2. PROBABILITY

Introduction:

Probability is a measure of the expectation that an event will occur or a statement is true. Probabilities are given a value between 0 (will not occur) and 1 (will occur). The higher the probability of an event, the more certain we are that the event will occur.

When dealing with **experiments** that are **random** and **well-defined** in a purely theoretical setting (like tossing a fair coin), probabilities describe the statistical number of outcomes considered divided by the number of all outcomes (tossing a fair coin twice will yield HH with probability 1/4, because the four outcomes HH, HT, TH and TT are possible). When it comes to practical application, however, the word ***probability*** does not have a singular direct **definition.**

The probability of an **event** *A* is written as P(*A*), p(*A*) or Pr(*A*). This mathematical definition of probability can extend to infinite sample spaces, and even uncountable sample spaces, using the concept of a measure.

The **opposite or complement** of an event *A* is the event [not *A*] (that is, the event of *A* not occurring); its probability is given by P(not *A*) = 1 - P(*A*). As an example, the chance of not rolling a six on a six-sided die is 1 – (chance of rolling a six) = $1 - {}^{1}/_{6} = {}^{5}/_{6}$

If both events *A* and *B* occur on a single performance of an experiment, this is called the intersection or **joint probability** of *A* and *B*, denoted as $P(A \cap B)$

Notes:

Probability is a value that represents the occurrence of an event when compared with the total number of trials. If P(A) represents the probability of A then P(A) lies in the interval from 0 to 1.

$\therefore \quad 0 \angle P(A) \leq 1$

$$P(A) = \frac{\text{Number of trials for event A}}{\text{Total number of trials in samples}}$$

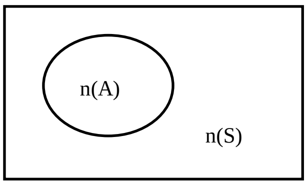
$$= \frac{n(A)}{n(S)}$$

Where A is an event with S possibilities.

If P(A) = 1, we are absolutely sure that event A will happen.

If P(A) = 0, then event A can never happen.

From $P(A) = \dfrac{n(A)}{n(S)}$



n(A) is greater or equal to zero.
n(S) is greater or equal to n(A)
Since A is a subset S.
$0 \leq n(A) \leq n(S)$ divide by n(S)

$$\frac{0}{n(S)} \leq \frac{n(A)}{n(S)} \leq \frac{n(S)}{n(S)}$$

$$0 \leq \frac{n(A)}{n(S)} \leq 1 \quad \text{but } \frac{n(S)}{n(S)} = 1$$

$$0 \leq P(A) \leq 1$$

There are mainly two types of probabilities.

Empirical or experimental probability: It arises out of carrying out experiments practically.

Theoretical probability: Where values for probability are obtained from daily experience.

Example: Probability of a head or a tail when a coin is tossed is $\frac{1}{2}$.

COMPLEMENT (EXHAUSTIVE) PROBABILITY

The complement of event A is A or A implying A does not occur.

$$P = (\bar{A}) = \frac{n(\bar{A})}{n(S)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

But $n(A) + n(\bar{A}) = n(S)$

$\therefore n(\bar{A}) = n(S) - n(A) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$

Substitute (ii) into (i)

$$P(\bar{A}) = \frac{n(S) - n(A)}{n(S)}$$

$$= \frac{n(S)}{n(S)} - \frac{n(A)}{n(S)}$$

$$P(\bar{A}) = 1 - \frac{n(A)}{n(S)} \quad \text{but } \frac{n(A)}{n(S)} = P(A)$$

$$P(\bar{A}) = 1 - P(A)$$

$$\mathbf{P(A) + P(\bar{A}) = 1}$$

LAWS OF PROBABILITY

They are mainly classified into two parts.

Additional law

It mainly includes:

Mutually exclusive events.

Non – mutually exclusive events.

Exhaustive events.

(ii) Multiplication law.

Independent events.

Dependent events or conditional probability.

Baye’s theorem.

(i) **ADDITIONAL LAW**

(a) **MUTUALLY EXCLUSIVE EVENTS**

They are events that cannot occur at the same time thus has no intersection. If the events are A and B then $P(A \cap B) = 0$

In probability OR is represented by U (union) while AND is represented by $\cap$ (intersection) also OR can be represented with + (addition sign) and can be represented with X (multiplication sign)

$$P(A \text{ or } B) = P(A + B)$$
$$= P(AUB)$$
$$= P(A) + P(B)$$

It can be extended to three events A, B and C

P(A or B or C) = P(A) + P(B) + P(C) for events A_1, A_2, A_3, ………., And then

$P(A, \text{ or } A_2 \text{ or } A_3 \text{ or } \ldots\ldots \text{ or } A_n) = P(A_1) + P(A_2) + P(A_3) + \ldots\ldots + P(A_n)$

(b) **NON-MUTUALLY EXCLUSIVE EVENTS**

They are events that occur at the same time. If the two events are A and B then $P(A \cap B) \neq 0$ then probability that A occurs plus the probability that B occurs Subtract probability that both A and B occur P(A or B) = P(AUB)

$$P(A + B) = P(A) + P(B) - P(A \cap B)$$

$$\therefore \quad P(AUB) = P(A) + P(B) - P(A \cap B)$$

If three events A, B and C are non-mutually exclusive then

$$\mathbf{P(AUBUC) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)}$$

(c) **EXHAUSTIVE EVENTS**

They are those events whose probability sum up to one. If A and B are mutually exhaustive, then

$$\begin{aligned} P(\text{ A or B }) &= P(AUB) \\ &= P(A) + P(B) \\ &= 1 \end{aligned}$$

$$\therefore P(AUB) = 1 \quad \text{or} \quad P(A) + P(B) = 1$$

If three events A, B and C are exhaustive events then

$$\begin{aligned} P(\text{A or B or C}) &= P(A) + P(B) + P(C) \\ &= 1 \end{aligned}$$

In general for events $A_1, A_2, A_3, \ldots\ldots A_n$, If they are exhaustive events then,

$$\begin{aligned} P(A_1 \text{ or } A_2 \text{ and } A_3 \text{ and } \ldots\text{and } A_n) &= P(A_1 \cap A_2 \cap A_3 \cap \ldots\ldots \cap A_n) \\ &= P(A_1) \times P(A_2) \times 2(A_3 \times \ldots \times P(A_n) \end{aligned}$$

MULTIPLICATION LAW

CONDITIONAL PROBABILTY (DEPENDENT EVENTS)

If A and B and are two events, where $P(A) \neq 0$ and $P(B) \neq 0$, then the probability of A, given that B has already occurred is written as P(A/B) and is probability of A given B.

$$\text{A and B}) = P(B) \times P(\text{A given} B)$$

$$P(A \cap B) = P(B) \times P(A \cap B)$$

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

(b) BAYE'S THEOREM

Suppose someone told you they had a nice conversation with someone on the bus. Not knowing anything else about this conversation, the probability that they were speaking to a woman is 50%. Now suppose they also told you that this person had long hair. It is now more likely they were speaking to a woman, since most long-haired people are women. Bayes' theorem can be used to calculate the probability that the person is a woman.

To see how this is done, let

W represent the event that the conversation was held with a woman, and
L denote the event that the conversation was held with a long-haired person.

It can be assumed that women constitute half the population for this example. So, not knowing anything else, the probability that W occurs is

$$P(W) = 0.5$$

Suppose it is also known that 75% of women have long hair, which we denote as

$P(L/W) \quad = \quad 0.75$

(read: the probability of event L given event W is 0.75).

Likewise, suppose it is known that 30% of men have long hair, or

$P(L/M) = \quad 0.30$,

where M is the **complementary event** of W, i.e., the event that the conversation was held with a man (assuming that every human is either a man or a woman).

Our goal is to calculate the probability that the conversation was held with a woman, given the fact that the person had long hair, or, in our notation, P(W/L) . Using the formula for Bayes' theorem, we have:

$$P(W/L) = \frac{P(L/W)P(W)}{P(L)} = \frac{P(L/W)P(W)}{P(L/W)P(W) + P(L/M)P(M)}$$

where we have used the **law of total probability.** The numeric answer can be obtained by substituting the above values into this formula. This yields

$$\mathbf{P(W/L) \approx 0.714}$$

i.e., the probability that the conversation was held with a woman, given that the person had long hair, is about 71%.

Statement and interpretation

Mathematically, Bayes' theorem gives the relationship between the **probabilities** of A and B, P(A)and P(B), and the **conditional probabilities** of A given Band B given A, P(A/B)
and P(B/A)
In its most common form, it is:

$$P(A/B) = \frac{P(B/A)P(A)}{P(B)}$$

The meaning of this statement depends on the **interpretation of probability** ascribed to the terms

Generally:
If A_1, A_2 ,A_n are n non-mutually exclusive and exhaustive events in a same face S. If B is another event in S then,

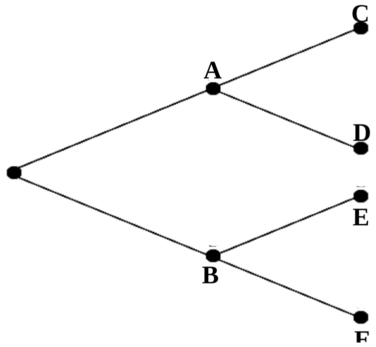
$$P(A_k/B) = \frac{P(B/A_k)}{\sum_{i=1}^{n}}$$

For $k = 1, 2, 3, \ldots, n$.

TREE DIAGRAM

Definition of 'Tree Diagram'

A diagram used in strategic decision making, valuation or probability calculations. The diagram starts at a single node, with branches emanating to additional nodes, which represent mutually exclusive decisions or events. In the diagram below, the analysis will begin at the first blank node. A decision or event will then lead to node A or B. From these secondary nodes, additional decisions or events will occur leading to the third level of nodes, until a final conclusion is reached.



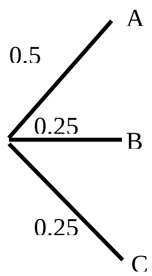
How to Use a Probability Tree for Probability Questions

Sometimes, you'll be faced with a probability question that just doesn't have a simple solution. Drawing a **probability tree** (or a **tree diagram**) is a way for you to visually see all of the possible choices, and to avoid making mathematical errors. Below is a step-by-step process of using a decision tree.

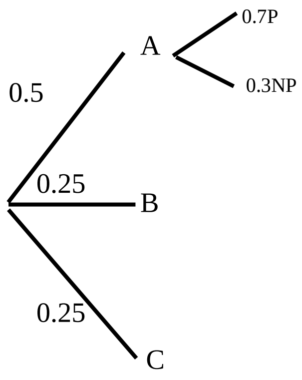
Example "**An airplane manufacturer has three factories A B and C which produce 50%, 25%, and 25%, respectively, of a particular airplane. Seventy percent of the airplanes produced in factory A are passenger airplanes, 25% of those produced in factory B are passenger airplanes, and 25% of the airplanes produced in factory C are passenger airplanes. If an airplane produced by the manufacturer is selected at random, calculate the probability the airplane will be a passenger plane.**"

Step 1:*Draw lines to represent the first set of options in the question* (in our case, 3 factories). Label them (our question list A B and C so that is what we'll use here).

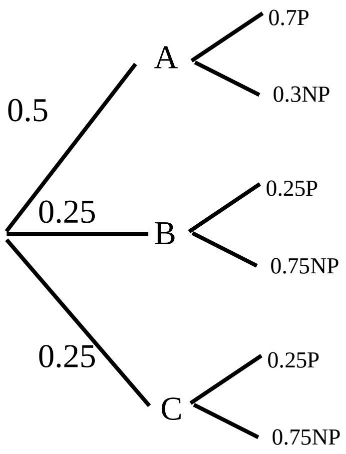
Step 2: ***Convert the percentages to decimals***, and place those on the appropriate branch in the diagram. For our example, 50% = 0.5, and 25% = 0.25.



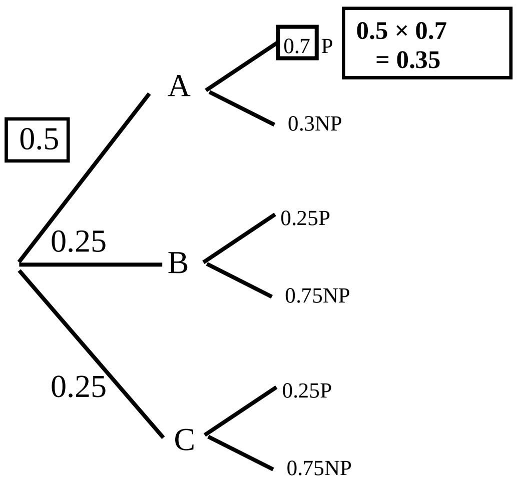
Step 3: ***Draw the next set of branches.*** In our case, we were told that 70% of factory A's output was passenger. Converting to decimals, we have 0.7 P ("P" is just my own shorthand here for "Passenger") and 0.3 NP ("NP" = "Not Passenger").



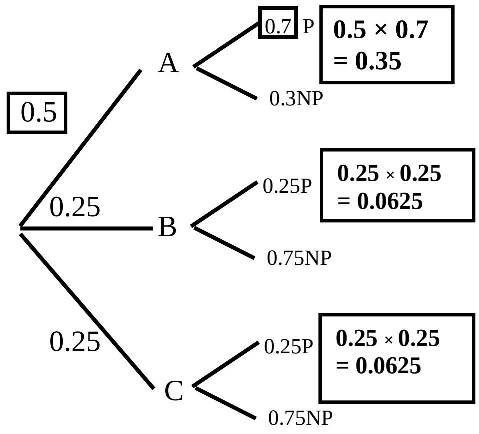
Step 4: ***Repeat step 3 for as many branches as you are given.***



Step 5: *Multiply the probabilities of the first branch that produces the desired result together*. In our case, we want to know about the production of passenger places, so we choose the first branch that leads to P.



Step 6: ***Multiply the remaining branches that produce the desired** result.* In our example there are two more branches that can lead to P.



Step 6: Add up all of the probabilities you calculated in steps and 6.

In our example, we had:

0.35 + 0.0625 + 0.0625 = 0**.475**

Note:

It solves problems that are mutually exclusive events and exclusive. The sum of probabilities from the same point on the tree diagram adds up to one. It solves two cases of problems.
(i) Picking with replacement, here the sample size in the second stage does not change since it is replaced.
(ii) Picking without replacement, the sample size in the second stage reduces by one since it is not replaced.

Example 1: Given the $P(A) = 0.3$, $P(B) = 0.4$ and $P(A \cap B) = 0.1$
Find: (i) $P(AUB)$
(ii) $P(AUB)^1$
(iii) $P(A^1UB)$

Solution:
(i) It's a non-mutually excusive event.

$$P(AUB) = P(A) + P(B) - P(A \cap B)$$
$$P(AUB) = 0.3 + 0.4 - 0.1$$
$$= 0.6$$

(ii) Are exhaustive events.

$$P(AUB)^1 + P(AUB) = 1$$
$$P(AUB)^1 = 1 - P(AUB)$$
$$= 1 - 0.6$$
$$= 0.4$$

(iii) Intersection = 0.3

$$P(A^1 \cap B) = 0.3$$
$$P(A^1UB) = P(A^1) + P(B) - P(A^1 \cap B)$$
$$= 0.7 + 0.4 - 0.3$$
$$= 0.8$$

Example 2: The probability that two events occur together is $\frac{2}{15}$

The probability that either or both events occur is $\frac{3}{5}$. Find the individual probabilities of the two events.

Solution:

Let the two events be A and B.

$$P(A \cap B) = P(A).P(B)$$

$$P(A).P(B) = \frac{2}{15}$$

$$P(A) = \frac{2}{15P(B)}$$

$$P(AUB) = P(A) + P(B) - P(A \cap B)$$

$$\therefore \quad P(A) + P(B) - P(A \cap B) = \frac{3}{5}$$

$$P(A \cap B) = \frac{2}{15}$$

$$P(A) + P(B) = \frac{3}{5} + \frac{2}{15}$$

$$= \frac{11}{15}$$

Substitute (i) into (ii)

$$\frac{2}{15P(B)} + P(B) = \frac{11}{15}$$

$$2 + 15P(B)^2 = 11P(B)$$

$$15P(B)^2 - 11P(B) + 2 = 0$$

$$P(B) = \frac{2}{5} \text{ or } \frac{1}{3}$$

When $P`(B) = \frac{2}{5}$;

$$P(A) = \frac{2}{15} \cdot \frac{5}{2}$$

$$= \frac{5}{15}$$

$$= \frac{1}{3}$$

When $P(B) = \frac{1}{3}$;

$$P(A) = \frac{2}{15} \cdot \frac{3}{1}$$

$$= \frac{2}{5}$$

Example 3: A die is tossed three times, what is the probability of getting.

exactly one 2

At least one 2.

Solution:

$$P(\text{exactly one } 2) = P(2).\ P(2).P(2) + P(\bar{2}).\ P(2).P(\bar{2}) + P(\bar{2}).\ P(\bar{2}).P(.2)$$

But, $P(2) = \frac{1}{6}$, $P(\bar{2}) = \frac{5}{6}$

P(exactly one 2) $= \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} + \frac{5}{6} \times \frac{1}{6} \times \frac{5}{6} + \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}$

$= \mathbf{0.3472}$

(b)P(At least one 2) = 1 – P(no six)

$= 1 - P(\bar{2}), P(\bar{2}), P(\bar{2})$

$= 1 - \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6}$

$= 1 - \frac{75}{216}$

$= \frac{91}{216}$

Example 4: If the two events A and B such that P(A) = $\frac{1}{3}$, P(B) - $\frac{1}{2}$

and P(A/B) = $\frac{P(A \cap B)}{P(B)}$. Find

(a) $P(A \cap B)$
(b) $P(A \cup B)$
(c) $P(B/\bar{A})$

Solution:

(i) $P(A \cap B) = P(B).P(A/B)$

$= \frac{1}{2} \times \frac{1}{4}$

$= \frac{1}{8}$

(ii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{1}{3} + \frac{1}{2} - \frac{1}{8}$

$= \frac{8 + 12 - 3}{24} = \frac{17}{24}$

(iii) $P(B/\bar{A}) = \frac{P(B \cap \bar{A})}{P(A)}$

$P(\bar{A}) + P(A) = 1$

$P(\bar{A}) = 1 - P(A)$

$= 1 - \frac{1}{3} = \frac{2}{3}$

$P(AUB) + P(AUB)^1 = 1$

$P(AUB)^1 = 1 - P(AUB)$

$= 1 - \frac{17}{24} = \frac{7}{24}$

But, $P(B \cap \bar{A}) = \frac{3}{8}$

$P(B/\bar{A}) = \frac{3/8}{2/3}$

$= \frac{3}{8} \times \frac{3}{2} = \frac{9}{16}$

Example 5:

Given that Aand B are two events such that P(A) = 0.5, P(B) = 0.7 and P(A∪B) = 0.8 Find.

(i) P(A∩B)

(ii) $P(A\cap\overline{B})$

Solution:

We have

P(A) = 0.5 , P(B) = 0.7 and P(A∪B) = 0.8

(i) $P(A\cap B) = P(A)+P(B) - P(A\cup B)$

$= 0.5 + 0.7 - 0.8$

$\therefore P(A\cap B) = 0.4$

(ii) $P(A\cap\overline{B}) = P(A) - P(A\cap B)$

$= 0.5 - 0.4$

$= 0.1$

$\therefore P(A\cap\overline{B}) = 0.1$

Example 6:

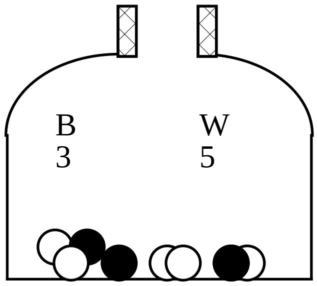
A bag contains 3 black and 5 white balls. Two balls are drawn at random one at a time without replacement.

Find. (i)the probability that the second ball is white.

(ii)the probability that the first ball is white given the second is white.

Solution:

(b)



Defining events,

B_1: a black ball is drawn first

W_1: a white ball is drawn first

B_2: ablack ball is drawn second,

and W_2: a white ball is drawn second

(i) we are interested in the union of the mutually exclusive events $(W_1\cap W_2)$ and $(B_1\cap W_2)$

Note the existence of W_2 in the two events above !

So $P(W_2) = P[(W_1\cap W_2)]\cup[(B_1\cap W_2)]$

$= P(W_1\cap W_2) + P(B_1\cap W_2)$

$= P(W_1).P(W_2/W_1)+P(B_1).P(W_2/B_1)$

But $P(W_1) = \frac{5}{8}$, $P(B_1) = \frac{3}{8}$

$P(W_2/W_1) = \frac{4}{7}$ and $P(W_2/B_1) = \frac{5}{7}$

Hence $P(W_2) = \frac{5}{8}\text{x}\frac{4}{7}+\frac{3}{8}\text{x}\frac{5}{7}$

$= \frac{35}{56}$

$\therefore P(W_2) = 0.625$ #

$$P(W_1/W_2) = \frac{P(W_1 \cap W_2)}{P(W_2)}$$

$$\therefore P(W_1/W_2) = \frac{P(W_1).P(W_2)W_1}{P(W_2)}$$

$$\therefore P(W_1/W_2) = \frac{\frac{5}{8}.\frac{4}{7}}{\frac{35}{56}} = 0.571$$

$$\therefore P(W_1/W_2) \approx 0.57$$

Example 7:
The probability that a student X can solve a certain problem is $^2/_3$ and that student y can solve it is ½ . Find the probability that the problem will be solve if both X and Y try to solve it independently.
Solution:
The problem shows two independent events
Defining events
X_1 'student x can solve the problem'
Y_1 : 'student y can solve the problem'
we are given

$$P(X_1) = \frac{2}{5}$$

$$P(Y_1) = \frac{1}{2}$$

we are asked $P(X_1 \cup Y)$
Now $P(X_1 \cup Y_1) = P(X_1) + P(Y_1) - P(X_1 \cap Y_1)$
Now for independent events ,

$$P(X_1 \cap Y_1) = P(X_1) \text{ x } P(Y_1)$$

$$\therefore P(X_1 \cap Y_1) = \frac{2}{5} + \frac{1}{2} - \frac{2}{5} \text{ x } \frac{1}{2}$$

$$\therefore P(X_1 \cap Y_1) = 0.7 \quad \#$$

Example 8 :
The probability that I have to wait at the traffic lights on my way to school is $^1/_4$.
Find the probability that, on two consecutive mornings, I have to wait on at least one morning.

