientific methods for collecting, organising, summarising, and analysing information. In this chapter, you will learn the meaning of statistics, how to calculate mean and mode from a set of data, construct frequency distribution tables and histograms. You will also learn how to calculate median from a set of data, frequency distribution tables, and cumulative frequency curves. The competencies developed will enable you to collect, analyse, and interpret information such as students' performances, census, crops production, number of people receiving vaccine, and in many other areas where the knowledge of statistics is applicable.

Activity 3.1: Identifying the convenient statistical method for interpreting a given set of data

Individually or in a group, perform the following tasks:

1. Measure and record the heights of students within the class or from other classes.
2. Find the total data values obtained in task 1 and divide the answer by number of data values.
3. Arrange the data in ascending or descending order and find the middle number.
4. Write the data value which has mostly occurred than other data values.
5. Compare the results you obtained in task 2, task 3, and task 4 and decide which result can conveniently describe the average height of students.
6. Give reasons for your answer in task 5 and share the conclusion to the rest of the class through a class discussion.

Measures of central tendency

Statistical data can be organised and presented in different ways such as graphs and tables. However, for easy understanding and precise interpretation of data, you may need to calculate certain statistical measures. One set of useful measures are the measures of central tendency. The measures of central tendency you will learn in this chapter include mean, median, and mode.

Activity 3.2: Identifying mean as a measure of central tendency

Perform the following tasks in a group:

1. Record the number of family members of each student in a group.
2. Record the total number of family members of all students in a group.
3. Find the average number of family members in task 2.
4. Compare the results you obtained in task 3 to the rest of the class through presentation. What did you observe? Give reasons.

Mean

Mean is one of the measures of central tendency. It is obtained by adding up all the values of the observations and then dividing the sum by the total number of observations. Mean is generally denoted by $\overline{x}$. In this section, you will learn how to calculate the mean for ungrouped and grouped data.

Mean of ungrouped data

Ungrouped data refers to a set of data presented in a list without any classification or categorization. Ungrouped data is also referred to as raw data.

In general, data are denoted by x_i for $i = 1,2,3, ..., N$, where N represents the number of observations. The mean of ungrouped data is given by

$$\text{Mean} = \frac{\text{Sum of all values in a set of data}}{\text{Total number of observations}}.$$

Thus,

$$\overline{x} = \frac{x_1 + x_2 + \cdots + x_n}{N} = \frac{\sum_{i=1}^{N} x_i}{N},$$

where the notation $\sum_{i=1}^{N} x_i$ represents the summation of observations (x_i) for $i = 1, 2, 3, . . ., N$.

Example 3.1

Calculate the mean of the following scores of students in a Mathematics test. Give the answer correct to two decimal places.

15, 35, 45, 25, 63, 42, 12.

Solution

Here, $N = 7$.

The mean is calculated as follows:

$$\bar{x} = \frac{\sum_{i=1}^{7} x_i}{7}$$

$$= \frac{15 + 35 + 45 + 25 + 63 + 42 + 12}{7}$$

$$= \frac{237}{7}$$

$$= 33.86.$$

Therefore, the mean score of students in the Mathematics test is 33.86.

For the case when the number of data is large and some of the individual data being repeated, a frequency distribution table for ungrouped data is used to summarize the data, and the mean can be calculated by using the formula

$$\bar{x} = \frac{\sum_{i=1}^{N} f_i x_i}{N} = \frac{\sum_{i=1}^{N} f_i x_i}{\sum_{i=1}^{N} f_i},$$

where f_i is the frequency of each observation (x_i) and $N = \sum_{i=1}^{N} f_i$ is the sum of the frequencies from $i = 1$ to $i = N$, $\sum_{i=1}^{N} f_i x_i$ is the sum of the product of each frequency and its respective observation.

Example 3.2

A researcher interviewed 60 families on the number of children each family had. The results were recorded as shown in Table 3.1.

Table 3.1: Number of children from 60 families

Number of children	0	1	2	3	4	5	6	7	8
Number of families	3	6	7	8	10	12	8	4	2

(a) Calculate the mean number of children per family.
(b) Give your comment on the number of children in the families interviewed.

Solution

(a) Let x_i represent the number of children and f_i represent the frequency for a single observation. Arrange the data in ascending order as shown in Table 3.2.

Table 3.2: Frequency distribution table of the number of children from 60 families

x_i	f_i	$f_i x_i$
0	3	0
1	6	6
2	7	14
3	8	24
4	10	40
5	12	60
6	8	48
7	4	28
8	2	16

From Table 3.2, the sum of frequencies and the sum of the product of frequency and observations are given by

$\sum_{i=1}^{9} f_i = 60$ and $\sum_{i=1}^{9} f_i x_i = 236$, respectively.

Hence,

$$\overline{x} = \frac{\sum_{i=1}^{9} f_i x_i}{\sum_{i=1}^{9} f_i}$$

$$= \frac{236}{60}$$

$$= 3.93 \simeq 4.$$

Therefore, the mean number of children per family is 4.

(b) Generally, there is a high possibility of finding 4 children in most of the interviewed families.

Mean for grouped data

A set of data can be presented in form of categories or classes. Organising data in classes involves identifying the class size and determining the frequencies which belong to the given range of values. Finding the mean for a set of grouped data follows similar procedures as for ungrouped data. When a set of grouped data is given in a frequency distribution table, the mean can be calculated by the formula

$$\overline{x} = \frac{\sum_{i=1}^{N} f_i x_i}{N} \text{ or } \overline{x} = \frac{\sum_{i=1}^{N} f_i x_i}{\sum_{i=1}^{N} f_i},$$

where;

f_i is the frequency for each data category,

x_i is the class mark for each class interval, and

N is the sum of all frequencies.

Remarks

1. The mean calculated from the grouped data is not an actual mean but an estimated mean.
2. The class mark is the average of the endpoints of the class interval.

Example 3.3

Table 3.3 shows the marks of 100 students in a Physics test. Calculate the mean mark and interpret the results.

Table 3.3: Marks for 100 students

Class interval	x_i	f_i
91 – 95	93	0
86 – 90	88	1
81 – 85	83	6
76 – 80	78	10
71– 75	73	15
66 – 70	68	34
61 – 65	63	22
56 – 60	58	10
51– 55	53	2
		$\sum f_i = 100$

Solution

Multiply the frequencies (f_i) by the class marks (x_i) and then use the formula to calculate the mean marks. The results are summarized in Table 3.4.

Table 3.4: Frequency distribution for Physics marks scored by 100 students

Class interval	x_i	f_i	$f_i x_i$
91 – 95	93	0	0
86 – 90	88	1	88
81– 85	83	6	498
76 – 80	78	10	780
71 – 75	73	15	1 095
66 – 70	68	34	2 312
61 – 65	63	22	1 386
56 – 60	58	10	580
51 – 55	53	2	106
		$\sum f_i = 100$	$\sum f_i x_i = 6\ 845$

Then,

$$\bar{x} = \frac{\sum f_i x_i}{\sum f_i}$$

$$= \frac{6\ 845}{100}$$

$$= 68.45.$$

Therefore, the mean marks is 68.45, which implies that on average, the majority of students scored around 68.5 marks.

Assumed mean

An alternative method of computing the mean is by using an assumed mean. This method helps to reduce the calculations and it results in working with small data values. It depends on estimating the mean and rounding it to an easy value to work with. This value is then subtracted from all the sample values. The assumed mean is obtained by choosing any value of the class mark (x_i) and applying the formula

$$\bar{x} = A + \frac{\sum f_i d_i}{\sum f_i} \quad or \quad \bar{x} = A + \frac{\sum f_i d_i}{N},$$

where A is the assumed mean,

x_i is the class mark of each class interval,

d_i is the difference between the data value x_i and the assumed mean A, that is, $d_i = x_i - A$,

f_i is the frequency of each class mark, and

$N = \sum f_i$ is the total number of frequencies or observations.

Example 3.4

Use the data in Table 3.4 to calculate the mean marks by using assumed mean method.

Solution

Start by choosing any arbitrary number from the column of class marks, say 78. In this case, the assumed mean $A = 78$. Subtract this number from each of the class marks as shown in Table 3.5.

Table 3.5: Frequency distribution for Physics marks scored by 100 students

x_i	f_i	$d_i = x_i - A$	$f_i d_i$
93	0	15	0
88	1	10	10
83	6	5	30
78	10	0	0
73	15	- 5	- 75
68	34	- 10	- 340
63	22	- 15	- 330
58	10	- 20	- 200
53	2	- 25	- 50
	$\sum f_i = 100$		$\sum f_i d_i = -955$

Thus, $\sum f_i = 100$ and $\sum f_i d_i = -955$

From the formula,

$$\bar{x} = A + \frac{\sum f_i d_i}{\sum f_i}.$$

Substitute the values:

$$\begin{aligned}\bar{x} &= 78 + \frac{-955}{100}\\ &= 78 - 9.55\\ &= 68.45.\end{aligned}$$

Therefore, the mean marks of the Physics test is 68.45.

It should be noted that for any value of the chosen assumed mean, the mean of the data will always be the same.

Example 3.5

Use the information given in Example 3.2 to calculate the mean when the assumed mean is 8.

Solution

The respective frequency distribution table is shown in Table 3.6.

Table 3.6: Frequency distribution of the number of children from 60 families

x_i	f_i	$d_i = x_i - A$	$f_i d_i$
0	3	–8	–24
1	6	–7	–42
2	7	–6	–42
3	8	–5	–40
4	10	–4	–40
5	12	–3	–36
6	8	–2	–16
7	4	–1	–4
8	2	0	0
	$\sum f_i = 60$		$\sum f_i d_i = -244$

Thus, $\sum f_i = 60$ and $\sum f_i d_i = -244$

The mean is given as

$$\bar{x} = A + \frac{\sum f_i d_i}{\sum f_i}.$$

Substituting the values into the formula gives

$$\begin{aligned}\bar{x} &= 8 + \frac{-244}{60}\\ &= 3.93\\ &\approx 4\end{aligned}$$

Therefore, the mean number of children per family is 4.

Note that, in some cases, the answers obtained by applying the two methods on the same set of data might differ slightly. However, both methods are correct.

Calculation of mean from histograms

A mean can be calculated from a set of data presented using histograms. In this case, list the class marks which are found on the horizontal axis and its respective frequencies on the vertical axis. Then, proceed with the other steps of finding the mean for grouped or ungrouped data.

Example 3.6

The mass of 100 students in kilograms was recorded and presented in a histogram as shown in Figure 3.1. Calculate the estimated mean mass of the students and use the mean to interpret the result.

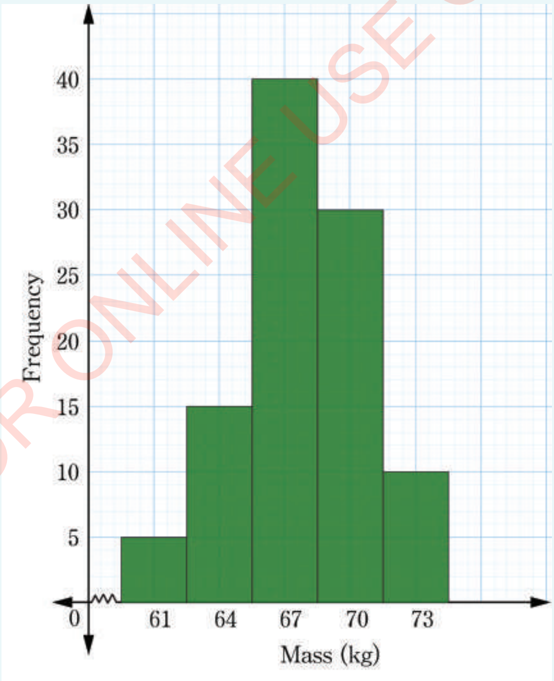


Figure 3.1: *Mass of students in kilograms*

Solution

From Figure 3.1, it can be observed that the class marks are 61, 64, 67, 70, and 73. The class marks corresponds to the frequencies 5, 15, 40, 30, and 10, respectively. The observations are summarized in Table 3.7. Let the assumed mean be 67.

Table 3.7: The frequency distribution table for the masses of 100 students in kilograms

x_i	f_i	$d_i = x_i - A$	$f_i d_i$	$f_i x_i$
61	5	–6	–30	305
64	15	–3	–45	960
67	40	0	0	2 680
70	30	3	90	2 100
73	10	6	60	730
	$N = 100$		$\sum f_i d_i = 75$	$\sum f_i x_i = 6775$

Thus, $\sum f_i x_i = 6775$ and $N = 100$.

From the formula,

$$\bar{x} = \frac{\sum f_i x_i}{N}.$$

Substituting the values in the formula gives

$$= \frac{6775}{100}$$

$$= 67.75.$$

Therefore, the mean mass of 100 students is 67.75 kilograms.

Using the assumed mean formula and taking the assumed mean A as 67, the mean is alternatively calculated as follows:

$$\bar{x} = A + \frac{\sum f_i d_i}{N}$$

$$= 67 + \frac{75}{100}$$

$$= 67.75$$

Therefore, the mean mass of 100 students is 67.75 kilograms. This implies that, the students have an average mass of 67.75 kilograms.

Activity 3.3: Comparing writing styles

It is known that a good and standard sentence contains 15 to 20 words. Find a friend and decide on a topic you can write about. Perform the following tasks with your friend.

1. Write a single page on the topic.
2. Use the mean to analyse your writing style.
3. By comparing with the recommended number of words in a sentence, who is the better writer?
4. How does the result help you in improving your writing style?

Exercise 3.1

1. Mpira Football Club has the following number of goals scored against them: 0, 1, 0, 2, 9, 0, 1, 2, 1.
   What is the mean number of goals scored against them?
2. Calculate the mean of each of the following data sets:
   (a) 83, 97, 89, 88, 84.
   (b) 1 512, 1 509, 1 519, 1 514, 1 505, 1 503.
3. An aircraft takes one hour to fly from town A to town B at 500 km/h. If it uses the same time and returns via the same route at 250 km/h, calculate its average speed and interpret the result.
4. In an athletic competition, the finishing time in seconds taken by 12 athletes in a 200 metres race recorded as follows:
   32.8, 28.2, 38.5, 37.4, 36.5, 29.5, 27.2, 25.6, 32.4, 33.8, 28.7, 25.2.
   Find their average finishing time.
5. Table 3.8 shows the number of beds in some hotels found in a certain country:

Table 3.8: Distribution of hotels by number of beds available daily

Number of beds	10–19	20–29	30–39	40–49	50–59	60–69	70–79	80–89
Number of hotels	27	15	12	12	14	5	5	10

Calculate the estimated mean number of beds per hotel.

6. The following table shows the amount of money spent by a certain small company as salaries for different categories of employees per month.

Employees category	**Monthly salaries (Tsh)**
Three unskilled workers	810 000
Technician	300 000
Secretary	350 000
Manager	700 000

What is the mean monthly salary of the employees? Based on the obtained mean monthly salary, what conclusion can you make about the salary offered by the company to the employees?

7. Table 3.9 shows the distribution of children's ages in months in a certain village.

Table 3.9: Distribution of children's ages in months

Ages (months)	41– 46	35– 40	29– 34	23–28	17–22	11–16	5– 10
Number of children	3	4	9	12	18	28	26

Calculate the mean age in months using any method of your choice and hence use the obtained mean to interpret the given data.

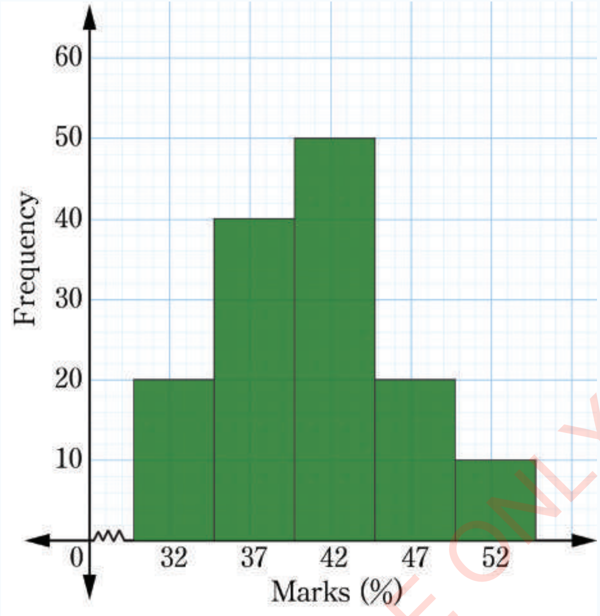
8. A survey of 200 children below 10 years was conducted for the purpose of discovering the number of visits they made to a paediatric clinic during the course of the year. The results were recorded as shown in Table 3.10.

Table 3.10: Number of children below 10 years who visited a paediatric clinic over the year

Number of visits	5	6	7	8	9	10	11	12	13	14	15
Number of children	16	33	47	54	31	10	4	2	0	2	1

What was the mean number of visits per child? What does this mean tell you about the number of children who visited the clinic?

9. The following figure is a histogram for 140 Basic Mathematics scores. Use it to calculate the mean score and interpret the result.



10. With examples, describe how mean is used in real life situation. Include the following in your answer:
    (a) Data collected from home or school that can be described using their mean values.
    (b) Your scores of Mathematics tests from Form One to date. Interpret your results using the mean score.

Median

In Activity 3.1, you employed different methods of interpreting data. One of the methods you used is the median. It is one of the measures of central tendency of given set of data.

Activity 3.4: Making sense of the median

Individually or in a group, identify the salaries of five employees in a small factory, and then perform the following tasks:

1. What is the mean salary of the employees and what does that mean tell you about the employees salary per year?
2. Does the mean salary precisely describe the salary of each employee? Why?
3. Based on your understanding, which measure of central tendency may precisely describe the average earning of these employees per year?
4. Arrange the salaries from the least to the highest.
5. Write the central value and state whether it represents the average salary of all the employees?
6. Share your findings to the rest of the class through presentations.

The average salaries of the employees in Activity 3.4 could be described appropriately using the median rather than the mean. A median is the value at the data point that divides a frequency distribution into two halves such that an equal number of observations or data values lie above and below that point. In other words, median is the middle value or middle score obtained by arranging the data in ascending or descending order.

In Activity 3.4 task 5, the central value obtained is the median. In this activity, it was much better to describe the average salary earned by the employees by using this value. This is because, median is not affected by isolated or extreme values (sometimes called outliers) that are much larger or smaller than the rest of the values in a given set of data.

Activity 3.5: Determining the shortest distance between regions

Perform the following tasks individually or in a group:

1. Taking the Region where you are now as the initial point, estimate the distances from that Region to other 10 Regions of your choice. You can estimate the distances by using a map of Tanzania, rulers, threads or any other convenient resources of your choice.

2. Find the mean distance of the data obtained in task 1.
3. Arrange the data values in ascending or descending order and find the middle number.
4. Record the data value which has occurred most frequently.
5. Compare the data values obtained in task 2, task 3, and task 4.
6. State with reasons if the value obtained in task 3 can be considered as the best option for describing the average distance between your Region and other Regions.
7. Share your observation to the rest of the class through a presentation.

Median of ungrouped data

The computation of the median of ungrouped data depends on whether the data set has odd or even number of observations. When the data set has an odd number of observations, the median is simply the middle value when the data are arranged in ascending or descending order.

For example, the data set consisting of 4, 10, 14, 6, and 8 when arranged by ascending order gives 4, 6, 8, 10, 14. In this case, 8 is at the middle and hence it is the median.

When the data set has an even number of observations, the median is the average of the two middle values when the data is arranged in ascending or descending order. For example, the median of the data set consisting of 12, 5, 8, and 6 is the average of 6 and 8 from the order 5, 6, 8, 12, which is given as

$$\begin{aligned}\text{Median} &= \frac{6+8}{2}\\ &= \frac{14}{2}\\ &= 7.\end{aligned}$$

Therefore, the median is 7.

Example 3.7

The heights of nine orange trees were recorded in metres as 1, 8, 3, 4, 7, 5, 10, 6, and 9. What is the median height of the orange trees? Interpret the result.

Solution

Arrange the data in ascending order to get: 1, 3, 4, 5, 6, 7, 8, 9, 10.
It can be observed that the middle value is 6. Therefore, the median is 6 metres. That is, 4 orange trees have height less than 6 metres and the other 4 orange trees have height greater than 6 metres.

Example 3.8

The masses of 8 watermelons in kilograms were recorded as 8, 3, 6, 4, 5, 1, 2, and 7. Calculate the median mass of the watermelons and interpret the result.

Solution

Arrange the data in ascending order to get 1, 2, 3, 4, 5, 6, 7, 8. Since there are 8 values, then the median is given by the average of the two middle values. The two middle values are 4 and 5. Thus,

$$\text{Median} = \frac{4+5}{2}$$
$$= \frac{9}{2}$$
$$= 4.5.$$

Therefore the median mass is 4.5 kilograms. That is, 4 watermelons each has mass less than 4.5 kilograms and the other 4 watermelons each has mass greater than 4.5 kilograms.

Example 3.9

The lengths of 8 trees in metres from a garden were recorded as 1, 2, 3, 4, 4, 5, 6, and 7. What is the median length of the trees?

Solution

In this case, the median is given by the average of the two middle observations:

$$\text{Median} = \frac{4+4}{2}$$
$$= \frac{8}{2}$$
$$= 4.$$

Therefore, the median length of the trees is 4 metres.

In general, when a data set with N values is arranged in ascending or descending order, the median is

(i) in the position $\left(\frac{N+1}{2}\right)$ if N is odd.

(ii) the average of the values in positions $\frac{N}{2}$ and $\frac{N}{2}+1$ if N is even.

Example 3.10

Table 3.11 shows the number of goals scored by a netball team in 31 matches. Find the median of the number of goals scored.

Table 3.11: Number of goals scored by a netball team in 31 matches

Number of goals	0	1	2	3	4	5
Number of matches	3	5	6	12	3	2

Solution

Table 3.12: Frequency distribution of the number of goals scored by a netball team

Number of goals	Frequency	Cumulative frequency
0	3	3
1	5	8
2	6	14
3	12	26
4	3	29
5	2	31

Since $N = 31$ which is odd, then the median position is

$$\left(\frac{N+1}{2}\right)^{\text{th}} \text{value} = \left(\frac{31+1}{2}\right)^{\text{th}} \text{value}$$

$$= 16^{\text{th}} \text{ value.}$$

Since the 16^{th} value in the cumulative frequency is 3, then the median is 3 goals.

Example 3.11

Table 3.13 shows the scores obtained by 40 students in a test. Find the median score and use it to interpret the data.

Table 3.13: Scores obtained by 40 students in a test

Test scores	40	42	64	68	70	72	74
Frequency	2	4	14	7	7	5	1

Solution

To get the median score for the test, use the cumulative frequency distribution as in Table 3.14

Table 3.14: Cumulative frequency distribution of the scores of 40 students in a Kiswahili test

Test scores	**Frequency**	**Cumulative frequency**
40	2	2
42	4	6
64	14	20
68	7	27
70	7	34
72	5	39
74	1	40

Since $N = 40$, which is even, the median position is the average of $\left(\frac{40}{2}\right)^{\text{th}}$ value $= 20^{\text{th}}$ value and $\left(\frac{40}{2}+1\right)^{\text{th}}$ value $= 21^{\text{st}}$ value.

Since the 20^{th} value is 64 and the 21^{st} value is 68, then,

$$\text{Median} = \frac{64+68}{2}$$
$$= 66.$$

Therefore, the median score is 66 which implies that majority of students scored 66 marks.

Median for grouped data

For grouped data, the median can be computed using the formula

$$\text{Median} = L + \frac{\left(\frac{N}{2} - n_b\right)}{n_w} i,$$

where,

L is the lower boundary of the median class,

N is the total number of observations or total frequency,

n_b is the cumulative frequency of the class before the median class,

n_w is the frequency of the median class, and

i is the class size or class width.

Example 3.12

Table 3.15 shows the distribution of the lengths of nails in millimetres. Calculate the median length and use it to interpret the data.

Table 3.15: Lengths of nails in millimetres

Length (mm)	Frequency	Cumulative frequency
88 – 96	3	3
97 – 105	5	8
106 – 114	9	17
115 – 123	12	29
124 – 132	5	34
133 – 141	4	38
142 – 150	2	40
		N = 40

Solution

Since $N = 40$, then $\frac{N}{2} = \frac{40}{2} = 20$, then the median class is 115 –123. Hence, L = 114.5, $n_b = 17$,

$n_w = 12$ and $i = 9$. Thus, substituting these values in the formula,

$$\text{Median} = L + \frac{\left(\frac{N}{2} - n_b\right)}{n_w} i$$

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$$\text{Median} = 114.5 + \frac{(20-17)9}{12}$$

$$= 114.5 + \frac{27}{12}$$

$$= 116.75.$$

Therefore, the median length is 116.75 mm, which implies that half of the length in a distribution is below 116.75 and the other half of the distribution is above 116.75 mm.

Example 3.13

Table 3.16 shows the distribution of marks scored by students in a Biology test.

Table 3.16: Distribution of marks scored by students in a Biology test

Marks (%)	Frequency	Cumulative Frequency
91 – 100	13	
81 – 90	22	
71 – 80	31	
61 – 70	30	
51 – 60	24	
41 – 50	6	
31 – 40	3	
21 – 30	1	

(a) Fill the table and then calculate the median.
(b) Use the results in (a) to interpret the data.

Solution

(a) Complete the frequency distribution table by finding the cumulative frequencies as shown in the following table:

Marks (%)	Frequency	Cumulative Frequency
91 – 100	13	130
81 – 90	22	117
71 – 80	31	95
61 – 70	30	64
51 – 60	24	34
41 – 50	6	10
31 – 40	3	4
21 – 30	1	1

Since N = 130, then $\frac{N}{2} = \frac{130}{2} = 65$. Therefore, the median class is 71 – 80.

Hence, L = 70.5, $n_b = 64$, $n_w = 31$, $i = 10$.

Thus, from Median $= L + \frac{\left(\frac{N}{2} - n_b\right)}{n_w} i$,

Substituting the values gives

$$\begin{aligned}\text{Median} &= 70.5 + \frac{(65-64)10}{31}\\ &= 70.5 + 0.32\\ &= 70.82\\ &\approx 71\%\end{aligned}$$

Therefore, the median is 71%.

(b) The results in (a) implies that half of the class scored below 71% and the other half scored above 71%.

Estimation of median from cumulative frequency curve

Median can be estimated from the cumulative frequency curve using the following steps:

Step 1: Determine the middle position of the data values by taking $\frac{N}{2}$.

Step 2: Locate the value obtained in step 1 on the vertical axis, and then draw a horizontal dotted line which intersects the curve.

Step 3: From a point where the horizontal dotted line intersects the curve, draw a vertical dotted line which intersects the horizontal axis (class marks).

Step 4: The value at the point where the vertical dotted line intersects the horizontal axis is the median of the distribution.

Example 3.14

Figure 3.2 shows a distribution of Chemistry marks of 70 students from Kiponda Secondary School. Calculate the median of the distribution and use it to interpret the data.

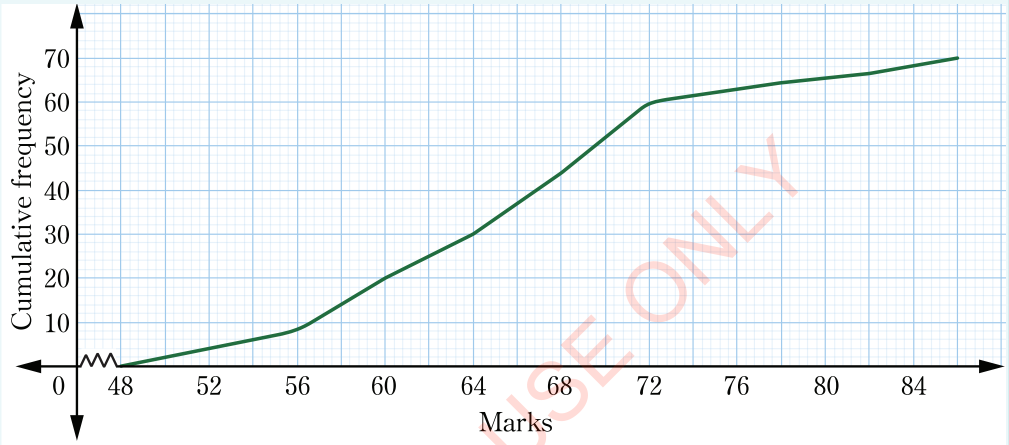


Figure 3.2: *Distribution of Chemistry marks of 70 students of Kiponda Secondary School*

Solution

The middle position = $\frac{N}{2} = \frac{70}{2} = 35.$

Locate 35 on the vertical axis, and then draw a horizontal dotted line which intersects with the curve from 35. From the point where the horizontal dotted line intersects the curve, draw a vertical dotted line which intersects the horizontal axis (class marks). The value at the point where the vertical dotted line intersects the horizontal axis is the median of the distribution as shown in Figure 3.3.

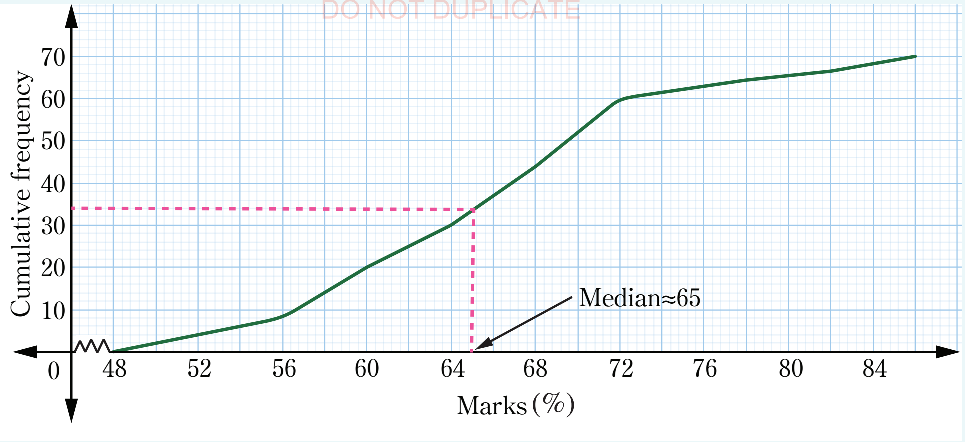


Figure 3.3: *Estimation of the median from a cumulative frequency curve*

From the cumulative frequency curve (Ogive), the approximated median is 65 marks. This implies that, the half of students scored below 65% marks and the other half scored above 65% marks.

Exercise 3.2

1. The following are the marks obtained by 8 students in a weekly test: 97, 64, 75, 63, 58, 91, 82, 85. Calculate the median mark and use it to interpret the scores.
2. Consider the following set of data: 46, 64, 87, 41, 58, 77, 35, 90, 55, 33, 92.
   (a) Calculate the median of the data.
   (b) If 92 is replaced by 19, and 41 by 43, determine the new median of the data.
3. If the median of 14, 18, $(x+2)$, $(x+4)$, 30, 34 arranged in ascending order is 24, find the value of x.
4. The heights of 6 maize plants measured in centimetres were as follows: 3, 12, 32, 24, 21, 8. Calculate the median height.
5. The following are masses in kilograms of 8 children: 3, 4, 5, 6, 6, 7, 8, 9. Calculate the median mass and interpret the data.
6. The following table shows the distribution of marks scored by 50 candidates in a test:

Marks	11– 20	21– 30	31– 40	41– 50	51 – 60	61 –70	71– 80
Frequency	1	3	10	21	6	5	4

Calculate the median mark and use the median to interpret the data.

7. The heights in centimetres of 100 people were recorded as shown in the following table.

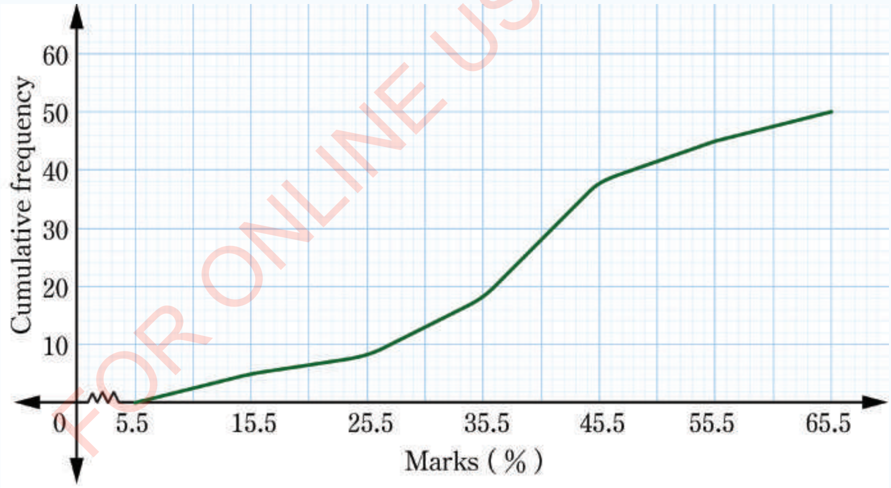
Height (cm)	160	165	170	175	180	185
Number of people	3	12	32	24	21	8

(a) Calculate the median height and use it to interpret the data.

(b) Draw a cumulative frequency curve and use it to estimate the median.

(c) Use the median to interpret the data.

8. The following figure is a cumulative frequency curve representing the test marks scored by 50 candidates. Use the curve to estimate the median marks of the candidates.



9. The mid-day temperatures over 80 days were recorded as shown in the following table.

Temperature (˚C)	20 – 21	22 – 23	24 – 25	26 – 27
Frequency	13	33	28	6

(a) Draw a cumulative frequency curve.
(b) Use the curve to estimate the median.

10. The ages of children who suffered from Cholera in Kagabilo village were as given in the following table.

Ages (years)	0 – 4	5 – 9	10 – 14	15 – 19
Frequency	12	15	20	3

(a) Draw a cumulative frequency curve.
(b) Use the curve in (a) to estimate the median.

Activity 3.6: Determining the most preferred sports and games by students

Individually or in a group, collect data on the type of sports and games preferred by students in your school by performing the following tasks:

1. List down 10 common sports and games that you are aware of.
2. Decide on the number of students you would like to collect data from, preferably not less than 20.
3. List 10 sports and games on a piece of paper and ask your fellow students to choose the games and sports they prefer.
4. Prepare a frequency distribution table which shows the sports and games and the frequency of students for each choice.
5. Find the most preferred sports and games. In statistics, what type of this data is refered to?
6. Share and compare the sports and games which were most listed by other groups.
7. Use the data to advise someone who is about to open a sports and games club around the school.

Mode

Mode is one of the measures of central tendency of a given set of data. It is the value that occurs most frequently in a set of data. It is possible for a set of data to have no mode or to have more than one mode. For example, a data set which has no repeated values shall have no mode, the data set with one mode is called unimodal, with two modes is called as bimodal and with three modes is called a trimodal and so on.

The mode is often important in various daily life activities. For example, a shoe shopkeeper would want to know the most popular shoe size. Also, the owner of a restaurant would want to know what type of food is mostly ordered. Under these situations, mode is the best option.

Mode of ungrouped data

The mode of ungrouped data can be found by picking a value or values with the highest frequency. For example:

1. The data set 2, 3, 3, 4, 5 has mode 3.
2. The data set 2, 3, 4, 5, 5, 5, 6, 6, 6, 8 has modes 5 and 6.
3. The data set 4, 5, 6, 7, 8, 9 has no mode.

Mode of grouped data

The modal class is the class with the highest frequency. The mode of ungrouped data can be calculated by using the formula derived from a histogram. Figure 3.4 represents a histogram of a certain distribution.

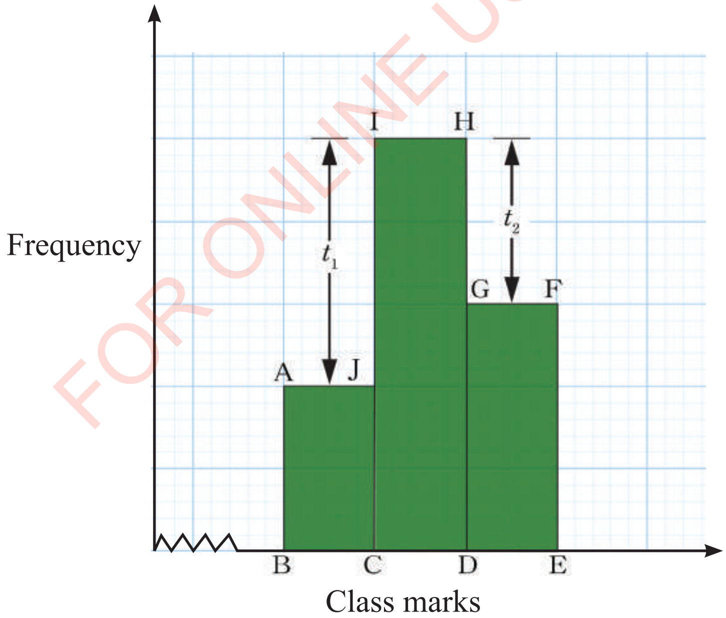


Figure 3.4: *The histogram from which the mode can be derived*

Since the rectangle CDHI has the highest frequency, then the class CD is the modal class, that is, it contains the mode. Let t_1 be the difference in frequency between rectangle CDHI and rectangle ABCJ and t_2 be the difference in frequency between rectangles CDHI and DEFG. The mode can be calculated using the formula:

$$\text{Mode} = L + \left(\frac{t_1}{t_1 + t_2}\right)i,$$

where L is the lower class limit or lower class boundary of the modal class,

t_1 is the difference between the modal frequency and the frequency of the class before the modal class,

t_2 is the difference between the modal frequency and the frequency of the class after the modal class, and

i is the class size or class width.

Example 3.15

The mass of 37 athletes were recorded in kilograms as shown in Table 3.17. Calculate the mode and use it to interpret the data.

Table 3.17: Mass of 37 athletes in kilograms

Mass (kg)	40 – 44	45 – 49	50 – 54	55 – 59	60 – 64	65 – 69
Frequency	7	8	11	10	4	2

Solution

The modal class is 50 – 54. Hence, L = 49.5, $t_1 = 11 - 8 = 3$, $t_2 = 11 - 10 = 1$, and $i = 5$. Thus,

$$\text{Mode} = L + \left(\frac{t_1}{t_1 + t_2}\right)i$$

Substituting the values,

$$\begin{aligned}\text{Mode} &= 49.5 + \left(\frac{3}{3+1}\right)5\\ &= 49.5 + 3.75\\ &= 53.25.\end{aligned}$$

Therefore, the mode is 53.25 kilograms, which implies that most of the athletes had a mass of 53.25 kilograms.

Estimation of mode from a histogram

In order to estimate the mode from a histogram, observe the following steps:

1. Identify a modal bar, that is, a rectangle which represents a value with the highest frequency.
2. Join by dotted straight lines the two corners of the modal bar with the immediate opposite corners of the bars adjacent to the modal bar.
3. Draw a dotted straight line from the intersection point of the two lines to meet the horizontal axis perpendicularly.
4. The point where the perpendicular dotted vertical line intersects the horizontal axis gives the mode.

Example 3.16

Figure 3.5 is the histogram for masses in kilograms of 100 people. Use the histogram to estimate the mode.

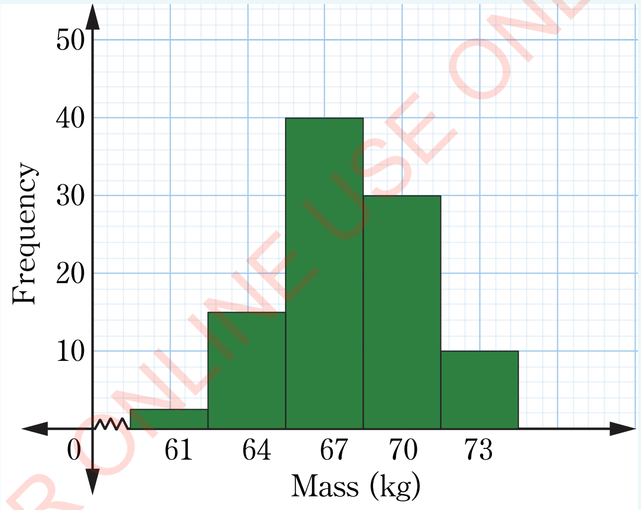


Figure 3.5: *Masses of 100 people in kilograms*

Solution

Follow the previous steps to estimate the mode from the histogram as shown in Figure 3.6.

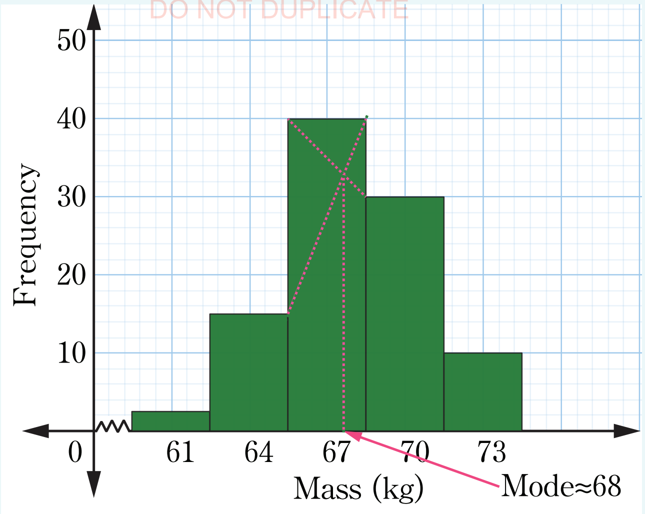


Figure 3.6: *Estimation of mode from a histogram*

Thus, based on the previous steps, the estimated mode is 68.

Exercise 3.3

1. The value of π to 20 decimal places is $\pi = 3.14159265358979323846$.
   (a) List the digits 0 to 9 and construct a frequency distribution table of the digits in π after the decimal point.
   (b) What are the most and least frequent occurring digits in π?

2. Find the mode of the following set of data:

   1, 3, 7, 3, 5, 4, 7, 2, 6, 7, 12, 10, 11, 3, 7, 8, 6, 7, 7, 4, 2, 11, 7, and 15.

3. If 18 is included in the following data set: 16, 13, 22, 16, 16, 18, 15, 25, give a reason whether the mode will increase or decrease.

4. The final scores in a History examination were recorded as shown in the following table.

Scores	65–69	70–74	75–79	80–84	85–89	90–94	95–99
Frequency	10	12	21	6	9	4	4

Calculate the mode of the scores and interpret the results.

5. The following data are the lengths of tobacco leaves to the nearest millimetres. Compute the mode and interpret the result.

138	168	132	147	135	150
146	140	176	150	144	135
150	146	147	138	136	125
140	161	142	142	163	148

6. In a village with 110 families, the number of children per family was found to be as shown in the following distribution. Calculate the mode of the distribution and use it to interpret the data.

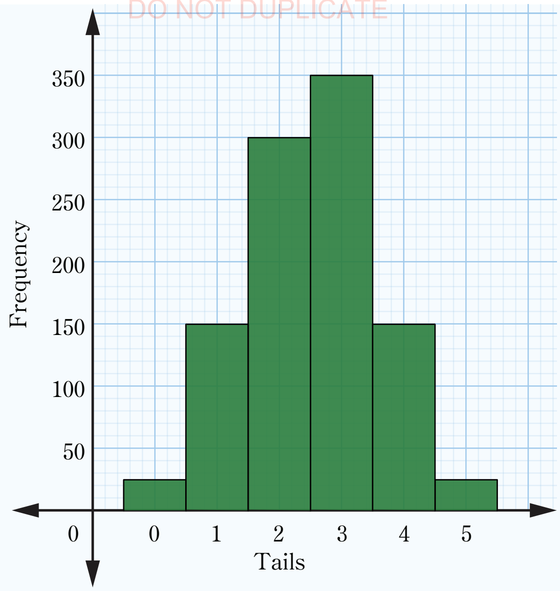
Number of children	0	1	2	3	4	5	6	7	8
Frequency	6	18	28	25	17	9	4	2	1

7. A factory sells needles in packets containing an average of 200 needles. The contents of 100 packets picked at random were counted and the results were tabulated as follows:

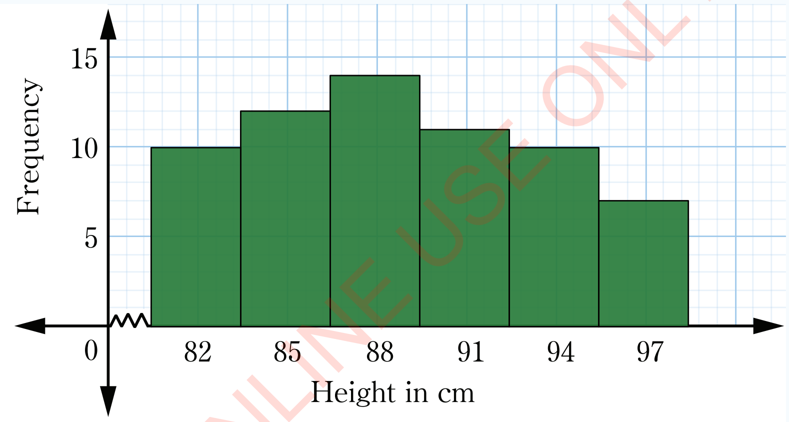
Number of needles	185–189	190–194	195–199	200–204	205–209	210–214
Frequency	4	14	32	28	17	5

Compute the mode of this distribution and use the mode to interpret the data.

8. Five 200 shillings coins were tossed together 1 000 times and in each toss, the number of tails were observed and recorded. The frequencies of tosses during which 0, 1, 2, 3, 4, and 5 tails were obtained are as shown in the following histogram. Calculate the mode of this distribution and use it to interpret the results.



9. The following figure shows a histogram for heights of children in centimetres. Calculate the mode and use it to interpret the data.



Activity 3.7: Using statistics in solving real life challenges

Perform the following tasks individually or in a group:

1. Identify a problem which can be solved through data collection and analysis.
2. Collect appropriate data which will enable you to solve the problem.
3. Represent the collected data using a frequency distribution table.
4. Depending on the nature of the data, use mean, mode, or meadian to analyse your data.

5. Use a large manila paper or any available materials to summarize your work.
6. Share the summary of your work to the rest of the class through a class presentation.

Chapter summary

1. The formula for calculating the mean of ungrouped data is

$$\text{Mean } (\bar{x}) = \frac{x_1 + x_2 + \cdots + x_N}{N} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

The formula for calculating the mean of grouped data is

$$\bar{x} = \frac{\sum f_i x_i}{\sum f_i} = \frac{\sum f_i x_i}{N}$$

where f_i is the frequency of each class mark, N is the total number of observations and x_i is the class mark of each class interval.

2. The formula for calculating the mean using the assumed mean is

$\bar{x} = A + \frac{\sum f_i d_i}{N}$, where A is the assumed mean, $d_i = x_i - A$, f_i is the frequency of each class mark, x_i are the data values (class marks), and $N = \sum f_i$ is the total number of frequencies or observations.

3. If the set of data is arranged in ascending or desceding order, the position of the middle data is $\frac{N}{2} + 1$ if N is odd or the average of $\frac{N}{2}$ and $\frac{N}{2} + 1$ if N is even.
4. The formula for finding the median of grouped data is

$$\text{Median} = L + \frac{\left(\frac{N}{2} - n_b\right)}{n_w} i,$$

where,

L is the lower boundary of the median class,
N is the total number of observations or total frequency,
n_b is the cumulative frequency of the class before the media class,
n_w is the frequency of the median class, and
i is the class size or class width.

5. Mode of a given dataset is simply a value that appears most frequently. A dataset may have no mode, have one mode or more than one mode. For example, a dataset with two modes is called bimodal data set and trimodal dataset for a dataset with three modes.
6. For grouped data, the mode is found by using the formula:

$$\text{Mode} = L + \left(\frac{t_1}{t_1 + t_2}\right)i,$$

where,

L is the lower class limit or lower class boundary of the modal class,

t_1 is the difference between the modal frequency and the frequency of the class before the modal class,

t_2 is the difference between the modal frequency and the frequency of the class after the modal class,

i is the class size or class width.

Revision exercise 3

1. The amount of annual rainfall, in centimetres, for a period of 15 years in a certain town was recorded as follows:
   25, 38, 27, 39, 42, 34, 27, 26, 24, 33, 32, 35, 44, 29, 27.
   Calculate the:
   (a) Median.
   (b) Mean.
   (c) Mode.
2. A darts player made the following scores in 12 throws:
   19, 10, 7, 14, 5, 1, 13, 17, 9, 6, 20, and 11. Calculate the mean score and use the obtained mean to interpret the given data.
3. The mean of the numbers a, 4, 10, 12, 16, b is 12. If $a = 4b$, calculate the mean of:
   (a) The first three numbers.
   (b) The last three numbers.

4. In a certain school, two streams A and B with 100 students each did a test which was marked out of 40 marks. The marks were recorded in the following table. By using histograms, which stream had better results?

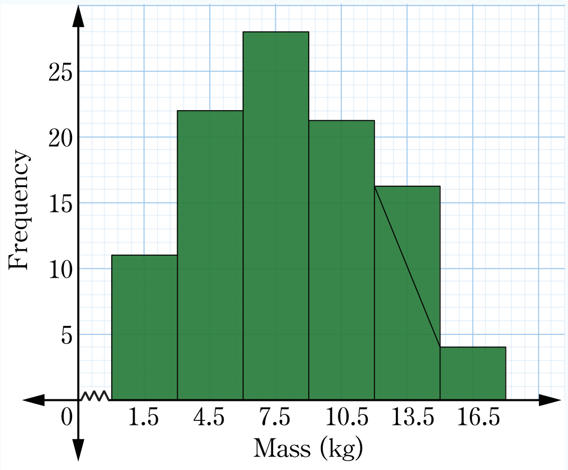
Marks	1–5	6–10	11–15	16–20	21–25	26–30	31–35	36–40
Stream A	1	26	28	27	5	8	3	2
Stream B	17	28	30	14	3	6	2	0

5. Using the data given in question 4, construct a cumulative frequence curve (ogive) for stream A and use it to estimate the median.
6. Eight families from a certain village were chosen at random and the number of people in each family was recorded as follows: 1, 2, 3, 4, 4, 4, 6, 7. Calculate the mean, median, and mode of the number of people per family and use them to interpret the data.
7. An association of tailors conducted a survey on the number of sewing machines that each primary association had and recorded as follows:
3, 8, 5, 4, 5, 5, 6, 7, 7, 4, 6, 6, 7, 4, 4, 6, 7, 5, 4, 3, 7, 5, 7, 8, 3.
Find the mode and use it to interpret the data.

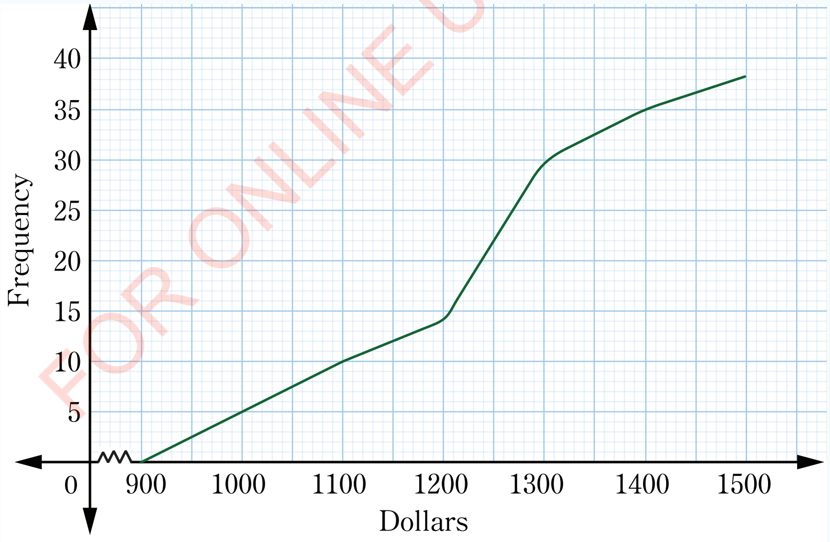
8. In a certain examination, the results were as follows:
3 students scored marks between 0 and 10,
5 students scored marks between 10 and 20,
6 students scored marks between 20 and 30,
4 students scored marks between 30 and 40, and
2 students scored marks between 40 and 50.
(a) How many students sat for the examination?
(b) How many students got less than 30 marks?
(c) Calculate the median mark if the examination weighs a total of 50 marks.

9. Using the data in question 8, construct a histogram and use the histogram to estimate the mode and interpret the data.

10. The masses of cassava harvested from a farm in kilograms were plotted in the following histogram. Estimate the median and mode from the histogram. Hence, use the mode to interpret the data.



11. Salaries of 36 foreign employees in dollars were given in a cumulative frequency curve as shown in the following figure.
    (a) How many employees had salaries of 1200 Dollars?
    (b) What was the salary of 15 employees?



12. The mean mark for 10 students in a Biology test is 75%. If the mean mark for 9 of them is 73%, find the marks for the tenth student.
13. The mean of five values is 8.2. If four of the values are 6, 10, 7 and 12, find the fifth value.
14. The following table represents the information about the percentage distribution of female employees in a certain company.

Percentage of female employees	Number of departments
15–25	2
26–36	4
37–47	4
48–58	7
59–69	11
70–80	6

(a) Find the mean percentage of female employees by the assumed mean method.
(b) Find the mode and median.

Chapter Four

Rates and variations

Introduction

Quantities of the same kind are compared by finding their ratios. For example comparing lengths of objects. Different quantities are compared by finding their rates. A good example is the speed which is the rate at which a body moves over time. Some variables relate in such a way that changing one variable causes change in its related variables. Such relationship is referred to as variation. In this chapter, you will learn to relate rates of different and same quantities, convert Tanzanian currency into other currencies, explain the concepts of direct and inverse variations, draw graphs of direct and inverse variations, explain joint variations, and solve problems on rates and variations. The competencies developed in this chapter will help you in determining payment rates, currency exchange, travelling speed, assigning duties depending on available resources, buying and selling commodities among many other applications.

Rates

Activity 4.1: Determining rates of performing different activities

Perform the following tasks in a group:

1. Identify any hands-on activity for which you can determine the rate of performance.
2. Perform the activity identified in task 1 individually at a time while other students are observing and recording the variables.
3. Record the results in a tabular form with four columns. The first column for student names, the second for first variable, the third for the second variable and the fourth for the rate of performance. The rate should be in the form of the first variable to the second variable.

4. Use the results to rank the performance from the highest to the lowest.
5. Share your findings to the rest of the class through a presentation.

When sets or quantities of different kinds are related, we use the word rate. For instance, the rate of payment in Tanzanian Shillings (Tsh) per hour (money-time), the price of milk per litre (money-litres) or the average speed (distance-time). In all these cases, the rate is a constant relation between the sizes or amounts of two quantities. Normally, rate deals with comparison of two quantities of different kinds. Therefore, rate is the ratio that compares two different units of quantities.

For example, in hiring a car at a rate of Tsh 2 000 per kilometre, a journey of 40 kilometres costs 40 × Tsh 2 000 = Tsh 80 000. Similarly, a journey of 100 kilometres costs 100 × Tsh 2 000 = Tsh 200 000. Thus, in stating the rate of two quantities, it is important to consider the unit of each quantity. For instance,

$$\text{Speed} = \frac{\text{Distance travelled in metres}}{\text{Time taken in seconds}}.$$

Rates can be written in a form similar to that of ratios. For example, 100 kilometres per 2 hours can be written in different ways as follows:

$$\frac{100 \text{ kilometres}}{2 \text{ hours}} \text{ or } \frac{50 \text{ km}}{1 \text{ h}} \text{ or } 50 \text{ km/h}.$$

Example 4.1

A man is paid Tsh 16 000 for 8 hours' work.

(a) What is the rate of payment?

(b) At this rate, how much would he receive for 20 hours of work?

(c) At this rate, for how long must he work in order to receive Tsh 30 000?

Solution

(a) To find the rate of payment, divide Tsh 16 000 by 8 hours, that is, $\frac{\text{Tsh } 16\ 000}{8 \text{ hours}} = \text{Tsh } 2\ 000$ per hour.

Therefore, the rate of payment is Tsh 2 000 per hour.

(b) Multiply Tsh 2 000 per hour by 20 hours to obtain the money he would receive by working for 20 hours.

Thus, Tsh 2 000 per hour $\times$ 20 hours = Tsh 40 000.

Therefore, for 20 hours work, he would receive Tsh 40 000.

(c) Divide Tsh 30 000 by Tsh 2 000 per hour.

$$= \frac{\text{Tsh } 30\ 000}{\text{Tsh } 2\ 000 \text{ per hour}}$$

$= 15$ hours

Therefore, he must work for 15 hours to receive Tsh 30 000.

Example 4.2

A student had two plant seedlings. She measured the rate at which the seedlings were growing. Seedling A grew 5 cm in 10 days and seedling B grew 8 cm in 12 days. Which seedling was growing more quickly?

Solution

The rates of growth of the two seedlings is computed as follows:

Rate of growth of seedling A $= \frac{5 \text{ cm}}{10 \text{ days}} = 0.5$ cm per day

Rate of growth of seedling B $= \frac{8 \text{ cm}}{12 \text{ days}} = 0.67$ cm per day

The growth rate of seedling B was greater than that of seedling A. Therefore, seedling B was growing more quickly than seedling A.

