Linear equations in one unknown

Linear equation is an equation of a straight line. It is an equation whose unknown variable has the highest power one. E.g. y = x + 5, 4y = 3x + 4, etc.

Linear equation in one unknown is the one which has got only one unknown variable, such as x + 2 = 5, $\frac{2x-3}{x+2} = 4$, …..

Solving for unknown of the equation is the same as making that variable the subject of the equation

Example 1

Solve the following equations

(i) 4x + 4 = 10

Solution

4x + 6 = 10

4x = 10 – 6 = 4

x= $\frac{4}{4} = 1$

(ii) 3x + 2 = x + 8
Solution
3x + 2 = x + 8
3x – x = 8 – 2
2x = 6
x = 3

(iii) $\frac{5x-3}{4} = \frac{4x-3}{3}$
Solution
$\frac{5x-3}{4} = \frac{4x-3}{3}$
3(5x – 3)= 4(4x – 3)
15x – 9 = 16x – 12
-x = -3
x = 3

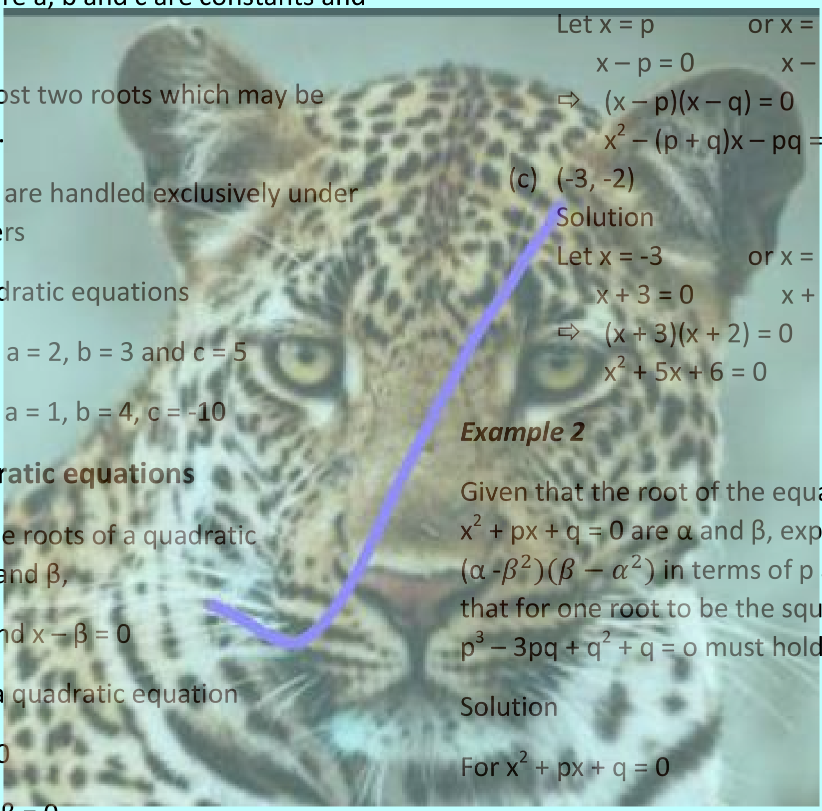
(iv) $\frac{1}{5}(2x-1) - \frac{1}{4}(3x-4) = 0$
$solution$
$\frac{1}{5}(2x-1) - \frac{1}{4}(3x-4) = 0$

Multiplying through by 20

$4(2x-1) - 5(3x-4) = 0$
8x – 4 – 15x + 20 = 0
-7x = - 16
x = $\frac{16}{7}$

Revision exercise

1. Solve the following equation
   (a) 2x + 4 = 0 [x=3]
   (b) 5x-6 = 24 [x =2]
   (c) 3x + 2 = x + 8 [x =3]
   (d) 2x + 5 = 29 -10x [x= 2]
   (e) 3(x – 8) + 2(4x – 1) = 3 $\left[x = \frac{29}{11}\right]$
   (f) 3(2x – 5) – 4(x – 2) = 5(x – 8) [x = 11]
2. Solve the following equations
   (a) $\frac{5x-3}{4} = \frac{4x-3}{3}$ [x = 3]
   (b) $\frac{7}{1-x} = \frac{3}{x+2}$ $\left[x = -\frac{17}{4}\right]$
   (c) 2x + 3 + $\frac{5x-1}{4} = \frac{3x-2}{8}$ $\left[x = -\frac{24}{23}\right]$
   (d) $\frac{2x-3}{4} + \frac{6x-4}{3} = \frac{2x+5}{6}$ $\left[\frac{35}{26}\right]$
   (e) $\frac{1}{5}(x-2) + \frac{1}{3}(5x-4) = \frac{1}{2}x$ $\left[x = \frac{52}{41}\right]$
   (f) $\frac{3x+1}{2x-3} = \frac{6x+1}{4x-5}$ $\left[x = \frac{2}{5}\right]$
   (g) $\frac{4-x}{2x+3} = \frac{x-1}{3-2x}$ $\left[x = \frac{5}{4}\right]$



Quadratic equations

These are equations expressed in the form $ax^2 + bx + c = 0$ where a, b and c are constants and $a \neq 0$

They have at most two roots which may be real or complex.

The latter roots are handled exclusively under complex numbers

Example of quadratic equations

$2y^2 + 3y + 5 = 0$; a = 2, b = 3 and c = 5

$x^2 + 4x - 10 = 0$; a = 1, b = 4, c = -10

Forming quadratic equations

Suppose that the roots of a quadratic equation are α and β,

then $x - \alpha = 0$ and $x - \beta = 0$

When forming a quadratic equation

$(x - \alpha)(x - \beta) = 0$

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

This means that if the roots of a quadratic equation are given, its equation in terms of x is x^2-(sum of roots)x + product of roots = 0

Example 1

Form quadratic equation I terms of x with roots

(a) (2, 3)
Solution
Let x = 2 or x = 3
$x - 2 = 0$ $x - 3 = 0$
⇨ $(x - 2)(x - 3) = 0$
$x^2 - 5x - 6 = 0$

(b) (p, q)
Solution
Let x = p or x = q
$x - p = 0$ $x - q = 0$
⇨ $(x - p)(x - q) = 0$
$x^2 - (p + q)x - pq = 0$

(c) (-3, -2)
Solution
Let x = -3 or x = -2
$x + 3 = 0$ $x + 2 = 0$
⇨ $(x + 3)(x + 2) = 0$
$x^2 + 5x + 6 = 0$

Example 2

Given that the root of the equation $x^2 + px + q = 0$ are α and β, express $(\alpha - \beta^2)(\beta - \alpha^2)$ in terms of p and q. Deduce that for one root to be the square of another $p^3 - 3pq + q^2 + q = o$ must hold

Solution

For $x^2 + px + q = 0$

$\alpha + \beta = -p$

$\alpha\beta = q$

$(\alpha - \beta^2)(\beta - \alpha^2) = \alpha\beta - \alpha^3 - \beta^3 + (\alpha\beta)^2$

$= \alpha\beta - (\alpha^3 + \beta^3) + (\alpha\beta)^2$

But $(\alpha^3 + \beta^3) = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$(\alpha - \beta^2)(\beta - \alpha^2)$

$= \alpha\beta - [(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)] + (\alpha\beta)^2$

$= q - [(-p)^3 - 3q(-p)] + q^2$

$= q + p^3 - 3pq + q^2$

$(\alpha - \beta^2)(\beta - \alpha^2) = p^3 - 3pq + q^2 + q$

If $\alpha = \beta^2$

$(\beta^2 - \beta^2)(\beta - \beta^4) = p^3 - 3pq + q^2 + q$

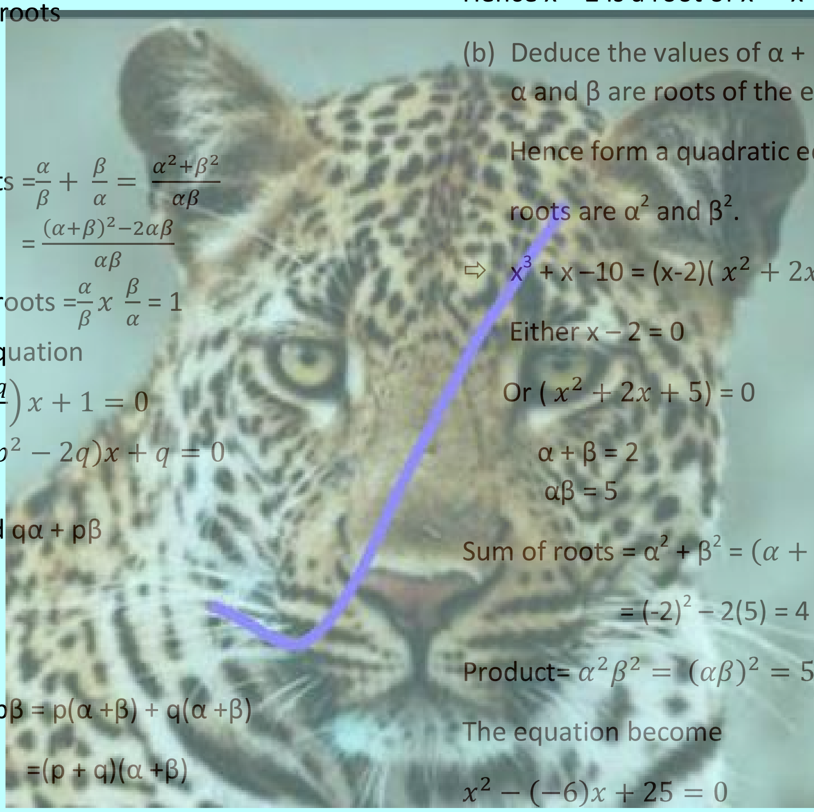
$0 = p^3 - 3pq + q^2 + q$

Or

$p^3 - 3pq + q^2 + q = 0$ As required

Example 3

Given that the root of the equation $x^2 + px + q = 0$ are α and β, Form quadratic equations with roots



(a) $\frac{\alpha}{\beta}$ *and* $\frac{\beta}{\alpha}$

Solution

Sum of roots $= \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2+\beta^2}{\alpha\beta}$

$= \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta}$

Product of roots $= \frac{\alpha}{\beta} x \frac{\beta}{\alpha} = 1$

Required equation

$x^2 - \left(\frac{p^2-2q}{q}\right)x + 1 = 0$

Or $qx^2 - (p^2 - 2q)x + q = 0$

(b) $p\alpha + q\beta$ and $q\alpha + p\beta$

Solution

Sum of roots

$p\alpha + q\beta + q\alpha + p\beta = p(\alpha + \beta) + q(\alpha + \beta)$

$= (p + q)(\alpha + \beta)$

$= p(p + q)$

Product of roots $= (p\alpha + q\beta)(q\alpha + p\beta)$

$= pq\alpha^2 + p^2\alpha\beta + q^2\alpha\beta + pq\beta^2$

$= pq(\alpha^2 + \beta^2) + \alpha\beta(p^2 + q^2)$

$= pq[(\alpha + \beta)^2 - 2\alpha\beta] + \alpha\beta(p^2 + q^2)$

$= pq(p^2 - 2q) + q(p^2 + q^2)$

Required equation

$x^2 - p(p + q)x + pq(p^2 - 2q) + q(p^2 + q^2) = 0$

Example 4

Given the equation $x^3 + x - 10 = 0$.

(a) Show that $x = 2$ is a root of the equation

Let $f(x) = x^3 + x - 10$

Substituting for $x = 2$

$f(2) = 2^3 + x - 10$

$= 8 + 2 - 10$

$= 10 - 10 = 0$

Hence $x = 2$ is a root of $x^3 + x - 10 = 0$

(b) Deduce the values of $\alpha + \beta$ and $\alpha\beta$ where α and β are roots of the equation.

Hence form a quadratic equation whose roots are α^2 and β^2.

⇨ $x^3 + x - 10 = (x-2)(x^2 + 2x + 5)$

Either $x - 2 = 0$

Or $(x^2 + 2x + 5) = 0$

$\alpha + \beta = 2$

$\alpha\beta = 5$

Sum of roots $= \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= (-2)^2 - 2(5) = 4 - 10 = -6$

Product $= \alpha^2\beta^2 = (\alpha\beta)^2 = 5^2 = 25$

The equation become

$x^2 - (-6)x + 25 = 0$

$x^2 + 6x + 25 = 0$

Solving quadratic equations

Quadratic equations may be solved by

(a) Factorization method
(b) Completing square method
(c) Graphical method

(a) Factorization method

It is used for quadratic equations that are easy to factorise

Example 5

(a) $4x^2 + 7x + 3 = 0$

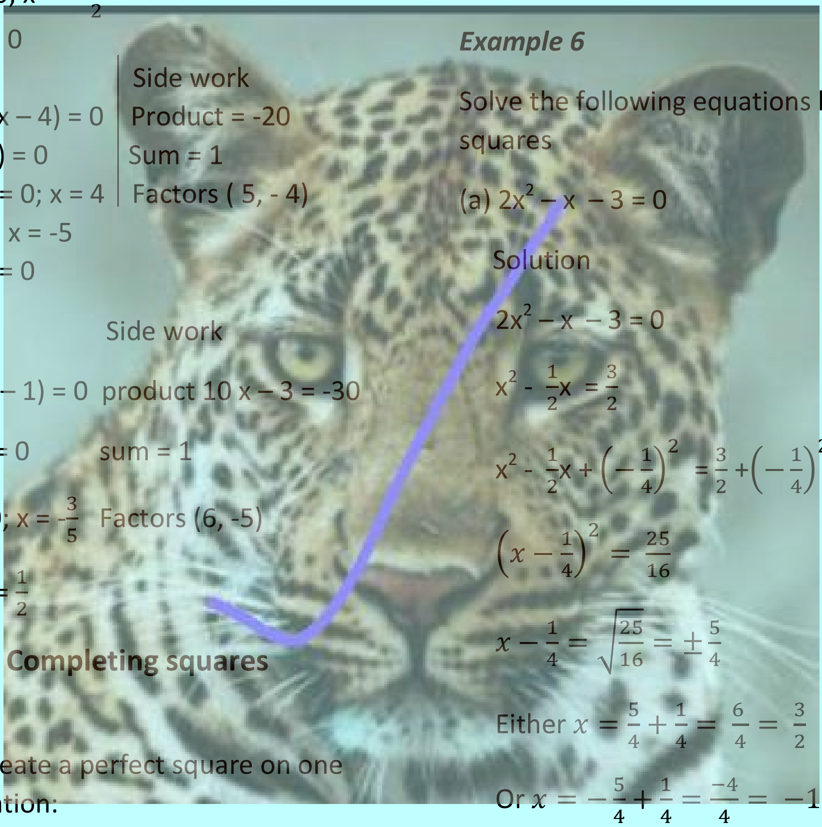
Solution	side work
$4x(x + 1) + 3(x + 1) = 0$	Product = 4 x 3 = 12
$(x + 1)(4x + 3) = 0$	Sum = 7
Either $x + 1 = 0$; $x = -1$	Factor = (4, 3)

Or $4x + 3 = 0$; $x = -\frac{3}{4}$

(b) $2x^2 + 5x + 3 = 0$

Solution	side work
$2x(x + 1) + 3(x + 1) = 0$	Product = 3 x 2 = 6
$(x + 1)(2x + 3) = 0$	Sum = 5
Either $x + 1 = 0$; $x = -1$	Factors 3, 2

Or $2x + 3 = 0$; $x = -\frac{3}{2}$



(c) $x^2 + x - 20 = 0$

Solution	Side work
$x(x - 4) + 5(x - 4) = 0$	Product = -20
$(x - 4)(x + 5) = 0$	Sum = 1
Either $x - 4 = 0$; $x = 4$	Factors (5, - 4)

Or $x + 5 = 0$; $x = -5$

(d) $10x^2 + x - 3 = 0$

Solution	Side work
$5x(2x - 1) + 3(2x - 1) = 0$	product 10 x – 3 = -30
$(5x + 3)(2x - 1) = 0$	sum = 1
Either $5x + 3 = 0$; $x = -\frac{3}{5}$	Factors (6, -5)

Or $2x - 1 = 0$; $x = \frac{1}{2}$

(b) Method II: Completing squares approach

The idea is to create a perfect square on one side of the equation:

Given the equation $ax^2 + bx + c = 0$

- Dividing the equation by and transposing the constant term to the RHS
  $x^2 + \frac{b}{a}x = -\frac{c}{a}$
- Marking the LHS a perfect square, add a half the coefficient of x squared on both sides

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

- Factorise the terms on the LHS
- $\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a} = \frac{b^2-4ac}{4a^2}$
- Taking square root on both sides of the equation
- $x + \frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$
- Solving $x + \frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2a}$
- $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

This is the general quadratic equation formula for finding the square root of any quadratic equation. This formula is locally known as ***bull dozer formula***

Example 6

Solve the following equations by completing squares

(a) $2x^2 - x - 3 = 0$

Solution

$2x^2 - x - 3 = 0$

$x^2 - \frac{1}{2}x = \frac{3}{2}$

$x^2 - \frac{1}{2}x + \left(-\frac{1}{4}\right)^2 = \frac{3}{2} + \left(-\frac{1}{4}\right)^2$

$\left(x - \frac{1}{4}\right)^2 = \frac{25}{16}$

$x - \frac{1}{4} = \sqrt{\frac{25}{16}} = \pm\frac{5}{4}$

Either $x = \frac{5}{4} + \frac{1}{4} = \frac{6}{4} = \frac{3}{2}$

Or $x = -\frac{5}{4} + \frac{1}{4} = \frac{-4}{4} = -1$

Hence $x = -1$ and $x = \frac{3}{2}$

(b) $18x^2 + 7x - 1 = 0$

Solution

$18x^2 + 7x - 1 = 0$

$x^2 + \frac{7}{18}x = \frac{1}{18}$

$x^2 + \frac{7}{18}x + \left(\frac{7}{36}\right)^2 = \frac{1}{18} + \left(-\frac{7}{36}\right)^2$

$\left(x + \frac{7}{36}\right)^2 = \frac{1}{18} + \frac{49}{1296} = \frac{121}{1296}$

$x + \frac{7}{36} = \sqrt{\frac{121}{1296}} = \pm\frac{11}{36}$

Either $x = \frac{11}{36} - \frac{7}{36} = \frac{4}{36} = \frac{1}{9}$

Or $x = -\frac{11}{36} - \frac{7}{36} = -\frac{18}{36} = -\frac{1}{2}$

Hence x = $\frac{1}{9}$ or x =$-\frac{1}{2}$

(c) $3x^2 + 7x + 2 = 0$

Solution

$3x^2 + 7x + 2 = 0$

$x^2 + \frac{7}{3}x = -\frac{2}{3}$

$x^2 + \frac{7}{3}x + \left(\frac{7}{6}\right)^2 = -\frac{2}{3} + \left(\frac{7}{6}\right)^2$

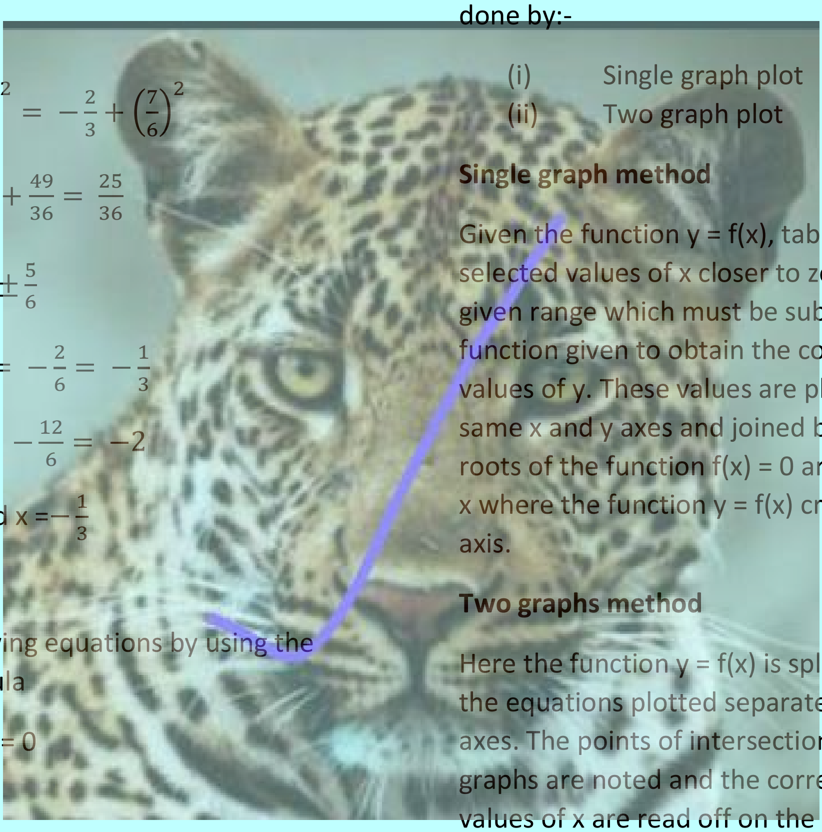
$\left(x + \frac{7}{6}\right)^2 = -\frac{2}{3} + \frac{49}{36} = \frac{25}{36}$

$x + \frac{7}{6} = \sqrt{\frac{25}{36}} = \pm\frac{5}{6}$

Either x =$\frac{5}{6} - \frac{7}{6} = -\frac{2}{6} = -\frac{1}{3}$

Or x =$-\frac{5}{6} - \frac{7}{6} = -\frac{12}{6} = -2$

Hence x = -2 and x =$-\frac{1}{3}$



Example 7

Solve the following equations by using the quadratic formula

(a) $7x^2 - 5x - 2 = 0$

Solution

$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{5\pm\sqrt{-5^2-4\,x\,7\,x-2}}{2\,x\,7} = \frac{5\pm9}{14}$

Either $x = \frac{14}{14} = 1$ or $x = \frac{-4}{14} = -\frac{2}{7}$

Hence $x = 1$ *and* $x = -\frac{2}{7}$

(b) $3x^2 - 7x - 6 = 0$

$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{7\pm\sqrt{-7^2-4\,x\,3\,x-6}}{2\,x\,3} = \frac{7\pm11}{6}$

Either $x = \frac{18}{6} = 3$ or $x = \frac{-4}{6} = -\frac{2}{3}$

Hence $x = 3$ *and* $x = -\frac{2}{3}$

(c) $6x^2 - 5x - 6 = 0$

$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{5\pm\sqrt{-5^2-4\,x\,6\,x-6}}{2\,x\,6} = \frac{5\pm13}{12}$

Either $x = \frac{18}{12} = \frac{3}{2}$ or $x = \frac{-8}{12} = -\frac{2}{3}$

Hence $x = \frac{3}{2}$ *and* $x = -\frac{2}{3}$

Method 3: Graphical approach

Here, the roots of the equations are established by plotting suitable graphs. It may done by:-

(i) Single graph plot

(ii) Two graph plot

Single graph method

Given the function y = f(x), tabulate the selected values of x closer to zero within a given range which must be substituted in the function given to obtain the corresponding values of y. These values are plotted on the same x and y axes and joined by a curve. The roots of the function f(x) = 0 are the values of x where the function y = f(x) crosses the x-axis.

Two graphs method

Here the function y = f(x) is split into two and the equations plotted separately on the same axes. The points of intersection of the two graphs are noted and the corresponding values of x are read off on the x – axis.

Example 8

Solve the following equation by using graphical approach

(a) $2x^2 + 3x - 3 = 0$

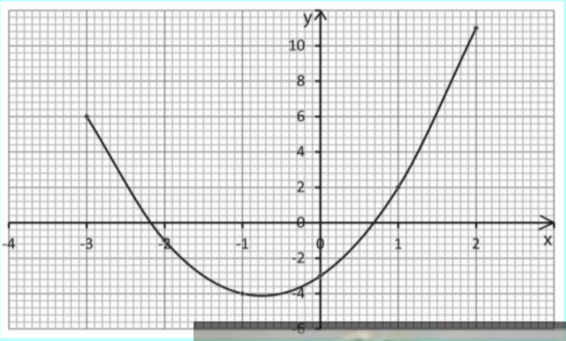
Solution

Using a single graphs

Let $y = 2x^2 + 3x - 3$ taking $-3 \leq x \leq 2$

Table of values

X	-3	-2	-1	0	1	2
y	6	-1	-4	-3	2	11



From the graph, the roots of the equation are $x = -2.2$ and $x = 0.7$

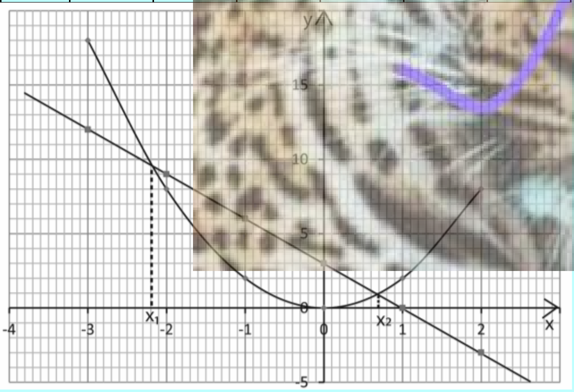
Using two graphs

By splitting the equation $2x^2 + 3x - 3 = 0$ into two we have $2x^2 = -3x + 3$

Let $y_1 = 2x^2$ and $y_2 = 3 - 3x$

Table of results

X	-3	-2	-1	0	1	2
y_1	18	8	2	0	2	8
y_1	12	9	6	3	0	-3



From the graph, the roots of the equation are $x = -2.2$ and $x = 0.7$

(b) $x^2 + x - 6 = 0$

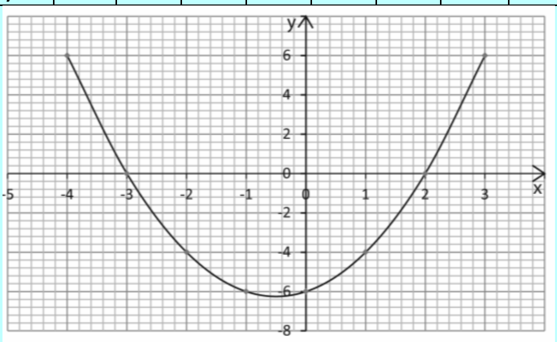
Solution

Using a single graphs

Let $y = x^2 + x - 6 = 0$, taking $-4 \leq x \leq 3$

Table of values

X	-4	-3	-2	-1	0	1	2	3
y	6	0	-4	-6	-6	-4	0	6



From the graph, the roots of the equation are $x = -3$ and $x = 2$

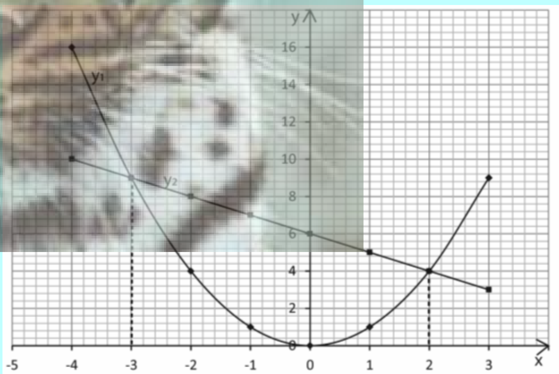
Using two graphs

By splitting the equation $x^2 + x - 6 = 0$ into two we have $x^2 = 6 - x$

Let $y_1 = x^2$ and $y_2 = 6 - x$

Table of results

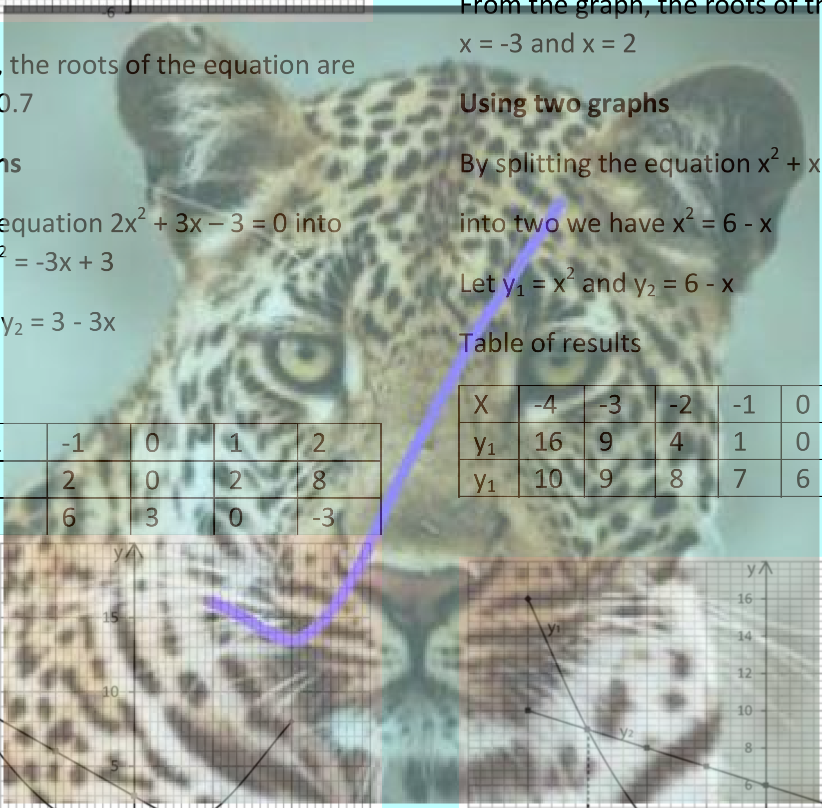
X	-4	-3	-2	-1	0	1	2	3
y_1	16	9	4	1	0	1	4	9
y_1	10	9	8	7	6	5	4	3



From the graph, the roots of the equation are $x = -3$ and $x = 2$

Minimum and maximum values of quadratic expression

The methods of completing squares and graphing can be used to obtain minimum and maximum values of quadratic expression



Completing square method

The general form of quadratic equation $y = ax^2 + bx + c$ can be expressed as

$y = ax^2 + bx + c$

$$y = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

$$y = a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2\right)$$

$$y = a\left(x + \frac{b}{2a}\right)^2 + \left(\frac{4ac-b^2}{4a}\right)$$

The minimum/maximum value is the constant $\frac{4ac-b^2}{4a}$ which is attained when $x + \frac{b}{2a} = 0$

Note

- if a < 0, the value of the function is maximum
- if a > 0, the value of the function is minimum

Example 9

Determine the minimum or maximum values of the following expressions using completing squares

(a) $2x^2 - x - 3 = 0$

Solution

Let $y = 2x^2 - x - 3 = 0$

$$y = 2\left(x^2 + \frac{2}{2}x - \frac{3}{2}\right)$$

$$y = 2\left(x^2 + x + \left(\frac{1}{2}\right)^2 - \frac{3}{2} - \left(\frac{1}{2}\right)^2\right)$$

$$y = 2\left(\left(x + \frac{1}{2}\right)^2 - \frac{7}{4}\right)$$

$$y = 2\left(x + \frac{1}{2}\right)^2 - \frac{7}{4}$$

Since a > 0, the function has got a minimum value at $x = -\frac{1}{2}$. Hence $y_{min} = -\frac{7}{4}$

(b) $-4 + 6x - x^2$

Let $y = -4 + 6x - x^2$

$y = -(x^2 - 6x + 4)$

$y = -(x^2 - 6x + (-3)^2 + 4 - (-3)^2)$

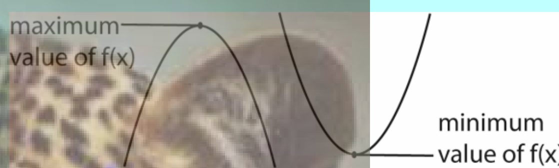
$y = -((x - 3)^2 + 4 - 9)$

$y = -((x - 3)^2 + -5)$

Since a = -1 < 0, the expression has got a maximum value when x = 3. Hence $y_{max} = 5$

Using graphical method

After graphing, y =f(x) the minimum value of the expression is lowest point if a trough/valley like or curving upwards and the maximum value is the maximum point if downwards



Example 10

Determine the maximum or minimum values of the following expression using graphical method

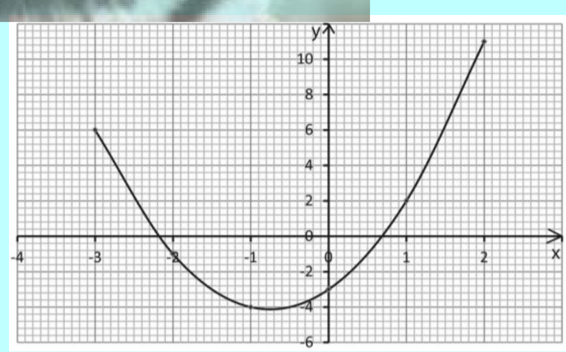
(a) $2x^2 + 3x - 3 = 0$

Solution

Let $y = 2x^2 + 3x - 3$ taking $-3 \leq x \leq 2$

Table of values

X	-3	-2	-1	0	1	2
y	6	-1	-4	-3	2	11



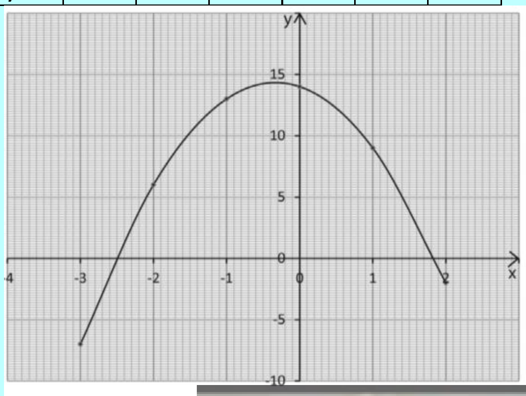
$y_{min} = -4.1$ at -0.75

(b) $14 - 2x - 3x^2$

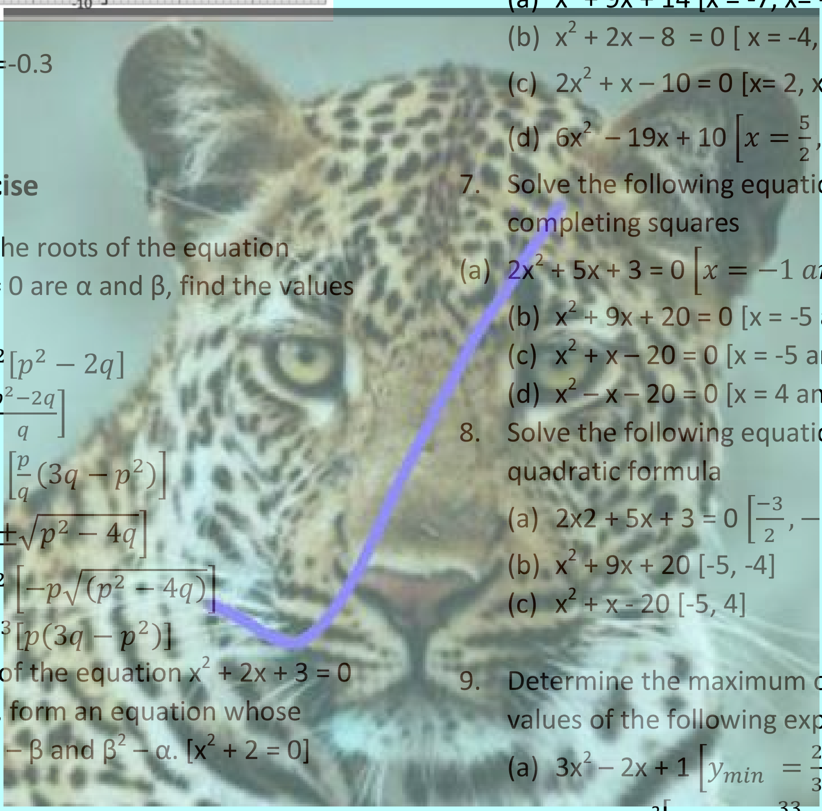
Ley $y = 14 - 2x - 3x^2$; $-3 \leq x \leq 2$

Table of results

x	-3	-2	-1	0	1	2
y	-7	6	13	14	9	-2



y_{max} = 14.3 at x = -0.3



Revision exercise

1. Given that the roots of the equation $x^2 + px + q = 0$ are α and β, find the values of:
   (a) $\alpha^2+\beta^2 [p^2-2q]$
   (b) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\left[\frac{p^2-2q}{q}\right]$
   (c) $\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}\left[\frac{p}{q}(3q-p^2)\right]$
   (d) $\alpha-\beta\left[\pm\sqrt{p^2-4q}\right]$
   (e) $\alpha^2-\beta^2\left[-p\sqrt{(p^2-4q)}\right]$
   (f) $\alpha^3+\beta^3[p(3q-p^2)]$
2. If the roots of the equation $x^2 + 2x + 3 = 0$ are α and β, form an equation whose roots are $\alpha^2 - \beta$ and $\beta^2 - \alpha$. [$x^2 + 2 = 0$]
3. Given that the roots of the equation $x^2 - 2x + 10 = 0$ are α and β, form an equation whose roots are

   $\frac{1}{(2+\alpha)^2}$ and $\frac{1}{(2+\beta)^2}$ $[324x^2+1=0]$
4. If α and β are roots of the equation $x^2 - px + q = 0$, find the equation whose roots are $\frac{\alpha^3-1}{\alpha}$ *and* $\frac{\beta^3-1}{\beta}$
   [$qx^2 - (p^2q - 2q^2 - p)x + (q^3 - p^3 + 3pq) + 1 = 0$]
5. The roots of the equation $3x^2 - ax + 6b = 0$ are α and β. Find the condition for one root to be
   (a) Twice the other [$81b = a^2$)
   (b) The cube of the other
   [$a^4 - 648(b - 1)b^2 - 18(4a^2 + 9)b = 0$]
6. By Factorization method solve the following quadratic equations
   (a) $x^2 + 9x + 14$ [x = -7, x= -2]
   (b) $x^2 + 2x - 8 = 0$ [x = -4, x = 2)
   (c) $2x^2 + x - 10 = 0$ [x= 2, x = $-\frac{5}{2}$]
   (d) $6x^2 - 19x + 10\left[x=\frac{5}{2},\ x=\frac{2}{3}\right]$
7. Solve the following equations by completing squares
   (a) $2x^2 + 5x + 3 = 0\left[x=-1\ and\ x=-\frac{3}{2}\right]$
   (b) $x^2 + 9x + 20 = 0$ [x = -5 and x = -4]
   (c) $x^2 + x - 20 = 0$ [x = -5 and x = 4]
   (d) $x^2 - x - 20 = 0$ [x = 4 and x = 5]
8. Solve the following equations using the quadratic formula
   (a) 2x2 + 5x + 3 = 0 $\left[\frac{-3}{2},-1\right]$
   (b) $x^2 + 9x + 20$ [-5, -4]
   (c) $x^2 + x - 20$ [-5, 4]
9. Determine the maximum or minimum values of the following expression
   (a) $3x^2 - 2x + 1\left[y_{min}=\frac{2}{3} at\ x=\frac{1}{3}\right]$
   (b) $4 - x - x^2\left[y_{max}=\frac{33}{8} at\ x=-\frac{1}{4}\right]$
10. Determine the maximum or minimum values of the following expression using graphical method
    (a) $14 - 2x - 2x^2$

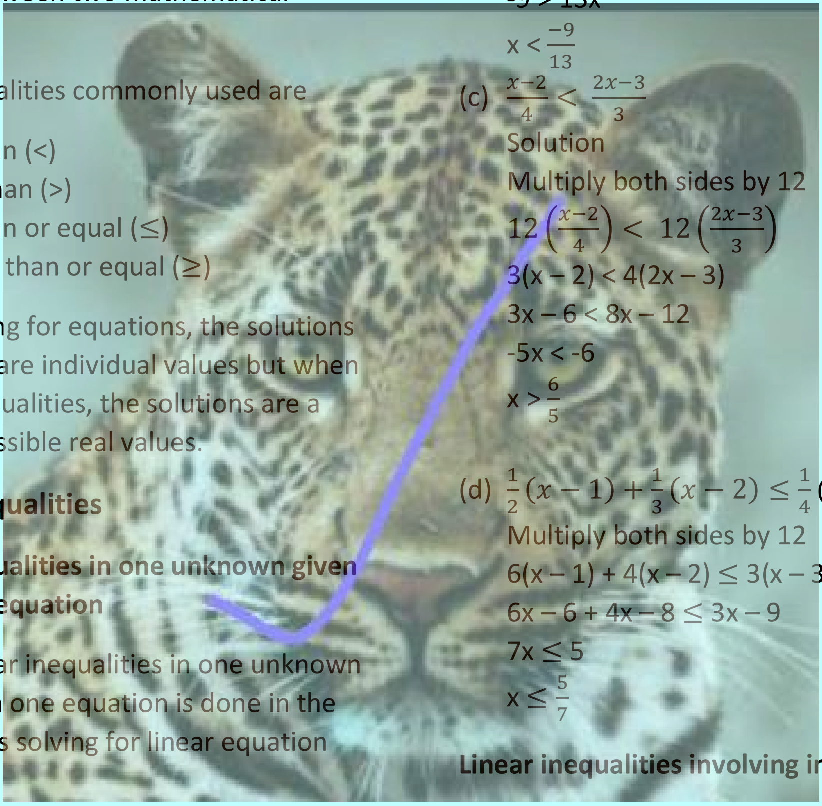
Inequalities

An inequality is a logical statement that states relationship between two mathematical expressions.

The basic inequalities commonly used are

- Less than (<)
- More than (>)
- Less than or equal (≤)
- Greater than or equal (≥)

When solving for equations, the solutions or answers are individual values but when solving inequalities, the solutions are a range of possible real values.



Linear inequalities

Linear inequalities in one unknown given one in one equation

Solving linear inequalities in one unknown given one in one equation is done in the same way as solving for linear equation except

- The inequality symbols must be maintained
- The inequality symbol changes when dividing both sides of inequality equation by a negative number.

Example 1

Solve the following inequalities

(a) $4x - 2 > x + 7$
Solution
$4x - x > 7 + 2$
$3x > 9$
$x > 3$

(b) $3(2 - x) > 5(3 + 2x)$
Solution
$6 - 3x > 15 + 10x$
$-9 > 13x$
$x < \frac{-9}{13}$

(c) $\frac{x-2}{4} < \frac{2x-3}{3}$
Solution
Multiply both sides by 12
$12\left(\frac{x-2}{4}\right) < 12\left(\frac{2x-3}{3}\right)$
$3(x - 2) < 4(2x - 3)$
$3x - 6 < 8x - 12$
$-5x < -6$
$x > \frac{6}{5}$

(d) $\frac{1}{2}(x-1) + \frac{1}{3}(x-2) \leq \frac{1}{4}(x-3)$
Multiply both sides by 12
$6(x - 1) + 4(x - 2) \leq 3(x - 3)$
$6x - 6 + 4x - 8 \leq 3x - 9$
$7x \leq 5$
$x \leq \frac{5}{7}$

Linear inequalities involving indices

When solving inequalities involving indices such as $a^x > b$, where a and b are positive integers, introduce natural logarithms to both sides of the inequality. i.e.

Ina^x > b

xlna > lnb

$x > \frac{lna}{lnb}$

Example 2

Solve the following inequalities correct to 3 decimal places

(a) $5^{2x} > 8$

Solution

$\ln 5^{2x} > \ln 8$

$2x\ln 5 > \ln 8$

$x > \frac{ln8}{2ln5} = 0.646$

$x > 0.646$

(b) $20^{-3x} < 15$

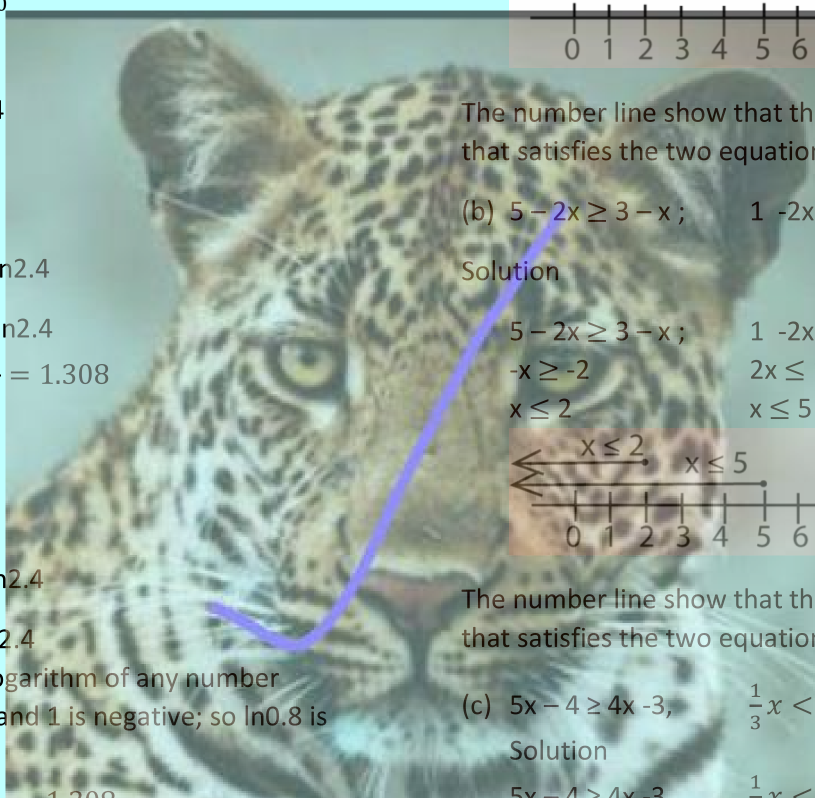
Solution

$\ln 20^{-3x} > \ln 15$

$-3x\ln 20 > \ln 15$

$x > -\frac{ln15}{3ln20} = -0.301$

$x > -0.301$



(c) $(0.8)^{-3x} > 2.4$

Solution

$\ln(0.8)^{-3x} > \ln 2.4$

$-3x\ln 0.8- > \ln 2.4$

$x > \frac{ln2.4}{-3ln0.8} = 1.308$

$x > 1.308$

(d) $(0.8)^{3x} > 2.4$

Solution

$\ln(0.8)^{3x} > \ln 2.4$

$3x\ln 0.8 > \ln 2.4$

Note that logarithm of any number between 0 and 1 is negative; so ln0.8 is negative

$x < \frac{ln2.4}{3ln0.8} = -1.308$

$x < -1.308$

Linear inequalities in one unknown given two inequalities equations.

Linear inequalities in one unknown given two inequalities equations

The solution to two linear inequalities can be best handled by use of a number line. When finding a set of integers that satisfy the equations, we only take on integral (discrete) values.

Example 3

Find the set of integers which satisfy simultaneously both of the following equations

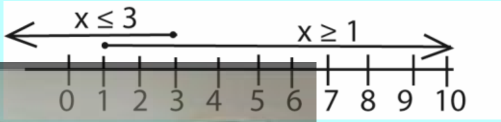
(a) $4x + 3 \geq 2x + 5$; $x + 4 \leq 7$

Solution

$4x + 3 \geq 2x + 5$; $x + 4 \leq 7$

$2x \geq 2$ $x \leq 3$

$x \geq 1$



The number line show that the set of integers that satisfies the two equations are {1, 2, 3}

(b) $5 - 2x \geq 3 - x$; $1 -2x \leq 11 - 4x$

Solution

$5 - 2x \geq 3 - x$; $1 -2x \leq 11 - 4x$

$-x \geq -2$ $2x \leq 10$

$x \leq 2$ $x \leq 5$

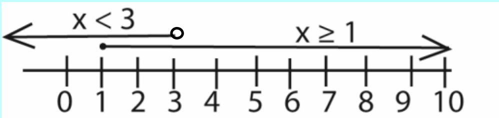
The number line show that the set of integers that satisfies the two equations are {x:x ≤ 2}

(c) $5x - 4 \geq 4x - 3$, $\frac{1}{3}x < 1$

Solution

$5x - 4 \geq 4x - 3$, $\frac{1}{3}x < 1$

$x \geq 1$ $x < 3$



The number line show that the set of integers that satisfies the two equations are {1, 2} (3 is not included

Example 4

Show that there is just one integer which simultaneously satisfies the three inequalities and find that number

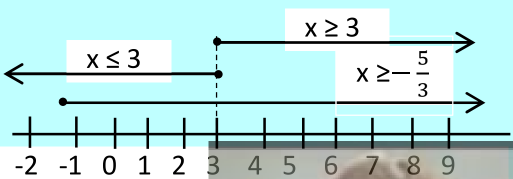
$\frac{1}{2}(x-1) \geq 1 \quad 2-3x \leq 7 \quad \frac{1}{3}x \leq 1$

Solution

$\frac{1}{2}(x-1) \geq 1 \quad 2-3x \leq 7 \quad \frac{1}{3}x \leq 1$

$x - 1 \geq 2 \qquad -3x \leq 5 \qquad x \leq 3$

$x \geq 3 \qquad x \geq -\frac{5}{3} \qquad x \leq 3$



From the number line, there is only one point of intersection of the three inequalities, which is 3

Hence the set of integers that satisfy the three inequalities is {3}

Non-linear inequalities in one unknown

The following methods are employed

- Sign change
- Graphical method

When using graphical method, the set of values above the axis are positive and those below are negative

When using sign change method, a table describing specific regions of inequalities is used and the necessary tests are performed

If the inequality symbol is ≥ or ≤, care must be taken, because the critical values of the function and the numerator in case of fractions will always satisfy the inequalities

Before solving inequality, all terms must be taken to one side preferably the LHS

Method I: Graphical method

Example 5

Solve the following inequalities

(a) $2x^2 - 3x + 1 \leq 0$

Solution

Let $y = 2x^2 - 3x + 1$

The curve cuts the x – axis when y = 0

⇨ $2x^2 - 3x + 1 = 0$

$(x - 1)(2x - 1) = 0$

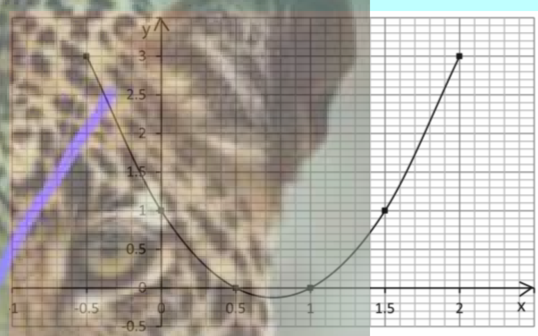
Either $x - 1 = 0$ or $2x - 1 = 0$;

$x = 1$ or $x = \frac{1}{2}$

the curve cuts the y – axis when x = 0

⇨ $y = 1$

Since the coefficient of x^2 is positive, that the curve is U shaped



The solution set is $0.5 \leq x \leq 1$

(b) $7x^2 > 1 - 6x$

Solution

$7x^2 > 1 - 6x$

$7x^2 + 6x - 1 > 0$

Let $y = 7x^2 + 6x - 1$

The curve cuts the x – axis when y = 0

⇨ $7x^2 + 6x - 1 = 0$

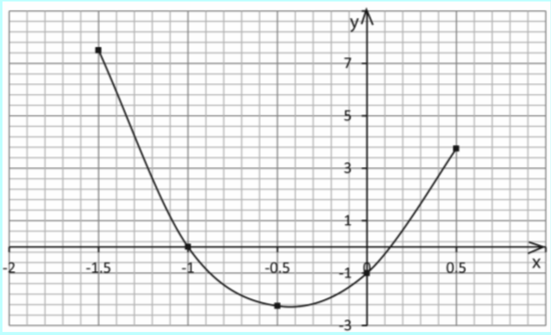
$(x + 1)(7x - 1) = 0$

Either $x + 1 = 0$ or $7x - 1 = 0$

$x = -1$ or $x = \frac{1}{7}$

the curve cuts the y – axis when x = 0

⇨ $y = -1$



From the graph, the solution is $x < -1$ and $x > \frac{1}{7}$

(c) $2x^3 + 3x^2 \geq 2x$

Solution

$2x^3 + 3x^2 - 2x \geq 0$

Let $y = 2x^3 + 3x^2 - 2x$

The curve cuts the x – axis when y = 0

⇨ $2x^3 + 3x^2 - 2x = 0$

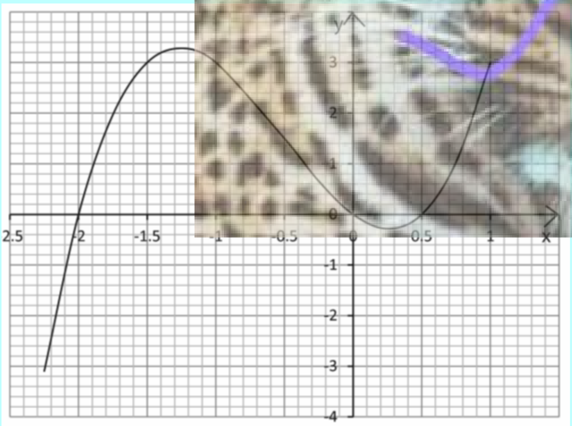
$x(2x - 1)(x + 2) = 0$

Either $x = 0$, $2x - 1 = 0$ or $x + 2 = 0$

$x = 0$, $x = \frac{1}{2}$ or $x = -2$

the curve cuts y – axis when x = 0

⇨ $y = 0$



From the graph the solution

$-2 \leq x \leq 0$ and $x \geq 0.5$

(d) Solve the inequality

$4x^2 + 2x < 3x + 6$ (06marks)

Method II: sign change

Example 6

(a) $2x^2 - 3x + 1 \leq 0$

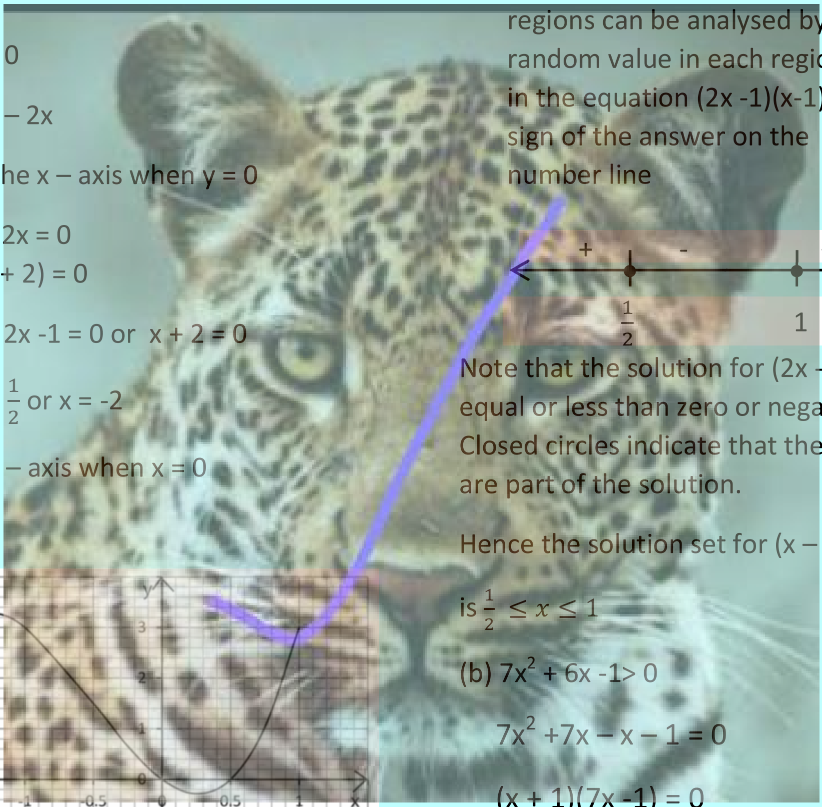
$(2x - 1)(x - 1) \leq 0$

The critical values of $(2x - 1)(x - 1) = 0$ are $x = 1$ and $x = \frac{1}{2}$ respectively

The above illustration shows that the numbers $\frac{1}{2}$ and 1 subdivide the number line into three regions namely

$x \leq \frac{1}{2}$, $\frac{1}{2} \leq x \leq 1$, $x \geq 1$

The corresponding sign in the respective regions can be analysed by choosing any random value in each region, substitute it in the equation $(2x - 1)(x - 1)$ and put the sign of the answer on the following number line



Note that the solution for $(2x - 1)(x - 1) \leq 0$ is equal or less than zero or negative.

Closed circles indicate that the critical values are part of the solution.

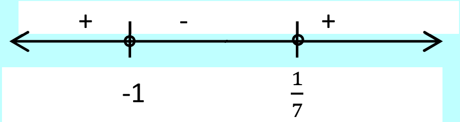
Hence the solution set for $(x - 1)(2x - 1) \leq 0$ is $\frac{1}{2} \leq x \leq 1$

(b) $7x^2 + 6x - 1 > 0$

$7x^2 + 7x - x - 1 = 0$

$(x + 1)(7x - 1) = 0$

The critical values $x = -1$ and $\frac{1}{7}$



The solution for $(x + 1)(7x - 1) > 0$ is positive . Open circles indicate that the critical values are not part of the solution

Hence the solution set for $(x + 1)(7x - 1) > 0$ is

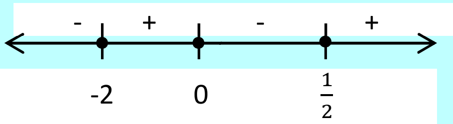
$x < -1$ and $x > \frac{1}{7}$

(c) $2x^3 + 3x^2 - 2x \geq 0$

$x(2x^2 + 3x - 2) \geq 0$

$x(x + 2)(2x - 1) \geq 0$

Critical values $x = 0$, $x = -2$, and $x = \frac{1}{2}$



The solution for $x(x + 2)(2x - 1) \geq 0$ is positive and the critical values are part of the solution

Hence the solution for $x(x + 2)(2x - 1) \geq 0$

$-2 \leq x \leq 0$ and $x \geq \frac{1}{2}$

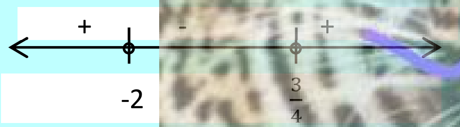
(d) $4x^2 + 5x - 6 < 0$

$4x^2 + 5x - 6 = 0$

Critical values

$x = \frac{-5 \pm \sqrt{5^2 - 4(4)(-6)}}{2(4)} = \frac{-5 \pm \sqrt{121}}{8} = \frac{-5 \pm 11}{8}$

$x = -2, \frac{3}{4}$



The solution $4x^2 + 5x - 6 < 0$ is negative and the critical values are not part of the solution

$\therefore$ the solution is $-2 < x < \frac{3}{4}$

(e) $\frac{3x^2 - 1}{x + 2} \geq 2$

$\frac{3x^2 - 1}{x + 2} - 2 \geq 0$

$\frac{3x^2 - 1 - 2(x + 2)}{x + 2} \geq 0$

$\frac{3x^2 - 1 - 2x - 4}{x + 2} \geq 0$

$\frac{(3x - 5)(x + 1)}{x + 2} \geq 0$

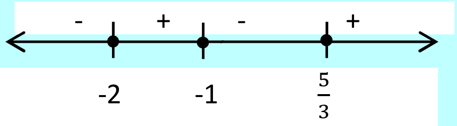
Finding critical values

$3x - 5 = 0$; $x = \frac{5}{3}$

$(x + 1) = 0$; $x = -1$

$(x + 2) = 0$, $x = -2$

Testing for correct region



The solution for $3x^2 - 2x - 5 \geq 0$ is positive and the critical values are part of the solution

Hence the solution for $3x^2 - 2x - 5 \geq 0$ is

$-2 \leq x \leq -1$ and $x \geq \frac{5}{3}$

The modulus of inequalities

The modulus of a number is the magnitude of that number (absolute value) which is always positive, e.g. $|1| = |-1| = 1$

When finding modulus of an inequality, the following must be considered

- The modulus on one side of the linear inequality is removed by introducing a negative number of the given value on the other side
  i.e. if $|x| < 3$, then $-3 < x < 3$
- The modulus on both sides of the linear inequality is removed by squaring both sides
- When the terms under modulus are fractional, square both sides of the inequality

Example 7

Solve the following inequalities

(a) $|x - 6| < 4$

$-4 < x - 6 < 4$

$-4 + 6 < x\ 4 + 6$

$2 < x < 10$

(b) $|3x + 4| < 6$

$-6 < 3x + 4 < 6$

$-6 - 4 < 3x < 6 - 4$

$-10 < 3x < 2$

$-\frac{10}{3} < x < \frac{2}{3}$

(c) $|2x-3| > |x+3|$

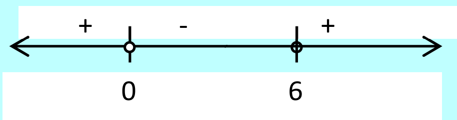
$(2x – 3)^2 > (x + 3)^2$

$4x^2 – 12x + 9 > x^2 + 6x + 9$

$3x^2 – 18x > 0$

$3x(x – 6) = 0$

Critical values are x = 0 and x = 6



The solution x < 0 and x > 6

(d) $|2x+5| < |x-3|$

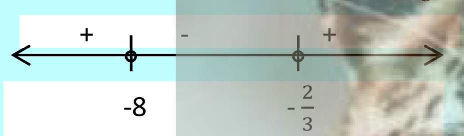
$(2x + 5)^2 < (x – 3)^2$

$4x^2 + 20x + 25 < x^2 – 6x + 9$

$3x^2 + 26x + 16 < 0$

$(3x + 2)(x + 8) < 0$

Critical values are x = -8 and x = $-\frac{2}{3}$



The solution is $-8 < x < -\frac{2}{3}$

Example 8

Find the range of value of x can take for the following inequality to be true

$\left|\frac{x}{x-3}\right| < 2$

Solution

Squaring both sides

$\frac{x^2}{x^2-6x+9} < 4$

$x^2 < 4(x^2 – 6x + 9)$

$x^2 < 4x^2 – 24x + 36$

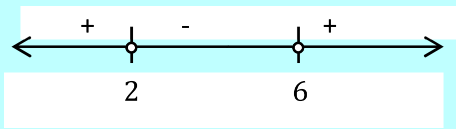
$0 < 3x^2 – 24x + 36$

Divide through by 3

$x^2 – 8x + 12 > 0$

$(x – 2)(x – 6) > 0$

Critical values are x = 2 and x = 6



Solution for (x – 2)(x – 6) > 0 (positive)

Hence the solution x < 2 and x > 6

Limits of inequality

This refers to interval within which the inequalities lies or does not lie.

This is done by expressing the function given as quadratic equation in x.

For real values of x, $b^2 \geq 4ac$

Example 9

(a) Given the function y = $\frac{3x-6}{x^2+6x}$, find the range of values within which y does not lies

Solution

y = $\frac{3x-6}{x^2+6x}$

$y(x^2 + 6x) = 3x – 6$

$yx^2 + (6y – 3)x + 6 = 0$

For real values of x, $b^2 \geq 4ac$

$(6y – 3)^2 = 24y$

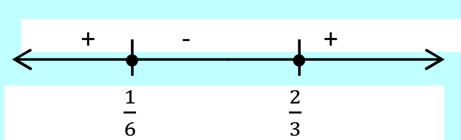
$36y^2 – 60y + 9 \geq 0$

Dividing through by 3

$12y^2 – 2y + 3 \geq 0$

$(6y – 1)(2y – 3) \geq 0$

The critical values are y = $\frac{1}{6}$ and y = $\frac{2}{3}$



Since the solution of the equation is positive; the required range $\frac{1}{6} < x < \frac{2}{3}$

(b) Find the range of values within which the function $y = \frac{3-2x}{4+x^2}$ lies

Solution

$y(4 + x^2) \geq 3 - 2x$

$yx^2 + 2x + 4y - 3 \geq 0$

For real values of x, $b^2 \geq 4ac$

$2^2 \geq 4y(4y - 3)$

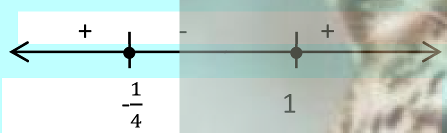
$1 \geq 4y^2 - 3y$

$0 \geq 4y^2 - 3y - 1$

$4y^2 - 3y - 1 \leq 0$

$(y - 1)(4y + 1) \leq 0$

Critical values $y = 1$ and $y = -\frac{1}{4}$



Solution for $(y - 1)(4y + 1) \leq 0$ is negative and critical values are part of the solution

Hence range of values is $-\frac{1}{4} \leq x \leq 1$

Simultaneous inequalities

Solving two simultaneous inequalities is best done by representing the inequalities on the graph. The unshaded (feasible) region represents the solution to the inequalities.

Example 10

Show by shading the unwanted regions; the region satisfying the inequalities $y \leq 2x + 1$ and $y \geq 3$

Solution

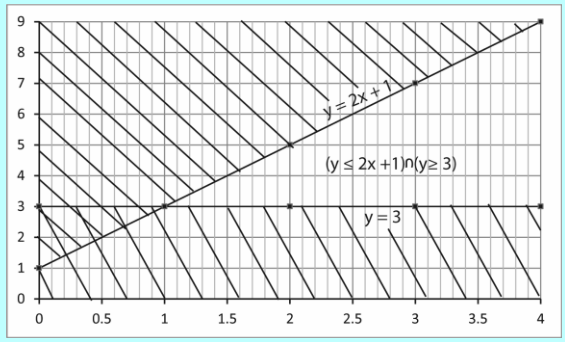
For $y \leq 2x + 1$, the boundary line is $y = 2x + 1$

If x= 0, y= 1, (x, y) = (0, 1)

If x= 2, y= 5, (x, y) = (2, 5)

Testing for wanted region using point (0,0); $0 \leq 1$. Hence this point is in wanted region.

For $y \geq 3$ boundary line is $y = 3$



Show by shading the unwanted regions; the region satisfying the inequalities $x + 2y \geq 6$, $y > x$, $x < 5$ and $3x + 5y \leq 30$

Solution

For $x + 2y \geq 6$ the boundary line is

$x + 2y = 6$

x	0	6
y	3	0

For $x > y$

The boundary line is $x = y$

x	0	5
y	0	5

For $x < 5$

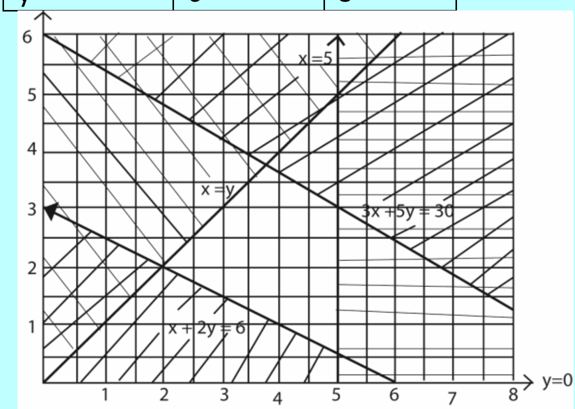
The boundary line is $x = 5$

For $3x + 5y \leq 30$

The boundary line

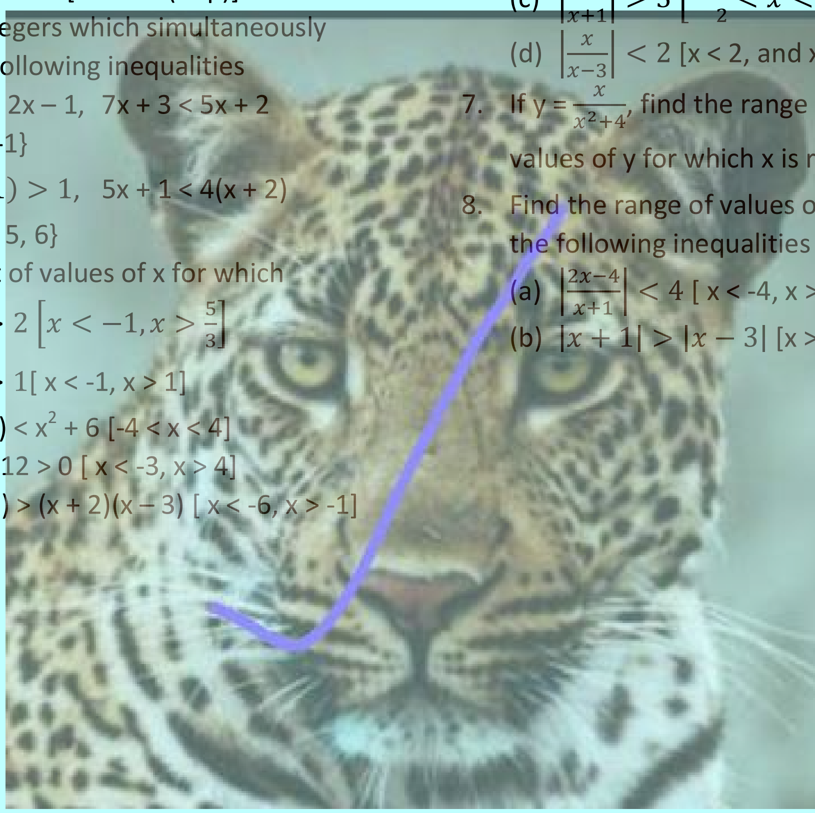
$3x + 5y = 30$

x	0	5
y	6	3



Revision exercise

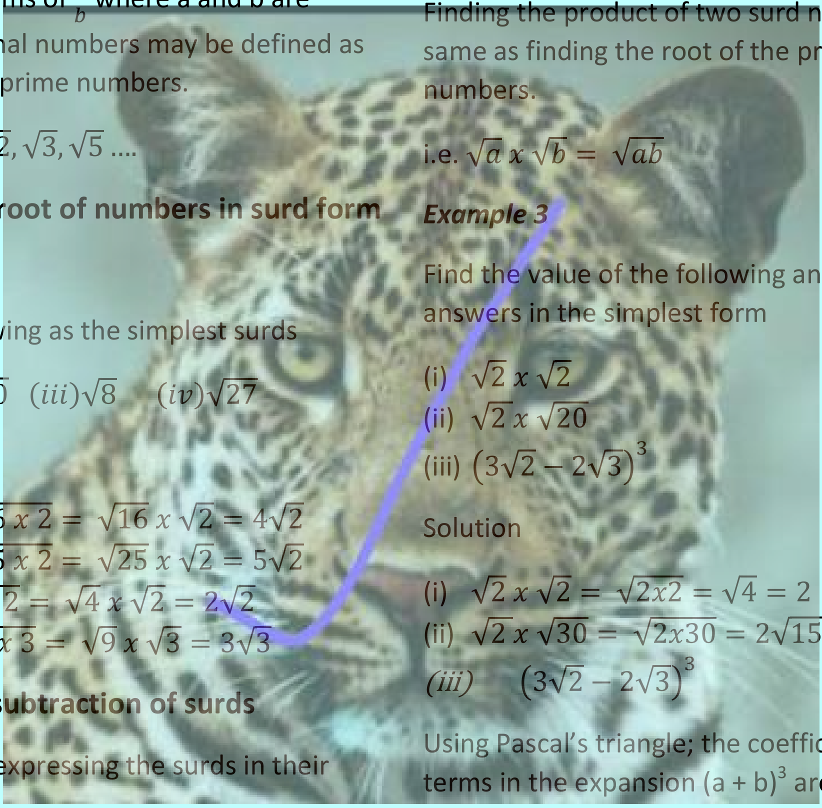
1. Solve the following inequalities
   (a) 7x – 3 ≥ 2x – 1 $\left[x \geq \frac{2}{5}\right]$
   (b) 5(2 – x) – 2(3 – 6x) + 2(x – 1) > 0
   $\left[x > -\frac{2}{9}\right]$
   (c) $\frac{1}{2}(x+3) \leq \frac{1}{3}(x-5)$ $[x \leq -19]$
   (d) $\frac{1}{3}(x - 3) + \frac{1}{2}(3x-1) > 2$ $\left[x \geq \frac{19}{11}\right]$
2. Solve the following inequalities. Correct 2 decimal places
   (a) (0.8)-3x > 4.0 [x > 2.07 (2dp)]
   (b) (0.6)-2x< 3.6 [x< 1.25 (2dp)]
3. Find the integers which simultaneously satisfy the following inequalities
   (a) 3x + 2 ≥ 2x – 1, 7x + 3 < 5x + 2
   {-3, -2, -1}
   (b) $\frac{1}{2}(x+1) > 1$, 5x + 1 < 4(x + 2)
   {2, 3, 4, 5, 6}
4. Find the set of values of x for which
   (a) $\frac{3x^2-1}{2+x} > 2$ $\left[x < -1, x > \frac{5}{3}\right]$
   (b) $\frac{3x^2-1}{1+x^2} > 1$[x < -1, x > 1]
   (c) 2(x^2 – 5) < x^2 + 6 [-4 < x < 4]
   (d) x^2 – x – 12 > 0 [x < -3, x > 4]
   (e) 2x(x + 3) > (x + 2)(x – 3) [x < -6, x > -1]
5. Solve the following inequalities
   (a) $\frac{x}{x+1} \leq \frac{x-2}{x+3}$
   $\left[x \leq -3 \; and -1 \leq x \leq -\frac{1}{2}\right]$
   (b) $\frac{x+2}{x-3} < \frac{x+5}{x-5}$ $[1 < x < 3 \; and \; x > 5]$
   (c) $\frac{x-1}{2+2} > 2x$
   $\left[x < -2 \; and -1 < x < -\frac{1}{2}\right]$
6. Solve the following inequalities
   (a) $|x-3| < |2x+3|$
   $[x < -18 \; and \; x > 0]$
   (b) $\left|\frac{2x-4}{x+1}\right| < 4$ [x < -4 and x> 0]
   (c) $\left|\frac{x-4}{x+1}\right| > 3$ $\left[-\frac{7}{2} < x < \frac{1}{4}\right]$
   (d) $\left|\frac{x}{x-3}\right| < 2$ [x < 2, and x > 6]
7. If y = $\frac{x}{x^2+4}$, find the range of possible values of y for which x is real $\left[\frac{-1}{4} \leq x \leq \frac{1}{4}\right]$
8. Find the range of values of x can take for the following inequalities to be true
   (a) $\left|\frac{2x-4}{x+1}\right| < 4$ [x < -4, x > 0]
   (b) $|x+1| > |x-3|$ [x > 1]



Surds

These are irrational numbers which cannot be expressed in terms of $\frac{a}{b}$ where a and b are rational. Irrational numbers may be defined as square roots of prime numbers.

Examples are $\sqrt{2}, \sqrt{3}, \sqrt{5}$ ....



Expression of root of numbers in surd form

Example 1

Write the following as the simplest surds

(i) $\sqrt{32}$ $(ii)\ \sqrt{50}$ $(iii)\sqrt{8}$ $(iv)\sqrt{27}$

Solution

(i) $\sqrt{32} = \sqrt{16\ x\ 2} = \sqrt{16}\ x\ \sqrt{2} = 4\sqrt{2}$
(ii) $\sqrt{50} = \sqrt{25\ x\ 2} = \sqrt{25}\ x\ \sqrt{2} = 5\sqrt{2}$
(iii) $\sqrt{8} = \sqrt{4\ x\ 2} = \sqrt{4}\ x\ \sqrt{2} = 2\sqrt{2}$
(iv) $\sqrt{27} = \sqrt{9\ x\ 3} = \sqrt{9}\ x\ \sqrt{3} = 3\sqrt{3}$

Addition and subtraction of surds

This is done by expressing the surds in their simplest form

Example 2

(i) $\sqrt{75} - 3\sqrt{27} + 2\sqrt{12}$

$= \sqrt{25\ x\ 3} - 3\sqrt{9\ x\ 3} + 2\sqrt{4\ x\ 3}$

$= \sqrt{25}\ x\ \sqrt{3} - 3x\sqrt{9}x\ \sqrt{3} + 2\ x\ \sqrt{4}x\sqrt{3}$

$= 5\ x\ \sqrt{3} - 3\ x\ 3\ x\ \sqrt{3} + 2\ x\ 2\ x\sqrt{3}$

$(5 - 9 + 4)\sqrt{3} = 0$

(ii) $\sqrt{50} + \sqrt{2} - 3\sqrt{18} + 2\sqrt{8}$
$= \sqrt{25\ x\ 2} + \sqrt{2} - 3\sqrt{9x\ 2} + 2\sqrt{4\ x\ 2}$
$=5\sqrt{2} + \sqrt{2} - 3\ x\ 3\sqrt{2} + 2x\ 2\sqrt{2}$

$= (5 + 1 - 9 + 4)\sqrt{2} = \sqrt{2}$

Multiplication of surds

Finding the product of two surd numbers is the same as finding the root of the product of two numbers.

i.e. $\sqrt{a}\ x\ \sqrt{b} = \sqrt{ab}$

Example 3

Find the value of the following and give your answers in the simplest form

(i) $\sqrt{2}\ x\ \sqrt{2}$
(ii) $\sqrt{2}\ x\ \sqrt{20}$
(iii) $(3\sqrt{2} - 2\sqrt{3})^3$

Solution

(i) $\sqrt{2}\ x\ \sqrt{2} = \sqrt{2x2} = \sqrt{4} = 2$
(ii) $\sqrt{2}\ x\ \sqrt{30} = \sqrt{2x30} = 2\sqrt{15}$
(iii) $(3\sqrt{2} - 2\sqrt{3})^3$

Using Pascal's triangle; the coefficients of the terms in the expansion $(a + b)^3$ are 1 3 3 1

$(3\sqrt{2} - 2\sqrt{3})^3 =$

$(3\sqrt{2})^3 + 3(3\sqrt{2})^2(-2\sqrt{3}) + 3(2\sqrt{3})(-2\sqrt{3})^2 + (-2\sqrt{3})^3$

$=(54\sqrt{2}) + 3(18)(-2\sqrt{3}) + 3(12)(3\sqrt{2}) - 24\sqrt{3}$

$= 162\sqrt{2} - 132\sqrt{3}$

Division of surds

There are two types of division of surds;

- A fraction whose denominators has a single term such as $\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{3}}$, etc.

- A fraction whose denominator has double terms, such as $\frac{1}{\sqrt{2}+\sqrt{3}}, \frac{2+\sqrt{3}}{2-\sqrt{3}}, \frac{1}{1+\sqrt{5}}$, etc.
- In both cases, first eliminate the surds from the denominator. The process of eliminating surds from the denominator is called rationalization.
- In the first case rationalize the fraction by multiplying the numerator and denominator by the surd term of the denominator.

Example 4

Rationalize the following

(i) $\frac{2}{3\sqrt{5}}$

(ii) $\frac{1}{\sqrt{2}}$

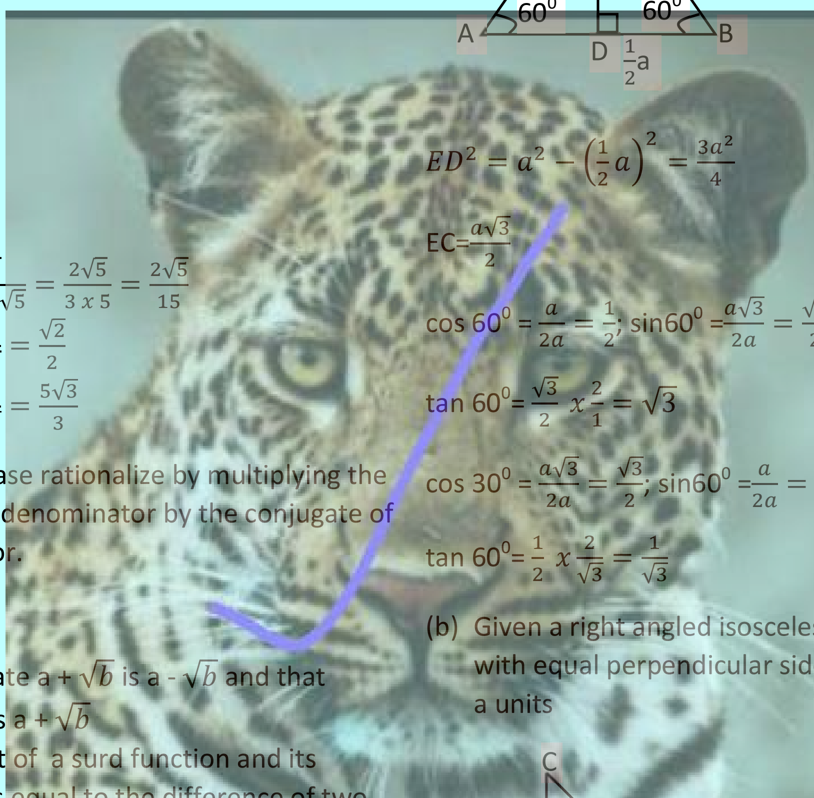
(iii) $\frac{5}{\sqrt{3}}$

Solution

(i) $\frac{2}{3\sqrt{5}} = \frac{2\sqrt{5}}{3\sqrt{5}\, x \,\sqrt{5}} = \frac{2\sqrt{5}}{3\, x\, 5} = \frac{2\sqrt{5}}{15}$

(ii) $\frac{1}{\sqrt{2}} = \frac{1\, x \sqrt{2}}{\sqrt{2}\, x\, \sqrt{2}} = \frac{\sqrt{2}}{2}$

(iii) $\frac{5}{\sqrt{3}} = \frac{5\sqrt{3}}{\sqrt{3}\, x\, \sqrt{3}} = \frac{5\sqrt{3}}{3}$



In the second case rationalize by multiplying the numerator and denominator by the conjugate of the denominator.

Note

(i) The conjugate a + $\sqrt{b}$ is a - $\sqrt{b}$ and that a - $\sqrt{b}$ is a + $\sqrt{b}$

(ii) The product of a surd function and its conjugate is equal to the difference of two squares. i.e. (a + $\sqrt{b}$)(a - $\sqrt{b}$) = $\left(a^2 - \left(\sqrt{b}\right)^2\right)$

Example 5

Rationalize the following

(i) $\frac{2}{2-\sqrt{2}}$

(ii) $\frac{1}{3-\sqrt{5}}$

(iii) $\frac{3-\sqrt{5}}{\sqrt{5}-3}$

solution

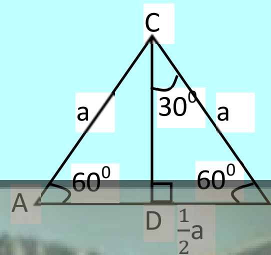
(i) $\frac{2}{2-\sqrt{2}} = \frac{2(2+\sqrt{2})}{(2-\sqrt{2})(2+\sqrt{2})} = \frac{2(2+\sqrt{2})}{4-2} = (2+\sqrt{2})$

(ii) $\frac{1}{3-\sqrt{5}} = \frac{1(3+\sqrt{5})}{(3-\sqrt{5})\left((3+\sqrt{5})\right)} = \frac{(3+\sqrt{5})}{9-5} = \frac{(3+\sqrt{5})}{4}$

(iv) $\frac{3-\sqrt{5}}{\sqrt{5}-3} x \frac{\sqrt{5}+3}{\sqrt{5}+3} = \frac{(3)^2-\left(\sqrt{5}\right)^2}{\left(\sqrt{5}\right)^2-(3)^2} = \frac{9-5}{5-9} = -1$

Set square angle (30^0, 45^0 and 60^0)

(a) Consider an equilateral triangle ABC of each side = a units



$ED^2 = a^2 - \left(\frac{1}{2}a\right)^2 = \frac{3a^2}{4}$

EC=$\frac{a\sqrt{3}}{2}$

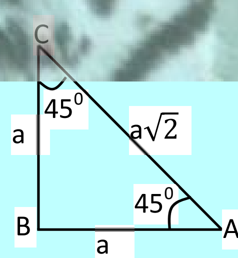
cos 60^0 = $\frac{a}{2a} = \frac{1}{2}$; sin$60^0$ =$\frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}$

tan 60^0=$\frac{\sqrt{3}}{2}\, x \,\frac{2}{1} = \sqrt{3}$

cos 30^0 = $\frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}$; sin$60^0$ =$\frac{a}{2a} = \frac{1}{2}$

tan 60^0=$\frac{1}{2}\, x\, \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}}$

(b) Given a right angled isosceles triangle ABC with equal perpendicular sides each of length a units



$AC^2 = a^2 + a^2 = 2a^2$

$AC = a\sqrt{2}$

cos 45^0 = sin45^0 = $\frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$

tan 45^0 = $\frac{a}{a} = 1$

Example 6

Express without a surd in the denominator each of the following

(a) $\frac{1+\tan 30^0}{1-\tan 30^0}$

(b) $\left(\frac{1+\cos 45^0}{2-\sin 60^0}\right)^2$

Solution

(a) $\frac{1+\tan 30^0}{1-\tan 30^0} = \frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}} = \left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)\left(\frac{\sqrt{3}+1}{\sqrt{3}+1}\right)$

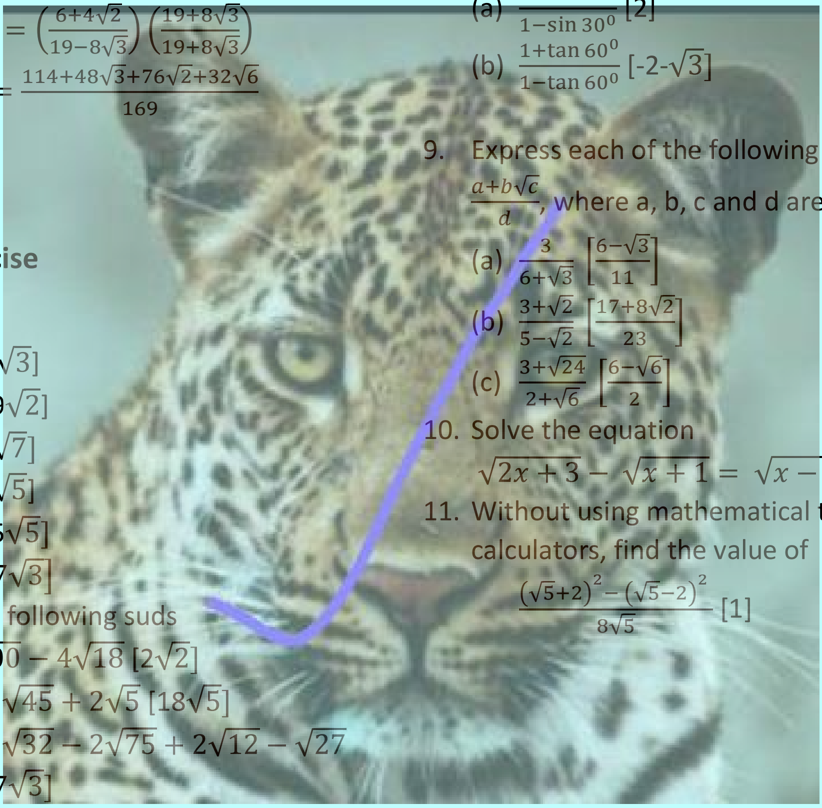
$= \frac{3+2\sqrt{3}+1}{3-1} = \frac{4+2\sqrt{3}}{2} = 2+\sqrt{3}$

(b) $\left(\frac{1+\cos 45^0}{2-\sin 60^0}\right)^2 = \left(\frac{1+\frac{\sqrt{2}}{2}}{2-\frac{\sqrt{3}}{2}}\right)^2 = \left(\frac{2+\sqrt{2}}{4-\sqrt{3}}\right)^2$

$= \frac{4+4\sqrt{2}+2}{16-8\sqrt{3}+3}$

$= \left(\frac{6+4\sqrt{2}}{19-8\sqrt{3}}\right)\left(\frac{19+8\sqrt{3}}{19+8\sqrt{3}}\right)$

$= \frac{114+48\sqrt{3}+76\sqrt{2}+32\sqrt{6}}{169}$

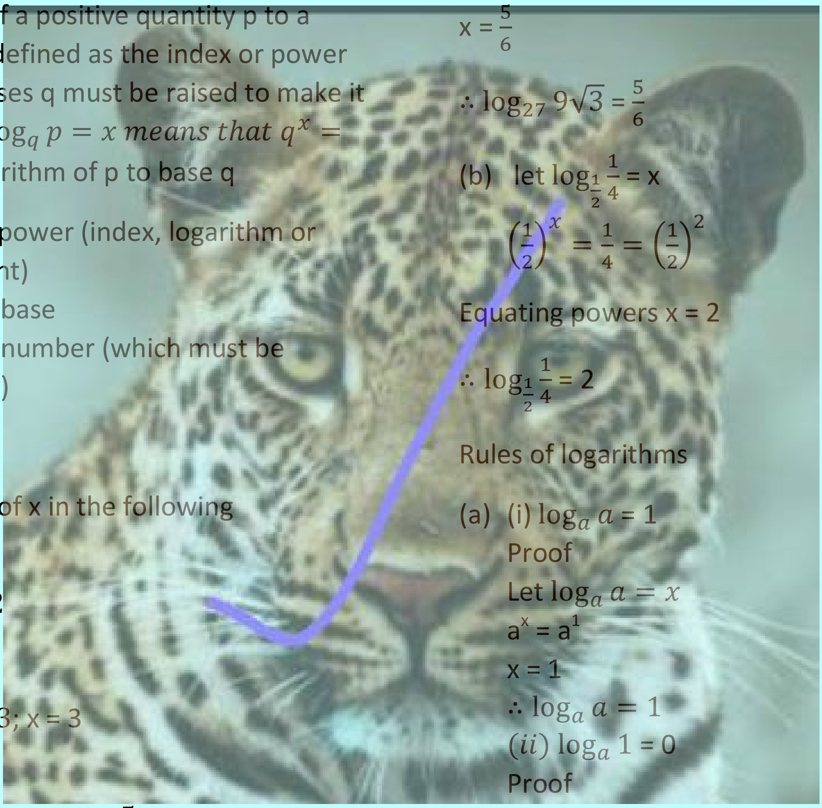


Revision exercise

1. Simplify
   (a) $\sqrt{48}$ [$4\sqrt{3}$]
   (b) $\sqrt{162}$ [$9\sqrt{2}$]
   (c) $\sqrt{28}$ [$2\sqrt{7}$]
   (d) $\sqrt{45}$ [$3\sqrt{5}$]
   (e) $\sqrt{125}$ [$5\sqrt{5}$]
   (f) $\sqrt{147}$ [$7\sqrt{3}$]
2. Simplify the following suds
   (a) $\sqrt{8}+\sqrt{200}-4\sqrt{18}$ [$2\sqrt{2}$]
   (b) $5\sqrt{20}+2\sqrt{45}+2\sqrt{5}$ [$18\sqrt{5}$]
   (c) $3\sqrt{50}+2\sqrt{32}-2\sqrt{75}+2\sqrt{12}-\sqrt{27}$ [$23\sqrt{2}-7\sqrt{3}$]
3. Given that $4\sqrt{20}+3\sqrt{5}-5\sqrt{125}=x\sqrt{5}$, find the value of x [-14]
4. Find the value of the following simplifying the answer as much as possible
   (a) $(5\sqrt{2}-\sqrt{5})(3\sqrt{5}-2\sqrt{2})$ [$5+13\sqrt{10}$]
5. Express each of the following in the form $\frac{a\sqrt{b}}{c}$ where a, b and c are integers
   (a) $\frac{2}{\sqrt{7}}$ $\left[\frac{2\sqrt{7}}{7}\right]$
   (b) $\frac{3}{\sqrt{2}}$ $\left[\frac{3\sqrt{2}}{2}\right]$
   (c) $\frac{14\sqrt{5}}{\sqrt{7}}$ [$2\sqrt{35}$]
   (d) $\frac{3\sqrt{50}}{5\sqrt{27}}$ $\left[\frac{\sqrt{6}}{3}\right]$
   (e) $\frac{4\sqrt{45}}{5\sqrt{8}}$ $\left[\frac{3\sqrt{10}}{5}\right]$
6. Express the following in the form a +b$\sqrt{c}$
   (a) $\frac{2\sqrt{3}-3\sqrt{2}}{2\sqrt{3}+3\sqrt{2}}$ [$-5+2\sqrt{6}$]
   (b) $\frac{4+3\sqrt{2}}{4-3\sqrt{2}}$ [$-17-12\sqrt{2}$]
   (c) $\frac{\sqrt{2}+\sqrt{3}}{\sqrt{2}+\sqrt{3}+1}$ $\left[\frac{1}{2}-\frac{1}{4}\sqrt{2}+\frac{1}{4}\sqrt{6}\right]$
7. Show that $\left(\frac{2+\sqrt{3}}{2-\sqrt{3}}\right)^2+\left(\frac{2-\sqrt{3}}{2+\sqrt{3}}\right)^2$=194
8. Express without a surd in the denominator each of the following
   (a) $\frac{1}{1-\sin 30^0}$ [2]
   (b) $\frac{1+\tan 60^0}{1-\tan 60^0}$ [-2-$\sqrt{3}$]
9. Express each of the following in the form $\frac{a+b\sqrt{c}}{d}$, where a, b, c and d are integers
   (a) $\frac{3}{6+\sqrt{3}}$ $\left[\frac{6-\sqrt{3}}{11}\right]$
   (b) $\frac{3+\sqrt{2}}{5-\sqrt{2}}$ $\left[\frac{17+8\sqrt{2}}{23}\right]$
   (c) $\frac{3+\sqrt{24}}{2+\sqrt{6}}$ $\left[\frac{6-\sqrt{6}}{2}\right]$
10. Solve the equation
    $\sqrt{2x+3}-\sqrt{x+1}=\sqrt{x-2}$ [3]
11. Without using mathematical tables or calculators, find the value of
    $\frac{(\sqrt{5}+2)^2-(\sqrt{5}-2)^2}{8\sqrt{5}}$ [1]

Surds

A logarithm is an exponent, an index or power



The logarithm of a positive quantity p to a given base q is defined as the index or power to which the bases q must be raised to make it equal to P. i.e. $\log_q p = x$ *means that* $q^x = p$ or x s the logarithm of p to base q

- x is the power (index, logarithm or exponent)
- q is the base
- p is the number (which must be positive)

Example 1

Find the values of x in the following

(a) $\log_2 8 = x$

(b) $\log_x 25 = 2$

Solution

(a) $8 = 2^3$

$\therefore \log_2 8 = 3$; x = 3

(b) $x^2 = 25 = 5^2$

$\therefore x = 5$

Example 2

Evaluate

(a) $\log_{27} 9\sqrt{3}$

(b) $\log_{\frac{1}{2}} \frac{1}{4}$

Solution

Let $\log_{27} 9\sqrt{3}$ = x

$27^x = 9\sqrt{3}$

$3^{3x} = 3^2.3^{\frac{1}{2}} = 3^{\frac{5}{2}}$

Equating powers

$3x = \frac{5}{2}$

$x = \frac{5}{6}$

$\therefore \log_{27} 9\sqrt{3} = \frac{5}{6}$

(b) let $\log_{\frac{1}{2}} \frac{1}{4} = x$

$\left(\frac{1}{2}\right)^x = \frac{1}{4} = \left(\frac{1}{2}\right)^2$

Equating powers x = 2

$\therefore \log_{\frac{1}{2}} \frac{1}{4} = 2$

Rules of logarithms

(a) (i) $\log_a a = 1$

Proof

Let $\log_a a = x$

$a^x = a^1$

x = 1

$\therefore \log_a a = 1$

(ii) $\log_a 1 = 0$

Proof

Let $\log_a 1 = x$

$a^x = a^0$

x = 0

$\therefore \log_a 1 = 0$

(b) The power rule

$\log_a P^q = \text{q}\log_a P$

Proof

Let $\log_a P = x$

$a^x = P$

Raising each to the power q

$a^{qx} = P^q$

⇨ $\log_a P = \log_a a^{qx} = qx$

$\therefore \log_a P^q = \text{q}\log_a P$

(c) The addition/multiplication rule

$\log_a pq = \log_a p + \log_a q$

Proof

Let $\log_a p = x$ *and* $\log_a q = y$

$p = a^x$ *and* $q = a^y$

$pq = a^x . a^y = a^{x+y}$

$\log_a pq = \log_a a^{x+y} = x + y$

$\therefore \log_a pq = \log_a p + \log_a q$

(d) The subtraction/division rule

$\log_a \left(\frac{p}{q}\right) = \log_a p - \log_a q$

Proof

Let $\log_a p = x$ *and* $\log_a q = y$

$p = a^x$ *and* $q = a^y$

$\frac{p}{q} = \frac{a^x}{a^y} = a^{x-y}$

$\log_a \left(\frac{p}{q}\right) = \log_a a^{x-y} = x - y$

$\therefore \log_a \left(\frac{p}{q}\right) = \log_a p - \log_a q$

(e) Change of base

$\log_a p = \frac{\log_q p}{\log_q a}$

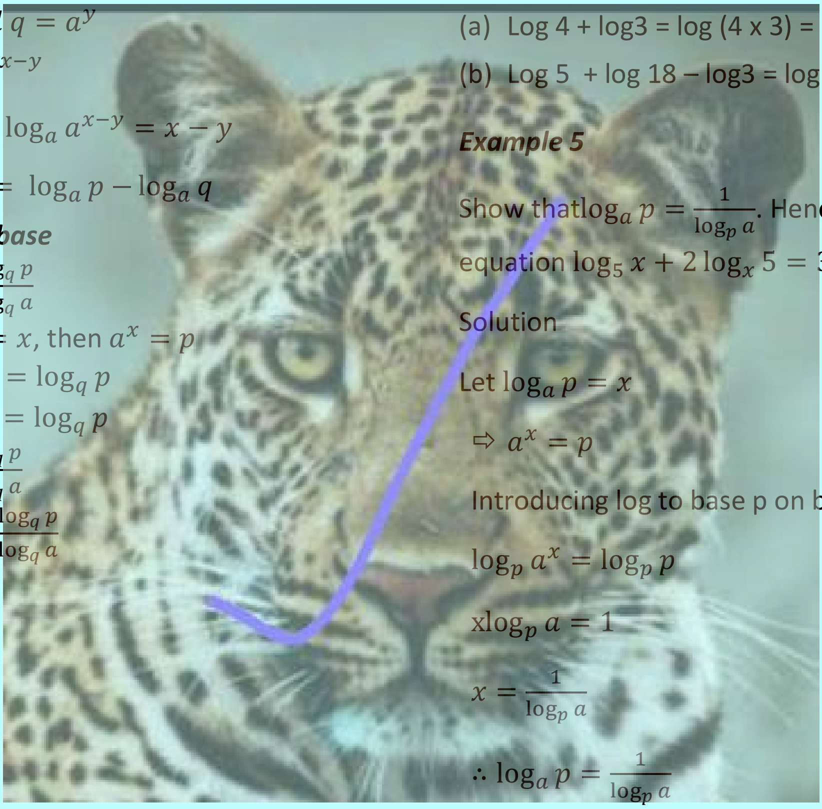
Let $\log_a p = x$, then $a^x = p$

⇨ $\log_q a^x = \log_q p$

$x\log_q a = \log_q p$

$x = \frac{\log_q p}{\log_q a}$

$\therefore \log_a p = \frac{\log_q p}{\log_q a}$



Example 3

Evaluate

(a) $\log_2 8\sqrt{2}$
(b) $\log_a \frac{1}{a}$

Solution

(a) Either: let $\log_2 8\sqrt{2}$= x

⇨ $2^x = 8\sqrt{2} = 2^3 . 2^{\frac{1}{2}} = 2^{\frac{7}{2}}$

$x = \frac{7}{2}$

$\therefore \log_2 8\sqrt{2} = \frac{7}{2}$

Or $\log_2 8\sqrt{2} = \log_2 2^3 . 2^{\frac{1}{2}} = \log_2 2^{\frac{7}{2}}$

$= \frac{7}{2}\log_2$

$= \frac{7}{2}$

(b) Let $\log_a \frac{1}{a}$ = x

$a^x = a^{-1}$

$x = -1$

$\therefore \log_a \frac{1}{a}$ = -1

Example 4

Express each of the following as a single logarithm

(a) Log 4 + log3
(b) Log 5 + log 18 – log3

Solution

(a) Log 4 + log3 = log (4 x 3) = log 12
(b) Log 5 + log 18 – log3 = $\log\left(\frac{5 \, x 18}{3}\right)$ = log30

Example 5

Show that $\log_a p = \frac{1}{\log_p a}$. Hence solve the equation $\log_5 x + 2\log_x 5 = 3$

Solution

Let $\log_a p = x$

⇨ $a^x = p$

Introducing log to base p on both sides

$\log_p a^x = \log_p p$

$x\log_p a = 1$

$x = \frac{1}{\log_p a}$

$\therefore \log_a p = \frac{1}{\log_p a}$

Then,

$\log_5 x + 2\log_x 5 = 3$

$\log_5 x + \frac{2}{\log_5 x} = 3$

Let y = $\log_5 x$

⇨ $y + \frac{2}{y} = 3$

$y^2 - 3y + 2 = 0$

(y – 1)(y – 2) = 0

Either y = 1 or y = 2

When y = 1: $\log_5 x = 1$; x = $5^1 = 5$

When y = 2: $\log_5 x = 2$; x = $5^2 = 25$

x = 5 and x = 25

Example 6

Solve $\log_x 5 + 4\log_5 x = 4$

Expressing terms on LHS to $\log_5$.

$\frac{\log_5 5}{\log_5 x} + 4\log_5 x = 4$

$\frac{1}{\log_5 x} + 4\log_5 x = 4$

Let $\log_5 x = y$

$\frac{1}{y} + 4y = 4$

$4y^2 - 4y + 1 = 0$

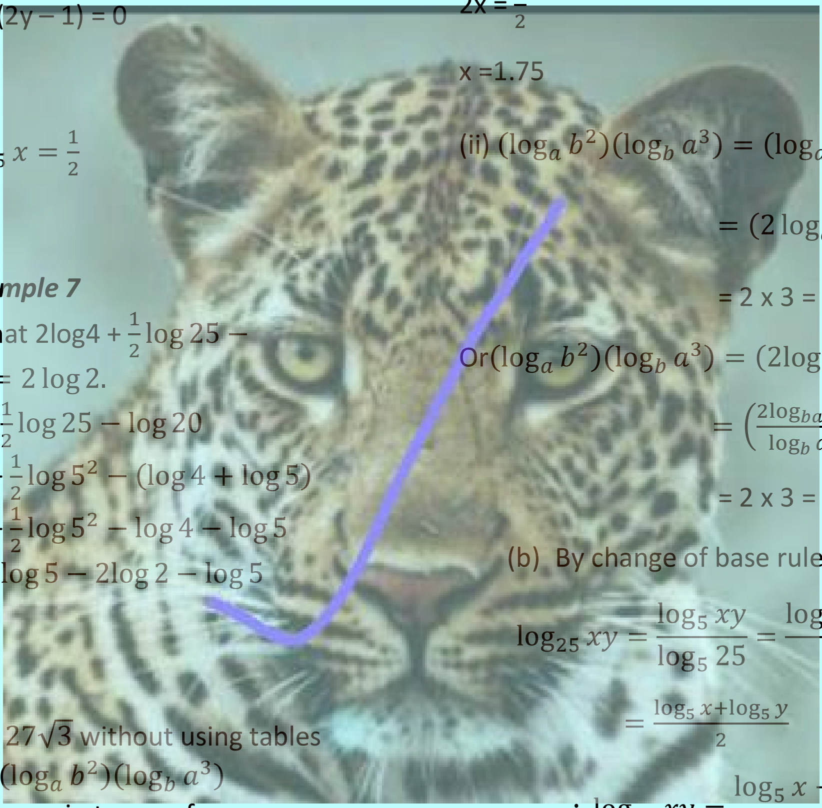
(2y – 1)(2y – 1) = 0

2y = 1

y = $\frac{1}{2}$

⇨ $\log_5 x = \frac{1}{2}$

x = $5^{\frac{1}{2}} = \sqrt{5}$



Example 7

Show that 2log4 + $\frac{1}{2}\log 25 -$ $\log 20 = 2\log 2$.

