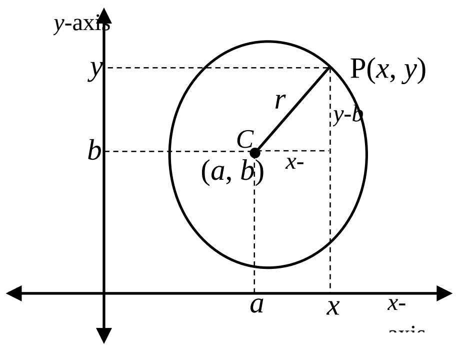
CIRCLES

A circle is a 2-dimensional shape in Euclidean geometry made by drawing a curve that is always the same distance from the center

A circle can also be defined as a locus of all points P(x, y) which are equidistant from the same given point fixed point C(a, b) [center]

Suppose that the distance of the points P from the given point C (a, b) is r



$$(x-a)^2+(y-b)^2=r^2$$

$(x-a)^2+(y-b)^2=r^2$ is the equation of the circle wit center (a, b) and radius r

If the center C is (0, 0) then the equation of the circle is $x^2+y^2=r^2$

For $(x-a)^2+(y-b)^2=r^2$

$$x^2-2ax+a^2+y^2-2by+b^2=r^2$$

$$x^2+y^2-2ax-2by+a^2+b^2-r^2=0$$

Suppose $-a=g, -b=f, C=a^2+b^2-r^2$

$$\Rightarrow C=g^2+f^2-r^2$$

The equation of the circle becomes

$$x^2+y^2+2gx+2fy+C=0$$

$x^2+y^2+2gx+2fy+C=0$ is the standard equation of a circle with center (-g, -f) and radius $r=\sqrt{g^2+f^2-C}$

Example I

Find the center and the radius of the circles below

(a) $(x-1)^2+(y-2)^2=9$
(b) $(x+1)^2+(y-3)^2=25$
(c) $x^2+y^2-4x-2y=4$
(d) $2x^2+2y^2-2x+2y=1$

Solution

(a) $(x-1)^2+(y-2)^2=9$

Comparing $(x-1)^2+(y-2)^2=9$ with $(x-a)^2+(y-b)^2=r^2$

$a=1, b=2, r^2=9$

$\Rightarrow$ The center is C (1, 2) and $r=3$

$(x-1)^2+(y-2)^2=9$ is a circle with radius 3 units and center (1, 2)

(b) $(x+1)^2+(y-3)^2=25$

Compare $(x+1)^2+(y-3)^2=25$ with

$$(x-a)^2+(y-b)^2=r^2$$

$$a=-1, b=3, r^2=25$$

$$r=5$$

The center is (-1, 3)

$\therefore (x+1)^2+(y-3)^2=25$ is the equation of the circle with center (-1, 3) and radius 5.

(c) $\boldsymbol{x^2+y^2-4x-2y=4}$

$x^2+y^2-4x-2y-4=0$

Comparing $x^2+y^2-4x-2y-4$ with

$x^2+y^2+2gx+2fy+C=0$

$$2g=-4, 2f=-2$$

$$C=-4$$

$$g=-2, f=-1$$

$$C=-4$$

Since the center is (-g, -f),

The center is (2, 1)

Radius $=\sqrt{g^2+f^2-C}$

$$r=\sqrt{(-2)^2+(-1)^2-(-4)}$$

$$r=\sqrt{4+1+4}$$

$$r=3$$

$x^2+y^2-4x-2y=4$ is a circle with radius 3 units and center (2, 1)

(d) $\boldsymbol{2x^2+2y^2-2x+2y=1}$

$$\frac{2x^2}{2}+\frac{2y^2}{2}-\frac{2x}{2}+\frac{2y}{2}=\frac{1}{2}$$

$$x^2+y^2-x+y-\frac{1}{2}=0$$

Comparing $x^2+y^2-x+y-\frac{1}{2}$ with

$x^2+y^2+2gx+2fy+C=0$

$2gx$ = -x, $2fy$ = y, $c=\frac{-1}{2}$

$g=\frac{-1}{2}$, $f=\frac{1}{2}$

Center (-g, -f)

Centre $\left(\frac{1}{2}, -\frac{1}{2}\right)$

Radius $= \sqrt{g^2 + f^2 - C}$

Radius $= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{-1}{2}\right)^2 - \frac{-1}{2}}$

Radius $= \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{2}}$

$= \sqrt{1}$

$= 1$

$2x^2 + 2y^2 - 2x + 2y = 1$ is the equation of the circle with center $\left(\frac{1}{2}, -\frac{1}{2}\right)$ and radius 1.

Example III

Find the equation of the circle with the following centers and radii

(a) Center (2, 3) radius 1

(b) Center (3, -4) radius 5

(c) Center $\left(\frac{-3}{2}, 2\right)$ and radius $\frac{1}{2}$

(d) Center $\left(\frac{-1}{4}, \frac{1}{2}\right)$ and radius $\frac{1}{2}\sqrt{2}$

(e) Center (0, -5) and radius 5

Solution

(a) Center (2, 3) radius 1

Given a circle of centre (a, b) and radius r. The equation of the circle is $(x - a)^2 + (y - b)^2 = r^2$.

Consider the equation of the circle

$(x - a)^2 + (y - b)^2 = r^2$ with center (a, b) and radius r

$$(x-2)^2 + (y-3)^2 = 1^2$$
$$(x-2)^2 + (y-3)^2 = 1$$
$$x^2 - 4x + 4 + y^2 - 6y + 9 = 1$$
$$x^2 + y^2 - 4x - 6y + 13 - 1 = 0$$
$$x^2 + y^2 - 4x - 6y + 12 = 0$$

The equation of the circle with center (2, 3) and radius 1 is $x^2 + y^2 - 4x - 6y + 12 = 0$

(b) Center (3, -4) radius 5

$$(x-a)^2 + (y-b)^2 = r^2$$
$$(x-3)^2 + (y--4)^2 = 5^2$$
$$(x-3)^2 + (y+4)^2 = 5^2$$
$$x^2 - 6x + 9 + y^2 + 8y + 16 = 25$$
$$x^2 + y^2 - 6x + 8y = 0$$

The equation of the circle with center (3, -4) and radius 5 is $x^2 + y^2 - 6x + 8y = 0$

(c) Center $\left(\frac{-3}{2}, 2\right)$ and radius $\frac{1}{2}$

$$\left(x - \frac{-3}{2}\right)^2 + (y-2)^2 = \left(\frac{1}{2}\right)^2$$

$$\left(x + \frac{3}{2}\right)^2 + (y-2)^2 = \frac{1}{4}$$
$$x^2 + 3x + \frac{9}{4} + y^2 - 4y + 4 = \frac{1}{4}$$

$$x^2 + y^2 + 3x - 4y + 6 = 0$$

Equation of the circle with center $\left(\frac{-3}{2}, 2\right)$ and radius $r = \frac{1}{2}$ is $x^2 + y^2 + 3x - 4y + 6 = 0$

(d) Center $\left(\frac{-1}{4}, \frac{1}{2}\right)$ and radius $\frac{1}{2}\sqrt{2}$

$$\left(x - \frac{-1}{4}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\sqrt{2}\right)^2$$
$$\left(x + \frac{1}{4}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}(2)$$
$$\left(x + \frac{1}{4}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}$$
$$x^2 + \frac{1}{2}x + \frac{1}{16} + y^2 - y + \frac{1}{4} = \frac{1}{2}$$
$$x^2 + y^2 + \frac{1}{2}x - y + \frac{1}{16} + \frac{1}{4} - \frac{1}{2} = 0$$
$$x^2 + y^2 + \frac{1}{2}x - y - \frac{3}{16} = 0$$

$$16x^2 + 16y^2 + 8x - 16y - 3 = 0$$

(e) Center (0, -5) and radius 5

$$(x-0)^2 + (y--5)^2 = 5^2$$
$$x^2 + (y+5)^2 = 5^2$$
$$x^2 + y^2 + 10y + 25 = 25$$
$$x^2 + y^2 + 10y = 0$$

Example III

State which of the following are equations of the circles

(a) $x^2 + y^2 - 5 = 0$

(b) $x^2 + y^2 + 10 = 0$

(c) $x^2 + y^2 + c = 0$

(d) $x^2 + y^2 + bxy = 1$

(e) $9x^2 + 9y^2 = 1$

(f) $7x^2 + 3x - y^2 + 2y = 16$

(g) $x^2 + 3x - y^2 = 7$

(h) $x^2 + y^2 + 2x - 8y = 1$

(i) $x^2 + 2xy + y^2 = 4$

Solution

(a) $x^2 + y^2 - 5 = 0$
$x^2 + y^2 = 5$
$(x-0)^2 + (y-0)^2 = \left(\sqrt{5}\right)^2$
$x^2 + y^2 - 5 = 0$ is an equation of a circle

(b) $x^2 + y^2 + 10 = 0$
$x^2 + y^2 = -10$
$(x-0)^2 + (y-0)^2 = \left(\sqrt{-10}\right)^2$
$x^2 + y^2 + 10 = 0$ is not an equation of a circle, since $r = \sqrt{-10}$ is not real.

(c) $x^2 + y^2 + c = 0$
$x^2 + y^2 = -c$
$(x-0)^2 + (y-0)^2 = \left(\sqrt{-c}\right)^2$
$x^2 + y^2 + c = 0$ is an equation of the circle when $c < 0$.

(d) $x^2 + y^2 + bxy = 1$

$$x^2 + y^2 + bxy = 1$$

Comparing $x^2 + y^2 + bxy = 1$ with $x^2 + y^2 + 2gx + 2fy + C = 0$
$\Rightarrow x^2 + y^2 + bxy = 1$ is not an equation of a circle because of the component of bxy

(e) $9x^2 + 9y^2 = 1$

$$\frac{9x^2}{9} + \frac{9y^2}{9} = \frac{1}{9}$$

$$x^2 + y^2 = \frac{1}{9}$$

$$(x-0)^2 + (y-0)^2 = \left(\frac{1}{3}\right)^2$$

$9x^2 + 9y^2 = 1$ is a circle

(f) $7x^2 + 3x - y^2 + 2y = 16$
Is not a circle because the co-efficient of $x^2 and\ y^2$ are not the same

(g) $x^2 + 3x - y^2 = 7$
Is not a circle because the co-efficient of x^2 and y^2 are not the same.

(h) $x^2 + y^2 + 2x - 8y = 1$
Is a circle

(i) $x^2 + 2xy + y^2 = 4$

Is not a circle

Example IV (UNEB Question)
The equation of the circle with center O is given by $x^2 + y^2 + Ax + By + C = 0$ where A, B and C are constants. Given that $4A = 3B$, $3A = 2C$ and $C = 9$

Determine
(a) The coordinates of the center of the circle
(b) The radius of the circle

Solution

4A = 3B …………………… (1)
3A = 2C …………………… (2)
C = 9 ……………………… (3)

Substituting eqn. (3) in eqn. (2)

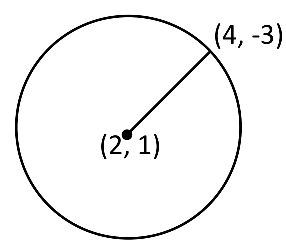
$$3A = 2(9)$$
$$3A = 18$$
$$A = 6$$

Substituting $A = 6$ in Eqn (1)
$4 \times 6 = 3B$
$B = 8$
$x^2 + y^2 + Ax + By + C = 0$
$x^2 + y^2 + 6x + 8y + 9 = 0$

Comparing $x^2 + y^2 + 6x + 8y + 9 = 0$ with $x^2 + y^2 + 2gx + 2fy + C = 0$
$2g = 6 \Rightarrow g = 3$
$2f = 8, \Rightarrow f = 4$
$C = 9$
Centre (-3, -4)
Radius = $\sqrt{g^2 + f^2 - C}$
Radius = $\sqrt{(-3)^2 + (-4)^2 - 9}$
$= \sqrt{9 + 16 - 9}$
$= 4$

Example V
Find the equation of a circle whose center is (2, 1) and passes through (4, -3)

Solution

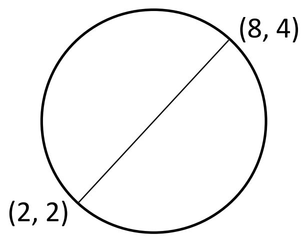


$$r = \sqrt{(2-4)^2 + (1--3)^2}$$
$$r = \sqrt{4+16}$$
$$r = \sqrt{20}$$
$$(x-a)^2 + (y-b)^2 = r^2$$
$$(x-2)^2 + (y-1)^2 = (\sqrt{20})^2$$
$$x^2 - 4x + 4 + y^2 - 2y + 1 = 20$$
$$x^2 + y^2 - 4x - 2y - 15 = 0$$

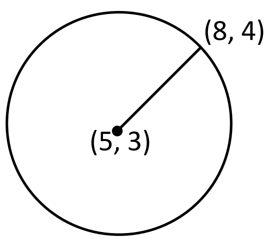
Example VI

The points (8, 4) and (2, 2) are end points of the diameter of the circle. Find the center, the radius and the equation of the circle

Solution



$$C\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$
$$C\left(\frac{2+8}{2}, \frac{2+4}{2}\right)$$
$$C(5, 3)$$



$$r = \sqrt{(5-8)^2 + (3-4)^2}$$
$$r = \sqrt{9+1}$$
$$r = \sqrt{10}$$
$$(x-a)^2 + (y-b)^2 = r^2$$
$$(x-5)^2 + (y-3)^2 = 10$$
$$x^2 - 10x + 25 + y^2 - 6y + 9 = 10$$
$$x^2 + y^2 - 10x - 6y + 34 - 10 = 0$$
$$x^2 + y^2 - 10x - 6y + 24 = 0$$

Example VI

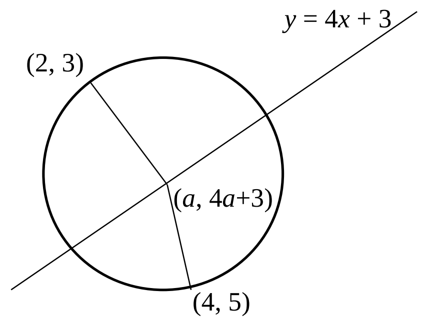
Find the equation of a circle passing through points (2, 3) and (4, 5) having its center on the line
$y = 4x + 3$