Solution:
. Let W_1 be the event "waiting on traffic lights on first day"
W_2 be the event "waiting on the traffic lights on 2^{nd} day"
Now , $P(W_1) = 1/4$, and
$P(W_2) = 1/4$.
W_1 and W_2 are independent events.
We are asked to find $P(W_1 \cup W_2)$.

$$\therefore P(W_1 \cup W_2) = P(W_1) + P(W_2) - P(W_1 \cap W_2)$$

$$= \frac{1}{4} + \frac{1}{4} - \frac{1}{4}.\frac{1}{4}$$

$$= \frac{1}{2} - \frac{1}{16}$$

$\therefore \quad P(W_1 \cup W_2) \quad = \quad \frac{7}{16}$ #

Example 9.
Two balls are randomly drawn without replacement from a bag containing 10 white and 6 red balls.
Find the probability that the second ball drawn is
(i) red given that the first one was white,
(ii) white

Solution:
Let R_1 be the event "a red ball is drawn first'
R_2 be the event "a red ball is drawn second'
W_1 be the event "a red ball is drawn first'
W_2 be the event "a red ball is drawn second'

W	R
10	6

Now , $P(R_1) \quad = \quad \frac{1}{16}$

$P(W_1) \quad = \quad \frac{10}{16}$

(i) $P\ (R_2/W_1) \quad = \quad \frac{6}{15} = \frac{2}{5}$ #

(ii) $P(W_2) = P\ (W_2 \cap W_1)$ or $P\ (W_2 \cap R_1)$

$= P(W_1) \times P(W_2/W_1) + P(R_1) \times P\ (W_2/R_1)$

$= \frac{10}{16} \times \frac{9}{15} + \frac{6}{16} \times \frac{10}{15}$

$= \frac{5}{8} \times \frac{3}{5} + \frac{3}{8} \times \frac{2}{3}$

$= \frac{3}{8} \times \frac{2}{8} = \frac{5}{8}$

$\therefore \quad P\ (W_2) = \frac{5}{8}$ #

Example 10:
Two events A and B a neither nor mutually exclusive. Given that P(B) = 1/3,
$P(A \cap \overline{B}) = 1/3$, find
(i) $P(\overline{A} \cup \overline{B})$
(ii) $P(\overline{A} / \overline{B})$

Solution:
Given that
$P(A) = ½ , P(B) = {}^1/_3$

$P(A \cap B) \quad = \quad {}^1/_3$

(i) $P(\overline{A} \cup \overline{B}) \quad = \quad P\ (A \cap B)'$

$= 1 - P\ (A \cap B)$

Now $P\ (A \cap \overline{B}) \quad = \quad P\ (A) - P\ (A \cap B)$

$\Rightarrow \quad P\ (A \cap B) = P(A) - P\ A \cap \overline{B}$

$= ½ - {}^1/_3 = {}^1/_6$

Hence $P(\overline{A} \cup \overline{B}) \quad = \quad 1 - {}^1/_6 = {}^5/_6$

(ii) $P\left(\overline{A}/\overline{B}\right) = P\,\frac{\left(\overline{A}\cap\overline{B}\right)}{P(\overline{B})}$

$$\frac{P\left(\overline{A\cup B}\right)}{1-P(B)} = \frac{P\left(\overline{A\cup B}\right)}{1-P(B)}$$

$P(A \cup B = P(A) + P(B) - P(A\cap B)$

$= \frac{1}{2} + \frac{1}{3} - \frac{1}{6}$

$= \frac{4}{6} = \frac{2}{3}$

Hence $P\left(\overline{A}/\overline{B}\right) = P\,\frac{\left(\overline{A}\cap\overline{B}\right)}{P(\overline{B})}$

$$= \frac{\left(1-\frac{2}{3}\right)}{\left(1-\frac{1}{3}\right)}$$

$$= \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

$\therefore P\left(\overline{A}/\overline{B}\right) = \frac{1}{2}$ #

Example 11

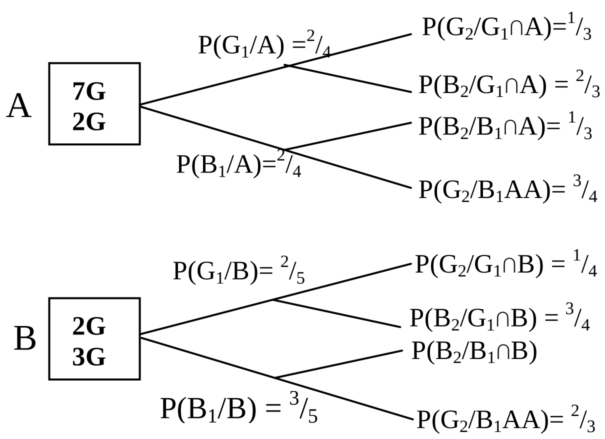
A bag A contains 2 green and 2 blue balls, while bag B contains 2 green and 3 blue balls. A bag is selected at random and two balls from it without replacement. Find the probability that the balls drawn are of different colours

Solution

Let G be the event green picked
Let B be the event Blue is picked
Let A and B be the event bag A and B are chosen

$P(A) = \frac{1}{2}$

$P(B) = \frac{1}{2}$



P(balls drawn are different colors)= $P(G_2\cap B_1\cap B)+P(B_2\cap B_1\cap B)+P(G_2\cap B_1\cap A)$
$(G_2\cap B_1\cap A)+P(B_2\cap G_1\cap A)$

$$= \left(\frac{2}{4}x\frac{3}{5}x\frac{1}{2}\right)+\left(\frac{3}{4}x\frac{2}{5}x\frac{1}{2}\right)+\left(\frac{2}{3}x\frac{2}{4}x\frac{1}{2}\right)+\left(\frac{2}{3}x\frac{2}{4}x\frac{1}{2}\right)$$

$$= \frac{6}{40}+\frac{6}{40}+\frac{4}{24}+\frac{4}{24}$$

$$= \frac{12}{40}+\frac{8}{24}$$

$$= \frac{3}{10} + \frac{1}{3}$$

$$= \frac{19}{30}$$

Example 12

A truth serum given to a suspect is known to be 90% reliable when a person is guilty and 99% reliable when innocent . In other words 10% of the guilty are judged innocent by the serum and 1% of the innocent are judged guilty . If the suspect was selected from a group of suspects of which only 5% have ever committed a crime, and the serum indicates that he is guilty , what is the probability that he is innocent ?

Solution:

We define the events :

Let ,

I : event that the suspect is innocent

G : the serum indicates guilt

Then we are asked P(I/G)

ie $P(I/G) = \frac{P(I \cap G)}{P(G)}$

Using Baye's rule

$$P(G) = P[(G \cap I) + (G \cap \bar{I})]$$

$$= P(I)P(G/I) + P(\bar{I})P(G/\bar{I})$$

From the given information

$P(I) = \frac{95}{100}$, $P(G/I) = \frac{1}{100}$

$P(\bar{I}) = \frac{5}{100}$, $P(G/\bar{I}) = \frac{90}{100}$

$$\text{So } P(G) = \frac{1}{100} \times \frac{95}{100} + \frac{90}{100} \times \frac{5}{100}$$

$$= \frac{109}{2000}$$

$$\text{Also } P(I \cap G) = P(I)P(G/I)$$

$$= \frac{95}{100} \times \frac{1}{100}$$

$$= \frac{19}{2000}$$

$$\text{Hence } P(I/G) = \frac{19}{2000} \div \frac{109}{2000}$$

$$= \frac{19}{109}$$

$\therefore$ $P(I/G) = 0.174$. #

Example 13: A die is loaded so that the chance of throwing 91 is $\frac{x}{4}$, the chance of a 6 is $\frac{1}{4}(1 - x$) and the chance of a 3, 4 0r 5 is $\frac{1}{6}$. The die is thrown twice.

(i)Prove that the chance of throwing a total 7 is $\frac{9x - 9x^2 + 10}{72}$

(ii)Find the value of X which make this chance a maxium and find this maximum probability.

Solution:

(i) $P(6) = \frac{1}{4}(1-x)$, $P(3) = (4) = P(5) = \frac{1}{6}$, $P(2) = \frac{1}{4}$ $P(1) = \frac{x}{4}$

Sum	1	2	3	4	5	6
1	11	12	13	14	15	16
2	21	22	23	24	25	26
3	31	32	33	34	35	36
4	41	42	43	44	45	46
5	51	52	53	54	55	56
6	61	62	63	64	65	66

Chance of sum of 7 = (6, 1) + (5, 2) + (4, 3) + (3, 4) + (2, 5) + (1. 6)
Since they are independent, substitute the values of each and multiply out.

$$= \frac{1}{4}(1-x)\ \frac{x}{4} + \frac{1}{6}.\frac{1}{4} + \frac{1}{6}.\frac{1}{6} + \frac{1}{6}.\frac{1}{6} + \frac{1}{4}.\frac{1}{6} + \frac{x}{4}.\frac{1}{4}\ (1-x)$$

$$= \frac{1}{16}(x-x^2) + \frac{1}{24} + \frac{1}{36} + \frac{1}{36} + \frac{1}{24} + \frac{1}{16}(x-x^2)$$

$$= \frac{1}{8}(x - x^2) + \frac{1}{12} + \frac{1}{18}$$

$$= \frac{9x - 9x^2 + 6 + 4}{72}$$

$$= \frac{9x - 9x^2 + 10}{72} \text{ hence proved}$$

(ii) Let $f(x) = \frac{9x - 9x^2 + 10}{72}$

$$= \frac{9x}{72} - \frac{9x^2}{72} + \frac{10}{72}$$

$$f^1(x) = \frac{9}{72} - \frac{18}{72}x = 0$$

$$\frac{18}{72}x = \frac{9}{72}$$

$$x = \frac{9}{18} = \frac{1}{2}$$

$$f^{11}(x) = -\frac{18}{72}$$

Since its negative value $x = \frac{1}{2}$ is a maximum point.

at $x = \frac{1}{2}$, $f\left(\frac{1}{2}\right) = \frac{9/2 - 9/4 + 10}{72}$

$$= \frac{49/4}{72}$$

$$= \frac{49}{288}$$

= **0.1701** is the maximum probability.

Example14.

A bag contains 7 black and 3 white marbles. Three marbles are chosen at random and in succession, each marble being replaced after it has been taken out of the bag, Draw a tree diagram to show all the possible selections.
From your diagram, or otherwise, calculate, to 2 significant figures, the probability of choosing:

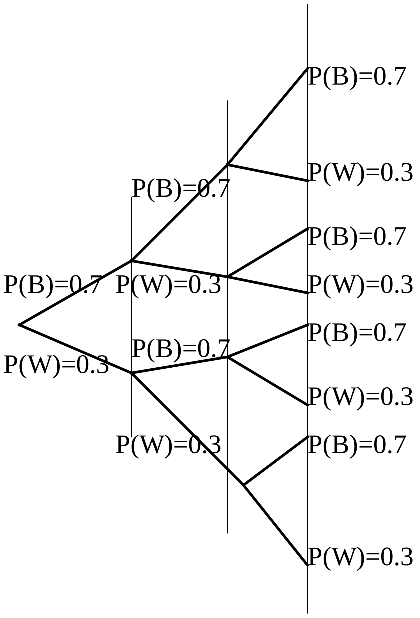
three black marbles

(b), a white marble, a black marble and a white marble in that order.

(c), two white marbles and a black marble in any order.

(d), at least one black marble.

Solution:



Using the probability tree above ,we have:

(a) P(three black marbles)

$P(B_1 \cap B_2 \cap B_3) = P(B_1).P(B_2).P(B_3)$

$= \frac{7}{10} \times \frac{7}{10} \times \frac{7}{10}$

$\therefore P(B_1 \cap B_2 \cap B_3) = 0.343$

(b) $P(W_1 \cap B_2 \cap W_3) = P(W_1).P(B_2).P(W_3)$

$= \frac{3}{10} \times \frac{7}{10} \times \frac{3}{10}$

$= 0.063$

So $P(W_1 \cap B_2 \cap W_3) = 0.063$ #

(c) $P(B_1 \cap W_1 \cap W_3) + P(W_1 \cap B_2 \cap W_3) + P(W_1 \cap W_2 \cap B_3) =$

$\frac{7}{10} \times \frac{3}{10} \times \frac{3}{10} + \frac{3}{10} \times \frac{7}{10} \times \frac{3}{10} + \frac{3}{10} \times \frac{3}{10} \times \frac{7}{10}$

$P(\overline{C}) = 0.189$

(d) P(at least one black marble) = 1 – (all white)

$= 1 - P(W_1 \cap W_2 \cap W_3)$

$= 1 - \frac{3}{10} \times \frac{3}{10} \times \frac{3}{10}$

$= 1 - \frac{27}{1000}$

$= \frac{973}{1000}$

$= 0.973$

Example 15: A bag contains 5 white, 3 red and 2 green counters. 3 counters are drawn without replacement. What is the probability that there

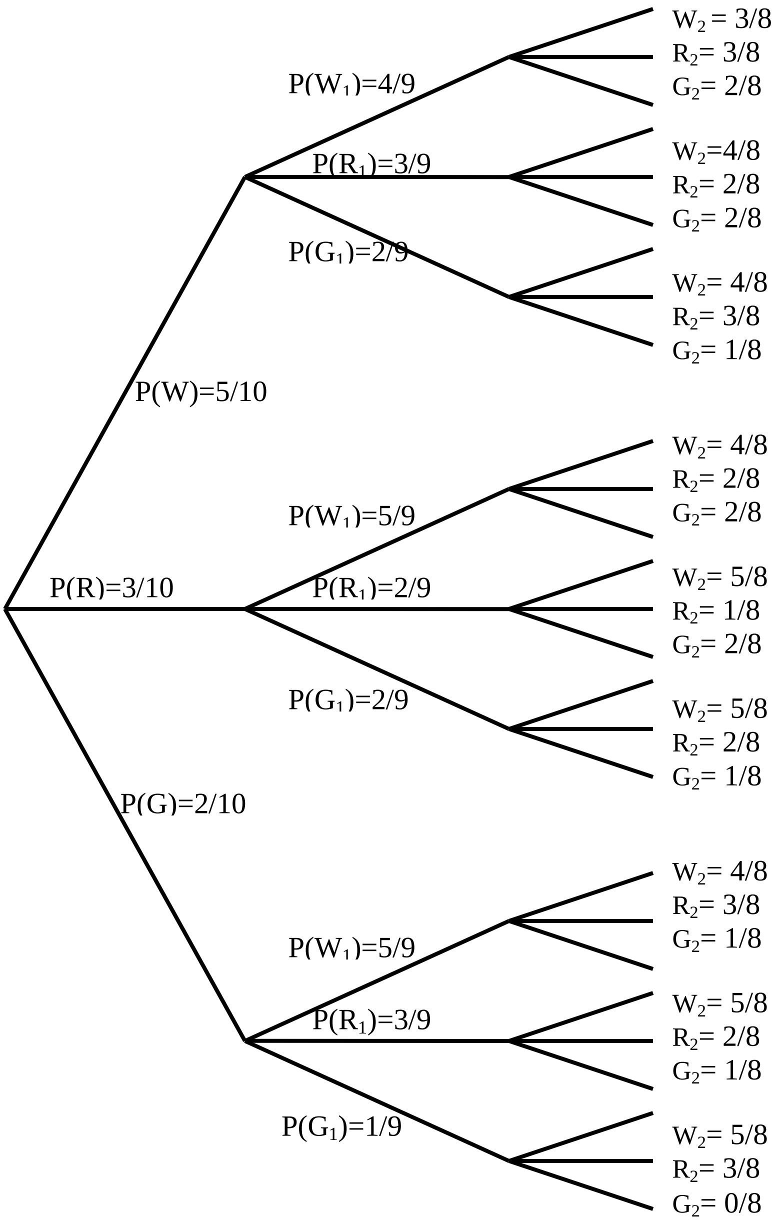
Is no green counter.

Are 2 white counters and a green counter?

Solution:

Let W = White, R = red and G = green.

EITHER



(i)P (no green counter) = $\frac{5}{10} \times \frac{4}{9} \times \frac{3}{8} + \frac{5}{10} \times \frac{4}{9} \times \frac{3}{8} + \frac{5}{10} \times \frac{3}{9} \times \frac{4}{8} + \frac{5}{10} \times \frac{3}{9} \times \frac{2}{8} + \frac{3}{10} \times \frac{5}{9} \times \frac{4}{8} + \frac{3}{10} \times \frac{5}{9} \times \frac{2}{8} + \frac{3}{10} \times \frac{2}{9} \times \frac{5}{8} + \frac{3}{10} \times \frac{2}{9} \times \frac{1}{8}$

$$= \frac{180 + 90 + 66}{720}$$

$$= \frac{336}{720}$$

$$= \frac{7}{15}$$

(ii)P (2 white counters and one green) = $\frac{5}{10} \times \frac{4}{9} \times \frac{2}{8} + \frac{5}{10} \times \frac{2}{9} \times \frac{4}{8} + \frac{2}{10} \times \frac{5}{9} \times \frac{4}{8}$

$$= \frac{40 + 40 + 40}{720}$$

$$= \frac{120}{720}$$

$$= \frac{12}{72}$$

$$= \frac{6}{36}$$

$$= \frac{1}{6}$$

OR (i) P(no green counter); P(G) $= \frac{2}{10}$, $P(\overline{G}) = \frac{8}{10}$

$P(\overline{G}) = \frac{8}{10}$ $\quad P(\overline{G}_2) = \frac{7}{9}$ $\quad P(\overline{G}_3) = \frac{6}{8}$

P (no green) $= P\,P(\overline{G}_1).P(\overline{G}_2).P(\overline{G}_3)$

$$= \frac{8}{10} \times \frac{7}{9} \times \frac{6}{8}$$

$$= \frac{336}{720}$$

$$= \frac{7}{15}$$

P(2W and G) = P(WWG or WGW or GWW)

P(W) $= \frac{5}{10}$, P(G) $= \frac{2}{10}$

$$= \frac{5}{10} \times \frac{4}{9} \times \frac{2}{8} + \frac{5}{10} \times \frac{2}{9} \times \frac{4}{8} + \frac{2}{10} \times \frac{5}{9} \times \frac{4}{8}$$

$$= \frac{120}{720}$$

$$= \frac{12}{72}$$

$$= \frac{1}{6}$$

Example 16.

(a) A box contains 7 red balls and 6 blue balls. Three balls are selected at random without replacement. Find the probability that:

(i) they are of the same colour.

(ii) at most two are blue.

(b) Two boxes P and Q contain white and brown cards. P contains 6 white cards and 4 brown cards. Q contains 2 white cards and three brown cards. A box is selected at random and a card is selected.

Find the probability that:

(i) a brown card is selected.

(ii) box Q is selected given that the card is white.

Solution

(a) (i) P(All are of same colour) $= P(R_1 \cap R_2 \cap R_3) + P(B_1 \cap B_2 \cap B_3)$

$$= \frac{7}{13}\times\frac{6}{12}\times\frac{5}{11} + \frac{6}{13} \text{ x } \frac{5}{12}\text{x}\frac{4}{11}$$

$$= \frac{210}{1716}+\frac{120}{1716}$$

$$= \frac{330}{1716}$$

$$= 0.1923$$

(ii) P (at most two blues) = 1 – P(All are blue)

$$= 1 - \frac{120}{1716}$$

$$= \frac{1596}{1716}$$

$$= 0.9301$$

(b)

(i) P(a Brown card is selected) = P(P ∩ B) + P(Q ∩ B)

$$= \frac{1}{2}\text{x}\frac{4}{10} + \frac{1}{2}\text{x}\frac{3}{5}$$

$$= \frac{2}{10}\text{x}\frac{3}{10}$$

$$= \frac{5}{10} = 0.5$$

$$P(Q/W) = \frac{P(Q\cap W)}{P(W)}$$

$$= \frac{P(Q\cap W)}{P(Q\cap W)+P(P\cap W)}$$

$$= \frac{\frac{1}{2}\text{x}\frac{2}{5}}{\frac{1}{2}\text{x}\frac{2}{5}+\frac{1}{2}\text{x}\frac{6}{10}}$$

$$= \frac{\frac{1}{5}}{\frac{2}{10}+\frac{3}{10}}$$

$$= \frac{1}{5}\text{x}\frac{10}{5}$$

$$= \frac{10}{25} \quad \mathbf{= 0.4}$$

Example 17:

When visiting a friend John may go by road, air or rail. The probabilities of using road, air and rail are 0.3, 0.8 and 0.6 respectively. The corresponding probabilities of arriving on an agreed time are 0.2, 0.8 and 0.1 respectively. Find the probability of having used the road given that he arrived on time.

Solution:

Let R, A, L and T denote Road, Air, Rail and arrival on time.

P(R) = 0.3, P(A) = 0.8, P(L) = 0.6

P(T/R) = 0.2, P(T/A) = 0.8, P(T/L) = 0.1

$$\therefore P(R/T)? = \frac{P(T/R).P(R)}{P(T/R).P(R) + P(A) + P(T/L).P(L)}$$

$$= \frac{0.3 \times 0.2}{0.3 \times 0.2 + 0.8 \times 0.8 + 0.6 \times 0.1}$$

$= \frac{0.06}{0.76}$

$= \frac{6}{76}$

$= \frac{3}{38}$

$= \mathbf{0.0789}$

Example 18: An industry manufactures a particular type of light bulb from three departments A_1, A_2 and A_3, 30% are manufactured by A_3. It is found that of those bulbs manufactured, in A_1, 1% is faulty, 1% in A_2 is faulty and 2% in A_3 are faulty. Suppose a bulb is selected and it is found to be faulty.
(i) Find the probability that it was one of the A3's bulb.
(ii) Find the probability that it is one of the A1's bulb.
(iii) Find the probability that it is one of the A_2's bulb.
(iv) Probability that the bulb is normal.

Solution: Let F = Faulty bulb , F^1 = Normal bulb

(i) $P(A_3/F) = \frac{P(A_3 \cap F)}{P(F)}$

$= \frac{P(A_3).P(F/A_3)}{P(F)}$

$P(F) = .\ P(A_1 \cap F) + P(A_2 \cap F) + P(A_3 \cap F)$

$= P(A_1).P(F/A_1) + P(A_2).P(F/A_2) + P(A_2).P(F/A_3)$

$= 0.3 \text{ x } 0.01 + 0.45 \text{ x } 0.01 + 0.25 \text{ x } 0.02$

$= 0.0125$

$P(A_3/F) = \frac{0.25 \text{ x } 0.02}{0.0125}$

$= \frac{0.005}{0.0125}$

$= \mathbf{0.4}$

(ii)$P(A_2/F) = \frac{P(A_2 \cap F)}{P(F)}$

$= \frac{P(A_2).P(F/A_2)}{P(F)}$

$= \frac{0.45 \text{ x } 0.01}{0.0125}$

$= \frac{0.0045}{0.0125}$

$= \frac{45}{125}$

$= 0.36$

(iii) $P(A_1/F) = \frac{P(A_1 \cap F)}{P(F)}$

$= \frac{P(A_1).P(F/A_1)}{P(F)}$

$= \frac{0.3 \text{ x } 0.01}{0.0125}$

$= \frac{0.003}{0.0125}$

$$= \frac{30}{125}$$

$$= \mathbf{0.25}$$

(iv) $P(F) + P(F^1) = 1$ (They are exhaustive events)

$P(F^1) = 1 - P(F)$

$= 1 - 0.0125$

$P(F^1) = \mathbf{0.9875}$

Example19.

A good football striker is nursing an injury in his leg. The probability that his team will win the next match when he is playing is $^4/_5$, otherwise it is $^2/_3$. The probability that he will have recovered by the time of the match is ¼.

Find the probability that his team will win the match

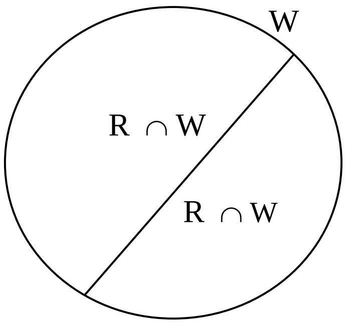
Solution:

Let W be event that the team wins the match and R be the event that the striker recovers.