2log4 + $\frac{1}{2}\log 25 - \log 20$

2log2^2 + $\frac{1}{2}\log 5^2 - (\log 4 + \log 5)$

2log2^2 + $\frac{1}{2}\log 5^2 - \log 4 - \log 5$

4log2 + $\log 5 - 2\log 2 - \log 5$

2log 2

Example 8

(a) (i) Find $\log_9 27\sqrt{3}$ without using tables

(ii) Simplify $(\log_a b^2)(\log_b a^3)$

(b) Express $\log_{25} xy$ in terms of $\log_5 x$ *and* $\log_5 y$. Hence solve the simultaneous equation s:

$$\log_{25} xy = 4\frac{1}{2}$$

$\frac{\log_5 x}{\log_5 y}$ = -10

Solution

(a)(i) Changing the base from 9 to 3

$$\log_9 27\sqrt{3} = \frac{\log_3 27\sqrt{3}}{\log_3 9}$$

$= \frac{\log_3 27 + \log_3 \sqrt{3}}{\log_3 9}$

= $\frac{\log_3 3^3 + \log_3 3^{\frac{1}{2}}}{\log_3 3^2} = \frac{3+\frac{1}{2}}{2} = \frac{7}{4} = 1.75$

Or

Let $\log_9 27\sqrt{3}$ = x

9^x =$27\sqrt{3}$

$(3^2)^x = 3^3.3^{\frac{1}{2}}$

$3^{2x} = 3^{\frac{7}{2}}$

Equating indices

2x = $\frac{7}{2}$

x =1.75

(ii) $(\log_a b^2)(\log_b a^3) = (\log_a b^2)\frac{(\log_a a^3)}{\log_a b}$

$= (2\log_a b)\frac{(3\log_a a)}{\log_a b}$

= 2 x 3 = 6

Or$(\log_a b^2)(\log_b a^3) = (2\log_a b)(3\log_b a)$

$= \left(\frac{2\log_{b} a\, b}{\log_b a}\right)(3\log_b a)$

= 2 x 3 = 6

(b) By change of base rule

$\log_{25} xy = \frac{\log_5 xy}{\log_5 25} = \frac{\log_5 x + \log_5 y}{\log_5 5^2}$

$= \frac{\log_5 x + \log_5 y}{2}$

$\therefore \log_{25} xy = \frac{\log_5 x + \log_5 y}{2}$

Hence solving

$\log_{25} xy = 4\frac{1}{2}$

$\frac{\log_5 x + \log_5 y}{2} = \frac{9}{2}$

$\log_5 x + \log_5 y = 9$ .... (i)

$\frac{\log_5 x}{\log_5 y}$ = -10

$\log_5 x = -10\log_5 y$ ....(ii)

Substituting eqn. (ii) into eqn. (i)

$-10\log_5 y + \log_5 y = 9$

$\log_5 y = -1$

$y = 5^{-1} = \frac{1}{5}$

Substituting $\log_5 y$ into equation (ii)

$\log_5 x = 10$

$x = 5^{10}$

$\therefore\ x = 5^{10}$ and $y = \frac{1}{5}$

Example 9

(a) Given that $\log_b a = x$ show that

b $= a^{\frac{1}{x}}$ and deduce $\log_b a = \frac{1}{\log_a b}$

(b) Find the value of x and y such that

(i) $\log_{10} x + \log_{10} y = 1.0$

$\log_{10} x - \log_{10} y = \log_{10} 2.5$

(ii) Simplify $2^x.2^y = 432$

(c) Simplify $\frac{1+\sqrt{2}+\sqrt{3}}{\sqrt{2}+\sqrt{3}}$

Solution

$\log_b a = x$

$b^x = a$

$\sqrt[x]{b^x} = \sqrt[x]{a}$

$b = a^{\frac{1}{x}}$

Taking log to bas a on both sides

$\log_a b = \ \log_a a^{\frac{1}{x}}$

$\log_a b = \frac{1}{x}\log_a a = \frac{1}{x}$

But x = $\log_b a$

$\therefore \log_{ba} b = \frac{1}{\log_b a}$

(b)(i) $\log_{10} x + \log_{10} y = 1.0$ ....(i)

$\log_{10} x - \log_{10} y = \log_{10} 2.5$ ...(ii)

Eqn. (i) + eqn. (ii)

2$\log_{10} x$ = $\log_{10} 10 + \log_{10} 2.5$

$\log_{10} x^2 = \log_{10} 25$

⇨ $x^2 = 25$

x = 5

Substituting x into eqn. (i)

$\log_{10} 5 + \log_{10} y = 1.0$

$\log_{10} y = \log_{10} 10 - \log_{10} 5$

$\log_{10} y = \log_{10} 10 \div 5 = \log_{10} 2$

y = 2

Hence x = 5 and y = 2

(ii) $2^x.2^y = 432 = 2^4.3^3$

Comparing

x = 4 and y = 3

(c) By rationalizing

$$\frac{(1+\sqrt{2}+\sqrt{3})(\sqrt{2}+\sqrt{3})}{(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})} = 1+\sqrt{3}-\sqrt{2}$$

Example 10

Prove that $\log_6 x = \frac{\log_3 x}{1+\log_3 2}$. Given that $\log_3 2 = 0.631$, find without using tables or calculator $\log_6 4$ correct to 3significant figures

Solution

$\log_6 x = \frac{\log_3 x}{\log_3 6} = \frac{\log_3 x}{\log_3 (2\ x 3)} = \frac{\log_3 x}{\log_3 3 + \log_3 2}$

$= \frac{\log_3 x}{1+\log_3 2}$

Substituting for $\log_3 2 = 0.631$

$\log_6 x = \frac{\log_3 2^2}{1+\log_3 2} = \frac{2\log_3 2}{1+\log_3 2} = \frac{2\ x\ 0.631}{1+0.631}$

$= 0.774$

Revision exercise

1. Evaluate

(a) $\log_{\frac{1}{5}} 25\sqrt{5}$ $\left[-\frac{5}{2}\right]$

(b) $\log_3 27$ [3]

2. Express the following as a single logarithm

(i) Log15 - $\frac{1}{2}log9$ [log 5]

(ii) 3log2 + 2log5 – log 20 [log 10]

3. Given that $\log_b a$ and $\log_c b = a$, show that $\log_c a = ac$

4. (a) solve the equation

(i) $\log_a 4 + \log_4 a^2$

$[a = 2\ and\ a = 4]$

(ii) $\log_{14} x = \log_7 4x$ $\left[\frac{1}{196}\right]$

5. Without using tables or calculator show that $\frac{2log9 + log8 - \log 375}{\frac{1}{3}log6 - log5^{\frac{1}{3}}}$= 9

6. If$\log_2 x + \log_4 x + \log_{16} x = \frac{21}{16}$. Find the value of x [x = 1.6818]

7. Given$\log_a b = \log_d c$, show that $\log_c a = \log_d b$. Hence or otherwise solve the equation$\log_{9x} 64 = \log_x 4$. [x=3]

8. Solve the simultaneous equations
$\log_{10}(y - x)$= 0
2$\log_{10}(21 + x)$ [(x,y) =(-5, -4) or (4,5)

9. Given that $\log_2 x + 2\log_4 y = 4$. Show that xy = 16. Solve simultaneous equations
$10\log_{10}(x + y) = 1$
$\log_2 x + 2\log_4 y = 4$. [(x, y) =(2,8)or (8,2)

10. (a) If $\log_b a = x$, show that b = $a^{\frac{1}{x}}$ and deduce that $\frac{1}{\log_a b}$.

(b) Solve

(i) $\log_x 4 + \log_4 x^2 = 3$ $[x = 2 \text{ } or \text{ } 4]$

(ii) $2^{2x-1} + \frac{3}{2} = 2^{x+1}$[x=0 or 1.585]

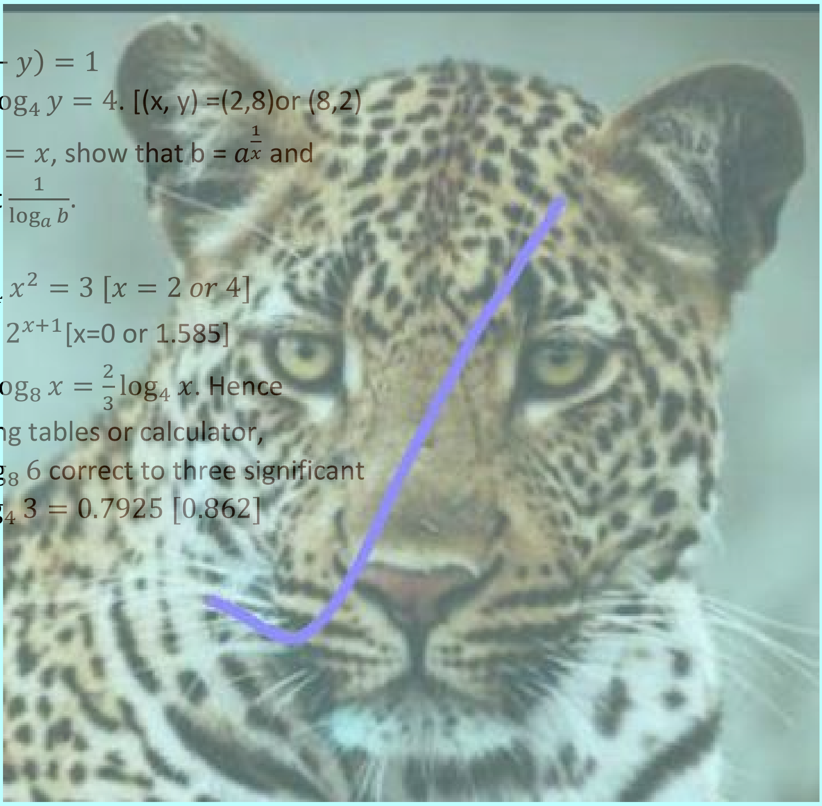
11. Prove that $\log_8 x = \frac{2}{3}\log_4 x$. Hence without using tables or calculator, evaluate $\log_8 6$ correct to three significant figure, if $\log_4 3 = 0.7925$ $[0.862]$

12. Given that $\log_3 x$ = p and $\log_{18} x = q$, show that $\log_6 3 = \frac{q}{p-q}$

13. Solve for x in the equation
$\log_4(6 - x) = \log_2 x$
[x = 2 since there is no negative log]

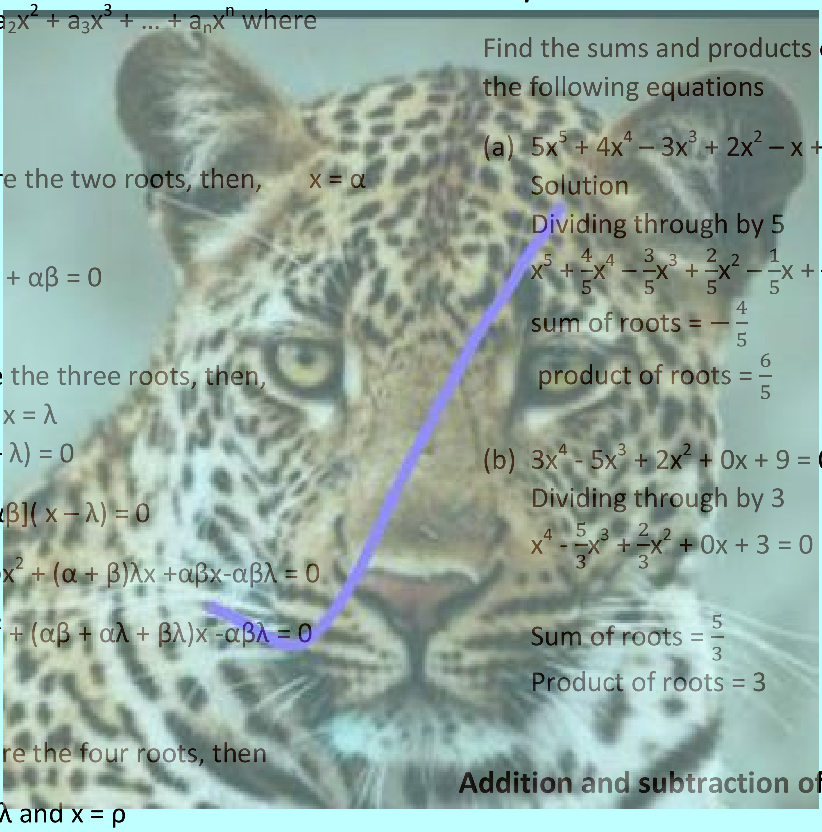
14. Solve the equation $\log_2 x - \log_x 8 = 2$
[x =8 or x = $\frac{1}{2}$]

15. Solve the equation
$\log_{25} 4x^2 = \log_5(3 - x^2)$ $[x = 1]$



Polynomials

A polynomial in x is a function in the form



$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n$ where $a \neq 0$

(a) Quadratics

If α and β are the two roots, then, x = α and x =β

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

(b) Cubic

If α, β and λ are the three roots, then,
x = α, x =β and x = λ
$(x - \alpha)(x - \beta)(x - \lambda) = 0$

$[x^2 - (\alpha + \beta)x + \alpha\beta](x - \lambda) = 0$

$x^3 - \lambda x^2 - (\alpha + \beta)x^2 + (\alpha + \beta)\lambda x + \alpha\beta x - \alpha\beta\lambda = 0$

$x^3 - (\alpha + \beta + \lambda)x^2 + (\alpha\beta + \alpha\lambda + \beta\lambda)x - \alpha\beta\lambda = 0$

(c) Qurtic

If α, β, λ and ρ are the four roots, then

x = α, x =β, x = λ and x = ρ

⇨ $(x - \alpha)(x - \beta)(x - \lambda)(x - \rho) = 0$

$[x^3 - (\alpha + \beta + \lambda)x^2 + (\alpha\beta + \alpha\lambda + \beta\lambda)x - \alpha\beta\lambda] (x - \rho) = 0$

$X^4 - (\alpha + \beta + \lambda + \rho)x^3 + (\alpha\beta + \alpha\lambda + \alpha\rho + \beta\lambda + \beta\rho + \lambda\rho)x^2 - (\alpha\beta\lambda + \alpha\beta\rho + \alpha\lambda\rho + \beta\lambda\rho)x + \alpha\beta\lambda\rho = 0$

The above illustration shows that:

(i) The signs of the terms alternate in the order: +, - , + ….. starting with the first term.
(ii) The coefficient of the second term is – (sum of the roots) and the last term with its appropriate sign is the product of the roots

Example 1

Find the sums and products of the roots of the following equations

(a) $5x^5 + 4x^4 - 3x^3 + 2x^2 - x + 6 = 0$
Solution
Dividing through by 5
$x^5 + \frac{4}{5}x^4 - \frac{3}{5}x^3 + \frac{2}{5}x^2 - \frac{1}{5}x + \frac{6}{5} = 0$
sum of roots = $-\frac{4}{5}$
product of roots = $\frac{6}{5}$

(b) $3x^4 - 5x^3 + 2x^2 + 0x + 9 = 0$
Dividing through by 3
$x^4 - \frac{5}{3}x^3 + \frac{2}{3}x^2 + 0x + 3 = 0$

Sum of roots = $\frac{5}{3}$
Product of roots = 3

Addition and subtraction of polynomial

Polynomial are added or subtracted if they are of the same degree. This is done by adding or subtracting the coefficients of the corresponding term

Example 2

(a) Given the polynomial
$f(x) = 5x^5 + 4x^4 - 3x^3 + 2x^2 - x + 6$ and
$g(x) = 3x^4 - 5x^3 + 2x^2 + 9$
Find (i) f(x) + g (x) (ii) f(x) – g(x)

Solution

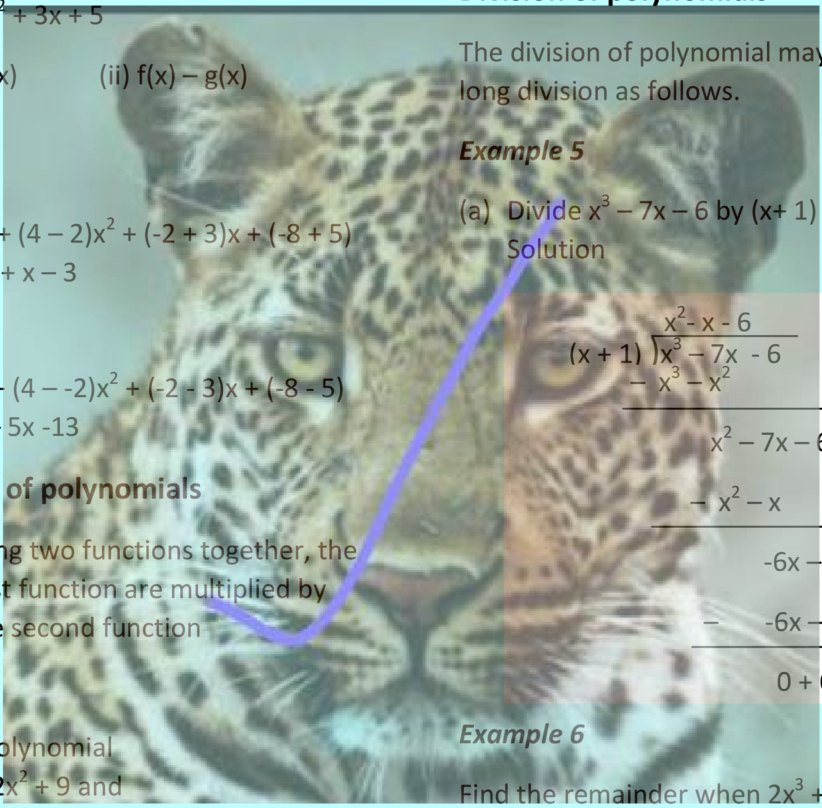
(i) f(x) + g(x)

$$\begin{array}{rl} & 5x^5 + 4x^4 - 3x^3 + 2x^2 - x + 6 \\ + & \quad\;\; 3x^4 - 5x^3 + 2x^2 + 0x + 9 \\ \hline = & 5x^5 + 7x^4 - 8x^3 + 2x^2 - x + 15 \end{array}$$

f(x) - g(x)

$$\begin{array}{rl} & 5x^5 + 4x^4 - 3x^3 + 2x^2 - x + 6 \\ - & \quad\;\; 3x^4 - 5x^3 + 2x^2 + 0x + 9 \\ \hline = & 5x^5 + x^4 + 2x^3 + 0x^2 - x - 3 \end{array}$$

(b) Given polynomials
$f(x) = 2x^3 + 4x^2 - 2x - 8$ and
$g(x) = x^3 - 2x^2 + 3x + 5$



Find (i) f(x) + g (x) (ii) f(x) – g(x)

Solution

(i) f(x) + g (x)
$= (2 + 1)x^3 + (4 - 2)x^2 + (-2 + 3)x + (-8 + 5)$
$= 3x^3 + 2x^2 + x - 3$

(ii) f(x) – g(x)
$= (2 - 1)x^3 + (4 - -2)x^2 + (-2 - 3)x + (-8 - 5)$
$= x^3 + 6x^2 - 5x - 13$

Multiplication of polynomials

When multiplying two functions together, the terms of the first function are multiplied by the terms of the second function

Example 3

(a) Given the polynomial
$f(x) = 5x^3 + 2x^2 + 9$ and
$g(x) = 4x^4 - 3x^3 + 2x^2 - x + 6$
Find f(x) x g (x)
Solution
$= 5x^3 (4x^4 - 3x^3 + 2x^2 - x + 6)$
$+ 2x^2 (4x^4 - 3x^3 + 2x^2 - x + 6)$
$+ 9 (4x^4 - 3x^3 + 2x^2 - x + 6)$
$= (20x^7 - 15x^6 + 10x^5 - 5x^4 + 30x^3)$
$+ (8x^6 - 6x^5 + 4x^4 - 2x^3 + 12x^2)$
$+ (36x^4 - 27x^3 + 18x^2 - 9x + 54)$
$= 20x^7 - 7x^6 + 4x^5 + 35x^4 + x^3 + 30x^2 - 9x + 54$

Hence f(x) .g(x) =

$20x^7 - 7x^6 + 4x^5 + 35x^4 + x^3 + 30x^2 - 9x + 54$

Example 4

Find coefficient of x^3 in the expansion

$(2x^3 + x^2 - 5x + 6)(2x + 4)$

Solution

$(2x^3 + x^2 - 5x + 6)(2x + 4)$

$= 2x(2x^3 + x^2 - 5x + 6) + 4(2x^3 + x^2 - 5x + 6)$

$= (2x^4 + 2x^3 - 10x^2 + 12x) + 8x^3 + 4x^2 - 20x + 16$

$= 2x^4 + 10x^3 - 6x^2 - 8x + 16$

The coefficient of x^3 is 10

Division of polynomials

The division of polynomial may be done by long division as follows.

Example 5

(a) Divide $x^3 - 7x - 6$ by $(x + 1)$
Solution

$$\begin{array}{r} x^2 - x - 6 \\ (x + 1)\overline{)\,x^3 - 7x - 6} \\ -\;x^3 - x^2 \\ \hline x^2 - 7x - 6 \\ -\;x^2 - x \\ \hline -6x - 6 \\ -\;-6x - 6 \\ \hline 0 + 0 \end{array}$$

Example 6

Find the remainder when $2x^3 + x^2 + 5x - 4$ is divided by 2x - 1

$$\begin{array}{r} x^2 + x + 3 \\ (2x - 1)\overline{)\,2x^3 + x^2 + 5x - 4} \\ -\;2x^3 - x^2 \\ \hline 2x^2 + 5x - 4 \\ -\;2x^2 - x \\ \hline 6x - 4 \\ -\;6x - 3 \\ \hline -1 \end{array}$$

Hence the remainder is -1

Example 7

Show that x = -2 is a root of the equation $2x^3 - x2 - 8x + 4 = 0$. Hence find the other roots

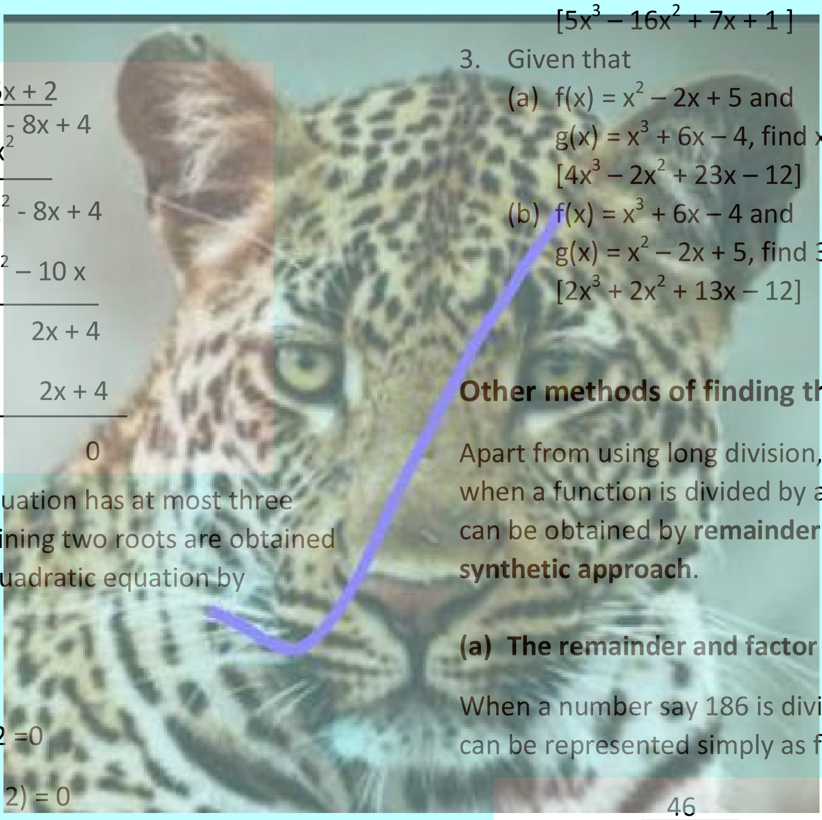
Solution

If x =-2 is a root of the function, then its

Remainder must be equal to zero

Hence x = -2 is a root of $2x^3 - x^2 - 8x + 4 = 0$

$x = -2 \Rightarrow x + 2 = 0$



$$
\begin{array}{r}
2x^2 - 5x + 2 \\
(x - 2)\,\overline{)2x^3 - x^2 - 8x + 4} \\
-\ 2x^3 + 4x^2 \qquad\qquad \\
\hline
-5x^2 - 8x + 4 \\
-\ 5x^2 - 10x \quad\ \ \\
\hline
2x + 4 \\
-\qquad 2x + 4 \\
\hline
0
\end{array}
$$

Since a cubic equation has at most three roots; the remaining two roots are obtained by solving the quadratic equation by factorization

2x2 – 5x + 2 = 0

2x2 – 4x – 2x + 2 = 0

$2x(x - 2) - 1(x - 2) = 0$

$(2x - 1)(x - 2) = 0$

$2x - 2 = 0$ or $x - 2 = 0$

$x = \frac{1}{2}$ or $x = 2$

∴ the other roots are $x = \frac{1}{2}$ and $x = 2$

Revision exercise 1

1. Find the degree of each of the following polynomials
   (a) $x^6 + 4x^4 - 2$ [6]
   (b) $2x^3 - x^2 - 8x + 4$ [3]
   (c) $5x^3 + 2x^2 + 9$ [3]
   (d) $2x^4 + 10x^3 - 6x^2 - 8x + 16$ [4]
2. Given that
   (a) $f(x) = x^3 + 2x^2 - 3x + 2$ and $g(x) = 2x^3 - x^2 + 5x - 4$, find $f(x) - g(x)$ $[-x^3 + 3x^2 - 8x + 6]$
   (b) $f(x) = x^3 + 2x^2 - 3x + 2$ and $g(x) = 2x^3 - x^2 + 5x - 4$, find $g(x) - f(x)$ $[x^3 - 3x^2 + 8x - 6]$
   (c) $f(x) = 2x^3 - 5x^2 + 6x$ and $g(x) = x^3 - 6x^2 + 5x + 1$, find $f(x) - g(x)$ $[x^3 + x^2 + x - 1]$
   (d) $f(x) = 2x^3 - 5x^2 + 6x$ and $g(x) = x^3 - 6x^2 + 5x + 1$, find $2f(x) + g(x)$ $[5x^3 - 16x^2 + 7x + 1]$
3. Given that
   (a) $f(x) = x^2 - 2x + 5$ and $g(x) = x^3 + 6x - 4$, find $xf(x) + 3g(x)$ $[4x^3 - 2x^2 + 23x - 12]$
   (b) $f(x) = x^3 + 6x - 4$ and $g(x) = x^2 - 2x + 5$, find $3f(x) + xg(x)$ $[2x^3 + 2x^2 + 13x - 12]$

Other methods of finding the remainder

Apart from using long division, the remainder when a function is divided by a certain factor can be obtained by **remainder theorem** and **synthetic approach**.

(a) The remainder and factor theorems

When a number say 186 is divided by 4, this can be represented simply as follow

$$
\begin{array}{r}
46 \\
4\,\overline{)186} \\
-\ 16\ \ \\
\hline
26 \\
-\ 24 \\
\hline
2
\end{array}
$$

∴ the quotient is 46 and the remainder is 2

The above alogarithm can be written as

$\frac{186}{4} = 46 + \frac{2}{4}$

Or simply $186 = 4Q + R$, where the quotient $Q = 46$ and the remainder, $R = 2$. This is referred to as the remainder theorem

The **remainder theorem** states that, when a function f(x) is divided by by (x – a) and leaves a remainder, then the remainder of the function is f(a)

From f(x) = (x –a)Q(x) + R

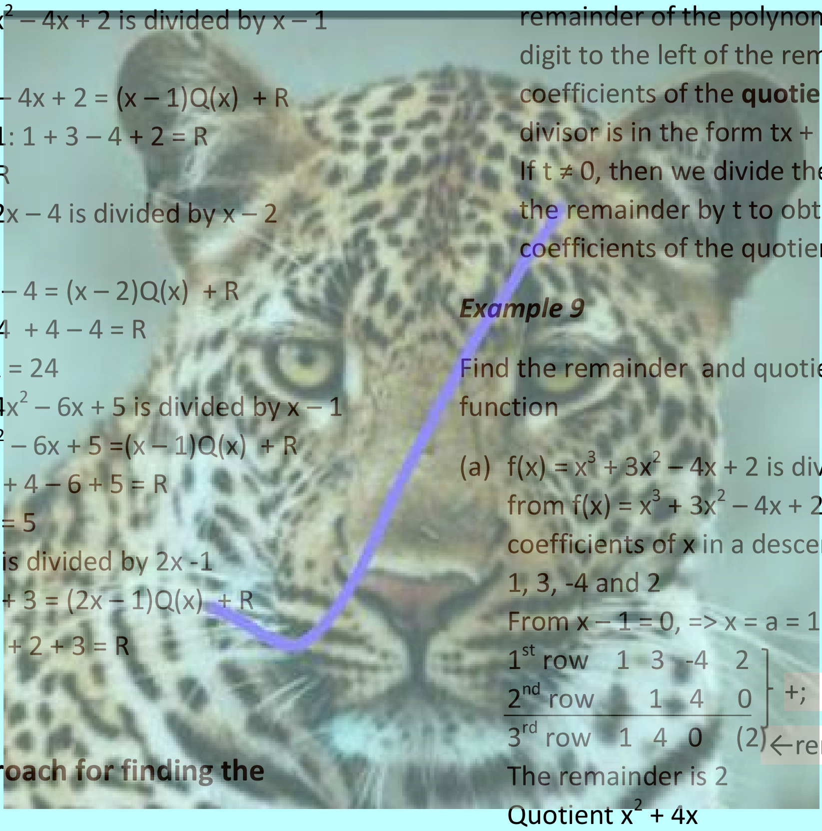
When we substitute for x = a; f(a) = R

When R = 0, => f(a) = 0. This is referred to as the **factor theorem**,

Example 8

Find the remainder when

(a) $f(x) = x^3 + 3x^2 – 4x + 2$ is divided by x – 1



Solution

Let $x^3 + 3x^2 – 4x + 2 = (x – 1)Q(x) + R$

Putting x = 1: 1 + 3 – 4 + 2 = R

2 = R

(b) $f(x) = 3x^3 + 2x – 4$ is divided by x – 2

Solution

Let $3x^3 + 2x – 4 = (x – 2)Q(x) + R$

Putting 2: 24 + 4 – 4 = R

R = 24

(c) $f(x) = 2x^3 + 4x^2 – 6x + 5$ is divided by x – 1

Let $2x^3 + 4x^2 – 6x + 5 =(x – 1)Q(x) + R$

Putting 1: 2 + 4 – 6 + 5 = R

R = 5

(d) $8x^3 + 4x + 3$ is divided by 2x -1

Let $8x^3 + 4x + 3 = (2x – 1)Q(x) + R$

Putting $\frac{1}{2}$: 1 + 2 + 3 = R

R = 6

Synthetic approach for finding the remainder

The synthetic approach can be illustrated as follows

$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n$ is divided by x – a, then the following steps are taken using a table

(i) The factor in question must be linear

(ii) The 1st row of the table contains the coefficients of x in f(x) in descending order.

(iii) The at most left hand side of the 3rd row contains a value a from the division, x – a i.e. if x – a = 0, then x = a

(iv) The first digit in the second row comes under the second digit in the first row and this digit is equal to a_0 x a, where a_0 is the coefficient of x^n.

(v) The second digit in the 3rd row is the same as the sum of the digits in the 1st and 2nd rows

(vi) The next digit in the 2nd row is equal to a_0 x the 2nd digit in the 3rd row

(vii) The corresponding numbers in the 1st and 2nd row are added to give the digit in the 3rd row and the process continues

(viii) The last digit in the 3rd row is the remainder of the polynomial and the digit to the left of the remainder are coefficients of the **quotient** provided the divisor is in the form tx + b, where t = 1. If t ≠ 0, then we divide the digit to left of the remainder by t to obtain the coefficients of the quotient.

Example 9

Find the remainder and quotient when the function

(a) $f(x) = x^3 + 3x^2 – 4x + 2$ is divided by x – 1

from $f(x) = x^3 + 3x^2 – 4x + 2$, the coefficients of x in a descending order are 1, 3, -4 and 2

From x – 1 = 0, => x = a = 1

1st row	1	3	-4	2	+; a=1
2nd row		1	4	0	
3rd row	1	4	0	(2)	←remainder

The remainder is 2

Quotient $x^2 + 4x$

(b) $x^5 + x – 9$ is divided by x + 1

$x^5 + x – 9 \equiv x^5 + 0x^4 + 0x^3 +0x^2 + x - 9$

1st row	1	0	0	0	1	-9	+; x = -1
2nd row		-1	1	-1	1	-2	
3rd row	1	-1	1	-1	2	(-11)	←remainder

Remainder: -11

Quotient: $x^4 – x^3 + x^2 –x + 2$

(c) $4x^5 – 3x^3 + 2x + 7$is divided by 2x – 1

$4x^5 - 3x^3 + 2x + 7 \equiv 4x^5 + 0x^4 - 3x^3 + 0x^2 + 2x + 7$

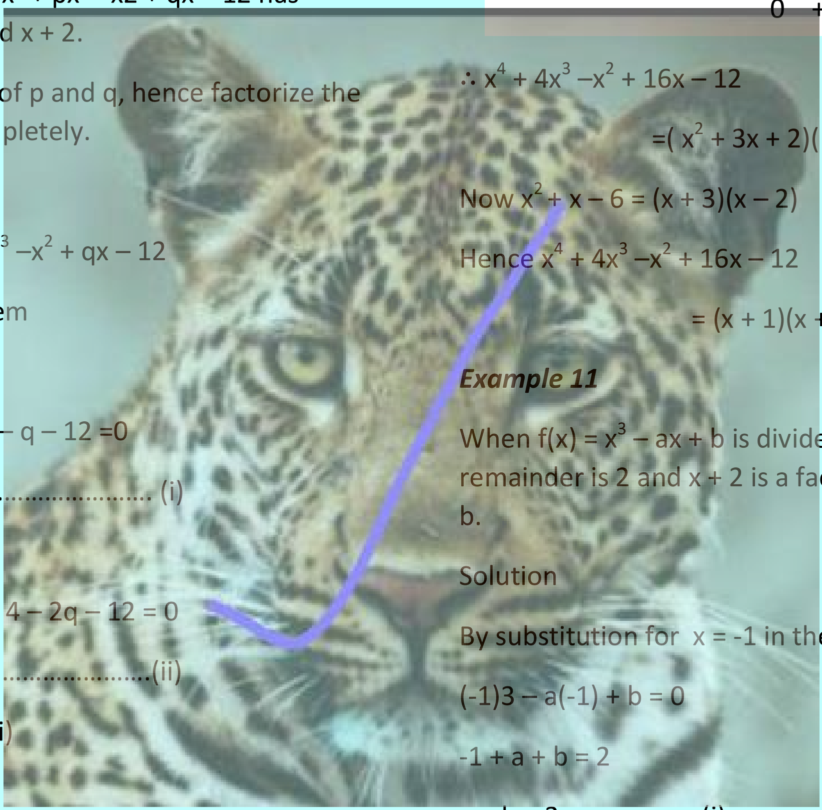
1^{st} row	4	0	-3	0	2	7	
2^{nd} row		2	1	-1	$-\frac{1}{2}$	$\frac{3}{4}$	+; $x = \frac{1}{2}$
3^{rd} row	4	2	-2	-1	$\frac{3}{2}$	$\left(\frac{31}{4}\right)$	←remainder

The quotient is $\frac{1}{2}[4x^4 + 2x^3 - 2x^2 - x + \frac{3}{2}] = 2x^4 + x^3 - x^2 - \frac{1}{2}x + \frac{3}{4}$ and remainder is $\frac{31}{4}$

Example 10

The polynomial $x^4 + px^3 - x2 + qx - 12$ has factors $x + 1$ and $x + 2$.

Find the values of p and q, hence factorize the polynomial completely.



Solution

Let $f(x) = x^4 + px^3 - x^2 + qx - 12$

By factor theorem

Putting $x = -1$

$f(-1) = 1 - p - 1 - q - 12 = 0$

$p + q = -12$ ………………………….. (i)

Putting $x = -2$

$f(-2) = 16 - 8p - 4 - 2q - 12 = 0$

$4p + q = 0$ …………………………..(ii)

Eqn. (ii) – eqn. (i)

$3p = 12$

$p = 4$

substituting for p into eqn. (i)

$4 + q = -12$

$q = -16$

$\therefore f(x) = x^4 + 4x^3 - x^2 + 16x - 12$

Now $(x + 1)(x + 2) = x^2 + 3x + 2$

Since $(x + 1)(x + 2)$ is a factor, then $x^2 + 3x + 2$ is also a factor of f(x)

By long division to find other factors

$$
\begin{array}{r}
x^2 + x - 6 \\
x^2 + 3x + 2\overline{)\,x^4 + 4x^3 - x^2 - 16x - 12} \\
-\ \underline{x^4 + 3x^3 + 2x^2} \\
x^3 - 3x^2 - 16x \\
-\ \underline{x^3 + 3x^2 + 2x} \\
-6x^2 - 18x - 12 \\
-\ \underline{-6x^2 - 18x - 12} \\
0 + 0 + 0
\end{array}
$$

$\therefore x^4 + 4x^3 - x^2 + 16x - 12$

$= (x^2 + 3x + 2)(x^2 + x - 6)$

Now $x^2 + x - 6 = (x + 3)(x - 2)$

Hence $x^4 + 4x^3 - x^2 + 16x - 12$

$= (x + 1)(x + 2)(x + 3)(x - 2)$

Example 11

When $f(x) = x^3 - ax + b$ is divided by $x + 1$, the remainder is 2 and $x + 2$ is a factor. Find a and b.

Solution

By substitution for $x = -1$ in the function

$(-1)3 - a(-1) + b = 0$

$-1 + a + b = 2$

$a + b = 3$ ………………. (i)

since $x + 2$ is a factor, substituting for $x = -2$ in the function gives zero

$(-2)^3 - a(-2) + b = 0$

$-8 + 2a + b = 0$

$2a + b = 8$ …………….(ii)

Eqn. (ii) – eqn. (i)

$a = 5$

substituting for a in eqn. (i)

$5 + b = 3 \Rightarrow b = -2$

Example 12

The function $f(x) = x^3 + px^2 – 5x + q$ has a factor $x – 2$ and has a value of 5 when $x = -3$. Find the values of p and q

Solution

By substitution for $x = 2$ in the function

$2^3 + p(2)^2 – 5(2) + q = 0$

$4p + q = 2$ ....(i)

By substitution for $x = -3$ in the function

$(-3)^3 + p(-3)^2 – 5(-3) + q = 5$

$9p + q = 17$ .... (ii)

Eqn. (ii) – eqn.(i)

$5p = 15$

$p = 3$

From eqn. (i)

$4 \times 3 + q = 2$

$q = -10$

Hence $p = 3$ and $q = -10$

Example 13

The polynomial $x^4 + px^3 – x^2 + qx – 12$ has a factor $x^2 + 3x + 2$.

Find the value of p and q and hence factorise completely

Solution

$x^2 + 3x + 2 = (x + 1) (x + 2)$

Let $f(x) = x^4 + px^3 – x^2 + qx – 12$

$f(-1) = (-1)^4 + p(-1)^3 – (-1)^2 + q(-1) – 12 = 0$

$1 – p – 1 + q – 12 = 0$

$-p + q = 12$................(i)

$f(-2) =(-2)^4 + p(-2)^3 – (-2)^2 + q(-2) – 12 = 0$

$16 – 8p – 4 – 2q – 12 = 0$

$-8p – 2q = 0$

$4p – q = 0$ ....(ii)

Eqn. (i) + eqn. (ii)

$3p = 12$

$p = 4$

From eqn. (i)

$q = -4 – 12 = -16$

Hence $p = 4$ and $q = -16$

$\therefore x^4 + px^3 – x^2 + qx – 12= x^4 + 4x^3 – x^2 – 16x – 12$

By long division to find other factors

$$\begin{array}{rl}
 & x^2 + x - 6 \\
x^2 + 3x + 2 & \overline{)\,x^4 + 4x^3 - x^2 - 16x - 12} \\
 & -\underline{\;x^4 + 3x^3 + 2x^2\;} \\
 & \qquad x^3 - 3x^2 - 16x \\
 & \quad -\underline{\;x^3 + 3x^2 + 2x\;} \\
 & \qquad\quad -6x^2 - 18x - 12 \\
 & \quad -\underline{\;-6x^2 - 18x - 12\;} \\
 & \qquad\quad 0 \;+ 0 \;+ 0
\end{array}$$

$\therefore x^4 + 4x^3 –x^2 + 16x – 12$

$=(x^2 + 3x + 2)(x^2 + x – 6)$

Now $x^2 + x – 6 = (x + 3)(x – 2)$

Hence $x^4 + 4x^3 –x^2 + 16x – 12$

$= (x + 1)(x + 2) (x + 3)(x – 2)$

Example 14

If $4x^3 + ax^2+ bx + 2$ is divisible by $x^2 + k^2$, show that $ab = 8$

Solution

By long division

$$\begin{array}{r} 4x + a \\ (x^2 + k^2)\overline{)4x^3 + ax^2 + bx + 2} \\ \underline{-\ 4x^3 + 4k^2} \qquad\qquad \\ ax^2 + (b - 4k^2)x + 2 \\ \underline{-\ ax^2 + ak^2} \qquad\qquad \\ (b - 4k^2)x + 2 - ak^2 \end{array}$$

Since the remainder is zero,

$(\lambda - 4k^2)x + 2 - ak^2 = 0$

Comparing coefficients of x:

$b - 4k^2 = 0$ ………………… (i)

$2 - ak^2 = 0$

$k^2 = \frac{2}{\mu}$……………….……… (ii)

Eqn. (ii) into eqn. (i)

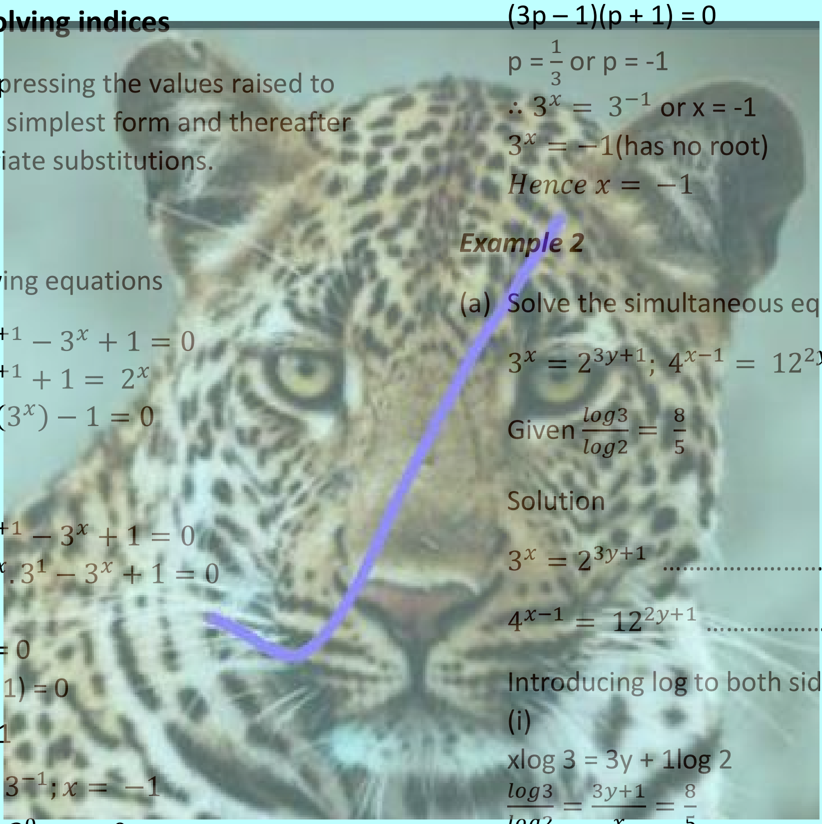
$b - 4\left(\frac{2}{a}\right) = 0$

⇨ $ab - 8 = 0$

∴ $ab = 8$

Revision exercise 2

1. Given that f(x) $3x^3 - 4x^2 - 5x + 2$, factorize f(x) completely and hence solve the equation f(x) = 0 $\left[x = -1, 2 \ or \ \frac{1}{3}\right]$
2. Prove that a – b is a factor of $a^2(b - c) + b^2(c - a) + c^2(a - b)$ and write down two other factors of the expression. Hence or otherwise factorize the expression completely
[-(a – b)(b – c)(c – a) or (b – a))(b – c)(c – a)]
3. The polynomial $ax^3 + bx^2 - cx - 2$ is divisible by x +2. When divided by x – 1 it leaves a remainder of 18 and when divided by x + 3, it leaves a remainder -50. Determine the values of a, b, and c. Hence factorize the polynomial completely
[a = 6, b = 13, c = -1; (x + 2)(2x +1)(3x-1)]
4. Factorize completely
$f(xyz) = (x + y)^3(x - y) + (y + z)^3(y - z) + (z + x)^3(z - x)$
[(x + y + z)(x – y)(y – z)(z – x)
5. When the quadratic expression ax2 + bx + c is divided by x – 1, x – 2 and x + 1, the remainders are 1, 1, and 25 respectively, determine the factors of the expression [(2x – 3) and (2x – 3)]
6. The remainder when $px^3 + 2x^2 - 5x + 7$ is divided by x – 2 is equal to the remainder when the same expression is divided by x + 1. Find the value of p [p = 1]
7. Given that x – 4 is a factor of $2x^3 - 3x^2 - 7x + k$, where k is a constant, find the remainder when the expression is divided by 2x -1 [k = - 52, remainder = -56]
8. The expression $px^3 + qx^2 + 3x + 8$ leaves a remainder of -6 when divided by x – 2 and a remainder of -34 when divided by x + 2. Find the values of constants p and q. [p = 1, q = -7]
9. Show that x – 2 is a factor of $x^3 - 9x^2 + 26x - 24$. Find the set of values of x for which $x^3 - 9x^2 + 26x - 24 < 0$
[x < 2 and 3< x < 4]
10. The remainder when $x^3 - 2x^2 + kx + 5$ is divided by x – 3 is twice when the same expression is divided by x + 1. Find the value of the constant k [k = -2]



Equations involving indices, logarithms and others

Equations involving indices

This involves expressing the values raised to the powers into simplest form and thereafter making appropriate substitutions.

Example 1

Solve the following equations

(a) $3^{2x+1} - 3^{x+1} - 3^{x} + 1 = 0$
(b) $2^{2x+1} - 2^{x+1} + 1 = 2^{x}$
(c) $3(3^{2x}) + 2(3^{x}) - 1 = 0$

Solution

(a) $3^{2x+1} - 3^{x+1} - 3^{x} + 1 = 0$
$3^{2x}.3^{1} - 3^{x}.3^{1} - 3^{x} + 1 = 0$
Let p = 3^{x}
$3p^2$ -4p + 1 = 0
(3p – 1)(p – 1) = 0
p = $\frac{1}{3}$ or p = 1
$\therefore 3^{x} = \frac{1}{3} = 3^{-1}; x = -1$
$\therefore 3^{x} = 1 = 3^{0}; x = 0$

(b) $2^{2x+1} - 2^{x+1} + 1 = 2^{x}$
$2(2^{x})^2 - 2(2^{x}) + 1 = 2^{x}$
$Let\ 2^{x} = q$
$2q^2 - 2q + 1 = q$
$2q^2 - 3q + 1 = 0$
$(2q - 1)$(q – 1) = 0
$q = \frac{1}{2}\ or\ q = 1$
$\therefore 2^{x} = \frac{1}{2} = 2^{-1}; x = -1$
$\therefore 2^{x} = 1 = 2^{0}; x = 0$
Hence x = 0 or x = -1

(c) $3(3^{2x}) + 2(3^{x}) - 1 = 0$
$3(3^{x})^2 + 2(3^{x}) - 1 = 0$
Let p = 3^{x}
$3p^2$ + 2p -1 = 0
(3p – 1)(p + 1) = 0
p = $\frac{1}{3}$ or p = -1
$\therefore 3^{x} = 3^{-1}$ or x = -1
$3^{x} = -1$(has no root)
$Hence\ x = -1$

Example 2

(a) Solve the simultaneous equations

$3^{x} = 2^{3y+1};\ 4^{x-1} = 12^{2y+1}$

Given $\frac{log3}{log2} = \frac{8}{5}$

Solution

$3^{x} = 2^{3y+1}$ ……………………… (i)

$4^{x-1} = 12^{2y+1}$ …………………(ii)

Introducing log to both sides of equation (i)
xlog 3 = 3y + 1log 2
$\frac{log3}{log2} = \frac{3y+1}{x} = \frac{8}{5}$
8x – 15y = 5 ……………………..(iii)
From eqn. (ii)

$4^{x-1} = 12^{2y+1}$

$2^{2(x-1)} = (3^{1}.2^{2})^{2y+1}$

$2^{2(x-1)} = 3^{2y+1}.2^{2(2y+1)}$

$3^{2y+1} = 2^{2x-4y-4}$

$\frac{log3}{log2} = \frac{2x-4y-4}{2y+1} = \frac{8}{5}$
10x – 36y = 28
5x – 18y = 14 ………………….(iv)
5eqn. (iii) – 8eqn (iv)

69y = -87

y = $\frac{-29}{23}$

substituting for y in equation (iii)

8x = 5+ 15$\left(\frac{-29}{23}\right)$

$x = \frac{320}{23 \ x \ 8} = \frac{-40}{23}$

∴ x = $\frac{-40}{23}$ and y = $\frac{-29}{23}$

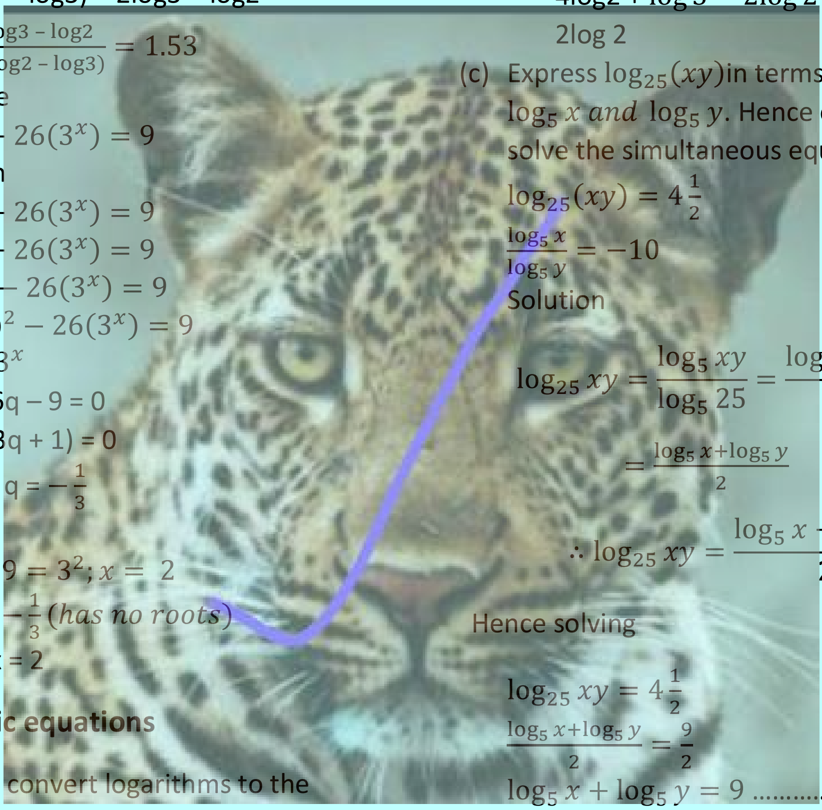
(b) Find the value of x to 2 decimal places if $2^{3x+1} = 3^{x+2}$

Solution

Introducing log

(3x + 1)log2 = (x +2)log3

x(3Log2 – log3) = 2log3 – log2



$x = \frac{2\log3 - \log2}{(3\text{Log}2 - \log3)} = 1.53$

(c) Evaluate

$3^{2x+1} - 26(3^x) = 9$

Solution

$3^{2x+1} - 26(3^x) = 9$

$3^{2x+1} - 26(3^x) = 9$

$3^1 . 3^{2x} - 26(3^x) = 9$

$3^1 . (3^x)^2 - 26(3^x) = 9$

Let q = 3^x

3q^2 – 26q – 9 = 0

(q – 9)(3q + 1) = 0

q = 9 0r q = $-\frac{1}{3}$

$\therefore\ 3^x = 9 = 3^2; x = 2$

$\therefore\ 3^x = -\frac{1}{3} (has\ no\ roots)$

Hence x = 2

Logarithmic equations

It convenient to convert logarithms to the same base before the calculations

Example 3

Solve the equations

(a) $\log_x 5 + 4\log_5 x = 4$

Expressing terms on LHS to log$_5$.

$\frac{\log_5 5}{\log_5 x} + 4\log_5 x = 4$

$\frac{1}{\log_5 x} + 4\log_5 x = 4$

Let $\log_5 x = y$

$\frac{1}{y} + 4y = 4$

$4y^2 - 4y + 1 = 0$

(2y – 1)(2y – 1) = 0

2y = 1

y = $\frac{1}{2}$

⇨ $\log_5 x = \frac{1}{2}$

x = $5^{\frac{1}{2}} = \sqrt{5}$

(b) Show that:

2log4 + $\frac{1}{2}\log 25 - \log 20 = 2\log 2$.

2log4 + $\frac{1}{2}\log 25 - \log 20$

2log2^2 + $\frac{1}{2}\log 5^2 - (\log 4 + \log 5)$

2log2^2 + $\frac{1}{2}\log 5^2 - \log 4 - \log 5$

4log2 + $\log 5 - 2\log 2 - \log 5$

2log 2

(c) Express $\log_{25}(xy)$in terms of $\log_5 x\ and\ \log_5 y$. Hence or otherwise solve the simultaneous equations:

$\log_{25}(xy) = 4\frac{1}{2}$

$\frac{\log_5 x}{\log_5 y} = -10$

Solution

$\log_{25} xy = \frac{\log_5 xy}{\log_5 25} = \frac{\log_5 x + \log_5 y}{\log_5 5^2}$

$= \frac{\log_5 x + \log_5 y}{2}$

$\therefore \log_{25} xy = \frac{\log_5 x + \log_5 y}{2}$

Hence solving

$\log_{25} xy = 4\frac{1}{2}$

$\frac{\log_5 x + \log_5 y}{2} = \frac{9}{2}$

$\log_5 x + \log_5 y = 9$ ………………. (i)

$\frac{\log_5 x}{\log_5 y}$ = -10

$\log_5 x = -10\log_5 y$ …………….(ii)

Substituting eqn. (ii) into eqn. (i)

$-10\log_5 y + \log_5 y = 9$

$\log_5 y = \ -1$

y = $5^{-1} = \frac{1}{5}$

Substituting $\log_5 y$ into equation (ii)

$\log_5 x = 10$

$x = 5^{10}$

$\therefore\ x = 5^{10}$ and y = $\frac{1}{5}$

Equations with repeated terms (s)

Whenever a term or terms appear repeated in an equation make use of appropriate substitution.

Example 4

(a) Solve the simultaneous equations

$2^x + 4^y = 12$

$3(2^x) - 2(2^y)$=16

$Solution$

$2^x + 4^y = 12$

$2^x = 12 - 4^y$..............(i)

$3(2^x) - 2(2^y)$=16 ....(ii)

Substituting eqn.(i) into eqn. (ii)

$3(12 - 4^y) - 2(2^y)$=16

$36 - 3(2^{2y}) - 2(2^y) = 16$

$5(2^{2y}) = 20$

$(2^{2y}) = 4 = 2^2$

$2y = 2; \gg y = 1$

Substituting y into eqn. (i)

$2^x = 12 - 4^1 = 8 = 2^3$

$x = 3$

Hence x = 3 and y = 1

(b) Solve the equations

(i) $9x^{\frac{2}{3}} + 5x^{-\frac{2}{3}} = 37$

Solution

$9x^{\frac{2}{3}} + \frac{4}{x^{\frac{2}{3}}} = 37$

Let q = $x^{\frac{2}{3}}$

9q+ $\frac{5}{q}$ = 37

$9q^2 - 37q + 4 = 0$

(9q – 1)(q – 4) = 0

q = $\frac{1}{9}$ or q = 4

When q = $\frac{1}{9}$;

$x^{\frac{2}{3}} = \frac{1}{9} = \frac{1}{3^2} = 3^{-2}$

$x = (3^{-2})^{\frac{3}{2}} = \frac{1}{27}$

When q = 4

$x^{\frac{2}{3}} = 4 = 2^2$

$x = (2^2)^{\frac{3}{2}} = 2^3 = 8$

Hence x = $\frac{1}{27}$ and x = 8

(ii) $(x^2 - 2x)^2 + 24 = 11(x^2 - 2x)$

Solution

Let t = $x^2 - 2x$

$t^2 - 11t + 24 = 0$

(t – 8)(t – 3) = 0

t = 8 or t = 3

When t = 8

$x^2 - 2x = 8$

$x^2 - 2x - 8 = 0$

(x – 4)(x + 2) = 0

x = 4 or x = -2

When t = 3

$x^2 - 2x = 3$

$x^2 - 2x - 3 = 0$

(x – 3)(x +1) = 0

x = 3 or x = -1

hence x = -2, -1, 3 or 4

Symmetrical equations

These are equations whose coefficients are symmetrical

Example 5

(a) (i) If $x + \frac{1}{x} = q$, $x^2 + \frac{1}{x^2}$, $x^3 + \frac{1}{x^3}$ and $x^4 + \frac{1}{x^4}$ in terms of q

Solution

$x + \frac{1}{x} = q$

Squaring both sides

$\left(x + \frac{1}{x}\right)^2 = q^2$

$x^2 + \frac{1}{x^2} + 2 = q^2$

$x^2 + \frac{1}{x^2} = q^2 - 2$

$x + \frac{1}{x} = q$

Cubing both sides

$\left(x + \frac{1}{x}\right)^3 = q^3$

$$x^3 + \frac{1}{x^3} + 3x^2\left(\frac{1}{x}\right) + 3x\left(\frac{1}{x}\right) = q^3$$
$$x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right) = q^3$$
$$x^3 + \frac{1}{x^3} = q^3 - 3q = q(q^2 - 3)$$

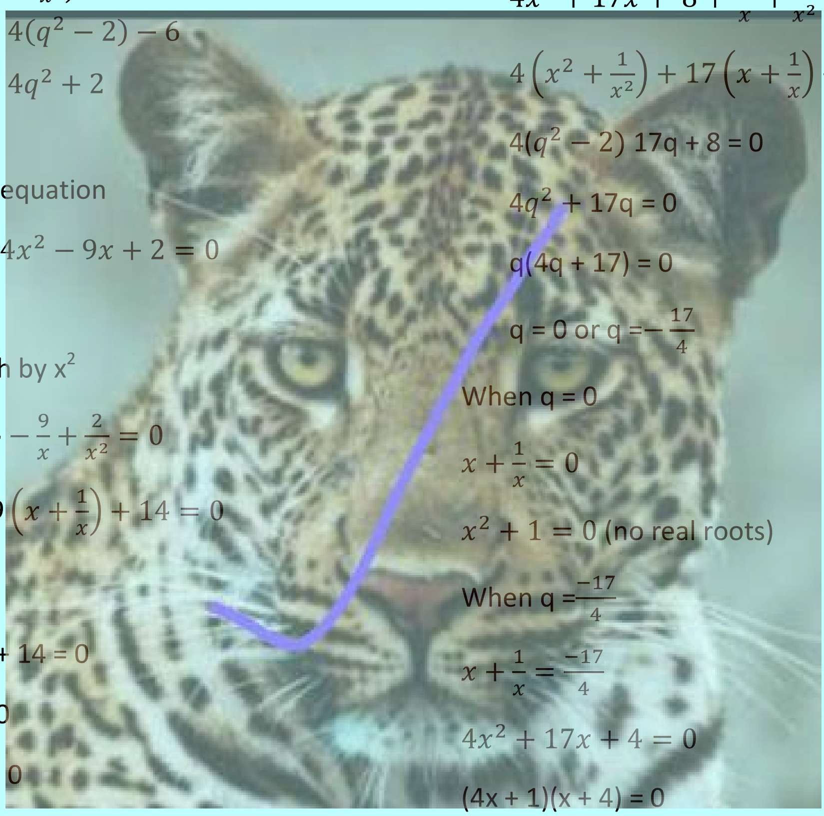
$x + \frac{1}{x} = q$

Raising both sides to power four

$$\left(x + \frac{1}{x}\right)^4 = q^4$$

$$x^4 + \frac{1}{x^4} + 4x^3\left(\frac{1}{x}\right) + 6x^2\left(\frac{1}{x}\right)^2 + 4x\left(\frac{1}{x}\right)^3 = q^4$$

$$x^4 + \frac{1}{x^4} + 4\left(x^2 - \frac{1}{x^2}\right) + 6 = q^4$$
$$x^4 + \frac{1}{x^4} = q^4 - 4(q^2 - 2) - 6$$
$$x^4 + \frac{1}{x^4} = q^4 - 4q^2 + 2$$



(a)(ii) Solve the equation

$2x^4 - 9x^3 + 14x^2 - 9x + 2 = 0$

Solution

Dividing through by x^2

$2x^2 - 9x + 14 - \frac{9}{x} + \frac{2}{x^2} = 0$

$2\left(x^2 + \frac{1}{x^2}\right) - 9\left(x + \frac{1}{x}\right) + 14 = 0$

Let q = $x + \frac{1}{x}$

$2(q^2 - 2) - 9q + 14 = 0$

$2q^2 - 9q + 10 = 0$

$(q - 2)(2q - 5) = 0$

q = 2 or q = $\frac{5}{2}$

When q = 2

$x + \frac{1}{x} = 2$

$x^2 - 2x + 1 = 0$

$(x - 1)(x - 1) = 0$

x = 1

When q =$\frac{5}{2}$

$x + \frac{1}{x} = \frac{5}{2}$

$2x^2 - 5x + 2 = 0$

$(2x - 1)(x - 2) = 0$

x = $\frac{1}{2}$ or x = 2

Hence [x:x = $\frac{1}{2}$, 1, or 2]

(b) By using the substitution q = $x + \frac{1}{x}$ solve the equation

$4x^4 + 17x^3 + 8x^2 + 17x + 4 = 0$

$4x^2 + 17x + 8 + \frac{17}{x} + \frac{4}{x^2} = 0$

$4\left(x^2 + \frac{1}{x^2}\right) + 17\left(x + \frac{1}{x}\right) + 8 = 0$

$4(q^2 - 2)$ 17q + 8 = 0

$4q^2 + 17q = 0$

q(4q + 17) = 0

q = 0 or q =$-\frac{17}{4}$

When q = 0

$x + \frac{1}{x} = 0$

$x^2 + 1 = 0$ (no real roots)

When q =$\frac{-17}{4}$

$x + \frac{1}{x} = \frac{-17}{4}$

$4x^2 + 17x + 4 = 0$

(4x + 1)(x + 4) = 0

x = $\frac{-1}{4}$ or x = -4

Hence x = $\frac{-1}{4}$ or x = -4

Equations with ratios

The basis of these types of equations it the ration theorem

Given $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = k$

Then $\frac{a+c+e}{b+d+f} = k$

This can

Then $\frac{a}{b} = k$ => a = bk

$\frac{c}{d} = k$ => c = dk

$\frac{e}{f} = k$ => e= fk

LHS: $\frac{bk+dk+fk}{b+d+f} = \frac{(b+d+f)k}{b+d+f} = k$

Example 6

Solve the equations

(a) $\frac{x+4z}{4} = \frac{y+z}{6} = \frac{3x+y}{5}; 4x + 2y + 5z = 30$

$Solution$

Let $\frac{x+4z}{4} = \frac{y+z}{6} = \frac{3x+y}{5} = k$

Then $\frac{x+4z+y+z+3x+y}{4+6+5} = k$

$\frac{4x+2y+5z}{15} = \frac{30}{15} = 2 = k$

$k = 2$

$\therefore \frac{x+4z}{4} = 2$; => x + 4z = 8 ……………….(i)

$\frac{y+z}{6} = 2$; => y + z = 12 ……………….(ii)

$\frac{3x+y}{5} = 2$; => 3x + y =10 …………….. (iii)

From eqn. (i): x = 8 – 4z

Substituting for x into eqn. (iii)

3(8 – 4z) + y = 10

y – 12z = -14 ……………………………….(iv)

eqn. (ii) – eqn. (iv)

13z = 26; z = 2

Substituting for z into eqn. (i)

x = 8 – 4 x 2 =0

substituting for x into eqn. (iii)

y = 10 – 3x = 10 – 0 = 10

Hence (x, y, z) = (0, 10, 2)

(b) $\frac{x+2y}{-3} = \frac{y+2z}{4} = \frac{2x+z}{5}; x + y + z = 2$

Solution

Let $\frac{x+2y}{-3} = \frac{y+2z}{4} = \frac{2x+z}{5} = k$

Then $\frac{3x+3y+3z}{6} = \frac{3(x+y+z)}{6} = \frac{3 \ x \ 2}{6} = 1 = k$

k = 1

$\therefore \frac{x+2y}{-3} = 1$; x + 2y = -3……………… (i)

$\frac{y+2z}{4} = 1$; y + 2z = 4 ………………(ii)

$\frac{2x+z}{5} = 1$; 2x + z = 5 ………………(iii)

From eqn. (i) x = -3 – 2y

Substituting x into eqn. (iii)

2(-3 – 2y) + z = 5

4y – z = -11…………………………….(iv)

From eqn. (ii): y = 4 – 2z

Substituting for y in eqn. (iv)

4(4 – 2z) –z = -11

9z = 27; z = 3

Substituting for z into eqn. (ii)

Y = 4 – 2 x 3 = -2

Substituting for y into eqn. (i)

x = -3 – 2(-2) = 1

Hence (x, y, z) = (1, -2, 3)

(c) 2x = 3y= -4z; $x^2 - 9y^2 - 4z = 0$

Solution

Let 2x = 3y= -4z =k, constant

$x = \frac{k}{2}; y = \frac{k}{3}; z = \frac{k}{-4}$

Substituting for x, y, z in $x^2 - 9y^2 - 4z = 0$

$\left(\frac{k}{2}\right)^2 - 9\left(\frac{k}{3}\right)^2 - 4\left(\frac{k}{-4}\right) = 0$

$3k^2 - 4k - 32 = 0$

(3k + 8)(k – 4) = 0

k = $-\frac{8}{3}$ or k = 4

When k =$-\frac{8}{3}$; x = 2, y = $\frac{4}{3}$, z = -1

When k = 4; x = $-\frac{4}{3}$, y = $-\frac{8}{9}$, z = $\frac{2}{3}$

∴ (x, y, z) = (2, $\frac{4}{3}$, -1) or ($-\frac{4}{3}$, $-\frac{8}{9}$, $\frac{2}{3}$)

Equations with repeated roots

If a function y = f(x) has a repeated root, then it is also a root of its derivative i.e. it is a root of $\frac{dy}{dx} f(x)$

In order to find other root, make use of the sum and product of roots

Note

Sum of roots = coefficient of the second term of the equation

Product of the roots= last term with the appropriate signs.

Example 7

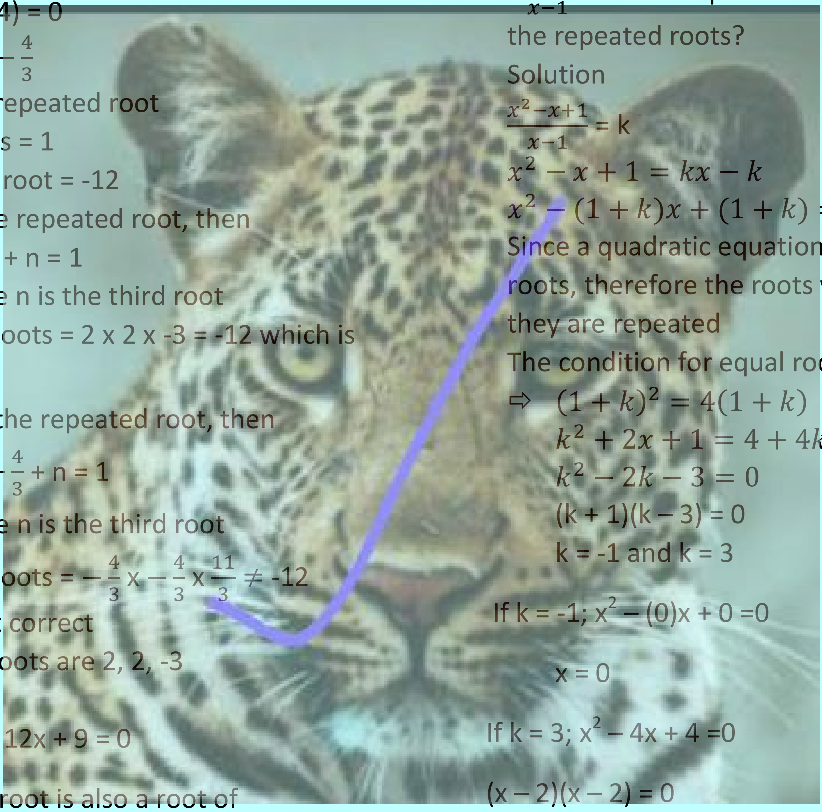
(a) Given that the following equations have repeated roots, solve them

(i) $X^3 - x^2 - 8x + 12 = 0$

A repeated root is also a root of

$\frac{d}{dx}(x^3 - x^2 - 8x + 12) = 0$

$3x^2 - 2x - 8 = 0$



$(x - 2)(3x + 4) = 0$

$x = 2$ or $x = -\frac{4}{3}$

Testing for repeated root

Sum of roots = 1

Products of root = -12

If x = 2 is the repeated root, then

Sum = 2 + 2 + n = 1

n = -3 where n is the third root

Product of roots = 2 x 2 x -3 = -12 which is correct

If $x = -\frac{4}{3}$ is the repeated root, then

Sum = $-\frac{4}{3} - \frac{4}{3} + n = 1$

$n = \frac{11}{3}$ where n is the third root

Product of roots = $-\frac{4}{3} \times -\frac{4}{3} \times \frac{11}{3} \neq -12$

which is not correct

Hence the roots are 2, 2, -3

(ii) $2x^3 - 11x^2 + 12x + 9 = 0$

A repeated root is also a root of

$\frac{d}{dx}(2x^3 - 11x^2 + 12x + 9) = 0$

$6x^2 - 22x + 12 = 0$

$3x^2 - 11x + 6 = 0$

$(x - 3)(3x - 2) = 0$

$x = 3$ or $x = \frac{2}{3}$

Testing for repeated root;

$2x^3 - 11x^2 + 12x + 9 = 0$

$x^3 - \frac{11}{2}x^2 + 6x + \frac{9}{2} = 0$

Sum of roots = $\frac{11}{2}$

Product of roots = $-\frac{9}{2}$

If x = 3 is the repeated root, then

Sum = 3 + 3 + n = $\frac{11}{2}$

$n = -\frac{1}{2}$ where n is the third root

Product of roots = $3 \times 3 \times -\frac{1}{2} = -\frac{9}{2}$ which is correct

Hence the roots are 3, 3, $-\frac{1}{2}$

(b) Find the value of k for which the equation $\frac{x^2-x+1}{x-1} = k$ has repeated roots. What are the repeated roots?

Solution

$\frac{x^2-x+1}{x-1} = k$

$x^2 - x + 1 = kx - k$

$x^2 - (1+k)x + (1+k) = 0$

Since a quadratic equation has only two roots, therefore the roots will be equal if they are repeated

The condition for equal roots: $b^2 = 4ac$

⇨ $(1+k)^2 = 4(1+k)$

$k^2 + 2x + 1 = 4 + 4k$

$k^2 - 2k - 3 = 0$

$(k + 1)(k - 3) = 0$

k = -1 and k = 3

If k = -1; $x^2 - (0)x + 0 = 0$

$x = 0$

If k = 3; $x^2 - 4x + 4 = 0$

$(x - 2)(x - 2) = 0$

$x = 2$

Hence repeated roots are 0 and 2

Example 8

Show that if the equations

$x^2 + px + q = 0$ and $x^2 + mx + k = 0$ have a repeated root, then

$(q - k)^2 = (m - p)(pk - mq)$

Let the common root be x_1

The two equations become

$x_1^2 + px_1 + q = 0$ ……………… (i)

$x_1^2 + mx_1 + k = 0$………………(ii)

Eqn. (i) – eqn. (ii)

$(p - m)x_1 + (q - k) = 0$

$x_1 = -\left(\frac{q-k}{p-m}\right)$

Substituting x^1 into equation (i)

$\left(\frac{q-k}{p-m}\right)^2$ - $p\left(\frac{q-k}{p-m}\right)$ + q = 0

Thus, $(q-k)^2 = (m-p)(pk - mq)$

Equations with square roots

(i) In this context, $\sqrt{x}$ means the positive square root of x.

(ii) The solution should be checked to discard off the unwanted root might emerge due to squaring.

Illustration

Suppose the root of the equation is x = 5

By squaring x2 = 52 i.e. $x^2 - 5^2 = 0$

(x -5)(x + 5) = 0

Now solution is x = -5 and x = 5 where -5 is unwanted.

Example 9

Solve the equations

(a) $\sqrt{(x-5)} + \sqrt{x} = 5$

Solution

$$\left(\sqrt{(x-5)}\right)^2 = \left(5 - \sqrt{x}\right)^2$$

$x - 5 = 25 - 10\sqrt{x} + x$

=> $10\sqrt{x} = 30$

$\sqrt{x} = 3$; x = 9 (upon squaring)

(c) $\sqrt{(1-20x)} - 2\sqrt{x+1} = 3$

Solution

$$\left(\sqrt{1-20x}\right)^2 = \left(3 + 2\sqrt{x+1}\right)^2$$

$1 - 20x = 9 + 12\sqrt{x+1} + 4(x + 1)$

$24x + 12 = -12\sqrt{x+1}$

$2x + 1 = -\sqrt{x+1}$

Squaring both sides

$4x^2 + 4x + 1 = x + 1$

$4x^2 + 3x = 0$

$x(4x + 3) = 0$

x = 0 or $x = -\frac{3}{4}$

When x = 0; $\sqrt{1} - 2\sqrt{1} \neq 3$; ∴x = 0 is not a root

When $x = -\frac{3}{4}$; $\sqrt{16} - 2\sqrt{\frac{1}{4}} = 3$ which is consistent. ∴$x = -\frac{3}{4}$

Other equations

Example 10

Given that $x^2 + 2xy + y^2 - 8x - 8y + 15 = 0$, find x in terms of y

Hence or otherwise solve the pair of equations:

$x^2 + 2xy + y^2 - 8x - 8y + 15 = 0$ and $x^2 + y^2 = 17$

Solution

$x^2 + 2xy + y^2 - 8x - 8y + 15 = 0$ …….. (i)

$(x + y)^2 - 8(x + y) + 15 = 0$

Let q = (x + y)

$q^2 - 8q + 15 = 0$

$(q - 5)(q - 3) = 0$

q = 5 and q = 3

When q = 5; x = 5 – y …………………(ii)

When q = 3; x = 3 – y ………………… (iii)

Now $x^2 + y^2 = 17$ ………………….(iv)

Eqn. (ii) into eqn. (iv)

$(5 - y)^2 + y^2 = 17$

$2y^2 - 10y + 8 = 0$

$(y - 4)(y - 1) = 0$; => $y = 4$ or $y = 1$

When $y = 4$: $x = 5 - 4 = 1$

When $y = 1$: $x = 5 - 1 = 4$

Eqn. (iii) into eqn. (iv)

$(3 - y)^2 + y^2 = 17$

$2y^2 - 6y - 8 = 0$

$(y - 4)(y + 1) = 0$; => $y = 4$, or $y = -1$

When $y = 4$: $x = 3 - 4 = -1$

When $y = -1$: $x = 3 + 1 = 4$

∴ $(x, y) = (1, 4), (4, 1), (-1, 4), (4, -1)$

Example 11

Solve the simultaneous equations

$\frac{1}{x} - \frac{1}{y} = \frac{1}{6}$, $x(5 - x) = 2y$

Solution

From $\frac{1}{x} - \frac{1}{y} = \frac{1}{6}$

$\frac{1}{y} = \frac{1}{x} - \frac{1}{6} = \frac{6-x}{6x}$

$y = \frac{6x}{6-x}$ ……………………..(i)

$x(5 - x) = 2y$ ………………(ii)

Substituting eqn. (i) into eqn. (ii)

$x(5 - x) = \frac{12x}{6-x}$

$x(5 - x)(6 - x) - 12x = 0$

$x[(5 - x)(6 - x) - 12] = 0$

$x(x^2 - 11x + 18) = 0$

$x(x - 2)(x - 9) = 0$; $x = 0$, $x = 2$, $x = 9$

Substituting for x in eqn. (i)

When $x = 0$; $y = 0$

When $x = 2$; $y = 3$

When $x = 9$; $y = -18$

∴$(x, y) = (0, 0), (2, 3)$ or $(9, -18)$

Example 12

Given that the equations $y^2 + py + q = 0$ and $y^2 + my + k = 0$ have common root. Show

$(q - k)^2 = (m - p)(pk - mq)$

Solution

Let α be the common root

For $y^2 + py + q = 0$

$\alpha^2 + p\alpha + q = 0$ ………………… (i)

For $y^2 + my + k = 0$

$\alpha^2 + m\alpha + k = 0$ ………………… (ii)

Eqn. (i) – eqn. (ii)

$(p - m)k + (q - k) = 0$

$\alpha = \frac{q-k}{m-p}$

Substituting α into eqn. (i)

$\left(\frac{q-k}{m-p}\right)^2 + \left(\frac{q-k}{m-p}\right)p + q = 0$

$\frac{(q-k)^2}{(m-p)^2} = -q - \left(\frac{q-k}{m-p}\right)p$

$(q - k)^2 = -q(m - p)^2 - (q - k)(m - p)p$

$(q - k)^2 = (m - p)[-q(m - p) - p(q - k)]$

$(q - k)^2 = (m - p)[-qm + qp - pq + pk]$

$(q - k)^2 = (m - p)(pk - qm)$ as required

Revision questions

1. Solve the equations
   (a) $\log_x 8 - \log_x 2 = 16$[x = 2]
(b) $\log_2 x + \log_4 x + \log_{16} x = \frac{21}{16}$
   $\left[x = 8^{\frac{1}{4}} = 1.6818\right]$
2. (a) Given that $\log_2 x + 2\log_4 y = 4$, show that xy = 16. Hence solve the simultaneous equations
   $\log_{10}(x + y) = 1$
   $\log_2 x + 2\log_4 y = 4$
   [(x, y) = (8, 2) or (2, 8)]
(b) Show that $\log_a b = \frac{1}{\log_b a}$. Hence solve the simultaneous equations
   $\log_a b + 2\log_b a = 3$
   $\log_9 a + 2\log_9 b = 3$
   $\therefore (x, y) = (27, 27) or\ (9, 81)$
3. Show that if the expressions
   (i) $x^2 + px + q$ and $3x^2 + q$ have a common root, then $3p^2 + 4q = 0$
   (ii) $x^2 + bx + c$ and $x^2 + px + q$ have a common root, then
   $(c - q)^2 = (b - p)(cp - bq)$
4. Solve the simultaneous equations
   (a) 2x+ y = 1
   $5x^2 + 2xy = y + 2x - 1$[(x, y)=(0,1), (-2, 5)]
   (b) x + 2y = 1
   3x2 + 5xy – 2y2 = 10 [(x, y) = (3, -1)
5. Solve the simultaneous equation
   $2^x + 4^y = 12$
   $3(2)^x - 2(2)^{2y} = 16$ [x= 2, y = 1]
   Hence show that $(4)^x + 4(3)^{2y} = 100$
6. Solve $4^x - 2^{x+1} - 15 = 0$ [x=2.322]
7. Show that if the expressions:
   $x^2 + bx + c$ and $x^2 + px + q$ have a common factor. Then $(c - q)^2 = (b - p)(cp - bq)$
8. Solve $2\sqrt{(x-1)} - \sqrt{(x+4)} = 1$
   [x = 5 or x =$\frac{13}{9}$]
9. Solve the simultaneous equations
   x + y + z = 2
   $\frac{x+2y}{-3} = \frac{y+2z}{4} = \frac{2x+z}{5}$ [x = 1, y = -2, z = 3]
10. Solve the simultaneous equations
   $x^2 - 10x + y^2 = 25$
   y – x = 1 [(x, y) =(6, 7) or $(-2, -1)$
11. Solve for x in the equation
   $\log_4(6 - x) = \log_2 x$ [x = 2]

Simultaneous equations

Simultaneous equations in two unknowns

These are equations containing two unknowns. The equations may all be linear (equations of straight lines) or one of them may be linear and the other non-linear.

Linear simultaneous equations

Simultaneous equations may solved by any of the three methods

- Elimination method
- Substitution method
- Graphical method
- Solving simultaneous equations using matrixes

Solving simultaneous equations using elimination method

This involves elimination of one of the unknown variables so as to be in position to find the other.

Example 1

(a) $5x + 3y = 7$

$2x - 4y = 3$

Solution

$5x + 3y = 7$ ………… (i)

$2x - 4y = 3$ …………(ii)

To eliminate y the equation are multiplied by relevant factors to make the coefficients of y in both equations equal. Thus eqn. (i) is multiplied by 4 and eqn. (ii) by 3

i.e. 4 x eqn. (i) + 3 x eqn. (ii)

$$\begin{array}{r} 20x + 12y = 28 \\ + \quad 6x - 12y = 9 \\ \hline 26x = 37 \end{array}$$

$x = \frac{37}{26}$

Substituting x in eqn.(i)

$5x + 3y = 7$

$3y = 7 - 5 \times \frac{37}{26}$

$Y = \frac{-1}{26}$

(b) $3x + 2y = 8$

$3y + 4x = 11$

Solution

Rearrange the eqns.

$3x + 2y = 8$ ………… (i)

$4x + 3y = 11$ ……….. (ii)

Eliminate x as follows

4 x eqn. (i) – 3 x eqn. (ii)

$$\begin{array}{r} 12x + 8y = 32 \\ - \quad 12x + 9y = 33 \\ \hline \end{array}$$

$y = 1$

Substituting y in eqn. (i)

3x + 2y = 8

3x + 2 x 1 = 8

3x = 6

x = 2

Solving simultaneous equations using substitution method

This involves the expression of one of the unknown variable in terms of the other.

Example 2

Solve the following pairs of simultaneous equation for x and y by substitution method

(a) 5x + 3y = 7

2x -4y = 3

Solution

5x + 3y = 7
5x = 7 – 3y
x = $\frac{7-3y}{5}$ ………… (i)

2x -4y = 3 ……… (ii)

Substituting x in eqn. (ii)

$2\left(\frac{7-3y}{5}\right) - 4y = 3$

Multiply 5 through

2(7 – 3y) – 20y = 15

14 – 6y – 20y = 15

-26y = 1

y = $\frac{-1}{26}$

Substituting y into eqn. (i)

x = $\frac{7-3y}{5} = \frac{7-3\left(\frac{-1}{26}\right)}{5} = \frac{26\ x\ 7+-3\ x-1}{5\ x\ 26} = \frac{37}{26}$

∴ $x = \frac{37}{26}$ and y =$\frac{-1}{26}$

(b) 3x + 2y = 8
y = $\frac{8-3x}{2}$………………… (i)

3y + 4x = 11 …………. (ii)

Substituting y in equation (ii)

$3\left(\frac{8-3x}{2}\right) + 4x = 11$

Multiplying 2 through

3(8 – 3x) + 8x = 22

24 – 9x + 8x = 22

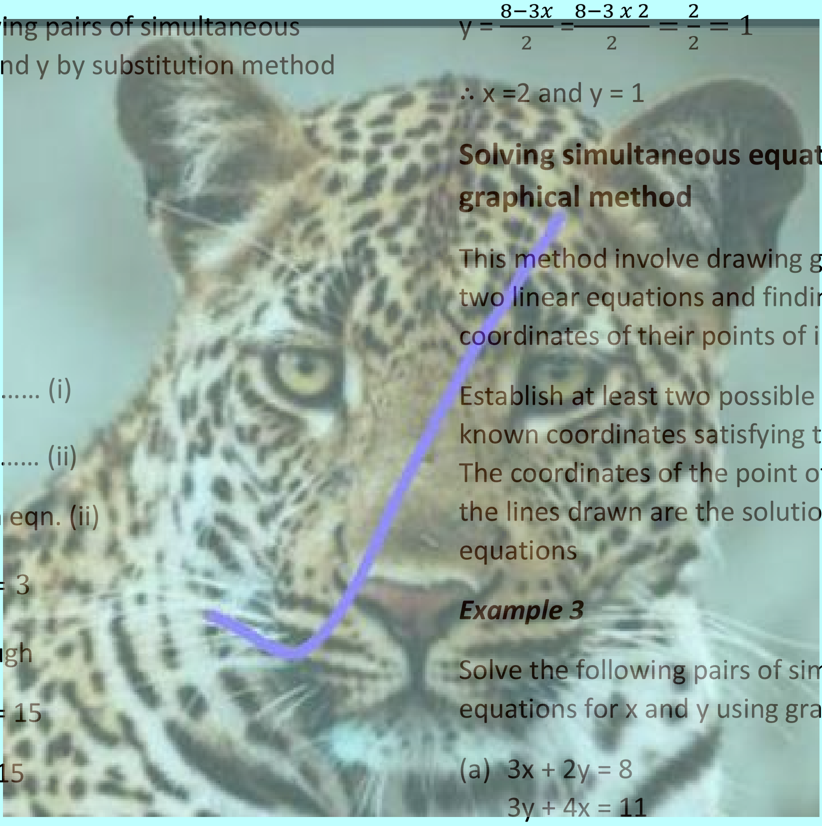
-1x = = -2

x = 2

substituting x into equation (i)

y = $\frac{8-3x}{2} = \frac{8-3\ x\ 2}{2} = \frac{2}{2} = 1$

∴ x =2 and y = 1



Solving simultaneous equations using graphical method

This method involve drawing graphs of the two linear equations and finding the coordinates of their points of intersection

Establish at least two possible points with known coordinates satisfying the equations. The coordinates of the point of intersection of the lines drawn are the solutions to the equations

Example 3

Solve the following pairs of simultaneous equations for x and y using graphical method

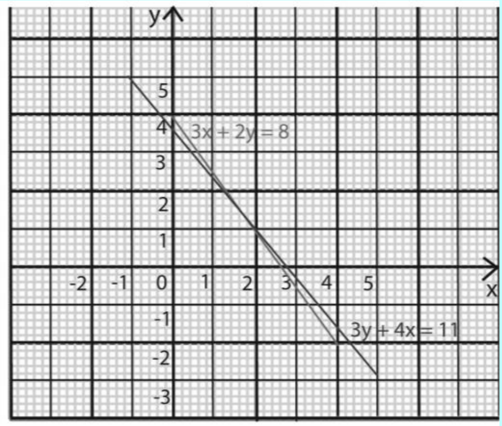
(a) 3x + 2y = 8
3y + 4x = 11

For 3x + 2y = 8

x	0	4
y	4	-2

For 3y + 4x = 11

x	-1	2
y	5	1



From the graph the point of intersection is (2, 1) Hence x = 2 and y = 1

(b) x + y = 4

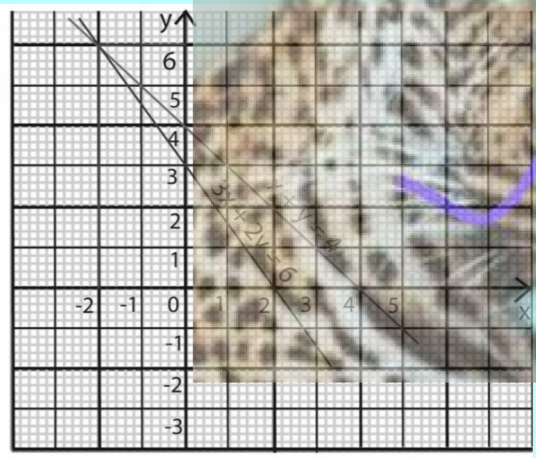
3x + 2y = 6

For x + y = 4

x	0	4
y	4	0

For 3x + 2y = 6

x	0	0
y	3	2



From the graph, the point of intersection is (-2, 6). Hence x = -2, y = 6

Solving simultaneous equations using matrixes

A. Determinant method

Consider the simultaneous equations

$a_1x + b_1y = c_1$

$a_2x + b_1y = d_1$

$$\Rightarrow \quad A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ d_1 \end{pmatrix} \text{.............. (i)}$$

$$B = \begin{pmatrix} c_1 & b_1 \\ d_2 & b_2 \end{pmatrix}$$

Matrix B is obtained by interchanging the coefficients of x in eqn. (i) with the column matrix on the right hand side

$$C = \begin{pmatrix} a_1 & c_1 \\ a_2 & d_1 \end{pmatrix}$$

Matric C is obtained by interchanging the coefficients of y in equation (i) with the column matrix on the right and side of eqn.(i)

For the determinant method

$x = \frac{|B|}{|A|}$ and $y = \frac{|C|}{|A|}$

Example 4

(a) x + 3y – 15 = 0

3x = 17 – 2y

Re-arranging the equation

x + 3y = 15

3x + 2y = 17

Expressing the equation in matrix form

$$\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 15 \\ 17 \end{pmatrix}$$

$$x = \frac{\begin{pmatrix} 15 & 3 \\ 17 & 2 \end{pmatrix}}{\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}}$$

$$x = \frac{(15\ x\ 2)-(17\ x\ 3)}{(1\ x\ 2)-(3\ x\ 3)} = \frac{-21}{-7} = 3$$

$$y = \frac{\begin{pmatrix} 1 & 15 \\ 3 & 17 \end{pmatrix}}{\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}}$$

$$y = \frac{(1\ x\ 17)-(3\ x\ 15)}{(1\ x\ 2)-(3\ x\ 3)} = \frac{-28}{-7} = 4$$

Hence x = 3 and y = 4

(b) 5x + 3y = 7

2x -4y = 3

Expressing the equation in matrix form

$$\begin{pmatrix} 5 & 3 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} \mathrm{x} \\ \mathrm{y} \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$$

$$x = \frac{\begin{pmatrix} 7 & 3 \\ 3 & -4 \end{pmatrix}}{\begin{pmatrix} 5 & 3 \\ 2 & -3 \end{pmatrix}}$$

$$x = \frac{(7\ x{-4})-(3\ x\ 3)}{(5\ x{-4})-(2\ x\ 3)} = \frac{-37}{-26} = \frac{37}{26}$$

$$y = \frac{\begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}}{\begin{pmatrix} 5 & 3 \\ 2 & 3 \end{pmatrix}}$$

$$y = x = \frac{(5\ x\ 3) - (2\ x\ 7)}{(5\ x - 4) - (2\ x - 3)} = \frac{-1}{26}$$

(c) 34 + 3y = 3x
    3x – 4y- 16 = 0

Rearranging the equations

-3x + 3y = -34

3x – 4y = 16

Expressing the equations in matrix form

$$\begin{pmatrix} -3 & 3 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -34 \\ 16 \end{pmatrix}$$

$$x = \frac{\left|\begin{pmatrix} -34 & 3 \\ 16 & -4 \end{pmatrix}\right|}{\left|\begin{pmatrix} -3 & 3 \\ 3 & -4 \end{pmatrix}\right|}$$

$$x = \frac{(-34\ x - 4) - (16\ x\ 3)}{(-3\ x - 4) - (3\ x\ 3)}$$

$$x = \frac{136 - 48}{12 - 9} = \frac{88}{3}$$

$$y = \frac{\left|\begin{pmatrix} -3 & -34 \\ 3 & 16 \end{pmatrix}\right|}{\left|\begin{pmatrix} -3 & 3 \\ 3 & -4 \end{pmatrix}\right|}$$

$$x = \frac{(-3\ x16) - (3\ x - 34)}{(-3\ x - 4) - (3\ x\ 3)}$$

$$x = \frac{-48 + 102}{12 - 9} = \frac{54}{3} = 18$$

Hence x = $\frac{88}{3}$ and y = 18

B. Adjunct method

Consider the simultaneous equations

$a_1x + b_1y = c_1$

$a_2x + b_1y = d_1$

Arranging in matrix form

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ d_1 \end{pmatrix} \text{...............} \text{(i)}$$

If A= $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$

Adjunct A = $\begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}$

To obtain the values of x and y, the adjunct is pre-multiplied on the both sides, i.e.

$$\begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}\begin{pmatrix} c_1 \\ d_1 \end{pmatrix}$$

Example 5

Solve the following equations using matrix method

(a) 4x – 3y = 2
    x + 2y = 1
(b) 4x + 3y = 17
    5x – 2y = 4
(c) 7x – y = -1
    3x – 2y = - 24

Solution

(a) 4x – 3y = 2
    x + 2y = 1

Express the equation in matrix form

$$\begin{pmatrix} 4 & -3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{.....................} \text{(i)}$$

Let $A = \begin{pmatrix} 4 & -3 \\ 1 & 2 \end{pmatrix}$

Adjunct $A = \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix}$

Pre- multiply both sides of eqn. (i) the adjunct matrix A

$$\begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 4 & -3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2\text{x}4 + 3\text{x}1 & 2x - 3 + 3x2 \\ -1x4 + 1x4 & -1x - 3 + 4x2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x2 + 3x1 \\ -1x2 + 4x1 \end{pmatrix}$$

$$\begin{pmatrix} 11 & 0 \\ 0 & 11 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$$

11x = 7

$$x = \frac{7}{11}$$

11y =2

$$y = \frac{2}{11}$$

Hence $= \frac{7}{11}, y = \frac{2}{11}$

(b) 4x + 3y = 17
5x – 2y = 4

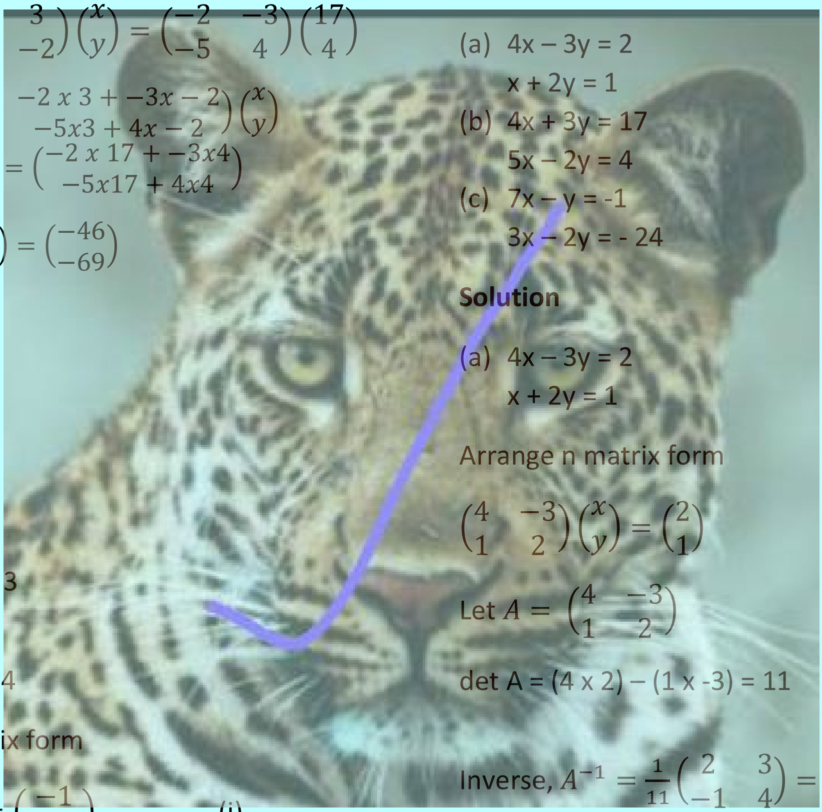
Express in matrix form

$\begin{pmatrix}4 & 3\\ 5 & -2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}17\\ 4\end{pmatrix}$ ………….. (i)

Let $A = \begin{pmatrix}4 & 3\\ 5 & -2\end{pmatrix}$

Adjunct A$\begin{pmatrix}-2 & -3\\ -5 & 4\end{pmatrix}$

Pre- multiply both sides of eqn. (i) the adjunct matrix A



$\begin{pmatrix}-2 & -3\\ -5 & 4\end{pmatrix}\begin{pmatrix}4 & 3\\ 5 & -2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}-2 & -3\\ -5 & 4\end{pmatrix}\begin{pmatrix}17\\ 4\end{pmatrix}$

$\begin{pmatrix}-2\,x4 + -3x5 & -2\,x\,3 + -3x-2\\ -5x4 + 4x5 & -5x3 + 4x-2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}-2\,x\,17 + -3x4\\ -5x17 + 4x4\end{pmatrix}$

$\begin{pmatrix}-23 & 0\\ 0 & -23\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}-46\\ -69\end{pmatrix}$

-23x = -46

$x = \frac{-46}{-23} = 2$

-23y = -69

$y = \frac{-69}{-23}$ = 3

Hence x = 2, y = 3

(c) 7x – y = -1
3x – 2y = - 24

Arrange in matrix form

$\begin{pmatrix}7 & -1\\ 3 & -2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}-1\\ -24\end{pmatrix}$ ……………..(i)

Let $A = \begin{pmatrix}7 & -1\\ 3 & -2\end{pmatrix}$

Adjunct A= $\begin{pmatrix}-2 & 1\\ -3 & 7\end{pmatrix}$

Pre- multiply both sides of eqn. (i) the adjunct matrix A

$\begin{pmatrix}-2 & 1\\ -3 & 7\end{pmatrix}\begin{pmatrix}7 & -1\\ 3 & -2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}-2 & 1\\ -3 & 7\end{pmatrix}\begin{pmatrix}-1\\ -24\end{pmatrix}$

$\begin{pmatrix}-2\,x\,7 + 1\,x\,3 & -2\,x - 1 + 1\,x - 2\\ -3\,x\,7 + 7\,x\,3 & -3\;x - 1 + 7\,x - 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}-2\,x - 1 + 1\,x - 24\\ -3\,x - 1 + 7x - 24\end{pmatrix}$

$\begin{pmatrix}-11 & 0\\ 0 & -11\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}-22\\ -165\end{pmatrix}$

-11x = -22

$x = \frac{-22}{-11} = 2$

-23y = -69

$y = \frac{-165}{-11}$ = 15

Hence x = 2 and y = 15

C. Inverse method

Example 6

(a) 4x – 3y = 2
x + 2y = 1

(b) 4x + 3y = 17
5x – 2y = 4

(c) 7x – y = -1
3x – 2y = - 24

Solution

(a) 4x – 3y = 2
x + 2y = 1

Arrange n matrix form

$\begin{pmatrix}4 & -3\\ 1 & 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}2\\ 1\end{pmatrix}$

Let $A = \begin{pmatrix}4 & -3\\ 1 & 2\end{pmatrix}$

det A = (4 x 2) – (1 x -3) = 11

Inverse, $A^{-1} = \frac{1}{11}\begin{pmatrix}2 & 3\\ -1 & 4\end{pmatrix} = \begin{pmatrix}\frac{2}{11} & \frac{3}{11}\\ \frac{-1}{11} & \frac{4}{11}\end{pmatrix}$

Pre-multiply both sides of eqn. (i) by the inverse matrix A

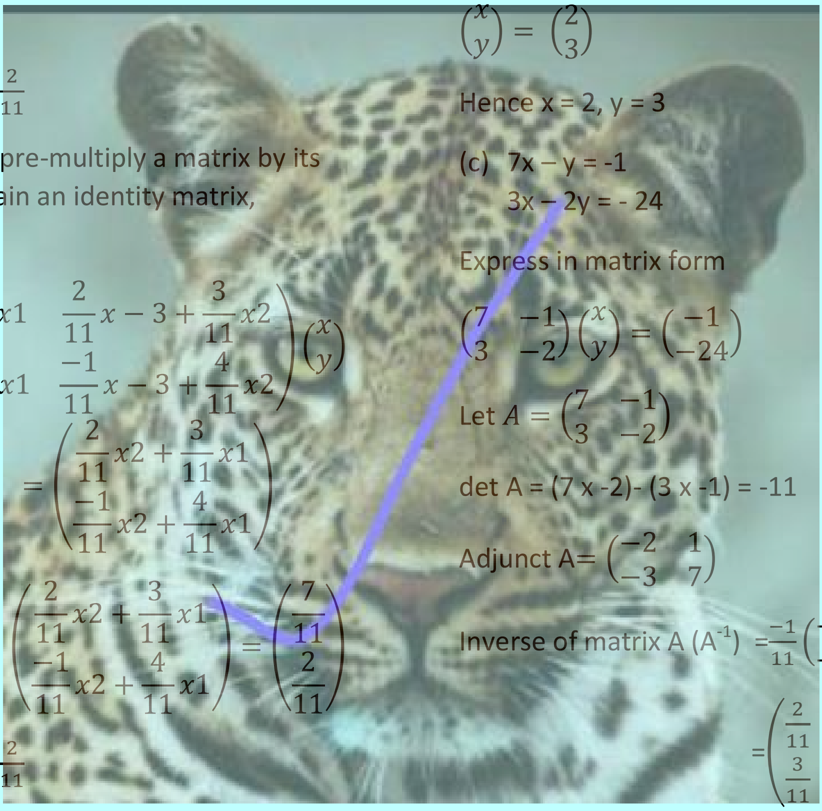
$\begin{pmatrix}\frac{2}{11} & \frac{3}{11}\\ \frac{-1}{11} & \frac{4}{11}\end{pmatrix}\begin{pmatrix}4 & -3\\ 1 & 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}\frac{2}{11} & \frac{3}{11}\\ \frac{-1}{11} & \frac{4}{11}\end{pmatrix}\begin{pmatrix}2\\ 1\end{pmatrix}$

$$\begin{pmatrix} \frac{2}{11}x4+\frac{3}{11}x1 & \frac{2}{11}x-3+\frac{3}{11}x2 \\ \frac{-1}{11}x4+\frac{4}{11}x1 & \frac{-1}{11}x-3+\frac{4}{11}x2 \end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix} \frac{2}{11}x2+\frac{3}{11}x1 \\ \frac{-1}{11}x2+\frac{4}{11}x1\end{pmatrix}$$

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\frac{7}{11}\\ \frac{2}{11}\end{pmatrix}$$

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\frac{7}{11}\\ \frac{2}{11}\end{pmatrix}$$

Hence x = $\frac{7}{11}$, y=$\frac{2}{11}$



Note: when we pre-multiply a matrix by its inverse, we obtain an identity matrix, i.e. $AA^{-1} = 1$

$$\begin{pmatrix} \frac{2}{11}x4+\frac{3}{11}x1 & \frac{2}{11}x-3+\frac{3}{11}x2 \\ \frac{-1}{11}x4+\frac{4}{11}x1 & \frac{-1}{11}x-3+\frac{4}{11}x2 \end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix} \frac{2}{11}x2+\frac{3}{11}x1 \\ \frac{-1}{11}x2+\frac{4}{11}x1\end{pmatrix}$$

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix} \frac{2}{11}x2+\frac{3}{11}x1 \\ \frac{-1}{11}x2+\frac{4}{11}x1\end{pmatrix} = \begin{pmatrix}\frac{7}{11}\\ \frac{2}{11}\end{pmatrix}$$

Hence x = $\frac{7}{11}$, y=$\frac{2}{11}$

(b) 4x + 3y = 17
5x – 2y = 4

Arrange in matrix form

$$\begin{pmatrix}4 & 3\\5 & -2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}17\\4\end{pmatrix}$$ ………………….(i)

Let $A = \begin{pmatrix}4 & 3\\5 & -2\end{pmatrix}$

det A = 4 x -2 – (5 x 3) = -23

Adjunct A = $\begin{pmatrix}-2 & -3\\-5 & 4\end{pmatrix}$

Inverse A (A^{-1})= $\frac{-1}{23}\begin{pmatrix}-2 & -3\\-5 & 4\end{pmatrix} = \begin{pmatrix}\frac{2}{23} & \frac{3}{23}\\ \frac{5}{23} & \frac{-4}{23}\end{pmatrix}$

Pre-multiply both sides of eqn. (i) by the inverse matrix A

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\frac{2}{23} & \frac{3}{23}\\ \frac{5}{23} & \frac{-4}{23}\end{pmatrix}\begin{pmatrix}17\\4\end{pmatrix}$$

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\frac{2}{23}\,x17+ & \frac{3}{23}\,x4\\ \frac{5}{23}x17+ & \frac{-4}{23}x4\end{pmatrix} = \begin{pmatrix}\frac{46}{23}\\ \frac{69}{23}\end{pmatrix}$$

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix}$$

Hence x = 2, y = 3

(c) 7x – y = -1
3x – 2y = - 24

Express in matrix form

$$\begin{pmatrix}7 & -1\\3 & -2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-1\\-24\end{pmatrix}$$

Let $A = \begin{pmatrix}7 & -1\\3 & -2\end{pmatrix}$

det A = (7 x -2)- (3 x -1) = -11

Adjunct A= $\begin{pmatrix}-2 & 1\\-3 & 7\end{pmatrix}$

Inverse of matrix A (A^{-1}) $=\frac{-1}{11}\begin{pmatrix}-2 & 1\\-3 & 7\end{pmatrix}$

$$=\begin{pmatrix}\frac{2}{11} & \frac{-1}{11}\\ \frac{3}{11} & \frac{-7}{11}\end{pmatrix}$$

Pre-multiply both sides of eqn. (i) by the inverse matrix A

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\frac{2}{11} & \frac{-1}{11}\\ \frac{3}{11} & \frac{-7}{11}\end{pmatrix}\begin{pmatrix}-1\\-24\end{pmatrix}$$

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\frac{2}{11}\,x-1+ & \frac{-1}{11}x-24\\ \frac{3}{11}x-1 & \frac{-7}{11}x-24\end{pmatrix} = \begin{pmatrix}\frac{22}{11}\\ \frac{165}{11}\end{pmatrix}$$

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2\\15\end{pmatrix}$$

Hence x = 2 and y 15

Non-linear simultaneous equation

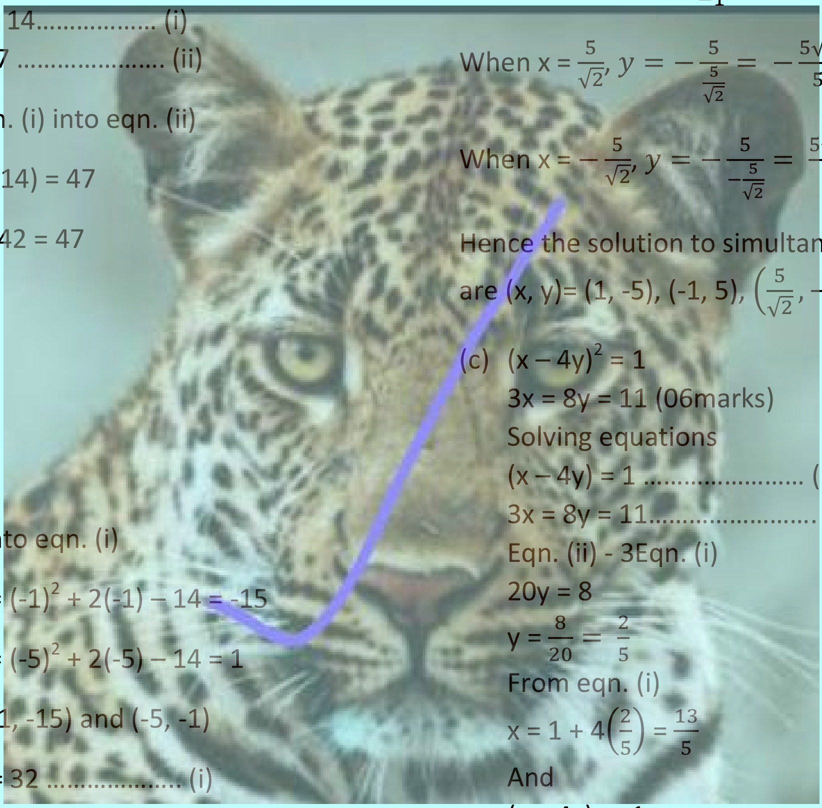
These are solved basically by using substitution method

Example 7

(a) $x^2 + 2x - y = 14$
$2x^2 - 3y = 47$

(b) $2x^2 - xy + y^2 = 32$
$y = -\frac{5}{x}$

Solution



(a) $x^2 + 2x - y = 14$
$y = x^2 + 2x - 14$………………. (i)
$2x^2 - 3y = 47$ ………………….. (ii)

Substituting eqn. (i) into eqn. (ii)

$2x^2 - 3(x^2 + 2x - 14) = 47$

$2x^2 - 3x^2 - 6x + 42 = 47$

$-x^2 - 6x - 5 = 0$

$x^2 + 6x + 5 = 0$

$(x+1)(x+5) = 0$

x = -1 or x = -5

Substituting x into eqn. (i)

When x = -1, $y = (-1)^2 + 2(-1) - 14 = -15$

When x = -5, $y = (-5)^2 + 2(-5) - 14 = 1$

Hence (x, y) = (-1, -15) and (-5, -1)

(b) $2x^2 - xy + y^2 = 32$ ……………….(i)
$y = -\frac{5}{x}$ …………………………….. (ii)

Substituting equation (ii) into eqn. (i)

$2x^2 - x\left(-\frac{5}{x}\right) + \left(-\frac{5}{x}\right)^2 = 32$

$2x^2 + 5 + \frac{25}{x^2} = 32$

$2x^2 + \frac{25}{x^2} - 27 = 0$

$2x^4 - 27x^2 + 25 = 0$

$(x^2 - 1)(2x^2 - 25) = 0$

$x^2 = 1$

$x = \pm 1$

$2x^2 - 25 = 0$

$2x^2 = 25$

$x \pm \frac{5}{\sqrt{2}}$

Substituting for x into eqn. (i)

$y = -\frac{5}{x}$

When x = 1, $y = -\frac{5}{1} = -5$

When x = -1, $y = -\frac{5}{-1} = 5$

When $x = \frac{5}{\sqrt{2}}$, $y = -\frac{5}{\frac{5}{\sqrt{2}}} = -\frac{5\sqrt{2}}{5} = -\sqrt{2}$

When $x = -\frac{5}{\sqrt{2}}$, $y = -\frac{5}{-\frac{5}{\sqrt{2}}} = \frac{5\sqrt{2}}{5} = \sqrt{2}$

Hence the solution to simultaneous equations are (x, y)= (1, -5), (-1, 5), $\left(\frac{5}{\sqrt{2}}, -\sqrt{2}\right)$, $\left(\frac{5}{\sqrt{2}}, \sqrt{2}\right)$,

(c) $(x - 4y)^2 = 1$
$3x = 8y = 11$ (06marks)
Solving equations
$(x - 4y) = 1$ …………………… (i)
$3x = 8y = 11$……………………. (ii)
Eqn. (ii) - 3Eqn. (i)
$20y = 8$
$y = \frac{8}{20} = \frac{2}{5}$
From eqn. (i)
$x = 1 + 4\left(\frac{2}{5}\right) = \frac{13}{5}$
And
$(x - 4y) = -1$ …………………… (i)
$3x = 8y = 11$……………………. (ii)
2(eqn (i)) + eqn. (ii)
$5x = 9$
$x = \frac{9}{5}$
From equation (i)
$4y = \frac{9}{5} + 1$
$y = \frac{7}{10}$
$\therefore$ (x,y) = $\left(\frac{13}{5}, \frac{2}{5}\right), \left(\frac{9}{5}, \frac{7}{10}\right)$

Three linear simultaneous equations n three unknown.