Solution

$$y = 4x + 3$$

Let the center be (x, y). Since it lies on the line $y = 4x + 3 = 0$, let the *x*-co-ordinate of the center be ***a***.
Then the *y*-co-ordinate

$$y = 4a + 3$$



$$r_1 = \sqrt{(a-2)^2 + (4a+3-3)^2}$$
$$r_1 = \sqrt{(a-2)^2 + (4a)^2}$$
$$r_1 = \sqrt{a^2 - 4a + 4 + 16a^2}$$
$$r_1 = \sqrt{17a^2 - 4a + 4}$$

$$r_2 = \sqrt{(a-4)^2 + (4a+3-5)^2}$$
$$r_2 = \sqrt{(a-4)^2 + (4a-2)^2}$$
$$r_2 = \sqrt{a^2 - 8a + 16 + 16a^2 - 16a + 4}$$
$$r_2 = \sqrt{17a^2 - 24a + 20}$$

$r_1 = r_2 = r$

$$\sqrt{17a^2 - 4a + 4} = \sqrt{17a^2 - 24a + 20}$$

$17a^2 - 4a + 4 = 17a^2 - 24a + 20$

$20a = 16$

$$a = \frac{16}{20} = \frac{4}{5}$$
$$r_1 = r = \sqrt{17 \times (\tfrac{4}{5})^2 - 4(\tfrac{4}{5}) + 4}$$
$$r = \sqrt{\tfrac{17\times16}{25} - \tfrac{16}{5} + 4}$$
$$r = \sqrt{\frac{292}{25}}$$

Centre$(a, 4a+3)$

centre $\left(\frac{4}{5}, \frac{4\times4}{5}+3\right)$

centre $\left(\frac{4}{5}, \frac{31}{5}\right)$

$$(x-\tfrac{4}{5})^2+(y-\tfrac{31}{5})^2=\left(\sqrt{\tfrac{292}{25}}\right)^2$$

$$(x-\tfrac{4}{5})^2+(y-\tfrac{31}{5})^2=\frac{292}{25}$$

$$x^2-\frac{8x}{5}+\frac{16}{25}+y^2-\frac{62y}{5}+\frac{961}{25}=\frac{292}{25}$$

$$x^2+y^2-\frac{8x}{5}-\frac{62y}{5}+\frac{977}{25}-\frac{292}{25}=0$$

$$x^2+y^2-\frac{8x}{5}-\frac{62y}{5}+\frac{685}{25}=0$$

$$\Rightarrow 5x^2+5y^2-8x-62y+137=0$$

Example

What is the equation of the circle whose center lies on the $x-2y+2=0$ which touches the positive axes.

Solution

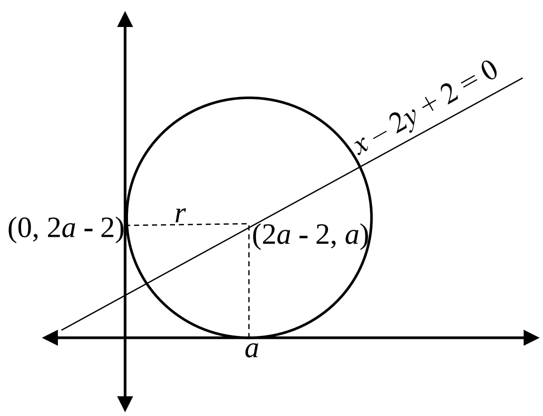
Let the y-coordinate of the centre be a

$x-2y+2=0$

$x-2a+2=0$

$x=2a-2$

$(2a-2, a)$



$2a-2=a$

$a=2$

The center is $(2, 2)$; radius $r=2$

$$(x-2)^2+(y-2)^2=2^2$$
$$x^2-4x+4+y^2-4y+4=4$$
$$x^2+y^2-4x-4y+4=0$$

Equation of circle passing through three points

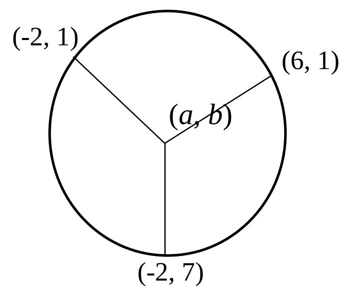
Example I

Find the equation of the circle passing through the points

(a) A(-2, 1) B(6, 1) and C(-2, 7)
(b) A(-1, 4) B(2, 5) and C(0, 1)
(c) A(3, 1) B(8, 2) and C(2, 6)
(d) A(5, 7) B(1, 6) and C(2, 2)

Solution

(a) A(-2, 1) B(6, 1) and C(-2, 7)



$$r_1=\sqrt{(a-{}^-2)^2+(b-1)^2}$$
$$r_2=\sqrt{(a-6)^2+(b-1)^2}$$
$$r_3=\sqrt{(a-{}^-2)^2+(b-7)^2}$$

Equating the radii; $r_1=r_2=r$

$$\sqrt{(a-{}^-2)^2+(b-1)^2}=\sqrt{(a-6)^2+(b-1)^2}$$
$$(a+2)^2+(b-1)^2=(a-6)^2+(b-1)^2$$
$$a^2+4a+4+b^2-2b+1$$
$$=a^2-12a+36+b^2-2b+1$$
$$a^2+b^2+4a-2b+5=a^2+b^2-12a-2b+37$$
$$4a-2b+5=-12a-2b+37$$
$$16a=32$$
$$a=2$$

Also $r_1=r_3=r$

$$\sqrt{(a--2)^2+(b-1)^2}=\sqrt{(a--2)^2+(b-7)^2}$$
$$(a+2)^2+(b-1)^2=(a+2)^2+(b-7)^2$$
$$a^2+4a+4+b^2-2b+1$$
$$=a^2+4a+4+b^2-14b+49$$
$$b^2-2b+1=b^2-14b+49$$
$$12b=48$$

$$b = 4$$

Center (a, b) = (2, 4)

$$radius = \sqrt{(a - -2)^2 + (b - 1)^2}$$
$$= \sqrt{(2 - -2)^2 + (4 - 1)^2}$$
$$= \sqrt{16 + 9}$$
$$= 5$$
$$(x - a)^2 + (y - b) = r^2$$
$$(x - 2)^2 + (y - 4)^2 = 5^2$$
$$x^2 - 4x + 4 + y^2 - 8y + 16 = 25$$
$$x^2 + y^2 - 4x - 8y + 20 = 25$$
$$x^2 + y^2 - 4x - 8y - 5 = 0$$

Alternatively; Consider the general equation of the circle $x^2 + y^2 + 2gx + 2fy + C = 0$

At (-2, 1)

$$-2^2 + 1^2 + 2g(-2) + 2f(1) + c = 0$$
$$-4g + 2f + c = -5 \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

At (6, 1), $6^2 + 1^2 + 2g(6) + 2f(1) + c = 0$

$$36 + 1 + 12g + 2f + c = 0$$
$$12g + 2f + c = -37 \ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}$$

At (-2, 7), $-2^2 + 7^2 + 2g(-2) + 2f(7) + c = 0$

$$4 + 49 - 4g + 14f + c = 0$$
$$-4g + 14f + c = -53 \ldots\ldots\ldots\ldots\ldots\ldots \text{(3)}$$

Solving equation (1), 2 and 3 simultaneously

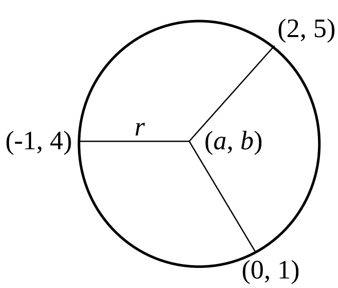
$$g = -2, f = -4, c = -5$$

Substituting $g = -2, f = -4, c = -5$ in the general equation of the circle $x^2 + y^2 + 2gx + 2fy + C = 0$

$$x^2 + y^2 + 2x(-2) + 2y(-4) + -5 = 0$$
$$x^2 + y^2 - 4x - 8y - 5 = 0$$

(As before)

(b) A(-1, 4) B(2, 5) and C(0, 1)



$$r_1 = \sqrt{(a - -1)^2 + (b - 4)^2}$$
$$r_2 = \sqrt{(a - 2)^2 + (b - 5)^2}$$
$$r_3 = \sqrt{(a - 0)^2 + (b - 1)^2}$$
$$r_1 = r_2 = r$$
$$\sqrt{(a+1)^2 + (b-4)^2} = \sqrt{(a-2)^2 + (b-5)^2}$$
$$(a + 1)^2 + (b - 4)^2 = (a - 2)^2 + (b - 5)^2$$
$$a^2 + 2a + 1 + b^2 - 8b + 16$$
$$= a^2 - 4a + 4 + b^2 - 10b + 25$$
$$2a - 8b + 17 = -4a - 10b + 29$$
$$6a + 2b = 12$$
$$\Rightarrow 3a + b = 6 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

Similarly, $r_1 = r_3 = r$

$$\sqrt{(a + 1)^2 + (b - 4)^2} = \sqrt{(a - 0)^2 + (b - 1)^2}$$
$$(a + 1)^2 + (b - 4)^2 = (a - 0)^2 + (b - 1)^2$$
$$a^2 + 2a + 1 + b^2 - 8b + 16 = a^2 + b^2 - 2b + 1$$
$$2a - 8b + 17 = -2b + 1$$
$$2a - 6b = -16$$
$$a - 3b = -8 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}$$

From eqn. (1)

$$b = 6 - 3a$$

Substituting $b = 6 - 3a$ in eqn. (2)

$$a - 3(6 - 3a) = -8$$
$$a - 18 + 9a = -8$$
$$10a = 10$$
$$a = 1$$
$$\Rightarrow b = 6 - 3 \times 1$$
$$b = 3$$

Center (1, 3)

$$r = \sqrt{(a + 1)^2 + (b - 4)^2}$$
$$r = \sqrt{(1 - -1)^2 + (3 - 4)^2}$$
$$r = \sqrt{4 + 1}$$
$$r = \sqrt{5}$$
$$(x - a)^2 + (y - b)^2 = r^2$$
$$(x - 1)^2 + (y - 3)^2 = (\sqrt{5})^2$$
$$x^2 - 2x + 1 + y^2 - 6y + 9 = 5$$
$$x^2 + y^2 - 2x - 6y + 5 = 0$$

Alternatively

$$x^2 + y^2 + 2gx + 2fy + C = 0$$

At (-1, 4): $(-1)^2 + 4^2 + 2g(-1) + 2f(4) + c = 0$

$$1 + 16 - 2g + 8f + c = 0$$
$$-2g + 8f + c = -17$$

At (2, 5): $(2)^2 + 5^2 + 2g(2) + 2f(5) + c = 0$

$$4 + 25 + 4g + 10f + c = 0$$
$$4g + 10f + c = -29 \ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}$$

At (0, 1): $0^2 + 1^2 + 2g(0) + 2f(1) + c = 0$

$$2f + c = -1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(3)}$$

Solving eqn. 1, 2 and 3 simultaneously

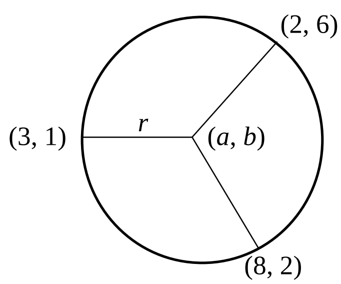
$$g = 1, f = -3, c = 5$$

Substituting $g = -1, f = -3, c = 5$ in the general equation of the circle $x^2 + y^2 + 2gx + 2fy + C = 0$

$$x^2 + y^2 + 2x(1) + 2y(-3) + 5 = 0$$

$$x^2 + y^2 - 2x - 6y + 5 = 0 \text{ (as before)}$$

(c) A(3, 1) B(8, 2) and C(2, 6)



$$r_1 = \sqrt{(a-3)^2 + (b-1)^2}$$
$$r_2 = \sqrt{(a-2)^2 + (b-6)^2}$$
$$r_2 = \sqrt{(a-8)^2 + (b-2)^2}$$
$$r_1 = r_2 = r$$
$$\sqrt{(a-3)^2 + (b-1)^2} = \sqrt{(a-2)^2 + (b-6)^2}$$

$$(a-3)^2 + (b-1)^2 = (a-2)^2 + (b-6)^2$$

$$a^2 - 6a + 9 + b^2 - 2b + 1 = a^2 - 4a + 4 + b^2 - 12b + 36$$
$$-6a - 2b + 10 = -4a - 12b + 40$$
$$-2a + 10b = 30$$
$$-a + 5b = 15$$
$$a = 5b - 15 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

Similarly; $r_1 = r_3 = r$

$$\sqrt{(a-3)^2 + (b-1)^2} = \sqrt{(a-8)^2 + (b-2)^2}$$

$$(a-3)^2 + (b-1)^2 = (a-8)^2 + (b-2)^2$$
$$a^2 - 6a + 9 + b^2 - 2b + 1 = a^2 - 16a + 64 + b^2 - 4b + 4$$
$$-6a - 2b + 10 = -16a - 4b + 68$$
$$10a + 2b = 58$$
$$\Rightarrow 5a + b = 29 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}$$

Substituting equation (1) in (2)

$$5(5b - 15) + b = 29$$

$$25b - 75 + b = 29$$
$$26b = 104$$
$$b = 4$$
$$a = 5b - 15$$
$$a = 5 \times 4 - 15$$
$$a = 5$$

Center (5, 4)

$$r = \sqrt{(5-3)^2 + (4-1)^2}$$
$$r = \sqrt{4 + 9}$$
$$r = \sqrt{13}$$
$$(x-5)^2 + (y-4)^2 = (\sqrt{13})^2$$
$$x^2 - 10x + 25 + y^2 - 8y + 16 = 13$$
$$x^2 + y^2 - 10x - 8y + 41 = 13$$
$$x^2 + y^2 - 10x - 8y + 28 = 0$$

Alternatively

$$x^2 + y^2 + 2gx + 2fy + C = 0$$

At (3, 1): $3^2 + 1^2 + 2g(3) + 2f(1) + c = 0$

$$9 + 1 + 6g + 2f + c = 0$$

$$6g + 2f + c = -10 \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

At (8, 2): $8^2 + 2^2 + 2g(8) + 2f(2) + c = 0$

$$64 + 4 + 16g + 4f + c = 0$$

$$16g + 4f + c = -68 \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}$$

At (2, 2): $2^2 + 2^2 + 2g(2) + 2f(2) + c = 0$

$$4 + 4 + 4g + 4f + c = 0$$
$$4g + 4f + c = -8 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(3)}$$

Solving eqn. 1, 2 and 3 simultaneously

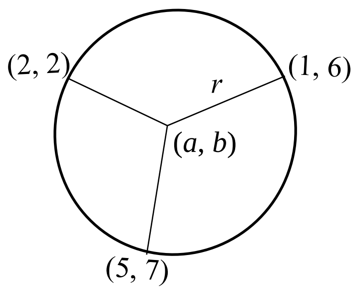
$$g = -5, f = -4, c = 28$$

Substituting the values of g, f and c in the general equation

$$x^2 + y^2 + 2gx + 2fy + C = 0$$
$$x^2 + y^2 + 2(-5)x + 2y(-4) + 28 = 0$$
$$x^2 + y^2 - 10x - 8y + 28 = 0$$