Now $P\left(W/R\right) = 4/5$

$P\left(W/\overline{R}\right) = 2/3$

$P(R) = 1/4$



$P(W) = P(R \cap W) + P(\overline{R} \cap W)$

$= P(R) \times P(W|R) + P(\overline{R}) \times P(W|\overline{R})$

$= \frac{1}{4} \times \frac{4}{5} + \frac{3}{4} \times \frac{2}{3}$

$= \frac{1}{5} + \frac{1}{2}$

$= \frac{2+5}{10}$

$= \frac{7}{10}$

Hence the probability that his team will win the match is $\frac{7}{10}$.

Example 20.

A and B two independent events with A twice as likely to occur as B.If P (A) = $\frac{1}{2}$**, find**

(i) P(A ∪ B),
(ii) P[(A ∩ B)/A)],

Solution:

$\Rightarrow \quad P(A) = 2x$

$$2x = \frac{1}{2}$$

$$x = \frac{1}{4}$$

Hence $P(B) = \frac{1}{4}$

(i) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A) \times P(B)$

$$= \frac{1}{2} + \frac{1}{4} - \frac{1}{2} \times \frac{1}{4}$$

$$= \frac{1}{2} + \frac{1}{4} - \frac{1}{8}$$

$$= \frac{5}{8}$$

(ii) $P[(A \cap B) \mid A] = \dfrac{P[(A \cap B) \cap A]}{P(A)}$

$$= \frac{P(A) \times P(B) \times P(A)}{P(A)}$$

$$= P(A) \times P(B)$$

$$= \frac{1}{2} \times \frac{1}{4}$$

$$= \frac{1}{8}$$

Example21.

At a bus park, 60% of the buses are Isuzu make , 25% are Styer type and the rest are of Tata make.

Of the Isuzu type , 50 % have radios , while for the Styer and Tata types only 5 % and 1% have radios, respectively .

If a bus is selected at random from the park , determine the probability that :

(i) it has a radio

(ii) a styer is selected given that it has a radio.

Solution:

Defining the events

Let I be 'the bus at park is Isuzu'

S be 'the bus at park is Steyer'

T be 'the bus at park is Tata

and R be 'the bus has a radio'.

We are given

$$P(I) = \frac{60}{100} = \frac{6}{10} = \frac{3}{5}$$

$$P(S) = \frac{25}{100} = \frac{5}{20} = \frac{1}{4}$$

$$P(T) = \frac{15}{100} = \frac{3}{20}$$

$$P(R/T) = \frac{50}{100} = \frac{1}{2}$$

$$P(R/S) = \frac{5}{100} = \frac{1}{20}$$

$$P(R/T) = \frac{1}{100}$$

(i) we find P(R) .

We need to consider the union of $R \cap I$, $R \cap S$ and $R \cap T$

i.e. $P(R) = P[(R \cap I) \cup (R \cap S) \cup (R \cap T)]$

$= P(R \cap I) + P(R \cap S) + P(R \cap T)$

From conditional probability

$P(R) = P(R/I) \times P(I) + P(R/S) \times P(S) + P(R/T)\ P(T)$

$\therefore$ **P(R) = 0.314** #.

(ii) $P(S/R) = \dfrac{P(S \cap R)}{P(R)}$

[Using Conditional probability]

$$P(S/R) = \frac{R(S) \times P(R/S)}{P(R)}$$

$$= \frac{\left(\frac{1}{4} \times \frac{1}{20}\right)}{0.314}$$

$$= \mathbf{0.0398}$$

Therefore the probability that a Steyer is selected given that it has a radio is 0.0398 #

Example22.

On certain day, fresh fish from lakes: Kyoga, Victoria, Albert and George were supplied to one of the central markets of Kampala in the ratios 30%, 40%, 20% and 10% respectively. Each lake had an estimated ratio of poisoned fish of 2%, 3% and 1% respectively. If a healthy inspector a fish at random,

i) What is the probability that the fish was poisoned?

ii) Given that the fish was poisoned, what was the probability that it was from Albert?

Solution:

i). Defining the event,

K = event, fish from Lake Kyoga

V = event, fish from Victoria

A = event, fish from Lake Albert

G = event, fish from Lake George

Also P = Fish is poisoned

Given

$P(K) = \dfrac{30}{100} = \dfrac{3}{10}$

$P(V) = \dfrac{40}{100} = \dfrac{4}{10}$

$P(A) = \dfrac{20}{100} = \dfrac{2}{10}$

$P(G) = \dfrac{10}{100} = \dfrac{1}{10}$

Also;

$P(P/K) = \dfrac{2}{100}$, $P(P/V) = \dfrac{3}{100}$

$P(P/A) = \dfrac{3}{100}$, $P(P/G) = \dfrac{1}{100}$

We are interested in the union of the mutually exclusive events

$P \cap K$, $P \cap V$, and $P \cap G$

Or $P(P) = P(P \cap K)$ or $P(P \cap V)$ or $P(P \cap A)$ or $P(P \cap G)$

$= P(P \cap K) + P(P \cap V) + P(P \cap A) + P(P \cap G)$

$= P(K) \times P(P/K) + P(V) \times P(P/V) + P(A) \times P(P/A) + P(G) \times P(P/G)$

$$= \frac{3}{10} \times \frac{2}{100} + \frac{4}{10} \times \frac{3}{100} + \frac{2}{10} \times \frac{3}{100} + \frac{1}{10} \times \frac{1}{100}$$

$\therefore P(P) = \dfrac{6+12+6+1}{1000}$

$= \dfrac{25}{100}$

$= \dfrac{1}{40}$

The probability that fish was poisoned is $\dfrac{1}{40}$ #

ii) We have $P(A/P) = \dfrac{P(A \cap P)}{P(P)}$

$= \dfrac{6}{1000} \div \dfrac{1}{40}$

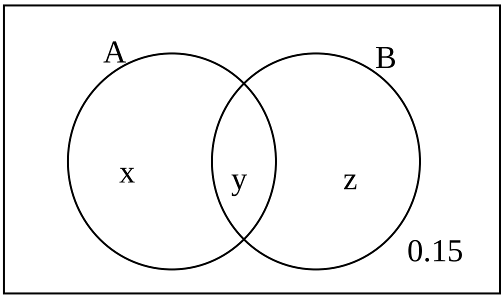
$= \dfrac{6}{1000} \times \dfrac{40}{1} = \dfrac{12}{50}$

$= \dfrac{6}{25}$

$\therefore P(A/P) = \dfrac{6}{25}$

Example 23.

(a) A and B are intersecting sets as shown in the Venn diagram below.



Given that P(A) = 0.6, P $(^{A'}/_{B}) = {}^{5}/_{7}$, and

P (A ∪ B) = 0.85, find

(i)The values of x, y and z

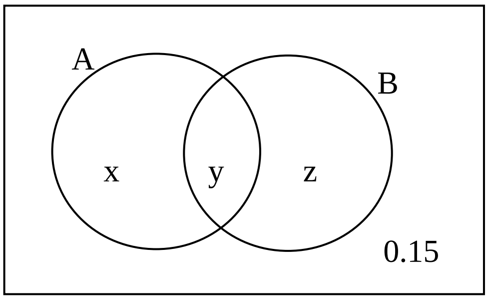
(ii)P $(^{A}/_{B'})$.

(b) A bag contains 4 white balls and 1 black ball. A second bag contains 1 white ball and 4 black balls. A ball is drawn at random from the first bag and put into the second bag.

Then a ball is taken from the second bag and put into the first bag. Find the probability that a white ball will be picked when a ball is selected from the first bag.

Solution:

We are given



$P(A) = 0.6$

$P(A/B) = {}^5/_7$

and $P(A \cup B) = 0.85$

(i) From the Venn – diagram

$x + y = P(A) = 0.6$

$\Rightarrow x + y = 0.6$ ……………..(1)

also $x + y + z = P(A \cup B) = 0.85$

$x + y + z = 0.85$ ……………..(2)

Solving equations (1) and (2)

$0.6 + z = 0.85$

$z = 0.85 - 060$

$\therefore$ $\mathbf{z = 0.25}$ #

Also $P(A^1/B) = \dfrac{P(A^1 \cap B)}{P(B)} = {}^5/_7$ ……(3)

Now $P(A^1 \cap B) = z$ (from the Venn diagram)

And $P(B) = y + z$

From above

$$\frac{z}{y+z} = \frac{5}{7}$$

Solving for y

$7z = 5y + 5z$

$5y = 2z$

$y = \dfrac{2}{5}z = \dfrac{2}{5} \times 0.25$

$\therefore$ $\mathbf{y = 0.1}$

From $x + y = 0.6$

$\Rightarrow x = 0.6 - y$

$= 0.6 - 0.1$

$= \mathbf{0.5}$

Hence ; $x = 0.5$, $y = 0.1$ and $z = 0.2.5$

(ii) $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$

$= \dfrac{x}{x + 0.15}$ (from Venn diagram)

$= \dfrac{0.50}{0.50 + 0.15}$

$= \dfrac{50}{65} = \dfrac{10}{13}$

$\therefore$ $\mathbf{P(A/B) = {}^{10}/_{13}}$ #

(b)

W B	W B
4 1	1 4
Bag 1	Bag 2

Let W_1 be a white ball drawn from bag 1
B_1 be a black ball drawn from bag 1
W_2 be a white ball drawn from bag 2
B_2 be a black ball drawn from bag 2
W be a white ball picked from bag 1 after a ball has been drawn from Bag2 and put in bag 1

Now $P(W_1) = {}^4/_5$
$P(B_1) = {}^1/_5$
$P(W_2/W_1) = {}^2/_6$
$P(W_2/B_1) = {}^1/_6$
$P(B_2/W_1) = {}^4/_6$
$P(B_2/B_1) = {}^5/_6$

Also
$P(W/W_1\cap W_2) = {}^4/_5$
$P(W/ W_1 \cap B_2) = {}^3/_5$
$P(W/B_1\cap W_2) = {}^5/_5$
$P(W/B_1\cap B_2) = {}^4/_5$

Also $P(W_1\cap W_2) = P(W_1)\times P(W_2/W_1)$
$= {}^4/_5\times{}^2/_6 = {}^8/_{30}$

$P(W_1 \cap B_2) = P(W_1)\times P(B_2/W_1)$
$= {}^4/_5\times{}^4/_6 = {}^{16}/_{30}$

$P(B_1 \cap W_2) = P(B_1)\times P(W_2/B_1)$
$= {}^1/_5 \times{}^1/_6 = {}^1/_{30}$

$P(B_1 \cap B_2) = P(B_1)\times P(B_2/B_1)$
$= {}^1/_5 \times {}^5/_6 = {}^5/_{30}$

We find P(W)

$P(W) = P(W \cap W_1 \cap W_2) + P(W \cap W_1 \cap B_2) + P(W \cap B_1\cap W_2) + P(W\cap B_1\cap B_2)$

$= P(W/W_1\cap B_2) \times P(W_1\cap W_2)+P(W/W_1\cap B_2) \times P(W_1\cap B_2) + P(W/B_1\cap W_2)\times P(B_1\cap W_2) + P(W/B_1\cap B_2) \times P(B_1\cap B_2)$

$$= \frac{4}{5}x\frac{8}{30}+\frac{3}{5}x\frac{16}{30}+\frac{5}{5}x\frac{1}{30}+\frac{4}{5}x\frac{5}{30}$$

$$= \frac{32}{150}+\frac{48}{150}+\frac{5}{150}+\frac{20}{150}$$

$$P(W) = \frac{105}{150} = \frac{21}{30} = \frac{7}{10}$$

The Probability that a white ball will be picked when a ball is selected from the first bag is $\frac{7}{10}$ #

Example24.

(a) Abel, Bob and Charles applied for the same job in a certain company. The probability that Abel will take the job is ${}^3/_4$. The probability that Bob will take it is ${}^1/_2$, while the probability that Charles won't take the job is ${}^1/_3$,

What is the probability that:

(i) None of them will take the job?

(ii)One of them will take the job?

(b)Two events A and B are independent. Given that $P(A \cap B') = ¼$ and $P(^{A'}/_{B}) = {}^{1}/_{6}$, use a Venn diagram to find the probabilities

(i) $P(A)$

(ii) $P(B)$

(iii) $P(A \cap B)$

(iv) $P(A \cup B)'$

Solution

Let A be the event "Abel will take the job"
Let B be the event "Bob will take the job"
Let C be the event "Charles will take the job"
A, B and C are independent events and we are given.

$P(A) = ¾ \Rightarrow P(A^1) = ¼$

$P(B) = ½ \Rightarrow P(B^1) = ½$

$P(C) = 1 – P(C^1)$
$= (1 – {}^{1}/_{3})$
$= {}^{2}/_{3}$

(i) P (none of the will take the job)

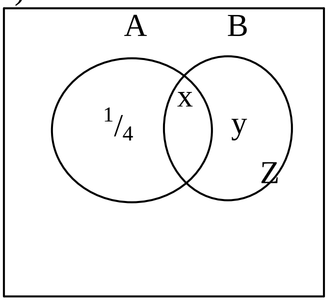
$= P(A \cap B \cap C)$
$= P(A) \times P(B) \times P(C)$
$= ¼ \times ½ \times {}^{2}/_{3} = {}^{2}/_{24}$ #

(ii) P (one of them will take the job)

$= P(A\cap B\cap C) + P(A\cap B\cap C) + P(A\cap B\cap C)$
$= ¾ \times ½ \times {}^{1}/_{3} + {}^{1}/_{4} \times ½ \times {}^{1}/_{3} + ¼ \times ½ \times {}^{2}/_{3}$
$= {}^{3}/_{24} + {}^{1}/_{24} + {}^{2}/_{24} = {}^{6}/_{24}$

$\therefore$ P (One of them will take the job) $= ¼$ #

(b)



Given $P(A \cap B) = ¼$
And $P(A/B) = {}^{1}/_{6}$
Let $P(A\cap B) = x$
$P(A\cap B) = y$
and $P(A\cup B)' = Z$

Now $P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{1}{6}$

$= \dfrac{y}{x + y} = \dfrac{1}{6}$

$\Rightarrow 6y = x + y$
$5y = x$

Also $(PA\cap B) = P(A) \times P(B)$
$\Rightarrow x = (x + ¼) \times (x + y)$
But $y = {}^{x}/_{5}$
$\Rightarrow x = (x + ¼)(x + {}^{x}/_{5})$

$$= \frac{4x-1}{4} \times \frac{6x}{5}$$

$20x = 6x(4x+1)$

$10x = 3x(4x+1)$

$10x = 12x^2 + 3x$

$12x^2 - 7x = 0$

$x(12x - 7) = 0$

$\therefore x = 0$ **OR** $x = {}^7/_{12}$

Now $5y = x = {}^7/_{12}$

$y = {}^7/_{60}$

Note : $¼ + x\ y + z = 1$

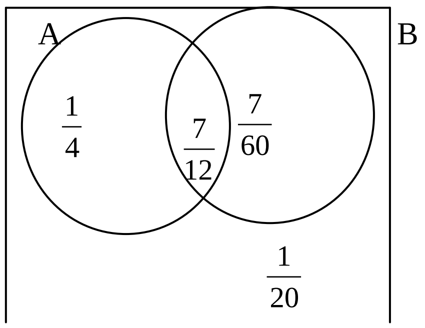
$¼ + 7/12 + 7/60 + z = 1$

$\frac{15+35+7}{60} + z = 1$

$\frac{57}{60} + z = 1$

$z = 1 - \frac{57}{60} = \frac{3}{60} = \frac{1}{20}$

Using the Venn diagram



(i) $P(A) = ¼ + x$

$= ¼ + {}^7/_{12} = \frac{3+7}{12}$

$= {}^{10}/_{12} = {}^5/_6$

$\therefore P(A) = {}^5/_6$ #

(ii) $P(B) = x + y$

${}^7/_{12} + {}^7/_{60} = \frac{35+7}{60}$

$= \frac{42}{60}$

$= {}^7/_{10}$

$\therefore P(B) = {}^7/_{10}$

(iii) $P(A \cap B) = P(A) \times P(B)$

$= {}^5/_6 \times {}^7/_{10} = {}^7/_{12}$ #

(iv) $P(A \cup B)^1 = z$

$= {}^1/_{20}$ #

Example 25:

In a survey of newspaper reading habits of members of staff of a university, it

is found that 60% read New Vision (N), 40% read monitor (M) and 30% read the East African (E)' . Further 20% read both M and N, 15% read both N and 10% read both M and E. 20% read New Vision only.

(a) If a member of staff is chosen at random from the university community find the probabilities:

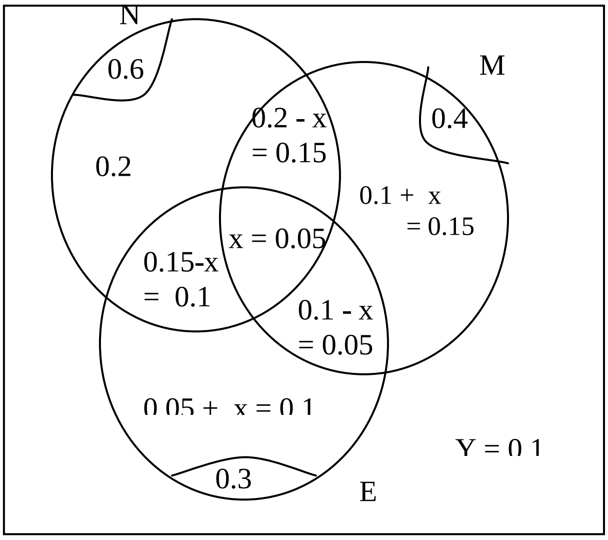
(i) that the member reads none of the three papers.

(ii) the member is one of those who read at least one of the three papers.

(b) Estimate the number of members of staff who read at least two papers if the total number is 500.

(c) What is the probability that, given that a member of staff reads at least two newspapers, he reads all the three.

Solution:



Let y = read none of the three

From the circle of New Vision

$$0.2 + 0.2 - x + x + 0.15 - x = 0.6$$

$$0.55 + x = 0.6$$

$$x = 0.05$$

(i) $y + 0.6 + 0.1 + 0.05 + 0.15 = 1$

$$y + 0.9 = 1$$

$$y = 0.1$$

(ii) P(those who read at least one) = 1 – P(those who read none)

$$= 1 - 0.1$$

$$= 0.9$$

(b) P(those who read at least two) = 0.05 + 0.15 + 0.05 + 0.1 = 0.35

$\therefore P = 0.35 \quad n = 500$

$E(n) = np$

$= 500 \times 0.35$

$= 175$ members.

(c) Let T = reads at least two

A = reads all the three

$P(T) = 0.35 \quad P(A) = 0.05$

$$P(A/T) = \frac{P(A \cap T)}{P(T)}$$

$P(A \cap T) = 0.05, \quad P(T) = 0.35$

$$P(A/T) = \frac{0.05}{0.35}$$

$$= \frac{5}{35}$$

$$= \frac{1}{7}$$

$$= \mathbf{0.1429}$$

Example26. The probability that two independent events occur together is $\frac{2}{15}$. The probability that either or both events occur is $\frac{3}{5}$.
Find the individual probabilities of the two events.

Solution:
Let the two events be A and B
Given

$$P(A \cup B) = \frac{3}{5}$$

$$P(A \cap B) = \frac{2}{15}$$

Since A and B are independent

$$P(A \cap B) = P(A) \times P(B) = \frac{2}{15}$$

For clarity in computation, let P(A) = x and P(B) = y
and using

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\Rightarrow \frac{3}{5} = x + y - \frac{2}{15} \quad (1)$$

But Also $P(A) \times P(B) = \frac{2}{15}$

$$\Rightarrow x \times y = \frac{2}{15}$$

$$\Rightarrow y = \frac{2}{15x} \quad (2)$$

Substituting for y in (1)

$$\frac{3}{5} = x + \frac{2}{15x} - \frac{2}{15} \quad (3)$$

$$9x = 15x^2 + 2 - 2x$$

$$15x^2 - 11x + 2 = 0 \quad (4)$$

Solving for x in equation (4)

$$15x^2 - 6x - 5x + 2 = 0$$

$$3x(5x - 2) - 1(5x - 2) = 0$$

$$(5x - 2)(3x - 1) = 0$$

Thus $x = {}^2/_5$ or $x = {}^1/_3$

When $x = \frac{2}{5}$, $y = \frac{2}{15x} = \frac{1}{3}$

When $x = \frac{1}{3}$, $y = \frac{2}{15x} = \frac{2}{5}$

Hence

$P(A) = \frac{2}{5}$ and `$P(B) = \frac{1}{3}$

or $P(A) = \frac{1}{3}$ and $P(B) = \frac{2}{5}$

***i.e.* the probabilities are interchangeable**

Example27:

The probability of two independent events P and Q occurring together is $^1/_8$. The probability that either or both events occur is $^5/_8$. Find:

(i)Prob (*P*)

(ii)Prob (*Q*)

Solution:

$P(A\cap B) = {}^1/_8$ (i)

$P(A\cup B) = {}^5/_8$ (ii)

A and B are independent

$P(A\cap B) = P(A) \times P(B)$

$= {}^1/_8$

Now, $P(A\cup B) = P(A) + P(B) - P(A\cap B)$

${}^5/_8 = P(A) + P(B) - P(A).P(B)$

$= P(A) + P(B) - {}^1/_8$

$P(A) + P(B) = {}^5/_8 + {}^1/_8 = {}^6/_8 = 3/4$

$= ¾$ (iii)

Let $P(A) = x$ and $P(B) = y$

$x + y = ¾$ (iv)

$xy = {}^1/_8$ (v)

$x({}^3/_4 - x) = {}^1/_8$

${}^3/_4x - x^2 = {}^1/_8$

$6x - 8x^2 = 1$

$8x^2 - 6x + 1 = 0$

$8x^2 - 4x - 2x + 1 = 0$

$4x(2x-1) - 1(2x-1) = 0$

$(2x-1)(4x-1) = 0$

$x = {}^1/_2$ or $x = ¼$

If $x = ½$, $y = ¼$

If $x = ¼$, $y = ½$

Example 28:

A survey of 500 students taking one or more courses in algebra, physics and Statistics during one of the semester revealed the following numbers of students in the indicated subjects.

Algebra 329, Algebra and Physics 83

Physics 186, Algebra and Statistics 217

Statistics 295 ,Physics and Statistics 63

Determine the probability that a student takes:

(a)all 3 subjects

(b)algebra but not statistics

(c)physics but not algebra

(d)statistics but not physics

(e)algebra or statistics but not physics

(f)algebra but not physics or statistics.

Solution:

(a) Let A = Algebra,

B = Physics and

C = Statistics.