When solving for three unknowns, there must be three equation that will be solved simultaneously. The methods that will be used to such equation are

- Elimination and substitution
- Row reduction to echelon

Elimination and substitution

This involves elimination of one unknown variable so as to remain two unknowns which can easily be solved

Example 7

(a) Solve the simultaneous equations

x – 2y - 2z = 0

2x + 3y + z = 1

3x –y – 3z = 3

X – 2y -2z = 0 ……………………… (i)

2x + 3y + z = 1………………… (ii)

3x –y – 3z = 3 ……………………..(iii)

Eqn. (i) + 2eqn. (ii)

4x + 4y = 2 ……………………………(iv)

3Eqn. (ii) + eqn. (iii)

9x + 8y = 6 ……………………….. (v)

2eqn. (iv) – eqn. (v)

x = -2

Substituting x = -2 into eqn. (iv)

5(-2) + 4y = 2

y = 3

Substituting x = -2 and y = 3 into eqn. (ii)

2(-2) + 3(3) + z =2

z = -4

∴ x = -2, y = 3 and z = -4

(b) 2x = 3y = 4z

$x^2 - 9y^2 – 4z + 8 = 0$

2x = 3y = 4z, substituting 4z = 2x and y = $\frac{2x}{3}$ into the equation $x^2 – 9y^2 – 4z + 8 = 0$

$x^2 – (2x)^2 – 2x + 8 = 0$

$-3x^2 – 2x + 8= 0$

x = $\frac{2\pm\sqrt{(-2)^2-4(-3)(8)}}{2(-3)}$; x = -2 or x = $\frac{4}{3}$

When x = -2; y = $\frac{2\ x\ (-2)}{3}$ = $\frac{-4}{3}$; z= $\frac{2\ x\ (-2)}{4}$ = -1

(x, y, z) = (-2, $\frac{-4}{3}$, -1)

When x = $\frac{4}{3}$; y = $\frac{2\ x\ (\frac{4}{3})}{3}$ = $\frac{8}{9}$; z= $\frac{2\ x\ (\frac{4}{3})}{4}$ = $\frac{2}{3}$

(x, y, z) = ($\frac{4}{3}$, $\frac{8}{9}$, $\frac{2}{3}$)

Alternatively

2x = 3y = 4z, substituting 4z = 3y and x = $\frac{3y}{2}$ into the equation $x^2 – 9y^2 – 4z + 8 = 0$

$\left(\frac{3}{2}y\right)^2 - 9y^2 – 3y + 8 = 0$

$9y^2 – 36y^2 – 12y + 32 = 0$

$-27y^2 – 12y + 32 = 0$

y = $\frac{12\pm\sqrt{(-12)^2-4(-27)(32)}}{2(-27)}$; y = $\frac{-4}{3}$ or x = $\frac{8}{9}$

When y = $\frac{-4}{3}$; x = $\frac{3}{2}\ x\ \frac{-4}{3} = -2$; z= $\frac{3}{4}\ x\ \frac{-4}{3}$ = -1

(x, y, z) = (-2, $\frac{-4}{3}$, -1)

When y = $\frac{8}{9}$; x = $\frac{3}{2}\ x\ \frac{8}{9} = \frac{4}{3}$; z= $\frac{3}{4}\ x\ \frac{8}{9} = \frac{2}{3}$

(x, y, z) = ($\frac{4}{3}$, $\frac{8}{9}$, $\frac{2}{3}$)

Row reduction to Echelon matrix form

This method involves the expression of the three equations into matrix form known as augmented matrix and thereafter transforming the augmented matrix to a unity triangular matrix (a matrix whose elements in the major diagonal are unity and zero below)

Example

Solve the simultaneous equations

(a) x – 2y - 2z = 0

2x + 3y + z = 1

3x –y – 3z = 3

Solution

Expressing the equation in matrix form

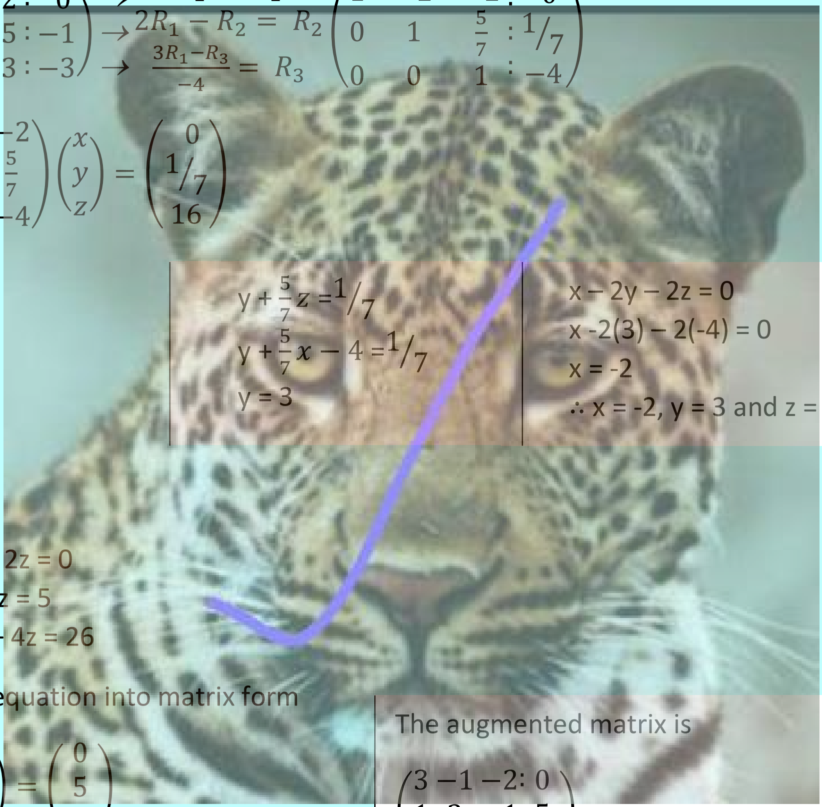
$$\begin{pmatrix} 1 & -2 & -2 \\ 2 & 3 & 1 \\ 3 & -1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

The augmented matrix is

$$\begin{pmatrix} 1 & -2 & -2 : 0 \\ 2 & 3 & 1\ : 1 \\ 3 & -1 & -3 : 3 \end{pmatrix}$$

Transforming augmented matrix to a unity triangular matrix

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\begin{pmatrix} 1 & -2 & -2 : 0 \\ 2 & 3 & 1\ : 1 \\ 3 & -1 & -3 : 3 \end{pmatrix}\begin{array}{r} \rightarrow \quad R_1 = R_1 \\ \rightarrow 2R_1 - R_2 = R_2 \\ \rightarrow 3R_1 - R_3 = R_3 \end{array}\begin{pmatrix} 1 & -2 & -2 : & 0 \\ 0 & -7 & -5 : & -1 \\ 0 & -5 & -3 : & -3 \end{pmatrix}$$



$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\begin{pmatrix} 1 & -2 & -2 : & 0 \\ 0 & -7 & -5 : & -1 \\ 0 & -5 & -3 : & -3 \end{pmatrix}\begin{array}{r} \rightarrow \quad R_1 = R_1 \\ \rightarrow 2R_1 - R_2 = R_2 \\ \rightarrow \frac{3R_1 - R_3}{-4} = R_3 \end{array}\begin{pmatrix} 1 & -2 & -2 : & 0 \\ 0 & 1 & \frac{5}{7} : & 1/7 \\ 0 & 0 & 1 : & -4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & \frac{5}{7} \\ 0 & 0 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1/7 \\ 16 \end{pmatrix}$$

z = -4

$y + \frac{5}{7}z = 1/7$

$y + \frac{5}{7}x - 4 = 1/7$

y = 3

x – 2y – 2z = 0

x -2(3) – 2(-4) = 0

x = -2

∴ x = -2, y = 3 and z = -4

(b) 3x – y – 2z = 0
x +3y – z = 5
2x – y + 4z = 26

Expressing the equation into matrix form

$$\begin{pmatrix} 3 & -1 & -2 \\ 1 & 3 & -1 \\ 2 & -1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 26 \end{pmatrix}$$

The augmented matrix is

$$\begin{pmatrix} 3 & -1 & -2: & 0 \\ 1 & 3 & -1: & 5 \\ 2 & -1 & 4\ : & 26 \end{pmatrix}$$

Transforming the augmented matrix into unity

$$\begin{pmatrix} 3 & -1 & -2: & 0 \\ 1 & 3 & -1: & 5 \\ 2 & -1 & 4\ : & 26 \end{pmatrix}\begin{array}{r} \rightarrow \quad R_1 = R_1 \\ \rightarrow R_1 - 3R_2 = R_2 \\ \rightarrow 2R_1 - 3R_3 = R_3 \end{array}\begin{pmatrix} 3 & -1 & -2 : & 0 \\ 0 & -10 & 1 : & -15 \\ 0 & 1 & -16 : & -78 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -1 & -2 : & 0 \\ 0 & -10 & 1 : & -15 \\ 0 & 1 & -16 : & -78 \end{pmatrix}\begin{array}{r} \rightarrow \quad R_1 \div 3 = R_1 \\ \rightarrow 10R_1 \div -10 = R_2 \\ \rightarrow \frac{R_2 + 10R_3}{159} \end{array}\begin{pmatrix} 1 & \frac{-1}{3} & \frac{-2}{3} & : 0 \\ 0 & 1 & -\frac{1}{10} & : \frac{15}{10} \\ 0 & 0 & 1 & : 5 \end{pmatrix}$$

Z = 5

y - $\frac{1}{10}$ $x5$ = $\frac{15}{10}$
10y - 5 = 15
y = 2

x – $\frac{1}{3}$y – $\frac{2}{3}$z = 0
x - $\frac{1}{3}$ (2) – $\frac{2}{3}$ (5) = 0
x = 4
∴ x = 4, y = 2 and z = 5

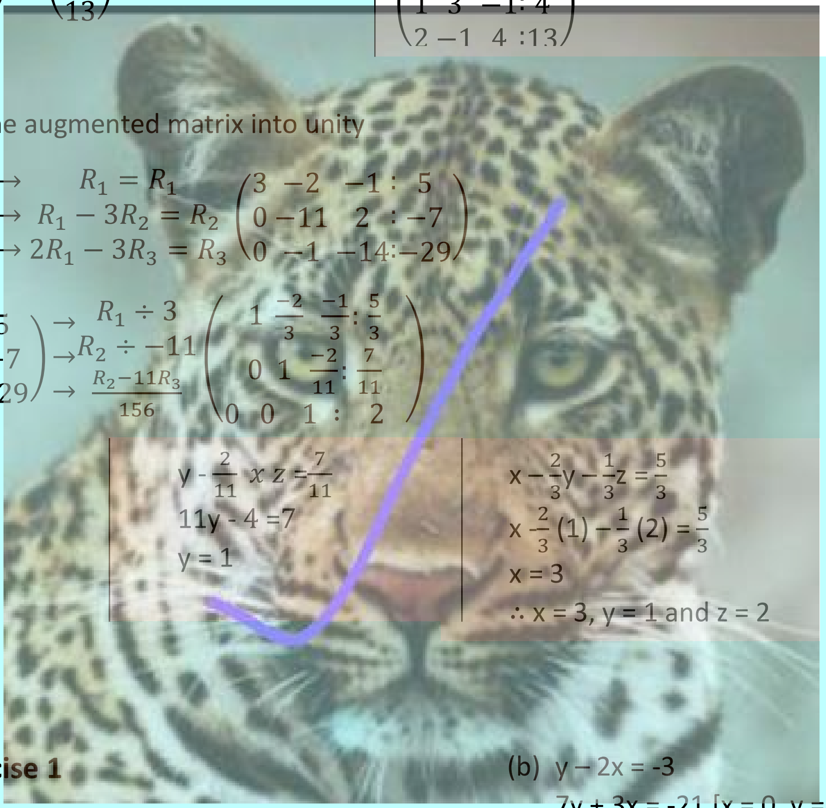
(c) 3x – 2y – z = 5
x + 3y – z = 4
2x – y + 4z = 13 [3, 1, 2]

Expressing the equation into matrix form

$$\begin{pmatrix} 3 & -2 & -1 \\ 1 & 3 & -1 \\ 2 & -1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 13 \end{pmatrix}$$

The augmented matrix is

$$\begin{pmatrix} 3 & -2 & -1 & : 5 \\ 1 & 3 & -1 & : 4 \\ 2 & -1 & 4 & :13 \end{pmatrix}$$



Transforming the augmented matrix into unity

$$\begin{pmatrix} 3 & -2 & -1 & : 5 \\ 1 & 3 & -1 & : 4 \\ 2 & -1 & 4 & :13 \end{pmatrix} \begin{matrix} \rightarrow R_1 = R_1 \\ \rightarrow R_1 - 3R_2 = R_2 \\ \rightarrow 2R_1 - 3R_3 = R_3 \end{matrix} \begin{pmatrix} 3 & -2 & -1 & : 5 \\ 0 & -11 & 2 & : -7 \\ 0 & -1 & -14 & :-29 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -2 & -1 & : 5 \\ 0 & -11 & 2 & : -7 \\ 0 & -1 & -14 & :-29 \end{pmatrix} \begin{matrix} \rightarrow R_1 \div 3 \\ \rightarrow R_2 \div -11 \\ \rightarrow \frac{R_2 - 11R_3}{156} \end{matrix} \begin{pmatrix} 1 & \frac{-2}{3} & \frac{-1}{3} & : \frac{5}{3} \\ 0 & 1 & \frac{-2}{11} & : \frac{7}{11} \\ 0 & 0 & 1 & : 2 \end{pmatrix}$$

Z = 2

y - $\frac{2}{11}$ $x\ z$ = $\frac{7}{11}$
11y - 4 = 7
y = 1

x – $\frac{2}{3}$y – $\frac{1}{3}$z = $\frac{5}{3}$
x - $\frac{2}{3}$ (1) – $\frac{1}{3}$ (2) = $\frac{5}{3}$
x = 3
∴ x = 3, y = 1 and z = 2

Revision exercise 1

1. Using elimination method solve the following pairs of simultaneous equation
   (a) -3x + 2y = -16
   x + 5y = 11 [x = 6, y = 1]
   (b) 3y – 2x = -18
   2y + 3x = -6 [x = 0, y = -3]
   (c) 2x - 3y = 7
   X + 4y = -2 [x = 2, y = -1]
   (d) 5x + 3y = 8
   3x + 2y = 6 [x = -2, y = 6)
2. Using substitution method solve the following pairs of simultaneous equation
   (a) -3x + 2y = 16
   5x + 3y = 33 [x = 6, y = 1]
   (b) y – 2x = -3
   7y + 3x = -21 [x = 0, y = -3]
   (c) 2x + 5y = 26
   3x + 2y = 6 [x = -2, y = 6]
3. Solve the following pairs of simultaneous equation using the matrix method
   (a) 2x – 3y = 7
   2x + 3y = 1 [x =2, y = -1]
   (b) 2x – 7y = 1
   3x + 3y = 15
   (c) 3x – 4y = 5
   6x – 3y = 0 [x = -1, y =-2]
   (d) 3x + 2y = 3
   x -6y = 1[x =1, y = 0]
   (e) 2x + 3y= 1

$3x + y = 5$ [$x = 2$, $y = -1$]

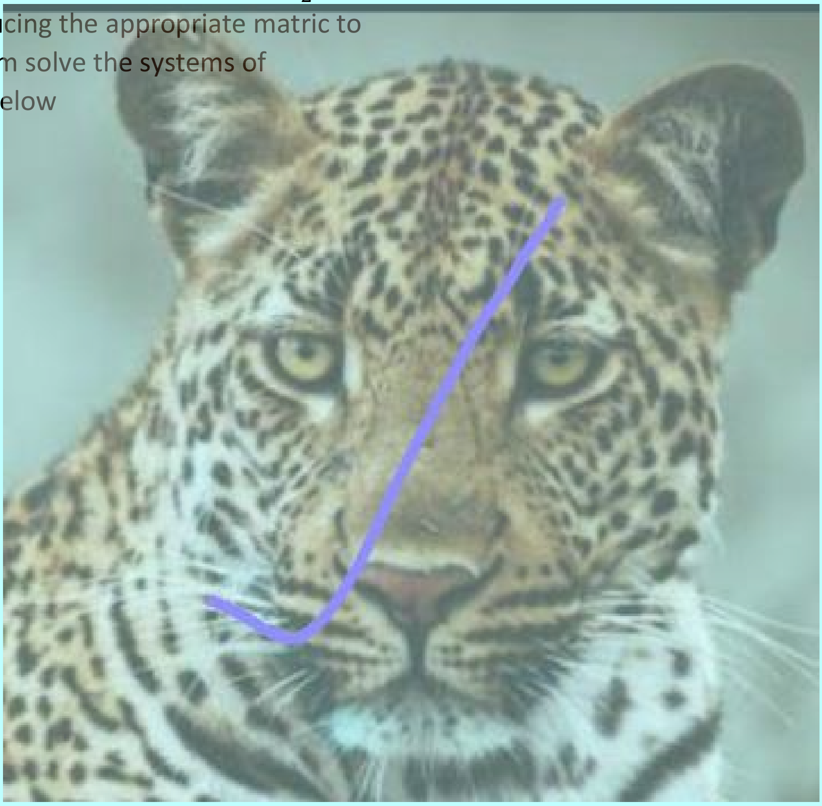
4. Solve the following simultaneous equations elimination and substitution method

(a) $3x - 2y - 2z = -2$
$x + 3y - 3z = -5$
$2x - y + 4z = 26$ [$(x, y, z)=(4, 2, 5)$]

(b) $2x + 2y - 3z = 1$
$3x + 3y - z = 5$
$4x - 2y + 2z = 4$ [$(x, y, z)=(1, 1, 1)$]

(c) $4x - y + 2z = 7$
$x + y + 6z = 2$
$8x + 3y - 10z = -3$ [$(x, y, z)=(1, -2, \frac{1}{2})$]

5. By row reducing the appropriate matric to echelon form solve the systems of equations below



(a) $x + 2y - 2z = 0$
$2x + y - 4z = -1$
$4x - 3y + z = 11$ [(x, y, z), $(3, 1, 2)$]

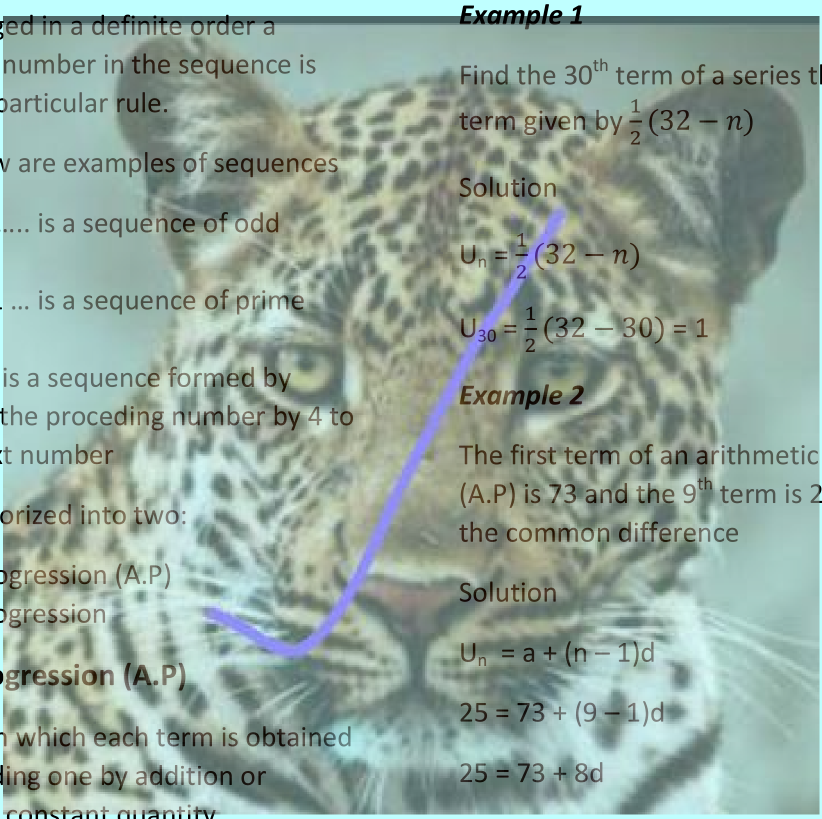
(b) $x - 2y + 3z = 6$
$3x + 4y - z = 3$
$4x + 6y - 5z = 0$ [$(x, y, z)=(2, \frac{-1}{2}, 1)$]

(c) $2x - y + 3z = 10$
$x + 2y - 5z = 9$
$5x + y + 4z$ [$x = 2$, $y = -2$, $z = 1$]

(d) $p + 2q - r = -1$
$3p - q + 2r = 16$
$2p + 3q + r = 3$ [$p = 4$, $q = -2$, $r = 1$]

Series

Introductions



Numbers arranged in a definite order a sequence. Each number in the sequence is derived from a particular rule.

The terms below are examples of sequences

(a) 1, 3, 5, 7, 9 ….. is a sequence of odd numbers
(b) 2, 3, 5, 7, 11 … is a sequence of prime number
(c) 4, 16, 64 …. is a sequence formed by multiplying the proceding number by 4 to give the next number

Series are categorized into two:

- Arithmetic progression (A.P)
- Geometric progression

Arithmetic progression (A.P)

This is a series in which each term is obtained from the preceding one by addition or subtraction of a constant quantity.

The series 1 + 3 + 5 + 7 + 9 ….. is an A.P

Note the following in an A.P

(i) The first term of an A.P is denoted a. the first letter of the English alphabet
(ii) There is a common difference d. in the progression, a = 1 and d = 2.
(iii) Given the first term, a and the common difference, d
1^{st} term = a
2^{nd} term = a + d
3^{rd} term = a + 2d
n^{th} term (U_n) = a + (n – 1)d

Example 1

Find the 30^{th} term of a series that has an n^{th} term given by $\frac{1}{2}(32-n)$

Solution

$U_n = \frac{1}{2}(32-n)$

$U_{30} = \frac{1}{2}(32-30) = 1$

Example 2

The first term of an arithmetic progression (A.P) is 73 and the 9^{th} term is 25. Determine the common difference

Solution

U_n = a + (n – 1)d

25 = 73 + (9 – 1)d

25 = 73 + 8d

d = -6

Example 3

The 3^{rd}, 5^{th} and 8^{th} terms of A.P are 3n + 8, n + 34, and n^3 + 15 respectively. Find the value of n and hence the common difference of the A.P

Solution

a + 2d = 3n + 8 ……………… (i)

a + 4d = n + 24 ………………(ii)

a + 7d = n^3 + 15 …………….(iii)

Eqn. (ii) – eqn. (i)

$2d = -2n + 16$

$d = -n + 8$ .... (iv)

eqn. (iii) – eqn. (i)

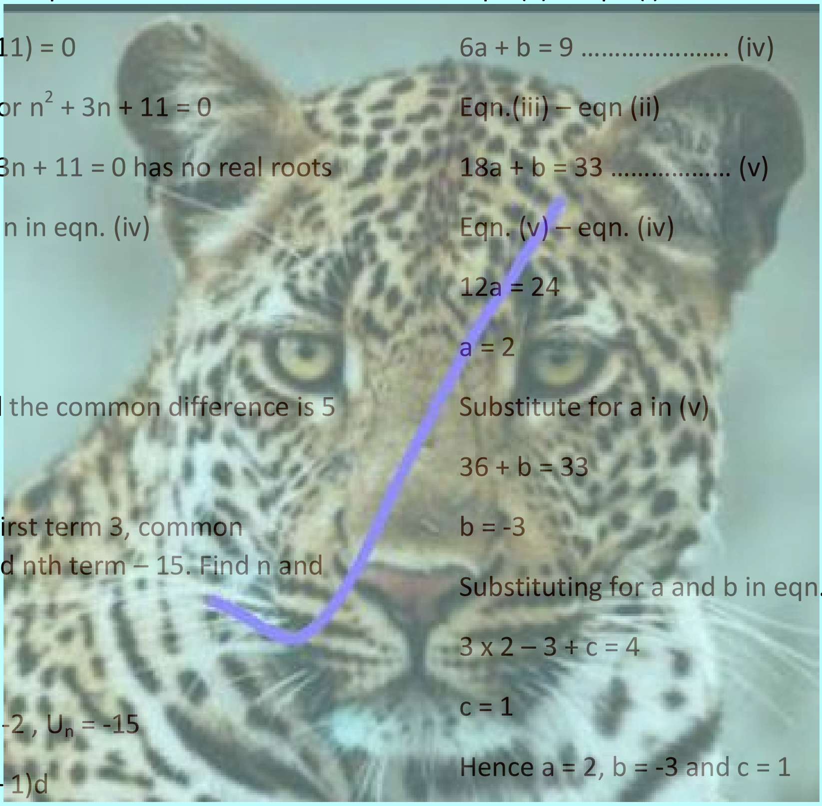
$5d = n^3 - 3n + 7$ .... (v)

Substituting eqn. (iv) into eqn. (v)

$5(-n + 8) = n^3 - 3n + 7$

$n^3 + 2n - 33 = 0$

By factorizing the equation



$(n- 3)(n^2 + 3n + 11) = 0$

Either $n - 3 = 0$ or $n^2 + 3n + 11 = 0$

$n = 3$ since $n^2 + 3n + 11 = 0$ has no real roots

Substituting for n in eqn. (iv)

$d = -n + 8$

$d = -3 + 8 = 5$

Hence n = 3 and the common difference is 5

Example 4

An A.P has the first term 3, common difference -2 and nth term – 15. Find n and the $(n -3)^{th}$ term

Solution

Given, $a = 3$, $d = -2$, $U_n = -15$

But $U_n = a + (n - 1)d$

$-15 = 3 + (n - 1) \times (-2)$

$n = 10$

substitute n – 3 for n

$U_{(n-3)} = 3 + [(10 - 3) - 1](-2)$

$= -9$

Example 5

The n^{th} term of a series is $Un = a3^n + bn + c$ given that $U_1 = 4$, $U_2 = 13$ and $U_3 = 46$, find the values of a, b and c.

Solution

By substituting for n

$n = 1$:

$3a + b + c = 4$ ....(i)

$n =2$

$9a + 2b + c = 13$ ....(ii)

$n = 3$

$27a + 3b + 2 = 46$ ....(iii)

Eqn. (ii) – eqn. (i)

$6a + b = 9$ .... (iv)

Eqn.(iii) – eqn (ii)

$18a + b = 33$ .... (v)

Eqn. (v) – eqn. (iv)

$12a = 24$

$a = 2$

Substitute for a in (v)

$36 + b = 33$

$b = -3$

Substituting for a and b in eqn. (i)

$3 \times 2 - 3 + c = 4$

$c = 1$

Hence a = 2, b = -3 and c = 1

The sum of the first n terms of an A.P

There are two formulas used for finding the sum of the first n terms of the A.P depending on the terms given

Formula A

If the first term, a and common difference are given, then sum (S_n) of the first n terms is given by

$S_n = \frac{n}{2}(2a + (n - 1)d)$

Formula B

If the first term is a and the last term is L, then sum of the first n terms (S_n)is given by

$S_n = \frac{n}{2}(a + L)$

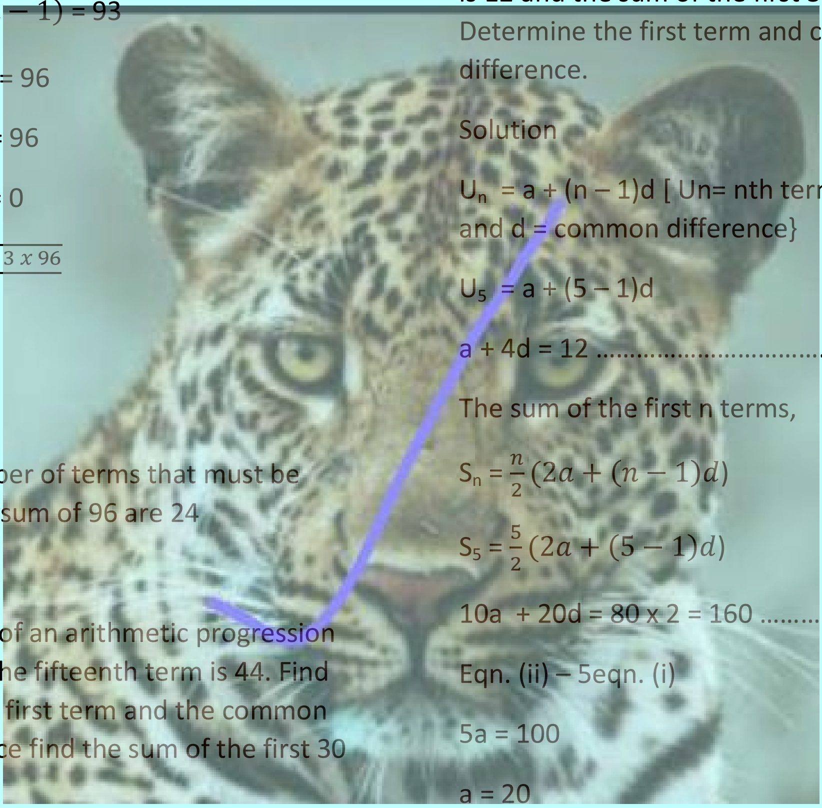
Example 6

The first term of A.P is 73 and the common difference is -6, find the number of terms that must be added to give a sum of 96

Solution

$S_n = \frac{n}{2}(2a + (n-1)d$

$\frac{n}{2}(2x\ 73 - 6(n-1) = 93$



n(73 – 3(n – 1)) = 96

73n – $3n^2$ + 3n = 96

$3n^2$ – 76n + 96 = 0

$n = \frac{76 \pm \sqrt{76^2 - 4x\ 3\ x\ 96}}{2\ x\ 3}$

$n = \frac{76 \pm 68}{6}$

n = 24

Hence the number of terms that must be added to give a sum of 96 are 24

Example 7

The tenth term of an arithmetic progression (A.P) is 29 and the fifteenth term is 44. Find the value of the first term and the common difference. Hence find the sum of the first 30 terms

Solution

U_n = a + (n – 1)d

U_{10} = a + (10 – 1)d

a + 9d = 29 …………….(i)

U_{15} = a + (15 – 1)d

a + 14d = 44 ……….. (ii)

Eqn (ii) – eqn. (i)

5d = 15

d = 3

Substituting d in eqn. (i0

a + 9 x 2 = 29

a = 2

Sum of the first 30 terms

$S_n = \frac{n}{2}(2a + (n-1)d)$

$S_n = \frac{30}{2}(2\ x\ 2 + 3(30-1))$ = 1365

Example 8

The 5th term of an arithmetic progression (A.P) is 12 and the sum of the first 5 terms is 80. Determine the first term and common difference.

Solution

U_n = a + (n – 1)d [Un= nth term, a = first term and d = common difference}

U_5 = a + (5 – 1)d

a + 4d = 12 …………………………….(i)

The sum of the first n terms,

$S_n = \frac{n}{2}(2a + (n-1)d)$

$S_5 = \frac{5}{2}(2a + (5-1)d)$

10a + 20d = 80 x 2 = 160 ………(ii)

Eqn. (ii) – 5eqn. (i)

5a = 100

a = 20

Substituting for a in eqn. (i)

20 + 4d = 12

d = -2

Hence the first term = 20 and the common difference = -2

Example 9

(a) Prove that $\sum_{r=1}^{n} r = \frac{n}{2}(n+1)$

Solution

$S_n = 1 + \quad 2 \quad + \quad 3 \quad + ... + n$

$+ \; S_n = n \; + (n - 1) + (n - 2) + ... + \; 1$

$2S_n = (n+1) + (n+1) + (n+1) + ... + (n+1)$

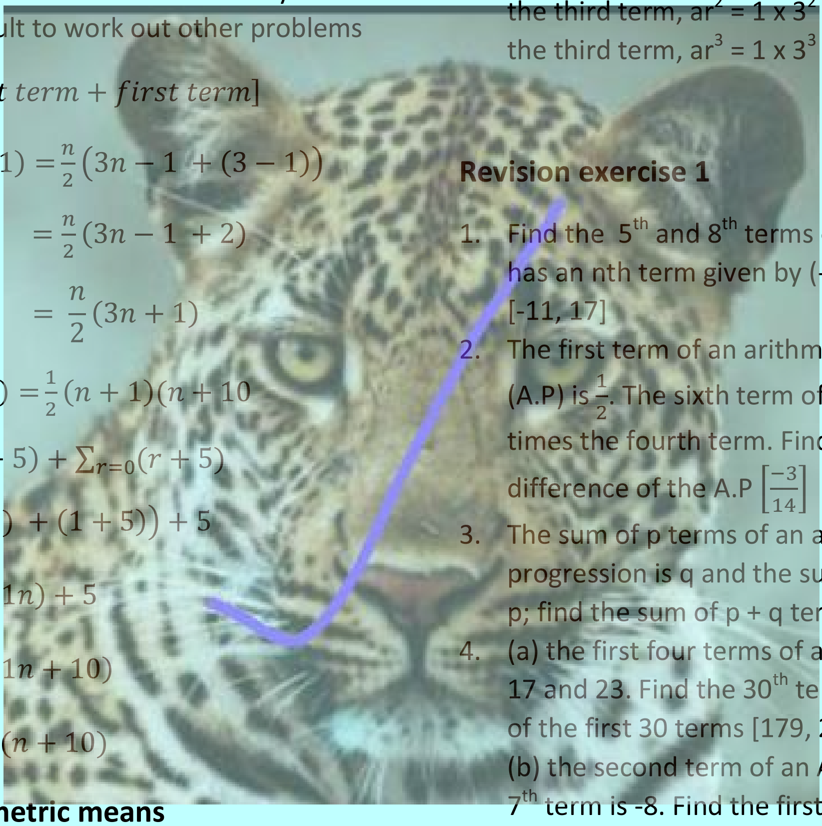
$2S_n = n(n + 1)$

$Sn = \frac{n}{2}(n+1)$

(b) Use your answer in (a) to deduce

(i) $\sum_{r=1}^{n}(3r-1) = \frac{n}{2}(3n+1)$

Note to deduce is to use the already existing result to work out other problems



$S_n = \frac{n}{2}[last\ term + first\ term]$

$\sum_{r=1}^{n}(3r-1) = \frac{n}{2}\left(3n - 1 + (3-1)\right)$

$= \frac{n}{2}(3n - 1 + 2)$

$= \frac{n}{2}(3n+1)$

(ii) $\sum_{r=0}^{n}(r+5) = \frac{1}{2}(n+1)(n+10$

$= \sum_{r=1}^{n}(r+5) + \sum_{r=0}(r+5)$

$= \frac{n}{2}\left((n+1) + (1+5)\right) + 5$

$= \frac{1}{2}(n^2 + 11n) + 5$

$= \frac{1}{2}(n^2 + 11n + 10)$

$= \frac{1}{2}(n+1)(n+10)$

Inserting geometric means

Like for A.Ps, the terms inserted between given two values of a G.P are known as geometric means.

If n terms are inserted, then the total number of terms will be n+ 2 with the two extreme values representing the first and last terms respectively

Example 10

(a) Insert two geometric means between 2 and 16

Solution

1^{st} term a = 2

4^{th} term, $ar^3 = 16$

$2(r^3) = 18$

$r = 2$

the second term, ar = 2 x 2 = 4

the third term, $ar^2 = 2 \times 2^2 = 8$

(b) Insert three geometric means between 1 and 81

a = 1

the 5^{th} term $ar^4 = 81$

$1(r^4) = 81$

$r = 3$

the second term, ar = 1 x 3 = 3

the third term, $ar^2 = 1 \times 3^2 = 9$

the third term, $ar^3 = 1 \times 3^3 = 27$

Revision exercise 1

1. Find the 5^{th} and 8^{th} terms of a series that has an nth term given by $(-1)^n(2n + 1)$ [-11, 17]
2. The first term of an arithmetic progression (A.P) is $\frac{1}{2}$. The sixth term of th A.P is four times the fourth term. Find the common difference of the A.P $\left[\frac{-3}{14}\right]$
3. The sum of p terms of an arithmetic progression is q and the sum of q terms is p; find the sum of p + q terms
4. (a) the first four terms of an A.P are 5, 11, 17 and 23. Find the 30^{th} term and the sum of the first 30 terms [179, 2760]
   (b) the second term of an A.P is 7 and the 7^{th} term is -8. Find the first term, common difference and the sum of the first 14 terms [10, -3, -133]
5. (a) An A.P has the first term of 2 and common difference 5. Given that the sum of the first n terms of the progression is 119, calculate n [7]
   (b) the sum of the first five terms of an A.P is $\frac{65}{2}$. Also , five times the 7^{th} term is the same as six times the second term. Find the first term and the common difference $\left[a = 6, d = \frac{1}{4}\right]$

6. The sum of the first n terms of the a series is n(n + 2). Find the first three terms [3, 5, 7]
7. In an A.P, the 1st term is 13 and 15th term is 11. Find the common difference and sum of the first 20 terms [7, 1590]
8. (a) Show that $\ln 2^r$, r = 1, 2,3, is an arithmetic progression
   (b) find the sum of the first 10 terms of the progression [38.1231]
   (c) Determine the least value of m for which the sum of the first 2m terms exceeds 883.7 [25]
9. In an arithmetic progression $u_1 + u_2 + u_3 + u_4 = 15$ and $u_{16} = -3$. Find the greatest integer N such that $U_N \geq 0$. Determine the sum of the first N terms of the progression. [N = 14, $S_{14} = 136.5$]

Geometric progression (G.P)

It is a series in which each term is obtained from the preceding one by multiplication or division by a constant quantity.

Observations

- The first term of G.P is also denoted, a
- The common ratio is r
- Given a and r
  1st term = a
  2nd term = ar
  3rd term = ar^2
  4th term = ar^3
  nth term = ar^{n-1}

The sum (S_n) of the first n terms of G.P

The sum (S_n) of the first n terms of G.P is given by

$S_n = \frac{a(r^n-1)}{r-1}$ for r > 1 or

$S_n = \frac{a(1-r^n)}{1-r}$ for r< 1

The geometric mean (G.M)

Suppose three numbers a, b and c are consecutive terms of GP, then, the middle term is the geometric mean.

The common ratio, $r = \frac{b}{a} = \frac{c}{b}$

⇨ $b^2 = ac$

$b = \sqrt{ac}$, the geometric mean

Example 10

A GP has 3rd term 7 and 5th term 847. Find the possible values of the common ratio and the corresponding 4th terms

Solution

$U_3 = ar^2 = 7$

⇨ $ar^2 = 7$ ……………….(i)

$U_5 = ar^4 = 847$

⇨ $ar^4 = 847$ …………..(ii)

Eqn. (ii) ÷ eqn. (i)

$r^2 = 121$

$r = \pm 11$

From eqn (i)

$a = \frac{7}{121}$

The 4th term, $U_4 = ar^3$

If r = 11, $U_4 = \frac{7}{121}(11)^3 = 77$

If r = -11, $U_4 = \frac{7}{121}(-11)^3 = -77$

Example 11

In a G.P the 2nd term is 15 and the 5th term is -405. Find the sum of the first 8 terms

Solution

$U_2 = ar = 15$

⇨ $ar = 15$ …………………….(i)

$U_5 = ar^4 = -405$

⇨ $ar^4 = -405$

Eqn. (ii) ÷ eqn. (i)

$r^3 = -27$

r = -3

From eqn. (i), a = -5

Since r < 1

$S_n = \frac{a(1-r^n)}{r-1} = \frac{-5(1-(-3)^8)}{1-(-3)} = \frac{32800}{4} = 8200 =$

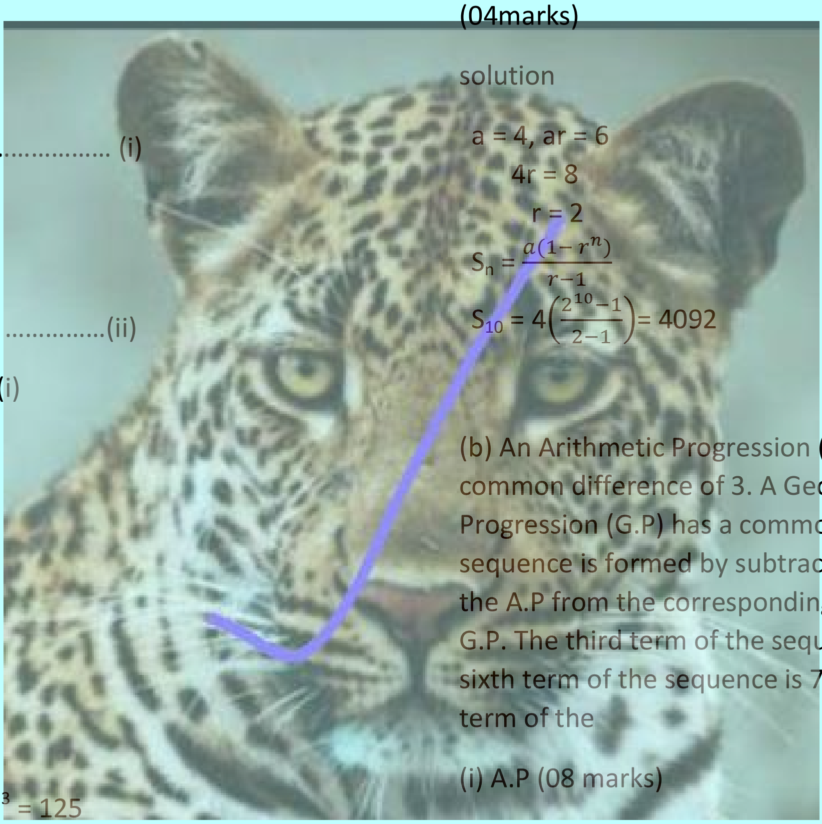
Example 12

In the geometric series $u_1 + u_2 + u_3 + \ldots$

$u_1 + u_3 = 26$ and $u_3 + u_5 = 650$.

Find the possible values of u_4

Solution



$u_1 + u_3 = 26$

$a + ar^2 = 26$

$a(1 + r^2) = 26$ ………………… (i)

$u_3 + u_5 = 650$.

$ar^2 + ar^4 = 650$

$ar^2(1 + r^2) = 650$ ……………(ii)

Eqn. (ii) ÷ eqn. (i)

$r^2 = 25$

$r = \pm 5$

From eqn. (i)

$a(1 + 25) = 26$

a = 1

$u_4 = ar^3$

If r = 5; $u_4 = a(5)^3 = 125$

If r = 5; $u_4 = a(-5)^3 = -125$

Example 13

In a Geometric Progression (G.P), the difference between the fifth and the second term is 156. The difference between the seventh and the fourth is 1404 Find the possible values of the common ratio.

$U_5 - U_2 = 156$
$ar^4 - ar = 156$
$ar(r^3 - 1) = 156$ ……………. (i)
$U_7 - U_4 = 156$
$Ar^6 - ar^3 = 1404$
$ar^3(r^3 - 1) = 156$ ……………. (ii)
Eqn. (ii) ÷ eqn. (i)
$\frac{ar^3(r^3-1)}{ar(r^3-1)} = \frac{1404}{156}$
$r^2 = 9$
$r = \pm 3$
∴ r = 3 and r = -3

Example 14

(a) The first three terms of a Geometric progression (G.P) are 4, 8 and 16. Determine the sum of the first ten terms of the G.P. (04marks)

solution

a = 4, ar = 6
4r = 8
r = 2
$S_n = \frac{a(1-r^n)}{r-1}$
$S_{10} = 4\left(\frac{2^{10}-1}{2-1}\right) = 4092$

(b) An Arithmetic Progression (A.P) has a common difference of 3. A Geometric Progression (G.P) has a common ration of 2. A sequence is formed by subtracting the term of the A.P from the corresponding terms of the G.P. The third term of the sequence is 4. The sixth term of the sequence is 79. Find the first term of the

(i) A.P (08 marks)

(ii) G.P (06 mars)

A.P
x, x+3, x +6, x + 9, x + 12, x+ 15, ……
G.P
y, 2y, 4y, 8y, 16y, 32y, ….
$4y - (x + 6) = 4$
$4y - x = 10$ ……………..(i)
$32y - (x + 15) = 79$
$32y - x = 94$…………. (ii)
Eqn. (ii) – Eqn. (i)
28y = 84, => y = 3
Substituting for y into eqn. (i)
12 – x = 10

x =2

(i) A.P, $U_1 = 2$

(ii) G.P, $U_1 = 3$

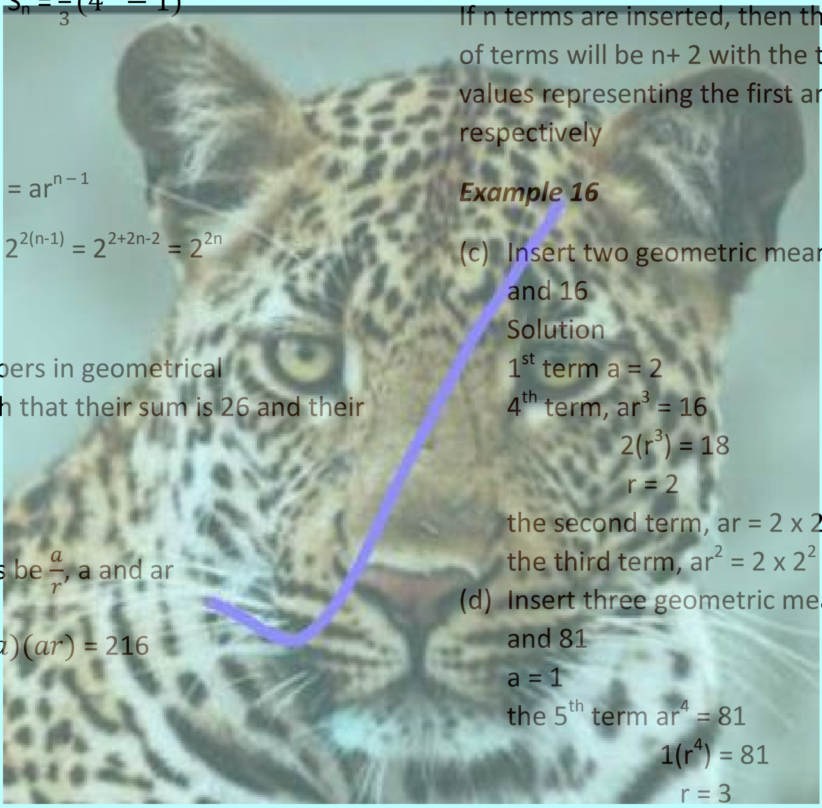
Example 15

The sum of the first n terms of a geometric progression (G.P) is $\frac{4}{3}(4^n - 1)$. Find the n^{th} term as an integral power of 2

Solution

$S_n = \frac{a(1-r^n)}{r-1}$

Comparing with $S_n = \frac{4}{3}(4^n - 1)$



a = 4

r – 1 = 3, r = 4

The n^{th} term, $U_n = ar^{n-1}$

$= 4 \times 4^{n-1} = 2^2 \times 2^{2(n-1)} = 2^{2+2n-2} = 2^{2n}$

Example 16

Find three numbers in geometrical progression such that their sum is 26 and their product is 216

Solution

Let the numbers be $\frac{a}{r}$, a and ar

Product = $\left(\frac{a}{r}\right)(a)(ar) = 216$

$a^3 = 216 = 6^3$

a = 6

$\therefore$ the terms are $\frac{6}{r}, 6 \text{ and } 6r$

Sum of terms $\frac{6}{r} + 6 + 6r = 26$

⇨ $6r^2 - 26r + 6 = 0$

$3r^2 - 13r + 3 = 0$

$(r - 3)(3r - 1) = 0$

Either $r - 3 = 0$; r = 3

Or $3r - 1 = 0$; $r = \frac{1}{3}$

When r = 3

the terms are $\frac{6}{3} = 2, 6 \text{ and } 6x3 = 18$

when $r = \frac{1}{3}$

the terms are $6 \div \frac{1}{3} = 18, 6 \text{ and } 6\ x\ \frac{1}{3} = 2$

Hence the terms in their order are 2, 6, 18 or 18, 6, 2

Inserting geometric means

Like for A.Ps, the terms inserted between given two values of a G.P are known as geometric means.

If n terms are inserted, then the total number of terms will be n+ 2 with the two extreme values representing the first and last terms respectively

Example 16

(c) Insert two geometric means between 2 and 16

Solution

1^{st} term a = 2

4^{th} term, $ar^3 = 16$

$2(r^3) = 18$

r = 2

the second term, ar = 2 x 2 = 4

the third term, $ar^2 = 2 \times 2^2 = 8$

(d) Insert three geometric means between 1 and 81

a = 1

the 5^{th} term $ar^4 = 81$

$1(r^4) = 81$

r = 3

the second term, ar = 1 x 3 = 3

the third term, $ar^2 = 1 \times 3^2 = 9$

the third term, $ar^3 = 1 \times 3^3 = 27$

Mixed terms of A.P and G.P

These are problems involving both A.Ps and G.Ps. when handling we make use of their respective properties.

Example 17

A geometric progression (G.P) and an arithmetic progression (A.P) have the same first term. The sum of their first, second and third terms are 6, 10.5 and 18 respectively. Calculate the sum of their 5^{th} terms.

Solution

Terms	G.P	A.P	Sum
1st	a	a	2a = 6 …………………(i)
2nd	a + d	ar	a + d + ar = 10.5 ….(ii)
3^{rd}	a + 2d	ar^2	$a + 2d + ar^2 = 18$ ..(iii)

From eqn. (i): 2a = 6; a = 3

From eqn. (ii): 3 + d + 3r = 10.5

d + 3r = 7.5 ………………. (iv)

From eqn. (iii) $3 + 2d + 3r^2 = 18$

$2d + 3r^2 = 15$ …………….(v)

Eqn. (v) – 2eqn. (iv)

$3r^2 – 6r = 0$

3r(r -2) = 0

r -2 = 0

r = 2

Substitute for r into eqn. (iv)

d + 6 = 7.5

d = 1.5

Sum of their fifth terms

$= (a + 4d) + ar^4$

$= (3 + 4 \times 1.5) + 3 \times 2^4 = 57$

Example 18

The 1^{st}, 4thand 8^{th} terms of A.P form a G.P. if the first term is 9, find the

(i) Common difference of the A.P
(ii) Common ratio of the G.P
(iii) Difference in sums of the first 6 terms of the progressions.

Solution

Given that a, a + 3d, a + 7d form a G.P

Substituting for a = 9, the terms are

9, 9 + 3d, 9 + 7d

For a G.P, r = $\frac{9+3d}{9} = \frac{9+7d}{9+3d}$

⇨ (9 + 3d)2 = 9(9 + 7d)
$81 + 54d + 9d^2 = 81 + 63d$
$9d^2 – 9d = 0$
9d(d – 1) = 0
Either d – 1 = 0; d = 1
Or d = 0

When d = 0 all terms of A.P are equal

Hence the common difference d = 1

Example 19

(a) The sum of the first m terms of a progression is m(2m + 11)
(i) Show that the progression is an A.P
(ii) Determine the nth term of the progression

Solution

Given $S_m = m(2m + 11)$

First term = $S_1 = 1(2 \times 1 + 11) = 13$

$S_2 = 2(2 \times 2 + 11) = 30$

Second term = 30 – 13 = 17

$S_3 = 3(2 \times 3 + 11) = 51$

Third term = 51 – 20 = 21

The progression is 13, 17, 21, ….. Hence A.P with the first term 13 and common difference, d= 4

(ii) $U_n = S_n – S_{n-1}$

$= n(2n + 11) – (n – 1)(2(n -1) + 11)$

= 9 + 4n

Example 20

(a) The first, second and last term of an A.P are a, b, c respectively. Prove that the sum of all terms is $\frac{(a+b)(b+c-2a)}{2(b-a)}$

Solution

If the 1^{st} term is a and the second term is b; the common difference, d = (b – a)

Last term (nth terms c = a + (n – 1)d

c = a + (n – 1)(b – 1)

i.e. n – 1 = $\frac{c-a}{b-a}$ => n = $\frac{b+c-2a}{b-a}$

but S_n = $\frac{1}{2}n(a+L)$

$=\frac{1}{2}\left(\frac{b+c-2a}{b-a}\right)(a+c)$

$=\frac{(a+b)(b+c-2a)}{2(b-a)}$

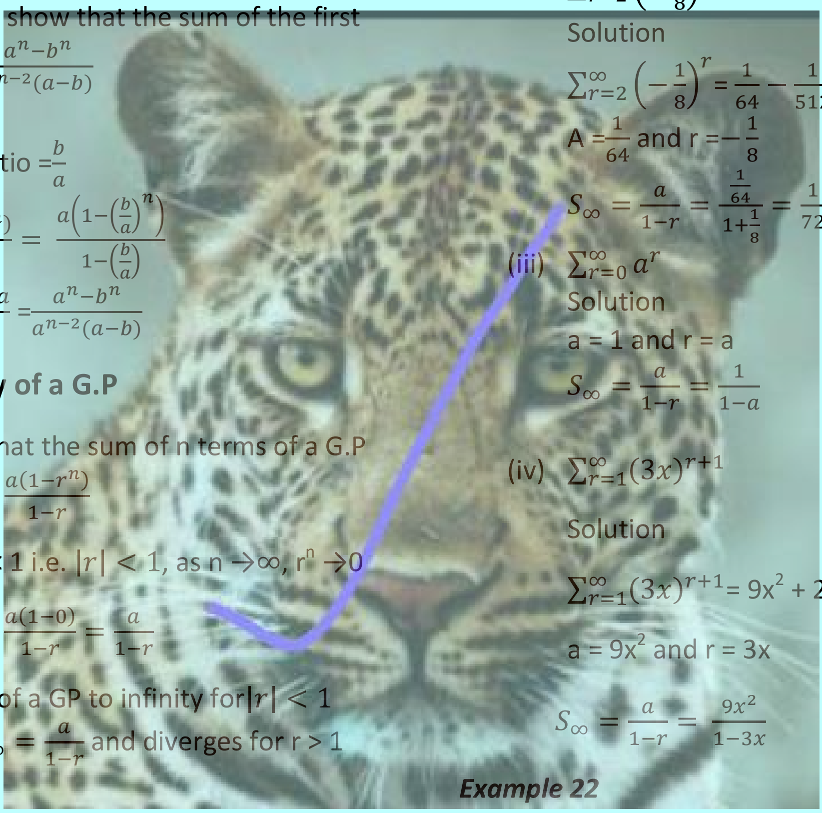
(b) The first, second and last terms of a GP are a and b. show that the sum of the first n terms is $\frac{a^n-b^n}{a^{n-2}(a-b)}$

Solution

Common ratio $=\frac{b}{a}$

$S_n=\frac{a(1-r^n)}{1-r}=\frac{a\left(1-\left(\frac{b}{a}\right)^n\right)}{1-\left(\frac{b}{a}\right)}$

$=\frac{a(a^n-b^n)a}{a^n(a-b)}=\frac{a^n-b^n}{a^{n-2}(a-b)}$



Sum to infinity of a G.P

We have seen that the sum of n terms of a G.P for r < 1 is $S_n=\frac{a(1-r^n)}{1-r}$

Now for -1 < n < 1 i.e. $|r|<1$, as n →∞, r^n →0

Therefore $S_n=\frac{a(1-0)}{1-r}=\frac{a}{1-r}$

Hence the sum of a GP to infinity for $|r|<1$ converges to $S_\infty=\frac{a}{1-r}$ and diverges for r > 1 and r < 1

Example 21

(a) Calculate the sum to infinity of the following terms

(i) $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots.$

Solution

a = 1 and r = $\frac{1}{2}$

$S_\infty=\frac{a}{1-r}=\frac{1}{1-\frac{1}{2}}=2$

(ii) $\frac{1}{5}+\frac{1}{25}+\frac{1}{125}+\cdots$

Solution

a = $\frac{1}{5}$ and r = $\frac{1}{5}$

$S_\infty=\frac{a}{1-r}=\frac{\frac{1}{5}}{1-\frac{1}{5}}=\frac{1}{4}$

(b) Work out the following

(i) $\sum_{r=0}^{\infty}\left(\frac{1}{3}\right)^r$

Solution

$\sum_{r=0}^{\infty}\left(\frac{1}{3}\right)^r=1+\frac{1}{3}+\frac{1}{9}+\cdots$

a = 1 and r = $\frac{1}{3}$

$S_\infty=\frac{a}{1-r}=\frac{1}{1-\frac{1}{3}}=\frac{3}{2}$

(ii) $\sum_{r=2}^{\infty}\left(-\frac{1}{8}\right)^r$

Solution

$\sum_{r=2}^{\infty}\left(-\frac{1}{8}\right)^r=\frac{1}{64}-\frac{1}{512}+\frac{1}{4096}-\cdots$

A = $\frac{1}{64}$ and r = $-\frac{1}{8}$

$S_\infty=\frac{a}{1-r}=\frac{\frac{1}{64}}{1+\frac{1}{8}}=\frac{1}{72}$

(iii) $\sum_{r=0}^{\infty}a^r$

Solution

a = 1 and r = a

$S_\infty=\frac{a}{1-r}=\frac{1}{1-a}$

(iv) $\sum_{r=1}^{\infty}(3x)^{r+1}$

Solution

$\sum_{r=1}^{\infty}(3x)^{r+1}$ = $9x^2+27x^3+81x^4+\ldots$

a = $9x^2$ and r = 3x

$S_\infty=\frac{a}{1-r}=\frac{9x^2}{1-3x}$

Example 22

(a) Express the following as fractions using approach of sum of a G.P to infinity

(i) $0.\dot{4}$

Solution

$0.\dot{4}=\frac{4}{10}+\frac{4}{100}+\frac{1}{1000}+\cdots$

$=\frac{4}{10}\left(1+\frac{1}{10}+\frac{1}{100}+\cdots\right)$

$=\frac{4}{10}\left(\frac{1}{1-\frac{1}{10}}\right)=\frac{4}{10}x\frac{10}{9}=\frac{4}{9}$

(ii) $3.1\dot{2}\dot{7}$

Solution

$3.1\dot{2}\dot{7}=3+\frac{1}{10}+\frac{27}{1000}+\frac{27}{10000}+\cdots$

$= 3 + \frac{1}{10} + \frac{27}{1000}\left(1 + \frac{1}{100} + \frac{1}{10000} + \cdots\right)$

$= \frac{31}{10} + \frac{27}{1000}\left(\frac{1}{1-\frac{1}{100}}\right) = \frac{31}{10} + \frac{27}{1000}\left(\frac{100}{99}\right)$

$= \frac{31}{10} + \frac{3}{110} = \frac{344}{110} = \frac{172}{55}$

Hence $3.1\dot{2}\dot{7} = \frac{172}{55}$

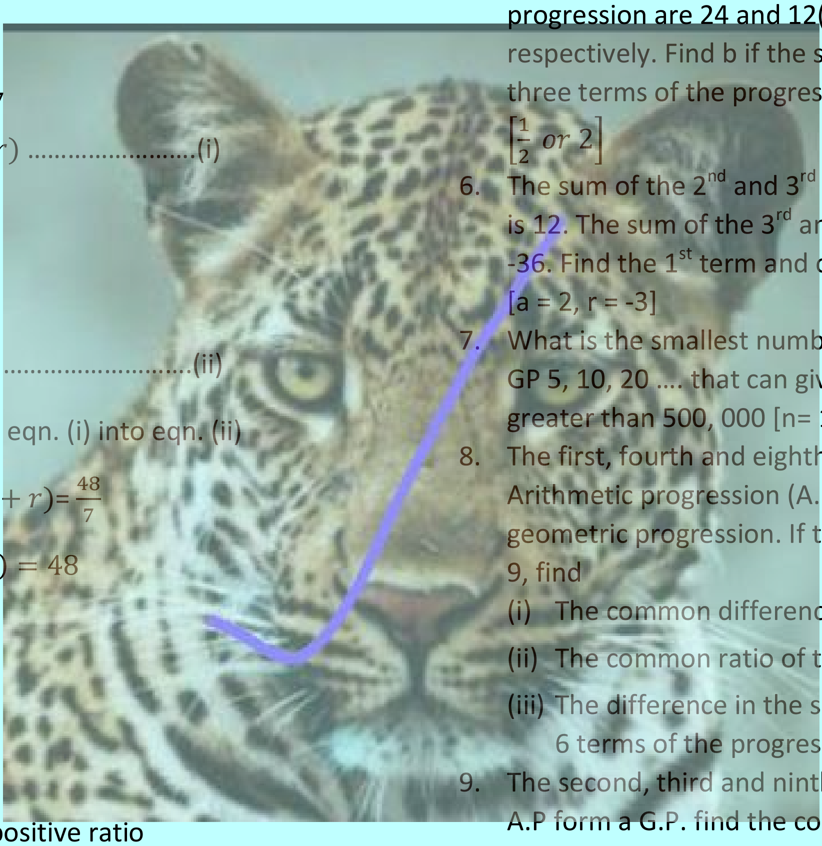
(b) The sum to infinity of a GP is 7 and the sum of the first two terms is $\frac{48}{7}$. Find the common ratio and the first term of the GP with positive common ratio

Solution

$S_\infty = 7$

$=> \frac{a}{1-r} = 7$

$a = 7(1-r)$ ……………………(i)



But $S_2 = \frac{48}{7}$

$a + ar = \frac{48}{7}$

$a(1 + r) = \frac{48}{7}$ ……………………….(ii)

substituting eqn. (i) into eqn. (ii)

$7(1-r)(1+r) = \frac{48}{7}$

$49\,(1 - r^2) = 48$

$49r^2 = 1$

$r^2 = \frac{1}{49}$

$r = \pm\frac{1}{7}$

Considering appositive ratio

From eqn. (i)

$a = 7(1 - \frac{1}{7}) = 6$

Revision exercise 2

1. The common ration of a GP is -5 and the sum of the first seven terms of the progression is 449. Find the first three terms. $\left[\frac{1}{29}, \frac{-5}{29}, \frac{25}{29}\right]$
2. In the geometrical series $\sum_{r=1}^{n} u_r$, $u_5 - u_2 = 156$ and $u_7 - u_4 = 1404$. Find the possible values of the common ratio and corresponding values of u_1
 [r = 3, a = 156; r = -3, a = $\frac{13}{7}$]
3. The sum of the second and third terms of a G.P is 9. It the seventh term is eight times the fourth term, find the
 (a) The first term and the common ration
 [a = $^3/_2$ and r = 2]
 (b) The sum of the fourth and first term
 [36]
4. Find the sum of ten terms of geometrical series 2, -4, 8 ….. [-682]
5. The second and the third terms of a G.P progression are 24 and 12(b + 1) respectively. Find b if the sum of the first three terms of the progression is 76
 $\left[\frac{1}{2}\ or\ 2\right]$
6. The sum of the 2nd and 3rd terms of a G.P is 12. The sum of the 3rd and 4th terms is -36. Find the 1st term and common ratio
 [a = 2, r = -3]
7. What is the smallest number of terms of GP 5, 10, 20 …. that can give a sum greater than 500, 000 [n= 17]
8. The first, fourth and eighth terms of Arithmetic progression (A.P) form a geometric progression. If the first term is 9, find
 (i) The common difference of A.P [1]
 (ii) The common ratio of the G.P $\left[\frac{4}{3}\right]$
 (iii) The difference in the sums of the first 6 terms of the progressions [55.7049]
9. The second, third and ninth terms of an A.P form a G.P. find the common ratio of the G.P [6]
10. (a) The sum of the first 10 terms of an AP is 120. The sum of the next 8 terms is 240. Find the sum of the next 6 terms [264]
 (b) the arithmetic mean of the a and b is three times their geometric mean. Show that $\frac{a}{b} = 7 \pm 12\sqrt{2}$
11. The first three terms of a geometric series are 1, p, and q. Given also that 10, q and p are the first three terms of an arithmetic series. Show that $2p^2 - p - 10 = 0$
 Hence find the possible values of p and q
 [p = -2 and q = 4 or p = $\frac{5}{2}$ and q = $\frac{25}{4}$]

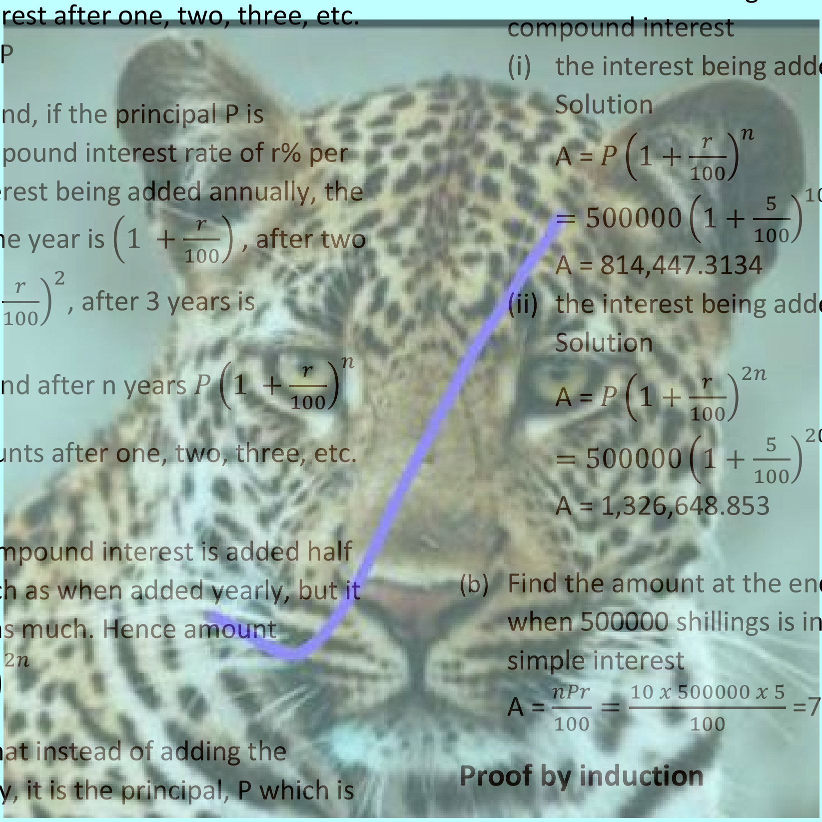
Application of A.Ps and G.Ps to interest rates

If a sum of money P is invested at a simple interest rate of r% per annum, the amount received after n years is given by A = P + I where I = $\frac{P\,x\,r\,x\,n}{100}$

By substitution we have

A = P + $\frac{P\,x\,r\,x\,n}{100}$ = P$\left(1 + \frac{nr}{100}\right)$

The interest for one year is $\frac{Pr}{100}$, for 2 years is $\frac{2Pr}{100}$, for n year = $\frac{nPr}{100}$. Therefore the various amounts of interest after one, two, three, etc. years form an AP



On the other hand, if the principal P is invested at compound interest rate of r% per annum, the interest being added annually, the amount after one year is $\left(1 + \frac{r}{100}\right)$, after two years is $P\left(1 + \frac{r}{100}\right)^2$, after 3 years is $P\left(1 + \frac{r}{100}\right)^3$ and after n years $P\left(1 + \frac{r}{100}\right)^n$

Hence the amounts after one, two, three, etc. years for a GP.

Note: if with compound interest is added half annually as much as when added yearly, but it is added twice as much. Hence amount A = $P\left(1 + \frac{r}{100}\right)^{2n}$

Now suppose that instead of adding the interest annually, it is the principal, P which is added annually,

Amount after 1st year = $P\left(1 + \frac{r}{100}\right)^1$

Amount after 2nd year = $P\left(1 + \frac{r}{100}\right)^2$

Amount after 3rd year = $P\left(1 + \frac{r}{100}\right)^3$

Amount after nth year = $P\left(1 + \frac{r}{100}\right)^n$

Total amount after n years

= $P\left(1 + \frac{r}{100}\right)^1 + P\left(1 + \frac{r}{100}\right)^2 + \ldots + P\left(1 + \frac{r}{100}\right)^n$

= P$\left[\left(1 + \frac{r}{100}\right)^1 + \left(1 + \frac{r}{100}\right)^2 + \cdots + \left(1 + \frac{r}{100}\right)^n\right]$

This is a G.P with:

first term = $\frac{100+r}{100}$ = (100 + r)%

And common ratio $\frac{100+r}{100}$ = (100 + r)%

$s_n = P$(100 + r)%$\left[\frac{(100 + r)\%-1}{(100 + r)\%-1}\right]$

Example 23

(a) Find the amount at the end of ten years when 500000 shillings is invested at 5% compound interest

(i) the interest being added annually

Solution

A = $P\left(1 + \frac{r}{100}\right)^n$

= $500000\left(1 + \frac{5}{100}\right)^{10}$

A = 814,447.3134

(ii) the interest being added twice a year

Solution

A = $P\left(1 + \frac{r}{100}\right)^{2n}$

= $500000\left(1 + \frac{5}{100}\right)^{20}$

A = 1,326,648.853

(b) Find the amount at the end of ten years when 500000 shillings is invested at 5% simple interest

A = $\frac{nPr}{100} = \frac{10\,x\,500000\,x\,5}{100}$ =750,000

Proof by induction

This is a mathematical technique that uses the reasoning that if a statement is true for a particular value say n = 1, then it must be true for n = 2, 3, 4, …. This involves the proof that the series on the LHS must be equal to the terms on the RHS

Example 24

Prove by induction that

1 + 2 + 3 + …+ n = $\frac{1}{2}n(n + 1)$

Solution

Here we need to show that the above series agrees for all values of n = 1, 2, 3 .., q and q +1

Suppose n = 1

LHS = 1 [taking only the 1st number]

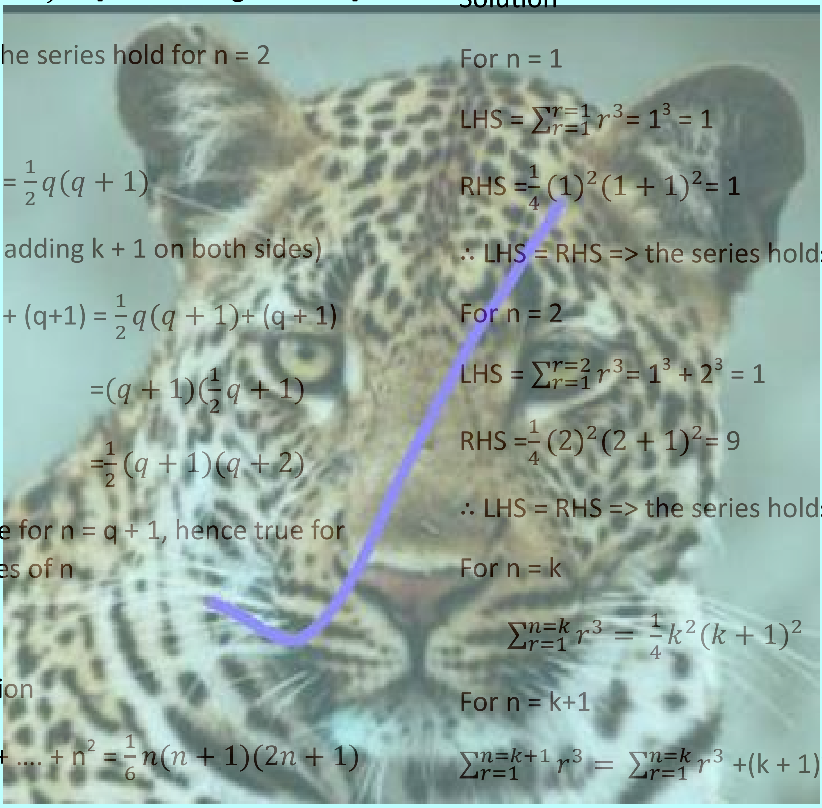
RHs = $\frac{1}{2}(1)((1)+1)$= 1[substituting for n = 1]

∴LHS = RHS => the series hold for n = 1

Suppose n = 2

LHS = 1+ 2 = 3 [taking first 2 numbers]

RHs = $\frac{1}{2}(2)((2)+1)$= 3[substituting for n = 2]



∴LHS = RHS => the series hold for n = 2

Suppose n = q

1 + 2 + 3 + …+ q = $\frac{1}{2}q(q+1)$

For n = q +1(i.e. adding k + 1 on both sides)

1 + 2 + 3 + …+ q + (q+1) = $\frac{1}{2}q(q+1)$+ (q + 1)

=$(q+1)(\frac{1}{2}q+1)$

=$\frac{1}{2}(q+1)(q+2)$

The result is true for n = q + 1, hence true for all positive values of n

Example 25

Prove by induction

(a) $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$

Solution

For n = 1

LHS = 1^2 = 1;

RHS = $\frac{1}{6}(1)((1)+1)(2(1)+1)$ = 1

∴ LHS = RHS => the series holds for n = 1

For n = 2

LHS = $1^2 + 2^2$ = 5;

RHS = $\frac{1}{6}(2)((2)+1)(2(2)+1)$ = 5

∴ LHS = RHS => the series holds for n = 2

For n = k

$1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{1}{6}k(k+1)(2k+1)$

For n = k + 1

$1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2$

$$= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2$$

=$[k\ +1]\left[\frac{1}{6}k(2k+1)+(k+1)\right]$

=$[k\ +1]\left[\frac{1}{6}(2k^2+7+6)\right]$

=$\frac{1}{6}(k+1)(k+2)(2k+3)$

∴ LHS = RHS => the series holds for n = k+1 hence for all positive values of n

(b) $\sum_{r=1}^{n=r} r^3 = \frac{1}{4}n^2(n+1)^2$

Solution

For n = 1

LHS = $\sum_{r=1}^{r=1} r^3 = 1^3 = 1$

RHS =$\frac{1}{4}(1)^2(1+1)^2$= 1

∴ LHS = RHS => the series holds for n = 1

For n = 2

LHS = $\sum_{r=1}^{r=2} r^3 = 1^3 + 2^3 = 1$

RHS =$\frac{1}{4}(2)^2(2+1)^2$= 9

∴ LHS = RHS => the series holds for n = 2

For n = k

$$\sum_{r=1}^{n=k} r^3 = \frac{1}{4}k^2(k+1)^2$$

For n = k+1

$\sum_{r=1}^{n=k+1} r^3 = \sum_{r=1}^{n=k} r^3 + (k+1)^3$

=$\frac{1}{4}k^2(k+1)^2 + (k+1)^3$

=$(k+1)^2\left[\frac{1}{4}k^2+k+1\right]$

=$\frac{1}{4}(k+1)^2[k^2+4k+4]$

=$\frac{1}{4}(k+1)^2(k+2)^2$

(c) p + pq + pq^2 +…+pq^{n-1} =p$\left(\frac{1-q^n}{1-q}\right)$

For n = 1,

LHS = p

RHS =p$\left(\frac{1-q^1}{1-q}\right)$ = p

∴the identity is true for n = 1

For n = 1,

LHS = p + pq =p(1+q)

RHS =p$\left(\frac{1-q^2}{1-q}\right)$ = p$\left(\frac{(1+q)(1-q)}{1-q}\right)$=p(1+q)

∴the identity is true for n = 2

For n = k

p + pq + pq^2 +…+pq^{k-1} =p$\left(\frac{1-q^k}{1-q}\right)$

For n = k+1

p + pq + pq^2 +…+pq^{k-1}+pq^k = p$\left(\frac{1-q^k}{1-q}\right)$+pq^k

= p$\left(\frac{1-q^k+q^k+p^{k+1}}{1-q}\right)$

= p$\left(\frac{1-q^{k+1}}{1-q}\right)$

∴the identity is true for n = k+1, hence true for all positive values of n

Revision exercise 3

1. Five millions shillings is invested each year at a rate of 15% compound interest by a certain bank.
   (a) Find how much he will receive at the end of ten years [116.7464m]
   (b) How many years will it take to accumulate to more than 50m [6]
2. John opened an account in the bank and deposited200,000 shillings every month for ten months without withdrawing. Find how much money he accumulated after 10 months if the bank offered 10% compound interest per month. [3,506,233.412]
3. Peter deposited sh. 100,000 at the beginning of every year for 5 years; find how much he got at the end of the fifth year at the compound interest rate of 2% per annum. [530,812.1]
4. Prove by induction
   (i) $\sum_{r=1}^{n} 3^{r-1} = \frac{3^n-1}{2}$
   (ii) $1^3 + 2^3 + 3^3 + \ldots + n^3$
   $= \frac{1}{4}n^2(3n+1)(5n+1)$
   (iii) $\sum_{r-1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$
   (iv) $\frac{1}{2} + \frac{1}{6} + \cdots + \frac{1}{n(n-1)} = 1 - \frac{1}{n}$
   (v) $\sum_{r=1}^{n} \frac{1}{r(1+1)} = \frac{n}{n+1}$
   (vi) $\sum_{r=1}^{n} ap^{r-1} = \frac{a(p^n-1)}{p-1}$
5. The sum of the first n terms of a Geometric Progression (G.P) is $\frac{4}{3}(4^n - 1)$. Find its nth term as an integral power of 2 [2^{2n}]
6. Prove by mathematical induction the $3^{2n} - 1$ is a multiple of 8 for all positive integers n
7. Use the method of induction to prove that 6n – 1is divisible by 5 for all positive integral values of n
8. Prove by induction that
   $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$
   hence evaluate $\sum_{r=1}^{20} r(r+1)$

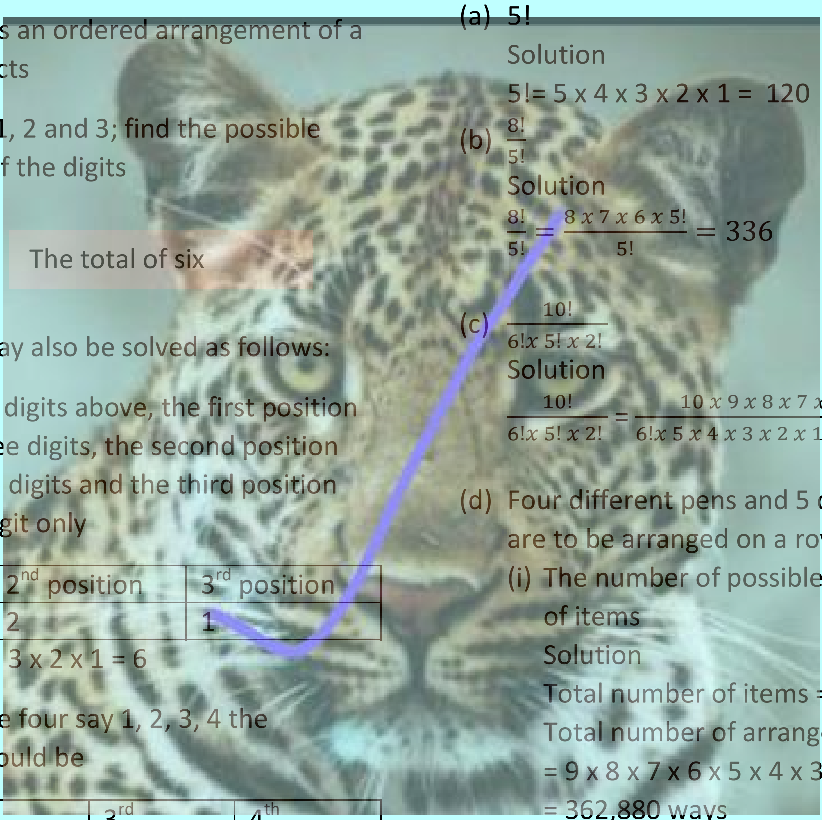
Permutations and combinations

Permutation

A permutation is an ordered arrangement of a number of objects

Consider digits 1, 2 and 3; find the possible arrangements of the digits

$\left.\begin{matrix}123, 132, 321\\231, 321, 321\end{matrix}\right\}$ The total of six



This problem may also be solved as follows:

Given the three digits above, the first position can take up three digits, the second position can take up two digits and the third position can take up 1 digit only

1^{st} position	2^{nd} position	3^{rd} position
3	2	1

The total is thus 3 x 2 x 1 = 6

If the digits were four say 1, 2, 3, 4 the arrangement would be

1^{st}	2^{nd}	3^{rd}	4^{th}
4	3	2	1

The total is thus 4 x 3 x 2 x 1 = 24

In summary the number of ways of arranging n different items in a row is given by n(n – 1)(n – 2)(n -3) x ……….2 x 1 and can be expressed as n!

If the total number of books is 6

The total number of arrangements = 6!

= 6 x 5 x 4 x 3 x 2 x 1 = 720 ways

Example 1

Find the values of the following expression

(a) 5!

Solution

5!= 5 x 4 x 3 x 2 x 1 = 120

(b) $\frac{8!}{5!}$

Solution

$\frac{8!}{5!} = \frac{8 \ x \ 7 \ x \ 6 \ x \ 5!}{5!} = 336$

(c) $\frac{10!}{6! x \ 5! \ x \ 2!}$

Solution

$\frac{10!}{6! x \ 5! \ x \ 2!} = \frac{10 \ x \ 9 \ x \ 8 \ x \ 7 \ x \ 6!}{6! x \ 5 \ x \ 4 \ x \ 3 \ x \ 2 \ x \ 1 \ x \ 2x \ 1} = 21$

(d) Four different pens and 5 different books are to be arranged on a row. Find

(i) The number of possible arrangements of items

Solution

Total number of items = 4 + 5 = 9

Total number of arrangements = 9!

= 9 x 8 x 7 x 6 x 5 x 4 x 3 x 2 x 1

= 362,880 ways

(ii) The number of possible arrangements if three of books must be kept together

Solution

The pens are taken to be one since they are to be kept together. So we consider total number of items to six.

The number of arrangements of six items = 6! = 6 x 5 x 4 x 3 x 2 x 1

= 720 ways

The arrangement of 4 pens = 4!

= 4 x 3 x 2 x 1 =24

Total number arrangements of all the items = 720 x 24 = 17, 280

Multiplication principle of permutation

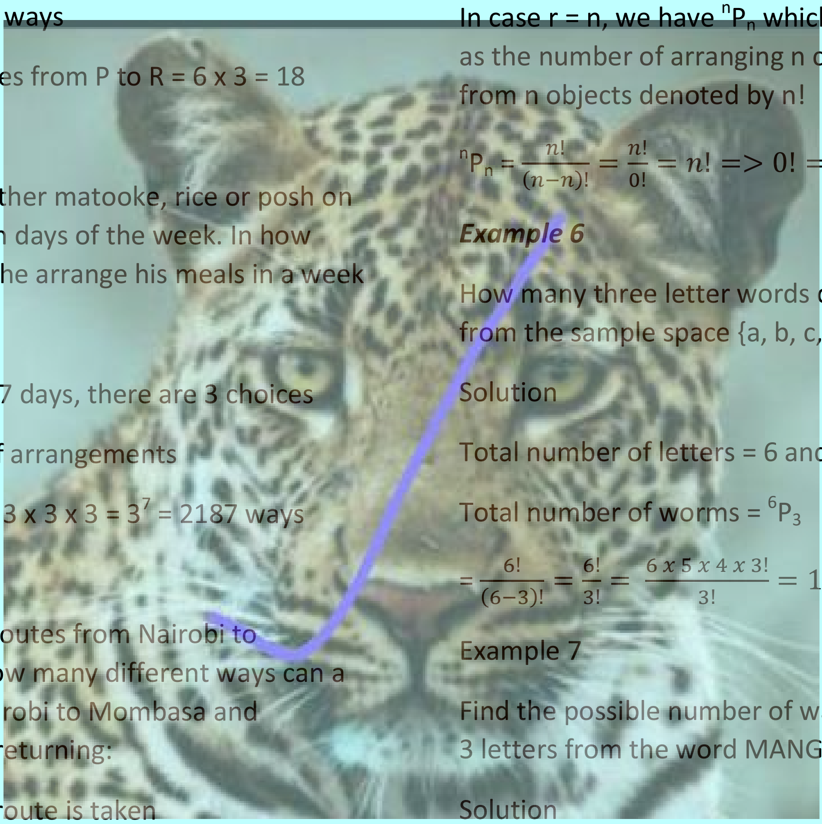
If one operation can be performed independently in a different ways and the second in b different ways, then either of the two events can be performed in (a + b) ways

Example 2

There are 6 roads joining P to Q and 3 roads joining Q to R. Find how many possible routes are from P to R

From P to Q = 6 ways

From Q to R = 3 ways



Number of routes from P to R = 6 x 3 = 18

Example 3

Peter can eat either matooke, rice or posh on any of the seven days of the week. In how many ways can he arrange his meals in a week

Solution

For each of the 7 days, there are 3 choices

Total number of arrangements

= 3 x 3 x 3 x 3 x 3 x 3 x 3 = 3^7 = 2187 ways

Example 4

There are four routes from Nairobi to Mombasa. In how many different ways can a taxi go from Nairobi to Mombasa and returning if for returning:

(a) any of the route is taken
= 4 x 4 = 16 ways
(b) the same route is taken
= 4 x 1 = 4 ways
(c) the same route is not taken
= 4 x 3 = 12 ways

Example 5

David can arrange a set of items in 5 ways and John can arrange the same set of items in 3 ways. In how many ways can either David or John arrange the items?

Solution

Number of ways in which David arranges = 5

Number of ways in which John arranges = 3

Number of ways in which either David or John arrange the items = 5 + 3 = 8 ways

The number of permutation of r objects taken from n unlike objects

The permutation of n unlike objects taking r at a time is denoted by ^nP_r which is defined as

$^nP_r = \frac{n!}{(n-r)!}$, where r ≤ n.

In case r = n, we have ^nP_n which is interpreted as the number of arranging n chosen objects from n objects denoted by n!

$^nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n! \Rightarrow 0! = 1$

Example 6

How many three letter words can be formed from the sample space {a, b, c, d, e, f}

Solution

Total number of letters = 6 and r = 3

Total number of worms = $^6P_3$

$= \frac{6!}{(6-3)!} = \frac{6!}{3!} = \frac{6 \times 5 \times 4 \times 3!}{3!} = 120\text{ways}$

Example 7

Find the possible number of ways of arranging 3 letters from the word MANGOES

Solution

Total number of letter in the word = 7

and r = 3

Number of ways $^7P^3 = \frac{7!}{(7-3)!}$

$= \frac{7 \times 6 \times 5 \times 4 \times 3!}{3!} = 840\text{ways}$

Example 8

Find number of ways of arranging six boys from a group of 13

Solution

Number of arrangements = ${}^{13}P_6$

$=\frac{13!}{(13-6)!} = \frac{13!}{7!}$

$= \frac{13x12x11x10\ x9\ x\ 8\ x\ 7!}{7!}$

= 1235520 ways

The number of permutations of n objects of which r are alike

The number of permutations of n objects of which r are alike is given by $\frac{n!}{r!}$

Example 9

Find the number of arranging in a line the letters B, C, C, C, C, C, C

The number of ways of arranging the seven letters of which of which 6 are alike $= \frac{7!}{6!} = \frac{7\ x\ 6!}{6!} = 7$ ways

The number of ways of permutations on n objects of which p of one type are alike, q of the second type are alike, r of the third type are alike, and so on.

The number of ways of permutations on n objects of which p of one type are alike, q of the second type are alike, r of the third type are alike given by $\frac{n!}{p!\ x\ q!\ x\ r!}$

Example 10

Find the possible number of ways of arranging the letter of the word MATHEMATICS in line

Solution

The word MATHEMATICS has 11 letters and contains 2 M, 2A and 2T repeated

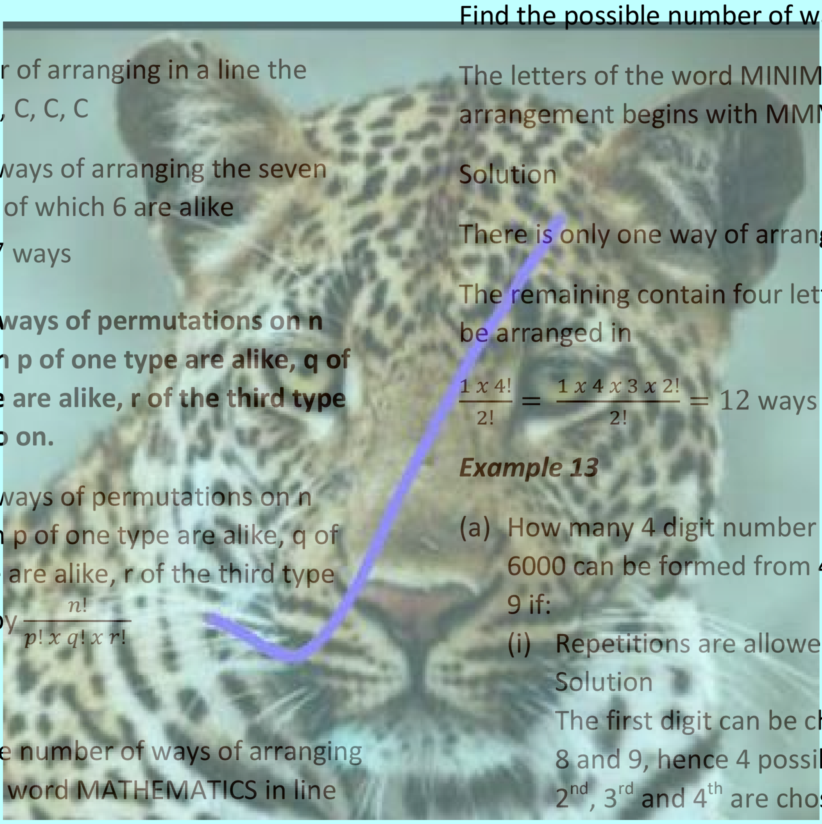
The number of ways = $\frac{11!}{2!x\ 2!\ x\ 2!}$

= $\frac{11\ x\ 10\ x\ 9\ \ x\ 8\ x\ 7\ x\ 6\ x\ 5\ x\ 4x3\ x\ 2\ x\ 1}{2\ x\ 2\ x\ 2}$

= 4,989,600

Example 11

Find the possible number of ways of arranging the letter of the word 'MISSISSIPPI' in line



Solution

The word 'MISSISSIPPI' has 10 letters with 4I, 4S, and 2P

The number of ways = $\frac{11!}{4!x\ 4!\ x\ 2!} = 34650$

The number of permutations of the like and unlike objects with restrictions

One should be cautious when handling these problems

Example 12

Find the possible number of ways of arranging

The letters of the word MINIMUM if the arrangement begins with MMM?

Solution

There is only one way of arranging MMM

The remaining contain four letters with 2I can be arranged in

$\frac{1\ x\ 4!}{2!} = \frac{1\ x\ 4\ x\ 3\ x\ 2!}{2!} = 12$ ways

Example 13

(a) How many 4 digit number greater than 6000 can be formed from 4, 5, 6, 7, 8 and 9 if:

(i) Repetitions are allowed

Solution

The first digit can be chosen from 6, 7, 8 and 9, hence 4 possible ways, the 2nd, 3rd and 4th are chosen from any of the six digits since repetitions are allowed

position	1st	2nd	3rd	4th
selections	4	6	6	6

Number of ways

= 4 x 6 x 6 x 6 = 864 ways

(ii) Repetition are not allowed

The first can be chosen from 6, 7, 8 and 9, hence 4 possible ways, the 2nd from 5, 3rd from 4 and 4th from 3 since no repetitions are allowed

position	1st	2nd	3rd	4th
selections	4	5	4	3

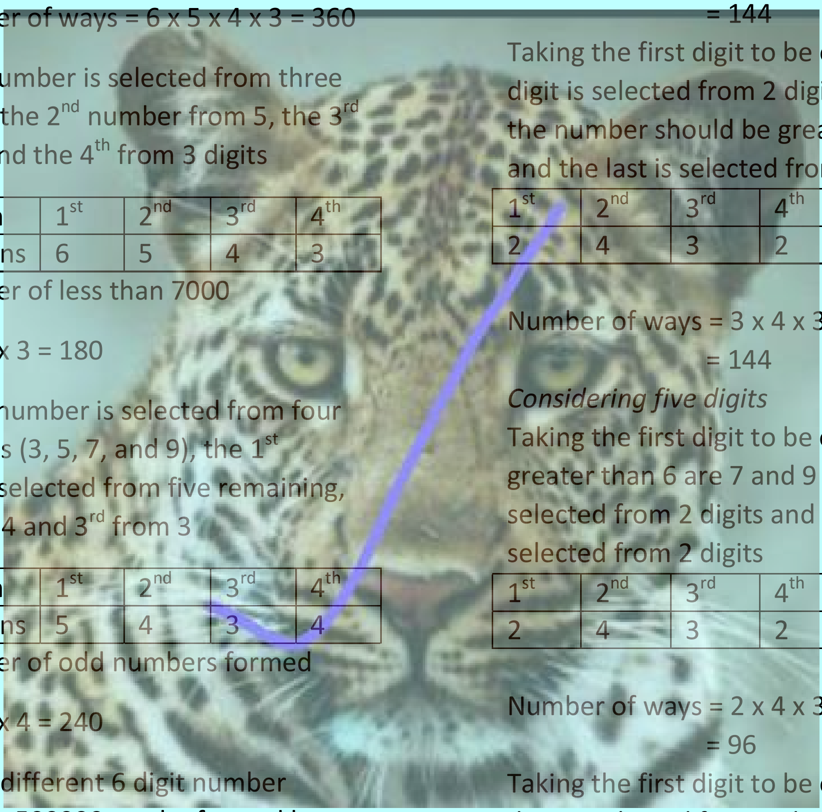
Number of ways
= 4 x 5 x 4 x 3 = 240 ways

(b) Find how many four digit numbers can be formed from the six digits 2, 3, 5, 7, 8 and 9 without repeating any digit.
Find also how many of these numbers
(i) Are less than 7000
(ii) Are odd

Solution

position	1^{st}	2^{nd}	3^{rd}	4^{th}
selections	6	5	4	3

Total number of ways = 6 x 5 x 4 x 3 = 360



(i) The 1^{st} number is selected from three (2, 3, 5), the 2^{nd} number from 5, the 3^{rd} from 4 and the 4^{th} from 3 digits

position	1^{st}	2^{nd}	3^{rd}	4^{th}
selections	6	5	4	3

Total number of less than 7000

= 3 x 5 x 4 x 3 = 180

(ii) The last number is selected from four odd digits (3, 5, 7, and 9), the 1^{st} number selected from five remaining, 2^{nd} from 4 and 3^{rd} from 3

position	1^{st}	2^{nd}	3^{rd}	4^{th}
selections	5	4	3	4

Total number of odd numbers formed

= 5 x 4 x 3 x 4 = 240

(c) How many different 6 digit number greater than 500000 can be formed by using the digits 1, 5, 7, 7, 7, 8
Solution
The 1^{st} digit is selected from five (5, 7, 7, 7, 8), the 2^{nd} from remaining five, 3^{rd} from four, 4^{th} from three, 5^{th} from two and 6^{th} from one

1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}	6th
5	5	4	3	2	1

Total number = $\frac{5 \, x \, 5 \, x \, 4 \, x \, 3 \, x \, 2 \, x \, 1}{3!}$ = 100
NB. The number is divided by 3! Because 7 appears three times

(d) How many odd numbers greater than 60000 can be formed from 0, 5, 6, 7, 8, 9, if no number contains any digit more than once
Solution
Considering six digits
Taking the first digit to be odd, the first digit is selected from 3 digits (5, 7, 9) and the last is selected from 2 digits

1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}	6th
3	4	3	2	1	2

Number of ways = 3 x 4 x 3 x 2 x 1 x 2
= 144
Taking the first digit to be even, the first digit is selected from 2 digits (6, 8) since the number should be greater than 60000 and the last is selected from 2 odd digits

1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}	6th
2	4	3	2	1	3

Number of ways = 3 x 4 x 3 x 2 x 1 x 2
= 144
Considering five digits
Taking the first digit to be odd, the digit greater than 6 are 7 and 9 so first digit is selected from 2 digits and the last is selected from 2 digits

1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}
2	4	3	2	2

Number of ways = 2 x 4 x 3 x 2 x 2
= 96
Taking the first digit to be even, the first digit is selected from 2 digits (6, 8) since the number should be greater than 60000 and the last is selected from 3 odd digits (5, 7, 9)

1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}
2	4	3	2	3

Number of ways = 2 x 4 x 3 x 2 x 3
= 144
The total number of selections
= 144 + 144 + 96 + 144 = 528

Example 14

The six letter of the word LONDON are each written on a card and the six cards are shuffled and placed in a line. Find the number of possible arrangements if

(a) The middle two cards both have the letter N on them

Solution

If the middle letter are NN, the we need to find the number of different arrangements of the letter LODO. With the 2O's, the number of arrangements = $\frac{4!}{2!} = 12$

(b) The two cards with letter O are not adjacent and the two cards with letter N are also not adjacent

Solution

If the two cards are not adjacent, the number of arrangements = Total number of arrangements of the word LONDON – number of arrangements when the two letters are adjacent

= = $\frac{6!}{2!2!} - 24 = 156$

Example 15

In how many different ways can letters of the word MISCHIEVERS be arranged if the S's cannot be together

Solution

There are 11 letters in the word MISCHIEVERS with 2S's, 2I's and 2E's

Total number of arrangements

= $\frac{11!}{2!2!2!}$ = 4989600

If S's are together, we consider them as one, so the number of arrangements

= $\frac{10!}{2!2!}$ = 907200

∴ the number of possible arrangements of the word MISCHIEVERS when S's are not together

= 4989600 – 907200 = 4082400

The number of permutation of n different objects taken r at a time, if repetition are permitted

Example 16

How many four digit numbers can be formed from the sample space {1, 2, 3, 4, 5} if repetitions are permissible

Solution

The 1st position has five possibilities, the 2nd five, the 3rd five, the 4th five

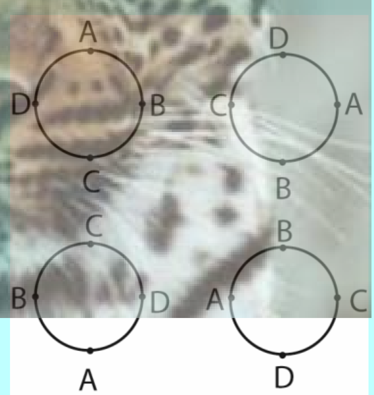
Number of permutations = 5 x 5 x 5 x 5 = 625

Circular permutations

Here objects are arrange in a circle

The number of ways of arranging n unlike objects in a ring when clockwise and anticlockwise are different.