(As before)

(d) A(5, 7) B(1, 6) and C(2, 2)



$$r_1 = \sqrt{(a-2)^2 + (b-2)^2}$$

$$r_2 = \sqrt{(a-1)^2 + (b-6)^2}$$

$$r_3 = \sqrt{(a-5)^2 + (b-7)^2}$$

Equating the radii

$$r_1 = r_2 = r$$

$$\sqrt{(a-2)^2 + (b-2)^2} = \sqrt{(a-1)^2 + (b-6)^2} \text{ .... (1)}$$

$$\sqrt{(a-2)^2 + (b-2)^2} = \sqrt{(a-5)^2 + (b-7)^2} \text{ .... (2)}$$

From equation (1)

$$(a-2)^2 + (b-2)^2 = (a-1)^2 + (b-6)^2$$

$$a^2 - 4a + 4 + b^2 - 4b + 4$$

$$= a^2 - 2a + 1 + b^2 - 12b + 36$$

$$-4a - 4b + 8 = -2a - 12b + 37$$

$$12b - 4b - 4a + 2a = 37 - 8$$

$$8b - 2a = 29 \text{ .... (3)}$$

From eqn. (2)

$(a-2)^2 + (b-2)^2 = (a-5)^2 + (b-7)^2$

$$a^2 - 4a + 4 + b^2 - 4b + 4$$

$$= a^2 - 10a + 25 + b^2 - 14b + 49$$

$$-4a - 4b + 8 = -10a - 14b + 74$$

$$6a + 10b = 74 - 8$$

$$6a + 10b = 66$$

$$3a + 5b = 33 \text{ .... (4)}$$

Solving Eqn (3) and (4) simultaneously

$$\Rightarrow a = \tfrac{7}{2}, \text{ and } b = \tfrac{9}{2}$$

Center $\left(\frac{7}{2}, \frac{9}{2}\right)$

$$r = \sqrt{\left(\frac{7}{2} - 2\right)^2 + \left(\frac{9}{2} - 2\right)^2}$$

$$r = \sqrt{\frac{9}{4} + \frac{25}{4}}$$

$$r = \frac{\sqrt{34}}{2}$$

$$(x-a)^2 + (y-b)^2 = r^2$$

$$\left(x - \frac{7}{2}\right)^2 + \left(y - \frac{9}{2}\right)^2 = \frac{34}{4}$$

$$x^2 - 7x + \frac{49}{4} + y^2 - 9y + \frac{81}{4} = \frac{34}{4}$$

$$x^2 + y^2 - 7x - 9y + \frac{96}{4} = 0$$

$$x^2 + y^2 - 7x - 9y + 24 = 0$$

Alternatively

$$x^2 + y^2 + 2gx + 2fy + C = 0$$

At (5, 7): $5^2 + 7^2 + 2g(5) + 2f(7) + c = 0$

$$25 + 49 + 10g + 14f + c = 0$$

$$10g + 14f + c = -74 \text{ .... (1)}$$

At (1, 6): $1^2 + 6^2 + 2g(1) + 2f(6) + c = 0$

$$1 + 36 + 2g + 12f + c = 0$$

$$2g + 12f + c = -37 \text{ .... (2)}$$

At (2, 2): $2^2 + 2^2 + 2g(2) + 2f(2) + c = 0$

$$4g + 4f + c = -8 \text{ .... (3)}$$

Solving eqn. 1, 2 and 3 simultaneously

$$g = \frac{-7}{2}, f = \frac{-9}{2}, c = 24$$

Substituting g, f and c in the general equation of the circle

$$x^2 + y^2 + 2gx + 2fy + C = 0.$$

$$x^2 + y^2 + 2\left(\frac{-7}{2}\right)x + 2\left(\frac{-9}{2}\right)y + C = 0$$

$$x^2 + y^2 - 7x - 9y + 24 = 0$$

Parametric Equations of circle

Consider a circle $(x-a)^2 + (y-b)^2 = r^2$ the parametric equations of the above circles are $x - a = r\cos\theta$ and $y - b = r\sin\theta$

$$\therefore x = a + r\cos\theta \text{ and } y = b + r\sin\theta$$

Example I

Find the parametric equation of the circle $(x-4)^2 + (y-3)^2 = 4$

Solution

$$(x-4)^2+(y-3)^2=2^2$$
$$x-4=\ r\cos\theta$$
$$y-3=\ r\sin\theta$$
$$r=2$$
$$x-4=\ 2\cos\theta$$
$$y-3=\ 2\sin\theta$$
$$x=4+2\cos\theta$$
$$y=3+2\sin\theta$$

The parametric equations of the circle $(x-4)^2+(y-3)^2=4$ are

$$x=4+2\cos\theta$$
$$y=3+2\sin\theta$$

Example II

Find the parametric equations of the circle

$$(x+1)^2+(y-2)^2=9$$

Solution

Comparing $(x+1)^2+(y-2)^2=9$ with the equation of the circle $(x-a)^2+(y-b)^2=r^2$

$$a=-1, b=2, r=3$$
$$x+1=\ r\cos\theta$$
$$y-2=r\sin\theta$$
$$x+1=\ 3\cos\theta$$
$$y-2=3\sin\theta$$
$$x=\ 3\cos\theta-1$$
$$y=2+3\sin\theta$$

Example III

Find the parametric equations of the circle

$$x^2+y^2-4x-2y+1=0$$

Solution

$$x^2+y^2-4x-2y+1=0$$

By completing squares;s

$$(x^2-4x+4)-4+y^2-2y+1=0$$
$$(x-2)^2+(y-1)^2=4$$
$$x-2=\ r\cos\theta$$
$$y-1=\ r\sin\theta$$
$$x-2=\ 2\cos\theta$$
$$y-1=\ 2\sin\theta$$
$$x=\ 2+2\cos\theta$$
$$y=1+\ 2\sin\theta$$

Example IV

Find the parametric equation of a circle

$x^2+y^2-6x+4y-12=0$

Solution

$$x^2+y^2-6x+4y-12=0$$
$$x^2-6x+y^2+4y-12=0$$

By completing squares;

$$x^2-6x+9-9+y^2+4y^2+4-4-12=0$$
$$(x-3)^2+(y+2)^2=25$$
$$(x-a)^2+(y-b)^2=5^2$$
$$a=3, b=-2, r=5$$
$$x-3=\ r\cos\theta$$
$$y+2=r\sin\theta$$
$$x-3=\ 5\cos\theta$$
$$y+2=5\sin\theta$$
$$x=3+\ 5\cos\theta$$
$$y=-2+5\sin\theta$$

Example V

Find the Cartesian equation of the circle with parametric equations

$x=-2+3\cos\theta$

$y=3+3\sin\theta$

Solution

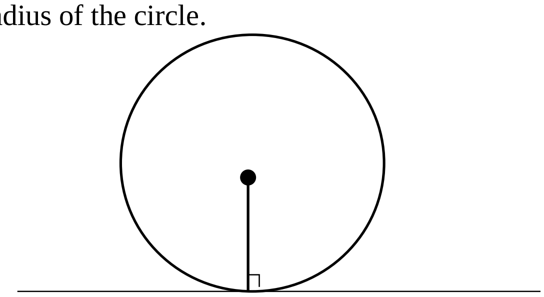
$$\frac{x+2}{3}=\cos\theta$$
$$\frac{y-3}{3}=\sin\theta$$
$$\cos^2\theta+\sin^2\theta=1$$
$$\frac{(x+2)^2}{9}+\frac{(y-3)^2}{9}=1$$

But $\frac{(x+2)^2}{9}+\frac{(y-3)^2}{9}=1$

$$(x+2)^2+(y-3)^2=9$$
$$x^2+4x+4+y^2-6y+9=9$$
$$x^2+4x+y^2-6y+4=0$$
$$x^2+y^2+4x-6y+4=0$$

Tangents to the Circle

A tangent to the circle is a line which touches the circle at only one point and makes 90^0 with the radius of the circle.



Length of the tangent to a circle

Example

Find the length of the tangent from (5, 7) to the circle

$$x^2 + y^2 - 4x - 6y + 9 = 0$$

Solution

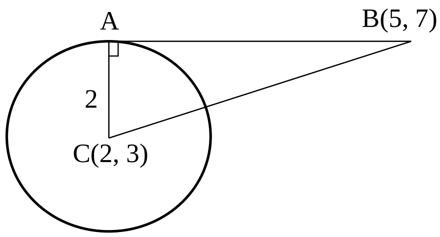
Comparing $x^2 + y^2 + 4x - 6y - 9 = 0$ with $x^2 + y^2 + 2gx + 2fy + C = 0.$

$$g = -2, f = -3, c = 9$$

Center (-*g*, -*f*)

Center (2, 3)

$$r = \sqrt{g^2 + f^2 - c}$$
$$r = \sqrt{4 + 9 - 9}$$
$$r = 2$$



$$CB = \sqrt{(2-5)^2 + (3-7)^2}$$
$$CB = \sqrt{9 + 16}$$
$$CB = 5$$
$$AB^2 + AC^2 = CB^2$$
$$AB^2 + 2^2 = 5^2$$
$$AB^2 = 5^2 - 2^2$$
$$AB^2 = 21$$
$$AB = \sqrt{21}\, units$$

The length of the tangent is $\sqrt{21}$ units

Example II

Find the lengths of the tangents from the given points to the following circles

(a) $x^2 + y^2 + 4x - 6y + 10 = 0, (0, 0)$

(b) $x^2 + y^2 + 6x + 10y - 2 = 0, (-2, 3)$

Solution

(a) $x^2 + y^2 + 4x - 6y + 10 = 0, (0, 0)$

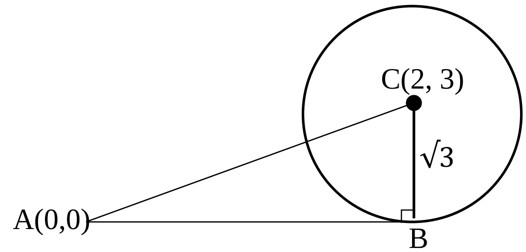
Comparing $x^2 + y^2 + 4x - 6y + 10 = 0$ with $x^2 + y^2 + 2gx + 2fy + C = 0.$

$$2gx = -4x$$
$$g = -2$$
$$2fy = -6y$$
$$f = -3$$

Center (-*g*, -*f*)

Center (2, 3)

$$r = \sqrt{g^2 + f^2 - c}$$
$$r = \sqrt{4 + 9 - 10}$$
$$r = \sqrt{3}$$



$$AC = \sqrt{(0-2)^2 + (0-3)^2}$$
$$AC = \sqrt{4 + 9}$$
$$AC = \sqrt{13}$$
$$AB^2 + CB^2 = AC^2$$
$$AB^2 + (\sqrt{3})^2 = (\sqrt{13})^2$$
$$AB^2 + 3 = 13$$
$$AB^2 = 10$$
$$AB = \sqrt{10}$$

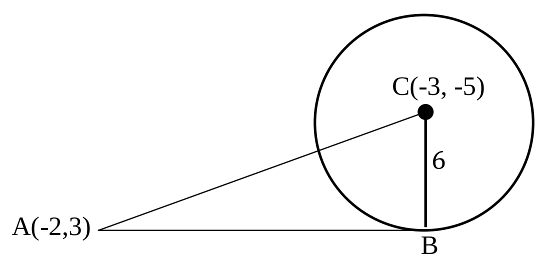
(b) $\boldsymbol{x^2 + y^2 + 6x + 10y - 2 = 0, (-2, 3)}$

Comparing $x^2 + y^2 + 6x + 10y - 2 = 0$ with $x^2 + y^2 + 2gx + 2fy + C = 0.$

$$g = 3, f = 5, c = -2$$

Center (-3, -5)

$$r = \sqrt{g^2 + f^2 - c}$$
$$r = \sqrt{9 + 25 - -2}$$
$$r = 6$$



$$AC = \sqrt{(-2 - -3)^2 + (3 - -5)^2}$$
$$AC = \sqrt{1 + 64}$$
$$AC = \sqrt{65}$$
$$AB^2 + 6^2 = (\sqrt{65})^2$$
$$AB^2 + 36 = 65$$
$$AB^2 = 65 - 36$$
$$AB^2 = 29$$
$$AB = \sqrt{29}$$

Alternative method of finding length of the tangent to a circle

The length of a tangent drawn from a point (x_1, y_1) to the circle $x^2+y^2+2gx+2fy+C=0$ is given by

$$L = \sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$$

$$= \sqrt{S_1} \text{ where } L = \text{length of the tangent}$$

The square of the length of the tangent from the point P is called a power point with respect to the circle.