$n(AUBUC) = n(A) + n(B) + n(C) - n(A\cap B) - n(B\cap C) - n(C\cap A) + n(A\cap B\cap C)$

$n(AUBUC) = 500,$

$n(A) = 329,$

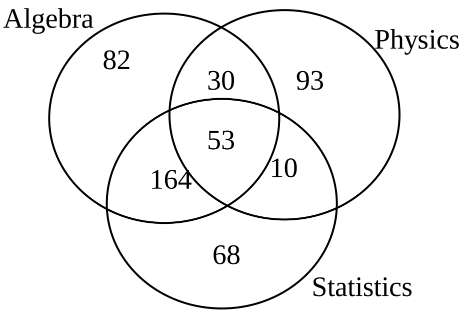
$n(B) = 186,$

n(C) = 295,
n(A∩B) = 83
n(A∩C) = 217
n(B∩C) = 63
500 = 329 + 186 + 295 – 83 – 217 – 63 + n(A∩B∩C)
n(A∩B∩C) = 53

$$P(A\cap B\cap C) = \frac{n(A \cap B \cap C)}{n(AUBUC)}$$

$$= \frac{53}{500}$$

= 0.106



n(Algebra but not statistics) = 82 + 30
= 112

$$P(\text{Algebra but not statistics}) = \frac{112}{500}$$

= 0.224

n(Physics but not algebra) = 93 + 10 = 103

$$P(\text{Physics but not algebra}) = \frac{103}{500} = 0.206$$

n(Statistics but not Physics) = 68 + 164 = 232

$$P(\text{Statistics but not physics}) = \frac{232}{500}$$

= 0.464

n(Algebra or statistics but not physics) = 164 + 82 + 68 = 314

n(Algebra but not physics or statistics) = 82

$$P(\text{Algebra but not physics or statistics}) = \frac{82}{500} = 0.164$$

Example29.
Events A and B are such that P(A) = ½ , P(B) = $^3/_8$ and P(A/B) = $^7/_{12}$.Find the
(i) P(A∩B
(ii) $P\left(B/\overline{A}\right)$

Solution:
P(A) = ½ , P (B) = $^3/_8$ P($^A/_B$) = $^7/_{12}$
(i) P (A∩B) = ?

$$P(^A/_B) = \frac{P(A \cap B}{P(B)}$$

P(A∩B) = P(A/B) x P(B)

$$= \frac{7}{12} \times \frac{3}{8} = \frac{7}{32}$$

(ii) $P(B/\bar{A}) = \dfrac{P(B \cap \bar{A})}{P(\bar{A})}$

$$= \frac{P(B) - P(B \cap A)}{1 - P(A)}$$

$$= \frac{3/8 - 7/32}{1 - \frac{1}{2}} = \frac{(12-7)/32}{½}$$

$$= \frac{5}{32} \times \frac{2}{1} = \frac{5}{16}$$

$\therefore \quad P(B/\bar{A}) = \dfrac{5}{16}$ #

Example 30:

(a)Aisha's chances of passing Physics are 0.60, of Chemistry 0.75 and of Mathematics 0.80

Determine the chances that she passes one subject only.

If it is known that she passes at least two subjects, what is the probability that she failed chemistry?

(b) Two biased tetrahedrons have each their face numbered 1 to 4. The chances of Getting any one face showing upper most is inversely proportional to the number on it.
If the two tetrahedrons are thrown and the number on the uppermost face noted,determine the probability that the faces show the same number.

Solution:

(a) (i) P(Passing only one subject) = P(Passing math only)
+ P(Passing Chemistry only) + P (Passing Physics only)

= 0.8 x 0.4 x 0.25 + 0.2 x 0.75 x 0.4 + 0.6 x 0.25 x 0.2

= 0.08 + 0.06 + 0.03

= 0.17

(ii)P(failed Chemistry / Passed at least two subjects) = $\dfrac{\text{P(Failed Chemistry only)}}{\text{P(Passed at least two subjects}}$

$$= \frac{0.25 \times 0.6 \times 0.8}{0.8 \times 0.75 \times 0.6 + 0.2 \times 0.75 \times 0.6 + 0.4 \times 0.6 \times 0.75 + 0.25}$$

$$= \frac{0.12}{0.36 + 0.09 + 0.24 + 0.12}$$

$$= \frac{0.12}{0.81}$$

$$= \frac{4}{7}$$

= **0.1481**

(b) $P(x) \; \alpha \; \dfrac{1}{x} \qquad P(x) = \dfrac{k}{x}$

at x = 1 P(1) = k, at x = 2 $P(2) = \dfrac{k}{2}$

at x = 3 $P(3) = \dfrac{k}{3}$, at x = 4 $P(4) = \dfrac{k}{4}$

$$\therefore \quad k + \frac{k}{2} + \frac{k}{3} + \frac{k}{4} = 1$$

$$k\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) = \frac{k(12 + 6 + 4 + 3)}{12}$$
$$= 1$$

$$\therefore \quad \frac{25k}{12} = 1$$

$$k = \frac{12}{25}$$

Number of tetrahedrons	1	2	3	4
Probabilities	$\frac{12}{25}$	$\frac{6}{25}$	$\frac{4}{25}$	$\frac{3}{25}$

$$\therefore \text{ P(Same number)} = \left(\frac{12}{25}\right)^2 + \left(\frac{6}{25}\right)^2 + \left(\frac{4}{25}\right)^2 + \left(\frac{3}{25}\right)^2$$
$$= \frac{205}{625}$$
$$= \frac{41}{125}$$
$$= \mathbf{0.328}$$

EXERCISE 2

1.A stack of 20 cards contains 4 cards of each of 5 different colours, namely white, black, green, red and blue. A part from the colour the cards are indistinguishable. A set of 3 cards is draw at random from the stack. Find the chances that the 3 cards are:

(i)all white

(ii)all of one colour

(iii) of different colours

2. Box A contains 3 white and 3 black balls and box B contains 4 white and 3 black balls. One ball is transferred from A to B. One ball is then drawn from B and is found to be white. What is the probability that the transferred ball was white?

3.A sample poll of 200 Voters revealed the following information concerning three candidates A, B and C of a certain party who were running for three different offices. 28 in favour of both A and B, 122 in favour of B or C but not A, 98 in favour of A or B but not C, 64 in favour of C but not A or B, 42 in favour of B but not C, 14 in favour of A and C but not B.

Find the probability of voters who were in favour of :

(a) all the three candidates

(b) A irrespective of B or C

(c) B irrespective of A or C

(d) C irrespective of A or B

(e) A and B but not C

(f) only one of the candidates.

4.Three cards are drawn from a deck of 52 cards. Find the probability that:

(a) two are jacks and one is a King

(b) all cards are of one suit.

(c) all cards are of different suits.
(d) at least two aces are drawn.

5.. If 10% of the rivets produced by a machine are defective, what is the probability that out of 5 rivets chosen at random:
(a) none will be defective
(b) one will be defective
(c) at least two will be defective?

6.. Two marbles are drawn in succession from the box containing 10 red, 30 white, 20 blue 15 orange marbles. Replacement being made after each drawing. Find the probability that:
(a)both are white.
(b)the first is red and the second is white.
(c)neither is orange
(d)they are either red or white or both (red and white)
(e)the second is not blue.
(f)the first is orange.
(g)at least one is blue.
(h)at most one is red.
(i)the first is white but the second is not.
(j)only one is red.

7..A factory has three machines 1,2 and 3 producing a particular type of item. One item is drawn at random from the factory is production. Let B denote the event that the chosen item is defective and let A_K denote the event that the item was produced on machine K, where K = 1,2 or 3. Suppose that the machines 1,2 and 3 produce respectively 35%, 45% and 20% of the total production of items and that $P(B/A_1) = 0.02$, $P(B/A_2) = 0.01$
$P(B/A_3) = 0.03$, Given that an item chosen at random is defective. Find which machine was the most likely to have produced it.

8..Three events A, B and c are defined in the sample space. The events A and C are mutually exclusive. The events A and B are independent.

Given that $P(A) = \frac{1}{3}$, $P(C) = \frac{1}{5}$, $P(AUB) = \frac{2}{3}$ Find:
(a)P(AUC)
(b)P(B)
(c)$P(A\cap B)$

Given also that $P(BUC) = \frac{3}{5}$, determine whether or not B are independent.

9.. A die with faces numbered 1 to 6 is biased so that P(score is r) =Kr
(r = 1, ……, 6). Find:
(i) the value of k
(ii) if the die is thrown twice, calculate the probability that the total score exceeds 10 .

10…The probabilities that a man make a certain dangerous journey by car, motor cycle or on foot are $\frac{1}{2}$, $\frac{1}{6}$ and $\frac{1}{3}$ respectively. If the probabilities of an accident when he uses these means of transport are $\frac{1}{5}$, $\frac{2}{5}$ and $\frac{1}{10}$ respectively, Find the probability of an accident occurring in a single journey . If an accident is known to have happened. Calculate the probabilities that the man was traveling:
(a)by car
(b)by motor cycle
(c)on foot

11..In a workshop three machines produce compression springs of a certain size Machine A produces 40% of the total, Machine B 25% and Machine the remainder. On average 6% of the springs produced by A do not conform to the tolerance requirements. The corresponding percentages for B and C are 4% and 3% respectively. One spring elected at random from the whole output, is found to be unsatisfactory. Calculate the probability that it was produced by A.

12..Events A and B are such that P(A) = $\frac{5}{12}$ $P\left(A/\overline{B}\right) = \frac{7}{12}$

$P(A\cap B) = \frac{1}{8}$ Determine:

(a)P(B)
(b)P(A/B)
(c)P(B/A)
(d)P(AUB)
State whether events A and B are:
(i)mutually exclusive
(ii) independent

13.In a large group of people it is known that 10% have a hot breakfast, 20% have lunch and 25% have a hot breakfast or a hot lunch. Find the probability that a person chosen at random from this group:

has a hot breakfast and a hot lunch
has a hot lunch, given that the person chosen had a hot breakfast.

14.Two cards are drawn successively from an ordinary deck of 52 well –shuffled cards. Find the probability that:

(a) the first card is not a ten of clubs or an ace.
(b) the first card is an ace but the second is not
(c) at least one card is a diamond
(d) the cards are not of the same sit
(e) not more than one card is a picture card (Jack, Queen, King)
(f) the second card is not a picture card
(g) the second card is not a picture card given that the first was a picture card
(h) the cards are picture cards or spade or both.

ANSWERS

1. (i) $\frac{1}{285}$ (ii) $\frac{1}{57}$ (iii) $\frac{32}{57}$

2. $\frac{5}{9}$

3. (a) $\frac{1}{25}$ (b) $\frac{39}{100}$ (c) $\frac{43}{100}$ (d) $\frac{51}{100}$ (e) $\frac{1}{10}$ (f) $\frac{71)}{100}$

4. (a) $\frac{6}{5525}$ (b) $\frac{22}{425}$ (c) $\frac{169}{425}$ (d) $\frac{73}{5525}$

5.((a) 0.59049 (b) 0.32805 (c) 0.08866)

6.(a) 4/25 (b) 4/75 (c) 16/25 (d) 64/225 (e) 11/15 (f) 1/5 (g) 104/225 (h) 221/225 (i) 6/25 (j) 52/225)

7.(Machine C probability = 0.4)

8.(a) $\frac{8}{15}$ (b) $\frac{1}{2}$ (c) $\frac{1}{6}$ Independent

9. (i) $\frac{1}{21}$ (ii) $\frac{32}{147}$)

10. $\frac{1}{5}$ (a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$

11. (48/89)

12. (a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{3}{10}$ (d) $\frac{19}{24}$ (i) No (ii) No

13. (a) 0.05 (b) 0.5)

14. (a) $\frac{47}{52}$ (b) $\frac{16}{221}$ (c) $\frac{31}{34}$ (d) $\frac{13}{17}$ (e) $\frac{210}{221}$ (f) $\frac{10}{13}$ (g) $\frac{40}{51}$ (h) $\frac{77}{442}$

3. DISCRETE PROBABILITY DISTRIBUTION.

Introduction:

A **discrete probability distribution** shall be understood as a *probability distribution* characterized by a **probability mass function**. Thus, the distribution of a **random variable** *X* is discrete, and *X* is then called a **discrete random variable**, if

$$\sum_{i=1}^{n}(X = x) = 1$$

as x runs through the set of all possible values of X. It follows that such a random variable can assume only a **finite or countably infinite** number of values.

In cases more frequently considered, this set of possible values is a topologically discrete set in the sense that all its points are **isolated points**. But there are discrete random variables for which this countable set is **dense** on the real line (for example, a distribution over **rational numbers).**

Among the most well-known discrete probability distributions that are used for statistical modeling are the Poisson distribution, the Bernoulli distribution, the binomial distribution, the geometric distribution, and the negative binomial distribution. In addition, the discrete uniform distribution is commonly used in computer programs that make equal-probability random selections between a number of choices.

Mathematically:

A discrete random variable X is defined as a probability function P(X = x), which gives the probability that X takes on values of x as a probability denoted by P(X = x) or P_i

Where $\sum_{i=1}^{n} P_i = 1$

(i) **EXPECTATION OR EXPECTED VALUE (E(X)) OR MEAN OF DISCRETE RANDOM VARIABLE**

$$\text{Mean } (\mu) = E(X)$$

$$= \sum_{\text{all } x}^{n} xP(X = x)$$

If we replace P(X = x) with Pi and for which X takes on the value xi for i = 1, 2, … S, n then,

$$\text{Mean}(\mu) = E(X)$$

$$= \sum_{i=1}^{n} P_i X_i \textbf{ for } i = 1, 2, , , , n$$

(ii) VARIANCE AND STANDARD DEVIATION OF A DISCRETE RANDOM VARIABLE

$$\text{Var}(X) = \sum_{\text{all } x} P_x (X - \mu)^2$$

$$= E(X^2) - [E(X)]^2$$

$= \quad E(X^2) - \mu^2$

Standard deviation $= \sqrt{\text{Var}X}$

$= \sqrt{E(X^2) - \mu^2}$

Where $E(X^2) = \sum_{\text{all } x} x^2 p(X = x)$

$= \sum_{i=1}^{n} P_i x_i^2$ for i = 1,2, …, n

(iii) MEDIAN (M) OF A DISCRETE RANDOM VARIABLE

Cumulate the probabilities from the top downwards until a value of 0.5 or slightly above 0.5 is obtained, also cumulate the probabilities from the bottom upwards until a value of 0.5 or slightly above 0.5 is obtained, for the values obtained read off the corresponding X values. If they are different find the average of the two. This explained as below.

$$\sum_{i=1}^{M} P_i \quad \geq \quad \frac{1}{2} \quad \leq \quad \sum_{i=M}^{N} P_i$$

If the two different values that satisfy the inequality are X_m and X_{m+1} then the median (m) $= \dfrac{X_m + X_{m+1}}{2}$

(iv) MODE OF A DISCRETE RANDOM VARIABLE

The mode of a discrete random variable is the value of X with the highest probability.

(v) DISCRETE UNIFORM DISTRIBUTION OR RECTANGULAR DISTRIBUTION

A discrete random variable X, taking on values 1, 2, 3,…k such that:

$$P(X = x) = \begin{cases} \dfrac{1}{k} & \text{for} \quad x = 1, 2, 3, \ldots k \\ 0 & \text{otherwise} \end{cases}$$

follows a rectangular or discrete uniform distribution.

Note:

$E(a) = a$

$E(ax) = a\,E(x)$,

$E(ax + b) = a\,E(x) + b$,

$\text{Var}(ax) = a^2\ \text{Var}(x)$

$\text{Var}(ax + b) = a^2\ \text{Var}(x)$

$E(x + y) = E(x) + E(y)$,

$E(x - y) = E(x) - E(y)$,

$E(ax + by) = a\,E(x) + b\,E(y)E(ax - by) = a\,E(x) - b\,E(y)$,

$\text{Var}(x + y) = \text{Var}(x - y) = \text{Var}(x) + \text{Var}(y)$

$\text{Var}(ax + by) = \text{Var}(ax - by) = a^2\,\text{Var}(x) + b^2\,\text{Var}(y)$

Example 1. A random variable x has probability function P(0) = 0.1. P(1) = 0.3, P(2) = 0.4 and P(4) = 0.2. Determine mean and standard deviation:

Solution.

X_i	0	1	2	4

P_i	0.1	0.3	0.4	0.2

Mean (μ) $= \sum X_i P_i$

$= 0 \text{ x } 0.1 + 1 \text{ x } 0.3 + 2 \text{ x } 0.4 + 4\text{x}0.2$

Mean (μ) = 1.9

Standard deviation $= \sqrt{E(X^2) - \mu^2}$

$E(X^2) = \sum X_i^{\,2} P_i$ = 02 x 0.1 +12 x 0.3 + 22 x 0.4 + 42 x 0.2

= 5.1

Standard deviation $= \sqrt{5.1 - (1.9)^2}$

= 1·2207

Example 2: The number of times a machine breaks down every month is a discrete random variable X with probability distribution.

$$\mathbf{P(x = x) =} \begin{cases} k\left(\frac{1}{4}\right)^x & x = 0, 1, 2, 3, \ldots. \\ 0 & \text{otherwise} \end{cases}$$

Where k is a constant.

Determine the probability that the machine will break down not more than two times a month.

Solution:

$$k\left[\left(\frac{1}{4}\right)^0 + \left(\frac{1}{4}\right)^1 + \left(\frac{1}{4}\right)^2 + \cdots + \left(\frac{1}{4}\right)^n\right] = 1$$

$$k\left[\,1 + \frac{1}{4} + \frac{1}{4^2} + \ldots + \frac{1}{4^n}\,\right] = 1 \qquad \ldots\ldots\ldots\ldots\ldots \quad *$$

It's a geometric progression (G.P) with a =1 and common ratio r = $\frac{1}{4}$

Sum of a G.P. = $\frac{a}{1-r}$

$$= \frac{1}{1 - \frac{1}{4}}$$

$$= \frac{1}{\frac{3}{4}} = \frac{4}{3} \qquad \ldots\ldots\ldots\ldots\ldots \quad **$$

Substitute ** into *

$$\frac{4k}{3} = 1$$

$$\therefore k = \frac{3}{4}$$

Probability that the machine will break down not more than two times.

$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$

$$= \frac{3}{4} + \frac{3}{4^2} + \frac{3}{4^3}$$

$$= \frac{48 + 12 + 3}{4^3}$$

$$= \frac{63}{64} = \mathbf{0.9844}$$

Example 3. A discrete random variable X has the following probability distribution.

X	**1**	**2**	**3**	**4**	**5**
P(X = x)	**k**	**2k**	**3k**	**4k**	**5k**

(i) **Determine the value of k.**
(ii) **Evaluate P(2 < X < 4)**
(iii) **Calculate mean, median, mode and standard deviation.**

Solution.

(i) $\sum P_i = 1$

$k + 2k + 3k + 4k + 5k = 1$

$15k = 1$

$\therefore k = \frac{1}{15}$

X	1	2	3	4	5
P(X = x)	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{5}{15}$

(ii) $P(2 < X < 4) = P(X = 3)$

$= \frac{3}{15}$

a. Mean (μ) $= \sum P_i X_i$

$= \frac{1}{15} \times 1 + \frac{2}{15} \times 2 + \frac{3}{15} \times 3 + \frac{4}{15} \times 4 + \frac{5}{15} \times 5$

$= \frac{55}{15} = \frac{11}{3}$

$=$ **3.6667**

X	P_i	Cumulative probability	
1	$\frac{1}{15}$	$\frac{1}{15}$	
2	$\frac{2}{15}$	$\frac{3}{15}$	
3	$\frac{3}{15}$ Location of median	$\frac{6}{15}$	
4	$\frac{4}{15}$	$\frac{10}{15}$	$\frac{9}{15}$
5	$\frac{5}{15}$		$\frac{5}{15}$

Median $=$ **4**

The highest frequency $= \frac{5}{15}$

$\therefore$ **Mode** $=$ **5**

Standard deviation $= \sqrt{\sum X^2_i P_i - (\mu)^2}$

$= \sqrt{1^2 \times \frac{1}{15} + 2^2 \times \frac{2}{15} + 3^2 \times \frac{3}{15} + 4^2 \times \frac{4}{15} + 5^2 \times \frac{5}{15} - \left(\frac{11}{3}\right)^2}$

$= \sqrt{1.5556}$

$= \mathbf{1.2471}$

Example 5. **A discrete random variable can assume values 0, 1, 2, 3, only. Given that (P(X $\le$ 2) = 0.9, P(X $\le$ 1) = 0.5 and E(X) = 1.4 determine:**

- **(i)** **P(X =1)**
- **(ii)** **P(X = 0)**
- **(iii)** **Median and mode**
- **(iv)** **Standard deviation**

Solution:

$P_0 + P_1 + P_2 + P_3 = 1$ (i)

$P_0 + P_1 + P_2 = 0.9$ (ii)

Substitute (ii) into (i)

$0.9 + P_3 = 1$

$\therefore$ $P(X = 3) = 0.1$

$P_0 + P_1 = 0.5$ (iv)

Substitute (iv) into (ii)

$0.5 + P_2 = 0.9$

$P_2 = 0.4$

$\mathbf{P(X = 4) = 0.4}$

$E(X) = X_oP_o + X_1P_1 + X_2P_2 + X_3P_3$

$= 1.4$

$0 \text{ x } P_0 + P_1 + 2 \text{ x } 0.4\ 3 \text{ x } 0.1 = 1.4$

$P_1 = 0.3$

$\therefore$ $\mathbf{P(X = 1) = 0.3}$

(ii) $P_0 + P_1 + P_2 + P_3 = 1$

$P_0 + 0.3 + 0.4 + 0.1 = 1$

$P_0 = 0.2$

$\mathbf{P(x = 0) = 0.2}$

(iii)

X	P	Cumulative probability
0	0.2	0.2
1	0.3	0.5
2	0.4	0.6
3	0.5	0.1

← Location of median

Median $= \frac{1+2}{2} = \frac{3}{2}$

$= 1.5$

The highest probability is 0.5

Mode = 2

(iv) Standard deviation $= \sqrt{\sum X^2 P_i - (\mu)^2}$

$= \sqrt{(0^2 \text{ x } 0.2 + 1^2 \text{ x } 0.3 + 2^2 \text{ x } 0.5 + 3^2 \text{ x } 0.1 - (1.4)^2}$

$= \sqrt{3.2 - 1.96} = \mathbf{1.1136}$

Example 6.Find the variance of the sum of the scores when an ordinary die is thrown 10 times.

Solution:

To find the variance of the sum of the scores when an ordinary die is thrown 10 times, we have;

$Var(x_1 + x_2 + x_3 + \dots\dots\dots + x_{10})$

But since it the same die, that is repeatedly thrown,

$x_1 = x_2 = \dots\dots\dots x_{10} = x$

$$Var(x_1 + x_2 + \dots + x_{10}) = Var(x_1) + Var(x_2) + \dots\dots + Var(x_{10})$$
$$= Var(x) + Var(x) + \dots\dots + Var(x_{10})$$
$$= 10\,Var(x)$$

x	1	2	3	4	5	6
x2	1	4	9	16	25	36
P(x = x)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$Var(x) = E(x^2) - [E(x)]^2$$

Now

$$E(x) = 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right)$$
$$= \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6}$$
$$= \frac{21}{6} = \frac{7}{2}$$
$$E(x^2) = 1\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 9\left(\frac{1}{6}\right) + 16\left(\frac{1}{6}\right) + 25\left(\frac{1}{6}\right) + 36\left(\frac{1}{6}\right)$$
$$= \frac{91}{6}$$
$$Var(x) = \frac{91}{6} - \left(\frac{7}{2}\right)^2$$

The required variance is therefore given by;

$$10\,Var(x) = 10\left(\frac{91}{6} - \frac{49}{4}\right) = 29\frac{1}{6}$$

Thus, the variance of the sum of the scores when an ordinary die is thrown is $29\frac{1}{6}$

Example 7. **A committee of 3 is to be chosen from 4 girls and 7 boys. Find the expected number of girls on the committee, if the members of the committee are chosen at random**.