Consider four people A, B, C and D seated at a round table. The possible arrangements are as shown below



With circular arrangements of this type, it is the relative positions of the objects being arranged which is important. The arrangements of the people above is the same. However, if the people were seated in a line the arrangements would not be the same, i.e. A, B, C, D is not the same as D, A, B, C. When finding the number of different arrangements, we fix one person say A and find the number of ways of arranging B, C and D.

Therefore, the number of different arrangements of four people around the table is 3!

Hence the number of different arrangements of n people seated around a table is (n – 1)!

Example 17

(a) Seven people are to be seated around a table, in how many ways can this be done

Solution

The number of ways = (7 – 1)! = 6!

= 720

(b) In how many ways can five people A, B, C, D and E be seated at a round table if

(i) A must be seated next to B

Solution

If A and B are seated together, they are taken as bound together. So four people are considered

The number of ways = (4 – 1)! = 3! = 6

The number of ways in which A and B can be arranged = 2

The total number of arrangements

= 6 x 2 = 12 ways

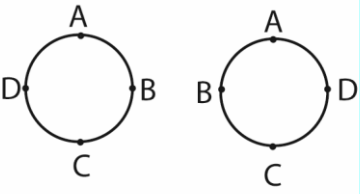
(ii) A must not seat next to B

If A and B are not seated together, then the number of arrangements = total number of arrangements – number of arrangements when A and B are seated together

= (5 – 1)! – 12 = 12 ways

The number of ways of arranging n unlike objects in a ring when clockwise and anticlockwise arrangements are the same

Consider the four people above, if the arrangement is as shown below



Then the above arrangements are the same since one is the other viewed from the opposite side

The number of arrangements = $\frac{3!}{2}$ = 3ways

Hence the number of ways of arranging n unlike objects in a ring when clockwise and anticlockwise arrangements are the same = $\frac{(n-1)!}{2}$

Example 18

A white, a blue, a red and two yellow cards at arranged on a circle. Find the number of arrangements if red and white cards are next to each other.

Solution

If red and white cards are next to each other, they are considered as bound together. So we have four cards. Since anticlockwise and clockwise arrangements are the same and there are two yellow cards, the number of arrangements = $\frac{(4-1)!}{2\ x\ 2!} = \frac{3!}{2\ x\ 2!}$

The number of ways of arranging red and white cards = 2

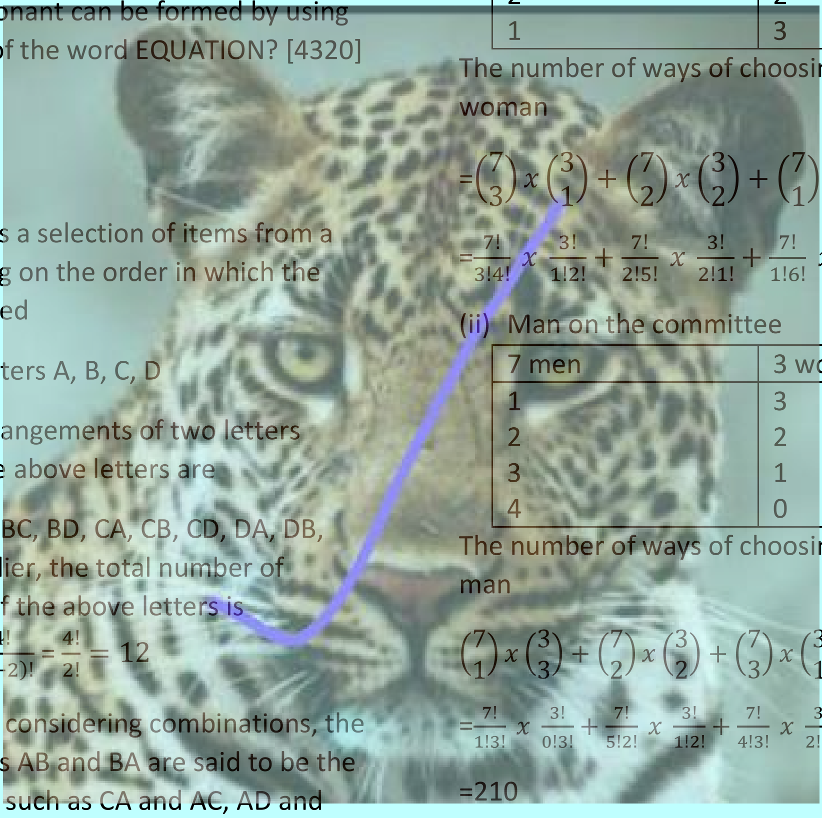
Total number of ways of arrangements

=$\frac{3!}{2\ x\ 2!}\ x\ 2$ = 3

Revision exercise 1

1. In how many ways can the letters of the words below be arranged
   (a) Bbosa (5!]
   (b) Precious [8!]
2. How many different arrangements of the letters of the word PARALLELOGRAM can be made with A's separate [83160000]
3. How many different arrangements of the letters of the word CONTACT can be made with vowels separated? [900]
4. How many odd numbers greater than 6000 can be formed using digits 2, 3, 4, 5 and 6 if each digit is used only once in each number [12]

5. Three boys and five girls are to be seated on a bench such that the eldest girl and eldest boy sit next to each other. In how many ways can this be done [2 x 7!]
6. A round table conference is to be held between delegates of 12 countries. In how many ways can they be seated if two particular delegates wish to sit together [2 x 10!]
7. In how many ways can 4 boys and 4 girls be seated at a circular table such that no two boys are adjacent [144]
8. How many words beginning or ending with a consonant can be formed by using the letters of the word EQUATION? [4320]



Combinations

A combination is a selection of items from a group not basing on the order in which the items are selected

Consider the letters A, B, C, D

The possible arrangements of two letters chosen from the above letters are

AB, AC, AD, BA, BC, BD, CA, CB, CD, DA, DB, DC. AS seen earlier, the total number of arrangements of the above letters is expressed as $\frac{4!}{(4-2)!} = \frac{4!}{2!} = 12$

However, when considering combinations, the grouping such as AB and BA are said to be the same groupings such as CA and AC, AD and DA, etc.

So the possible combinations are AB, AC, AD, BC, BD, CD which is six ways.

Thus the number of possible combinations of n items taken r at a time is expressed as ^nC_r or $\binom{n}{r}$ which is defined as $^nC_r = \frac{n!}{(n-r)!r!}$ where $r \le n$

Hence the number of combinations of the above letters taken two at a time is

$\binom{4}{2} = \frac{4!}{2!2!} = 6$

Example 19

A committee of four people is chosen at random from a set of seven men and three women

How many different groups can be chosen if there is at least one

(i) Woman on the committee

Solution

Possible combinations

7 men	3 women
3	1
2	2
1	3

The number of ways of choosing at least one woman

$=\binom{7}{3} x \binom{3}{1} + \binom{7}{2} x \binom{3}{2} + \binom{7}{1} x \binom{3}{3}$

$=\frac{7!}{3!4!} x \frac{3!}{1!2!} + \frac{7!}{2!5!} x \frac{3!}{2!1!} + \frac{7!}{1!6!} x \frac{3!}{3!0!} = 175$

(ii) Man on the committee

7 men	3 women
1	3
2	2
3	1
4	0

The number of ways of choosing at least one man

$\binom{7}{1} x \binom{3}{3} + \binom{7}{2} x \binom{3}{2} + \binom{7}{3} x \binom{3}{1} + \binom{7}{4} x \binom{3}{0}$

$=\frac{7!}{1!3!} x \frac{3!}{0!3!} + \frac{7!}{5!2!} x \frac{3!}{1!2!} + \frac{7!}{4!3!} x \frac{3!}{2!1!} + \frac{7!}{3!4!} x \frac{3!}{3!0!}$

=210

Example 20

A group of nine has to be selected from ten men and eight women. It can consist of either five men and four women or four men and five women. How many different groups can be chosen?

Solution

Possible combination

10 men	8 women
5	4
4	5

Number of groups = $\binom{10}{5} x \binom{8}{4} + \binom{10}{4} x \binom{8}{5}$

= $\frac{10!}{5!5!}$ x $\frac{8!}{4!4!} + \frac{10!}{6!4!}$ x $\frac{8!}{5!3!}$ = 29400

Example 21

A team of six is to be formed from 13 boys and 7 girls. In how many ways can the team be selected if it must consist of

(a) 4 boy and 2 girls

13 boys	7 girls
4	2

$\binom{13}{4} \cdot \binom{7}{2} = \frac{13!}{9!4!} x \frac{7!}{5!2!}$ = 15015

(b) At least one member of each sex

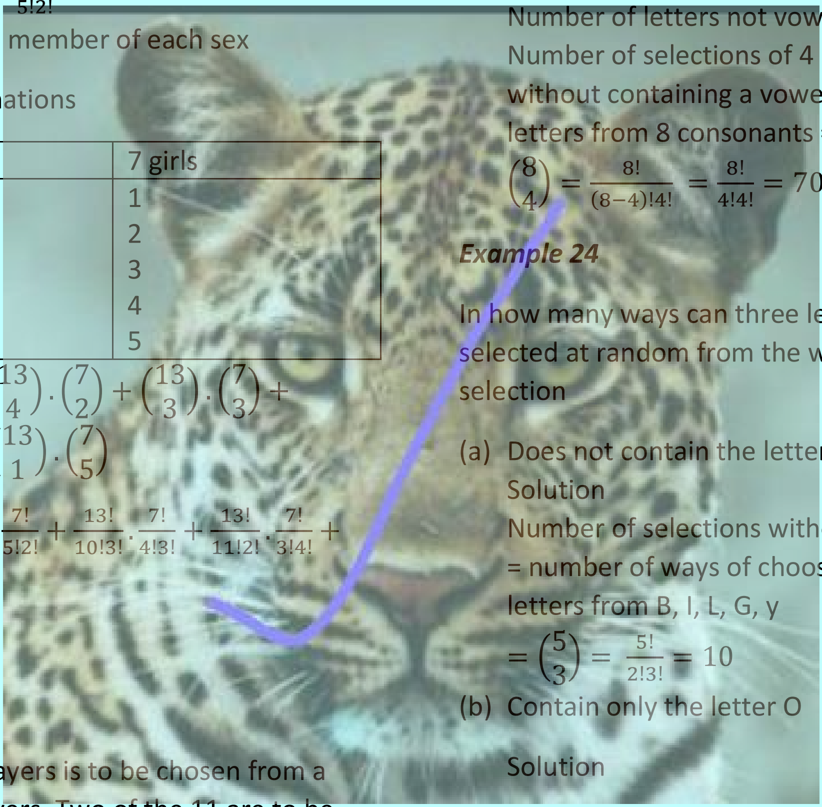
Possible combinations

13 boys	7 girls
5	1
4	2
3	3
2	4
1	5

$=\binom{13}{5} \cdot \binom{7}{1} + \binom{13}{4} \cdot \binom{7}{2} + \binom{13}{3} \cdot \binom{7}{3} + \binom{13}{2} \cdot \binom{7}{4} + \binom{13}{1} \cdot \binom{7}{5}$

$=\frac{13!}{8!5!} \cdot \frac{7!}{6!1!} + \frac{13!}{9!4!} \cdot \frac{7!}{5!2!} + \frac{13!}{10!3!} \cdot \frac{7!}{4!3!} + \frac{13!}{11!2!} \cdot \frac{7!}{3!4!} + \frac{13!}{12!1!} \cdot \frac{7!}{2!5!}$

= 37037



Example 22

A team of 11 players is to be chosen from a group of 15 players. Two of the 11 are to be randomly elected a captain and vice-captain respectively. In how many ways can this be done?

Number of ways of choosing 11 players from 15 = $\binom{15}{11}$

A captain will be elected from 11 players and a vice-captain from 10 players

Total number of selection = $\binom{15}{11}\ x\ 11\ x\ 10$

= $\frac{15!}{4!11!}$ x 11 x 10 = 150150

Example 23

(a) Find the number of different selections of 4 letters that can be made from the word UNDERMATCH.

Solution

There are 10 letters which are all different

Number of selections of 4 letters from 10 is given by $\binom{10}{4} = \frac{10!}{(10-4)!4!} = \frac{10!}{6!4!} = 210$

(b) How many selections do not contain a vowel?

Solution

Number of vowels in the word = 2

Number of letters not vowels = 8

Number of selections of 4 letters from 10 without containing a vowel = selecting 4 letters from 8 consonants =

$\binom{8}{4} = \frac{8!}{(8-4)!4!} = \frac{8!}{4!4!} = 70$

Example 24

In how many ways can three letters be selected at random from the word BIOLOGY is selection

(a) Does not contain the letter O

Solution

Number of selections without the letter O = number of ways of choosing three letters from B, I, L, G, y

$= \binom{5}{3} = \frac{5!}{2!3!} = 10$

(b) Contain only the letter O

Solution

Number of selections with one letter O = number of ways of choosing two letters from B, I, L, G, y

$= \binom{5}{2} = \frac{5!}{3!2!} = 10$

(c) Contains both of the letters O

Solution

Number of selections with two letter O = number of ways of choosing one letter from B, I, L, G, y

$= \binom{5}{1} = \frac{5!}{4!1!} = 5$

Example 25

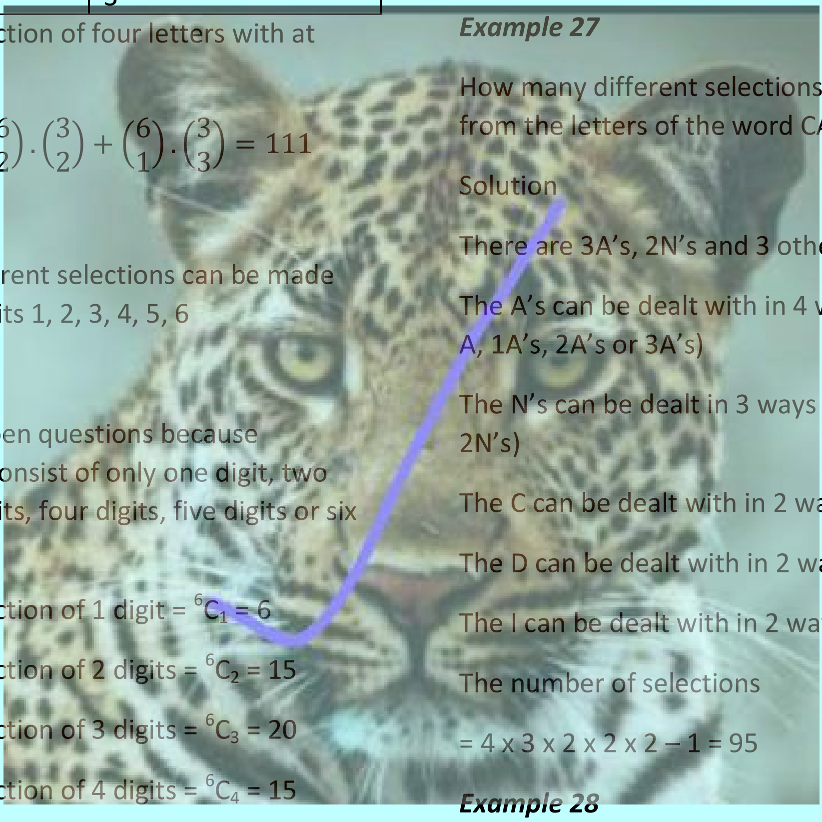
In how many ways can four letters be selected at random from the word BREAKDOWN if the letters contain at least one vowel?

Solution

Vowels: E, A, O (3)

Consonants: B, R, K, D, W, N (6)

Consonants (6)	Vowels (3)
3	1
2	2
1	3



Number of selection of four letters with at least one vowel

$$= \binom{6}{3}\cdot\binom{3}{1} + \binom{6}{2}\cdot\binom{3}{2} + \binom{6}{1}\cdot\binom{3}{3} = 111$$

Example 26

How many different selections can be made from the six digits 1, 2, 3, 4, 5, 6

Solution

Note: this an open questions because selections can consist of only one digit, two digits, three digits, four digits, five digits or six digits

Number of selection of 1 digit = ${}^{6}C_{1}$ = 6

Number of selection of 2 digits = ${}^{6}C_{2}$ = 15

Number of selection of 3 digits = ${}^{6}C_{3}$ = 20

Number of selection of 4 digits = ${}^{6}C_{4}$ = 15

Number of selection of 5 digits = ${}^{6}C_{5}$ = 6

Number of selection of 6 digits = ${}^{6}C_{1}$ = 1

Total number of selections

= 6 + 15 + 20 + 15 +6 + 1 = 63

This approach is tedious for a large group of objects.

The general formula for selection from n unlike objects is given by $2^n - 1$.

For the above problems, number of selections $= 2^6 - 1 = 63$

Example 26

How many different selections can be made from 26 different letters of the alphabet?

Number of selection = $2^{26} - 1$

= 67,108,863

Cases involving repetitions

Suppose we need to find the number of possible selections of letters from a word containing repeated letters, we take the selections mutually exclusive

Example 27

How many different selections can be made from the letters of the word CANADIAN?

Solution

There are 3A's, 2N's and 3 other letters

The A's can be dealt with in 4 ways (either no A, 1A's, 2A's or 3A's)

The N's can be dealt in 3 ways (no N, 1N, or 2N's)

The C can be dealt with in 2 ways (no C, 1C)

The D can be dealt with in 2 ways (no D, 1D)

The I can be dealt with in 2 ways (no I, 1I)

The number of selections

= 4 x 3 x 2 x 2 x 2 – 1 = 95

Example 28

How many different selections can be made from the letters of the word POSSESS?

Solution

There are 4S's and 3 other letters

The S's can be dealt in 5 ways (no S, 1S, 2N's, 3S's, 4S's, or 5S's)

The P can be dealt with in 2 ways (no P, 1P)

The O can be dealt with in 2 ways (no O, 1O)

The E can be dealt with in 2 ways (no E, 1E)

Total number of selections = 5 x 2 x 2 x 2 – 1

= 39

Cases involving division into groups

The number of ways of dividing n unlike objects into say two groups of p and q where p + q = n is given by $\frac{n!}{p!q!}$

For three groups of p, q and r provided p + q + r = n

Number of ways of division = $\frac{n!}{p!q!r!}$

However, for the two groups above, if p = q then the number of ways of division = $\frac{n!}{p!p!2!}$

For three groups where p = q = r

then the number of ways of division = $\frac{n!}{p!p!p!3!}$

Example 29

The following letters a, b, c, d, e, f, g, h, I, j, k, l are to be divided into groups containing

(a) 3, 4, 5
(b) 5, 7
(c) 6, 6
(d) 4, 4, 4 letters. In how many ways can this be done?

Solution

(a) Number of ways = $\frac{12!}{3!4!5!}$ = 27720
(b) Number of ways = $\frac{12!}{5!7!}$ = 792
(c) Number of ways = $\frac{12!}{6!6!2!}$ = 462
(d) Number of ways = $\frac{12!}{4!4!4!3!}$ =5775

Example 30

Find the number of ways that 18 objects can be arranged into groups if there are to be

(a) Two groups of 9 objects each
(b) Three groups of 6 objects each
(c) 6 groups of 3 objects each
(d) Three groups of 5, 6 and 7 objects each

Solution

(a) Number of ways = $\frac{18!}{9!9!2!}$ = 24310
(b) Number of ways = $\frac{18!}{6!6!6!3!}$ = 2858856
(c) Number of ways = $\frac{18!}{3!3!3!3!3!3!6!}$ =190590400
(d) Number of ways = $\frac{18!}{5!6!7!}$ = 14702688

Example 31

(a) Find how many words can be formed using all letters in the word MINIMUM.

Solution

Number of ways of arranging the letters = 7!

There are 3M's and 2I's

Number of words formed = $\frac{7!}{3!2!} = 420$

(b) Compute the sum of four-digit numbers formed with the four digits 2, 5, 3, 8 if each digit is used only once in each arrangement

Solution

Number of ways of arranging a four digit number = 4!

Sum of any four digit number formed = 2 + 5 + 3 + 8 = 18

Total sum of four digit numbers formed

= 18 x 4! = 432

(c) A committee consisting of 2 men and 3 women is to be formed from a group of 5 men and 7 women. Find the number of different committees that can be formed. If two of the women refuse to serve on the same committee, how many committees can be formed?

Solution

The committees formed = ${}^5C_2.{}^7C_3$

= 10 x 35 = 350

Suppose two women are to serve together, we take them as glued together, so the number of committees =${}^5C_2.{}^6C_3$ = 200

Number of committees in which two women refuse to serve together = 350 – 200 = 150

Revision exercise 2

1. (a) Find the number of different selection of 3 letters that can be made from the word PHOTOGRAPH. [53]
   (b) How many of these selections contain no vowel [18]
   (c) How many of these selections contain at least one vowel? [35]
2. (a) find the number of different selections of 3 letters that can be made from the letters of the word SUCCESSFUL.[36]
   (c) How many of these selections contain only consonants [11]
   (d) How many of these selections contain at least one vowel [25]
3. (a) Find the value of n if $^nP_4 = 30\,^nC_5$ [8]
   (b) How many arrangement can be made from the letters of the name MISSISSIPPI
      (i) when all the letters are taken at a time [34650]

      (ii) If the two letters PP begin every word [630 ways]

   (c) Find the number of ways in which a one can chose one or more of the four girls to join a discussion group [15 ways]
4. Find in how many ways 11 people can be divided into three groups containing 3, 4, 4 people each. [5775]
5. A group of 5 boys and 8 girls. In how many ways can a team of four be chosen, if the team contains
   (a) No girl [5]
   (b) No more than one girl [85]
   (c) At least two boys [365]
6. Calculate the number of 7 – letter arrangements which can be made with the letters of the word MAXIMUM. In how many of these do all the 4 consonants appear next to each other? [840, 96]
7. In how many ways can a club of 5 be selected from 7 boys and 3 girls if it must contain
   (a) 3 boys and 3 girls [105]
   (b) 2 men and 3 girls [21]
   (c) At least one girl [231]
8. How many different 6 digit numbers greater than 400,000 can be formed form the following digits 1, 4, 6, 6, 6 7? [100]

Binomial theorem

Pascal's triangle

The Pascal's triangle below and its further extension is used to determine the coefficients of the expansion of $(p + q)^n$

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

Observations

- The coefficients are symmetrical; they are the same irrespective of which side they are read from, for instance the coefficients of $(p + q)^6$ are
  1 6 15 20 15 6 1
- The coefficient of the 2nd term in the expansion is the index of a given expansion. E.g. in the expansion of $(p + q)^6$, the coefficient of the 2nd term is 6.
- The number of terms in the expansion exceeds the index by one, e.g. $(p + q)^4$ with index 4, has 5 terms
- The index of the first term of the expansion decreases by one, from the index given till zero, whereas, the index of the second term increases by one from zero to the given index
  For $(p + q)^3 = 1p^3q^0 + 3p^2q^1 + 3p^1q^2 + 1p^0q^3$
  $=p^3 + 3p^2q + 3pq^2 + q^3$
- The sum of indices in each term s constant and equal to the index of the expansion.

Example 1

Use Pascal's triangle to expand

(a) $(p + q)^4$

The coefficients are 1 4 6 4 1

Terms are $1p^4q^0 + 4p^3q^1 + 6p^2q^2 + 4p^1q^3 + 1q^4$

$= p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4$

(b) $(2x + 3y)^3$

The coefficients are 1 3 3 1

Terms are

$1(2x)^3(3y)^0 + 3(2x)^2(3y)^1 + 3(2x)^1(3y)^2 + 1(2x)^0(3y)^3$

$= 8x^3 + 36x^2y + 54xy^2 + 27y^3$

(c) $(a + b + c)^2$

The terms can be grouped into two ways:

Either a and $(b + c)$ or $(a + b)$ and c

The coefficients are 1 2 1

Either

Terms are: $1a^2(b + c)^0 + 2a^1(b + c)^1 + 1a^0(b + c)^2$

$= a^2 + 2a(b + c) + (b + c)^2$

$= a^2 + 2ab + 2ac + b^2 + 2bc + c^2$

Or

$(a + b + c)^2$

$= 1(a + b)^2c^0 + 2(a + b)_1c^1 + 1c^2$

$= (a + b)^2 + 2c(a + b) + c^2$

$=a^2 + 2ab + b^2 + 2ac + 2bc + c^2$

Example 2

Expand $(a + b)^4$ using Pascal's triangle. Hence find $(1.996)^4$ correct to 3 decimal places

Solution

From Pascal's triangle the coefficients are
1 4 6 4 1

$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$(1.996)^4 = (2- 0.004)^4$

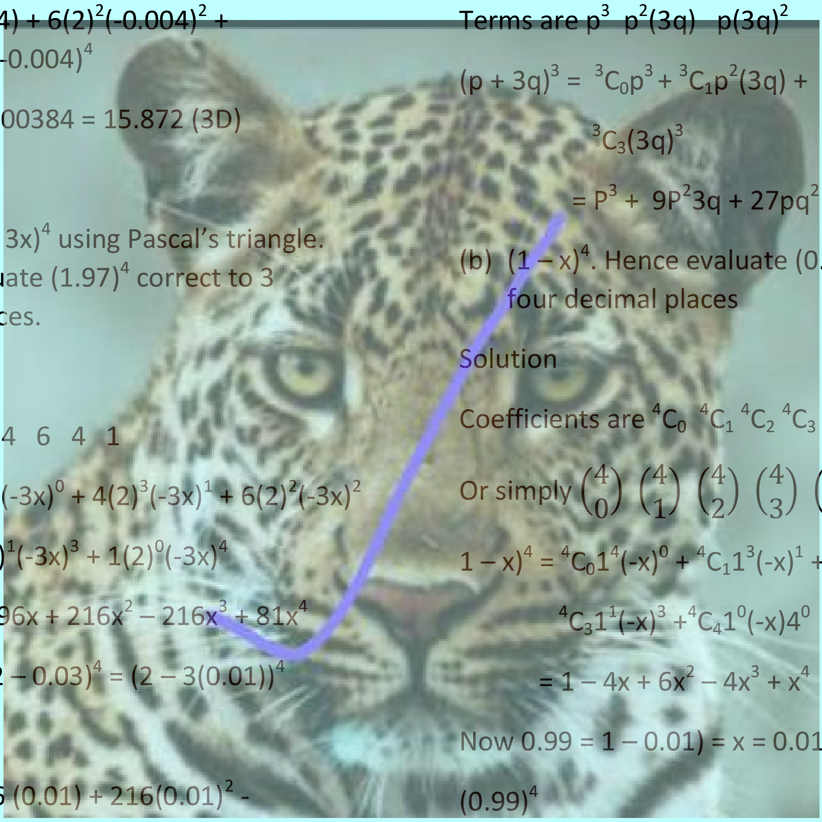
Substituting a = 2 and b = -0.004

$24 + 4(2)^3(-0.004) + 6(2)^2(-0.004)^2 + 4(2)(-0.004)^3 + (-0.004)^4$

16 – 0.128 + 0.000384 = 15.872 (3D)

Example 3

(a) Expand $(2 - 3x)^4$ using Pascal's triangle. Hence evaluate $(1.97)^4$ correct to 3 decimal places.



Solution

Coefficients: 1 4 6 4 1

$(2 - 3x)^4 = 1\,(2)^4(-3x)^0 + 4(2)^3(-3x)^1 + 6(2)^2(-3x)^2$

$\quad + 4(2)^1(-3x)^3 + 1(2)^0(-3x)^4$

$\quad = 16 - 96x + 216x^2 - 216x^3 + 81x^4$

Now $(1.97)^4 = (2 - 0.03)^4 = (2 - 3(0.01))^4$

⇨ x = 0.01

$(1.97)^4 = 16 - 96\,(0.01) + 216(0.01)^2 -$

$\quad 216(0.01)^3 + 81(0.01)^4 = 15.0613848$

$\quad = 15.061$ (3d.p)

(b) Use Pascal's triangle to evaluate $(1.02)^3$ correct 5 significant figures

Solutions

$(1.02)^3 = (1 + 02)^3$

Coefficients are : 1 3 3 1

$(1 + 02)^3 = 1^3 + 3(1)^2(0.02) + 3(1)(0.02)^2 +$

$\quad (0.02)^3 = 1.061208$

= 1.0612 (5 significant figures)

The idea of factorial and combination can also be used to determine the coefficients of the expansions

Example 3

Expand

(a) $(p + 3q)^3$

Solution

The coefficients are ${}^3C_0$ ${}^3C_1$ ${}^3C_2$ ${}^3C_3$

Terms are p^3 $p^2(3q)$ $p(3q)^2$ $(3q)^3$

$(p + 3q)^3 = {}^3C_0p^3 + {}^3C_1p^2(3q) + {}^3C_2p(3q)^2 +$

$\quad {}^3C_3(3q)^3$

$\quad = P^3 + 9P^2 3q + 27pq^2 + 27q^3$

(b) $(1 - x)^4$. Hence evaluate $(0.99)^4$ correct to four decimal places

Solution

Coefficients are ${}^4C_0$ ${}^4C_1$ ${}^4C_2$ ${}^4C_3$ ${}^4C_4$

Or simply $\binom{4}{0}$ $\binom{4}{1}$ $\binom{4}{2}$ $\binom{4}{3}$ $\binom{4}{4}$

$1 - x)^4 = {}^4C_0 1^4(-x)^0 + {}^4C_1 1^3(-x)^1 + {}^4C_2 1^2(-x)^2 +$

$\quad {}^4C_3 1^1(-x)^3 + {}^4C_4 1^0(-x)4^0$

$\quad = 1 - 4x + 6x^2 - 4x^3 + x^4$

Now 0.99 = 1 – 0.01) = x = 0.01

$(0.99)^4$

$= 1 - 4(0.01) + 6(0.01)^2 - 4(0.01)^3 + (0.01)^4$

= 0.96059601

= 0.9606 (4d.p)

The binomial theorem for positive integral index

Consider the expansion of $(1 + x)^4$, the result is

$(1 + x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$

Observations

(i) The indices of x increase by 1 from term to term

(ii) The index of the last term being the same as the power to which (1 + x) s raised

(iii) The coefficients of the terms of expansion are $^4C_0$ $^4C_1$ $^4C_2$ $^4C_3$ $^4C_4$

Hence the expansion of $(1 + x)^n$ is

$$(1 + x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \dots x^n$$

The expansion of $(a + x)^n$ is

$$(a + x)^n = a^n\left(1 + \frac{x}{a}\right)^n$$

$$= a^n\left[1 + \binom{n}{1}\left(\frac{x}{a}\right) + \binom{n}{2}\left(\frac{x}{a}\right)^2 + \cdots x\left(\frac{x}{a}\right)^n\right]$$

$$= a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + \cdots x^n$$

Example 4

Expand $(1 + 4x)^{14}$ in ascending power of x up to and include the 4^{th} term. Hence evaluate $(1.0004)^{14}$ correct to four decimal places

Solution

$(1 + 4x)^{14}$

$= 1 + \binom{14}{1}(4x) + \binom{14}{2}(4x)^2 + \binom{14}{3}(4x)^3$

$= 1 + 56x + 1456x^2 + 23296x^3$

Now $(1.0004)^{14} = (1 + 0004)^{14}$ => x = 0.0001

$(1 + 4x)^{14}$

$= 1 + 56(0.0001) + 1456(0.0001)^2 +$

$23296(0.0001)^3$

= 1.005614583

= 1.0056 (4d.p)

Note the next term in the expansion is

$\binom{14}{4}(4x)^4 = 256256x^4$

Its value = $256256(0.0001)^4 = 2.56256 \times 10^{-11}$

Which when added to the above answer there will negligible change in value

Example 5

Expand $(3 – 2x)^{12}$ in ascending powers of x up to and including the term x^3. Hence evaluate $(2.998)^{12}$ correct to the nearest whole number.

Solution

$(3 – 2x)^{12} = 3^{12} + \binom{12}{1}(3)^{11}(-2x)^1 +$

$\binom{12}{2}(3)^{10}(-2x)^2 + \dots$

$= 531441 – 4251528x + 1588936x^2$

Now $(2.998)^{12} = (3 – 0.002)^{12}$ => x = 0.001

$(2.998)^{12} = 531441 – 4251528(0.001) +$

$1588936(0.001)^2$

= 527205.0609

=527205 (nearest whole number)

Particular terms of binomial expansion

As earlier seen, the expansion of

$(a + x)^5 = a^5x^0 + 5a^4x^1 + 10a^3x^2 + 10a^2x^3 + 5ax^4 +$

x^5

In general if r is the power of x in the expansion $(a + x)^n$, then

the U_{r+1} term of x = $^nC_ra^{n-r}x^r$

Example 6

Find the term of x^4 in the expression of

(i) $(1 + x)^9$

By using U_{n+1}; term of x = $^nC_ra^{n-r}x^r$

n = 9, r = 4, a = 1

$U_5 = {}^9C_4 1^{9-4}x^4 = {}^9C_4x^4 = 126x^4$

(ii) (3 + x)7

n = 7, a = 3, r = 4

$U_5 = {}^7C_4 3^{7-4}x^4 = {}^7C_4(3)^3x^4 = 945x^4$

(iii) $\left(2 - \frac{x}{2}\right)^{12}$

$n = 12, a = 2$

$U_5 = {}^{12}C_4 2^{12-4}\left(-\frac{x}{2}\right)^4 = {}^{12}C_4(2)^8\left(-\frac{x}{2}\right)^4 = 7920x^4$

Example 7

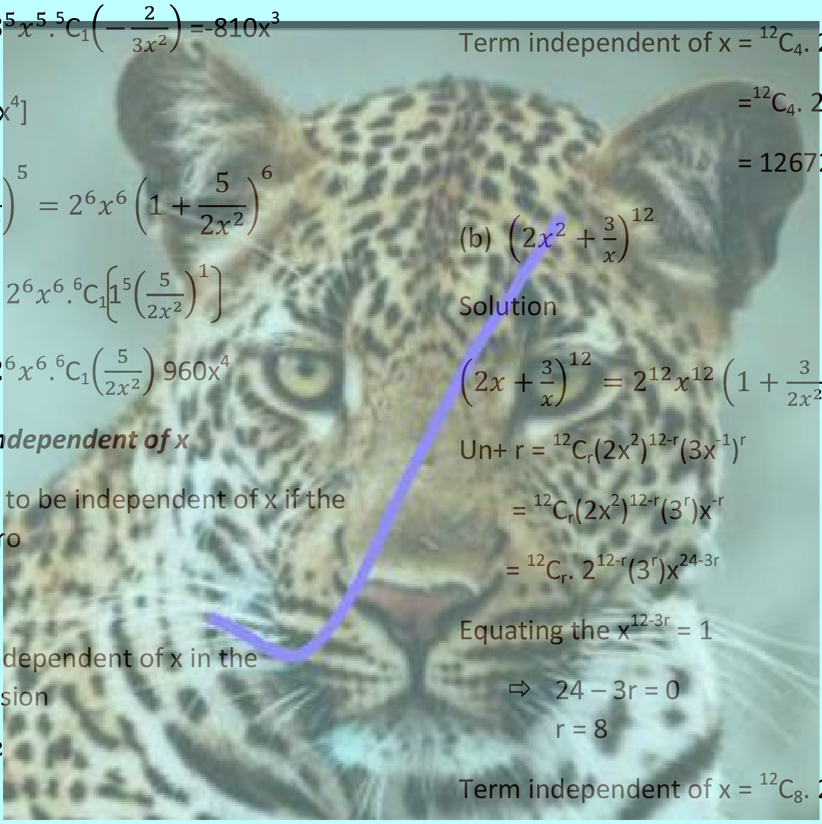
Find the term indicated in expansion of the following expression

(i) $\left(3x - \frac{2}{x}\right)^5$ $[x^3]$

$$\left(3x - \frac{2}{x}\right)^5 = 3^5x^5\left(1 - \frac{2}{3x^2}\right)^5$$

The term in $x^3 = 3^5x^5.{}^5C_1\left[1^4\left(-\frac{2}{3x^2}\right)^1\right]$

$= 3^5x^5.{}^5C_1\left(-\frac{2}{3x^2}\right) = -810x^3$



(ii) $\left(2x + \frac{5}{x}\right)^6$ $[x^4]$

$$\left(2x + \frac{5}{x}\right)^5 = 2^6x^6\left(1 + \frac{5}{2x^2}\right)^6$$

The term in $x^3 = 2^6x^6.{}^6C_1\left[1^5\left(\frac{5}{2x^2}\right)^1\right]$

$= 2^6x^6.{}^6C_1\left(\frac{5}{2x^2}\right)$ $960x^4$

Finding terms independent of x

The term is said to be independent of x if the power of x is zero

Example 8

Find the term independent of x in the following expansion

(a) $\left(2x + \frac{1}{x^2}\right)^{12}$

Solution

$$\left(2x + \frac{1}{x^2}\right)^{12} = 2^{12}x^{12}\left(1 + \frac{1}{2x^3}\right)^{12}$$

The term independent of x = $2^{12}x^{12}$ multiplied by the term in x^{-12} in the expansion $\left(2x + \frac{1}{x^2}\right)^{12}$

⇨ The term independent of x

$= 2^{12}x^{12}.{}^{12}C_4\left(\frac{1}{2x^3}\right)^4 = 2^{12}x^{12} \text{ x } {}^{12}C_4 \text{ x } \frac{1}{2^4x^{12}}^4$

$= 2^8 \text{ x } 495 = 126720$

Alternatively

By using $U_{r+1} = {}^nC_r a^{n-r}x^r$

The term independent of x is got by equating the index of x to zero

$U_{r+1} = {}^{12}C_r(2x)^{12-r}\left(\frac{1}{x^2}\right)^r$

$= {}^{12}C_r(2x)^{12-r}x^{-2r}$

$= {}^{12}C_r.\ 2^{12-r}.x^{12-3r}$

Equating the $x^{12-3r} = 1$

⇨ $12 - 3r = 0$

$r = 4$

Term independent of x = ${}^{12}C_4.\ 2^{12-4}$

$= {}^{12}C_4.\ 2^8$

$= 126720$

(b) $\left(2x^2 + \frac{3}{x}\right)^{12}$

Solution

$$\left(2x + \frac{3}{x}\right)^{12} = 2^{12}x^{12}\left(1 + \frac{3}{2x^2}\right)^{12}$$

$Un + r = {}^{12}C_r(2x^2)^{12-r}(3x^{-1})^r$

$= {}^{12}C_r(2x^2)^{12-r}(3^r)x^{-r}$

$= {}^{12}C_r.\ 2^{12-r}(3^r)x^{24-3r}$

Equating the $x^{12-3r} = 1$

⇨ $24 - 3r = 0$

$r = 8$

Term independent of x = ${}^{12}C_8.\ 2^{12-4}.3^8$

$= {}^{12}C_8.\ 2^4.3^8$

$= 51, 963,120$

Binomial expansion of terms with fractional or negative powers

As noted earlier

$(1 + x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \dots x^n$

The following is noted

(i) For positive integral value of n, i.e. n ≥ 1, the series above terminates at the term x^n and its sum is $(1 + x)^n$.

(ii) For fractional or negative values of n, the series above does not terminate but instead converges to $(1 + x)^n$ as the limit of its sum only $-1 < x < 1$ or $|x| < 1$

Example 9

Expand $\sqrt{(1-2x)}$ up to the term x^3. Hence evaluate $\sqrt{0.98}$ correct to four decimal places.

Solution

$$\sqrt{(1-2x)} = (1-2x)^{\frac{1}{2}}$$

Using

$$(1 + x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots\ x^n$$

Comparing with $(1-2x)^{\frac{1}{2}}$ => $x \equiv -2x$ and $n = ½$

$$\sqrt{(1-2x)} = 1 + \frac{1}{2}(-2x) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)(-2x)^2}{2!} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-2x)^3}{3!}$$

$$= 1 - x - \frac{1}{2}x^2 - \frac{1}{2}x^3$$

Now $\sqrt{0.98} = \sqrt{(1-0.02)}$

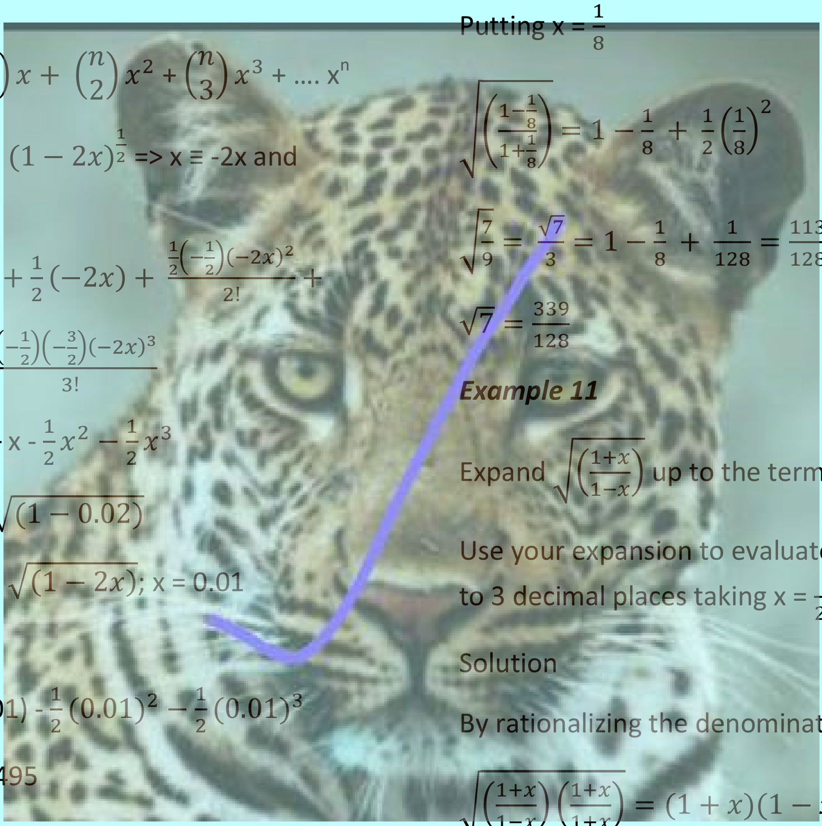
Comparing with $\sqrt{(1-2x)}$; $x = 0.01$

Substituting

$$\sqrt{0.98} = 1 - (0.01) - \frac{1}{2}(0.01)^2 - \frac{1}{2}(0.01)^3$$

$$= 0.9899495$$

$\therefore \sqrt{0.98} = 0.9899$ (4d.p)



Example 10

Given that x is very small that its cube and higher powers can be neglected, show that

$$\sqrt{\left(\frac{1-x}{1+x}\right)} = 1 - x + \frac{1}{2}x^2$$

By putting $x = \frac{1}{8}$ show that $\sqrt{7} = \frac{339}{128}$

Solution

By rationalizing the denominator

$$\sqrt{\left(\frac{1-x}{1+x}\right)\left(\frac{1-x}{1-x}\right)} = \sqrt{\frac{(1-x)^2}{(1-x^2)}}$$

$$= (1-x)(1-x^2)^{\frac{1}{2}}$$

But $(1-x^2)^{\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)(-x^2) + \ldots$

$$= 1 + \frac{1}{2}x^2$$

$$\sqrt{\left(\frac{1-x}{1+x}\right)} = (1-x)\left(1 + \frac{1}{2}x^2\right)$$

$$= 1 - x + \frac{1}{2}x^2$$

Putting $x = \frac{1}{8}$

$$\sqrt{\left(\frac{1-\frac{1}{8}}{1+\frac{1}{8}}\right)} = 1 - \frac{1}{8} + \frac{1}{2}\left(\frac{1}{8}\right)^2$$

$$\sqrt{\frac{7}{9}} = \frac{\sqrt{7}}{3} = 1 - \frac{1}{8} + \frac{1}{128} = \frac{113}{128}$$

$$\sqrt{7} = \frac{339}{128}$$

Example 11

Expand $\sqrt{\left(\frac{1+x}{1-x}\right)}$ up to the term x^3

Use your expansion to evaluate $\sqrt{23}$ correct to 3 decimal places taking $x = \frac{1}{24}$.

Solution

By rationalizing the denominator

$$\sqrt{\left(\frac{1+x}{1-x}\right)\left(\frac{1+x}{1+x}\right)} = (1+x)(1-x^2)^{\frac{1}{2}}$$

But $(1-x^2)^{\frac{1}{2}} = 1 + \frac{1}{2}x^2 + \cdots$

$$\sqrt{\left(\frac{1+x}{1-x}\right)} = (1+x)\left(1 + \frac{1}{2}x^2\right)$$

$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{2}x^3$$

Hence putting $x = \frac{1}{24}$

$$\sqrt{\left(\frac{1+\frac{1}{24}}{1-\frac{1}{24}}\right)} = 1 + \frac{1}{24} + \frac{1}{2}\left(\frac{1}{24}\right)^2 + \frac{1}{2}\left(\frac{1}{24}\right)^3$$

$\sqrt{\frac{25}{23}} = \frac{5}{\sqrt{23}} = \frac{28825}{27648}$

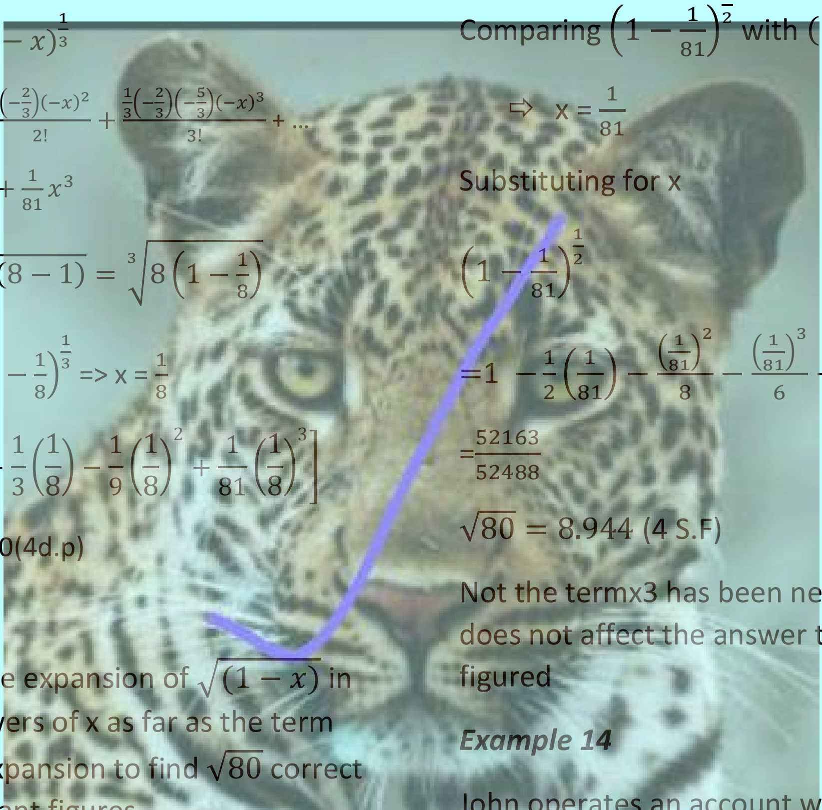
$\sqrt{23} = \frac{5 \times 27648}{28825} = 4.7958$

$\therefore \sqrt{23} = 4.7958$ (4d.p)

Example 12

Use the binomial theorem to expand $\sqrt[3]{(1-x)}$ up to x^3. Use your expansion to evaluate $\sqrt[3]{7}$ correct to four decimal places

Solution



$\sqrt[3]{(1-x)} = (1-x)^{\frac{1}{3}}$

$= 1 + \frac{1}{3}(-x) + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)(-x)^2}{2!} + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)(-x)^3}{3!} + \ldots$

$= 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{1}{81}x^3$

Hence $\sqrt[3]{7} = \sqrt[3]{(8-1)} = \sqrt[3]{8\left(1-\frac{1}{8}\right)}$

$= 2\left(1-\frac{1}{8}\right)^{\frac{1}{3}}$ => $x = \frac{1}{8}$

$\sqrt[3]{7} = 2\left[1 + \frac{1}{3}\left(\frac{1}{8}\right) - \frac{1}{9}\left(\frac{1}{8}\right)^2 + \frac{1}{81}\left(\frac{1}{8}\right)^3\right]$

=1.9130(4d.p)

Example 13

Write down the expansion of $\sqrt{(1-x)}$ in ascending powers of x as far as the term x^4. Use your expansion to find $\sqrt{80}$ correct to four significant figures.

Solution

$\sqrt{(1-x)} = (1-x)^{\frac{1}{2}}$

For binomial expansion

$(1 + x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots x^n$

Comparing $(1-x)^{\frac{1}{2}}$ with$(1 + x)^n$

$n = \frac{1}{2}$ and $x = -x$

$(1-x)^{\frac{1}{2}} = 1 + \frac{1}{2}(-x) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)(-x)^2}{2!} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-x)^3}{3!} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(-x)^4}{4!}$

$(1-x)^{\frac{1}{2}} = 1 - \frac{1}{2}x - \frac{x^2}{8} - \frac{x^3}{6} - \frac{x^4}{24}$

Now $\sqrt{80} = \sqrt{81-1} = \sqrt{81\left(1-\frac{1}{81}\right)}$

$= 9\left(1-\frac{1}{81}\right)^{\frac{1}{2}}$

Comparing $\left(1-\frac{1}{81}\right)^{\frac{1}{2}}$ with $(1-x)^{\frac{1}{2}}$

⇨ $x = \frac{1}{81}$

Substituting for x

$\left(1-\frac{1}{81}\right)^{\frac{1}{2}}$

$= 1 - \frac{1}{2}\left(\frac{1}{81}\right) - \frac{\left(\frac{1}{81}\right)^2}{8} - \frac{\left(\frac{1}{81}\right)^3}{6} - \frac{\left(\frac{1}{81}\right)^4}{24}$

$= \frac{52163}{52488}$

$\sqrt{80} = 8.944$ (4 S.F)

Not the termx3 has been neglected as it does not affect the answer to 4 significant figured

Example 14

John operates an account with a certain bank which pays a compound interest rate of 13.5% per annum. He opened the account at the beginning of the year with sh. 500,000 and deposits the same amount of money at the beginning of every year. Calculate how much he will accumulate at the end of 9 years. After how long will the money have accumulated to sh. 3.32 millions?

Solution

The 1st deposit will grow to

$500000\left(1+\frac{13.5}{100}\right)$= 500000 x 1.135

2^{nd} deposit will grow to 500000 x 1.135^2

n^{th} deposit will grow to 500000 x 1.135^n

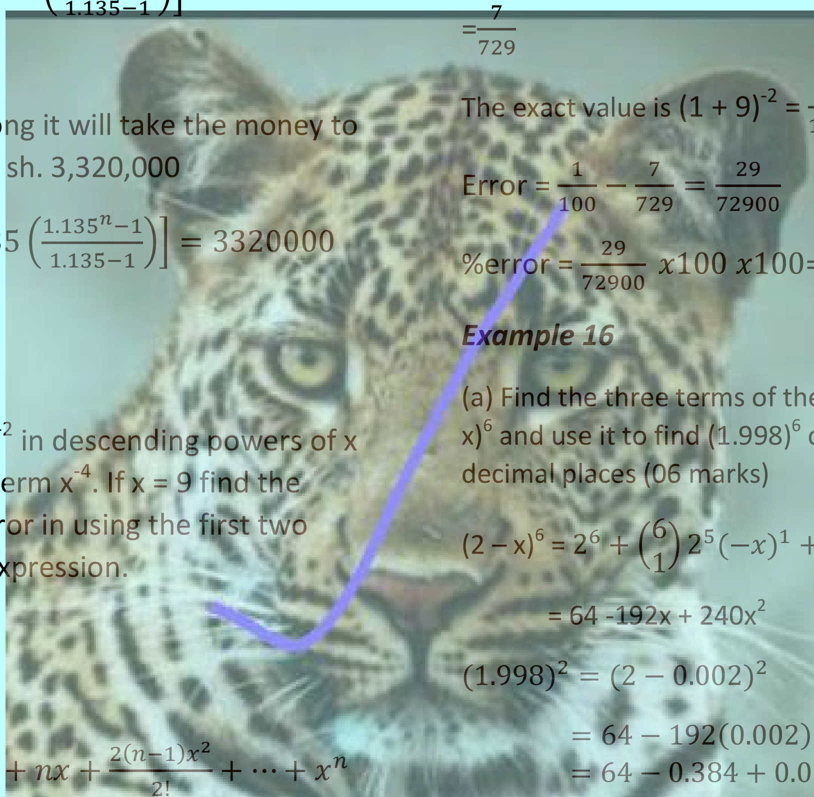
9^{th} deposit will grow to 500000 x 1.135^9

The total = 500,000[1.135 +1.135^2 + … + 1.135^9]

=$500000\left[a\left(\frac{r^n-1}{r-1}\right)\right]$

=$500000\ \left[1.135\left(\frac{1.135^9-1}{1.135-1}\right)\right]$

=8,936,381



Finding how long it will take the money to accumulate to sh. 3,320,000

$500000\ \left[1.135\left(\frac{1.135^n-1}{1.135-1}\right)\right]=3320000$

n = 4.6years

Example 15

Expand $(1 + x)^{-2}$ in descending powers of x including the term x^{-4}. If x = 9 find the percentage error in using the first two terms of the expression.

Solution

From

$$(1+x)^n=1+nx+\frac{2(n-1)x^2}{2!}+\cdots+x^n$$

Now $(1 + x)^{-2}$ = $x^{-2}\left(1+\frac{1}{x}\right)^{-2}$

$x^{-2}\left(1+\frac{1}{x}\right)^{-2}$

$=x^{-2}\left[1+(-2)\frac{1}{x}+\frac{(-2)(-3)}{2!}\left(\frac{1}{x}\right)^2\right]$

=$x^{-2}\left[1-\frac{2}{x}+\frac{3}{x^2}\right]$

=$x^{-2}[1-2x^{-1}+3x^{-2}]$

=$x^{-2}-2x^{-3}+3x^{-4}$

If x = 9

$(1 + x)^{-2}$=$9^{-2}-2(9)^{-3}+3(9)^{-4}$

=$\frac{1}{81}-\frac{2}{729}$ (using the first 2 terms)

=$\frac{7}{729}$

The exact value is $(1 + 9)^{-2}$ = $\frac{1}{100}$

Error = $\frac{1}{100}-\frac{7}{729}=\frac{29}{72900}$

%error = $\frac{29}{72900}\ x100\ x100$= 3.978%(3d.p)

Example 16

(a) Find the three terms of the expansion $(2 – x)^6$ and use it to find $(1.998)^6$ correct to two decimal places (06 marks)

$(2 – x)^6$ = 2^6 + $\binom{6}{1}2^5(-x)^1+\binom{6}{2}4(-x)^2$

= 64 -192x + $240x^2$

$(1.998)^2=(2-0.002)^2$

$$=64-192(0.002)+240(0.002)^2$$
$$=64-0.384+0.00096$$
$$=63.61696$$
$$=63.62\ (2D)$$

Example 17

(a) Prove by induction

1.3 + 2.4 + … + n(n + 2) = $\frac{1}{6}n(n+1)(2n+7)$ for all values of n.

Suppose n = 1

L.H.S = 1 x 3 = 3

R.H.S = $\frac{1}{6}x1(1+1)(2+7)=3$

L.H.S = R.H.S, hence the series holds for n =1

Suppose n = 2

L.H.S= 1 x 3 + 2 x 4 = 11

R.H.S = $\frac{1}{6}\ x\ 2(2+1)(4+7)=11$

L.H.S = R.H.S, hence the series holds for n =2

Suppose n = k

1.3 + 2.4 + … + k(k + 2) = $\frac{1}{6}k(k+1)(2k+7)$

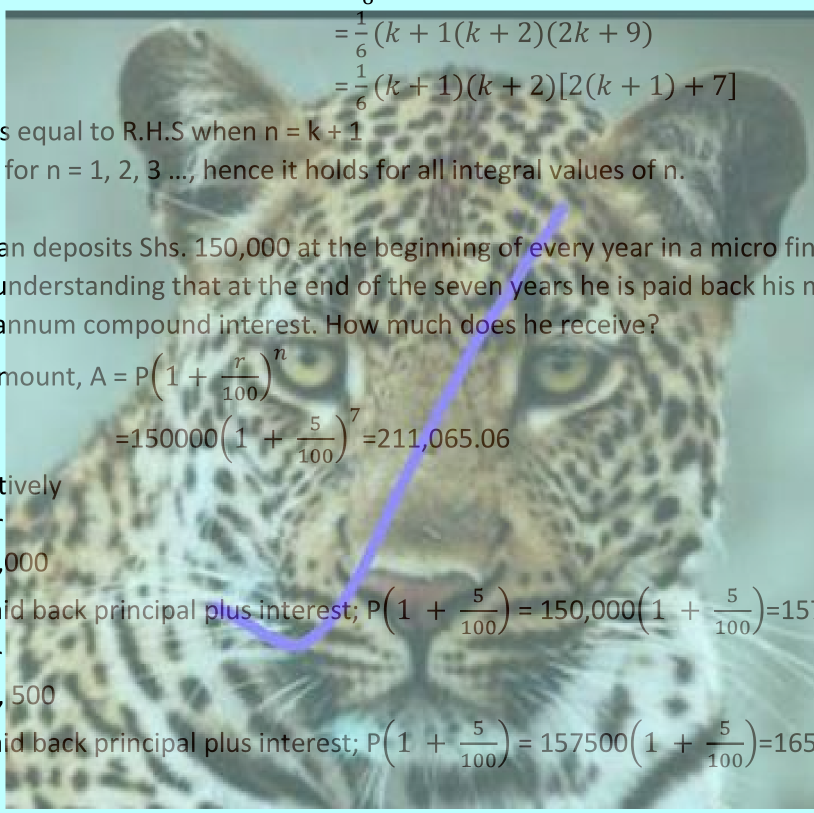
For n = k + 1

1.3 + 2.4 + … + k(k + 2), (k + 1)(k + 3) = $\frac{1}{6}k(k+1)(2k+7)$+(k +1)(k + 3)

$=(k+1)\left[\frac{1}{6}k(2k+7)+(k+3\right]$

$=\frac{1}{6}(k+1)(2k^2+13k+18)$

$=\frac{1}{6}(k+1)(2k^2+4k+9k+18)$



$=\frac{1}{6}(k+1(k+2)(2k+9)$

$=\frac{1}{6}(k+1)(k+2)[2(k+1)+7]$

Which is equal to R.H.S when n = k + 1

It holds for n = 1, 2, 3 …, hence it holds for all integral values of n.

(b) A man deposits Shs. 150,000 at the beginning of every year in a micro finance bank with the understanding that at the end of the seven years he is paid back his money with 5% per annum compound interest. How much does he receive?

Using amount, A = $P\left(1+\frac{r}{100}\right)^n$

$=150000\left(1\ +\ \frac{5}{100}\right)^7$=211,065.06

Alternatively

1st year

P = 150,000

He is paid back principal plus interest; $P\left(1\ +\ \frac{5}{100}\right)$ = 150,000$\left(1\ +\ \frac{5}{100}\right)$=157,500

2nd year

P = 157, 500

He is paid back principal plus interest; $P\left(1\ +\ \frac{5}{100}\right)$ = 157500$\left(1\ +\ \frac{5}{100}\right)$=165375

3rd year

P = 165375

Interest = $\frac{5}{10}\ x\ 165375=8268.75$

He is paid back principal plus interest; $P\left(1\ +\ \frac{5}{100}\right)$ = 165375 $\left(1\ +\ \frac{5}{100}\right)$=173643.75

4th year

P=173643.75

He is paid back principal plus interest; $P\left(1\ +\ \frac{5}{100}\right)$ = 173643.75$\left(1\ +\ \frac{5}{100}\right)$=182325.94

5th year

P =182325.94

His paid back principal plus interest; $P\left(1\ +\ \frac{5}{100}\right)$ = 182325.94$\left(1\ +\ \frac{5}{100}\right)$=191442.23

6th year

P = 191442.23

He is paid back principal plus interest; $P\left(1\ +\ \frac{5}{100}\right)$ = 191442.23$\left(1\ +\ \frac{5}{100}\right)$=201014.35

7th year

P = 201014.35

He is paid back principal plus interest; $P\left(1 + \frac{5}{100}\right) = 201014.35\left(1 + \frac{5}{100}\right)$=211,065.06

∴ by the 7th year he has accumulated shs. **211,065.06**

Example 18

Expand $\sqrt{\left(\frac{1+2x}{1-x}\right)}$ up to the term x^2. Hence find the value of $\sqrt{\left(\frac{1.04}{0.98}\right)}$ to four significant figures. (12marks)

$$\sqrt{\left(\frac{1+2x}{1-x}\right)} = (1+2x)^{\frac{1}{2}}(1-x)^{\frac{-1}{2}}$$

$$using\ (1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \cdots$$

$$\sqrt{\left(\frac{1+2x}{1-x}\right)} = \left(1 + x - \frac{1}{2}x^2\right)\left(1 + \frac{1}{2}x + \frac{3}{8}x^2\right)$$

$$= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + x + \frac{1}{2}x^2 - \frac{1}{2}x^2$$

$$= 1 + \frac{3}{2}x + \frac{3}{8}x^2$$

$$\therefore \sqrt{\left(\frac{1+2x}{1-x}\right)} \approx 1 + \frac{3}{2}x + \frac{3}{8}x^2$$

$$substituting\ for\ x = 0.02$$

$$\sqrt{\left(\frac{1.04}{0.98}\right)} = \sqrt{\frac{1+2(0.02)}{1-0.02}}$$

$$= 1 + \frac{3}{2}(0.02) + \frac{3}{8}(0.02)^2 = 1.030$$

Revision exercise

1. Given that the ratio of the 3rd to the 4th term of the expansion (2+3x)n is 5:14, find the value of n when $x = \frac{2}{5}$. [n= 16]
2. Expand $(3 - 4x)^5$ in ascending order of x up to and including the term x^3. Hence evaluate $(4.96)^5$ correct to 2 d.p. [3001.98]
3. (a) Find the coefficient of x2 in the expansion of $(1 - 2x)^n$ is 24 [4]
   (b) Find the term independent of x in the expansion of $\left(2x^3 - \frac{1}{x}\right)^{20}$ [-496128]
   (c) Use the binomial expansion to expand $\sqrt[4]{(1+2x)}$ up to the term x^3. Hence evaluate $\sqrt[4]{83}$ correct to three decimal places [3.018]
4. Expand $\sqrt{\left(\frac{1+2x}{1-2x}\right)}$ up to and including the term x3. Hence find the value of $\sqrt{\frac{1.02}{0.98}}$ to four significant figures. Deduce the value of $\sqrt{51}$ to 3 significant figures [1.0202, 7.14]
5. Five millions shillings are invested each year at a rate of 15%. In how many years will it accumulate to more than 50millions? [6years]
6. Expand $\left(1 - \frac{x}{3}\right)^{\frac{1}{2}}$ as far as x3. Hence evaluate $\sqrt{8}$ correct to 3 decimal places [2.829]
7. A man deposits sh. 800,000 into his saving account on which interest is 15% per annum. If he makes no withdraws, after how many years will his balance exceed sh. 8millions? [16.5 years]
8. Determine the binomial expansion of $\left(1 + \frac{x}{2}\right)^4$. Hence evaluate $(2.1)^4$ correct to 2 decimal places.[19.45]
9. Determine the binomial expansion of $\left(1 - \frac{x}{2}\right)^5$. Hence evaluate $(0.875)^5$ correct to four decimal places [0.5129]

10. A financial credit society give a compound interest of 2% per annum to its members. If Bbosa deposits sh. 10000 at the beginning of every year. How much would he accumulate at the end of the fifth year if no withdraws within this period [sh. 530812]

11. Expand $\sqrt{\left(\frac{1+x}{1-x}\right)}$ in ascending powers of x up to a term x^2. $\left[1 + x + \frac{x^2}{x} + \cdots\right]$

12. (a) Using the expansion $(1 + x)^{\frac{1}{2}}$ up to the term x^3, find the value of $\sqrt{1.08}$ to four decimal places [1.0392]

(b) Express $\sqrt{1.08}$ in the form $\frac{a}{b}\sqrt{c}$. Hence evaluate $\sqrt{3}$ correct to 3 significant figure $\left[\frac{3}{5}\sqrt{3}, 1.73\right]$

13. (a) obtain the first four non – zero terms of the binomial expansion in ascending powers of x of $(1 - x^2)^{-\frac{1}{2}}$ give that$|x| < 1$ $\left[1 + \frac{x^2}{2} + \frac{3x^4}{8} + \frac{5x^6}{16}\right]$

(b) show that, when x = $\frac{1}{3}$;

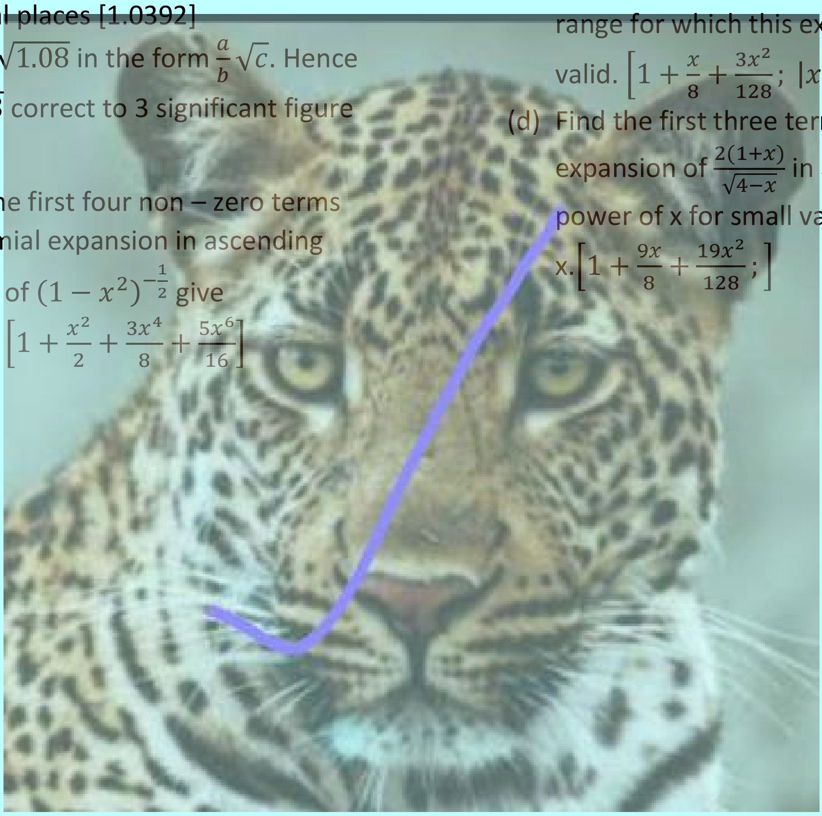
$(1 - x^2)^{-\frac{1}{2}} = \frac{3\sqrt{2}}{4}$

(c) substitute x = $\frac{1}{3}$ into your expansion and hence obtain an approximation of $\sqrt{2}$, give your answer to five decimal places [1.41415]

14. (a) show that $\frac{1}{\sqrt{4-x}} = \frac{1}{2}\left(1 - \frac{x}{4}\right)^{-\frac{1}{2}}$

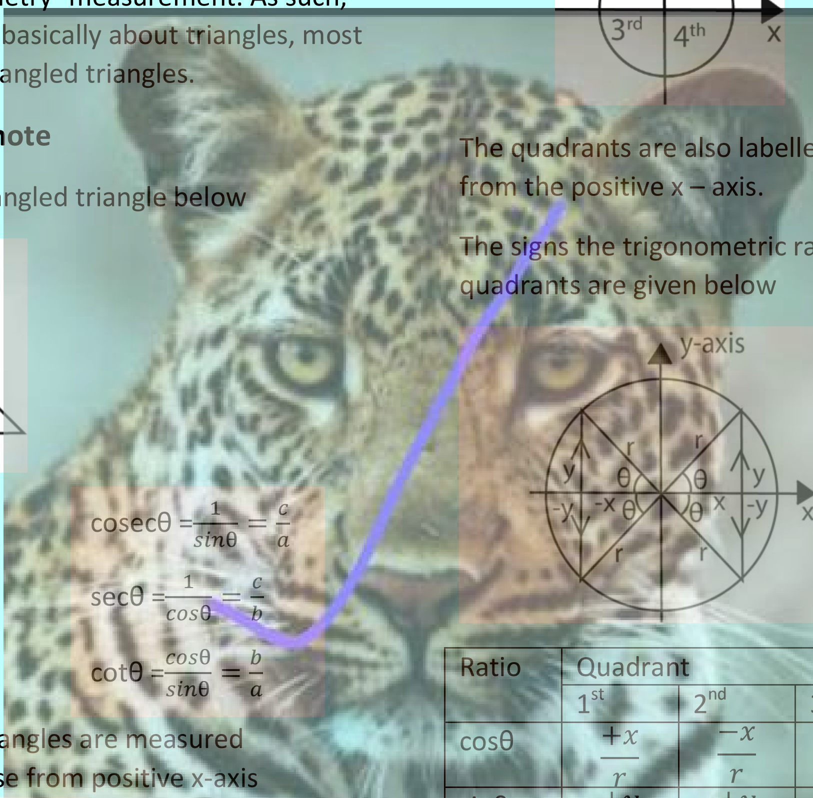
(c) Write the first three terms in the binomial expansion of $\left(1 - \frac{x}{4}\right)^{-\frac{1}{2}}$ in ascending power of x stating the range for which this expansion is valid. $\left[1 + \frac{x}{8} + \frac{3x^2}{128};\ |x| < 4\right]$

(d) Find the first three terms in the expansion of $\frac{2(1+x)}{\sqrt{4-x}}$ in ascending power of x for small values of x.$\left[1 + \frac{9x}{8} + \frac{19x^2}{128};\right]$



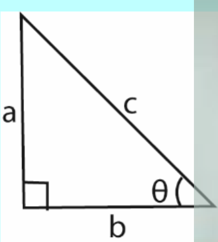
Trigonometry

The word 'trigonometry' suggests 'tri'-three, 'gono'-angle, 'metry'-measurement. As such, trigonometry is basically about triangles, most especially right-angled triangles.



Important to note

(a) For a right angled triangle below



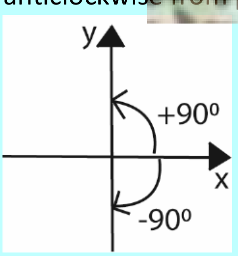
- $\sin\theta = \frac{a}{c}$
- $\cos\theta = \frac{b}{c}$
- $\tan\theta = \frac{\sin\theta}{\cos\theta} \frac{a}{b}$

$\text{cosec}\theta = \frac{1}{\sin\theta} = \frac{c}{a}$

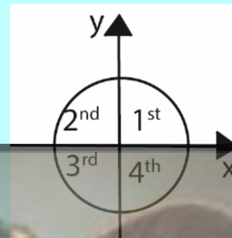
$\sec\theta = \frac{1}{\cos\theta} = \frac{c}{b}$

$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{b}{a}$

(b) All positive angles are measured anticlockwise from positive x-axis



(c) A circle drawn with the centre O, divides the co-ordinate axis into four equal parts called quadrants



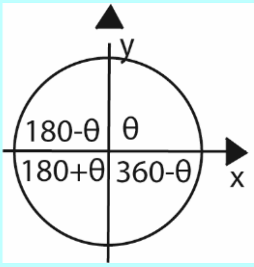
The quadrants are also labelled anti-clockwise from the positive x – axis.

The signs the trigonometric ratios in the quadrants are given below

Ratio	Quadrant			
	1st	2nd	3rd	4th
cosθ	$\frac{+x}{r}$	$\frac{-x}{r}$	$\frac{-x}{r}$	$\frac{+x}{r}$
sinθ	$\frac{+y}{r}$	$\frac{+y}{r}$	$\frac{-y}{r}$	$\frac{-y}{r}$
tanθ	$\frac{+y}{x}$	$\frac{y}{-x}$	$\frac{y}{x}$	$\frac{-y}{x}$
secθ	$\frac{+r}{x}$	$\frac{-r}{x}$	$\frac{-r}{x}$	$\frac{+r}{x}$
cosecθ	$\frac{+r}{y}$	$\frac{+r}{y}$	$\frac{-r}{y}$	$\frac{-r}{y}$
cotθ	$\frac{+x}{y}$	$\frac{-x}{y}$	$\frac{+x}{y}$	$\frac{-x}{y}$

Note

- If θ is the angle in the 1st quadrat
- In the 2nd quadrat the angle is (180 – θ)
- In the 3rd quadrat the angle is (180 + θ)
- In the 4th quadrat the angle is (360 –θ)



Solving equations

We make use of the quadrants to find the ranges of values within which the angle follows

Example 1

Solve the following equations for $0^0 \leq \theta \leq 360^0$

(i) $3\cos\theta + 2 = 0$

Solution

$\cos\theta = -\frac{2}{3}$

But cos is negative in the 2^{nd} and 3^{rd} quadrants.

Ignoring the negative sign, the angle obtained is referred to as the key or principle angle, i.e.

key angle = $\cos^{-1}\frac{2}{3} = 48.2^0$ (1d.p)

In the 2^{nd} quadrant, $\theta = 180 - 48.2 = 131.8^0$

In the 3^{rd} quadrant, $\theta = 180 + 48.2 = 228.2^0$

$\therefore\{\theta: \theta = 131.8^0, 228.2^0\}$

Note that: the key angle s not part of the solution but only a guide.

(ii) $4\cos^2\theta - 1 = 0$

Solution

$\cos\theta = \sqrt{\frac{1}{4}} = \pm\frac{1}{2}$

Key angle, $\theta = \cos^{-1}\frac{1}{2} = 60^0$

When $\cos\theta = ½$ (positive is 1^{st} and 4^{th} quadrants)

1^{st} quadrant $\theta = 60^0$

4^{th} quadrant $\theta = 360 - 60 = 300^0$

When $\cos\theta = -½$ (positive is 2^{nd} and 3^{rd} quadrants)

3^{rd} quadrant $\theta = 180 - 60 = 120^0$

4^{th} quadrant $\theta = 180 + 60 = 240^0$

$\therefore\{\theta: \theta = 60^0, 120^0, 240^0, 300^0\}$

(iii) $\text{cosec}\theta + 2 = 0$

Solution

$\text{cosec}\theta = -2 \Rightarrow \sin\theta = -½$ (taking reciprocal)

Key angle = $\sin^{-1}\frac{1}{2} = 30^0$

In the 3^{rd} quadrant $\theta = 180 + 30 = 210^0$

In the 4^{th} quadrant $\theta = 360 - 30 = 330^0$

$\therefore\{\theta: \theta = 210^0, 330^0\}$

(iv) $3\sec^2\theta - 4 = 0$

Solution

$\sec\theta = \pm\frac{2}{\sqrt{3}} \Rightarrow \cos\theta = \pm\frac{\sqrt{3}}{2}$

Key angle = $\cos^{-1}\frac{\sqrt{3}}{2} = 30^0$

For $\cos\theta = \frac{\sqrt{3}}{2}$; $\theta = 30^0, 330^0$

For $\cos\theta = -\frac{\sqrt{3}}{2}$; $\theta = 120^0, 210^0$

$\therefore\{\theta: \theta = 30^0, 120^0, 210^0, 330^0\}$

(d) Definitions of angle

(i) **Acute angle** is an angle between 0^0 and 90^0. It lies in the 1^{st} quadrant

(ii) **Right angle** is an angle = 90^0

(iii) **Obtuse angle** is an angle between 90^0 and 180^0. It lies in the 2^{nd} quadrant

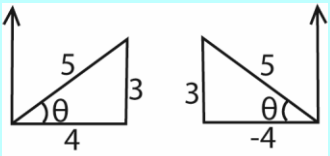
(iv) **Reflex angle** is an angle between 180^0 and 360^0. It lies in the 3^{rd} and 4^{th} quadrant

Example 2

(a) If $\sin\theta = \frac{3}{5}$ and $0^0 \leq \theta \leq 360^0$. Find the possible values of $3\tan\theta - \cot\theta$

Solution

If $\sin\theta = \frac{3}{5}$; θ lies in 1st or 2nd quadrants

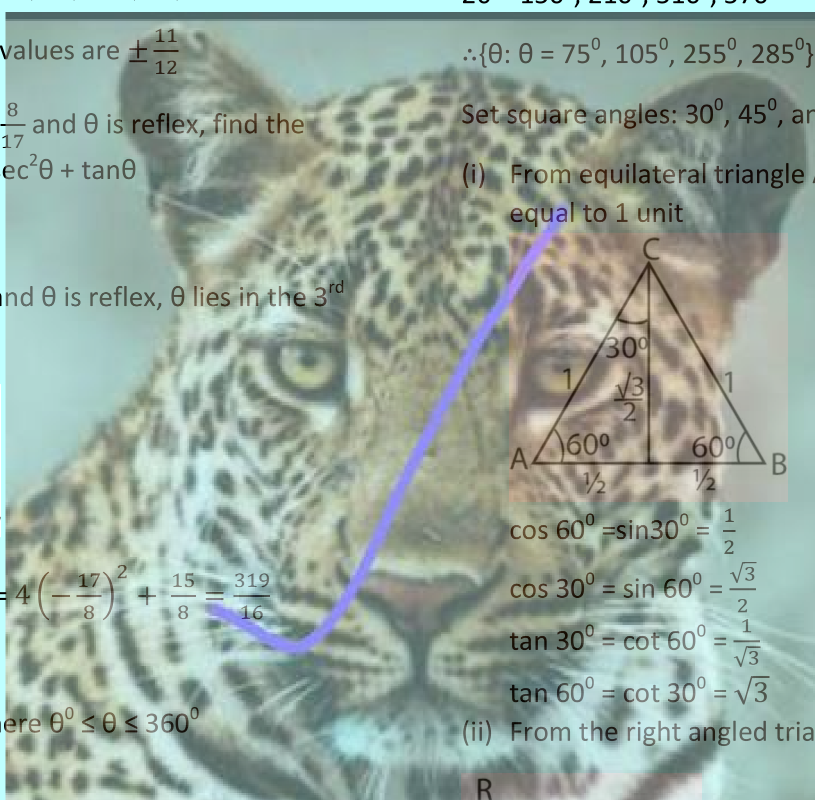


In 1st quadrant

$3\tan\theta - \cot\theta = 3\left(\frac{3}{4}\right) - \left(\frac{4}{3}\right) = \frac{11}{12}$

In 2nd quadrant

$3\tan\theta - \cot\theta = 3\left(-\frac{3}{4}\right) - \left(-\frac{4}{3}\right) = -\frac{11}{12}$

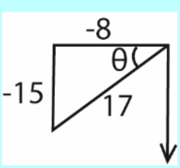


∴ the possible values are $\pm\frac{11}{12}$

(b) If $\cos\theta = -\frac{8}{17}$ and θ is reflex, find the value of $4\sec^2\theta + \tan\theta$

Solution

If $\cos\theta = -\frac{8}{17}$ and θ is reflex, θ lies in the 3rd quadrant



$4\sec^2\theta + \tan\theta = 4\left(-\frac{17}{8}\right)^2 + \frac{15}{8} = \frac{319}{16}$

Example 3

Solve for θ, where $0^0 \leq \theta \leq 360^0$

(i) $3\tan^2 3\theta = 1$

Solution

$\tan 3\theta = \pm\frac{1}{\sqrt{3}}$

taking $\tan 3\theta = \frac{1}{\sqrt{3}}$

⇨ $3\theta = 30^0, 210^0, 390^0, 570^0, 750^0, 930^0$
$\theta = 10^0, 70^0, 130^0, 190^0, 250^0, 310^0$

taking $\tan 3\theta = -\frac{1}{\sqrt{3}}$

⇨ $3\theta = 150^0, 330^0, 510^0, 690^0, 870^0, 1050^0$
$\theta = 50^0, 110^0, 170^0, 230^0, 290^0, 350^0$

$\therefore \{\theta: \theta = 10^0, 50^0, 70^0, 110^0, 130^0, 170^0, 190^0, 230^0, 250^0, 290^0, 310^0, 350^0\}$

Note

- If $0^0 \leq \theta \leq 360^0$ then $0^0 \leq 3\theta \leq 1080^0$ [multiply the interval through by 3]

(ii) $2\cos 2\theta + \sqrt{3} = 0$

Solution

$\cos 2\theta = -\frac{\sqrt{3}}{2}$ and $0^0 \leq 2\theta \leq 720^0$

$2\theta = 150^0, 210^0, 510^0, 570^0$

$\therefore \{\theta: \theta = 75^0, 105^0, 255^0, 285^0\}$

Set square angles: 30^0, 45^0, and 60^0

(i) From equilateral triangle ABC with side equal to 1 unit

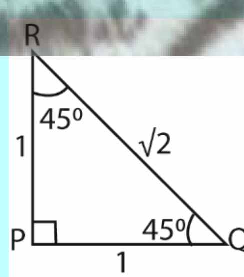
$\cos 60^0 = \sin 30^0 = \frac{1}{2}$

$\cos 30^0 = \sin 60^0 = \frac{\sqrt{3}}{2}$

$\tan 30^0 = \cot 60^0 = \frac{1}{\sqrt{3}}$

$\tan 60^0 = \cot 30^0 = \sqrt{3}$

(ii) From the right angled triangle PQR below



$\cos 45^0 = \sin 45^0 = \frac{1}{\sqrt{2}}$

$\tan 45^0 = 1$

Example 4

Without using tables or calculators find the value of

(i) $\cos 240^0$

Solution

$\cos 240^0 = -\cos(240 - 180)^0 = -\cos 60^0 = -\frac{1}{2}$

(ii) $\tan 3990^0$

Solution

$\tan 3990^0 = \tan [(360 \times 11) + 30]^0 = \tan 30^0 = \frac{1}{\sqrt{3}}$

(iii) $\sin 570^0$

Solution

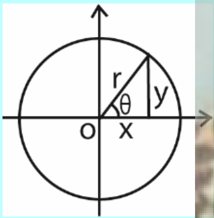
$\sin 570^0 = \sin \{(360 \times 1) + 210\}^0 = -\sin 30 = -\frac{1}{2}$

(iv) $\sec 225^0$

Solution

$\sec 225^0 = \sec (225 - 180)0 = \sec 45^0 = -\sqrt{2}$

The Pythagoras theorem



For any acute angle θ

$x = r\cos\theta$ and $y = r\sin\theta$

By Pythagoras theorem

$x^2 + y^2 = r^2$

Substituting for x and y

$(r\cos\theta)^2 + (r\sin\theta)^2 = r^2$

$r^2(\cos^2\theta + \sin^2\theta) = r^2$

$\therefore \cos^2\theta + \sin^2\theta = 1$

Now $\tan\theta = \frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \frac{\sin\theta}{\cos\theta}$

$\therefore \frac{\sin\theta}{\cos\theta} = \tan\theta$

Identities

$\cos^2\theta + \sin^2\theta = 1$ …………………(i)

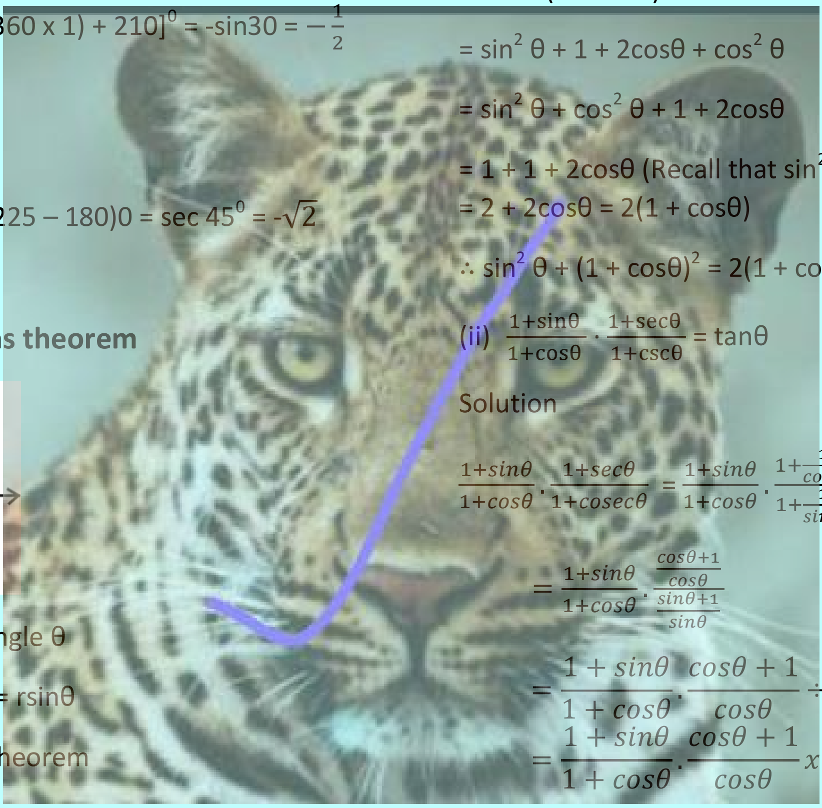
Identity (i) ÷ $\cos^2\theta$

$1 + \tan^2\theta = \sec^2\theta$ ……………… (ii)

Identiy (i) ÷ $\sin^2\theta$

$1 + \cot^2\theta = \text{cosec}^2\theta$ …………(iii)

Example 5



Show that

(i) $\sin^2\theta + (1 + \cos\theta)^2 = 2(1 + \cos\theta)$

Solution

$\sin^2\theta + (1 + \cos\theta)^2$

$= \sin^2\theta + 1 + 2\cos\theta + \cos^2\theta$

$= \sin^2\theta + \cos^2\theta + 1 + 2\cos\theta$

$= 1 + 1 + 2\cos\theta$ (Recall that $\sin^2\theta + \cos^2\theta = 1$)

$= 2 + 2\cos\theta = 2(1 + \cos\theta)$

$\therefore \sin^2\theta + (1 + \cos\theta)^2 = 2(1 + \cos\theta)$

(ii) $\frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{1+\sec\theta}{1+\csc\theta} = \tan\theta$

Solution

$$\frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{1+\sec\theta}{1+\text{cosec}\theta} = \frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{1+\frac{1}{\cos\theta}}{1+\frac{1}{\sin\theta}}$$

$$= \frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{\frac{\cos\theta+1}{\cos\theta}}{\frac{\sin\theta+1}{\sin\theta}}$$

$$= \frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{\cos\theta+1}{\cos\theta} \div \frac{\sin\theta+1}{\sin\theta}$$

$$= \frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{\cos\theta+1}{\cos\theta}\, x\, \frac{\sin\theta}{\sin\theta+1}$$

$$= \frac{\sin\theta}{\cos\theta} = \tan\theta$$

$\therefore \frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{1+\sec\theta}{1+\csc\theta} = \tan\theta$

(iii) $(\tan\theta + \sec\theta)^2 = \frac{1+\sin\theta}{1-\sin\theta}$

Solution

$(\tan\theta + \sec\theta)^2 = \left(\frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta}\right)^2 = \left(\frac{\sin\theta+1}{\cos\theta}\right)^2$

$$= \frac{(1+\sin\theta)^2}{\cos^2\theta} = \frac{(1+\sin\theta)^2}{1-\sin^2\theta}$$

$= \frac{(1+sin\theta)(1+sin\theta)}{(1+sin\theta)(1-sin\theta)} = \frac{1+sin\theta}{1-sin\theta}$

$\therefore(\tan\theta + \sec\theta)^2 = \frac{1+sin\theta}{1-sin\theta}$

Example 6

Solve the following equations for $-180^0 \le x \le 180^0$

(i) $2\cos^2\theta + \sin\theta - 1 = 0$

Solution

$2(1 - \sin^2\theta) + \sin\theta - 1 = 0$

$2\sin2\theta - \sin\theta - 1 = 0$

$(\sin\theta - 1)(2\sin\theta + 1) = 0$

Either $\sin\theta = 1$ or $\sin\theta = -\frac{1}{2}$

When $\sin\theta = 1$; $\theta = 90^0$

When $\sin\theta = -\frac{1}{2}$; $\theta = -150^0, -30^0, 210^0, 330^0$

[θ: $\theta = -150^0, -30^0, 90^0$ for given range]

(ii) $\cos\theta + \sqrt{3}\sin\theta = 1$

Solution

1^st approach

$\sqrt{3}\sin\theta = 1 - \cos\theta$

Squaring both sides

$3\sin^2\theta = 1 - 2\cos\theta + \cos^2\theta$

$3(1 - \cos^2\theta) = 1 - 2\cos\theta + \cos^2\theta$

$4\cos^2\theta - 2\cos\theta - 1 = 0$

$(2\cos\theta + 1)(\cos\theta - 1) = 0$

$\cos\theta = -\frac{1}{2}$	$\cos\theta = 1$
$\theta = \pm 120^0$	$\theta = 0^0$

$\therefore$[θ: $\theta = 0^0, \pm 120^0$]

2^nd approach

$\sqrt{3}\sin\theta = 1 - \cos\theta$

Dividing through by $\cos\theta$

$\sqrt{3}\tan\theta = \sec\theta - 1$

Squaring both sides

$3\tan^2\theta = \sec^2\theta - 2\sec\theta + 1$

$3\tan^2\theta = \sec^2\theta - 2\sec\theta + 1$

$3[\sec^2\theta - 1] = \sec^2\theta - 2\sec\theta + 1$

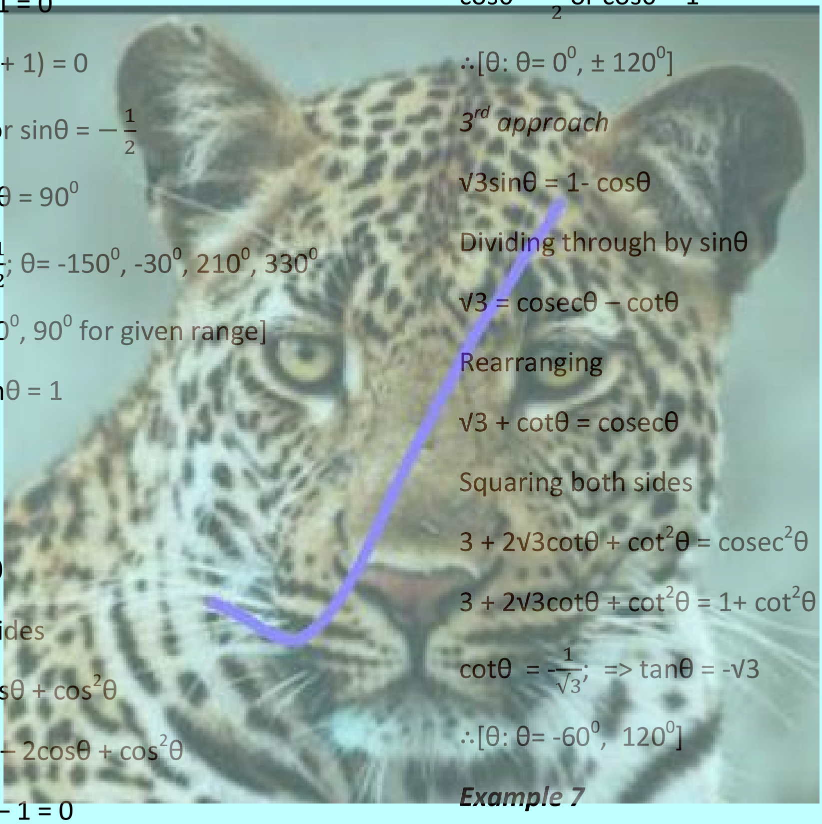
$2\sec^2\theta + 2\sec\theta - 4 = 0$

$\sec^2\theta + \sec\theta - 2 = 0$

$(\sec\theta + 2)(\sec\theta - 1) = 0$

$\sec\theta = -2$ or $\sec\theta = 1$

$\cos\theta = -\frac{1}{2}$ or $\cos\theta = 1$



$\therefore$[θ: $\theta = 0^0, \pm 120^0$]

3^rd approach

$\sqrt{3}\sin\theta = 1 - \cos\theta$

Dividing through by $\sin\theta$

$\sqrt{3} = \csc\theta - \cot\theta$

Rearranging

$\sqrt{3} + \cot\theta = \csc\theta$

Squaring both sides

$3 + 2\sqrt{3}\cot\theta + \cot^2\theta = \csc^2\theta$

$3 + 2\sqrt{3}\cot\theta + \cot^2\theta = 1 + \cot^2\theta$

$\cot\theta = -\frac{1}{\sqrt{3}}$; $\Rightarrow \tan\theta = -\sqrt{3}$

$\therefore$[θ: $\theta = -60^0, 120^0$]

Example 7

(a) Given that $7\tan\theta + \cot\theta = 5\sec\theta$, derive a quadratic equation for $\sin\theta$. Hence or otherwise, find all values of θ in the interval $0^0 \le \theta \le 180^0$ which satisfy the equation, giving your answer to the nearest 0.10 where necessary

Solution

$7\tan\theta + \cot\theta = 5\sec\theta$

$7\frac{sin\theta}{cos\theta} + \frac{cos\theta}{sin\theta} = \frac{5}{cos\theta}$

$7\frac{sin^2\theta + cos^2\theta}{cos\theta sin\theta} = \frac{5}{cos\theta}$

$7\sin^2\theta + \cos^2\theta = 5\sin\theta$

$7\sin^2\theta + (1 - \sin^2\theta) = 5\sin\theta$

$6\sin^2\theta - 5\sin\theta + 1 = 0$

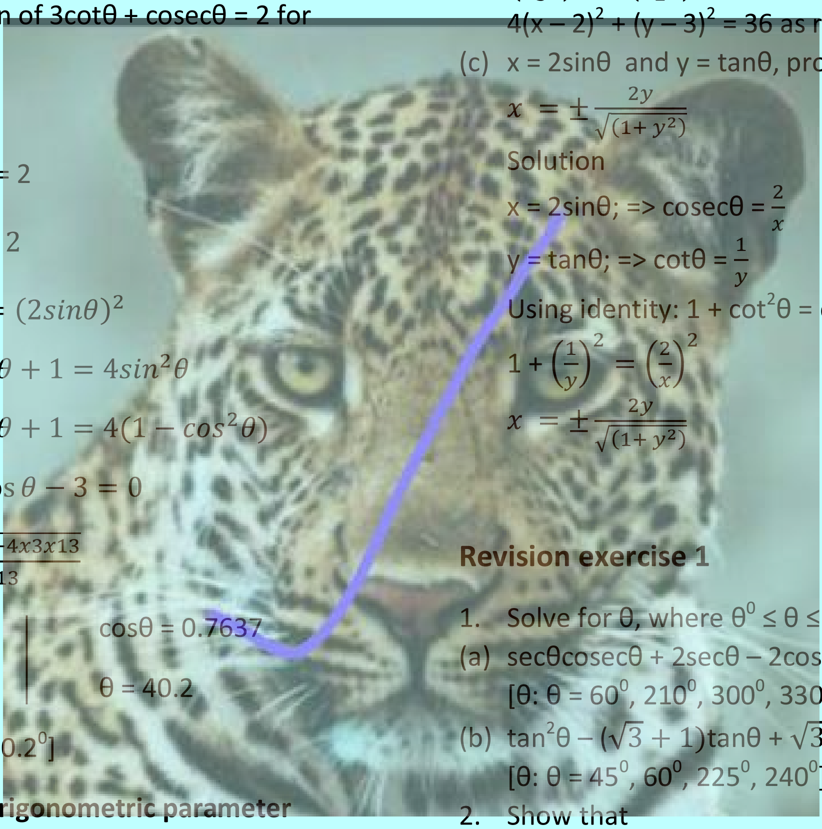
$(3\sin\theta - 1)(2\sin\theta - 1) = 0$

$\sin\theta = \frac{1}{3}$	$\sin\theta = \frac{1}{2}$
$\theta = 19.5^0, 160.5^0$	$\theta = 30^0, 150^0$

$\therefore[\theta: \theta = 19.5^0, 30^0, 150^0, 160.5^0]$

Example 8

Find the solution of $3\cot\theta + \operatorname{cosec}\theta = 2$ for $0^0 \le \theta \le 180^0$.



Solution

$3\cot\theta + \operatorname{cosec}\theta = 2$

$3\frac{\cos\theta}{\sin\theta} + \frac{1}{\sin\theta} = 2$

$(3\cos\theta + 1)^2 = (2\sin\theta)^2$

$9\cos^2\theta + 6\cos\theta + 1 = 4\sin^2\theta$

$9\cos^2\theta + 6\cos\theta + 1 = 4(1 - \cos^2\theta)$

$13\cos^2\theta + 6\cos\theta - 3 = 0$

$\cos\theta = \frac{-6 \pm \sqrt{6^2 + 4x3x13}}{2x13}$

$\cos\theta = 0.3021$	$\cos\theta = 0.7637$
$\theta = 72.40$	$\theta = 40.2$

$\therefore[\theta: \theta = 72.4^0, 40.2^0]$

Elimination of trigonometric parameter

This involves the use of identities to eliminate the trigonometric values in equation

Example 9

(a) If $x = \tan\theta + \sec\theta$ and $y = \tan\theta - \sec\theta$; show that $xy + 1 = 0$

Solution

$x + y = \tan\theta$

$x - y = 2\sec\theta$

$\sec\theta = \frac{1}{2}(x - y)$

Using identity: $1 + \tan^2\theta = \sec^2\theta$

$1 + (x + y)\ 2 = \left[\frac{1}{2}(x - y)\right]^2$

$4 + x^2 + 2xy + y^2 = x^2 - 2xy + y^2$

$4xy + 4 = 0$

$xy + 1 = 0$ as required

(b) $x = 2 + 3\sin\theta$ and $y = 3 + 2\cos\theta$ show that $4(x - 2)^2 + (y - 3)^2 = 36$

Solution

$x = 2 + 3\sin\theta \Rightarrow \sin\theta = \frac{x-2}{3}$

$y = 3 + 2\cos\theta \Rightarrow \cos\theta = \frac{y-3}{2}$

Using identity $\sin^2\theta + \cos^2\theta = 1$

$\left(\frac{x-2}{3}\right)^2 + \left(\frac{y-3}{2}\right)^2 = 1$

$4(x - 2)^2 + (y - 3)^2 = 36$ as reqyured

(c) $x = 2\sin\theta$ and $y = \tan\theta$, prove that $x = \pm\frac{2y}{\sqrt{(1 + y^2)}}$

Solution

$x = 2\sin\theta; \Rightarrow \operatorname{cosec}\theta = \frac{2}{x}$

$y = \tan\theta; \Rightarrow \cot\theta = \frac{1}{y}$

Using identity: $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

$1 + \left(\frac{1}{y}\right)^2 = \left(\frac{2}{x}\right)^2$

$x = \pm\frac{2y}{\sqrt{(1 + y^2)}}$

Revision exercise 1

1. Solve for θ, where $0^0 \le \theta \le 360^0$

(a) $\sec\theta\operatorname{cosec}\theta + 2\sec\theta - 2\operatorname{cosec}\theta - 4 = 0$

$[\theta: \theta = 60^0, 210^0, 300^0, 330^0]$

(b) $\tan^2\theta - (\sqrt{3} + 1)\tan\theta + \sqrt{3} = 0$

$[\theta: \theta = 45^0, 60^0, 225^0, 240^0]$

2. Show that

(a) $\frac{1 - \cos\theta + \sin\theta}{1 - \cos\theta} = \frac{1 + \cos\theta + \sin\theta}{\sin\theta}$

(b) $\tan\theta + \cot\theta = \sec\theta\operatorname{cosec}\theta$

(c) $\cos^4\theta - \sin^4\theta + 1 = 2\cos^2\theta$

(d) $\frac{\sin\theta}{1 + \cos\theta} + \frac{1 + \cos\theta}{\sin\theta} = 2\operatorname{cosec}\theta$

(e) $\sqrt{\frac{1 + \sin\theta}{1 - \sin\theta}} = \sec\theta + \tan\theta$

3. Solve the following equations for $-180^0 \le x \le 180^0$

(i) $2\cos^2\theta + \sin\theta - 1 = 0$

$[\theta: \theta = -150^0, -30^0, 90^0]$

(ii) $\sin 2\theta + 5\cos 2\theta = 3$

$[\theta: \theta = \pm 45^0, \pm 135^0]$

(iii) $4\cot^2\theta + 24\operatorname{cosec}\theta + 39 = 0$

[θ: θ = 16.6^0, 23.6^0, 156.4^0, 163.4^0]

4. Solve each of the following equations in the stated range

(a) $4\cos^2\theta + 2\sin\theta = 4$ $0^0 \le \theta \le 360^0$

[θ: θ = 0^0, 48.6^0, 131.4^0, 180^0, 360^0]

(b) $2\sec^2\theta - 4\tan\theta - 2 =$ $-180^0 \le \theta \le 360^0$

[θ: θ = -135^0, -161.6^0, 18.4^0, 45^0]

(c) $5\cos^2 3\theta = 3(1 + \sin 3\theta)$, $0^0 \le \theta \le 360^0$

[θ: θ = 7.9^0, 52.1^0, 90^0, 127.9^0, 172.1^0]

5. Solve for θ; 00 ≤ θ ≤ 3600

(a) $\tan\theta + 3\cot\theta = 4$

[θ: θ = 45^0, 71.6^0, 225^0, 251.6^0]

(b) $4\cos\theta - 3\sin\theta = 2$

[θ: θ = 29.50, 256.70]

6. Solve

(a) $\cos\theta + \sqrt{3}\sin\theta = 2$ $0 \le \theta \le \pi$

$\left[\theta = \frac{\pi}{3}\right]$

(b) $2\cos\theta - \operatorname{cosec}\theta = 0$ $0^0 \le \theta \le 270^0$

[θ: θ = 45^0, 225^0]

(c) $2\sin^2\theta + 3\cos\theta = 0$ $0^0 \le \theta \le 360^0$

[θ: θ = 240^0, 120^0]

(d) $3\sin\theta + 4\cos\theta = 2$ $-180^0 \le \theta \le 180^0$

[θ: θ = -29.55^0, 103.29^0]

(e) $3\tan^2\theta + 2\sec^2\theta = 2(5 - 3\tan\theta)$ for $0^0 < \theta < 180^0$

[θ: θ = 38.66^0, 116.57^0]

7. Without using a tables or calculator, show that $\tan 15^0 = 2 - \sqrt{3}$

8. Solve equation

$8\cos^4\theta - 10\cos^2\theta + 3$ for $0^0 \le \theta \le 180^0$

[θ: θ = 30^0, 45^0, 135^0, 150^0]

9. Eliminate θ from the following equation

(a) x = asecθ and y = b + ccosθ

[ac = x(y – b)]

(b) x = secθ +tanθ and y = secθ – tanθ

[xy = 1]

10. Solve the simultaneous equation

Cos x + 4siny = 1

4secx – 3cosecy = 5 for values of x and y between 0^0 and 360^0

[x = 78.8^0, 281.5^0; y = 11.5^0, 168.5^0]

11. Prove each of the following identities

(a) Sinxtanx + cosx =secx

(b) Cosecx + tanxsecx = cosecxsec^{2}x

(c) Cosecx – sinx = cotxcosx

(d) $(\sin x + \cos x)^2 - 1 = 2\sin x\cos x$

12. Eliminate θ from each of the following pairs of relationships

(a) x =3sinθ, y = cosecθ [xy = 3]

(b) 5x = sinθ, y = 2cosθ [$100x^2 + y^2 - 4 = 0$]

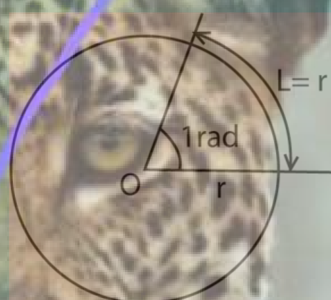
(c) x = 3 + sinθ, y = cosθ [$(x-3)^2 + y^2 = 1$]

(d) x = 2 + sinθ, cosθ = 1+y

[$(x-2)^2 + (y+1)^2 = 1$]

Measuring angles in radians

A radian is defined as an angle subtended at the centre of a circle by an arc that is equal to the radius of the circle. One radian is represented by π, where $\pi = \frac{22}{7}$



How to convert between degrees and radians

1 revolution = circumference of a circle

But circumference of a circle subtends an angle 2π at the centre.