Exercise 4.1

1. What rate in metres per second is equivalent to a speed of 45 kilometres per hour?
2. A car covers a distance of 200 kilometres in 60 minutes. What is the speed of the car in metres per second?
3. A water tap takes 10 minutes to fill a 500 litres tank. Find the flow rate of water in litres per second.
4. If premium petrol costs Tsh 2 500 per litre, how many litres can be purchased with Tsh 10 000?

5. Find the rate in kilometres per hour at which a car travels if it covers $115\frac{1}{2}$ kilometres in 35 minutes.

Exchange rates

Activity 4.2: Exchanging Tanzania currency with other currencies

Individually or in a group, perform the following tasks:

1. Find a chart of foreign currency exchange rates from valid sources such as banks.
2. Determine the price in US dollars of a commodity of your choice such as a car, bicycle, shoes, watch, or smart phone.
3. Use the chart to convert the price of a commodity chosen in task 2 into Tanzania currency.
4. Convert the price of the item to any other three currencies of your choice.
5. Among the currencies you have converted, which one has the highest value, and which one has the lowest value?
6. Share your final work with the whole class through a class discussion.

In any country, people expect to do transanctions in the currency of their own country. When money from country A is to be used in country B, it is necessary to exchange the currency of country A to the currency of country B. Various currencies in the world are linked together by exchange rates. This enables smooth transfer of money and payments to take place between countries.

Table 4.1 shows foreign mean exchange rates as supplied by one of the commercial banks in Tanzania on 15th March, 2021. These are mean rates in Tanzanian shillings per unit of a foreign currency.

Table 4.1: Mean exchange rates of different currencies on 15th March, 2021

Country	Currency	Tanzanian shillings
United States	1 Dollar	2 339.0
Europe	1 Euro	2 830.0
Japan	1 Yen	23.79
Britain	1 Pound Sterling	3 288
Switzerland	1 Franc	2 516.81
Canada	1 Dollar	1 863.89
Australia	1 Australian Dollar	1 811.66
Sweden	1 Kronor	282.56
Denmark	1 Kronor	381.91
Norway	1 Kronor	283.97
Kenya	1 Shilling	24.15
Uganda	1 Shilling	0.66
South Africa	1 Rand	161.00
Zambia	1 Kwacha	110.00
Malawi	1 Kwacha	3.21
Mozambique	1 Metical	0.09
Botswana	1 Pula	214.97
New Zealand	1 New Zealand Dollar	1 669.79
Saudi Arabia	1 Riyal	619.52
India	1 Rupee	34.37

Example 4.3

A tourist from Sweden wished to exchange 1 000 Kronor into Tanzanian shillings. How many Tanzanian shillings did the tourist receive if the mean exchange rate was as shown in Table 4.1?

Solution

Let x be the amount of money in Tanzanian shillings the tourist received. Table 4.1 shows that 1 Swedish Kronor was equivalent to Tsh 282.56, that is,

$$\begin{aligned} 1 \text{ Kronor} &= \text{Tsh } 282.56 \\ 1\,000 \text{ Kronor} &= x \end{aligned}$$

Cross multiplication gives

$$x = \frac{1\ 000 \text{ Kronor} \times \text{Tsh } 282.56}{1 \text{ Kronor}}$$

$$= \text{Tsh } 282.56 \times 1\ 000$$

$$= \text{Tsh } 282\ 560$$

Therefore, the tourist received Tsh 282 560.

Example 4.4

How many US Dollars worth Tsh 600 000 on 15th March, 2021 as far as exchange rates in Table 4.1 is concerned?

Solution

Let y be the amount of money in US Dollars. Table 4.1 shows that 1 US Dollar was equivalent to Tsh 2 339.0. That is,

$$1 \text{ US Dollar} = \text{Tsh } 2\ 339.0$$

$$y = \text{Tsh } 600\ 000$$

Cross multiplication gives

$$y = \frac{1 \text{ US Dollar} \times \text{Tsh } 600\ 000}{\text{Tsh } 2\ 339.0}$$

$$= 256.5 \text{ US Dollars}$$

Therefore, 256.5 US Dollars worth Tsh 600 000 on 15th March, 2021.

Example 4.5

A British traveler had £800 on arriving in Tanzania. How many Tanzanian shillings did this amount of money exchanged for on 15th March, 2021 using the exchange rates shown in Table 4.1?

Solution

Let x be the amount in Tanzanian shillings that was exchanged for.

Table 4.1 shows that £1 was equivalent to Tsh 3 288, that is,

$$£1 = \text{Tsh } 3\ 288$$

$$£\,800 = x$$

Cross multiplication gives

$$x = \frac{£800 \times \text{Tsh } 3\ 288}{£1}$$

$$= \text{Tsh } 3\ 288 \times 800$$

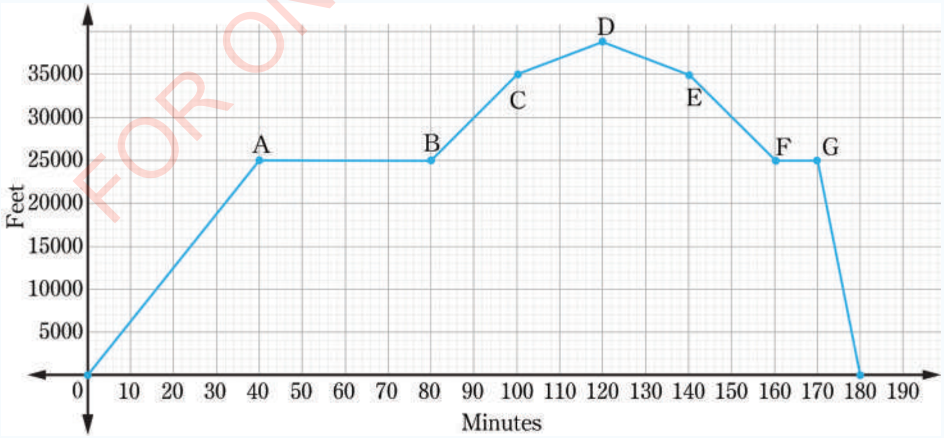
$$= \text{Tsh } 2\ 630\ 400$$

Therefore, £800 was exchanged for Tsh 2 630 400.

Exercise 4.2

Use Table 4.1 to answer questions 1 to 8.

1. How much is Tsh 20 600 worth in Indian Rupees?
2. Convert 1 New Zealand Dollar into Saudi Arabia Riyal.
3. How much is Tsh 500 000 worth in Euro?
4. How many Yen are equivalent to Tsh 1?
5. How many Tanzanian shillings can a visitor from Kenya exchange for 930 Kenyan shillings?
6. Find the amount in Tanzanian shillings for each of the following:
   (a) 30 000 Euros (b) 4 200 Pula
   (c) 640 Rands (d) 12 000 Riyal
7. Exchange Tsh 300 000 into the following currencies:
   (a) Metical
   (b) Malawian Kwacha
   (c) Franc
   (d) Rupee
8. How much is Tsh 6 000 000 worth in Pound Sterling?
9. Study the following figure which shows different altitudes attained by an ATCL plane and then answer the questions that follow.



(a) During its first forty minutes of flight, the plane took off to the altitude shown by point A. Estimate the rate of change in altitude during this time.
(b) Between points E and F, the plane is descending. Estimate the rate of change of altitude during this time.
(c) Is the rate of change in (b) positive or negative? Give reasons for your answer.
(d) Find the rate of change in altitude between points C and D. Is this rate greater or less than the rate between points D and E? How does the graph visually show that the two rates are different?
(e) Can the rate of change between any two points be 0? Briefly explain and give an example from the figure.

Variations

There are basically three types of variations, namely; direct, inverse, and joint variations.

Direct variations

Activity 4.3: Finding the relationship between volume and height

Individually or in a group, perform the following tasks:

1. Collect some round containers such as cups, cylinders, beakers, glasses or any other available containers within your environment. Collect as well a certain amount of water.
2. Measure the diameter (cm) and find the base area (cm^2) of the round containers and record them.
3. Pour same amount of water in each container and measure the height (cm) attained by water in each container. Make sure you use 5 different containers of different diameters.
4. Calculate the volume based on base area and each height of the water and record your data as shown in the following table.

Container label	Diameter	Area	Height	Volume

5. Use the table to plot a graph of volumes against heights.
6. What does the graph tell you about the relationship between volume and height?
7. What happens to the volume when you increase or decrease the height?
8. Use the graph to formulate an equation which describes the variation.
9. What generally can you say about the relationship observed between volume and height?
10. Share your findings to the rest of the class through a class discussion.

Some quantities are related in such a way that they increase or decrease together at the same rate. Quantities with this relationship are directly proportional, or vary directly. If a car travels at a constant speed, the distance it covers is directly proportional to the time taken. Similarly, the amount of maize you buy is directly proportional to the amount of money you pay.

If y is directly proportional to x, mathematically, it is written as $y \propto x$, where $\propto$ is a symbol of proportionality. This expression can be written as an equation by introducing equal sign and a constant of proportionality instead of the proportionality sign. For instance, if y varies directly as the square of x, then it can be written as $y \propto x^2$ and the corresponding mathematical equation is $y = kx^2$, where k is the constant of proportionality.

For any two pairs of quantities x and y, say, (x_1, y_1) and (x_2, y_2), two equations $y_1 = kx_1^2$ and $y_2 = kx_2^2$ are obtained. This implies that $k = \frac{y_1}{x_1^{\,2}} = \frac{y_2}{x_2^{\,2}}$. So, it is said that x and y vary directly if the ratios of the values of y to the values of x are proportional.

Example 4.6

If y varies directly as x and $x = 15$ when $y = 4$, find the value of y when $x = 12$.

Solution

Given that $y \propto x$. This implies that $y = kx$, where k is the constant of proportionality.

Making k the subject of the equation gives

$$k = \frac{y}{x}.$$

But, for any two pairs of quantities, $k = \frac{y_1}{x_1}$ and $k = \frac{y_2}{x_2}$.

Thus, $\frac{y_1}{x_1} = \frac{y_2}{x_2}$.

Given $x_1 = 15$, $x_2 = 12$, and $y_1 = 4$, the value of y_2 is obtained as follows:

$$y_2 = \frac{y_1 x_2}{x_1}$$

Substituting the given values gives

$$y_2 = \frac{4 \times 12}{15} = \frac{16}{5}.$$

Therefore, the value of y is $\frac{16}{5}$ when $x = 12$.

Example 4.7

If x varies directly as the square of y, and $x = 4$ when $y = 2$, find the value of x when $y = 8$.

Solution

Let $x_1 = 4$, $y_1 = 2$ and $y_2 = 8$. The variation equation is given by

$$\frac{x_1}{x_2} = \frac{{y_1}^2}{{y_2}^2}$$

Substituting the values of x_1, y_1, and y_2 gives

$$\frac{4}{x_2} = \frac{2^2}{8^2}$$

$$\frac{4}{x_2} = \frac{4}{64}$$

Cross multiplication gives

$$4 \times x_2 = 4 \times 64$$

$$x_2 = \frac{4 \times 64}{4}$$

$$= 64$$

Therefore, the value of x is 64 when y is 8.

Example 4.8

A car travels 60 kilometres using 5 litres of diesel. How many litres of diesel are needed to travel 150 kilometres?

Solution

Let x denote the number of litres of diesel and y denote the number of kilometres. The equation for the variation becomes

$$\frac{x_1}{x_2} = \frac{y_1}{y_2}.$$

Given $x_1 = 5$ litres, $y_1 = 60$ km and $y_2 = 150$ km, the value of x_2 is given by:

$$x_2 = \frac{x_1 y_2}{y_1}$$

$$= \frac{5 \text{ litres} \times 150 \text{ km}}{60 \text{ km}}$$

$$= 12.5 \text{ litres}$$

Therefore, 12.5 litres of diesel are needed to travel 150 kilometres.

Example 4.9

A precious stone worth 15 600 000 Tanzanian shillings is accidentally dropped and broken into three pieces. The weights of the pieces are in the proportions of 2:3:5, respectively. If the value of the piece of stone varies directly as the cube of its weight, calculate the value of the remaining stone in percentage.

Solution

Let V be the value in Tanzanian shillings of the stone of weight W.

The relation between V and W is given by $V \propto W^3$.

The variation equation of this relation is given by;

$$V = kW^3, \qquad (1)$$

where k is the constant of proportionality.

Since the weights of the three broken pieces of stones are in proportions of 2:3:5, let their weights be $2W$, $3W$ and $5W$, respectively.

Thus, the weight of unbroken stone $= 2W + 3W + 5W = 10W$

Substitution in the variation equation (1) gives

$$V = k(10W)^3$$

$$V = 1000kW^3$$

Hence, Tsh 15 600 000 $= 1\,000\, kW^3$

$$kW^3 = \frac{15\,600\,000}{1\,000}$$

$$kW^3 = 15\,600$$

Let V_1, V_2, and V_3 be the values in Tanzanian shillings of the three broken pieces of stones weighing $2W$, $3W$ and $5W$, respectively.

So, the result is

$$V_1 = k(2W)^3 = 8kW^3$$
$$V_2 = k(3W)^3 = 27kW^3$$
$$V_3 = k(5W)^3 = 125kW^3$$

The total value of the three pieces = Tsh V_1 + Tsh V_2 + Tsh V_3

= Tsh $8kW^3$ + Tsh $27kW^3$ + Tsh $125kW^3$

= Tsh $160kW^3$

= Tsh $160 \times 15\ 600$

= Tsh 2 496 000

The value of the remaining stone:

= Total value of the original stone – Total value of the three pieces of stones

= Tsh 15 600 000 – Tsh 2 496 000

= Tsh 13 104 000

Thus, the percentage value of the remaining stone

$$= \frac{\text{Remaining value (Tsh)}}{\text{Total value of the original stone}} \times 100\%$$

$$= \frac{\text{Tsh } 13\ 104\ 000}{15\ 600\ 000} \times 100\%$$

$$= 84\%$$

Therefore, the percentage value of the remaining stone is 84%.

If x and y represent variables such that $y \propto x$, then $y = kx$. The form of the equation $y = kx$ is a straight line passing through the origin, k being the gradient (slope) of the line. The graph in Figure 4.1 shows the relation $y \propto x$ for $k = 1$ and Figure 4.2 shows a graph of $y \propto (x + 1)$ for $k = 1$.

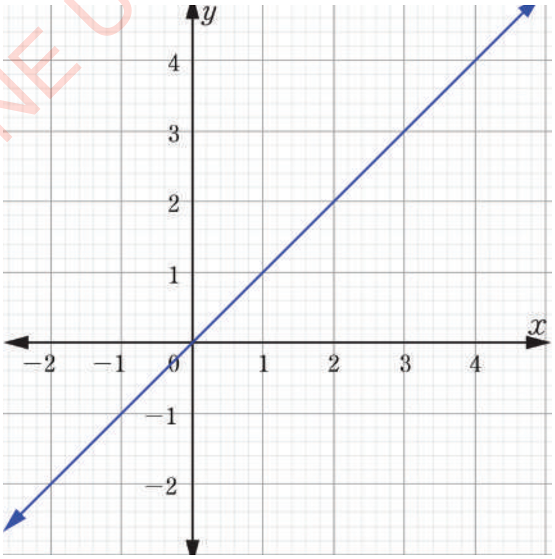


Figure 4.1: *Graph of the relation* $y \propto x$

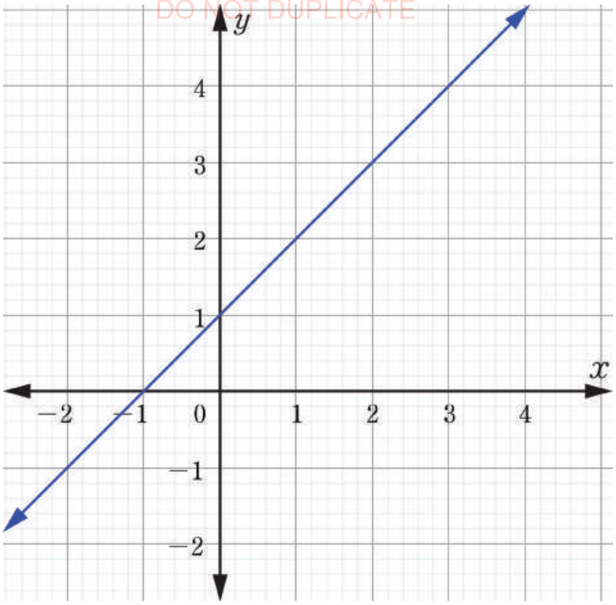


Figure 4.2: *Graph of the relation* $y \propto (x+1)$

Exercise 4.3

1. If x varies directly as y and $x = 16$ when $y = 10$, find the value of y when $x = 20$.
2. The surface area of a circular object varies directly as the square of its radius. If its surface area is 78.5 cm^2, when the radius is 5 cm, find the surface area of the circular object when the radius is 7 cm.
3. If x varies directly as y and $x = 30$ when $y = 40$, find the value of x when $y = 16$.
4. A mason can build 100 metres of fence in 20 hours. How long will it take 5 masons with the same ability to build 875 metres of fence?
5. If x varies directly as $2y + 7$ and $x = 5$, when $y = 4$, find the value of y when $x = 6$.
6. If 8 men can assemble16 machines in 12 days, how long will it take 15 men of the same ability to assemble 100 machines?
7. If y varies directly as the square root of x and $y = 12$ when $x = 4$, find the value of y when $x = 9$.
8. If y varies directly as x and $y = 8$ when $x = 3$, find the value of y when $x = 18$.

9. Two variables x and y which vary directly have corresponding values as shown in the following table.

x	3	5	6
y	17		34

   (a) Find the value of y when $x = 5$.
   (b) Find the rule connecting x and y.
   (c) Draw a graph which shows that $y \propto x$ for $k = 1$.

10. Study the following table and answer the questions that follow.

Hours worked	2	3	4	5
Earning (Tsh)	1150	1 725	2 300	2 875

   (a) Do earnings vary directly as the number of hours worked?
   (b) If so, calculate the constant of variation and write an equation that describes the relationship.

11. The volume of the sphere varies as the cube of its radius. Three solid spheres of diameters $\frac{3}{2}$m, 2m, and $\frac{5}{2}$m are melted and combined to form a new solid sphere. Find the diameter of the new sphere.

12. When watching at two buildings at the same time, the length of the buildings' shadows vary directly as their heights. If a 5 storey building has a shadow of length 20 m, how many storey buildings would form a shadow of length 32m?

Inverse variations

Activity 4.4: Examining the relationship between area and depth of liquids in containers

Individually or in a group, perform the following tasks:

1. Collect any round containers such as cylinders, beakers, glasses or any other available container within your environment and label them. Collect as well a certain volume of water such as 100 cm^3, 200 cm^3 or any volume of your choice.
2. Measure the diameter (cm) and find the base area (cm^2) of each round container and record it.

3. Pour the amount of water in each container and measure the height (cm) attained by water in each container.
4. Collect your data as shown in the following table

Container label	Diameter	Area	Height	Volume

5. Use the table to plot a graph of the heights (*y-axis*) against the base areas (*x-axis*).
6. What does the graph tell you about the relationship between area and height?
7. What happens to the height when you increase or decrease the area?
8. What generally can you say about the relationship observed between area and height?
9. Share your findings to the rest of the class through a class discussion.

In some cases, one quantity increases at the same rate as another quantity decreases. In this case, the quantities vary inversely, or they are in inverse proportion. Inverse proportion is sometimes called indirect proportion.

For example, the number of men employed to cultivate a farm and time it takes for them to complete the work are inversely related. Likewise, the time to travel to a certain place and the speed are inversely related. The proportionality symbol $\propto$ is also used for inverse proportions.
The statement y is inversely proportional to x is written as $y \propto \frac{1}{x}$. The variation equation is obtained by introducing a constant of proportionality to the relation. If $y \propto \frac{1}{x}$, then $y = \frac{k}{x}$, where k is the constant of propotionality.

Other cases of inverse variations

One quantity may vary inversely as some power of another.

1. If y varies inversely as the square of x, the equation connecting x and y is $y = \frac{k}{x^2}$ or $k = yx^2$.
2. If y varies inversely as the square root of x, the equation connecting x and y is $y = \frac{k}{\sqrt{x}}$ or $k = y\sqrt{x}$.

A sketch of a relation $y \propto \frac{1}{x^2}$ for $k = 1$ is shown in Figure 4.3. Note that, in the curve representing $y = \frac{k}{x^2}$ approaches both axes but does not touch them. This is because division by 0 is undefined.

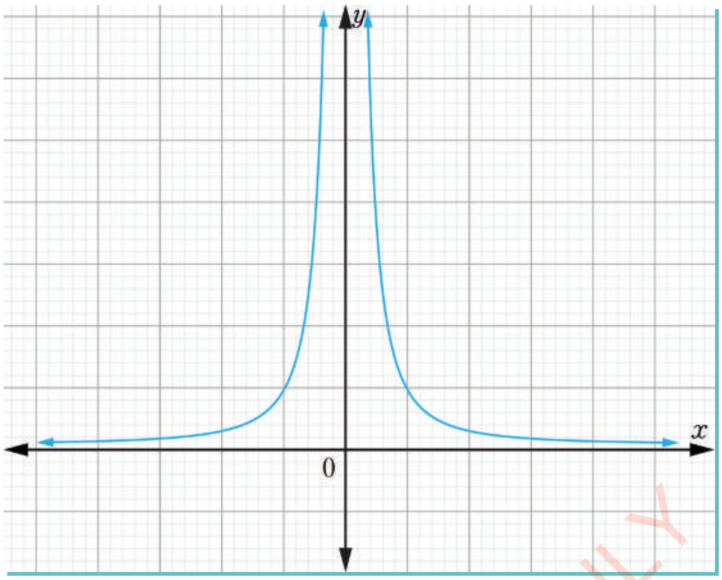


Figure 4.3: *Graph of the relation* $y \propto \frac{1}{x^2}$ for $k = 1$

Furthermore, if the product of two variables x and y is a constant, then it is said that x and y vary inversely. That is, if $xy = k$, then x and y vary inversely.

If $x_1y_1 = k$ and $x_2y_2 = k$, then $x_1y_1 = x_2y_2$ or $\frac{x_1}{x_2} = \frac{y_2}{y_1}$.

Example 4.10

If x varies inversely as y and $x = 2$ when $y = 3$, find the value of y when $x = 18$.

Solution

The statement $x \propto \frac{1}{y}$ implies that $x = \frac{k}{y}$, where k is the constant of proportionality. Making k the subject of the equation gives

$xy = k$, which implies that

$x_1y_1 = x_2y_2$.

When $x_1 = 2$, $y_1 = 3$, and $x_2 = 18$, the value of y_2 is

$$y_2 = \frac{x_1 y_1}{x_2} = \frac{2 \times 3}{18} = \frac{1}{3}.$$

Therefore, the value of y is $\frac{1}{3}$ when the value of x is 18.

Example 4.11

If it takes 12 days for 10 men to assemble a machine, how long does it take 15 men with the same ability to assemble the same machine?

Solution

Let m represent the number of men and d represent the number of days. It is obvious that 15 men will take less time to assemble the machine than 10 men. Thus, m and d vary inversely, that is,

$m \propto \frac{1}{d}$, which implies that $m = \frac{k}{d}$.

Thus, $k = md$.

For two pairs of quantities (m_1, d_1) and (m_2, d_2), we have

$m_1 d_1 = m_2 d_2$.

Therefore, $d_2 = \frac{m_1 d_1}{m_2}$.

Given $d_1 = 12$ days, $m_1 = 10$ men and $m_2 = 15$ men, the value of d_2 is

$$d_2 = \frac{10 \text{ men} \times 12 \text{ days}}{15 \text{ men}}$$
$$= 8 \text{ days}$$

Therefore, it takes 8 days for 15 men to assemble the same machine.

Recall that, direct variation is a linear function which can be described by an equation of the form $y = kx$, where k is a constant of proportionality which is not equal to zero. In direct variation, if one quantity increases the other quantity increases as well, and if one quantity decreases the other quantity decreases. Inverse variation is described by the equation $y = \frac{k}{x}$, which implies that, when one quantity increases the other quantity decreases and vice versa.

Exercise 4.4

1. If y varies inversely a x and if $y = 3$ when $x = 4$, find y when $x = 6$.
2. If x varies inversely as the square of y and if $x = 3$ when $y = 4$, what is the value of x when $y = 8$?
3. If x varies inversely as y and if $x = 4$ when $y = \frac{3}{2}$, then find the value of y when $x = 8$.
4. If y is inversely proportional to x and $x = 2\frac{1}{2}$ when $y = 2$, find the value of y when $x = 4$.
5. If x varies inversely as $2y - 1$ and $x = 2$ when $y = 3$, find the value of y when $x = 6$.
6. If q varies inversely as the square of p and $p = 8$ when $q = 2$, find the value of q when $p = 4$.
7. If y is inversely proportional to x and if $x = 5$ when $y = 6$, find the value of x when $y = 20$.
8. If y is inversely proportional to x and if $y = 3$ when $x = 4$, find the value of y when $x = 6$.
9. If y varies inversely as the cube root of x, and if $y = 3$ when $x = 64$, find the formula connecting the variables. Hence, find the value of x when $y = \frac{15}{4}$.
10. The time t in seconds that Mary takes to return home from school varies inversely with her average speed v in metres per second. If Mary gets back home in half an hour at an average speed of 10 m/s;
    (a) write and graph an equation expressing t in terms of v.
    (b) if Mary wants to get back home in 15 minutes, what must be her average speed?
    (c) why does the graph in (a) never crosses either axes?
    (d) what happens to t as v increases? As v decreases? How do t and v vary?
11. The intensity of light varies inversely as the square of the distance from the light source. If the intensity from a light source 90 cm away is 12 lumen, how far should the light source be so that the intensity is 4 lumen?

Joint variations

A given quantity can relate directly or inversely with two or more quantities. This kind of relationship is called joint variation. Joint variation can involve variables which both varry either directly or inversely. Also, a joint variation can involve a variable which vary directly with some other variables as well as inversely with other variables.

If x varies directly as y and inversely as z, then $x \propto y$ and $x \propto \frac{1}{z}$. This implies that $x \propto \frac{y}{z}$. Therefore, the variation equation of this joint variation is given by $x = \frac{ky}{z}$.

An example of joint variation is the increase in price of petrol (x) which leads to an increase in transportation costs (y) and eventually causes an increase in the cost of commodities (z).

Activity 4.5: Discovering joint variations in real life

Perform the following tasks individually or in a group:

1. Identify two real life events or situations in which two or more variables relate jointly.
2. Write the relationship of the joint variation by using proportionality sign.
3. Obtain values of the variables through mesurements and find the proportionality constant.
4. Find different values of one of the variables if the values of other variables obtained through measurements are given.
5. Share with the rest of the class on how joint variation can be useful in solving daily life problems.

Example 4.12

Suppose y varies directly as x and z. Given $x = 4$, $z = 2$ and $y = 24$, find:

(a) The variation equation connecting x, y, and z.

(b) The value of y when $x = 5$ and $z = 6$.

Solution

(a) Since $y \propto x$ and $y \propto z$, it follows that $y \propto xy$.

The variation equation is $k = \frac{y}{xz}$, where k is a constant of proportionality.

Given $x = 4$, $z = 2$, and $y = 24$.

$$k = \frac{24}{4 \times 2}$$
$$= 3$$

Therefore, the variation equation becomes $y = 3xz$.

(b) With $y = 3xz$, if $x = 5$ and $z = 6$, then

$$y = 3 \times 5 \times 6$$
$$= 90$$

Therefore, the value of y is 90.

Example 4.13

If x varies directly as y and inversely as z, and $x = 8$ when $y = 12$ and $z = 6$, find the value of x when $y = 16$ and $z = 4$.

Solution

If x varies directly as y and inversely proportional as z, that is, $x \propto y$ and $x \propto \frac{1}{z}$. The joint variation becomes $x \propto \frac{y}{z}$. The variation equation is given by $x = \frac{ky}{z}$. It implies that

$$\frac{x_1 z_1}{y_1} = \frac{x_2 z_2}{y_2}$$

Given $x_1 = 8$, $y_1 = 12$, $z_1 = 6$, $y_2 = 16$ and $z_2 = 4$, then

$$x_2 = \frac{x_1 z_1 y_2}{y_1 z_2}$$
$$= \frac{8 \times 6 \times 16}{12 \times 4}$$
$$= 16$$

Therefore, the value of x is 16.

Example 4.14

Three tailors can sew 15 clothes in 5 days. How long will it take 5 tailors working at the same speed to sew 20 clothes?

Solution

Let t, d and c represent the numbers of tailors, days and clothes, respectively. The number of tailors is inversely proportional to number of days, that is, $t \propto \frac{1}{d}$ and directly proportional to number of clothes, that is, $t \propto c$. Combining the two proportions gives a joint variation given by

$$t \propto \frac{c}{d}.$$

The variation equation is then given as

$$t = k\frac{c}{d}.$$

The value of k is computed as follows:

Given $t = 3$, $d = 5$, and $c = 15$,

$$k = \frac{td}{c}$$
$$= \frac{3 \times 5}{15}$$
$$= 1.$$

So, the variation equation is $t = \frac{c}{d}$.

Using $t = 5$ and $c = 20$, then

$$5 = \frac{20}{d}$$
$$5d = 20$$
$$d = 4.$$

Therefore, it will take 4 days for 5 tailors to sew 20 clothes.

Example 4.15

Nine workers work 8 hours a day to complete a piece of work in 52 days. How long will it take 13 workers to complete the same job by working 6 hours a day?

Solution

Let w, h, and d represent the number of workers, hours, and days, respectively. From the question, it follows that the number of workers varies inversely as the number of hours and days, that is,

$$w \propto \frac{1}{h} \text{ and } w \propto \frac{1}{d}.$$

The joint variation is given by the relation

$$w \propto \frac{1}{hd}.$$

The variation equation is

$$w = \frac{k}{hd}, \text{ where } k \text{ is a constant of proportionality.}$$

Thus,

$$w_1 = \frac{k}{h_1 d_1}$$

$$w_2 = \frac{k}{h_2 d_2}$$

Hence, $\frac{w_1}{w_2} = \frac{h_2}{h_1} \times \frac{d_2}{d_1}$.

Given $d_1 = 52$ days, $h_1 = 8$ hours, $w_1 = 9$ workers, $w_2 = 13$ workers, and $h_2 = 6$ hours, then

$$\frac{9}{13} = \frac{6}{8} \times \frac{d_2}{52},$$

$$d_2 = \frac{9 \times 52 \times 8}{13 \times 6}.$$

$$d_2 = 48 \text{ days}.$$

Therefore, it will take 48 days for 13 workers to complete the piece of work.

Exercise 4.5

1. If y varies directly as the square of x and inversely as z, find the percentage change in y when x is increased by 10% and z is decreased by 20%.
2. Suppose P varies directly as V and inversely as the square root of R. If $P = 180$ when $R = 25$ and $V = 9$, find the value of P when $V = 6$ and $R = 36$.
3. The height h of a cone varies directly as its volume V and inversely as the square of its radius r. Write a formula for the height of the cone.
4. If y^2 varies directly as $x - 1$ and inversely as $x + d$ and $x = 2$, $d = 4$ when $y = 1$, find the value of x when $y = 2$ and $d = 1$.
5. If 2 typists in a typing pool can type 210 pages in 3 days, how many typists working at the same speed will be needed to type 700 pages in 2 days?
6. Suppose x varies directly as y^2 and inversely as p. If $x = 2$, when $y = 3$ and $p = 1$, find the value of y when $x = 4$ and $p = 5$.
7. If V varies directly as the square of x and inversely as y, and if $V = 18$ when $x = 3$ and $y = 4$, find the value of V when $x = 5$ and $y = 2$.
8. Sketch the following curves. Assume that the constant of proportionality is 1.
   (a) $y \propto \frac{1}{x}$ (b) $y \propto \frac{1}{x^2}$ (c) $y \propto \frac{1}{\sqrt{x}}$.

9. The following table shows the values of y for some selected values of x. The variables x and y are connected by the relation, 'y varies inversely as x'. Calculate the missing values of y.

x	5	10	15	20
y		3		1.5

10. Express each of the following relations as an equation using k as constant of proportionality.
    (a) c varies directly as p and q, and inversely as s.
    (b) d varies jointly as t and r^2.
    (c) d varies directly as y and the square root of z.

Chapter summary

1. A rate gives the change of one quantity with respect to another.
2. Exchange rate is the conversion rate among different currencies.
3. If $y = kx$, then y varies directly with x, or y is directly proportional to x. The constant k is the constant of proportionality.
4. If y varies as $\frac{1}{x}$, then y is inversely proportional to x.
5. If a quantity varies as the product of two or more quantities, then it varies jointly with other quantities.

Revision exercise 4

1. If $y = kx$, and $y = 8$ when $x = 7$, find the value of k and the value of y when $x = 40$.
2. If y is directly proportional to x and $y = 10$ when $x = 4$, find the value of y when $x = 15$ and the value of x when $y = 8.4$.
3. If $y \propto x$ and $y = 16.5$ when $x = 3.5$, find the equation connecting x and y. Hence, find the value of x when $y = 21$.
4. If y is proportional to x^2 and if $x = 15$ when $y = 200$, find the equation connecting x and y. Find the value of y when $x = 8.5$.
5. If $y \propto \sqrt{x}$ and $y = 3.5$ when $x = 4$, express y in terms of x. What is the value of y when $x = 25$?
6. If $y \propto \frac{1}{x}$, fill in the gaps in the following table.

x		1.2	8	
y	6		1.5	0.8

7. Given that y varies directly as x and inversely as z. If $y = 10$ when $x = 8$ and $z = 5$, find the equation connecting x, y and z. Find the value of y when $x = 6$ and $z = 2.5$.

8. If y varies jointly as x and z^2, and if $y = 13\frac{1}{3}$ when $x = 2.5$ and $z = \frac{4}{3}$, find the equation connecting the three variables. Find the value of x when $z = \frac{3}{2}$ and $y = 54$.

9. Suppose y varies directly as x^2 and inversely as $\sqrt{z}$. If $x = 8$, $y = 16$, and $z = 25$, find the value of y when $x = 5$ and $z = 9$.

10. Determine whether the data in the following tables have an inverse variation relationship. If they do, find the missing values.

(a)

x	y
7	10
9	12
12	15
	6

(b)

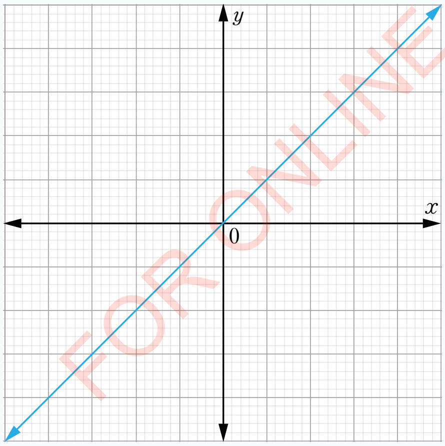
x	y
12	4
6	2
21	7
	3

(c)

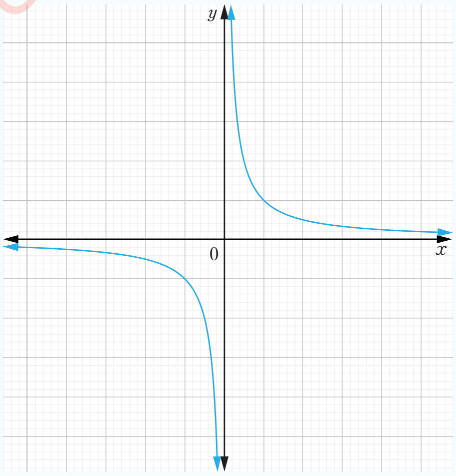
x	y
−15	−8
−8	−15
	10

11. For each of the following figures, write an equation that can represent the function. What type of variation does each figure represent?

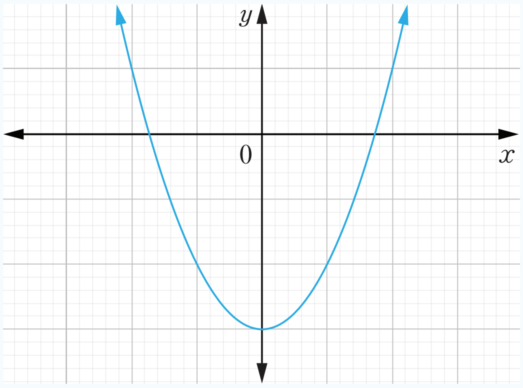
(a)



(b)



(c)



12. For a fixed amount of simple interest I in one year, the yearly interest rate r and the principal P vary inversely. Which of the following equations do not show this relationship?

(a) $P = \frac{I}{r}$

(b) $r = \frac{P}{I}$

(c) $r = \frac{I}{P}$

(d) $Pr = 1$

13. Do speed and time vary directly or inversely? Explain.
14. Do the distance and speed vary directly or inversely? Explain.
15. If x and y vary inversely, use the given pair of values to find an equation which in each case relate the variables:

(a) $x = 6, y = 4$

(b) $x = 8, y = 12$

(c) $x = 10, y = \frac{1}{3}$

Chapter Five

Sequences and series

Introduction

In daily life, there are various events and situations occuring in a coordinated manner, such that one event leads to the occurance of another event. In Mathematics, the occurance of such patterns is referred to as sequences and series. In this chapter, you will learn the concepts of sequences and series, identify Arithmetic Progression (AP), and Geometric Progression (GP). You will also learn how to find the general term of an AP and GP, derive formulae for the sum of AP and GP, calculate arithmetic mean, geometric mean, and compound interest using formulae. The competencies developed in this chapter are applied in different real life situations such as in determining the compound interest, predicting population growth and decay, determining the age of living and non-living things, predicting events which occur at different times and many other applications.

Sequences

Activity 5.1: Making shapes with objects

Individually or in a group perform the following tasks:

1. Use matchsticks to make shapes similar to that figure shown in Figure 5.1.

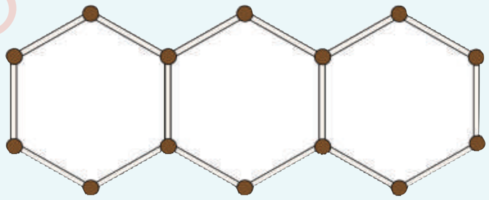


Figure 5.1: *Hexagons formed by arranging matchsticks*

2. Make three more hexagons to have a total of six and use the resulting figures to complete Table 5.1.

Table 5.1: Number of hexagons formed by different number of matchsticks

Number of hexagons	1	2	3	4	5	6
Number of matchsticks	6	11				

3. How many matchsticks are required to make a shape similar to that in Figure 5.1 with:
   (i) 8 hexagons?
   (ii) 16 hexagons?
   (iii) 100 hexagons?
4. Explain how you arrived at the answers in task 1 and task 2.
5. Explain using an equation if possible, how you can find the number of matchsticks for any number of hexagons you can make.
6. Present your final work to the rest of the class for further discussion and inputs.

There are sets of numbers with simple patterns. Examples of such sets are; the set of natural numbers 1, 2, 3, 4, 5, . . ., the set of even numbers 2, 4, 6, 8, 10, 12, . . ., the set of odd numbers 1, 3, 5, 7, . . . and so on. Given any of these sets, it is easy to determine the next number from the previous one. For example, the pattern 1, 2, 4, 7, 11, . . . shows that the first number is 1, the second is 2, the third is 4, the fourth is 7 and the fifth is 11 and so on. The pattern is such that the difference between two consecutive numbers follows the pattern of natural numbers. Hence, the sixth number which follows after 11 is 16.

In general, a pattern is the order or rule which help to find other numbers in a given set. A member of a particular pattern is called a term. Each term has its particular position; a term in the first position is called the first term, a term in the second position is called the second term, and so on. Using such naming, a term in the n^{th} position is called the n^{th} term or the general term.

The order in which terms appear in a certain pattern is important. For example, the pattern 1, 2, 3, 4, . . . is completely different from the pattern 4, 3, 2, 1, . . . because they have different ordering.

A sequence is a set of terms which are ordered and have a well defined pattern of formation. Sequences are either finite or infinite. The terms of finite and infinite sequences can be organized in order of increasing or decreasing values.
A sequence in which its terms are increasing in values is called an increasing sequence. For instance, the sequence 11, 18, 25, 32, 39, . . . is an example of an increasing sequence.

A sequence in which its terms are decreasing in values is called a decreasing sequence. For example; 64, 16, 4, 1, . . . is a decreasing sequence.
A sequence with a known number of terms is known as a finite sequence. For example; $-9,\ -6,\ -3,\ 0,\ 3,\ 6,\ 9$ is a finite sequence.
A sequence with unlimited number of terms is called an infinite sequence. For example; $\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \frac{1}{54}, \ldots$ is an infinite sequence.

Example 5.1

List down the terms of the sequences obtained from the following descriptions:
(a) Multiples of 3 between 0 and 20.
(b) Squares of the counting numbers from 1 to 10.
(c) Counting numbers less than 50 but divisible by 9.

Solution
(a) The terms are 3, 6, 9, 12, 15, 18.
(b) The terms are 1, 4, 9, 16, 25, 36, 49, 64, 81, 100.
(c) The terms are 9, 18, 27, 36, 45.

The terms of a sequence can be extended if the pattern governing the terms is known. The pattern is obtained by examining the relationship of three or more consecutive terms. For example, the next term of the sequence 1, 2, 3, 4, . . . must be 5. This is because every term differs from the previous term by 1. Similarly, the next two terms for the sequence 2, 1, 4, 3, 6, 5, . . . are 8 and 7 because the second term is obtained by subtracting 1 from the first term and the third term is obtained by adding 3 to the second term and the process repeats.

Example 5.2

Find the fifth term of the sequence 1, 4, 7, 10, . . .

Solution

From the sequence, the difference between two consecutive terms is 3. Hence, every term is obtained by adding 3 to the previous one. Therefore, the fifth term is $10 + 3 = 13$.

Example 5.3

Given the sequence 2, 4, 6, 8, . . . , find:

(a) The fifth term

(b) The nth term

Solution

(a) The difference between two consecutive terms is 2. Hence, every term is obtained by adding 2 to the previous term. Thus, the 5th term $= 8 + 2 = 10$.

(b) The 1st term is $2 = 2 \times 1$

The 2nd term is $4 = 2 \times 2$

The 3rd term is $6 = 2 \times 3$

The 4th term is $8 = 2 \times 4$

$\vdots \quad \vdots$

$2 \times n$

Therefore, the nth term is $2 \times n$ or $2n$.

Example 5.4

Find the next two terms in each of the following sequences:

(a) 22, 37, 52, 67, . . .

(b) 4, 2, 0, −2, . . .

(c) $\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \frac{1}{24}, \ldots$

Solution

(a) In the sequence 22, 37, 52, 67, . . . every two consecutive terms differ by 15. Thus, the fifth term is 67+15 = 82, and the sixth term is 82 + 15 = 97. Therefore, the next two terms are 82 and 97.

(b) In the sequence 4, 2, 0, −2, . . . every two consecutive terms differ by −2. Therefore, the next two terms are −4 and −6.

(c) In this sequence, $\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \frac{1}{24}, \ldots$ the denominators have the following sequence, 3, 6, 12, 24, . . . Every term is obtained by multiplying the previous term by 2, so that the next two denominators are 48 and 96. Therefore, the next two terms are $\frac{1}{48}$ and $\frac{1}{96}$.

Example 5.5

If the nth term of the sequence is $3n - 5$, find the third term.

Solution

If the nth term is $3n - 5$, then the 3rd term is obtained when $n = 3$.

The 3rd term $= 3 \times 3 - 5$

$= 4$

Therefore, the third term is 4.

Exercise 5.1

1. Find the next two terms of each of the following sequences:

   (a) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$

   (b) $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots$

   (c) 1, 2, 6, 24, 120, . . .

   (d) 1, −1, 1, −1, . . .

2. Find the nth term of the sequence 1, 3, 5, 7, . . .
3. Find the nth term of the sequence 3, 6, 9, 12, . . .
4. If the nth term of a sequence is given by $2n + 1$, find the tenth term.
5. If the nth term of a sequence is $4n - 1$, write the set of the first five terms.
6. Find the general term of the sequence 1, −1 , 1, −1, 1,…
7. The nth term of a certain sequence is 2^{n-1}. Find the sum of the first three terms.
8. Find the general term of the sequence 3, 7, 11, 15, 19, . . .

9. Find the next two terms of the sequence $\frac{5}{2}, \frac{13}{4}, 4, \frac{19}{4}, \ldots$

10. The general term of a certain sequence is $\frac{1}{2+(-1)^n}$. Find the first four terms. Is the sequence increasing or decreasing? Give reasons.

Series

Activity 5.2: Deducing the general pattern of a series

Perform the following tasks individually or in a group:

1. Put matchsticks or toothpicks on your desk.
2. Use the matchsticks to form figures as shown in Figure 5.2.

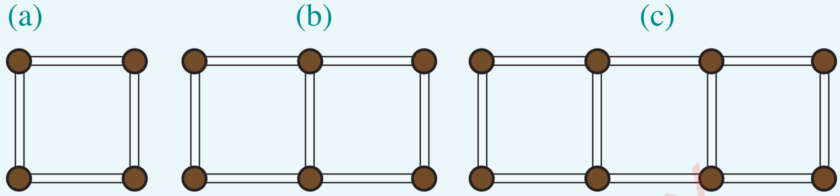


Figure 5.2: *Pattern formed by matchsticks*

3. As you form the shapes, count the number of matchsticks used. You can keep on forming the shapes and study the nature of the pattern as you increase the number of shapes and the number of matchsticks.
4. Use the pattern to determine how many matchsticks you would need to make a figure of 10-squares.
5. If you are to make a figure of n–squares, how many matchsticks would you need?
6. Share your final work to the rest of the class through class discussion.

In the previous section, you have learnt that, when a set of terms is written in a definite order and there is a rule by which the terms are obtained, then the set of terms is called a sequence. When the terms of sequence are connected by addition or subtraction signs, the resulting expression is known as a series or a progression. Some examples of series are:

(a) $1+2+3+4+\cdots+50$.

(b) $2+4+6+8+\cdots+20$.

(c) $1+-1+1+-1+\cdots$

(d) $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots$

(e) $-2-4-6-8-10-\cdots$

If a series ends after a finite numbers of terms, the series is said to be finite. However, a series is infinite if it does not have an end. Thus, $1+2+3+4+5$ is a finite series while $1+2+3+4+5+\cdots$ is an infinite series.

The sum of n^{th} terms of a series

Consider a series having n terms, the sum of all the n terms of the series is usually denoted by S_n. Thus, the sum of $1+2+3+4+5+6+\cdots+n$ can be written as $S_n = 1+2+3+4+5+6+\cdots+n$. In this case, n is the general term or the last term of the finite series $1+2+3+4+5+6+\cdots+n$.

In a simple infinite series, the general term can be found by studying few terms in the given series. For instance, the series $10+20+30+40+\cdots$ has its terms as multiples of 10, where

(i) the first term is 1×10, with $n = 1$.

(ii) the second term is 2×10, with $n = 2$.

(iii) the third term is 3×10, with $n = 3$.

This implies that the n^{th} term is $n\times 10$.

Therefore, the series $10+20+30+40+\cdots$ can also be written as $10+20+30+40+\cdots+10n+\cdots$

Example 5.6

Consider the sequence $2,\ 4,\ 6,\ 8,\ 10,\ 12,\ \ldots$

(a) Write the series corresponding to the sequence.

(b) Find the sum of the first four terms.

Solution

(a) The series corresponding to the given sequence is $2+4+6+8+10+12+\cdots$

(b) Since the first four terms are 2, 4, 6 and 8, the sum is $2+4+6+8 = 20$.

Example 5.7

Consider the sequence 1, 3, 5, 7, . . .

(a) Find the general term of the sequence.

(b) Write its corresponding series.

(c) Find the sum of the first five terms.

Solution

(a) From the sequence,

The first term: $1 = 2 \times 1 - 1$

The 2^{nd} term: $3 = 2 \times 2 - 1$

The 3^{rd} term: $5 = 2 \times 3 - 1$

The 4^{th} term: $7 = 2 \times 4 - 1$

$\vdots \quad \vdots$

The n^{th} term: $2 \times n - 1 = 2n - 1$

Therefore, the general term is $2n - 1$.

(b) The corresponding series is $1 + 3 + 5 + 7 + \cdots + (2n - 1) + \cdots$

(c) The first five terms are 1, 3, 5, 7, 9 whose sum is $1 + 3 + 5 + 7 + 9 = 25$.

Exercise 5.2

1. If the general term of a certain sequence is $2(-1)^n$,
   (a) write its series.
   (b) find the sum of the series up to the sixth term.
2. Find the n^{th} term of the series $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots$
3. Find the sum of the first ten terms of the series $-4 - 1 + 2 + 5 + \cdots$
4. The first term of a certain series is k, the second term is $2k$, and the third term is $3k$. Find:
   (a) The n^{th} term.
   (b) The sum of the first ten terms.
5. Find the next two terms of the following series:
   (a) $3 + 9 + 27 + 81 + \cdots$
   (b) $1 + -1 + 1 + -1 + \cdots$
6. The first term of a certain series is a, the second term is ar, and the third term is ar^2. Find:
   (a) The 6^{th} term.
   (b) The n^{th} term.

7. If the n^{th} term of a certain series is $\frac{1}{n(n+1)}$,
   (a) write its corresponding series.
   (b) find the sum of the first three terms.
8. Consider the following series $a + (a + d) + (a + 2d) + (a + 3d) + \cdots$ Find the,
   (a) difference between the second and the first term.
   (b) difference between the third and the second term.
   (c) difference between the fifth and the fourth term.
   (d) expression of the general term.
9. If a sequence is given as $a, ar, ar^2, ar^3, ...$
   (a) What is the value when the second term is divided by the first term.
   (b) What is the value when the fifth term is divided by the fourth term.
   (c) Write down the corresponding series.
   (d) Form the n^{th} term of the corresponding series.
10. A lecture hall has 20 rows of seats, with 100 seats in the back row. Each row has 2 fewer number of seats than the row immediately behind it. How many seats are there in the lecture hall?

Arithmetic Progression (AP)

Activity 5.3: Deducing an arithmetic series

1. Individually or in a group, collect an electronic geoboard from your subject teacher or you can make one from locally available materials such as nails and boards.
2. Make a pattern of nested squares by arranging rubber bands on a geoboard or arranging matchsticks or toothpicks on the surface of a table as shown in Figure 5.3.

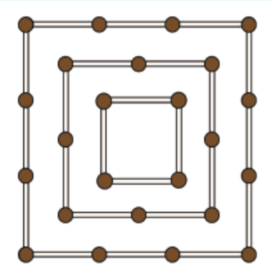


Figure 5.3: *A pattern of nested squares*

3. Study the pattern carefully to determine the number of sticks or rubber bands used in each square.
4. Study carefully the pattern when increasing the number of rubber bands or sticks as you increase the number of nested squares.
5. From the pattern observed in task 4, derive a general rule for obtaining the number of rubber bands or sticks which may be used as you increase the number of the nested squares.
6. If you have 312 sticks or rubber bands, how many squares will the final pattern contain?
7. Prepare a poster showing a summary of your work and use it in a class discussion.

A sequence of numbers in which each term after the first is obtained by adding a fixed number to the preceding term is known as an Arithmetic Progression (AP). The fixed number which is the difference between any two consecutive terms is called the common difference, denoted by d.

An arithmetic series is obtained by adding the terms of an arithmetic progression. Examples of arithmetic series with common differences 1, -2 and $4x$, are respectively:

1. $1 + 2 + 3 + 4 + \cdots + 99$
2. $-1 - 3 - 5 - 7 - \cdots$
3. $x + 5x + 9x + 13x + \cdots$

The nth term of an arithmetic progression (AP)

If n is the number of terms of an arithmetic progression (AP) with the first term A_1, then the nth term is denoted by A_n and the common difference by d.

Consider a problem of counting in 2's starting from one. It means the first term A_1 is 1 and the common difference d is 2. To find the 23rd term, for instance, do the following steps:

$$A_1 = 1 \text{ and } d = 2$$
$$A_2 = 1 + d = 1 + 2$$
$$A_3 = 1 + d + d = 1 + 2d = 1 + (2 \times 2)$$
$$A_4 = 1 + d + d + d = 1 + 3d = 1 + (3 \times 2)$$
$$A_5 = 1 + d + d + d + d = 1 + 4d = 1 + (4 \times 2) \text{ and so on.}$$

From the above pattern, the 23th term is $A_{23} = 1 + (22 \times 2) = 45$.

Similarly, the n^th^ term is obtained by adding $n-1$ times the common difference 2 to the first term. Then, A_n is given by:

$$A_n = 1 + 2(n-1) = 2n-1.$$

Therefore, we obtain the series $1 + 3 + 5 + 7 + \cdots + (2n-1)$.

In general, if d is the common difference between two successive terms of an arithmetic progression, we use the same procedure to generate the n^th^ term as follows:

$$\begin{aligned}
A_1 &= A_1 \\
A_2 &= A_1 + d \\
A_3 &= A_2 + d = (A_1 + d) + d = A_1 + (3-1)d \\
A_4 &= A_3 + d = (A_1 + 2d) + d = A_1 + (4-1)d \\
A_5 &= A_4 + d = (A_1 + 3d) + d = A_1 + (5-1)d \\
A_6 &= A_5 + d = (A_1 + 4d) + d = A_1 + (6-1)d
\end{aligned}$$

The n^th^ term A_n is obtained by adding A_1 to the product of $(n-1)$ and the common difference d. Therefore, the n^th^ term is given by:

$$A_n = A_1 + (n-1)d.$$

Example 5.8

Identify the arithmetic series from the following expressions and write down the common difference of each arithmetic series:

(a) $1 + \frac{3}{2} + 2 + \frac{5}{2}$

(b) $2 + 4 + 8 + 16$

(c) $10 + 8 + 6 + 4 + 2$

(d) $-2 - 5 - 8 - 11$

(e) $1^2 + 2^2 + 3^2 + 4^2$

(f) $1 - 3 + 5 - 7 + 9$

Solution

(a) It is an arithmetic series with a common difference, $d = \frac{1}{2}$.

(b) It is not an arithmetic series since it does not have a common difference.

(c) It is an arithmetic series with a common difference, $d = -2$.
(d) It is an arithmetic series with a common difference, $d = -3$.
(e) It is not an arithmetic series because it has no common difference.
(f) It is not an arithmetic series as it does not have a common difference.

Example 5.9

The first term of an arithmetic progression is 6 and the common difference is 5. Find:

(a) The third term.   (b) The nth term.

Solution

(a) Given $A_1 = 6, \ d = 5, n = 3$

From $A_n = A_1 + (n-1)d$

$$A_3 = 6 + (3-1)\times 5$$
$$= 6 + 10$$
$$= 16.$$

Therefore, the third term is 16.

(b) $$A_n = A_1 + (n-1)d = 6 + (n-1)\times 5$$
$$= 5n + 1.$$

Therefore, the nth term is $5n + 1$.

Example 5.10

The second term of an arithmetic progression is 15 and the fifth term is 21. Find the common difference and the first term.

Solution

Given $A_2 = 15$ and $A_5 = 21$.

Since $A_n = A_1 + (n-1)\, d$, we get

$$A_2 = A_1 + d = 15 \qquad (1)$$
$$A_5 = A_1 + 4d = 21 \qquad (2)$$

To obtain A_1 and d solve equations (1) and (2) simultaneously

$$\begin{cases} A_1 + d = 15 & (3) \\ A_1 + 4d = 21 & (4) \end{cases}$$

Subtract (3) from (4) to get $3d = 6$, from which $d = 2$. Substituting $d = 2$ in equation (3) gives $A_1 = 13$. Therefore, the common difference is 2 and the first term is 13.

Exercise 5.3

1. Which of the following expressions are arithmetic progressions? Write the common difference for those which are arithmetic progressions.
   (a) $\frac{1}{4} + \frac{1}{3} + \frac{1}{2} + 1$
   (b) $26 + 19 + 12 + 5 - 2$
   (c) $2 + 2.2 + 2.22 + 2.222$
   (d) $-5 + 4 + 12 + 5 - 2$
   (e) $1.3 + 2 + 2.7 + 3.4 + \cdots$
   (f) $-\frac{1}{4} - \frac{7}{8} - \frac{3}{2} - \frac{17}{8}$
2. Write the first 5 terms of the arithmetic progression and find the formula for the n$^{\text{th}}$ term obtained from counting in 7 beginning with 5.
3. The fourth and fifth terms of an arithmetic progression are 47 and 52, respectively. Find,
   (a) the common difference.
   (b) the first term.
   (c) the thirteenth term.
4. If the first term of an arithmetic progression is 3 and the common difference is 4, find the n$^{\text{th}}$ term.
5. The third term of an arithmetic progression is 9 and the common difference is 2. Find,
   (a) the first term.
   (b) the 200$^{\text{th}}$ term.
   (c) the n$^{\text{th}}$ term.