Example I

Find the length of the tangent drawn from the point (5, 1) to the circle $x^2+y^2+6x-4y-3=0$

Solution

Comparing $x^2+y^2+6x-4y-3=0$ with $x^2+y^2+2gx+2fy+C=0$

$$g=3, f=-2, c=3$$

$$(x_1, y_1) = (5, 1)$$

$$L = \sqrt{x_1^{\,2}+y_1^{\,2}+2gx_1+2fy_1+c}$$

$$L = \sqrt{5^2+1^2+2g(5)+2f(1)+c}$$

$$L = \sqrt{5^2+1^2+2 \times 3(5)+2(-2)(1)-3}$$

$$L = \sqrt{25+1+30-4-3}$$

$$L = 7\ Units$$

Example II

If the length of the tangent from the point (f, g) to the circle $x^2+y^2=4$ is four times the length of the tangent from (f_1, g_1) it to the circle $x^2+y^2=4x$, show that $15f_1^{\,2}+15g_1^{\,2}-64f_1+4=0$

Solution

$$x^2+y^2-4=0$$
$$g=0, f=0, c=-4$$

$$L_1 = \sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$$

$$L_1 = \sqrt{g_1^2+f_1^2+0+0+-4}$$

$$L_1 = \sqrt{g_1^{\,2}+f_1^{\,2}-4}$$

For $x^2+y^2-4x=0$, $g=-2$ and $f=0$

$$L_2 = \sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$$

$$L_2 = \sqrt{g_1^{\,2}+f_1^{\,2}-2(-2)g_1+0+0}$$

$$L_2 = \sqrt{g_1^{\,2}+f_1^{\,2}+4g_1}$$

But $L_1 = 4L_2$

$$\sqrt{g_1^{\,2}+f_1^{\,2}-4} = 4\sqrt{g_1^{\,2}+f_1^{\,2}+4g_1}$$

$$g_1^{\,2}+f_1^{\,2}-4 = 16\left(g_1^{\,2}+f_1^{\,2}+4g_1\right)$$

$$g_1^{\,2}+f_1^{\,2}-4 = 16g_1^{\,2}+16f_1^{\,2}+64g_1$$

$$15g_1+15f_1+64g_1+4=0 \text{ (as required)}$$

Equation of a Tangent

Example I

Find the equation of the tangent to the circle $x^2+y^2+2x-2y-8=0\ at\ (2,2)$

Solution

$$\frac{d}{dx}(x^2+y^2+2x-2y-8) = \frac{d}{dx}(0)$$
$$2xdx+2ydy+2dx-2dy=0$$
$$2x+2y\frac{dy}{dx}+2-2\frac{dy}{dx}=0$$
$$\frac{dy}{dx}(2y-2) = -2-2x$$
$$\frac{dy}{dx} = \frac{-2-2x}{2y-2}$$

$$\frac{dy}{dx} = \frac{-2(1+x)}{2(y-1)}$$

$$\frac{dy}{dx} = \frac{-1(1+x)}{y-1}$$

$$\frac{dy}{dx} = \frac{-1-x}{y-1}$$

$$\left.\frac{dy}{dx}\right|_{(2,2)} = \frac{-1-2}{2-1}$$

$$\frac{dy}{dx} = -3$$

$$\frac{y-2}{x-2} = -3$$

$$y-2 = -3(x-2)$$

$$y-2 = -3x+6$$

$$y = -3x+8$$

Alternatively

Note: The equation of the tangent to the circle $x^2 + y^2 = a^2\ at\ (x_1, y_1)$ is $xx_1 + yy_1 = a^2$

The equation of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + C = 0$ at x_1, y_1 is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

We can now find the equation of the tangent to the $x^2 + y^2 + 2x - 2y - 8 = 0\ at\ (2, 2)$

Comparing $x^2 + y^2 + 2x - 2y - 8 = 0$ with $x^2 + y^2 + 2gx + 2fy + C = 0$

$$g = 1, f = -1, c = -8$$

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

$$x(2) + y(2) + g(x + 2) + f(y + 2) - 8 = 0$$

$$2x + 2y + 1(x + 2) - 1(y + 2) - 8 = 0$$

$$2x + 2y + x + 2 - y - 2 - 8 = 0$$

$$3x + y = 8$$

$$y = -3x + 8\ (as\ before\ )$$

Example II

Find the equation of the tangent to the circle $2x^2 + 2y^2 - 8x - 5y - 1 = 0$ at C (1, -1)

Solution

$$2x^2 + 2y^2 - 8x - 5y - 1 = 0$$
$$4xdx + 4ydy - 8dx - 5dy = 0$$
$$4x + 4y\frac{dy}{dx} - 8 - 5\frac{dy}{dx} = 0$$
$$\frac{dy}{dx}(4y - 5) = 8 - 4x$$
$$\frac{dy}{dx} = \frac{8 - 4x}{4y - 5}$$
$$\left.\frac{dy}{dx}\right|_{(1,-1)} = \frac{8-4(1)}{4\times{}^{-}1-5} = \frac{4}{-9}$$
$$\frac{y - -1}{x - 1} = \frac{-4}{9}$$
$$9(y + 1) = -4(x - 1)$$
$$9y + 9 = -4x + 4$$
$$9y = -4x - 5$$

Alternative method

From $2x^2 + 2y^2 - 8x - 5y - 1 = 0$,

$$x^2 + y^2 - 4x - \frac{5y}{2} - \frac{1}{2} = 0$$

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

$$\Rightarrow g = -2,\ f = \frac{-5}{4},\ c = \frac{-1}{2}$$

$x_1 = 1, y_1 = -1$

The equation of the tangent is given by

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

$$x(1) + y(-1) + -2(x + 1) + \frac{-5}{4}(y - 1) + \frac{-1}{2} = 0$$

$$x - y - 2x - 2 + \frac{-5y}{4} + \frac{5}{4} - \frac{1}{2} = 0$$

$$-x - \frac{9y}{4} - \frac{5}{4} = 0$$

$-4x - 9y - 5 = 0$

$4x + 9y + 5 = 0$

Example III

The tangent to the circle $x^2 + y^2 - 4x + 6y - 77 = 0$ at the point (5, 6) meets the axes at A and B. find A and B

Solution

$$x^2 + y^2 - 4x + 6y - 77 = 0$$
$$2xdx + 2ydy - 4dx + 6dy = 0$$
$$2x + 2y\frac{dy}{dx} - 4 + 6\frac{dy}{dx} = 0$$
$$(2y + 6)\frac{dy}{dx} = 4 - 2x$$
$$\frac{dy}{dx} = \frac{4 - 2x}{2y + 6}$$
$$\left.\frac{dy}{dx}\right|_{(5,6)} = \frac{4-2(5)}{2(6)+6}$$
$$\frac{dy}{dx} = \frac{-6}{18}$$
$$= \frac{-1}{3}$$
$$\frac{y - 6}{x - 5} = \frac{-1}{3}$$
$$3(y - 6) = -1(x - 5)$$
$$3y - 18 = -x + 5$$
$$3y = -x + 23$$
$$x + 3y = 23$$

Alternative method

Comparing $x^2 + y^2 - 4x + 6y - 77 = 0$ with $x^2 + y^2 + 2gx + 2fy + c = 0$

$g = -2, f = 3, c = -77$

$x_1 = 5,\ y_1 = 6$

The equation of the tangent is

$$x_1x + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

$\Rightarrow 5x + 6y + -2(x + 5) + 3(y + 6) - 77 = 0$

$5x + 6y - 2x - 10 + 3y + 18 - 77 = 0$

$3x + 9y = 69$

$\Rightarrow x + 3y = 23$, as before.

At the x- axis (A), $y = 0$

$$0 = -x + 23$$
$$x = 23$$

The tangent meets the x- axis at (23, 0)

At the y- axis (B), $x = 0$

$$3y = 23$$
$$y = \frac{23}{3}$$

The curve cuts the y- axis at $(0, \frac{23}{3})$

Example VII

Find the equation of the tangent to the circle $x^2 + y^2 - 30x + 6y + 109 = 0\ at\ (4, -1)$

Solution

$$x^2 + y^2 - 30x + 6y + 109 = 0$$
$$\frac{d}{dx}(x^2 + y^2 - 30x + 6y + 109) = \frac{d}{dx}(0)$$
$$2xdx + 2ydy - 30dx + 6dy = 0$$
$$2x + 2y\frac{dy}{dx} - 30 + 6\frac{dy}{dx} = 0$$
$$(2y + 6)\frac{dy}{dx} = 30 - 2x$$
$$\frac{dy}{dx} = \frac{30 - 2x}{2y + 6}$$
$$\frac{dy}{dx} = \frac{15 - x}{y + 3}$$
$$\left.\frac{dy}{dx}\right|_{(4,-1)} = \frac{15-4}{-1+3} = \frac{11}{2}$$
$$\frac{y - -1}{x - 4} = \frac{11}{2}$$
$$2y + 2 = 11x - 44$$
$$2y = 11x - 46$$
$$0 = 11x - 2y - 46$$

Alternatively

Given a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ the equation of the tangent at (x_1, y_1) is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

Comparing $x^2 + y^2 - 30x + 6y + 109 = 0$ with $x^2 + y^2 + 2gx + 2fy + c = 0$

$g = -15, f = 3, c = 109$, $x_1 = 4$, $y_1 = -1$

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$
$$\Rightarrow x(4) + y(-1) + -15(x+4) + 3(y + -1) + 109 = 0$$
$$4x - y - 15x - 60 + 3y - 3 + 109 = 0$$
$$-11x + 2y + 46 = 0$$
$$11x - 2y - 46 = 0\ (as\ before)$$

Example IV

Show that $y = mx + c$ is a tangent to the circle $x^2 + y^2 = a^2$ if $c^2 = a^2(1 + m^2)$

Solution

$$y = mx + c$$
$$x^2 + y^2 = a^2$$
$$c + (mx + c)^2 = a^2$$
$$x^2 + m^2x^2 + 2mcx + c^2 = a^2$$
$$x^2 + m^2x^2 + 2mcx + c^2 - a^2 = 0$$
$$(1 + m^2)x^2 + (2mc)x + c^2 - a^2 = 0$$
$$B^2 = 4AC\ (for\ tangency)$$
$$(2mc)^2 = 4(1 + m^2)[c^2 - a^2]$$
$$4m^2c^2 = 4(1 + m^2)[c^2 - a^2]$$
$$m^2c^2 = (1 + m^2)[c^2 - a^2]$$
$$m^2c^2 = c^2 - a^2 + m^2c^2 - m^2a^2$$
$$c^2 = a^2 + m^2a^2$$
$$c^2 = a^2(1 + m^2)$$

Example V

Show that the line $y = x + 1$ touches the circle $x^2 + y^2 - 8x - 2y + 9 = 0$.

Solution

$$x^2 + y^2 - 8x - 2y + 9 = 0$$
$$y = x + 1$$
$$x^2 + (x + 1)^2 - 8x - 2(x + 1) + 9 = 0$$
$$x^2 + x^2 + 2x + 1 - 8x - 2x - 2 + 9 = 0$$
$$2x^2 - 8x + 8 = 0$$
$$x^2 - 4x + 4 = 0$$

For the line to touch the circle

$$B^2 = 4AC$$
$$(-4)^2 = 4(4)(1)$$
$$16 = 16$$

The line y = x + 1 touches the circle $x^2 + y^2 - 8x - 2y + 9 = 0$

Note:

> If $y = mx + c$ is a line and $x^2 + y^2 = a^2$ is a circle then
> (i) $C^2 > a^2(1 + m^2)$ the line is a secant to

	the circle
(ii)	If $C^2 = a^2(1+m^2)$ the line touches the circle
(iii)	If $C^2 < a^2(1+m^2)$ the line doesn't meet the circle

Example VI

For what values of c will the line $y = 2x + c$ be tangent to the circle $x^2+y^2=5^2$

Solution

$$y = 2x + c$$
$$x^2+y^2=5^2$$
$$x^2+(2x+c)^2=5$$
$$x^2+4x^2+4xc+c^2=5$$
$$5x^2+4xc+c^2-5=0$$

For tangency $B^2=4AC$

$$(4c)^2=4(5)(c^2-5)$$
$$16c^2=20c^2-100$$
$$100=4c^2$$
$$25=c^2$$
$$c=5$$

Example VII

For what values of α, does the line $3x+4y=\alpha$ touch the circle $x^2+y^2-10x=0$?

Solution

$$3x+4y=\alpha \text{ ………….. (i)}$$
$$x^2+y^2-10x=10 \text{ ………. (ii)}$$

Substituting $y=\frac{\alpha-3x}{4}$ in Eqn (ii)

$$\Rightarrow x^2+\left(\frac{\alpha-3x}{4}\right)^2-10x=0$$
$$x^2+\frac{\alpha^2-6\alpha x+9x^2}{16}-10x=0$$
$$16x^2+\alpha^2-6x\alpha+9x^2-160x=0$$
$$25x^2+(-6\alpha-160)x+\alpha^2=0$$

For tangency $B^2=4AC$

$$(-6\alpha-160)^2=4\times 25\,(\alpha^2)$$
$$36\alpha^2+1920\alpha+25600=100\alpha^2$$
$$64\alpha^2-1920\alpha-25600=0$$
$$\alpha^2-30\alpha-400=0$$
$$(\alpha-40)(\alpha+10)=0$$
$$\alpha=40, \alpha=-10$$

Example VIII

Find the equation of the tangents to the circle $x^2+y^2-6x+4y-12=0$ which are parallel to the line $4x+3y+5=0$

Solution

Let the tangent be $y=mx+c$

Since the tangent is parallel to $4x+3y+5=0$ $(y=-\frac{4x}{3}-\frac{5}{3})$

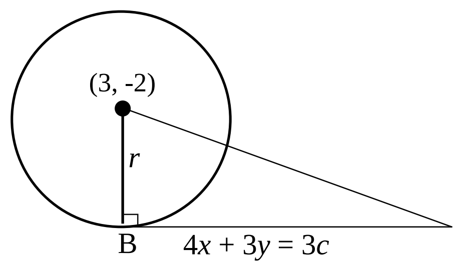
$$m=\frac{-4}{3}$$
$$y=-\frac{4x}{3}+c$$

$3y+4x=3c$ is equation of the tangent

Comparing $x^2+y^2-6x+4y-12=0$ with $x^2+y^2+2gx+2fy+c=0$

$$g=-3, f=2, c=-12$$

Center (+3, -2)



$$r=\sqrt{g^2+f^2-c}$$
$$r=\sqrt{9+4--12}$$
$$r=5$$

But we can obtain r using the formula for perpendicular distance of a point from a line

$$d=\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$$
$$r=\left|\frac{4(3)+3(-2)+-3c}{\sqrt{4^2+3^2}}\right|$$
$$5=\left|\frac{12-6-3c}{5}\right|$$
$$5=\pm\left(\frac{6-3c}{5}\right)$$
$$25=6-3c$$
$$3c=6-25$$
$$3c=-19$$
$$5=-\left(\frac{6-3c}{5}\right)$$

$$25 = -6 + 3c$$
$$31 = 3c$$

Since the tangents to the circle are given by

$$4x + 3y = 3c$$

$\Rightarrow$ The equations of the tangents are $4x + 3y = -19$ and $4x + 3y = +31$

Example ix

(i) Find the equation of the tangents to the circle $x^2 + y^2 - 2x - 4y - 4 = 0$ which are parallel to line $3x - 4y - 1 = 0$

(ii) Which are perpendicular to the line $3x–4y – 1=0$

Solution

Comparing $x^2 + y^2 + 2gx + 2fy + c = 0$ with $x^2 + y^2 - 2x - 4y - 4 = 0$

$$g = -1, f = -2, c = -4$$

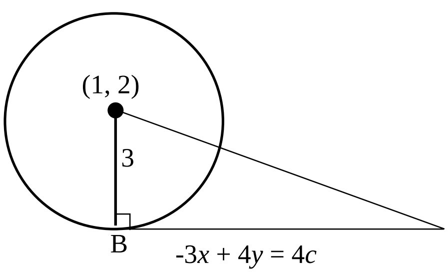
Center (1, 2)

$$r = \sqrt{g^2 + f^2 - c}$$
$$r = \sqrt{1 + 4 - -4}$$
$$r = 3$$
$$3x - 4y - 1 = 0$$
$$\frac{3x}{4} - \frac{1}{4} = y$$