⇨ 1 revolution = 2π = 360^0

$\pi = 180^0$

$1^0 = \frac{\pi}{180}$ radians

$x^0 = \frac{\pi}{180}x$ radians

Example 10

Convert the following angles to radians

(a) 330^0

(b) 90^0

(c) 30^0

Solution

(a) $330^0 = \frac{\pi}{180} x\ 330 = \frac{11\pi}{6}$ radians

(b) $90^0 = \frac{\pi}{180} x\ 90 = \frac{\pi}{2}$ radians

(c) $30^0 = \frac{\pi}{180} x\ 30 = \frac{\pi}{6}$ radians

Converting radians to degrees

2π radians = 360^0

1 radian = $\frac{180^0}{\pi}$

x radians = $\frac{180^0}{\pi} x$

Example 11

Convert each of the following radians to degrees

(i) $\frac{\pi}{3}$ radians

(ii) $\frac{2\pi}{5}$ radians

(iii) π radians

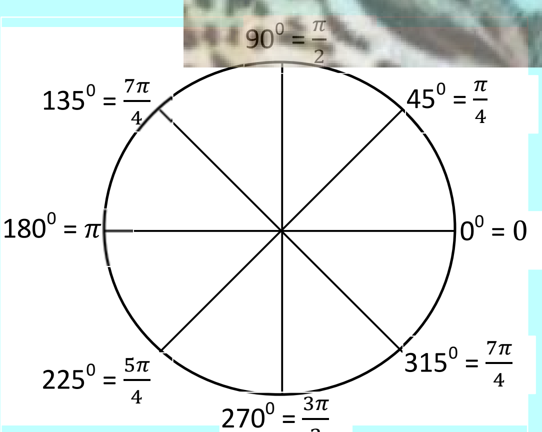
Solution

(i) $\frac{\pi}{3}$ radians = $\frac{180^0}{\pi}\ x \frac{\pi}{3} = 60^0$

(ii) $\frac{2\pi}{5}$ radians= $\frac{180^0}{\pi}\ x \frac{2\pi}{5} = 72^0$

(iii) π radians = $\frac{180^0}{\pi}\ x\ \pi = 180^0$

Some equivalent angles in degrees and radians



Example 12

Find each of the following values

(a) $\sin\left(\frac{2\pi}{3}\right)$

(b) $\cos\left(\frac{4\pi}{3}\right)$

(c) $\tan\left(\frac{7\pi}{4}\right)$

Solution

Convert the angles from radian to degrees

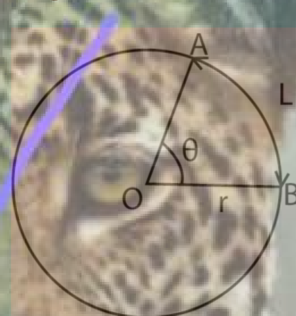
(a) $\sin\left(\frac{2\pi}{3}\right) = \sin\left(\frac{2\ x\ 180}{3}\right) = \sin 120^0 = \frac{\sqrt{3}}{2}$

(b) $\cos\left(\frac{4\pi}{3}\right) = \cos\left(\frac{4\ x\ 180}{3}\right) = \cos 240^0 = -\frac{1}{2}$

(d) $tan\left(\frac{7\pi}{4}\right) = \tan\left(\frac{7x\ 180}{4}\right) = \tan 60^0 = \sqrt{3}$

Length of an arc

Suppose that the angle subtended by the length L of an arc AB of a circle is θ as shown.



$\frac{L}{\theta} = \frac{2\pi r}{2\pi}$

L = rθ where θ must be in radians

Example 13

Find the length of an arc of a circle of radius 14 if it subtends an angle

(i) $\frac{\pi}{4}$

(ii) 150^0

Solution

(i) L = rθ = $14\ x\ \frac{\pi}{4} = 11$cm

(ii) Convert degrees to radians

$150^0 = \frac{\pi}{180} x 150 = \frac{5\pi}{6}$ radians

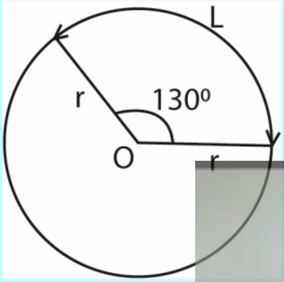
L = $14\ x\ \frac{5\pi}{6} = 36.67$cm

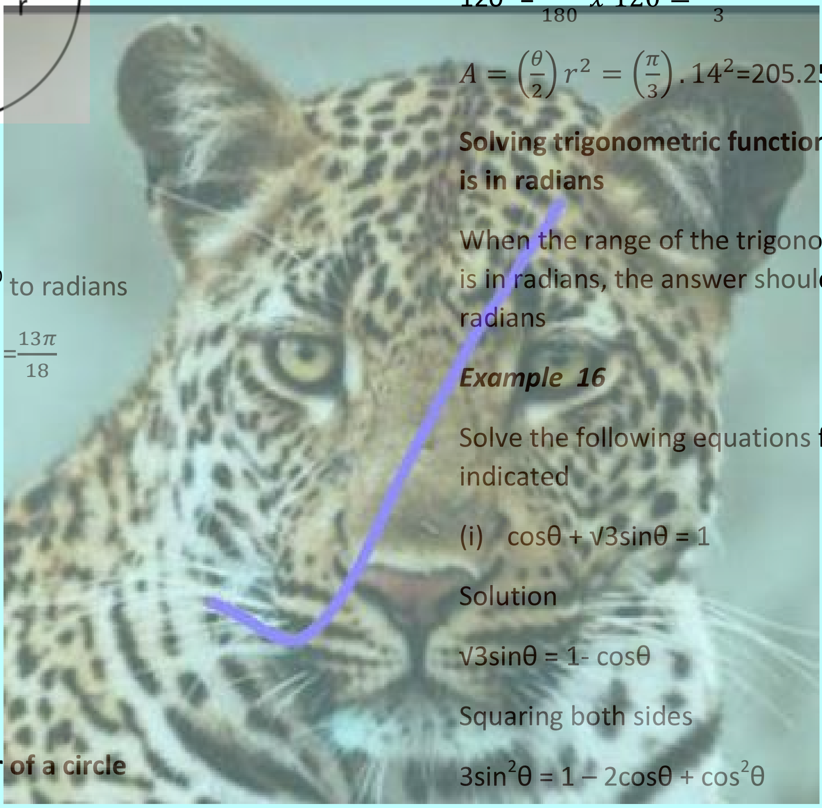
Example 14

A sector was drawn which had a perimeter of 80cm, and centre angle of 130^0. Calculate the radius

Solution

The sides of a sector are composed of an arc, and two more sides which are radii of a circle.





2r + L = 80

L = 80 – 2r

Converting 130^0 to radians

$130^0 = \frac{\pi}{180} x 130 = \frac{13\pi}{18}$

But L = rθ

$80 - 2r = \frac{13\pi r}{18}$

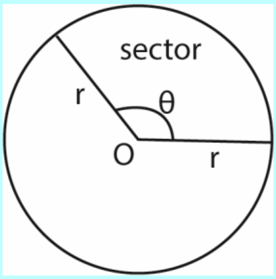
$2r + \frac{13\pi r}{18} = 80$

$\frac{(36+13\pi)r}{18} = 80$

r = 18.74cm

Area of a sector of a circle

A sector of a circle is a portion of the interior of a circle intercepted by a central angle.



The area of a sector of a circle of radius r and central angle θ is given by

$A = \left(\frac{\theta}{2\pi}\right)\pi r^2 = \left(\frac{\theta}{2}\right) r^2$

Where θ must be in radians

Example 15

Find the area of a sector with radius 14cm and angle (i) $\frac{\pi}{4}$ (ii) 1200

Solution

(i) $A = \left(\frac{\theta}{2}\right) r^2 = \left(\frac{\pi}{8}\right). 14^2$=77cm^3

(ii) Converting 120^0 to radians

$120^0 = \frac{\pi}{180} x\ 120 = \frac{2\pi}{3}$

$A = \left(\frac{\theta}{2}\right) r^2 = \left(\frac{\pi}{3}\right). 14^2$=205.25cm^3

Solving trigonometric functions whose range is in radians

When the range of the trigonometric function is in radians, the answer should be given in radians

Example 16

Solve the following equations for the ranges indicated

(i) cosθ + √3sinθ = 1 $\quad 0 \le \theta \le \pi$

Solution

√3sinθ = 1- cosθ

Squaring both sides

$3\sin^2\theta = 1 - 2\cos\theta + \cos^2\theta$

$3(1 - \cos^2\theta) = 1 - 2\cos\theta + \cos^2\theta$

$4\cos^2\theta - 2\cos\theta - 1 = 0$

(2cosθ + 1)(cosθ – 1) = 0

$\cos\theta = -\frac{1}{2}$	cosθ = 1
$\theta = \pm 120^0$	$\theta = 0^0$

$\pm 1200 = \pm\frac{\pi}{180} x\ 120 = \pm\frac{2\pi}{3}$ Radians

0^0 = 0 radians

$\therefore \left[\theta: \theta = 0, \pm\frac{2\pi}{3}\right]$

(ii) $2\cos^2\theta + \sin\theta - 1 = 0 \qquad 0 \le \theta \le \pi$

Solution

$2(1 - \sin^2\theta) + \sin\theta - 1 = 0$

$2\sin 2\theta - \sin\theta - 1 = 0$

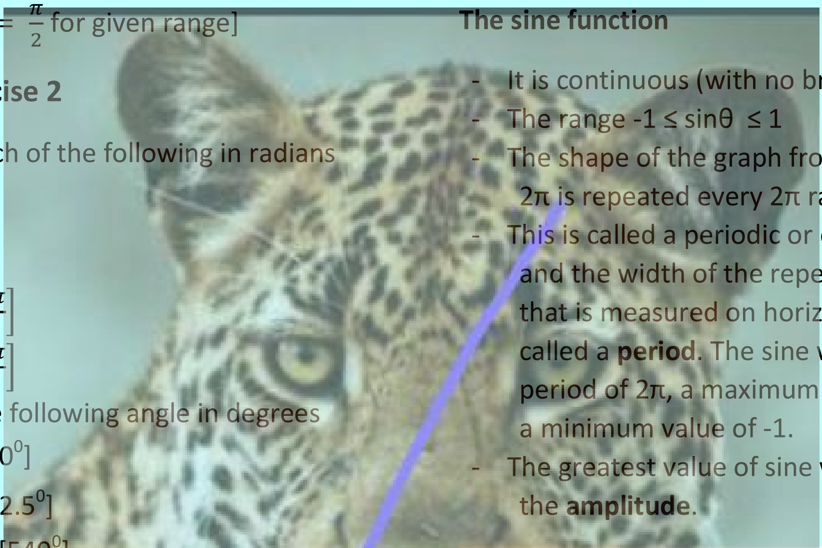
$(\sin\theta - 1)(2\sin\theta + 1) = 0$

Either $\sin\theta = 1$ or $\sin\theta = -\frac{1}{2}$

When $\sin\theta = 1$; $\theta = 90^0$

When $\sin\theta = -\frac{1}{2}$; $\theta = -150^0, -30^0, 210^0, 330^0$

[θ: $\theta = \frac{\pi}{180} x\ 90 = \frac{\pi}{2}$ for given range]



Revision exercise 2

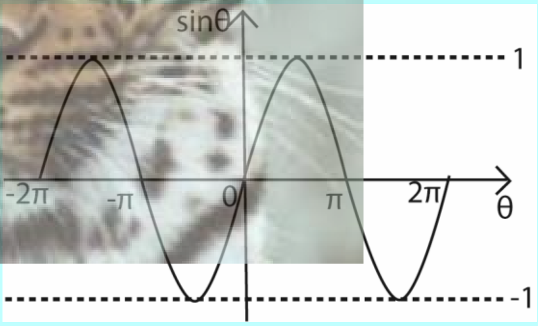
1. Express each of the following in radians
   (a) $30^0 \left[\frac{\pi}{6}\right]$
   (b) $45^0 \left[\frac{\pi}{4}\right]$
   (c) $120^0 \left[\frac{2\pi}{3}\right]$
   (d) $300^0 \left[\frac{5\pi}{3}\right]$
2. Express the following angle in degrees
   (a) $\frac{\pi}{3}$ rad $[60^0]$
   (b) $\frac{\pi}{8}$ rad $[22.5^0]$
   (c) 3π rad $[540^0]$
   (d) 5.2π rad $[936^0]$
3. A sector of the circle of radius 7 cm subtends an angle $\frac{\pi}{3}$ radians at the centre. Calculate the
   (a) Length of the arc $\left[6\frac{2}{3}cm\right]$
   (b) Perimeter of the sector $\left[20\frac{2}{3}cm\right]$
   (c) Area of the sector $\left[\frac{77}{3}cm^2\right]$
4. AOB is a sector of a circle, centre O, and is such that OA = OB = 7cm and angle AOB is 300. Calculate the
   (a) Perimeter of sector AOB $\left[17\frac{2}{3}cm\right]$
   (b) The area of AOB $\left[\frac{77}{6}cm^2\right]$
5. Find the value each of the following
   (a) $\sin\pi$ [0]
   (b) $\cos 3\pi$ [-1]
   (c) $\tan\frac{\pi}{3}$ [$\sqrt{3}$]
6. Solve the following equations for the ranges indicated
   (a) $2\sec^2\theta = 3 + \tan\theta$ for $0 \le \theta \le 2\pi$
   [θ: $\theta = 0.25\pi, 0.85\pi, 1.25\pi, 1.85\pi$]
   (b) $2\sin^2 x\cos x + \cos x - 1$ for $0 \le \theta \le 2\pi$
   [θ: $\theta = 0.38\pi, 1.62\pi, 2\pi$]
   (c) $2\tan\theta + 4\cot\theta = \csc\theta$ for $-\pi \le \theta \le \pi$
   $\left[\theta: \theta = \pm\frac{1}{3}\pi, \pm 0.73\pi\right]$

Graphs of trigonometric functions

The following are the characteristic of the three major trigonometric functions

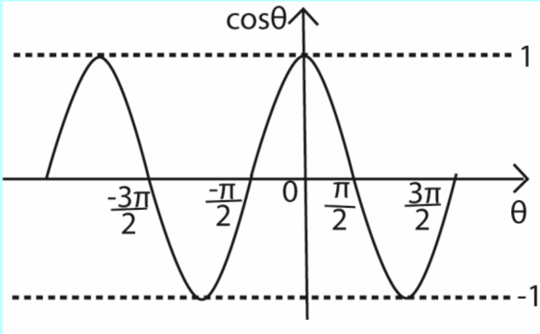
The sine function

- It is continuous (with no breaks)
- The range $-1 \le \sin\theta \le 1$
- The shape of the graph from θ = 0 to θ = 2π is repeated every 2π radians
- This is called a periodic or cyclic function and the width of the repeating pattern that is measured on horizontal axis is called a **period**. The sine wave has a period of 2π, a maximum value of +1 and a minimum value of -1.
- The greatest value of sine wave is called the **amplitude**.



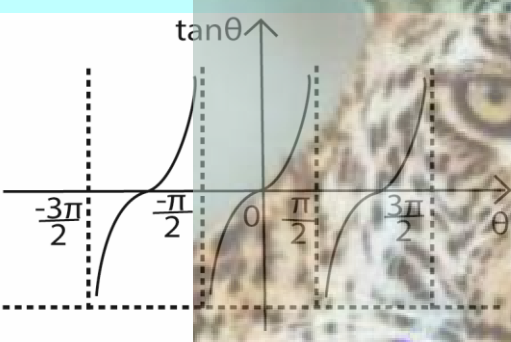
The coosine function

- It is continuous (with no breaks)
- The range $-1 \le \sin\theta \le 1$
- Has a period of 2π
- The shape is the same as the sine wave but displaced a distance $\frac{\pi}{2}$ to the left on the horizontal axis. This is called a **phase shift**



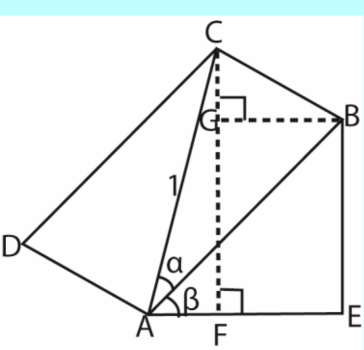
The tan function

- The tan function is found using; $tan\theta = \frac{sin\theta}{cos\theta}$. It follows that tanθ = 0 when sin θ = 0; and tanθ is undefined when cosθ = 0
- The graph is continuous, but undefined when θ= $-\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$
- The range of values for tanθ is unlimited
- It has a period π



Compound angles

Consider a cardboard ABCD of unit diagonal that stands on the edge A, making an angle β with the horizontal ground. Let the unit diagonal AC be inclined at an angle α to the side AB (see diagram)



Angles EAB= ABG(Alternative angles)

∴ Angle ABG = β

Angle [ABG + GBC] = 90^0

∴ Angle GBC = 90 – β

From triangle GBC,

Angle BCG = 180 – (90 + 90 – β)

∴ Angle BCG = β

From

(1) Triangle ABC:
cosα = $\frac{AB}{AC} = \frac{AB}{1}$; => AB = cosα

(2) Triangle ABE:
cosβ = $\frac{AE}{AB} = \frac{AE}{cos\alpha}$; => AE = cosβcosα
sinβ = $\frac{BE}{AB} = \frac{BE}{cos\alpha}$; => BE = cosαsinβ

(3) Triangle BCG:
cosβ = $\frac{CG}{BC} = \frac{CG}{sin\alpha}$; => CG = sinαcosβ
sinβ = $\frac{BG}{BC} = \frac{BG}{sin\alpha}$; => BG = sinαsinβ

(4) Triangle ACF:
cos(α + β) = $\frac{AF}{AC} = \frac{AF-BG}{1}$ = AE –BG

∴ **cos(α + β) = cosαcosβ – sinαsinβ**

sin(α + β) = $\frac{CF}{AC} = \frac{CG-GF}{1}$ = CG + GF

∴ **sin(α + β) = sinαcosβ + cosαsinβ**

It follows that

(i) cos(α + β) = cosαcosβ – sinαsinβ
(ii) cos(α - β) = cosαcosβ + sinαsinβ

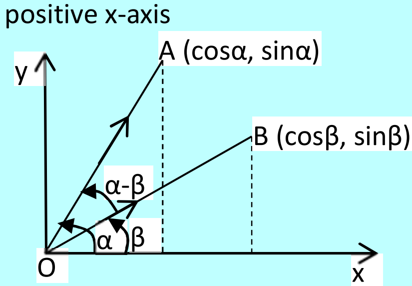
[substituting –β for β)

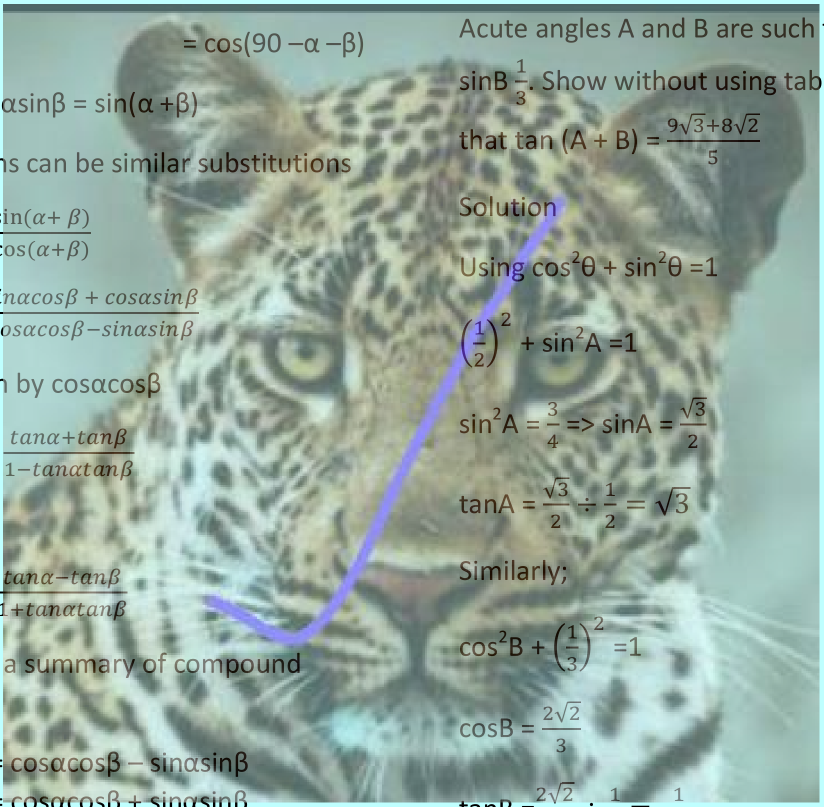
(iii) sin(α + β) = sinαcosβ + cosαsinβ
(iv) sin(α - β) = sinαcosβ - cosαsinβ

[substituting –β for β)

These can also be derived using vector approach.

Consider two unit vectors $\underline{OA}$ and $\underline{OB}$ each inclined at angles α and β, respectively to the positive x-axis





Using the definition of a vector product:

$\underline{OA}.\underline{OB} = |\underline{OA}|.|\underline{OB}|\cos(\alpha - \beta)$

Since $\underline{OA}$ and $\underline{OB}$ are unit vectors,

$|\underline{OA}| = |\underline{OB}| = 1$

$\therefore \underline{OA}.\underline{OB} = \cos(\alpha - \beta)$

⇨ $(\cos\alpha\underline{i} + \sin\alpha\underline{j}).(\cos\beta\underline{i} + \sin\beta\underline{j}) = \cos(\alpha - \beta)$

$\therefore \cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha - \beta)$

Substituting $90 - \alpha$ for α

$\cos(90 - \alpha)\cos\beta + \sin(90 - \alpha)\sin\beta$

$= \cos(90 - \alpha - \beta)$

$\therefore \sin\alpha\cos\beta + \cos\alpha\sin\beta = \sin(\alpha + \beta)$

Other expansions can be similar substitutions

i.e. $\tan(\alpha + \beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$

$= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$

Dividing through by $\cos\alpha\cos\beta$

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

Similarly

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

The following is a summary of compound angles

1. $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
2. $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
3. $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$
4. $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$
5. $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$
6. $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

Example 17

Calculate the value of $\sin 15^0$ given that $\sin 45^0 = \cos 45 = \frac{1}{\sqrt{2}}$, $\sin 30^0 = \frac{1}{2}$ and $\cos 30^0 = \frac{\sqrt{3}}{2}$

$\text{Sin}15^0 = \sin(45^0 - 30^0)$

$= \sin 45^0\cos 30^0 - \sin 30^0\cos 45^0$

$= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} - \frac{1}{2}.\frac{1}{\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}} = 0.2588$

Example 18

Prove that $\tan(45^0 + A) = \frac{1+\tan A}{1-\tan A}$

From $\tan(\alpha + \beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

$\tan(45^0 + A) = \frac{\tan 45^0 + \tan\beta}{1 - \tan 45^0\tan\beta}$

$= \frac{1+\tan A}{1-\tan A}$

Example 19

Acute angles A and B are such that: $\cos A = \frac{1}{2}$, $\sin B \frac{1}{3}$. Show without using tables or calculator that $\tan(A + B) = \frac{9\sqrt{3}+8\sqrt{2}}{5}$

Solution

Using $\cos^2\theta + \sin^2\theta = 1$

$\left(\frac{1}{2}\right)^2 + \sin^2 A = 1$

$\sin^2 A = \frac{3}{4} \Rightarrow \sin A = \frac{\sqrt{3}}{2}$

$\tan A = \frac{\sqrt{3}}{2} \div \frac{1}{2} = \sqrt{3}$

Similarly;

$\cos^2 B + \left(\frac{1}{3}\right)^2 = 1$

$\cos B = \frac{2\sqrt{2}}{3}$

$\tan B = \frac{2\sqrt{2}}{3} \div \frac{1}{3} = \frac{1}{2\sqrt{2}}$

But

From $\tan(\alpha + \beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

$= \frac{\sqrt{3}+\frac{1}{2\sqrt{2}}}{1-\sqrt{3}.\frac{1}{2\sqrt{2}}}$

$= \frac{(2\sqrt{2}\sqrt{3}+1)(2\sqrt{2}+\sqrt{3})}{(2\sqrt{2}-\sqrt{3})(2\sqrt{2}+\sqrt{3})}$

$\tan(A + B) = \frac{9\sqrt{3}+8\sqrt{2}}{5}$

Example 20

Solve cos(θ + 35^0) = sin(θ + 25^0)
for 0^0 ≤ θ ≤ 360^0

Cosθcos35^0 – sinθsin35^0 =sinθcos25^0 + cosθsin25^0

Dividing through by cosθ

Cos35^0 – tanθsin35^0 = tanθcos25^0 + sin25^0

tanθ = $\frac{cos35^0 - sin25^0}{cos35^0 + sin25^0} = \frac{0.3965337825}{1.479884223}$

θ = 15^0, 195^0 for 0^0 ≤ θ ≤ 360^0

Example 21

(a) Prove than $\frac{2tan\theta}{1+tan^2\theta} = sin2\theta$

Solution

$\frac{2tan\theta}{1+tan^2\theta} = \frac{2sin\theta}{cos\theta} \div \left(1 + \frac{sin^2\theta}{cos^2\theta}\right)$

$= \frac{2sin\theta}{cos\theta} \div \left(\frac{cos^2\theta + sin^2\theta}{cos^2\theta}\right)$

$= \frac{2sin\theta}{cos\theta} \div \left(\frac{1}{cos^2\theta}\right)$

$= 2sin\theta cos\theta = sin2\theta$

(b) Solve sin2θ = cosθ for 0^0 ≤ θ ≤ 90^0

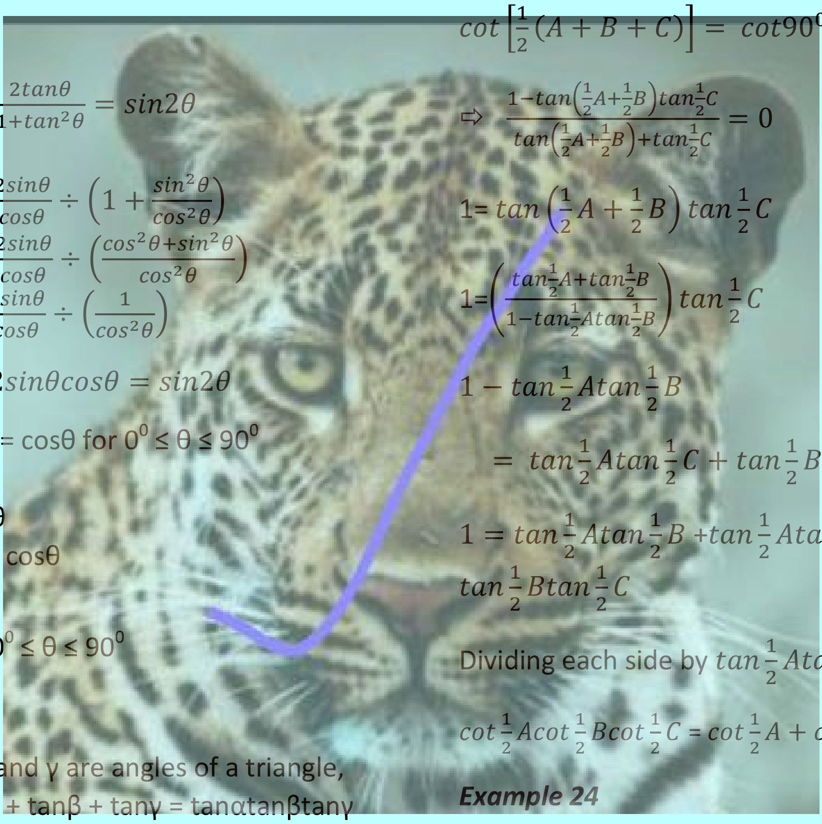
Solution

sin2θ = cosθ

2sinθcosθ = cosθ

sinθ = $\frac{1}{2}$

θ = 30^0 for 0^0 ≤ θ ≤ 90^0



Example 22

Given that α, β and γ are angles of a triangle, show that tanα + tanβ + tanγ = tanαtanβtanγ

Hence find tanγ if tanα = 1 and tanγ = 2.

Solution

α + β + γ= 180^0 (angle sum of a triangle)

tan (α + β + γ) = tan180^0 = 0

tan[(α + β) + γ] = 0

$\frac{tan(\alpha+\beta) + tan\gamma}{1 - tan(\alpha+\beta)\ tan\gamma} = 0$

⇨ $tan(\alpha + \beta) + tan\gamma = 0$

$\frac{tan\alpha + tan\beta}{1 - tan\alpha\ tan\beta} = -tan\gamma$

$tan\alpha + tan\beta = -tan\gamma +$ tanαtanβtanγ

∴ tanα + tanβ + tanγ = tanαtanβtanγ

Example 23

In a triangle ABC, prove that
$cot\frac{1}{2}A + cot\frac{1}{2}B + cot\frac{1}{2}C$

$= cot\frac{1}{2}Acot\frac{1}{2}Bcot\frac{1}{2}C$

Solution

$\frac{1}{2}(A + B + C) = \frac{1}{2}(180^0) = 90^0$

$cot\left[\frac{1}{2}(A + B + C)\right] = cot90^0 = 0$

⇨ $\frac{1 - tan\left(\frac{1}{2}A + \frac{1}{2}B\right)tan\frac{1}{2}C}{tan\left(\frac{1}{2}A + \frac{1}{2}B\right) + tan\frac{1}{2}C} = 0$

$1 = tan\left(\frac{1}{2}A + \frac{1}{2}B\right)tan\frac{1}{2}C$

$1 = \left(\frac{tan\frac{1}{2}A + tan\frac{1}{2}B}{1 - tan\frac{1}{2}Atan\frac{1}{2}B}\right)tan\frac{1}{2}C$

$1 - tan\frac{1}{2}Atan\frac{1}{2}B$

$= tan\frac{1}{2}Atan\frac{1}{2}C + tan\frac{1}{2}Btan\frac{1}{2}C$

$1 = tan\frac{1}{2}Atan\frac{1}{2}B + tan\frac{1}{2}Atan\frac{1}{2}C + tan\frac{1}{2}Btan\frac{1}{2}C$

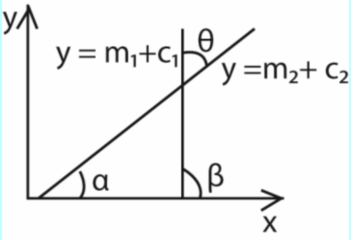
Dividing each side by $tan\frac{1}{2}Atan\frac{1}{2}Btan\frac{1}{2}C$

$cot\frac{1}{2}Acot\frac{1}{2}Bcot\frac{1}{2}C = cot\frac{1}{2}A + cot\frac{1}{2}B + cot\frac{1}{2}C$

Example 24

Prove that the angle θ, between the straight line y = m_1x + c_1 and the straight line y = m_2x + c_2 is given by tanθ = $\frac{m_2 - m_1}{1 + m_2 m_1}$

Let the lines be inclines at angles α and β with the x-axis respectively



From the diagram above

θ = β – α

⇨ $\tan\theta = \tan(\beta - \alpha)$

$= \frac{tan\beta - tan\alpha}{1+tan\beta tan\alpha}$

$\tan\theta = \frac{m_2 - m_1}{1+m_2 m_1}$

Revision exercise 3

1. (a) show that sin(α +β) - sin(α - β) = 2cosαsinβ
   (b) If sin(α +β) = 5cos(α - β) show that $\tan\alpha = \frac{5-tan\beta}{1+tan\beta}$
   (c) Without using tables or calculator, show that $\cos 15^0 = \sin 75^0$
   (d) If $\alpha + \beta = 45^0$, show that $\tan\alpha = \frac{1-tan\beta}{1+tan\beta}$
2. Prove that:
   (i) $\frac{\sin(\alpha+\beta)}{\cos(\alpha-\beta)} + 1 = \frac{(1+tan\beta)(1+cot\alpha)}{cos\alpha+tan\beta}$
   (ii) $tan\alpha - tan\beta = \frac{\sin(\alpha-\beta)}{cos\alpha cos\beta}$
   (iii) $cot\alpha + cot\beta = \frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}$
   (iv) $\frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)} = \frac{tan\alpha - tan\beta}{tan\alpha+tan\beta}$
   (v) $\frac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)} = \frac{cot\alpha cot\beta+1}{cot\alpha cot\beta-1}$
   (vi) $\cot(A+B) = \frac{cotAcotB-1}{cotA+cotB}$
3. (a) Determine solution of tan2x + 2sinx = 0 for $0^0 \le x \le 180^0$ [x: x= 0^0, 60^0, 120^0, 180^0]
   (vii) Show that in triangle ABC, tanA + tanB + tanC = tanAtanBtanC
4. Find the values of tan α for each of the following
   (a) $\sin(\alpha - 30^0) = \cos\alpha$ [√3]
   (b) $\sin(\alpha + 45^0) = \cos\alpha$ $[\sqrt{2} - 1]$
   (c) $\cos(\alpha + 60^0) = \sin\alpha$ [2 - √3]
   (d) $\sin(\alpha + 60^0) = \cos(\alpha - 60^0)$ [1]
   (e) $\cos(\alpha + 60^0) = 2\cos(\alpha + 30^0)$ [4 + 3√3]
   (f) $\sin(\alpha + 60^0) = \cos(45^0 - \alpha)$ $\left[\frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}-1}\right]$
5. Given that
   (a) tan(α –β) = ½ and tanα = 3 find the value of tanβ [1]
   (b) tan(α +β) = 5 and tanβ = 2 find the value of tanα $\left[\frac{3}{11}\right]$
6. Given that
   (a) $\tan(\theta - 45^0) = 4$, find the value of θ $\left[-\frac{5}{3}\right]$
   (b) tan $(\theta + 60^0)$ find the value of cotθ $[8 + 5\sqrt{3}]$

Double angles and half angles

(b) From cos (θ + θ) = cosθcosθ – sinθsinθ

⇨ $\cos 2\theta = \cos^2\theta - \sin^2\theta$............ (i)

Either

$\cos 2\theta = \cos^2\theta - 1 + \cos^2\theta$ $(\cos^2\theta + \sin^2\theta = 1]$

$\cos 2\theta = 2\cos^2\theta - 1$ .... (ii)

Or

$\cos 2\theta = 1 - \sin^2\theta - \sin^2\theta$

⇨ $\cos 2\theta = 1 - 2\sin^2\theta$ ....(iii)

It follows that

$\cos^2\theta = \frac{1}{2}(1 + cos2\theta)$ ....(iv)

$\sin^2\theta = \frac{1}{2}(1 - cos2\theta)$ ....(iv)

The identities imply

$\cos 6\theta = \cos^2 3\theta - \sin^2 3\theta$

$= 2\cos^2 3\theta - 1 = 1 - 2\sin^2 3\theta$

$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$

$= 2\cos^2\frac{\theta}{2} - 1 = 1 - 2\sin^2\frac{\theta}{2}$

(c) sin(θ + θ) = sinθcosθ + cosθsinθ

⇨ sin2θ = 2sinθcosθ

It follows that

sin6θ = 2sin3θcos3θ

$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

(d) $\tan(\theta + \theta) = \frac{tan\theta + tan\theta}{1 - tan\theta tan\theta}$

⇨ $tan2\theta = \frac{2tan\theta}{1 - tan^2\theta}$

It follows that

$$tan\theta = \frac{2tan\frac{\theta}{2}}{1 - tan^2\frac{\theta}{2}}$$

$$tan6\beta = \frac{2tan3\beta}{1 - tan^2 3\beta}$$

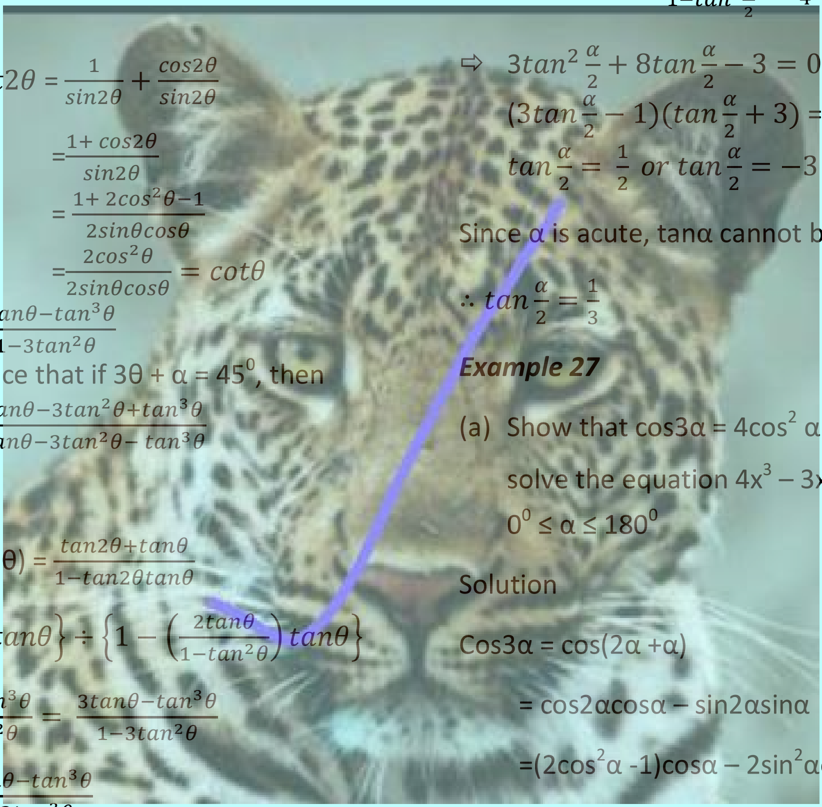
Note that in all cases, the angles on the right hand side are half the angles on the left hand side [***half angle formulae]***

Example 25

Show that

(a) cosec2θ + cot2θ = cotθ

Solution



$$cosec2\theta + cot2\theta = \frac{1}{sin2\theta} + \frac{cos2\theta}{sin2\theta}$$

$$= \frac{1 + cos2\theta}{sin2\theta}$$

$$= \frac{1 + 2cos^2\theta - 1}{2sin\theta cos\theta}$$

$$= \frac{2cos^2\theta}{2sin\theta cos\theta} = cot\theta$$

(b) $tan3\theta = \frac{3tan\theta - tan^3\theta}{1 - 3tan^2\theta}$

Hence deduce that if 3θ + α = 45^0, then

tanα = $\frac{1 - 3tan\theta - 3tan^2\theta + tan^3\theta}{1 + 2tan\theta - 3tan^2\theta - tan^3\theta}$

Solution

tan3θ=tan(2θ + θ) = $\frac{tan2\theta + tan\theta}{1 - tan2\theta tan\theta}$

$$= \left\{\left(\frac{2tan\theta}{1 - tan^2\theta}\right) + tan\theta\right\} \div \left\{1 - \left(\frac{2tan\theta}{1 - tan^2\theta}\right) tan\theta\right\}$$

$$= \frac{2tan\theta + tan\theta - tan^3\theta}{1 - tan^2\theta - 2tan^2\theta} = \frac{3tan\theta - tan^3\theta}{1 - 3tan^2\theta}$$

$$\therefore tan3\theta = \frac{3tan\theta - tan^3\theta}{1 - 3tan^2\theta}$$

Hence 3θ + α = 45^0 => α = 45^0 - 3θ

Tanα = tan(45^0 - 3θ)

$$= \frac{\tan 45^0 - tan3\theta}{1 + tan45^0 tan3\theta} = \frac{1 - tan3\theta}{1 + tan3\theta}$$

$$= \frac{1 - \left(\frac{3tan\theta - tan^3\theta}{1 - 3tan^2\theta}\right)}{1 + \left(\frac{3tan\theta - tan^3\theta}{1 - 3tan^2\theta}\right)}$$

$$= \frac{1 - 3tan\theta - 3tan^2\theta + tan^3\theta}{1 + 2tan\theta - 3tan^2\theta - tan^3\theta}$$

$$\therefore tan\alpha = \frac{1 - 3tan\theta - 3tan^2\theta + tan^3\theta}{1 + 2tan\theta - 3tan^2\theta - tan^3\theta}$$

Example 26

If tanα = $\frac{3}{4}$ and α is acute, without using tables or calculator work out the value of

(a) tan2α

$$tan2\alpha = \frac{2tan\alpha}{1 - tan^2\alpha} = \frac{2x\frac{3}{4}}{1 - \left(\frac{3}{4}\right)^2} = \frac{\frac{2}{2}}{\frac{3}{4}} = \frac{24}{7}$$

(b) tan$\frac{\alpha}{2}$

similarly $tan\alpha = \frac{2tan\frac{\alpha}{2}}{1 - tan^2\frac{\alpha}{2}} = \frac{3}{4}$

⇨ $3tan^2\frac{\alpha}{2} + 8tan\frac{\alpha}{2} - 3 = 0$

$(3tan\frac{\alpha}{2} - 1)(tan\frac{\alpha}{2} + 3)$ =0

$tan\frac{\alpha}{2} = \frac{1}{2}$ or $tan\frac{\alpha}{2} = -3$

Since α is acute, tanα cannot be negative

$\therefore tan\frac{\alpha}{2} = \frac{1}{3}$

Example 27

(a) Show that cos3α = 4cos^2 α -3cos α. Hence solve the equation $4x^3 - 3x - \frac{\sqrt{3}}{3} = 0$ for $0^0 \leq \alpha \leq 180^0$

Solution

Cos3α = cos(2α +α)

= cos2αcosα – sin2αsinα

=(2cos^{2}α -1)cosα – 2sin^{2}αcosα

=(2cos^{2}α -1)cosα – 2(1-cos^{2}α)cosα

= 2cos^{3}α – cosα -2cosα + 2cos^{3}α

=4cos^2 α -3cos α

Hence $4x^3 - 3x = \frac{\sqrt{3}}{3}$

i.e. 4cos^2 α -3cos α =$\frac{\sqrt{3}}{3}$

$0^0 \leq \alpha \leq 180^0$; cos3α = $\frac{\sqrt{3}}{3}$

For the range $0^0 \leq \alpha \leq 180^0$

⇨ $0^0 \leq 3\alpha \leq 540^0$

$3\alpha = 54.7^0, 414.7^0$

$\alpha = 18.23^0, 138.23^0$ (2d.p)

[α: α= $18.23^0, 138.23^0$]

(b) Given that t = $\tan 22\frac{1}{2}$, show that $t^2 + 2t - 1 = 0$,

Hence show that $\tan 22\frac{1}{2}^0 = -1+ \sqrt{2}$

Solution

$\tan 45^0 = \frac{2\tan 22\frac{1}{2}^0}{1-\tan^2 22\frac{1}{2}^0}$

$1 = \frac{2t}{1-t^2}$

$1 - t^2 = 2t$

$t^2 + 2t - 1 = 0$ (as required)

solving

$t = \frac{-2\pm\sqrt{2^2-(4x1\ x-1)}}{2x1}$

$t = \frac{-2\pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}$

Since $22\frac{1}{2}^0$ is an acute angle,

$\tan 22\frac{1}{2}^0 = -1+ \sqrt{2}$ is positive

$\therefore \tan 22\frac{1}{2}^0 = -1+ \sqrt{2}$

Example 28

(a) Show that $3\sin\theta = 3\sin\theta - 4\sin^3\theta$. Hence solve the equation $\sin 3\theta + \sin\theta = 0$ for $0^0 \le \theta \le 360^0$

Solution

$\text{Sin}3\theta = \sin(2\theta + \theta)$

$= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$

$= 2\sin\theta\cos^2\theta + (1- 2\sin^2\theta)\sin\theta$

$= 2\sin\theta(1- \sin^2\theta) + (1- 2\sin^2\theta)\sin\theta$

$= 3\sin\theta - 4\sin^3\theta$

Hence $\sin 3\theta + \sin\theta = 0$

$3\sin\theta - 4\sin^3\theta + \sin\theta = 0$

$4\sin\theta - 4\sin^3\theta = 0$

$4\sin\theta(1-\sin^2\theta) = 0$

$4\sin\theta(1-\sin\theta)(1+\sin\theta) = 0$

$\sin\theta = 0$; $\theta = 0^0, 180^0, 360^0$

$\sin\theta = 1$; $\theta = 90^0$

sino = -1; $\theta = 270^0$

$\therefore \theta$: $\theta = 0^0, 90^0, 180^0, 270^0, 360^0$

(b) Prove that $\cot 2\theta = \frac{\cot^2\theta - 1}{2\cot\theta}$. Hence solve the equation $\cot 2\theta + 2\cot\theta = 2$ for $0^0 \le \theta \le 360^0$

Solution

$\cot 2\theta = \frac{\cos 2\theta}{\sin 2\theta} = \frac{\cos^2\theta - \sin^2\theta}{2\sin\theta\cos\theta}$

dividing through by $\sin^2\theta$

$\cot 2\theta = \frac{\cot^2\theta - 1}{2\cot\theta}$

Hence, $\cot 2\theta + 2\cot\theta = 0$

$\frac{\cot^2\theta - 1}{2\cot\theta} + 2\cot\theta = 0$

$5\cot^2\theta - 4\cot\theta - 1 = 0$

$(5\cot\theta + 1)(\cot\theta - 1) = 0$

$\cot\theta = -\frac{1}{5}$ or $\cot\theta = 0$

⇨ $\tan\theta = -5$ ot $\tan\theta = 1$

When $\tan\theta = -5$; $\theta = 101.3^0, 281.3^0$

When $\tan\theta = 1$, $\theta = 45^0, 225^0$

$\therefore\{\theta$: $\theta = 45^0, 101.3^0, 225^0, 281.3^0\}$

Revison exercise 4

1. Prove that

(a) $\sin\alpha\operatorname{cosec}\beta + \cos\alpha\sec\beta = 2\sin(\alpha + \beta)\operatorname{cosec}2\beta$

(b) $\cos^6\theta + \sin^6\theta = 1- \frac{3}{4}\sin^2 2\theta$

(c) $\frac{\sin 3\alpha}{1+2\cos 2\alpha} = \sin\alpha$ and hence deduce that $\sin 15^0 = \frac{\sqrt{3}-1}{2\sqrt{2}}$

2. (a) Solve the equation for θ, $0^0 \le \theta \le 360^0$

$\sin^2\theta - 2\sin\theta\cos\theta - 3\cos^2\theta = 0$

[θ: $\theta = 71.6^0, 135^0, 251.6^0, 315^0$]

(b) show that $\frac{\cos\theta}{1+\sin\theta} = \cot(\frac{\theta}{2} + 45^0)$.

Hence or otherwise solve the equation $\frac{\cos\theta}{1+\sin\theta} = \frac{1}{2}$ for $0^0 \le \theta \le 360^0$ [θ= 36.8^0]

3. (a) solve the equation $4\cos 2\theta - 2\cos\theta + 3 = 0$ for $0^0 \le \theta \le 360^0$

[θ: θ= $60^0, 104.5^0, 255.5^0, 300^0$]

(c) Solve the equation $\sin\theta + \sin\frac{\theta}{2} = 0$ for $0^0 \le \theta \le 360^0$

[θ: θ= -360^0, -240^0, 0^0, 2405^0, 360^0]

4. (a) Prove that $\tan\left(\frac{\pi}{4}+\theta\right)$ - $\tan\left(\frac{\pi}{4}-\theta\right)$ = 2tan2θ
   (b) By expressing 2sinθsin(θ + α) as difference of cosines of two angles or otherwise, where α is constant, find its least value $\left[\frac{-a}{2}\right]$
   (c) Solve for θ in the equation Cosθ – cos(θ + 60^0) = 0.4 for $0^0 \le \theta \le 360^0$ [θ = 126.4^0, 353.6^0]
5. (a) Show that cos3θ = $4\cos^3\theta$ – 3cosθ. Hence solve the equation $4x^3 - 3x - \frac{\sqrt{3}}{3} = 0$ [x: x = -0.746, -0.204, 0.959]
   (b) Find all solutions of the equation 5cosx – 4sinx = 6 in the range $-180^0 \le x \le 180^0$ [x: x = -59.1^0, -18.3^0]
6. (a) Express $\sqrt{\left(\frac{sin2\theta-cos2\theta-1}{2-2sin2\theta}\right)}$ in terms of tanθ$\left[\frac{1}{\sqrt{(tan\theta-1)}}\right]$
   (b) Find the solution of the equation $\sqrt{3sin\theta} - cos\theta + 1 = 0$ for 0≤θ≤2π $\left[\theta: \theta = \frac{4}{3}\pi, 2\pi\right]$

(c) Factorize cosθ – cos3θ – cos7θ + cos9θ in form Acospθsinqθsinrθ where A, p, q and r are constants [A= -4, p =5, q = 5, r = 2]

7. (a) Given that sinα + sinβ = p and cosα + cosβ = q show that
   (i) $\tan\left(\frac{\alpha+\beta}{2}\right) = \frac{p}{q}$
   (ii) cos(α + β)= $\frac{q^2-p^2}{q^2+p^2}$
   (b) Solve the simultaneous equation: cosα + 4sinβ = 1 4secα – 3cosecβ = 5 [θ= 78.5^0, 281.5^0]
8. (a) Express sinθ + sin3θ in form pcosθsinqθ where p and q are constant [p =2, q = 2]
   (b) Find the solution of cos7θ + cos5θ = 2cosθ for $0^0 \le \theta \le 360^0$ [0^0, 60^0, 270^0, 360^0]
   (c) Prove that $\frac{sinA+sin4A+sin7A}{cosA+cos4A+cos7A} = tan4A$
9. Eliminate θ from each of the following pairs of expression
   (a) x+1 = cos2θ, y = sinθ [$x + 2y^2 = 0$]
   (b) x = cos2θ, y = cosθ -1 [$x = 2y^2 + 4y + 1$]
   (c) y – 3 = cos2θ, x = 2 – sinθ [$2x^2 - 8x + y + 4 = 0$]
10. Solve the following equations for $-180^0 \le \theta \le 180^0$
   (a) Sin2θ + sinθ = 0 [$\pm120^0$, $\pm180^0$]
   (b) Sin2θ – $2\cos^2\theta$ = 0 [-135^0, 45^0, $\pm90^0$]
   (c) 3cos2θ + 2+ cosθ = 0 [$\pm70.5^0$, $\pm120^0$]
   (d) sin2θ = tanθ [0^0, $\pm45^0$, $\pm135^0$, $\pm180^0$]
11. Solve the following equations for $-360^0 \le \theta \le 360^0$, giving your answer correct to 1 decimal place
   (a) sinθ = $\sin\left(\frac{\theta}{2}\right)$[$0^0$, $\pm120^0$, $\pm360^0$]
   (b) $3\cos\left(\frac{\theta}{2}\right)$ = 2sinθ [$\pm180^0$, 97.2^0, 262.8^0]
   (c) 2sinθ = $\tan\left(\frac{\theta}{2}\right)$ [0^0, $\pm120^0$, $\pm240^0$, $\pm360^0$]
   (d) 2cosθ = $15\cos\left(\frac{\theta}{2}\right) + 2$ [$\pm209^0$]
12. Prove the following identities
   (a) $2\cos^2\theta$ – cos2θ = 1
   (b) 2cosec2θ = cosecθsecθ
   (c) $2\cos^3\theta$ + sin2θsinθ =2cosθ
   (d) tanθ + cotθ = 2cosec2θ
   (e) $\cos^4\theta - \sin^4\theta$ = cos2θ
   (f) $\frac{1-cos2\theta}{1+cos2\theta} = tan^2\theta$
   (g) Cotθ – tanθ = 2cot2θ
   (h) cot2θ + cosecθ = cotθ
   (i) $\frac{cos2\theta}{cos\theta+sin\theta} = cos\theta - sin\theta$
   (j) $\frac{sin2\theta}{1-cos2\theta} = cot\theta$
   (k) $cos2\theta = \frac{1-tan^2\theta}{1+tan^2\theta}$
   (l) $tan3\theta = \frac{3tan\theta - tan^3\theta}{1-3tan^2\theta}$
   (m) $tan\left(\frac{\theta}{2}\right) = \frac{sin\theta}{1+cos\theta}$
13. Express $\tan 22\frac{1}{2}^0$ in the form a + b√2 where a and b are integers [a = -1, b= ±1]
14. Solve the equation

(i) 4cosθ – 2cos2θ = 3 for 0 ≤ θ ≤ π $\left[\frac{\pi}{3}\right]$

(ii) Cos2θ + cos3θ + cosθ = 0 for $0^0 \le \theta \le 360^0$ [θ = 45^0, 120^0, 135^0]

(iii) cosθ + sin2θ = 0 for $0^0 \le \theta \le 360^0$ [θ = 90^0, 210^0, 270^0, 330^0]

(iv) 2sin2θ = 3cosθ for $-180^0 \le \theta \le 180^0$ [θ = -90^0, 48.6^0, 90^0, 132.4^0]

(v) Sinθ – 4sin4θ = sin2θ – sin3θ for $-\pi \le \theta \le \pi$ $\left[\frac{-\pi}{5}, \frac{-\pi}{2}, \frac{-3\pi}{5}, 0, \frac{\pi}{2}, \frac{\pi}{5}, \frac{3\pi}{5}\right]$

Harmonic form

These are trigonometric functions expressed in the form of **Rcos(x ± α) and Rsin(x ±α).** They are in two ways

(i) solving equations in the form
$a\cos\theta + b\sin\theta + c = 0$

(ii) determining the maximum and minimum values of the function
$a\cos\theta + b\sin\theta + c = 0$
where a, b and c are constants

A: Solving equations

Example 29

(a) Express $3\cos\theta - 4\sin\theta$ in the form $R\cos(\theta + \alpha)$, where R and α are constants

Solution

Let $3\cos\theta - 4\sin\theta = R\cos(\theta + \alpha)$
$= R(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$
$= R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

Comparing coefficient of cosθ and sinθ

$R\cos\alpha = 3$ ……….. (i)

$R\sin\alpha = 4$ ………… (ii)

Eqn (ii) ÷ eqn (i)

$\tan\alpha = \frac{4}{3}$; $\alpha = 53.1^0$

Eqn. $(i)^2$ + eqn. $(ii)^2$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 3^2 + 4^2 = 25$

$R^2[\cos^2\alpha + \sin^2\alpha] = 25$

$R^2 = 25$

$R = 5$

$\therefore 3\cos\theta - 4\sin\theta = 5\cos(\theta + 53.1^0)$

(b) Solve the equation $3\cos\theta - 4\sin\theta = 5$ for $0^0 \le \theta \le 360^0$.

Solution

$3\cos\theta - 4\sin\theta = 5\cos(\theta + 53.1^0)$

⇨ $5\cos(\theta + 53.1^0) = 5$
$\cos(\theta + 53.1^0) = 1$
$x + 53.1^0 = 0^0, 360^0$
$x = -53.1^0, 306.9^0$

Hence $x = 306.9^0$

Example 30

(a) Express $\sin\theta - \sqrt{3}\cos\theta$ in the form $R\sin(\theta - \alpha)$

Solution

Let $\sin\theta - \sqrt{3}\cos\theta = R\sin(\theta - \alpha)$
$= R(\sin\theta\cos\alpha - \cos\theta\sin\alpha)$

Equating coefficients

$R\cos\alpha = 1$ ………………… (i)

$R\sin\alpha = \sqrt{3}$………………. (ii)

Eqn. (ii) ÷ eqn. (i)

$\tan\alpha = \sqrt{3}$; => $\alpha = 60^0$

$R^2[\cos^2\alpha + \sin^2\alpha] = 4$

$R^2 = 4$; $R = 2$

$\therefore \sin\theta - \sqrt{3}\cos\theta = 2\sin(\theta - 60^0)$

(b) Solve the equation
$\sin\theta - \sqrt{3}\cos\theta + 1 = 0$ for $0^0 \le \theta \le 360^0$

$\sin\theta - \sqrt{3}\cos\theta = 2\sin(\theta - 60^0)$

⇨ $2\sin(\theta - 60^0) + 1 = 0$

$\sin(\theta - 60^0) = -\frac{1}{2}$

$\theta - 60^0 = 210^0, 330^0$

$\theta = 270^0, 390^0$

Hence $\theta = 270^0$ for the given range

Example 31

(a) Express $4\cos\theta - 5\sin\theta$ in the form $A\cos(\theta + \beta)$, where A is constant and β is an acute angle

Let $4\cos\theta - 5\sin\theta = A\cos(\theta + \beta)$
$= A(\cos\theta\cos\beta - \sin\theta\sin\beta)$
$= A\cos\theta\cos\beta - R\sin\theta\sin\beta$

Comparing coefficient of cosθ and sinθ

$A\cos\beta = 4$ ……….. (i)

$A\sin\beta = 5$ ………… (ii)

Eqn (ii) ÷ eqn (i)

$\tan\alpha = \frac{5}{4}$; $\alpha = 51.3^0$

Eqn. $(i)^2$ + eqn. $(ii)^2$

$A^2\cos^2\beta + A^2\sin^2\beta = 4^2 + 5^2 = 41$

$A^2[\cos^2\alpha + \sin^2\alpha] = 41$

$A^2 = 41$

$A = \sqrt{41}$

$\therefore 3\cos\theta - 4\sin\theta = \sqrt{41}\cos(\theta + 51.3^0)$

(b) Solve the equation $3\cos\theta - 4\sin\theta = 2.2$ for $0^0 \le \theta \le 360^0$

Solution

$3\cos\theta - 4\sin\theta = \sqrt{41}\cos(\theta + 51.3^0)$

⇨ $\sqrt{41}\cos(\theta + 51.3^0) = 2.2$

$\cos(\theta + 51.3^0) = \frac{2.2}{\sqrt{41}} = 0.3436$

$(\theta + 51.3^0) = 69.9^0,\ 290.1^0$

$\therefore \theta = 18.6^0,\ 238.3^0$

B: Maximum and minimum values

The maximum and minimum values of a circular function may be obtained using three methods

(i) Express the given function either in for Rcos (θ ± α) or Rsin(θ ± α) if possible, where R and α are constants.

(ii) Differentiating the given function with respect to the given function say θ

(iii) Sketching the graphs of the function given and noting their maximum and minimum points.

In this chapter approach I will be considered.

Example 32

Determine the maximum and minimum values of the following, stating the value of θ for which they occur

(a) $\sqrt{3}\sin\theta + \cos\theta + 7$

Let $\sqrt{3}\sin\theta + \cos\theta = R\sin(\theta + \alpha)$

$= R(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$

Equating coefficients

$R\sin\alpha = 1$ ……………………. (i)

$R\cos\alpha = \sqrt{3}$………………… (ii)

Eqn. (i) ÷ eqn. (ii)

$\tan\alpha = \frac{1}{\sqrt{3}}$; => $\alpha = 30^0$

$R^2[\cos^2\alpha + \sin^2\alpha] = \left[1^2 + \left(\sqrt{3}\right)^2\right] = 2$

$R^2 = 4$; $R = 2$

$\therefore \sqrt{3}\sin\theta + \cos\theta = 2\sin(\theta + 30^0)$

⇨ $\sqrt{3}\sin\theta + \cos\theta + 7 = 2\sin(\theta + 30^0) + 7$

The minimum value occurs when

$\sin(\theta + 30^0) = -1$

⇨ Minimum value = 2(-1) + 7 = 5

Now for $\sin(\theta + 30^0) = -1$

$\theta + 30^0 = 270^0$

$\theta = 240^0 = \frac{4\pi}{3}$

The minimum value is $\left(\frac{4\pi}{3}, 5\right)$

And maximum value occurs when

$\sin(\theta + 30^0) = 1$

⇨ Minimum value = 2(1) + 7 = 9

Now for $\sin(\theta + 30^0) = 1$

$\theta + 30^0 = 90^0$

$\theta = 60^0 = \frac{\pi}{3}$

The maximum value is $\left(\frac{\pi}{3}, 9\right)$

(b) $5\cos\theta - 12\sin\theta - 13$

Solution

Let $5\cos\theta - 12\sin\theta = R\cos(\theta - \alpha)$

$= R(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$

$= R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

Comparing coefficient of cosθ and sinθ

$R\cos\alpha = 5$ ……….. (i)

$R\sin\alpha = 12$ ……….. (ii)

Eqn (ii) ÷ eqn (i)

$\tan\alpha = \frac{12}{5}$; $\alpha = 67.4^0$

Eqn. $(i)^2$ + eqn. $(ii)^2$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 5^2 + 12^2 = 169$

$R^2[\cos^2\alpha + \sin^2\alpha] = 169$

$R^2 = 169$

$R = 13$

$\therefore 2\cos\theta - 12\sin\theta = 13\cos(\theta - 67.4^0)$

⇨ $5\cos\theta - 12\sin\theta - 13 = 13\cos(\theta - 67.4^0) - 13$

The minimum value occurs when

$\cos(\theta - 67.4^0) = -1$

⇨ Minimum value = 13(-1) -13 = -26

Now for $\cos(\theta - 67.4^0) = -1$

$\theta - 67.4^0 = 180^0$

$\theta = 247.4^0$

The minimum value is $(247.4^0, -26)$

And maximum value occurs when

$\cos(\theta - 67.4^0) = 1$

⇨ Minimum value = 13(1) - 13 = 0

Now for $\cos(\theta - 67.4^0) = 1$

$\theta - 67.4^0 = 0^0$

$\theta = 67.4^0$

The maximum value is $(67.4^0, 0)$

Example 33

(a) Given that $p = 2\cos\theta + 3\cos2\theta$ and $q = 2\sin\theta + 3\sin2\theta$, show that $1 \leq p^2 + q^2 \leq 25$
If $p^2 + q^2 = 19$ and θ is acute, find θ and show that $pq = \frac{-5\sqrt{3}}{4}$

Solution

$p^2 = 4\cos^2\theta + 12\cos\theta\cos2\theta + 9\cos^2 2\theta$ ....(i)

$q^2 = 4\sin^2\theta + 12\sin\theta\sin2\theta + 9\sin^2 2\theta$ ....(ii)

Eqn. (i) + eqn. (ii)

$p^2 + q^2 = 4 + 12(\cos\theta\cos2\theta + \sin\theta\sin2\theta) + 9$

$p^2 + q^2 = 13 + 12\cos\theta$ [$\cos(-\theta) = \cos\theta$]

But $-1 \leq \cos\theta \leq 1$

Multiplying through by 12

$-12 \leq 12\cos\theta \leq 12$

Adding 13 throughout

$1 \leq 12\cos\theta + 12 \leq 25$

$\therefore 1 \leq p^2 + q^2 \leq 25$ as required

If $p^2 + q^2 = 19$, $\Rightarrow 13 + 12\cos\theta = 19$

$\cos\theta = \frac{1}{2}$; $\theta = 60^0$ [θ is acute]

⇨ $p = 2\cos60^0 + 3\cos120^0 = 1 - \frac{3}{2} = -\frac{1}{2}$

$q = 2\sin60^0 + 3\sin120^0 = \sqrt{3} + 3\frac{\sqrt{3}}{2} = \frac{5\sqrt{3}}{2}$

$\therefore pq = \left(-\frac{1}{2}\right)\left(\frac{5\sqrt{3}}{2}\right) = \frac{5\sqrt{3}}{4}$

(b) Express $f(x) = 5\sin^2\theta - 3\sin\theta\cos\theta + \cos^2\theta$ in the form $p + q\cos(2\theta - \alpha)$
Hence show that $\frac{1}{2} \leq f(x) \leq 5\frac{1}{2}$

Solution

Using $\sin^2\theta = \frac{1}{2}(1 - \cos2\theta)$ and

$\cos^2\theta = \frac{1}{2}(1 + \cos2\theta)$

$f(x) = \frac{5}{2}(1 - \cos2\theta) - 3\sin\theta\cos\theta + \frac{1}{2}(1 + \cos2\theta)$

$= \frac{5}{2} - \frac{5}{2}\cos2\theta - 3.\frac{2}{2}\sin\theta\cos\theta + \frac{1}{2} + \frac{1}{2}\cos2\theta$

$= 3 - 2\cos2\theta - \frac{3}{2}\sin2\theta$

$= 3 - [2\cos\theta + \frac{3}{2}\sin2\theta]$

Now:

$3 - [2\cos\theta + \frac{3}{2}\sin2\theta] \equiv p + q\cos(2\theta - \alpha)$

$= 3 + [q\cos2\theta\cos\alpha + q\sin2\theta\sin\alpha]$

By comparing: $p = 3$, $q\sin\alpha = \frac{3}{2}$ and $q\cos\alpha = 2$

⇨ $\tan\alpha = \frac{3}{4}$; $\alpha = 36.9^0$

And $q = \sqrt{\left\{\left(\frac{3}{2}\right)^2 + (2)^2\right\}} = \frac{5}{2}$

⇨ $3 - [2\cos\theta + \frac{3}{2}\sin2\theta] = 3 - \frac{5}{2}\cos(2\theta - 36.9^0)$

But $-1 \leq \cos(2\theta - 36.9^0) \leq 1$

Multiplying through by $-\frac{5}{2}$

$\frac{5}{2} \geq -\frac{5}{2}\cos(2\theta - 36.9^0) \geq -\frac{5}{2}$

Adding 3 throughout

$3 - \frac{5}{2} \leq 3 - \frac{5}{2}\cos(2\theta - 36.9^0) \leq 3 + \frac{5}{2}$

$\therefore \frac{1}{2} \leq f(x) \leq 5\frac{1}{2}$

(c) Find the maximum and minimum points of the function; $f(x) = 3\cos\theta - 4\sin\theta + 20$ for $0^0 \leq \theta \leq 360^0$

Solution

Let $3\cos\theta - 4\sin\theta = R\cos(\theta + \alpha)$

$= R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

Comparing coefficient of $\cos\theta$ and $\sin\theta$

$R\cos\alpha = 3$ .... (i)

$R\sin\alpha = 4$ .... (ii)

Eqn (ii) ÷ eqn (i)

$\tan\alpha = \frac{4}{3}$; $\alpha = 53.1^0$

Eqn. $(i)^2$ + eqn. $(ii)^2$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 3^2 + 4^2 = 25$

$R^2[\cos^2\alpha + \sin^2\alpha] = 25$

$R^2 = 25$

$R = 5$

$\therefore 3\cos\theta - 4\sin\theta = 5\cos(\theta - 53.1^0)$

⇨ $3\cos\theta - 4\sin\theta + 20 = 5\cos(\theta - 53.1^0) + 20$

The minimum value occurs when

$\cos(\theta - 53.1^0) = -1$

⇨ Minimum value = 5(-1) +20 = 15

Now for $\cos(\theta - 53.1) = -1$

$$\theta - 53.1^0 = 180^0$$

$$\theta = 126.8^0$$

The minimum value is $(126.8^0\ ,15)$

And maximum value occurs when

$\cos(\theta - 53.1^0) = 1$

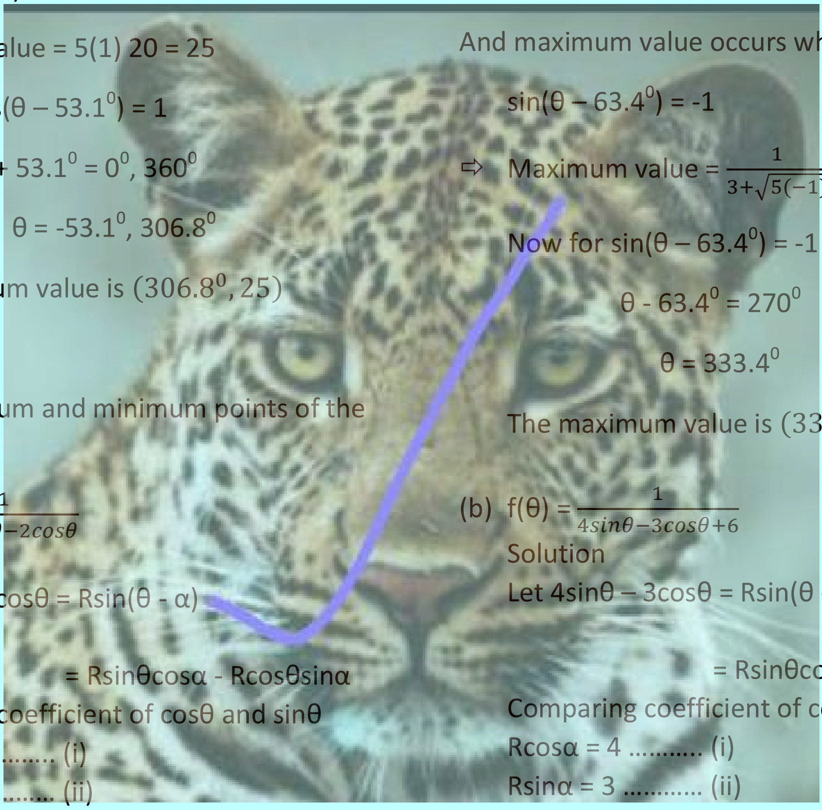
⇨ Minimum value = 5(1) 20 = 25

Now for $\cos(\theta - 53.1^0) = 1$

$$\theta + 53.1^0 = 0^0, 360^0$$

$$\theta = -53.1^0, 306.8^0$$

The maximum value is $(306.8^0, 25)$



Example 34

Find the maximum and minimum points of the following

(a) $f(\theta) = \frac{1}{3+\sin\theta-2\cos\theta}$

Solution

Let $\sin\theta - 2\cos\theta = R\sin(\theta - \alpha)$

$= R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$

Comparing coefficient of $\cos\theta$ and $\sin\theta$

$R\cos\alpha = 1$ ………. (i)

$R\sin\alpha = 2$ ………. (ii)

Eqn (ii) ÷ eqn (i)

$\tan\alpha = 2$; $\alpha = 63.4^0$

Eqn. $(i)^2$ + eqn. $(ii)^2$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 1^2 + 2^2 = 5$

$R^2[\cos^2\alpha + \sin^2\alpha] = 5$

$R^2 = 5$

$R = \sqrt{5}$

$\therefore \sin\theta - 2\cos\theta = \sqrt{5}\sin(\theta - 63.4^0)$

⇨ $3 + \sin\theta - 2\cos\theta = \sqrt{5}\sin(\theta - 63.4^0) + 3$

⇨ $f(\theta) = \frac{1}{3+\sqrt{5}\sin(\theta-63.4^0)}$

Note: for a fractional function, a maximum point is obtained when the denominator is minimum and the vice versa for the maximum point

The minimum value occurs when

$\sin(\theta - 63.4^0) = 1$

⇨ Minimum value $= \frac{1}{3+\sqrt{5}} = 0.31$

Now for $\sin(\theta - 63.4) = 1$

$$\theta - 63.4^0 = 90^0$$

$$\theta = 153.4^0$$

The minimum value is $(153.4^0. 0.31)$

And maximum value occurs when

$\sin(\theta - 63.4^0) = -1$

⇨ Maximum value $= \frac{1}{3+\sqrt{5(-1)}} = 1.31$

Now for $\sin(\theta - 63.4^0) = -1$

$$\theta - 63.4^0 = 270^0$$

$$\theta = 333.4^0$$

The maximum value is $(333.4^0, 1.31)$

(b) $f(\theta) = \frac{1}{4\sin\theta-3\cos\theta+6}$

Solution

Let $4\sin\theta - 3\cos\theta = R\sin(\theta - \alpha)$

$= R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$

Comparing coefficient of $\cos\theta$ and $\sin\theta$

$R\cos\alpha = 4$ ………. (i)

$R\sin\alpha = 3$ ………. (ii)

Eqn (ii) ÷ eqn (i)

$\tan\alpha = 0.75$; $\alpha = 36.9^0$

Eqn. $(i)^2$ + eqn. $(ii)^2$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 3^2 + 4^2 = 25$

$R^2[\cos^2\alpha + \sin^2\alpha] = 25$

$R^2 = 25$

$R = 5$

$\therefore 4\sin\theta - 3\cos\theta = 5\sin(\theta - 36.9^0)$

⇨ $4\sin\theta - 3\cos\theta + 6 = 5\sin(\theta - 36.9^0) + 6$

⇨ $f(\theta) = \frac{1}{5\sin(\theta-36.9^0)+6}$

The minimum value occurs when

$\sin(\theta - 36.9^0) = 1$

⇨ Minimum value = $\frac{1}{5(1)+6} = \frac{1}{11}$

Now for sin(θ – 63.4) = 1

$\theta - 36.9^0 = 90^0$

$\theta = 126.9^0$

The minimum value is $(126.9^0. \frac{1}{11})$

And maximum value occurs when

$\sin(\theta - 36.9^0) = -1$

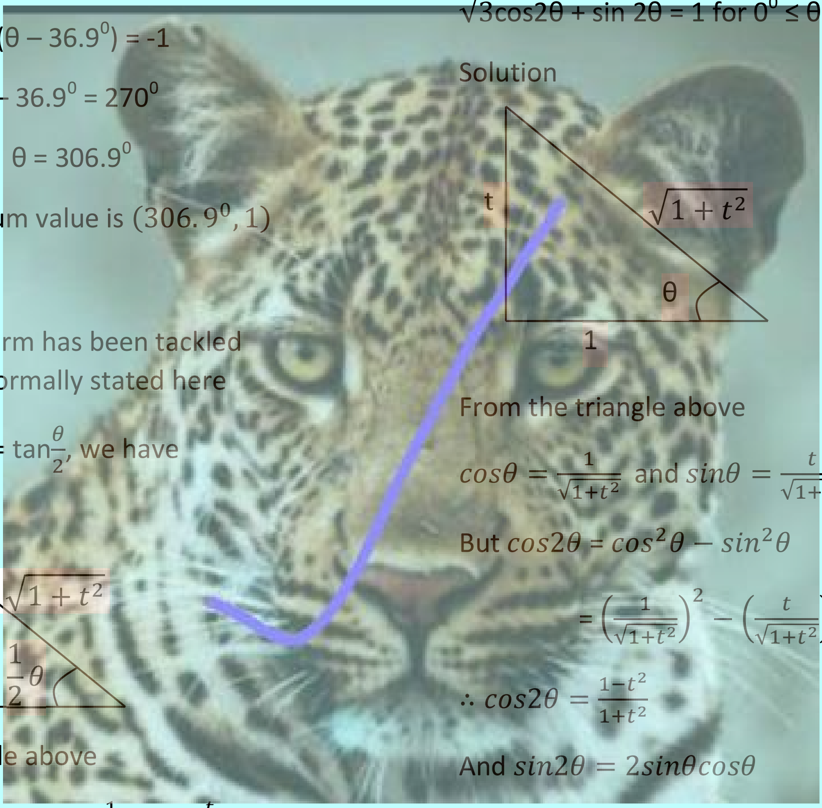
⇨ Maximum value = $\frac{1}{5(-1)+6} = 1$

Now for $\sin(\theta - 36.9^0) = -1$

$\theta - 36.9^0 = 270^0$

$\theta = 306.9^0$

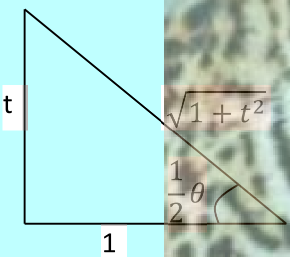
The maximum value is $(306.9^0, 1)$



The t-formula

Although this form has been tackled indirectly, it is formally stated here

Suppose that t = $\tan\frac{\theta}{2}$, we have



From the triangle above

$\cos\frac{1}{2}\theta = \frac{1}{\sqrt{1+t^2}}$ and $\sin\frac{1}{2}\theta = \frac{t}{\sqrt{1+t^2}}$

But $\cos\theta = \cos^2\frac{1}{2}\theta - \sin^2\frac{1}{2}\theta$

$= \left(\frac{1}{\sqrt{1+t^2}}\right)^2 - \left(\frac{t}{\sqrt{1+t^2}}\right)^2$

$\therefore \cos\theta = \frac{1-t^2}{1+t^2}$

And $\sin\theta = 2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$

$= 2\left(\frac{t}{\sqrt{1+t^2}}\right)\left(\frac{1}{\sqrt{1+t^2}}\right)$

$\therefore \sin\theta = \frac{2t}{1+t^2}$

The t- formula is used widely in solving equations and proving trigonometric identities. These can be extended as follows

(i) For t= tanθ, sin2θ =$\frac{2t}{1+t^2}$ and $\cos 2\theta = \frac{1-t^2}{1+t^2}$

(ii) For t= $\tan\left(\frac{5x}{4}\right)$, $\sin\left(\frac{5x}{2}\right) = \frac{2t}{1+t^2}$ and

$\cos\left(\frac{5x}{2}\right) = \frac{1-t^2}{1+t^2}$

Example 35

Show that if t = tanθ, then sin2θ =$\frac{2t}{1+t^2}$ and $2\theta = \frac{1-t^2}{1+t^2}$. Hence solve the equation $\sqrt{3}\cos 2\theta + \sin 2\theta = 1$ for $0^0 \leq \theta \leq 360^0$.

Solution

From the triangle above

$\cos\theta = \frac{1}{\sqrt{1+t^2}}$ and $\sin\theta = \frac{t}{\sqrt{1+t^2}}$

But $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= \left(\frac{1}{\sqrt{1+t^2}}\right)^2 - \left(\frac{t}{\sqrt{1+t^2}}\right)^2$

$\therefore \cos 2\theta = \frac{1-t^2}{1+t^2}$

And $\sin 2\theta = 2\sin\theta\cos\theta$

$= 2\left(\frac{t}{\sqrt{1+t^2}}\right)\left(\frac{1}{\sqrt{1+t^2}}\right)$

$\therefore \sin 2\theta = \frac{2t}{1+t^2}$

Hence $\sqrt{3}\cos 2\theta + \sin 2\theta = 1$

⇨ $\sqrt{3\left(\frac{1-t^2}{1+t^2}\right) + \left(\frac{2t}{1+t^2}\right)} = 1$

$\sqrt{3} - \sqrt{3t^2} + 2t = 1 + t^2$

$(1+\sqrt{3})t^1 - 2t + 1 - \sqrt{3} = 0$

$t = \frac{2\pm\sqrt{2^2-4(1+\sqrt{3})(1-\sqrt{3})}}{2(1+\sqrt{3})} = \frac{2\pm\sqrt{12}}{2(1+\sqrt{3})} = \frac{1\pm\sqrt{3}}{1+\sqrt{3}}$

$t = \frac{1+\sqrt{3}}{1+\sqrt{3}}$ = 1 or

$t = \left(\frac{1-\sqrt{3}}{1+\sqrt{3}}\right)\left(\frac{1-\sqrt{3}}{1-\sqrt{3}}\right)$ = -2 + √3

If tanθ = 1; θ = 450, 2250

If tanθ = -2 + √3; θ = 165^0, 345^0

∴ θ: θ = 45^0, 165^0, 225^0, 345^0

Example 36

Find all the solutions of the equation
5cosθ – 4sinθ = 6 for $-180^0 \le \theta \le 180^0$

Solution

Let t = $\tan\frac{\theta}{2}$ then

$$cos\theta = \frac{1-t^2}{1+t^2}$$

$$sin\theta = \frac{2t}{1+t^2}$$

⇨ $5\left(\frac{1-t^2}{1+t^2}\right) - 4\left(\frac{2t}{1+t^2}\right) = 6$

$5(1-t^2) - 8t = 6(1+t^2)$

$5 - 5t^2 - 8t = 6 + 6t^2$

$11t^2 + 8t + 1 = 0$

$t = \frac{-8\pm\sqrt{8^2-4\ x\ 11\ x\ 1}}{2\ x\ 11} = \frac{-8\pm4.4721}{22}$

t = $\frac{-8+4.4721}{22}$ = −0.1604 or

t = $\frac{-8-4.4721}{22}$ = -0.5669

Taking t = -0.1604

$tan\frac{\theta}{2} = -0.1604;\ \theta = -18.2^0$

Taking t = -0.5669

$tan\frac{\theta}{2} = -0.5669;\ \theta = -59.1^0$

∴θ = -59.1^0, -18.2^0

Example 37

Solve the equation

$3\tan^2\theta + 2\sec^2\theta = 2(5 - 3\tan\theta)$ for
$0^0 < \theta < 180^0$

Let t = tan θ

$3t^2 - 2(1 + t^2) = 2(5 - 3t)$

$5t^2 + 6t - 8 = 0$

t = $\frac{-6\pm\sqrt{6^2-4(5)(-8)}}{2(5)} = \frac{-6\pm14}{10}$ = -2 or $\frac{4}{5}$

Taking t = -2; θ = $\tan^{-1}(-2) = 116.57^0$

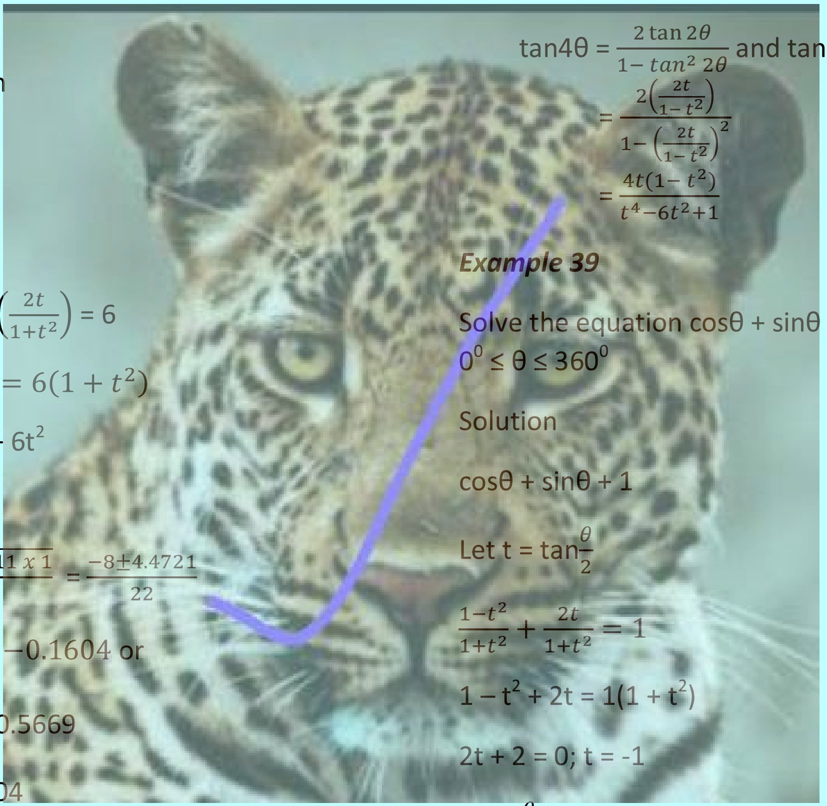
Taking t = $\frac{4}{5}$; θ = $\tan^{-1}(\frac{4}{5}) = 38.66^0$

Hence θ = 38.66^0, 116.57^0

Example 38

Show that tan4θ = $\frac{4t(1-t^2)}{t^4-6t^2+1}$, where t = tanθ.

Solution



$$\tan 4\theta = \frac{2\tan 2\theta}{1-tan^2\ 2\theta} \text{ and } \tan 2\theta = \frac{2\tan\theta}{1-tan^2\ \theta}$$

$$= \frac{2\left(\frac{2t}{1-t^2}\right)}{1-\left(\frac{2t}{1-t^2}\right)^2}$$

$$= \frac{4t(1-t^2)}{t^4-6t^2+1}$$

Example 39

Solve the equation cosθ + sinθ + 1 = 0 for
$0^0 \le \theta \le 360^0$

Solution

cosθ + sinθ + 1

Let t = $\tan\frac{\theta}{2}$

$\frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2} = 1$

$1 - t^2 + 2t = 1(1 + t^2)$

2t + 2 = 0; t = -1

∴ $\tan\frac{\theta}{2}$ = -1

$\frac{\theta}{2} = 135^0, 315^0$

θ = 270^0, 630^0

Hence θ = 270^0

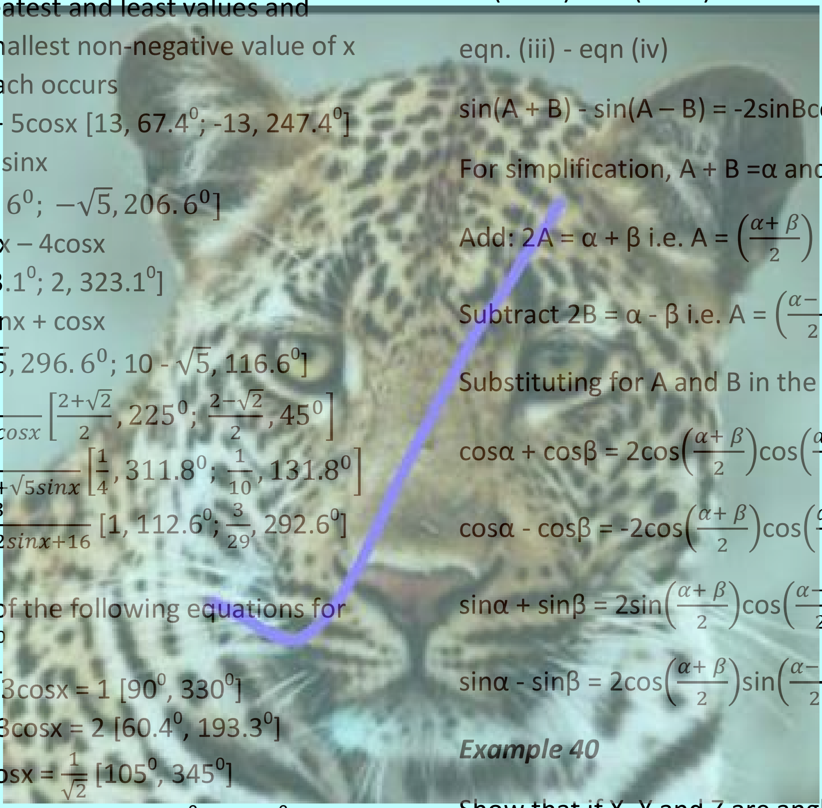
Revision exercise 5

1. Solve equation 3cosθ + 4sinθ = 2 for $0^0 \le\ \le 360^0$ [119.6^0, 346.7^0]
2. (a) Show that cos4x = $\frac{tan^4x-6tan^2x+1}{tan^4x+2tan^2x+1}$

   (b) Show that if q = cos2x + sin2x, then $(1+q)\tan^2x - 2\tan x + q - 1 = 0$.

Deduce that if the roots of the above equation are $\tan x_1$ and $\tan x_2$, the $\tan(x_1 + x_2) = 1$

3. Find the values of R and tanα in each of the following equations
   (a) 2cosθ + 5sinθ = Rsin(θ + α) $\left[\sqrt{29}, \frac{2}{5}\right]$
   (b) 2cosθ + 5sinθ = Rcos(θ - α) $\left[\sqrt{29}, \frac{5}{2}\right]$
   (c) √3cosθ + sinθ = Rcos(θ - α) $\left[2, \frac{1}{\sqrt{3}}\right]$
   (d) 5sinθ – 12cosθ = Rsin(θ - α) $\left[13, \frac{12}{5}\right]$
   (e) Cosθ – sinθ = Rcos(θ + α) $\left[\sqrt{2}, 1\right]$



4. Find the greatest and least values and state the smallest non-negative value of x for which each occurs
   (i) 12sinx + 5cosx [13, 67.4^0; -13, 247.4^0]
   (ii) 2cosx + sinx
   $[\sqrt{5}, 26.6^0; -\sqrt{5}, 206.6^0]$
   (iii) 7 + 3sinx – 4cosx
   [12, 143.1^0; 2, 323.1^0]
   (iv) 10 – 2sinx + cosx
   $[10 + \sqrt{5}, 296.6^0; 10 - \sqrt{5}, 116.6^0]$
   (v) $\frac{1}{2+sinx+cosx}$ $\left[\frac{2+\sqrt{2}}{2}, 225^0; \frac{2-\sqrt{2}}{2}, 45^0\right]$
   (vi) $\frac{1}{7-2cosx+\sqrt{5}sinx}$ $\left[\frac{1}{4}, 311.8^0; \frac{1}{10}, 131.8^0\right]$
   (vii) $\frac{3}{5cosx-12sinx+16}$ $\left[1, 112.6^0; \frac{3}{29}, 292.6^0\right]$

5. Solve each of the following equations for $0^0 \le x \le 360^0$
   (a) sinx + √3cosx = 1 [90^0, 330^0]
   (b) 4sinx – 3cosx = 2 [60.4^0, 193.3^0]
   (c) sinx + cosx = $\frac{1}{\sqrt{2}}$ [105^0, 345^0]
   (d) 5sinx + 12cosx = 7 [80.0^0, 325.2^0]
   (e) 7sinx – 4cosx = 3 [51.6^0, 187.9^0]
   (f) cosx – 3sinx = 2 [237.7^0, 339.2^0]
   (g) 5cosx + 2sinx = 4[63.8^0, 339.8^0]
   (h) 9cox2x – 4sin2x = 6 [14.2^0, 141.8^0, 194.2^0, 321.8^0]
   (i) 7cosx + 6sinx = 2 [118.1^0, 323.1^0]
   (j) 9cosx – 8sinx = 12 [313.6^0, 323.1^0]

The factor formulae

The following identities were developed from compound angles

cos (A + B) = cosAcosB – sinAsinB ……….(i)

cos(A – B) = coaAcosB + sinAsinB ………… (ii)

sin(A + B) = sinAcosB + sinBcosA …………(iii)

sin(A – B) = sinAcosB – sinBcosA …………(iv)

eqn. (i) + eqn (ii)

cos(A + B) + cos(A – B) = 2cosAcosB

eqn. (i) – eqn (ii)

cos(A + B) - cos(A – B) = -2cosAcosB

eqn. (iii) + eqn (iv)

sin(A + B) + sin(A – B) = 2sinAcosB

eqn. (iii) - eqn (iv)

sin(A + B) - sin(A – B) = -2sinBcosA

For simplification, A + B =α and A – B = β

Add: 2A = α + β i.e. A = $\left(\frac{\alpha+\beta}{2}\right)$

Subtract 2B = α - β i.e. A = $\left(\frac{\alpha-\beta}{2}\right)$

Substituting for A and B in the above equation

cosα + cosβ = 2cos$\left(\frac{\alpha+\beta}{2}\right)cos\left(\frac{\alpha-\beta}{2}\right)$

cosα - cosβ = -2cos$\left(\frac{\alpha+\beta}{2}\right)cos\left(\frac{\alpha-\beta}{2}\right)$

sinα + sinβ = 2sin$\left(\frac{\alpha+\beta}{2}\right)cos\left(\frac{\alpha-\beta}{2}\right)$

sinα - sinβ = 2cos$\left(\frac{\alpha+\beta}{2}\right)sin\left(\frac{\alpha-\beta}{2}\right)$

Example 40

Show that if X, Y and Z are angles of a triangle, then

(a) cosX + cosY + cosZ – 1 = 4sin$\frac{X}{2}$sin$\frac{Y}{2}$sin$\frac{Z}{2}$

solution

LHS cosX + cosY + cosZ – 1

= 2cos $\frac{X+Y}{2}$ cos $\frac{X-Y}{2}$ + 1 – 2sin$^2\frac{Z}{2}$ – 1

(to eliminate -1)

= 2cos $\frac{180^0 - Z}{2}$ cos $\frac{X-Y}{2}$ – 2sin$^2\frac{Z}{2}$

(since X + Y = 180^0 – Z)

= 2sin $\frac{Z}{2}$ cos $\frac{X-Y}{2}$ −2 sin$^2\frac{Z}{2}$

(Since cos(90^0 – A) = sinA)

= 2sin $\frac{Z}{2}\left[\cos\frac{X-Y}{2} - 2sin^2\left\{\frac{180^0-(X+Y)}{2}\right\}\right]$

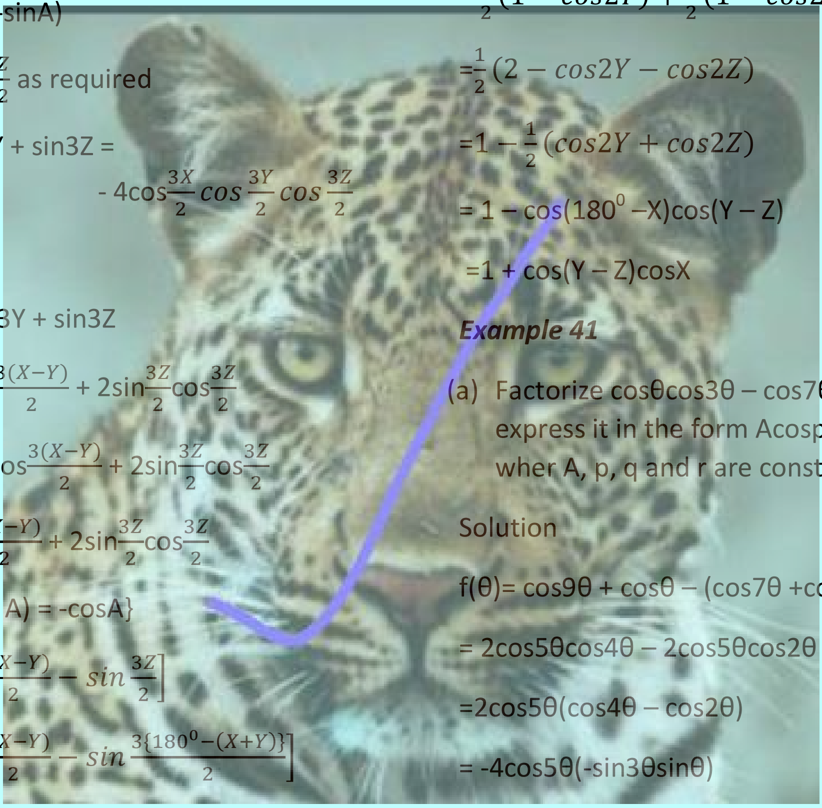
= 2sin $\frac{Z}{2}\left[\cos\frac{X-Y}{2} - cos\left\{\frac{X+Y)}{2}\right\}\right]$

(Since sin (90^0 – A) = cos A)

= 2sin$\frac{Z}{2}\left[-2sin\frac{X}{2}sin\frac{-y}{2}\right]$

=2sin$\frac{Z}{2}\left[2sin\frac{X}{2}sin\frac{Y}{2}\right]$

(Since sin(-A) = -sinA)



$4sin\frac{X}{2}sin\frac{Y}{2}$ sin $\frac{Z}{2}$ as required

(b) Sin3X + sin3Y + sin3Z =

- 4cos$\frac{3X}{2}$ $cos\frac{3Y}{2}$ $cos\frac{3Z}{2}$

Solution

LHS: Sin3X + sin3Y + sin3Z

= 2sin$\frac{3(X+Y)}{2}$cos$\frac{3(X-Y)}{2}$ + 2sin$\frac{3Z}{2}$cos$\frac{3Z}{2}$

=2sin$\frac{3(180^0-Z)}{2}$ cos$\frac{3(X-Y)}{2}$ + 2sin$\frac{3Z}{2}$cos$\frac{3Z}{2}$

= -2cos$\frac{3Z}{2}$ cos$\frac{3(X-Y)}{2}$ + 2sin$\frac{3Z}{2}$cos$\frac{3Z}{2}$

Since sin(270^0 – A) = -cosA}

=-2cos$\frac{3Z}{2}\left[\cos\frac{3(X-Y)}{2} - sin\frac{3Z}{2}\right]$

=-2cos$\frac{3Z}{2}\left[\cos\frac{3(X-Y)}{2} - sin\frac{3\{180^0-(X+Y)\}}{2}\right]$

=-2cos$\frac{3Z}{2}\left[\cos\frac{3(X-Y)}{2} - cos\frac{3(X+Y)}{2}\right]$

=-2cos$\frac{3Z}{2}\left[2cos\frac{3X}{2} + cos\frac{-3Y}{2}\right]$

Since cos(-A) = cosA

=- 4cos$\frac{3X}{2}$ $cos\frac{3Y}{2}$ $cos\frac{3Z}{2}$

(c) cos4X + cos4Y + cos4Z + 1

= 4cos2Xcos2Ycos2Z

Solution

LHS: cos4X + cos4Y + cos4Z + 1

= 2cos2(X + Y)cos2(X – Y) + 2cos^{2}2Z -1 + 1

= 2cos2(180^0 – Z)cos2(X – Y) + 2cos^{2}2Z

= 2cos2Z[cos2(X – Y) + cos2{180^0 – (X + Y)}]

= 2cos2Z[cos2(X – Y) + cos2 (X + Y)]

= 2cos2Z[2cos2X cos-2Y]

Since cos(-A) = cosA

=4cos2Z2cos2X cos2Y

(d) sin^{2}Y + sin^{2}Z = 1 + cos(Y – Z)cosX

LHS: sin^{2}Y + sin^{2}Z

= $\frac{1}{2}(1 - cos2Y) + \frac{1}{2}(1 - cos2Z)$

=$\frac{1}{2}(2 - cos2Y - cos2Z)$

=1 $-\frac{1}{2}(cos2Y + cos2Z)$

= 1 – cos(180^0 –X)cos(Y – Z)

=1 + cos(Y – Z)cosX

Example 41

(a) Factorize cosθcos3θ – cos7θ + cos9θ and express it in the form Acospθsinqθsinrθ wher A, p, q and r are constants

Solution

f(θ)= cos9θ + cosθ – (cos7θ +cos3θ)

= 2cos5θcos4θ – 2cos5θcos2θ

=2cos5θ(cos4θ – cos2θ)

= -4cos5θ(-sin3θsinθ)

=-4cos5θsin3θsinθ

⇨ A = -4, p = 5, q = 3, r = 1

(b) Given that

p = sinα + sinβ

q = cosα + cosβ. Show that

$\frac{2pq}{p^2+q^2}$ = sin(α + β)

Solution

$\frac{2pq}{p^2+q^2}$

=$\frac{2(sin\alpha+sin\beta)(cos\alpha+cos\beta)}{sin^2\alpha+2sin\alpha sin\beta+sin^2\beta+cos^2\alpha+2cos\alpha cos\beta+cos^2\beta}$

$$=\frac{2\left[2sin\frac{\alpha+\beta}{2}cos\frac{\alpha-\beta}{2}\right]\left[2cos\frac{\alpha+\beta}{2}cos\frac{\alpha-\beta}{2}\right]}{2+2(cos\alpha cos\beta+sin\alpha sin\beta}$$

$$=\frac{2\left[2sin\frac{\alpha+\beta}{2}cos\frac{\alpha+\beta}{2}\right]\left[2cos^2\frac{\alpha-\beta}{2}\right]}{2+2cos(\alpha-\beta)}$$

$$=\frac{2[\sin(\alpha+\beta)][1+\cos(\alpha-\beta)]}{2[1+\cos(\alpha-\beta)]}$$

$= \sin(\alpha + \beta)$

Example 42

Solve $5\cos^2 3\theta = 3(1 + \sin 3\theta)$ for $0^0 \leq \theta \leq 90^0$.

Solution

$5\cos^2 3\theta = 3(1 + \sin 3\theta)$

$5(1 - \sin^2 3\theta) = 3(1 + \sin 3\theta)$

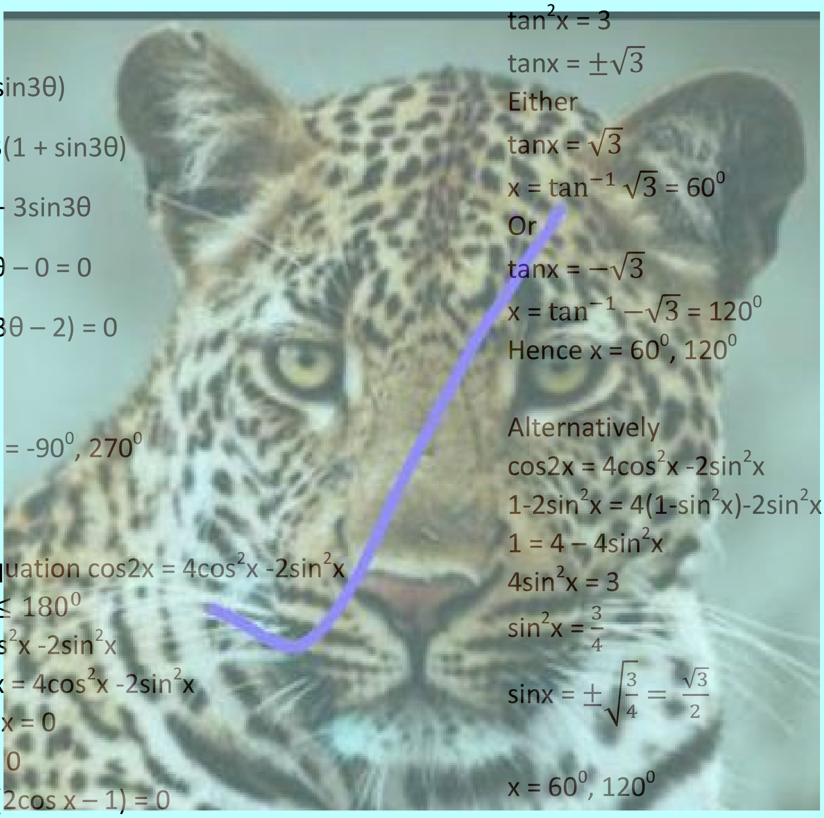
$5 - 5\sin^2 3\theta = 3 + 3\sin 3\theta$

$5\sin^2 3\theta + 3\sin 3\theta - 0 = 0$

$(\sin 3\theta + 1)(5\sin 3\theta - 2) = 0$

$\sin 3\theta + 1 = 0$

$3\theta = \sin^{-1}(-1) = -90^0, 270^0$



Example 43

(a) solve the equation $\cos 2x = 4\cos^2 x - 2\sin^2 x$ for $0 \leq \theta \leq 180^0$

$\cos 2x = 4\cos^2 x - 2\sin^2 x$

$\cos^2 x - \sin^2 x = 4\cos^2 x - 2\sin^2 x$

$3\cos^2 x - \sin^2 x = 0$

$4\cos^2 x - 1 = 0$

$(2\cos x + 1)(2\cos x - 1) = 0$

Either

$2\cos x + 1 = 0$

$\cos x = -\frac{1}{2}$

$x = \cos^{-1}(-\frac{1}{2}) = 120^0$

Or

$2\cos x - 1 = 0$

$\cos x = \frac{1}{2}$

$x = \cos^{-1}(\frac{1}{2}) = 60^0$

$\therefore x(60^0, 120^0)$

Alternatively

$\cos 2x = 4\cos^2 x - 2\sin^2 x$

$= \frac{4}{2}(1 + \cos 2x) - \frac{2}{2}(1 - \cos 2x)$

$= 2 + 2\cos 2x - 1 + \cos 2x$

$2\cos 2x + 1 = 0$

$\cos 2x = -\frac{1}{2}$

$2x = \cos_{-1}(-\frac{1}{2}) = 120^0, 240^0$

$x = 60^0, 120^0$

Alternatively

$\cos 2x = 4\cos^2 x - 2\sin^2 x$

$\cos^2 x - \sin^2 x = 4\cos^2 x - 2\sin^2 x$

$3\cos^2 x - \sin^2 x = 0$

$\sin^2 x = 3\cos^2 x$

$\tan^2 x = 3$

$\tan x = \pm\sqrt{3}$

Either

$\tan x = \sqrt{3}$

$x = \tan^{-1}\sqrt{3} = 60^0$

Or

$\tan x = -\sqrt{3}$

$x = \tan^{-1} -\sqrt{3} = 120^0$

Hence $x = 60^0, 120^0$

Alternatively

$\cos 2x = 4\cos^2 x - 2\sin^2 x$

$1 - 2\sin^2 x = 4(1 - \sin^2 x) - 2\sin^2 x$

$1 = 4 - 4\sin^2 x$

$4\sin^2 x = 3$

$\sin^2 x = \frac{3}{4}$

$\sin x = \pm\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$

$x = 60^0, 120^0$

Alternatively

$\cos 2x = 4\cos^2 x - 2\sin^2 x$

$1 - 2\sin^2 x = 4\cos^2 x - 2\sin^2 x$

$4\cos^2 x = 1$

$\cos x = \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2}$

$x = 60^0, 120^0$

(b) Show that if $\sin(x + \alpha) = p\sin(x - \alpha)$ then $\tan x = \left(\frac{p+1}{p-1}\right)\tan\alpha$.