6. Find the number of terms in the following arithmetic series:
   (a) $2 - 9 - 20 - 31 - \cdots - 130$.
   (b) $6\frac{1}{4} + 7\frac{1}{2} + 8\frac{3}{4} + \cdots + 17\frac{1}{2}$.
   (c) $407 + 401 + 395 + \cdots + 131$.
7. Find the n^{th} term of an arithmetic progression whose first term is $x + 2$ and the common difference is 3.
8. The fourth term of an arithmetic progression is 11 and the sixth term is 17. Find the tenth term.
9. The n^{th} term of an arithmetic progression $2.3 + 4.2 + 6.1 + \cdots$ is 36.5. Find the value of n.
10. The 5^{th} term of an arithmetic progression is 21 and the 8^{th} term is 30. Find the 20^{th} term.

The sum of the first *n* terms of an Arithmetic Progression (AP)

Consider the sum of counting numbers up to 100 in both ascending and descending order

$$S_{100} = 1 + 2 + 3 + \cdots + 98 + 99 + 100 \quad (1)$$
$$S_{100} = 100 + 99 + 98 + \cdots + 3 + 2 + 1 \quad (2)$$

Adding equations (1) and (2) gives

$$2S_{100} = 101 + 101 + 101 + \cdots + 101 + 101 + 101 \quad (3)$$

Since we have 100 terms, then;

$$2S_{100} = 100 \times 101$$
$$2S_{100} = 10\,100 \quad (4)$$

Divide equation (4) both sides by 2 to get

$$S_{100} = 5\,050$$

Therefore, the sum of the counting numbers from 1 to 100 is 5 050.

For any arithmetic progression with the first term A_1, the last term A_n, and the common difference d, the sum S_n can be obtained using the same method used in the previous example:

$$S_n = A_1 + (A_1 + d) + (A_1 + 2d) + \cdots + (A_n - 2d) + (A_n - d) + A_n \quad (5)$$
$$S_n = A_n + (A_n - d) + (A_n - 2d) + \cdots + (A_1 + 2d) + (A_1 + d) + A_1 \quad (6)$$

Adding equations (5) and (6) gives,

$$2S_n=(A_1+A_n)+(A_1+A_n)+(A_1+A_n)+\cdots+(A_1+A_n)+(A_1+A_n)+(A_1+A_n)+\cdots \quad (7)$$

Since we have n terms each with (A_1+A_n), then, equation (7) becomes to

$$2S_n = n(A_1+A_n) \quad (8)$$

Dividing equation (8) by 2 both sides gives,

$$S_n = \tfrac{n}{2}\left(A_1+A_n\right) \quad (9)$$

Therefore, the formula $S_n = \tfrac{n}{2}\left(A_1+A_n\right)$ is the sum of the first n terms of an arithmetic progression where A_1 is the first term and A_n is the last term. In other words, the sum equals the number of terms times the average of the first and the last terms. The sum of the first n terms of an arithmetic progression can be expressed by another formula obtained by replacing A_n in equation (9) with $A_n = A_1+(n-1)d$. Thus, equation (9) becomes

$$\begin{aligned} S_n &= \tfrac{n}{2}\left(A_1+(A_1+(n-1)d)\right) \\ &= \tfrac{n}{2}\left(2A_1+(n-1)d\right) \end{aligned}$$

Therefore, $S_n = \tfrac{n}{2}\left(2A_1+(n-1)d\right)$.

Example 5.11

Find the sum of the first sixteen terms of the arithmetic series $3+10+17+21+\cdots$

Solution

Given $d = 7$, $A_1 = 3$, $n = 16$

Therefore, from

$S_n = \tfrac{n}{2}\left(2A_1+(n-1)d\right)$.

Substituting the values gives

$$\begin{aligned} S_{16} &= \frac{16}{2}\left(2\times 3+(16-1)\times 7\right) \\ &= \frac{16}{2}\left(6+105\right) \\ &= 8\times 111 \\ &= 888 \end{aligned}$$

Therefore, the sum of the first sixteen terms is 888.

Example 5.12

The first term of an arithmetic progression is 2 and the last term is 46. If the arithmetic progression consists of 23 terms, find the sum of all the terms.

Solution

Given $A_1 = 2$, $A_{23} = 46$, $n = 23$, required to find the sum of first 23 terms.
The formula for the sum of the first n terms is given as

$$S_n = \frac{n}{2}\left(A_1 + A_n\right).$$

Substituting the given values gives

$$\begin{aligned} S_{23} &= \frac{23}{2}\left(2+46\right) \\ &= \frac{23}{2} \times 48 \\ &= 552 \end{aligned}$$

Therefore, the sum of the first 23 terms is 552.

Example 5.13

The first term of an arithmetic progression is 2 and the common difference is 5. If the sum of the first n terms is 245, find the number of terms of the progression.

Solution

Given $A_1 = 2$, $d = 5$, $S_n = 245$.
Required to find the number of terms, n.
From $S_n = \frac{n}{2}\left(2A_1 + (n-1)d\right)$.
Substituting the values into the formula gives

$$\begin{aligned} 245 &= \frac{n}{2}\left(2 \times 2 + 5\,(n-1)\right). \\ 5n^2 - n - 490 &= 0 \end{aligned}$$

Solving for values of n, we obtain $n = 10$ or $n = -9.8$.
Since the number of terms cannot be negative, then $n = 10$. Therefore, there are 10 terms in the arithmetic progression.

Example 5.14

Alinda saved Tsh 2 860 000 in the first year of employment and each year afterwards she saves Tsh 445 250 more than the previous year.

(a) How much money will she have saved in the first 24 years?

(b) How much money will she have saved in thirteenth year?

Solution

(a) Given A_1 = Tsh 2 860 000, d = Tsh 445 250, n = 24 years,

Required: Amount of money to be saved in the first 24 years.

The formula for the sum of first n terms of an AP is given by

$$S_n = \frac{n}{2}\left(2A_1 + (n-1)d\right)$$

Substituting the given values into the formula gives

$$S_{24} = \frac{24}{2}\left(2 \times 2\ 860\ 000 + (24-1) \times 445\ 250\right)$$

$$= 12(5\ 720\ 000 + 10\ 240\ 750)$$

$$= 191\ 529\ 000$$

Therefore, in the first 24 years Alinda will have saved Tsh 191 529 000

(b) Recall the formula for the n$^{\text{th}}$ term of an AP:

$$A_n = A_1 + (n-1)d$$

Substituting the given values gives

$$A_{13} = 2\ 860\ 000 + (13-1) \times 445\ 250$$

$$= 2\ 860\ 000 + 5\ 343\ 000$$

$$= 8\ 203\ 000$$

Therefore, in the thirteenth year she will have saved Tsh 8 203 000.

The arithmetic mean

If a, M and b are three consecutive terms of an arithmetic progression, the common difference equals to $M - a$ or $b - M$

Therefore, $M - a = b - M$

$$2M = a + b, \text{ or}$$

$$M = \frac{a+b}{2}$$

In ordinary language, the arithmetic mean is the average of a and b and therefore, M is called the arithmetic mean of the terms a and b.

Example 5.15

Find the arithmetic mean of 3 and 27.

Solution

The arithmetic mean of 3 and 27 is $M = \frac{3+27}{2}$

$= \frac{30}{2}$

$= 15$

Therefore, the arithmetic mean of 3 and 27 is 15.

Example 5.16

An arithmetic mean and one of the terms of an arithmetic progression are 14 and 7, respectively. Find the other term.

Solution

Arithmetic mean $(M) = \frac{a+b}{2}$

Given $M = 14$ and one term $= 7$

Thus, $14 = \frac{7+b}{2}$

$28 = 7 + b$

$b = 21$

Therefore, the other term is 21.

Exercise 5.4

1. Find the sum of the following arithmetic series:
   (a) $-10 - 7 - 4 - \cdots + 50$
   (b) $a + 3a + 5a + \cdots + 21a$
   (c) $2.01 + 2.02 + 2.03 + \cdots + 3.00$
2. The numbers 24, m, y and 72 are such that m is the arithmetic mean of 24 and y, while y is the arithmetic mean of m and 72. Find the values of m and y.
3. The first term of an arithmetic progression is 6 and the common difference is 1.5. If the sum of the first n terms is 90, find the number of terms of the progression.

4. Find the arithmetic mean of:
   (a) 22 and 28 (b) $2-\sqrt{3}$ and $2+\sqrt{3}$
   (c) 55 and 65 (d) $(K-M)^2$ and K^2-M^2
   (e) -3 and $+3$ (f) -7 and 9
5. If Ali earns a salary of 1 256 000 Tanzanian shillings per month and his annual increment is 45 500 Tanzanian shillings, determine;
   (a) his monthly earning during the 16^{th} year of employment.
   (b) his total earning after 18 years.
6. The first term of an arithmetic progression is -12 and the last term is 40. If the sum of n terms is 196, find the number of terms and the common difference.
7. The first term of an arithmetic progression is 13 and the fifth term is 21. Find the common difference and the sum of the first ten terms.
8. Find the difference between the sum of the first five terms of two arithmetic progressions whose first terms are 12 and 8, and their common differences are 2 and 3, respectively.
9. How many integers are there between 14 and 1 000 which are divisible by 17?
10. Find:
    (a) A_1 and S_9 given that $A_9 = 40$ and $d = 4$.
    (b) d given that $A_1 = 2$ and $S_n = 100$.
    (c) S_n given that $A_1 = 19$ and $d = -2$.

Geometric Progression (GP)

Activity 5.4: Formulating a general rule to determine the amount of money saved for a certain period of time

Individually or in a group, think of any amount of money you would like to save. Your plan should be to save a certain amount of money every week. Use the following steps:

1. Prepare a table with two columns. The first column should be for weeks and the second for amount of money saved.
2. Enter the amount of money you want to save in the first week.
3. For the following weeks, double the amount you have saved from the previous week.

4. Fill in the table with the amount saved for 5 weeks.
5. Study carefully how the amount of money saved varies from the first to the fifth week.
6. Provide a brief explanation of the pattern which you think may lead to such an increase of the amount of money saved.
7. Use the pattern obtained from previous steps to formulate a rule which may allow you to find the amount of money saved at any time.
8. Use the rule you have formulated to determine the amount of money that will have been saved up to the 10^{th} week.
9. Share your rule with the class through a presentation.

A sequence in which a new term is obtained by multiplying the preceding term by a fixed number is known as a geometric progression (GP). The fixed number is called the common ratio, denoted by r.

A geometric series is obtained by adding the terms of a geometric progression. Examples of geometric series are:

(a) $1 + 2 + 4 + 8 + \cdots$ with a common ratio 2.

(b) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$ with a common ratio $\frac{1}{3}$.

(c) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ with a common ratio $\frac{1}{2}$.

The n^th^ term of a geometric progression (GP)

If n is the number of terms of a geometric progression, the n^th^ term is denoted by G_n and the common ratio is denoted by r.

Suppose that 3 is the first term G_1 of a geometric progression whose common ratio r is 2, then the first four terms are $G_1 = 3$, $G_2 = 3\,(2)^1$, $G_3 = 3\,(2)^2$, and $G_4 = 3(2)^3$. The n^th^ term deduced from the above pattern is given by :

$$G_n = 3(2^{n-1})$$

Therefore, the geometric series in this case is

$$3 + 6 + 12 + 24 + \cdots + 3(2^{n-1})$$

In general, we can derive the n^th^ term G_n of any geometric progression whose first term is G_1 and the common ratio is r using the following procedure:

Terms	Exponent of r
$G_1 = G_1$	0
$G_2 = G_1 r$	1
$G_3 = G_1 r^2$	2
$G_4 = G_1 r^3$	3
$\vdots$	$\vdots$
$G_n = G_1 r^{n-1}$.	n–1

Therefore, the n$^{\text{th}}$ term is $G_n = G_1 r^{n-1}$.

Example 5.17

Find the 8$^{\text{th}}$ term of each of the following geometric series:

(a) $2 + 4 + 8 + \cdots$

(b) $12 + 6 + 3 + \cdots$

Solution

(a) Given $G_1 = 2, r = 2$ and $n = 8$.

From $G_n = G_1 r^{n-1}$

Substituting the values into the formula gives

$$G_8 = 2(2)^7$$
$$= 2^8$$
$$= 256.$$

Therefore, the 8$^{\text{th}}$ term is 256.

(b) Given, $G_1 = 12, \ r = \frac{1}{2}, \ n = 8$

$$G_n = G_1 r^{n-1}$$

Substituting the values into the formula gives

$$G_8 = 12\left(\frac{1}{2}\right)^7$$
$$= \frac{12}{128}$$
$$= \frac{3}{32}$$

Therefore, the 8$^{\text{th}}$ term is $\frac{3}{32}$.

Example 5.18

The first term of a geometric progression is 4 and the fourth term is 108. Find the common ratio.

Solution

Given, $G_1 = 4$, $G_4 = 108$, $n = 4$

From $G_n = G_1 r^{n-1}$.

Substituting the values into the formula gives

$108 = 4r^3$.

Dividing by 4 both sides gives

$r^3 = 27$, which implies that $r^3 = 3^3$

Comparing the exponents, the value of $r = 3$.

Therefore, the common ratio is 3.

Example 5.19

Find the number of terms in the following geometric series.

$$1 + 2 + 4 + 8 + 16 + \cdots + 512$$

Solution

Given, $G_1 = 1$, $r = 2$, $G_n = 512$

From $G_n = G_1 r^{n-1}$, it implies that

$$512 = 1 \times 2^{n-1}$$
$$2^n = 512 \times 2$$
$$2^n = 2^{10}$$
$$n = 10$$

Therefore, the number of terms is 10.

Example 5.20

The value of a machine depreciates by 25% every year. If its value is 75 000 000 Tanzanian shillings when it is new, what is its value after 21 years?

Solution

Given, $G_1 = 75\ 000\ 000$ Tanzanian shillings, $n = 21$ years,

$r = (100 - 25)\ \% = 75\%$

The n^{th} term is given by $G_n = G_1 r^{n-1}$.

Substituting the values into the formula gives

$$G_{20} = 75\ 000\ 000\ (0.75)^{20} \text{ Tanzanian shillings}$$

$$= 237\ 840 \text{ Tanzanian shillings}$$

Therefore, its value after 21 years will be approximately 237 840 Tanzanian shillings.

Exercise 5.5

1. If the fourth term of a geometric progression is 9 and the sixth term is 81, find
   (a) the common ratio.
   (b) the first term.
2. The first three terms of a geometric progression are -4, 1, and $-\frac{1}{4}$. Find the fifth term.
3. If the n^{th} term of the geometric series $4 + 8 + 16 + \ldots$ is 1024, find n.
4. The value of a machine depreciates each year by 15% of its value at the beginning of that year. If its value when new is 2 000 000 Tanzanian shillings, what is its value after 15 years?
5. Find the eighth term of the geometric progression $2\ \ -6, 18, -54, \ldots$
6. Find the fifth term of a geometric progression with the first term 100 and common ratio 0.5.
7. Which of the following are geometric series? Write down their common ratios.
   (a) $3 + 9 + 27 + \cdots$
   (b) $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{36} + \cdots$
   (c) $-1 + 2 - 4 + 8\ - \cdots$
   (d) $2 + 4 - 8 - 16 + \cdots$
8. A certain school spends 18 500 000 Tanzanian shillings every year for stationeries. The school administration decided that each year the school expenditure should be reduced by 10% of the expenditure of the preceding year. Find the school expenditure at the end of the 8^{th} year.

9. The second and fifth terms of a geometric progression are 24 and 5184, respectively. Find;
   (a) the first term.
   (b) the common ratio.
   (c) the seventh term.

10. Find the number of terms in each of the following geometric series:
   (a) $81 + 27 + 9 + \cdots + \frac{1}{27}$.
   (b) $5 + 10 + 20 + \cdots + 5120$.
   (c) $0.03 + 0.06 + 0.12 + \cdots + 1.92$.

Sum of the first *n* terms of a geometric progression

Let the sum of the first n terms of a geometric progression be denoted by S_n. Then
$S_n = G_1 + G_2 + G_3 + \cdots + G_n$.

Since it is a geometric progression,
$G_1 = G_1, \quad G_2 = G_1 r, \quad G_3 = G_1 r^2, \quad G_n = G_1 r^{n-1}$.

Therefore, $S_n = G_1 + G_1 r + G_1 r^2 + \cdots + G_1 r^{n-1}$. (1)
Multiply both sides of equation (1) by r to get
$rS_n = G_1 r + G_1 r^2 + G_1 r^3 + \cdots + G_1 r^n$. (2)

Subtract equation (2) from equation (1) to obtain
$S_n - rS_n = G_1 - G_1 r^n$. (3)
Factorization gives
$S_n(1 - r) = G_1(1 - r^n)$.

Make S_n the subject to obtain

$$S_n = \frac{G_1(1 - r^n)}{1 - r} \qquad (4)$$

where r is a fraction such that $|r| < 1$.

Also, subtracting equation (1) from equation (2) gives;
$rS_n - S_n = G_1 r^n - G_1$. (5)

Factorize equation (5) to get $S_n(r - 1) = G_1(r^n - 1)$.

Divide both sides by $r - 1$ to obtain

$$S_n = \frac{G_1(r^n - 1)}{r - 1} \text{ for } |r| > 1. \quad (6)$$

Therefore, given a geometric progression with the first term G_1 and the common ratio r, the sum of the first n terms is given by $S_n = \frac{G_1(1 - r^n)}{1 - r}$, for $|r| < 1$ or $S_n = \frac{G_1(r^n - 1)}{r - 1}$, for $|r| > 1$.

Example 5.21

If the sum of the first four terms of a geometric progression is 80 and its common ratio is 3. Find the first term.

Solution

Given $S_4 = 80$, $n = 4$, $r = 3$. Thus,

$$S_n = \frac{G_1(r^n - 1)}{r - 1}.$$

Substituting the values into the formula gives

$$80 = \frac{G_1(81 - 1)}{3 - 1}$$
$$40G_1 = 80$$
$$G_1 = 2$$

Therefore, the first term is 2.

Example 5.22

Find the sum of the first six terms of a geometric series $-81 - 27 - 9 - \cdots$

Solution

$$G_1 = -81, \; n = 6, \; r = \frac{1}{3}.$$

The formula for the sum of geometric series is given by:

$$S_n = \frac{G_1(1 - r^n)}{1 - r}$$

Substituting the given values into the formula gives

$$S_6 = \frac{-81\left(1-\left(\frac{1}{3}\right)^6\right)}{1-\frac{1}{3}}$$

$$= \frac{-81\left(1-\left(\frac{1}{729}\right)\right)}{\frac{2}{3}}$$

$$= \frac{-364}{3}.$$

Therefore, the sum of the first six terms is $\frac{-364}{3}$.

Example 5.23

The n^th term of a geometric progression is given by $24(5)^{n-1}$. Determine the;

(a) first five terms of the geometric progression.

(b) sum of the first 10 terms of the geometric progression.

(c) general formula for the sum of first n terms of the geometric progression.

(d) smallest value of n for which the sum, $S_n > 90\ 000$.

Solution

(a) From $G_n = G_1 r^{n-1}$

The first term, $G_1 = 24(5)^{1-1} = 24(5)^0 = 24$

The second term, $G_2 = 24(5)^{2-1} = 24(5)^1 = 120$

The third term, $G_3 = 24(5)^{3-1} = 24(5)^2 = 600$

The fourth term, $G_4 = 24(5)^{4-1} = 24(5)^3 = 3\ 000$

The fifth term, $G_5 = 24(5)^{5-1} = 24(5)^4 = 15\ 000$.

Therefore, the first five terms are 24, 120, 600, 3 000 and 15 000.

(b) Given $G_1 = 24, r = 5, n = 10$.

From $S_n = \frac{G_1(r^n - 1)}{r-1}$,

Substitution of the values into the formula gives

$$S_{10} = \frac{24(5^{10}-1)}{5-1}$$

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$$= \frac{24 \times 9\ 765\ 624}{4}$$

$$= 58\ 593\ 744.$$

Therefore, the sum of the first 10 terms is 58 593 744.

(c) $$S_n = \frac{G_1(r^n - 1)}{r - 1}$$

$$= \frac{24(5^n - 1)}{5 - 1}$$

$$= 6(5^n - 1).$$

Therefore, the formula for the sum of the first n terms of the sequence is $S_n = 6(5^n - 1)$.

(d) $S_n > 90\ 000$

$6(5^n - 1) > 90\ 000$

$5^n - 1 > 15\ 000$

$5^n > 15\ 001.$

Solving for n gives $n > 5.9747$.

Therefore, the smallest value of n is 6.

Geometric Mean

If a, M and b are three consecutive terms of a geometric progression, M is called the geometric mean of a and b.

The common ratio in each case is $\frac{M}{a}$ or $\frac{b}{M}$. Therefore,

$$\frac{M}{a} = \frac{b}{M} \text{ which gives}$$

$$M^2 = ab \text{ or } M = \pm\sqrt{ab}$$

Therefore, the geometric mean of a and b is $M = \pm\sqrt{ab}$.

Example 5.24

Find the geometric mean of 2 and 8, hence write down its geometric series.

Solution

Geometric mean $= \pm\sqrt{ab}$ where $a = 2$ and $b = 8$.

$$= \pm\sqrt{2\times 8}$$

$$= \pm\sqrt{16}$$

$$= \pm 4$$

Therefore, the geometric mean of 2 and 8 is ± 4.

The geometric series can either be $2 + 4 + 8 + \cdots$ or $2 - 4 + 8 - \cdots$

Example 5.25

The geometric mean of 12 and x is 6. Find the value of x.

Solution

The geometric mean M of three numbers is $\pm\sqrt{ab}$.

Given $M = 6$ and one term $= 12$

$$M = \pm\sqrt{ab}$$

$$6 = \pm\sqrt{12x}$$

$$36 = 12x$$

$$x = 3$$

Therefore, the value of x is 3.

Example 5.26

Find the geometric mean of -64 and -289 and write down its geometric series.

Solution

From $M = \pm\sqrt{ab}$.

$$= \pm\sqrt{-64\times(-29)}$$

$$= \pm\sqrt{18496}$$

$$= \pm 136$$

Therefore, the geometric mean of -64 and -289 is ± 136.

The geometric series can be either $-64 + 136 - 289 + \cdots$ or $-64 - 136 - 289 - \cdots$

Exercise 5.6

1. Given the geometric series $2 + 6 + 18 + \cdots$ Find the sum of the first eight terms.
2. Find the sum of the first twenty terms of the geometric series $4 + 8 + 16 + \cdots$
3. Find the sum of the first ten terms of the geometric series $2 - 6 + 18 - 54 + \cdots$
4. Find the geometric mean of each of the following sets of numbers:
   (a) 49 and 81 (b) -361 and -1024 (c) 4 and 9
5. If the sum of the first n terms of a geometric progression with the first term 1 and the common ratio $\frac{1}{2}$ is $\frac{31}{16}$, find the number of terms.
6. If the n[th] term of a geometric progression is given by $G_n = 2^n$, find the sum of the first five terms.
7. If the sum of the first n terms of a sequence is $3^n - 1$, show that the sequence is a geometric progression.
8. Find the sum of the first n terms of each of the following geometric series:
   (a) $10 + 50 + 250 + \cdots$ (b) $1 - 2 + 4 + \cdots$
   (c) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ (d) $\frac{1}{81} - \frac{1}{27} + \frac{1}{9} - \frac{1}{3} + \cdots$
9. The geometric mean of two numbers 4 and p is 24. Find the value of p.
10. A rectangular plot of land has sides of lengths 90 m and 250 m. Use the concept of geometric mean to find the length of one side of a square having the same area as that of a rectangular plot.

Compound interest

When money is invested or borrowed from a bank, the interest is usually calculated at the end of the year. The interest accrued is added to the original principal at the end of the first year and the obtained amount becomes the new principal at the beginning of the second year. This process may continue for a number of years. This process is called compound interest.

Activity 5.5: Deducing the formula for finding compound interest

Have you ever thought that sequences and series could be useful in financial matters? Banks and other financial institutions use the knowledge of sequences and series to determine the amount of interest in loans.

In principle, banks calculate their interest on time basis based on the current principal. The following tasks will guide you to develop a formula for computing compound interest. Individually or in a group, perform the tasks:

1. Write down the formula for obtaining the amount of money A_1 after one year, that is, the sum of the principal and simple interest for one year. Factorize the expression obtained. Remember $I = \frac{PRT}{100}$.
2. Use the amount of money obtained at the end of year one as the new principal in the second year and write the amount of money at the end of the second year. Factorize the expression obtained.
3. Use the same procedure to get the amount of money at the end of the third year.
4. Study carefully the final result at each year to discover the pattern in the increase of the amount of money at the bank. Is the pattern an arithmetic or a geometric progression? Why?
5. Write a general rule for the increase of the amount of money after any given period of time, say n years.
6. Use the rule to determine the amount of money that will be available in the bank after 10 years.
7. Share the pattern you have discovered and the general rule to the rest of the class through a class discussion.

The amount of money accumulated when invested at a certain interest rate is a good example of arithmetic and geometric progression. Simple interest varies directly as the time and is given by the formula:

$$I = \frac{PRT}{100},$$

where P is the principal, R is the interest rate and T is the period.
For example, the simple interest payable on Tsh 1 000 at a rate of 15% per annum for 1, 2, 3, … years will be Tsh 150, Tsh 300, Tsh 450,..., respectively.
The sum of Tsh 150 + Tsh 300 + Tsh 450 + · · · indicates an arithmetic series generated by the simple interests.

For compound interest, after each interest period, the interest is added to the principal to obtain a new principal for the next interest year.

Suppose Tsh 8 000 is invested at 5% interest rate compounded annually. Then, the amount at the end of each year is computed as follows:

If Tsh 8 000 is the principal at the beginning of the first year, then the amount at the end of the first year is:

$8\ 000 + 8\ 000 \times 0.05 = 8\ 000(1 + 0.05) = 8\ 000 \times 1.05$.

The amount at the end of the second year is:

$$8\ 000 \times 1.05 + ((8\ 000 \times 1.05) \times 0.05) = (8\ 000 \times 1.05)(1 + 0.05)$$
$$= \text{Tsh } 8\ 000 \times (1.05)^2$$

The amount at the end of the third year is:

$$= (8\ 000 \times (1.05)^2 (1 + 0.05) = \text{Tsh } 8\ 000 \times (1.05)^3.$$

Similarly, the amount at the end of the n$^{\text{th}}$ year is Tsh $8\ 000 \times (1.05)^n$.

The amounts Tsh $8\ 000$, Tsh $8\ 000 \times (1.05)$, Tsh $8\ 000 \times (1.05)^2$ and so on are terms of a geometric progression whose first term G_1 equals to Tsh 8 000 and common ratio r equals to 1.05.

Generally, if P is the principal, R is the rate of interest, I is the interest, A is the amount after receiving the interest, and T is the period, then

$$A = P + I$$

But, $I = \dfrac{PRT}{100}$

Thus,

$$A = P + \frac{PRT}{100}$$
$$= P\left(1 + \frac{RT}{100}\right).$$

If time is in years, then the amount at the end of the first year is given by

$$A_1 = P\left(1 + \frac{RT}{100}\right)$$

The amount at the end of the second year is $A_1 + I$, where the interest now is

$$I = \frac{A_1 RT}{100} = P\left(1 + \frac{RT}{100}\right)\frac{RT}{100}$$

Therefore,

$$A_2 = A_1 + P\left(1 + \frac{RT}{100}\right)\frac{RT}{100}$$

$$= P\left(1 + \frac{RT}{100}\right) + P\left(1 + \frac{RT}{100}\right)\frac{RT}{100}$$

$$= P\left(1 + \frac{RT}{100}\right)\left(1 + \frac{RT}{100}\right)$$

$$= P\left(1 + \frac{RT}{100}\right)^2.$$

Similarly, the amount of money at the end of the third year A_3 is given by

$$A_3 = P\left(1 + \frac{RT}{100}\right)^3.$$

In general, if n is the number of years, the amount of money at the end of n^{th} year is given by:

$$A_n = P\left(1 + \frac{RT}{100}\right)^n.$$

Example 5.27

Find the compound interest earned after three years on Tsh 200 000 invested at the interest rate of 8% per annum.

Solution

Given P = Tsh 200 000, $R = 8, n = 3$, and $T = 1$ year.

From, $A_n = P\left(1 + \frac{RT}{100}\right)^n$

$$A_3 = 200\ 000\left(1 + \frac{8}{100}\right)^3$$

$$= 200\ 000(1 + 0.08)^3$$

$$= 200\ 000(1.08)^3$$

$$= \text{Tsh } 251\ 942.4$$

Therefore, the amount after 3 years is approximately Tsh 252 000.

Interest I $= A_3 - P$

$= \text{Tsh } 252\ 000 - \text{Tsh } 200\ 000$

$= \text{Tsh } 52\ 000.$

Therefore, the compound interest is Tsh 52 000.

Example 5.28

Paula invested a certain amount of money in a savings account which provides an interest rate of 10% compounded annually. After two years, she had a total of Tsh 50 000.

(a) How much did she invest at the beginning?

(b) How much did she receive as interest at the end of the second year?

Solution

(a) Given that A_2 = Tsh 50 000, R = 10, T = 1 year and n = 2 years.

The formula for computing the amount is given by

$$A_n = P\left(1+\frac{RT}{100}\right)^n$$

Substituting the values into the formula results into

$$50\ 000 = P\left(1+\frac{10}{100}\right)^2$$

Solving for P gives

$$P(1.1)^2 = 50\ 000$$

$$P = \frac{50\ 000}{(1.1)^2}$$

$$= \text{Tsh } 41\ 322.30$$

Therefore, Paula invested Tsh 41 322.30.

Since the amount after two years was Tsh 50 000,

$$\text{Interest I} = \text{Amount} - \text{Principal}$$
$$= \text{Tsh } 50\ 000 - \text{Tsh } 41\ 322.30$$
$$= \text{Tsh } 8\ 677.70$$

Therefore, the compound interest was Tsh 8 677.70.

Example 5.29

Find the amount of money at the end of 5 years after investing 50 000 Tanzanian shillings at a compound interest rate of 5% annually.

Solution

Given that $P = $ Tsh 50 000, $R = 5\%$, $T = 1$ year, and $n = 5$ years, then recall the formula for computing the amount A_n at a given period of time

$$A_n = P\left(1 + \frac{RT}{100}\right)^n$$

$$= 50\ 000\left(1 + \frac{5}{100}\right)^5$$

$$= 50\ 000(1 + 0.05)^5$$

$$= 63\ 814.08$$

Hence, the amount at the end of 5 years will be Tsh 63 814.08.

Note that, it is not necessary for the interest to be compounded annually. It can be compounded semi-annually, quarterly, monthly and so on. In this case, when the interest is not compounded annually, the following formula is used:

$$A_n = P\left(1 + \frac{R}{t}\right)^{nt}$$

where A_n = Amount

P = Principal

R = Interest rate

n = Number of years of compounding the interest

t = Number of times the interest is compounded per year

Example 5.30

A business person deposited 800 000 Tanzanian shillings in a bank account paying 6% interest compounded semi-annually. What will be the total amount after 5 years?

Solution

Given $P = $ Tsh 800 000, $n = 5$ years, $R = 6\%$, $t = 2$.

From $A_n = P\left(1 + \frac{R}{t}\right)^{nt}$

Substituting the values into the formula gives

$$A_5 = 800\ 000\left(1 + \frac{6}{100 \times 2}\right)^{2\times5}$$

$$= 800\ 000(1.03)^{10}$$

$$= 1\ 075\ 133.$$

Therefore, the amount after 5 years is Tsh 1 075 133.

Exercise 5.7

1. Find the amount of money at the end of two years and three years if 1 200 000 Tanzanian shillings is invested at 6% compounded annually.
2. Find the interest for a principal of 1 000 000 Tanzanian shillings at 4% interest compounded annually after 10 years.
3. Find the amount of money accumulated after investing for 6 years a principal of 5 650 000 Tanzanian shillings at 8% compounded annually.
4. Find the difference in the amount of money accumulated at the end of the third and fifth years if 120 000 Tanzanian shillings is invested at 8% interest rate compounded annually.
5. Find the amount of money accumulated at the end of two years if a principal of 15 000 000 Tanzanian shillings is invested at 7.5% interest rate compounded annually.
6. The amount of 843 000 Tanzanian shillings was collected after investing a certain amount of money at an interest rate of 10% compounded semi-annually for 3 years. What was the initial principal invested?
7. Find the amount of money at the end of n years if 1 000 000 000 Tanzanian shillings is invested at 9.5% interest rate compounded annually.
8. Find the amount of money at the end of 3 years, if a principal of 4 000 000 Tanzanian shillings is invested at 6% interest rate compounded semi-annually.
9. How long will it take for a sum of money to double itself at 5% per annum compound interest rate?
10. How many years will an investment of 250 000 Tanzanian shillings double itself if it is invested at 8% compounded annually?
11. How much money would you need to deposit at 9% annual interest compounded monthly to have Tsh 120 000 in the account after 6 years.
12. Suppose that Tsh 500 000 is deposited into an account paying 6% annual interest compounded quarterly. How long will the amount be Tsh 800 000?

Chapter Summary

1. An arithmetic sequence or arithmetic progression is a sequence in which the difference between any two consecutive terms is the same.
2. The general term of an AP is given by $A_n = A_1 + (n-1)d$, where A_n is the n^{th} term, A_1 the first term, n is the number of terms, and d is the common difference.
3. An arithmetic series is the sum of terms of an arithmetic progression.

4. The formula for computing the sum of the first n terms of an AP is given by

$$S_n = \frac{n}{2}\left(2A_1 + (n-1)d\right) \text{ or } S_n = \frac{n}{2}\left(A_1 + A_n\right)$$

5. A geometric progression is a sequence in which a new term is obtained by multiplying the previous term by a common ratio. The n[th] term is $G_n = G_1 r^{n-1}$, where G_1 is the first term, r is the common ratio, and n is the number of terms.
6. A geometric series is the sum of terms of a geometric progression.
7. The formula for computing the sum of the first n terms of a GP is given by

$$S_n = \frac{G_1(r^n - 1)}{r-1}, \text{ for } |r| > 1 \text{ and } S_n = \frac{G_1(1-r^n)}{1-r}, \text{ for } |r| < 1.$$

8. Compound interest is interest earned by investing an amount of money in a bank which pays an annual rate of $R\%$, compounded n times per year.
9. The amount of money A_n after n years when the interest is compounded annually is given by $A_n = P\left(1 + \frac{RT}{100}\right)^n$.
10. The amount of money A_n after n years when the interest is not compounded annually is given by $A_n = P\left(1 + \frac{R}{t}\right)^{nt}$

Revision exercise 5

1. Find the general term of the sequence formed by counting the natural numbers in three's starting from 1.
2. The general term of a certain sequence is $\frac{5n-1}{2}$. Find the sum of the first five terms.
3. What is the sum of the first n odd numbers?
4. Each year, a coconut tree produces 3 more coconuts than it did the previous year. If it produced 10 coconuts in1985,
   (a) how many coconuts were produced in the year 2 000?
   (b) find the total number of coconuts produced from the year 1985 to year 2000.

5. The starting salary of a worker in a certain company is Tsh 2 400 000 per annum. If the annual increment is Tsh 20 000, what is the total amount of money earned by the worker in four years?

6. Find the nth term of the geometric sequence $1, \frac{1}{2}, \frac{1}{4}, \ldots$

7. Find the geometric mean and arithmetic mean of each of the following sets of numbers:
   (a) 18 and 72 (b) -625 and -2304 (c) -4 and -16

8. The second, fourth, and eighth terms of an arithmetic progression form the first three terms of a geometric progression while the sum of the third and fifth terms of the geometric progression is 20. Find the first four terms of the geometric progression.

9. What is the difference between the sum of the first ten terms of the arithmetic progression and geometric progression whose terms are $-2, 4, \ldots$?

10. Find the sum of the first n terms of the following geometric progression $x, -x^2, x^3, \ldots$

11. A geometric progression has $G_1 = 2$ and $G_2 = 4$. Find the product of the first 100 terms.

12. Find the amount of money at the end of the second and third year if Tsh 200 000 is invested at 5% interest compounded annually.

13. In how many years would one's investment double, if 5 000 000 Tanzanian shillings is invested at 5.5% interest compounded semi-annually?

14. A company borrows Tsh 2 000 000 from a bank at 7% interest rate compounded annually, and repays 500 000 Tanzanian shillings at the end of each year. How much does the company still owe the bank at the end of two years?

15. The amount of money A_n accumulated from the principal P invested at $R\%$ interest rate compounded annually for n years is given by the formula $A_n = P\left(1 + \frac{RT}{100}\right)^n$. Find A_n if $P =$ Tsh 1 260 000, $T = 1$ year, $R = 4\%$, and $n = 5$ years.

Chapter Six

Circles

Introduction

If you look around the environment, you can see objects with circular shapes. Round plates, buckets, bottles, round tables, circular house decorations, and structures of many other objects are constructed based on the knowledge of circles. A circle is a close curve whose every point is at an equal distance from a fixed point. In this chapter, you will learn about basic concepts of a circle, derivation of a formula for an arc length, concept of radian measure, and circle theorems of inscribed angles and chords. The competencies developed in this chapter will be useful in various real life situations such as in building constructions, manufacturing of goods, decorations, road designs, study of the universe, and many other applications.

Parts of a circle

Activity 6.1: Identifying parts of a circle

Individually or in a group, perform the following tasks:

1. Collect circular objects such as lids of plastic buckets, circular coins and bottles, or other objects with circular bottoms.
2. Take one object and locate its centre using a piece of chalk or a marker pen. Label the point with the letter O.
3. Use a piece of chalk or a marker pen to mark three points on the circular edge of the object. Label the points with letters A, B, and C.
4. Use a piece of thread and a metre ruler to measure the length from the centre O to the points A, B, and C on the circular edge. Record the lengths OA, OB, and OC.

5. Compare the lengths you obtained. What is the name of the lengths *OA*, *OB* or *OC* as related to the circle?
6. Draw a line from point *A* to *O* and extend it to touch the circular edge of the object. Label the point with letter *D*.
7. Measure the length *AD* and compare it with the lengths *OA*, *OB*, and *OC*. From your previous knowledge about circles, what is the name of length *AD*?
8. Put a piece of thread around the circular edge of the object (from point *A* around and back to point *A*), then stretch the thread on a metre ruler to obtain the length. What do you call this length? Are there some other ways of finding the same length? How?
9. Draw a large circle and indicate all the parts of a circle you have identified in this activity.
10. Share your results with the rest of the class through discussion.

In Activity 6.1, you have identified the common parts of a circle. Other parts of a circle include a segment, a chord, a sector, an arc, and a tangent line. These parts are summarised in Figure 6.1.

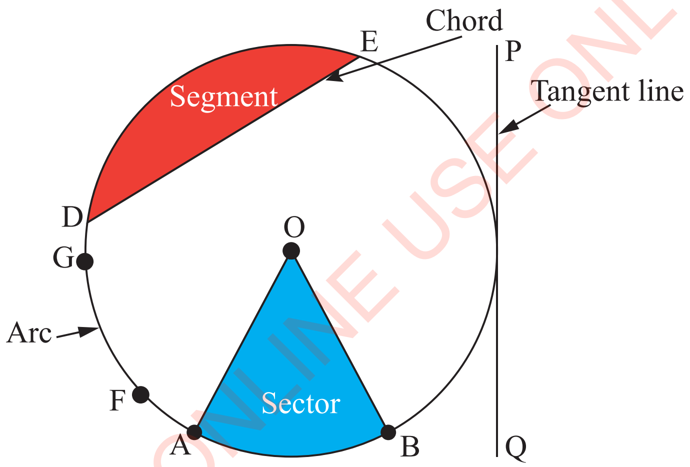


Figure 6.1: *Parts of a circle*

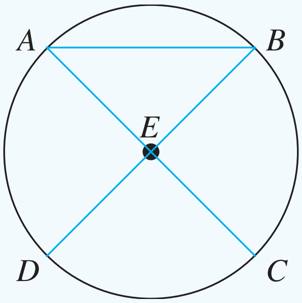
A tangent to a circle is a line which touches the circle at one point. In Figure 6.1, line *PQ* is a tangent. A chord is a line segment which joins two points on the circle. In Figure 6.1, line *DE* is a chord of a circle.

An arc is any part of the circumference of a circle. Parts *AB*, *FG*, and *DA* are examples of the arcs of a circle as shown in Figure 6.1. A segment is part of a circle which is bounded by a chord and an arc. In Figure 6.1, a part in red is a segment of a circle.

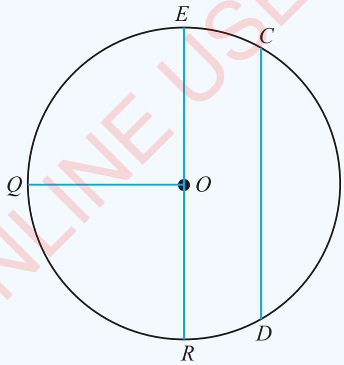
A sector is a part of a circle which is bounded by two radii and an arc. In Figure 6.1, part $AOBA$ is an example of a sector which is formed by two radii AO and BO, and an arc AB.

Exercise 6.1

1. Write an equation that shows how the diameter of a circle is related to the radius of the circle.
2. In the following figure, if E is the centre of the circle, answer the questions that follow.



   (a) Find the radius of the circle given $\overline{BD} = 12$ cm.
   (b) Find the diameter of the circle given that $\overline{CE} = 6$ cm.
3. Study the following circle with centre O, then identify the radius, diameter, and chord.



4. Given the radius r, diameter d or circumference C of a circle, calculate the missing measures. Round the answer to the nearest hundredth.
   (a) $r = 7$ cm, $d =$ ______, $C =$ ______,
   (b) $C = 26\pi$ cm, $d =$ ______, $r =$ ______,
   (c) $d = 2a$ cm, $r =$ ______, $C =$ ______,
   (d) $r = \frac{a}{6}$ cm, $d =$ ______, $C =$ ______.

Central angles

Activity 6.2: Identifying the central angle of a circle

Perform the following tasks individually or in a group:

1. Use a pair of compasses to draw a circle with centre O and radius of your choice.
2. From the centre, draw two radii with end points P and Q on the circumference of the circle.
3. The two radii form an angle at O. What is the name of this angle?
4. Use a protractor to measure the angle.
5. Using one of the radius as a starting point, draw three more different angles and compare the magnitude of each angle with its respective arc.
6. Again, using the same centre of the circle, increase the radius and draw another circle. Repeat this task two times and study the relationship between the angle and the arc of the circle. What happens to the angle and the arc when you increase or decrease the radius?
7. What is your conclusion in tasks 5 and 6? Share the results with the rest of the class through discussion.

In addition to what you discovered in Activity 6.1, Figure 6.2 (a) and Figure 6.2 (b) show that the length of an arc of a circle is proportional to the measure of the central angle, provided that the radius of the circle is not changed.

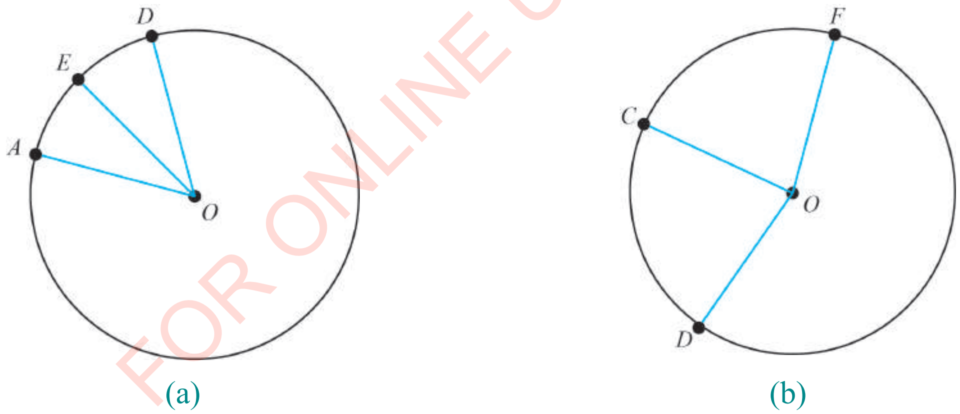


Figure 6.2: *Relationship between central angle and length of an arc*

The length of an arc depends also on the size of the circle. For example, taking a central angle of, say 20°, we find that the larger the circle the larger is the arc, as shown in Figure 6.3. In this case, arc CD is larger than arc AB.

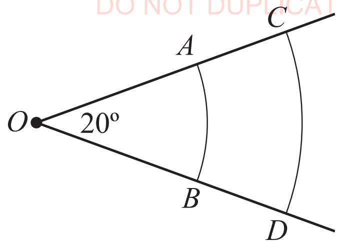


Figure 6.3: *Relationship between the radius of a circle and the length of an arc*

Arc length

Activity 6.3: Deducing a formula for finding the length of an arc

Individually or in a group with the assistance of your teacher, prepare a round geoboard by using a hard board, nails, and a hammer. You can also use a softboard and softboard pins. A circle formed on the geoboard should appear as shown in Figure 6.4.

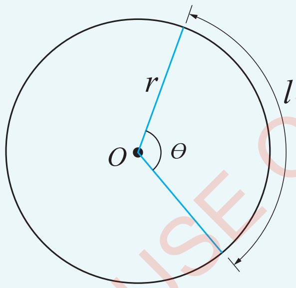


Figure 6.4: *The arc length l of a circle*

1. Use a rubber band to trace a circle on the geoboard. Then, use another rubber band to trace a sector on the circle.
2. Measure the angle formed by the sector and the arc length of the sector. You can measure the length of the arc by first tracing the arc with a thread and mark the end points. Then, use a ruler to measure the length between the marked points.
3. Measure the radius of the circle and find its circumference.
4. Having the angle of the sector and the circumference of the circle, is it possible to find the length of the arc? Give reasons.
5. Formulate the general formula for finding the length of the arc. Use the formula you have formed to verify the answer you obtained in task 3.
6. Share your findings with the rest of the class through a presentation.

From Activity 6.3, you have observed that the formula for computing the circumference of a circle is $C = 2\pi r$. This means that, the circumference is the length of the arc subtended by a central angle 360°.

The length of an arc is proportional to the measure of its central angle. Thus, if the central angle is 1°, then the length of the arc can be computed by using the formula

$$l = \frac{2\pi r}{360°}.$$

Generally, if θ is a central angle, then the length l of an arc is given by

$$l = \frac{\pi r}{180°}\theta.$$

For example, the length of the arc which forms a quarter of a circle, subtending a central angle of 90° (right angle) is

$$l = \frac{90°\pi r}{180°}$$

$$= \frac{\pi r}{2}.$$

Also, the arc subtending a central angle of 30° has an arc length given by

$$l = \frac{30°\pi r}{180°}$$

$$= \frac{\pi r}{6}.$$

From the formula for finding the arc length, the general formula for finding the central angle θ is

$$\theta = \frac{180° l}{\pi r}.$$

For example, the central angle for a circle of radius $r = 2$ cm and arc length $l = 1$ cm is given by:

$$\theta = \frac{180° l}{\pi r}, \text{ given } \pi = 3.14$$

$$= \frac{180° \times 1 \text{ cm}}{3.14 \times 2 \text{ cm}}$$

$$= 28.7°.$$

Therefore, the central angle is 28.7°.

Example 6.1

An arc of a circle whose radius is 3 cm subtends a central angle whose measure is 60°. Calculate the length of the arc (leave your answer in terms of π).

Solution

Given $r = 3$ cm, $\theta = 60°$. Required to find the length l.

The arc length of a circle is given by:

$$l = \frac{\pi r}{180°}\theta.$$

Substituting the given values:

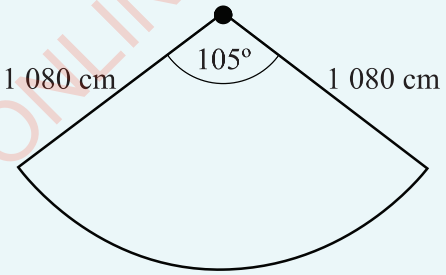
$$l = \frac{3\text{ cm} \times \pi}{180°} \times 60°.$$

$$= \pi \text{ cm}.$$

$\therefore$ The length of the arc is π cm.

Example 6.2

The following figure represents the shape of a garden plot. If its central angle is 105° and the radius is 1080 cm, calculate the arc length of the garden plot. Use $\pi = \frac{22}{7}$.



Solution

Given $r = 1\,080$ cm, $\theta = 105°$.

Required to find the arc length, l.

Recall the formula for computing the arc length:

$$l = \frac{\pi r}{180°}\theta.$$

Substituting the given values into the formula gives

$$l = \frac{22 \times 1\,080 \text{ cm} \times 105°}{7 \times 180°},$$

$$= 1\,980 \text{ cm}.$$

Therefore, the arc length of the garden plot is 1 980 cm.

Exercise 6.2

1. A circular loaf of bread has a mass of 2kg. If 0.75 kg of the bread is to be given to a friend, explain how are you going to divide the bread without using a weighing balance.
2. What is the size of the central angle for a circle of radius 9 cm and an arc length of 3 cm?
3. Find the length of an arc of a circle of radius 1cm with a central angle of 90°. Use $\pi = 3.14$.
4. A roundabout at a certain road junction has a radius of 30 m. A car drives around the roundabout, turning through an angle of 140°. What distance does the car travel?
5. What is the length of an arc of a circle of radius10 cm if the central angle is:
   (a) 45°? (b) 90°? (c) 54°? Leave your answer in terms of π.
6. Find the angle in degrees through which the minute hand of a wall clock turns between noon and the following times:
   (a) 12:40 pm (b) 3:00 pm (c) 9:00 am.
7. Show that the radius of a circle with an arc of length π cm and central angle 30° is 6 cm.
8. What is the length of an arc of the circle of radius 40 mm which subtends an angle of 25°at the centre?
9. Find the length of an arc of a circle with radius 32 cm which subtends an angle of 155° at the centre of the circle. Leave your answer in terms of π.
10. An arc of length 30 m is formed from a circle of radius 50 m. What angle does the arc subtends?

Radian measure

A radian is a measure of an angle formed at the centre of a circle by an arc whose length is equal to that of the radius of the same circle. The relationship between the arc length l, central angle θ, and radius r can be used to compare the measurements of an angle in radians with measurements in degrees.

Thus, if $l = \frac{\pi r\theta}{180°}$, then $\frac{l}{r} = \frac{\pi\theta}{180°}$.

The expression $\frac{l}{r}$ is called the radian and is denoted by 's'.
The formula for the angle in radians is $s = \frac{\pi\theta}{180°}$.

Therefore, if s is a radian measure of an angle and θ is its degree measure, then

$$s = \frac{\pi\theta}{180°}.$$

For example, for an angle of $270°$, the radian measure can be obtained as follows:

$$s = \frac{270°\pi}{180°}$$

$$= \frac{3\pi}{2}.$$

Therefore, the radian measure is $\frac{3\pi}{2}$.

Similarly, $\frac{\pi}{2}$ radians can be converted into degrees using the formula:

$$\theta = \frac{180° s}{\pi}.$$

Substituting $\frac{\pi}{2}$ for s gives

$$\theta = \frac{180° \times \frac{\pi}{2}}{\pi}$$

$$= 90°.$$

Therefore, $\frac{\pi}{2}$ radians is equivalent to $90°$.

If $s = 1$, then $\theta = \frac{180° \times 1}{\pi} = 57.3°$, which implies that, 1 radian is equivalent to $57.3°$.

Angles can be used to measure the size of turning. Turns of the minute hand of a clock face and a wheel can both be measured in degrees and in radians. For example, the minute hand of a clock face turns through an angle of $90°$ or $\frac{\pi}{2}$ radians between noon and 12:15 pm as shown in Figure 6.5. The minute hand has covered an angular distance of $\frac{\pi}{2}$ radians.

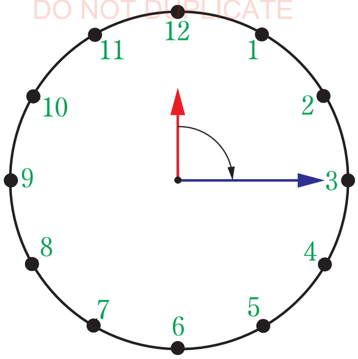


Figure 6.5: *A clock face showing an angle of* 90° *or* $\frac{\pi}{2}$ *radians.*

From noon to 12:45 pm, the minute hand turns through an angle of 270° or $\frac{3\pi}{2}$ radians. The angle 270° is a reflex angle as shown in Figure 6.6.

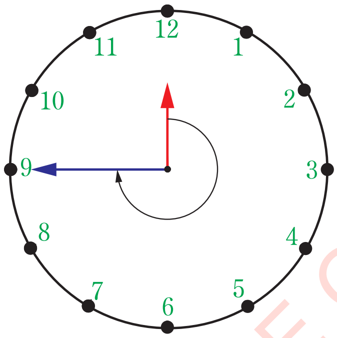


Figure 6.6: *Turning of the minute hand through an angle of* 270° *or* $\frac{3\pi}{2}$ *radians*

One complete turn of a hand of a clock represents an angle of 360° or 2π radians. Measures of angles of more than 360° or 2π radians can be obtained if the minute hand of a clock makes more than one complete turn. For example, from noon to 1:15 pm, the minute hand turns through $\frac{5}{4}$ turns. Since one turn is 360°or 2π radians, $\frac{5}{4}$ turns is $360° \times \frac{5}{4}$ or $2\pi \times \frac{5}{4}$ radians, which gives 450° or $\frac{5}{2}\pi$ radians. Therefore, from noon to 1:15 pm, the minute hand turns through 450° or $\frac{5}{2}\pi$ radians.

Example 6.3

Convert each of the following degrees into radians as a multiple of π:

(a) 165° (b) 225°

Solution

(a) Recall the formula for computing the radian measure:

$$s = \frac{\pi\theta}{180^\circ}.$$

Given $\theta = 165^\circ$, substituting 165° into the formula gives

$$s = \frac{165^\circ \times \pi}{180^\circ}.$$

$$= \frac{11\pi}{12} \text{ radians.}$$

Therefore, 165° is equivalent to $\frac{11\pi}{12}$ radians.

(b) Substituting 225° for θ into $s = \frac{\pi\theta}{180^\circ}$ gives

$$s = \frac{225^\circ \times \pi}{180^\circ},$$

$$= \frac{5\pi}{4} \text{ radians.}$$

Therefore, 225° is equivalent to $\frac{5\pi}{4}$ radians.

Example 6.4

Convert each of the following radians into degrees:

(a) 1.7 radians (b) $\frac{19\pi}{4}$ radians

Solution

(a) Given s = 1.7 radians

From $s = \frac{\pi\theta}{180^\circ}$,

$$\theta = \frac{180^\circ \times 1.7}{\pi},$$

$$= 97.4^\circ.$$

$\therefore$ 1.7 radians is equivalent to 97.5°.

(b) Given $s = \frac{19\pi}{4}$ radians

$$\theta = \frac{180^\circ s}{\pi},$$

$$\theta = \frac{180^\circ \times 19\pi}{\pi \times 4},$$

$$\theta = 855^\circ.$$

$\therefore$ $\frac{19\pi}{4}$ radians is equivalent to 855°.

Exercise 6.3

1. Convert each of the following degrees into radians:
   (a) 10° (b) 45° (c) 160°

2. Convert the following radians into degrees. Give your answers correct to two decimal places.

   (a) 1 radian (b) 0.3 radians (c) 5 radians

3. Find the angles in radians as multiples of π for each of the following degrees:

   (a) 80° (b) 600° (c) 240° (d) 315°

4. What is the length of an arc of a circle of radius 10 cm if the central angle is:

   (a) 1 radian? (b) 0.14 radians? (c) 2.5 radians?

5. Convert each of the following angles in radians into degrees:

   (a) $\frac{\pi}{4}$ (b) $\frac{2}{3}\pi$ (c) $\frac{7}{4}\pi$ (d) $\frac{3}{2}\pi$

6. Calculate the size of angles in radians through which the minute hand of a clock face turns between 12.00 noon and the following times:

   (a) 12:40 pm (b) 3:00 pm (c) 9:00 am

7. Find the size of angles in radians through which the minute hand of a clock turns between 12.00 noon and the following times. Leave your answer in terms of π.

   (a) 12:20 pm (b) 2:15 pm (c) midnight

8. An automobile tyre has an outside diameter of 6 decimetres. What angle, in radians, does the wheel turn when the car travels a distance of one kilometre?

Angle properties of circles

Angles formed in a circle under certain specific conditions are uniquely related. These relationships are summarized in general rules known as theorems.

Relationship between the central angle and the angle formed at the circumference of a circle

In Figure 6.7, APB and $AXQYB$ are two arcs of a circle with centre O. The diameter $\overline{XY}$ divides the circle into two equal arcs, XQY and $XAPBY$.