Since the tangents are parallel to the line

$\Rightarrow$ $m = \frac{3}{4}$ for the tangent $y = mx + c$

$y = \frac{3x}{4} + C$, $(4y - 3x) = 4C$ are the equations of the tangents



$$r = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$
$$3 = \left|\frac{-3(1) + 4(2) + 4c}{\sqrt{-3^2 + 4^2}}\right|$$
$$3 = \left|\frac{5 + 4c}{5}\right|$$
$$3 = \pm\left(\frac{5 + 4c}{5}\right)$$
$$3 = \frac{5 + 4c}{5}$$
$$15 = 5 + 4c$$
$$4c = 10$$
$$3 = -\left(\frac{5 + 4c}{5}\right)$$
$$15 = -5 - 4c$$
$$4c = -20$$

Since the equations of the tangent that are parallel to the line $3x - 4y - 1 = 0$ are $-3x + 4y = 4c$

$\Rightarrow$ The required tangents are:

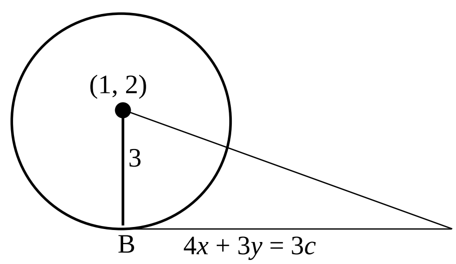
$$-3x + 4y = 10$$
$$-3x + 4y = -20$$

(ii) Let the tangents that are perpendicular to the line $3x - 4y - 1 = 0$ be $y = mx + c$

$$3x - 4y - 1 = 0$$
$$4y = 3x - 1$$
$$y = \frac{3x}{4} - \frac{1}{4}$$
$$m = \frac{3}{4}$$
$$\Rightarrow m_1 = \frac{-4}{3}$$
$$y = \frac{-4x}{3} + c$$
$$3y + 4x = 3c$$

Center (1, 2)

$$r = 3$$



$$r = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$
$$3 = \left|\frac{4(1) + 3(2) - 3c}{\sqrt{4^2 + 3^2}}\right|$$

$$3 = \left|\frac{4 + 6 - 3c}{5}\right|$$
$$3 = \frac{10 - 3c}{5}$$
$$15 = 10 - 3c$$
$$3c = -5$$
$$3 = \frac{-(10 - 3c)}{5}$$
$$15 = -10 + 3c$$
$$3c = 25$$

Since the tangent are;

$$3x + 4y = 3c$$
$$3x + 4y = 25$$
$$3x + 4y = -5$$

Director Circle

The locus of the point of intersection of two perpendicular tangents is called the Director circle of a given circle. The Director circle of a circle is a concentric circle having radius equal to $\sqrt{2}$ times the original radius.

Example

Find the equation of the director circle of the circle $(x - 2)^2 + (y + 1)^2 = 2$

Solution

$$(x - 2)^2 + (y + 1)^2 = 2$$

Center (2, -1)

Radius $r = \sqrt{2}$

The center of the director circle is (2, -1) and the radius of the director circle is

$$\sqrt{2} \times r$$
$$= \sqrt{2}\ X\ \sqrt{2}$$
$$= 2$$

The equation of the director circle is

$$(x - 2)^2 + (y + 1)^2 = 2^2$$
$$x^2 - 4x + 4 + y^2 + 2y + 1 = 4$$
$$x^2 + y^2 - 4x + 2y + 1 = 0$$

Example II

Find the equation of a director circle of the circle whose diameters are $2x - 3y + 12 = 0$ and $x + 4y - 5 = 0$ and has an area of 154.

Solution

$$2x - 3y + 12 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$
$$x + 4y - 5 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Solving eqn. (1) and (2) simultaneously

$$x = -3, y = 2$$

The center of a circle is (-3, 2)

$$\pi r^2 = 154$$
$$\frac{22}{7}\ X\ r^2 = 154$$
$$r = 7$$

Radius of the director circle is $7\sqrt{2}$

The equation of the director circle is

$$(x - 3)^2 + (y - 2)^2 = (7\sqrt{2})^2$$
$$(x - 3)^2 + (y - 2)^2 = 98$$
$$x^2 + 6x + 9 + y^2 - 4y + 4 = 98$$

Therefore, $x^2 + y^2 + 6x - 4y - 85 = 0$ is the equation of the director circle.

Equation of a common chord of two circles

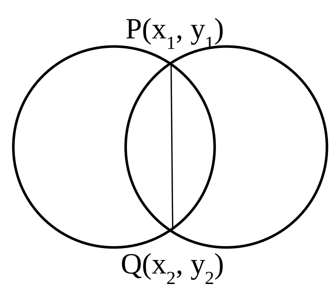
Let the equations of two intersecting circles be

$$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \ldots\ldots (1)$$

And

$$x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0 \ldots\ldots. (2)$$

Intersect at $P(x_1, y_1)$ and $Q(x_2, y_2)$



Now we observe from the figure that $P(x_1, y_1)$ lies on both given equations therefore, we get

$$x_1^2 + y_1^2 + 2g_1x_1 + 2f_1y_1 + c_1 = 0 \ldots\ldots (3)$$
$$x_1^2 + y_1^2 + 2g_2x_1 + 2f_2y_1 + c_2 = 0 \ldots.. (4)$$

Eqn. (3) $-$ Eqn. (4)

$$2(g_1 - g_2)x_1 + 2(f_1 - f_2)y_1 + c_1 - c_2 = 0 \ldots (5)$$

Again we observe from the above figure that point $Q(x_2, y_2)$ lies on both circles

$$x_2^2 + y_2^2 + 2g_1x_2 + 2f_1y_2 + c_1 = 0 \ldots... (6)$$
$$x_2^2 + y_2^2 + 2g_2x_2 + 2f_2y_2 + c_2 = 0 \ldots... (7)$$

Eqn. 6 – eqn. 7

$$2(g_1 - g_2)x_2 + 2(f_1 - f_2)y_2 + c_1 - c_2 = 0 \;\ldots\ldots\; (8)$$

From eqn. 5 and 8, it's evident that the points $P(x_1, y_1)$and $Q(x_2, y_2)$ lie on $2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0$ which is a linear equation in x and y.

Note: While finding the equation of the common chord of two given intersecting circle, we fast express each equation in the form

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Example

Determine the equation of the chord of the two intersecting circles $x^2 + y^2 - 4x - 2y - 31 = 0$ and $2\,x^2 + 2y^2 - 6x + 8y - 35 = 0$ and prove that the common chord is perpendicular to the line joining the two centres of the circles.

Solution

$x^2 + y^2 - 4x - 2y - 31 = 0$……….. (1)

$$2\,x^2 + 2y^2 - 6x + 8y - 35 = 0$$

$x^2 + y^2 - 3x + 4y - \frac{35}{2} = 0$………… (2)

Eqn. (1) – Eqn. (2)

$$-x - 6y + \frac{27}{2} = 0$$

$$-2x - 12y + 27 = 0$$

$$y = \frac{-2x}{12} + \frac{27}{12}$$

The equation of the chord:

The gradient of the chord is $\frac{-1}{6}$

Comparing $x^2 + y^2 - 4x - 2y - 31 = 0$ with $x^2 + y^2 + 2gx + 2fy + c = 0$

$$g = -2, f = -1$$

Center (2, 1) = C_1

Comparing $x^2 + y^2 - 3x + 4y - \frac{35}{2} = 0$ with $x^2 + y^2 + 2gx + 2fy + c = 0$

$$g = -\frac{3}{2}, f = 2$$

Center ($\frac{3}{2}, -2$) = C_2

The gradient joining the two centers

$$= \frac{-2 - 1}{\frac{3}{2} - 2}$$

$$= \frac{-3}{-\frac{1}{2}} = 6$$

Gradient of chord × gradient of line joining the two centres

$$6 \times \frac{-1}{6} = -1$$

The chord is perpendicular to the line joining the two centers

Example

Show that the common chord of the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 4x - 2y - 4 = 0$ passes through the origin

Solution

$x^2 + y^2 = 4$

$x^2 + y^2 - 4 = 0$ …………………….. (1)

$x^2 + y^2 - 4x - 2y - 4 = 0$ ………... (2)

Eqn. (2) – eqn. (1)

$$4x + 2y = 0$$

$y = -2x$ is the equation of the common chord

At (0, 0), $x = 0, y = 0$

$$0 = -2 \times 0$$

$$0 = 0$$

The common chord passes through the origin.

Example

Find the equation of the common chord of the circles

$x^2 + y^2 - 4x - 2y + 1 = 0$

$x^2 + y^2 + 4x - 16y - 10 = 0$

Solution

$x^2 + y^2 - 4x - 2y + 1 = 0$ ………… (1)

$x^2 + y^2 + 4x - 16y - 10 = 0$ ………. (2)

Eqn. (2) – eqn. (1)

$+8x - 14y - 11 = 0$

$$14y = -11 + 8x$$

Example

Find the point of intersection of the two circles

$x^2 + y^2 - 2x - 6y + 6 = 0$ and

$x^2 + y^2 - 6x - 6y + 14 = 0$

Solution

When we are finding the point of intersection, we first find the equation of the common chord and then we solve it simultaneously with one of the equations of the circles

$x^2 + y^2 - 2x - 6y + 6 = 0$ ……….. (1)

$x^2 + y^2 - 6x - 6y + 14 = 0$ ……… (2)

Eqn. (1) – eqn. (2)

$$4x - 8 = 0$$

$$x = 2$$

$x = 2$ is the equation of the common chord

Substituting $x = 2$, in eqn. (1)

$$2^2 + y^2 - 2 \times 2 - 6y + 6 = 0$$

$$y^2 - 6y + 6 = 0$$

$$y = \frac{6 \pm \sqrt{36 - 4 \times 1 \times 6}}{2 X 1}$$

$$y = \frac{6 \pm \sqrt{12}}{2}$$

$$y = 3 \pm \sqrt{3}$$

$$\left(2, 3 - \sqrt{3}\right) and \left(2, 3 + \sqrt{3}\right)$$

The point of intersection of both circles is $\left(2, 3 - \sqrt{3}\right) and \left(2, 3 + \sqrt{3}\right)$

Example

Find the point of intersection of the circles

$x^2 + y^2 + 2x + 2y - 23 = 0$ and

$x^2 + y^2 - 10x - 7y + 31 = 0$

Solution

$x^2 + y^2 + 2x + 2y - 23 = 0$…………. (1)

$x^2 + y^2 - 10x - 7y + 31 = 0$ ………. (2)

Eqn (1) – Eqn (2)

$$12x + 9y - 54 = 0$$

$$4x + 3y = 18$$

$$y = \frac{18 - 4x}{3}$$

$$x^2 + \left(\frac{18 - 4x}{3}\right)^2 + 2x + 2\left(\frac{18 - 4x}{3}\right) - 23 = 0$$

$$x^2 + \frac{324 - 144x + 16x^2}{9} + 2x + \frac{36 - 8x}{3} - 23 = 0$$

$$9x^2 + 16x^2 - 144x + 18x + 108 - 24x + 324 + 108 - 207 = 0$$

$$25x^2 - 150x + 225 = 0$$

$$x^2 - 6x + 9 = 0$$

$$(x - 3)^2 = 0$$

$$x = 3$$

$$y = \frac{18 - 4 \times 3}{3}$$

$$y = 2$$

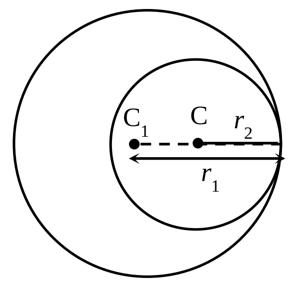
The point of intersection is (3, 2)

Types of intersecting circles

(1) Touching each other internally

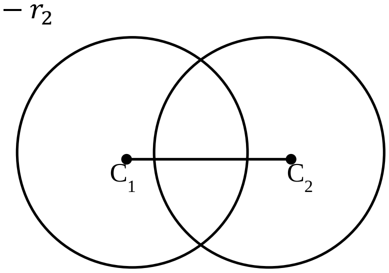
Two circles touch each other internally if the distance between their centers is equal to the distance between their radii

$$C_1C_2 = r_1 - r_2$$

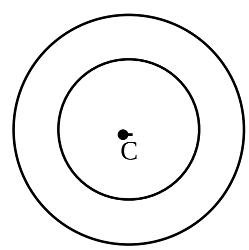


$$C_1C_2 = r_1 - r_2$$

(2) Circle intersect at two distinct points when $C_1C_2 < r_1 - r_2$

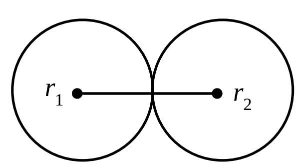


(3) Concentric circles



These are circles with the same center.

(4) Circle which touches each other externally if the distance between their centers is equal to the sum of their radii.



Example

Prove that the circles $x^2+y^2-10x-7y+31=0$ and $x^2+y^2+2x+2y-23=0$ touch each other externally.