Hence solve the equation $\sin(x + \alpha) = p\sin(x - \alpha)$ for $p = 2$ and $\alpha = 20^0$.

sinxcosα + coxsinα = p(sinxcosα - coxsinα)

cosxsinα (p + 1) = sinxcosα (p - 1)

cosxsinα$\left(\frac{p+1}{P-1}\right) =$ sinxcosα

$\frac{cosxsin\alpha}{\text{sinxcos}\alpha}\left(\frac{p+1}{p-1}\right) = \frac{sinxcos\alpha}{\text{sinxcos}\alpha}$

$tanx = \left(\frac{p+1}{P-1}\right)\tan\alpha$

For sin(x + 20^0) = 2sin(x – 20^0)

tanx = $\frac{2+1}{2-1}tan20^0$ = 3tan20^0

x = $\tan^{-1}(3tan20^0)$ = 47.52^0

Example 44

Prove that in any triangle ABC,

$\frac{\sin(A-B)}{\sin(A+B)} = \frac{a^2-b^2}{c^2}$

Solution

$\frac{a^2-b^2}{c^2} = \frac{(2Rsin\ A)^2-(2Rsin\ B)^2}{(2Rsin\ C)^2}$

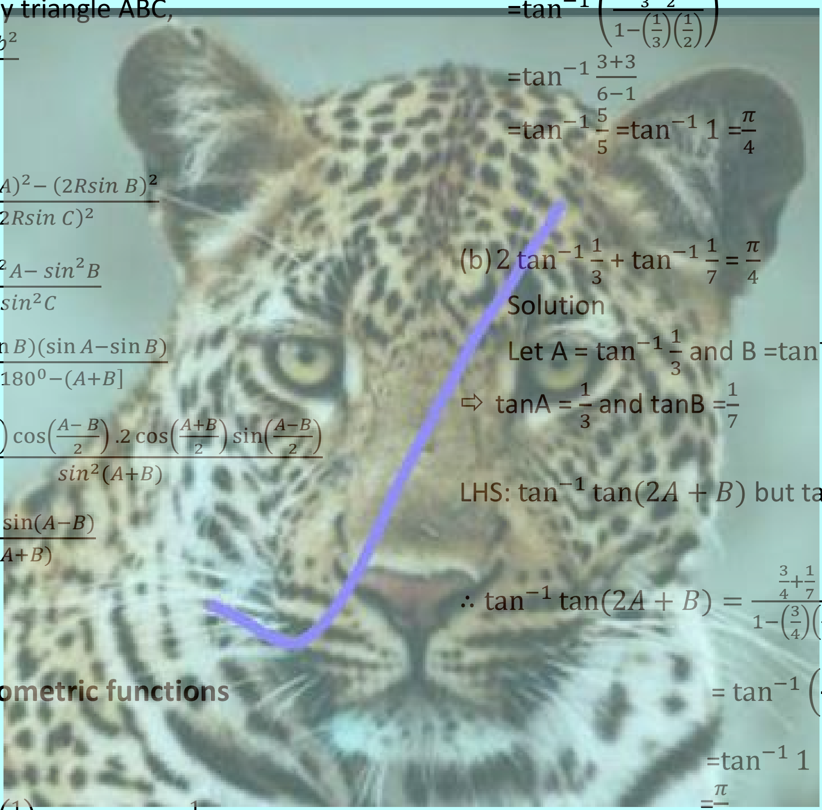
$= \frac{4R^2(sin^2A-sin^2B}{4R^2sin^2C}$

$= \frac{(\sin A+\sin B)(\sin A-\sin B)}{sin^2[180^0-(A+B]}$

$= \frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right).2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}{sin^2(A+B)}$

$= \frac{\sin(A+B)\sin(A-B)}{sin^2(A+B)}$

$= \frac{\sin(A-B)}{\sin(A+B)}$



Inverse trigonometric functions

Note that

(a) If θ = $\cos^{-1}\left(\frac{1}{2}\right)$ then cosθ = $\frac{1}{2}$

(b) $\tan^{-1}(tan\alpha)$= tan($\tan^{-1}\alpha$) =α

(c) $\cos^{-1}[\cos(x+y)]$

= cos[$\cos^{-1}(x+y)$ = x+y

(d) $sin(\sin^{-1}\theta) = \sin^{-1}(\sin^{-1}\theta)$

To avoid errors test the values

Example 45

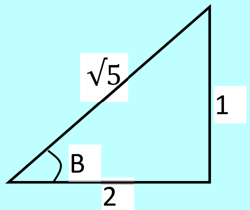
Show that

(a) $\tan^{-1}\frac{1}{3} + \sin^{-1}\frac{1}{\sqrt{5}} = \frac{\pi}{4}$

Solution

A = $\tan^{-1}\frac{1}{3}$ and B =$\sin^{-1}\frac{1}{\sqrt{5}}$

⇨ tanA = $\frac{1}{3}$ and B = $\frac{1}{\sqrt{5}}$



⇨ tan B = $\frac{1}{2}$

LHS = $\tan^{-1}\frac{1}{3} + \sin^{-1}\frac{1}{\sqrt{5}}$ = A + B

$=\tan^{-1}[\tan(A+B)]$

$=\tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{2}}{1-\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)}\right)$

$=\tan^{-1}\frac{3+3}{6-1}$

$=\tan^{-1}\frac{5}{5} = \tan^{-1}1 = \frac{\pi}{4}$

(b) $2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \frac{\pi}{4}$

Solution

Let A = $\tan^{-1}\frac{1}{3}$ and B =$\tan^{-1}\frac{1}{7}$

⇨ tanA = $\frac{1}{3}$ and tanB = $\frac{1}{7}$

LHS: $\tan^{-1}\tan(2A+B)$ but tan2A =$\frac{2\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2} = \frac{3}{4}$

∴ $\tan^{-1}\tan(2A+B) = \frac{\frac{3}{4}+\frac{1}{7}}{1-\left(\frac{3}{4}\right)\left(\frac{1}{7}\right)}$

$= \tan^{-1}\left(\frac{21+4}{28-3}\right)$

$=\tan^{-1}1$

$=\frac{\pi}{4}$

(c) $\cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}$

Solution

Let θ= $\cos^{-1}x$; => x = cosθ = sin$\left(\frac{\pi}{2}-\theta\right)$

sinx = $\frac{\pi}{2}-\theta$

∴ $\cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}$

Example 46

Solve the equations

(a) $\tan^{-1}(2\theta + 1) + \tan^{-1}(2\theta - 1) = \tan^{-1}(2)$

Solution

Let A = $\tan^{-1}(2\theta + 1)$ and B =$\tan^{-1}(2\theta - 1)$
⇨ tanA = $2\theta + 1$ and tanB =$2\theta - 1$

∴ A + B = $\tan^{-1} 2$ or tan(A +B) =2

$$\frac{2\theta+1+2\theta-1}{1-(2\theta+1)(2\theta-1)} = 2$$

4θ = 2(1 - 4θ^2 – 1)

2θ^2 + θ – 1 = 0

(2θ – 1)(θ + 1) = 0

θ = $\frac{1}{2}$ or θ = -1

(b) $\tan^{-1}(1+\theta) + \tan^{-1}(1-\theta) = 32$
Let A = $\tan^{-1}(1+\theta)$ and B =$\tan^{-1}(1-\theta)$
⇨ tanA = $1+\theta$ and tanB =$1-\theta$

∴ A + B = 32 or tan(A +B) = tan32

Introducing tangents

$$\frac{1+\theta+1-\theta}{1-(1+\theta)(1-\theta)} = tan32$$

θ^{2}tan32 = 2

θ = $\sqrt{2cot32} = \pm 1.789$

Example 47

If x = $\tan^{-1}\alpha$ and y = $\tan^{-1}\beta$;

Show that x + y = $\tan^{-1}\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)$

Solution

tanx = α; tany = β

(x + y) = tan[$\tan^{-1}$(x + y)]

$= \tan^{-1}\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)$

Example 48

Solve the equation

$\tan^{-1}\left(\frac{1}{x-1}\right) + \tan^{-1}(x + 1) = \tan(-2)$

Solution

Let A = $\tan^{-1}\left(\frac{1}{x-1}\right)$ and B = $\tan^{-1}(x + 1)$

⇨ A + B = $\tan^{-1}(-2)$

$$\frac{\frac{1}{x-1}+(x+y)}{1-\left(\frac{1}{x-1}\right)(x+y)} = -2$$

$$\frac{1+x^2-1}{x-1-x-1} = -2$$

$$\therefore x^2 = 4; x = \pm 2$$

Example 50

Without using tables or calculators determine the values of $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}$.

Solution

$\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}$.

$=\frac{\frac{1}{2}+\frac{1}{5}}{1-\left(\frac{1}{2}\right)\left(\frac{1}{5}\right)} + \tan^{-1}\frac{1}{8}$

$=\tan^{-1}\frac{7}{9} + \tan^{-1}\frac{1}{8}$

$=\frac{\frac{7}{9}+\frac{1}{8}}{1-\left(\frac{7}{9}\right)\left(\frac{1}{8}\right)} = \tan^{-1}\left(\frac{65}{65}\right) = \frac{\pi}{4}$

Example 51

Solve equations

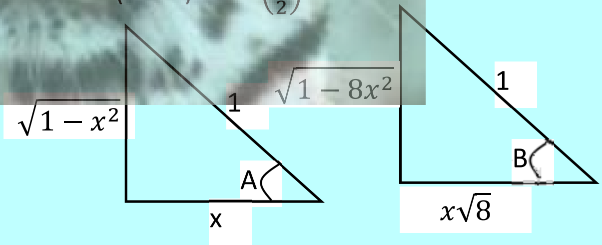
(a) $\cos^{-1}x + \cos^{-1}x\sqrt{8} = \frac{\pi}{2}$

Solution

Let A = $\cos^{-1}x$ and B =$\cos^{-1}x\sqrt{8}$

A + B = $\frac{\pi}{2}$

Cos (A + B) = $\cos\left(\frac{\pi}{2}\right)$



$$x(x\sqrt{8}) - \left(\sqrt{1-x^2}\right)\left(\sqrt{1-8x^2}\right) = 0$$

$$x(x\sqrt{8}) = \left(\sqrt{1-x^2}\right)\left(\sqrt{1-8x^2}\right)$$

8x^4 = $(1-x^2)(1-8x^2)$

1- 9x^2 = 0

(1 – 3x)(1 + 3x) = 0

Either x = $\frac{1}{3}$ or x = $-\frac{1}{3}$

We discard the negative value, so the root is x = $\frac{1}{3}$

(b) $2\sin^{-1}\left(\frac{x}{2}\right) + \sin^{-1}(x\sqrt{2}) = \frac{\pi}{2}$

Solution

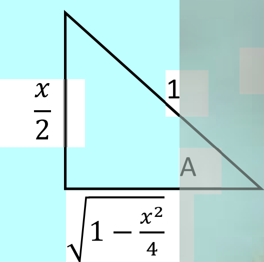
Let A = $\sin^{-1}\left(\frac{x}{2}\right)$ and B = $\sin^{-1}(x\sqrt{2})$

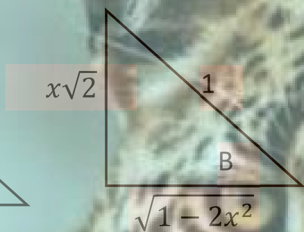
2A + B = $\frac{\pi}{2}$

2A = $\frac{\pi}{2} - B$

Sin(2A) = sin $\left(\frac{\pi}{2} - B\right)$

2SinAcosA = cosB





$2\left(\frac{x}{2}\right).\sqrt{1-\frac{x^2}{4}} = \sqrt{1-2x^2}$

$x.\sqrt{\frac{4-x^2}{4}} = \sqrt{1-2x^2}$

$\frac{x}{2}.\sqrt{4-x^2} = \sqrt{1-2x^2}$

$\frac{x^2}{4}.(4-x^2) = (1-2x^2)$

$X^4 - 12x^2 + 4 = 0$

$x^2 = \frac{12\pm\sqrt{144-4(4\ x\ 1)}}{2\ x 1}$

$x^2 = \frac{12\pm\sqrt{128}}{2} = 6 \pm 4\sqrt{2}$

$x = \sqrt{6 \pm 4\sqrt{2}}$

After testing for $x = \sqrt{6+4\sqrt{2}}$ and for $x = \sqrt{6-4\sqrt{2}}$, the value that satisfies the equation is $x = \sqrt{6-4\sqrt{2}}$ = 0.5858

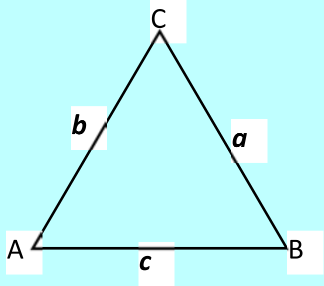
Hence the value of x = 0.5858

Revision exercise 6

1. If p = sinα + sin β and q = cosα + cosβ show that $\frac{p}{q} = tan\,\frac{\alpha+\beta}{2}$
2. (a) Prove that:
   (i) (sin2x – sinx)(1+ 2cosx) = sin3x
   (ii) cos4θ = $\frac{tan^4\theta-6tan^2\theta+1}{tan^4\theta+2tan^2\theta+1}$
   (iii) $\frac{sinx+2sin2x+sin3x}{sinx+2sinx+sin3x} = tan^2\frac{x}{2}$
3. Solve the equation for $0^0 \le x \le 180^0$:
   (a) sinx + sn3x + sin5x + sin7x = 0
   [x: x = 0^0, 45^0, 90^0, 135^0, 180^0]
   (b) sin7x + sinx + sin5x + sin3x = 0
   [x: x = 60^0, 180^0]
   (c) sinx + sin4x = 0
   [x: x= 0^0, 60^0, 72^0, 144^0, 180^0]
   (d) cos (x + 10^0) – -cos(x + 30^0) = 0
   [70^0]
   (e) cos5x – sin2x = cosx
   [x: x = 0^0, 70^0, 90^0, 110^0,180^0]
   (f) sin2x +sin10x + cos4x = 0
   [x: x = 22.5^0, 35^0, 55^0, 67.5^0, 95^0, 112.5^0, 115^0, 157.5^0, 175^0]
4. Show that
   (a) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$
   (b) $2\sin^{-1}\left(\frac{1}{2}\right) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$
   (c) the positive value that satisfies the equation $\tan^{-1}3x + \tan^{-1}2x = \frac{\pi}{4}$ is $\frac{1}{6}$
   (d) $\tan^{-1}(-x) = -\tan^{-1}x$
   (e) $\cos^{-1}\left(\frac{63}{65}\right) + 2\tan^{-1}\left(\frac{1}{5}\right) = \sin^{-1}\left(\frac{3}{5}\right)$
5. Prove that
   (a) $\frac{sinA-sinB}{sinA+sinB} = tan\left(\frac{A-B}{2}\right)cot\left(\frac{A+B}{2}\right)$
   (b) Sin3x + sinx = $4sinxcos^2x$
   (c) $\frac{sinx+sin2x+sin3x}{cosx+cos2x+cos3x} = tan2xx$
   (d) sin(A + B) – sin(A – B) =2cosAsinB
   (e) $\frac{sin5x+sinx}{sin4x+sin2x} = 2cosx - secx$
   (f) Cos3x + cosx = $4cos^2x$ – 2cosx

Solution to triangles

In a triangle ABC



(a) Six elements are considered: three angles and three sides

Capital letters denote angles and ***small bold and italics letters*** sides

(b) The opposite side of angle A is a, of angle B is b and of angle C is c.

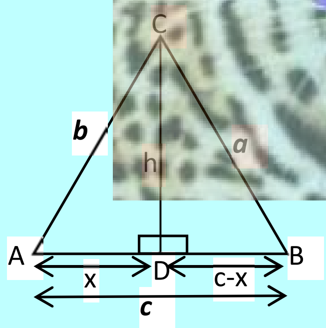
(c) The angle sum of a triangle is two right angles i.e. A + B + C = 180^0

(d) The sides are independent except that the sum of the two sides of the triangle should be equal to or greater than the third side

How to deal with triangles

1. The cosine rule

   (a) Given an acute angle A



From triangle

ACD: $x^2 + h^2 = \boldsymbol{b}^2$ ....(i)

BCD: $(\boldsymbol{c} - x)^2 + h^2 = \boldsymbol{a}^2$

$\boldsymbol{c}^2 - 2\boldsymbol{c}x + x^2 + h^2 = \boldsymbol{a}^2$ .... (ii)

Substituting eqn. (i) into eqn. (ii)

$\boldsymbol{c}^2 - 2\boldsymbol{c}x + \boldsymbol{b}^2 = \boldsymbol{a}^2$

But

$x = \boldsymbol{b}\cos A$

⇨ $\boldsymbol{b}^2 + \boldsymbol{c}^2 - 2\boldsymbol{bc}\cos A = a^2$

$\boldsymbol{a}^2 = \boldsymbol{b}^2 + \boldsymbol{c}^2 - 2\boldsymbol{b}\boldsymbol{c}\cos A$ (1)

Similarly;

$\boldsymbol{b}^2 = \boldsymbol{a}^2 + \boldsymbol{c}^2 - 2\boldsymbol{ac}\cos B$ (2)

$\boldsymbol{c}^2 = \boldsymbol{a}^2 + \boldsymbol{b}^2 - 2\boldsymbol{ab}\cos C$ (3)

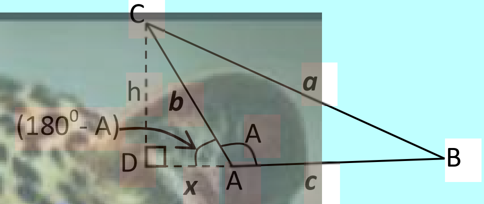
It follows that

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

$$\cos B = \frac{a^2+c^2-b^2}{2ac}$$

$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

(b) Given an obtuse angle A



In triangle ABC, A is an obtuse angle and CD is the altitude.

From triangle

ACD: $x^2 + h^2 = \boldsymbol{b}^2$ ....(i)

BCD: $(\boldsymbol{c} - x)^2 + h^2 = \boldsymbol{a}^2$

$\boldsymbol{c}^2 - 2\boldsymbol{c}x + x^2 + h^2 = \boldsymbol{a}^2$ .... (ii)

Substituting eqn. (i) into eqn. (ii)

$\boldsymbol{c}^2 - 2\boldsymbol{c}x + \boldsymbol{b}^2 = \boldsymbol{a}^2$

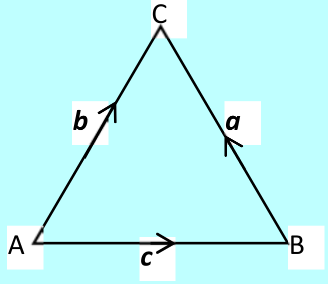
But

$x = \boldsymbol{b}\cos(180^0 - A) = -\boldsymbol{bc}\cos A$

From triangle ACD

$\boldsymbol{a}^2 = \boldsymbol{b}^2 + \boldsymbol{c}^2 - 2\boldsymbol{bc}\cos A$ as before

The cosine rule can be derived using the vector approach.



Given a triangle above with BC = ***a***, AC = ***c*** and AB = ***b***

BC = BA + AC = AC – AB

a = ***b*** – ***c***

⇨ $\boldsymbol{a}.\boldsymbol{a} = (\boldsymbol{b} - \boldsymbol{c})(\boldsymbol{b} - \boldsymbol{c})$

$= \boldsymbol{b}.\boldsymbol{b} - 2\boldsymbol{b}.\boldsymbol{c} + \boldsymbol{c}.\boldsymbol{c}$

$= \boldsymbol{b}.\boldsymbol{b} + \boldsymbol{c}.\boldsymbol{c} - 2\boldsymbol{b}.\boldsymbol{c}$

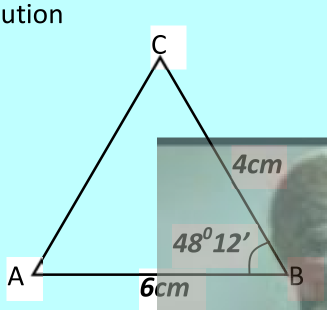
$\therefore \boldsymbol{a}^2 = \boldsymbol{b}^2 + \boldsymbol{c}^2 - 2\boldsymbol{bc}\cos A$

since $\boldsymbol{b}.\boldsymbol{c} = |bc|\cos A$

Example 52

Solve the triangle in which AB = 6cm, BC = 4cm and angle ACB = $48^0 12'$

Solution



Using: $\boldsymbol{b}^2 = \boldsymbol{a}^2 + \boldsymbol{c}^2 - 2\boldsymbol{ac}\cos B$

$= 6^2 + 4^2 - 2(6)(4)\cos 48.2^0$

1^0 (degree) = 60′ (minutes)

b = 4.47cm

Using: $\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{20.0+36-16}{2(4.47)(6)}$

A =41.8^0

But A + B + C = 180^0

$41.8^0 + 48.2^0 + C = 180^0$

C = 90^0

∴ AC = 4.47cm, angles BAC = 41.8^0 and ACB = 90^0

Example 53

In a triangle ABC, prove that

(a) $a^2 = (b - c)^2 + 4bc\sin^2\left(\frac{A}{2}\right)$ hence that

$a = (b - c)\sec\alpha$ where $\tan\alpha = \frac{\sqrt{bc}\sin\left(\frac{A}{2}\right)}{b-c}$

From $\cos A = 1 - 2\sin^2\left(\frac{A}{2}\right)$

Substituting for cosA into the cosine formula $\boldsymbol{a}^2 = \boldsymbol{b}^2 + \boldsymbol{c}^2 - 2\boldsymbol{bc}\cos A$

$\boldsymbol{a}^2 = \boldsymbol{b}^2 + \boldsymbol{c}^2 - 2\boldsymbol{bc}[1 - 2\sin^2\left(\frac{A}{2}\right)]$

$\boldsymbol{a}^2 = \boldsymbol{b}^2 + \boldsymbol{c}^2 - 2\boldsymbol{bc} + 4\sin^2\left(\frac{A}{2}\right)$

$a^2 = (b - c)^2 + 4bc\sin^2\left(\frac{A}{2}\right)$

Hence, substituting for $\sin^2\left(\frac{A}{2}\right)$ into tanα expression we get

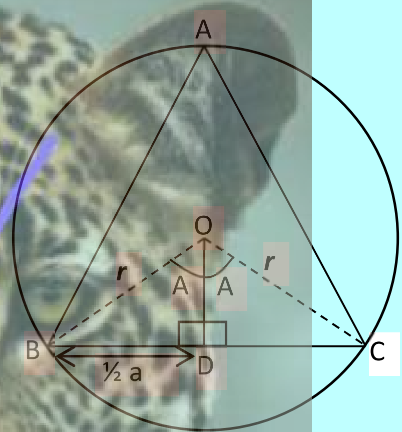
$a^2 = (b - c)^2 + (b - c)^2\tan^2\alpha$

$a^2 = (b - c)^2(1 + \tan^2\alpha)$

$a^2 = (b - c)^2\sec^2\alpha$

$a = (b - c)\sec\alpha$

2. The Sine Rule



The figure shows a circle with centre O and radius r circumscribing triangle ABC

Angle BOC = 2A [angle subtended by the same arc at the centre of the circle is twice the angle formed at any point on the circumference]

Triangle BOC is isosceles

OD bisects angle BOC and side BC

∴BD = ½a

From triangle BOD

$\sin A = \frac{a}{2r}$ i.e. $\frac{a}{\sin A} = 2r$

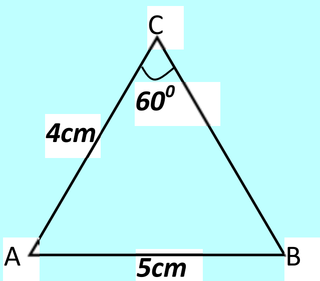
if instead we consider triangles AOC and AOB, we obtain $\frac{b}{\sin B} = 2r$ and $\frac{c}{\sin C} = 2r$

In general: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Example 54

Solve the triangle in which AB = 5cm, AC = 4cm and angle ACB = 600

Solution



Using sine rule

$\frac{b}{sinB} = \frac{c}{sinC}$ => B $= \sin^{-1}\left(\frac{bsinC}{c}\right)$

B = $\sin^{-1}\left(\frac{4}{5}sin60^0\right)$ = 43.9^0

From A + B + C+ = 180^0

A = $(180 - 60 - 43.9)^0 = 76.1^0$

Similarly a = $\frac{bsinA}{sinB} = \frac{4sin76.1^0}{sin43.9^0}$ = 5.6cm

$\therefore\ \overline{AB} = 5.6cm, B\hat{A}C = 76.1^0, A\hat{B}C = 43.9^0$

Example 55

Prove that in any triangle

$\frac{a^2-b^2}{c^2} = \frac{\sin(A-B)}{\sin(A+B)}$

Solution

From sine rule formula;
a = 2rsinA, b = 2rsinB, c = 2rsinC

By substitution

$\frac{a^2-b^2}{c^2} = \frac{(2rsinA)^2-(2rsinB)^2}{(2rsinC)^2} = \frac{sin^2A-sin^2B}{sin^2C}$

But A + B + C = 180^0

C = 180^0 – (A + B)

sinC = sin[180^0 – (A + B)] = sin (A + B)

By substitution

$\frac{a^2-b^2}{c^2} = \frac{sin^2A-sin^2B}{\sin(A+B)} = = \frac{(sinA+\sin B)(sinA-sinB)}{\sin(A+B)}$

$= \frac{2sin\frac{1}{2}(A+B)cos\frac{1}{2}(A-B).2cos\frac{1}{2}(A+B)sin\frac{1}{2}(A-B)}{\sin(A+B)}$

$= \frac{2sin\frac{1}{2}(A-B)cos\frac{1}{2}(A-B)}{} = \frac{\sin(A-B)}{\sin(A+B)}$

Hence $\frac{a^2-b^2}{c^2} = \frac{\sin(A-B)}{\sin(A+B)}$

Example 56

Prove that in any triangle ABC,

$sin\frac{1}{2}(B-C) = \frac{b-c}{a}cos\frac{1}{2}A$

Solution

From sine rule formula;
a = 2rsinA, b = 2rsinB, c = 2rsinC

By substitution

$\frac{b-c}{a} = \frac{2rsinB-2rsinC}{2rsinA} = \frac{SinB-sinC}{sinA}$

But A + B + C = 180^0

A = 180^0 – (B + C)

sinA = sin[180^0 – (B + C)] = sin(B + C)

By substitution

$\frac{b-c}{a} = \frac{SinB-sinC}{sinA} = \frac{SinB-sinC}{\sin(B+C)}$

$= \frac{2cos\frac{1}{2}(B+C)sin\frac{1}{2}(B+C)}{2cos\frac{1}{2}(B+C)cos\frac{1}{2}(B+C)}$

$= \frac{sin\frac{1}{2}(B+C)}{cos\frac{1}{2}(B+C)}$

From A + B + C = 180^0

B + C = 180^0 – A

$\frac{1}{2}(B+C) = \left(90^0 - \frac{1}{2}A\right)$

$sin\frac{1}{2}(B+C) = sin\left(90^0 - \frac{1}{2}A\right) = cos\frac{1}{2}A$

By substitution

$\frac{b-c}{a} = \frac{sin\frac{1}{2}(B+C)}{cos\frac{1}{2}A}$

$\therefore\ sin\frac{1}{2}(B-C) = \frac{b-c}{a}cos\frac{1}{2}A$

3. The Tangent Rule

It states that in a triangle ABC

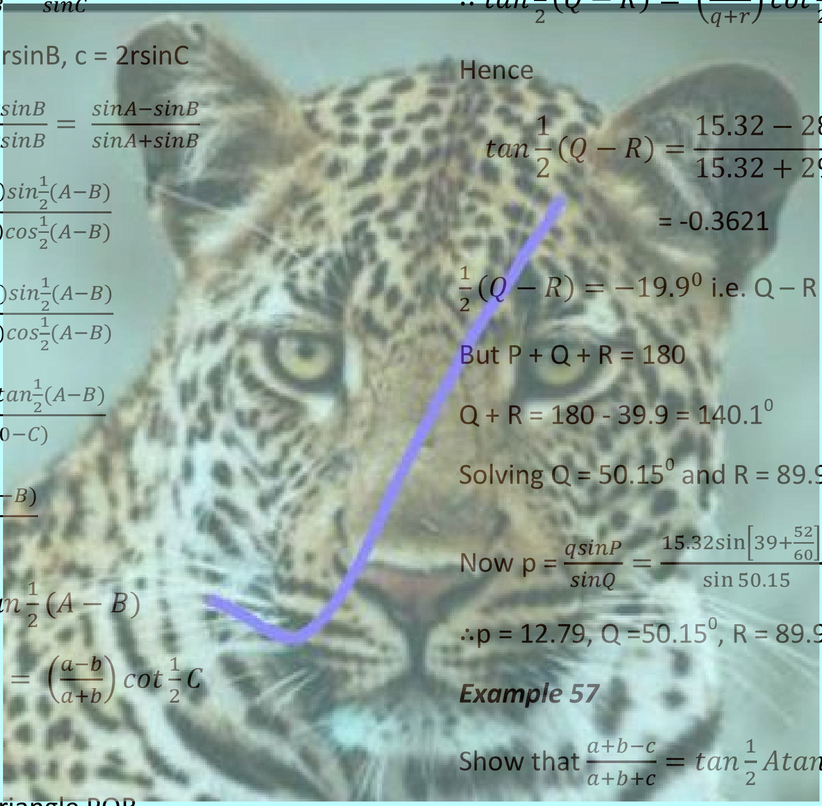
$$tan\frac{1}{2}(A-B) = \left(\frac{a-b}{a+b}\right)cot\frac{1}{2}C$$

$$tan\frac{1}{2}(C-A) = \left(\frac{c-a}{c+a}\right)cot\frac{1}{2}B$$

$$tan\frac{1}{2}(b-c) = \left(\frac{b-c}{b+c}\right)cot\frac{1}{2}A$$

Proof

From $\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC}$



a = 2rsinA, b = 2rsinB, c = 2rsinC

$$\frac{a-b}{a+b} = \frac{2rsinA-2rsinB}{2rsinA+2rsinB} = \frac{sinA-sinB}{sinA+sinB}$$

$$= \frac{2cos\frac{1}{2}(A+B)sin\frac{1}{2}(A-B)}{2sin\frac{1}{2}(A+B)cos\frac{1}{2}(A-B)}$$

$$= \frac{2\cos(90-\frac{1}{2}C)sin\frac{1}{2}(A-B)}{2\sin(90-\frac{1}{2}C)cos\frac{1}{2}(A-B)}$$

$$= \frac{\cos(90-\frac{1}{2}C)tan\frac{1}{2}(A-B)}{\sin\frac{1}{2}(90-C)}$$

$$= \frac{\sin\frac{1}{2}Ctan\frac{1}{2}(A-B)}{\cos\frac{1}{2}C}$$

$$\frac{a-b}{a+b} = tan\frac{1}{2}Ctan\frac{1}{2}(A-B)$$

$$\therefore tan\frac{1}{2}(A-B) = \left(\frac{a-b}{a+b}\right)cot\frac{1}{2}C$$

Example 56

Show that in a triangle PQR

$$tan\frac{1}{2}(Q-C) = \left(\frac{q-r}{q+r}\right)cot\frac{1}{2}P$$

Hence solve the triangle in which q =15.32, r = 28.6 and P = $39^0 52'$

Solution

From $\frac{p}{sinP} = \frac{q}{sinQ} = \frac{r}{sinR}$

p = 2rsinP, q = 2rsinQ, r = 2rsinR

$$\frac{q-r}{q+r} = \frac{2rsinQ-2rsinR}{2rsinQ+2rsinR} = \frac{sinQ-sinR}{sinQ+sinR}$$

$$= \frac{2cos\frac{1}{2}(Q+R)sin\frac{1}{2}(Q-R)}{2sin\frac{1}{2}(Q+R)cos\frac{1}{2}(Q-R)}$$

$$= \frac{2\cos(90-\frac{1}{2}P)sin\frac{1}{2}(Q-R)}{2\sin(90-\frac{1}{2}P)cos\frac{1}{2}(Q-R)}$$

$$= \frac{\cos(90-\frac{1}{2}P)tan\frac{1}{2}(Q-R)}{\sin\frac{1}{2}(90-P)}$$

$$= \frac{\sin\frac{1}{2}Ptan\frac{1}{2}(Q-R)}{\cos\frac{1}{2}P}$$

$$\frac{q-r}{q+r} = tan\frac{1}{2}Ptan\frac{1}{2}(Q-R)$$

$$\therefore tan\frac{1}{2}(Q-R) = \left(\frac{q-r}{q+r}\right)cot\frac{1}{2}P$$

Hence

$$tan\frac{1}{2}(Q-R) = \frac{15.32-28.6}{15.32+29.6}cot39^0 52'$$

= -0.3621

$\frac{1}{2}(Q-R) = -19.9^0$ i.e. Q – R = -39.9^0

But P + Q + R = 180

Q + R = 180 - 39.9 = 140.1^0

Solving Q = 50.15^0 and R = 89.95^0

Now p = $\frac{qsinP}{sinQ} = \frac{15.32\sin\left[39+\frac{52}{60}\right]^0}{\sin 50.15}$ = 12.79

∴p = 12.79, Q = 50.15^0, R = 89.95^0

Example 57

Show that $\frac{a+b-c}{a+b+c} = tan\frac{1}{2}Atan\frac{1}{2}B$

Solution

LHS = $\frac{a+b-c}{a+b+c}$

$$= \frac{2rsinA+2rsinB-2rsinC}{2rsinA+2rsinB+2rsinC}$$

$$= \frac{sinA+sinB-sinC}{sinA+sinB+sinC}$$

$$= \frac{2sin\frac{1}{2}(A+B)cos\frac{1}{2}(A-B)-2sin\frac{1}{2}Ccos\frac{1}{2}C}{2sin\frac{1}{2}(A+B)cos\frac{1}{2}(A-B)-2sin\frac{1}{2}Ccos\frac{1}{2}C}$$

$$= \frac{2sin\left(90-\frac{1}{2}C\right)cos\frac{1}{2}(A-B)-2sin\frac{1}{2}Ccos\frac{1}{2}C}{2sin\left(90-\frac{1}{2}C\right)cos\frac{1}{2}(A-B)-2sin\frac{1}{2}Ccos\frac{1}{2}C}$$

$$= \frac{2cos\frac{1}{2}Ccos\frac{1}{2}(A-B)-2sin\frac{1}{2}Ccos\frac{1}{2}C}{2os\frac{1}{2}Ccos\frac{1}{2}(A-B)-2sin\frac{1}{2}Ccos\frac{1}{2}C}$$

$$= \frac{cos\frac{1}{2}(A-B)-sin\frac{1}{2}C}{cos\frac{1}{2}(A-B)+sin\frac{1}{2}C}$$

$$=\frac{\cos\frac{1}{2}(A-B)-\sin\left(90-\frac{1}{2}(A+B)\right)}{\cos\frac{1}{2}(A-B)+\sin\left(90-\frac{1}{2}(A+B)\right)}$$

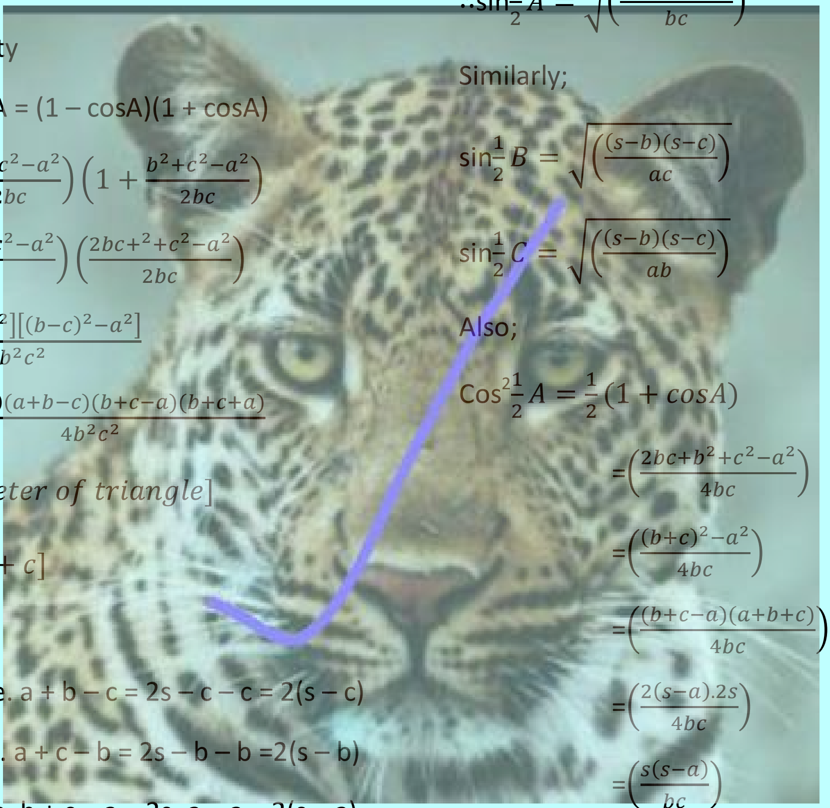
$$=\frac{\cos\frac{1}{2}(A-B)-\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)+\cos\frac{1}{2}(A+B)}$$

$$=\frac{-2\sin\frac{1}{2}A\sin\left(-\frac{1}{2}B\right)}{\cos\frac{1}{2}A+\cos\frac{1}{2}B}$$

$$=\tan\frac{1}{2}A\tan\frac{1}{2}B$$

Expressions for sinA, sin$\frac{1}{2}A$ and cos$\frac{1}{2}A$ in terms of the sides of the triangle

(a) sinA



From the identity

$\sin^2A = 1 - \cos^2A = (1 - \cos A)(1 + \cos A)$

$$=\left(1-\frac{b^2+c^2-a^2}{2bc}\right)\left(1+\frac{b^2+c^2-a^2}{2bc}\right)$$

$$=\left(\frac{2bc-b^2+c^2-a^2}{2bc}\right)\left(\frac{2bc+^2+c^2-a^2}{2bc}\right)$$

$$=\frac{[a^2-(b-c)^2][(b-c)^2-a^2]}{4b^2c^2}$$

$$\therefore \sin^2A=\frac{(a+c-b)(a+b-c)(b+c-a)(b+c+a)}{4b^2c^2}$$

Let s $=\frac{1}{2}[perimeter\ of\ triangle]$

$$=\frac{1}{2}[a+b+c]$$

2s =[a + b + c]

a + b = 2s – c; i.e. a + b – c = 2s – c – c = 2(s – c)

a + c= 2s – b; i.e. a + c – b = 2s – b – b =2(s – b)

b + c = 2s – a; i.e. b + c – a = 2s- a – a = 2(s – a)

$$\therefore \sin^2A=\frac{2(s-b).2(s-c).2(s-a).2s}{4b^2c^2}$$

$$\sin A=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

Similarly, $\sin B=\frac{2}{ac}\sqrt{s(s-a)(s-b)(s-c)}$

$$\sin C=\frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}$$

(b) sin$\frac{1}{2}A$ and cos$\frac{1}{2}A$

From $\sin^2\frac{1}{2}A=\frac{1}{2}(1-\cos A)$

$$=\frac{1}{2}\left(1-\frac{b^2+c^2-a^2}{2bc}\right)$$

$$=\left(\frac{2bc-b^2+c^2-a^2}{4bc}\right)$$

$$=\left(\frac{a^2-(b-c)^2}{4bc}\right)$$

$$=\left(\frac{(a+c-b)(a+b-c)}{4bc}\right)$$

$$=\left(\frac{2(s-b).2(s-c)}{4bc}\right)$$

$$=\left(\frac{(s-b)(s-c)}{bc}\right)$$

$$\therefore \sin\frac{1}{2}A=\sqrt{\left(\frac{(s-b)(s-c)}{bc}\right)}$$

Similarly;

$$\sin\frac{1}{2}B=\sqrt{\left(\frac{(s-b)(s-c)}{ac}\right)}$$

$$\sin\frac{1}{2}C=\sqrt{\left(\frac{(s-b)(s-c)}{ab}\right)}$$

Also;

$$\text{Cos}^2\frac{1}{2}A=\frac{1}{2}(1+\cos A)$$

$$=\left(\frac{2bc+b^2+c^2-a^2}{4bc}\right)$$

$$=\left(\frac{(b+c)^2-a^2}{4bc}\right)$$

$$=\left(\frac{(b+c-a)(a+b+c)}{4bc}\right)$$

$$=\left(\frac{2(s-a).2s}{4bc}\right)$$

$$=\left(\frac{s(s-a)}{bc}\right)$$

$$\therefore \cos\frac{1}{2}A=\sqrt{\left(\frac{(s-a)}{bc}\right)}$$

Similarly;

$$\cos\frac{1}{2}B=\sqrt{\left(\frac{(s-b)}{ac}\right)}$$

$$\cos\frac{1}{2}C=\sqrt{\left(\frac{(s-c)}{ab}\right)}$$

The expression for $tan\frac{1}{2}A$ can be deduced as follows

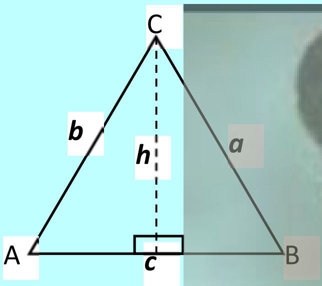
$\tan\frac{1}{2}A = \frac{sin\frac{1}{2}A}{cos\frac{1}{2}A} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

Similarly;

$\tan\frac{1}{2}B = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$

$\tan\frac{1}{2}C = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

Area of a triangle



Area, $\Delta = \frac{1}{2}(base)(perpendicular\ height)$

$= \frac{1}{2}ch$

$= \frac{1}{2}cbsinA$

Substituting for

$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$

$\Delta = \frac{1}{2}bcx\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$

$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

This a convenient form given the three sides of a triangle. The formula is called Hero's formula from the first mathematician who suggested it.

Example 58

The area of a triangle is 336m^2. The sum of the three sides is 84m and one side is 28m. Calculate the length of the remaining two sides

Solution

Given Δ = 336, a + b + c =84 and a = 28

$s = \frac{1}{2}(a+b+c) = \frac{1}{2}(84) = 42$

28 + b + c = 84

b + c = 56, or c = 56 – b

But $\Delta^2 = s(s-a)(s-b)(s-c)$

336^2 = 42(42–28)(42– b)(42 – 56 + b)

b^2 – 56b + 780 = 0

$b = \frac{56\pm\sqrt{56^2-4x\ 1\ x\ 780}}{2\ x\ 1}$

b = 30 or 26

substituting for c = 56 – b

c = 26 or 30

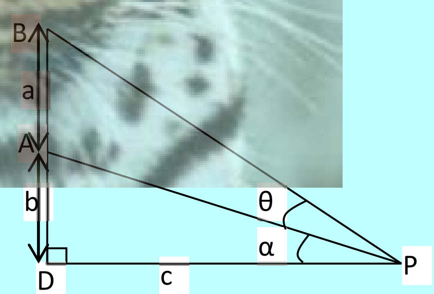
∴the remaining sides are 30m and 26m

Applications of trigonometry in finding distances and bearings

Example 59

A vertical pole BAD stands with its base D on a horizontal plane where BA = a and AD = b. A point P is situated on the horizontal plane at a distance C from D and the angle APB = θ. Prove that $\theta = \tan^{-1}\left(\frac{ac}{b^2+ab+c^2}\right)$

Solution



Let angle APD = α

For triangle APD: $\tan\alpha = \frac{b}{c}$

For triangle DPB: $\tan(\theta+\alpha) = \frac{a+b}{c}$

⇨ $\frac{tan\theta+tan\alpha}{1-tan\theta tan\alpha} = \frac{a+b}{c}$

Substituting for tanα

$$\Rightarrow \quad \frac{\tan\theta+\frac{b}{c}}{1-\left(\frac{b}{c}\right)\tan\theta} = \frac{a+b}{c}$$

$c^2\tan\theta + bc = ac + bc - ab\tan\theta - b^2\tan\theta$

$(b^2 + ab + c^2)\tan\theta = ac$

$\tan\theta = \frac{ac}{b^2+ab+c^2}$

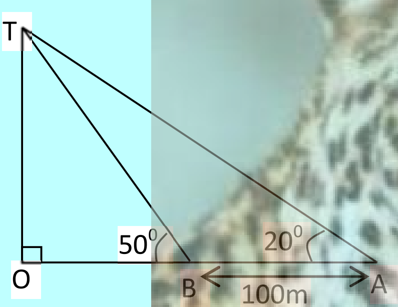
$\therefore \theta = \tan^{-1}\left(\frac{ac}{b^2+ab+c^2}\right)$

Example 60

The angle of the top of a vertical tower from a point A is 20^0 and from another point B is 50^0. Given that A and B lie on the same horizontal plane in the same direction where AB = 100m. Find the height of the tower

Solution

Let OT be the height of the tower



$A\hat{T}B = 50 - 30 = 30^0$

Using sine rule

$$\frac{TB}{\sin 20^0} = \frac{100}{\sin 30^0}$$

$$TB = \frac{100 \sin 20^0}{\sin 30^0}$$

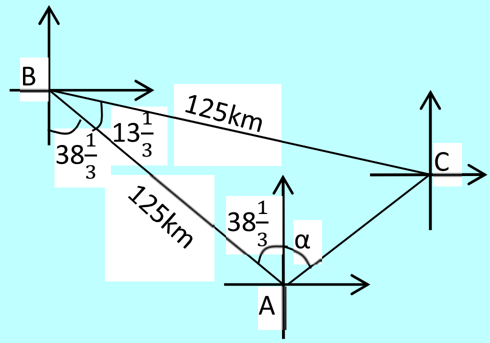
But $OT = TB\sin 50^0$

$$OT = \frac{100 \sin 20^0 \sin 50^0}{\sin 30^0} = 26.2m$$

Example 61

From a point A, a pilot flies in the direction N$38^0$20'W to point B 125km from A. He then flies in the direction S$50^0$40'E for 125km. He wishes to return to A from this point. How far and in what direction must he fly.

Solution



From the digram

Let $B\hat{A}C = B\hat{C}A = \theta$

$\Rightarrow \quad 2\theta + 13\frac{1}{3}^0 = 180^0$

$\theta = 83\frac{1}{3}^0$

But $38\frac{1}{3} + \alpha = \theta$

$38\frac{1}{3} + \alpha = 83\frac{1}{3}^0$

$\alpha = 45^0$

From the sine rule

$$\frac{AC}{\sin 13\frac{1}{3}} = \frac{125}{\sin 83\frac{1}{3}}$$

AC = 29km

$\therefore$ he has to fly 29km in the direction S45^0W

Example 62

(a) Prove that $\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\tan B}$

$$\tan(A-B) = \frac{\sin(A-B)}{\cos(A-B)}$$

$$= \frac{\sin A\cos B - \cos A\sin B}{\cos A\sin B + \sin A\sin B}$$

Diving numerator and denominator on the R.H.S by cosAcosB

$$\tan(A-B) = \frac{\frac{\sin A\cos B}{\cos A\cos B} - \frac{\cos A\sin B}{\cos A\cos B}}{\frac{\cos A\cos B}{\cos A\cos B} + \frac{\sin A\sin B}{\cos A\cos B}}$$

$$= \frac{\tan A - \tan B}{1+\tan A\tan B}$$

Hence show that $\frac{1-\tan 15^0}{1+\tan 15^0} = \frac{1}{\sqrt{3}}$

$$\frac{1-\tan 15^0}{1+\tan 15^0} = \frac{\tan 45^0 - tan15^0}{1+\tan 45^0 tan15^0}$$

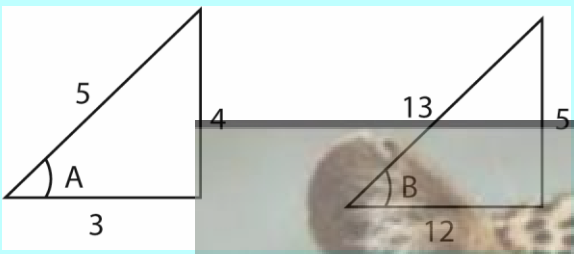
$= \tan(45^0 - 15^0)\ \tan30^0 = \frac{1}{\sqrt{3}}$

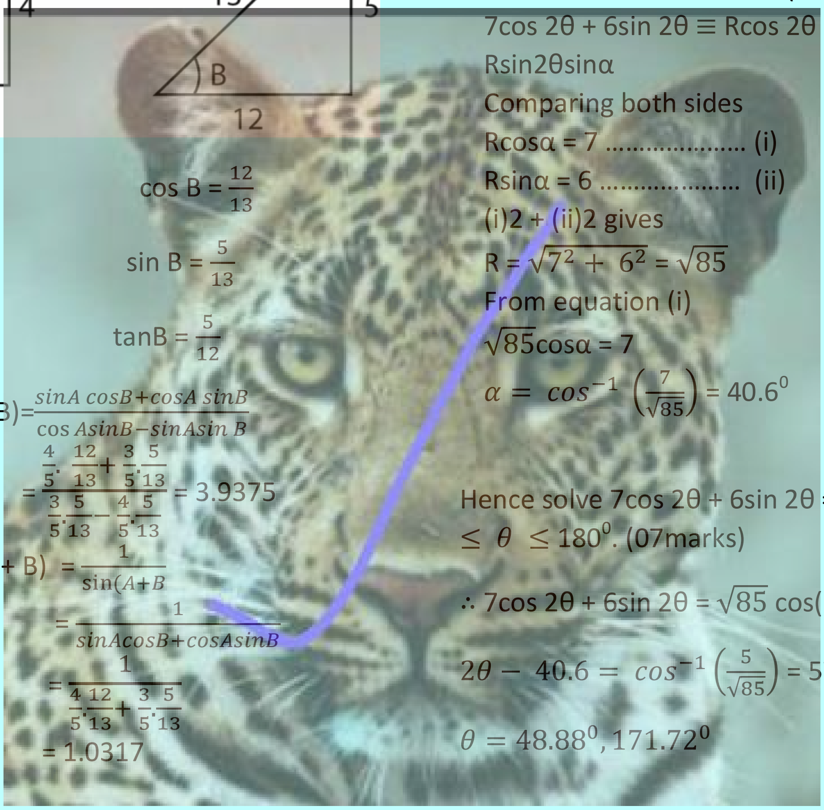
(b) Given that cos A = $\frac{3}{5}$ and cos B = $\frac{12}{13}$ where A and B are acute, find the values of

(i) tan (A + B)

(ii) cosec (A + B)

Solution





cos A = $\frac{3}{5}$ cos B = $\frac{12}{13}$

sin A = $\frac{4}{5}$ sin B = $\frac{5}{13}$

tan A = $\frac{4}{3}$ tanB = $\frac{5}{12}$

(i) tan (A + B) $= \frac{sinA\ cosB + cosA\ sinB}{\cos AsinB - sinAsin\ B}$

$= \frac{\frac{4}{5}\cdot\frac{12}{13} + \frac{3}{5}\cdot\frac{5}{13}}{\frac{3}{5}\cdot\frac{5}{13} - \frac{4}{5}\cdot\frac{5}{13}} = 3.9375$

(ii) cosec (A + B) $= \frac{1}{\sin(A+B)}$

$= \frac{1}{sinAcosB + cosAsinB}$

$= \frac{1}{\frac{4}{5}\cdot\frac{12}{13} + \frac{3}{5}\cdot\frac{5}{13}}$

= 1.0317

Example 63

Express cos $(\theta + 30)^0$ – cos $(\theta + 48)^0$ in the form RsinPsinQ, where R is constant.

Hence solve th3 equation

cos $(\theta + 30)^0$ – cos $(\theta + 48)^0$ = 0.2

Solution

cos $(\theta + 30)^0$ – cos $(\theta + 48)^0$

$= -2\sin\left(\frac{\theta + 30^0 + \theta + 48^0}{2}\right)\sin\left(\frac{\theta + 30^0 - \theta - 48^0}{2}\right)$

$= -2\sin(\theta + 39^0)\sin(-9^0)$

cos $(\theta + 30)^0$ – cos $(\theta + 48)^0$ = 0.

⇨ $-2\sin(\theta + 39^0)\sin(-9^0) = 0.2$

$\sin(\theta + 39^0) = 0.63925$

$\theta + 39^0 = 39.74^0$

$\theta = 0.74^0$

Example 64

Express 7cos 2θ + 6sin 2θ in form Rcos (2θ – α), where R is a constant and α is an acute angle.

7cos 2θ + 6sin 2θ ≡ Rcos (2θ – α)

7cos 2θ + 6sin 2θ ≡ Rcos 2θ cos α + Rsin2θsinα

Comparing both sides

Rcosα = 7 .... (i)

Rsinα = 6 .... (ii)

(i)2 + (ii)2 gives

$R = \sqrt{7^2 + 6^2} = \sqrt{85}$

From equation (i)

$\sqrt{85}\cos\alpha = 7$

$\alpha = cos^{-1}\left(\frac{7}{\sqrt{85}}\right) = 40.6^0$

Hence solve 7cos 2θ + 6sin 2θ = 5 for $0^0 \le \theta \le 180^0$. (07marks)

∴ 7cos 2θ + 6sin 2θ = $\sqrt{85}$ cos$(2\theta - 40.6^0)$ =5

$2\theta - 40.6 = cos^{-1}\left(\frac{5}{\sqrt{85}}\right) = 57.16^0, 302.84^0$

$\theta = 48.88^0, 171.72^0$

Revision exercise 7

1. Solve the triangles

(a) a = 17m, b = 21.42m, B = $51^034'$
[A =38.44^0, C = 90^0, c = 27.34m]

(b) b = 107.2m, c= 76.69m, B =$102^025'$
[A =33.26^0, C = 44.32^0, a = 60.21m]

(c) a = 7m, b = 3.59m, C= 47^0
[A= $103^02'$, B = $29^052'$, c= 5.25m]

(d) A = 60^0, b = 8m, C = 15
[a = 13, B= 32.2^0, C = 87.8^0]

2. Show that for all values of x

$$cosx + cos\left(x + \frac{2\pi}{3}\right) + cos\left(x + \frac{3\pi}{3}\right) = 0$$

3. (a) Simplify $\frac{sin3\theta}{sin\alpha} - \frac{cos3\theta}{cos\alpha}\left[\frac{2\sin(3\theta-\alpha)}{sin2\alpha}\right]$

(b) Express 5sinθ + 12cosθ in the form rsin(θ + α) where r and α are constant. Hence determine the minimum value of 5sinθ + 12cosθ + 7.
[r =13, α = 67.4^0, -6]

(c) Given that tanθ = $\frac{3}{4}$, where θ is acute, find values of tan2θ and tan$\frac{\theta}{2}$
[tan2θ =$\frac{24}{7}$ and tan$\frac{\theta}{2}$ = $\frac{1}{3}$]

4. (a) Show that $2\tan^{-1}\left(\frac{1}{3}\right) + tan\left(\frac{1}{7}\right) = \frac{\pi}{4}$

(b) Find x given that
$\tan^{-1}(1+x) + \tan^{-1}(1-x) = 32$
[x = ±1.789]

(c) Given that sinα+ sinβ = p and cosα + cos β = q
Show that sin(α + β) = $\frac{2pq}{p^2+q^2}$

5. (a) By expressing 2sinθsin(θ + α) as a difference of cosines of two angles or otherwise, where αis constant, find the least value [minimum value = cosα – 1. It occurs when θ = $\frac{-\alpha}{2}$]

(b) Solve for x in the equation cosx – cos(x + 60^0) = 0.4 for $0^0 \leq x \leq 360^0$ [x: x = 126.4^0, 353.6^0]

6. (a) Prove that in any triangle ABC
$\frac{b^2-c^2}{a^2} = \frac{\sin(B-C)}{\sin(B+C)}$

(b) Show that for any isosceles triangle ABC with AB = c the base, is given by Δ =$\frac{1}{2}c\sqrt{s(s-c)}$ where s is the perimeter of the triangle
Given that Δ= $\sqrt{3}$ and s = 4, determine the sides of the triangle [1, 3.5, 3.5]

7. Given that $\tan^{-1}\alpha = x$ and $\tan^{-1}\beta = y$, by expressing α and β as tangents ratio of x and y and manipulating the ratios show that x + y = $\tan^{-1}\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)$
Hence or otherwise

(i) Solve for x in

$$\tan^{-1}\left(\frac{1}{x-1}\right) + \tan(x+1) = \tan(-2)$$
[x = ±2]

(ii) Without using tables of calculators determine the value of

$$\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}\left[\frac{\pi}{4}\right]$$

8. (a) Prove that $\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC} = 2R$ where ABC has all angles acute and R is the radius of the circumcircle.

(b) From the top of a vertical cliff 10m high, the angle of depression of ship A is 10^0 and ship B is 15^0. The Bearings of A and B from the cliff are 162^0 and 202.5^0 respectively. Find the bearing of B from A [301.5^0]

9. (a) Prove that
(sin2x –sinx)(1 + 2cosx) = sin3x

(b) A vertical pole BAO stands with its base O on a horizontal plane, where BA = c and AO = b, a point P is situated on horizontal plane at a distance x from O and angle APB = θ
Prove that tanθ =$\frac{cx}{x^2+b^2+bc}$
As P takes different positions on the horizontal plane, find the value of x for which θ is greatest.
[$18^0 26'$, when x = b = c]

10. (a) Prove that sin3x = 3sinx – $4\sin^2$x.

(b) Find all the solutions to $2\sin^2$x = 1 for 00≤ x≤ 360^0. [x =10^0, 50^0, 130^0, 170^0, 250^0, 290^0]

11. Solve cosx +$\sqrt{3}$sinx =2 for $0^0 \leq x \leq 360^0$
[x = 60^0]

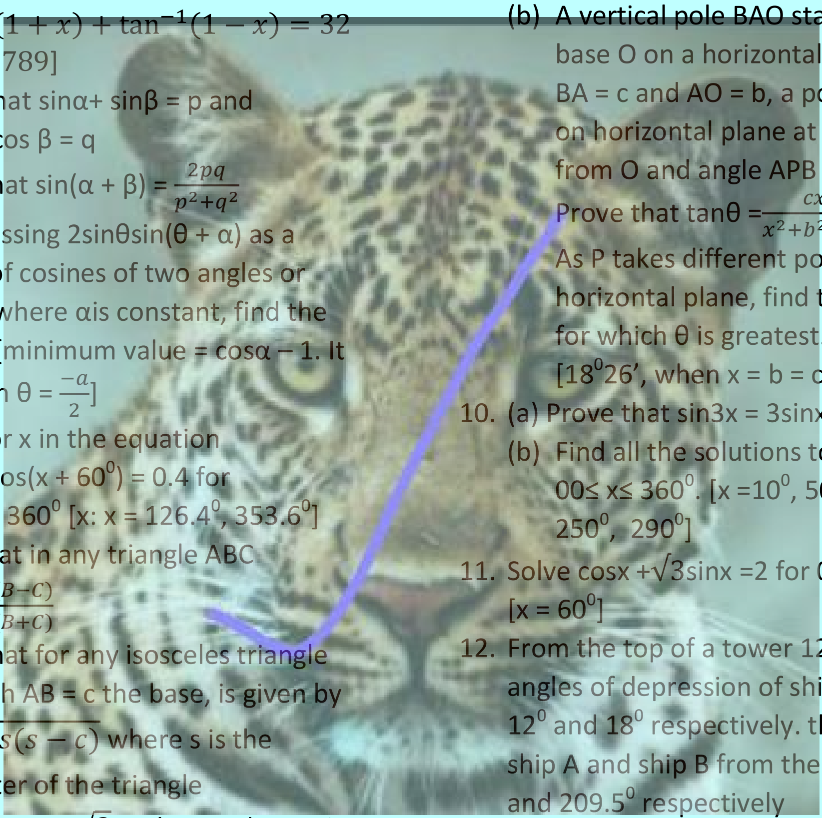
12. From the top of a tower 12.6m high, the angles of depression of ship A and B are 12^0 and 18^0 respectively. the bearing of ship A and ship B from the tower are 148^0 and 209.5^0 respectively
Calculate

(i) How far the ships are from each other [53.14m]

(ii) The bearing of ship A from ship B[108.1^0]

13. (a) Solve sin3x + $\frac{1}{2}$ = $2\cos^2$x for $0^0 \leq x \leq 360^0$
[x = 30^0, 60^0, 120^0, 150^0, 240^0, 300^0]

(b) Given that in any triangle ABC, $\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}cot\left(\frac{A}{a}\right)$ solve the triangle with two sides 5 and 7 and the included angle 45^0.
[A = 45^0, B = 89.4^0, C= 45.6^0]



14. (a) Solve $\cot^2 x = 5(\text{cosc}x + 1)$ for $0^0 \leq x \leq 360^0$ [9.6^0, 170.4^0, 270^0]

(b) Use $\tan\frac{\theta}{2} = t$ to solve $5\sec\theta - 2\sin\theta = 2$ for $0^0 \leq x \leq 360^0$ [46.4^0, 270^0]

15. Given that $\sin 2x = \cos 3x$, fins the values of $\sin\theta$, $0 \leq x \leq \pi$ [0.309 3dp]

16. (a) Show that $\tan\left(\frac{A+B}{2}\right) - \tan\left(\frac{A-B}{2}\right) = \frac{2sinB}{cosA+cosB}$

(b) Find in radians the solution of the equation $\cos\theta + \sin 2\theta = \cos 3\theta$ for $0 \leq \theta \leq \pi$ $\left[0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\right]$

17. (a) Show that cotA + tan2A = cotAsec2A

(b) Show that $tan3x = \frac{3t-t^3}{1-3t^2}$, where t = tanx. Hence or otherwise show that $\tan^{-1}\left(\frac{\pi}{12}\right) = 2 - \sqrt{3}$

18. (a) Find all the values θ, 00 ≤ θ ≤ 3600, which satisfies the equation $\sin^2\theta - \sin 2\theta - 3\cos^2\theta = 0$ [θ =135^0, 315^0]

(b) Show that $\frac{cosx}{1+sinx} = cot\left(\frac{x}{2} + 45^0\right)$.

Hence or otherwise solve $\frac{cosx}{1+sinx} = \frac{1}{2}$; $0^0 \leq x \leq 360^0$ [x = 36.8^0]

19. (a) Given that X, Y and Z are angles of a triangle XYZ. Prove that $tan\left(\frac{X-Y}{2}\right) = \frac{x-y}{x+y} cot\frac{Z}{2}$.

Hence solve the triangle if x = 9cm, y = 5.7cm and Z = 57^0. [z = 7.6cm, X= 84.4^0]

(b) Use the substitution $t = \tan\left(\frac{\theta}{2}\right)$ to solve the equation $3\cos\theta - 5\sin\theta = -1$ for $0^0 \leq \theta \leq 360^0$ [40.84^0, 201.1^0]

20. Prove that $tan\left(\frac{\pi}{4} + \theta\right) - tan\left(\frac{\pi}{4} - \theta\right) = 2tan2\theta$

21. (a) Solve the equation 3cosx + 4sinx = 2 for $0^0 \leq x \leq 360^0$ [x = 119.5^0, 346.7^0]

(b) If A, B, C are angles of a triangle. Show that

cos2A + cos2B + cos2C = -1 –4cosAcosB

22. (a) Solve 2sin2θ = 3 for $-180^0 \leq x \leq 180^0$ [-90^0, 48.6^0, 90^0, 131.4^0]

(b) Solve sinx – sin4x = sin2x – sin3x for –π ≤ x ≤ π

$\left[-\frac{\pi}{5}, -\frac{\pi}{2}, -\frac{3\pi}{5}, 0, \frac{\pi}{5}, \frac{\pi}{5}, \frac{3\pi}{5}\right]$

23. Without using tables or calculator, show that $\tan 150 = 2 - \sqrt{3}$

24. (a) Solve the equation cosx + cos2x = 1 for $0^0 \leq x \leq 360^0$ [x = 38.67^0, 321.33^0]

(b) (i) Prove that $\frac{cosA+cosB}{sinA+sinB} = cot\frac{A+B}{2}$

(ii) $\frac{cosA+cosB}{sinA+sinB} = tan\frac{C}{2}$ where A, B and C are angles of a triangle

25. Given that $\sin(\theta - 45^0) = 3\cos(\theta + 45^0)$ show that tanθ = 1. Hence find θ if $0^0 \leq \theta \leq 360^0$ [45^0, 225^0]

26. (a) Use the factor formula to show that $\frac{\sin(A+2B)+sinA}{coa(A+2B)+cosA} = \tan(A+B)$

(b) Express y = 8cosx + 6sinx in the form Rcos(x – α) where R is positive and α is acute

Hence find the maximum and minimum values of $\frac{1}{8cosx+6sinx+15}$ [0.2, 0.04]

27. Express sinx + cosx in the form Rcos(x – α). Hence, find the greatest value of sinx + cos x – 1. [0.4142]

28. (a) Solve cosx + cos3x= cos2x, $0 \leq x \leq 360^0$ [x = 45^0, 60^0, 135^0, 225^0, 300^0, 315^0]

(b) Show that $tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \frac{1+sin\theta}{cos\theta}$

29. Show that $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} = \tan^{-1}\frac{7}{9}$

30. (a) Solve 3sinx + 4cosx = 2 for $-180^0 \leq x \leq 180^0$. [-29.55^0, 103.29^0]

(b) Show that in any triangle ABC $\frac{a^2-b^2}{c^2} = \frac{\sin(A-B)}{\sin(A+B)}$

31. (a) Prove that $\frac{2tanx}{1+tan^2x} = sin2x$

(b) Solve sin2x = cosx; $0^0 \leq x \leq 90^0$ [x = 30^0, 90^0]

32. (a) Solve the equation $8\cos^4 x - 10\cos^2 x + 3 = 0$; $0^0 \leq x \leq 180^0$ [30^0, 45^0, 135^0, 150^0]

(b) Prove that cos4A – cos4B – Cos4C = 4sin2Bsin2Ccos2A -1 given that A, B and C are angles of a triangle

33. Given that cos 2A - cos 2B = -p and sin2A – sin2B = q, prove that sec (A + B) = $\frac{1}{q}\sqrt{p^2 + q^2}$

34. Solve

(a) $4\sin^2\theta - 12\sin 2\theta + 35\cos^2\theta = 0$; for $0^0 \leq \theta \leq 90^0$ [74.0^0]

(b) $3\cos\theta - 2\sin\theta = 2$, for $0^0 \leq \theta \leq 360^0$
$[\theta: \theta = 22.62^0, 270.00^0]$

35. Solve the equation $\sin 2\theta + \cos 2\theta\cos 4\theta = \cos 4\theta\cos 6\theta$ for $0 \leq \theta \leq \frac{\pi}{4}$. $\left[\theta = 0, \frac{3\pi}{16}\right]$

36. (a) solve the equation $\cos 2x = 4\cos^2 x - 2\sin^2 x$ for $0 \leq \theta \leq 180^0$ $[\theta = 60^0, 120^0]$

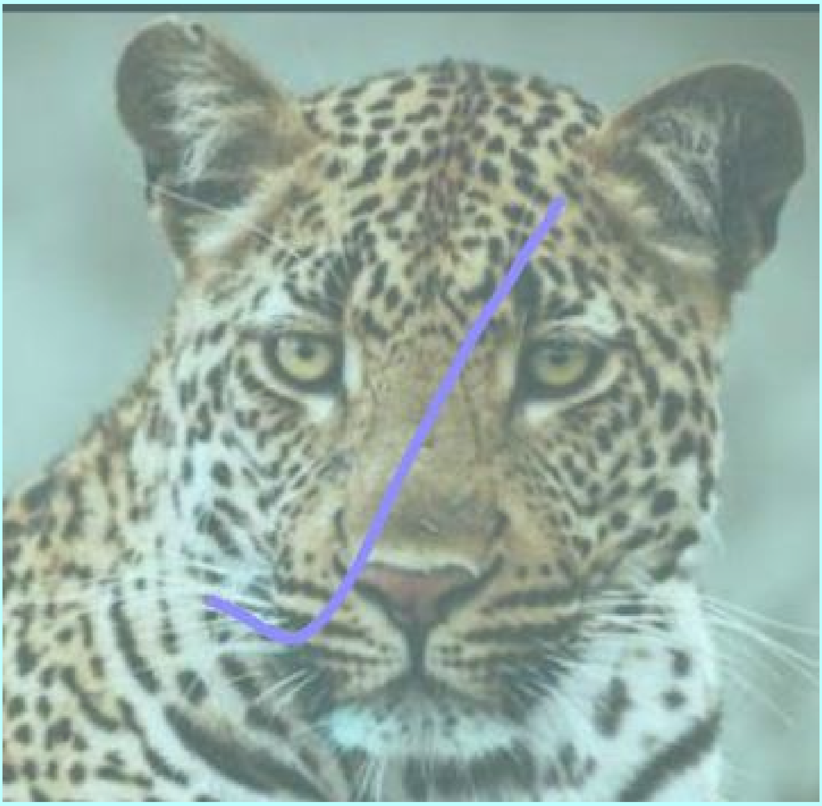
(b) Show that if $\sin(x + \alpha) = p\sin(x - \alpha)$ then $\tan x = \left(\frac{p+1}{p-1}\right)\tan\alpha$. Hence solve the equation $\sin(x + \alpha) = p\sin(x - \alpha)$ for $p = 2$ and $\alpha = 20^0$. $[x = 47.52^0]$

37. Solve the equation
$3\tan^2\theta + 2\sec^2\theta = 2(5 - 3\tan\theta)$
for $0^0 < \theta < 180^0$ $[\theta = 38.66^0, 116.57^0]$

38. (a) Show that $\tan 4\theta = \frac{4t(1-t^2)}{t^4-6t^2+1}$, where $t = \tan\theta$

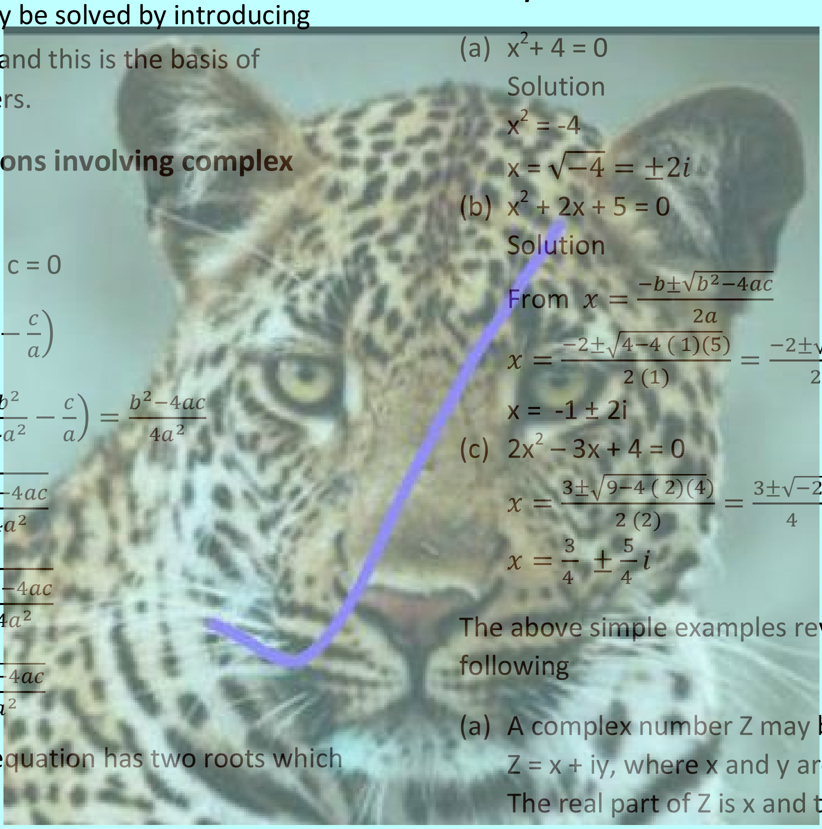
(b) Solve the equation
$\sin x + \sin 5x = \sin 2x + \sin 4x$
for $0^0 < x < 90^0$. $[x = 60^0]$

39. Solve $2\cos 2\theta - 5\cos\theta = 4$
for $0^0 \leq \theta \leq 360^0$. $[\theta = 138.59^0, 221.41^0]$



Complex numbers

Equations $ax^2 + bx + c = 0$ where $a \neq 0$ and $b^2 < 4ac$ can only be solved by introducing $\sqrt{(-1)}$ = i or j; and this is the basis of complex numbers.



Solving equations involving complex numbers

Given $ax^2 + bx + c = 0$

$$x^2 + \left(\frac{b}{a}\right)x = \left(-\frac{c}{a}\right)$$

$$\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b^2}{4a^2} - \frac{c}{a}\right) = \frac{b^2-4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2-4ac}{4a^2}}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{4a^2}$$

If $b^2 > 4ac$; the equation has two roots which are both real.

If $b^2 = 4ac$; the equation has got two roots which are repeated and real.

If $b^2 < 4ac$; the equation has got two roots which are not real.

In complex numbers equations whose roots are not real are considered

Hence

$$x = -\frac{b}{2a} \pm i\frac{\sqrt{b^2-4ac}}{4a^2}$$

We let x = p ± iq

Where $p = -\frac{b}{2a}$ and $q = \frac{\sqrt{b^2-4ac}}{4a^2}$

Example 1

(a) $x^2 + 4 = 0$

Solution

$x^2 = -4$

$x = \sqrt{-4} = \pm 2i$

(b) $x^2 + 2x + 5 = 0$

Solution

From $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$x = \frac{-2\pm\sqrt{4-4\,(1)(5)}}{2\,(1)} = \frac{-2\pm\sqrt{-16}}{2} = \frac{-2\pm 4i}{2}$$

x = -1 ± 2i

(c) $2x^2 - 3x + 4 = 0$

$$x = \frac{3\pm\sqrt{9-4\,(2)(4)}}{2\,(2)} = \frac{3\pm\sqrt{-25}}{4} = \frac{3\pm 5i}{4}$$

$$x = \frac{3}{4} \pm \frac{5}{4}i$$

The above simple examples reveal the following

(a) A complex number Z may be written as Z = x + iy, where x and y are real numbers. The real part of Z is x and the imaginary part is y. i.e. *x = Re(Z)* and *y = im(Z).*
(b) Solution of such equations whose roots are complex occur in pairs, called conjugates. For instance, compare Z = $\pm 2i$, -1 ± 2i and $\frac{3}{4} \pm \frac{5}{4}i$
    (i) If Z = -1 + 2i is one root of an equation, then its conjugate $\bar{Z}$ = -1 – 2i is another root.
    (ii) An equation cannot have odd number of complex roots
(c) If x + iy = 0, then x = 0 and y = 0

This implies that two complex numbers are equal if and only if their corresponding real and imaginary parts are equal

i.e. $Z_1 = x_1 + iy_1$ and $Z_2 = x_2 + iy_2$, then

$Z_1 = Z_2$, only and only if $x_1 = x_2$ and $y_1 = y_2$.

Proof: If Z1 = Z2

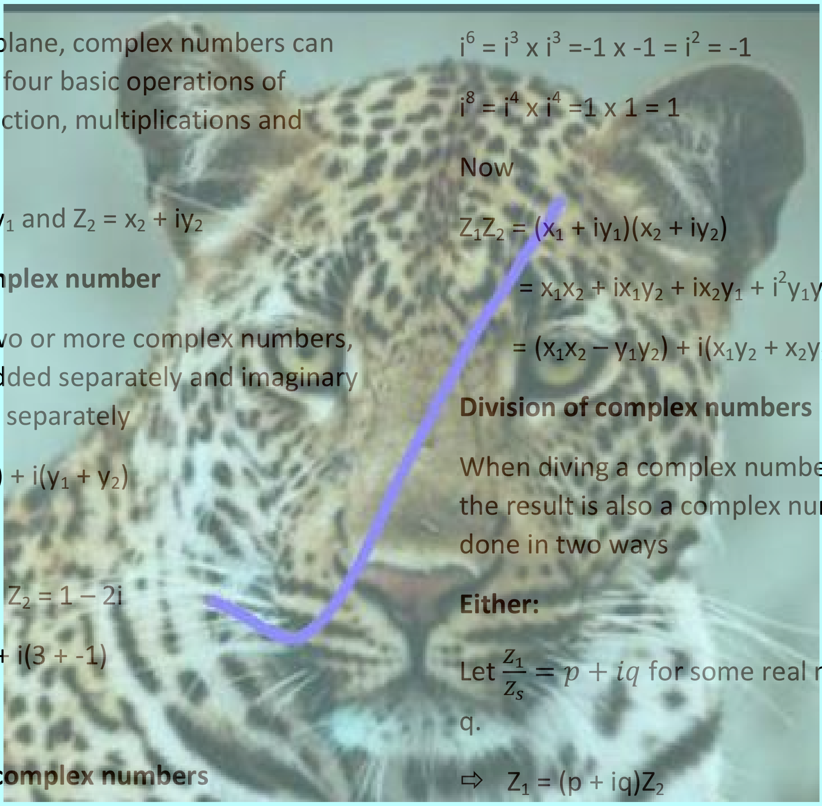
$x_1 + iy_1 = x_2 + iy_2$

i.e. $(x_1 – x_2) + (y_1 – y_2) = 0$

=> $x_1 – x_2 = 0$ and $y_1 – y_2 = 0$

$x_1 = x_2$ and $y_1 = y_2$

Algebra of complex numbers



Like in the real plane, complex numbers can be subjected to four basic operations of addition, subtraction, multiplications and division.

Using $Z_1 = x_1 + iy_1$ and $Z_2 = x_2 + iy_2$

Addition of complex number

When adding two or more complex numbers, real parts are added separately and imaginary parts are added separately

$Z_1 + Z_2 = (x_1 + x_2) + i(y_1 + y_2)$

Illustration

If $Z_1 = 2 + 3i$ and $Z_2 = 1 – 2i$

$Z_1 + Z_2 = (2 + 1) + i(3 + -1)$

$= 3 + 2i$

Subtraction of complex numbers

When subtracting two or more complex numbers, real parts are subtracted from real parts and imaginary parts subtracted from imaginary parts.

$Z_1 – Z_2 = (x_1 – x_2) + i(y_1 – y_2)$

Illustration

If $Z_1 = 2 + 3i$ and $Z_2 = 1 – 2i$

$Z_1 + Z_2 = (2 – 1) + i(3 – -1)$

$= 1 + 4i$

Geometrically, the operations of addition and subtraction of complex numbers are diagonals of a parallelogram. (see the Argand diagrams)

Multiplication of complex numbers

When multiplying complex numbers the following properties should be observed

$i^2 = \sqrt{-1} x \sqrt{-1} = -1$;

$i^3 = -1 x i = -1$

$i^4 = i^2 x i^2 = -1 x -1 = 1$

$i^5 = i^3 x i^2 =-1 x -1 = 1$

$i^6 = i^3 x i^3 =-1 x -1 = i^2 = -1$

$i^8 = i^4 x i^4 =1 x 1 = 1$

Now

$Z_1Z_2 = (x_1 + iy_1)(x_2 + iy_2)$

$= x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2$

$= (x_1x_2 – y_1y_2) + i(x_1y_2 + x_2y_1)$

Division of complex numbers

When diving a complex number with another, the result is also a complex number. It can be done in two ways

Either:

Let $\frac{Z_1}{Z_s} = p + iq$ for some real numbers p and q.

⇨ $Z_1 = (p + iq)Z_2$

$x_1 + iy_1 = (p + iq)(x_2 + iy_2)$

$= px_2 + ipy_2 + iqx_2 +i2qy_2$

$= px_2 – qy_2 + i(py_2 + qx_2)$

Equating the corresponding real and imaginary parts of the resulting complex number

$x_1 = px_2 – qy_2$ and $x_2 = py_2 + qx_2$

Solving these equation simultaneously for p and q p = $\frac{x_1x_2+y_1y_2}{x_2{}^2+y_2{}^2}$ and q = $\frac{x_1x_2-y_1y_2}{x_2{}^2+y_2{}^2}$

Or:

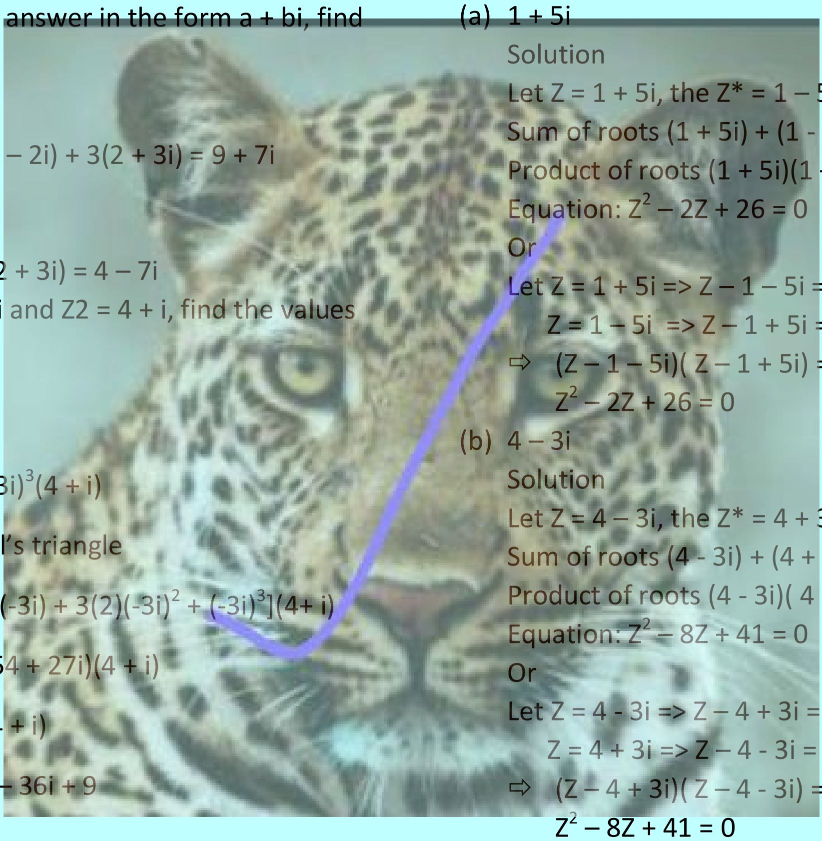
'Realizing' the denominator by multiplying through by it conjugate

$$\frac{Z_1}{Z_S} = \frac{(x_1+iy_1)(x_2-iy_2)}{(x_2+iy_2)(x_2-iy_2)} = \frac{x_1x_2-ix_1y_2-ix_2y_1\ y_1y_2}{x_2{}^2+y_2{}^2}$$

$$= \frac{x_1x_2+y_1y_2}{x_2{}^2+y_2{}^2} + i\frac{(x_1x_2-y_1y_2)}{x_2{}^2+y_2{}^2}$$

Example 2

Given that Z_1 = 3 – 2i and Z_2= 2 + 3i, by expressing your answer in the form a + bi, find



(a) $Z_1 + Z_2$
Solution
$Z_1 + 3Z_2$ = (3 – 2i) + 3(2 + 3i) = 9 + 7i

(b) $2Z_1$ – Z2
Solution
2(3 – 2i) – (2 + 3i) = 4 – 7i

(c) If Z1 = 2 – 3i and Z2 = 4 + i, find the values of

(i) $Z_1{}^3Z_2$
Solution

$Z_1{}^3Z_2$ = $(2 – 3i)^3(4 + i)$

Using Pascal's triangle

= $[2^3 + 3(2)^2(-3i) + 3(2)(-3i)^2 + (-3i)^3](4+ i)$

=(8 – 36i – 54 + 27i)(4 + i)

=(-46 – 9i)(4 + i)

= -184 – 46i – 36i + 9

= -175 – 82i

(ii) $\frac{Z_2{}^1}{Z_1}$

Solutions

$$\frac{Z_2{}^1}{Z_1} = \frac{(4+i)^2}{(2-3i)} = \frac{16+8i-1}{(2-3i)} = \frac{15+8i}{(2-3i)}$$

$$= \frac{(15+8i)(2+3i)}{(2-3i)(2+3i)} = \frac{30+45i+16i-24}{4+9}$$

$$= \frac{6}{13} + i\frac{61}{13}$$

Forming quadratic equations given roots

When given roots to a complex number, the quadratic equation say Z can be obtained by finding the sum and the products of the roots

i.e. the equation is given by
Z^2 – (sun of roots)Z + product of roots

Not that if Z is a root of the equation, then its conjugate $\bar{Z}$ is also a root

Example 3

Form a quadratic equation in Z given the following roots

(a) 1 + 5i
Solution
Let Z = 1 + 5i, the Z* = 1 – 5i
Sum of roots (1 + 5i) + (1 - 5i) = 2
Product of roots (1 + 5i)(1 – 5i)= 26
Equation: Z^2 – 2Z + 26 = 0
Or
Let Z = 1 + 5i => Z – 1 – 5i = 0
Z = 1 – 5i => Z – 1 + 5i = 0
⇨ (Z – 1 – 5i)(Z – 1 + 5i) = 0
Z^2 – 2Z + 26 = 0

(b) 4 – 3i
Solution
Let Z = 4 – 3i, the Z* = 4 + 3i
Sum of roots (4 - 3i) + (4 + 3i) = 8
Product of roots (4 - 3i)(4 - 3i)= 41
Equation: Z^2 – 8Z + 41 = 0
Or
Let Z = 4 - 3i => Z – 4 + 3i = 0
Z = 4 + 3i => Z – 4 - 3i = 0
⇨ (Z – 4 + 3i)(Z – 4 - 3i) = 0
Z^2 – 8Z + 41 = 0

Solving cubic equations

It is quite difficult to solve cubic equations, however, we use the approach of inspection, i.e. we substitute the assumed factors into the given equation. If the unction is f(x) = 0 and x – a = 0 is a root, then f(a) = 0.

We then get another equation (quadratic) using long division which can easily be solved to obtain other factors.

Example 4

Solve the following equations

(a) $x^3 - 4x^2 + x + 26 = 0$

Solution

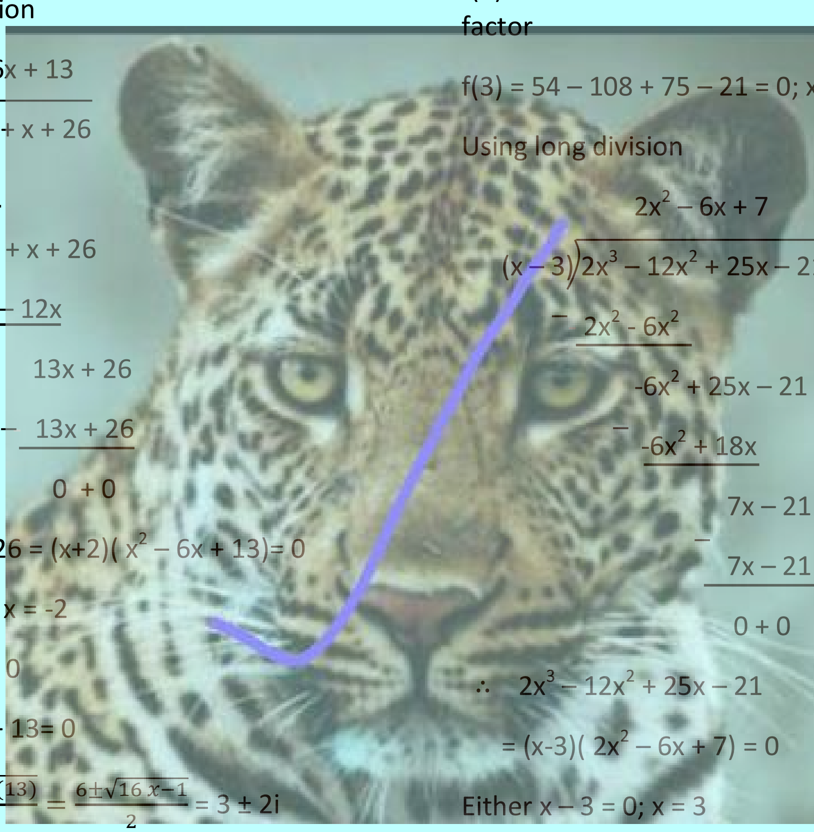
Let $f(x) = x^3 - 4x^2 + x + 26 = 0$

$f(1) = 2 - 4 + 1 + 26 \neq 0$; => $x - 1$ is not a factor

$f(2) = 8 - 8 + 2 + 26 \neq 0$; => $x - 2$ is not a factor

$f(-2) = -8 - 16 - 2 + 26 = 0$; $x + 2$ is a factor

Using long division



$$\begin{array}{rl} & x^2 - 6x + 13 \\ (x+2) \big) & x^3 - 4x^2 + x + 26 \\ - & \underline{x^3 + 2x^2} \\ & -6x^2 + x + 26 \\ - & \underline{-6x^2 - 12x} \\ & 13x + 26 \\ - & \underline{13x + 26} \\ & 0 + 0 \end{array}$$

$\therefore x^3 - 4x^2 + x + 26 = (x+2)(x^2 - 6x + 13) = 0$

Either $x + 2 = 0$; $x = -2$

Or $x^2 - 6x + 13 = 0$

Solving $x^2 - 6x + 13 = 0$

$$x = \frac{6 \pm \sqrt{36 - 4(1)(13)}}{2(1)} = \frac{6 \pm \sqrt{16\ x - 1}}{2} = 3 \pm 2i$$

The roots of the equation are $x = -2, 3 \pm 2i$

Alternatively

Given the equation $x^2 - 6x + 13 = 0$

Let the roots be $x \pm 2i$

Sum of roots $x + iy + x - iy = 6$

$2x = 6$; $x = 3$

Product of roots $= (x + iy)(x - iy) = 13$

$x^2 + y^2 = 13$

Substituting for x;

$9 + y^2 = 13$; $y = \pm 2$

The roots of the equation are $x = -2, 3 \pm 2i$

(b) $2x^3 - 12x^2 + 25x - 21 = 0$

Solution

Let $f(x) = 2x^3 - 12x^2 + 25x - 21 = 0$

$f(1) = 2 - 12 + 25 - 12 = -6 \neq 0$; $x - 1$ is not a factor

$f(2) = 16 - 48 + 50 - 21 = -3 \neq 0$; $x - 2$ is not a factor

$f(3) = 54 - 108 + 75 - 21 = 0$; $x - 3$ is a factor

Using long division

$$\begin{array}{rl} & 2x^2 - 6x + 7 \\ (x-3) \big) & 2x^3 - 12x^2 + 25x - 21 \\ - & \underline{2x^3 - 6x^2} \\ & -6x^2 + 25x - 21 \\ - & \underline{-6x^2 + 18x} \\ & 7x - 21 \\ - & \underline{7x - 21} \\ & 0 + 0 \end{array}$$

$\therefore 2x^3 - 12x^2 + 25x - 21 = (x-3)(2x^2 - 6x + 7) = 0$

Either $x - 3 = 0$; $x = 3$

Or $2x^2 - 6x + 7 = 0$

$$x = \frac{6 \pm \sqrt{36 - 4(2)(7)}}{2(2)} = \frac{6 \pm \sqrt{4\ x\ 5\ x - 1}}{4} = \frac{3}{2} \pm \frac{\sqrt{5}}{2}$$

The roots of the equation are $x = 3, \frac{3}{2} \pm \frac{\sqrt{5}}{2}$

Alternatively

Given the equation $2x^2 - 6x + 7 = 0$

Or $x^2 - 3 + \frac{7}{2} = 0$

Let the roots be $x \pm 2i$

Sum of roots $x + iy + x - iy = 3$

$$2x = 3; x = \frac{3}{2}$$

Product of roots = $(x + iy)(x - iy) = \frac{7}{2}$

$$x^2 + y^2 = \frac{7}{2}$$

Substituting for x;

$\frac{9}{4} + y^2 = \frac{7}{2}$; $y = \frac{\pm\sqrt{5}}{2}$

The roots of the equation are $x = 3, \frac{3}{2} \pm \frac{\sqrt{5}}{2}$

Finding the square root of a complex number

Suppose that the square root of Z is a + bi

Then $a + bi = Z^{\frac{1}{2}}$ => $(a + bi)^2 = Z$

Example 5

Find the square roots of

(a) -5 + 12i

Solution

Approach 1

Let $\sqrt{(-5 + 12i)} = x + iy$

By squaring both sides

$-5 + 12i = x^2 - y^2 + i2xy$

Equating corresponding parts

$x^2 - y^2 = -5$ .... (i)

$2xy = 12$

$x = \frac{6}{y}$................. (ii)

Substituting eqn.(ii) into eqn. (i)

$$\left(\frac{6}{y}\right)^2 - y^2 = -5$$

$36 - y^4 = -5y^2$

Let k = y2

$k^2 - 5k - 36 = 0$

$(k - 9)(k + 4) = 0$ i.e. k = 9 or k = -4

When k = -4, $y^2 = -4$ which is inadmissible since y must be real.

When k = 9; $y^2 = 9$ => $y = \pm3$

⇨ $x = \frac{6}{\pm3} = \pm2$

$\therefore \sqrt{(-5 + 12i)} = 2 + 3i$ or $-2 - 3i \equiv \pm(2 + 3i)$

Approach 2

Let $\sqrt{(-5 + 12i)} = x + iy$

$x^2 - y^2 + i2xy = -5 + 12i$

Equating corresponding parts

$x^2 - y^2 = -5$ .... (i)

$2xy = 12$............... (ii)

Eqn. $(i)^2$ + eqn. $(ii)^2$

$$\begin{array}{rl} & x^4 - 2x^2y^2 + y^4 = 25 \\ + & 4x^2y^2 = 144 \\ \hline & x^4 + 2x^2y^2 + y^4 = 169 \end{array}$$

$(x^2 + y^2)^2 = 169$

$x^2 + y^2 = 13$ ....(iii)

Eqn.(i) + eqn. (iii)

$2x^2 = 8; x = \pm2$

Substituting for x in eqn. (iii)

$4 + y^2 = 13; y = \pm3$

$\therefore \sqrt{(-5 + 12i)} = 2 + 3i$ or $-2 - 3i \equiv \pm(2 + 3i)$

(b) 3 + 4i

Let $\sqrt{(3 + 4i)} = x + iy$

$x^2 - y^2 + i2xy = 3 + 4i$

Equating corresponding parts

$x^2 - y^2 = 3$ .... (i) and $2xy = 4$............... (ii)

Eqn. $(i)^2$ + eqn. $(ii)^2$

$$\begin{array}{rl} & x^4 - 2x^2y^2 + y^4 = 9 \\ + & 4x^2y^2 = 16 \\ \hline & x^4 + 2x^2y^2 + y^4 = 25 \end{array}$$

$(x^2 + y^2)^2 = 25$

$x^2 + y^2 = 5$ …………….(iii)

Eqn.(i) + eqn. (iii)

$2x^2 = 8; x = \pm 2$

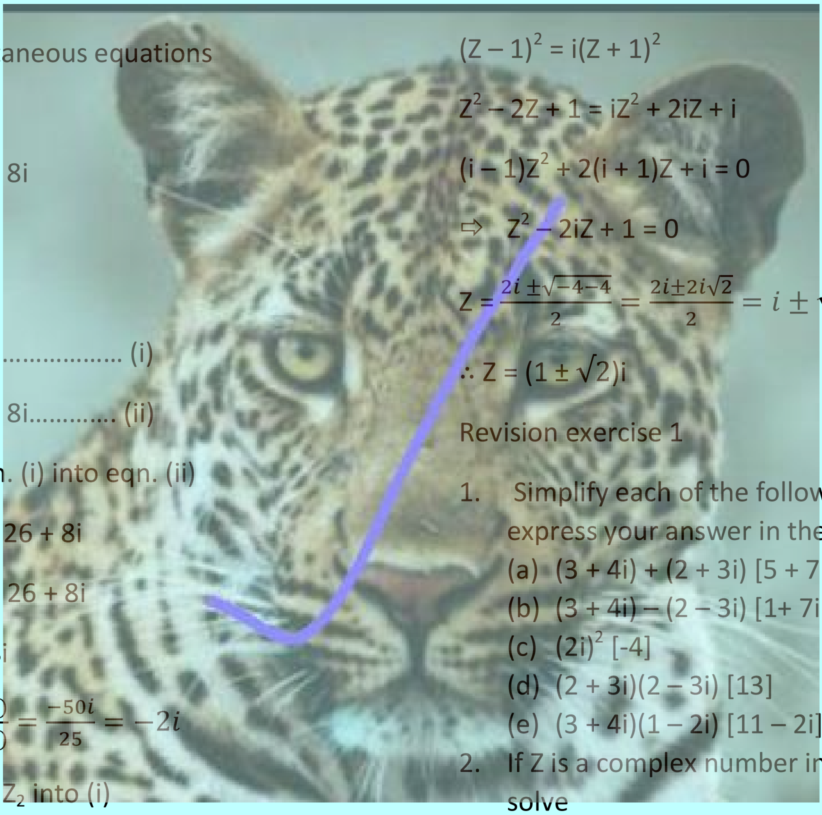
Substituting for x in eqn. (iii)

$4 + y^2 = 5; y = \pm 1$

∴ $\sqrt{(3+4i)}$ = 2 + i or -2 – i ≡ $\pm(2+i)$

Other equations

Example 6



Solve the simultaneous equations

$Z_1 + Z_2 = 8$

$4Z_1 - 3iZ_2 = 26 + 8i$

Solution

$Z_1 + Z_2 = 8$

$Z_1 = 8 - Z_2$ ………………………. (i)

$4Z_1 - 3iZ_2 = 26 + 8i$………… (ii)

Substituting eqn. (i) into eqn. (ii)

$4(8 - Z_2) + 3iZ_2 = 26 + 8i$

$32 - 4Z_2 - 3iZ_2 = 26 + 8i$

$(4 + 3i)Z_2 = 6 - 8i$

$Z_2 = \frac{(6-8i)(4-3i)}{(4+3i)(4-3i)} = \frac{-50i}{25} = -2i$

Substituting for Z_2 into (i)

$Z_1 = 8 + 2i$

∴ $Z_1 = 8 + 2i$ and $Z_2 = -2i$

Example 7

Solve the equation 6x2 – 2(1+ 2i)x – 1 = 0

Solution

Using $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$x = \frac{2(1+2i) \pm \sqrt{4(1+2i)^2 - 4(6)(-1)}}{2(6)}$$

$= \frac{1+2i\pm\sqrt{(3+4i)}}{6}$

But $\sqrt{(3+4i)} = \pm(2+i)$

⇨ $x = \frac{1+2i\pm(2+i)}{6}$

Either $x = \frac{1}{2} + \frac{1}{2}i$ or $x = -\frac{1}{6} + \frac{1}{6}i$

Example 8

Solve the equation $\left(\frac{Z-1}{Z+1}\right)^2 = i$, If Z is a complex number in form a + bi

Solution

$(Z - 1)^2 = i(Z + 1)^2$

$Z^2 - 2Z + 1 = iZ^2 + 2iZ + i$

$(i - 1)Z^2 + 2(i + 1)Z + i = 0$

⇨ $Z^2 - 2iZ + 1 = 0$

$Z = \frac{2i \pm\sqrt{-4-4}}{2} = \frac{2i\pm2i\sqrt{2}}{2} = i \pm \sqrt{2}$

∴ $Z = (1 \pm \sqrt{2})i$

Revision exercise 1

1. Simplify each of the following and express your answer in the form a + bi
   (a) (3 + 4i) + (2 + 3i) [5 + 7i]
   (b) (3 + 4i) – (2 – 3i) [1+ 7i]
   (c) $(2i)^2$ [-4]
   (d) (2 + 3i)(2 – 3i) [13]
   (e) (3 + 4i)(1 – 2i) [11 – 2i]
2. If Z is a complex number in form (a + bi) solve
   (a) $\left(\frac{Z-1}{Z+1}\right)^2 = i$ [$Z = (1 \pm \sqrt{2})i$]
   (b) $\frac{Z}{Z+1} = 1 + 2i$ $\left[-1 + \frac{1}{2}i\right]$

(c) Express each of the following complex numbers in the form a + bi
   (a) $Z_1 = (1 - i)(1 + 2i)$ [3 + i]
   (b) $Z_2 = \frac{2+6i}{3-i}$ [2i]
   (c) $Z_3 = \frac{-4i}{1-i}$ [2 – 2i]

(d) Find the square root of
   (a) 12i – 5 [2 +3i or -2 – 3i]
   (b) 5 + 2i [±(3 +2i)]
   (c) 15 + 8i [±(4 + i)]

(d) $7 - 24i$ [$\pm(4 - 3i)$]

(e) Given that $Z = 3 + 4i$, find the value of the expression $Z + \frac{25}{Z}$ [6]

(f) Given that the complex number Z* and its complex conjugate satisfy the equation $ZZ^* + 2iZ = 12 + 6i$, find the possible values of Z [$3 + 3i$; $3 - i$]

(g) Solve the simultaneous equation

$Z_1 + Z_2 = 8$

$4Z_1 - 3iZ_2 = 26 + 8i$

[$Z_1 = 8 + 2i$ and $Z_2 = -2i$]

(h) Express each of the following in the form a + bi

(a) $\frac{2+3i}{1-i}\left[-\frac{1}{2}+\frac{5}{2}i\right]$

(b) $\frac{3-i}{1+2i}\left[\frac{1}{5}-\frac{7}{5}i\right]$

(c) $\frac{4}{1+i}$ [$-1 + i$]

(d) $\frac{3+4i}{3-4i}-\frac{3-4i}{4+4i}\left[\frac{48i}{25}\right]$

(i) Solve the following equations

(a) $2x^2 + 32 = 0$ [$\pm 4i$]

(b) $4x^2 + 9 = 0\left[\pm\frac{3}{2}i\right]$

(c) $x^2 + 2x + 5 = 0$ [$-1 \pm 2i$]

(d) $x^2 - 4x + 5 = 0$ [$2 \pm i$]

(e) $2x^2 + x + 1 = 0\left[\frac{-1\pm\sqrt{71}}{4}\right]$

(j) Form quadratic equations having roots

(a) 3i, -3i [$x^2 + 9 = 0$]

(b) 1 + 2i, 1- 2i [$x^2 - 2x + 5 = 0$]

(c) 2 + I, 2 – i[$x^2 - 4x + 5 = 0$]

(d) 3+4i, 3 – 4i [$x^2 - 6x + 25 = 0$]

(k) Solve the following equations if each has at least one real root

(a) $x^3 - 7x^2 + 19x - 13 = 0$ [$1, 3 \pm 2i$]

(b) $2x^3 - 2x^2 - 3x - 2 = 0\left[2, -\frac{1}{2}\pm\frac{1}{2}i\right]$

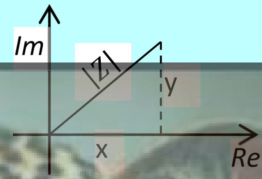
(c) $x^3 + 3x^2 + 5x + 3 = 0$ [$-1, \pm\sqrt{2}i$]

The Argand diagram

Complex numbers can be represented graphically just as coordinates are. However, instead of the x – axis and the y – axes, thereal (*Re*) and the imaginary (*Im*) exes are used respectively instead. This representation was first suggested by a mathematician named R. Argand, hence the Argand diagram. Each complex number is represented by a line of certain length in a particular direction. Thus each complex number is shown as a vector on Argand diagram.

The sum and the difference of the two complex numbers can be shown on an Argand diagram in the same way as we show vectors which are normally added or subtracted.