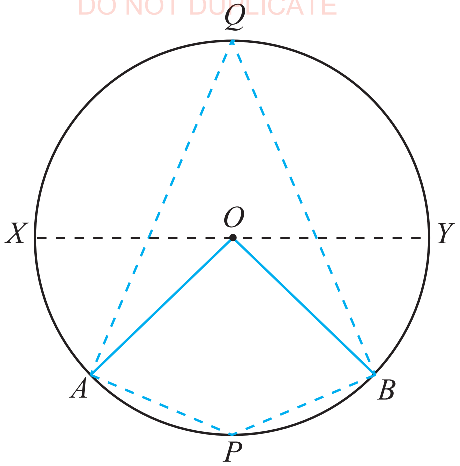


Figure 6.7: *Angle subtended at the centre and at the circumference of a circle*

Arc *APB* subtends angle *AOB* at the centre and angle *AQB* at the circumference. Similarly, arc *AXQYB* subtends the reflex angle *AOB* at the centre and angle *APB* at the circumference. Angles *AOB* and *AQB,* and reflex angle *AOB* and angle *APB* have a unique relationship.

Activity 6.4: Deducing the relationship between a central angle and the angle formed at the circumference of a circle

Individually or in a group, perform the following tasks:

1. Use a geoboard, either an electronic geoboard or the one prepared from locally available materials. You can prepare a geoboard by placing nails 1 cm apart on a hard board or a piece of flat wood of 30 cm by 30 cm. A common geoboard appears as shown in Figure 6.8. The blue dots represent nails and the boundary represents the edge of the hard board.

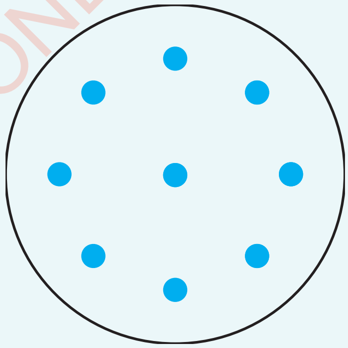


Figure 6.8: *Picture of a round geoboard*

2. Use rubber bands to form shapes on the geoboard as shown in Figure 6.9.

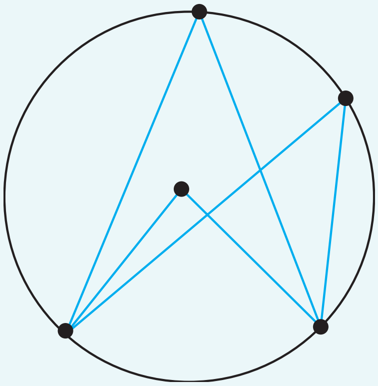


Figure 6.9: *Angles formed at the centre and on the circumference of a circle*

3. Using a protractor, measure the angles formed on the circumference and at the centre of the circle.
4. Comment on the relationship between the angles measured in task 3.
5. Using a pair of compasses, ruler, and protractor, draw a similar shape and measure the central angle and the angles formed on the circumference.
6. Compare the results in task 3 and task 5 to discover the relationship.
7. Write a general rule for the relationship you have discovered.
8. Use the knowledge of congruence, similarity, and algebra to prove what you have discovered in task 7.
9. Share your findings to the rest of the class through a whole class discussion.

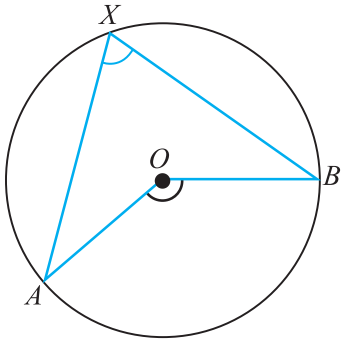
The general rule you have discovered in Activity 6.4 is the central angle theorem of circles. Generally, the angle subtended by an arc at the centre of a circle is twice the angle subtended by the same arc at the circumference. This result is useful in solving numerous angle problems related to circles.

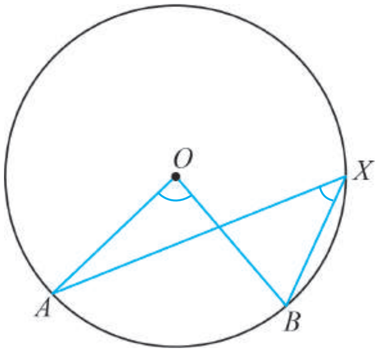
Theorem 6.1
The angle which an arc subtends at the centre of a circle is twice the angle which it subtends at any point on the remaining part of the circumference.

Theorem 6.1 is described as follows:

In the following figures, $A\hat{O}B$ is the angle subtended by arc AB at the centre. Also, $A\hat{X}B$ is an angle subtended at point X on the circumfererence of the circle. According to the central angle theorem,

$$A\hat{O}B = 2A\hat{X}B.$$

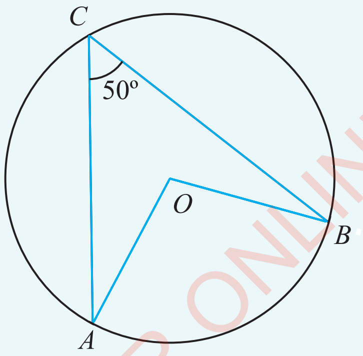


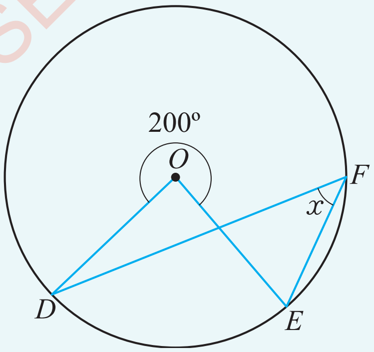


Example 6.5

Use the following figures to find:

(a) $A\hat{O}B$ (b) $D\hat{O}E$ (c) the value of x.





Solutions

(a) $A\hat{C}B = \frac{1}{2} A\hat{O}B$ (angle at the centre is twice the angle subtended at the circumference by the same arc).

$$2A\hat{C}B = A\hat{O}B$$
$$2(50°) = A\hat{O}B$$
$$\therefore A\hat{O}B = 100°.$$

(b) $D\hat{O}E + 200° = 360°$ (sum of angles in a circle)

$D\hat{O}E = 360° - 200°,$

$= 160°.$

Therefore, $D\hat{O}E$ is $160°$.

(c) $D\hat{F}E = \frac{1}{2} D\hat{O}E$ (angle at the centre is twice the angle subtended at the circumference by the same arc)

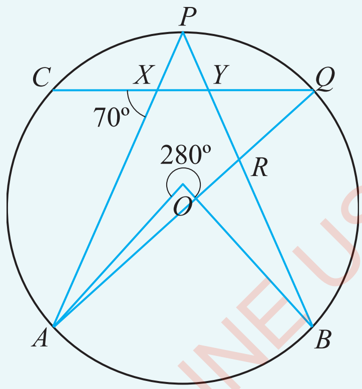
$D\hat{F}E = \frac{1}{2} \times 160°$

$= 80°.$

Therefore, $x = 80°$.

Example 6.6

In the following figure, O is the centre of a circle with arc AB = arc AC. The reflex angle $AOB = 280°$ and $A\hat{X}C = 70°$. Calculate $Y\hat{R}Q$ and hence, show that $\overline{QY} = \overline{QR}$



Solution

$A\hat{O}B + 280° = 360°$ (sum of angles in a circle). Thus, $A\hat{O}B = 80°$

But $A\hat{O}B = 2A\hat{P}B$ (angle at centre is twice the angle subtended at the circumference by the same arc).

$A\hat{P}B = \frac{1}{2} A\hat{O}B,$

$= \frac{1}{2} \times 80°.$

Hence, $A\hat{P}B = 40°$.

Since, $A\hat{P}B = A\hat{Q}C$ (angles subtended by equal arcs on the circumference).

Hence, $A\hat{Q}C = 40°$.

Similarly, $A\hat{X}C = Q\hat{A}X + A\hat{Q}X$ (exterior angle of a ΔAQX)

Thus,

$Q\hat{A}X = A\hat{X}C - A\hat{Q}X$

$= 70° - 40°$ $(A\hat{Q}X = A\hat{P}B)$

$= 30°$

Therefore, $Q\hat{A}X = 30°$.

$A\hat{R}B = Y\hat{R}Q$ (vertically opposite angles)

$A\hat{R}B = P\hat{A}R + A\hat{P}R$ (exterior angle of ΔAPR)

$Y\hat{R}Q = P\hat{A}R + A\hat{P}R$

$= Q\hat{A}X + A\hat{P}B$

$= 30° + 40°$

$= 70°$

In ΔYRQ, we find that $Q\hat{Y}R = 180° - (Y\hat{R}Q + R\hat{Q}Y)$

$Q\hat{Y}R = 180° - (70° + 40°)$

$= 180° - 110°$

$= 70°$

But, $Q\hat{Y}R = Y\hat{R}Q = 70°$

Therefore, $Y\hat{R}Q = 70°$.

Since $Q\hat{Y}R$ and $Y\hat{R}Q$ are base angles of an isosceles ΔYRQ, then $\overline{QY} = \overline{QR}$.

Relationship between angles formed in the same segment of a circle

Figure 6.10 shows a circle with centre O. The segment $PRQP$ of the circle is bounded by the arc PRQ and the chord PQ. Since X, Y, and Z are points on the arc PRQ, we say that $P\hat{X}Q$, $P\hat{Y}Q$ and $P\hat{Z}Q$ are angles in the same segment $PRQP$. The angles formed by the same arc within the segment $PRQP$ have a unique relationship to each other. Engage in Activity 6.5 to study this relationship.

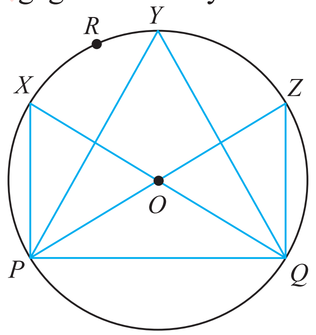


Figure 6.10: *Angles formed in the same segment of a circle*

Activity 6.5: Deducing the relationship between angles formed in the same segment

Perform the following tasks individually or in a group:

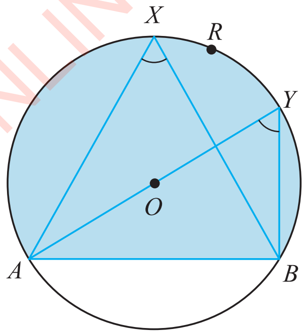
1. Using the geoboard prepared in Activity 6.4, trace Figure 6.10 by rubber bands.
2. Measure the sizes of all the angles formed on the circumference of a circle.
3. Compare the size of each angle measured in task 2 and conclude about the relationship between the angles.
4. Draw a similar figure in your exercise book and measure the angles.
5. Compare the results in task 2 and task 4.
6. What is the general conclusion can you draw about the nature of the angles formed by the same arc in the same segment?
7. Use the knowledge of congruence, similarity, algebra, and circle theorems to prove the fact you have established in task 6.
8. Share your findings with the rest of the class through a discussion.

In Activity 6.5, you have established a relationship between the angles formed by the same arc in the same segment. These angles are always equal.

Theorem 6.2

Angles formed by the same arc in the same segment of a circle are equal.

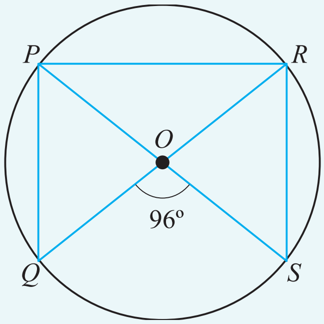
Theorem 6.2 is illustrated in the following figure:



Angles *AXB* and *AYB* are formed in the same segment (shaded blue) by the same arc *AB*. Therefore, according to this theorem, $A\hat{X}B = A\hat{Y}B$.

Example 6.7

In the following figure, O is the centre of the circle.



Find:

(a) $Q\hat{P}S$ (b) $Q\hat{R}S$ (c) $P\hat{O}Q$ (d) $P\hat{R}Q$

Solution

(a) $Q\hat{P}S = \frac{1}{2}Q\hat{O}S$ (angle at the centre is twice the angle subtended at the circumference by the same arc)

$$Q\hat{P}S = \frac{1}{2} \times 96°.$$
$$= 48°.$$

Therefore, $Q\hat{P}S$ is $48°$.

(b) $Q\hat{R}S = Q\hat{P}S$ (angles subtended by the same arc in the same segment)

$Q\hat{P}S = 48°$. Therefore, $Q\hat{R}S = 48°$.

(c) $P\hat{O}Q + Q\hat{O}S = 180°$(angles on a straight line)

But, $Q\hat{O}S = 96°$.

Thus, $P\hat{O}Q = 180° - Q\hat{O}S$

$P\hat{O}Q = 180° - 96°$

$= 84°$.

Therefore, $P\hat{O}Q = 84°$.

(d) $P\hat{R}Q = \frac{1}{2}P\hat{O}Q$ (angle at the centre is twice the angle subtended at the circumference by the same arc)

$$= \frac{1}{2} \times 84°$$
$$= 42°.$$

Therefore, $P\hat{R}Q = 42°$.

Relationship between opposite angles of a cyclic quadrilateral

Given a circle, if two or more points are lying on the circumference of a circle, they are said to be concyclic. Figure 6.11 shows concyclic points.

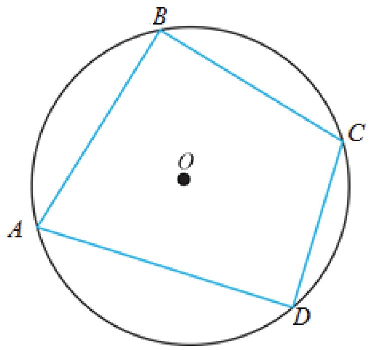


Figure 6.11: *Concyclic points and cyclic quadrilateral*

The points A, *B*, *C*, and *D* in Figure 6.11 are concyclic points. By joining these points in order, a quadrilateral *ABCD* inscribed in the circle with centre *O* is formed. Quadrilaterals such as *ABCD* whose vertices are points on the circle, are called cyclic quadrilaterals. Thus, *ABCD* is a cyclic quadrilateral. The opposite angles of a cyclic quadrilateral are uniquely related as you will deduce in Activity 6.6.

Activity 6.6: Deducing the relationship between angles of the cyclic quadrilateral

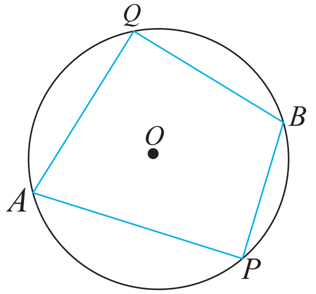
Individually or in a group, perform the following tasks:

1. Use a pair of compasses to draw a circle of convenient radius.
2. Form a cyclic quadrilateral using the circle you have drawn.
3. Measure all four angles formed inside the cyclic quadrilateral.
4. Sum the pairs of opposite angles obtained in task 3.
5. Draw other four circles of different radii and repeat tasks 1 to 4.
6. Study carefully the results obtained in task 4 and write a single statement which summarizes your findings.
7. Use congruence and similarity theorems, circle theorems, algebra, and any other method of your choice to prove the findings you have obtained in task 6.
8. Write a general statement which summarizes your findings.
9. Use algebra, congruence of polygons, similarity theorems, and any other algebraic method of your choice to prove your discovery in task 8.
10. Share your findings with the rest of the class through a discussion.

The conclusion you have made in Activity 6.6 is the circle theorem which describes the relationship between the opposite angles of a cyclic quadrilateral. The opposite angles of a cylic quadrilateral sum up to 180 degrees. In other words, the angles are supplementary.

Theorem 6.3
Opposite angles of a cyclic quadrilateral are supplementary.

Theorem 6.3 is described as follows:
In the following figure, $AQBP$ is a cyclic quadrilateral inscribed in a circle with centre O.

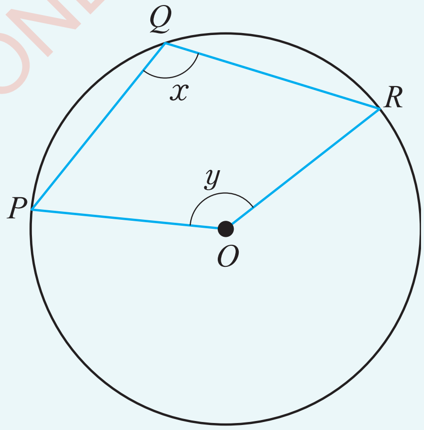


$A\hat{Q}B$ and $A\hat{P}B$ are opposite angles of a cyclic quadrilateral. Likewise, $P\hat{A}Q$ and $P\hat{B}Q$ are opposite angles of the cyclic quadrilateral. According to Theorem 6.3, we get

$$P\hat{A}Q + P\hat{B}Q = 180° \text{ and } A\hat{Q}B + A\hat{P}B = 180°.$$

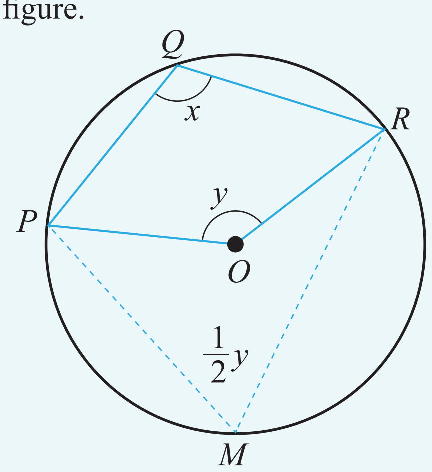
Example 6.8

Let O be the centre of the circle as shown in the following figure. Find an equation relating x and y.



Solution

From the given figure, construct lines PM an RM as shown in the following figure.



$P\hat{O}R = 2\, P\hat{M}R$ (Aangle at the centre is twice the angle subtended by the same arc at the circumference).

But $P\hat{O}R = y$. Therefore, $P\hat{M}R = \frac{1}{2}y$.

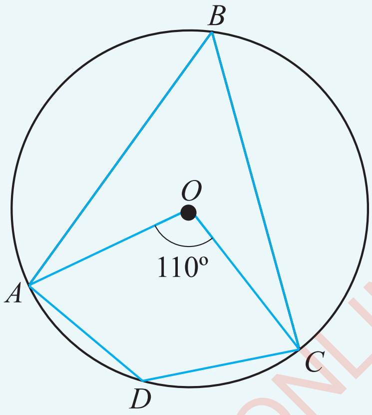
Thus, $\frac{1}{2}y + x = 180°$

(Opposite angles of a cyclic quadrilateral).

Therefore, $y + 2x = 360°$.

Example 6.9

In the following diagram, O is the centre of the circle. If $\angle AOC = 110°$, find $\angle ADC$.



Solution

$A\hat{B}C = \frac{1}{2}A\hat{O}C$ (angle at the centre is twice the angle subtended at the circumference by the same arc).

$A\hat{B}C = \frac{1}{2} \times 110°$

$= 55°$.

But, $\angle ABC + \angle ADC = 180°$ (opposite angles of a cylic quadrilateral).

$\angle ADC = 180° - \angle ABC$

$\angle ADC = 180° - 55°$

$= 125°$.

Therefore, $\angle ADC$ is $125°$.

Relationship between the interior angle of a cyclic quadrilateral and its opposite exterior angle

You have learnt that the sum of two opposite angles of a cyclic quadrilateral is 180 degrees. The interior angle of a cyclic quadrilateral and its corresponding exterior angle shares a unique relationship as well. Engage in the Activity 6.7 to study this relationship.

Activity 6.7: Deducing the relationship between interior angle of a cyclic quadrilateral and its opposite exterior angle

Perform the following tasks individually or in a group:

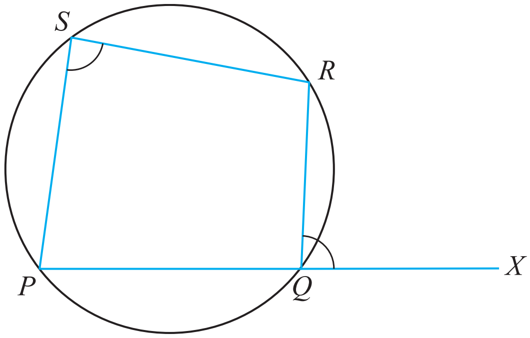
1. Draw a cyclic quadrilateral of any radius.
2. Label each corner of the cyclic quadrilateral formed in task 1.
3. Identify the exterior angle formed by the extended side together with its opposite interior angle.
4. Use a protractor to measure each interior angle and its corresponding opposite exterior angle.
5. Repeat tasks 1 to 4 using two cyclic quadrilaterals of different convenient radii.
6. What have you observed from the angles you have measured in each quadrilateral?
7. Write down a general rule which summarizes your findings.
8. Use a method of your choice to prove your findings in task 7.
9. Share your findings with the rest of the class through a discussion.

The conclusion you have drawn in Activity 6.7 is another circle theorem which describes the relationship between an interior angle of a cyclic quadrilateral and its opposite exterior angle. These angles are always equal.

Theorem 6.4
The exterior angle of a cylic quadrilateral is equal to its opposite interior angle.

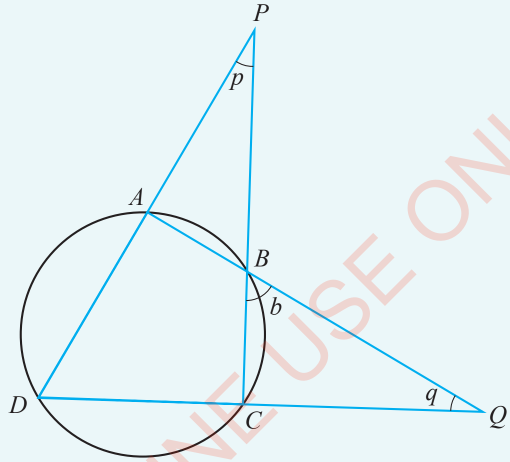
Theorem 6.4 is described as follows:
In the following figure, angle *RQX* is an exterior angle of the cyclic quadrilateral *PSRQ*.

Its opposite interior angle is $P\hat{S}R$. Hence, using theorem 6.4, $P\hat{S}R = R\hat{Q}X$



Example 6.10

In the following figure, $Q\hat{B}C = b$, $B\hat{Q}C = q$. Find $A\hat{P}B$ in terms of q and b.



Solution

Given, $A\hat{P}B = p$.

Using ΔBCQ, we have $B\hat{C}Q + b + q = 180°$ (sum of angles of ΔBCQ).

Thus, $B\hat{C}Q = 180° - (b + q)$.

$B\hat{A}D = A\hat{B}P + p$ (exterior angle of ΔABP).

But, $B\hat{C}Q = B\hat{A}D$ (exterior angle of a cyclic quadrilateral is equal to its opposite interior angle).

Hence, $A\hat{B}P + P = 180° - (b + q)$.

But, $A\hat{B}P = C\hat{B}P = b$ (vertically opposite angles).

Thus, $b + p = 180° - (b + q)$.

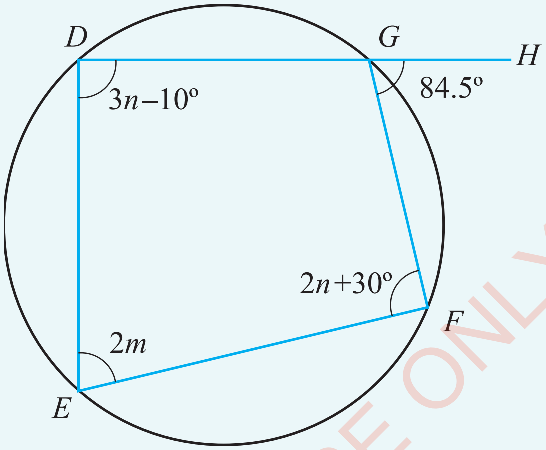
$$p = 180° - b - q - b$$

$$p = 180° - 2b - q$$

Therefore, $A\hat{P}B = 180° - 2b - q$.

Example 6.11

In the following figure, $DEFG$ is a cyclic quadrilateral. If $FGH = 84.5°$, find the values of m and n.



Solution

$2m = 84.5°$(the exterior angle of a quadrilateral is equal to its opposite interior angle)

$$m = \frac{1}{2} \times 84.5°$$

$$= 42.25°.$$

$$m = 42.25°.$$

$3n - 10° + 2n + 30° = 180°$(opposite angles of a cylic quadrilateral are supplementary)

$$5n + 20° = 180°$$

$$5n = 160°$$

$$n = 32°.$$

Therefore, $m = 42.25°$and $n = 32°$.

Angles formed in a semi circle

Activity 6.8: Deducing the nature of the angle formed in a semi-circle

Individually or in a group, perform the following tasks:

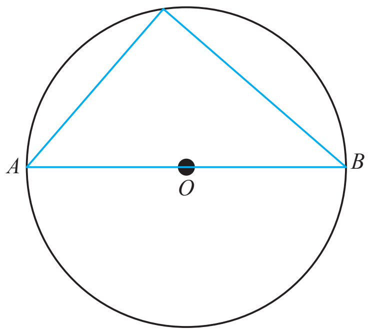
1. Trace a circle of convenient radius on the geoboard.
2. Trace an angle on the circumference of the circle formed by two intersecting line segments passing through the end points of the diameter.
3. Measure the angle traced in task 2.
4. Repeat tasks 2 and 3 several times, each time changing the position of the point of intersection of the two lines.
5. Record all the angles measured and note down any unique result.
6. Draw similar figures in your exercise book and measure the angles.
7. What have you generally noticed on the nature of all the angles formed?
8. Write a general rule regarding the nature of the angle formed in a semi circle.
9. Using the previous circle theorems, algebra, congruence and similarity theorems, prove the findings established in task 8.
10. Use any method of your choice to share the findings with the rest of the class.

An inscribed angle in a semi circle measures 90°. This angle is formed at a point on the circumference of a circle by two chords which originate from each end of the diameter.

Theorem 6.5
The angle subtended at the circumference by the diameter of a circle is a right angle.

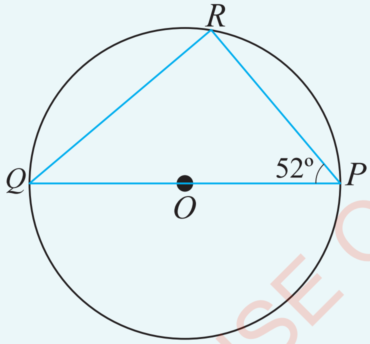
Theorem 6.5 is described as follows:

In the following figure, $A\hat{C}B$ is formed on the circumference in a semi circle by two chords AC and BC originating from each end of the diameter. According to Theorem 6.5, this angle is always 90 degrees.



Example 6.12

In the following figure, $\overline{PQ}$ is a diameter of a circle and R is a point on the circumference. If $\angle QPR = 52°$, find $\angle PQR$.



Solution

$\angle PRQ = 90°$ (angle in a semi circle). Hence,

$\angle PQR + \angle PRQ + 52° = 180°$ (sum of angles in a triangle)

$\angle PQR + 90° + 52° = 180°$

$\angle PQR = 180° - 142°$

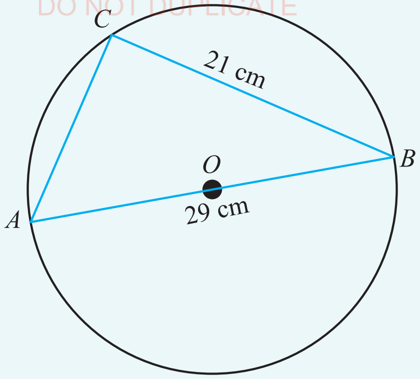
$= 38°$

Therefore, $\angle PQR = 38°$.

Example 6.13

In the following figure, $\overline{AB}$ is the diameter of the circle and C is a point on the circumference. If $\overline{AB} = 29$ cm and $\overline{BC} = 21$ cm, find:

(a) $\overline{AC}$ (b) $\angle ABC$



Solution

(a) The angle subtended by the diameter AB is $90°$. Hence, the side $\overline{AC}$ can be obtained by the Pythagoras' theorem.

$$(\overline{AC})^2 + (\overline{CB})^2 = (\overline{AB})^2$$

$$\overline{AC} = \sqrt{(\overline{AB})^2 - (\overline{CB})^2}$$

$$= \sqrt{(29\text{ cm})^2 - (21\text{cm })^2}$$

$$= \sqrt{400\text{ cm}^2}$$

$$= 20\text{ cm.}$$

Therefore, $\overline{AC} = 20$ cm.

(b) From trigonometric ratios:

$$\cos A\hat{B}C = \frac{\overline{BC}}{\overline{BA}}$$

$$= \frac{21\text{ cm}}{29\text{ cm}}$$

$$A\hat{B}C = \cos^{-1}\left(\frac{21}{29}\right)$$

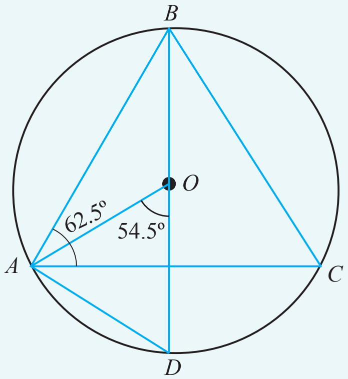
$$= 43.6°.$$

Therefore, $\angle ABC = 43.6°$.

Example 6.14

In the following figure, $\overline{BD}$ is a diameter of a circle whose centre is at O. If $B\hat{A}C = 62.5°$ and $A\hat{O}D = 54.5°$, find the values of:

(a) $C\hat{A}D$ (b) $A\hat{B}D$ (c) $A\hat{D}B$ (d) $A\hat{C}B$



Solution

(a) $B\hat{A}D = 90°$ (angle in a semi circle)

Therefore,

$C\hat{A}D + B\hat{A}C = 90°$

$C\hat{A}D + 62.5° = 90°$

$C\hat{A}D = 90° - 62.5°$

$C\hat{A}D = 27.5°$.

Therefore, $C\hat{A}D = 27.5°$.

(b) $A\hat{B}D = \frac{1}{2}A\hat{O}D$ (angle subtended at the circumference is half the angle subtended at the centre by the same arc).

$A\hat{B}D = \frac{1}{2} \times 54.5°$

$= 27.25°$.

Therefore, $A\hat{B}D = 27.25°$.

(c) $A\hat{D}B + B\hat{A}D + A\hat{B}D = 180°$

$A\hat{D}B + 90° + 27.25° = 180°$

$A\hat{D}B = 180° - 117.25°$.

$A\hat{D}B = 62.75°$.

Therefore, $A\hat{D}B = 62.75°$.

(d) $A\hat{C}B = A\hat{D}B$ (angles formed by the same arc AB on the circumference)

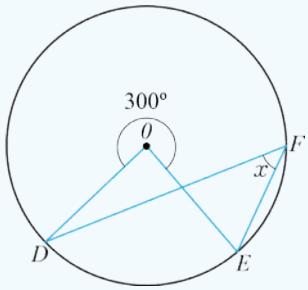
$A\hat{D}B = 62.75°$ (calculated)

Therefore, $A\hat{C}B = 62.75°$.

Exercise 6.4

In problems 1 to 6, if O is the center of the circle, find the value of x.

1.



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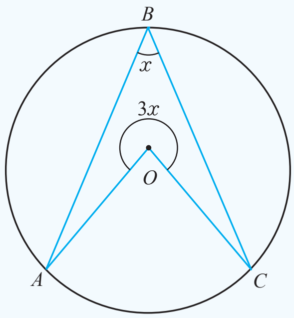

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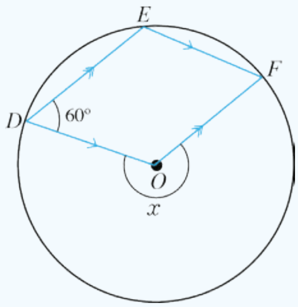

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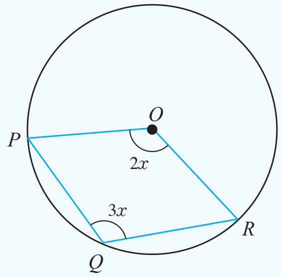

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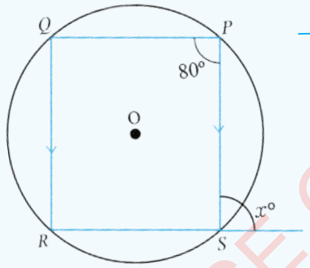
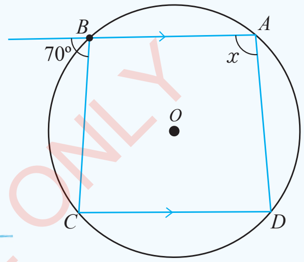
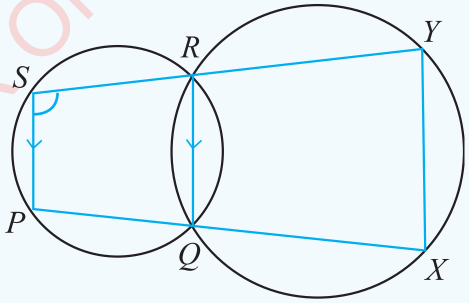


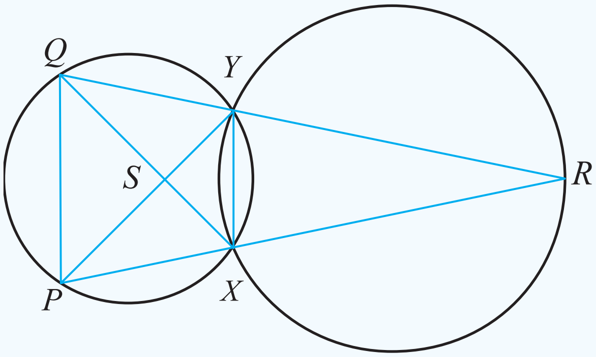
6.



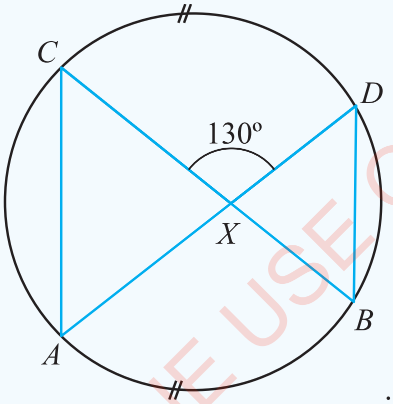
7. If $APQB$ and $CPQD$ are two intersecting circles where APC and BQD are straight lines and $A\hat{P}Q = 70°$, find $C\hat{D}Q$.
8. In the following figure, $\overline{SP} // \overline{RQ}$, PQX and SRY are straight lines and $P\hat{S}R = 100°$. Find $Q\hat{X}Y$.



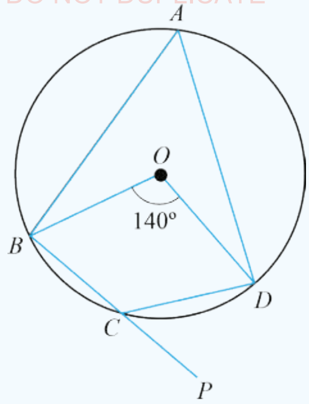
9. In the following figure, PR and QR are straight lines. If $\overline{XR} = \overline{YR}$, prove that:
   (a) $\overline{PQ} \parallel \overline{XY}$ (b) $\overline{PX} = \overline{QY}$ (c) $\overline{SX} = \overline{SY}$



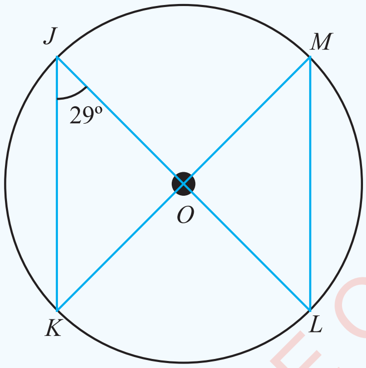
10. In the following figure, arc AB = arc CD and $C\hat{X}D = 130°$. Calculate $A\hat{C}X$ and prove that $\overline{AC}$ is parallel to $\overline{BD}$.



11. If $\overline{AB}$ is a diameter of a circle with centre O and C is a point on the circle, show that $A\hat{C}B = 90°$.
12. A circle with centre O has a diameter $\overline{AB}$. If C is a point on the circle such that $\overline{BC} = 2\overline{AC}$, find the angle subtended by the arc BC on the circumference.
13. The points A, B, C, and D are concyclic and arc AB = arc BC = arc CD. If $B\hat{D}C = 45°$, prove that $ABCD$ is a square.
14. In the following figure, O is the centre of the circle. The angle subtended by arc BCD at the centre is 140°. If $\overline{BC}$ is extended to P, determine $B\hat{A}D$ and $D\hat{C}P$.



15. In the following figure, if O is the centre of the circle, find:
(a) $L\hat{M}K$ (b) $K\hat{O}L$ (c) $J\hat{L}M$ (d) $L\hat{O}M$



Chord properties of circles

Just like angles, chords of circles have unique characteristics under some conditions. In this section, you will learn the properties of lines of symmetry of circles, perpendicular bisector of a chord, parallel chords, and equal chords.

Properties of the lines of symmetry of a circle

Regular figures are symmetrical. Symmetry generally refers to an exact correspondence between opposite halves of a figure. Circles, like any other regular figure, are symmetrical. Engage in Activity 6.9 to learn more about the symmetry of circles.

Activity 6.9: Deducing the line of symmetry of a circle

Perform the following tasks in a group or individually:

1. Draw a large circle of a convenient radius on a manila paper and cut out the circle.

2. Fold the circle along a line through the centre and unfold it.
3. Study carefully the two segments formed by the line of folding.
4. How similar or dissimilar are the two segments of the circle?
5. What can you say about the line you observed in task 4?
6. What is the name of the line you observed in task 4?
7. Briefly describe the relationship between the line of symmetry and a diameter of the circle.
8. Share your findings with the rest of the class through a presentation.

Properties of the perpendicular bisector of a chord

A perpendicular bisector is a line which intersects another line perpendicularly and divides it into two equal parts. Thus, the perpendicular bisector of a chord is a line which intersects a chord perpendicularly and divides it into two equal parts. Also, it is a line passing through the centre of a circle.

Activity 6.10: Examining the properties of a perpendicular bisector of a chord

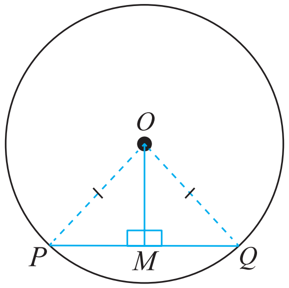
Perform the following tasks individually or in a group:

1. Use a pair of compasses to draw a circle of a convenient radius with the centre O.
2. Draw a chord PQ anywhere across the circle.
3. Draw the perpendicular bisector of chord PQ at R.
4. Use a ruler to measure the length of $\overline{PR}$ and $\overline{QR}$.
5. What is the relationship between the length $\overline{PR}$ and $\overline{QR}$?
6. Does the perpendicular bisector of a chord pass through the centre?
7. Write a general statement which summarizes your observations.
8. Prove the findings in task 7 using the knowledge of algebra, congruence of polygons, similarity, and other methods of your choice.
9. Share your findings with the rest of the class through a presentation.

A bisector is a line which divides another line into two equal parts. When a chord of a circle is bisected by a line, its perpendicular bisector always passes through the centre of the circle.

Theorem 6.6
A perpendicular bisector of a chord passes through the centre of the circle.

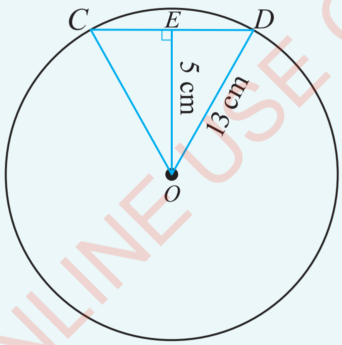
Theorem 6.6 is described as follows:
In the following figure, O is the centre of the circle, PQ is a chord and $\overline{MO}$ is the perpendicular bisector of the chord PQ.



Theorem 6.6 postulates that, the bisector must pass through the centre of the circle as seen in the figure.

Example 6.15

Find the length of the chord CD, given that O is the centre of a circle, $\overline{OE} = 5\text{cm}$, and $\overline{OD} = 13\text{cm}$.



Solution

Consider a right-angled triangle ODE. Using Pythagoras' theorem, we have

$$\begin{aligned}
(\overline{OE})^2 + (\overline{ED})^2 &= (\overline{OD})^2 \\
(5\text{ cm})^2 + (\overline{ED})^2 &= (13\text{ cm})^2 \\
25\text{ cm}^2 + (\overline{ED})^2 &= 169\text{ cm}^2 \\
(\overline{ED})^2 &= (169 - 25)\text{ cm}^2 \\
(\overline{ED})^2 &= 144\text{ cm}^2 \\
\overline{ED} &= \sqrt{144\text{ cm}^2} \\
&= 12\text{ cm.}
\end{aligned}$$

Since the perpendicular bisector $\overline{OE}$ divides the chord CD into equal parts, then,

$$\overline{CE} = \overline{ED}$$

$$\begin{aligned}\overline{CD} &= \overline{CE} + \overline{ED} \\ &= 12\text{ cm} + 12\text{ cm} \\ &= 24\text{ cm}.\end{aligned}$$

Therefore, the length of the chord is 24 cm.

Properties of parallel chords of a circle

Parallel lines are lines which are separated by a constant distance as they extend. Parallel chords have the same feature, they are separated by a constant distance between them. Due to this property, parallel chords have unique properties as revealed in Activity 6.11.

Activity 6.11: Deducing the property of parallel chords

Perform the following tasks individually or in a group:

1. Cut a circle of a convenient radius from a manila paper.
2. Draw two parallel chords on the circle.
3. Mark the arcs formed between the two chords.
4. Fold the circle along a chord which joins the opposite ends of the two chords.
5. Study the lengths of the arcs between the parallel chords and write a general statement regarding your observation.
6. Use the previous knowledge of circle theorems, congruence and similarity or algebra to prove your conclusion in task 5.
7. Share your findings with the rest of the class through presentation.

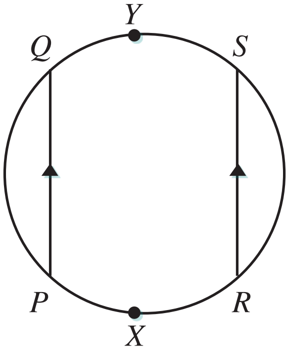
Congruent arcs are arcs with the same measure. Congruent arcs can be formed between chords of a circle which are parallel.

Theorem 6.7

Parallel chords of a circle intersect congruent arcs.

Theorem 6.7 is described as follows:

In the following figure, QYS and PXR are congruent arcs intersected by the parallel chords PQ and RS.



Hence, according to Theorem 6.7, arcs PXR and QYS are congruent or equal.

Properties of equal chords of a circle

In the previous section, you learnt that parallel chords intercept congruent arcs. Given some conditions, such chords may be equal, in this case, equal chords have unique chord properties.

Activity 6.12: Deducing the properties of equal chords of a circle

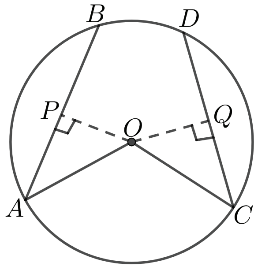
Perform the following tasks individually or in a group:

1. Use a pair of compasses to draw a circle of a convenient radius centred at O.
2. Draw any two chords PQ and RS of equal length.
3. Locate the midpoints A and B of the chords $\overline{PQ}$ and $\overline{RS}$, respectively.
4. Join $\overline{AO}$ and $\overline{BO}$.
5. Using a ruler, measure the length of $\overline{AO}$ and $\overline{BO}$.
6. What is the relationship between the length $\overline{AO}$ and $\overline{BO}$?
7. Write a general statement which summarizes your observation in task 6.
8. Use the concepts of congruence of polygons, similarity, algebra, and previous circle theorems to prove your conclusion in task 7.
9. Share your findings with the rest of the class through a presentation.

From Activity 6.11 and Activity 6.12, you will have discovered that, equal chords of a circle have the same length. Chords with the same length are separated by the same distance from the centre of a circle.

Theorem 6.8
Equal chords of a circle are equidistant from the centre.

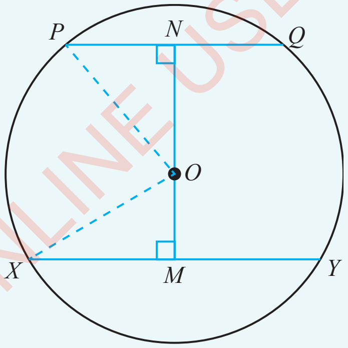
Theorem 6.8 is described as follows:
In the following figure, O is the centre of the circle.



If the chords AB and CD are equal, then the two chords must be separated by the same perpendicular distance from the centre. In this case, since $\overline{AB} = \overline{CD}$, it implies that $\overline{OP} = \overline{OQ}$.

Example 6.16

In the following figure, $\overline{XY}$ and $\overline{PQ}$ are parallel chords in a circle with centre O and radius 5 cm. If $\overline{XY} = 8$ cm and $\overline{PQ} = 4$ cm, find the distance between the chords.



Solution

Given a circle with centre O, radius 5 cm, $\overline{PQ}//\overline{XY}$, $\overline{PQ} = 4$ cm, $\overline{XY} = 8$ cm.
Required to find the distance between $\overline{PQ}$ and $\overline{XY}$ (this is the distance MN).
Construction: In the figure, the two parallel chords are on either side of the centre O. $\overline{OM}$ is perpendicular to $\overline{XY}$ and $\overline{ON}$ is perpendicular to $\overline{PQ}$.
Draw $\overline{OX}$ and $\overline{OP}$.

$\overline{XM} = \frac{1}{2}\overline{XY}$ (since $\overline{XY} \perp \overline{MN}$)

$\overline{XM} = \frac{1}{2} \times 8$ cm

$= 4$ cm.

$\overline{PN} = \frac{1}{2}\overline{PQ}$ (since $\overline{PQ} \perp \overline{MN}$)

$= \frac{1}{2} \times 4$ cm

$= 2$ cm.

In a triangle OXM,

$(\overline{OM})^2 = (\overline{OX})^2 - (\overline{XM})^2$ (Pythagoras' theorem)

$(\overline{OM})^2 = (5 \text{ cm})^2 - (4 \text{ cm})^2$

$= 25 \text{ cm}^2 - 16 \text{ cm}^2.$

$= 9 \text{ cm}^2.$

Hence, $\overline{OM} = 3$ cm.

In triangle OPN,

$(\overline{ON})^2 = (\overline{OP})^2 - (\overline{PN})^2$

$(\overline{ON})^2 = (5 \text{ cm})^2 - (2 \text{ cm})^2$

$= 25 \text{ cm}^2 - 4 \text{ cm}^2$

$= 21 \text{ cm}^2$

$= \sqrt{21 \text{ cm}^2}$

$= 4.58$ cm.

Hence, $\overline{ON} = 4.58$ cm.

Thus, $\overline{MN} = \overline{OM} + \overline{ON}$

$= 3 \text{ cm} + 4.58 \text{ cm}$

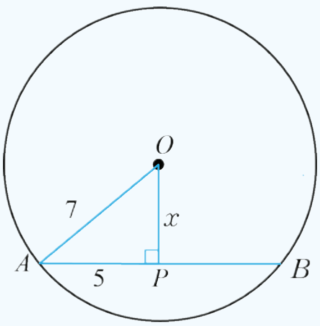
$= 7.58$ cm.

Therefore, the distance between the chords is 7.58 cm.

Exercise 6.5

In questions 1 to 4, O is the centre of each circle. Find the value of x (all measurements are in centimetres).

1.



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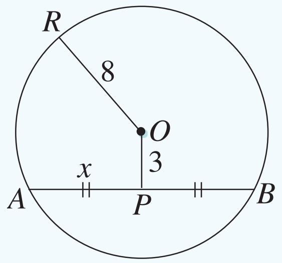

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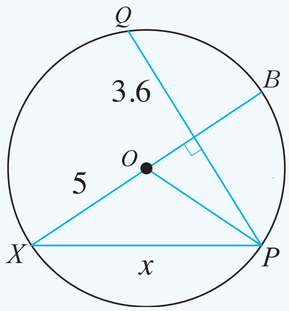

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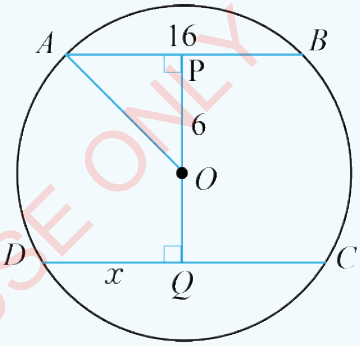
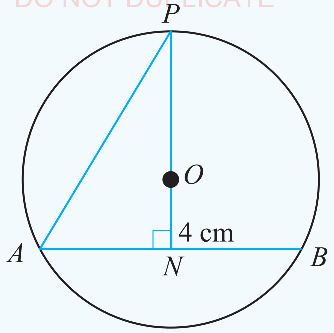
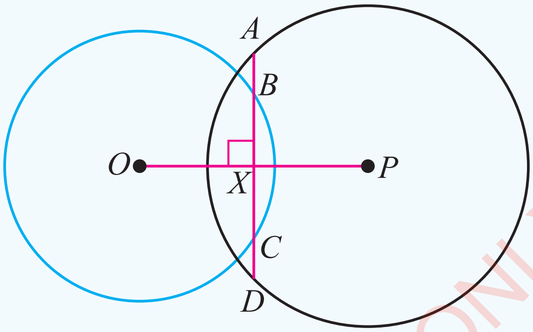


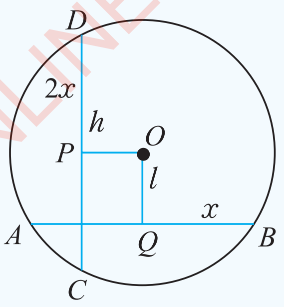
5. Two chords, AB and CD of a circle whose radius is 13 cm are equal and parallel. If each chord is 12 cm long, find the distance between them.
6. Two circles with centres A and B have radii 13 cm and 10 cm, respectively, and intersect at P and Q. If $\overline{PQ} = 12\,\text{cm}$, find the distance between the centres, giving your answer correct to 1 decimal place.
7. Two circles with centres X and Y intersect at R and S. If $\overline{RS} = 24\,\text{cm}$, $\overline{XY} = 14\,\text{cm}$ and the radius of one circle is 15 cm, find the radius of the other circle.
8. In the following figure, O is the centre of the circle, $\overline{AB} = 6\,\text{cm}$, and $\overline{ON} = 4\,\text{cm}$. Show that $\overline{AP} = 3\sqrt{10}\,\text{cm}$.



9. In the following figure, O and P are centres of the two circles and $ABCD$ is perpendicular to $\overline{OP}$ at X. Prove that $\overline{CD} = \overline{AB}$.



10. In the following figure, O is the centre of the circle, $\overline{DP} = 2x$ cm, $\overline{BQ} = x$ cm, $\overline{OP} = h$ cm and $\overline{OQ} = l$ cm. If $\overline{OP}$ and $\overline{OQ}$ are perpendicular to $\overline{CD}$ and $\overline{AB}$, respectively, show that $\overline{AB} = \frac{2}{3}\sqrt{3(l^2 - h^2)}$ cm.



Tangent properties

In the previous sections, you learnt about properties of angles and chords of a circle. In this section, you will learn the tangent properties of a circle.

Activity 6.13: Deducing the nature of the angle formed by radius of a circle and a tangent at the point of tangency

Individually or in a group, perform the following tasks:

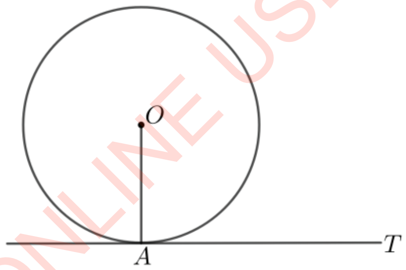
1. Draw a circle of any convenient radius with centre W.
2. Draw a tangent to the circle and label the point of contact by X.
3. Join the points W and X by a line segment WX to form a radius.
4. Measure the angle formed between the radius and the tangent.
5. Repeat tasks 1 to 4 by using two more circles with different radii.
6. What is your observation about the angles you have measured?
7. Write a general statement regarding what you have discovered about these angles.
8. Share your findings with others by any means of your choice.

Theorem 6.9

The angle between a tangent and the radius at the point of contact is a right angle.

Theorem 6.9 is described as follows:

In the following figure, the line segment AT is a tangent which touches the circle at point A and $\overline{OA}$ is its radius.



According to Theorem 6.9, since the tangent and the radii meet at point A on the circle, the angle OAT is a right angle.

Activity 6.14: Deducing properties of tangents from a point outside the circle

Perform the following tasks individually or in a group:

1. Use a pair of compasses to draw a circle with centre O of a convenient radius.
2. Draw two tangent lines from an external point P.

3. Label the points of contact of the tangents and the circle by R and S.
4. Join $\overline{OS}$ and $\overline{OR}$.
5. By using a ruler, measure the length of the tangents $\overline{PR}$ and $\overline{PS}$.
6. Compare the size of line segments $\overline{PR}$ and $\overline{PS}$.
7. Repeat tasks 1 to 6 by drawing 3 more circles of different radii.
8. Write a general statement to summarize your findings.
9. Use algebra, congruence of polygons, similarity or other method of your choice to prove the conclusion you have drawn in task 8.
10. Share your conclusion with the rest of the class through a discussion.

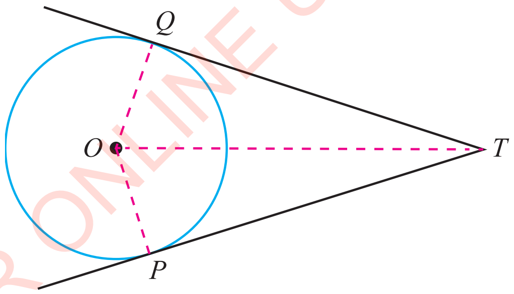
In Activity 6.13, you have learned that at a point of tangency, the tangent and the radius meet at a right angle. In Activity 6.14 you have discovered that two tangents drawn from a common external point are congruent. These properties are useful especially in determining lengths of chords such as diameter and other chords related to circles.

Theorem 6.10

Two tangents from a common external point are equal.

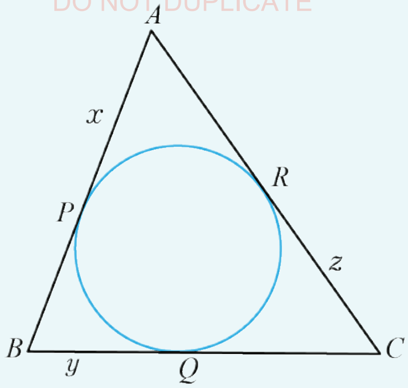
Theorem 6.10 is described as follows:

In the following figure, lines TQ and TP are tangents to a circle with centre O. Since the two tangents are from the same external point T, then, $\overline{TP} = \overline{TQ}$.



Example 6.17

In the following figure, $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ are tangents to the circle. Calculate the values of x, y, and z given that $\overline{AB} = 8$ cm, $\overline{BC} = 7$ cm, and $\overline{CA} = 9$ cm.



Solution

$\overline{AP} = \overline{AR} = x$ (equal tangents from an external point)

$\overline{BP} = \overline{BQ} = y$ (equal tangents from an external point)

$\overline{CQ} = \overline{CR} = z$ (equal tangents from an external point)

Hence,

$$x + y = 8 \qquad (1)$$

$$y + z = 7 \qquad (2)$$

$$x + z = 9 \qquad (3)$$

Subtracting (2) from (1) gives

$$x - z = 1 \qquad (4)$$

Adding (3) and (4) gives

$$2x = 10$$

$$x = 5.$$

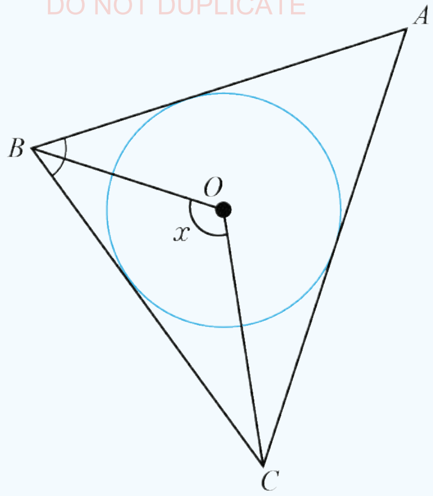
Substituting $x = 5$ in (1) and (3), gives

$$y = 3 \text{ and } z = 4.$$

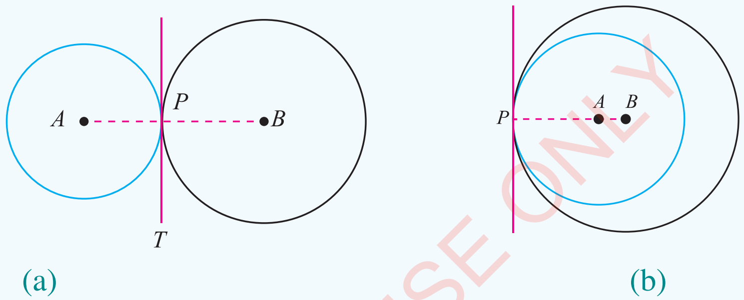
Therefore, $x = 5$ cm, $y = 3$ cm, and $z = 4$ cm.