Solution.

This is a discrete random variable.

The number of ways of choosing 3 committee members from a total of 11 people is

$$\binom{11}{3} = \frac{11!}{3!8!} = 165$$

The probability that the committee contains only x girls is

$$P(X = x) = \frac{\binom{4}{x}\binom{7}{3-x}}{\binom{11}{3}} = \frac{\binom{4}{x}\binom{7}{3-x}}{165}$$

When $x = 0$

$$P(X=0) = \frac{\binom{4}{0}\binom{7}{3}}{165} = \frac{35}{165}$$

When x = 1

$$P(X=1) = \frac{\binom{4}{1}\binom{7}{2}}{165} = \frac{84}{165}$$

When x = 2

$$P(X=2) = \frac{\binom{4}{2}\binom{7}{1}}{165} = \frac{42}{165}$$

When x = 3

$$P(X=3) = \frac{\binom{4}{3}\binom{7}{0}}{165} = \frac{4}{165}$$

The probability distribution function is then given below

x	0	1	2	3
P(X = x)	$\frac{35}{165}$	$\frac{84}{165}$	$\frac{42}{165}$	$\frac{4}{165}$

The expected number of girls E(X) is

$$E(X) = \sum_{x=0}^{3} XP(X=x)$$

$$= 0\left(\frac{35}{165}\right) + 1\left(\frac{84}{165}\right) + 2\left(\frac{42}{165}\right) + 3\left(\frac{4}{165}\right)$$

$$\mathbf{E(X)} = \frac{12}{11}.$$

Example 8. A random variable X has the probability function.

$$\mathbf{f(x)} = \begin{cases} k2^x & ; \quad x = 0, 1, 2, \ldots, 6 \\ 0 & , \quad \text{else where} \end{cases}$$

Determine: (i) the value of k

(ii) E(X)

(iii) P(X < 4/ X > 1)

Solution:

(i) $k\{2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^4 + 2^6 = 1$

$k\{1 + 2 + 4 + 8 + 16 + 32 + 64\} = 1$

$127k = 1$

$$\mathbf{k} = \frac{1}{127}$$

(ii) $E(X) = \sum P_i X_i$

$$= 0\times\frac{1}{127} + 1\times\frac{2}{127} + 2\times\frac{4}{127} + 3\times\frac{8}{127} + 4\times\frac{16}{127} + 5\times\frac{32}{127} + 6\times\frac{64}{127}$$

$$= \frac{642}{127}$$

$= \mathbf{5.0551}$

(iii) Let A = $x < 4$ $\quad$ B = $x > 1$

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A\cap B) = \frac{4}{127} + \frac{8}{127}$$
$$= \frac{12}{127}$$
$$P(B) = 1 - \frac{2}{127}$$
$$= \frac{125}{127}$$
$$= \frac{12/127}{125/127}$$
$$= \frac{12}{127} = \mathbf{0.096}$$

Example 9. **A random variable X has probability density function P(X = x) given below.**

$$\mathbf{P(X = x) =} \begin{cases} \frac{x}{2k} & , x = 1, 2, 3, \ldots n \\ 0 & , \text{else where} \end{cases}$$

Given that E(X) = 3, Find:

- **(i)** **the value of k**
- **(ii)** **the value of n**
- **(iii)** **the median and mode**
- **(iv)** **variance**

Solution:

X	1	2	3	…..	n
P(X = x)	$\frac{1}{2K}$	$\frac{2}{2K}$	$\frac{3}{2K}$		$\frac{n}{2k}$

From $\sum P_i = 1$

$$\frac{1}{2k} + \frac{2}{2k} + \frac{3}{2k} + \ldots + \frac{n}{2k} = 1$$
$$\frac{1}{2k}(1 + 2 + 3 + \ldots + n) = 1$$
$$\sum_1^n n = \frac{n}{2}(n + 1)$$
$$\frac{n(n+1)}{4k} = 1 ,$$
$$k = \frac{n(n+1)}{4} \quad \ldots\ldots\ldots\ldots\text{(i)}$$
$$E(X) = \sum P_i X_i$$
$$= \frac{1}{2k} + \frac{2^2}{2k} + \frac{3^2}{2k} + \ldots + \frac{n^2}{2k}$$
$$\sum^n n^2 = \frac{n}{6}(n + 1)(2n + 1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots * $$
$$E(X) = \frac{n}{2k}(1 + 2^2 + 3^2 + \ldots + n^2) = 3 \quad \ldots. **$$

Substitute * into **

$$\frac{n}{12k}(n+1)(2n+1) = 3$$

$$\frac{n(n+1)(2n+1)}{36} = k \quad \ldots\ldots\ldots\ldots\text{(ii)}$$

Equate equation (i) and (ii)

$$\frac{n(n+1)}{4} = \frac{n(n+1)(2n+1)}{36}$$

$$\frac{36}{4} = 2n+1$$

$$2n+1 = 9$$

$$2n = 8$$

$$\mathbf{n = 4}$$

Using equation (i)

$$k = \frac{n(n+1)}{4}$$

$$= \frac{4\times 5}{4}$$

$$= 5$$

(i) **k = 5**

(ii)

X	1	2	3	4
P(X =x)	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{4}{10}$

X	P_i	Cumulative probability	
1	$\frac{1}{10}$	$\frac{1}{10}$	
2	$\frac{2}{10}$	$\frac{3}{10}$	
3	$\frac{3}{10}$	$\frac{6}{10}$	$\frac{7}{10}$
4	$\frac{4}{10}$		$\frac{4}{10}$

Location of median

Median = 3

The highest probability $= \frac{4}{10}$

:. Mode = 4

$$\text{Var}(X) = \sum X_i^2 P_i - (E(x))^2$$

$$= 1^2 \times \frac{1}{10} + 2^2 \times \frac{2}{10} + 3^2 \times \frac{3}{10} + 4^2 \times \frac{4}{10}$$

$$= 10 - (3)^2 \quad = \quad 1$$

:. Var(X) = 1

Example 10. A discrete random variable X, has the following probability distribution.

X	**1**	**2**	**3**	**4**
P(X)	$\frac{1}{16}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{16}$

Find: (i) the mean

(ii) median and mode

(iii) variance of x

Solution:

(i) Mean E(X) $= \sum xP(X)$

$$= 1 \times \frac{3}{16} + 2 \times \frac{8}{16} + 3 \times \frac{4}{16} + 4 \times \frac{1}{16}$$

$$= \frac{35}{16}$$

$= \mathbf{2.25}$

(ii)

x	P	Cumulative probability	
1	$\frac{3}{16}$	$\frac{3}{16}$	
2	$\frac{8}{16}$	$\frac{11}{16}$	$\frac{12}{16}$
3	$\frac{4}{16}$		$\frac{4}{16}$
4	$\frac{1}{16}$		$\frac{1}{16}$

Location of median

:. Median = 2

Highest probability $= \frac{8}{16}$

:. Mode = 2

(iii) Var(X) $= \sum x^2 P(X) - (E(X))^2$

$$= \left[1\times\frac{3}{16} + 4\times\frac{8}{16} + 9\times\frac{4}{16} + 16\times\frac{1}{16}\right] - \left(\frac{35}{16}\right)^2$$

$$= \frac{87}{16} - \left(\frac{35}{16}\right)^2$$

$$= \frac{167}{256}$$

= 0.6523

Example 11: A discrete random variable X is represented by the probability function

$$\textbf{P(X = x)} = \begin{cases} \frac{1+x}{kx} & ; \quad \text{for } x\, 1, 2, 3, \ldots, 6 \\ 0 & , \quad \text{elsewhere} \end{cases}$$

Find **(i) the value of k**
(ii) Expectation of X
(iii) Mode and median
(iv) P(X $\geq$ 3 / X $\leq$ 4)

Solution

(i)

X	1	2	3	4	5	6
$P(X = x) = \frac{1+x}{kx}$	$\frac{2}{k}$	$\frac{3}{k}$	$\frac{4}{k}$	$\frac{5}{k}$	$\frac{6}{k}$	$\frac{7}{k}$

$$\text{From } \sum P_i = 1$$

$$\frac{2}{k} + \frac{3}{k} + \frac{4}{k} + \frac{5}{k} + \frac{6}{k} + \frac{7}{k} = 1$$

$$\frac{1}{k}\left(\frac{120+90+80+75+72+70}{60}\right) = 1$$

$$\frac{507}{60k} = 1$$

$$k = \frac{507}{60}$$

X	1	2	3	4	5	6
P(X = x)	$\frac{120}{507}$	$\frac{90}{507}$	$\frac{90}{507}$	$\frac{75}{507}$	$\frac{72}{507}$	$\frac{70}{507}$

(ii) mean, (E(X)) $= \frac{120+180+240+300+360+420}{507}$

$$= \frac{1620}{507}$$

E(X) = 3.1953

(iii) the highest probability is $\frac{120}{507}$,

The mode is 1

X	P_i	Cumulative probability	
1	$\frac{120}{507}$	$\frac{120}{507}$	
2	$\frac{210}{507}$	$\frac{210}{507}$	
3	$\frac{290}{507}$	$\frac{290}{507}$	$\frac{297}{507}$
4	$\frac{75}{507}$		$\frac{217}{507}$
5	$\frac{72}{507}$		$\frac{142}{507}$

1	$\frac{120}{507}$	

← Location of median

Median = 3

(iv) let A = $X \geq 3$ B = $X \leq 4$

X	P_i	$W_i = 3$	$Z_i = 2x + 4$

$P(A/B) = \frac{P(A \cap B}{P(B)}$

$P(A \cap B) = P(X = 3) +$

P(X = 4)

$$= \frac{80}{507} + \frac{70}{507}$$

$$= \frac{155}{507}$$

$$P(B) = 1 - (P(X = 6))$$

$$= 1 - \left(\frac{72}{507} + \frac{70}{507}\right)$$

$$= 1 - \frac{142}{507}$$

$$= \frac{365}{507}$$

$$P(X \geq 3 / X \leq 4) = \frac{155/507}{365/507}$$

$$= \frac{155}{365}$$

$$= \mathbf{0.4247}$$

Example

12. The table below shows a random variable X with the following probability distribution.

X	**1**	**2**	**3**	**4**	**5**
P(X = x)	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

(i) Construct tables for the distributions W and Z, such that W = 3x and Z = 2x + 4.
(ii) Find the expectations of W and Z
(iii) Calculate the variance of Z.

Solution:

(i)

1	$\frac{1}{5}$	3	6
2	$\frac{1}{5}$	6	8
3	$\frac{1}{5}$	9	10
4	$\frac{1}{5}$	12	12
5	$\frac{1}{5}$	15	14

(ii) E(W) $= \sum W_iP_i$

$= \frac{1}{5}x3 + \frac{1}{5}x6 + \frac{1}{5}x9 + \frac{1}{5}x12 + \frac{1}{5}x15$

$= \frac{3+6+9+12+15}{5}$

$= \frac{45}{5} = 9$

E(W) = 9

E(Z) $= \sum Z_iP_i$

$= \frac{1}{5} \text{ x } 6 + \frac{1}{5} \text{ x } 8 + \frac{1}{5} \text{ x } 10 + \frac{1}{5} \text{ x } 12 + \frac{1}{5} \text{ x } 14$

$= \frac{6+8+10+12+14}{5}$

$= \frac{50}{5}$

= **10**

Example 13. **A random variable X has a probability density function f (X) given as**

X	**-1**	**0**	**1**
P(X = x)	**a**	$\frac{1}{2}$	**b**

Where a and b are the probability of P(x = -1) and P(x = 1) respectively.

Given that E(X) = $\frac{1}{6}$**, Determine**

(i) **the value of a and b**
(ii) **the variance and standard deviation**
(iii) **P(x > -1)**

Solution

(i) $\sum P_i = 1$

$a + \frac{1}{2} + b = 1$

$\therefore a + b = \frac{1}{2}$

or $2a + 2b = 1$ (i) x3

$6a + 6b = 3$ (ii)

$E(X) = \sum P_iX_i = \frac{1}{6}$

$-a + 0 + b = \frac{1}{6}$

$6b - 6a = 1$ (iii)

$6a + 6b = 3$ (from (ii))

$12b = 4$ (adding (ii) and (iii))

$b = \frac{4}{12}$

$= \frac{1}{3}$

$a = \frac{1}{2} - \frac{1}{3}$

$= \frac{3-2}{6}$

$= \frac{1}{6}$

:. a $=$ $\frac{1}{6}$ and **b** $=$ $\frac{1}{3}$

X	-1	0	1
P(X = x)	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$

(ii) Var(x) $= \sum X^2P_i - (E(X))^2$

$= (-1)^2 \times \frac{1}{6} + 0^2 \times \frac{3}{6} + 1^2 \times \frac{1}{2} - \left(\frac{1}{6}\right)^2$

$= \frac{3}{6} - \frac{1}{36}$

$= \frac{17}{36}$

$=$ **0.4722**

Standard deviation $= \sqrt{\text{Var}(X)}$

$= \sqrt{0.4722}$

$=$ **0.6872**

(iii) $P(x > -1) = P(X = 0) + P(X = 1)$

$= \frac{3}{6} + \frac{2}{6}$

$= \frac{5}{6}$

$=$ **0.83**

Example 14: The probability mass function P(X) of the random variable X is given by:

$$P(X) = \begin{cases} kx + d & ; \ x = -2, -1, 0, 1, 2 \\ 0 & , \text{ elsewhere} \end{cases}$$

Where k and d are constants.

(i) If P(x = 2) = 2P(x = -2), determine the values of k and d.

(ii) Compute the mean and variance of x.

(iii) What is the probability that x $\neq$ 0?

Solution:

(i) $\sum_{i=1}^{n} P_i = 1$

At $x = -2$ $P(-2) = -2k + d$

$x = -1$ $P(-1) = -k + d$

x = 0 P(0) = d

x = 1 P(1) = k + d

x = 2 P(2) = 2k + d

$\Rightarrow$ P(-2) + P(-1) + P(0) + P(1) + P(2) = 1

-2k + d – k +d + k + d + 2k + d = 1

5d = 1

$d = \frac{1}{5}$

P(X = 2) = 2P(X = -2)

-2k + d = 2(-2k + d)

6k = d

$6k = \frac{1}{5}$

$\mathbf{k} = \frac{1}{30}$

$$P(X) = \begin{cases} \frac{x}{30} + \frac{1}{5} & x = -2, -1, 0, 1, 2 \\ 0 & \text{, elsewhere} \end{cases}$$

(ii) Mean E(X) = $\sum_{i=1}^{n} X_i P_i$ substitute the values of x in P(x)

X	-2	-1	0	1	2
P	0.1333	0.1666	0.2	0.2333	0.2666

E(X) = -2 x 0.1333 – 1 x 0.1666 +0.0.2 +1 1 x 0.2333 + 2 x 0.2666

= **0.3333**

(iii) Either : $P(X \neq 0)$ = P(X = -2) + P(X = -1) + P(X = 1) + P(X = 2)

= 0.1333 + 0.1666 + 0.2333 + 0.2666

= 0.7998

$\approx$ **0.8**

Or : $P(X \neq 0)$ = 1 – P(x = 0)

= 1 – 0.2

= **0.8**

Example 15; A discrete random variable X has a probability density function.

P(X = x) =k $|x|$ where x takes the values -3,-2, -1, 0, 1, 2, 3. Find;

(a) the value of the constant K

(b) E(X)

(c) E(X^2)

(d) Standard deviation

Solution

(a)

X_i	-3	-2	-1	0	1	2	3
P_i	3k	2k	k	0	k	2k	3k

$\sum_{i=1}^{n} X_i P_i$ = 1

3k + 2k + k + k + 2k + 3k = 1

12k = 1

$\mathbf{k} = \frac{1}{12}$

X_i	-3	-2	-1	0	1	2	3

P_i	$\frac{3}{12}$	$\frac{2}{12}$	$\frac{1}{12}$	0	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{3}{12}$

(b) $E(X) = \sum_{i=1}^{n} X_i P_i$

$$E(X) = -3 \times \frac{3}{12} - 2 \times \frac{2}{12} - \frac{1}{12} \times 1 + 0 + 1 \times \frac{1}{12} + 2 \times \frac{2}{12} + 3 \times \frac{3}{12}$$

$$E(X) = \frac{9-4-1+0+1+4+9}{12}$$

$$= \frac{0}{12}$$

$$E(X) = 0$$

(c) $E(X^2) = \sum_{i=1}^{n} X^2_i P_i$

$$E(X^2) = (-3)^2 x \frac{3}{12} + (-2)^2 x \frac{2}{12} + (-1)^2 x \frac{1}{12} + 0 + 1^2 x \frac{1}{12} + 2^2 x \frac{2}{12} + 3^2 x \frac{3}{12}$$

$$= \frac{27}{12} + \frac{8}{12} + \frac{1}{12} + \frac{1}{12} + \frac{8}{12} + \frac{27}{12}$$

$$= \mathbf{6}$$

(d) Standard deviation $= \sqrt{\text{Variance}}$

Variance $= E(X^2) - (E(X))^2$

$= 6 - (0)^2$

$= 6$

Standard deviation $= \sqrt{6}$

$= \mathbf{2.4445}$

Example 16. The probability distribution for the number of heads that show up when a coin is tossed 3 times is given by $P(X = x) = \left\{\frac{1}{k}\binom{3}{x}\right., x = 0, 1,2, 3,.$

Find:

(i) The value of k,

(ii) E(X).

Solution:

We have

$$P(X = x) = \left\{\frac{1}{k}\binom{3}{x}\right., x = 0,1,2,3.$$

This is a discrete random variable.

'It should be noted that

$$\binom{n}{x} = \frac{n!}{(n-x)!x!}$$

Hence when

$X = 0,\quad P(x = 0) = \frac{1}{k}\left(\frac{3!}{3!0!}\right) = \frac{1}{k}$

$X = 1,\quad P(x = 1) = \frac{1}{k}\left(\frac{3!}{2!1!}\right) = \frac{3}{k}$

$X = 2,\quad P(x = 2) = \frac{1}{k}\left(\frac{3!}{1!2!}\right) = \frac{3}{k}$

$X = 3, \quad P(x = 3) \quad = \frac{1}{k}\left(\frac{3!}{0!3!}\right) \quad = \frac{1}{k}$

In table form,

X	0	1	2	3
P (x = x)	$\frac{1}{k}$	$\frac{3}{k}$	$\frac{3}{k}$	$\frac{1}{k}$

(i) To find the value of k

Now $\sum_{x=0}^{3} P(X = x) \quad = 1$

$\Rightarrow \frac{1}{k} + \frac{3}{k} + \frac{3}{k} + \frac{1}{k} = 1$

$\frac{8}{k} = 1$

$\therefore \quad \mathbf{k} = \mathbf{8}$ #

(ii)Table is now in form

X	0	1	2	3
P(x = x)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Now

$$E(x) = \sum_{x=0}^{3} xP(x = x)$$

$$= 0\left(\frac{1}{8}\right) + 1\left(\frac{3}{8}\right) + 2\left(\frac{3}{8}\right) + 3\left(\frac{1}{8}\right)$$

$$= \frac{3}{8} + \frac{6}{8} + \frac{3}{8}$$

$$= \frac{12}{8} = \frac{3}{2} = 1.5$$

$\therefore \quad E(x) = 1.5$ #

Example 17. A balanced coin is tossed three times and the number of times X a 'Head' appear is recorded. Complete the following table.

n	**0**	**1**	**2**	**3**
Event	**(TTT)**		**HHT,HTT,THH**	
P(X=n)	**1/8**			

Determine the average of the expected number of heads to appear.

Solution:

n	0	1	2	3
Event	TTT	HTT THT TTH	HHT THH HTH	HHH
P(X = n)	1/8	3/8	3/8	1/8

Expected number of heads to appear

$$E(X) = \sum n\, P(X = n)$$
$$= (0 \times 1/8) + (1 \times 3/8) + (2 \times 3/8) + (3 \times 1/8)$$
$$= 0 + \tfrac{3}{8} + \tfrac{6}{8} + \tfrac{3}{8}$$
$$= \frac{12}{8} = \tfrac{3}{2} = 1.5$$
$$E(X) = 1.5$$

Hence the average of the expected number of heads to appear is 1.5

EXERCISE 3

1.. A discrete random variable X has distribution function F(X) where

$$F(X) = 1 - \left(1 - \frac{x}{4}\right)^{x} \qquad \text{for } x = 1, 2, 3, 4$$

(a)Show that $F(3) = \dfrac{63}{64}$ and $F(2) = \dfrac{3}{4}$

(b)Obtain a probability distribution of x

(c)Find E(X) and Var(X).

(d)Find P(X > E(X))

2. The random variable X takes integer values only and has p d f

$P(X = x) = kx \qquad x = 1, 2, 3, 4, 5$

$P(X = x) = k(10 - x) \qquad x = 6, 7, 8, 9$

Find a) the value of the constant k.
b) E(X)
c) Var(X)
d) E(2x – 3)
e) Var(2x – 3)

3. If X is a random variable on "a biased die" and the probability density function of X is as shown.

X	1	2	3	4	5	6
P(X = x)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{5}$	y	$\frac{1}{5}$	$\frac{1}{6}$

Find (a)the value of y
(b)E(X)
(c)$E\left(X^2\right)$
(d)Var(X)
(e)Var(4X)

4. A random variable R takes the integer value r with probability.

$P(r) = Kr^2 \qquad r = 1, 2, 3$

$P(r) = K(7 - r)^2 \qquad r = 4, 5, 6$

$P(r) = 0 \qquad$ otherwise

Find; (a) the value of y
(b) The mean
(c) The variance

5. A discrete random variable X can take only the values 0, 1, 2, or 3 and its probability distribution is given by P(X = 0) = k, P(X = 1) = 3k, P(X =2) = 4k and P(X = 3) = 5k, where K is a constant. Find;

(a) The value of K.

(b) Mean and variance of X.

6. A discrete random variable X has probability function given by;

$$P(X) = \begin{cases} \left(\frac{1}{2}\right)^2 & x = 1, 2, 3, 4, 5 \\ c & x = 6 \\ 0 & \text{otherwise} \end{cases}$$

Where C is a constant.

Determine

(i)the value of C

(ii)the mode

(iii)the arithmetic mean

7. The discrete random variable X has a probability density function given by;

$$P(X = x) = \frac{3x+1}{22} \quad \text{for } x = 0, 1, 2, 3.$$

Find (a) E(X)

(b) E(X^2)

(c) E(3x - 2)

(d) E($2x^2$ + 4x - 3)

8. A discrete random variable X has a probability density function.

X	0	1	2	3
P(X = x)	a	a^2	a^2+a	$3a^2+2a$

Determine; (a) the constant a

(b)E(X)

9. A curiously shaped six-faced die produces a score X, for which the probability distribution is given in the following table.