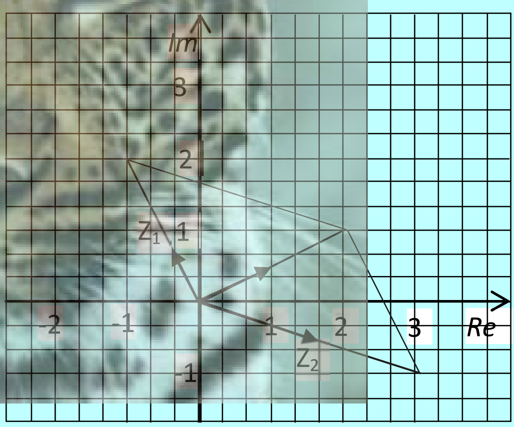
The complex number x = x + iy is represented as shown below



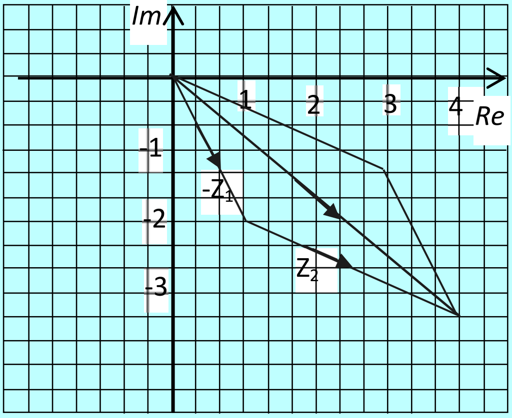
Example 8

Given $Z_1 = -1 + 2i$ and $Z_2 = 3 - i$ show $Z_1 + Z_2$ and $Z_2 - Z_1$ using Argand diagram

(a) $Z_1 + Z_2 = -1 + 2i + 3 - i = 2 + i$



(b) $Z_2 - Z_1 = (3 - i) - (-1 + 2i) = 4 - 3i$



The modulus and argument of a complex number

From the above diagram

$r^2 = |Z|^2 = x^2 + y^2$

i.e. $r = |Z| = \sqrt{x^2 + y^2}$ which is modulus Z = mod(Z) and $\tan\theta = \frac{x}{y}$, i.e. $\theta = \tan^{-1}\frac{y}{x}$, which is the argument, arg(Z) or amplitude, amp(Z).

Note: for principal values, $-\pi \le Arg(Z) \le \pi$

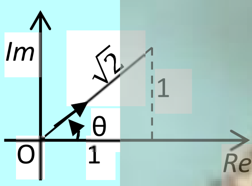
Example 9

Find the modulus and argument of the following

(a) $Z_1 = 1 + i$

Solution

$|Z_1| = \sqrt{(1^2 + 1^2)} = \sqrt{2}$

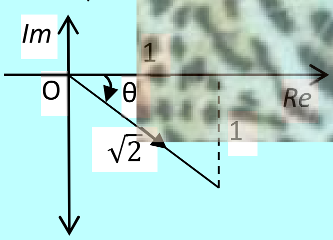


Arg $(Z_1) = \tan^{-1}\left(\frac{1}{1}\right) = 45^0$ *or* $\frac{\pi}{4}$

(b) $Z_2 = 1 - i$

Solution

$|Z_1| = \sqrt{(1^2 + (-1)^2)} = \sqrt{2}$

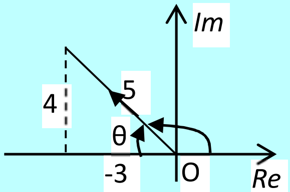


$Arg(Z_2) = \tan^{-1}(-1) = 45^0$ *or* $-\frac{\pi}{4}$

(c) $Z_3 = -3 + 4i$

Solution

$|Z_2| = \sqrt{((-3)^2 + 4^2)} = 5$

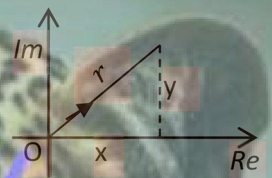


$Arg(Z_3) = \tan^{-1}\left(\frac{4}{3}\right) = 180^0 - 53.1^0 = 126.9^0$

Note: Due care must be taken in obtaining the argument of complex numbers as above example shows. Although the moduli for Z_1 and Z_2 are the same, the arguments differ because the fall in different quadrants. A sketch may be thus may be a necessary pre-requisite.

The polar (modulus – argument) form of a complex number.

Given Z = x + iy (the Cartesian form)



$x = r\cos\theta$ and $y = r\sin\theta$

$\therefore Z = x + iy = r\cos\theta + r\sin\theta$

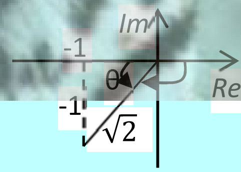
i.e. $Z = r\cos\theta + r\sin\theta$ or simply Z = rcis. This form is referred to as polar form of Z.

Example 10

Express $Z_1 = -1 - i$ and $Z_2 = -\sqrt{3} + 3i$

Solution

$(a)|Z_1| = \sqrt{((-1)^2 + (-1)^2)} = \sqrt{2}$

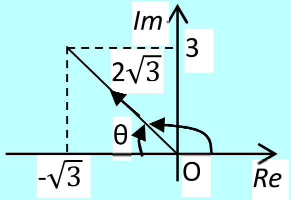


$Arg(Z_1) = \tan^{-1}\left(\frac{-1}{-1}\right) = 225^0 = -135^0 or -\frac{\pi}{4}$

$Z_1 = \sqrt{2}(\cos(-135^0) + isin135^0)$

$= \sqrt{2}\left(cos\frac{3\pi}{4} - isin\frac{3\pi}{4}\right)$

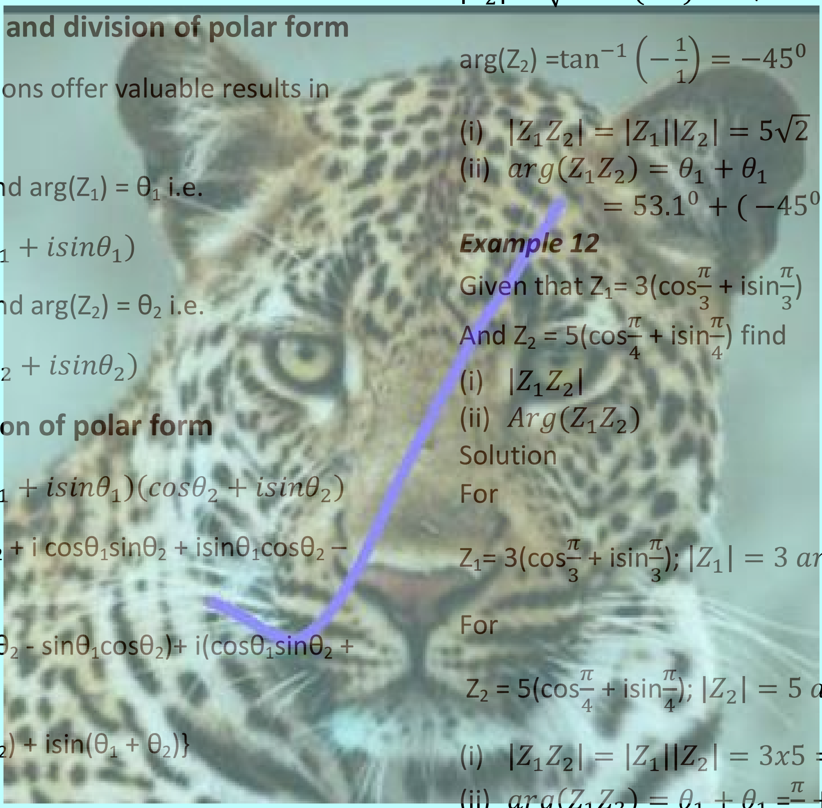
$(b)|Z_2| = \sqrt{((-\sqrt{3})^2 + 3^2)} = 2\sqrt{3}$



Arg(Z_2) = $\tan^{-1}\left(\frac{3}{-\sqrt{3}}\right)$= 120^0 = $\frac{2\pi}{3}$

Z_2 = $2\sqrt{3}(\cos 120^0 + i\sin 120^0)$

Z_2 = $2\sqrt{3}(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$



Multiplication and division of polar form

The two operations offer valuable results in complex plane

Let $|Z_1| = r_1$ and arg(Z_1) = θ_1 i.e.

$|Z_1| = r_1(cos\theta_1 + isin\theta_1)$

Let $|Z_2| = r_2$ and arg(Z_2) = θ_2 i.e.

$|Z_2| = r_2(cos\theta_2 + isin\theta_2)$

(a) Multiplication of polar form

$Z_1Z_2 = r_1r_2(cos\theta_1 + isin\theta_1)(cos\theta_2 + isin\theta_2)$

$= r_1r_2(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - \sin\theta_1\cos\theta_2)$

$= r_1r_2\{(\cos\theta_1\cos\theta_2 - \sin\theta_1\cos\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\}$

$= r_1r_2\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$

Deductions:

(i) $|Z_1Z_2| = r_1r_2 = |Z_1||Z_2|$
(ii) $arg(Z_1Z_2) = \theta_1 + \theta_1$
$= \arg(Z_1) + \arg(Z_2)$
(iii) $|Z^2| = |Z||Z| = |Z|^2$
(iv) $arg(Z^2) = \arg(Z) + \arg(Z) = 2\arg(Z)$

In general

(i) $|Z_1Z_2Z_3 \dots Z_n| = |Z_1||Z_2||Z_2| \dots . . |Z_n|$
(ii) $|Z^n| = |Z||Z||Z| \dots |Z| = |Z|^n$
(iii) $arg(Z_1Z_2Z_3 \dots Z_n)$

$= \arg(Z_1) + \arg(Z_2) + \arg(Z_3) \dots \arg(Z_n)$

(iv) $arg(Z^n) = n\arg(Z)$

Example 11

Given that Z_1 = 3 + 4i and Z_2 = 1 – i, find

(i) $|Z_1Z_2|$
(ii) $Arg(Z_1Z_2)$

Solution

$|Z_1| = \sqrt{3^2 + 4^2} = 5$

arg(Z_1) =$\tan^{-1}\left(\frac{4}{3}\right) = 53.1^0$

$|Z_2| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

arg(Z_2) =$\tan^{-1}\left(-\frac{1}{1}\right) = -45^0$

(i) $|Z_1Z_2| = |Z_1||Z_2| = 5\sqrt{2}$
(ii) $arg(Z_1Z_2) = \theta_1 + \theta_1$
$= 53.1^0 + (-45^0) = 8.1^0$

Example 12

Given that Z_1= $3(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$

And Z_2 = $5(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$ find

(i) $|Z_1Z_2|$
(ii) $Arg(Z_1Z_2)$

Solution

For

Z_1= $3(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$; $|Z_1| = 3$ *and* $\arg(Z_1) = \frac{\pi}{3}$

For

Z_2 = $5(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$; $|Z_2| = 5$ *and* $\arg(Z_2) = \frac{\pi}{4}$

(i) $|Z_1Z_2| = |Z_1||Z_2| = 3x5 = 15$
(ii) $arg(Z_1Z_2) = \theta_1 + \theta_1 = \frac{\pi}{3} + \frac{\pi}{4} = \frac{7\pi}{12}$

(b) Division of polar form

$\frac{Z_1}{Z_2}$

$= \frac{r_1}{r_2}\left\{\frac{cos\theta_1cos\theta_2 - icos\theta_1sin\theta_2 + isincos\theta_2 - sin\theta_1sin\theta}{(cos^2\theta_2 + sin^2\theta_2)}\right.$

$=\frac{r_1}{r_2}\{cos\theta_1cos\theta_2 - icos\theta_1sin\theta_2 + isincos\theta_2 - sin\theta_1sin\theta_2\}$

$=\frac{r_1}{r_2}\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$

Deduction

(a) $\frac{Z_1}{Z_2} = \frac{r_1}{r_2} = \frac{|Z_1|}{|Z_2|}$
(b) $arg\left(\frac{Z_1}{Z_2}\right) = \theta_1 - \theta_2$

$$= \arg(Z_1) - \arg(Z_2)$$

Example 13

Given that $Z_1 = 3 + 4i$ and $Z_2 = 1 - i$, find

(i) $\left|\frac{Z_1}{Z_2}\right|$

(ii) $Arg\left(\frac{Z_1}{Z_2}\right)$

Solution

$|Z_1| = \sqrt{3^2 + 4^2} = 5$

$\arg(Z_1) = \tan^{-1}\left(\frac{4}{3}\right) = 53.1^0$

$|Z_2| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$\arg(Z_2) = \tan^{-1}\left(-\frac{1}{1}\right) = -45^0$

(i) $\left|\frac{Z_1}{Z_2}\right| = \frac{|Z_1|}{|Z_2|} = \frac{5}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$

(ii) $arg\left(\frac{Z_1}{z_2}\right) = \theta_1 - \theta_1$

$$= 53.1^0 - (-45^0) = 98.1^0$$

Example 14

Three complex numbers are given as

$Z_1 = -1 - i$; $Z_2 = 3 - i\sqrt{3}$; $Z_3 = -1 + i\sqrt{3}$

Find

(a) $\left|\frac{Z_1}{Z_2}\right|$

(b) $arg\left(\frac{Z_2}{z_1 Z_3}\right)$

(c) $\left|\frac{{Z_1}^2}{{Z_2}^3 Z_3}\right|$

(d) $arg\left(\frac{{Z_3}^3 {Z_2}^2}{Z_1}\right)$

Solution

$|Z_1| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

$\arg(Z_1) = \tan^{-1}\left(\frac{-1}{-1}\right) = -135^0 = -\frac{3\pi}{4}$

$|Z_2| = \sqrt{3^2 + (\sqrt{3})^2} = 2\sqrt{3}$

$\arg(Z_2) = \tan^{-1}\left(-\frac{\sqrt{3}}{3}\right) = -30^0 = -\frac{\pi}{6}$

$|Z_3| = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$

$\arg(Z_1) = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right) = 120^0 = -\frac{2\pi}{3}$

(a) $\left|\frac{Z_1}{Z_2}\right| = \frac{|Z_1|}{|Z_2|} = \frac{\sqrt{2}}{2}$

(b) $arg\left(\frac{Z_2}{z_1 Z_3}\right) = argZ_2 - arg(z_1 Z_3)$

$= argZ_2 - argZ_1 - argZ_3$

$= -30^0 + 135^0 - 120^0 = -15 = -\frac{\pi}{12}$

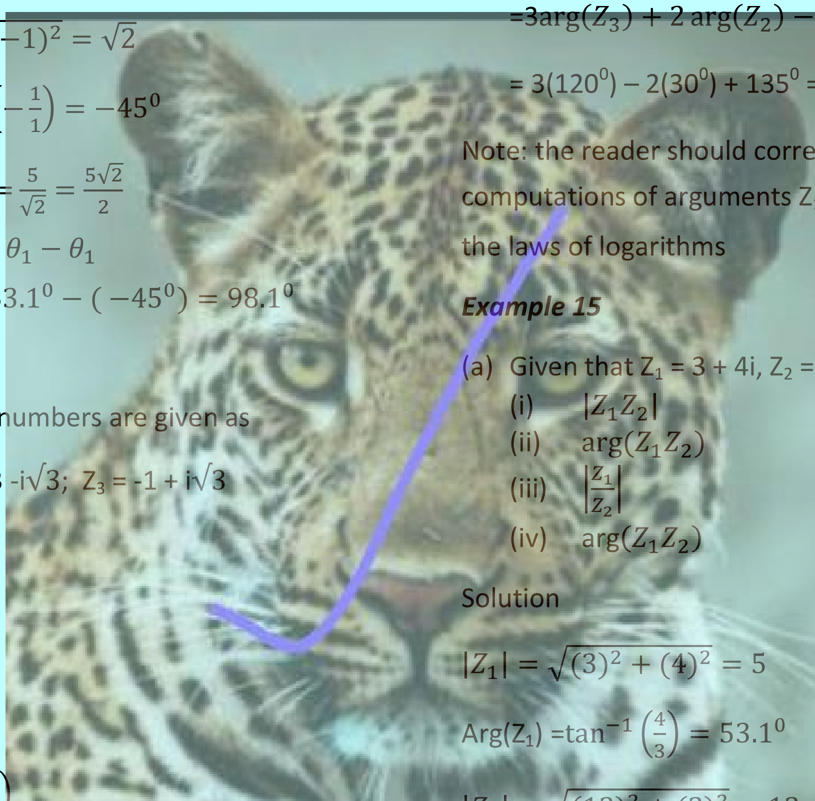
(c) $\left|\frac{{Z_1}^2}{{Z_2}^3 Z_3}\right| = \frac{|Z_1|^2}{|Z_2|^3|Z_3|} = \frac{2}{24\sqrt{3}(2)} = \frac{\sqrt{3}}{72}$

(d) $arg\left(\frac{{Z_3}^3 {Z_2}^2}{Z_1}\right)$

$= arg({Z_3}^2) + \arg({Z_2}^2) - \arg(Z_1)$

$= 3\arg(Z_3) + 2\arg(Z_2) - \arg(Z_1)$

$= 3(120^0) - 2(30^0) + 135^0 = 75^0 = \frac{5\pi}{12}$



Note: the reader should correlate the computations of arguments Z_1Z_2 and $\frac{Z_1}{Z_2}$ with the laws of logarithms

Example 15

(a) Given that $Z_1 = 3 + 4i$, $Z_2 = 12 + 5i$ find
   (i) $|Z_1 Z_2|$
   (ii) $\arg(Z_1 Z_2)$
   (iii) $\left|\frac{Z_1}{Z_2}\right|$
   (iv) $\arg(Z_1 Z_2)$

Solution

$|Z_1| = \sqrt{(3)^2 + (4)^2} = 5$

$Arg(Z_1) = \tan^{-1}\left(\frac{4}{3}\right) = 53.1^0$

$|Z_3| = \sqrt{(12)^2 + (3)^2} = 13$

$Arg(Z_1) = \tan^{-1}\left(\frac{5}{12}\right) = 22.6^0$

(i) $|Z_1 Z_2| = |Z_1||Z_2| = 5 \, x \, 13 = .65$

(ii) $\arg(Z_1 Z_2) = \arg(Z_1) + \arg(Z_2)$

$= 53.1^0 + 22.6^0 = 75.7^0$

(iii) $\left|\frac{Z_1}{Z_2}\right| = \frac{|Z_1|}{|Z_2|} = \frac{5}{13}$

(iv) $\arg\left(\frac{Z_1}{Z_2}\right) = \arg(Z_1) - \arg(Z_2)$

$= 53.1^0 - 22.6^0 = 30.5^0$

(b) Express Z = -1 -i$\sqrt{3}$ in modulus-urgument form. Hence find $\frac{1}{Z}$ in form a + bi where a and b are real numbers

Solution

$$|Z| = \sqrt{(-1)^2 + (-\sqrt{3})^2} = 2$$

Arg(Z) $=\tan^{-1}\left(\frac{-\sqrt{3}}{-1}\right) = -120^0 = -\frac{2}{3}\pi$

$\therefore Z = 2\left(cos\frac{2\pi}{3} - isin\frac{2\pi}{3}\right)$

Hence, $\frac{1}{Z} = Z^{-1} = 2^{-1}\left(cos\frac{2\pi}{3} - isin\frac{2\pi}{3}\right)^{-1}$

$.= \frac{1}{2}\left(cos\left(-\frac{2\pi}{3}\right) - isin\left(-\frac{2\pi}{3}\right)\right)$

$= \frac{1}{2}\left(cos\left(\frac{2\pi}{3}\right) + isin\left(\frac{2\pi}{3}\right)\right)$

$= -\frac{1}{4} + i\frac{\sqrt{3}}{4}$

Solving equations of higher order

When one or more roots of a complex number is(are) given, their conjugates are also the roots of the same equation

To prove that a certain root is a factor of a complex function, we either use long division to show that the quotient of division is zero or we substitute the factor given into the equation to note the remainder or we get the product of the already existing roots and then carry on long division to observe the remainder or we use the synthetic approach

Example 16

Given that (1+ 3i) and (2 – i) are roots of the equation

$aZ^4 + bZ^3 + cZ^2 + dZ + e = 0$, find

(i) The other two roots
(ii) The sum of the four roots
(iii) The product of the four roots

Solution

(i) Since 1 + 3i and 2 – I are roots of the equation, then their conjugates are also roots of the same equations. Hence the other two roots are 1 – 3i and 2 + i.
(ii) Sum of roots = (1 + 3i) + (1 – 3i) + (2 – i) + (2 + i) = 6
(iii) Products of roots
= (1 + 3i)(1 – 3i)(2 – i (2 + i) = 6
= $(1 - 9i^2)(4 - i^2)$
=(1 + 9)(4 + 1)= 50

Example 17

Show that Z_1 = 2 + 3i is a root of the equation $Z^4 - 5Z^3 + 18Z^2 - 17Z + 13 = 0$

Hence find the other remaining roots

Solution

Approach 1

Given 2 + 3i is a root, its conjugate 2 - 3i is also a root of the equation

The equation of these two roots

Sum of roots 2 + 3i + 2 – 3i = 4

Product of roots = (2 + 3i)(2 – 3i) = 13

The equation is $Z^2 - 4Z + 13 = 0$

Using long division

Let $f(Z) = Z^4 - 5Z^3 + 18Z^2 - 17Z + 13$

$$\begin{array}{r l}
 & Z^2 - Z + 1 \\
Z^2 - 4Z + 13 \,\big) & Z^4 - 5Z^3 + 18Z^2 - 17Z + 13 \\
 & \underline{Z^4 - 4Z^3 + 13Z^2} \\
 & -Z^3 + 5Z^2 - 17Z + 13 \\
 & \underline{-Z^3 + 4Z^2 - 13Z} \\
 & Z^2 - 4Z + 13 \\
 & \underline{Z^2 - 4Z + 13} \\
 & 0 + 0 + 0
\end{array}$$

Since the remainder is zero, $Z^2 - Z + 1$is also a factor of f(Z)

⇨ $f(Z) = (Z^2 - 4Z + 13)(Z^2 - Z + 1)$

Either $Z^2 - 4Z + 13 = 0$ or $Z^2 - Z + 1 = 0$

Factors of $Z^2 - Z + 1$

$$Z = \frac{1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = \frac{1 \pm i\sqrt{3}}{2}$$

The roots are $2 \pm 3i$ and $\frac{1 \pm i\sqrt{3}}{2}$

Approach 2

Taking Z = 2 + 3i and substituting in the given equation

LHS = $(2 + 3i)^4 - 5(2 + 3i)^3 + 18(2 + 3i)^2 - 17(2 + 3i) + 13 = 0$ (RHS)

$Z1 = 2 + 3i$ is a root

This means $Z_2 = 2 - 3i$ is also a root of the equation as it is a conjugate of the given root.

Given $f(Z) = Z^4 - 5Z^3 + 18Z^2 - 17Z + 13 = 0$

Sum of roots = 5 and

Product of roots = 13

Let the other roots be α and β

Sum of the roots = $2 + 3i + 2 - 3i + \alpha + \beta = 5$

$4 + \alpha + \beta = 5$

$\alpha + \beta = 1$

$\alpha = (1 - \beta)$ ………………………………. (i)

Product of roots = $(2 + 3i)(2 - 3i)\alpha\beta = 13$

$13\alpha\beta = 13$

$\alpha\beta = 1$ ………………………………….. (ii)

Substituting (i) into (ii)

$(1 - \beta)\beta = 1$ i.e. $\beta 2 - \beta + 1 = 0$

$$\beta = \frac{1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = \frac{1 \pm i\sqrt{3}}{2}$$

∴ The roots are $2 \pm 3i$ and $\frac{1 \pm i\sqrt{3}}{2}$

Approach 3: using synthetic method

2 + 3i	1	-5	18	-17	13
	0	2+3i	-15-3i	15+3i	-13
	1	-3 + 3i	3 -3i	-2+3i	0

Since the last value in the table is zero, therefore 2 + 3i is a root of the equation. Other roots can be obtained as shown above.

Illustration of the synthetic method

Procedure

(a) Write down the root and coefficients of the expression in the first row (as shown above)
(b) Write zero immediately below the first coefficient (this is the only entry in the second row that is simply written, the other are to be obtained by multiplication)
(c) At the first coefficient to zero to get the first entry in the third row
(d) Obtain the second entry in the second row by multiplying 1 by 2 + 3i, then add to -5 to get the second entry in the third row i.e. -3 + 3i
(e) Repeat (d) by multiplying -3 + 3i to get the third entry in the second row, this entry is added to 18 to get to get the third row i.e. 3 – 3i. continue with this trend up to the last entry. If the last entry in the table is zero, then the value being tested is a root.

Example 18

Show that Z1 = 1 + I is a root of the equation $Z^4 - 6Z^3 + 25Z^2 - 34Z + 26 = 0$

Solution

Using synthetic division

1 + i	1	-6	25	-34	26
	0	2+i	-6-4i	21+13i	-26
	1	-5 + i	17 - 4i	-3+ 13i	0

Since the last entry in the table is zero, 1 + I is a root.

The conjugate 1 – i is also a root

Let the other roots be α and β

Sum of roots = $1 + i + 1 - i + \alpha + \beta = 6$

$2 + \alpha + \beta = 6$

$\alpha + \beta = 4$

α = 4 – β ……………… (i)

Product of roots (1 + i)(1 – i)αβ = 26

2αβ = 26

Αβ = 13 ……………… (ii)

Substituting eqn.(i) into eqn. (ii)

(4 – β)β = 13

β^2 - 4β + 13 = 0

$\beta = \frac{4 \pm \sqrt{16-4(1)(13)}}{2(1)} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2}$

= 2 ± 3i

Hence the other roots are 2 + 3i and 2 – 3i

Revision exercise 2

1. Given Z_1 = 7 – 4i and Z_2 = -1 + 3i, express in the modulus-argument form the following
   (a) Z_1Z_2
   {r = 25.45 [cos(0.437π) + isin(0.437π)]}
   (b) $\left|\frac{Z_1}{Z_2}\right|$
   $\{p = 2.55[cos(1.233\pi) + isin(1.233\pi)]$
2. Find the modulus and the principal arguments of
   (a) $\frac{1-i}{1+i}\left[1, \frac{-\pi}{3}\right]$
   (b) $\frac{-1-7i}{4+3i}\left[\sqrt{2}, \frac{-3\pi}{4}\right]$
   (c) $\frac{1+i}{2-i}\left[\frac{\sqrt{10}}{5}, 125\right]$
   (d) $\frac{(3+i)^2}{1-i}\left[2\sqrt{5}, 143\right]$
3. If z_1 and z_2 are complex numbers, solve the simultaneous equation, giving your answer in the form a + bi.
   $4z_1 + 3z_2$= 23
   $z_1 + iz_2$ = 6 [2 +3i, 5 – 4i]
4. Given that 2 + I is a root of the equation z^3 – 11z + 20 = 0. Find the remaining roots [2 – i, -4]
5. Show that 2+ i is a root of the equation $2z^3 - 9z^2$ + 14z – 5 = 0. Hence find other roots [2- i, ½]
6. Find equation whose roots are -1 ± i, where i = $\sqrt{-1}$ [z^2 +2z + 2 = 0]
7. Show that z = 1 is a root of the equation $z^3 - 5z^2$ + 9z – 5 = 0 [2 – i, 2 + i]
8. The complex number satisfies $\frac{z}{z+2} = 2 - i$ find the real and imaginary parts of z, and the modulus and argument of z.
   [*Re*(z) = -3,*Im*(z) = -1, $\sqrt{10}$, -2.82rads]
9. One root of the equation z2 + az + b = 0 where a and b are real constants is 2 + 3i. Find the values of a and b [a = -4, b = 13]
10. If z_1 = 2 $\left(cos\frac{2\pi}{3} + isin\frac{2\pi}{3}\right)$ and
    z_2 =6 $\left(cos\left(\frac{-3\pi}{4}\right) + isin\left(\frac{-3\pi}{4}\right)\right)$ find
    (a) $\left|\frac{z_1}{z_2}\right|\left[\frac{1}{3}\right]$
    (b) $arg\left(\frac{z_1}{z_2}\right)\left[\frac{17\pi}{12}\right]$
    (c) $\left|\frac{z_2}{z_1}\right|$[3]
    (d) $arg\left(\frac{z_2}{z_1}\right)\left[\frac{-17\pi}{12}\right]$
11. If z1 and z2 are two complex numbers such that $|z_1 - z_2| = |z_1 + z_2|$. Show that the difference of their arguments is $\frac{\pi}{2}$ or $\frac{3\pi}{2}$.
12. Find the modulus and argument of $\frac{(2-i)^2(3i-1}{i+3}$ [5, 0.6435 rad]
13. If $z_1 = \frac{1+7i}{1-i}$ and $z_2 = \frac{17-7i}{2+2i}$ find the moduli of
    (a) z_1 [5]
    (b) z_2 [6.5]
    (c) $z_1 + z_2$ [2.061]
    (d) z_1z_2 [32.5]

Demoivre's theorem

This states that for all rational integers

r(cosθ + isinθ)^n = r(cos*n*θ + isin*n*θ)

The theorem can be proved by induction approach or otherwise

By induction

Given that (cosθ+ isinθ)^n = cos*n*θ + isin*n*θ

Suppose n = 1

cosθ+ isinθ = cos*1*θ + isin*1*θ

⇨ cosθ+ isinθ = cosθ+ isinθ i.e. the proof holds for n = 1

Suppose n = k

$(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

For n = k + 1

$(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)^1$

$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$

$= \cos k\theta\cos\theta + i\cos k\theta\sin\theta + i\sin k\theta\cos\theta + i^2\sin k\theta\sin\theta$

$= \cos k\theta\cos\theta - \sin k\theta\sin\theta + i(\cos k\theta\sin\theta + \sin k\theta\cos\theta)$

$= \cos(k+1)\theta + i\sin(k+1)\theta$

which is true when n = k + 1

Therefore the proof holds for all values of n

Using otherwise

Given $z = \cos\theta + i\sin\theta$

$Z^2 = (\cos\theta + i\sin\theta)^2$

$= \cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta$

$= \cos^2\theta - \sin^2\theta + i2\sin\theta\cos\theta$

$= \cos 2\theta + i\sin 2\theta$

$Z^3 = (\cos 2\theta + i\sin 2\theta)^2(\cos\theta + i\sin\theta)$

$= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta + i(\sin 2\theta\cos\theta + \cos 2\theta\sin\theta)$

$= \cos 3\theta + i\sin 3\theta$

⇨ $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ hold for all positive values of n

Also, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ holds for fractional and negative indices

(a) $z^{-1} = \frac{1}{Z} = \frac{1}{\cos\theta + i\sin\theta}$

$= \left(\frac{1}{\cos\theta + i\sin\theta}\right)\left(\frac{(\cos\theta - i\sin\theta)}{(\cos\theta - i\sin\theta)}\right)$

$= \frac{(\cos\theta - i\sin\theta)}{(\cos^2\theta - i^1\sin^2\theta)} = \cos\theta - i\sin\theta$

⇨ $z^{-1} = \cos(-1\theta) + i\sin(-1\theta)$

(b) suppose n = -m

$z^{-m} = \frac{1}{Z^m} = \frac{1}{(\cos\theta + i\sin\theta)^m}$

$= \left(\frac{1}{\cos m\theta + i\sin m\theta}\right)\left(\frac{(\cos m\theta - i\sin m\theta)}{(\cos m\theta - i\sin m\theta)}\right)$

⇨ $(\cos\theta + i\sin\theta)^{-m}$

$= \cos(-m\theta) + i\sin(-m\theta)$

Hence the theorem holds for all values of n.

Applications of Demoivre's theorem

It is mainly employed in proving trigonometric functions and finding roots of complex function.

Proving identities

Example 19

By using Demoivre's theorem, show that

(a) $\tan 5x = \frac{5\tan x - 10\tan^3 x + \tan^5 x}{1 - 10\tan^2 x + \tan 4x}$

Solution

$\cos 5x + i\sin 5x = (\cos x + \sin x)5$

$= \cos^5 x + 5i\sin x\cos^4 x - 10\sin^2 x\cos^3 x - 10i\sin^3 x\cos^2 x + \sin^4 x\cos x + i\sin^5 x$

Equating real parts

$\cos 5x = \cos^5 x - 10\cos^3 x\sin^2 x + \cos x\sin^4 x$

Equating imaginary parts

$\sin 5x = 5\cos^4 x\sin x - 10\cos^2 x\sin^3 x + \sin^5 x$

$\tan 5x = \frac{\sin 5x}{\cos 5x} = \frac{5\cos^4 x\sin x - 10\cos^2 x\sin^2 x + \sin^5 x}{\cos^5 x - 10\cos^3 x\sin^2 x + \cos x\sin^4 x}$

Dividing terms of the R.H.S by $\cos^5 x$

$\tan 5x = \frac{5\tan x - 10\tan^3 x + \tan^5 x}{1 - 10\tan^2 x + \tan 4x}$

(b) $\frac{\cos 3x + i\sin 3x}{\cos 5x + i\sin 5x} = \cos 8x + i\sin 8x$

Solution

$\frac{\cos 3x + i\sin 3x}{\cos 5x + i\sin 5x} = \frac{(\cos x + i\sin x)^3}{(\cos x + i\sin x)^5}$

$= (\cos x + i\sin x)^{3-(-5)}$

$= (\cos x + i\sin x)^8$

$= \cos 8x + i\sin 8x$

(c) $\frac{(\cos x + i\sin x)(\cos 2x + i\sin 2x)}{\cos\frac{x}{2} + i\sin\frac{x}{2}} = \cos\frac{5x}{2} + i\sin\frac{5x}{2}$

Solution

$\frac{(\cos x + i\sin x)(\cos 2x + i\sin 2x)}{\cos\frac{x}{2} + i\sin\frac{x}{2}}$

$= \frac{(\cos x + i\sin x)^1(\cos x + i\sin x)^2}{(\cos x + i\sin x)^{\frac{1}{2}}}$

$=(cosx + isinx)^{1+2-\frac{1}{2}}$

$=(cosx + isinx)^{\frac{5}{2}} = cos\frac{5x}{2} + isin\frac{5x}{2}$

(d) $16\sin^5 x$ = sin5x -5sin3x + 10sinx

Given Z = cosx + isinx

$\bar{Z}$ = cosx – isinx

(Z – $\bar{Z}$) = 2isinx

This means that $(Z - \bar{Z})^n$ = 2isin*n*x

$(Z - Z^*)^5$ = 2isin5x = $32i\sin^5 x$

$(Z - \bar{Z})^5$

= $Z^5 - 5Z^4\bar{Z} + 10Z^3\bar{Z}^2 + 10Z^2(-\bar{Z})^3 + 5Z\bar{Z}^4 - \bar{Z}^5$

= $Z^5 - 5Z^3 + 10Z - 10(\bar{Z}) + 5\bar{Z}^3 - \bar{Z}^5$

= $[Z^5 - \bar{Z}^5] - 5[Z^3 - \bar{Z}^3] + 10[Z - \bar{Z}]$

= 2isin5x – 10isin3x + 20isinx

⇨ $32i\sin^5 x$ = 2isin5x – 10isin3x + 20isinx

∴$16\sin^5 x$ = sin5x -5sin3x + 10sinx

Finding roots

Suppose that z = r(cosθ + isinθ)

Then $z^n = r^n(\cos n\theta + i\sin n\theta)$

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(cos\frac{1}{n}\theta + isin\frac{1}{n}\theta\right)$$

The general expression for finding the roots of a complex function is given by

$z = r[cos(\theta + 2\pi k) + isin(\theta + 2\pi k)]$ where k = 0, 1, 2, …k and θ is usually in radians.

Now for n^{th} root, we have

$z^n = r^n[cosn(\theta + 2\pi k) + isin\,n(\theta + 2\pi k)]$

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(cos\frac{1}{n}(\theta + 2\pi k) + isin\frac{1}{n}(\theta + 2\pi k)\right)$$

Example 20

Use Demoivre's theorem to find the square root of

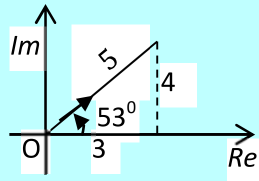
(a) 3 + 4i

Solution

$|z| = \sqrt{(9 + 16} = 5$

$\theta = \tan^{-1}\frac{4}{3} = 53.1^0 = 0.295\pi$

The principal angle lies in the first quadrant



z = $5[cos(0.295\pi + 2\pi k) + isin(0.295\pi + 2\pi k)]$

$z^{\frac{1}{2}}$

= $5^{\frac{1}{2}}\left[cos\left(\frac{0.295\pi + 2\pi k}{2}\right) + isin\left(\frac{0.295\pi + 2\pi k}{2}\right)\right]$

Substituting for k = 0

$z_1^{\frac{1}{2}} = 5^{\frac{1}{2}}\left[cos\left(\frac{0.295\pi}{2}\right) + isin\left(\frac{0.295\pi}{2}\right)\right]$= -2 - i

$\therefore \sqrt{3 + 4i} = 2$±i

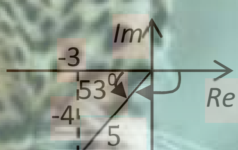
(b) -3 – 4i

Solution

$|z| = \sqrt{(9 + 16} = 5$

$\theta = \tan^{-1}\frac{-4}{-3} = -127^0 = -0.7\pi$

The principal angle lies in the third quadrant



z = $5[cos(0.295\pi + 2\pi k) + isin(0.295\pi + 2\pi k)]$

$z^{\frac{1}{2}} = 5^{\frac{1}{2}}\left[cos\left(\frac{0.295\pi + 2\pi k}{2}\right) + isin\left(\frac{0.295\pi + 2\pi k}{2}\right)\right]$

Substituting for k = 0

$z_1^{\frac{1}{2}} = 5^{\frac{1}{2}}\left[cos\left(\frac{0.295\pi}{2}\right) + isin\left(\frac{0.295\pi}{2}\right)\right]$= -2 - i

Substituting for k = 1

$z_2 = 5^{\frac{1}{2}}\left[cos\left(\frac{0.295\pi + 2\pi}{2}\right) + isin\left(\frac{0.295\pi + 2\pi}{2}\right)\right]$

= ±(1-2i)

$\therefore \sqrt{-3 - 4i}$ = ±(1 – 2i)

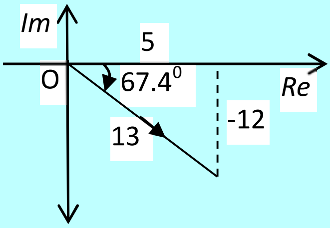
(c) 5 – 12i

Solution

$|z| = \sqrt{(25 + 144} = 13$

$\theta = \tan^{-1}\frac{-12}{5} = -67.4^0 = -0.374\pi$

The principal angle lies in the third quadrant



z = $13[cos(-0.374\pi + 2\pi k) + isin(-0.374\pi + 2\pi k)]$

$z^{\frac{1}{2}} = 13^{\frac{1}{2}}\left[cos\left(\frac{-0.374\pi+2\pi k}{2}\right) + isin\left(\frac{-0.374\pi+2\pi k}{2}\right)\right]$

Substituting for k = 0

$z_1 = 13^{\frac{1}{2}}\left[cos\left(\frac{-0.374\pi}{2}\right) + isin\left(\frac{-0.374\pi}{2}\right)\right]$

= 3 - 2i

Substituting for k = 1

$z_2 =$
$13^{\frac{1}{2}}\left[cos\left(\frac{-0.374\pi+2\pi}{2}\right) + isin\left(\frac{-0.374\pi+2\pi}{2}\right)\right]$

= -3 + 2i

$\therefore \sqrt{5 - 12i} = \pm(3 - 2i)$

Example 21

(a) Use roots 1, α and α^2 of unity. Hence show that $1 + \alpha + \alpha^2 = 0$

Solution

Let $\sqrt[1]{1} = z$

z = $1^{\frac{1}{3}} = (cos0^0 + isin0^0)$ (In mod-Arg form

$=[cos(0 + 2\pi k) + isin\ (0 + 2\pi k)]^{\frac{1}{3}}$, k = 0,1,2

$= cos(\frac{2\pi k}{3}) + isin\ (\frac{2\pi k}{3})$

When k = 0

$z_1 = cos0^0 + isin\ 0^0 = 1$

When k = 1

$z_2 = cos(\frac{2\pi}{3}) + isin\ (\frac{2\pi}{3}) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

When k = 2

$z_3 = cos(\frac{4\pi}{3}) + isin\ (\frac{4\pi}{3}) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

∴the cube roots are 1, $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

Using otherwise

Taking

z = $1^{\frac{1}{3}}$, then $z^3 - 1 = 0$

Let f(z) = $z^3 - 1$

f(1) = $1^3 - 1 = 0$

∴ z – 1 is a factor.

Using long division

$$\begin{array}{r} z^2 + z + 1 \\ z-1\,\overline{)\, z^3 - 1} \\ \underline{z^3 - z^2} \\ z^2 - 1 \\ \underline{z^2 - 1} \end{array}$$

$z^3 - 1 = (z - 1)(z^2 + z + 1)$

Either z – 1 = 0, z = 1

Or $z^2 + z + 1 = 0$

$z = \frac{-1\pm\sqrt{1^2-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

∴the cube roots are 1, $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

Hence

Letting α = $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$, the

$\alpha^2=\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

Letting α = $-\frac{1}{2} - i\frac{\sqrt{3}}{2}$, the

$\alpha^2=\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

Hence the roots can be written as 1, $\alpha + \alpha^2$

Adding these roots

$1 - \frac{1}{2} - i\frac{\sqrt{3}}{2} - \frac{1}{2} + i\frac{\sqrt{3}}{2} = 0$

(b) Evaluate $(-16)^{\frac{1}{4}}$

Solution

Let z = $(-16)^{\frac{1}{4}} = 16^{\frac{1}{4}}(-1)^{\frac{1}{4}}$ = $2(-1)^{\frac{1}{4}}$

$= [cos180^0 + isin180^0]^{\frac{1}{4}}$

=2 $\left[cos\left(\frac{\pi+2\pi k}{4}\right) + isin\left(\frac{\pi+2\pi k}{4}\right)\right]$, k = 0, 1, 2, 3

When k = 0

z_1 = $[cos45^0 + isin45^0]$=$\sqrt{2} + i\sqrt{2}$

When k = 1

Z_2 = $[cos135^0 + isin135^0]$=-$\sqrt{2} + i\sqrt{2}$

When k = 2

Z_3 = $[cos225^0 + isin225^0]$=$\sqrt{2} - i\sqrt{2}$

When k = 3

Z_3 = $[cos315^0 + isin315^0]$=-$\sqrt{2} - i\sqrt{2}$

$\therefore (-16)^{\frac{1}{4}} = \pm(\sqrt{2} \pm i\sqrt{2})$

Example 23

Evaluate $(1+\sqrt{3}i)^{\frac{2}{3}}$

Solution

$(1+\sqrt{3}i)^{\frac{2}{3}} = \left[(1+\sqrt{3}i)^2\right]^{\frac{1}{3}}$

$= [(1+\sqrt{3}i)(1+\sqrt{3}i)]^{\frac{1}{3}}$

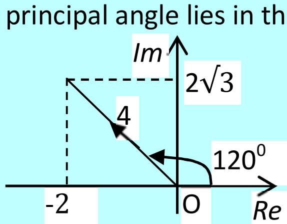
$= [-2+2\sqrt{3}i]^{\frac{1}{3}}$

Let z = -2 + 2$\sqrt{3}$i

$|z| = \sqrt{(-2)^2 + (2\sqrt{3})^2} = 4$

Arg(z) = $\tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = -60^0 = 120^0 = \frac{2\pi}{3}$

The principal angle lies in the second quadrant



$z = 4\left[cos\left(\frac{2\pi}{3} + 2\pi k\right) + isin\left(\frac{2\pi}{3} + 2\pi k\right)\right]$

$z^{\frac{1}{3}} = 4^{\frac{1}{3}}\left[cos\left(\frac{2\pi+6\pi k}{9}\right) + isin\left(\frac{2\pi+6\pi k}{9}\right)\right]$

When k = 0

$z_1 = 4^{\frac{1}{3}}\left[cos\left(\frac{2\pi}{9}\right) + isin\left(\frac{2\pi}{9}\right)\right]$

=1.2160 +1.0204i

When k = 1

$z_2 = 4^{\frac{1}{3}}\left[cos\left(\frac{8\pi}{9}\right) + isin\left(\frac{8\pi}{9}\right)\right]$

= -1.4917 + 0.5429i

When k = 2

$z_3 = 4^{\frac{1}{3}}\left[cos\left(\frac{14\pi}{9}\right) + isin\left(\frac{14\pi}{9}\right)\right]$

= 0.2756 – 1.5633i

$\therefore (1+\sqrt{3}i)^{\frac{2}{3}}$ =1.2160 +1.0204i, -1.4917 + 0.5429i, 0.2756 – 1.5633i

Example 24

Find the root of $z^4 + 4 = 0$ using Demoivre's theorem

$z^4 = -4 = 4(-1 + 0i)$

arg(z) =$\tan^{-1}\left(\frac{0}{-1}\right) = 180^0 = \pi$

⇨ $z^4 = 4[cos180^0 + sin180^0]$

$z = 4^{\frac{1}{4}}\left[cos\left(\frac{\pi+2\pi k}{4}\right) + sin\left(\frac{\pi+2\pi k}{4}\right)\right]$

When k = 0

$z_1 = 4^{\frac{1}{4}}\left[cos\left(\frac{\pi}{4}\right) + sin\left(\frac{\pi}{4}\right)\right]$

$= \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = 1 + i$

When k = 1

$z_2 = 4^{\frac{1}{4}}\left[cos\left(\frac{3\pi}{4}\right) + sin\left(\frac{3\pi}{4}\right)\right]$

$= \sqrt{2}\left(\frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = -1 + i$

When k = 2

$$z_5 = 4^{\frac{1}{4}}\left[cos\left(\frac{5\pi}{4}\right) + sin\left(\frac{5\pi}{4}\right)\right]$$

$$= \sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = -1 - i$$

When k = 3

$$z_4 = 4^{\frac{1}{4}}\left[cos\left(\frac{7\pi}{4}\right) + sin\left(\frac{7\pi}{4}\right)\right]$$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = 1 - i$$

∴ the roots of $z^4 + 4 = 0$ are 1 ± i and -1 ± i

The locus of a complex number

This is an equation of a set of point representing a variable complex number or a path traced. It may be in form of an equation of a circle or a straight line.

Example 25

Determine the locus of z if

(a) $|z - 4| = 3$
(b) Arg(z+2) = $\frac{\pi}{6}$

Solution

Let z = x + iy

(a) $|x + iy - 4| = 3$
$|(x - 4) + iy| = 3$
$(x - 4)^2 + y^2 = 9$
$x^2 + y^2 - 8x + 7 = 0$
The locus of z is a circle with centre (4, 0)

(b) Arg(x + iy +2) = $\frac{\pi}{6}$

Arg((x+2) + iy) = $\frac{\pi}{6}$

$\tan^{-1}\left(\frac{y}{x+2}\right) = \frac{\pi}{6}$

$\frac{y}{x+2} = tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$

$\frac{y}{x+2} = \frac{\sqrt{3}}{3}$

$y = \frac{\sqrt{3}}{3}(x + 2)$

The locus of z is a straight line

Example 26

Determine the locus of z if:

(a) $|z| = 2$ and sketch it
(b) $\arg(z - 1) = \frac{\pi}{4}$ and sketch it
(c) $2|z - 1| = |z + i|$
(d) $\arg\left(\frac{z-i}{z+1}\right) = \frac{\pi}{3}$

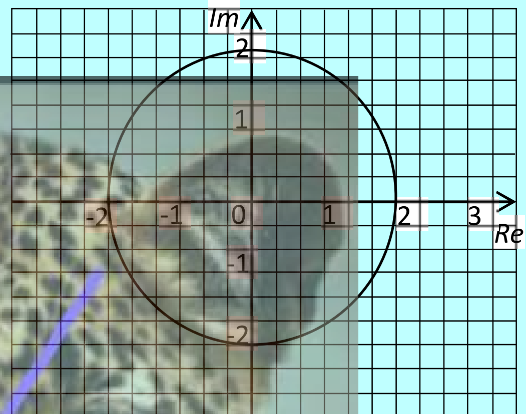
Solution

Let z = x + iy

(a) $|x + iy| = 2$ i.e. $\sqrt{x^2 + y^2} = 2$

$x^2 + y^2 = 4$ *or* $(x - 0)^2 + (y - 0)^2 = 2^2$

The locus of z is a circle with centre (0, 0) and radius 2units



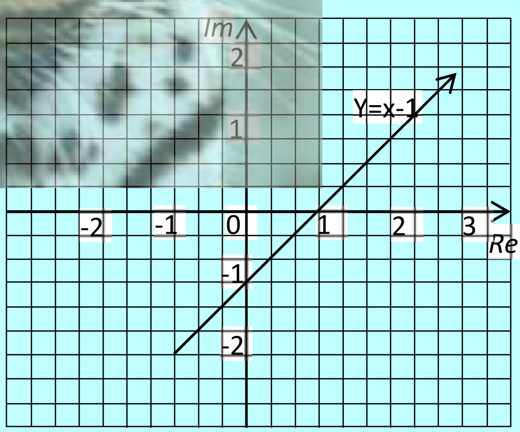
(b) Arg[(x – 1) + iy] = $\frac{\pi}{4}$

$\tan^{-1}\left(\frac{y}{x-1}\right) = \frac{\pi}{4}$

$\frac{y}{x-1} = tan\frac{\pi}{4} = 1$

$y = x - 1$

The locus is a straight line



(c) $2|(x - 1) + iy| = |x + i(y + 1)|$

$$\left[2\sqrt{(x-1)^2 + y^2}\right]^2 = \left[\sqrt{x^2 + (y+1)^2}\right]^2$$

$3x^2 + 3y^2 - 8x - 2y + 3 = 0$

$x^2 + y^2 - \frac{8}{3}x - \frac{2}{3}y + 1 = 0$

$= \left(x - \frac{4}{3}\right)^2 + \left(y - \frac{1}{3}\right)^2 = -1 + \frac{16}{9} + \frac{1}{9}$

$= \left(x - \frac{4}{3}\right)^2 + \left(y - \frac{1}{3}\right)^2 = \frac{8}{9}$

The locus of Z is a circle with the centre $\left(\frac{4}{3}, \frac{1}{3}\right)$ and radius $\sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$

(d) $\arg\left(\frac{x+iy-i}{x+iy+1}\right) = \frac{\pi}{3}$

$\arg\left(\frac{x+i(y-1)}{(x+1)+iy}\right) = \frac{\pi}{3}$

$\arg[x + i(y-1)] - \arg[x+1) + iy] = \frac{\pi}{3}$

$\tan^{-1}\left(\frac{y-1}{x}\right) - \tan^{-1}\left(\frac{y}{x+1}\right) = \frac{\pi}{3}$

$\frac{\left(\frac{y-1}{x}\right) - \left(\frac{y}{x+1}\right)}{1 + \left(\frac{y-1}{x}\right)\left(\frac{y}{x+1}\right)} = \sqrt{3}$

$\frac{y-x-1}{x^2+y^2+x-y} = \sqrt{3}$

$\sqrt{3}x^2 + \sqrt{3}y^2 + (\sqrt{3}+1)x - (\sqrt{3}+1)y + \frac{\sqrt{3}}{3} = 0$

$x^2 + y^2 + \left(\frac{3+\sqrt{3}}{3}\right)x - \left(\frac{3+\sqrt{3}}{3}\right)y + \frac{\sqrt{3}}{3} = 0$

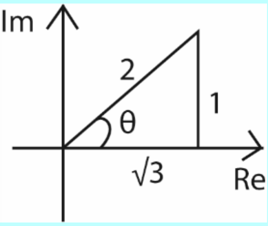
The locus is a circle with centre at $\left[-\frac{1}{6}(3+\sqrt{3}) - \frac{1}{6}(3+\sqrt{3})\right]$ and radius $\sqrt{\frac{2}{3}}$

Example 27

The complex number $z = \sqrt{3} + i$.
$\bar{Z}$ *is the conjugate of* Z.

(a) Express Z in the modulus argument form

$|z| = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = \sqrt{3+1} = 2$



$\tan\theta = \frac{1}{\sqrt{3}}$

$\theta = \text{Arg}(z) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

Hence $z = 2\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$

or $z = 2[\cos 30^0 + i\sin 30^0]$

(i) On the same Argand diagram plot $\bar{Z}$ and $2\bar{Z} + 3i$

$\bar{z} = \sqrt{3} - i$

$|z| = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2} = \sqrt{3+1} = 2$

$2\bar{Z} + 3i = 2(\sqrt{3} - i) + 3i = 2\sqrt{3} + i$

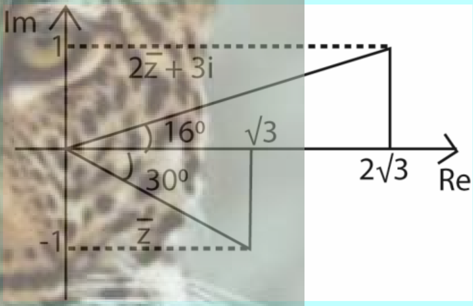
$|2\bar{Z} + 3i| = \sqrt{\left(2\sqrt{3}\right)^2 + 1^2} = \sqrt{13}$

Finding $\arg(\bar{z})$:

$\text{Arg}(\bar{z}) = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -30^0$

Finding $\arg(2\bar{Z} + 3i)$:

$\text{Arg}(2\bar{Z} + 3i) = \tan^{-1}\left(\frac{1}{2\sqrt{3}}\right) = 16.2^0$



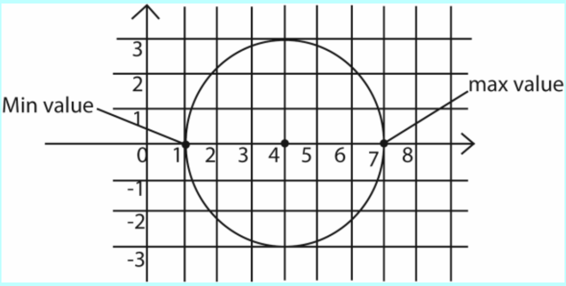
(b) What are the greatest and least values of $|Z|$ *if* $|Z - 4| \leq 3$?

$|z - 2| \leq 3$

$|x - 4 + iy| \leq 3$

$(x-4)^2 + y^2 \leq 3$

This is an equation of the circle with centre (4, 0) and radius 3.



Greatest value of $|z| = 4 + 3 = 7$

Lowest value of $|z| = 4 - 3 = 1$

Example 28

(a) Given that the complex number Z and its conjugate $\bar{Z}$ satisfy the equation

$$Z\bar{Z} + 2iZ = 12 + 6i. \text{ Find Z.}$$

Let z = x + iy then $\bar{z} = x - iy$

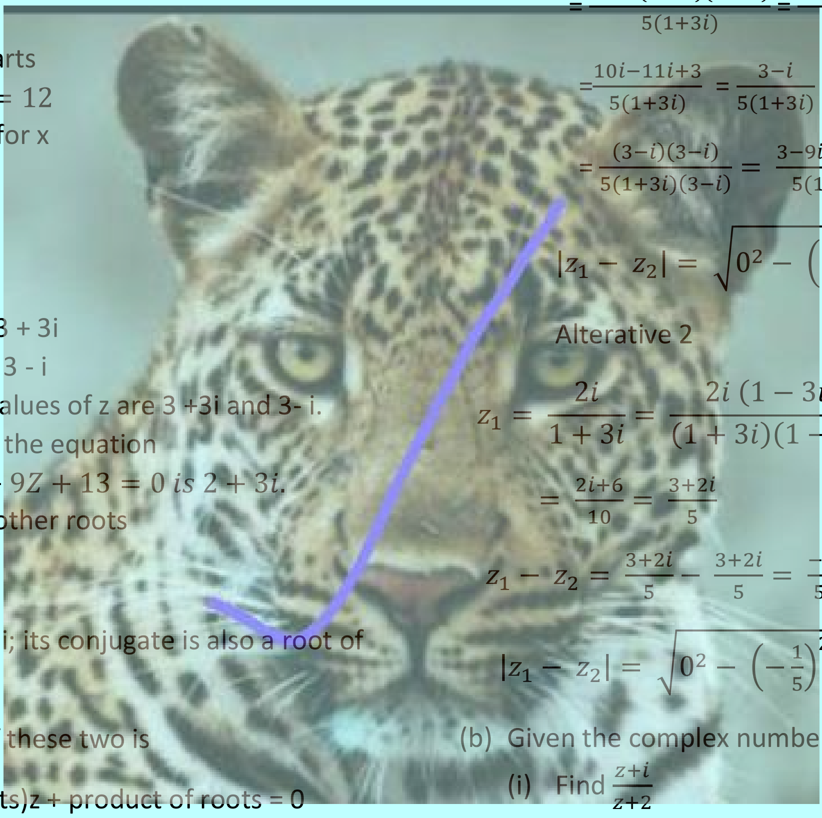
(x + iy)(x – iy) +2i(x + iy) = 12 + 6i

$x^2 + y^2 - 2y + 2xi = 12 + 6i$

Equating imaginary parts

2xi = 6i

x = 3



Equating real parts

$x^2 + y^2 - 2y = 12$

By substituting for x

9 + y^2 -2y = 12

y^2 -2y -3 = 0

(y – 3)(y+1) = 0

y = 3 or y = -1

when y = 3; z = 3 + 3i

when y = -1; z = 3 - i

∴ the possible values of z are 3 +3i and 3- i.

(b) One root of the equation $Z^3 - 3Z^2 + 9Z + 13 = 0$ *is* $2 + 3i$. Determine other roots

Solution

Given root 2 + 3i; its conjugate is also a root of the equation

The equation of these two is

z^2 – (sum of roots)z + product of roots = 0

Sum = 2 + 3i + 2 – 3i = 4

Product of roots =(2 + 3i)(2 – 3i) = 2 + 9 = 13

$$z^2 - 4z + 13 = 0$$

$$\begin{array}{r|l} & z+1 \\ \hline z^2 - 4z + 13 & z^3 - 3z^2 - 9z + 13 \\ & -z^3 - 4z^2 - 13z \\ \hline & z^2 - 4z + 13 \\ & -z^2 - 4z + 13 \\ \hline \end{array}$$

$$(z^2 - 4z + 13)(z + 1) = 0$$

z + 1= 0

z = -1

So the roots are 2±3i and -1

Example 29

(a) If $z_1 = \frac{2i}{1+3i}$ and $z_2 = \frac{3+2i}{5}$, find $|z_1 - z_2|$

Solution

$$z_1 - z_2 = \frac{2i}{1+3i} - \frac{3+2i}{5},$$

$$= \frac{10i-(1+3i)(3+2i)}{5(1+3i)} = \frac{10i-[3+2i+9i-6]}{5(1+3i)}$$

$$= \frac{10i-11i+3}{5(1+3i)} = \frac{3-i}{5(1+3i)}$$

$$= \frac{(3-i)(3-i)}{5(1+3i)(3-i)} = \frac{3-9i-i-3}{5(1+9)} = \frac{-i}{5}$$

$$|z_1 - z_2| = \sqrt{0^2 - \left(-\frac{1}{5}\right)^2} = \frac{1}{5}$$

Alterative 2

$$z_1 = \frac{2i}{1+3i} = \frac{2i\,(1-3i)}{(1+3i)(1-3i)} = \frac{2i+6}{1+9}$$

$$= \frac{2i+6}{10} = \frac{3+2i}{5}$$

$$z_1 - z_2 = \frac{3+2i}{5} - \frac{3+2i}{5} = \frac{-i}{5}$$

$$|z_1 - z_2| = \sqrt{0^2 - \left(-\frac{1}{5}\right)^2} = \frac{1}{5}$$

(b) Given the complex number z = x + iy

(i) Find $\frac{z+i}{z+2}$

$$\frac{z+i}{z+2} = \frac{x+i(1+y)}{(x+2)+iy} = \frac{[x+i(1+y)][(x+2)-iy]}{[(x+2)+iy][(x+2)-iy]}$$

$$= \frac{x[(x+2)-iy]-i(1+y)[(x+2)-iy]}{(x+2)^2+y^2}$$

$$= \frac{x^2+2x-ixy+i(x+2+xy+2y)+y+y^2}{(x+2)^2+y^2}$$

$$= \frac{x^2+2x+y^2+y+i(2+x+2y)}{(x+2)^2+y^2}$$

(ii) Show that the locus of $\frac{z+i}{z+2}$ is a straight line when its imaginary part is zero. State the gradient of the line.

If imaginary part is zero

$(2 + x + 2y) = 0$

$2y = -x - 2$

y = $-\frac{1}{2}x$ + 1

Comparing with y = mx + c

The gradient = $-\frac{1}{2}$

Example 30

(a) Given that a complex number Z* and its conjugate satisfy the equation

ZZ* + 2iZ = 12 + 6i, find the possible values of Z.

Solution

Let z* = x + iy, then z = x –iy

(x + iy)(x – iy) + 2i(x – iy) + 6i = 12 + 6i

$x^2 + y^2 + 2y + 2xi = 12 + 6i$

Equating real parts

$x^2 + y^2 + 2y = 12$

Equating imaginary parts

2x = 6 => x = 3

$9 + y^2 + 2y = 12$

$y^2 + 2y – 3 = 0$

(y – 1)(y + 3) = 0 => y = 1 or y = -3

When y = -3; z = 3 + 3i and

When y = 1, z = 3 –i

The possible values of z are 3 + 3i and 3 - i

(b) Find the Cartesian equation, in its simplest form of the curve described by $|z - 3 + 6i| = 2|z|$ where z is the complex number x + iy.

Hence sketch an Argand diagram the region satisfying $|z - 3 + 6i| \leq 2|z|$

Solution

Substituting z = x = iy

$|(x - 3) + i(y - 6)| = 2|x + iy|$

$(x - 3)^2 + (y + 6)^2 = 4(x^2 + y^2)$

$x^2 + y^2 – 6x + 12y + 45 = 4x^2 + 4y^2$

$x^2 + y^2 + 2x – 4y – 15 = 0$

Hence sketching the region satisfying $|z - 3 + 6i| \leq 2|z|$

i.e. $x^2 + y^2 + 2x – 4y – 15 \geq 0$

substituting (0, 0)

LHS = - 15 ; RHS = 0

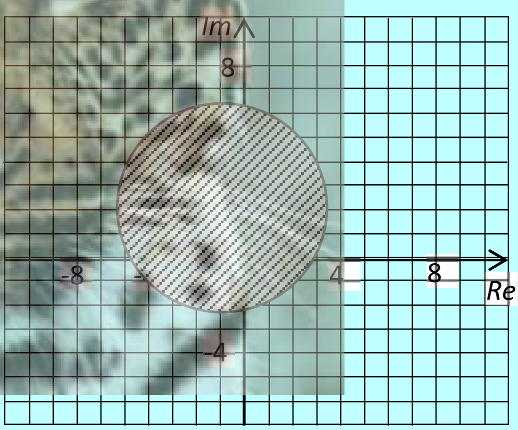
Since LHS< RHS, the point (0, 0) does not lie in the region $|z - 3 + 6i| \leq 2|z|$ therefore it lies in unwanted region.

Now $x^2 + y^2 + 2x – 4y – 15 \geq 0$

$= (x + 1)^2(y - 2)^2 \geq 20$

The locus is a circle centre (-1, 2) and radius $\sqrt{20}$

The plot of $|z - 3 + 6i| \leq 2|z|$



Example 31

If n is a variable and z = 4n + 3i(1 – n), show that the locus of Z is a straight line. Determine the minimum value of $|z|$

Solution

Let z = x = iy

⇨ x + iy = 4n +3i(1 – n)

Equating real parts

x = 4n => n = $\frac{x}{4}$

Equating imaginary parts

y = 3(1 – n)

⇨ y = $3\left(1-\frac{x}{4}\right)$

y =$-\frac{3x}{4}+3$ which is a straight line with gradient$-\frac{3}{4}$ and intercept 3

Given z = 4n + 3i(1 – n)

$|z|^2 = 16n^2 + 9(1-n)^2$

$|z|^2 = 25n^2 - 18n + 9$

The minimum value can be obtained by completing squares or differentiation

By completing squares

$$|z|^2 = 25(n^2 - \frac{18}{25}n + \frac{9}{25})$$

$$= 25\left[\left(n - \frac{9}{25}\right)^2 + \frac{9}{25} - \frac{81}{625}\right]$$

$$= 25\left[\left(n - \frac{9}{25}\right)^2 + \frac{144}{625}\right]$$

$$= 25\left(n - \frac{9}{25}\right)^2 + \frac{144}{25}$$

$|z|^2$ is minimum when $n = \frac{9}{25}$

$|z|^2{}_{min}$ = $\frac{144}{25}$

⇨ $|z|_{min} = \frac{12}{5}$

By differentiation

$$2|z|\left|\frac{dz}{dn}\right| = 50n - 18 = 0$$

n = $\frac{18}{50} = \frac{9}{25}$

$$|z_{min}|^2 = 25\left(\frac{9}{25}\right)^2 - 18\left(\frac{9}{25}\right) + 9 = \frac{3600}{625}$$

⇨ $|z|_{min} = \frac{60}{25} = \frac{12}{5}$

Example 32

Show that the locus of z if $\frac{z-3}{z+i}$ is wholly imaginary is a circle with centre at $\frac{3}{2}-\frac{1}{2}i$ and radius $\frac{1}{2}\sqrt{10}$

Solution

Let z = x + iy

$$\frac{z-3}{z+i} = \frac{(x-3+iy)}{(x+i(y+1))}\cdot\frac{(x-i(y+1))}{(x-i(y+1))}$$

$$= \frac{x(x-3)+y(y+1)+i[(xy-(x-3)(y+1)}{x^2+(y+1)^2}$$

If the above expression is wholly imaginary, then the real part must be zero

i.e. $\frac{x(x-3)+y(y+1)}{x^2+(y+1)^2} = 0$

=> $x^2 + y^2 - 3x + y = 0$

$$\left(x-\frac{3}{2}\right)^2 + \left(y+\frac{1}{2}\right)^2 = \frac{9}{4}+\frac{1}{4} = \frac{10}{4}$$

The locus is a circle with centre with $\frac{3}{2}-\frac{1}{2}i$ and radius $\frac{1}{2}\sqrt{10}$

Revision exercise 4

1. Use Demoivre's theorem to show that
(a) cos6x = $\cos^6$x -15$\cos^4$xsinx + 15coxsin^{4}x – sin^{6}x
(b) tan3x = $\frac{3tanx-tan^3x}{1-3tan^2x}$
2. (a) Evaluate $(1+i\sqrt{3})^{\frac{2}{3}}$
[1.2160 + 1.0204i, -1.4917 +0.5429i]
(b) Find the roots of $z^4 + 4 = 0$ using Demoivre's theorem
[1± i, -1± i]
3. (a) Given that the complex number Z and its conjugate $\overline{Z}$ satisfy the equation
$Z\overline{Z} - 2Z + 2\overline{Z} = 5 - 4i$. Find possible values of Z. (06marks)
[Z = ±2 + 1]
(b) Prove that if $\frac{Z-6i}{Z+8}$ is real, then the locus of the point representing the complex number Z is a straight line. (06marks) [y = $\frac{3}{4}x$ + 6]
4. (a)Express 5 + 12i in polar form
[5 + 12i = 13(cos 67.38^0 + isin67.38^0)]
Hence, evaluate$\sqrt[3]{5+12i}$, giving your answer in the form a + ib where a and b are corrected to two decimal places. (12marks)

[z = 2.17 + 0.90i, -1.86 + 1.43i, -0.31 – 2.33i]

5. (a) The total impedance z in an electric circuit with two branches z_1 and z_2 is given by $\frac{1}{z} = \frac{1}{z_1}+\frac{1}{z_2}$. Given that z_1= 3 + 4i and z_2

= 5 + 5i where i =$\sqrt{-1}$, calculate the total impedance z in form a + bi
$\left[z = \frac{275}{145} + i\frac{325}{145}\right]$

(b) If n is a variable and z = 4n + 3i(1 – n), show that the locus of Z is a straight line. Determine the minimum value of $|z|$ $\left[|z|_{min} = \frac{12}{5}\right]$

6. Given that z_1 = 3 + 4i and z_2 = -1 + 2i
   (a) Express z_1, z_2, $z_1 + z_2$ and $z_1 - z_2$ in the form r(cosθ + isinθ)
   [z_1= 5(scos53.1^0+ isin53.1^0),
   z_2 =$\sqrt{5}$(cos116.6^0 + isin116.6^0),
   $z_1 + z_2$ = 2$\sqrt{10}$(cos71.6^0 + isin71.6^0)
   $z_1 - z_2$ = 2$\sqrt{5}$(cos-153.4^0 + isin-153.4^0)]
   (b) Determine the angle between $z_1 + z_2$ and z_1- z_2 [225^0]
7. Solve the simultaneous equation
   $z_1 + z_2 = 8$
   $4z_1 - 3iz_2 = 26 + 8i$
   Using the values of z_1 and z_2, find the modulus and argument of $z_1 + z_2 - z_1z_2$.
   [z_1 = 8 +2i, z_2 = -2i; 6.49, 75.96^0]
8. (a) Given that z = 3 + 4i, find the value of expression z + $\frac{25}{z}$ [6]
   Given that $\left|\frac{z-1}{z+1}\right| = 2$, show that the locus of the complex number is
   $x^2 + y^2 + \frac{10x}{3} + 1 = 0$. Sketch the locus
   [The centre of the circle is $\left(\frac{-5}{3}, 0\right)$ it passes through -3 but does not touch the y-axis]
9. (a)(i) Express each of the following complex numbers in the form a + bi
   z1 = (1 –i)(1 + 2i) [3 + i]
   z2 = $\frac{2+6i}{3-i}$ [2i]
   z3 = $\frac{-4i}{1-i}$ [2 – 2i]
   (b) Find the modulus and argument of $z_1z_2z_3$ [17.889, 63.4349^0]
   (c) Find the square root of 12i – 5
   [1 + 3i or -2 – 3i]
10. (a) given that z = $\sqrt{3}$+ i, find the modulus and argument
   (i) z^2 $\left[4, \frac{\pi}{3}\right]$
   (ii) $\frac{1}{z}\left[\frac{1}{2}, -\frac{\pi}{6}\right]$
   (iii) Show in an Argand diagram the points representing complex numbers z, z^2 and $\frac{1}{z}$.
   (b) In an Argand diagram, P represents a complex number z such that $2|z-2| = |z-6i|$ show that P lies on a circle, find
   (i) The radius of this circle[4.2164]
   (ii) The complex number represented by it centre
   $\left[z = \frac{8}{3} - 2i\right]$
11. (a) given the complex number z_1 = 1 – i, z_2 = 7 + I represent z_1z_2 and $z_1 - z_2$ on the Argand diagram.
   Determine the modulus and argument of $\frac{z_1-z_2}{z_1z_2}$ [0.6325, -124.7^0]
   (b) If z is a complex number of form a + bi, solve $\left(\frac{z-1}{z+1}\right)^2 = i$ [z = 1 ±i$\sqrt{2}$]xz
12. (a) show the region represented by $|z-2+i| < 1$ on an Argand diagram
   [it is a circle of centre(2, -1) and radius 1 unit, the circle is drawn with dotted line since the equation is less than, and the unwanted region is the area outside the circle]
   (b) Express the complex number z = 1 - i$\sqrt{3}$ in modulus argument form and hence find z^2 and $\frac{1}{z}$ in the form a + bi
   $$\left[\begin{array}{c} z = 2\left(cos\left(-\frac{\pi}{3}\right) + isin\left(-\frac{\pi}{3}\right)\right), \\ z^2 = -2 - i2\sqrt{3}, \frac{1}{z} = \frac{1}{4} + i\frac{\sqrt{3}}{4} \end{array}\right]$$
13. (a)Given that z_1 = -i + 1, z_2 = 2 + i, and z_3 = 1 + 5i, represent z_2z_3, z_2-z_1 $\frac{1}{z_1}$and $\frac{z_2z_3}{z_2-z_1} + \frac{1}{z_1}$ o, Argand diagram
   (b) Prove that for positive integers $(cos\theta + sin\theta)^n = cosn\theta + sinn\theta$.
   Deduce that this formula is also true for negative values of n.
14. If z is a complex number, describe and illustrate on the Argand diagram the locus given by each of the following

(a) $\left|\frac{z+i}{z-2}\right| = 3$

[it is a circle centre $\left(\frac{9}{4}, \frac{1}{8}\right)$ radius 0. 8385]

(ii) Arg (z + 3) = $\frac{\pi}{6}$ [is it a straight line represented by equation y = $\frac{\sqrt{3}}{3}x + 3$]

15. (a) Use Demoivre's theorem or otherwise to simplify $\frac{(cos\theta+isin\theta)(cos2\theta+isin2\theta}{cos\frac{\theta}{2}+isin\frac{\theta}{2}}$

$\left[cos\left(\frac{5\theta}{2}\right) + isin\left(\frac{5\theta}{2}\right)\right]$

(b) Express $\frac{i}{4+6i}$ in modulus-argument form.

{z = 0.1387[cos(0.187π)+ isin(0.187π)]}

(c) Solve (z + zz*)z = 5 + 2z where z* is the complex conjugate of z.
[z = 1 + 2i, z* = 1 -2i]

16. Show that 2 + I is a root of the equation $2z^3 - 9z^2 + 14 - 5 = 0$. Hence find the other roots 2 – i and ½

17. (a) find the equation whose root are -1 ± i where i = $\sqrt{-1}$ [$z^2 + 2z + 2 = 0$]

(b) Find the sum of the first 10 terms of the series 1 + 2i+ -4 – 8i + 16 + ….
[205 + 410i]

(c) Prove by induction that
$(cos\theta + sin\theta)^n = cosn\theta + sinn\theta$.

18. Show that z = 1 is a root of the equation $z^3 - 5z2 + 9z - 5 = 0$.
Hence solve the equation for other roots
[2 ± i]

19. (a) Use Demoivre's theorem to express tan5θ in terms of tanθ.

(b) Solve the equation $z^3 + 1 = 0$
$\left[-1, \frac{1}{2} + i\frac{\sqrt{3}}{2}, \frac{1}{2} - i\frac{\sqrt{3}}{2}\right]$

20. (a) Without using tables or calculators, simplify$\frac{\left(cos\frac{\pi}{17}+isin\frac{\pi}{17}\right)^8}{\left(cos\frac{\pi}{17}-isin\frac{\pi}{17}\right)^9}$ [-1]

(b) Given that x and y are real, find the values of x and y which satisfy the equation: $\frac{2y+4i}{2x+y} - \frac{y}{x-i} = 0$ [x = -1 when y = -2 and x = 1 when y = 2]

21. Given the complex number z =$\frac{(3i+1)(i-2)^2}{i-3}$

(i) Determine z in form a + bi where a and b are constant [-4,- 3i]

(ii) arg(z) [-143.13^0]

22. (a) Express the complex numbers z_1 =4i and z_2 = 2 – 2i in trigonometric form r(cosθ + isinθ).
[z_1 =4(cos90^0 + isin90^0),
z_2 = 2$\sqrt{2}$(cos-45^0 + isin-45^0)]
Hence or otherwise evaluate $\frac{z_1}{z_2}$ $\left[\frac{1}{2}\right]$

(b) Find the values of x and y in
$\frac{x}{3+3i} - \frac{y}{(2-3i)} = \frac{(6+2i)}{(1+8i)}$[x = 2.8, y =0.4]

23. Find the fourth root of 4 + 3i

[±(1.4760 + 0.23971i), ±(0.2397 – 1.4760i)]

24. (a) Given that $\frac{ix}{1+iy} = \frac{3x+4i}{x+3y}$; find the values of x and y
[x = 2, y= 1.5 or x = -2 and y = -1.5]

(b) If z = x +iy, find the equation of the locus $\left|\frac{z+3}{z-1}\right| = 4$
$\left[x^2 + y^2 - \frac{38}{15}x + \frac{7}{15} = 0\right]$

25. (a) given that the complex number z and its conjugate Z*, satisfy the equation zz*+ 3z* = 34 – 12i, find the value of z.
[z = 3 - 4i and z = -6 + 4i]

(b) Find the Cartesian equation of the locus of a point P represented by equation$\left|\frac{z+3}{z+2-4i}\right| = i$
[the locus P is an equation of line 8y + 2x = 11]

26. (a) form a quadratic equation having -3 + 4i as one of its roots. [$z^2 + 6z + 25$]

(b) Given z_1 = -1 + i$\sqrt{3}$ and z_2 = -1 - i$\sqrt{3}$

(i) Express $\frac{z_1}{z_2}$ in form a + i$\sqrt{b}$, where a and b are real number
$\left[\frac{-1}{2} - i\frac{\sqrt{3}}{2}\right]$

(ii) Represent $\frac{z_1}{z_2}$ on an Argand diagram

(iii) find$\left|\frac{z_1}{z_2}\right|$ [1]

27. If z = $\frac{(2-i)(5+12i)}{(1+2i)^2}$

(a) Find the

(i) Modulus of z [5.814]
(ii) Argument z [-86.055^0]

(b) Represent z on a complex plane

(c) Write z in the polar form
[z = 5.814(cos(-86.055^0) + isin(-86.055^0) or z = 5.814(cos (0.47π) + isin(0.47π))]

28. (a) the complex number z = $\sqrt{3} + i$.
$\bar{Z}$ *is the conjugate of* Z.
(i) Express Z in the modulus argument form
[z = $2\left[\cos\left(\frac{\pi}{6}\right) + isin\left(\frac{\pi}{6}\right)\right]$ or z = $2[\cos 30^0 + isin30^0]$]
(ii) On the same Argand diagram plot $\bar{Z}$ and $2\bar{Z} + 3i$

(b) What are the greatest and least values of $|Z|$ *if* $|Z-4| \le 3$?

[Greatest value of $|z| = 7$, Lowest value of $|z| =$ = 1]

29. (a) Given that the complex number Z and its conjugate $\bar{Z}$ satisfy the equation
$Z\bar{Z} + 2iZ = 12 + 6i$. Find Z.
[possible values of z are 3 +3i and 3- i]
(b)One root of the equation $Z^3 - 3Z^2 + 9Z + 13 = 0$ *is* $2 + 3i$. Determine other roots
[other roots are 2-3i and -1]

30. (a) If $z_1 = \frac{2i}{1+3i}$ and $z_2 = \frac{3+2i}{5}$, find $|z_1 - z_2|$
$\left[\frac{1}{5}\right]$

(b) Given the complex number z = x + iy
(i) Find $\frac{z+i}{z+2}$ $\left[\frac{x^2+2x+y^2+y+i(2+x+2y)}{(x+2)^2+y^2}\right]$
(ii) Show that the locus of $\frac{z+i}{z+2}$ is a straight line when its imaginary part is zero. State the gradient of the line. [y = $-\frac{1}{2}x + 1$]

31. (a) Given that the complex number Z and its conjugate $\overline{Z}$ satisfy the equation
$Z\overline{Z} - 2Z + 2\overline{Z} = 5 - 4i$. Find possible values of Z. [Z = $\pm 2 + 1$]

(b) Prove that if $\frac{Z-6i}{Z+8}$ is real, then the locus of the point representing the complex number Z is a straight line. [y = $\frac{3}{4}x$ + 6]

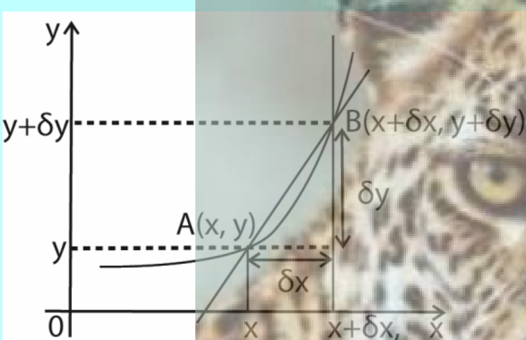
32. (a) Express 5 + 12i in polar form
Hence, evaluate $\sqrt[3]{5 + 12i}$, giving your answer in the form a + ib where a and b are corrected to two decimal places. (12marks)

[$\sqrt[3]{5+12i} = \sqrt[3]{13}\left[\cos\left(\frac{67.38+2\pi k}{3}\right) + isin\left(\frac{67.38+2\pi k}{3}\right)\right]$ *taking* $k = 0, 1, 2$;
2.17 + 0.90i; -1.86 + 1.43i; -0.31 – 2.33i]

33. Show that the modulus of $\frac{(1-i)^6}{(1+i)}$=$4\sqrt{2}$

Differentiation

Consider point A(x, y) lying on a curve drawn below, if another point B(x + δx, y + δy) lies in the same curve, where δx and δy are small increments in x and y respectively, the straight line AB, drawn through the curve is called a chord of the curve.



As the distance δx becomes smaller and smaller, point B moves close to A and the chord AB approaches the position of the target at A

Now, Gradient, $M_{AB} = \frac{y+\delta y-y}{x+\delta x-x}$

$M_{AB} = \frac{\delta y}{\delta x}$

As δx tends to zero, i.e.δx →0.

$\frac{\delta y}{\delta x}$ approaches the value of the gradient of the target line at A. This value is called limiting value of $\frac{\delta y}{\delta x}$ and is written as $\lim_{\delta x \to 0} \frac{\delta y}{\delta x}$.

The limiting value of $\frac{\delta y}{\delta x}$ is called a differential coefficient or first derivative of y with respect to x which is denoted by $\frac{dy}{dx}$.

Note: the process of finding this limiting value is called differentiation.

Direct differentiation of explicit functions

Explicit functions are functions where one variable is expressed in terms of the other variable. Examples $y = x^2$, $y = x^4 + 2x$ etc.

Given the function $x = x^n$, the derivative of y with respect to x, denoted by either y' or $\frac{dy}{dx}$ is given by $y' = \frac{dy}{dx} = nxn^{-1}$.

This result applies for all rational values of n. this means that multiply the term given by the give power index and then reduce the power by one.

Note: If

(i) $y = f(x) + g(x) + h(x)$, then

$$\frac{dy}{dx} = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x)) + \frac{d}{dx}(h(x))$$

(ii) If y = a, this is written as $y = 0a^0$,

$$\frac{dy}{dx} = 0(ax^{-1}) = 0$$

Example 1

Find the derivatives of the following with respect to x

(a) $y = x^3$

solution

$$\frac{dy}{dx} = 3x^{3-2} = 3x^2$$

(b) $y = 2x^2 + 3$

Solution

$y = 2x^2 + 3x^0$

$$\frac{dy}{dx} = \frac{d}{dx}(2x^2) + \frac{d}{dx}(3x^0)$$
$$= 2(\,2x^{2-1}) + 0(3x^{0-1})$$
$$= 4x + 0 = 4x$$

(c) $y = \frac{1}{x}$

Solution

y = x^{-1}

$\frac{dy}{dx} = -1x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$

(d) y = $\sqrt{x}$

Solution

$y = x^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$

(e) y = $\frac{-2}{x}$

Solution

y = $-2x^{-1}$

$\frac{dy}{dx} = -2(-1x^{-1-1}) = 2x^{-2} = \frac{2}{x^2}$

(f) y = $x^4 + 3x^2 + 2$

Solution

y = $x^4 + 3x^2 + 2x^0$

$\frac{dy}{dx} = 4x^{4-1} + 2(3x^{2-1}) + 0(2x^{0-1})$

= $4x^3 + 6x + 0$

=$4x^3 + 6x$

(g) y = $\frac{3}{\sqrt{x}} - 2\sqrt{x}$

Solution

$y = 3x^{-\frac{1}{2}} - 2x^{\frac{1}{2}}$

$\frac{dy}{dx} = -\frac{1}{2}\left(3x^{-\frac{1}{2}-1} - \frac{1}{2}\left(2x^{\frac{1}{2}-1}\right)\right)$

$-\frac{3}{2}x^{\frac{-3}{2}} - x^{-\frac{1}{2}} = -\frac{3}{2x^{\frac{2}{3}}} + \frac{1}{x^{\frac{1}{2}}}$

(h) y = x4(x + 1)

solution

y= $x^5 + x^4$

$\frac{dy}{dx} = 5x^{5-1} + 4x^{4-1} = 5x^4 + 4x^3$

(i) y =$6\sqrt{x}(x^2 - 2x)$

Solution

$y = 6x^{\frac{5}{2}} - 12x^{\frac{3}{2}}$

$\frac{dy}{dx} = \frac{5}{2}\left(6x^{\frac{5}{2}-1}\right) - \frac{3}{2}\left(12x^{\frac{3}{2}-1}\right)$

$15x^{\frac{3}{2}} - 18x^{\frac{1}{2}}$

Revision exercise 1

Find the derivatives of the following with respect to x

(a) y = $3x^2$ [6x]

(b) y = $2x^4 + 2$ [$8x^3$]

(c) y = b [0]

(d) y = $\frac{9}{2x^3}$ $\left[-\frac{27}{2x^4}\right]$

(e) y = $2x^{-2}$ [$-4x^{-3}$]

(f) y = $\frac{-3}{4x^4}$ $\left[\frac{3}{x^5}\right]$

(g) y = $\sqrt[1]{x}$ $\left[\frac{1}{4x^{\frac{3}{4}}}\right]$

(h) y = $\frac{4}{5\sqrt{x}}$ $\left[\frac{2}{5x^{\frac{3}{2}}}\right]$

(i) y = $\frac{-6}{\sqrt[3]{x}}$ $\left[\frac{2}{x^{\frac{4}{3}}}\right]$

(j) $6\sqrt{x}(x^3 - 2x + 1)$

$\left[21x^{\frac{5}{2}} - 18x^{\frac{1}{2}} + \frac{3}{x^{\frac{1}{2}}}\right]$

Differentiation of functions from first principles

There are four basic steps followed when differentiating functions from first principles.

Given the function y = f(x), the steps are

(i) Add small changes in x and y to the function y = f(x) i.e. y + δy = f(x = δx)
(ii) Subtract y = f(x) from the established function in step one above i.e. δy = f(x+δx)- f(x)
(iii) Divide the function in step (ii) by δx i.e. $\frac{\delta y}{\delta x} = \frac{f(x+\delta x)-f(x)}{\delta x}$
(iv) Find the limit of the above quotient when δx→0. This is the derivative required

Differentiation of polynomial functions from first principles

These are functions in terms of y = ax^n where n is both rational and irrational numbers.

Example 2

Differentiated the following with respect to x from first principles

(a) y = x^2

Solution

y = x^2

y + δy = $(x + δx)^2$

δy = $(x + δx) – x^2$ ………………….. (i)

Eqn. (i) is difference of two squares expression

δy = (x +δx + x)(x + δx – x)

$\delta y = (2x + \delta x)\delta x = 2x\delta x + (\delta x)^2$
$\frac{\delta y}{\delta x} = 2x + \delta x$
$\frac{\delta y}{\delta x} = \max_{\delta x \to 0} \frac{dy}{dx} = 2x$
$\therefore \frac{dy}{dx} = 2x$

(b) y = $\sqrt{x}$

Solution

y = $\sqrt{x}$

y + $\delta y = \sqrt{x + \delta x}$

$\delta y = \sqrt{x + \delta x} - y$

$\delta y = \sqrt{x + \delta x} - \sqrt{x}$

Dividing through by δx

$\frac{dy}{dx} = \frac{\sqrt{x+\delta x}-\sqrt{x}}{\delta x}$

Rationalizing the numerator on the RHS

$\frac{\delta y}{\delta x} = \frac{\sqrt{x+\delta x}-\sqrt{x}}{\delta x}\left(\frac{(\sqrt{x+\delta x}+\sqrt{x})}{(\sqrt{x+\delta x}+\sqrt{x})}\right)$

$\frac{\delta y}{\delta x} = \frac{x+\delta x-x}{\delta x(\sqrt{x+\delta x}+\sqrt{x})} = \frac{\delta x}{\delta x(\sqrt{x+\delta x}+\sqrt{x})}$

$\frac{\delta y}{\delta x} = \frac{1}{(\sqrt{x+\delta x}+\sqrt{x})}$

$\frac{\delta y}{\delta x} = \max_{\delta x \to 0} \frac{dy}{dx} = \frac{1}{(\sqrt{x}+\sqrt{x})} = \frac{1}{2\sqrt{x}}$

$\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

(c) y = $\frac{1}{x^2}$

Solution

y = $\frac{1}{x^2}$

y + δy = $\frac{1}{(x + \delta x)^2}$

$\delta y = \frac{1}{(x + \delta x)^2} - y$

$\delta y = \frac{1}{(x + \delta x)^2} - \frac{1}{x^2}$

$\delta y = \frac{x^2-(x + \delta x)^2}{x^2(x + \delta x)^2} = \frac{(x+x+dx)(x-x-\delta x)}{x^2(x + \delta x)^2}$

$\delta y = \frac{(2x+\delta x)(-\delta x)}{x^2(x + \delta x)^2} = \frac{-2x\delta x-(\delta x)^2}{x^2(x + \delta x)^2}$

Dividing through by δx

$\frac{\delta y}{\delta x} = \frac{-2x-\delta x}{x^2(x + \delta x)^2}$

$\frac{\delta y}{\delta x} = \max_{\delta x \to 0} \frac{dy}{dx} = \frac{-2x}{x^4} = \frac{-2}{x^3}$

$\therefore \frac{\delta y}{\delta x} = \frac{-2}{x^3}$

(d) y = $2x^3$

Solution

y = $2x^3$

y + $\delta y = 2(x + \delta x)^3$

$\delta y = 2(x + \delta x)^3 - 2x^3$

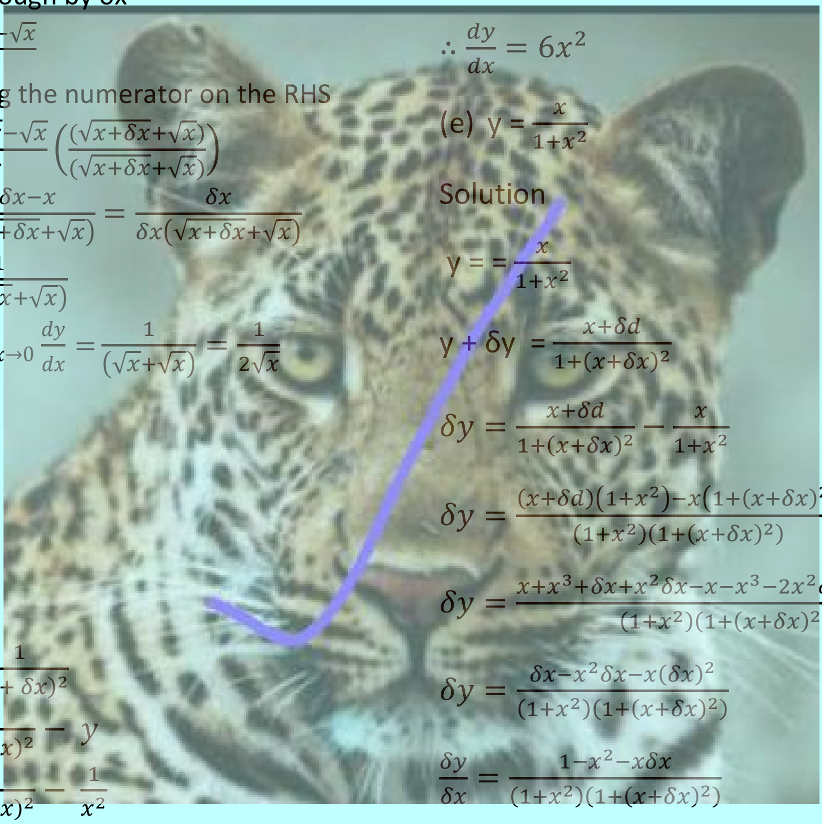
$\delta y = 2x^3 + 6x^2\delta x + 6x(\delta x)^2 - 2x^3$

$\delta y = 6x^2\delta x + 6x(\delta x)^2$

$\frac{\delta y}{\delta x} = 6x^2 + 6x\delta x$

$\frac{\delta y}{\delta x} = \max_{\delta x \to 0} \frac{dy}{dx} = 6x^2$

$\therefore \frac{dy}{dx} = 6x^2$



(e) y = $\frac{x}{1+x^2}$

Solution

y = = $\frac{x}{1+x^2}$

y + δy = $\frac{x+\delta d}{1+(x+\delta x)^2}$

$\delta y = \frac{x+\delta d}{1+(x+\delta x)^2} - \frac{x}{1+x^2}$

$\delta y = \frac{(x+\delta d)(1+x^2)-x(1+(x+\delta x)^2)}{(1+x^2)(1+(x+\delta x)^2)}$

$\delta y = \frac{x+x^3+\delta x+x^2\delta x-x-x^3-2x^2\delta x-x(\delta x)^2}{(1+x^2)(1+(x+\delta x)^2)}$

$\delta y = \frac{\delta x-x^2\delta x-x(\delta x)^2}{(1+x^2)(1+(x+\delta x)^2)}$

$\frac{\delta y}{\delta x} = \frac{1-x^2-x\delta x}{(1+x^2)(1+(x+\delta x)^2)}$

$\frac{\delta y}{\delta x} = \max_{\delta x \to 0} \frac{dy}{dx} = \frac{1-x^2}{(1+x^2)(1+x^2)} = \frac{1-x^2}{(1+x^2)^2}$

$\therefore \frac{dy}{dx} = \frac{1-x^2}{(1+x^2)^2}$

(f) y=x^n

Solution

y=x^n

$y + \delta y = (x + \delta x)^n$

$\delta y = (x + \delta x)^n - x^n$

Since n is assumed to be positive, we expand $(x+\delta x)^n$ using binomial expansion

$$\delta y = x^n + \binom{n}{1} x^{n-1}\delta x + \binom{n}{2} x^{n-2}(\delta x)^2 + \cdots + -x^n$$

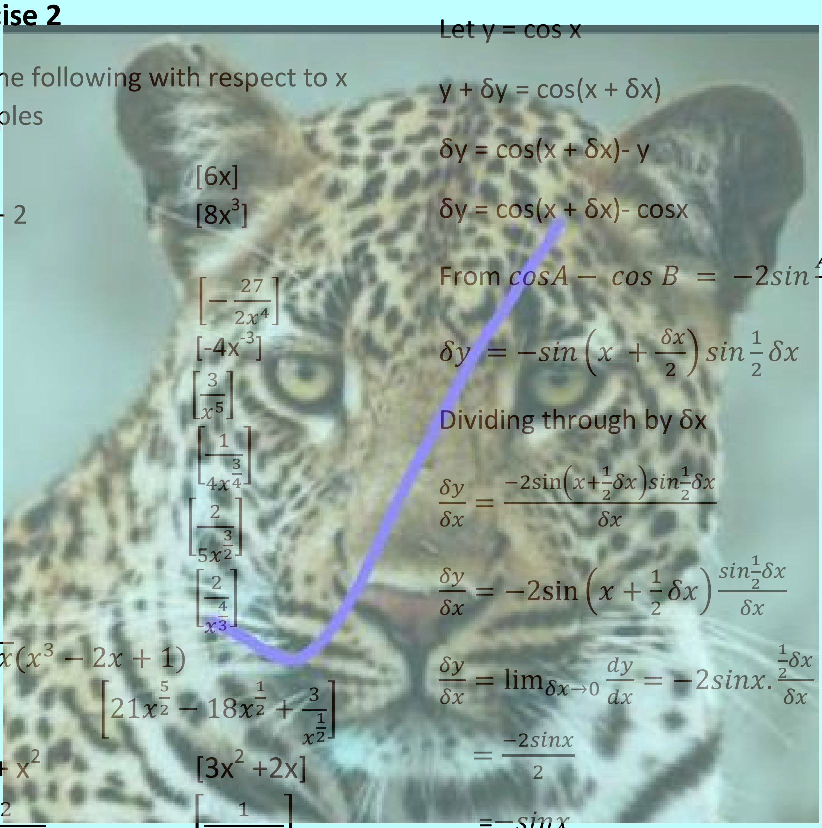
$$\delta y = nx^{n-1}\delta x + \binom{n}{2} x^{n-2}(\delta x)^2 + \cdots + (\delta x)^n$$

$$\frac{\delta y}{\delta x} = nx^{n-1} + \binom{n}{2} x^{n-2}\delta x + \cdots + (\delta x)^{n-1}$$

$$\frac{\delta y}{\delta x} = \max_{\delta x \to 0} \frac{dy}{dx} = nx^{n-1}$$

$$\therefore \frac{dy}{dx} = nx^{n-1}$$

Revision exercise 2



Differentiated the following with respect to x from first principles

(a) $y = 3x^2$ [6x]

(b) $y = 2x^4 + 2$ [$8x^3$]

(c) y = b [0]

(d) $y = \frac{9}{2x^3}$ $\left[-\frac{27}{2x^4}\right]$

(e) $y = 2x^{-2}$ [$-4x^{-3}$]

(f) $y = \frac{-3}{4x^4}$ $\left[\frac{3}{x^5}\right]$

(g) $y = \sqrt[1]{x}$ $\left[\frac{1}{4x^{\frac{3}{4}}}\right]$

(h) $y = \frac{4}{5\sqrt{x}}$ $\left[\frac{2}{5x^{\frac{3}{2}}}\right]$

(i) $y = \frac{-6}{\sqrt[3]{x}}$ $\left[\frac{2}{x^{\frac{4}{3}}}\right]$

(j) $y = 6\sqrt{x}(x^3 - 2x + 1)$

$\left[21x^{\frac{5}{2}} - 18x^{\frac{1}{2}} + \frac{3}{x^{\frac{1}{2}}}\right]$

(k) $y = X^3 + x^2$ [$3x^2 + 2x$]

(l) $y = \frac{2}{\sqrt{(x+2)}}$ $\left[\frac{1}{\sqrt{(x+2)}}\right]$

(m) $y = 4x + 2x^2$ $[4 + 4x]$

Differentiation of trigonometric functions from first principles

These include trigonometric functions with single or multiple angles and those with higher or fractional powers.