Exercise 6.6

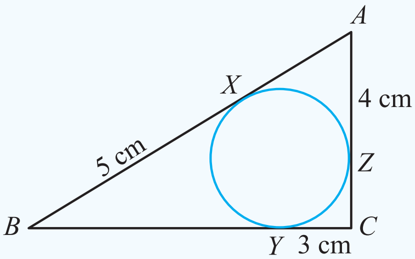
1. In the following figure, O is the centre of the circle inscribed in ΔABC. If $B\hat{A}C = 50°$, and $A\hat{B}C = 70°$, find the value of x.



2. In the following two figures, the radius of the circle with centre A is r cm and that of the circle with centre B is R cm, show that the distance between A and B is $(r + R)$ cm in figure (a) and $(R - r)$ cm in figure (b).



3. Suppose $\overline{TA}$ and $\overline{TB}$ are tangents from an external point T to a circle with centre O. If $A\hat{T}O = 35°$, find $A\hat{O}B$.
4. In the following figure, the three sides of the triangle ABC are tangents to the circle at X, Y, and Z. If $\overline{BX} = 5$ cm, $\overline{CY} = 3$ cm and $\overline{AZ} = 4$ cm, find $\overline{AX}$, $\overline{BY}$, and $\overline{CZ}$.



Angles in alternate segments

In Figure 6.12 (a), the chord $\overline{AB}$ divides the circular region into two segments. In Figure 6.12 (b), $A\hat{X}B$ is in the first segment while in Figure 6.12 (c), $A\hat{Y}B$ is in the second segment (or alternate segment). An angle subtended in the alternate segment by a chord which forms another angle with a tangent have a unique relationship.

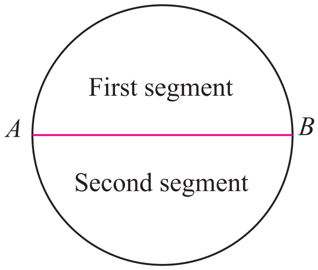


Figure 6.12 (a): *Two segments of a circle*

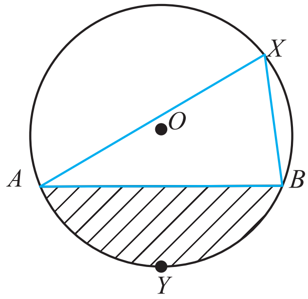


Figure 6.12 (b): *An angle formed in one segment*

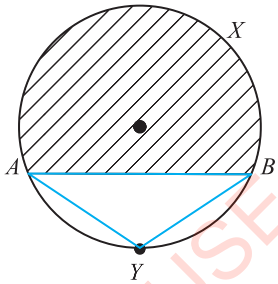


Figure 6.12 (c): *An angle formed in another (alternate segment)*

Activity 6.15: Deducing the relationship between angles in alternate segments

Perform the following tasks individually or in a group:

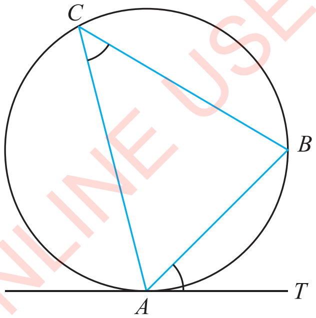
1. Draw a circle with centre O of any convenient radius.
2. Draw a chord anywhere on the circle.
3. Label from outside the circle, the two segments formed.
4. Draw a tangent line in such a way that a point of contact is at one end of the chord.
5. Identify and label the angle formed between the chord and the tangent. Also, identify the segment where the angle you have labelled lies.
6. Measure the angle you have labelled in task 5.

7. On the other segment, draw two line segments each from one end of the chord and meet on the circle.
8. Label and measure the angle formed between the two line segments.
9. On the same circle drawn in task 1, repeat tasks 7 and 8 several times, each time the lines meet the circle at different points.
10. By comparing the angles you have measured, what unique characteristics do you observe from these angles?
11. Write a general statement which summarizes your observation.
12. Use the knowledge of algebra, congruence and similarity theorems, and the previous circle theorems to prove what you have concluded in task 11.
13. Present your final work to the rest of the class.

The conclusion you have drawn in Activity 6.15 is known as the alternate segment circle theorem.

Theorem 6.11
Angles formed in the alternate segments of a circle are equal.

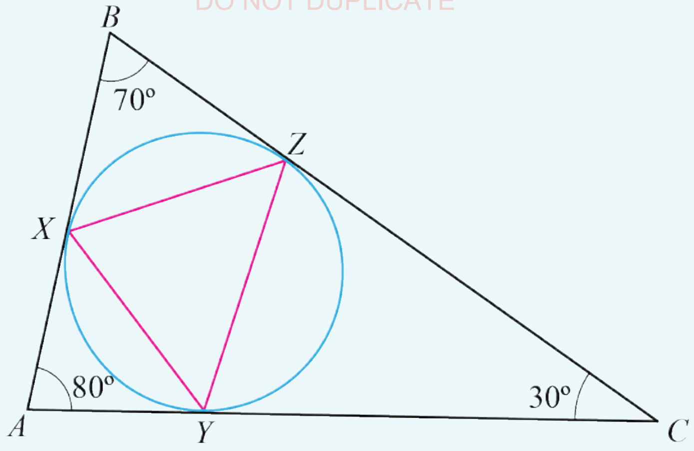
Theorem 6.11 is described as follows:
In the following figure, angle BAT is formed between the chord AB and the tangent AT. Angle ACB is formed by an arc which forms a segment with chord AB.



Since the angle ACB is formed in the alternate (opposite) segment, Theorem 6.11 suggests that $A\hat{C}B = B\hat{A}T$.

Example 6.18

In the following figure, a circle is inscribed in a ΔABC touching it at X, Y, and Z. If the angles of ΔABC are $70°$, $80°$, and $30°$, find the angles of triangle XYZ.



Solution

ΔAXY is isosceles (tangents from a common external point are equal)

$\angle AXY = \frac{1}{2}(180° - 80°) = 50°$

$\angle XZY = \angle AXY$ (alternate segment theorem). Hence, $\angle XZY = 50°$.

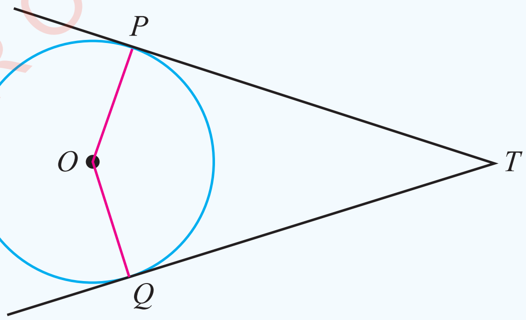
Similarly,

$\angle ZXY = 75°$ and $\angle XYZ = 55°$.

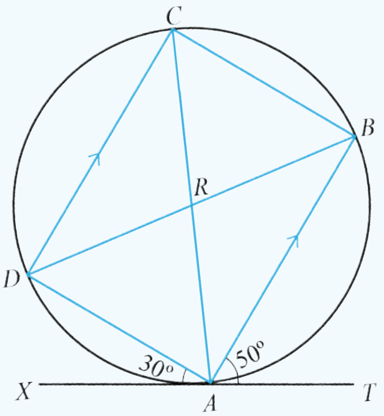
Therefore, the angles of ΔXZY are 50°, 75°, and 55° (angles formed in the alternate segments of a circle are equal).

Exercise 6.7

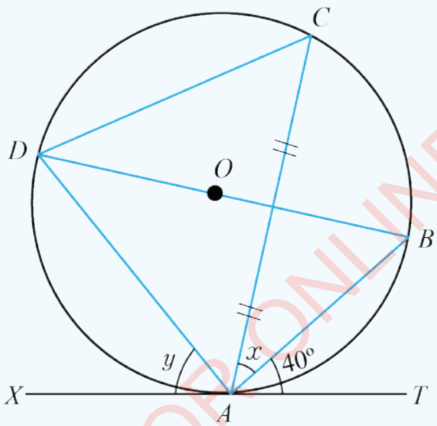
1. In the following figure, $\overline{TP}$ and $\overline{TQ}$ are tangents to the circle with centre O. Prove that $\overline{TO}$ bisects the angle between the tangents as well as the angle between the radii.



2. In the following figure, $\overline{TAX}$ is a tangent, $\overline{AB}$ is parallel to $\overline{DC}$, $T\hat{A}B = 50°$, and $X\hat{A}D = 30°$. Calculate $A\hat{B}C$.



3. In the following figure, $T\hat{A}X$ is a tangent and $\overline{DB}$ is a perpendicular bisector of $\overline{AC}$. If $T\hat{A}B = 40°$, calculate the values of x and y.

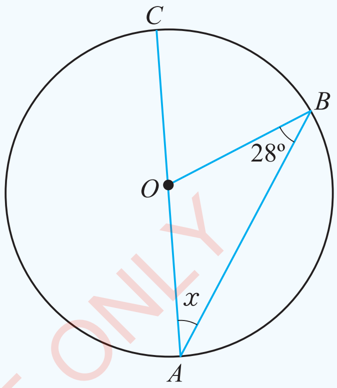


4. The distance of a chord PQ from the centre of a circle is 5 cm. If the radius of the circle is 13 cm, find the length of PQ.

5. A chord of length 32 cm is 12 cm away from the centre of a circle. Find the radius of the circle.

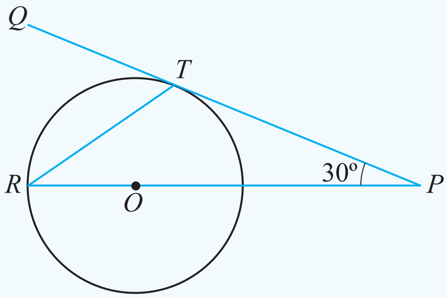
6. Two chords AB and CD intersect internally at X. If $\overline{AX} = 5$ cm, $\overline{BX} = 8$ cm, $\overline{CX} = 4$ cm, and $\overline{DX} = 10$ cm, prove that A, B, C, and D are concyclic points.

7. In the following figure, find the value of x if O is the centre of the circle.

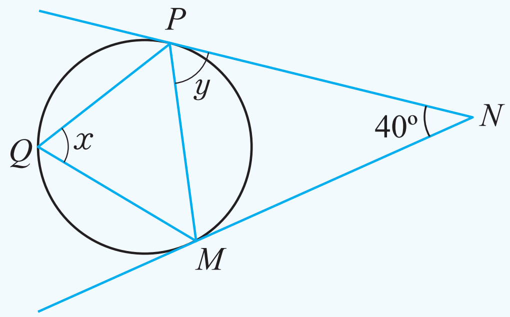


8. The chord AB of a circle with centre O and radius 3.3 cm is 2.15 cm long. Find the length of $\overline{AB}$ from O.

9. In the following figure, if $\overline{TP}$ is a tangent to the circle with centre O, and $T\hat{P}O = 30°$, find the size of each of the following angles:
   (a) $P\hat{O}T$ (b) $O\hat{R}T$ (c) $R\hat{T}Q$



10. In the following figure, find the sizes of the angles marked x and y.



Chapter summary

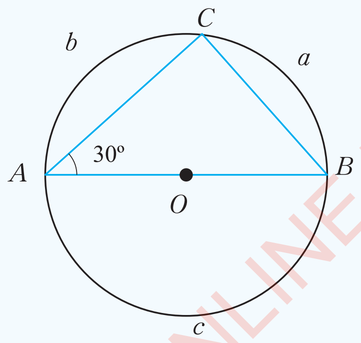
1. If an arc subtends an angle x at the centre of a circle of radius r, then its length is $\frac{x}{360°} \times 2\pi r$ or $\frac{\pi r x}{180°}$.
2. If r is a radius of a circle and AB is an arc length l, the ratio $\frac{l}{r}$ is the angle at the centre of the circle measured in radians.
3. To convert an angle from degrees into radians, multiply the given angle by $\frac{\pi}{180°}$, which is about 0.01745. To convert an angle from radians to degrees, multiply the given angle by $\frac{180°}{\pi}$, which is about 57.3°.
4. Suppose $\overline{AB}$ is a chord of a circle, O is the centre, and C is a point on the circumference, then $\angle AOB = 2 \times \angle ACB$.
5. Suppose $\overline{AB}$ is a diameter of a circle and C is a point on the circumference, then $\angle ACB = 90°$.
6. Suppose $\overline{AB}$ is a chord of a circle, and C and D are points on the circumference in the same segment, then, $\angle ACB = \angle ADB$.
7. A cyclic quadrilateral has all its vertices on a circle.
8. Suppose $ABCD$ is a cyclic quadrilateral, then $\angle DAB + \angle DCB = 180°$.
9. Suppose $\overline{AB}$ is a chord of a circle. The perpendicular bisector of AB passes through the centre of the circle.
10. Suppose $\overline{AB}$ and $\overline{CD}$ are parallel chords. The two arcs between them are congruent.
11. A tangent to a circle touches the circle at one point. A tangent is perpendicular to the radius at the point of contact. The tangents from a point to a circle are equal in length.

12. Suppose two tangents are drawn from point T to a circle with centre C. Then, $\overline{TC}$ bisects the angle between the tangents, as well as the angle between the two radii.
13. Suppose $\overline{AB}$ is a chord of a circle and $\overline{AT}$ is a tangent. If C is in the alternate segment, then $\angle TAB = \angle ACB$.

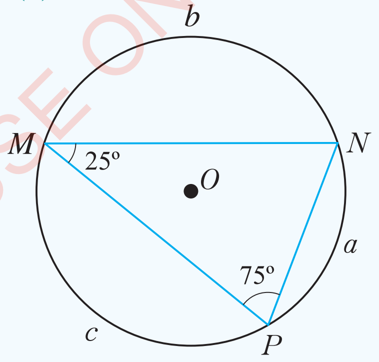
Revision exercise 6

1. Find the angles subtended at the circumference of a circle of radius 4.8 cm by a chord 3.0 cm long.
2. A chord XY of a circle of radius 6.8 cm subtends an angle of 42° at the centre of the circle. Find the difference in length between the chord XY and the minor arc XY.
3. Find in terms of π the lengths of the arcs marked a, b, and c in each of the following figures if the radii of the two circles is 6 cm.

(a)



(b)



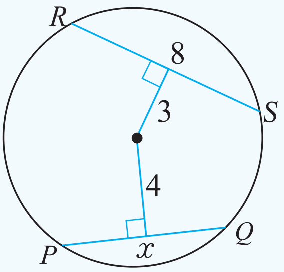
4. (a) Convert each of the following degrees into radians. Express your answers as multiples of π.
   (i) 45°
   (ii) 120°
   (iii) 135°
   (iv) 150°

   (b) Convert the following radians into degrees:
   (i) $\frac{\pi}{2}$ (ii) $\frac{\pi}{6}$ (iii) $\frac{5}{4}\pi$ (iv) $\frac{10}{3}\pi$

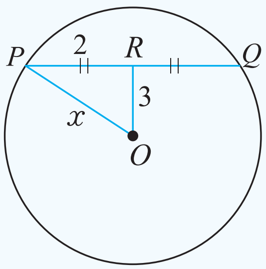
5. Give the size in degrees and radians of angles through which the minute hand of a clock has turns between noon and the following times:
   (a) 1220 hr
   (b) 0100 hr
   (c) 0215 hr
   (d) 2400 hr

6. Find the value of x in each of the following figures. All measurements are in centimetres, and O is the centre of the circles.

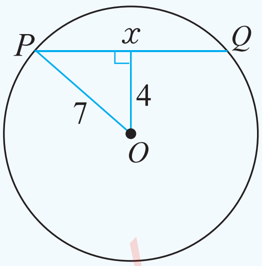
(a)



(b)

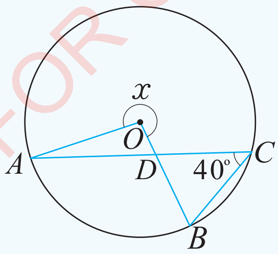


(c)

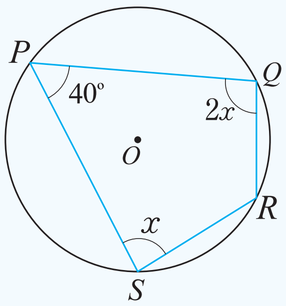


7. A circle has a centre C and radius 10 cm. If $\overline{AB}$ is a chord for which $\angle ACB$ is 140°,find the length of the chord AB.
8. The radii of two concentric circles are 17 cm and 10 cm. A straight line $ABCD$ crosses one circle at A and D, and the other circle at B and C. If $\overline{BC} =$ 12 cm, calculate $\overline{AB}$.
9. A tangent from an external point T to a circle of radius 4.5 cm is 16 cm long. Find the distance from T to the nearest point on the circumference.
10. If $\overline{AC}$ is a diameter of the circle $ABCD$ and $B\hat{D}C = 25°$, find $A\hat{C}B$.
11. In each of the following figures, find the value of x if O is the centre of each circle.

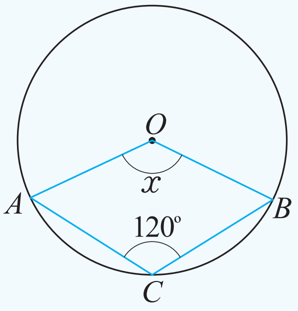
(a)



(b)



(c)

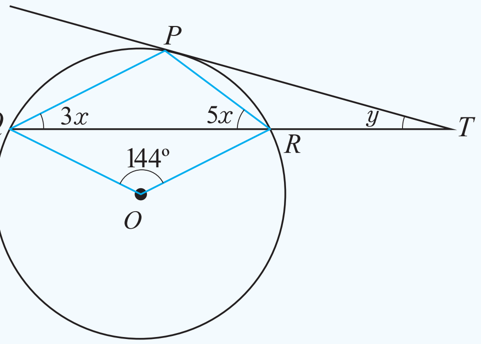


12. Two tangents from point T touches the circle at A and B. If $\overline{TA} = 9$ cm and $\angle ATB = 52°$, find $\overline{AB}$.
13. $PQRS$ is a cyclic quadrilateral. If $\angle PQR = 112°$, find $\angle PSR$.
14. A point T is 6 cm from the centre C of a circle of radius 3 cm. Find;
    (a) the length of the tangent from T to the circle.
    (b) the angle between the tangent and $\overline{TC}$.

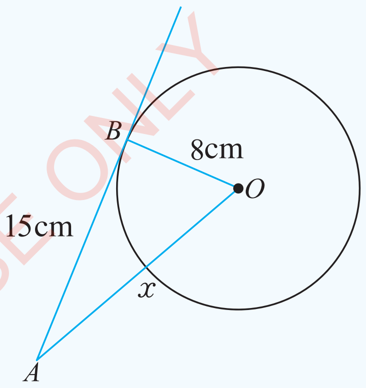
3. If C is the centre of a circle and AB is a chord, $\angle CAB = 50°$, and $\overline{AC} = 6$ cm, find:
    (a) The length $\overline{AB}$.
    (b) The distance $\overline{AB}$ from C.

16. Let chord PQ of a circle be extended to R and $\overline{RT}$ be a tangent. If $\overline{TQ} = \overline{QR}$, and $Q\hat{R}T = 35°$, calculate $P\hat{Q}T$ and $P\hat{T}Q$.

17. In the following figure, O is the centre of the circle. Find the values of x and y, hence find $P\hat{R}Q$ and $P\hat{T}R$.



18. In the following figure, find the value of x.



Chapter Seven

Earth as a sphere

Introduction

The shape of the Earth is very close to that of a sphere. It is not exactly spherical, but mathematicians, astronomers, and navigators have found it useful to describe the Earth as a sphere. In this chapter, you will learn about features of the Earth, location of places and calculate distance between places along great and small circles. The competencies developed in this chapter will help you in various daily life activities such as in navigation, locating places around the world, estimating distances between places, and many other applications.

Features and location of places on the Earth's surface

Scientists consider the Earth as a sphere although it is not a perfect sphere. The north and south poles are said to be slightly flat. Consider a sphere representing the shape of the Earth as shown in Figure 7.1.

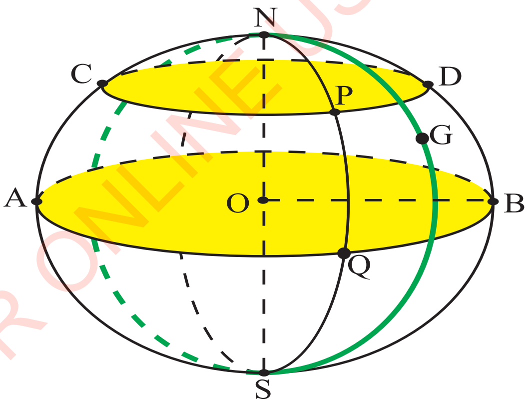


Figure 7.1: *A sphere representing the shape of the Earth*

The Earth rotates once a day about a line called the Earth’s axis. The axis passes through the centre of the Earth, and meets the Earth’s surface at the north pole (N) and south pole (S) . The line NS is the axis of the Earth and O is the centre of the Earth.

All points on the surface of the Earth are assumed to have equal distance from the centre. For example, $\overline{OB}$ is the radius of the Earth and it is taken to be approximately 6 370 kilometres.

The Earth has two types of circles, namely; great circles and small circles. A Great circle is the circle formed on the surface of the Earth by a plane which passes through the centre of the Earth *O*. Its radius is equal to the radius of the Earth. The number of great circles is infinite. In Figure 7.1, the circles through points *A, Q, B* and *A, N, B,* are examples of great circles.

The Equator is a great circle that lies on a plane perpendicular to the Earth's axis *NOS*. The circle through *A, Q,* and *B* is the Equator.
All great circles which pass through the Earth`s poles *N* and *S* are called Meridians. The circles through *N, D, B, S; N, G, S*, and *N, P, Q, S* are meridians. The meridian which passes through the Greenwich in the United Kingdom is specifically known as the Prime Meridian.

A small circle is formed on the surface of the Earth by a plane that cuts through the Earth, but does not pass through its centre. In Figure 7.1, a circle that goes through *C, P,* and *D* is a small circle.

Activity 7.1: Constructing a globe and locating small and great circles

Individually or in a group, perform the following tasks:
1. Collect wires, bamboo sticks or related materials.
2. Construct a large enough globe.
3. Use the constructed globe from task 2 to demonstrate great and small circles in front of the class.

Latitudes and longitudes

A parallel of latitude is a section of the Earth's surface formed by a plane parallel to the Equator. All circles formed on the surface of the Earth which are parallel to the Equator form parallels of latitudes. Figure 7.2 shows that the circles formed by points *A, B*, *C,* and *D,* and *E*, *F* and *G* are parallels of latitudes.

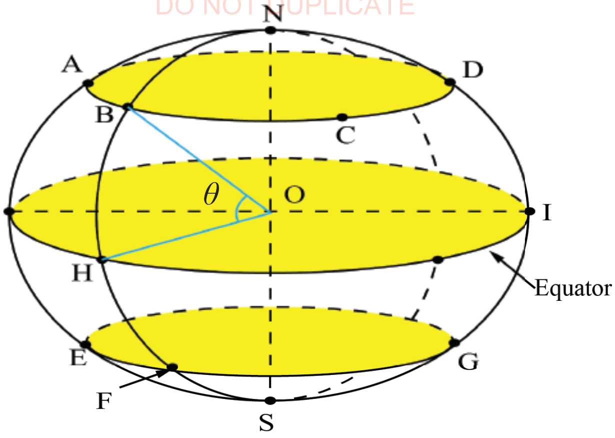


Figure 7.2: *Parallels of latitudes*

Suppose that any meridian through N, B, H, F, S cross the Equator at H and parallels of latitudes at B and F, then the angle which $\overline{OB}$ makes with the plane of the Equator is called the latitude of B. In Figure 7.2, θ is the latitude of point B. Points $A, B, C,$ and D are on the same parallel of latitude. Therefore, all places on the same parallel of latitude on the Earth's surface have the same latitude. Latitudes range from 0° along the Equator to 90°N at the North Pole or 90° S at the South Pole. Points H and I which are both on the Equator, have latitude 0°.

In Figure 7.3, if *NEAS* is the Prime Meridian, the angle between the planes *NEAS* and *NFGHS* is called the longitude of F. It is represented by the angle *EPF* or by the angle *AOH*.

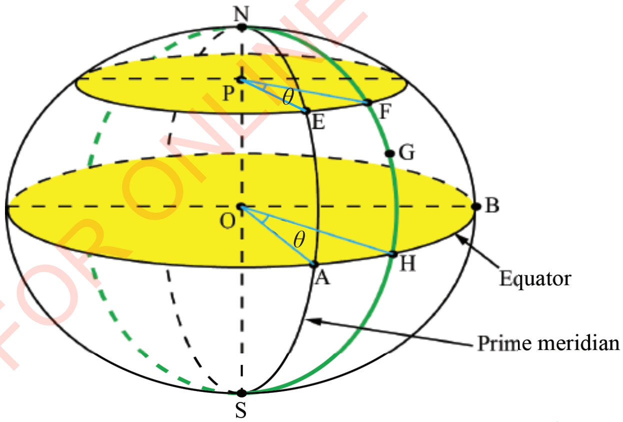


Figure 7.3: *Locating longitudes*

The points E and A have the same longitude θ since they lie on the same meridian. Thus, all points on the same meridian have the same longitude.

The longitudes vary from 0° along the Prime Meridian to 180°*E* or 180° *W*. If *G* in Figure 7.3 represents the Greenwich, then the Prime Meridian that passes through *N, F, G, H,* and *S* has the longitude of 0°.

Latitudes and longitudes are used to locate places on the Earth's surface. The latitudes are usually written first followed by the longitudes. For example, Q(25°N, 30°E) means point Q is 25° North of the equator and 30° East of the Prime Meridian.

If *A* and *B* are any two points having the same latitudes but different longitudes, or the same longitudes but different latitudes, the difference between the latitudes or longitudes gives the angle subtended at the centre of the Earth by an arc *AB*. For example, if point *A* is (25°N, 40°E) and point *B* is (66°N, 40°E), then the points differ only in their latitudes. Figure 7.4 shows the angle subtended by the arc *AB* at the centre of the Earth as $A\hat{O}B$.

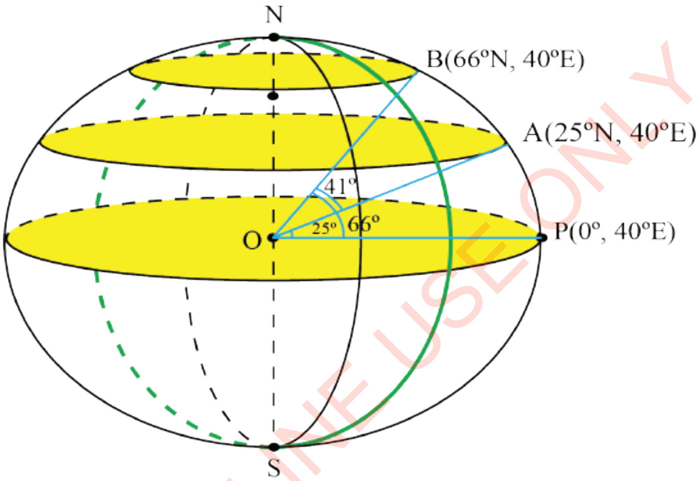


Figure 7.4: *Angle subtended at the centre of the Earth*

From Figure 7.4,

$$A\hat{O}B = B\hat{O}P - A\hat{O}P$$
$$= 66° - 25°$$
$$= 41°$$

Therefore, $A\hat{O}B$ is 41°.

The difference between the latitudes gives the angle subtended by the arc *AB* at the centre of the Earth, because the points *A* and *B* along the meridian are on the same side of the Equator. If the points are on different sides of the Equator, the angle formed at the centre is obtained by adding the latitudes.

Activity 7.2: Representing places on the globe

Individually or in a group, use a world map from an Atlas to perform the following tasks:

1. Identify three towns on the map that lie on different latitudes and longitudes.
2. Write the locations of the towns in task 1.
3. Use the globe you constructed in Activity 7.1 to demonstrate the location of the places.
4. Use the globe to demonstrate how to find differences in angles between places.
5. Share with the class your observations through presentation.

Example 7.1

Find the angle subtended at the centre of the Earth by the arc AB if A is $(25°N, 40°E)$ and B is $(66°S, 40°E)$.

Solution

The points $A(25°N, 40°E)$ and $B(66°S, 40°E)$ are shown in Figure 7.5.

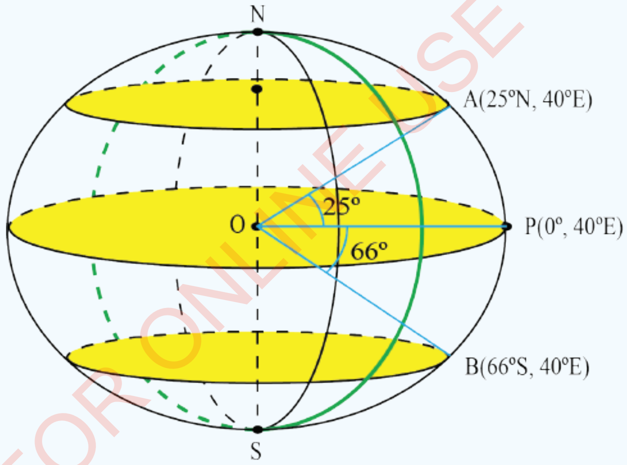


Figure 7.5: *Representation of points* $A(25°N, 40°E)$ *and* $B(66°S, 40°E)$

In Figure 7.5, angle AOB is the angle subtended by arc AB at the centre of the Earth. Since A and B lie along the same meridian and they are on different sides of the Equator, then the angle subtended by the arc AB at the centre is the sum of the latitudes.

Thus, $A\hat{O}B = A\hat{O}P + B\hat{O}P$
$= 25° + 66°$
$= 91°$

Therefore, the angle subtended by the arc AB is 91°.

Example 7.2

Two towns A and B are located on the Equator. The longitude of A is 10°E and that of B is 42°E. Find the angle subtended by the arc AB at the centre of the Earth.

Solution

The points $G(0°, 0°)$, $A(0°, 10° E)$ and $B(0°, 42°E)$ are shown in Figure 7.6. The circle through N, G, S is the Prime Meridian whose longitude is 0°.

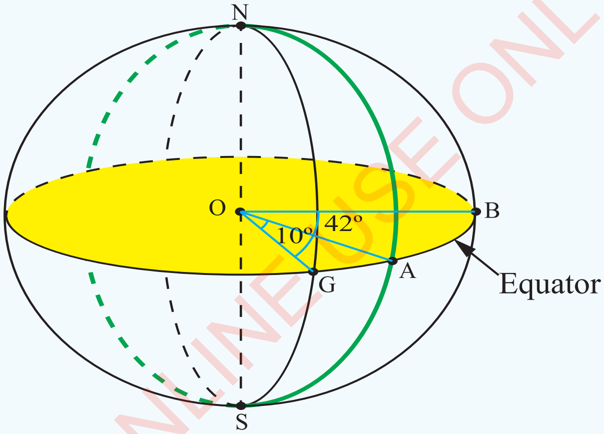


Figure 7.6: *Representation of points* $G(0°, 0°)$, A $(0°, 10° E)$ *and* B $(0°, 42°E)$

Angle AOB is the angle subtended by the arc AB at the centre of the Earth. Since town A and town B are on the same side of the prime meridian, the angle subtended by the arc AB at the centre is the difference between the longitudes.
Thus, $A\hat{O}B = G\hat{O}B - G\hat{O}A = 42° - 10° = 32°$.

Therefore, the angle subtended by the arc AB is 32°.

Example 7.3

Two towns A and B are on the Equator. The longitude of A is $35°\ E$ and that of B is $72°\ W$. Find the angle subtended by the arc AB at the centre of the Earth.

Solution

The points $G(0°, 0°)$, $A(0°, 35°E)$, and $B(0°, 72°W)$ are indicated in Figure 7.7. In this case, NGS is the Prime Meridian whose longitude is $0°$.

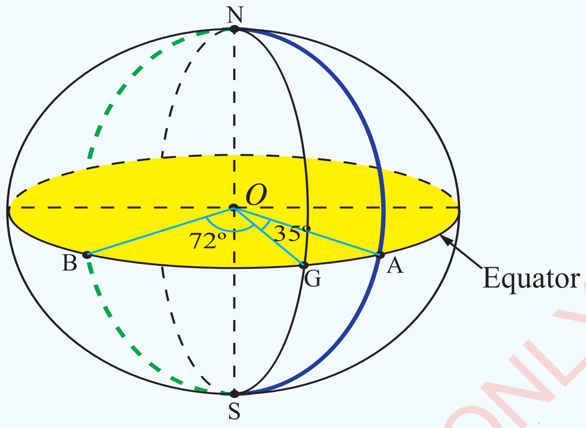


Figure 7.7: *Representation of the points* $G(0°, 0°)$, $A(0°, 35°E)$ *and* $B(0°, 72°W)$

In Figure 7.7, angle AOB is subtended by the arc AB at the centre of the Earth. Since town A and town B are on either side of the prime meridian, the angle subtended by the arc AB at the centre is the sum of the longitudes. Thus,

$$A\hat{O}B = G\hat{O}A + G\hat{O}B = 35° + 72° = 107°.$$

Therefore, the angle subtended by the arc AB is $107°$.

Exercise 7.1

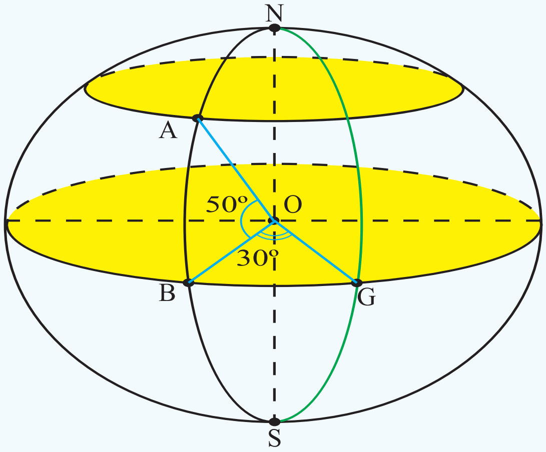
In questions 1 to 4, consider the towns and cities indicated in Table 7.1, then answer the questions that follow.

Table 7.1: Locations of different towns and cities

Tabora (5°S, 33°E)	Dar es Salaam $(7°S, 39°E)$	Mbeya $(9°S, 33°E)$
Chake chake $(5°S, 40°E)$	Tanga $(5°S, 39°E)$	Moshi $(3°S, 37°E)$
Zanzibar $(6°S, 40°E)$	Mwanza $(3°S, 33°E)$	Morogoro $(7°S, 38°E)$
Nakuru $(0°\ 36°E)$	Kampala $(0°, 33°E)$	Gulu $(3°N, 32°E)$

1. Which towns or cities have latitudes like that of:
   (a) Moshi? (b) Chake chake?
2. Which towns or cities have longitudes like that of:
   (a) Mbeya? (b) Dar es Salaam?
3. Find the angle subtended at the centre of the Earth by the arc AB if A is Mwanza City and B is Mbeya City.
4. Find the angle subtended at the centre of the Earth by the arc XY if X is Nakuru and Y is Kampala.
5. Consider the following list of places: Nakuru $(0°, 36°\ E)$, Kampala $(0°, 33°E)$, Accra $(6°N, 0°)$, London $(52°N, 0°)$, Hull $(54°N, 0°)$.
   (a) Which of these places lie on the Prime Meridian?
   (b) Which of these places lie on the Equator?
6. Given that Morogoro is $(7°S, 38°E)$ and Moscow is $(56°N, 38°E)$, find the angle subtended at the centre of the circle by the arc which connects the two places.
7. Which of the following places lie on the same meridian? Tanga $(5°S, 39°E)$, Mbeya $(9°S, 33°E)$, Dar es Salaam $(7°S, 39°E)$, Morogoro $(7°S, 38°E)$, Nairobi $(1°\ S, 37°E)$, Addis Ababa $(9°N, 39°E)$, Cairo $(30°N, 31°E)$, New York $(41°N, 74°W)$.

8. In the following figure, if the circle through N, G, and S is the Prime Meridian, O is the centre of the Earth and the Equator passes through B and G, give the longitude and latitude of A.



9. Draw a figure to illustrate the position of point $H(60°S, 45°E)$.
10. Suppose P and Q are two towns on the latitude 0°. If the longitude of P is 116° E and that of Q is 105°W, find the angle subtended by the arc connecting the two places at the centre of the Earth. Draw a figure to illustrate their positions.

Distance along great circles

Activity 7.3: Finding the distance along great circles using a globe

Individually or in a group, use the globe you prepared in Activity 7.1 to perform the following tasks:

1. Use wires, threads or any other related materials to indicate a great circle.
2. Use a world map from an Atlas and choose two cities which are on the same meridian but on different latitudes.
3. Locate the two cities on your globe and indicate the angle formed by the arc passing through the locations of the cities.
4. Use the results in task 3 to draw a figure which represents the locations of the cities. Use the globe and the figure to find the difference in angles between the places.
5. Repeat the tasks (2) to (4) using different cities located on the equator.
6. With these two cases, use your prior knowledge of circles to find the distance between the two places you have located in each case.
7. Share your final work to the rest of the class through presentation.

Distances on the surface of the Earth are usually expressed in kilometres or nautical miles. A nautical mile is the length of an arc of a great circle on the Earth's surface that subtends an angle of 1 minute ($1'$) at the centre of the Earth. For example, in Figure 7.8, the length of the arc AB is 1 nautical mile.

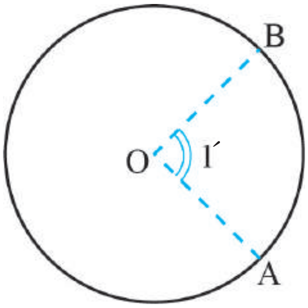


Figure 7.8: *An arc length of 1 nautical mile*

One degree ($1°$) is equal to 60 minutes and therefore $1°$ is equal to 60 nautical miles (nm). This means that 1 minute is equal to 1 nm. That is,
$1° = 60$ minutes and $1° = 60$ nm
Hence, 1 minute = 1nm.

But, 1nm is equal to the length of an arc along a great circle which subtends an angle of 1 minute at the centre of the great circle. Since the angle at the centre is $360°$, then the distance along the great circle is given by;

Distance along the great circle = 360×60 nm
$= 21\ 600$ nm.

The distance along the great circle represents the circumference of the equator.

But, length of an arc $= \frac{\theta}{360°} \times 2\pi R$, where θ is the angle subtended at the centre

$$= 2\pi R\left(\frac{1'}{360°}\right)$$

$$= 2\pi R\left(\frac{1'}{360 \times 60'}\right), \text{ since } 1° = 60'$$

$$= \frac{2\pi \times 6\ 370 \text{ km}}{360 \times 60}, \text{ since } R = 6\ 370 \text{ km}$$

$$= 1.852 \text{ km}$$

Thus, 1 nm = 1.852 km.

So, the circumference C of a great circle is given by

$$\begin{aligned} C &= 21\,600 \times 1.852 \text{ km} \\ &= 40\,003.2 \text{ km} \\ &\simeq 40\,000 \text{ km.} \end{aligned}$$

Therefore, the circumference of the Earth is approximately 40 000 km. This length is approximately the same as 21600 nautical miles.

The circumference of any great circle can also be obtained using the formular $C = 2\pi R$, where R is the radius of the Earth. The radius of the Earth is approximately 6 370 km.

Thus, $C = 2\pi R$

$$\begin{aligned} &= 2 \times 3.14 \times 6\,370 \text{ km} \\ &= 40\,003.6 \text{ km.} \end{aligned}$$

Therefore, the circumference of the Earth is approximately 40 000 km.

Example 7.4

An aeroplane starts at (20˚S, 30˚E), and flies north for 4 000 km. Find its new latitude and longitude.

Solution

The aeroplane flies north, hence its longitude is unchanged. It starts south of the equator, and flies north. It may cross the equator, and so end up north of the equator. Suppose the plane flies along x latitudes.

Length of an arc $= \dfrac{\theta}{360°} \times 2\pi R$, $R = 6\,370$ km, $\theta = x$.

Length of an arc = 4 000 km

Substituting the values into the formula gives

$$4\,000 = \frac{2 \times 3.14 \times 6\,370 \times x}{360°}$$

Solving for x gives

$$\begin{aligned} x &= \frac{4\,000 \times 360°}{2 \times 3.14 \times 6\,370} \\ &= 35.99° \approx 36°. \end{aligned}$$

Since, $36° > 20°$, then, its new position is at north of the equator, that is (y N, 30° E).

Therefore,

$$\begin{aligned} y + 20° &= 36° \\ y &= 36° - 20° \\ &= 16° \end{aligned}$$

Hence, the new latitude is 16°N. Therefore, the new position is (16°N, 30°E).

Exercise 7.2

In the following questions, take the radius of the Earth $R = 6370$ km and $\pi = 3.14$.

1. Calculate the length of the Prime Meridian from South to North pole in:
   (a) Nautical miles (b) Kilometres.
2. The city of Kampala lies on the Equator. Calculate the distance in kilometres from Kampala to the South pole.
3. How far is point B from point A, if A is $(0°, 0°)$ and B is $(0°, 180° E)$?
4. If the latitude of Nakuru is $0°$, find the distance in nautical miles from Nakuru to the North pole.
5. An aeroplane flies south from $(12°N, 36°E)$ to $(7°S, 36°E)$ in 21 hours. Find its speed, giving your answer both in nautical miles per hour and in kilometres per hour.
6. A ship sails south at 18 km/h. If it starts at $(15°N, 150°W)$, find its position after 40 hours.
7. A ship sails at 20 km/h. How long does it take to sail from $(31°N, 150°W)$ to $(22°S, 150°W)$?
8. A ship sailed north from $(45°S, 28°W)$ to $(39°S, 28°W)$, taking 30 hours. What was its speed? Give your answer both in nautical miles per hour and in kilometres per hour.

Distance between points along the meridian

All the meridians are great circles. Thus, finding the distance between two points located on the same meridian is the same as finding an arc length of a circle with the radius of the Earth.

Activity 7.4: Finding the distance along a meridian by using a globe

Individually or in a group, use the globe you already prepared in Activiy 7.1 to perform the following tasks:

1. Use wires, threads or any other related materials to indicate a great circle.
2. Use a world map from an Atlas and choose two cities which are on the same meridian but different latitudes.
3. Locate two cities in task 2 on your globe and indicate the angle formed at the centre of the Earth between the cities.
4. Use the result in task 3 to draw a figure which represents the locations of the cities. Use the globe and the figure to find the difference in degrees between the cities.

5. Use your prior knowledge of circles to find the distance between the two places you have located.
6. Share your final work to the rest of the class through presentation.

Figure 7.9 shows points A and B which are along a meridian on the same side of the Equator.

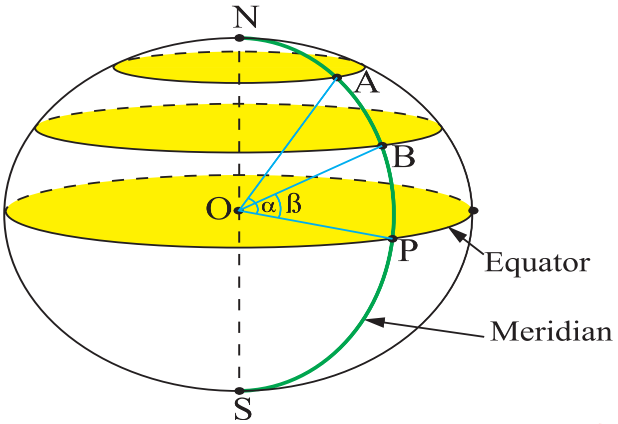


Figure 7.9: *Points A and B along a meridian*

If the latitude of point A is α and that of point B is β, where $\alpha > \beta$, then the distance between A and B along the meridian is $(\alpha - \beta) \times 60$ nautical miles.
A difference of 1° along the meridian is equivalent to a distance of 60 nautical miles.

Also, since 1 nautical mile is taken to be 1.852 km, the length of the arc AB is given by:

Length of arc $AB = (\alpha - \beta) \times 60 \times 1.852$ km

$= 111.12\,(\alpha - \beta)$ km.

The length of the arc AB in kilometres, or distance between the points A and B is given by:

Length of arc $AB = \frac{(\alpha - \beta)}{360°} \times 2\pi R$ or Length of arc $AB = \frac{\pi R(\alpha - \beta)}{180°}$,

where R is the radius of the Earth in kilometres.

Similarly, the distance between two points along a meridian on either side of the Equator can be calculated.

Consider Figure 7.10 which shows two points A and B located on either side of the Equator.

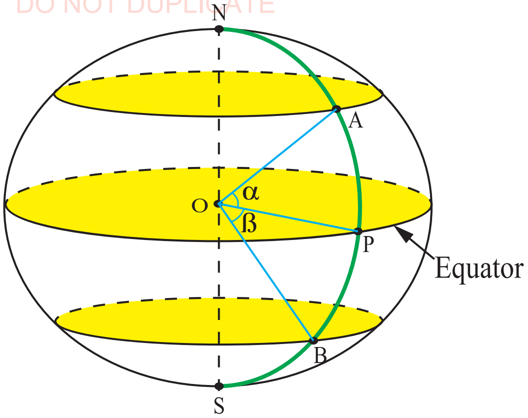


Figure 7.10: *Points A and B located on either sides of the equator*

The angle subtended by an arc AB at the centre of the Earth is $\alpha + \beta$. The distance between the points A and B along the meridian is therefore given by:

$$(\alpha + \beta) \times 60° \text{ nautical miles.}$$

The length of the arc AB is given by:

Length of arc $AB = (\alpha + \beta) \times 60 \times 1.852$ km

$= 111.12\ (\alpha + \beta)$ km.

The length of the arc AB, or the distance between points A and B in kilometres is given by length of arc $AB = \frac{(\alpha + \beta)}{360^0} \times 2\pi R$, where R is the radius of the Earth in kilometres.

Usually, the speed of ships is expressed in terms of knots. A knot is a speed of one nautical mile per hour. Therefore, 1 knot = 1.852 km/h.

For example, when a ship's speed is given as 20 knots, it is actually sailing at 20 nautical miles per hour, or approximately 37 kilometres per hour.

Example 7.5

Find the distance between A(30°N, 139°E) and B(45°N, 139°E) in:

(a) Nautical miles.

(b) Kilometres.

Solution

Since A and B have the same longitude, they are on the same meridian. The difference between the latitudes is $(\alpha - \beta) = 45° - 30° = 15°$.

(a) Distance between A and B = 15° × 60 nautical miles = 900 nautical miles.
Therefore, the distance is 900 nautical miles.

(b) Since 1 nautical mile = 1.852 km, then
900 nautical miles = 1.852 × 900 km
= 1 666.8 km

Therefore, the distance is approximately 1 667 km.

Alternatively;

Length of an arc $AB = \frac{(\alpha - \beta)}{360°} \times 2\pi R$, where $R = 6\ 370$ km.

$$= \frac{(45° - 30°)}{360°} \times 2 \times 3.14 \times 6\ 370 \text{ km}$$

$$= 1\ 666.8 \text{ km}$$

Therefore, the distance in is approximately 1 667 kilometres.

Example 7.6

Find the distance between $A(29°N, 72°W)$ and $B(10°S, 72°W)$,

(a) in nautical miles (b) in kilometres.

Solution

(a) Both points have the same longitude. So they are on the same meridian.
The sum of latitudes is given by:
$(\alpha + \beta) = 29° + 10° = 39°$.
Distance between A and B in nautical miles = $(\alpha + \beta) \times 60$ minutes
Distance = 39 × 60 minutes, but $1' = 1$ nm
= 2 340 nautical miles.

Therefore, the distance between A and B is 2 340 nautical miles.

(b) Since 1 nautical mile = 1.852 km, then we get
2 340 nautical miles = 2 340 × 1.852 km.

The distance between A and B = 4 334 km.
Alternatively,
Given $\alpha = 29°$, $\beta = 10°$, $R = 6\ 370$ km, $\pi = 3.14$.

The length of arc $AB = \frac{(\alpha+\beta)}{360°} \times 2\pi R$

$$= \frac{(29°+10°)}{360°} \times 2 \times 3.14 \times 6\,370 \text{ km}$$

$$= 4\,334 \text{ km.}$$

Therefore the distance between points A and B is 4 334 km.

Example 7.7

A ship sails from $J(0°, 20°W)$ to $K(10°N, 20°W)$ at 16 knots. If it leaves point J at 8 am on Tuesday, at what time will it arrive at point K?

Solution

Both places have the same longitude. So, points J and K are on the same meridian. The angle subtended by the arc JK at the centre of the Earth is 10°.

The distance $JK = 10 \times 60$ nautical miles

$= 600$ nautical miles.

Since the speed of the ship is 16 knots or 16 nautical miles per hour, the time of voyage is calculated as follows:

$$= \frac{600}{16} \text{ hours}$$

$$= 37.5 \text{ hours}$$

$$= 37 \text{ h}, 30 \text{ min.}$$

From Tuesday 8 am to Wednesday 8 am, the duration is 24 hours. Since there are 13h 30 min left, the ship will arrive at point K at 9.30 pm on Wednesday.

Therefore, the ship's arrival time is 9.30 pm on Wednesday.

Exercise 7.3

1. Find the distance in nautical miles between two places on the same meridian with latitudes:
   (a) $10°N$ and $35°N$ (b) $20°N$ and $42°S$

2. Find the distance in kilometres between two places on the same meridian with latitudes:
   (a) $15°S$ and $13°N$ (b) $8°S$ and $24°S$
3. Find the distances in nautical miles between each of the following pairs of points:
   (a) $A(18°N, 12°E)$ and $B(65°N, 12°E)$
   (b) $H(31°S, 76°W)$ and $G(22°N, 76°W)$
4. Find the distance in kilometres between the following pairs of points which lie on the Equator:
   (a) $A(0°, 17°W)$ and $B(0°, 52°W)$
   (b) $X(0°, 22°W)$ and $Y(0°, 11°E)$
5. Find the distance in kilometres between each of the following cities:
   (a) Tanga $(5°S, 39°E)$ and Addis Ababa $(9°N, 39°E)$
   (b) Mbeya $(9°S, 33°E)$ and Tabora $(5°S, 33°E)$
6. The Equator crosses the western coast of Africa at longitude $11°W$ and the eastern coast at longitude $43°E$. What is the distance in kilometres across the African continent along the Equator?
7. Two towns A and B are 1 800 nautical miles apart, B being North of A. If the latitude A is $20°N$, find the latitude of B.
8. Two towns, P and Q are 2 500 kilometres apart, P being due South of Q. If the latitude of P is $5°S$, find the latitude of Q.
9. A ship sails northwards to Tanga $(5°S, 39°E)$ at an average speed of 12 knots. If the ship's starting point is Dar es Salaam $(7°S, 39°E)$ at 12:00 noon, when will the ship reach Tanga?
10. Two towns $P(60°N, 15°W)$ and $Q(60°N, 30°E)$ are on the same parallel of latitude. An aircraft flying along the parallel of latitude takes 4 hours 10 minutes to fly from P to Q. Taking the radius of the Earth to be 6 370 km, calculate to the nearest whole number:
    (a) The distance covered by the aircraft from P to Q.
    (b) The speed of the aircraft.

Distances along small circles

Activity 7.5: Finding the distance along small circles using a globe

Individually on in a group, use the globe you prepared in Activity 7.1 to perform the following tasks:

1. Use wires, threads or any other related materials to indicate a great circle.
2. Using world map from an Atlas, choose two cities which are on the same latitude but different longitudes.
3. Locate two cities on your globe and indicate the angle formed by the arc through the two cities.
4. Use the result in task 3 to draw a figure which represents the locations of the cities. Use the globe and the figure to find the angle difference between the two places.
5. Use the prior knowledge of circles to find the distance between the two cities you have located.
6. Share your final work with the rest of the class through presentations and discussions.

All latitudes are small circles except the equator which is a great circle. The radius of small circles decrease as one move either south or north of the equator. Thus, all small circles have radii which are shorter than that of the Earth or any great circle. Therefore, in order to obtain the radius of any small circle, you need to resolve the radius of the Earth. Let P be any point on the surface of the Earth. Through this point, a parallel of latitude can be drawn as shown in Figure 7.11.

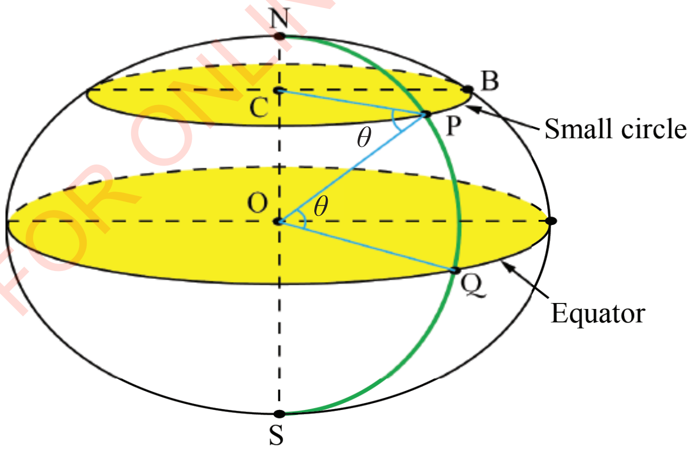


Figure 7.11: *A parallel of latitude through point P on the surface of the Earth*

The radius of the Earth on the plane of the Equator is perpendicular to the polar axis passing through the centre of the Equator. Since the latitude is parallel to the Equator, then $\overline{OQ}$ is parallel to $\overline{CP}$ (see Figure 7.11). Therefore,

$$P\hat{O}Q = C\hat{P}O = \theta° \text{(alternate interior angles).}$$

The radius of a small circle r, radius of the Earth R, and the angle θ are shown in Figure 7.12.

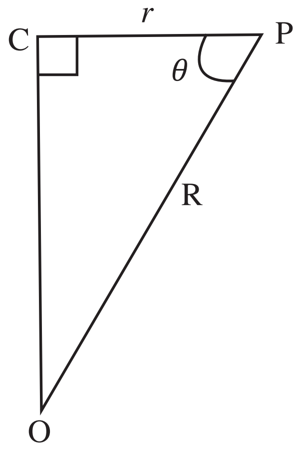


Figure 7.12: *Relationship between the radius of the Earth and the radius of a small circle*

The relationship between r, R, and θ is given by

$$r = R\cos\theta.$$

Therefore, the length of the parallel of latitude is given by:

$$\begin{aligned} C &= 2\pi r \\ &= 2\pi R\cos\theta. \end{aligned}$$

This is the circumference of a small circle of the parallel of latitude. For example, if the latitude of point P is $60°N$, the length of the parallel of latitude through P is given by:

$$\begin{aligned} C &= 2\pi R\cos\theta \\ &= 2\pi \times 6\,370 \cos 60° \text{ km} \end{aligned}$$

But, $\cos 60° = 0.5$. It follows that,

$$\begin{aligned} C &= 2 \times 3.14 \times 6\,370 \times 0.5 \text{ km} \\ &= 20\,001.8 \text{ km.} \end{aligned}$$

Therefore, the length of the parallel of latitude is approximately 20 002 kilometres.

Consider two points A and B having the same latitude as shown in Figure 7.13.

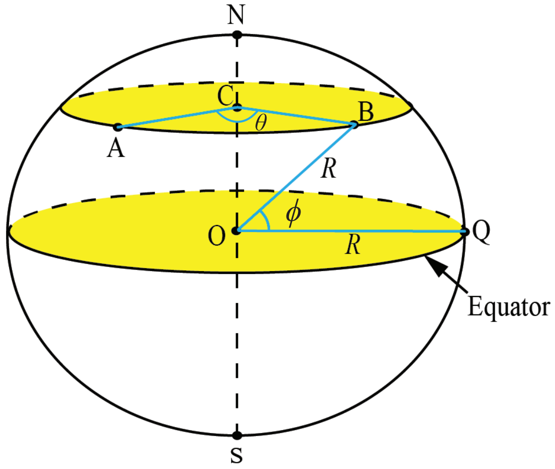


Figure 7.13: *Points A and B on the same latitude*

The angle and positions of A and B along a parallel of latitude is shown in Figure 7.14.

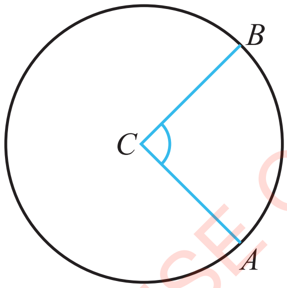


Figure 7.14: *The angle and position of points A and B along the the parallel of latitude*

Using Figure 7.14, we have

$$\frac{\text{Length of arc AB}}{\text{Circumference of a small circle}} = \frac{\theta}{360°}, \text{ where } \theta \text{ is the measure of } A\hat{C}B.$$

$$\text{Length of arc AB} = \frac{\theta}{360°} \times \text{circumference of a small circle}$$

$$= \frac{\theta}{360°} \times 2\pi r = \frac{\theta}{180°} \times \pi r.$$

But, the radius of a small circle is $r = R\cos\phi$, where R is the radius of the Earth and ϕ is the angle subtended at the centre of the Earth. This indicates that, the length of arc AB along a parallel of latitude is given by:

$$\text{Length of arc } AB = \frac{\theta}{180°} \times \pi R \cos\phi.$$

Example 7.8

Two towns L and M are on latitude $40°N$. If L is on the meridian $25°E$ and M is on $55°E$, find the length of arc LM in:

(a) Kilometres.