R	1	2	3	4	5	6
P(X = r)	C	$\frac{c}{2}$	$\frac{c}{3}$	$\frac{c}{4}$	$\frac{c}{5}$	$\frac{c}{6}$

(i)Show that the constant c is $\frac{20}{49}$

(ii)Find the mean and variance

(iii)The die is thrown twice. Show that the probability of obtaining equal scores is approximately $\frac{1}{4}$

10. A discrete random variable has a probability density function of

P(x = r) = k(n – r) for r = 1, 2, 3 …, n where K is a constant.

show that

(i) k is $\frac{2}{n(n-1)}$

(ii) E(X) = $\frac{1}{3}$(n + 1)

(iii) Var(X) = $\frac{1}{18}(n + 1)(n - 2)$

11. A disc is drawn from a bag containing 10 disc numbered 0, 1, 2, …., 9. The random variable X is defined as the square of the number drawn. Find:
 (i)E(X)
 (ii)Var(X)

12. A bag contains one 50p coins, three 20p coins, seven 10p coins and several 5p coins. Given when one coin is selected at random the expectation is 10p. Find.
 (i) The number of 5p coins
 (ii) Find also the expectation when two coins are selected at random without replacement.

13. A random variable R takes values 1, 2, …, n with equal probabilities. Determine;
 (i) The expectation μ of R
 (ii) Show that the variance δ^2 is given by $12\delta^2 = n^2 - 1$
 (iii)find $P(|R - \mu|) > \delta)$ in the case n = 100

14. A cubical die is biased in such away that the probability of scoring n where n = 1, 2, 3, 4, 5, 6 is proportional to n.
Determine; (i) the mean value
 (iii)Variance of the score obtained in a single score.
 (iii)The mean and variance if the score is doubled.

EXERCISE 3

1.. (b)

X	1	2	3	4
P(X = x)	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{15}{64}$	$\frac{1}{16}$

(c) $2\frac{1}{16}$ (d) $\frac{1}{4}$

2. (a) 0.04 (b) 5 (c) 4 (d) 7 (e) 16)

3. (a) $\frac{1}{10}$ (b) $3\frac{1}{2}$ (c) $15\frac{7}{30}$ (d) $2\frac{59}{60}$ (e) $47\frac{11}{15}$)

4. (a) $\frac{1}{28}$ (b) 3.4 (c) 1.25)

5. (a) $\frac{1}{31}$ (b) $2\frac{12}{13}$)

6.((i) $\frac{1}{32}$ (ii) 1 (iii)$1\frac{31}{32}$)

7. (a) $\frac{24}{11}$ (b) $\frac{61}{11}$ (c) $\frac{50}{11}$ (d) $16\frac{9}{11}$

8. (a) 0.2 (b) 2.08

9. (a) $\frac{120}{49}$ (b) 2.57)

10.Proof.

11. (a) 28.5 (b) 721.05

12. (a) 14 (b) 20p

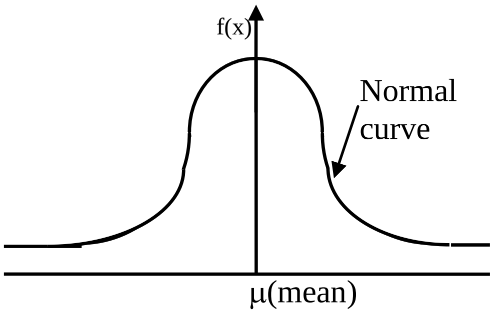
13. ((i) ½(n + 1) (ii) 0.42)

14. (i) $4\frac{1}{3}$ (b) $2\frac{2}{9}$ (c) $8\frac{2}{3}$ (d) $8\frac{8}{9}$

6.NORMAL DISTRIBUTION

Introduction:

The normal distribution (bell shaped structure) is considered the most prominent probability distribution in statistics. There are several reasons for this:



First, the normal distribution arises from the **central limit theorem,** which states that under mild conditions, the mean of a large number of **random variables** independently drawn from the same distribution is distributed approximately normally, irrespective of the form of the original distribution. This gives it exceptionally wide application in, for example, sampling.

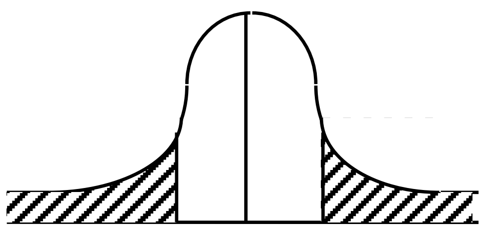
Secondly, the normal distribution is very tractable analytically, that is, a large number of results involving this distribution can be derived in explicit form.

For these reasons, the normal distribution is commonly encountered in practice, and is used throughout statistics, the natural sciences, and the social sciences as a simple model for complex phenomena. For example, the observational error in an experiment is usually assumed to follow a normal distribution, and the propagation of uncertainty is computed using this assumption.

- Note that a normally distributed variable has a symmetric distribution about its mean

Mathematically:

This is a continuous random variable that is equally skewed on both sides, resulting into a bell shaped structure usually referred to as a normal curve and the variable is called the **normal random variable**. A convenient way of denoting that X is normally distributed with mean μ and variance S_2 is X ~N(n, S_2)



CHARACTERISTICS OF THE NORMAL CURVE

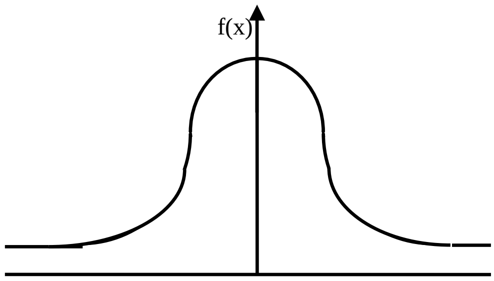
(i) the curve is symmetrical about the mean.
(ii) The x-axis is an asymptote to the curve.
(iii) The area under the curve is 1.

(iv) As x $\longrightarrow + \infty$, f(x) 0 $\longrightarrow$

(v) The mode occurs at the maximum point (P) on the vertical axis.

AREA UNDER THE NORMAL CURVE

The area under the two co-ordinates X_1 and X_2, represent the probability that a random variable X lies between the two values
X_1 and X_2.



Area = P(X1 <X<X2) = P(X <X2) – P(X <X1).

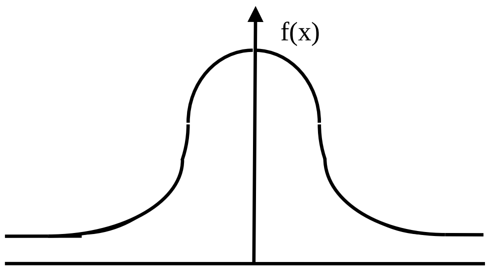
STANDARD NORMAL DISTRIBUTION

This is a special type of normal distribution with

(i) mean (μ) equal to zero ($\mu = 0$).

(ii) Standard deviation (δ) is equal to 1. ($\delta = 1$)

(iii) The total area under the curve is 1.



STANDARDISATION

This is a process of transforming a normal variable into a standard normal variable following a normal distribution with mean (μ) and standard deviation (So, the corresponding standard normal variable is given by $Z = \dfrac{X-\mu}{\sigma}$

If X_1 and X_2 are two normal variable with mean (μ) and standard deviation (σ) to obtain the probability of a random variable X that lies betweens X_1 and X_2

$$P(X_1 < X < X_2) = P(X < X_2) - P(X < X_1)$$

Standardizing X_1 and X_2, where Z_1 is the standard normal variable for X_1 and Z_2 is the standard normal variable for X_2 then from $Z = \dfrac{X-\mu}{\sigma}$

at $X = X_1$, $Z_1 = \dfrac{X_1-\mu}{\sigma}$

$P(X_1 < X < X_2) = P(Z_1 < Z < Z_2)$

$$= P\left(\frac{X_1-\mu}{\sigma} \angle Z \angle \frac{X_2-\mu}{\sigma}\right)$$

$\therefore P(Z_1 < Z < Z_2) \quad = \quad P(Z < Z_2) - P(Z < Z_1)$

NORMAL DISRIBUTION TABLES

There are mainly two types of tables.

(i) One with areas under both negative and positive values of Z. It is the most convenient but not common.

(ii) One with the positive values of Z. These are of two types.

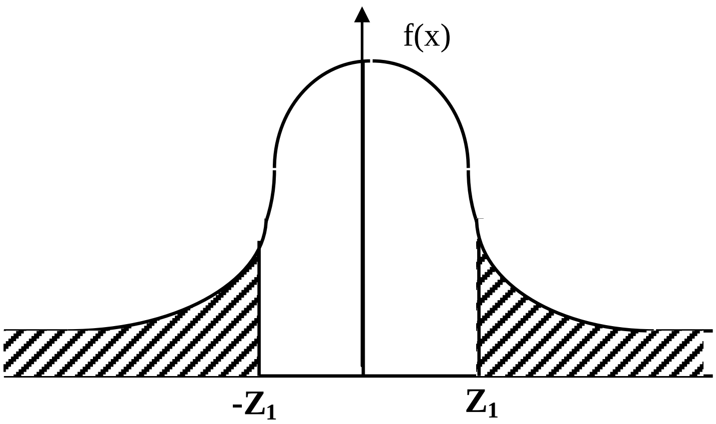
(a) one with $P(Z < 0) = 0.5000$

(b) one with $P(Z < 0) = 0.0000$

with (ii) (b) we must first of all add 0.5 to all the Z values.

For (ii) (a) and (b) in case the Z value is negative there is need to use the formulae for conversion.

300



$P(Z < -Z1) = P(Z > Z1)$ (i)

(i) Since the area below –Z on left hand side is equal to area above Z1.

Also $P(Z > Z1) + P(Z < Z_1) \quad = \quad$ Total area under the curve

$P(Z > Z_1) + P(Z < Z_1) = 1$

$P(Z > Z_1) \quad = \quad 1 - P(Z < Z_1)$ (ii)

Substitute (ii) into (i)

$P(Z < -Z_1) \quad = \quad 1 - (Z < Z_1)$. is the formulae for conversion.

NOTE: Normal distribution tables values of Z that in the form of less than.
That is $P(Z < |Z1|)$

DE-STANDARDISING

Sometimes it is necessary to find a value X which corresponds to the standardized value Z.

If $Z = \dfrac{X-\mu}{S}$ then $X = \mu + \sigma Z$.

DETERMINING THE Z – VALUE WHEN THE AREA IS GIVEN (PROBABILITY)

In some cases a Z value may be needed when the area (Probability) is given.

$P(Z < Z1) = 0.6772$.

Check for 0.6772 inside the table, and after identifying it read off the value of Z in full including its decimal value if it is

$P(Z > -Z_1) = 0.6331$ convert it into less than from.

$P(Z > -Z_1) + P(Z < -Z_1) = 1$

$P(Z > -Z_1) = 1 - P(Z < -Z_1) = 0.6331.$

$P(Z < -Z_1) = 0.3669.$

$P(Z < -Z_1) = P(Z > Z_1) = 1 - P(Z < Z_1)$

$P(Z < Z_1) = 0.6331$

$Z_1 = 0.34$

$\therefore$ **$-Z1$ = -0.34**

THE NORMAL APPROXIMATION TO A BINOMIAL DISTRIBUTION

Normal distribution is used in estimation of discrete random variable. These include approximation of normal distribution to Binomial distribution. Normal distribution is used to estimate Binomial distribution when.

(i).. if the number of independent trials in a binomial distribution is large.

(ii) When the probability of success p is close to $\frac{1}{2}$.

Let X be a random variable following a binomial distribution.

$$E(X) = \mu = np$$

$$Var(X) = npq$$

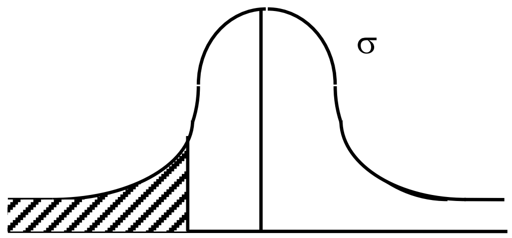
$$\text{Standard deviation} = \sqrt{npq}$$

If n is large, the value of Z changes from

$$Z = \frac{x - \mu}{S} \text{ to } Z = \frac{(X \pm 0.5) - np}{\sqrt{npq}} \qquad (1)$$

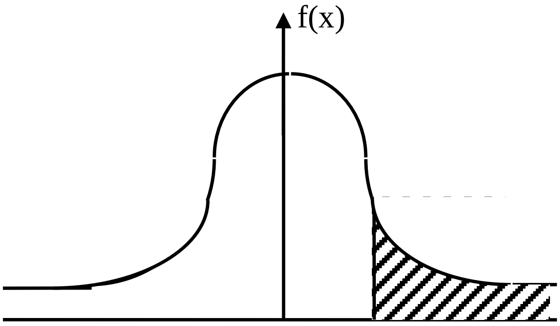
0.5 included in (1) is called the continuity collection. Since we are approximating a discrete distribution function to a continuous distribution function.

If X_1 and X_2 correspond to Z_1 and Z_2.



$$Z_1 = \frac{(X_1 + 0.5) - np}{\sqrt{npq}} \text{ for at most X cases.}$$

(ii)



$P(X > X_2) \quad = \quad P(Z > Z_2)$.

$P(X>X_2) \quad = \quad P(\text{ at least } X_2) = P(\text{at least } Z_2)$

$Z_2 \quad = \quad \dfrac{(Z - 0.5) - np}{\sqrt{npq}}$ for at least X cases.

Where n is large

p = probability of failure

q = 1 - p = probability of failure

n = number of independent trials.

NOTE: Add 0.5 if we want the probability of a t most X successes and subtract 0.5 for at least X successes.

THE DISTRIBUTION OF THE SAMPLE MEAN

Consider X1, X2, X3 ……. Xn has a random sample of size n.

Taken from a population with mean (μ) and variance ($\sigma 2$), the sample mean ($\bar{x}$) = $\frac{1}{n}\sum X_i$, has mean u and variance $\left(\frac{\sigma^2}{n}\right)$

SAMPLING FROM A NORMAL POPULATION

If X_1, X_2, X_3, ………Xn is a random sample of size n(n < 30) taken from a population that is normally distributed then the mean = u and variance = $\frac{\sigma^2}{n}$

Standard deviation = $\frac{\sigma}{\sqrt{n}}$ it is termed as standard error of mean.

(i)**When sample size n is large ($n \geq 30$)**

If X_1, X_2, X_3, ……., X_n is a random sample of size $n(n \geq 30)$ taken from a population that is normally distributed, then the mean = μ

Variance = $\frac{\sigma^2}{n}$

Standard deviation = $\frac{\sigma}{\sqrt{n}}$ also called standard error of mean,

(iii) **The sample proportion**.

If a population has P as a proportion of success and q as failure with sample size n, if it is a normal approximation to the binomial distribution then.

E(P) = nP = mean

Variance = Var(P) = npq

Standard deviation = $\sqrt{npq}$

Normal approximation to Binomal, the continuity correction is $\pm \frac{1}{2}$

$$Z = \frac{(X + 0.5 - np)}{\sqrt{npq}}$$

$$Z = \frac{(X + 0.5) - np}{\sqrt{npq}}$$

$$Z = \frac{(X - 0.5) - np}{\sqrt{npq}}$$

Example 1: A certain type of cabbage has a mass which is normally distributed with mean 1 kg and standard deviation 0.15 kg. In a lorry load of 800 of these cabbages, estimate how many will have mass:

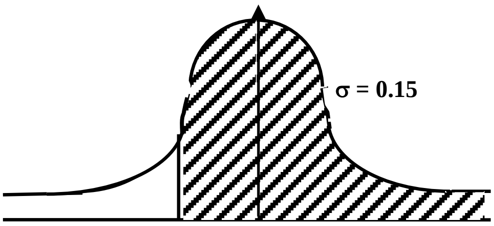
(a) greater than 0.79 Kg

(b) less than 1.13 kg.

(c) between 0.85 Kg and 1.15 kg.

(d) between 0.75 kg and 1.29 kg.

Solution:

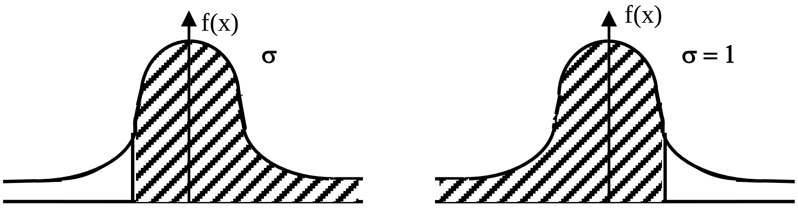


$$P(X > 0.79) = P\left(Z > \frac{(0.79 - 1))}{0.15}\right)$$

$$= P(Z > -1.4)$$

$$P(Z > -1.4) = P(Z < 1.4)$$

NOTE:



$P(X > -X_1) = P(X < X_1)$

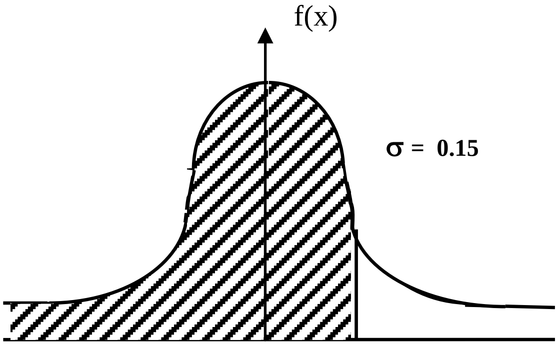
$P(Z > -1.4) = 0.9192.$

The expected number $= np.$ *(n = 800, p = 0.9192.)*

$= 800 \times 0.9192$

$=$ **735 cabbages.**

(b) $P(X < 1.13)$



P(X < 1.13) = $P\left(Z \angle \frac{1.13-1}{0.15}\right)$

= P (Z < 0.8667)

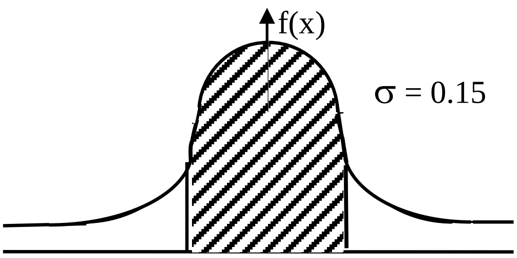
= 0.8097

The expected number of cabbages = np

n = 800 p = 0.8078

= 800 x 0.8078 = 646 cabbages.

(c) P(0.85 < X < 1.15)



/

P(0.85 < X < 1.15) = P (X < 1.15) – P(X < 0.85)

P(- 1 < Z < 1) = $P\left(Z < \frac{(1.15-1)}{0.15}\right) - P\left(Z < \frac{0.85-1}{0.15}\right)$

= P(Z < 1) - P(Z < -1)

= P(Z < 1) – P(Z > 1))

= P(Z < 1) – (1 – P(Z < 1))

= 2P(Z < 1) – 1

= 2 x 0.8413 – 1

P(-1 < Z < 1) = 0.6826

The expected number of cabbages = np.

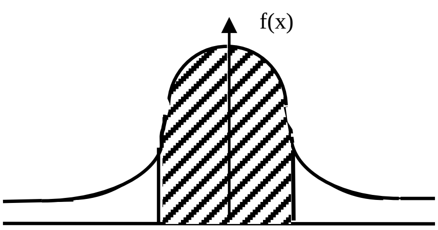
n = 800, p = 0.6826

= 800 x 0.6826

= 546 cabbages

(e)P(0.75 < X < 1.29).

Solution:



$P(0.75 < X < 1.29) = P(X < 1.29) - P(X < 0.75)$

$P(-1.6667 < Z < 1.9333) = P\left(Z < \frac{(1.29-1)}{0.15}\right) - P\left(Z < \frac{(0.75-1)}{0.15}\right)$

$= P(Z < 1.9333) - P(Z < -1.6667)$

$= 0.9734 - (P(Z > 1.6667)$

$= 0.9734 - (1 - P(Z < 1.6667))$

$= 0.9734 - 1 + 0.9522$

$= \mathbf{0.9256}$

Expected number of cabbages $= np$

$n = 800$, $p = 0.9256$

$= 800 \times 0.9256$

= **740 cabbages**

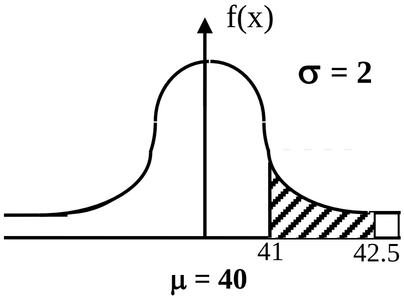
Example 2: A certain firm sells maize flour in bags of mean weight 40Kg and standard deviation of 2kg, Given that the weight is normally distributed, find:

(i) the probability that the weight of any bay taken at random will lie between 410 and 42.5Kg.

(ii) the percentage of bags whose weight exceeds 43 Kg.

(iii)the number of bags rejected out of a 500 bags purchased by a retailer whose consumers cannot accept a bag whose weight is below 38 5Kg.

Solution: (i) $P(41.0 < X < 42.5)$, $\mu = 40$, $\sigma = 2$



$P(41 < X < 42.5) = P(X < 42.5) - P(X < 41)$

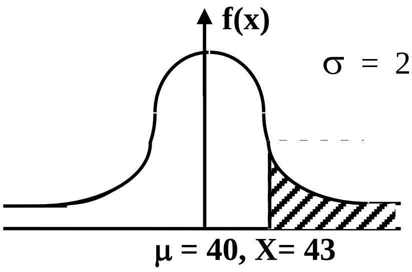
$P(0.5 < 2 < 1.25) = P\left(Z < \frac{(42.5-40))}{2}\right) - P\left(Z < \frac{(41-40)}{2}\right)$

$= P(Z < 1.25) - P(Z < 0.5)$

$= 0.89644 - 0.6915$

$= \mathbf{0.2029}$

(ii) $P(X > 43)$



$P(X > 43) = 1 - P(X < 43)$

$= 1 - P\left(Z \angle \frac{43-40}{2}\right)$

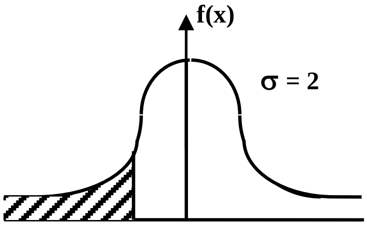
$= 1 - P(X < 1.5) = 1 - 09332$

$P(X > 43) = 0.0668$

Percentage of $P(X > 43) = 0.0668 \times 100\%$

= **6.68%**

(iv) P(X < 38.5)



$$P(X<38.5) = P\left(Z < \frac{38.5-40}{2}\right) = P(Z - 0.75)$$

$$P(Z < -0.75) = P(Z > 0.75)$$
$$= 1 - P(Z < 0.75)$$
$$= 1 - 0.7734$$
$$= 0.2266$$

Number of bags rejected = nP.

n = 500, P = 0.2266

= 500 x 0.2266

= 113.3 bags

≃ **113 bags**

Example 3: A machine which automatically packs potatoes into bags is known to operate with mean 25Kg and standard deviation 0.5Kg. Assuming it's a normal distribution find the percentage of bags which weigh:

(a) more than 26Kg.

(b) between 24 and 25Kg

To what new target mean weight should the machine be set so that 90% of the bags weigh more than 26Kg? In this case what weight would be exceeded by 0.1% of the bags?