Solution

$$x^2+y^2-10x-7y+31=0$$
$$x^2+y^2+2x+2y-23=0$$

Comparing $x^2+y^2-10x-7y+31=0$ with $x^2+y^2+2gx+2fy+c=0$

$$g=-5, f=-\frac{7}{2}, c=31$$

Center $\left(5,\frac{7}{2}\right)$

Radius $=\sqrt{g^2+f^2-c}$

$$r=\sqrt{(-5)^2+\left(\frac{-7}{2}\right)^2-31}$$

$$r=\frac{5}{2}$$

Comparing $x^2+y^2+2x+2y-23=0$ with $x^2+y^2+2gx+2fy+c=0$

$$g=1, f=1, c=-23$$

Center (-1, -1)

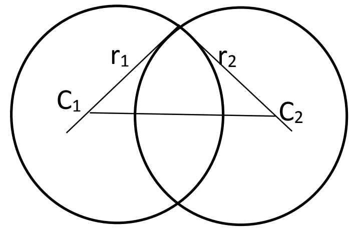
$$radius=\sqrt{(-1)^2+(1)^2--23}$$
$$r=5$$
$$C_1\left(5,\frac{7}{2}\right)\ and\ C_2(-1,-1)$$
$$C_1C_2=\sqrt{(5-{}^-1)^2+\left(\tfrac{7}{2}--1\right)^2}$$
$$=\sqrt{36+\frac{81}{4}}$$
$$C_1C_2=\frac{15}{2}$$
$$C_1C_2=7.5$$
$$r_1+r_2=7.5$$

Since $C_1C_2=r_1+r_2$

The two circles touch each other externally

Orthogonal Circle

Two circles are said to be orthogonal if the tangents at their point of intersection cut at right angles as illustrated below.



$$r_1^2+r_2^2=(C_1C_2)^2$$

Example

Prove that the circles $x^2+y^2+4x-2y-11=0$ and $x^2+y^2-4x-8y+11=0$ are orthogonal

Solution

Comparing $x^2+y^2+2gx+2fy+c=0$ with $x^2+y^2+4x-2y-11=0$

$$g=2, f=-1, c=-11$$

Center $C_1(-2,1)$

$$r_1=\sqrt{2^2+-1^2--11}$$
$$r_1=4$$

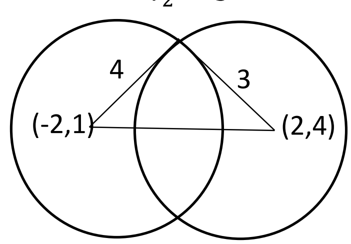
Similarly

Comparing $x^2+y^2-4x-8y+11=0$ with $x^2+y^2+2gx+2fy+c=0$

$$g=-2, f=-4, c=11$$

Center $C_2(2,4)$

$$r_2=\sqrt{(-2)^2+(-4)^2-11}$$
$$r_2=\sqrt{4+16-11}$$
$$r_2=3$$



$$C_1C_2=\sqrt{(-2-2)^2+(1-4)^2}$$
$$\overline{C_1C_2}=5$$

Since $r_1^2+r_2^2=\overline{C_1C_2}^2$

The two circles are orthogonal

Example (UNEB Question)

13. a) Form the equation of a circle that passes through the points A (-1, 4), B (2, 5) and C (0, 1)

b) The line $x + y = c$ is a tangent to the circle $x^2 + y^2 - 4y + 2 = 0$. Find the coordinates of the point of contact of the tangent for each value of c.

Solution

General equation of the circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

At A(-1, 4),

$$-2g + 8f + c = -17 \text{.............................. (i)}$$

At B(2, 5);

$$4g + 10f + c = -29 \text{.............................. (ii)}$$

At C(0, 1):

$$2f + c = -1 \text{.. (iii)}$$

2 Eqn (i) + Eqn (ii)

$10c = 50$

$c = 5$

From Eqn (iii);

$2f + 5 = -1$

$2f = -6$

$f = -6$

From Eqn (i)

$-2g + 8(-3) + 5 = -17$

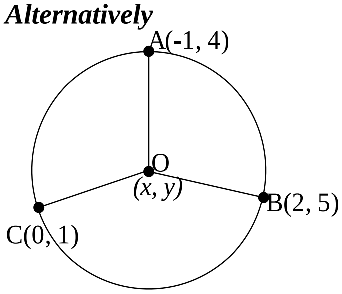
$-2g = 24 - 17 - 5$

$g = -1$

Hence the equation of the circle is

$\mathbf{x^2 + y^2 - 2x - 6y + 5 = 0}$

Alternatively



$(0 - x)^2 + (1 - y)^2 = (-1 - x)^2 + (4 - y)^2$

$$3y - x = 8 \text{.................................. (i)}$$

Also

$(0 - x)^2 + (1 - y)^2 = (2 - x)^2 + (5 - y)^2$

$$2y + x = 7 \text{...................................... (ii)}$$

Eqn (i) + Eqn (ii)

$5y = 15$

$y = 3$

$3(3) - x = 8$

$x = 1$

Centre of the circle = (1, 3) and the radius is

$\sqrt{(0-1)^2 + (1-3)^2} = \sqrt{5}$

Equation of the circle is $\mathbf{x^2 + y^2 - 2x - 6y + 5 = 0}$

b) $x^2 + y^2 - 4y + 2 = 0$,

And $y = c - x$

At the point of contact,

$x^2 + (c - x)^2 - 4(c - x) + 2 = 0$

$2x^2 + (4 - 2c)x + (c^2 - 4c + 2) = 0$

For tangency, $b^2 = 4ac$

$(4 - 2c)^2 = 4 \times 2 \times (c^2 - 4c + 2)$

$4(2 - c)^2 = 8(c^2 - 4c + 2)$

$(2 - c)^2 = 2(c^2 - 4c + 2)$

$4 - 4c + c^2 = 2c^2 - 8c + 4$

$c^2 - 4c = c(c - 4) = 0$

Either $c = 0$ or $c = 4$

If $c = 0$, $y = -x$

$\Longrightarrow \quad x^2 + x^2 + 4x + 2 = 0$

$2x^2 + 4x + 2 = 0$

$x^2 + 2x + 1 = (x + 1)^2 = 0$

$\Longrightarrow \quad x = -1$

Therefore $y = 1$

The point is (-1, 1)

If $c = 4$, $y = 4 - x$

$(4 - x)^2 + x^2 - 4(4 - x) + 2 = 0$

$16 - 8x + x^2 + x^2 - 16 + 4x + 2 = 0$

$2x^2 - 4x + 2 = 0$

$x^2 - 2x + 1 = (x - 1)^2 = 0$

$x = 1$, $y = 3$

The point is (1, 3)

Example (UNEB Question)

a) Find the equation of a circle which passes through the points (5, 7), (1, 3) and (2, 2).

b) i) If $x = 0$ and $y = 0$ are tangents to the circle, $x^2 + y^2 + 2gx + 2fy + c = 0$, show that $c = g^2 = f^2$.

ii) Given that the line $3x - 4y + 6 = 0$ is also a tangent to the circle in (b) (i) above, determine the equation of the circle lying in the first quadrant. (06 *marks*)

Solution

(a) The equation of the circle is given by;

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Substituting for (5, 7),

$25 + 49 + 10g + 14f + c = 0$

$74 + 10g + 14f + c = 0$

$10g + 14f + c = -74$ …………..(i)

Substituting for (1, 3)

$1 + 9 + 2g + 4f + c = 0$

$2g + 6f + c = -10$ ……………(ii)

Substituting for (2, 2)

$4 + 4 + 4g + 4f + c = 0$

$$4g + 4f + c = -8 \text{........................... (iii)}$$

Eqn (i) − Eqn (ii)

$6g + 10f = -64$

$$g + 6f = -8 \text{.................................. (iv)}$$

Eqn (i) − Eqn (iii)

$6g + 10f = -66$

$$3g + 5f = -33 \text{................................ (v)}$$

3 Eqn (iv) − Eqn (v)

$3g + 3f = -24$

$3g + 5f = -33$

$-2f = 9$

$$f = \frac{-9}{2}$$

Substituting for f in Eqn (iv)

$$g = \frac{9}{2} - 8 = -7/2$$

Substituting for f and g in Eqn (iii)

$$4\left(-7/2\right) + 4\left(\frac{-9}{2}\right) + c = -8$$

$-28 - 36 + 2c = -16$

$2c = 64 - 16$

$$c = \frac{48}{2} = 24$$

The equation of the circle is $x^2 + y^2 - 7x - 9y + 24 = 0$

b) Given $x^2 + y^2 + 2gx + 2fy + c = 0$

When $y = 0$, $\quad x^2 + 2gx + c = 0$

For tangency, $b^2 = 4ac$

$(2g)^2 = 4c$

$4g^2 = 4c$

$g^2 = c$

When $x = 0$, $y^2 + (2f)^2 + c = 0$

For tangency, $b^2 = 4ac$

$(2f)^2 = 4c$

$4f^2 = 4c$

$f^2 = c$

Hence $c = g^2 = f^2$

ii) From the line $3x - 4y + 6 = 0$

$4y = 3x + 6$

$$y = \frac{3x+6}{4}$$

$$y^2 = \frac{(3x+6)^2}{16}$$

Substituting for y and y^2 in the equation

$x^2 + y^2 + 2gx + 2fy + c = 0$

$$x^2 + \frac{(3x+6)^2}{16} + 2fx + 2f\left(\frac{3x+6}{4}\right) + f^2 = 0$$

$16x^2 + (3x + 6)^2 + 32fx + 8f(3x + 6) + 16f^2 = 0$

$16x^2 + 9x^2 + 36x + 36 + 32fx + 24fx + 48f + 32f^2 = 0$

$25x^2 + (36 + 54f)x + (36 + 48f + 16f^2) = 0$

For tangency, $\quad b^2 = 4ac$

$(36 + 54f)^2 = 4 \times 24(36 + 48f + 16f^2)$

$(36 + 54f)^2 = 100(36 + 48f + 16f^2)$

By opening brackets and simplifying we obtain

$2f^2 - f - 3 = 0$

$2f^2 - 3f + 2f - 3 = 0$

$f(2f - 3) + 1(2f - 3) = 0$

$(2f - 3)(f + 1) = 0$

Either $\quad 2f - 3 = 0$

$2f = 3$

$f = 3/2$

Or $\quad f + 1 = 0$

$f = -1$

Now $\quad f = g$

$\Longrightarrow \quad g = 3/2$ or -1

Centre of the circle is $(-g, -f)$. Since it is in the first quadrant, then the centre is (1, 1)

But $c = g^2 = f^2 = 1$

The equation of the circle is $x^2 + y^2 - 2x - 2y + 1 = 0$

LOCI

When a point moves in the plane according to some given conditions, the path along which it moves is called a locus.

A locus is a set of points which satisfy certain geometric conditions. Many geometric shapes are most naturally and easily described as a loci. For example a circle is a set of points in the plane which are fixed at distance r from a given point *P* (center).

Problems involving describing a certain locus is often solved by explicitly finding equations for the coordinates of the points in the locus. Here is a step by step procedure for finding plane loci

Step I: If possible, choose a coordinate system that will make computations and equations as simple as possible

Step II: Write the given conditions in mathematics from involving the coordinates (x, y).

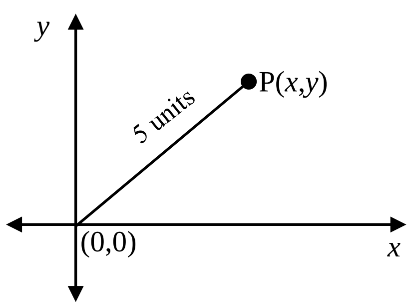
Step III: Simplify the equations.

Step IV: Identify the shape out by the equations.

Example I

Find the locus of a circle with center at the origin and radius 5 units.

Solution



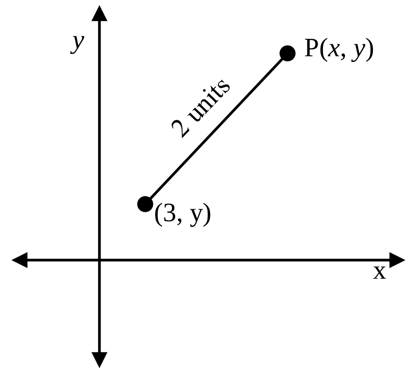
$$\sqrt{(x-0)^2+(y-0)^2}=5$$
$$x^2+y^2=5$$

The locus is $x^2 + y^2 = 5$

Example II

What is the locus of a point which moves so that its distance from the point (3, 1) is 2 units?

Solution



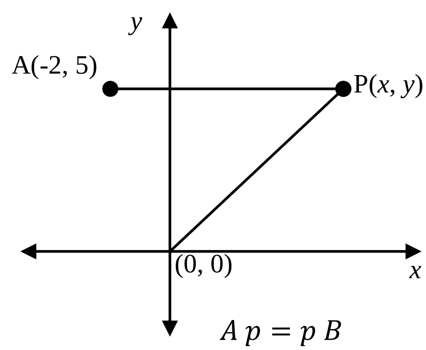
$$\sqrt{(x-3)^2+(y-1)^2}=2$$
$$(x-3)^2+(y-1)^2=2^2$$
$$x^2-6x+9+y^2-2y+1=4$$
$$x^2+y^2-6x-2y+6=0$$

The locus is a circle with center (3, 1) and radius 2

Example III

What is the locus of point which is equidistant from the origin (0, 0) and the point (-2, 5)

Solution



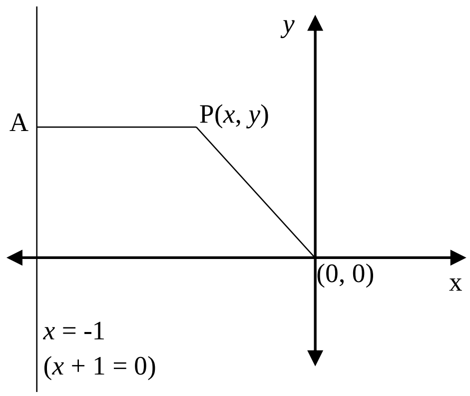
$$A\,p=p\,B$$
$$\sqrt{(x+2)^2+(y-5)^2}=\sqrt{x^2+y^2}$$
$$(x+2)^2+(y-5)^2=x^2+y^2$$
$$x^2+4x+4+y^2-10y+25=x^2+y^2$$
$$4x-10y+29=0$$

The locus is a straight line with a positive gradient.

Example IV

Find the locus of a point which is equidistant from the line $x=-1$ and the origin.