(b) Nautical miles.

Solution

(a) The angle subtended by arc LM at the centre of the parallel of latitude $40°N$ is $\theta = 55° - 25° = 30°$, $\phi = 40°$.

The length of arc $LM = \frac{\theta}{360°} \times 2\pi R \cos\phi$

$$= \frac{30°}{360°} \times 2 \times 3.14 \times 6\ 370 \cos 40° \text{ km}$$

$$= 2\ 553.7 \text{ km}$$

Therefore, the length of arc LM is 2 553.7 km.

(b) The length of an arc subtending 1° at the centre of the parallel of latitude 40° is $60\cos 40°$ nautical miles.

The difference in longitudes is $\theta = 55° - 25° = 30°$.

The length of arc $LM = \theta \times 60 \cos \phi$

$$= 30° \times 60\cos 40°$$

$$= 1\ 800\cos 40° \text{ nautical miles}$$

$$= 1\ 378.9 \text{ nautical miles.}$$

Therefore, the length of arc LM is approximately 1 379 nautical miles.

Example 7.9

The points P and Q are located on latitude $50°N$. The distance between P and Q measured along the parallel of latitude is 1 429 km. Find the difference between their longitudes.

Solution

Given $\phi = 50°$, distance of an arc $PQ = 1\ 429$ km, we find the difference between their longitudes, θ.

The length of arc PQ = $\theta \times \cos\phi$, where $\phi = 50°$ and length of arc PQ = 1 429 km

$$1\,429 \text{ km} = \theta \times 60 \cos 50° \times 1.852 \text{ km}$$

But cos 50° = 0.6428, thus,

$$1\,429 \text{ km} = \theta \times 60 \times 0.6428 \times 1.852 \text{ km}$$

Solving for θ gives

$$\theta = \frac{1\,429 \text{ km}}{60 \times 0.6428 \times 1.852 \text{ km}}$$

$$= 20.006°$$

$$\simeq 20°$$

Therefore, the difference between their longitudes is 20°.

Example 7.10

A ship is steaming at 15 knots from point Q in a western direction to point P. If the position of point P is $(40°S, 178°E)$ and that of point Q is $(40°S, 172°E)$, how long will the journey take?

Solution

Given speed = 15 knots, $\alpha = 178°$, $\beta = 172°$, and $\phi = 40°$, then we get

$$\theta = \alpha - \beta$$
$$= 178° - 172°$$
$$= 6°.$$

The points P and Q are on the same latitude. The speed of the ship is 15 knots, or 15 nautical miles per hour. The difference in their longitudes is 6°.

Now, the length of arc PQ = $6° \times 60 \cos 40°$ nautical miles.

$= 6° \times 60 \times 0.766$ nautical miles.

$= 275.8$ nautical miles

$$\text{Time taken} = \frac{\text{The length of arc } PQ}{\text{The speed of the ship}}$$

$$= \frac{275.8 \text{ nautical miles}}{15 \text{ knots}}$$

$$= 18.4 \text{ hours.}$$

Therefore, the journey will take approximately 18 hours and 24 minutes.

Exercise 7.4

1. Two points on latitude $50°N$ lie on longitudes $35°E$ and $40°W$. What is the distance between them in nautical miles?
2. The location of Morogoro is ($7°S$, $38°E$) and that of Dar es Salaam is ($7°S$, $39°E$). Find the distance between them in kilometres.
3. A ship after sailing for 864 nautical miles eastwards, finds that her longitude has altered by 30°. Along what parallel of latitude is the ship sailing?
4. Two places A and B on latitude $54°N$ are 450 nautical miles apart. Find the difference in their longitudes.
5. An aeroplane flies westwards along the parallel of latitude $20°N$ from town F on longitude $40°E$ to town G on longitude $10°W$. Find the distance between the two towns in kilometres.
6. An aeroplane flies from town P to town Q along a parallel of latitude $30°S$. If the longitude of P is $118°E$ and that of Q is $150°E$, find the length of arc PQ in nautical miles.
7. Find the parallel of latitude South of the Equator whose length is half that of the Equator.
8. Three towns, P, Q, and R are on the parallel of latitude $20°N$. The distance between P and Q is 1 482 kilometres and between Q and R is 926 kilometres. Town Q, whose longitude is $10°E$, is between town P and town R. Find the longitudes of P and R.
9. An aeroplane flies from Tabora ($5°S$, $33°E$) to Tanga ($5°S$, $39°E$) at 332 kilometres per hour along a parallel of latitude. If it leaves Tabora airport at 3:00 pm, find the arrival time at Tanga airport.
10. Prove that the length of the parallel of latitude of $60°N$ is equal to the distance from the North Pole to the South Pole measured along any meridian.
11. Calculate the circumference of a small circle in kilometres along the parallel of latitude $10°S$.
12. Calculate the length of the parallel of latitude through Mumbai which is located at ($19°N$, $73°E$). Give your answer in:
    (a) Nautical miles (b) Kilometres.
13. What is the latitude of a point P North of the Equator if the length of the parallel of latitude through P is 28 287 km? (Give your answer to the nearest degree).
14. Find the radius of a small circle of latitude $70°N$.

Chapter Summary

1. The position of any point on the Earth's surface is given by its latitude and longitude. The latitude measures the degrees north or south of the Equator, and the longitude measures the degrees east or west of the Greenwich meridian.
2. When finding the difference between the latitudes of two different places on the Earth's surface, take into account whether they are on the same side of the Equator or on opposite sides. When finding the difference between longitudes, take into account whether they are on the same side of the Greenwich meridian or on opposite sides.
3. The distance between two points along a great circle is $\frac{2\pi R\theta}{360}$ km or 60θ nm, where θ is the difference in latitudes and R is the radius of the Earth.
4. A circle of latitude α has a radius $R\cos\alpha$. A distance between two points along this circle is $\frac{2\pi R\theta}{360°}\cos\alpha$ km or $60\theta\cos\alpha$ nm, where θ is the difference in longitudes.

Revision exercise 7

1. Two points X and Y are located on the Equator. The longitude of X is $8°W$ and that of Y is $8°E$. Find:
   (a) The angle subtended by arc XY at the centre of the Earth.
   (b) The length of arc XY in kilometres.
2. Find the length in kilometres of the parallel of latitude through Dodoma which is located at $(6°S, 36°E)$.
3. Find the radius of a small circle parallel to the Equator along latitude $23.5°S$.
4. An aeroplane flies from point A with longitude $32°E$ to point B with longitude $78°E$, both located along parallel of latitude $30°N$.
   (a) Locate the positions of points A and B using a diagram of a sphere indicating angles at the centre.
   (b) Find the speed at which an aeroplane flies if it takes 10 hours to reach point B.
5. A ship sails from point $T(10°S, 30°W)$ to point $U(10°N, 30°W)$ at 20 knots. If it leaves point T at 12:00 midnight on Monday, when will it arrive at point U?

6. A straight tunnel through the centre of the Earth from Botswana would come out in the Hawaii island.
   (a) How long would the tunnel be in kilometres?
   (b) How far is Botswana from the Hawaii island on the surface of the Earth along a great circle? Give your answer in kilometres.
7. How long is the distance around the Earth in kilometres on the parallel of latitude 30° South of the Equator?
8. A ray from the centre of the Earth to a point P on the surface of the Earth makes an angle of 75° with the South Pole. How long is the parallel of latitude through P?
9. The points X and Y are located on the Equator, Y being due east of X. If the length of the arc XY along the Equator is 3 000 km, and the longitude of Y is 18°E, find the longitude of X.
10. Ships C and D are 250 nautical miles apart, D being due South of C. If the latitude of C is 5°S, find the latitude of D.
11. Find the parallel of latitude South of the Equator whose length is double that of latitude 75°N.
12. If O is the centre of the Earth, C is the centre of a small circle of latitude 60°N, and R is the radius of the Earth, show that the distance between O and C measured along the polar axis is $\frac{R}{2}\sqrt{3}$.
13. The points A and B are on the surface of the Earth on the same parallel of latitude. The difference in their longitudes is 90°. If the shortest distance from point A to point B along the parallel of latitude is 1 600 km, what is the latitude of point A and point B? Take the radius of the Earth as 6 400 km.
14. Given the points K(50°N, 30°E) and L(50°N,150°W), two aeroplanes fly from point K to point L. One flies non-stop along a great circle over the North Pole at an average speed of 300 knots. How long does the flight take? The second aeroplane flies along the parallel of latitude 50°. What is the average speed of this aeroplane if it takes the same time of the flight as the first aeroplane?

Chapter Eight

Accounting

Introduction

The knowledge of accounting is vital to every individual as everyday activities involve exchange of money with goods and other services. In this chapter, you will learn the double entry system, describe types of ledgers, prepare and post entries in ledger accounts, balance accounts, prepare a trial balance, financial statements (final accounts), income statements (trading, profit and loss accounts), and statements of financial position (balance sheet), and interpret information from a statement of a financial position. The competencies developed in this chapter will help you in managing transactions related to various activities such as domestic requirements like buying foods, clothes, construction of houses, managing banking services, and business, among many other applications.

Concept of accounting

Accounting is the process of recording financial transactions pertaining to a business. The accounting process involves analysing and summarizing business financial transactions and preparation of financial statements at the end of the trading period. The following are some of the importances of accounting.

Keeping records of business transactions

Accounting helps in keeping up-to-date records of financial information of a business for the purpose of assessing its peformance over a period of time.

Decision making

Accounting is vital in providing financial information to the owners of a business or an organisation for effective decision making. Activities such as expanding a business, hiring, dismissing workers, improving the business or organisation plans are facilitated by detailed financial information which is a result of accounting.

Sharing of business results

Credibility of a business, company or an organisation is highly determined by its ability of sharing credible accounting information to the intended users. Accounting helps in carefully recording, analysing and finally sharing of important financial information to external users such as prospective investors, government (Tax inspectors), financial institutions and creditors.

Meeting legal requirements

Organisations and businesses are subject to providing justification for the allocation of resources. Effective accounting system provides a legal means of justification on how the resources are allocated and utilised.

Double entry

Any business needs to keep records of its transactions. The simplest form of recording is by using single entry in which every transaction is recorded once in the account. This form of recording has several shortcomings such as;

(a) If an arithmetic error is made, it is difficult to identify and correct it.
(b) Difficultiness in identifying omitted transactions.
(c) Inaccuracy as only one aspect of the transaction is recorded.
(d) Difficultiness in preparing financial statements by using the single entry system.
(e) Easy manipulation of accounts due to lack of cross-checking option.

A good method of recording business transactions is by using double entry. A double entry is a system whereby business transactions are recorded twice in the account. The principle of double entry states that, every business transaction should be recorded twice, that is, every debit entry must have its corresponding credit entry of the same amount. Generally, one side of the account receives while the other side gives depending on the nature of transaction.

For example, when a shopkeeper sells a television set, he or she receives money and records the transaction on the debit side of the cash account and on the credit the sales account to complete the double entry. On the other hand, when a shopkeeper has made a purchase, the shopkeeper pays money and records the transactions on the credit side of the cash account and on the debit side of the purchases account to complete the double entry. The following are some of the terms related to accounting.

Transactions

A transaction is an exchange between two parties, a buyer and a seller. There are two types of transactions, namely; cash transactions and credit transactions. All transactions are recorded in the accounts.

Cash transactions

These are transactions done by cash basis, that is, exchange between two parties involve the prompt payment. Usually, cash transactions are recorded in the cash account before being recorded to the ledger accounts.

Credit transactions

These are exchanges between two parties whereby, the payment is done in the future. Credit transactions are normally recorded in the customer's personal account and sales account.

Account

It is a section in the ledger where different transactions are recorded. All accounts have two sides, the debit side (Dr) and the credit side (Cr). The debit side is the left-handside of the account while the credit side is the right-handside of the account. Examples of accounts are cash account, sales account, purchases account, wages account, rent account, and motor vehicle account. Any account is characterised by the following structure:

Name of account

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount

Date column

This column is used to record the date when a transaction takes place. The date is taken in terms of the day, month, and the year of transaction.

Particulars column

This column is used to record short narration/description of the entry of a transaction.

Folio column

All accounts in a ledger are given numbers known as folio. Therefore, folio column is used to indicate the number of accounts in a ledger.

Amount column

This column is used to record the amount of money received or paid.

Cash account

A cash account is the account in which all details of business transactions made in cash are recorded according to a double entry system. The cash account is used to record inflows (receipts or incomes) which are recorded on a debit side and outflows (expenditure or payments) which are recorded on a credit side.

Remark: Cash accounts exist in two forms; as an account and as a book of original entry. As an account, it is a part of the double entry. A book of original entry means that, all cash transactions are recorded in cash account before being posted in ledger accounts.

Ledger

A ledger is a book which contains different accounts of the same nature. There are four types of ledgers, which are; sales, purchases, general, and private ledgers.

Sales ledger (SL)

The sales ledger is a book that records the accounts of debtors or customers. It is also known as debtor's ledger. A debtor is a person to whom the business person sells goods on credit.

Purchases ledger (PL)

The purchases ledger is a book that records the accounts of creditors. It is also known as the creditor's ledger. A creditor is a person to whom the business person buys goods on credit.

Private ledger (PVL)

It is a book that records the confidential accounts of a business. Examples of confidential accounts are capital and drawing accounts.

General ledger (GL)

The general ledger is a book that records all accounts other than debtors, creditors and confidential accounts. Examples of accounts that are recorded in the general ledger are purchases account, sales account, stock account, and expenses accounts such as rent and wages.

Posting entries in ledger accounts

Posting entries is the process of recording transactions in ledger accounts by debiting and crediting respective accounts.

Example 8.1

Fumu started a business on 1st April, 2015 with a capital of	Tsh 600 000
April 2: Purchased goods for cash	Tsh 400 000
April 3: Bought goods from Mwajuma on credit	Tsh 100 000
April 4: Sold goods for cash	Tsh 300 000
April 5: Paid cash to Mwajuma	Tsh 20 000
April 5: Paid salary for cash	Tsh 150 000
April 8: Sold goods on credit to Bakari	Tsh 250 000
April 8: Paid transport expenses for cash	Tsh 120 000

Record the above transactions in respective ledger accounts.

Solution

From the transactions, the accounts are cash, capital, purchases, sales, transport, salary, Mwajuma, and Bakari. Cash account has its own book, capital account is recorded in the private ledger, purchases, sales, salary, transport, and salary are recorded in the general ledger. Mwajuma account is recorded in the purchases ledger and Bakari account is recorded in the sales ledger.

CASH ACCOUNT (1)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2015				2015			
April 1	Capital	PVL 2	600 000	April 2	Purchases	GL 3	400 000
April 4	Sales	GL 4	300 000	April 5	Mwajuma	PL 5	20 000
				April 5	Salary	GL 6	150 000
				April 8	Transport	GL 7	120 000

PRIVATE LEDGER

CAPITAL ACCOUNT (2)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
				2015			
				April 1	Cash	CA 1	600 000

GENERAL LEDGER

PURCHASES ACCOUNT (3)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2015							
April 2	Cash	CA 1	400 000				
April 3	Mwajuma	PL 5	100 000				

SALES ACCOUNT (4)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
				2015 April 4	Cash	CA 1	300 000
				April 8	Bakari	SL 8	250 000

SALARY ACCOUNT (6)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2015 April 5	Cash	CA 1	150 000				

TRANSPORT ACCOUNT (7)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2015 April 8	Cash	CA 1	120 000				

PURCHASES LEDGER

MWAJUMA ACCOUNT (5)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2015 April 5	Cash	CA 1	20 000	2015 April 3	Purchases	GL 3	100 000

SALES LEDGER

BAKARI ACCOUNT (8)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2015 April 8	Sales	GL 4	250 000				

Example 8.2

Wasandunga traders had the following transaction on August 2018. Post the transactions in their respective ledger accounts.

August 1: Started a business with capital of Tsh 500 000

August 3: Bought furniture for cash Tsh 100 000

August 4: Purchased goods for cash	Tsh 20 000
August 5: Bought goods on credit from Bupe	Tsh 40 000
August 6: Sold goods for cash	Tsh 30 000
August 6: Paid rent for cash	Tsh 50 000
August 8: Sold goods on credit to Masalo	Tsh 50 000
August 10: Bought goods on credit from Juma	Tsh 60 000
August 11: Paid for electricity in cash	Tsh 10 000
August 12: Sold goods on credit to Peter	Tsh 90 000
August 13: Received cash from Masalo	Tsh 20 000
August 14: Withdrew cash for home use	Tsh 70 000
August 15: Paid cash to Juma	Tsh 30 000

Solution

In the first transaction, money is taken from capital account and placed to cash account, which means, the capital account is credited and the cash account is debited.

In second transaction, Wasandunga traders bought furniture, cash account is credited and the furniture account is debited. Furniture, purchases, sales, rent, and electricity are recorded in the general ledger.
Capital and drawing accounts are recorded in the private ledger while cash account has its own book.

WASANDUNGA TRADERS

CASH ACCOUNT (1)

Dr				**Cr**			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2018 August 1	Capital	PVL2	500 000	2018 August 3	Furniture	GL 3	100 000
August 6	Sales	GL 5	30 000	August 4	Purchases	GL 4	20 000
August 13	Masalo	SL 8	20 000	August 6	Rent	GL 6	50 000
				August 11	Electricity	GL 7	10 000
				August 14	Drawings	PVL9	70 000
				August 15	Juma	PL 10	30 000

PRIVATE LEDGER

CAPITAL ACCOUNT (2)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
				2018 August 1	Cash	CA 1	500 000

DRAWING ACCOUNT (9)

Dr				Cr			
Date	Particular	Folio	Amount	Date	Particular	Folio	Amount
2018 August 14	Cash	CA 1	70 000				

GENERAL LEDGER

FURNITURE ACCOUNT (3)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2018 August 3	Cash	CA 1	100 000				

PURCHASES ACCOUNT (4)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2018 August 4	Cash	CA 1	20 000				
August 5	Bupe	PL 11	40 000				
August 10	Juma	PL 10	60 000				

SALES ACCOUNT (5)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
				2018 August 6	Cash	CA 1	30 000
				August 8	Masalo	SL 8	50 000
				August 12	Peter	SL 12	90 000

RENT ACCOUNT (6)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2018 August 6	Cash	CA 1	50 000				

ELECTRICITY ACCOUNT (7)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2018 August 11	Cash	CA 1	10 000				

SALES LEDGER

MASALO ACCOUNT (8)

Dr				Cr			
Date	Particular	Folio	Amount	Date	Particular	Folio	Amount
August 8	Sales	GL 5	50 000	August 13	Cash	CA 1	20 000

PETER ACCOUNT (12)

Dr				Cr			
Date	Particular	Folio	Amount	Date	Particular	Folio	Amount
August 12	Sales	GL 5	90 000				

PURCHASES LEDGER

JUMA ACCOUNT (10)

Dr				Cr			
Date	Particulars	Folio	Amounts	Date	Particulars	Folio	Amount
2018 August 15	Cash	CA 1	30 000	2018 August 10	Purchases	GL 4	60 000

BUPE ACCOUNT (11)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
				2018 August 5	Purchases	GL 4	40 000

Activity 8.1: Preparing and posting accounts in a ledger

Individually or in a group, perform the following tasks:

1. Prepare a list of 10 accounts of your choice of different ledgers and assign them to the respective ledgers.
2. Consult either a business person or a school accountant to collect the actual figures of the transactions based on the list of accounts. You may feel free to change the accounts in your list and accept those given by the accountant. Remember to ask for all necessary information for preparing ledger accounts.
3. Prepare a ledger on a large manila sheet and use it as a guide in your class presentation.
4. Present your final work to the rest of the class for further discussion and inputs.

Closing accounts

An account must be balanced at the end of a trading period or at any time when it is convenient. The entries on debit and credit sides of an account are unlikely to be equal. Balancing the account which is also known as closing the account involves the following procedures:

1. Find the total amounts in the debit and credit side.
2. Find the difference between the sum of debit and credit side.
3. Write the difference to the side which is less than the other so that both sides will have the same amount. The amount placed in the side with smaller amount is referred to as the balance carried down (c/d). The balance carried down (c/d) shows the amount remains in the account on the closing date.
4. Write the total on both sides and close the accounts by drawing a single line above and double lines below the totals.
5. The balance is then posted on the opposite side as a balance brought down (b/d) to the next trading period. If an account has one entry, then the entry itself is the balance.

Example 8.3

Balance the cash account in Example 8.2.

CASH ACCOUNT

Dr				**Cr**			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2018 August 1	Capital	PVL 2	500 000	2018 August 3	Furniture	GL 3	100 000
August 6	Sales	GL 5	30 000	August 4	Purchases	GL 4	20 000
August13	Masalo	SL 8	20 000	August 6	Rent	GL 6	50 000
				August 11	Electricity	GL 7	10 000
				August 14	Drawing	PVL 9	70 000
				August 15	Juma	PL 10	30 000

Solution

The credit side is less by Tsh 270 000. Place this amount in the credit side using the word 'balance c/d' meaning balance carried down in the particulars column. The balanced cash account is as follows.

CASH ACCOUNT

Dr				**Cr**			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2018 August 1	Capital	PVL2	500 000	2018 August 3	Furniture	GL 3	100 000
August 6	Sales	GL 5	30 000	August 4	Purchases	GL 4	20 000
August 13	Masalo	SL 8	20 000	August 6	Rent	GL 6	50 000
				August 11	Electricity	GL 7	10 000
				August 14	Drawing	PVL9	70 000
				August 15	Juma	PL 10	30 000
				August 31	Balance	c/d	270 000
			550 000				**550 000**
Sept 1	Balance	b/d	270 000				

The total debit is Tsh 550 000.
The total credit is Tsh 280 000.
The difference is Tsh 270 000.

Example 8.4

Ngosha started a business on 1st April, 2015 with a capital of Tsh 600 000.

April 2: Purchased goods for cash Tsh 400 000

April 3: Bought goods for cash Tsh 100 000

April 4: Sold goods for cash Tsh 300 000

April 5: Paid salary for cash Tsh 150 000

April 8: Sold goods for cash Tsh 250 000

April 8: Paid transport expenses for cash Tsh 120 000

Enter and balance the above transactions in respective ledger accounts.

Solution

All transactions in this example fall in the book of original entry (cash account) before being posted to the respective accounts in different ledgers.

CASH ACCOUNT (1)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2015				2015			
April 1	Capital	PVL2	600 000	April 2	Purchases	GL3	400 000
April 4	Sales	GL4	300 000	April 3	Purchases	GL3	100 000
April 8	Sales	GL4	250 000	April 5	Salary	GL5	150 000
				April 8	Transport	GL6	120 000
				April 30	Balance	c/d	380 000
			1 150 000				**1 150 000**
May 1	Balance	b/d	380 000				

PRIVATE LEDGER

CAPITAL ACCOUNT (2)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2015 April 30	Balance	c/d	600 000	April 1	Cash	CA 1	600 000
			600 000				**600 000**
				May 1	Balance	b/d	600 000

GENERAL LEDGER

PURCHASES ACCOUNT (3)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2015 April 2	Cash	CA 1	400 000	2015 April 30	Balance	c/d	500 000
April 3	Cash	CA 1	100 000				
			500 000				**500 000**
May 1	Balance	b/d	500 000				

SALES ACCOUNT (4)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2015 April 30	Balance	c/d	550 000	2015 April 4	Cash	CA 1	300 000
				April 8	Cash	CA 1	250 000
			550 000				**550 000**
				May 1	Balance	b/d	550 000

SALARY ACCOUNT (5)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2015							
April 5	Cash	CA 1	150 000	April 30	Balance	c/d	150 000
			150 000				**150 000**
May 1	Balance	b/d	150 000				

TRANSPORT ACCOUNT (6)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2015				2015			
April 8	Cash	CA 1	120 000	April 30	Balance	c/d	120 000
			120 000				**120 000**
May 1	Balance	b/d	120 000				

Activity 8.2: Balancing accounts

1. Individually or in a group, close the accounts you prepared in Activity 8.1.
2. Use a large manila sheet to display the balanced accounts.
3. Present your final work to the class for further discussion and inputs.

Exercise 8.1

1. On 1st June 2000, Urasa started a business with a capital of Tsh 2 000 000. Record the following transaction in the respective ledger accounts.

June 2	Bought goods for cash	Tsh 1 000 000
June 3	Bought shelves for cash	Tsh 300 000
June 4	Sold goods for cash	Tsh 800 000
June 8	Paid cash for cleaning expenses	Tsh 100 000
June 12	Paid wages for cash	Tsh 200 000
June 15	Paid wages for cash	Tsh 300 000
June 17	Bought goods for cash	Tsh 600 000
June 20	Sales for cash	Tsh 450 000

2. Fatuma started a business on 1st July, 2012 with a capital of Tsh 11 000 000 in cash. Record the following transactions in the respective ledger accounts and balance the accounts.

July 2	Purchased goods for cash	Tsh 4 000 000
July 3	Bought furniture from Lang'ata Traders on credit	Tsh 1 200 000
July 7	Sold goods for cash	Tsh 4 000 000
July 11	Purchased goods for cash	Tsh 1 400 000
July 13	Paid rent for cash	Tsh 400 000
July 16	Sold goods to Mwanaisha on credit	Tsh 1 900 000
July 20	Paid transport for cash	Tsh 160 000
July 22	Received cash from Mwanaisha	Tsh 1 250 000
July 23	Paid cash to Lang'ata Traders	Tsh 550 000
July 26	Paid salaries for cash	Tsh 320 000
July 27	Paid cash to Lang'ata Traders	Tsh 300 000

3. On 1st June 2002, Ikhomo started a business with a capital of Tsh 1 500 000. Record the following transactions in the respective ledger accounts.

June 2	Bought goods for cash	Tsh 900 000
June 5	Sold goods for cash	Tsh 600 000
June 6	Sales for cash	Tsh 100 000
June 11	Bought furniture for cash	Tsh 300 000
June 15	Paid rent for cash	Tsh 100 000
June 18	Bought goods for cash	Tsh 250 000
June 25	Paid salary for cash	Tsh 200 000

4. On May 1st 2020 Mr. Katwana started a business with a cash of Tsh 400 000.

2nd May	Purchased goods for cash	Tsh 300 000
4th May	Sold goods for cash	Tsh 200 000
5th May	Paid carriage on goods sold for cash	Tsh 75 000
15th May	Bought goods on credit from Amani	Tsh 200 000
20th May	Paid wages for cash	Tsh 60 000
22nd May	Paid transport for cash	Tsh 20 000
27th May	Sold goods in cash	Tsh 150 000
28th May	Paid Amani for cash	Tsh 150 000

Use the following transactions to prepare and balance the ledger accounts

5. On 1st August, 2001, Mnyampanda started a business with a capital of Tsh 700 000. Record the given transaction to the relevant ledger accounts.

August 3	Purchased goods for cash	Tsh 400 000
4	Sold goods for cash	Tsh 300 000
8	Bought furniture from Gab on credit	Tsh 350 000
12	Paid cash for advertisement	Tsh 50 000
18	Paid rent for cash	Tsh 60 000
18	Purchased goods for cash	Tsh 230 000
29	Sold goods to XYZ traders on credit	Tsh 210 000
30	Paid wages for cash	Tsh 120 000

6. On 1st January, 2019, ABC Traders started a business with a capital of Tsh 1 000 000.

January 2	Bought furniture for cash	Tsh 200 000
January 3	Purchased goods for cash	Tsh 130 000
January 4	Bought goods from Hapakazi Ltd on credit	Tsh 180 000
January 4	Sold goods for cash	Tsh 350 000
January 5	Paid rent for cash	Tsh 50 000
January 6	Sold goods on credit to Mr Tugawane	Tsh 100 000
January 10	Paid electricity bill for cash	Tsh 30 000
January 15	Received cash from Mr Tugawane	Tsh 80 000
January 18	Withdrew cash for allowance payments	Tsh 150 000
January 23	Paid cash to Hapakazi Ltd	Tsh 140 000

Record the above transactions in the relevant ledger accounts and balance them.

7. On 15th July, 2004, Mnyampaa started a business with a capital of Tsh 800 000. Use these transactions to post and balance entries in their respective ledger accounts.

July 16	Bought goods for cash	Tsh 400 000.
17	Sold goods for cash	Tsh 200 000
19	Bought furniture for cash	Tsh 250 000
19	Bought goods for cash	Tsh 300 000
25	Paid salary for cash	Tsh 150 000
27	Paid rent for cash	Tsh 220 000
31	Sold goods for cash	Tsh 350 000

8. The following represents the transactions made by the UPENDO GROUP Traders 2016. Record these transactions in a ledger accounts and balance them.

March 1st 2016: Started a business with capital of Tsh 3 500 000

2nd March	Purchased goods for cash	Tsh 500 000
3rd March	Bought furniture for cash	Tsh 100 000
4th March	Bought a machine for cash	Tsh 1 500 000
6th March	Sold goods for cash	Tsh 400 000
7th March	Paid rent for cash	Tsh 150 000
10th March	Bought goods from Ngosha Ltd on credit	Tsh 1 500 000
12th March	Sold goods for cash	Tsh 1 300 000
18th March	Bought goods for cash	Tsh 1 000 000
18th March	Paid insurance for cash	Tsh 400 000
18th March	Sold goods on credit to Mazengo Company	Tsh 2 000 000
20th March	Paid cash to Ngosha Ltd	Tsh 1 300 000
22nd March	Sold goods for cash	Tsh 1 200 000
24th March	Received cash from Mazengo Company	Tsh 800 000

9. January 1, 2006 Mr. Njiku started a business with capital of Tsh 1 200 000 in cash. Record the entries in their respective ledger accounts by using the following transactions and then close the accounts.

January 2	Purchased goods for cash	Tsh 800 000
3	Purchased shelves for cash	Tsh 250 000
5	Sold goods for cash	Tsh 600 000
8	Paid rent for cash	Tsh 240 000
10	Bought goods from Musa on credit	Tsh 400 000
13	Paid wages for cash	Tsh 60 000
18	Bought goods from Amina on credit	Tsh 350 000
25	Sold goods to Hamisi on credit	Tsh 300 000
26	Received cash from Hamisi	Tsh 300 000
26	Paid cash to Musa	Tsh 200 000
28	Paid Insurance for cash	Tsh 100 000

Trial balance

A trial balance is a statement which contains a list of both debit and credit balances extracted from all ledger accounts at a particular date. It checks the arithmetic accuracy of the double entry recording of business transactions at any given date, normally at the end of each month or year. The sum of debit balances in the trial balance must be equal to the sum of credit balances.

When recording in the trial balance, the debit balances brought down (b/d) taken from the ledger accounts should be listed in the debit column while the credit balances brought down (b/d) should be listed in the credit column.

Importances of trial balance

(a) It is used to verify the arithmetic accuracy of transactions recorded in the accounts. That is, it is used to check if the double entry made in the accounts are correct or not.

(b) It is also used as a summary of accounts and thus used in preparation of financial statements.

(c) It assists in detecting errors when the debit total and credit total are not balanced, and hence enable an accountant to detect and rectify errors.

(d) It assist in comperative analysis. Trial balances help to compare balances of the current year with past years balances along with peer analysis. This helps the business to make important decisions on income, expenses and production costs.

(e) It is used in preparation of audit reports. Trial balance helps the auditors to locate entries in the original books of accounts.

Constructing a trial balance

When preparing a trial balance, it is necessary to include the following:

(a) Title of the statement which contains name of a business, name of a statement and a date in which the trial balance is prepared, usually at the end of trading period.

(b) The main columns are the name of account, debit, and credit. Note that serial numbers and folio are not necessary in the trial balance but may be included. A structure of trial balance is presented as follows:

MOHAMED TRIAL BALANCE AS AT 31ST JUNE, 2020

Name of account	Dr	Cr
Capital		xxx
Purchases	xxx	
Sales		xxx
Machine and plants	xxx	
Motor vehicle/Motor van	xxx	
Fixture and Fitting	xxx	
Expenses such as;		
(a) Electricity	xxx	
(b) Rent	xxx	
(c) Water bills	xxx	
	xxx	
(d) Transport	xxx	
Drawing(s)	xxx	
Creditor(s)		xxx
Debtor(s)	xxx	
Discount allowed	xxx	
Interest or commission received		xxx
Cash in hand or at Bank	xxx	
Opening stock	xxx	
Discount received		xxx
Return inwards	xxx	
Return outwards		xxx
Carriage inwards	xxx	
Carriage outwards	xxx	
Bad debts	xxx	
Loan from Bank		xxx
Total	**xxx**	**xxx**

Example 8.5

Prepare the trial balance for Wasandunga Traders in Example 8.2.

Solution

First, balance the accounts of Wasandunga traders as follows:

CASH ACCOUNT (1)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2018 August 1	Capital	PVL2	500 000	2018 August 3	Furniture	GL 3	100 000
August 6	Sales	GL 5	30 000	August 4	Purchases	GL 4	20 000
August 13	Masalo	SL 8	20 000	August 6	Rent	GL 6	50 000
				August 11	Electricity	GL 7	10 000
				August 14	Drawings	PVL9	70 000
				August 15	Juma	PL 10	30 000
				August 31	Balance	c/d	270 000
			550 000				**550 000**
Sept 1	Balance	b/d	270 000				

PRIVATE LEDGER

CAPITAL ACCOUNT (2)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2018 Aug 31	Balance	c/d	500 000	2018 Aug 1	Cash	CA 1	500 000
			500 000				**500 000**
				Sept 1	Balance	b/d	500 000

DRAWINGS ACCOUNT (9)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2018 Aug 14	Cash	CA 1	70 000	2018 Aug 31	Balance	c/d	70 000
			70 000				**70 000**
Sept 1	Balance	b/d	70 000				

GENERAL LEDGER

FURNITURE ACCOUNT (3)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2018 Aug 3	Cash	CA 1	100 000	2018 Aug 31	Balance	c/d	100 000
			100 000				**100 000**
Sept 1	Balance	b/d	100 000				

PURCHASES ACCOUNT (4)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2018 Aug 4	Cash	CA 1	20 000	2018 Aug 31	Balance	c/d	120 000
Aug 5	Bupe	PL 11	40 000				
Aug 10	Juma	PL 10	60 000				
			120 000				**120 000**
Sept 1	Balance	b/d	120 000				

SALES ACCOUNT (5)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2018 Aug 31	Balance	c/d	170 000	2018 Aug 6	Cash	CA 1	30 000
				Aug 8	Masalo	SL 8	50 000
				Aug 12	Peter	SL12	90 000
			170 000				**170 000**
				Sept 1	Balance	b/d	170 000

RENT ACCOUNT (6)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2018 Aug 6	Cash	CA 1	50 000	2018 Aug 31	Balance	c/d	50 000
			50 000				**50 000**
Sept 1	Balance	b/d	50 000				

ELECTRICITY ACCOUNT (7)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2018 Aug 11	Cash	A1	10 000	2018 Aug 31	Balance	c/d	10 000
			10 000				**10 000**
Sept 1	Balance	b/d	10 000				

PURCHASES LEDGER

BUPE ACCOUNT (11)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particular	Folio	Amount
2018 Aug 31	Balance	c/d	40 000	2018 Aug 5	Purchases	GL 4	40 000
			40 000				**40 000**
				Sept 1	Balance	b/d	40 000

JUMA ACCOUNT (10)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2018 Aug 15	Cash	CA 1	30 000	2018 Aug 10	Purchases	GL 4	60,000
August 31	Balance	c/d	30 000				
			60 000				**60 000**
				Sept 1	Balance	b/d	30 000

SALES LEDGER

MASALO ACCOUNT (8)

Dr				**Cr**			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2018 Aug 8	Sales	GL 5	50 000	2018 Aug 13	Cash	CA 1	20 000
				Aug 31	Balance	c/d	30 000
			50 000				**50 000**
Sept 1	Balance	b/d	30 000				

PETER ACCOUNT (12)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2018 Aug 12	Sales	GL 5	90 000	2018 Aug 31	Balance	c/d	90 000
			90 000				**90 000**
Sept 1	Balance	b/d	90 000				

Then the trial balance comprises of balances of all accounts.

WASANDUNGA TRADERS
TRIAL BALANCE AS AT 31 AUGUST 2018

Name of account	Dr	Cr
Capital		500 000
Cash	270 000	
Furniture	100 000	
Purchases	120 000	
Sales		170 000
Rent	50 000	
Electricity	10 000	
Drawings	70 000	
Creditors: Bupe		40 000
Juma		30 000
Debtors: Masalo	30 000	
Peter	90 000	
	740 000	**740 000**

Example 8.6

On 1st July, 2017, Neema started a business with the capital in cash of Tsh 1 000 000 and made the following transactions;

July 2	Purchased goods for cash	Tsh 700 000
July 3	Sold goods for cash	Tsh 500 000
July 6	Purchased goods for cash	Tsh 300 000
July10	Bought goods for cash	Tsh 235 000
July 15	Paid cash for rent	Tsh 110 000
July 26	Paid cash for wages	Tsh 55 000
July 28	Sold goods for cash	Tsh 310 000

Use these transactions to prepare the accounts, balance them and hence prepare the trial balance.

Solution

NEEMA'S CASH ACCOUNT (1)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2017 July 1	Capital	PVL 2	1 000 000	2017 July 2	Purchases	GL 3	700 000
July 3	Sales	GL 4	500 000	July 6	Purchases	GL 3	300 000
July 28	Sales	GL 4	310 000	July 10	Purchases	GL 3	235 000
				Jul y15	Rent	GL 5	110 000
				July 26	Wages	GL 6	55 000
				July 31	Balance	c/d	410 000
			1 810 000				**1 810 000**
Aug 1	Balance	b/d	410 000				

PRIVATE LEDGER

CAPITAL ACCOUNT (2)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2017 July 31	Balance	c/d	1 000 000	2017 July 1	Cash	CA 1	1 000 000
			1 000 000				**1 000 000**
				Aug 1	Balance	b/d	1 000 000

GENERAL LEDGER

PURCHASES ACCOUNT (3)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2017 July 2	Cash	CA 1	700 000	2017 July 31	Balance	c/d	1 235 000
July 6	Cash	CA 1	300 000				
July 10	Cash	CA 1	235 000				
			1 235 000				**1 235 000**
Aug 1	Balance	b/d	1 235 000				

SALES ACCOUNT (4)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2018 July 31	Balance	c/d	810 000	2017 July 3	Cash	CA 1	500 000
				July 28	Cash	CA 1	310 000
			810 000				**810 000**
				Aug 1	Balance	b/d	810 000

RENT ACCOUNT (5)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2017 July 15	Cash	CA 1	110 000	2017 July 31	Balance	c/d	110 000
			110 000				**110 000**
Aug 1	Balance	b/d	110 000				

WAGES ACCOUNT (6)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2017 July 26	Cash	CA 1	55 000	2017 July 31	Balance	c/d	55 000
			55 000				**55 000**
Aug 1	Balance	b/d	55 000				

The Trial balance from Neema's business is as follows;

NEEMA'S TRIAL BALANCE AS AT 31st JULY, 2017

Account name	Dr	Cr
Cash	410 000	
Capital		1 000 000
Purchases	1 235 000	
Sales		810 000
Rent	110 000	
Wages	55 000	
	1 810 000	**1 810 000**

Activity 8.3: Preparing a trial balance from the given data

1. In pairs, record the following transactions in respective ledger accounts, close them and extract the trial balance.

June 1, 2018	John started a business with capital of	Tsh 3 000 000
June 7, 2018	Bought goods in cash for	Tsh 1 500 000
June 12, 2018	Paid cash for Transport	Tsh 300 000
June 15, 2018	Sold goods in cash	Tsh 500 000
June 20, 2018	Paid wages for cash	Tsh 200 000
June 24, 2018	Paid rent for cash	Tsh 150 000
June 26, 2018	Sold goods for cash	Tsh 400 000
June 28, 2018	Bought goods for cash	Tsh 700 000
June 29, 2018	Sold goods for cash	Tsh 300 000
June 30, 2018	Paid cash for advertisement	Tsh 200 000

2. Exchange your work in pairs, assess the procedures used by others and provide recommendations for improvement of the work, where necessary.
3. Use the feedback to participate in class discussion on the process of preparing a trial balance.

Activity 8.4: Preparing a trial balance

In a group, perform the following tasks:

1. At your home, you live with parents and other relatives who in one way or another engage in business. Each student from the group, collect 10 transactions from any of your relative who own business.
2. Use transactions from three different businesses, record the transactions in respective ledger accounts and close them.
3. Extract the trial balance in each business.
4. Compare the accuracy of transaction records among the three types of businesses.
5. Share your work with others through class discussion.

Exercise 8.2

1. Mwanane started a business on 1st May, 2005 with a capital in cash of Tsh 4 000 000.

May	2	Bought goods for cash	Tsh 1 500 000
May	3	Bought furniture for cash	Tsh 500 000
May	8	Sold goods for cash	Tsh 1 500 000
May	10	Cash purchases	Tsh 600 000
May	12	Paid rent for cash	Tsh 100 000
May	15	Cash sales	Tsh 900 000
May	17	Paid transport for cash	Tsh 50 000
May	20	Sold goods for cash	Tsh 200 000
May	25	Paid salary for cash	Tsh 100 000

Record these transactions in the respective ledger accounts and extract a trial balance.

2. Record the following transactions in the cash account, post the transactions in respective legder accounts and then prepare a trial balance.
On 1st March, 2004 Latifa started a business with the capital in cash of Tsh 250 000.

March 5	Bought office machines for cash	Tsh 75 000
March 9	Bought goods for cash	Tsh 100 000
March 12	Paid wages for cash	Tsh 7 500
March 15	Sold good for cash	Tsh 42 500
March 20	Received loan from bank in cash	Tsh 100 000
March 24	Sold goods for cash	Tsh 60 000
March 27	Paid electricity bill for cash	Tsh 22 500
March 30	Paid rent for cash	Tsh 15 000

3. Mrs. Chakubanga started a business on February, 15th, 2005 with a capital in cash Tsh 1 055 000

February	16	Bought goods for cash	Tsh 500 000
February	18	Bought shelves for cash	Tsh 55 000
February	19	Sold goods for cash	Tsh 450 000
February	20	Purchases for cash	Tsh 400 000
February	21	Sold goods for cash	Tsh 700 000
February	25	Paid rent for cash	Tsh 150 000

Record the given transactions in the respective ledger accounts and extract a trial balance.

4. Michael started his business on June 1, 2013 with a capital Tsh 900 000. Use the following transactions to extract a trial balance from Michael's business.

June 2	Bought goods for cash	Tsh 400 000
June 3	Paid transport services	Tsh 150 000
June 4	Sold goods for cash	Tsh 400 000
June 8	Cash sales	Tsh 100 000
June11	Purchased goods for cash	Tsh 180 000
June12	Paid salary for cash	Tsh 38 000
June 20	Bought goods for cash	Tsh 150 000
June 22	Cash sales	Tsh 200 000
June 25	Paid rent for cash	Tsh 50 000
June 26	Cash sales	Tsh 250 000
June 29	Bought stationery for cash	Tsh 10 000

5. Njia Panda Traders started a business on April 1, 2005 with a capital in cash Tsh 2 675 000

April 2	Bought goods for cash	Tsh 1 500 000
April 3	Bought furniture for cash	Tsh 305 000
April 4	Purchases shelves for cash	Tsh 270 000
April 5	Sold goods for cash	Tsh 1 300 000
April 9	Paid wages for cash	Tsh 54 000
April 12	Cash purchases	Tsh 342 000
April 13	Sold goods for cash	Tsh 511 000
April 16	Paid rent for cash	Tsh 114 000
April 20	Bought goods for cash	Tsh 425 000
April 25	Cash sales	Tsh 498 000
April 27	Paid salary for cash	Tsh 67 000

Record the above transactions in a cash account and post the entries to respective ledger account hence prepare a trial balance.

Financial statements

These are statements prepared at the end of a trading period for the purpose of finding the profit or loss generated by a business as well as to show the financial position of the business. A financial statement comprises of two statements which are; the income statement (trading, profit, and loss accounts) and statement of financial position (balance sheet). Generally, financial statements are prepared in order to assess the performance of the business.

Income statement (Trading, Profit and Loss accounts)

An income statement is a statement prepared at the end of the trading period in order to determine the gross profit or gross loss and net profit or net loss generated by a business. Income statement has two sections which are trading, and income and expenses.

Trading section is used to determine the gross profit or gross loss generated by a business by comparing the sales and cost of goods sold. On the other hand, income and expenses section is used to find out the net profit or net loss by comparing the income and the expenses incurred by a business within a year.

Terms used in income statements

Sales and purchases

These are transactions between two or more parties in which the seller receives money and the buyer receives tangible or intangible goods or services.

Inventories (stocks)

Inventories are goods available in the business for resale within a trading period. There are two types of inventories namely; opening inventories and closing inventories.

Opening inventories are goods available in the business for resale at the beginning of the trading period. On the other hand, closing inventories are goods available in the business for resale at the end of a trading period.

Carriage

It is the cost of transporting goods from one place to another. There are two types of carriages namely; carriage inwards and carriage outwards.

Carriage inwards

Carriage inwards is the cost of transporting purchased goods from the supplier to the trader's premises. It should be added to purchases costs of the goods bought. Usually this is done in the trading section of the income statement.

Carriage outwards

Carriage outwards is the cost a business pays to deliver purchased goods to the customer's premises. In this case, carriage outwards are expences to the business and should be recorded as expenses in the income and expenses section of the income statement.

Returns

This is the act of sending back goods to a seller by a buyer due to some reasons. Such reasons include but not limited to goods expired, damage in transit, and supplying goods contrary to the agreement. There are two types of returns namely return inwards and return outwards.

Return inwards (return on sales)

These are goods sent back by a customer to the business. Return inwards are shown as a deduction from the sales in the trading section of the income statement.

Return outwards

These are goods sent back by the business to the creditors or suppliers. Returns outwards are shown as deductions from purchases in the trading section of the income statement.

Income

Income is a gain received by the business from different sources such as a gross profit generated by business, discount received after paying the commodity bought from a supplier, and commission received from another business. Generally, whatever the business receives as payment is an income.

Expenses

These are costs incurred by running a business in a day to day activities of the business. Examples of the expenses are; paying wages and salaries, transport, rent, bills and gross loss.

Discounts

It is a common practice in business to deduct a certain percentage from the price of a commodity. Thus a discount is a reduction in the original price of a good. There are two types of discounts; trade discount and cash discount.

Trade discounts

These are much larger deductions which are allowed to buyers who purchase in large quantities. A trade discount is offered irrespective of whether the goods are bought on credit or not. Trade discounts normally appears during the calculation of the price of a commodity bought.

Cash discounts

In order to prompt customers to settle their accounts, cash discounts are usually offered. This means that a business will accept a lower sum in full settlement provided that the payment is made within a short period. There are two categories of cash discounts, namely; discount allowed and discount received.

Discount allowed

A business may allow cash discount to its customers. This allowance is offered to encourage debtors to settle their accounts in time. A discount allowed is the expense of the business and it is treated as an expense to the income and expenses section in the income statement.

Discount received

In the same manner, cash discounts may be received by the business from the suppliers. This allowance is offered by suppliers (or creditors) to the business for payments settled in time. A discount received is an income to the business and is recorded as an income to the income and expenses section of the income statement.

Gross profit and loss

Gross profit is calculated in the trading section of the income statement by taking the excess of sales over the cost of goods sold during the trading period. If the sales exceed the cost of goods sold, it is a gross profit and if the cost of goods sold exceeds sales, it is a gross loss. That is, gross profit is calculated as follows:

Gross profit = Sales – Cost of goods sold.

Gross loss on the other hand is calculated as follows:

Gross loss = Cost of goods sold – Sales.

Cost of goods sold is given by:

Cost of good sold = Cost of goods available for sale – Closing stock

But, Cost of goods available for sale = Opening stock + Net purchases.

Net profit and loss

Net profit is the excess of income over the expenses, that is, the difference between income and expenses of the business. If the income is less than the expenses, then there will be a net loss. Net profit and loss are calculated as follows:

Net profit = Income – Expenses.

Net loss = Expenses – Income.

Determination of gross profit or loss using income statement

Income statement is one of the approaches used in determining the gross profit and loss generated by a business. It is prepared based on the following structure:

INCOME STATEMENT AS AT DAY/MONTH/YEAR

Sales					xxx
Less	Return inwards				xxx
Net sales					xxx
Less	Cost of goods sold				
	Opening stock			xxx	
	Add: Purchases	xxx			
	Add: Carriage inwards	xxx	xxx		
	Less: Return outwards		xxx		
	Net purchases			xxx	
	Cost of goods available for sale			xxx	
	Less: Clossing stock			xxx	xxx
	Gross profit or Loss				xxx

Example 8.7

On 31st December, 2017, Mafuru's accounts disclosed the following information:

Purchases	Tsh 200 000;
Sales	Tsh 400 000;
Stock on 1st January, 2017	Tsh 80 000;
Stock on 31st December, 2017	Tsh 90 000.

Show the income statement for the year ended 31st December, 2017

Solution

Mafuru's income statement for the year ended 31st December, 2017

Sales			400 000
Less :	Cost of goods sold		
	Opening stock	80 000	
	Add: Purchases	200 000	
	Cost of goods availale for sale	280 000	
	Less: Closing stock	90 000	190 000
	Gross profit		210 000

Activity 8.5: Preparing an income statement

1. In pairs, use the following transactions from Rashid's business to prepare Rashid's income statement for the year ended on 31st December, 2019.

Opening stock on January 1st, 2019	Tsh 480 000
Closing stock on December 31st, 2019	Tsh 660 000
Purchases	Tsh 4 350 000
Sales	Tsh 6 000 000

2. Share with your collegue the procedures you used in preparing Rashid's income statement
3. Present your final work to the rest of the class through discussion.

Activity 8.6: Preparing an income statement from self collected data

1. Use transactions collected in Activity 8.4 or collect new set of transactions from any source of your choice and prepare an income statement at a given period of time.
2. Share the procedures you used in preparing the income statement to your fellows.
3. Present the prepared income statement to the rest of the class for further discussion and inputs.

Determination of net profit or loss using income statement

Just like gross profit and loss, net profit and loss can be calculated from an income statement. The income statement for net profit and loss is prepared based on the following structure:

Income statement as at date/month/year

Income		xxx
Less:	Expenses	xxx
	Net profit or loss	xxx

Example 8.8

Use the following information to prepare an income statement for the year ended 31st December, 2015.

Gross profit	Tsh 880 000
Electricity	Tsh 220 000
Printing	Tsh 360 000
General expenses	Tsh 290 000
Salary	Tsh 340 000
Interest received	Tsh 120 000

Solution

Income statement for the year ended 31st December, 2015

Income		
Gross profit		880 000
Interest received		120 000
Total income		1 000 000
Less: Expenses		
General expenses	290 000	
Salary	340 000	
Electricity	220 000	
Painting	360 000	
Total expenses		1 210 000
Net loss		210 000

Example 8.9

Prepare the income statement from the following transactions.

31.12.2016 – Total purchases	Tsh 2 500 000
31.12.2016 - Total sales	Tsh 5 500 000

Solution

Income statement for the year ended 31.12.2016

Sales	5 500 000
Less: Cost of goods sold	
Purchases	2 500 000
Gross profit	3 000 000

Example 8.10

Using the following trial balance of Jatu PLC, prepare an income statement for the year ended December 31st, 2016.

Particulars	Dr	Cr
Stock on 1/12/2016	236 800	
Carriage outwards	20 000	
Carriage inwards	31 000	
Returns inwards	20 500	
Returns outwards		32 200
Sales		1 860 000
Purchases	1 187 400	
Salaries and wages	386 200	
Rent and Rates	30 400	
Insurance	7 800	
Motor expenses	66 400	
Office expenses	21 600	
Lightining expenses	16 600	
General expenses	31 400	
Premises and equipment	500 000	
Motor vehicles	180 000	
Fixture and fittings	35 000	
Debtors	389 600	
Creditors		173 100
Water bills	48 200	
Drawings	120 000	
Capital		1 263 600
	3 328 900	**3 328 900**

Additional information: Stock at December 31st, 2016 was Tsh 294 600.

Solution

INCOME STATEMENT FOR THE YEAR ENDED DECEMBER, 2021

Sales					1 860 000
Less	Return inwards				20 500
Net sales					1 839 500
Less	Cost of goods sold				
	Opening stock			236 800	
	Add: purchases	1 187 400			
	Add: Carriage inwards	31 000	1 218 400		
	Less: Return outwards		32 200		
	Net purchases			1 186 200	
	Cost of goods available for sale			1 423 000	
	Less: Clossing stock			294 600	1 128 400
	Gross profit or Loss				711 100
	Less: Expenses				
	Carriage outwards		20 000		
	Salaries and wages		386 200		
	Rents and rates		30 400		
	Insurance		7 800		
	Motor expenses		66 400		
	Office expenses		21 600		
	Lighting expenses		16 600		
	General expenses		31 400		
	Total expenses				580 400
	Net profit				130 700

Example 8.11

Opening stock at 1st January, 2005 Tsh 34 500
Closing stock at December 31st, 2015 Tsh 26 700
Net purchases in 2015 Tsh 219 300

Gross profit made- up Tsh 138 550
Expenses for the year Tsh 45 800

Use the given information to find:

(a) Cost of goods sold (b) Average stock
(c) Sales (d) Net profit

Solution

(a) Cost of goods sold = Opening stock + Net purchase – Closing stock
= Tsh 34 500 + Tsh 219 300 – Tsh 26 700
= Tsh 227 100

Therefore, the cost of goods sold was Tsh 227 100

(b) Average stock $= \dfrac{\text{Opening stock + Closing stock}}{2}$

Average stock $= \dfrac{\text{Tsh } 34\,500 + \text{Tsh } 26700}{2}$

= Tsh 30 600

Therefore, the average stock was Tsh 30 600.

(c) Sales = Cost of goods sold + Gross profit
= Tsh 227 100 + Tsh 138 550
= Tsh 365 650

Therefore the sales was Tsh 365 650.

(d) Net profit = Gross profit – Expenses
= Tsh 138 550 – Tsh 45 800
= Tsh 92 750

Therefore, the net profit was Tsh 92 750.

Exercise 8.3

1. On 1st January, 2014, Mnyangaa Trader, commenced a business with capital in cash of Tsh 10 000 000. Record the following transactions in cash account, complete double entry, extract a trial balance, and prepare the income statement.

Jan 2	Bought goods for cash	Tsh 5 000 000
Jan 2	Bought furniture for cash	Tsh 1 000 000
Jan 3	Sold goods for cash	Tsh 4 000 000

Jan 3	Sold goods on credit to David which cost	Tsh 1 000 000
Jan 5	Paid transport for cash	Tsh 100 000
Jan 7	Received cash from David	Tsh 500 000
Jan 10	Bought motor vehicle for cash	Tsh 2 230 000
Jan 12	Paid wages for cash	Tsh 77 000
Jan 15	Paid rent for cash	Tsh 300 000
Jan 17	Cash purchases	Tsh 1 884 000
Jan 18	Cash sales	Tsh 2 500 000
Jan 23	Sold goods for cash	Tsh 1 450 000
Jan 25	Bought goods on credit from Hangida	Tsh 400 000
Jan 27	Paid cash to Hangida	Tsh 200 000

Note: Closing stock on January 31^{st}, 2014 was Tsh 255 000.

2. Use the following trial balance to prepare an annual income statement for JZN Trading Company.

JZN Trading Company Trial Balance as at 31^{st} August, 2019

Name of account	Dr	Cr
Capital		51 300
Cash	34 000	
Sales		63 750
Purchases	22 800	
Opening stock	900	
Advertisement	2 000	
Office expenses	5 000	
Transport	8 000	
Machinery	4 000	
Debtors	2 250	
Creditors		42 700
Bank	78 800	
	157 750	**157 750**

Note: Closing stock on 31^{st} August 2019 was Tsh 7 400.

3. From the following trial balance of Njia Panda Traders, prepare income statement as on 31st December, 2017.

NJIA PANDA TRADER'S TRIAL BALANCE AS AT 31st DEC. 2017

Account name	Dr	Cr
Capital		2 675 000
Cash	1 900 000	
Purchases	2 274 000	
Rent	114 000	
Furniture	305 000	
Shelves	270 000	
Sales		2 309 000
Salaries	67 000	
Wages	54 000	
	4 984 000	**4 984 000**

Note: Closing stock on 31st December, 2017 was Tsh 100 000

Statement of financial position (Balance sheet)

A statement of financial position is a statement which shows the efficiency of a business as at a certain period. Usually, it is prepared at the end of the trading period. It shows the capital, assets, and liabilities of the business. The basic accounting equation states that, assets equal to the sum of liabilities and capital, that is,

Assets = Liabilities + Capital.

Assets

Assets are properties or anything of value possessed by a business. There are two categories of assets which are; non-current assets (fixed assets) and current assets.

Non current assets

These are assets owned for more than a year for the purpose of creating future income. Examples of non-current assets are land, buildings, plants and machinery, motor vehicles, furniture and fixture, and fittings.

Current Assets

These are assets held in a business for a short period usually within a year. They are either in form of cash or are easily convertible into cash in short period. Examples of current assets are stock at close, debtors, cash in hand, and cash at bank.