Solution: (a) $\mu = 25$ $\sigma = 0.5$Kg.

$$P(X > 25) = P\left(z > \frac{26-25}{0.5}\right)$$
$$= P(Z > 2)$$
$$= 1 - P(Z < 2)$$
$$= 1 - 0.9772$$
$$= \mathbf{0.0228}$$

Percentage of P(X > 25) = 0.0228 x 100%

= **2.28%**

(b) $P(24 < X < 25) = P(X < 25) - P(X < 24)$

$$= P\left(z > \frac{26-25}{0.5}\right) - P\left(z > \frac{24-25}{0.5}\right)$$
$$= P(Z < 0) - P(Z < -2)$$
$$= 0.5 - (P(Z > 2))$$
$$= 0.5 - \{1 - P(Z < 2)\}$$
$$= P(Z < 2) - 0.5$$
$$= 0.9772 - 0.5$$
$$= \mathbf{0.4772}$$

Percentage of P (24 < X < 25) = 0.4772 x 100%

= **47.72%**

Let the new mean = μ

$$P(X > 26) = 0.9$$

$$P\left(z > \frac{26-\mu}{0.5}\right) = 0.9$$

$$\frac{26 - \mu}{0.5} = -1.28$$

$$\mu = 26 + 1.28 \times 0.5$$
$$= \mathbf{26.64}$$

Let the required weight be x.

$$P(X > x) = \frac{0.1}{100} = 0.001$$

$$P\left(z > \frac{x - 26.64}{0.5}\right) = 0.001$$

$$\frac{X - 26.64}{0.5} = 3.01$$

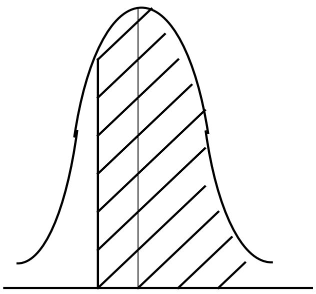
$$x = 26.64 + 3.01 \times 0.5 = 28.145$$

Problem: A certain type of cabbage has a mass which is normally distributed with mean 1 kg and standard deviation 0.15 kg. In a lorry of 800 of these cabbages, estimate how many will be

(a). Greater than 0.79 kg
(b). Less than 1.13 kg
(c). Between 0.85 kg and 1.15 kg

10.

σ = 0.15 kg
800 cabbages



(a) The number of cabbages that will weigh greater than 0.79 kg is given by;
P (x > 0.79) x 800

But P (x > 0.79) = 1 - P (x ≤ 0.79)

Using z - values,

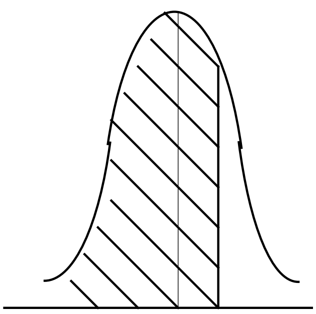
$$z = \frac{0.79 - 1}{0.15} = -1.4$$

P (x > 0.79) = 1 - P (z ≤ 0.79)
= P (z > -1.4)
= 0.5 + 0.4192
P (x > 0.79) = 0.9192

Hence the number of cabbages that weigh greater than 0.79 kg are;
= 800 x 0.9192
= 735.36

So, the number of cabbages that will weigh more than 0.79 kg are approximately 735.

(b)



Those that will weigh less than 1.13 kg are;
P(x < 1.13) x 800
Using z - values,

$$z = \frac{1.13 - 1}{0.15} = 0.867$$

P(z < 0.867) x 800
Using tables,
P(z < 0.867) = 0.5 + 0.2794
= 0.7794

The number of cabbages that will weigh less than 1.13 kg are;
0.7794 x 800 = 623 cabbages.

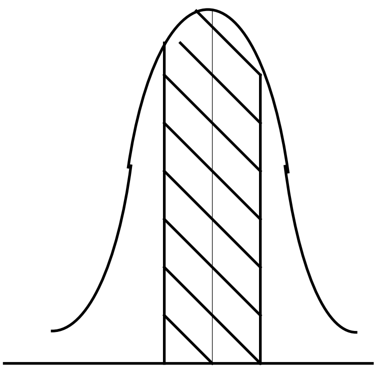
(c) Those that will weigh between 0.85 kg and 1.15 kg are;
P (0.85 ≤ x ≤ 1.15) x 800
Using z values,

$$z_1 = \frac{0.85 - 1}{0.15} = -1$$

$$\& \quad z_2 = \frac{1.15 - 1}{0.15} = 1$$

σ = 0.15

we find P (z_1 < z < z_2)



P (-1 < z < 1) = P (z < 1) + P(z - 1)
= 0.3413 + 03413
= 0.6826
Hence the required probability is 0.6826
The number of cabbages is;

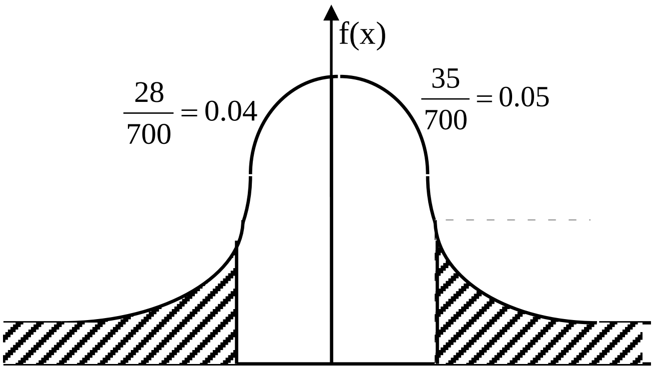
0.6826 x 800 = 546

Example 4: A total population of 700 students sat an examination for which the pass mark was 50. Their marks were normally distributed. 28 students scored below 40 marks while 35 scored above 60.

(a) Find the mean mark and standard deviation of the students.

(b) What is the probability that a student chosen at random passed the examination ?

Solution: (a)



$$P(X < 40) = P\left(z > \frac{40-\mu}{\sigma}\right) = 0.04. \quad \text{Let } \frac{40-\mu}{\sigma} = Z1$$

$$P\left(z < -\left(\frac{40-\mu}{\sigma}\right)\right) = P\left(z > \frac{40-\mu}{\sigma}\right)$$

$$= -\ P\left(z > \frac{40-\mu}{\sigma}\right)$$

$$\therefore \quad P\left(z < \frac{40-\mu}{\sigma}\right) = 0.96$$

Z1 = 1.75 **-Z1 = -1.75**

$$\therefore \quad \frac{40-\mu}{\sigma} = -1.75$$

$40 - \mu = -1.75$ S (i)

$P(X > 60) = 1 - P(X < 60) = 0.05$

$$P\left(z < \frac{40-\mu}{\sigma}\right) = 0.95$$

Z2 = 1.645

$$\frac{60-\mu}{\sigma} = 1.645$$

$60 - \mu = 1.645\sigma$ (ii)

$40 - \mu = -1.75\sigma$

$20 = 3.395\sigma$

$\sigma = 5.89$

$\mu = 60 - 1.645 \times 5.89 = 50.31$

$\boldsymbol{\mu}$ **= 50.31.**

(b) $P(X > 50) = 1 - P(X < 50)$

$$= 1 - P\left(Z < \frac{50 - 50.31}{5.89}\right)$$

$= 1 - P(Z < -0.0526)$

$= 1 - P(Z > 0.0526)$

$= 1 - (1 - P(Z < 0.0526)$

$= P(Z < 0.0526)$

$=$ **0.5211**

Number of students who passed = np n = 700, p = 0.5211

$= 700 \times 0.5211$

$= 365$ Students

$P(X < 48) = 1 - P(X < 48)$

$= 1 - P\left(Z < \frac{48 - 50.31}{5.89}\right)$

$= 1 - P(Z < -0.3926)$

$= 1 - P(Z > 0.3926)$

$= 1 - (1 - P(Z < 0.3926)$

$= P(Z < 0.3926)$

$=$ **0.6527**

Number of students $P(X > 48)$ = np

n = 700 p = 0.6527

$= 700 \times 0.6527$

$= 457$ students.

The increase in number of students who passed

= (457 - 365) students.

= **92 students.**

Examples 5: The mean length of 500 Tropical forest leaves from a certain forest is 151mm and the standard deviation is 15mm.

Assuming that the length are normally distributed find how many leaves measures:

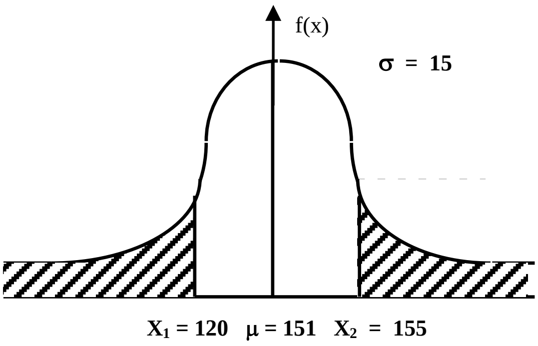
(a) between 120 and 155mm.

(b) more than 185mm

(c) less than 128mm

(d) 128mm

Solution (a)



$P(120 < X < 155) = P(X < 155) - P(X<120)$

$= P\left(z < \frac{155 - 151}{15}\right) - P\left(z < \frac{120 - 151}{15}\right)$

$= P(Z < 0.27) - P(Z < -2.07).$

$= 0.6064 - P(Z > 2.07)$

$= 0.6064 - (1 - P(Z < 2.07).$

$= 0.6064 - (1 - 0.9808)$

$= 0.5872$

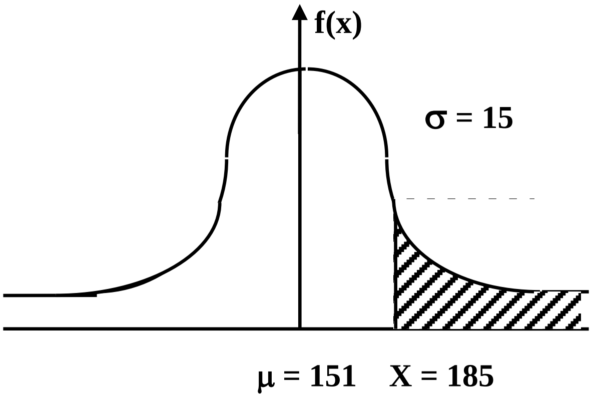
Number of leaves having the length between 120 and 155mm = nP

n = 500 P = 0.5872

$= 500 \times 0.5872$

$= \underline{294}$

(b) P(X >185)



$P(X > 185) = 1 - P(X < 185)$

$= 1 - P\left(z < \frac{185 - 151}{15}\right)$

$= 1 - P(Z < 2.27).$

$= 1 - 0.9884$

$= 0.0116.$

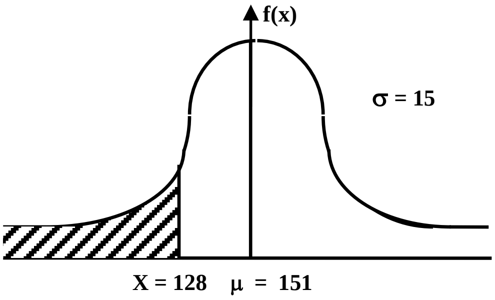
Number of leaves having the length greater than 185mm = np.

$n = 500 \quad p = 0.0116$

$= 500 \times 0.0116$

$= 6.$

(c) P(X < 128)



$(X < 128) = P\left(z < \frac{128 - 151}{15}\right)$

$= P(Z < -1.53)$

$= P(Z > 1.53)$

$= 1 - P(Z < 1.53)$

$= 1 - 0.9370$

$= \underline{\mathbf{0.063}}$

Number of leaves having the length less than 128 = nP

$n = 500, \quad P = 0.063$

$= 500 \times 0.063$

$= \mathbf{32.}$

(d) $P(X = 128) = P(X<128.5) - P(X <127.5)$

$= P\left(z < \frac{128.5 - 151}{15}\right) - P\left(z < \frac{127.5 - 151}{15}\right)$

$= P(Z < -1.5) - P(Z < -1.57).$

$= P(Z > 1.5) - P(Z > 1.57)$

$= 1 - P(Z < 1.5) - 1 + P(Z < 1.57)$

$= \quad 1 - 0.9332 - 1 + 0.9418$

$= \quad \mathbf{0.0086}$

Number of leaves having length equal to 128 = np.

n $= \quad 500, \quad p = 0.0086$

$= \quad 500 \times 0.0086$

$= \quad 4$

Example 6: The diameter of a sample of oranges to the nearest cm were:

Diameter (cm)	**8**	**9**	**10**	**11**	**12**	**13**	**14**
Frequency	**9**	**15**	**21**	**32**	**19**	**13**	**11**

(i) Calculate the mean and standard deviation.

(ii) Assuming the distribution is normal, find the minimum diameter if the smallest 10% of the oranges are rejected for being too small.

Solution:

x	**f**	**fx**	**fx2**
8	9	72	576
9	15	135	1215
10	21	210	2100
11	32	352	3872
12	19	228	2736
13	13	169	2197
14	11	154	2156

$\sum f = 120$

$\sum f x = 1320 \quad , \quad \sum fx^2 = 14,852$

(i) Mean (μ) $= \dfrac{\sum fx}{\sum f}$

$= \dfrac{1320}{120}$

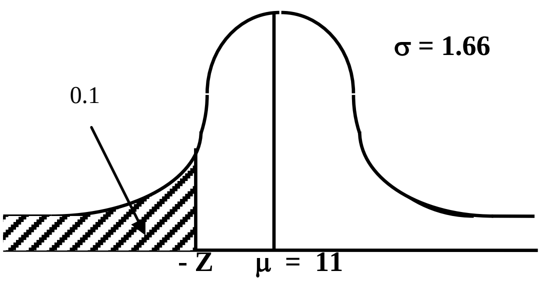
$= \mathbf{11}$

Standard deviation $\sigma = \sqrt{\dfrac{\sum fx^2}{\sum f} - \mu^2}$

$= \sqrt{\dfrac{14852}{120} - (11)^2}$

$= \underline{\mathbf{1.6633}}$

(iii) 10% being too small.



P(Z<-Z1) $= 0.1$

$P(Z > Z_1) = 0.1$

$1 - P(Z < Z_1) = 0.1$

$P(Z < Z_1) = 0.9$

$Z_1 = 1.282.$ $\therefore$ $-Z_1 = -1.282$

$$-Z_1 = \frac{X_1 - \mu}{\sigma}$$

$$-1.282 = \frac{X_1 - 11}{1.6633}$$

$X_1 = 11 - 1.282 \times 1.6633$

$= 8.876$

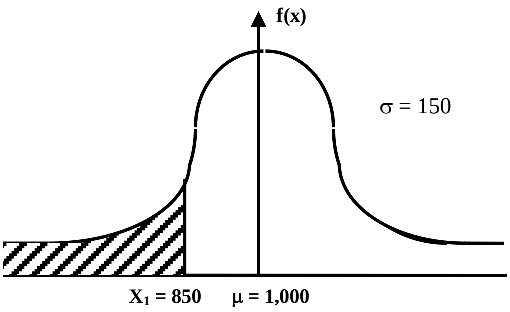
$\mathbf{X_1 \simeq 8.87}$

Example 7: The number of AIDS Victims in a certain town is said to be normally distributed with mean of 1,000 and standard deviation 150. Find :

(a) the probability that in a group of less than 850 people chosen at random, you will find a victim.

(b) the probability that in a randomly selected group of people between 750 and 1200 there will be an AIDS free person.

(c)Assuming that the total population of town is 62,000. Find the expected number of people who are AIDS victims in a group of more than 1375 chosen at random.



$$P(X < 850) = P\left(Z < \frac{850 - 1{,}000}{150}\right)$$

$= P(Z < -1)$

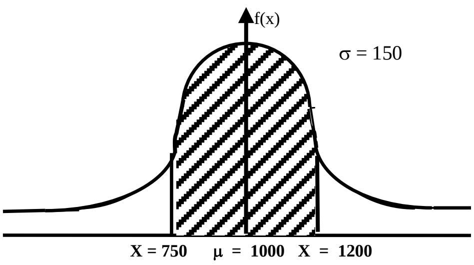
$= P(Z > 1)$

$= 1 - P(Z < 1)$

$= 1 - 0.3413$

$= 0.1587$

(b)



$P(750 < X < 1200) = P(X < 1200) - P(X < 750)$

$$= P\left(Z < \frac{1200 - 1000}{150}\right) - P\left(Z < \frac{750 - 1000}{150}\right)$$

$= P(Z < 1.33) - P(Z < -1.67)$

$= 0.9082 - 1 + P(Z < 1.67)$

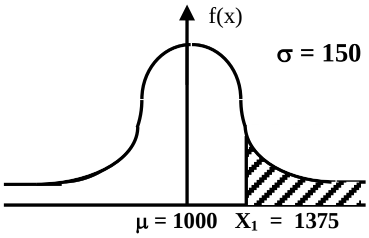
$= 0.9082 - 1 + 0.9525$

$= 0.8607$

$\therefore$ The probability of normal people $= 1 - 0.8607$

$= \mathbf{0.1393}$

(c) $n = 62{,}000$, $P(X > 1375)$.



$$P(X > 1375) = P\left(Z \angle \frac{1375 - 1000}{150}\right)$$
$= P(Z > 2.5)$
$= 1 - P(Z < 2.5)$
$= 1 – 0.9938$
$= \mathbf{0.0062}$

Expected number of AIDS victims = np, n = 62,000 P = 0.0062

6. On a certain farm 20% of the cows are infected by a tick disease. If a random sample of 50 cows is selected from the farm, find the probability that not more than 10% of the cows are infected.

Solution:

Let x be the random variable for the number of cows infected.

$\Rightarrow x \tilde{} B (n, p)$

$n = 50$

$p = P (A) = 0.2$

$q = 1 – q = 0.8$

Since n > 30 X – N (np. npq)

Mean = np $= 50 \times 0.2 = 10$

S.d $= \sqrt{npq} = \sqrt{50 \times 0.2 \times 0.8} = \sqrt{8}$

10% of 50 cows = 5 cows

$\Rightarrow P(x \le 5) = 1 – P (x \ge 6).$

$\Rightarrow P (x \ge 6) = P (x > 6 – 0.5)$
$= P (x > 5.5)$

Standardizing

$$\Rightarrow \text{We have } P\left(Z > \frac{5.5 - 10}{\sqrt{8}}\right)$$

$$\Rightarrow P(Z > \frac{-4.5}{2.828}) = P (x > - 1.591)$$

$P (Z > - 1.591) = 0.5 + P(-1.591 < z < 0)$
$= 0.5 + P(0<z<1.591$
$= 0.5 + 0.4442$
$= 0.9442.$

$\Rightarrow P(x \ge 5) = 0.9442$

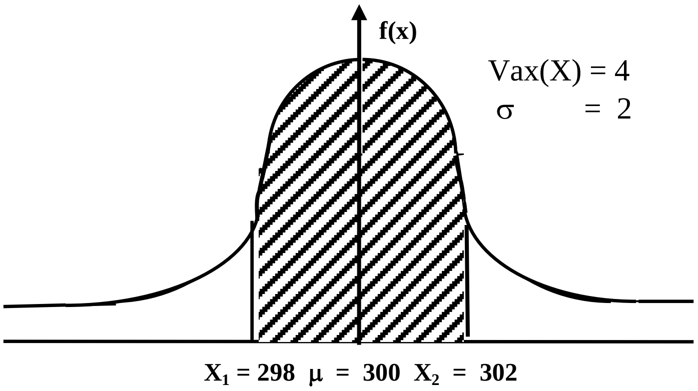
Thus $p (x \le 5) = 1- 0.9442$
$= 0.0558$

The probability that not more than 10% of the cows are infected is 0.0558. (4dps)

Example 8:

The distribution of the volumes of Mirinda fruity in a bottle is normally Distributed with mean 300mls and variance 4. If a random sample of 4 bottles is taken, find the probability that one of the bottles has a volume of less than 302mls but more than 298mls.

Solution:



$P(298 < X < 302) = P(X < 302) - P(X < 298)$

$= P\left(Z \angle \frac{302-300}{2}\right) - P\left(Z \angle \frac{298-300}{2}\right)$

$= P(Z < 1) - P(Z < 1)$

$= P(Z < 1) - P(Z < 1)$

$= P(Z < 1) - 1 + P(Z < 1)$

$= 2P(Z < 1) - 1$

$= 2 \times 0.8413 - 1$

$= \mathbf{0.6826}$

$P = 0.6826 \quad q = 0.3174$

$P(X = 1) = {}^4C_1 \ P1 \quad q3 = {}^4C_1 \ (0.6826)1 \ (0.3174)3$

$\overline{X} = np = 200 \times \frac{1}{2} = 100 \qquad p = \frac{1}{2} \qquad q = \frac{1}{2} \quad n = 200$

$\delta = \sqrt{npq}$

$= \sqrt{200 \times \frac{1}{2} \times \frac{1}{2}}$

$= 7.0711$

$= P\left(Z \angle \frac{114.5-100}{7.0711}\right)$

$= P(Z < 2.05)$

$= \mathbf{0.9798}$

(iii) $P(X > 95) = P\left(Z \angle \frac{94.5-100}{7.0711}\right)$

$= P(Z > -0.78)$

$= P(Z < 0.78)$

$= \mathbf{0.7823}$

$P(92 < X < 112) = P(X < 112) - P(X < 92)$

$= P\left(Z \angle \frac{112.5-100}{7.0711}\right) - P\left(Z \angle \frac{92.5-100}{7.0711}\right)$

$= P(Z < 1.77) - P(Z < -1.06)$

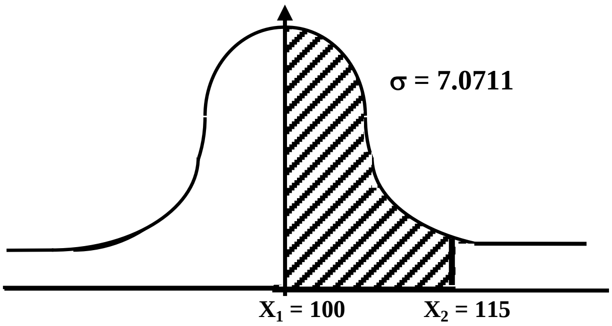
$= P(Z < 1.77) - P(Z > 1.06)$

$= P(Z < 1.77) - 1 + P(Z < 1.06)$

$= 0.9616 - 1 + 0.8554$

$= \mathbf{0.817}$

(iv) $P(100 < X < 115)$



P(100 < X < 115) = P(X < 115) – P(X < 100)

$= P\left(Z < \frac{115.5 - 100}{7.0711}\right) - P\left(Z < \frac{100.5 - 100}{7.0711}\right)$

= P(Z < 2.19) - P(Z < 0.07)

= 0.9857 – 0.5279

= **0.4578**

Example 10:

A sub-committee of European Economic Community has 15% of the members from Belgium, 25% from Germany, 10% from Spain, 30% from England and 20% from Holland attend a conference sample of 100 members is randomly selected from the conference, determine the probability that

(i) Exactly 25 members come from Holland.
(ii) At most 30 members are from England
(iii) Between 10 and 15 members are from Spain.
(iv) At least 12 members are from Belgium.