Solution



The perpendicular distance of the line $ax + by + c=0$ from (x_1, y_1) is $\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$

Comparing $x + 1 = 0$ with $ax + by + c = 0$

$$a = 1, b = 0, c = 1$$

The perpendicular distance of the point (x, y) from the line $x + 1 = 0$ is

$$AP = \left|\frac{1(x) + 0(y) + 1}{\sqrt{(1)^2 + 0^2}}\right| = x + 1$$

$$Ap = x + 1$$

$$pB = \sqrt{x^2 + y^2}$$

$$Ap = pB$$

$$x + 1 = \sqrt{x^2 + y^2}$$

$$(x + 1)^2 = x^2 + y^2$$

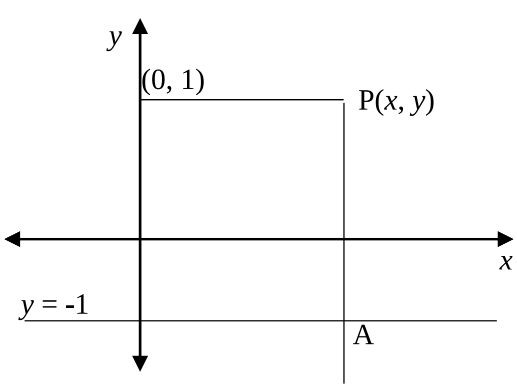
$$x^2 + 2x + 1 = x^2 + y^2$$

$$y^2 = 2x + 1$$

The locus is a parabola

Example

Find the locus of a point which is equidistant from the point (0, 1) and the line $y = -1$



Comparing $y + 1 = 0$ $(y = -1)$ with general equation of the line $ax + by + c = 0$

$$a = 0, b = 1, c = 1$$

The perpendicular distance of the point $P(x, y)$ from the line is $y = -1$ $(y + 1 = 0)$

$$\Rightarrow \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$

$$\left|\frac{0(x) + 1(y) + 1}{\sqrt{0^2 + 1^2}}\right| = y + 1$$

$$AP = y + 1$$

$$PB = \sqrt{(x - 0)^2 + (y - 1)^2}$$

$$PB = \sqrt{(x^2 + (y - 1)^2}$$

$$AP = PB$$

$$y + 1 = \sqrt{x^2 + (y - 1)^2}$$

$$(y + 1)^2 = x^2 + (y - 1)^2$$

$$y^2 + 2y + 1 = x^2 + y^2 - 2y + 1$$

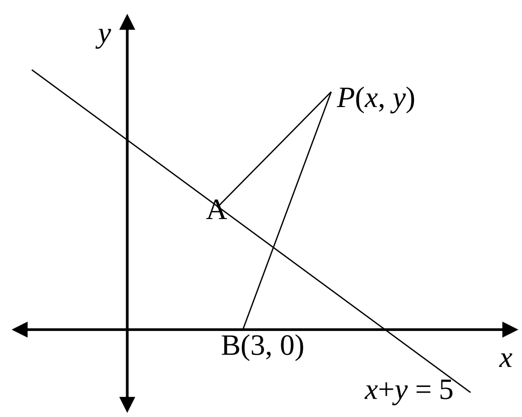
$$x^2 = 4y$$

The locus is a parabola

Example VI (UNEB Question)

A point p is twice as far from the line $x + y = 5$ as from the point (3, 0). Find the locus of P.

Solution



$AP = PB$

The perpendicular distance of point (x, y) from the line $x + y - 5 = 0$ is

$$\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$

$$\Rightarrow \left|\frac{1(x)+1(y)-5}{\sqrt{1^2+1^2}}\right|$$

$$AP = \left|\frac{x + y - 5}{\sqrt{2}}\right|$$

$$PB = \sqrt{(x - 3)^2 + y^2}$$

$$AP = 2PB$$

$$\frac{x + y - 5}{\sqrt{2}} = 2\sqrt{(x - 3)^2 + y^2}$$

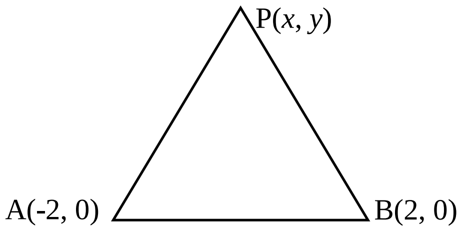
$$x + y - 5 = 2\sqrt{2}\left[\sqrt{(x - 3)^2 + y^2}\right]$$

$$(x + y - 5)^2 = 8[(x - 3)^2 + y^2]$$

$$7x^2+7y^2-2xy-58x+10y+47=0$$

Example VII

Find the locus of a point which moves so that the sum of squares of its distances from $(-2,0)$ and (2, 0) is 26



Solution

$$(\overline{AP})^2+(\overline{PB})^2=26$$

$$\left(\sqrt{(x+2)^2+(y-0)^2}\right)^2+\left(\sqrt{(x-2)^2+y^2}\right)^2=26$$

$$(x+2)^2+y^2+(x-2)^2+y^2=26$$

$$x^2+4x+4+y^2+x^2-4x+4+y^2$$
$$=26$$

$$2x^2+2y^2=18$$

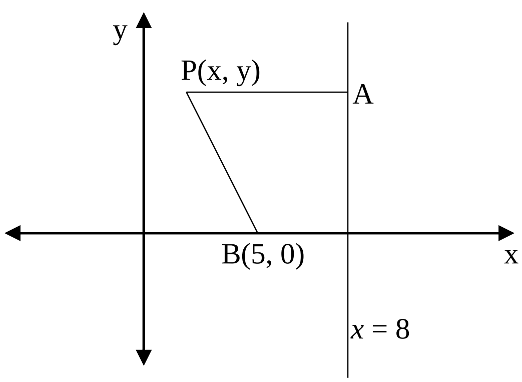
$$x^2+y^2=9$$

The locus is a circle with center $(0,0)$ and radius 3 units

Example VIII

Find the locus of the point P which moves so that its distance from the point $(5,0)$ is a half its distance from the line $x-8=0$

Solution



$$PB=\frac{1}{2}PA$$

$$2PB=PA$$

$$PA=8-x$$

$$PB=\sqrt{(x-5)^2+y^2}$$

$$2PB=PA$$

$$2\sqrt{(x-5)^2+y^2}=(8-x)$$

$$4[(x-5)^2+y^2]=(8-x)^2$$

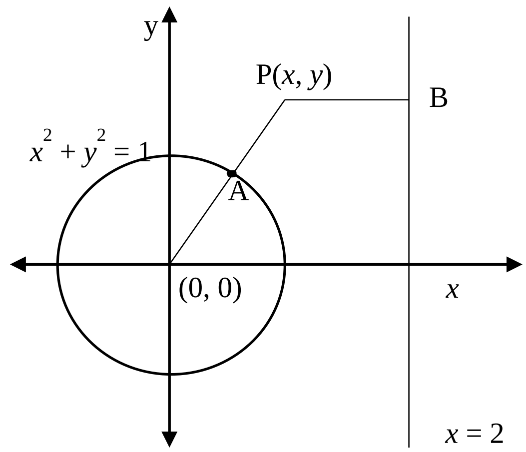
$$4[x^2-10x+25+y^2]$$
$$=64-16x+x^2$$

$$3x^2+4y^2-24x+36=0$$

Example IX

Find the locus of a point which is equidistant from the line x = 2 and the circle $x^2+y^2=1$

Solution



$$AP=PB$$

$$\sqrt{x^2+y^2}-1=2-x$$

$$\sqrt{x^2+y^2}=3-x$$

$$x^2+y^2=(3-x)^2$$

$$x^2+y^2=9-6x+x^2$$

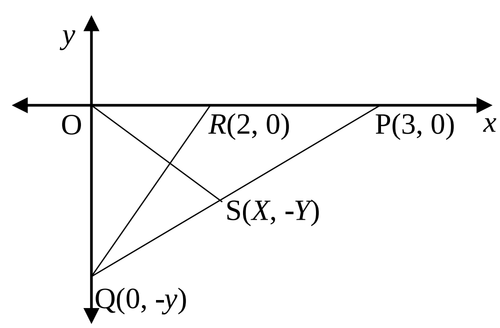
$$y^2+6x-9=0$$

The locus is a parabola

Example X

The points $R(2,\ 0)$ and $P(3,0)$ lie on the x-axis and $Q(0,\ -y)$ on the y- axis. The perpendicular from the origin to QR meets PQ at point S(X, -Y). Find the locus of S.

Solution



Since S is in terms of X and Y

Then the locus of S must be in terms of X and Y

From the figure above,

(The gradient of PQ) = (Gradient of SQ)

$$\frac{0--y}{3-0}=\frac{-Y--y}{X-0}$$

$$\frac{y}{3}=\frac{-Y+y}{X}$$

$$yX=-3Y+3y \text{ …………….. (1)}$$

(Gradient of RQ) X (Gradient of OS) = -1

$$\left(\frac{-Y}{X}\right) X \left(\frac{y}{2}\right)=-1$$

$$\frac{-Yy}{2X}=-1$$

$$y=\frac{2X}{Y} \text{……………………...(2)}$$

Substituting eqn. 2 in (1)

$$\frac{2X^2}{Y}=-3Y+3\left(\frac{2X}{Y}\right)$$

$$2X^2=-3Y^2+6X$$

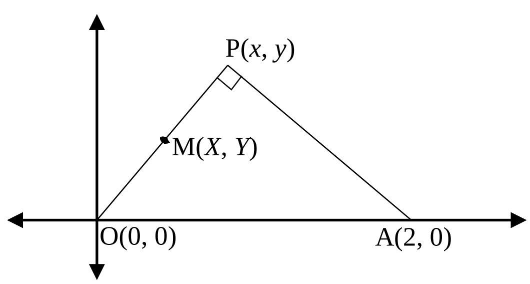
$$3Y^2=-2X^2+6X$$

$$3Y^2=2X(3-X) \text{ is the locus of S}$$

Example XI

Variable lines through the point O(0, 0) and A(2, 0) intersect at right angles at the point *P*. Show that the locus of the midpoint of OP is $y^2+x(x-1)=0$

Solution



(The gradient of *OP*) × (Gradient *AP*) = -1

$$\left(\frac{y}{x}\right) \times \frac{-y}{2-x}=-1$$

$$y^2=x(2-x) \text{………………………… (1)}$$

Let the midpoint Op be M(*X*, *Y*)

$$X=\frac{0+x}{2}$$

$$x=2X$$

$$Y=\frac{0+y}{2}$$

$$2Y=y$$

$$y=2Y$$

But *x* and *y* satisfy the above equation.

Substituting *x* = 2*X* and *y* = 2*Y* in Eqn (1);

$$(2Y)^2=2X[(2-(2X)]$$

$$4Y^2=2X[2-2X]$$

$$Y^2=+X[1-X]$$

$$Y^2=-X[X-1]$$

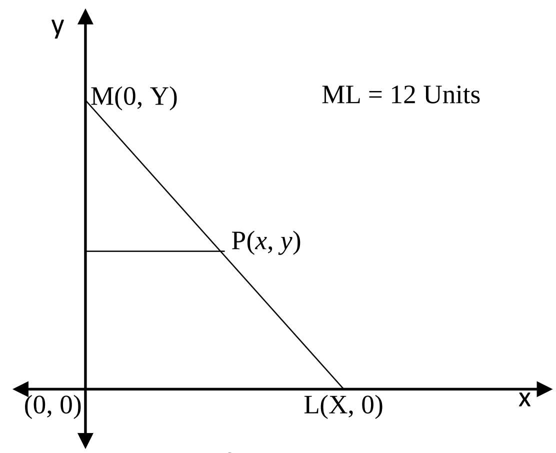
$$Y^2+X[X-1]=0$$

Example XII

P is a point on a line of length 12 units, which moves so that it's length lie on the axes. Find the locus of P when its

(a) The midpoint of line,
(b) The point of trisection near the *y*-axis.

Solution



Since P is a midpoint of LM

$$x=\frac{0+X}{2}$$

$$2x=X$$

Similarly, $y=\frac{0+Y}{2}$

$$2y=Y$$

Applying Pythagoras theorem on triangle OLM

$$X^2+Y^2=12^2$$

$$X^2+Y^2=144$$

$$X^2+Y^2=144 \text{…………………… (1)}$$

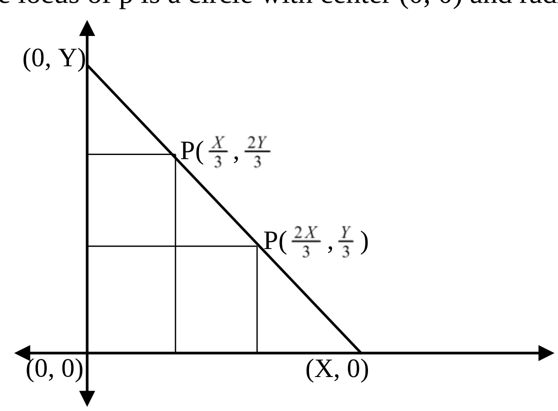
Substituting $X=2x \text{ and } Y=2y$ in equation (1)

$$(2x)^2+(2y)^2=144$$

$$4x^2+4y^2=144$$

$$x^2+y^2=36$$

The locus of p is a circle with center (0, 0) and radius 6



Since P(x, y) is a point of intersection near the y-axis

$$x = \frac{1}{3}X, y = \frac{2}{3}Y$$
$$3x = X, \frac{3y}{2} = Y$$

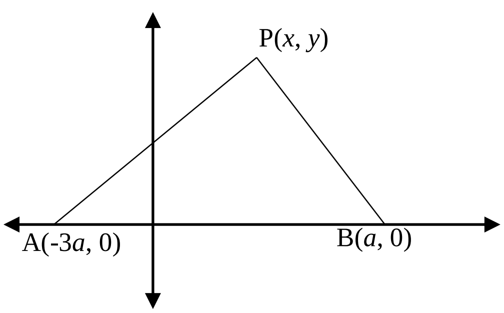
Substituting $X = 3x \; and \; Y = \frac{3y}{2}$ in Eqn (1)

$$(3x)^2 + \left(\frac{3y}{2}\right)^2 = 144$$
$$9x^2 + \frac{9y^2}{4} = 144$$
$$x^2 + \frac{y^2}{4} = 16$$
$$4x^2 + y^2 = 64$$

Example XIII

The fixed points A and B have coordinates (-3a, 0) and (a, 0) respectively. Find the locus of P which moves in the coordinate plane so that AP = 3pB. Show that the locus is a circle, S which touches the axis of y and has a center at the point$\left(\frac{3a}{2}, 0\right)$. A point Q moves in such a way that the perpendicular distance of Q from the y-axis is equal to the length of the tangent from Q to the circle S. find the equation of the locus of Q. show that this locus is also a locus of points which are equidistant from the line 4x + 3a = 0 and the point $\left(\frac{3a}{4}, 0\right)$.