Liabilities

A liability is any amount of which a business is legally bound to pay. It is a claim by an outsider on the assets of the business. Liabilities can be classified as long term liabilities and short term liabilities (current liabilities). Long term liabilities are business debts which are payable at a period longer than a year. For example, a loan from a bank. Short term liabilities are the short term debts of the business payable within a year. For instance, creditors and bank overdraft.

Capital

This is the amount invested by the owner to start a business. There are four main types of capital which are equity, loan, fixed, and working capital.

Equity capital is a capital invested by the owner together with any profit made. Loan capital on the one hand is an amount borrowed to start a business, contrary to the fixed capital which is the worth value of all fixed assets. Working capital on the other hand is the capital obtained by taking the difference between the total current assets and total current liabilities. It is given by the formula.

$$\text{Working capital} = \text{Total current assets} - \text{Total current liabilities}$$

Preparation of a statement of a financial position

When preparing the statement of financial position, it is necessary to include the following:

(i) Title of the statement which include name of the business, name of statement, and date at which a statement is prepared.
(ii) Assets of the business which include non-current assets and currents assets.
(iii) Liabilities of the business which involve long term liabilities and short term liabilities.
(iv) Owner's capital section.

A structure of a statement of financial position is as follows:

NAME OF COMPANY-STATEMENT OF FINANCIAL POSITION
DATE/MONTH/YEAR

Non current assets (Fixed assets)			
Buildings			xxx
Furnitures			xxx
Total non-current assets			xxx
Current Assets			
Stocks	xxx		
Debtors	xxx		
Cash	xxx		
Bank	xxx		
Total current assets			xxx
Total assets			xxx
Less: Liabilities			
Long term liabilities			
Loan from bank		xxx	
Short term liabilities			
Creditors	xxx		
Bank overdraft	xxx		
Total short term liabilities		xxx	
Total liabilities			xxx
			xxx
Financed by :			
Capital			xxx
Add: Net profit or less net loss		xxx	
Less: Drawings		xxx	xxx
			xxx

Example 8.12

Find the capital and prepare statement of financial position as on 31/12/2016 from the following information (in Tsh):

Loan from bank	1 500 000
Cash	30 000
Bank	86 000
Creditors	150 000
Debtors	120 000
Stock	140 000
Machinery	1 000 000
Motor van	400 000
Buildings	600 000
Bank overdraft	35 000

Solution

Capital = Total assets – Total liabilities

Total Assets (Tsh)

Cash	30 000
Bank	86 000
Debtors	120 000
Stock	140 000
Machinery	1 000 000
Motor Van	400 000
Buildings	+ 600 000
Total	2 376 000

Total Liabilities (Tsh)

Creditors	150 000
Bank ofverdrafts	35 000
Loan from Bank	+ 1 000 000
Total	1 685 000

Thus,

Capital = Tsh 2 376 000 – Tsh 1 685 000

= Tsh 691 000

Therefore, the capital is Tsh 691 000.

STATEMENT OF FINANCIAL POSITION AS AT 31ST DECEMBER, 2016

Non-current assets (Fixed assets)			
Buildings			600 000
Machinery			1 000 000
Motor van			400 000
Total non-current assets			2 000 000
Current Assets			
Stocks	140 000		
Debtors	120 000		
Bank	86 000		
Cash	30 000		
Total current assets			376 000
Total assets			2 376 000
Less: Liabilities			
Long term liabilities			
Loan from bank		1 500 000	
Short term liabilities			
Creditors	150 000		
Bank overdraft	35 000		
Total short term liabilities		185 000	
Total liabilities			1 685 000
			691 000
Financed by :			
Capital			691 000

Example 8.13

The following list represents the assets and liabilities of Mwajuma's business as on 31st December, 2016:

Creditors Tsh 100 000
Bank overdraft Tsh 50 000
Stock Tsh 85 000
Premises Tsh 250 000
Debtors Tsh 150 000
Cash in hand Tsh 15 000
Furniture Tsh 42 000
Capital Tsh 392 000

Use this information to find:

(a) Total current assets
(b) Total current liabilities
(c) Working capital
(d) Total fixed assets

Solution

(a) Total current assets
= Stock + Debtors + Cash in hand
= Tsh 85 000 + Tsh150 000 + Tsh 15 000
= Tsh 250 000
Therefore, the total current assets was Tsh 250 000.

(b) Total current liabilities
= Creditors + Bank overdraft
= Tsh 100 000 + Tsh 50 000
= Tsh 150 000
Therefore, the total current liabilities was Tsh 150 000.

(c) Working capital
= Total current assets – Total current liabilities
= Tsh 250 000 – Tsh 150 000
= Tsh 100 000
Therefore, the total working capital was Tsh 100 000.

(d) Total fixed assets
= Premises + Furniture
= Tsh 250 000 + Tsh 42 000
= Tsh 292 000
Therefore, the total fixed assets was Tsh 292 000.

MWAJUMA'S STATEMENT OF FINANCIAL POSITION AS AT 31ST DEC, 2016

Non-current assets (Fixed assets)			
Premises		250 000	
Furniture		42 000	292 000
Current Assets			
Stock	85 000		
Debtors	150 000		
Cash	15 000	250 000	
Less: Current liabilities			
Creditors	100 000		
Bank overdraft	50 000	150 000	100 000
			392 000
Financed by:			
Capital			392 000

Interpreting information from the statement of financial position

The information from the statement of financial position should be analysed and interpreted to reveal the useful information to the internal and external users. The users of the financial statements are owners of the business for making decisions on matters of the business, government (tax inspectors) for tax purposes , financial institutions for providing the financial credits to the traders, prospective investors (prospective partners) for joining in the business, and creditors for guaranteeing credits to the traders. Analysis of the statement of financial position is done by considering several factors such as the working capital (circulating capital), current ratio, quick ratio and other ratios.

(a) **Working capital**

It is a capital obtained from the difference between the total current assets and total current liabilities.

Working capital or circulating capital is given by:

Working capital = Current assets – Current liabilities.

(b) **Current ratio**

Current ratio measures the ability of the business to pay the current liabilities out of the current assets. If the answer is one and above, the current assets are capable to pay current liabilities.

$$\text{Current ratio} = \frac{\text{Current assets}}{\text{Current liabilities}}.$$

(c) **Quick ratio (Acid test ratio)**

Quick ratio (Acid test ratio) measures the ability of a business to pay current liabilities by using current assets while excluding stock which is not cash. The quick ratio is given by the following formula.

$$\text{Quick ratio} = \frac{\text{Current assets} - \text{Stock}}{\text{Current liabilities}}.$$

Example 8.14

The following list represent assets and liabilities of Mwajuma's business as on 31st December, 2016:

Creditors	Tsh 100 000	Cash in hand	Tsh 15 000
Bank overdraft	Tsh 50 000	Furniture	Tsh 42 000
Stock	Tsh 85 000	Capital	Tsh 392 000
Premises	Tsh 250 000		
Debtors	Tsh 150 000		

Use this information to find:

(a) Total current assets.
(b) Total current liabilities.
(c) Working capital.
(d) Total fixed assets or Total non-current assets.
(e) Current ratio.
(f) Acid test ratio.

Solution

(a) Total current assets

= Stock + Debtors + Cash in hand
= Tsh 85 000 + Tsh150 000 + Tsh 15 000
= Tsh 250 000.

Therefore, the total current assets was Tsh 250 000

(b) Total current liabilities

= Creditors + Bank overdraft
= Tsh 100 000 + Tsh 50 000
= Tsh 150 000.

Therefore, the total current liabilities was Tsh 150 000

(c) Working capital

= Total current assets – Total current liabilities
= Tsh 250 000 –Tsh 150 000
= Tsh 100 000.

Therefore, the working capital was Tsh 100 000

(d) Total fixed assets or Total non-current assets

$$= \text{Premises} + \text{Furniture}$$
$$= \text{Tsh } 250\,000 + \text{Tsh } 42\,000$$
$$= \text{Tsh } 292\,000.$$

Therefore, the total fixed assets was Tsh 292 000.

(e) $$\text{Current ratio} = \frac{\text{Current assets}}{\text{Current liabilities}}$$
$$= \frac{250\,000}{150\,000}$$
$$= 1.6$$

Therefore, the current assets of the Mwajuma's business is capable to pay the current liabilities of the business.

(f) $$\text{Acid test ratio} = \frac{\text{Current assets} - \text{Stocks}}{\text{Current liabilities}}$$
$$= \frac{250\,000 - 85\,000}{150\,000}$$
$$= \frac{165\,000}{150\,000}$$
$$= 1.1$$

Thus, the acid test ratio was 1.1

Therefore, the current assets excluding the stock of the Mwajuma's business was capable to pay the current liabilities of the business.

Exercise 8.4

1. Prepare a balance sheet as at 31st January, 2002 by using the following information:

Capital	Tsh 300 000
Furniture	Tsh 80 000
Cash in hand	Tsh 160 000
Net profit	Tsh 240 000
Machinery	Tsh 200 000
Stock (closing)	Tsh 100 000
Drawings	Tsh 40 000
Land and buildings	Tsh 180 000
Loan from NMB Bank	Tsh 120 000
Debtors	Tsh 140 000
Creditors	Tsh 100 000
Bank overdraft	Tsh 140 000

2. Juma had an opening stock of Tsh 300 000 and his net purchases was Tsh 220 000. Find the cost of goods sold.
3. If Ashura's business had an opening stock of Tsh 500 000 and an average stock of Tsh 1 200 000, what was the closing stock for her business?

Chapter summary

1. Double entry system describes the procedures to use in the process of recording the financial transactions to the ledger accounts.
2. The concept of a trial balance helps to record debit and credit balances as a summary of all accounts and checking the arithmetic accuracy of the double enrty records.
3. Income statement is used to find gross and net profit or loss, that is,
   i. Gross profit = Net sales – Cost of goods sold
   ii. Gross loss = Cost of goods sold – Net sales
   iii. Net profit = Total income – Total expenses
   iv. Net loss = Total expenses – Total income
   v. Cost of goods sold = Cost of goods available for sale – Closing stock
   vi. Cost of goods available for sale = Opening stock + Net purchase
   vii. Net purchases = Purchases + Carriage inwards – Return outwards
   viii. Average stock $= \dfrac{\text{Opening stock + Closing Stock}}{2}$
   ix. Net sales = Sales – Return inwards
4. Statement of financial position (Balance sheet) is a statement showing the progress of a business. It helps a business person to know the efficiency of the business.

Revision exercise 8

1. (a) Explain an advantage of double entry bookkeeping over single entry bookkeeping.
   (b) Give three accounts that would be kept in the general ledger.
2. The XYZ Traders had the following transactions on June, 2019. Post the transactions in the relevant ledgers, balance the accounts and prepare a trial balance.

June 1:	Started business with capital Tsh 500 000
June 2:	Bought furniture for cash Tsh 100 000
June 4:	Purchased goods for cash Tsh 20 000
June 5:	Bought goods from John on credit Tsh 40 000
June 5:	Sold goods for cash Tsh 30 000
June 6:	Paid rent Tsh 5 000
June 10:	Sold goods to Juakali on credit Tsh 56 000
June 13:	Bought goods from Masoud on credit Tsh 60 000
June 18:	Paid electricity bill in cash Tsh 10 000
June 20:	Sold goods to Ali on credit Tsh 80 000
June 21:	Received cash from Juakali Tsh 20 000
June 25:	Withdrew cash for personal use Tsh 70 000
June 30:	Paid cash to Masoud Tsh 30 000

3. The following information is a part of a certain income statement:

Opening stock at 1^{st} Jan, 2014	Tsh 3 000 000
Closing stock at 31^{st} Dec, 2014	Tsh 2 000 000
Net purchases in 2014	Tsh 20 000 000
Gross profit	Tsh 10 500 000
General expenses for the year	Tsh 4 000 000

Use the information to find:
   (a) Cost of goods sold
   (b) Average stock
   (c) Sales
   (d) Net profit

4. The following represents the ABC statement of financial position prepared on 31st December, 2020.

ABC BALANCE SHEET AS AT DECEMBER 31st, 2020

Capital	179 000		Assets:	
Add: Net profit	30 250		Furniture	40 000
	209 250		Machinery	30 000
Less: drawings	18 000	191 250	Debtors	10 000
Creditors		7 000	Cash at bank	3 000
Salaries accrued		5 000	Rates prepaid	500
Tel. Outstanding		250	Closing stock	120 000
		203 500		203 500

Use the information to find the:

(a) Total fixed assets
(b) Total current assets
(c) Total current liabilities
(d) Working capital

5. Use the following trial balance to prepare the income statement and statement of financial position as at 31st December, 2019.

MUNA'S TRIAL BALANCE AS AT 31st DECEMBER, 2019

Account Name	Dr	Cr
Cash	1 907 000	
Capital		2 675 000
Purchases	2 267 000	
Rent	114 000	
Furniture	305 000	
Shelves	270 000	
Sales		2 309 000
Salaries	67 000	
Wages	54 000	
	4 984 000	4 984 000

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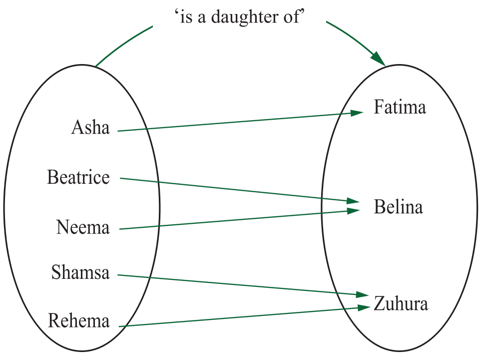
ANSWERS TO ODD-NUMBERED QUESTIONS

Chapter One

Exercise 1.1

1.(a)



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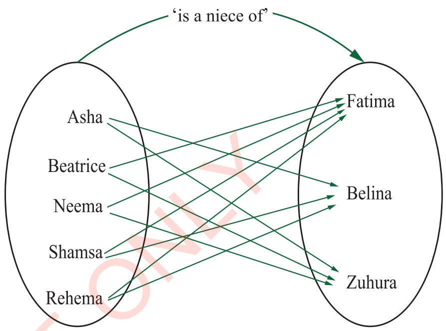

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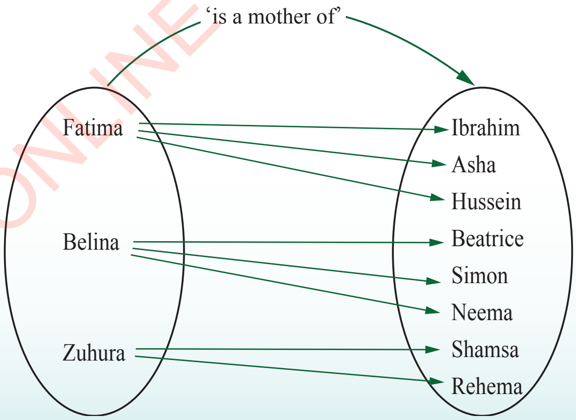

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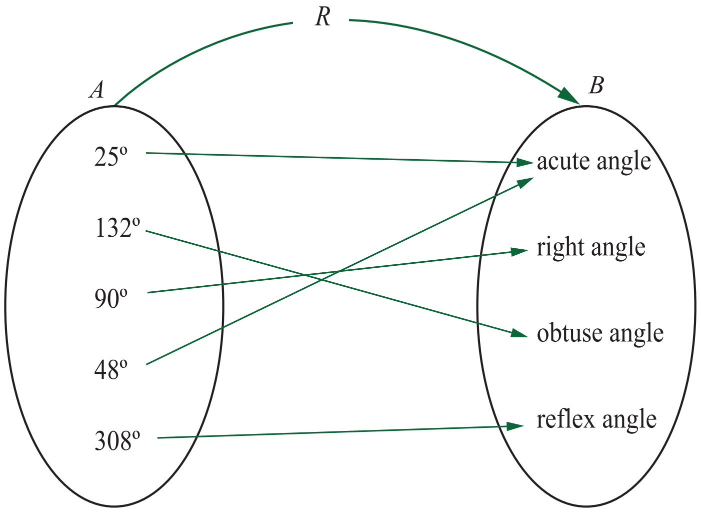
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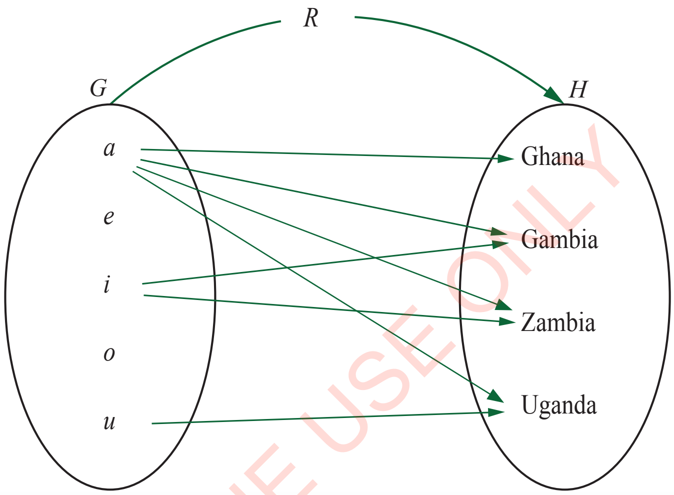
acute angle
right angle
obtuse angle
reflex angle

5.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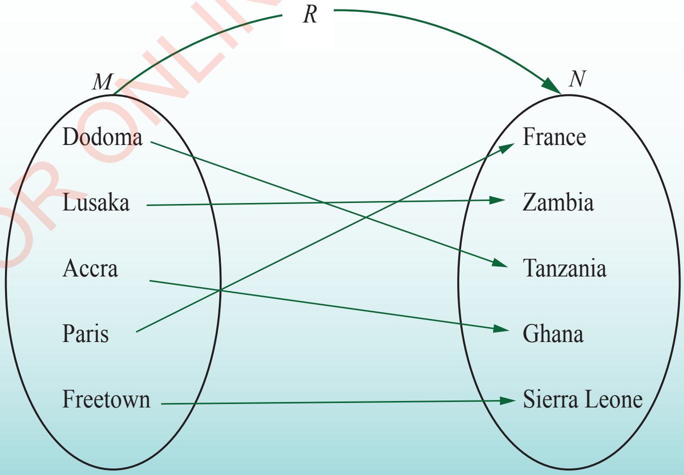
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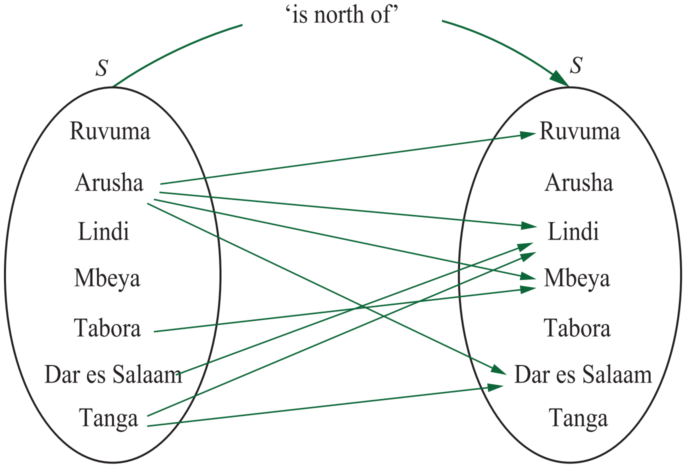
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Arusha
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Mbeya
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Dar es Salaam
Tanga
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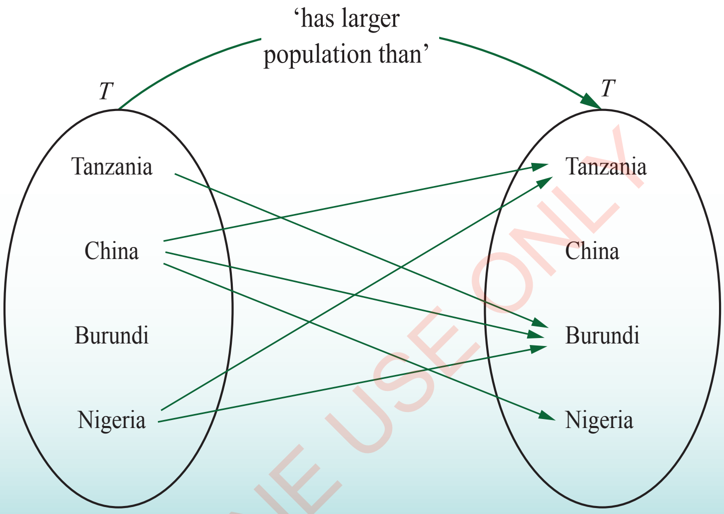
Mbeya
Tabora
Dar es Salaam
Tanga

11.


'has larger population than'
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Nigeria
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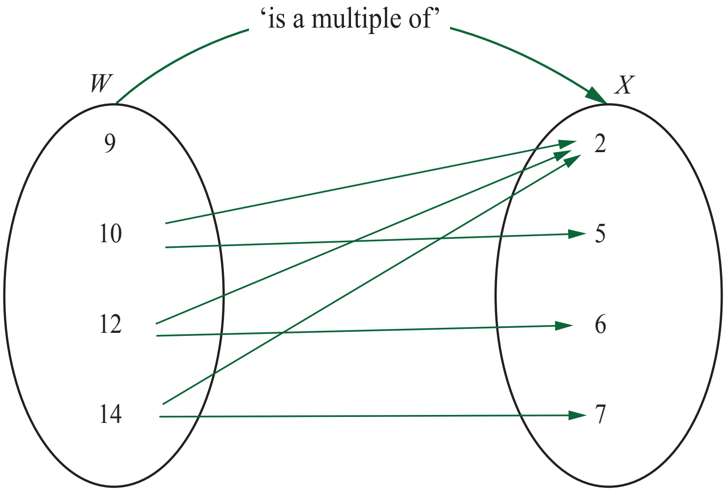
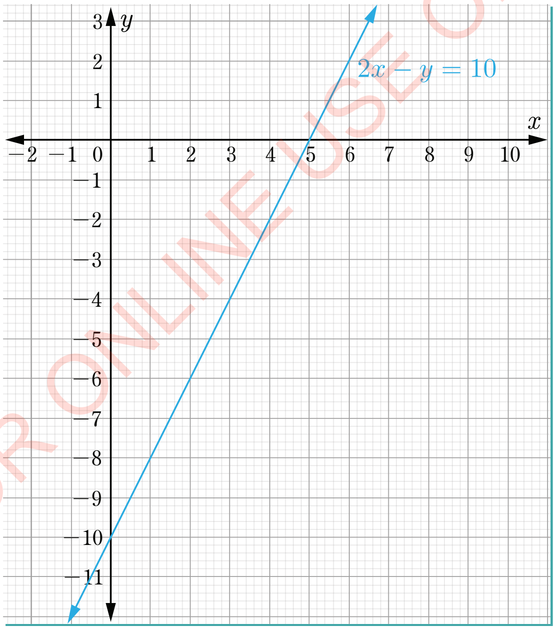
China
Burundi
Nigeria

13.

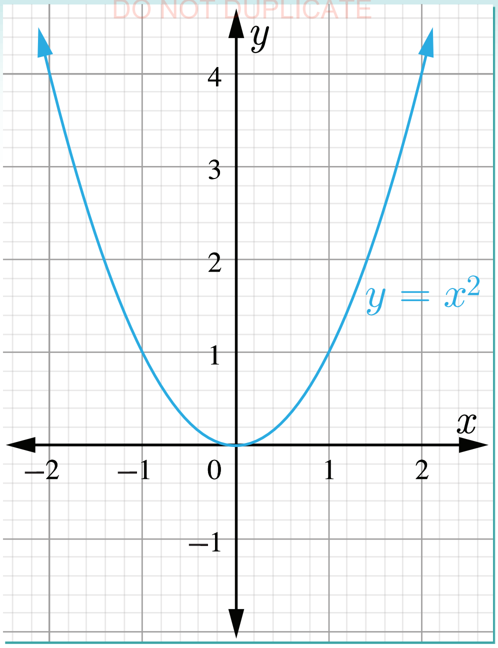


Exercise 1.2

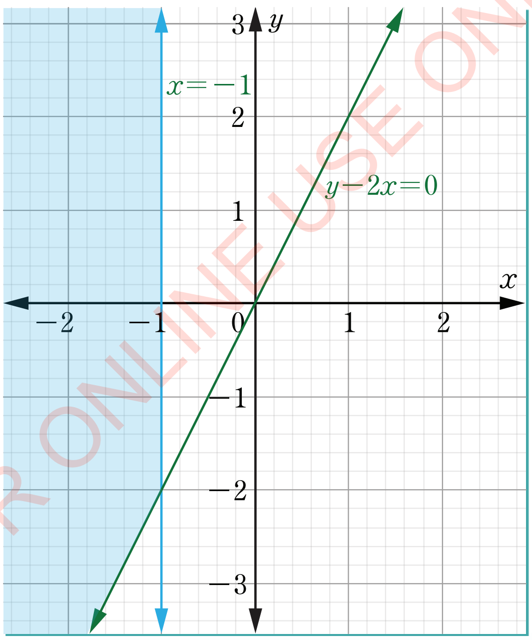
1. Domain $=\{x : x \in \mathbb{R}\}$, Range$=\{y : y \in \mathbb{R}\}$



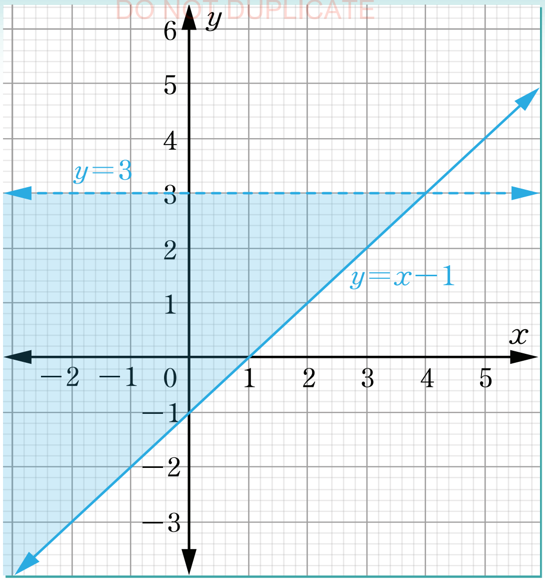
3. Domain $=\{x : x \in \mathbb{R}\}$, Range$=\{y : y \geq 0\}$



5. Domain $=\{x : x \leq -1\}$, Range $=\{y : y \in \mathbb{R}\}$



7. Domain $=\{x : x \leq 4\}$, Range $=\{y : y < 3\}$



9. $R=\{(x,y) : x+y \leq 1 \text{ and } y \geq 1\}$
Domain= $\{x: x \leq 0\}$, Range = $\{y: y \geq 1\}$

11. $R=\{(x,y) : -2 \leq x \leq 2 \text{ and } -1 \leq y \leq 1\}$
Domain = $\{x: -2 \leq x \leq 2\}$ and Range = $\{y: -1 \leq y \leq 1\}$

13. $R=\{(x,y) : y \leq x+2 \text{ and } y \geq x-2\}$
Domain={All real numbers} and Range ={All real numbers}.

Exercise 1.3

1. (a) Domain $=\{x : x \geq 0\}$, Range $=\{y : y \in \mathbb{R}\}$

(b) Domain=$\{x : x \geq 1\}$, Range=$\{y : y \geq 1\}$

3. (a) $\left\{(0,-1),(0,1),\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right),\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)\right\}$

(b) Domain $=\left\{0,\ \frac{1}{2},\ \frac{\sqrt{2}}{2}\right\}$ (c) Range $=\left\{-1,\ 1,\ \frac{\sqrt{3}}{2},\ \frac{\sqrt{2}}{2}\right\}$

5. (a) Domain $=\{x : x \in \mathbb{R}\}$, Range = $\{y: y \geq 3\}$

(b) Domain = $\{x: x \geq 0\}$, Range $=\{y : y \in \mathbb{R}\}$

Exercise 1.4

1. $R^{-1} = \{(x, y): (y, x)$, where x and y are real numbers$\}$

3. $R^{-1} = \{(x, y): x < 4\}$

5. (a) $R^{-1} = \left\{(x, y): y = \frac{x}{2} - \frac{1}{2}\right\}$ (b) Domain $= \{x : x \in \mathbb{R}\}$, Range $= \{y : y \in \mathbb{R}\}$

7. (a) $R^{-1} = \left\{(x, y): y = \pm\sqrt{x}\right\}$ (b) Domain $= \{x : x \geq 0\}$, Range $= \{y : y \in \mathbb{R}\}$

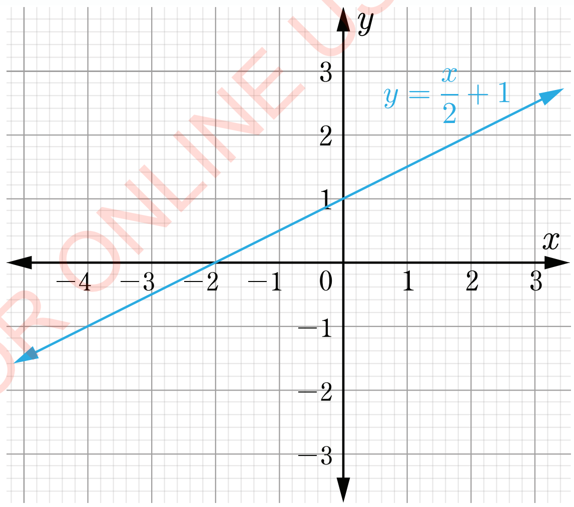
9. (a) $R^{-1} = \left\{(x, y): y > \frac{x}{3}\right\}$ (b) Domain $= \{x : x \in \mathbb{R}\}$, Range $= \{y : y \in \mathbb{R}\}$

11 (a) $R^{-1} = \left\{(x, y): y = \frac{2}{x} - 5\right\}$ (b) Domain $= \{x : x \in \mathbb{R},\ x \neq 0\}$

Range $= \{y : y \in \mathbb{R},\ y \neq -5\}$

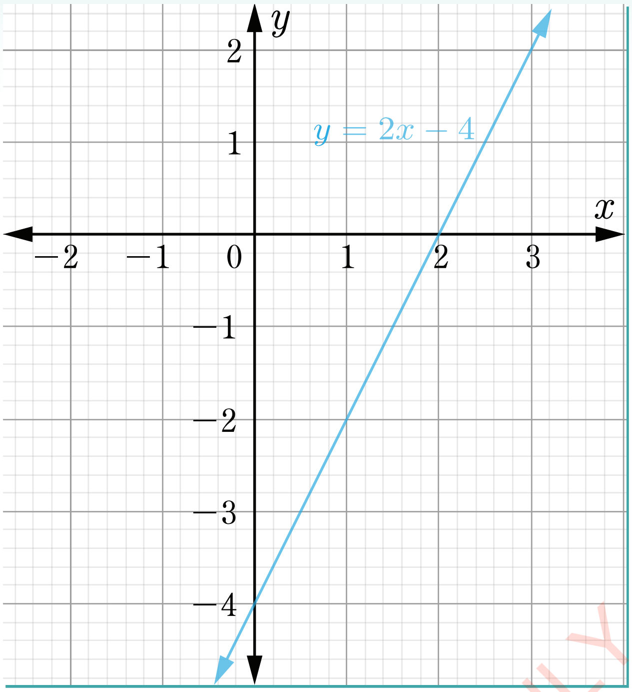
Exercise 1.5

1.



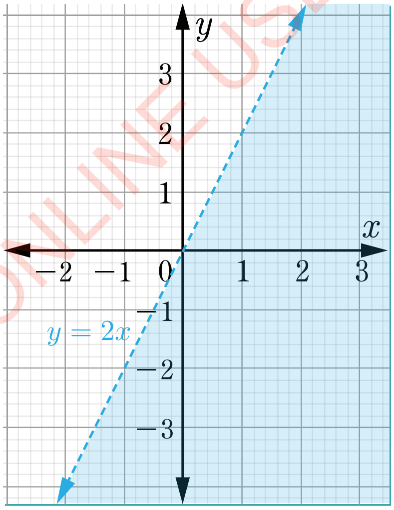
Domain $= \{x : x \in \mathbb{R}\}$; Range $= \{y : y \in \mathbb{R}\}$

3.



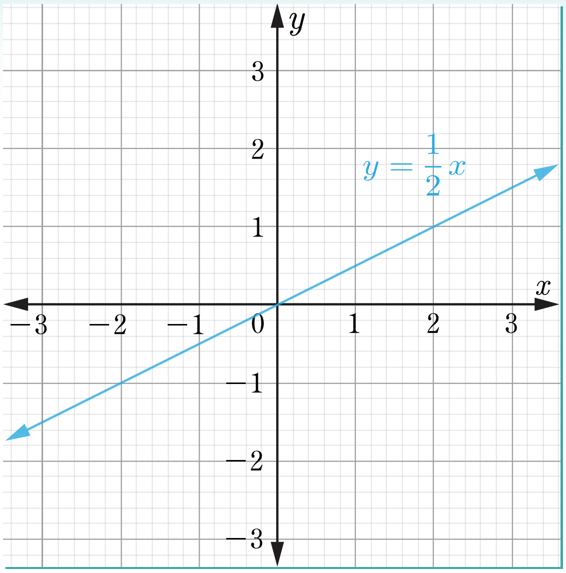
Domain $= \{x : x \in \mathbb{R}\}$; Range $= \{y : y \in \mathbb{R}\}$

5.



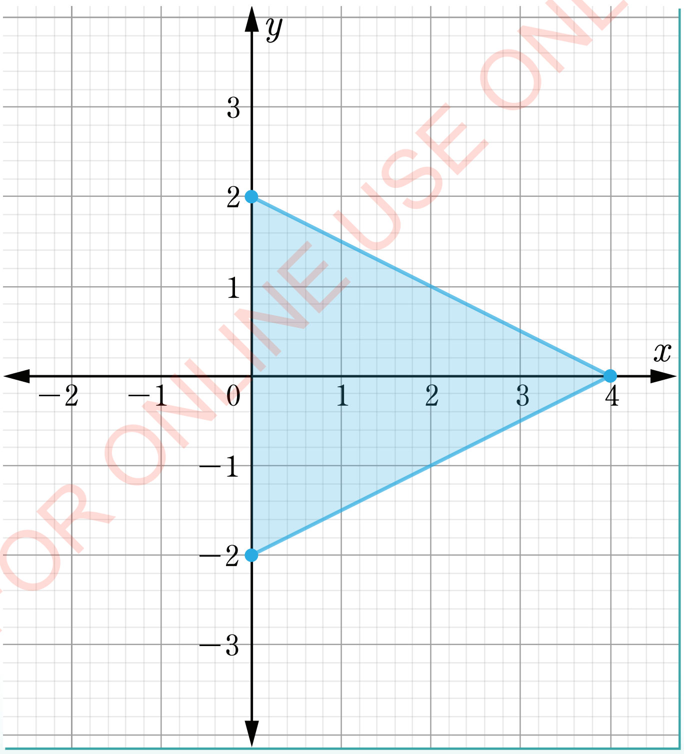
Domain $= \{x : x \in \mathbb{R}\}$; Range $= \{y : y \in \mathbb{R}\}$

7.



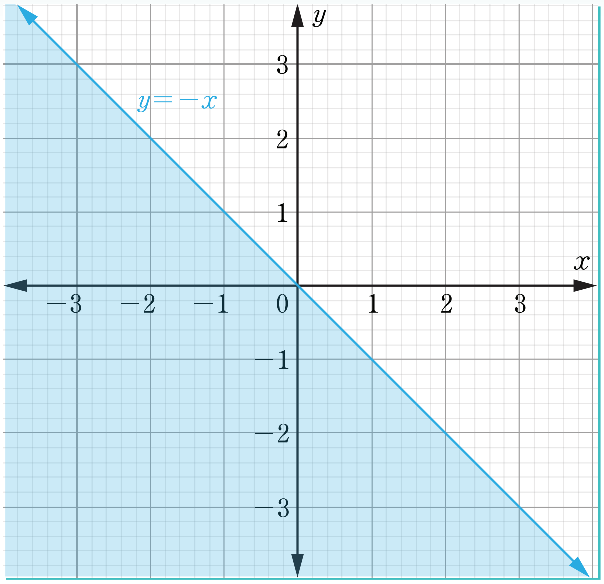
Domain $=\{x: x \in \mathbb{R}\}$ Range $=\{y: y \in \mathbb{R}\}$

9.



Domain of $R^{-1} = \{x: 0 \le x \le 4\}$ Range of $R^{-1} = \{y: -2 \le y \le 2\}$

11.

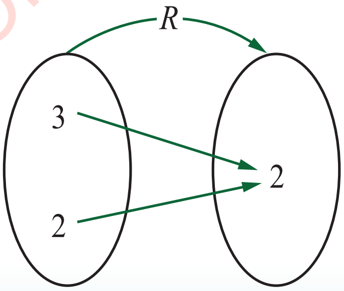


Domain $=\{x : x \in \mathbb{R}\}$ Range $=\{y : y \in \mathbb{R}\}$

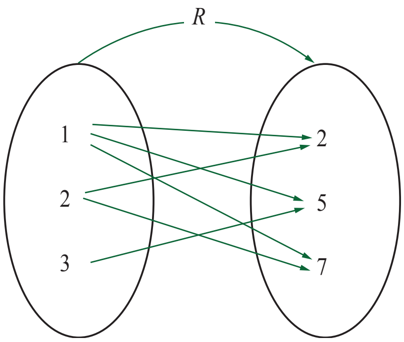
Revision exercise 1

1. Range = {2, 4, 6, 8}
3. Range = {9, 13, 17, 21}
5. (a) {(4, a), (4, 1), (3, a), (3, 1), (8, a), (8, 1)}
   (b) {(a, 4), (1, 4), (a,3), (1, 3), (a, 8), (1, 8)}
   (c) {(4, 4), (4, 3), (4, 8), (3, 4), (3, 3), (3, 8), (8, 4), (8, 3), (8, 8)}
   (d) {(a, a), (a, 1), (1, a), (1, 1)}

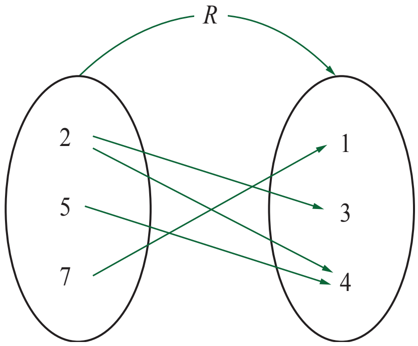
7.(a)



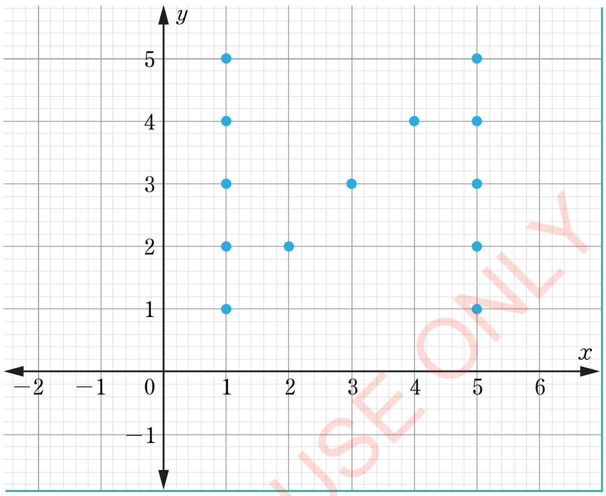
(b)



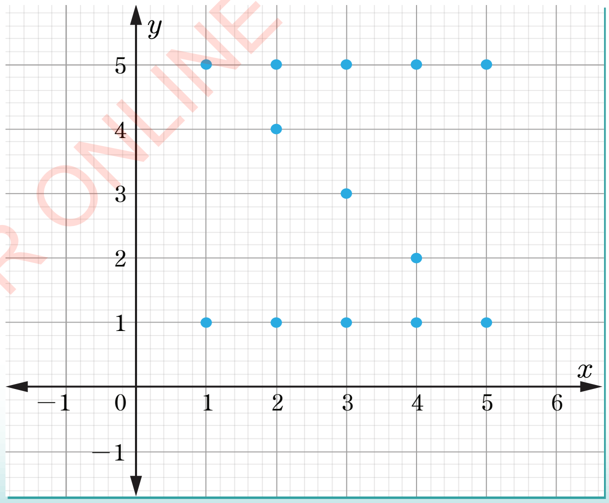
(c)



9 (a)



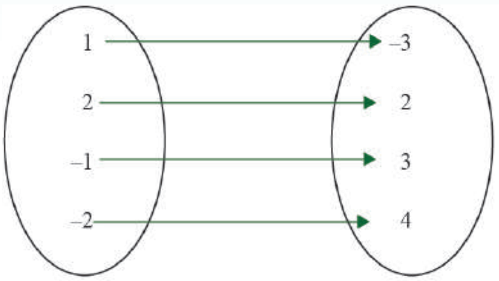
(b)



Chapter Two

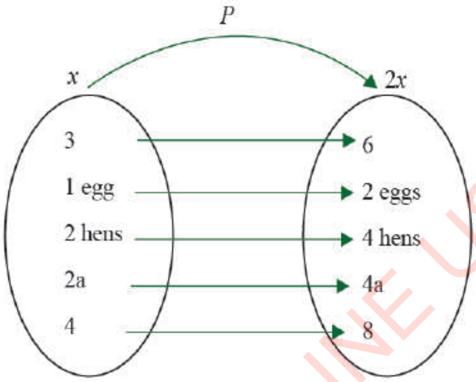
Exercise 2.1

1.



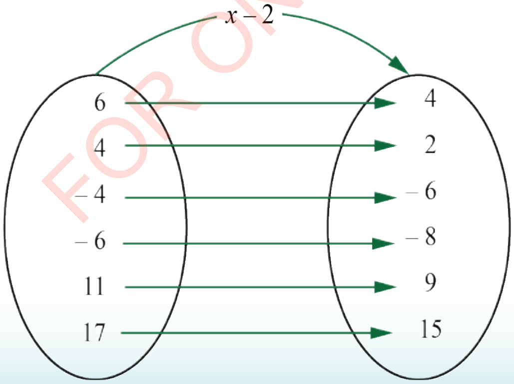
3. b

5.



7. (a), (b), (d)

9.



Exercise 2.2

1. $f : x \to \frac{x}{5} + 2$

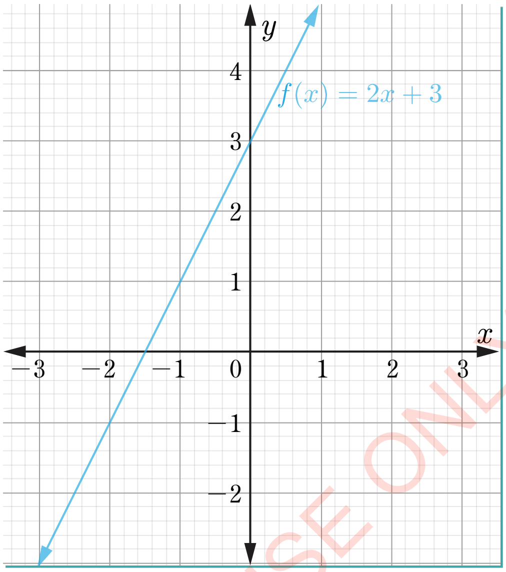
3. $f : x \to x^2 - 7$

5. $f : x \to 8 - x$

7. $f(1) = 5, \quad f(-2) = -1, \quad f(a) = 2a + 3$

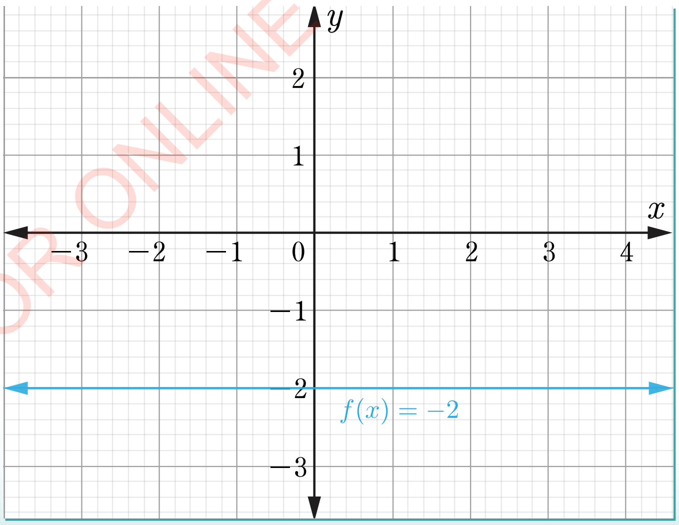
9. $S(-1) = 1, \; S(5) = 25, \; S(4) = 16$

11.



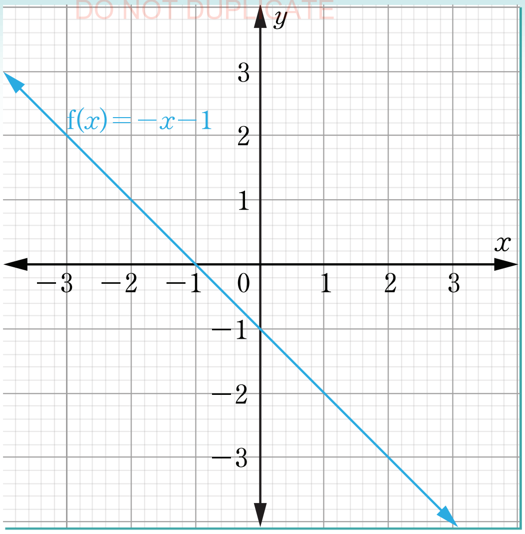
It is a one-to-one function

13.



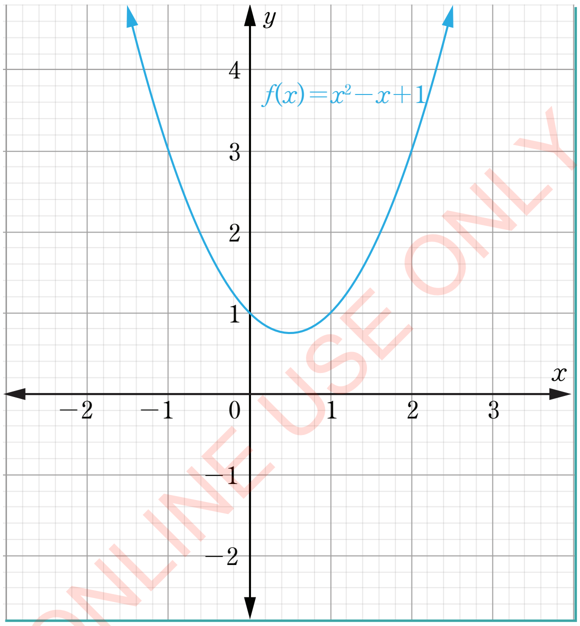
It is not a one-to-one function

15.



It is a one-to-one function

17.



It is not a one-to-one function

19. Not one-to-one

21. One-to-one

23. One-to-one

Exercise 2.3

1. Domain $=\{x: -10 \leq x \leq 10\}$; Range $=\{y: -33 \leq y \leq 47\}$

3. Domain $=\{x: 1 \leq x \leq 2\}$; Range $=\left\{y: \frac{1}{2} \leq y \leq 1\right\}$

5. Domain $= \{x : x \in \mathbb{R}\}$; Range $= \{y : y \in \mathbb{R}\}$

7. Domain $= \{t : t \in \mathbb{R}\}$; Range $= \{y : y \leq 2\}$

9. Domain $= \{x : x \in \mathbb{R}\}$; Range $= \{y : y \in \mathbb{R}\}$

11. Domain $= \{x : x \in \mathbb{R}\}$; Range $= \{y : y = 4\}$

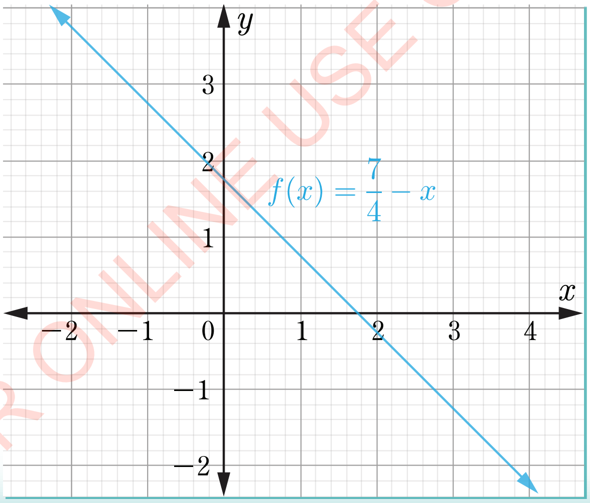
Exercise 2.4

1. Not a polynomial
3. Not a polynomial
5. Not a polynomial
7. Polynomial
9. Polynomial
11. Not a Polynomial
13. Polynomial
15. Not a polynomial
17. 3
19. 2
21. 2

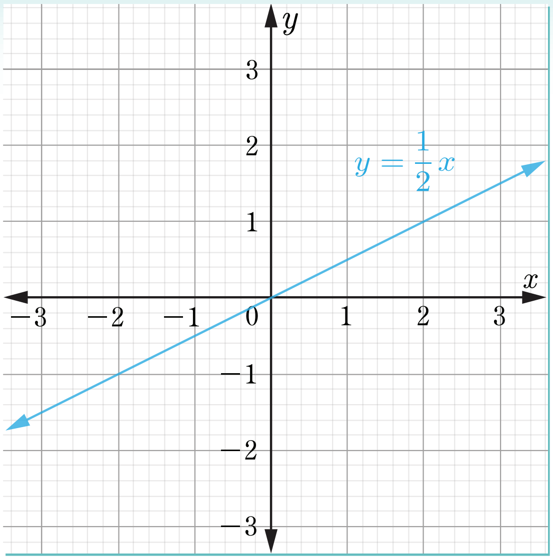
Exercise 2.5

1. $f(x) = -3x + 6$
3. $f(x) = -2x$
5. $f(x) = 4x+8$

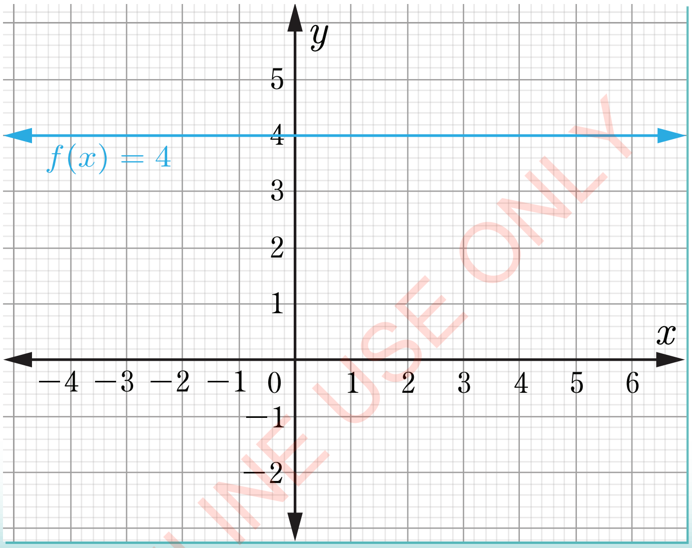
7.



9.

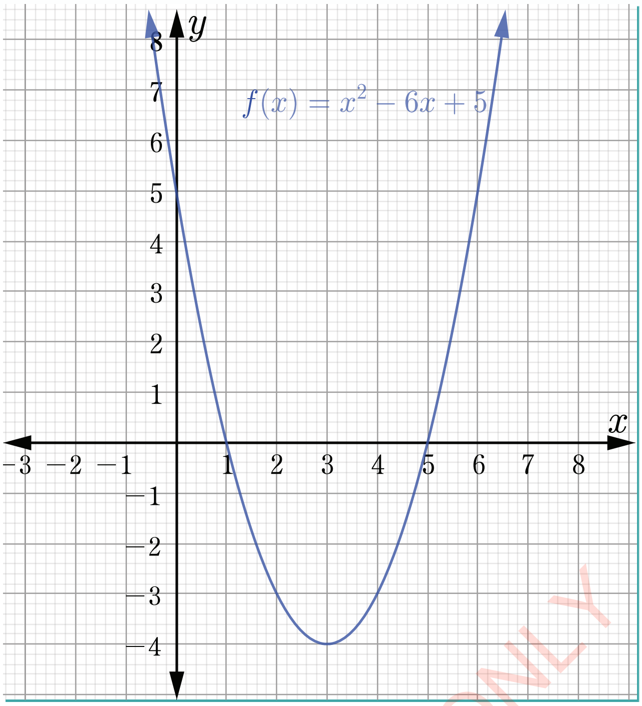


11.



Exercise 2.6

1.



Minimum value is $f(x) = -4$ at $x = 3$.

3. $y = (x+4)^2 - 11$

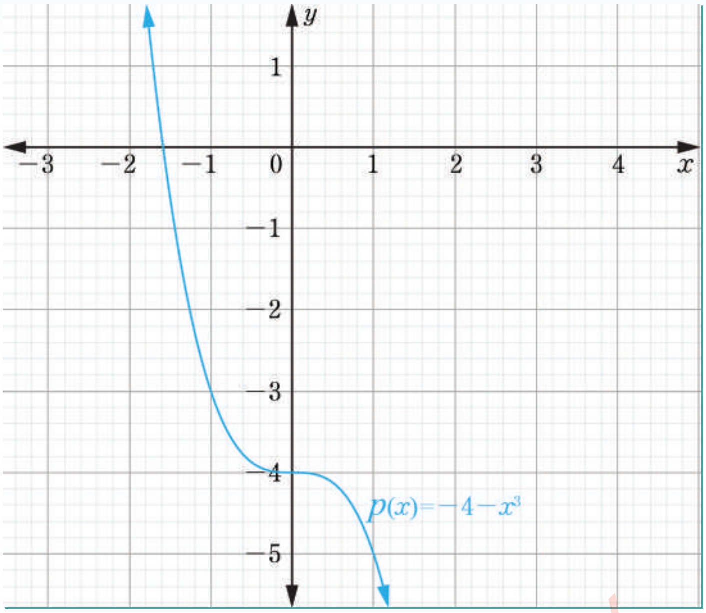
5. $y = -9\left(x + \frac{1}{3}\right)^2 + 6$

7. (a) Maximum value is 4 (b) axis of symmetry is $x = 1$

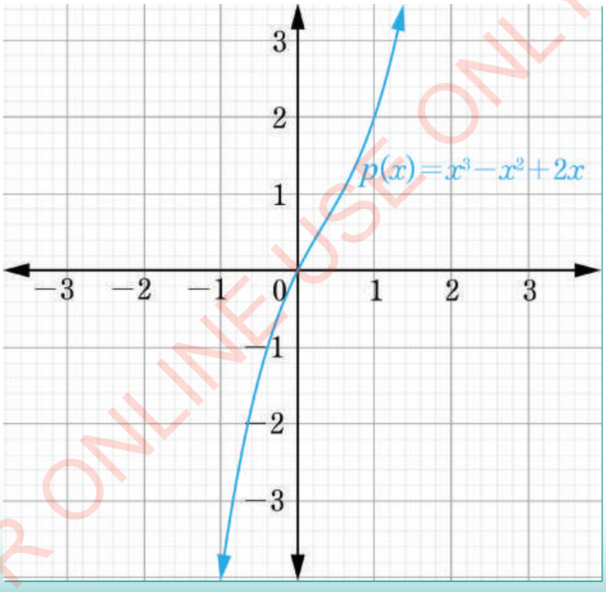
9. (a) Maximum value is $\frac{9}{4}$ (b) axis of symmetry is $x = -\frac{1}{2}$

Exercise 2.7

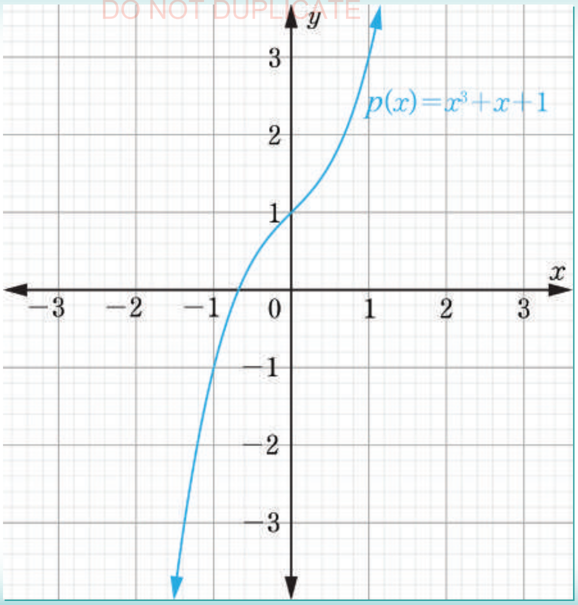
1.



3.