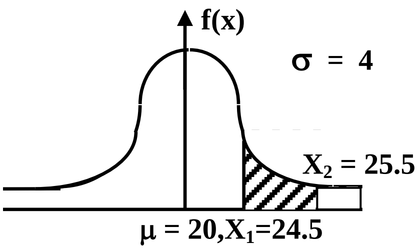
Solution:

(i) P(X = 25) = P(24.5 < **X** < 25.5), Holland

n = 100 , $p = \frac{200}{100} = 0.2$ q = 0.8

E(H) = np
= 100 x 0.2
= 20

$\sigma = \sqrt{20 \times 0.8}$
= **4**



P(24.5 < X < 25.5) = P(X < 25.5) - P(X < 24.5)

$= P\left(Z < \frac{26 - 20}{4}\right) - P\left(Z < \frac{25 - 20}{4}\right)$

= P(Z < 1.5) – P(Z < 1.25)

= 0.9332 - 0.8944

= 0.0388

(ii) P(X < 30) from England

$P(E) = \frac{30}{100} = 0.3 = P,$

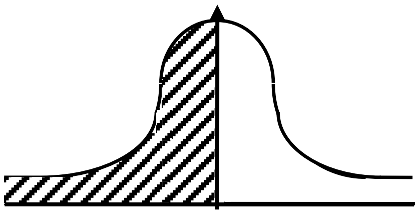
$q = 0.7,\ n = 100$

$E(E) = nP$

$= 100 \times 0.3 = 30$

$\delta = \sqrt{npq} = \sqrt{30 \times 0.7}$

$= 4.58$



$\mu = 30$

$\sigma = 4.58$

$P(X < 30) = P\left(Z < \frac{(30 + 0.5) - 30}{4.58}\right) = P(Z < 0.11)$

(iii) $P(10 < X < 15)$ from Spain.

$P(S) = \frac{10}{100}$

$= 0.1 = p$

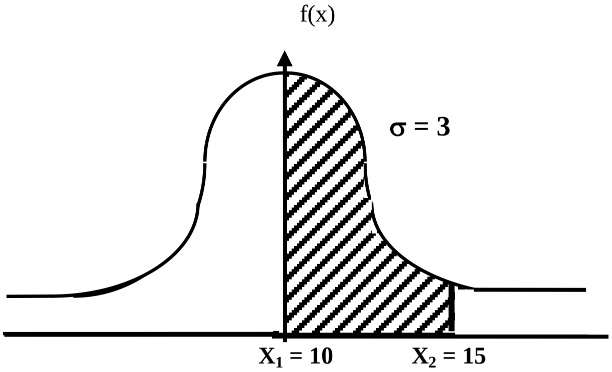
$q = 0.9$

$E(S) = 0.1 \times 100$

$= 10$

$\sigma = \sqrt{10 \times 0.9}$

$= 3$



$P(10 < x < 15) = P(X, 15) - P(x, 10)$

$= P\left(Z < \frac{15.5 - 10}{3}\right) - P\left(Z < \frac{10.5 - 10}{3}\right)$

$= P(Z < 1.83) - P(< 0.17)$

$= 0.9664 - 0.5675$

$= \mathbf{0.3989}$

(iv) $P(X > 12)$ from Belgium.

$P(B) = \frac{15}{100}$

$= 0.15 = P,$

$q = 0.85 \quad n = 100$

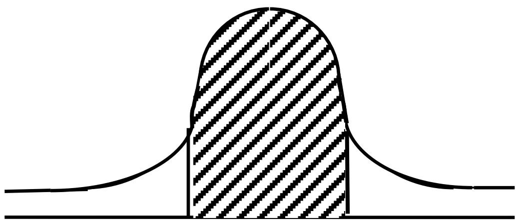
$E(B) = nP$

$= 100 \times 0.15$

$= 15$

$\sigma = \sqrt{15 \times 0.85}$

$= 3.57$



$$P(X > 12) = P\left(Z > \frac{(12 - 0.5) - 15}{3.57}\right)$$

$= P(Z > -0.98)$

$= P(Z < 0.98)$

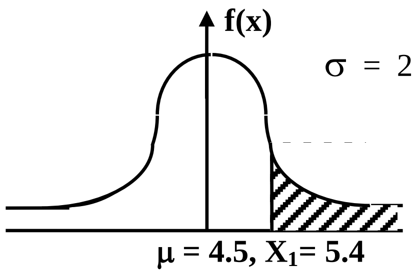
$= \mathbf{0.8365}$

Example 11: Boxes in a factory have weights which are normally distributed with a mean of 4.5Kg and a standard deviation of 2.0Kg. Find the probability of there being a box with a weight of more than 5.4Kg when a box is chosen at random. If a sample of 16 boxes is drawn, find the probability that the mean is between:

(i) 4.6 and 4.7Kg.

(ii) 4.3 and 4.7Kg.

Solution: $u = 4.5$ $\sigma = 2$ $P(X > 5.4)$



$$P(X > 5.4) = 1 - P(X < 5.4)$$

$$= 1 - P\left(Z > \frac{5.4 - 4.5}{2}\right)$$

$= 1 - P(Z < 0.45)$

$= 1 - 0.6736$

$= \mathbf{0.3262}$

(i) Sample of size (n) = 16.

$E(X) = \mu$, Standard deviation $= \dfrac{\sigma}{\sqrt{n}}$

$$E(X) = \mu = 4.5 \quad = \frac{2}{\sqrt{16}} = \frac{1}{2}$$

$= 0.5$

$$P(4.6 \le X \le 4.7) = P\left(\frac{4.6 - 4.5}{0.5} \le Z \le \frac{4.7 - 4.5}{0.5}\right)$$

$= P(0.2 \le Z \le 0.4)$

$= P(Z \le 0.4) - P(Z \le 0.2)$

$= 0.6554 - 0.5793$

$= \mathbf{0.0761}$

(ii) $P(4.3 \le X \le 4.7) = P\left(\frac{4.3-4.5}{0.5} \le Z \le \frac{4.7-4.5}{0.5}\right)$

$= P(-0.4 \le Z \le 0.4)$

$= P(Z \le 0.4) - P(Z \le -0.4)$

$= P(Z \le 0.4) - P(Z \ge 0.4)$

$= P(Z \le 0.4) - 1 + P(Z \le 0.4)$

$= 2P(Z \le 0.4) - 1$

$= 2 \times 0.6554 - 1$

$= \mathbf{0.3108}$

Example 12: Certain tubes manufactured by a company have a mean life time of 800 hours and standard deviation of 60 hours. Find the probability that a random sample of 16 tubes taken from the group will have a mean life time.

(a) between 790 and 810 hours.

(b) Less than 785 hours

(c) more than 820 hours

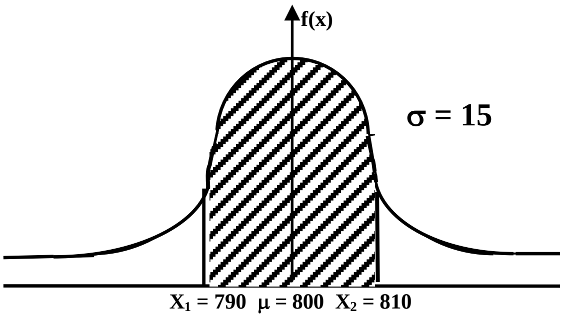
(d) between 770 and 830 hours

Solution:

(a) Sample size (n) = 16 is small, $\mu = 800$, $\sigma = 60$

$E(X) = \mu = 800$ Standard deviation $= \frac{\sigma}{\sqrt{n}}$

$= \frac{60}{\sqrt{16}} = \frac{60}{4} = 15$



$P(790 \le X \le 810) = P(X \le 810) - P(X \le 790)$

$= P\left(Z \le \frac{810-800}{15}\right) - P\left(Z \le \frac{790-800}{15}\right)$

$= P(Z \le 0.671) - P(Z \le -0.67)$

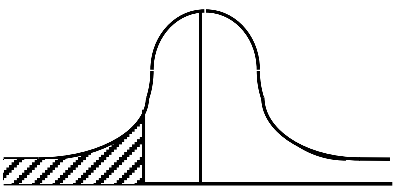
$= P(Z \le 0.67) - P(Z \ge 0.67)$

$= P(Z \le 0.67) - 1 + P(Z \le 0.67)$

$= 2P(Z \le 0.67) - 1$

$= 2 \times 0.7486 - 1$

$= \mathbf{0.4972}$

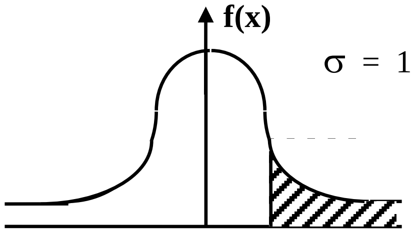


P(X < 785) = $P\left(Z < \frac{785-800}{15}\right)$

= P(Z < -1) = P(Z < 1)

= 1 – P(Z < 1) = 1 – 0.8413

= 0.1587

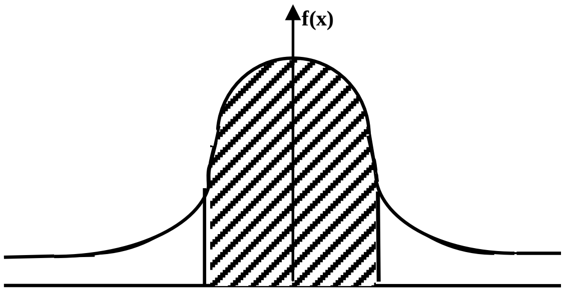


$P\left(Z > \frac{820-800}{15}\right)$ = P(Z > 1.33)

= 1 – P(Z < 1.33)

= 1- 0.9082

= **0.0918**



P (770 ≤ X ≤ 830) = P(X ≤ 830) – P(X ≤ 770)

= $P\left(Z \le \frac{830-800}{15}\right) - P\left(Z \le \frac{770-800}{15}\right)$

= P(Z ≤ 2) – P(Z ≤ -2)

= P(Z ≤ 2) – P(Z ≥ 2)

= P(Z ≤ 2) – 1 + P(Z ≤ 2)

= 2P(Z ≤ 2) – 1

= 2 x 0.9772 – 1

= **0.9544**

Example 13:

A sample of 100 apples is taken from a load. The apples have the following distribution of sizes.

Diameter to nearest cm	**6**	**7**	**8**	**9**	**10**
Frequency	**11**	**21**	**38**	**17**	**13**

Determine:

(i) the mean and standard deviation of these diameters.

(ii) Assume the distribution is approximately normal with this mean and standard deviation if 5% are to be rejected as too small and 5% are to be rejected as too large.

Solution:

x	f	fx	fx^2
6	11	66	396
7	21	147	1029
8	38	304	2432
9	17	153	1377
10	13	130	1300

$\sum f = 100 \quad \sum fx = 800 \quad \sum fx^2 = 6534$

(i) Mean (μ) $= \dfrac{\sum fx}{\sum f}$

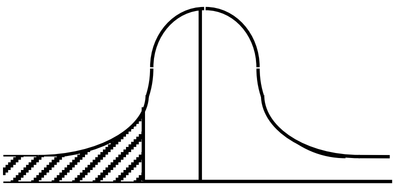
$= \dfrac{800}{100} = 8$

Standard deviation $= \sqrt{\dfrac{\sum fx^2}{\sum f} - (\mu)^2}$

$= \sqrt{\dfrac{6534}{100} - \mu^2}$

$=$ **1.1576.**

(ii) (a)



$P(Z < -Z_1) = 0.05$

$P(Z > Z_1) = 0.05$

$1 - P(Z < Z_1) = 0.05$

$P(Z < Z_1) = 0.95$

$Z_1 = 1.645$

$-Z_1 = -1.645$

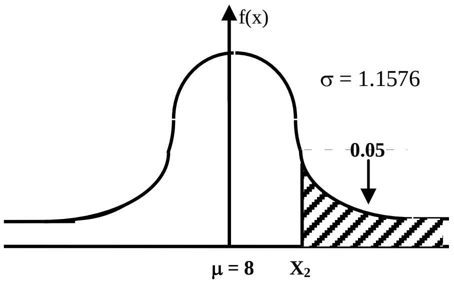
$-Z1 = \dfrac{X_1 - \mu}{\sigma} = -1.645$

$X_1 - 8 = -1.645 \text{ x } 1.1576$

$X_1 = 8 - 1625 \text{ x } 1.1576$

$X_1 = 6.0957$

$\mathbf{X_1 = 6.0957}$



$P(Z > Z_2) = 0.05$

$1 - P(Z < Z_2) = 0.05$

$P(Z < Z_2) = 0.95$

$Z2 = 1.645$

$\dfrac{X_2 - \mu}{\sigma} = Z_2$

$$\frac{X_2 - 8}{1.1576} = 1.645$$

$X2 = 8 + 1.645 \times 1.1576$

$X2 = 9.9043$

$\therefore$ range is **6.0957 – 9.9043**

Example 14: The bulbs manufactured by a company have a mean life of 800 hours and standard deviation of 60 hours. Find the probability that a random sample of 64 bulbs taken from a group will have a mean life time of :

(a) between 790 and 810 hours.
(b) Less than 785 hours
(c) More than 820 hours
(d) Between 770 and 830 hours

Solution:

$\mu = 800 \qquad \sigma = 60$

Sample selected (n) = 64 is large.

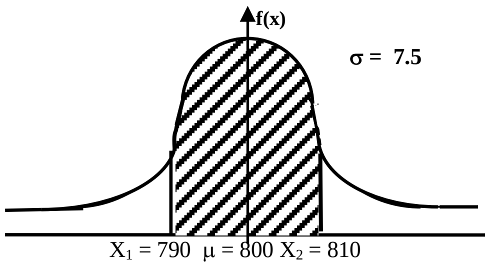
$E(X) = \mu = 800.$

$$\text{Standard deviation} = \frac{\sigma}{\sqrt{n}}$$

$$= \frac{60}{\sqrt{64}}$$

$$= 7.5$$

(a) $P(790 \le X \le 810) = P(X \le 810) - P(X \le 790)$



$$= P\left(Z \le \frac{810 - 800}{7.5}\right) - P\left(Z \le \frac{790 - 800}{7.5}\right)$$

$= P(Z \le 1.33) - P(Z \le -1.33)$

$= P(Z \le 1.33) - P(Z \ge 1.33)$

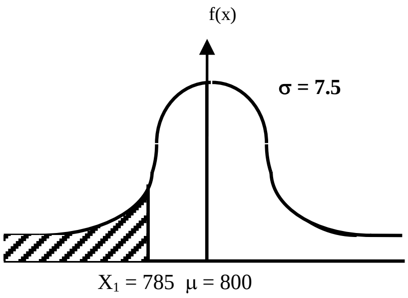
$= P(Z \le 1.33) - 1 + P(Z \le 1.33)$

$= 2P(Z \le 1.33) - 1$

$= 2 \times 0.9082 - 1$

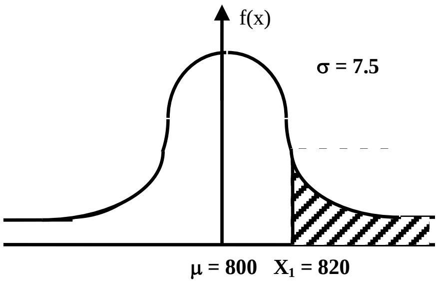
$= \mathbf{0.8164}$

(b) $P(X < 785)$



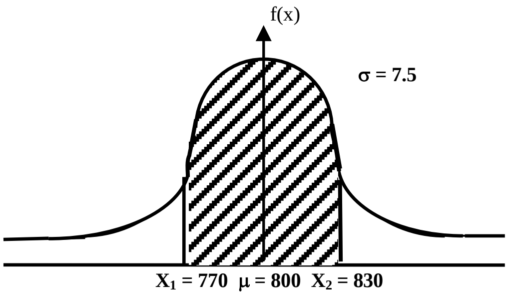
$$P(X < 785) = P\left(Z \le \frac{785 - 800}{7.5}\right)$$
$= P(Z < -2)$
$= P(Z > 2)$
$= 1 - P(Z < 2)$
$= 1 - 0.9772$
$= \mathbf{0.0228}$

(c) P (X > 820)



$$P(X > 820) = P\left(Z \le \frac{820 - 800}{7.5}\right) = P(Z > 2.67)$$
$= 1 - P(Z < 2.67)$
$= 1 - 09962$
$= \mathbf{0.0038}$

(d) $P(770 \le X \le 830) = P(X \le 830) - P(X \le 770)$



$$= P\left(Z \le \frac{830 - 800}{7.5}\right) - P\left(Z \le \frac{770 - 800}{7.5}\right)$$
$= P(Z < 4.0) - P(Z < -4.0)$
$= P(Z < 4.0) - P(Z > 4.0)$
$= P(Z < 4.0) - 1 + P(Z < 4.0)$
$= 2P(Z < 4.0) - 1$
$= 2 - 1$
$= \mathbf{1}$

Example 15: It is known that 3% of the eggs arriving at the shop are broken, what is the probability that on a morning when 500 eggs arrive.
(a) 5% or more will be broken?
(b) 3% or less will be broken?

Solution: $\mu = np \quad n = 500 \quad p = \frac{3}{100} = 0.03$

Use normal approximation to binomial
` $q = 1 - 0.03 = 0.97$

Standard deviation $= \sqrt{500 \times 0.03 \times 0.97}$
$= 3.814$

(a) $P(X \ge 25) = P\left(Z \le \frac{24.5 - 15}{3.814}\right)$

(b) $P(X \le 15) = P\left(Z \le \frac{15.5 - 15}{3.814}\right)$

$= P(\le 0.13)$

$= \underline{\mathbf{0.5521}}$

EXERCISE

1… The weights of Broilers are normally distributed with mean 1.55Kg and standard deviation 0.2kg. Determine the percentage of Broilers with weights.:

(i) less than or equal to 1 kg.

(ii) between 1.2 and 1.3kg.

(iii) between 1.5 and 1.75kg.

(iv) greater than or equal to 2Kg.

2. The marks obtained by 500 candidates by 500 candidates in an examination are normally distributed with mean of 45 marks and a standard deviation of 20 marks.

(i) Given that the pass mark is 41, estimate the number of candidates who passed the examination.

(ii) If 5% of the candidates obtain a distinction by scoring x marks or more, estimate the value of x.

(iii) Estimate the interquartile range of the distribution.

3. A horse can jump and clear a height of 1.68m once in five attempts and a height of 1.52m nine times out of ten attempts. Assuming the heights the horse clears are normally distributed estimate:

(i) the mean

(ii) standard deviation of the distribution.

4. A sample of 100 passion fruits is taken from a load. The passion fruits have distribution of size that are as follows.

Diameter to nearest cm	6	7	8	9	10
Frequency	11	21	38	17	13

Determine: (i) mean

(ii) standard deviation of these diameters.

Assuming the distribution is approximately normal with this mean and this standard deviation find the size of passion fruits for packing.

(iii)If 5% are to be rejected as too small.

(iii) If 5% are to rejected as too large.

5. The distribution of the length mm of 717 eggs of hens was found to i.e as below.

Length central value)	19.25	20.25	21.25	22.25	23.25
Frequency	3	22	123	300	201

Length central Value)	24.25	25.25	26.25
Frequency	61	6	1

Show that

(i) mean is 22.49

(ii) standard deviation is 1,00

The eggs are classified into three categories described as long, medium and short in such away that one third of the eggs might be expected to fall in each category. Determine where the divisions on the length scale should be made.

6. In the Bushenyi District Joint Mock examination the marks in Mathematics were normally distributed with mean 72 and standard deviation 8.
(a) Find the minimum mark of the top 20% of the students.
(b) Find the probability that in a random sample of 100 students the minimum mark of the to 20% will be less than 76.
7. The coins produced by a certain machine have mean diameter of 30.0Imm with standard deviation 0.08mm. A sample of 100 coins is taken. Find the probability that the mean diameter of the coins in this sample is less than 30mm.

8. The diameters x of 110 iron rods were measured in centimeters and the results were summarized as follows $\sum x = 36.5$ and $\sum x^2 = 12.49$
Find: (i) mean
(iii) standard deviation of these measurements.
Assuming these measurements are a sample from a normal distribution with this mean and this standard deviation find:
(iii) the probability that the mean diameter of a sample of size 110 is greater than 0.345cm

9. A bus arrives at Mbarara at 09.30 daily . On ten successive days the number of minutes by which the train was late were as follows:
3, 0, 4, -2, -3, 13, 8, -2, 6, 3.
Show that: (i) the mean of arrival is 09.33
(ii) Calculate the standard deviation.
Assuming the time of arrival is normally distributed with the above mean and standard deviation estimate the probability that the bus will arrive.
(i) on or before 09.30
(ii)more than 8 minutes late.

10. In a subsidiary Mathematics U.N.E.B. examination last year 30% of the candidates failed and 10% obtained distinctions, the pass mark was 84 out of 200 and the minimum mark required for the distinction was 154 out of 200. Assuming the candidates marks were normally distributed determine.
(i) mean
(ii) standard deviation.
11. If an unbiased coin is tossed 1000 times what is the probability that:
(a) there will be more than 600 heads?
(b) there will be at least 450 and at most 550 heads?
(c) there will be fewer than 520 heads?

12. A confectionary firm produces there types of toffees, liquorices, nut and plain, and mixes them together in the ratio 1:2:5 before packing them into boxes. If there are 80 toffees in a box what percentage of boxes will contain:
(i) more than 25 nut toffees?
(ii) fewer than 58 Plain toffees?
(iii) more nut and liquorices than plain toffees?
13. (a) The heights of applicants to the police force are normally distributed with mean 170cm and standard deviation 3.8cm, if 30% of applicants are rejected because they are too short, what is the minimum height for the police force
(c) The diameter of washers produced by a machine has standard deviation of 0.1mm. What should the mean diameter be if there is to be probability of only 3% that diameter exceeds 2.0mm?

14. The volume in litres of whisky in bottles filled by a machine is N(1.01, 0.012)
(a) What is the probability that a bottle contains less than 1 litre?
(b) To what value must the mean be altered to reduce the probability in (a) to 1% assuming the standard deviation is unaltered?

15. (a) The life of a certain make of electric light bulb is known to be normally distributed with mean life of 2000 hours and standard deviation of 120 hours. Estimate the probability that the life of such a bulb will be:
(i) greater than 2150 hours.
(ii) greater than 1910 hours.
(iii) with in the range 1850 hours to 2090 hours.
(b)The masses of packets of sugar are normally distributed in a large consignment of packets of sugar. It is found that 5% of them have a mass greater than 510g and 2% have a mass greater than 515g.

Estimate: (i) the mean.
(ii) the standard deviation of this distribution.

ANSWERS

1. (i) 0.3% (ii) 6.6% (iii) 44% (iv) 1.2%)
2.(i) 290 (ii) 78 (iii) 27)
3.(i) $\mu = 1.616$ (ii) S = 0.075
4.(i) 8 (ii) 1.16 (iii) 6.1 (iv) 9.9)
5.22.92, 22.06mm
6.(a) 78.7 (b) 0.0090)
7.(0.1056)
8.(i) 0.332 (ii) 0.0587 (iii) 0.0092
9.(ii) 4.8min (iii) 0.27 (iv) 0.15
10. (i) $\mu = 104$ (ii) $\sigma = 38.8$
11.(a) 0 (b) 0.999 (c) 0.89)
12. (i) 7.8% (ii) 95.8% (iii) 0.8%)
13.(a) 168cm (b) 1.8119mm)
14.(a) 0.1587 (b) 1.023 litres).
15.(a) (i) 0.1056 (ii) 0.7734 (iii) 0.6678
(b) (i) $\mu = 50.154$ (iii) $\sigma = 4$)