Solution



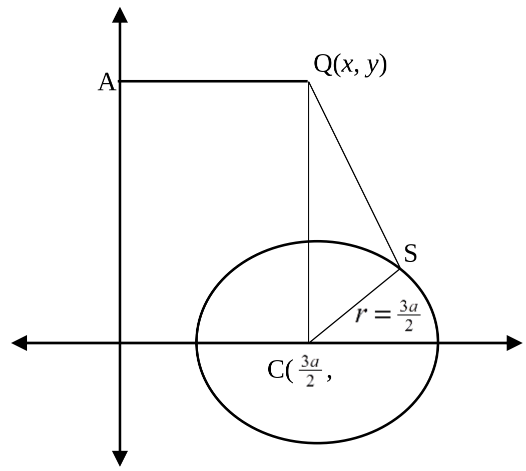
$$AP = 3PB$$
$$\Rightarrow \sqrt{(x+3a)^2 + y^2} = 3\sqrt{(x-a)^2 + y^2}$$
$$(x+3a)^2 + y^2 = 3[(x-a)^2 + y^2]$$
$$x^2 + 6ax + 9a^2 + y^2 = 9[x^2 + y^2 - 2ax + a^2]$$
$$x^2 + 6ax + 9a^2 + y^2 = 9x^2 + 9y^2 - 18ax + 9a^2$$
$$8x^2 + 8y^2 - 24ax = 0$$
$$x^2 + y^2 - 3ax = 0$$

Comparing $x^2 + y^2 + 2gx + 2fy + c = 0$

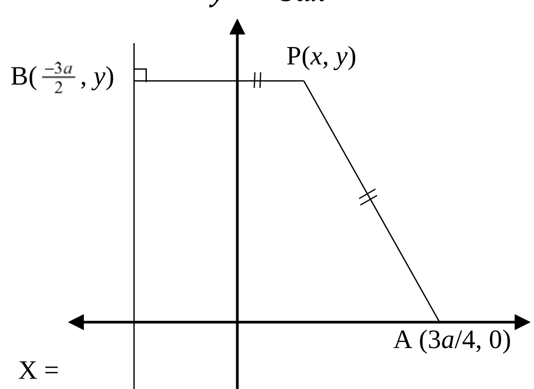
$$2g = -3a$$
$$g = -\frac{3a}{2}$$
$$f = 0$$

Center $\left(\frac{3a}{2}, 0\right)$

$$r = \sqrt{\left(\frac{3a}{2}\right)^2 + 0^2 - 0}$$
$$r = \frac{3a}{2}$$



$$CS^2 + QS^2 = CQ^2$$
$$\frac{9a^2}{4} + SQ^2 = \left(x - \frac{3a}{2}\right)^2 + y^2$$
$$SQ^2 = \left(x - \frac{3a}{2}\right)^2 + y^2 - \frac{9a^2}{4}$$
$$SQ^2 = x^2 - 3ax + \frac{9a^2}{4} + y^2 - \frac{9a^2}{4}$$
$$SQ^2 = x^2 + y^2 - 3ax$$
$$AQ = SQ$$
$$AQ^2 = SQ^2$$
$$x^2 = x^2 + y^2 - 3ax$$
$$y^2 = 3ax$$



$$x = \frac{-3a}{4}$$
$$4x + 3ax = 0$$
$$PB = d = \left|\frac{4(x) + 3a}{\sqrt{4^2}}\right|$$

$$PB = \frac{4x + 3a}{4}$$

$$\frac{4x + 3a}{4} = \sqrt{\left(x - \frac{3a}{4}\right)^2 + y^2}$$

$$\frac{16x^2 + 24ax + 9a^2}{16} = x^2 - \frac{3}{2}ax + \frac{9}{16}a^2 + y^2$$

$$x^2 + \frac{3}{2}ax + \frac{9}{16}a^2 = x^2 - \frac{3}{2}ax + \frac{9}{16}a^2 + y^2$$

$$y^2 = 3ax$$

Revision Exercise

1. Find the equation of the circle which passes through the origin and the points (2, 0), (3, -1).

2. Fid the radii and coordinates of the centres of the following circles.
(a) $x^2 + y^2 + 4x - 6y + 12 = 0$
(b) $x^2 + y^2 - 2x - 4y + 1 = 0$
(c) $x^2 + y^2 - 3x = 0$
(d) $x^2 + y^2 + 3x - 4y - 6 = 0$

3. Find the equations of the circle with the following centres and radii:
(a) (3, 2), 4
(b) (-1, -2), 1
(c) (0, 0), 5
(d) (½, 0), $\frac{3}{2}$
(e) (4, -1), $\sqrt{3}$

4. Find the equation of the circle which has the points (0, -1) and (2, 3) as ends of its diameter.

5. What is the equation of the circle with centre (2, -3) and touches the x-axis?

6. Find the equation of the curve having AB as diameter where A is the point (1, 8) and B(3, 14).

7. Find the range of the values of k for which each of the following represents a circle with non-zero radius.
(a) $x^2 + y^2 = k$
(b) $x^2 + ky^2 - 2x - 8 = 0$
(c) $kx^2 + y^2 + 4y + 9 = 0$
(d) $2x^2 + 2y^2 + kxy - 9 = 0$

8. Find the equation of the diameter of the circle $x^2 + y^2 - 6x + 2y = 15$, which when produced passes through the point (8, -2).

9. Find the radii of the two circles with centres at the origin which touch the circle $x^2 + y^2 - 8x - 6y + 24 = 0$

10. Find the equation of the tangent to the circle $(x - 2)^2 + (y - 3)^2 = 16$ at the general point $((2 + 4\cos\theta), (3 + 4\sin\theta))$. Hence find the equation of the tangent at the point $(4, 3+2\sqrt{3})$.

11. Find the equation of the tangents to the following circles at the given points:
(a) $x^2 + y^2 = 5$, (-2, 1)
(b) $x^2 + y^2 - 4x + 2y = 3$. (0, -3)
(c) $x^2 + y^2 + 6y - 1 = 0$, (3, -4)
(d) $2x^2 + 2y^2 + 9x - 4y + 4 = 0$, (-2, 3)

12. Find the equation of the circle whose centres lies on the line $y = 3x - 7$ and which passes through the points (1, 1) and (2, -1).

13. Show that the distance of the centre of the circle $x^2 + y^2 - 6x - 4y + 4 = 0$ from the y-axis is equal to the radius. What does this prove about the y-axis and the centre?

14. Prove that the circles $x^2 + y^2 - 4x - 6y = 0$ and $x^2 + y^2 - 4x - 6y = 3$ are concentric. Find the radius of the common centre.

15. Find the lengths of the tangents drawn from the following points to the given circles:
(a) (6, -1), $x^2 + y^2 = 12$
(b) (-1, 3), $x^2 + y^2 - 8x + 4y + 19 = 0$
(c) (4, -2), $x^2 + y^2 - 10y - 4 = 0$
(d) (3, -4), $x^2 + y^2 + x - 3y = 0$.

16. If O is the origin and P, Q are the intersections of the circle $x^2 + y^2 + 4x + 2y - 20 = 0$ and the straight line
$x - 7y + 20 = 0$. Show that OP and OQ are perpendicular. Find the equation to the circle through O, P and Q.

17. The circle x2 + y2 + 2gx + 2fy + c = 0 passes through the points A(-1, -2), B(1, 2), C(2, 3). Write down three equations which must be satisfied by g, f, c. Solve these equations and write down the equations of the circle ABC.

18. Prove that the line $y = 3x - 1$ neither cuts nor touches the circle $(x - 1)^2 + (y - 1)^2 = 9$

19. Find the greatest and least distance of a point P from the origin as it moves round the circle
(a) $x^2 + y^2 - 24x - 10y + 48 = 0$
(b) $x^2 + y^2 + 6x - 8y - 24 = 0$

20. A circle which passes through the origin cuts off intercepts of lengths 4 and 6 units on the positive x and y-axes respectively. Find the equation to the circle and the equations to the tangents to the circle at the points other than the origin where it cuts the axes.

21. A is the point (3, -1) and B is the point (5, 3). Show that the locus of the point P, which moves so that $PA^2 + PB^2 = 28$ is a circle Find its centre and radius.

22. Prove that the line $y = 2x - 3$ is a tangent to the circle $(x - 5)^2 + (y - 2)^2 = 5$

23. Find the equation of the circle which has the points
(-7, 3) and (1, 9) at the end of a diameter. Find also the equation of the tangents to the circle which are parallel
(a) to the x-axis
(b) to the y-axis

24. The point (a, b) is the midpoint of a chord of the circle $x^2 + y^2 = R2$. Show that the equation to the chord is $ax + by = a^2 + b^2$.

25. A circle touches the x-axis and cuts off a constant length $2a$ from the y-axis. Show that the equation to the locus of its centre is a curve $y^2 - x^2 = a^2$.

26. Find the length of the chord joining the points in which the straight line $(x/a) + (y/b) = 1$ meets the circle $x^2 + y^2 = R^2$.

27. Show that the line $2x - 3y + 26 = 0$ is a tangent to the circle $x^2 + y^2 - 4x + 6y - 104 = 0$ and find the equation to the diameter through the point of contact.

28. Find the length of the tangent to the circle $x^2 + y^2 - 4 = 0$ from the point (x, y) and deduce the equation of the locus of P, when it moves so that the length of the tangents to the circle is always equal to the distance of P from the point (1, 0).

29. Prove that the line $x - y - 3 = 0$ is a common tangent to the circles $x^2 + y^2 - 2x - 4y - 3 = 0$ and $x^2 + y^2 + 4x - 7y - 13 = 0$. What are the coordinates of the point in which it meets the other common tangent?

30. Show that the common chord of the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 4x - 7y - 4 = 0$ passes through the origin.

31. Show that the following pair of circle are orthogonal:
(a) $x^2 + y^2 - 6x - 8y + 9 = 0$, $x^2 + y^2 = 9$
(b) $x^2 + y^2 - 4x + 2 = 0$, $x^2 + y^2 + 6y - 2 = 0$
(c) $x^2 + y^2 - 6y + 8 = 0$, $x^2 + y^2 - 4x + 2y - 14 = 0$
(d) $x^2 + y^2 + 10x - 4y - 3=0$, $x^2 + y^2 - 2x - 6y + 5=0$

32. Prove that the line $y = 2x$ is a tangent to the curve $x^2 + y^2 - 8x - y + 5 = 0$ and find the coordinates of the point of contact.

33. A and B have coordinates (-3, 0) and (3, 0). Show that the locus of a point P which moves such that $PB = 2PA$ is a circle with centre (-5, 0) and radius 4.

34. A triangle has vertices (0, 6), (4, 0), (6, 0). Find the equation of the circle through the midpoint of the sides and show that it passes through the origin.

35. Prove that the following pairs of circles touch each other and state whether the contact is external or internal.
(a) $x^2 + y^2 - 2x = 0$, $x^2 + y^2 - 8x + 12 = 0$
(b) $x^2 + y^2 - 2x - 2y = 18$, $x^2 + y^2 - 14x - 8y + 60 = 0$
(c) $x^2 + y^2 - 12x - 2y = 12$, $x^2 + y^2 - 4x + 4y + 4 = 0$
(d) $x^2 + y^2 - 4x + 2y = 8$, $x^2 + y^2 + 6x - 13y + 22 = 0$

36. Prove that the circle $x^2 + y^2 - 2x - 6y + 1 = 0$ cuts the circle $x^2 + y^2 - 8x - 8y + 31 = 0$ in two distinct places and find the equation of the common chord.

37. Points A(0, 2) and B(4, -2) lie on the circumference of a given circle. Points C(-3, -3) and D(7, 2) lie outside the circle but the centres of the circle lie on the CD. Find the equation of the circle.

38. Show that the $x^2 + y^2 + 4x - 2y - 11 = 0$ and $x^2 + y^2 - 4x - 8y + 11 = 0$ intersect at right angles.

39. Show that the line $x + 3y - 1 = 0$ touches the circle $x^2 + y^2 - 3x - 3y + 2 = 0$

40. Show that the locus of a point which moves such that the square of its distance from the point (3, 4) is proportional to its distance from the line $x + y = 0$, one of the locus being the point (1, 2), is a circle and find its centre and radius.

Answers

1. $x^2 + y^2 - 2x + 4y = 0$
2. (a) 1, (-2, 3) (b) 2, (1, 2)
   (c) $\frac{3}{2}$, ($\frac{3}{2}$, 0) (d) $\frac{7}{2}$, ($\frac{-7}{2}$, 2)
3. (a) $x^2 + y^2 - 6x - 4y - 3 = 0$
   (b) $x^2 + y^2 + 2x + 4y + 4 = 0$
   (c) $x^2 + y^2 = 25$
   (d) $x^2 + y^2 - x - 2 = 0$
4. $x^2 + y^2 - 2x - 2y - 3 = 0$
5. $x^2 + y^2 - 4x + 6y + 4 = 0$
6. $x^2 + y^2 - 4x - 22y + 115 = 0$
7. (a) $k > 0$ (b) $k = 1$
   (b) No value of k (c) $k = 0$
8. $x + 5y + 2 = 0$
9. (4, 6)
10. $(y - 3)\sin\theta + (x - 2)\cos\theta = 4$, $y\sqrt{3} + x = 10 + 3\sqrt{3}$.
11. (a) $y = 2x + 5$ (b) $x + y + 3 = 0$
    (c) $x + y + 3 = 0$ (d) $x + 8y = 32$.
12. $x^2 + y^2 - 5x - y + 4 = 0$
13.
14. (2, 3)
15. (a) 5 (b) 7 (c) 6 (d) $2\sqrt{10}$
16. $x^2 + y^2 + 5x - 5y = 0$
17. $x^2 + y^2 - 16x + 8y - 5 = 0$
18.
19. (a) 24, 2 (b) 12, 2
20.
21. (4, 1), 3
22.
23. $x^2 + y^2 - 4x - 6y = 0$
    $2x - 3y = 8$
    $y = 1, y = 11$
    $x = 2, x = -8$
24.
25.
26. $2\sqrt{\left(R^2 - \frac{a^2b^2}{a^2 + b^2}\right)}$
27. $3x + 2y = 0$
28. $\sqrt{(x^2 + y^2 - 4}$, $2x - 5 = 0$
29. (7, 4)
30. (1, 2)
31.
32. $x^2 + y^2 - 5x - y = 0$
33.
34. $3x + y = 15$
35.
36.
37. $x^2 + y^2 - 2x + 2y = 8$

Locus

1. L and M are the feet of perpendiculars from a point P onto the axes. Find the locus of P when it moves so that LM is length 4 units.
2. A variable line through the point (3, 4) cuts the axes at Q and R. and the perpendiculars to the axes at Q and R intersect at P. What is the locus of the point P?
3. A variable joint P lies on the curve $xy = 12$. Q is the midpoint of the line joining P to the origin. Find the locus of Q.
4. P is a variable line on the curve $y = 2x^2 + 3$ and O is the origin. Q is the point of intersection of OP nearer the origin. Find the locus of Q.
5. A line parallel to the x-axis cuts the curve $y^2 = 4x$ at P and the line $x = -1$ at Q. Find the locus of the midpoint of PQ.
6. Find the locus of a point which moves so that the sum of the squares of its distance from the lines $x + y = 0$ and $x - y = 0$ is 4.
7. A is the point (1, 0), B is the point (2, o) and O is the origin. A point P moves so that the angle BPO is a right angle and Q is the midpoint of AP. What is the locus of Q?
8. A line parallel to the y-axis meets the curve $y = x^2$ at P and the line $y = x + 2$ at Q. Find the locus of the midpoint of PQ.

Answers to Locus Questions

1. $x^2 + y^2 = 16$
2. $xy = 3y + 4x$
3. $xy = 3$
4. $y = 6x^2 + 1$
5. $y^2 = 8x + 4$
6. $x^2 + y^2 = 4$
7. $4x^2 + 4y^2 - 8x + 3 = 0$
8. $2y = x^2 + x + 2$