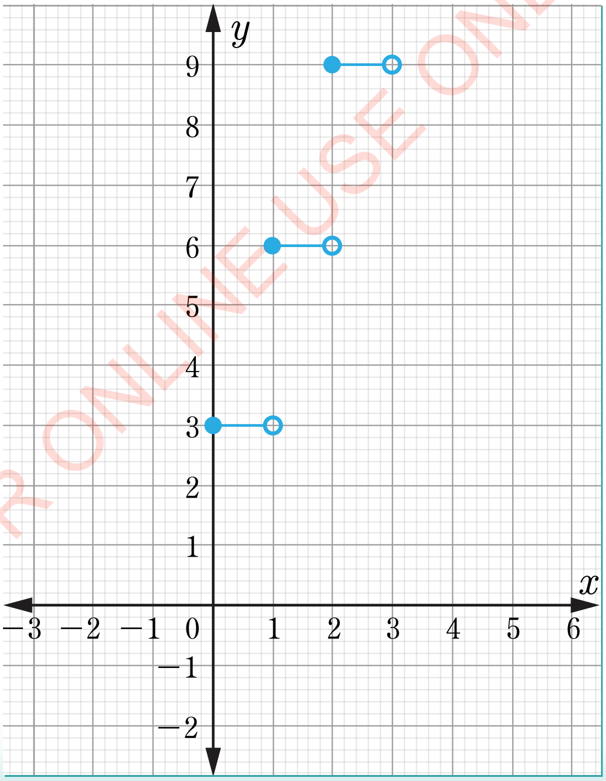
5.



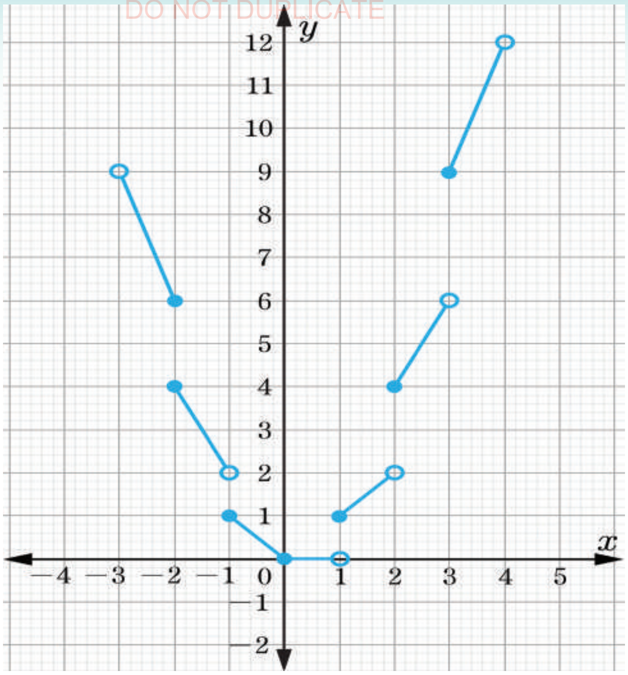
Exercise 2.8

1. (a) 5 (b) 1 (c) –1 (d) 102

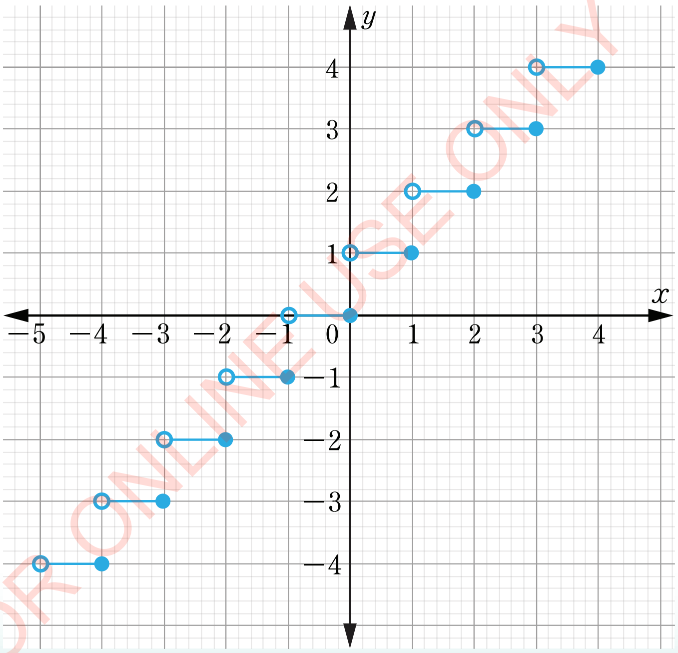
3.



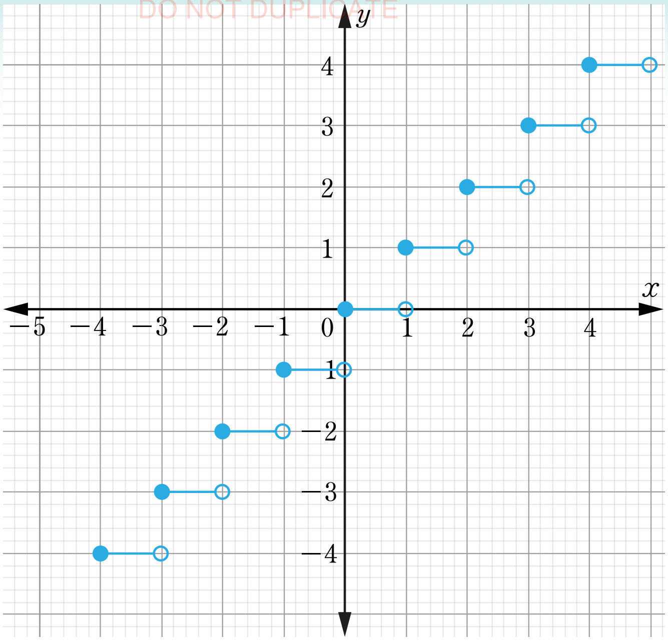
5.



7(a)

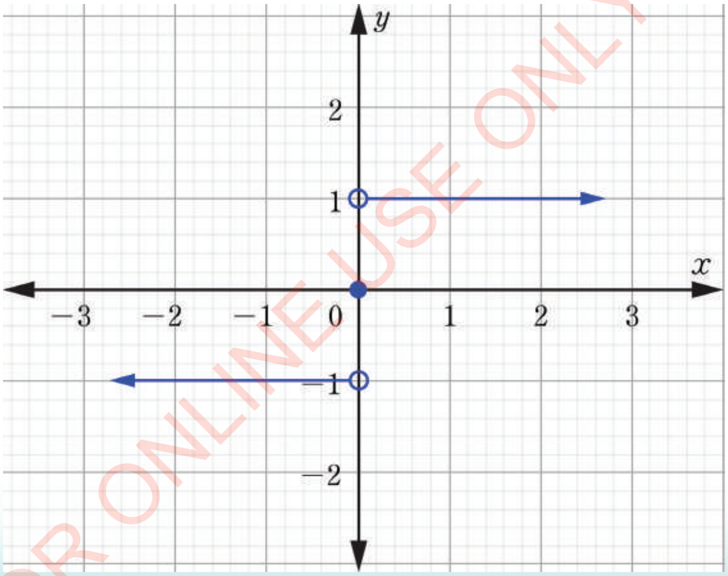


(b)

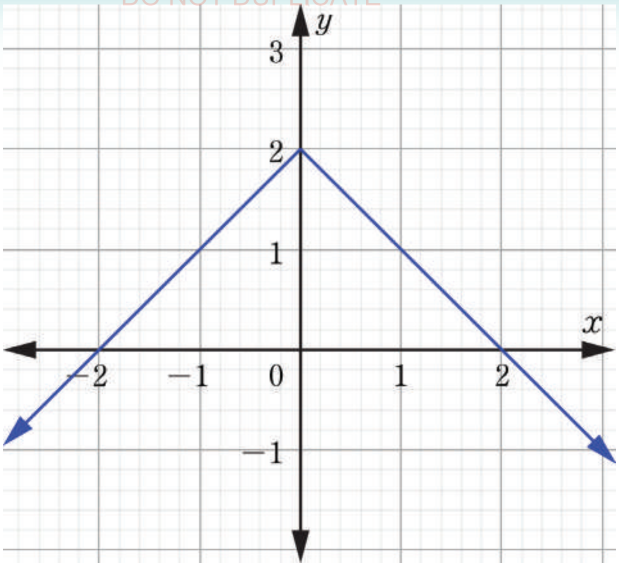


9. (a) $f(3) = 1$ and $f(-2) = -1$

(b)



11.



Domain $= \{x : x \in \mathbb{R}\}$ Range $= \{y : y \leq 2\}$

Exercise 2.9

1. (a) $f^{-1}(x) = -1 \pm \sqrt{x}$ (b) Not a function

3. (a) $f^{-1}(x) = \dfrac{x+2}{3}$ (b) A function

5. $f^{-1}(x) = x^2$ for $x \geq 0$

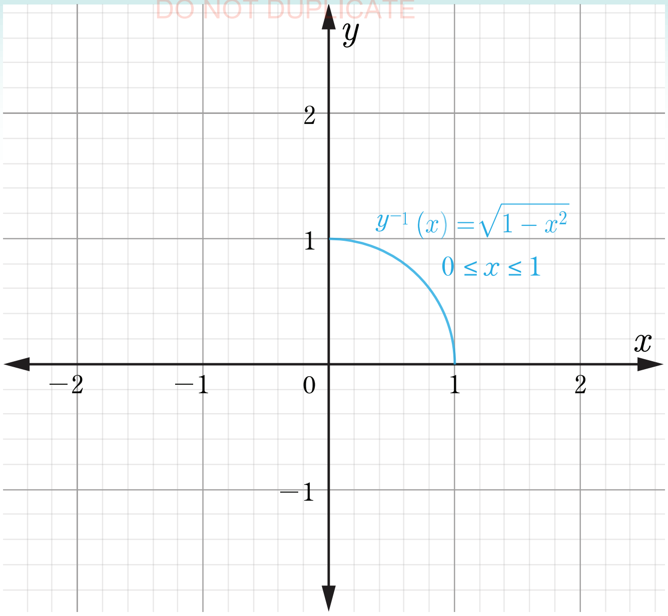
7. $F^{-1} = \{(3,2), (4,3), (5,4), (6,7)\}$

9. $f^{-1}(x) = \pm\sqrt{x+2}$

Revision exercise 2

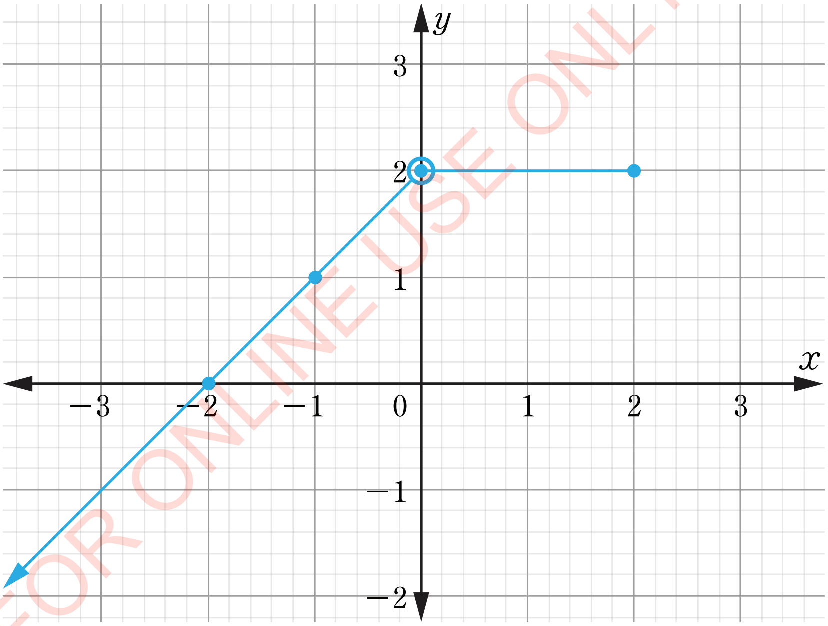
1. 3 3. 2 5. (b), (c), (d)

7.



9. (a) (i) g(−1)=1 (ii) g(−4) = −2 (iii) g (1.6) = 2

(b)



(c) Domain $=\{x : x \le 2\}$ Range $=\{y : y \le 2\}$

11. (a) $\left(\frac{7}{4}, -\frac{1}{8}\right)$ (b) $\left(\frac{-1}{8}, \frac{15}{16}\right)$ (c) $\left(\frac{-3}{4}, \frac{23}{8}\right)$

13. (a) It is a polynomial of degree 4. The powers are all non-negative integers.

Chapter Three

Exercise 3.1

1. $\bar{x} = 2$ 3. 375 km/h 5. $\bar{x} = 40$

7. $\bar{x} = 17.94$ which implies that a child chosen randomly from the village would have an average age of 17.9 months.

9. $\bar{x} = 41$ which implies that the average performance of the students was 41 marks.

Exercise 3.2

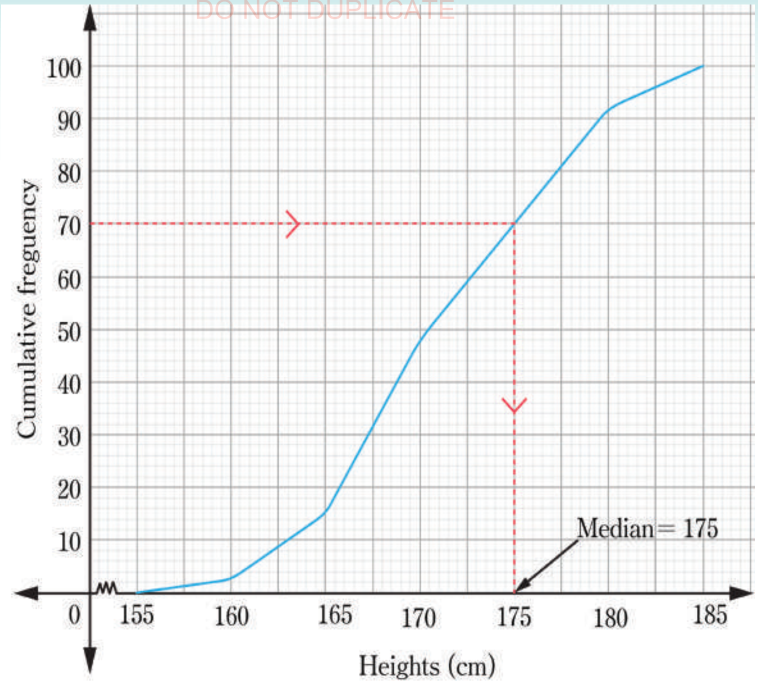
1. Median = 78.5 which implies that the typical score of 8 students in the weekly test was 78.5 marks

3. $x = 21$

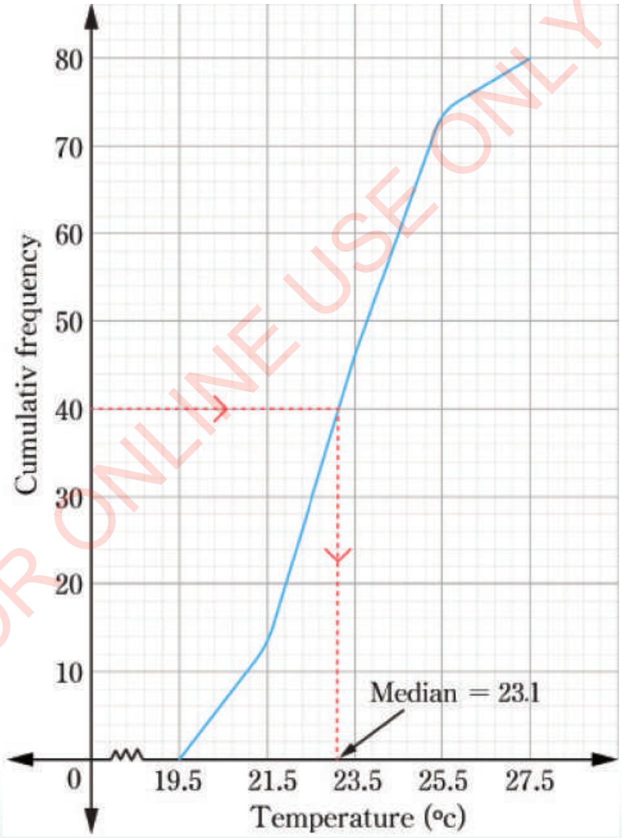
5. Median = 6 kilograms which implies that a child chosen at random from the group of 8 children would have a typical mass of 6 kilograms.

7. (a) Median 175 cm which implies that a person chosen at random from 100 people would have a typical height of 175 cm.

7(b)



9 (a)



(b) Median = 23.14

Basic Mathematics Form Three

Exercise 3.3

1. (a)

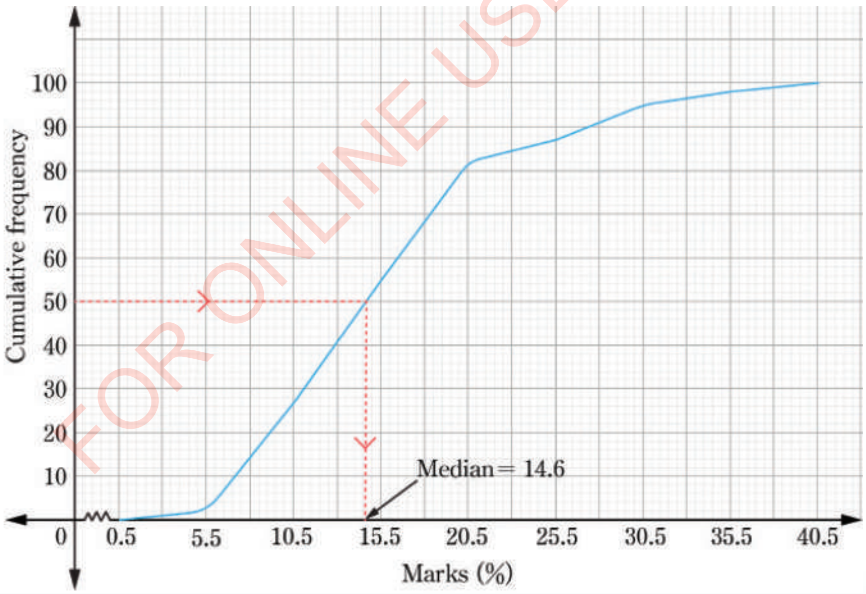
x	0	1	2	3	4	5	6	7	8	9
f	0	2	2	3	2	3	2	1	2	3

(b) The most frequent occurring digits are 3, 5 and 9, the least frequent occuring digit is 7.

3. The mode will remain the same
5. Mode = 150 which implies that typical tobacco leaf would have a length of 150 cm.
7. Mode = 199. The mode shows that most of the packets contained 199 needles
9. Mode = 87.7 cm. This mode shows that majority of the children have height of 87.7 centimetres.

Revision exercise 3

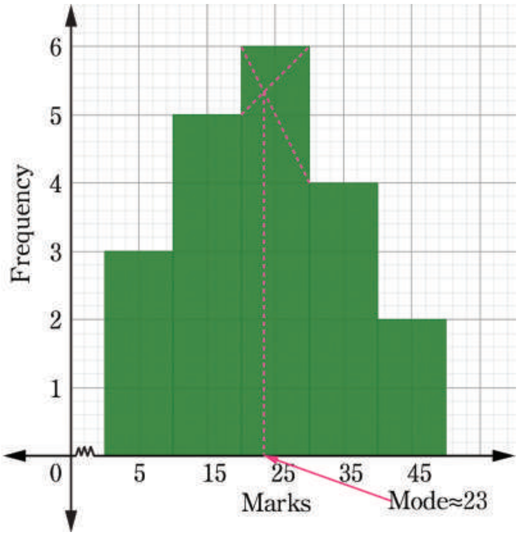
1. (a) 32 (b) 32.13 (c) 27
3. (a) 12.67 (b) 11.33
5. (a)



5. (b) 14.6

7. Mode = 7 which shows that most associations had 7 sewing machines.

9. (a)



9. (b) Mode ≈ 23 which shows that students scored an average of 23 marks.

11. (a) 14 (b) 1210 Dollars

13. The fifth value is 6

Chapter Four

Exercise 4.1

1. 12.5 metres per second 3. 0.83 litres per second 5. 198 km/h

Exercise 4.2

1. 599.36 Rupees 3. 176.68 Euros 5. Tsh 22 459.5

7. (a) 3 333 333.3 Metical (b) 93 457.9 Malawian Kwacha (c) 119.2 Francs
   (d) 8 728.5 Rupee

9. (a) 625 feet per minute
   (b) 500 feet per minute

(c) Negative because the altitude is decreasing

(d) 150 feet per minute. The rates are equal, one is positive and the other is negative.

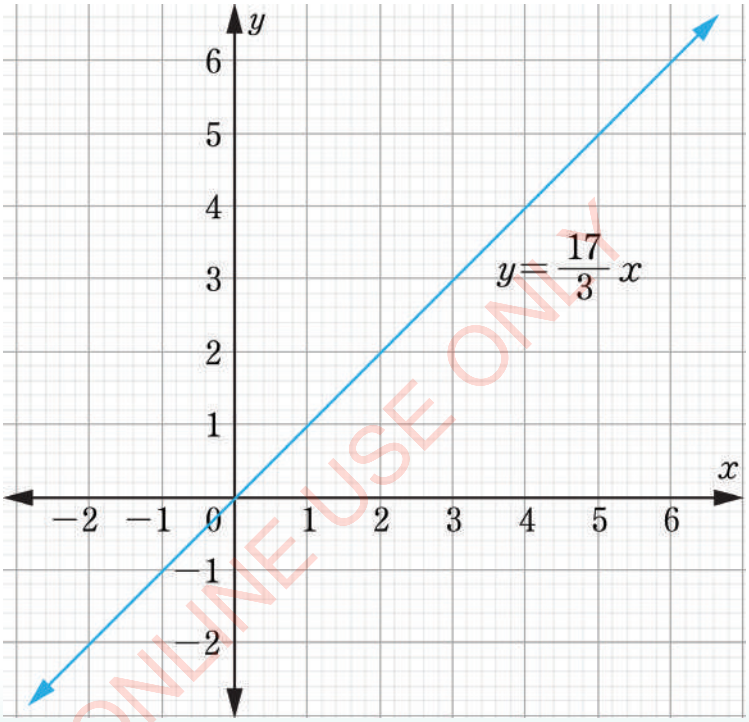
(e) Yes, if there is no change in altitude. An example is from A to B.

Exercise 4.3

1. 12.5   3. 12   5. 5.5   7. 18

9. (a) $\frac{85}{3}$ (b) direct variation; that is $x = \frac{3}{17}y$

(c)



11. 3m

Exercise 4.4

1. 2   3. $\frac{3}{4}$ 5. $\frac{4}{3}$ 7. $\frac{3}{2}$

9. $y = \frac{12}{\sqrt[3]{x}}$ $x = 32.8$ 11. 155.9 cm

Exercise 4.5

1. 51.25%
3. $h = \frac{kV}{r^2}$
5. 10 typists
7. V = 100
9. $y_1 = 6$ and $y_3 = 2$

Revision exercise 4

1. $k = \frac{8}{7}$, $y = \frac{320}{7}$
3. $y = \frac{33}{7}x$, $x = \frac{49}{11}$
5. $y = \frac{7}{4}\sqrt{x}$, $y = \frac{35}{4}$
7. $y = \frac{25x}{4z}$, $y = 15$
9. $y = \frac{125}{12}$
11. (a) $y = x$ Direct variation
    (b) $y = \frac{1}{x}$ Inverse variation
    (c) $y = x^2 - 3$. It is neither direct nor inverse variation
13. Speed and time vary inversely, because the speed increases as the time decreases and vice versa.
15. (a) $x = \frac{24}{y}$ (b) $x = \frac{96}{y}$ (c) $y = \frac{10}{3y}$

Chapter Five

Exercise 5.1

1. (a) $\frac{4}{5}, \frac{5}{6}$ (b) $\frac{1}{16}, \frac{-1}{32}$ (c) 720, 5040 (d) 1, −1

3. $3n$ 5. 3, 7, 11, 15, 19 7. 7 9. $\frac{11}{2}, \frac{25}{4}$

Exercise 5.2

1. (a) $-2+2-2+2-2+2-\ldots$ (b) 0 3. 95

5. (a) 243, 729 (b) 1, −1

7. (a) $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\ldots$ (b) $\frac{3}{4}$

9. (a) r (b) r (c) $a+ar+ar^2+\ldots$ (d) ar^{n-1}

Exercise 5.3

1. (b) $26+19+12+5-2,\ d=-7$

   (e) $1.3+2+2.7+3.4+\cdots,\ d=0.7$

   (f) $\frac{1}{4}-\frac{7}{8}+\frac{1}{12}-\frac{19}{8}-\ldots\ ,\ d=-\frac{5}{8}$

3. (a) $d=5$ (b) $A_1=32$ (c) $A_{13}=92$

5. (a) $A_1=5$ (b) $A_{200}=403$ (c) $A_n=2n+3$

7. $A_n=x+3n-1$

9. $n=19$

Exercise 5.4

1. (a) $S_{21} = 420$ (b) $S_{11} = 121a$ (c) $S_{100} = 250.5$

3. 8 terms

5. (a) 1 938 500 Tanzanian shillings (b) 29 569 500 Tanzanian shillings

7. $d = 2$ $S_{10} = 220$ 9. n = 58

Exercise 5.5

1. (a) common ratio, $r = 3$ (b) $G_1 = \frac{1}{3}$ 3. $n = 9$

5. $G_8 = -4374$

7. (a) It's a geometric series, common ratio, $r = 3$

   (b) It's a geometric series, common ratio, $r = \frac{1}{3}$

   (c) Its a geometric series common ratio, $r = -2$

   (d) It is not a geometric series, it has no common ratio

9. (a) $G_1 = 4$ (b) $r = 6$ (c) $G_7 = 186\ 624$

Exercise 5.6

1. $S_8 = 6\ 560$ 3. $S_{10} = -29\ 524$ 5. $n = 5$

7. Is a geometric progression with $r = 3$ 9. $p = 144$

Exercise 5.7

1. $A_2 = \text{Tsh } 1\ 348\ 320$ $A_3 = \text{Tsh } 1\ 429\ 219.20$

3. $A_6 = \text{Tsh } 8\ 965\ 839.93$ 5. $A_2 = \text{Tsh } 17\ 334\ 375$

7. $A_n = 1\ 000\ 000\ 000(1.095)^n$ 9. 14 years 11. Tsh 70 072.15

Revision exercise 5

1. $A_n = 3n - 2$ 3. $S_n = n^2$ 5. Tsh 9 720 000

7. (a) 36; 45 (b) -1200; $-1\,464.5$ (c) ± 8 9. 432 11. 2^{5050}

13. 13 years 15. $A_6 =$ Tsh 1 532 983

Chapter Six

Exercise 6.1

1. $d=2r$

3. Radius = $\overline{OQ} = \overline{OR} = \overline{OE}$, Diameter $= \overline{ER}$, chords $\overline{CD}$ and $\overline{ER}$

Exercise 6.2

3. 1.57 cm 5. (a) $\frac{5}{2}\pi$ cm (b) 5π cm (c) 3π cm 9. $\frac{248}{9}\pi$ cm

Exercise 6.3

1. (a) $\frac{\pi}{18}$ (b) $\frac{\pi}{4}$ (c) $\frac{8}{9}\pi$
3. (a) $\frac{4}{9}\pi$ (b) $\frac{10\pi}{3}$ (c) $\frac{4}{3}\pi$ (d) $\frac{7}{4}\pi$
5. (a) 45° (b) 120° (c) 315° (d) 270°
7. (a) $\frac{2\pi}{3}$ (b) $\frac{9\pi}{2}$ (c) 24π

Exercise 6.4

1. $x = 30°$ 3. $x = 240°$ 5. $x = 100°$ 7. $C\hat{D}Q = 70°$

15. (a) 29° (b) 58° (c) = 29° (d) 122°

Exercise 6.5

1. $x = 4.9$ cm 3. $x = 9.2$ cm 5. 23 cm 7. 13 cm

Exercise 6.6

1. $x = 115°$ 3. $A\hat{O}B = 110°$

Exercise 6.7

3. $x = 40°$ and $y = 50°$ 5. 20 cm 7. $x = 28°$

9. (a) $P\hat{Q}T = 60°$ (b) $O\hat{R}T = 30°$ (c) $R\hat{T}Q = 60°$

Revision exercise 6

1. 18.2°, 161.8

3. (a) $a = 2\pi$ $b = 4\pi$ $c = 6\pi$ (b) $a = \frac{5}{3}\pi$ $b = 5\pi$ $c = \frac{16}{3}\pi$

5. (a) 120°; $\frac{2}{3}\pi$ (b) 360°; 2π (c) 810°; $\frac{9}{2}\pi$ (d) 4320°; 24π 7. $\overline{AB} = 18.8$ cm

9. 10.9 cm 11. (a) $x = 280°$; (b) $x = 60°$; (c) $x = 120°$

13. $P\hat{S}R = 68$, $C\hat{Q}D = 10°$

15. (a) 7.7cm (b) 4.6 cm

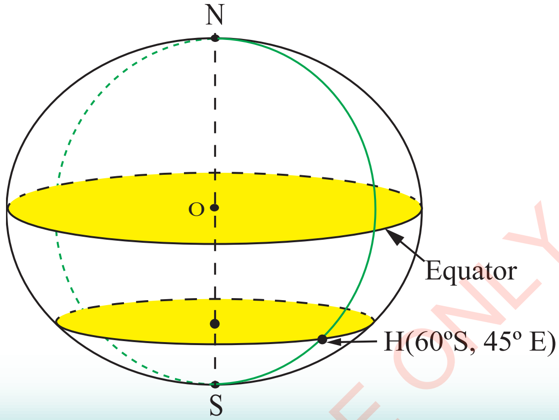
17. $x = 9°$, $y = 18°$, $P\hat{R}Q = 45°$, and $P\hat{T}R = 18°$

Chapter Seven

Exercise 7.1

1. (a) Mwanza (b) Tabora and Tanga
3. Angle subtended by the arc AB is $6°$.
5. (a) Accra, London, and Hull (b) Nakuru, and Kampala
7. Tanga, Dar es Salaam, and Addis Ababa.

9



Exercise 7.2

1. (a) 10 800 nautical miles (b) 20 001.8 km
3. 20 001.8 km
5. (a) 54.3 nm/h (b) 100.5 km/h
7. 294 hrs

Exercise 7.3

1. (a) 1 500 nautical miles (b) 3 720 nautical miles
3. (a) 2 820 nautical miles (b) 3 180 nautical miles
5. (a) 1 555.7 km (b) 444.5 km
7. 50°N
9. 2 200 hrs or 10:00 pm

Exercise 7.4

1. 2 892.5 nautical miles
3. 61.3°
5. 5 221 km
7. 60°S
9. 5:00 pm.
11. 39 395.9 km
13. 45°N

Revision exercise 7

1. (a) 16° (b) 1 777.9 km
3. 5 842 km
5. Thursday 12:00 noon
7. 34 644 km
9. 9°W
11. 59°
13. $\theta = 81°$

Chapter Eight

Exercise 8.1

1. CASH ACCOUNT (1)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2000				2000			
June 1	Capital	PVL2	2 000 000	June 2	Purchases	GL3	1 000 000
June 4	Sales	GL5	800 000	June 3	Shelves	GL4	300 000
June 20	Sales	GL5	450 000	June 8	Cleaning expenses	GL6	100 000
				June 12	Wages	GL7	200 000
				June 15	Rent	GL8	300 000
				June 17	Purchases	GL3	600 000

PRIVATE LEDGER

CAPITAL ACCOUNT (2)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
				2000 June 1	Cash	CA1	2 000 000

GENERAL LEDGER

PURCHASES ACCOUNT (3)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2000							
June 2	Cash	CA1	1 000 000				
June 17	Cash	CA1	600 000				

SHELVES ACCOUNT (4)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2000 June 3	Cash	CA1	300 000				

SALES ACCOUNT (5)

DR				CR			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
				2000			
				June 4	Cash	CA1	800 000
				June 20	Cash	CA1	450 000

CLEANING EXPENSES ACCOUNT (6)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2000 June 8	Cash	CA1	100 000				

WAGES ACCOUNT (7)

DR				CR			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2000 June 12	Cash	CA1	200 000				

RENT ACCOUNT (8)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2000 June 15	Cash	CA1	300 000				

3. CASH ACCOUNT (1)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2002				2002			
June 1	Capital	PVL2	1 500 000	June 2	Purchases	GL3	900 000
June 5	Sales	GL4	600 000	June 11	Furniture	GL5	300 000
June 6	Sales	GL4	100 000	June 15	Rent	GL6	100 000
				June 18	Purchases	GL3	250 000
				June 25	Salary	GL7	200 000

PRIVATE LEDGER

CAPITAL ACCOUNT (2)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
				2002			
				June 1	Cash	CA1	1 500 000

GENERAL LEDGER

PURCHASES ACCOUNT (3)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2002							
June 2	Cash	CA1	900 000				
June 18	Cash	CA1	250 000				

SALES ACCOUNT (4)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
				2002			
				June 5	Cash	CA1	600 000
				June 6	Cash	CA1	100 000

FURNITURE ACCOUNT (5)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2002 June 11	Cash	CA1	300 000				

RENT ACCOUNT (6)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2002 June 15	Cash	CA1	100 000				

SALARY ACCOUNT (7)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2002 June 25	Cash	CA1	200 000				

5. CASH ACCOUNT (1)

Dr				Cr			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
2001				2001			
August 1	Capital	PVL2	700 000	August 3	Purchases	GL3	400 000
August 4	Sales	GL4	300 000	August 12	Advertisement	GL5	50 000
				August 18	Rent	GL6	60 000
				August 28	Purchases	GL3	230 000
				Aug. 30	Wages	GL7	120 000

PRIVATE LEDGER

CAPITAL ACCOUNT (2)

DR				CR			
Date	Particulars	Folio	Amount	Date	Particulars	Folio	Amount
				2001 August 1	Cash	CA1	700 000

GENERAL LEDGER

PURCHASES ACCOUNT (3)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	Folio	**Amount**
2001							
Aug3	Cash	CA1	400 000				
Aug28	Cash	CA1	230 000				

SALES ACCOUNT (4)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
				2001			
				August 4	Cash	CA1	300 000
				Aug29	XYZ trader	SL10	210 000

ADVERTISEMENT ACCOUNT (5)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2001 Aug.12	Cash	CA1	50 000				

RENT ACCOUNT (6)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2001 August 18	Cash	CA1	60 000				

WAGES ACCOUNT (7)

Dr				Cr			
Date	**Particulars**	**Date**	**Particulars**	**Date**	**Particulars**	**Date**	**Particulars**
2001 Aug 30	Cash	CA1	120 000				

FURNITURE ACCOUNT (8)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2001 Aug.8	Gab	PL9	350 000				

PURCHASES LEDGER

GAB ACCOUNT (9)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
				2001 Aug. 8	Furniture	GL8	350 000

SALES LEDGER

XYZ TRADERS ACCOUNT (10)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2001 Aug. 29	Sales	GL4	210 000				

7. CASH ACCOUNT (1)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2004				2001			
July15	Capital	PVL2	800 000	July 16	Purchases	GL3	400 000
July17	Sales	GL4	200 000	July 19	Furniture	GL5	250 000
July31	Sales	GL4	350 000	July 19	Purchases	GL3	300 000
				July 25	Salaries	GL6	150 000
				July 27	Rent	GL7	220 000
				July 31	Balance	c/d	30 000
			1 350 000				**1 350 000**
August 1	Balance	b/d	30 000				1 350 000

PRIVATE LEDGER

CAPITAL ACCOUNT (2)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2004 July 31	Balance	c/d	800 000	2004 July 15	Cash	CA1	800 000
			800 000				**800 000**
				August 1	Balance	b/d	800 000

GENERAL LEDGER

PURCHASES ACCOUNT (3)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2004				2004			
July 16	Cash	CA1	400 000	July 31	Balance	c/d	700 000
July 19	Cash	CA1	300 000				
			700 000				**700 000**
Aug. 1	Balance	b/d	700 000				

SALES ACCOUNT (4)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2004				2004			
July 31	Balance	c/d	550 000	July17	Cash	CA1	200 000
				July 31	Cash	CA1	350 000
			550 000				**550 000**
				Aug1	Balance	b/d	550 000

FURNITURE ACCOUNT (5)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2004				2004			
July19	Cash	CA1	250 000	July 31	Balance	c/d	250 000
			250 000				**250 000**
August 1	Balance	b/d	250 000				

SALARY ACCOUNT (6)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2004				2004			
July 25	Cash	CA1	150 000	July 31	Balance	c/d	150 000
			150 000				150 000
August 1	Balance	b/d	150 000				

RENT ACCOUNT (7)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2004				2004			
July 27	Cash	CA1	220 000	July 31	Balance	c/d	220 000
			220 000				220 000
August 1	Balance	b/d	220 000				

9. CASH ACCOUNT (1)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2006 January 1	Capital	PVL2	1 200 000	2006 Jan 2	Purchases	GL3	800 000
January 5	Sales	GL5	600 000	Jan 3	Shelves	GL4	250 000
January 26	Hamisi	SL8	300 000	Jan 8	Rent	GL6	240 000
				Jan 13	Wages	GL7	60 000
				Jan 26	Musa	PL9	200 000
				Jan 27	Insurance	GL10	100 000
				Jan 31	Balance	c/d	450 000
			2 100 000				**2 100 000**
Feb 1	Balance	b/d	450 000				

PRIVATE LEDGER

CAPITAL ACCOUNT (2)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2006 Jan 31	Balance	c/d	1 200 000	2006 Jan 1	Cash	CA1	1 200 000
			1 200 000				**1 200 000**
				Feb 1	Balance	b/d	1 200 000

GENERAL LEDGER

PURCHASES ACCOUNT (3)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2006 Jan 2	Cash	CA1	800 000	2006 Jan 31	Balance	c/d	1 550 000
Jan 10	Musa	PL9	400 000				
Jan 18	Amina	PL11	350 000				
			1 550 000				**1 550 000**
Feb 1	Balance	b/d	1 550 000				

SHELVES ACCOUNT (4)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	Folio	Amount
2006 Jan 3	Cash	CA1	250 000	2006 Jan 1	Balance	c/d	250 000
			250 000				**250 000**
Feb 1	Balance	b/d	250 000				

SALES ACCOUNT (5)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2006 Jan 31	Balance	c/d	900 000	2006 Jan 5	Cash	CA 1	600 000
				Jan 25	Hamisi	SL 9	300 000
			900 0000				**900 000**
				Feb 1	Balance	b/d	900 000

RENT ACCOUNT (6)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2006 Jan 8	Cash	CA1	240 000	2006 Jan 31	Balance	c/d	240 000
			240 000				**240 000**
Feb 1	Balance	b/d	240 000				

WAGES ACCOUNT (7)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2006 Jan 13	Cash	CA1	60 000	2006 Jan 31	Balance	c/d	60 000
			60 000				**60 000**
Feb 1	Balance	b/d	60 000				

INSURANCE ACCOUNT (10)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2006 Jan 27	Cash	CA1	100 000	2006 Jan 31	Balance	c/d	100 000
			100 000				**100 000**
Feb 1	Balance	b/d	100 000				

SALES LEDGER

HAMISI (DEBTOR'S) ACCOUNT (8)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2006 Jan 25	Sales	GL5	300 000	2006 26 Jan	Cash	CA1	300 000
			300 000				**300 000**

PURCHASES LEDGER

MUSA (CREDITOR'S) ACCOUNT (9)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2006 Jan 26	Cash	CA1	200 000	2006 Jan 10	Purchases	GL3	400 000
Jan 31	Balance	c/d	200 000				
			400 000				**400 000**
				Feb 1	Balance	b/d	200 000

AMINA (CREDITOR'S) ACCOUNT (11)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2006 Jan 31	Balance	c/d	350 000	2006 Jan 18	Purchases	GL3	350 000
			350 000				**350 000**
				Feb 1	Balance	b/d	350 000

Exercise 8.2

1. CASH ACCOUNT (1)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2005				2005			
May 1	Capital	PVL2	4 000 000	May 2	Purchases	GL3	1 500 000
May 8	Sales	GL5	1 500 000	May3	Furniture	GL4	500 000
May15	Sales	GL5	900 000	May10	Purchases	GL3	600 000
May20	Sales	GL5	200 000	May12	Rent	GL6	100 000
				May17	Transport	GL7	50 000
				May25	Salaries	GL8	100 000
				May31	Balance	c/d	3 750 000
			6 600 000				**6 600 000**
June 1	Balance	b/d	3 750 000				

PRIVATE LEDGER

CAPITAL ACCOUNT (2)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2005				2005			
May 31	Balance	c/d	4 000 000	May 1	Cash	CA1	4 000 000
			4 000 000				**4 000 000**
				June 1	Balance	b/d	4 000 000

GENERAL LEDGER

PURCHASES ACCOUNT (3)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2005				2005			
May 3	Cash	CA1	1 500 000	May 31	Balance	c/d	2 100 000
10	Cash	CA1	600 000				
			2 100 000				**2 100 000**
June 1	Balance	b/d	2 100 000				

FURNITURE ACCOUNT (4)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2005				2005			
May 3	Cash	CA1	5 00 000	May 31	Balance	c/d	500 000
			500 000				**500 000**
June 1		b/d	500 000				

SALES ACCOUNT (5)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2005				2005			
May 31	Balance	c/d	2 600 000	May 8	Cash	CA1	1 500 000
				15	Cash	CA1	900 000
				20	Cash	CA1	200 000
			2 600 000				**2 600 000**
				June, 1	Balance	b/d	2 600 000

RENT ACCOUNT (6)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2005				2005			
May 12	Cash	CA1	100 000	May 31	Balance	c/d	100 000
			100 000				**100 000**
June 1	Balance	b/d	100 000				

TRANSPORT ACCOUNT (7)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2005 May 17	Cash	CA1	50 000	2005 May 31	Balance	c/d	50 000
			50 000				50 000
June1	Balance	b/d	50 000				

SALARIES ACCOUNT (8)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2005 May 25		CA1	100 000	2005 May 31	Balance	c/d	100 000
			100 000				**100 000**
June 1	Balance	b/d	100 000				

MRS MWANANE TRIAL BALANCE AS AT 31^{ST} MAY, **2005**

Account Name	**Dr**	**Cr**
Cash	3 750 000	
Capital		4 000 000
Purchases	2 100 000	
Furniture	500 000	
Sales		2 600 000
Rent	100 000	
Transport	50 000	
Salaries	100 000	
	6 600 000	**6 600 000**

3. CASH ACCOUNT (1)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2005				2005			
Feb. 15	Capital	PVL2	1 055 000	Feb. 16	Purchase	GL3	500 000
19	Sales	GL5	450 000	18	Shelves	GL4	55 000
21	Sales	GL5	700 000	20	Purchase	GL3	400 000
				25	Rent	GL6	150 000
				Feb. 28	Balance	c/d	1 100 000
			2 205 000				**2 205 000**
March 1	Balance	b/d	1 100 000				

PRIVATE LEDGER

CAPITAL ACCOUNT (2)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2005				2005			
Feb 28	Balance	c/d	1 055 000	Feb. 15	Cash	CA1	1 055 000
			1 055 000				**1 055 000**
				March. 1	Balance	b/d	1 055 000

GENERAL LEDGER

PURCHASES ACCOUNT (3)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2005				2005			
Feb 16	Cash	CA1	500 000	Feb. 29	Balance	c/d	900 000
20	Cash	CA1	400 000				
			900 000				**900 000**
March 1	Balance	b/d	900 000				

SHELVES ACCOUNT (4)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2005				2005			
Feb 18	Cash		55 000	Feb. 28	Balance	c/d	55 000
			55 000				**55 000**
March1	Balance	b/d	55 000				

SALES ACCOUNT (5)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2005				2005			
Feb. 29	Balance	c/d	1 150 000	Feb. 19	Cash	CA1	450 000
				Feb. 21	Cash	CA1	700 000
			1 150 000				**1 150 000**
				March 1	Balance	b/d	1 150 000

RENT ACCOUNT (6)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2005				2005			
Feb 25	Cash	CA1	150 000	Feb. 29	Balance	c/d	150 000
			150 000				**150 000**
March. 1	Balance	b/d	150 000				

MRS. CHAKUBANGA'S TRIAL BALANCE AS AT 28TH FEBRUARY 2005

Account Name	Dr	Cr
Cash	1 100 000	
Capital		1 055 000
Purchases	900 000	
Sales		1 150 000
Shelves	55 000	
Rent	150 000	
	2 205 000	**2 205 000**

5. CASH ACCOUNT (1)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2005				2005			
April 1	Capital	PVL2	2 675 000	April 2	Purchases	GL3	1 500 000
5	Sales	GL6	1 300 000	3	Furniture	GL4	305 000
13	Sales	GL6	511 000	4	Shelves	GL5	270 000
25	Sales	GL6	498 000	9	Wages	GL7	54 000
				12	Purchases	GL3	342 000
				16	Rent	GL8	114 000
				20	Purchases	GL3	425 000
				27	Salaries	GL9	67 000
				30	Balance	c/d	1 907 000
			4 984 000				**4 984 000**
May 1	Balance	b/d	1 907 000				

PRIVATE LEDGER

CAPITAL ACCOUNT (2)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2005				2005			
April 30	Balance	c/d	2 675 000	April, 1	Cash	CA1	2 675 000
			2 675 000				**2 675 000**
				May, 1		b/d	2 675 000

GENERAL LEDGER

PURCHASES ACCOUNT (3)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2005				2005			
April. 2	Cash	CA1	1 500 000	April, 30	Balance	c/d	2 267 000
12	Cash	CA1	342 000				
20	Cash	CA1	425 000				
			2 267 000				**2 267 000**
May, 1		b/d	2 267 000				

FURNITURE ACCOUNT (4)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2005				2005			
April, 3	Cash	CA1	305 000	April, 30	Balance	c/d	305 000
			305 000				**305 000**
May, 1	Balance	b/d	305 000				

SHELVES ACCOUNT (5)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2005				2005			
April, 4	Cash	CA1	270 000	April, 30	Balance	c/d	270 000
			270 000				**270 000**
May, 1		b/d	270 000				

SALES ACCOUNT (6)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2005				2005			
April, 30	Balance	c/d	2 309 000	April, 5	Cash	CA1	1 300 000
				13	Cash	CA1	511 000
				25	Cash	CA1	498 000
			2 309 000				**2 309 000**
				May, 1	Balance	b/d	2 309 000

WAGES ACCOUNT (7)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2005				2005			
April, 9	Cash	CA1	54 000	April, 30	Balance	c/d	54 000
			54 000				**54 000**
May, 1	Balance	b/d	54 000				

RENT ACCOUNT (8)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2005				2005			
April 16	Cash	CA1	114,000	April 30	Balance	c/d	114,000
			114,000				**114,000**
May 1	Balance	b/d	114,000				

SALARIES ACCOUNT (9)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2005				2005			
April.27	Cash	CA1	67 000	April, 30	Balance	c/d	67 000
			67 000				**67 000**
May, 1	Balance	b/d	67 000				

NJIAPANDA TRADER'S TRIAL BALANCE AS AT 30TH APRIL, 2005

Account Name	Dr	Cr
Cash	1 907 000	
Capital		2 675 000
Purchases	2 267 000	
Rent	114 000	
Furniture	305 000	
Shelves	270 000	
Sales		2 309 000
Salaries	67 000	
Wages	54 000	
	4 984 000	**4 984 000**

Exercise 8.3

1. CASH ACCOUNT (1)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2014				2014			
Jan. 1	Capital	PVL2	10 000 000	Jan. 2	Purchases	GL3	5 000 000
3	Sales	GL5	4 000 000	2	Furniture	GL4	1 000 000
7	David	SL7	500 000	5	Transport	GL6	100 000
18	Sales	GL5	2 500 000	10	Motor Vehicle	GL8	2 230 000
23	Sales	GL5	1 450 000	12	Wages	GL9	77 000
				15	Rent	GL10	300 000
				17	Purchases	GL3	1 884 000
				27	Hangida	PL11	200 000
				31	Balance	c/d	7 659 000
			18 450 000				**18 450 000**
Feb. 1	Balance	b/d	7 659 000				

PRIVATE LEDGER

CAPITAL ACCOUNT (2)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2014				2014			
Jan. 31	Balance	c/d	10 000 000	Jan.1	Cash	CA1	10 000 000
			10 000 000				**10 000 000**
				Feb. 1	Balance	b/d	10 000 000

GENERAL LEDGER

PURCHASES ACCOUNT (3)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2014				2014			
Jan. 2	Cash	CA1	5 000 000	Jan. 31	Balance	c/d	7 284 000
17	Cash	CA1	1 884 000				
25	Hangida	PL11	400 000				
			7 284 000				**7 284 000**
Feb. 1	Balance	b/d	7 284 000				

FURNITURE ACCOUNT (4)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2014				2014			
Jan 2		CA1	1 000 000	Jan. 31	Balance	c/d	1 000 000
			1 000 000				**1 000 000**
Feb. 1	Balance	b/d	1 000 000				

SALES ACCOUNT (5)

Cr				Dr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2014				2014 Jan 3	David	SL7	1 000 000
Jan. 31	Balance c/d		8 950 000	Jan. 3	Cash	CA1	4 000 000
				18	Cash	CA1	2 500 000
				23	Cash	CA1	1 450 000
			8 950 000				**8 950 000**
				Feb. 1	Balance	b/d	8 950 000

TRANSPORT ACCOUNT (6)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2014				2014			
Jan. 5	Cash	CA1	100 000	Jan. 31	Balance	c/d	100 000
			100 000				**100 000**
Feb. 1	Balance	b/d	100,000				

MOTOR VEHICLE ACCOUNT (8)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2014				2014			
Jan 10	Cash	CA1	2 230 000	Jan 31	Balance	c/d	2 230 000
			2 230 000				**2 230 000**
Feb 1	Balance	b/d	2 230 000				

WAGES ACCOUNT (9)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2014				2014			
Jan. 12	Cash	CA1	77 000	Jan. 31	Balance	c/d	77 000
			77 000				**77 000**
Feb. 1	Balance	b/d	77 000				

RENT ACCOUNT (10)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2014				2014			
Jan. 15	Cash	CA1	300 000	Jan. 31	Balance	c/d	300 000
			300 000				**300 000**
Feb. 1	Balance	b/d	300 000				

SALES LEDGER

DAVID ACCOUNT (7)

Dr				Cr			
Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2014				2014			
Jan 3	Sales	SL5	1 000 000	Jan 7	Cash	CA1	500 000
				Jan 31	Balance	c/d	500 000
			1 000 000				**1 000 000**
Feb 1	Balance	b/d	500 000				

PURCHASES LEDGER

HANGIDA ACCOUNT (11)

Date	**Particulars**	**Folio**	**Amount**	**Date**	**Particulars**	**Folio**	**Amount**
2014				2014			
Jan. 27	Cash	CA1	200 000	Jan. 31	Purchases	GL3	400 000
Jan 31	Balance	c/d	200 000				
			400 000				**400 000**
				Feb. 1	Balance	b/d	200 000

MNYANGAA TRADER'S TRIAL BALANCE AS AT 31ST JANUARY, 2006

Account Name	Dr	Cr
Cash	7 659 000	
Capital		10 000 000
Purchases	7 284 000	
Furniture	1 000 000	
Sales		8 950 000
Transport	100 000	
David (Debtor)	500 000	
Motor Vehicle	2 230 000	
Wages	77 000	
Rent	300 000	
Hangida (Creditor)		200 000
	19 150 000	**19 150 000**

MNYANGAA TRADERS INCOME STATEMENT FOR THE YEAR ENDED 31 JANUARY, 2014

Sales		8 950 000
Less: Cost of goods sold		
Purchases	7 284 000	
Less: Closing stock	255 000	7 029 000
Gross profit		1 921 000
Less: Expenses		
Transport	100 000	
Wages	77 000	
Rent	300 000	
Total expenses		477 000
Net profit		1 444 000

3.

NJIA PANDA TRADER'S INCOME STATEMENT FOR THE YEAR ENDED 31ST DECEMBER, 2017

Sales		2 309 000
Less: Cost of goods sold		
Purchases	2 267 000	
Less: Closing stock	100 000	2 167 000
Gross profit		142 000
Less: Expenses		
Rent	114 000	
Salaries	67 000	
Wages	54 000	
Total expenses		235 000
Net loss		93 000

Exercise 8.4

1.

STATEMENT OF FINANCIAL POSITION AS AT 31ST JANUARY 2002.

Non-Current Assets (Fixed Assets)			
Furniture			80 000
Machinery			200 000
Land and Buildings			180 000
			460 000
Current Assets			
Stocks		100 000	
Debtors		140 000	
Cash in hand		160 000	400 000
			860 000
Less: Liabilities			
Long term liabilities			
Loan from NMB bank		120 000	
Current liabilities			
Creditors	100 000		
Bank overdraft	140 000	240 000	360 000
			500 000
Financed by:			
Capital			300 000
Add: Net profit		240 000	
Less: Drawings		40 000	200 000
			500 000

3. Tsh 1 900 000

Revision exercise 8

3. (a) Tsh 21 000 000 (c) Tsh 31 500 000

   (b) Tsh 2 500 000 (e) Tsh 6 500 000

5.

NUNA'S INCOME STATEMENT FOR THE YEAR ENDED 31 DECEMBER 2013

Sales		2 309 000
Less: Cost of goods sold		
Purchases		2 267 000
Gross profit		42 000
Less: Expenses		
Rent	114 000	
Salaries	67 000	
Wages	54 000	
Total expenses		235 000
Net Loss		193 000

STATEMENT OF FINANCIAL POSITION AS AT 31ST DECEMBER 2013

Non-Current Assets (Fixed Assets)	
Furniture	305 000
Shelves	270 000
	575 000
Current Assets	
Cash	1 907 000
	2 482 000
Financed by:	
Capital	2 675 000
Less: Net loss	193 000
	2 482 000

Glossary

Accounting entry	A record that documents a transaction.
Angle	A space between two meeting lines or surfaces which is normally measured in degrees or radians.
Congruent figures	Figures with exactly the same shape and size.
Cumulative frequency	A total of a frequency and all frequencies in a frequency distribution until a certain defined class interval.
Data	Set of values of a variable which represent information recorded for the purpose of analysis.
Exchange rate	The value through which a currency is converted to another.
Frequency	Represent the number of times a particular observation has appeared in a distribution.
Hexagon	A figure with six sides.
Histogram	A representation of frequency distribution by means of rectangles which are obtained by plotting frequencies against class intervals or class marks.
Knots	A distance of one nautical mile per hour.
Mapping	A process of linking each element of one set to another set, usually by using arrows.
Mapping diagram	A diagram which consists of two sets of a relation in which its elements are connected using arrows. Another term for a mapping diagram is pictorial diagram, arrow diagram or pictorial representation.

Maximum point	A point of a graph with the highest value.
Minimum point	A point of a graph with the lowest value.
Notation	A system of symbols used to represent facts or quantities.
Ordered pairs	A set of two corresponding elements of a relation, one from a set of domain and another from a set of range. Ordered pairs are represented using round brackets ( ).
Per annum	A period of one year.
Perpendicular bisector	A line which divides a line segment into two equal parts at $90°$.
Quadrilateral	A four sided figure.
Quadrant	Is one of four parts into which a circle or other shape has been devided.
Radian	An angle subtended at the centre of the circle by an arc length equal to its radius.
Similar figures	Figures whose corresponding sides are proportional.
Theorem	A formal statement about a phenomenon which can be proved.
Variable	A factor that changes in relation to how other related factors change. In mathematics, a variable is represented by a letter or symbol.

Bibliography

Fardon, M & Cox, D. (2004). *AS Accounting for AQA*. *London*. Osborne Books Limited.

Institute of Curriculum Studies. (1991). *Secondary basic mathematics*. *Book Two*. Dar es Salaam. Institute of curriculum studies.

Institute of Curriculum Studies. (1993). *Secondary basic mathematics*. *Book Three*. Dar es Salaam. Institute of curriculum studies.

Solomon, B & Buckwell, G. (2009). *Pupil's Book 3*. *Mathematics for Tanzanian secondary schools*. Dar es Salaam. Macmillan Aidan Ltd.

Sancheti, D.C & Kapoor, V.K. (1993). *Business mathematics*. *Newdelhi, India*. Sultan Chand & Sons Educational Publishers.

Tanzania Institute of Education (2002). *Secondary basic mathematics book 1*. *Dar es Salaam*. Tanzania Institute of Education.

Tanzania Institute of Education (2002). *Secondary basic mathematics book 2*. *Dar es Salaam*. Tanzania Institute of Education.

Tanzania Institute of Education (2002). *Secondary basic mathematics book 3*. *Dar es Salaam*. Tanzania Institute of Education.

Tanzania Institute of Education (2005). *Secondary basic mathematics*. *Book 3*. *Dar es Salaam*. Tanzania Institute of Education.

Tanzania Institute of Education (2011). *Basic mathematics teacher's manual for ordinary secondary education*. *Dar es Salaam*. Tanzania Institute of Education

Wood, F & Sangster, A. (2002). *Business Accounting*. *Edinburgh, England*. Pearson Education limited.

Wood, F & Sangster, A. (2005). *Business Accounting One*. *(10th ed)*. *China*. Pearson Education.

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