Note the following formula as well

$$cosA + cos B = 2cos\frac{A+B}{2}cos\frac{A-B}{2}$$

$$cosA - cos B = -2sin\frac{A+B}{2}sin\frac{A-B}{2}$$

$$sinA + sinB = 2sin\frac{A+B}{2}cos\frac{A-B}{2}$$

$$sinA - sinB = 2cos\frac{A+B}{2}sin\frac{A-B}{2}$$

Example 3 single angle

Differentiate the following functions from first principle

(a) cosx

Solution

Let y = cos x

$y + \delta y = \cos(x + \delta x)$

$\delta y = \cos(x + \delta x) - y$

$\delta y = \cos(x + \delta x) - \cos x$

From $cosA - cos B = -2sin\frac{A+B}{2}sin\frac{A-B}{2}$

$$\delta y = -sin\left(x + \frac{\delta x}{2}\right) sin\frac{1}{2}\delta x$$

Dividing through by δx

$$\frac{\delta y}{\delta x} = \frac{-2\sin\left(x+\frac{1}{2}\delta x\right)sin\frac{1}{2}\delta x}{\delta x}$$

$$\frac{\delta y}{\delta x} = -2\sin\left(x+\frac{1}{2}\delta x\right)\frac{sin\frac{1}{2}\delta x}{\delta x}$$

$$\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\frac{dy}{dx} = -2sinx.\frac{\frac{1}{2}\delta x}{\delta x}$$

$$= \frac{-2sinx}{2}$$

$$= -sinx$$

$$\therefore \frac{d}{dx}cosx = -sinx$$

(b) sinx

let y = sinx

$y + \delta y = \sin(x + \delta x)$

$\delta y = \sin(x + \delta x) - \sin x$

From $sinA - sinB = 2cos\frac{A+B}{2}sin\frac{A-B}{2}$

$$\delta y = 2\cos\left(x + \frac{1}{2}\delta x\right) sin\frac{1}{2}\delta x$$

Dividing through by δx

$$\frac{\delta y}{\delta x} = \frac{2\cos\left(x+\frac{1}{2}\delta x\right)\sin\frac{1}{2}\delta x}{\delta x}$$

$$\frac{\delta y}{\delta x} = 2\cos\left(x+\frac{1}{2}\delta x\right)\frac{\sin\frac{1}{2}\delta x}{\delta x}$$

$$\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{dy}{dx} = 2\cos x.\frac{\frac{1}{2}\delta x}{\delta x}$$

$$= \frac{2\cos x}{2} = \cos x$$

$$\therefore \frac{d}{dx}sinx = cosx$$

(c) tanx

$let\ y = tanx$

$y + \delta y = \tan(x+\delta x)$

$\delta y = \tan(x+\delta x) - \tan x$

$$= \frac{\sin(x+\delta x)}{\cos(x+\delta x)} - \frac{sinx}{cosx}$$

$$= \frac{\sin(x+\delta x)cosx - \cos(x+\delta x)sinx}{\cos(x+\delta x)cosx}$$

$$= \frac{sin\delta x}{\cos(x+\delta x)cosx}$$

Divide by δx

$$\frac{\delta y}{\delta x} = \frac{sin\delta x}{\cos(x+\delta x)cosx}\cdot\frac{1}{\delta x}$$

$$\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{dy}{dx} = \frac{s\delta x}{cosxcosx}\cdot\frac{1}{\delta x}$$

$$= \frac{1}{\cos^2 x} = sec^2 x$$

$$\therefore \frac{d}{dx}tanx = sec^2 x$$

(d) sec x

Let y = sec x

$$y = \frac{1}{cosx}$$

$$y + \delta y = \frac{1}{\cos(x+\delta x)}$$

$$\delta y = \frac{1}{\cos(x+\delta x)} - \frac{1}{cosx}$$

$$= \frac{cox - \cos(x+\delta x)}{\cos x\cos(x+\delta x)}$$

$$= \frac{-2\sin\left(x+\frac{1}{2}\delta x\right)\sin\frac{1}{2}\delta x}{\cos x\cos(x+\delta x)}$$

Dividing by δx

$$\frac{\delta y}{\delta x} = \frac{-2\sin\left(x+\frac{1}{2}\delta x\right)\sin\left(-\frac{1}{2}\delta x\right)}{\cos x\cos(x+\delta x)\delta x}$$

$$\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{dy}{dx} = \frac{+2sinx.\frac{1}{2}\delta x}{cosx.cosx.\delta x}$$

$$= \frac{sinx}{cosx}\cdot\frac{1}{cosx} = tanxsecx$$

$$\therefore \frac{d}{dx}secx = tanxsecx$$

(e) cot x

Let y = cot x

$$y = \frac{\cos x}{\sin x}$$

$$y + \delta y = \frac{\cos(x+\delta x)}{\sin(x+\delta x)}$$

$$\delta y = \frac{\cos(x+\delta x)}{\sin(x+\delta x)} - \frac{\cos x}{\sin x}$$

$$= \frac{\text{sinx}\cos(x+\delta x) - cosx\sin(x+\delta x)}{\text{sinxsin}(x+\delta x)}$$

$$= \frac{sin\{x-(x+\delta x)\}}{\text{sinxsin}(x+\delta x)}$$

$$= \frac{sin(-\delta x)}{\text{sinxsin}(x+\delta x)} = \frac{-sin\delta x}{\text{sinxsin}(x+\delta x)}$$

Dividing by δx

$$\frac{\delta y}{\delta x} = \frac{-sin\delta x}{\text{sinxsin}(x+\delta x)}\cdot\frac{1}{\delta x}$$

$$\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{dy}{dx} = \frac{-\delta x}{\text{sinxsin}\,x}\cdot\frac{1}{\delta x} = \frac{-1}{sin^2 x}$$

$$= -cosec^2 x$$

$$\therefore \frac{d}{dx}cot = -cosec^2 x$$

Example 4 double angle

Differentiate the following functions from first principle

(a) cos 2x

Let y = cos2x

$y + \delta x = \cos 2(x+\delta x)$

$\delta y = \cos 2(x+\delta x) - cos2x$

$= -2sin(2x+\delta x)sin\delta x$

Dividing through by δx

$$\frac{\delta y}{\delta x} = -2sin(2x+\delta x)\frac{sin\delta x}{\delta x}$$

$$\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{dy}{dx} = -2sin2x.\frac{\delta x}{\delta x} = -2sin2x$$

$$\therefore \frac{d}{dx}cos2x = -2sin2x$$

(b) sin2x

(c) Let y = sin2x

$y + \delta x = \sin 2(x+\delta x)$

$\delta y = \sin 2(x+\delta x) - sin2x$

$= 2cos2(x+\delta x)sin\delta x$

Divide by δx

$$\frac{\delta y}{\delta x} = 2cos2(x+\delta x)\frac{sin\delta x}{\delta x}$$

$$\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\frac{dy}{dx} = 2cos2x\frac{\delta x}{\delta x} = 2cos2x$$

$$\therefore \frac{d}{dx}sin2x = 2cos2x$$

(d) $cos\frac{1}{2}x$

Let $y = cos\frac{1}{2}x$

$y + \delta y = cos\frac{1}{2}(x+\delta x)$

$\delta y = cos\frac{1}{2}(x+\delta x) - cos\frac{1}{2}x$

$= -2sin\left(\frac{x}{2}+\frac{\delta x}{4}\right)sin\left(\frac{\delta x}{4}\right)$

Dividing through by δx

$$\frac{\delta y}{\delta x} = -2sin\left(\frac{x}{2}+\frac{\delta x}{4}\right)\frac{sin\left(\frac{\delta x}{4}\right)}{\delta x}$$

$$\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\frac{dy}{dx} = -2sin\left(\frac{x}{2}\right)\frac{\frac{\delta x}{4}}{\delta x}$$

$$= -\frac{2}{4}sin\left(\frac{1}{2}x\right) = -\frac{1}{2}sin\left(\frac{1}{2}x\right)$$

$$\therefore \frac{d}{dx}cos\frac{1}{2}x = -\frac{1}{2}sin\left(\frac{1}{2}x\right)$$

(e) tan2x

let y = tan2x

y + δy = tan2(x+ δx)

δy = tan2(x+ δx)-tan2x

$$=\frac{sin2(x+\delta x)}{cos2(x+\delta x)} - \frac{sin2x}{cos2x}$$

$$= \frac{cos2xsin2(x+\delta x) - sinxcos2(x+\delta x)}{cos2xcos2(x+\delta x)}$$

$$= \frac{sin2\delta x}{cos2xcos2(x+\delta x)}$$

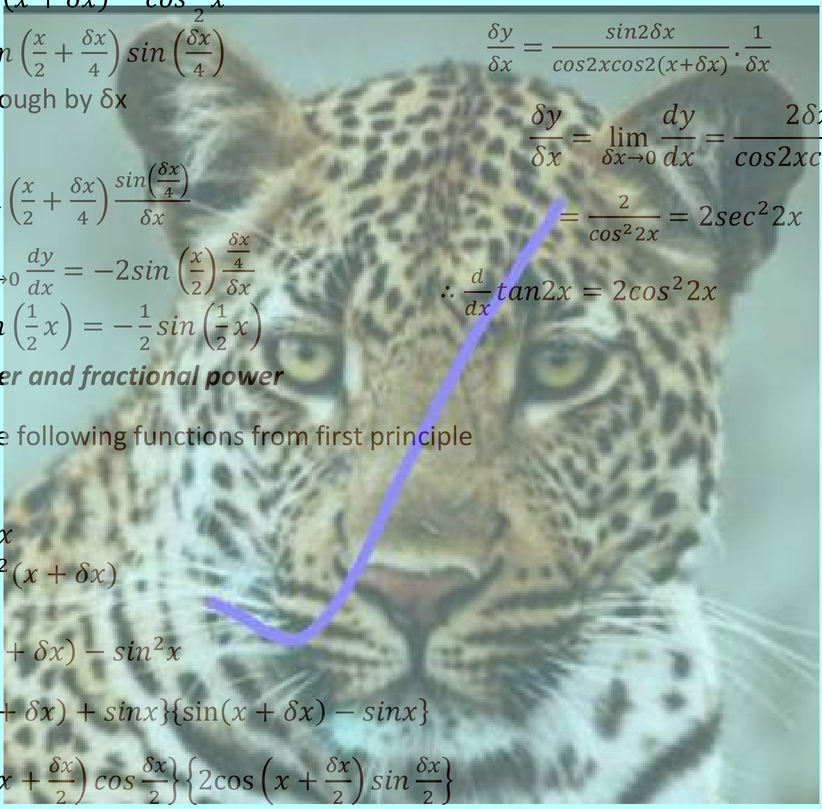
Dividing through by δx

$$\frac{\delta y}{\delta x} = \frac{sin2\delta x}{cos2xcos2(x+\delta x)}.\frac{1}{\delta x}$$

$$\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\frac{dy}{dx} = \frac{2\delta x}{cos2xcos2x}.\frac{1}{\delta x}$$

$$= \frac{2}{cos^2 2x} = 2sec^2 2x$$

$$\therefore \frac{d}{dx}tan2x = 2cos^2 2x$$



Example 5 higher and fractional power

Differentiate the following functions from first principle

(a) $sin^2 x$

Let y = $sin^2 x$

y + δy = $sin^2(x+\delta x)$

δy = $sin^2(x+\delta x) - sin^2 x$

= $\{\sin(x+\delta x) + sinx\}\{\sin(x+\delta x) - sinx\}$

= $\left\{2\sin\left(x+\frac{\delta x}{2}\right)cos\frac{\delta x}{2}\right\}\left\{2\cos\left(x+\frac{\delta x}{2}\right)sin\frac{\delta x}{2}\right\}$

δy = $2\sin\left(x+\frac{\delta x}{2}\right)2\cos\left(x+\frac{\delta x}{2}\right)cos\frac{\delta x}{2}sin\frac{\delta x}{2}$

=$4\sin\left(x+\frac{\delta x}{2}\right)\cos\left(x+\frac{\delta x}{2}\right)cos\frac{\delta x}{2}sin\frac{\delta x}{2}$

Dividing through by δx

$$\frac{\delta y}{\delta x} = 4\sin\left(x+\frac{\delta x}{2}\right)\cos\left(x+\frac{\delta x}{2}\right)cos\frac{\delta x}{2}\frac{sin\frac{\delta x}{2}}{\delta x}$$

$$\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\frac{dy}{dx} = 4sinxcosx.\frac{\frac{\delta x}{2}}{\delta x}$$

$$= \frac{4}{2}sinxcosx = 2sinxcosx$$

$$\therefore \frac{d}{dx}sin^2 x = 2sinxcosx$$

(b) cos^2x

Let y =cos^2x

y + δy = $cos^2(x+\delta x)$

δy = $cos^2(x+\delta x) = cos^2x$

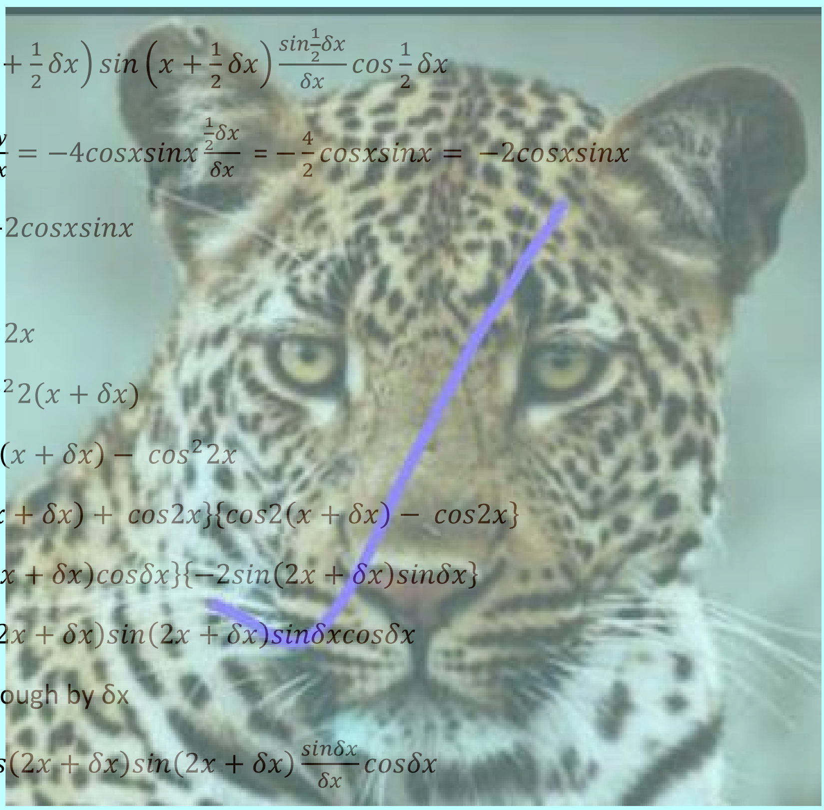
= $\{\cos(x+\delta x)+\cos x\}\{\cos(x+\delta x)-cosx\}$

= $\left\{2cos\left(x+\frac{1}{2}\delta x\right)cos\frac{1}{2}\delta x\right\}\left\{-2sin\left(x+\frac{1}{2}\delta x\right)sin\frac{1}{2}\delta x\right\}$

= $-4cos\left(x+\frac{1}{2}\delta x\right)sin\left(x+\frac{1}{2}\delta x\right)sin\frac{1}{2}\delta x cos\frac{1}{2}\delta x$

Dividing through by δx

Dividing through by δx



$\frac{\delta y}{\delta x} = -4cos\left(x+\frac{1}{2}\delta x\right)sin\left(x+\frac{1}{2}\delta x\right)\frac{sin\frac{1}{2}\delta x}{\delta x}cos\frac{1}{2}\delta x$

$\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{dy}{dx} = -4cosxsinx\frac{\frac{1}{2}\delta x}{\delta x} = -\frac{4}{2}cosxsinx = -2cosxsinx$

$\therefore \frac{d}{dx}cos^2x = -2cosxsinx$

(c) $cos^2 2x$

Let y = $cos^2 2x$

y + δy = $cos^2 2(x+\delta x)$

δy = $cos^2 2(x+\delta x) - cos^2 2x$

= $\{cos2(x+\delta x) + cos2x\}\{cos2(x+\delta x) - cos2x\}$

=$\{2cos(2x+\delta x)cos\delta x\}\{-2sin(2x+\delta x)sin\delta x\}$

=$-4cos(2x+\delta x)sin(2x+\delta x)sin\delta x cos\delta x$

Dividing through by δx

$\frac{\delta y}{\delta x} = -4cos(2x+\delta x)sin(2x+\delta x)\frac{sin\delta x}{\delta x}cos\delta x$

$\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{dy}{dx} = -4cos2x\text{sin2x}.\frac{\delta x}{\delta x}$

$= -4cos2xsin2x$

$\therefore \frac{d}{dx}cos^2 2x = -4cos2xsin2x$

(d) $\sqrt{cosx}$

Let y = $\sqrt{cosx}$

y + δy=$\sqrt{cos(x+\delta x)}$

δy = $\sqrt{cos(x+\delta x)} - \sqrt{cosx}$

by rationalizing

δy = $\frac{\sqrt{cos(x+\delta x)}-\sqrt{cosx}}{1}.\frac{\sqrt{cos(x+\delta x)}+\sqrt{cosx}}{\sqrt{cos(x+\delta x)}+\sqrt{cosx}}$

$= \dfrac{\cos(x+\delta x)-\cos x}{\sqrt{\cos(x+\delta x)}+\sqrt{\cos x}}$

$= \dfrac{-2\sin\left(x+\frac{1}{2}\delta x\right)\sin\frac{1}{2}\delta x}{\sqrt{\cos(x+\delta x)}+\sqrt{\cos x}}$

Dividing through by δx

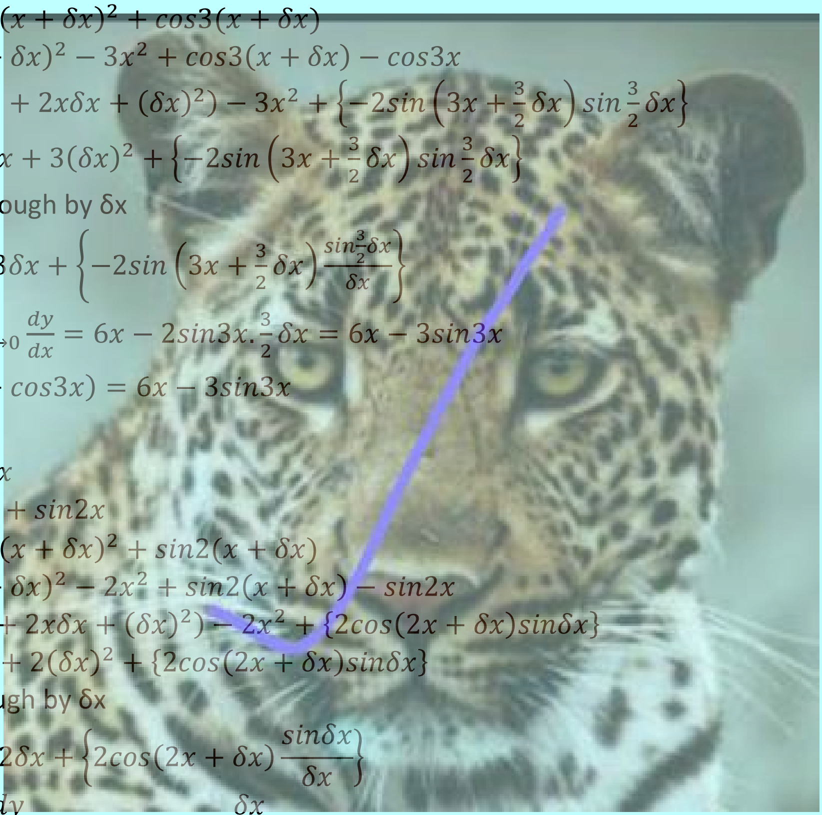
$\dfrac{\delta y}{\delta x} = \dfrac{-2\sin\left(x+\frac{1}{2}\delta x\right)\sin\frac{1}{2}\delta x}{\left(\sqrt{\cos(x+\delta x)}+\sqrt{\cos x}\right)\delta x}$

$\dfrac{\delta y}{\delta x} = \lim_{\delta x\to 0}\dfrac{dy}{dx} = \dfrac{-\sin x\,.\frac{1}{2}\delta x}{2\sqrt{\cos x}.\delta x} = \dfrac{-\sin x}{2\sqrt{\cos x}}$

$\therefore \dfrac{d}{dx}\sqrt{\cos x} = -\dfrac{\sin x}{2\sqrt{\cos x}}$

(e) $3x^2 + \cos 3x$

Let $y = 3x^2 + \cos 3x$



$y + \delta y = 3(x+\delta x)^2 + \cos 3(x+\delta x)$

$\delta y = 3(x+\delta x)^2 - 3x^2 + \cos 3(x+\delta x) - \cos 3x$

$\delta y = 3(x^2 + 2x\delta x + (\delta x)^2) - 3x^2 + \left\{-2\sin\left(3x + \frac{3}{2}\delta x\right)\sin\frac{3}{2}\delta x\right\}$

$\delta y = 6x\delta x + 3(\delta x)^2 + \left\{-2\sin\left(3x + \frac{3}{2}\delta x\right)\sin\frac{3}{2}\delta x\right\}$

Dividing through by δx

$\dfrac{\delta y}{\delta x} = 6x + 3\delta x + \left\{-2\sin\left(3x + \frac{3}{2}\delta x\right)\dfrac{\sin\frac{3}{2}\delta x}{\delta x}\right\}$

$\dfrac{\delta y}{\delta x} = \lim_{\delta x\to 0}\dfrac{dy}{dx} = 6x - 2\sin 3x.\frac{3}{2}\delta x = 6x - 3\sin 3x$

$\therefore \dfrac{d}{dx}(3x^2 + \cos 3x) = 6x - 3\sin 3x$

(f) $2x^2 + \sin 2x$

let $y = 2x^2 + \sin 2x$

$y + \delta y = 2(x+\delta x)^2 + \sin 2(x+\delta x)$

$\delta y = 2(x+\delta x)^2 - 2x^2 + \sin 2(x+\delta x) - \sin 2x$

$\delta y = 2(x^2 + 2x\delta x + (\delta x)^2) - 2x^2 + \{2\cos(2x+\delta x)\sin\delta x\}$

$\delta y = 4x\delta x + 2(\delta x)^2 + \{2\cos(2x+\delta x)\sin\delta x\}$

Divide through by δx

$\dfrac{\delta y}{\delta x} = 4x + 2\delta x + \left\{2\cos(2x+\delta x)\dfrac{\sin\delta x}{\delta x}\right\}$

$\dfrac{\delta y}{\delta x} = \lim_{\delta x\to 0}\dfrac{dy}{dx} = 4x + 2\cos 2x.\dfrac{\delta x}{\delta x} = 4x + 2\cos 2x$

$\therefore \dfrac{d}{dx}(2x^2 + \sin 2x) = 4x + 2\cos 2x$

(g) Given that x = θ – sin θ and y = 1- cosθ. Show that $\dfrac{\delta y}{\delta x} = \cot\dfrac{\theta}{2}$

Solution

x = θ - sinθ

$\dfrac{\delta x}{\delta \theta} = 1 - \cos\theta$

y= $1 - \cos\theta$

$\dfrac{\delta y}{\delta \theta} = \sin\theta$

$\dfrac{\delta y}{\delta x} = \dfrac{\delta y}{\delta \theta}.\dfrac{\delta \theta}{\delta x}$

$= \sin\theta.\dfrac{1}{1-\cos\theta}$

$= \dfrac{\sin\theta(1+\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}$

$= \dfrac{\sin\theta(1+\cos\theta)}{(1-\cos^2\theta)}$

$= \dfrac{\sin\theta(1+\cos\theta)}{\sin^2\theta}$

$= \dfrac{(1+\cos\theta)}{\sin\theta}$

$= \dfrac{1+2\cos^2\frac{\theta}{2}-1}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$

$= \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}$

$= \cot\frac{\theta}{2}$

Example 6

If y = $\sqrt{x}$, show that $\frac{\delta y}{\delta x} = \frac{1}{\sqrt{(x+\delta x)}+\sqrt{x}}$. Hence deduce $\frac{dy}{dx}$.

y+ δy = $\sqrt{(x+\delta x)}$

$\delta y = \sqrt{(x+\delta x)} - \sqrt{x}$

Dividing through by δx

$\frac{\delta y}{\delta x} = \frac{\sqrt{(x+\delta x)}-\sqrt{x}}{\delta x}$

By rationalizing the numerator

$\frac{\delta y}{\delta x} = \frac{\sqrt{(x+\delta x)}-\sqrt{x}}{\delta x} \cdot \frac{\sqrt{(x+\delta x)}+\sqrt{x}}{\sqrt{(x+\delta x)}+\sqrt{x}}$

$\frac{\delta y}{\delta x} = \frac{\left(\sqrt{(x+\delta x)}\right)^2-\left(\sqrt{x}\right)^2}{\delta x\left(\sqrt{(x+\delta x)}+\sqrt{x}\right)} = \frac{\delta x}{\delta x\left(\sqrt{(x+\delta x)}+\sqrt{x}\right)}$

$\therefore \frac{\delta y}{\delta x} = \frac{1}{\left(\sqrt{(x+\delta x)}+\sqrt{x}\right)}$

As δx →0; $\frac{\delta y}{\delta x} \rightarrow \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{(\sqrt{x}+\sqrt{x})} = \frac{1}{2\sqrt{x}}$

Example 7

Differentiate y= $\sqrt{\frac{1+sinx}{1-sinx}}$

y= $\sqrt{\frac{(1+sinx)(1+sinx)}{(1-sinx)(1+sinx)}} = \sqrt{\frac{(1+sinx)^2}{1-sin^2x}} = \sqrt{\frac{(1+sinx)^2}{cos^2x}}$

⇨ y = $\frac{1+x}{cosx}$

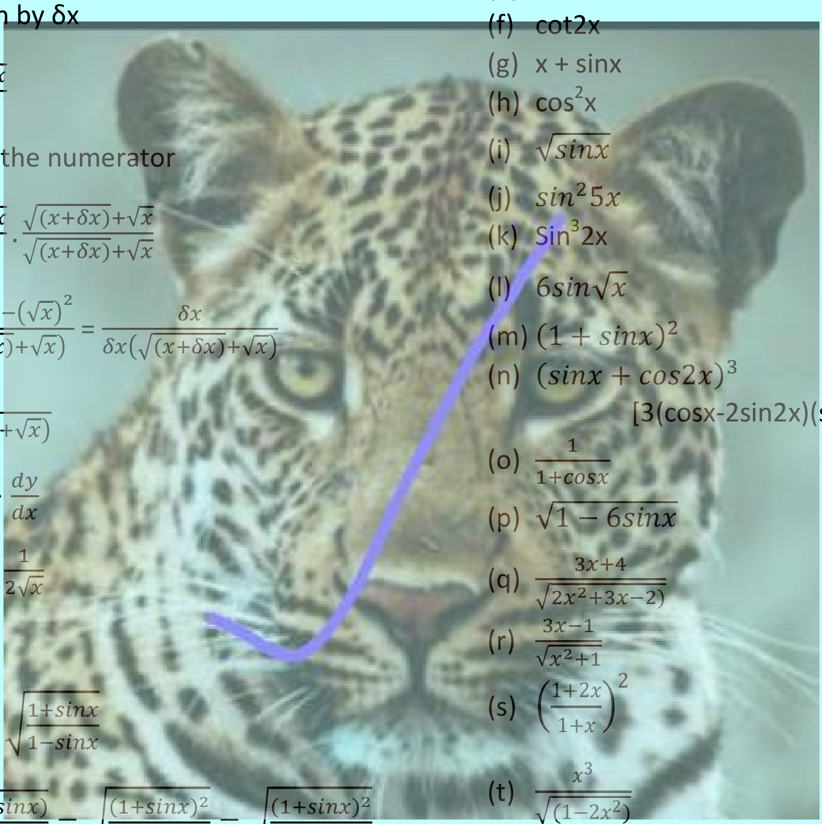
$\frac{dy}{dx} = \frac{cosx(cosx)-(1+sinx)(1-sinx)}{cos^2x}$

$= \frac{cos^2x+sinx+sin^2x}{cos^2x} = \frac{1+sinx}{cos^2x} = \frac{1+sinx}{1-sin^2x}$

$= \frac{1+sinx}{(1+sinx)(1-sinx)}$

$= \frac{1}{1-sinx}$

Hence $\frac{d}{dx}\sqrt{\frac{1+sinx}{1-sinx}} = \frac{1}{1-sinx}$



Revision exercise 3

1. Differentiate with respect to x

(a) $5x^3$	$[15x^2]$
(b) $1-x^2$	$[-2x]$
(c) x - $\frac{3}{x}$	$\left[1+\frac{3}{x^2}\right]$
(d) $\sqrt{x}$	$\left[\frac{1}{2}\frac{1}{\sqrt{x}}\right]$
(e) cos3x	[3sin3x]
(f) cot2x	$[-2cosec^2 2x]$
(g) x + sinx	[1 + cosx]
(h) cos^2x	[-cosxsinx]
(i) $\sqrt{sinx}$	$\left[\frac{cosx}{2\sqrt{sinx}}\right]$
(j) $sin^2 5x$	[10sin5xcos5x]
(k) $Sin^3 2x$	$[6sin^2 2xcosx]$
(l) $6sin\sqrt{x}$	$\left[\frac{3cos\sqrt{x}}{\sqrt{x}}\right]$
(m) $(1+sinx)^2$	[2cosx(1+sinx)]
(n) $(sinx+cos2x)^3$	$[3(cosx-2sin2x)(sinx + cos2x)^2]$
(o) $\frac{1}{1+cosx}$	$\left[\frac{sinx}{(1+cosx)^2}\right]$
(p) $\sqrt{1-6sinx}$	$\left[\frac{-3cosx}{\sqrt{1-6sinx}}\right]$
(q) $\frac{3x+4}{\sqrt{2x^2+3x-2)}}$	$\left[\frac{-7x-24}{2\sqrt{2x^2+3x-2)}}\right]$
(r) $\frac{3x-1}{\sqrt{x^2+1}}$	$\left[\frac{x+3}{\sqrt{x^2+1}}\right]$
(s) $\left(\frac{1+2x}{1+x}\right)^2$	$\left[\frac{2(1+2x)}{(1+x)^3}\right]$
(t) $\frac{x^3}{\sqrt{(1-2x^2)}}$	$\left[\frac{3x^2-4x^4}{(1-2x^2)^{\frac{3}{2}}}\right]$

2. Given that $y = \sqrt{\frac{1+sinx}{1-sinx}}$ show that $\frac{dy}{dx} = \frac{1}{1-sinx}$
3. Show from first principles that $\frac{d}{dx}(tanx) = sec^2x$

Differentiation of product and quotient of a function

Given the function y = uv and that u and v are functions of x, the derivatives of y with respect to x is done from first principles.

Let δx be a small increment in x and let δu, δv and δy be the resulting small increment in u, v and y

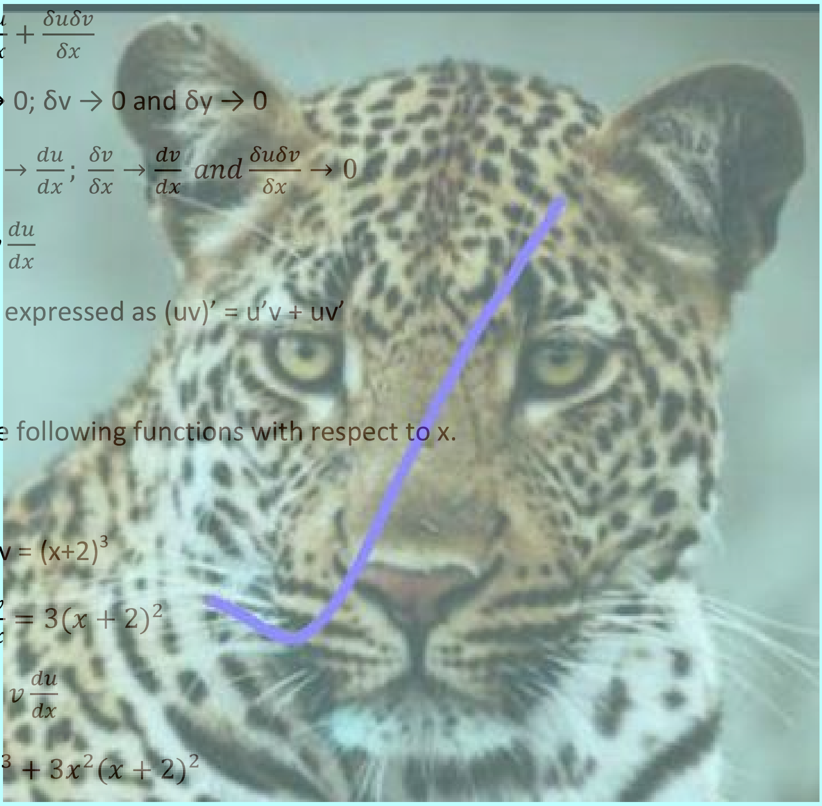
y = uv

$y + \delta y = (u + \delta u)(v + \delta v)$

$\delta y = (u + \delta u)(v + \delta v) - uv$

$\quad = u\delta v + v\delta u + \delta u\delta v$

Dividing through by δx



$\frac{\delta y}{\delta x} = u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \frac{\delta u\delta v}{\delta x}$

As δx → 0; δu → 0; δv → 0 and δy → 0

⇨ $\frac{\delta y}{\delta x} \to \frac{dy}{dx}; \frac{\delta u}{\delta x} \to \frac{du}{dx}; \frac{\delta v}{\delta x} \to \frac{dv}{dx}$ and $\frac{\delta u\delta v}{\delta x} \to 0$

$\therefore \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

This can also be expressed as (uv)' = u'v + uv'

Example 8

Differentiate the following functions with respect to x.

(a) $x^2(x+2)^3$

Here u = x^2 and v = $(x+2)^3$

$\frac{du}{dx} = 2x \; and \; \frac{dv}{dx} = 3(x+2)^2$

But $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{\delta y}{\delta x} = 2x(x+2)^3 + 3x^2(x+2)^2$

$\quad = (x+2)^2(2x^2 + 4x + 3x^2)$

$\quad = (x+2)^2(5x^2 + 4x)$

$\quad = x(x+2)^2(5x+4)$

$\therefore \frac{\delta}{\delta x}(x^2(x+2)^3 = x(x+2)^2(5x+4)$

(b) $(x+2)^3(1-x^2)^4$

u = $(x+2)^3$ and $v =$ $(1-x^2)^4$

$\frac{du}{dx} = 3(x+2)^2 \; and$

$\frac{dv}{dx} = 4(1-x^2)^3(-2x) = -8x(1-x^2)^3$

But $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = -8x(x+2)^3(1-x^2)^3 + 3(1-x^2)^4(x+2)^2$

$= (1-x^2)^3(x+2)^2\,[-8x(x+2) + 3(1-x^2)]$

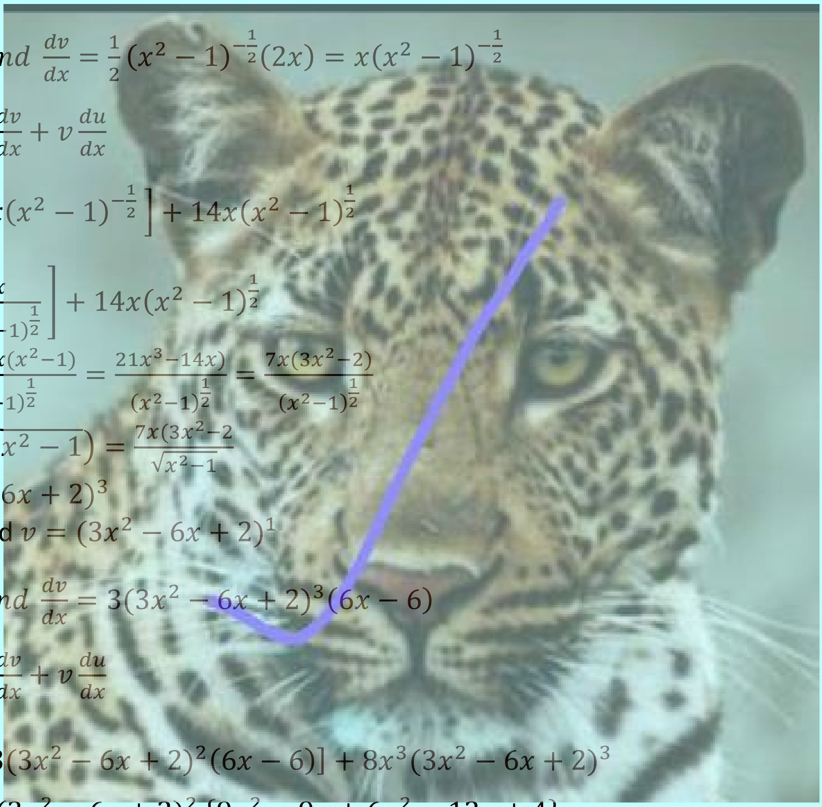
$= (1-x^2)^3(x+2)^2\,[-8x^2 - 16x + 3 - 3x^2]$

$= (1-x^2)^3(x+2)^2\,(3 - 16x - 11x^2)$

$\therefore \frac{\delta}{\delta x}\{(x+2)^3(1-x^2)^4\} = (1-x^2)^3(x+2)^2\,(3 - 16x - 11x^2)$

(c) $7x^2\sqrt{x^2-1}$

u = $7x^2$ and v = $(x^2-1)^{\frac{1}{2}}$



$\frac{du}{dx} = 14x\ and\ \frac{dv}{dx} = \frac{1}{2}(x^2-1)^{-\frac{1}{2}}(2x) = x(x^2-1)^{-\frac{1}{2}}$

But $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = 7x^2\left[x(x^2-1)^{-\frac{1}{2}}\right] + 14x(x^2-1)^{\frac{1}{2}}$

$= 7x^2\left[\frac{x}{(x^2-1)^{\frac{1}{2}}}\right] + 14x(x^2-1)^{\frac{1}{2}}$

$= \frac{7x^3+14x(x^2-1)}{(x^2-1)^{\frac{1}{2}}} = \frac{21x^3-14x)}{(x^2-1)^{\frac{1}{2}}} = \frac{7x(3x^2-2)}{(x^2-1)^{\frac{1}{2}}}$

$\therefore \frac{\delta}{\delta x}\left(7x^2\sqrt{x^2-1}\right) = \frac{7x(3x^2-2}{\sqrt{x^2-1}}$

(d) $2x^4(3x^2-6x+2)^3$

$u = 2x^4$ and $v = (3x^2-6x+2)^1$

$\frac{du}{dx} = 8x^3\ and\ \frac{dv}{dx} = 3(3x^2-6x+2)^3(6x-6)$

But $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = 2x^4[3(3x^2-6x+2)^2(6x-6)] + 8x^3(3x^2-6x+2)^3$

$= 4x^3\,(3x^2-6x+2)^2\,\{9x^2 - 9x + 6x^2 - 12x + 4\}$

$= 4x^3\,(3x^2-6x+2)^2\,\{15x^2 - 21x + 4\}$

$\therefore \frac{\delta}{\delta x}(2x^4(3x^2-6x+2)^3) = 4x^3\,(3x^2-6x+2)^2\,(15x^2-21x+4)$

(e) $\sqrt{(6+x)}\sqrt{(3-2x)}$

$u = (6+x)^{\frac{1}{2}}$ and $v = (3-2x)^{\frac{1}{2}}$

$\frac{du}{dx} = \frac{1}{2}(6+x)^{-\frac{1}{2}}\ and\ \frac{dv}{dx} = \frac{1}{2}(3-2x)^{-\frac{1}{2}}(-2) = -(3-2x)^{-\frac{1}{2}}$

But $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = \frac{-\sqrt{(6+x)}}{\sqrt{3-2x}} + \frac{\sqrt{(3-2x)}}{2\sqrt{(6+x)}} = \frac{-12-2x+3-2x}{2\sqrt{3-2x}\sqrt{(6+x)}} = \frac{-(9x+4)}{2\sqrt{3-2x}\sqrt{(6+x)}}$

$\therefore \frac{d}{dx}\left(\sqrt{(6+x)}\sqrt{(3-2x)}\right) = \frac{-(9x+4)}{2\sqrt{3-2x}\sqrt{(6+x)}}$

(f) $sin^2xcos2x$

$u = sin^2x \text{ and } v = cos2x$

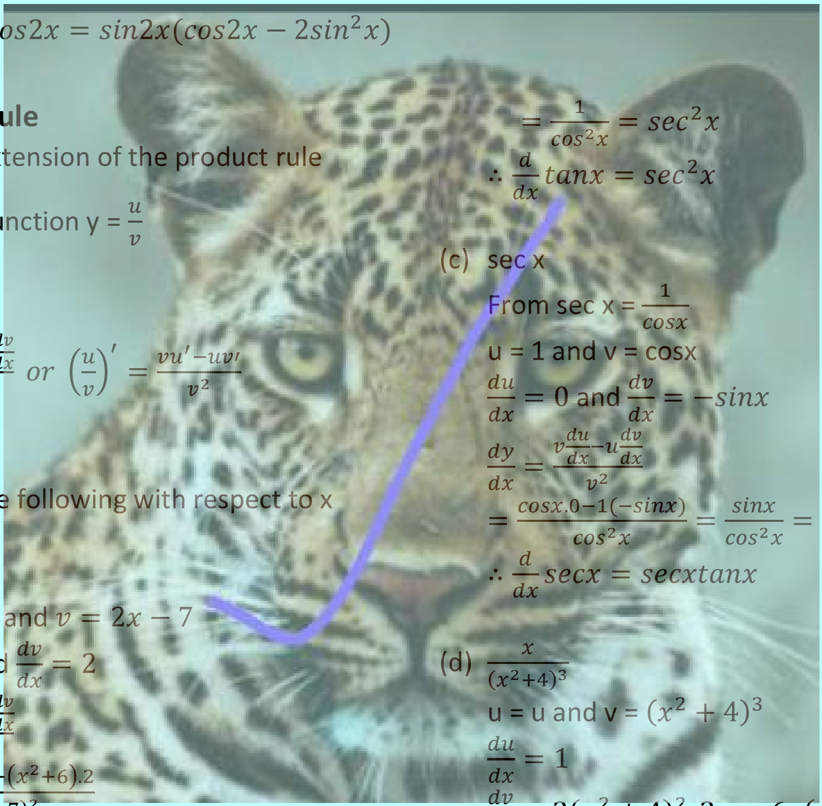
$u = sin^2x$ and $v = cos2x$

$\frac{du}{dx} = 2sinxcosx \text{ and } \frac{dv}{dx} = -2sin2x$

But $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\frac{dy}{dx} = -2sin2xsin^2x + cos2x(2cosxsinx)$$
$$= -2sin2xsin^2x + cos2xsin2x$$
$$= sin2x(cos2x - 2sin^2x)$$

$\therefore \frac{d}{dx}sin^2xcos2x = sin2x(cos2x - 2sin^2x)$



Quotient rule

This is an extension of the product rule

Given the function y = $\frac{u}{v}$

Then

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \text{ or } \left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$$

Example 9

Differentiate the following with respect to x

(a) $\frac{x^2+6}{2x-7}$

$u = x^2 + 6$ and $v = 2x - 7$

$\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = 2$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$= \frac{(2x-7).x-(x^2+6).2}{(2x-7)^2}$$
$$= \frac{2(2x^2-7x-x^2-6)}{(2x-7)^2} = \frac{2(x^2-7x-6)}{(2x-7)^2}$$

$\therefore \frac{d}{dx}\left(\frac{x^2+6}{2x-7}\right) = \frac{2(x^2-7x-6)}{(2x-7)^2}$

(b) tanx

From tan x =$\frac{sinx}{cosx}$

$sinx$ and $v = cosx$

$\frac{du}{dx} = cosx$ and $\frac{dv}{dx} = -sinx$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$= \frac{sinxcosx-(-sinxcosx)}{cos^2x}$$
$$= \frac{sin^2x+cos^2x}{cos^2x}$$
$$= \frac{1}{cos^2x} = sec^2x$$

$\therefore \frac{d}{dx}tanx = sec^2x$

(c) sec x

From sec x = $\frac{1}{cosx}$

u = 1 and v = cosx

$\frac{du}{dx} = 0$ and $\frac{dv}{dx} = -sinx$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$= \frac{cosx.0-1(-sinx)}{cos^2x} = \frac{sinx}{cos^2x} = secxtanx$$

$\therefore \frac{d}{dx}secx = secxtanx$

(d) $\frac{x}{(x^2+4)^3}$

u = u and v = $(x^2+4)^3$

$\frac{du}{dx} = 1$

$\frac{dv}{dx} = 2(x^2+4)^2.2x = 6x(x^2+4)^2$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$= \frac{(x^2+4)^3 - 6x^2(x^2+4)^2}{((x^2+4)^3)^2}$$
$$= \frac{(4-5x^2)}{(x^2+4)^6}$$

Revision exercise 4

1. Find the derivatives of each of the following

a. $\frac{sinx}{x}$ $\left[\frac{xcosx-sinx}{x^2}\right]$

b. $\frac{cosx}{x^2}$ $\left[\frac{-(xsinx+2cosx}{x^3}\right]$

c. $\frac{2x+1}{3x-4}$ $\left[\frac{-11}{(3x-4)^2}\right]$

d. $\frac{3x-4}{2x+1}$ $\left[\frac{11}{(2x+1)^2}\right]$

e. $\frac{x^2-3}{2x+1}$ $\left[\frac{-2(x^2+1+3)}{(2x+1)^2}\right]$

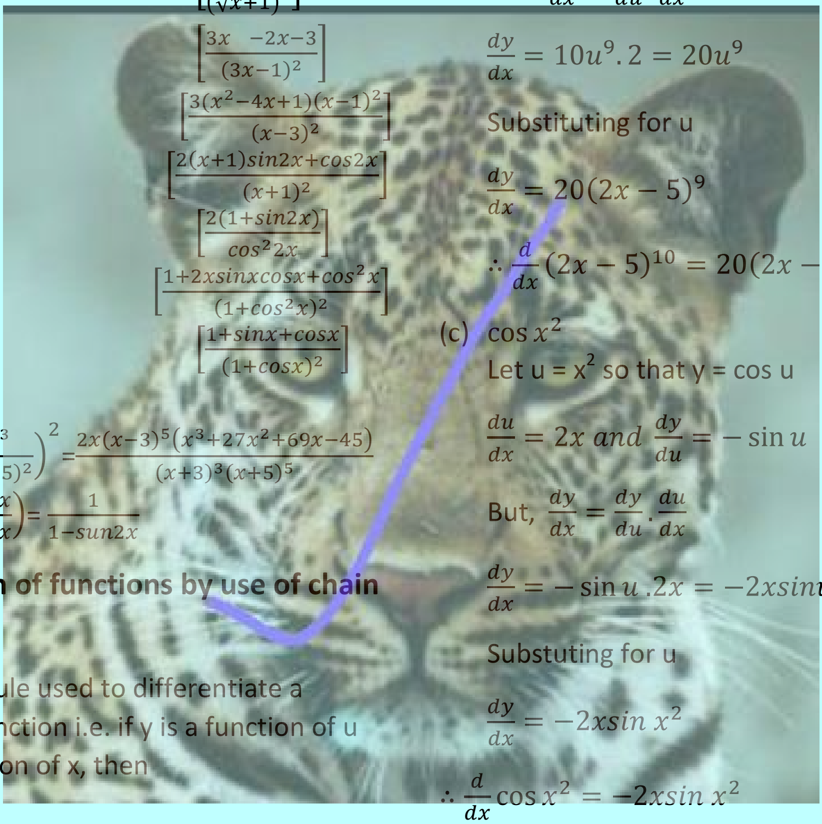
f. $\frac{2x+1}{x^2-3}$ $\left[\frac{-2(x^2+1+3)}{(x^2-3)^2}\right]$

g. $\sqrt{\frac{x^3}{x^2-1}}$ $\left[\frac{\sqrt{x}(x^2-3)}{2\sqrt{x^2-1}}\right]$

h. $\sqrt{\frac{3+x}{2-3x}}$ $\left[\frac{11}{2\sqrt{(3+x)}\sqrt{(2-3x)}}\right]$

i. $\frac{\sqrt{x}+1}{\sqrt{x}-1}$ $\left[-\frac{1}{\sqrt{x}(\sqrt{x}-1)^2}\right]$

j. $\frac{2x}{\sqrt{x}+1}$ $\left[\frac{\sqrt{x}+2}{(\sqrt{x}+1)^2}\right]$



k. $\frac{x^2+1}{3x-1}$ $\left[\frac{3x\quad -2x-3}{(3x-1)^2}\right]$

l. $\frac{x(x-1)^3}{x-3}$ $\left[\frac{3(x^2-4x+1)(x-1)^2}{(x-3)^2}\right]$

m. $\frac{cos2x}{x+1}$ $\left[\frac{2(x+1)sin2x+cos2x}{(x+1)^2}\right]$

n. $\frac{1=sin2x}{cos2x}$ $\left[\frac{2(1+sin2x)}{cos^2 2x}\right]$

o. $\frac{x}{1+cos^2x}$ $\left[\frac{1+2xsinxcosx+cos^2x}{(1+cos^2x)^2}\right]$

p. $\frac{1+sinx}{1+cosx}$ $\left[\frac{1+sinx+cosx}{(1+cosx)^2}\right]$

2. Show that

(a) $\frac{d}{dx}\left(\frac{x(x-3)^3}{(x+3)(x+5)^2}\right)^2 = \frac{2x(x-3)^5(x^3+27x^2+69x-45)}{(x+3)^3(x+5)^5}$

(b) $\frac{d}{dx}\left(\frac{cosx+sinx}{cosx-sinx}\right) = \frac{1}{1-sun2x}$

Differentiation of functions by use of chain rule

Chain rule is a rule used to differentiate a function of a function i.e. if y is a function of u and u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

Example 10

Find $\frac{dy}{dx}$ of each of the following using chain rule

(a) $(x+5)^3$

Let u = (x + 5); thus y = u^3

$\frac{dy}{du} = 3u^2\ and\ \frac{du}{dx} = 1$

Using chain rule; $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

$\frac{dy}{dx} = 3u^2.1 = 3u^2$

Substituting for u

$\frac{dy}{dx} = 3(x+5)^2$

$\therefore \frac{d}{dx}(x+5)^3 = 3(x+5)^2$

(b) $(2x-5)^{10}$

Let $u = 2x - 5$ so that y = u^{10}

$\frac{du}{dx} = 2\ and\ \frac{dy}{du} = 10u^9$

But, $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

$\frac{dy}{dx} = 10u^9.2 = 20u^9$

Substituting for u

$\frac{dy}{dx} = 20(2x-5)^9$

$\therefore \frac{d}{dx}(2x-5)^{10} = 20(2x-5)^9$

(c) $\cos x^2$

Let u = x^2 so that y = cos u

$\frac{du}{dx} = 2x\ and\ \frac{dy}{du} = -\sin u$

But, $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

$\frac{dy}{dx} = -\sin u\ .2x = -2xsinu$

Substuting for u

$\frac{dy}{dx} = -2xsin\ x^2$

$\therefore \frac{d}{dx}\cos x^2 = -2xsin\ x^2$

(d) cos^2x

Since cos^2x = $(cosx)^2$

Let u = cosx so that y =u^2

$\frac{du}{dx} = -sinx\ and\ \frac{dy}{du} = 2u$

But, $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

$\frac{dy}{dx} = 2u.-sinx = -2usinx$

Substituting for u

$\frac{dy}{dx} = -2cosxsinx$

$\therefore \frac{d}{dx}cos^2x$=$-2cosxsinx$

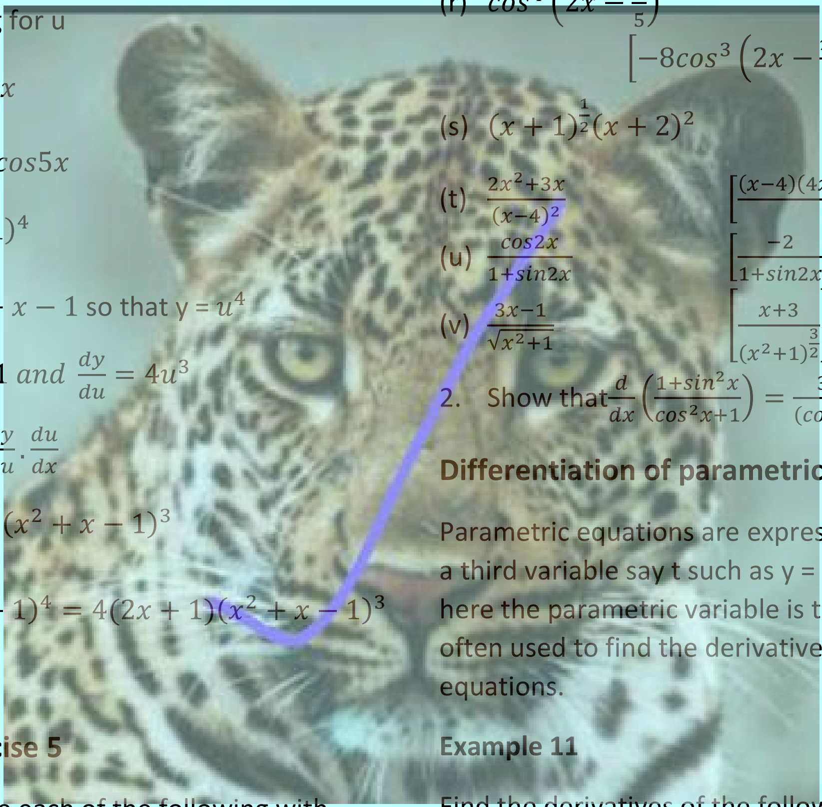
(e) sin 5x

Let u = 5x so that y = sinu

$\frac{du}{dx}5 \ and \ \frac{dy}{du} = cosu$

But, $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

$\frac{dy}{dx} = cosu.5 = 5cosu$



Substituting for u

$\frac{dy}{dx} = 5cos5x$

$\therefore \frac{d}{dx}sin5x = 5cos5x$

(f) $(x^2 + x - 1)^4$

Let u = $x^2 + x - 1$ so that y = u^4

$\frac{du}{dx} = 2x + 1 \ and \ \frac{dy}{du} = 4u^3$

But, $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

$\frac{dy}{dx} = 4(2x+1)(x^2 + x - 1)^3$

$\therefore \frac{d}{dx}(x^2 + x - 1)^4 = 4(2x+1)(x^2 + x - 1)^3$

Revision exercise 5

1. Differentiate each of the following with respect to x using chain rule

(a) $2(1-x)^5$ $[-10(1-x)^4]$

(b) $(x^2+3)^4$ $[8x(x^2+3)^3]$

(c) $\frac{1}{3-7x}$ $\left[\frac{7}{(3-7x)^2}\right]$

(d) $\sqrt{6x+1}$ $\left[\frac{3}{\sqrt{6x+1}}\right]$

(e) $(6x^2-5)^4$ $[48x(6x^2-5)^3]$

(f) $(2x-5)^{-3}$ $[-6(2x-5)^{-4}]$

(g) $(3x+2)^{-1}$ $[-3(3x+2)^{-2}]$

(h) $(x^2+3)^{-2}$ $[-4x(x^2+3)^{-3}]$

(i) $(5-2x^3)^{-1}$ $[6x^2(5-2x^3)^{-2}]$

(j) $\frac{1}{3+4x}$ $\left[\frac{-4}{(3+4x)^2}\right]$

(k) $(2x-1)^{\frac{1}{2}}$ $\left[\frac{1}{\sqrt{2x-1}}\right]$

(l) $(6-x)^{\frac{1}{3}}$ $\left[\frac{-1}{3(6-x)^{\frac{2}{3}}}\right]$

(m) $(x^3-2)^{\frac{2}{3}}$ $\left[\frac{2x^2}{(6-x)^{\frac{1}{3}}}\right]$

(n) $(4-x^5)^{-\frac{1}{5}}$ $\left[\frac{x^4}{(4-x^5)^{\frac{6}{5}}}\right]$

(o) $\sqrt{x^3-6x}$ $\left[\frac{3(x^2-2)}{2\sqrt{x^3-6x}}\right]$

(p) $\frac{1}{x^2-3x+5}$ $\left[\frac{3-2x^2}{(x^2-3x+5)^2}\right]$

(q) $sin\left(4x-\frac{\pi}{5}\right)$ $\left[4cos\left(4x-\frac{\pi}{5}\right)\right]$

(r) $cos^4\left(2x-\frac{\pi}{5}\right)$ $\left[-8cos^3\left(2x-\frac{\pi}{5}\right)sin\left(2x-\frac{\pi}{5}\right)\right]$

(s) $(x+1)^{\frac{1}{2}}(x+2)^2$ $\left[\frac{(5x+6)(x+2)}{2(x+1)^{\frac{1}{2}}}\right]$

(t) $\frac{2x^2+3x}{(x-4)^2}$ $\left[\frac{(x-4)(4x+3)-2(2x^2+3x)}{(x-4)^3}\right]$

(u) $\frac{cos2x}{1+sin2x}$ $\left[\frac{-2}{1+sin2x}\right]$

(v) $\frac{3x-1}{\sqrt{x^2+1}}$ $\left[\frac{x+3}{(x^2+1)^{\frac{3}{2}}}\right]$

2. Show that $\frac{d}{dx}\left(\frac{1+sin^2x}{cos^2x+1}\right) = \frac{3sin2x}{(cos^2x+1)^2}$

Differentiation of parametric equations

Parametric equations are expressed in terms of a third variable say t such as y = t^2 and x = 2t +1, here the parametric variable is t. Chain rule is often used to find the derivatives of these equations.

Example 11

Find the derivatives of the following in terms of parameter t.

(a) y = $3t^2$ + 2t, x= 1-2t

$\frac{dy}{dt} = 6t + 2$ and $\frac{dx}{dt} = -2$

$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$

$= (6t+2).\frac{1}{-2}$

$= -(3t+1)$

(b) y = $(1 + 2t)^3$, x= t^3

$\frac{dy}{dt} = 6(1+2t)^2$ and $\frac{dx}{dt} = 3t^2$

$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$

$=\frac{6(1+2t)^2}{3t^2} = \frac{2(1+2t)^2}{t^2}$

(c) $x = t^2$, $y = 4t-1$

$\frac{dy}{dt} = 4$ and $\frac{dx}{dt} = 2t$

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

$= \frac{4}{2t} = \frac{2}{t}$

(d) $x = \frac{2}{3+\sqrt{t}}$, $y = \sqrt{t}$

$x = 2\left(3 + t^{\frac{1}{2}}\right)^{-1}$

$\frac{dx}{dt} = -2\left(3 + t^{\frac{1}{2}}\right)^{-2} \cdot \frac{1}{2} t^{-\frac{1}{2}} = \frac{-1}{\left(3+t^{\frac{1}{2}}\right)^2 t^{\frac{1}{2}}}$

$\frac{dy}{dt} = \frac{1}{2\sqrt{t}}$

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

$= \frac{1}{2\sqrt{t}} \cdot -\left(3 + t^{\frac{1}{2}}\right)^2 t^{\frac{1}{2}} = \frac{-\left(3+t^{\frac{1}{2}}\right)^2}{2}$

(e) $x = a\cos t$ and $y = b\sin t$ when $t = \frac{\pi}{4}$

$\frac{dx}{dt} = -a\sin t$ and $\frac{dy}{dt} = b\cos t$

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

$= \frac{b\cos t}{-a\sin t}$

At $t = \frac{\pi}{4}$

$\frac{dy}{dx} = \frac{b\cos\frac{\pi}{4}}{-a\sin\frac{\pi}{4}} = -\frac{b}{a}$

(f) $x = a\sec t$ and $y = b\tan t$ when $t = \frac{\pi}{6}$

$\frac{dx}{dt} = a\sec t \tan t$ and $\frac{dy}{dt} = b\sec^2 t$

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

$= \frac{b\sec^2 t}{a\sec t\tan t} = \frac{b}{a\sin t}$

At $t = \frac{\pi}{6}$

$\frac{dy}{dx} = \frac{b}{a\sin\frac{\pi}{6}} = \frac{2b}{a}$

Revision exercise 6

Find $\frac{dy}{dx}$ for each of the following

(a) $x = 2\sqrt{t}, y = 5t - 4$ $\quad [5\sqrt{t}]$

(b) $x = 4\sqrt{t} - t, y = t^2 - 2\sqrt{t}$ $\quad \left[\frac{2\sqrt{t^3}-1}{2-\sqrt{t}}\right]$

(c) $x = \frac{2}{\sqrt[3]{3t-4}}, y = \sqrt[3]{6t+1}$ $\quad \left[\sqrt[3]{\frac{(3t-4)^4}{(6t+1)^2}}\right]$

(d) $y = \tan^2(3x+1)$

$[6\tan(3x+1)\sec^2(3x+1)]$

(e) $x = t + 5$, $y = t^2 - 2t$ $\quad [2(t-1)]$

(f) $x = t^6$, $y = 6t^3 - 5$ $\quad [3t^{-3}]$

(g) $x = \sqrt{t-1}, y = \frac{1}{t}$ $\quad \left[\frac{-2\sqrt{t-1}}{t^2}\right]$

(h) $x = t^2(3t-1), y = \sqrt{3t+4}$

$\left[\frac{3}{2\sqrt{3t+4}(9t^2-2t)}\right]$

(i) $x = 3(2\theta - \sin\theta), y = 3(1 - \cos 2\theta)$

$[\cot\theta]$

(j) $x = \cos 2\theta, y = \cos\theta$ $\quad \left[\frac{1}{4}\sec\theta\right]$

(k) $x = t^2 \sin 3t, y = t^2 \cos 3t$ $\quad \left[\frac{2-3t\sin 3t}{2\tan 3t+3t}\right]$

(l) $x = t + 2\cos t, y = t + 2\cos t$ $\quad \left[\frac{1-2\sin t}{3+\cos t}\right]$

(m) $x = 1 + 2\sin t, y = \sin t + \cos t$

$\left[\frac{1-2\sin t}{3+\cos t}\right]$

Differentiation of implicit functions

The functions given in the form y = f(x) such as $y = 2x$, $y = x^5 + 3x$ etc. are known as explicit functions whereas functions that cannot be expressed in the form y = f(x) such as $y^2+2xy = 5$, $x^2 + 5xy + y^2 = 4$ etc. are known as implicit functions because y cannot be expressed easily in terms of x.

When differentiating such functions with respect to x or y, we consider each of the individual terms in the equation given

Example 12

Find $\frac{dy}{dx}$ for each of the following functions.

(a) $x^2 - 6y^3 + y = 0$

$\frac{d}{dx}(x^2) - \frac{d}{dx}(6y^2) + \frac{d}{dx}(y) = 0$

$2x - 18y^2\frac{dy}{dx} + \frac{dy}{dx} = 0$

$\frac{dy}{dx}(18y^2 - 1) = 2x$

$\frac{dy}{dx} = \frac{2x}{18y^2-1}$

(b) $x^2 y = 5x + 2$

$\frac{d}{dx}(x^2 y) = \frac{d}{dx}(5x) + \frac{d}{dx}(2)$

$\frac{d}{dx}(x^2 y)$ is done by use of product rule

$x^2\frac{d}{dx}(y) + y\frac{d}{dx}(x^2) = \frac{d}{dx}(5x) + \frac{d}{dx}(2)$

$$x^2\frac{dy}{dx}+2xy=5$$
$$\frac{dy}{dx}=\frac{5-2xy}{x^2}$$

(c) $(x+y)^5-7x^2=0$

$$\frac{d}{dx}(x+y)^5-\frac{d}{dx}7x^2=0$$
$$5(x+y)^4\frac{d}{dx}(x+y)-14x=0$$
$$5(x+y)^4\left(1+\frac{dy}{dx}\right)=14x$$
$$\frac{dy}{dx}=\frac{14x}{5(x+y)^4}-1$$
$$=\frac{14x-5(x+y)^4}{5(x+y)^4}$$

(d) $Siny\ +\ x^2y^3-cosx\ =\ 2y$

$$\frac{d}{dx}siny+\ x^2\frac{d}{dx}y^3+y^3\frac{d}{dx}x^2-\frac{d}{dx}cox=\frac{d}{dx}2y$$
$$\text{cosy}\frac{dy}{dx}+2y^2x^2\frac{dy}{dx}+2xy^3+sinx=2\frac{dy}{dx}$$
$$\frac{dy}{dx}(\cos y+2y^2x^2-2)=-(2xy^3+sinx)$$
$$\frac{dy}{dx}=\frac{-(2xy^3+sinx)}{(\cos y+2y^2x^2-2)}$$

(e) $y^2+x^3-y^3+6=3y$

$$\frac{d}{dx}y^2+\frac{d}{dx}x^3-\frac{d}{dx}y^3+\frac{d}{dx}6=\frac{d}{dx}3y$$
$$2y\frac{dy}{dx}+3x^2-3y^2\frac{dy}{dx}+0=3\frac{dy}{dx}$$
$$3x^2=\frac{dy}{dx}(3y^2-2y+3)$$
$$\frac{dy}{dx}=\frac{3x^2}{(3y^2-2y+3)}$$

(f) $y^2+x^3-xy+cosy=0$

$$\frac{d}{dx}y^2+\frac{d}{dx}x^3-x\frac{d}{dx}y-y\frac{d}{dx}x+\frac{d}{dx}cosy=0$$
$$2y\frac{dy}{dx}+2x^2-x\frac{dy}{dx}-y-siny\frac{dy}{dx}=0$$
$$\frac{dy}{dx}(2y-x-siny)=y-2x^2$$
$$\frac{dy}{dx}=\frac{y-2x^2}{(2y-x-siny)}$$

Revision exercise 7

1. Find $\frac{dy}{dx}$ for each of the following functions

(a) $\frac{x^3}{x+y}=2$ $\left[\frac{3x^2-2}{2}\right]$

(b) $2x-y^3=3xy$ $\left[\frac{2-3y}{3x+3y^2}\right]$

(c) $x^6-5xy^3=9xy$ $\left[\frac{6x^5-y^2-9y}{3x(3+5y^3)}\right]$

(d) $\frac{x^2}{x+y}=2x$ $\left[\frac{x+y}{x}\right]$

(e) $\frac{y}{x^2-7y^3}=x^5$ $\left[\frac{7x^4(x^2-5y^3)}{1+21x^5y^2}\right]$

(f) $\sqrt{x}+\sqrt{y}$ $\left[\sqrt{\frac{y}{x}}\right]$

(g) $\frac{y}{x}+\frac{x}{y}=1$ $\left[\frac{y}{x}\right]$

(h) $siny+\ x^2+4y=cosx$ $\left[\frac{-\sin x-2x}{4+cosy}\right]$

(i) $3xy^2+\cos y^2=2x^3+5$ $\left[\frac{6x^2-3y^2}{6xy-2ysiny^2}\right]$

(j) $5x^2-x^3siny+5xy=10$ $\left[\frac{10x-3x^2siny+5y}{x^3cosy-5x}\right]$

(k) $x-\cos x^2+\frac{y^2}{x}+3x^5=4x^3$ $\left[\frac{12x^4-15x^6+y^2-2x^3\sin x^2-x^2}{2xy}\right]$

(l) $\tan 5y-ysinx+3xy^2=9$ $\left[\frac{ycosx-3y^2}{5sec^2 5y-sinx-6xy}\right]$

(m) $x^2+xy+y^2-3x-y=3$ $\left[\frac{3-2x-y}{x+2y-1}\right]$

(n) $y^2-5xy+8x^2=2$ $\left[\frac{5y-16x}{2y-5x}\right]$

2. For each of the following find the gradient of the stated curve at the point specified,

(a) $xy^2-6y=8\ at\ (2,1)$ $\left[\frac{1}{10}\right]$

(b) $3y^4-7xy^2-12y=5\ at(-2,1)$ $\left[\frac{1}{4}\right]$

(c) $\frac{x^2}{x-y}=8\ at\ (4,2)$ $[0]$

(d) $\frac{2}{x}+\frac{5}{y}=2xy\ at\ \left(\frac{1}{2},5\right)$ $[-15]$

(e) $(x+2y)^4=1\ at\ (5,-2)$ $\left[-\frac{1}{2}\right]$

(f) $x^2+6y^2=10\ at\ (2,-1)$ $\left[\frac{1}{3}\right]$

(g) $x^3+4xy=15+y^2\ at\ (2,1)$ $\left[-2\frac{2}{3}\right]$

Differentiation of inverse trigonometric functions

Example 13

Differentiate the following functions with respect to x

(a) $cos^{-1}x$

Let y =$cos^{-1}x$

cosy = x

$-siny\frac{dy}{dx} = 1$

$-(1 - cos^2x)^{\frac{1}{2}}\frac{dy}{dx} = 1$

$-(1 - x^2)^{\frac{1}{2}}\frac{dy}{dx} = 1$

$\frac{dy}{dx} = -\frac{1}{\sqrt{(1-x^2}}$

(b) $sin^{-1}x$

Let y = $sin^{-1}x$

siny = x

$cosy\frac{dy}{dx} = 1$

$(1 - sin^2x)^{\frac{1}{2}}\frac{dy}{dx} = 1$

$(1 - x^2)^{\frac{1}{2}}\frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\sqrt{(1-x^2}}$

(c) $tan^{-1}x$

Let y = $tan^{-1}x$

$tan\ y = x$

$sec^2y\frac{dy}{dy} = 1$

$(1 + tan^2y)\frac{dy}{dx} = 1$

$(1 + x^2)\frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{1+x^2}$

(d) $cos^{-1}(-2x^2)$

Let y = $cos^{-1}(-2x^2)$

$cos\ y = (-2x^2)$

$-siny\frac{dy}{dy} = \frac{-4x}{2} = -4x$

$\sqrt{(1 - cos^2y)}\frac{dy}{dx} = 4x$

$\sqrt{(1 - (-2x^2)^2)}\frac{dy}{dx} = 4x$

$\frac{dy}{dx} = \frac{4x}{\sqrt{1-4x^4}}$

(e) $sin^{-1}\left(\frac{1-x}{1+x}\right)$

Let y = $sin^{-1}\left(\frac{1-x}{1+x}\right)$

siny =$\left(\frac{1-x}{1+x}\right)$

$cosy\frac{dy}{dx} = \frac{-(1+x)-(1-x)}{(1+x)^2}$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{1-x}{1+x}\right)^2}}.\frac{-(1+x)-(1-x)}{(1+x)^2}$

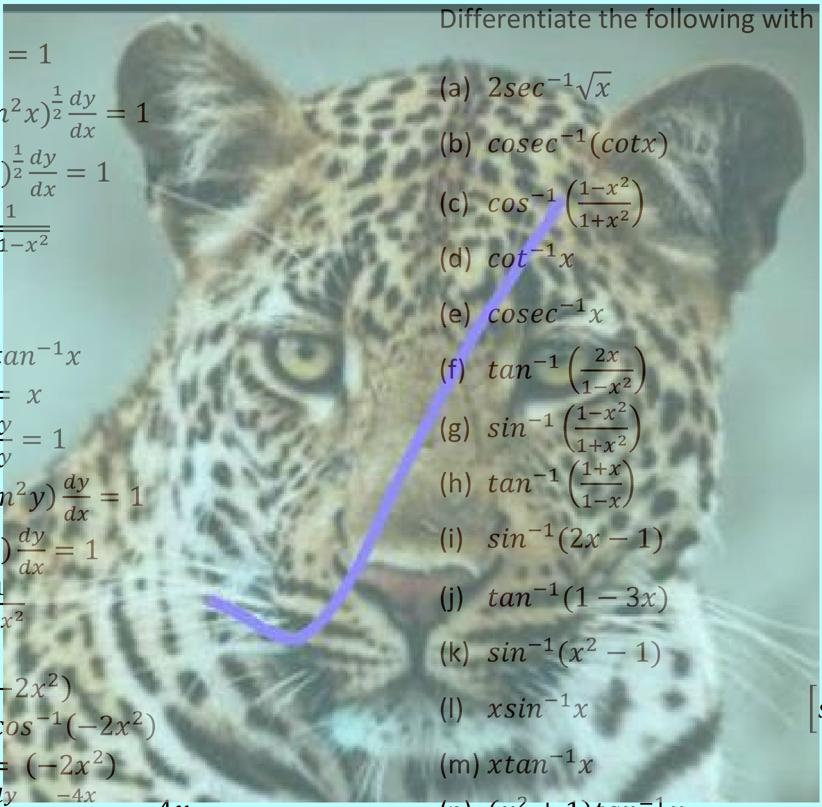
$= \frac{1+x}{\sqrt{(1+x)^2-(1-x)^2}}.\frac{-(1+x)-(1-x)}{(1+x)^2}$

$= \frac{1}{\sqrt{(1+x)^2-(1-x)^2}}.\frac{-2}{(1+x)}$

$= \frac{1}{\sqrt{4x}}.\frac{-2}{(1+x)}$

$= \frac{-1}{\sqrt{x}(1 + x)}$

Revision exercise 8



Differentiate the following with respect to x

(a) $2sec^{-1}\sqrt{x}$ $\left[\frac{1}{x\sqrt{1-x^2}}\right]$

(b) $cosec^{-1}(cotx)$ $\left[\frac{1}{cosx\sqrt{cos2x}}\right]$

(c) $cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ $\left[\frac{2}{1+x^2}\right]$

(d) $cot^{-1}x$ $\left[\frac{1}{1+x^2}\right]$

(e) $cosec^{-1}x$ $\left[\frac{-1}{\sqrt{x^2-1}}\right]$

(f) $tan^{-1}\left(\frac{2x}{1-x^2}\right)$ $\left[\frac{2}{1+x^2}\right]$

(g) $sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ $\left[\frac{-2}{1+x^2}\right]$

(h) $tan^{-1}\left(\frac{1+x}{1-x}\right)$ $\left[\frac{1}{1+x^2}\right]$

(i) $sin^{-1}(2x - 1)$ $\left[\frac{1}{\sqrt{x}(1-x)}\right]$

(j) $tan^{-1}(1 - 3x)$ $\left[\frac{-3}{2-6x+9x^2}\right]$

(k) $sin^{-1}(x^2 - 1)$ $\left[\frac{2}{\sqrt{2-x^2}}\right]$

(l) $xsin^{-1}x$ $\left[sin^{-1}x + \frac{x}{\sqrt{1-x^2}}\right]$

(m) $xtan^{-1}x$ $\left[tan^{-1}x + \frac{x}{1+x^2}\right]$

(n) $(x^2 + 1)tan^{-1}x$ $[2xtan^{-1}x + 1]$

Second derivatives

Suppose y is a function of x, the first derivative of y with respect to x is denoted as $\frac{dy}{dx}$ or f'(x)

The result of differentiating $\frac{dy}{dx}$ with respect to x is the second derivative denoted by $\frac{d^2y}{dx^2}$ or f''(x)

Note that If $\frac{d^2y}{dx^2}$ is used to determine the natures of stationary points

A stationary point on a curve occurs when $\frac{dy}{dx}=0$ Once you have established where there is a stationary point, the type of stationary point (maximum, minimum or point of inflexion) can be determined using the second derivative.

If $\frac{d^2y}{dx^2}$ is positive, then it is a minimum point

If $\frac{d^2y}{dx^2}$ is negative, then it is a maximum point

If $\frac{d^2y}{dx^2}$ =0 then it could be maximum, maximum or point of inflection

Example 14

Determine the second derivative of each of the following

(a) x^4

$\frac{dy}{dx}=4x^3$

$\frac{d^2y}{dx^2}=\frac{d}{dx}(4x^3)=12x^2$

(b) cos2x

$\frac{dy}{dx}=-2sin2x$

$\frac{d^2y}{dx^2}=\frac{d}{dx}(-2sin2x)=-4cos2x$

(c) $x^2(1-x)^2$

$x^2(1-x)^2=x^2(1-2x+x^2)$

$=x^2-2x^3+x^4$

$\frac{dy}{dx}=2x-6x^2+4x^3$

$\frac{d^2y}{dx^2}=2-12x+12x^2$

(d) $xsinx$

$\frac{dy}{dx}=sinx+xcosx$

$\frac{d^2y}{dx^2}=cosx+cosx-xsinx$

$=2xosx-xsinx$

(e) x^3sinx

$\frac{dy}{dx}=3x^2sinx+x^3cosx$

$\frac{d^2y}{dx^2}$

$=6xsinx+3x^2cosx+3x^2cosx-x^3sinx$

$=(6x-x^3)sinx+6x^2cosx$

(f) $xtan^{-1}$x

$\frac{dy}{dx}=tan^{-1}x+\frac{x}{1+x^2}$

$\frac{d^2y}{dx^2}=\frac{x}{1+x^2}+\frac{(1+x^2)(1)-x(2x)}{(1+x^2)^2}$

$=\frac{2}{(1+x^2)^2}$

(g) If $x^2+3xy-y^2=3$, find $\frac{dy}{dx}$and $\frac{d^2y}{dx^2}$ at the point (1,1)

$2x+3y+3x\frac{dy}{dx}-2y\frac{dy}{dx}=0$

$\frac{dy}{dx}=\frac{2x+3y}{2y-3x}$

At (1,1)

$\frac{dy}{dx}=\frac{2(1)+3(1)}{2(1)-3(1)}=-5$

$\frac{d^2y}{dx^2}=\frac{(2y-3x)\left(2+3\frac{dy}{dx}\right)-(2x+3y)\left(2\frac{dy}{dx}-3\right)}{(2y-3x)^2}$

Substituting for x =1, y= 1and $\frac{dy}{dx}=-5$

$\frac{d^2y}{dx^2}=\frac{(2-3)(2+3(-5))-(2+3)(2(-5)-3)}{(2-3)^2}$

$=\frac{(-1)(-13)-(5)(-13)}{(-1)^2}$

$=\frac{13+65}{1}=78$

Example 15 (parametric equation)

Find$\frac{d^2y}{dx^2}$in terms of t if

(a) $x=a(t^2-1)and\ y=2a(t+1)$,

$\frac{dx}{dt}=2at$

$\frac{dy}{dt}=2a$

$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}$

$=\frac{2a}{2at}=\frac{1}{t}$

$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)\cdot\frac{dt}{dx}$

$=\frac{d}{dx}(t^{-1})\cdot\frac{dt}{dx}$

$=\frac{-1}{t^2}\cdot\frac{1}{2at}$

$=\frac{-1}{2at^3}$

(b) $x=cost+sint\ and\ y=sint-cost$

$\frac{dx}{dt}=-sint+cost$

$\frac{dy}{dt}=cost+sint$

$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}$

$=\frac{-sint+cost}{sint+cost}$

$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{-sint+cost}{sint+cost}\right)\frac{dx}{dt}$

$= \frac{(-sint+cost)(-sint+cost)-(cost+sint)(-cost-sint}{(-sint+cost)^2(-sint+cost)}$

$= \frac{2}{(-sint+cost)^3}$

Revision exercise 9

1. Find $\frac{d^2y}{dx^2}$ of each of the following

(a) $\frac{x^2}{1+x}$ $\left[\frac{2}{(1+x)^3}\right]$

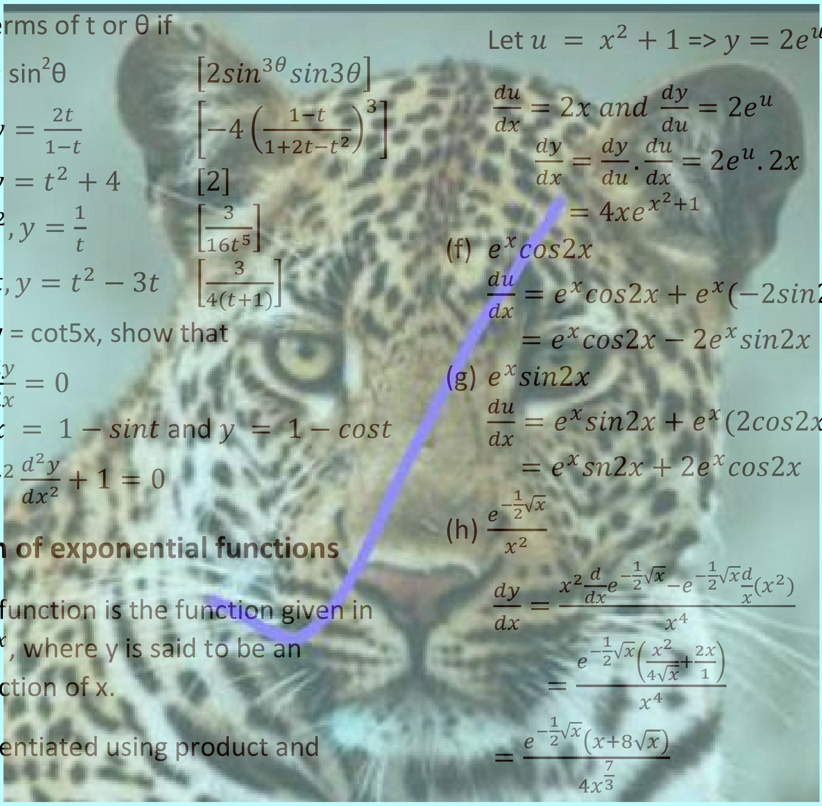
(b) $\frac{sinx}{x^2}$ $\left[\frac{(6-x^2)sinx-4cosx}{x^4}\right]$

(c) tan^2x $[4y(1+y)^2]$

(d) $tan3x$ $[18y(1+y^2)]$

(e) $xtanx$ $\left[\frac{2(x^2+y^2)(1+y)}{x^2}\right]$

(f) $sec2x$ $[4y(2y^2-1)]$



2. Find $\frac{d^2y}{dx^2}$ in terms of t or θ if

(a) x = cotθ, y = sin²θ $[2sin^{3\theta}sin3\theta]$

(b) $x = \frac{1+t^2}{1-t}$, $y = \frac{2t}{1-t}$ $\left[-4\left(\frac{1-t}{1+2t-t^2}\right)^3\right]$

(c) $x = t+3, y = t^2+4$ $[2]$

(d) $x = 3-2t^2, y = \frac{1}{t}$ $\left[\frac{3}{16t^5}\right]$

(e) $x = t^2+2t, y = t^2-3t$ $\left[\frac{3}{4(t+1)}\right]$

3. Given that y = cot5x, show that $\frac{d^2y}{dx^2} + 10y\frac{dy}{dx} = 0$

4. Given that $x = 1 - sint$ and $y = 1 - cost$ show that $y^2\frac{d^2y}{dx^2} + 1 = 0$

Differentiation of exponential functions

An exponential function is the function given in the form $y = e^x$, where y is said to be an exponential function of x.

These are differentiated using product and quotient rules.

Example 16

Differentiate each of the following with respect to x

(a) e^x

$\frac{d}{dx}(e^x) = e^x$

(b) e^{3x^2}

$Let\ u = 3x^2 and\ y = e^u$

$\frac{du}{dx} = 6x\ and\ \frac{dy}{du} = e^u$

$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = 6x.e^u$

$= 6xe^{3x^2}$

(c) e^{sinx}

Let $u = tanx \Rightarrow y = e^u$

$\frac{du}{dx} = cosx\ and\ \frac{dy}{du} = e^u$

$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = e^u.cosx$

$e^{tanx}cosx$

(d) e^{3x}

Let $u = 3x \Rightarrow y = e^u$

$\frac{du}{dx} = 3\ and\ \frac{dy}{du} = e^u$

$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = e^u.3$

$= 3e^{3x}$

(e) $y = 2e^{x^2+1}$

Let $u = x^2+1 \Rightarrow y = 2e^u$

$\frac{du}{dx} = 2x\ and\ \frac{dy}{du} = 2e^u$

$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = 2e^u.2x$

$= 4xe^{x^2+1}$

(f) $e^x cos2x$

$\frac{du}{dx} = e^x cos2x + e^x(-2sin2x)$

$= e^x cos2x - 2e^x sin2x$

(g) $e^x sin2x$

$\frac{du}{dx} = e^x sin2x + e^x(2cos2x)$

$= e^x sn2x + 2e^x cos2x$

(h) $\frac{e^{-\frac{1}{2}\sqrt{x}}}{x^2}$

$\frac{dy}{dx} = \frac{x^2\frac{d}{dx}e^{-\frac{1}{2}\sqrt{x}} - e^{-\frac{1}{2}\sqrt{x}}\frac{d}{x}(x^2)}{x^4}$

$= \frac{e^{-\frac{1}{2}\sqrt{x}}\left(\frac{x^2}{4\sqrt{x}} + \frac{2x}{1}\right)}{x^4}$

$= \frac{e^{-\frac{1}{2}\sqrt{x}}(x+8\sqrt{x})}{4x^{\frac{7}{3}}}$

Differentiation of logarithmic functions

Logarithms of numbers to base e is called natural logarithm or napeilian logarithm.

The natural logarithm of a number say x is denoted by $\log_e x$ or Inx

Let y $= \log_e x$

$e^y = x$

$\frac{d}{dx}(e^y) = \frac{d}{dx}(x)$

$\frac{dy}{dx} = \frac{\frac{d}{dx}(x)}{e^y} = \frac{\frac{d}{dx}(e^y)}{e^y}$

Example 17

Differentiate with respect to x

(a) Inx

Let y = Inx

$\frac{dy}{dx} = \frac{\frac{d}{dx}(x)}{x} = \frac{1}{x}$

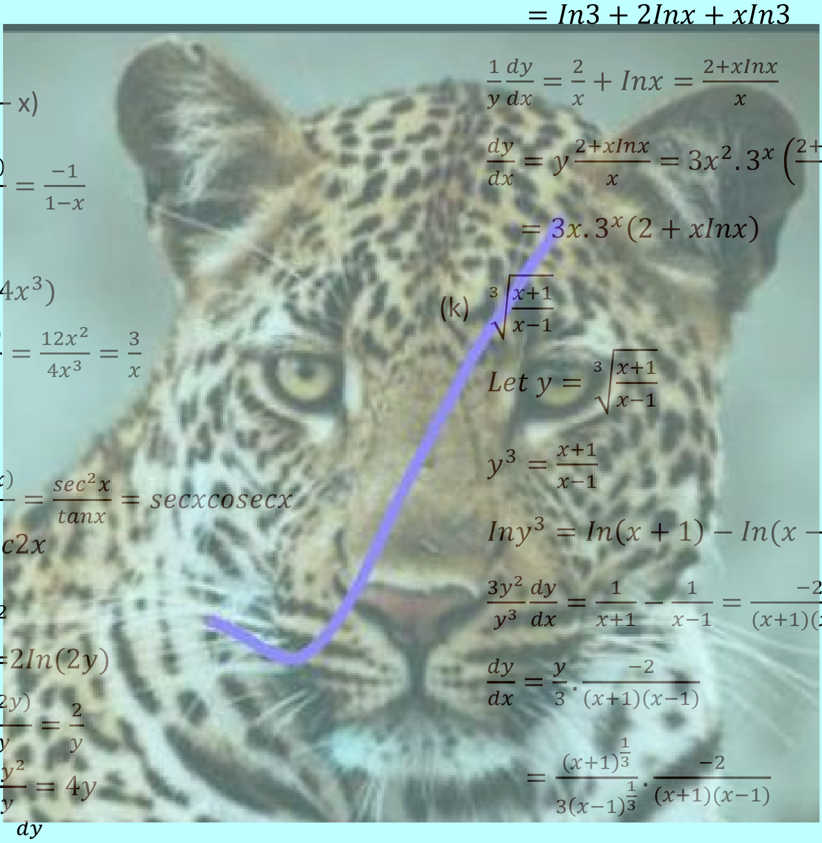
(b) In(1+2x)

Let y = In(1 + 2x)

$\frac{dy}{dx} = \frac{\frac{d}{dx}(1+2x)}{1+2x} = \frac{2}{1+2x}$

(c) In(1- x)

Let y = In(1 – x)



$\frac{dy}{dx} = \frac{\frac{d}{dx}(1-x)}{1-x} = \frac{-1}{1-x}$

(d) $In(4x^3)$

Let $y = In(4x^3)$

$\frac{dy}{dx} = \frac{\frac{d}{dx}(4x^3)}{4x^3} = \frac{12x^2}{4x^3} = \frac{3}{x}$

(e) In(tanx)

$\frac{dy}{dx} = \frac{\frac{d}{dx}(tanx)}{tanx} = \frac{sec^2x}{tanx} = secxcosecx$

$= 2cosec2x$

(f) $2y^2$

Let $q = 2y^2$

$Inq = 2y^2 = 2In(2y)$

$\frac{1}{q}\frac{dt}{dy} = 2\frac{\frac{d}{dy}(2y)}{2y} = \frac{2}{y}$

$\frac{dq}{dy} = \frac{2q}{y} = \frac{4y^2}{y} = 4y$

But $\frac{dq}{dy} = \frac{dq}{dy}.\frac{dy}{dx}$

$\frac{dq}{dx} = 4y\frac{dy}{dx}$

(g) Iny

Let q = Iny

$\frac{dq}{dy} = \frac{1}{y}$

But $\frac{dq}{dy} = \frac{dq}{dy}.\frac{dy}{dx}$

$\frac{dq}{dy} = \frac{1}{y}.\frac{dy}{dx}$

(h) 2^x

Let y = 2^x

Iny =In2^x=xIn2

$\frac{1}{y}\frac{dy}{dx} = In2$

$\frac{dy}{dx} = yIn2 = 2^x In2$

(i) 2^{x^2}

$Iny = In2^{x^2} = x^2 In2$

$\frac{1}{y}\frac{dy}{dx} = 2xInx$

$\frac{dy}{dx} = y2xInx = 2^{x^2}2xIn2$

(j) $3x^2.3^x$

Let y =$3x^2.3^x$

$Iny = In3x^2.3^x$

$= In3 + Inx^2 + In3^x$

$= In3 + 2Inx + xIn3$

$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + Inx = \frac{2+xInx}{x}$

$\frac{dy}{dx} = y\frac{2+xInx}{x} = 3x^2.3^x\left(\frac{2+xInx}{x}\right)$

$= 3x.3^x(2 + xInx)$

(k) $\sqrt[3]{\frac{x+1}{x-1}}$

$Let\ y = \sqrt[3]{\frac{x+1}{x-1}}$

$y^3 = \frac{x+1}{x-1}$

$Iny^3 = In(x+1) - In(x-1)$

$\frac{3y^2}{y^3}\frac{dy}{dx} = \frac{1}{x+1} - \frac{1}{x-1} = \frac{-2}{(x+1)(x-1)}$

$\frac{dy}{dx} = \frac{y}{3}.\frac{-2}{(x+1)(x-1)}$

$= \frac{(x+1)^{\frac{1}{3}}}{3(x-1)^{\frac{1}{3}}}.\frac{-2}{(x+1)(x-1)}$

$= \frac{-2}{3(x+1)^{\frac{2}{3}}(x-1)^{\frac{4}{3}}}$

Revision exercise 10

1. Differentiate with respect x

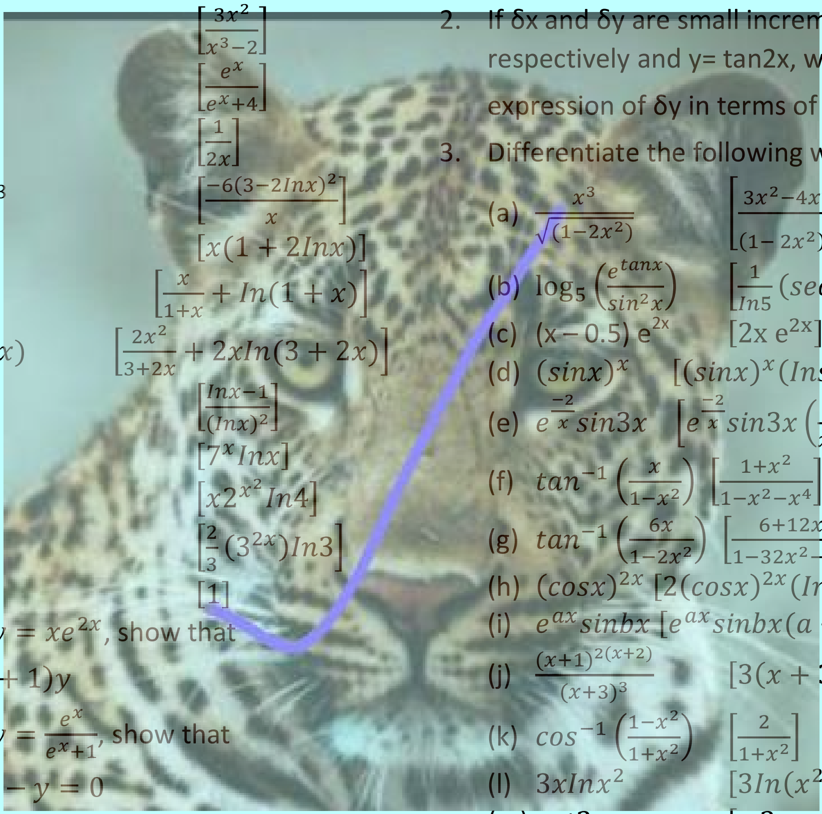
(a) e^{2y} $\left[2e^{2y}\frac{dy}{dx}\right]$

(b) e^{siny} $\left[cosye^{siny}\frac{dy}{dx}\right]$

(c) $4x^2 + \frac{2}{e^{x^2}}$ $\left[8x - \frac{4x}{e^{x^2}}\right]$

(d) xe^{-x} $[e^{-x} - xe^{-x}]$

(e) $Insinx$ $[cotx]$



(f) In(tany) $\left[secxcosec\frac{dy}{dx}\right]$

(g) $\frac{\sqrt{x^2+1}}{(2x-1)^2}$ $\left[\frac{2x^2+x+4}{(x^2+1)^{\frac{1}{2}}(2x-1)^3}\right]$

(h) $\frac{x^2e^x}{(x-1)^3}$ $\left[\frac{xe^x(x^2-2x-2)}{(x-1)^4}\right]$

(i) $\frac{sin4x}{5^{2x}}$ $\left[\frac{2sin^4x^2(4xcotx^2-In5)}{5^{2x}}\right]$

(j) $\frac{e^{x^2}\sqrt{\cos x}}{(2x+1)^3}$ $\left[\frac{e^{x^2}\sqrt{\cos x}}{(2x+1)^3}\left(2x-\frac{1}{2}tanx-\frac{6}{2x+1}\right)\right]$

(k) $\frac{2e^{-x}}{2^xcosx}$ $\left[\frac{2e^{-x}}{2^xcosx}(tanx-In2-1)\right]$

(l) $\frac{(x-1)(2-3x)}{(1+x)(x+2)}$ $\left[\frac{2(8-4x-7x^2)}{(1+x)^2(x+2)^2}\right]$

(m) $In(1+x^2)$ $\left[\frac{2x}{1+x^2}\right]$

(n) $In(x^3-2)$ $\left[\frac{3x^2}{x^3-2}\right]$

(o) $In(e^x+4)$ $\left[\frac{e^x}{e^x+4}\right]$

(p) $In(\sqrt{x})$ $\left[\frac{1}{2x}\right]$

(q) $(3-2Inx)^3$ $\left[\frac{-6(3-2Inx)^2}{x}\right]$

(r) x^2Inx $[x(1+2Inx)]$

(s) $xIn(1+x)$ $\left[\frac{x}{1+x}+In(1+x)\right]$

(t) $x^2In(3+2x)$ $\left[\frac{2x^2}{3+2x}+2xIn(3+2x)\right]$

(u) $\frac{x}{Inx}$ $\left[\frac{Inx-1}{(Inx)^2}\right]$

(v) 7^x $[7^xInx]$

(w) 2^{x^2} $[x2^{x^2}In4]$

(x) 3^{2x-1} $\left[\frac{2}{3}(3^{2x})In3\right]$

(y) e^{Inx} $[1]$

2. Given that $y=xe^{2x}$, show that $x\frac{dy}{dx}=(2x+1)y$
3. Given that $y=\frac{e^x}{e^x+1}$, show that $(1+e^x)\frac{dy}{dx}-y=0$
4. Given that $y=\frac{e^{x^2}}{x}$, show that $\frac{dy}{dx}=\frac{2e^{x^2}-y}{x}$
5. Given that e^x-e^{-x}, show that $\left(\frac{dy}{dx}\right)^2-y^2=4$
6. Given that $Ae^{4x}+Be^{-4x}$, where A and B are constants show that $\frac{d^2y}{dx^2}-16y=0$
7. Given that $y=In(Inx)$, show that $(Inx)\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\frac{1}{x^2}=0$
8. Given that $y=In\left(\frac{1+x}{1-x}\right)$, show that $(1-x^2)\frac{dy}{dx}-2=0$
9. Given that $y=\frac{In(1+x)}{x^2}$, show that $x^2\frac{dy}{dx}+2xy=\frac{1}{1+x}$
10. Given that $y=In(I1+e^x)$, show that $\frac{d^2y}{dx^2}=e^x\left(1-\frac{dy}{dx}\right)^2$
11. Given that $y=e^{3x}sin2x$, show that $\frac{d^2y}{dx^2}+13y=6\frac{dy}{dx}$

Revision exercise 11

1. Find the derivative of $y=sin^2x$ from the first principles [2sinxcosx]
2. If δx and δy are small increment in x and y respectively and y= tan2x, write down an expression of δy in terms of x and δx. $\left[\frac{2\delta x}{cos^2x}\right]$
3. Differentiate the following with respect to x

(a) $\frac{x^3}{\sqrt{(1-2x^2)}}$ $\left[\frac{3x^2-4x^4}{(1-2x^2)^{\frac{3}{2}}}\right]$

(b) $\log_5\left(\frac{e^{tanx}}{sin^2x}\right)$ $\left[\frac{1}{In5}(sec^2x-2cotx)\right]$

(c) $(x-0.5)e^{2x}$ $[2xe^{2x}]$

(d) $(sinx)^x$ $[(sinx)^x(Insinnx+xcotx)]$

(e) $e^{\frac{-2}{x}}sin3x$ $\left[e^{\frac{-2}{x}}sin3x\left(\frac{2}{x^2}+3cot3x\right)\right]$

(f) $tan^{-1}\left(\frac{x}{1-x^2}\right)$ $\left[\frac{1+x^2}{1-x^2-x^4}\right]$

(g) $tan^{-1}\left(\frac{6x}{1-2x^2}\right)$ $\left[\frac{6+12x^2}{1-32x^2-4x^4}\right]$

(h) $(cosx)^{2x}$ $[2(cosx)^{2x}(Incosx-xtanx)]$

(i) $e^{ax}sinbx$ $[e^{ax}sinbx(a+bcosbx)]$

(j) $\frac{(x+1)^{2(x+2)}}{(x+3)^3}$ $[3(x+3)^2]$

(k) $cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ $\left[\frac{2}{1+x^2}\right]$

(l) $3xInx^2$ $[3In(x^2+2)]$

(m) $cot2x$ $[-2cosec^2 2x]$

(n) $(sinx)^x$ $[(sinx)^x(xcotx+Insinx)]$

(o) $\frac{(x+1)^2}{(x+4)^3}$ $\left[\frac{(5-x)(x+1)}{(x+4)^4}\right]$

(p) $\frac{3x+4}{\sqrt{2x^2+3x-2}}$ $\left[\frac{-(7x+4)}{(2x^2+3x-2)^{\frac{3}{2}}}\right]$

(q) $\log_e\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}$ $\left[\frac{1}{1-x^2}\right]$

(r) $\frac{1+sin^2x}{cos^2x+1}$ $\left[\frac{3sin2x}{(cos^2x+1)^2}\right]$

(s) $tan^{-1}\left(\frac{x^2}{2}+2x^3\right)$ $\left[\frac{4x(1+6x)}{4+(x^2+4x^3)^2}\right]$

(t) e^{ax^2} $\left[2e^{ax^2}\right]$

(u) $(1-2x)^{-\frac{1}{2}}$ $\left[\frac{2x}{1-2x^2}\right]$

(v) $(x+1)^{\frac{1}{2}}(x+2)^2$ $\left[\frac{(5x+6)(x+2)}{2(x+1)^{\frac{1}{2}}}\right]$

(w) $\frac{2x^2+3x}{(x-4)^2}$ $\left[\frac{(x-4)(4x+3)-2(2x^2+3x)}{(x-4)^3}\right]$

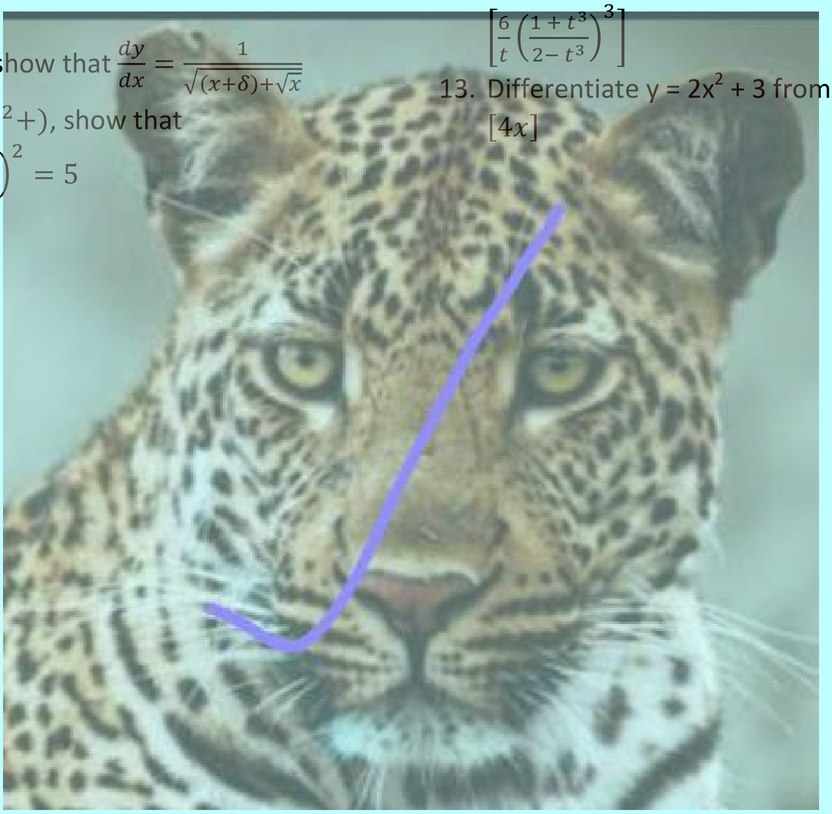
(x) $\frac{3x-1}{\sqrt{x^2+1}}$ $\left[\frac{x+3}{(x^2+1)^{\frac{3}{2}}}\right]$

(y) $\frac{cos2x}{1+sin2x}$ $\left[\frac{-2}{1+sin2x}\right]$

(z) $In(secx+tanx)$ $[secx]$

(aa) $\left(\frac{1+2x}{1+x}\right)^2$ $\left[\frac{2(1+2x)}{(1+x)^3}\right]$

4. If $y = tan\left(\frac{x+1}{2}\right)$ show that $\frac{d^2y}{dx^2} - y\frac{dy}{dx} = 0$
5. Given that $y = e^{tanx}$, show that $\frac{d^2y}{dx^2} = 6\frac{dy}{dx}$
6. If $y = \sqrt{x}$ show that $\frac{dy}{dx} = \frac{1}{\sqrt{(x+\delta)}+\sqrt{x}}$
7. If $y = \sqrt{(5x^2+)}$, show that $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 5$



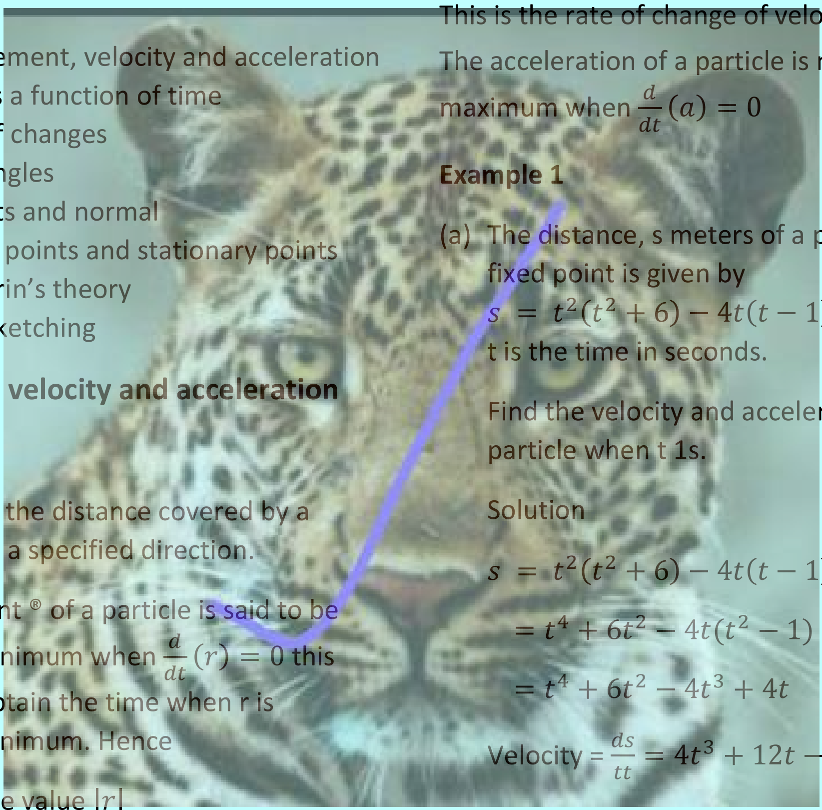
8. Given $y = In\left(1-\frac{1}{u}\right)^{\frac{1}{2}}$, $2u = \left(x-\frac{1}{x}\right)$, show that $\frac{dy}{dx} = \frac{(x+1)}{(x^2+1)(x-1)}$
9. If $y = e^{-t}\cos(t+\beta)$, show that $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$
10. Given that $y = \sqrt{\frac{1+sinx}{1-sinx}}$, show that $\frac{dy}{dx} = \frac{1}{1-sinx}$
11. Show from first principles that $\frac{d}{dx}(tanx) = sec^2x$
12. Given that x = $\frac{t^2}{1+t^3}$ and y =$\frac{t^3}{1+t^3}$, find $\frac{d^2y}{dx^2}$.

    $\left[\frac{6}{t}\left(\frac{1+t^3}{2-t^3}\right)^3\right]$
13. Differentiate y = $2x^2 + 3$ from first principles

    $[4x]$

Application of Differentiation

Differentiation is helpful in various application including



- Displacement, velocity and acceleration given as a function of time
- Rates of changes
- Small angles
- Tangents and normal
- Turning points and stationary points
- Maclaurin's theory
- Curve sketching

Displacement, velocity and acceleration

Displacement

Displacement is the distance covered by a particle/body in a specified direction.

The displacement ® of a particle is said to be maximum or minimum when $\frac{d}{dt}(r) = 0$ this enables us to obtain the time when r is maximum or minimum. Hence

$r_{\max\ or\ r_{min}}$ is the value $|r|$

Velocity

This is the rate of change of displacement or $v = \frac{d}{dt}(r)$ where r is displacement.

The velocity of a partice is maximum or minimum when $\frac{d}{dt}(v) = 0$, this enables us to obtain the time when v is maximum or minimum. Hence

$v_{\max\ or\ v_{min}}$ is the value $|v|$

Acceleration, a

This is the rate of change of velocity or $a = \frac{dv}{dt}$. The acceleration of a particle is minimum or maximum when $\frac{d}{dt}(a) = 0$

Example 1

(a) The distance, s meters of a particle from a fixed point is given by $s = t^2(t^2+6) - 4t(t-1)(t+1)$, where t is the time in seconds.

Find the velocity and acceleration of the particle when t 1s.

Solution

$$s = t^2(t^2+6) - 4t(t-1)(t+1)$$

$$= t^4 + 6t^2 - 4t(t^2-1)$$

$$= t^4 + 6t^2 - 4t^3 + 4t$$

Velocity $= \frac{ds}{tt} = 4t^3 + 12t - 12t^2 + 4$

Whet t = 1

$v = 4 + 12 - 12 + 4 = 8ms^{-1}$

Acceleration $= \frac{dv}{dt} = 12t^2 + 12 - 24t$

When t= 1

$a = 12 + 12 - 24 = 0ms^{-2}$

(b) A particle moves along a straight line OX so that its displacement x meters from the origin, O at time t second is given by

$$x = 4t^3 - 18t^2 + 24t$$

Find

(i) when and where the velocity of the particle is zero

$x = 4t^3 - 18t^2 + 24t$

$v = \frac{dx}{dt} = 12t^2 - 36t + 24$

For v = 0

$12t^2 - 36t + 24 = 0$

$t^2 - 3t + 2 = 0$

$(t-1)(t-2) = 0$

Either t = 1 or t= 2

$\therefore velocity = 0\, when$

$t = 1s\ or\ t = 2s$

When t = 1s

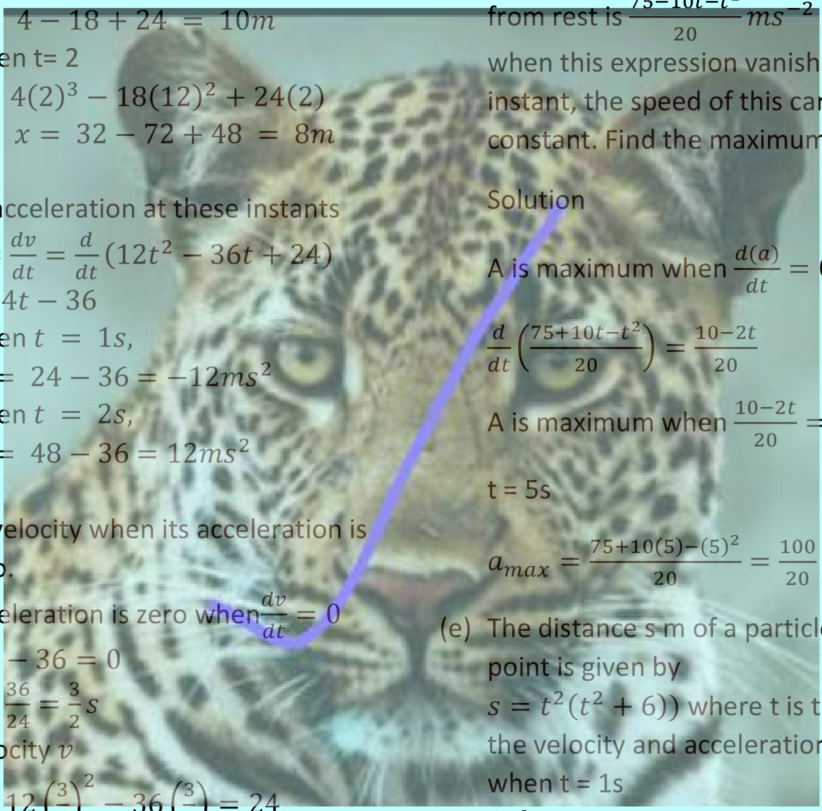
$x = 4(1)^3 - 18(1)^2 + 24(1)$

$x = 4 - 18 + 24 = 10m$

When t= 2

$x = 4(2)^3 - 18(12)^2 + 24(2)$

$x = 32 - 72 + 48 = 8m$



(ii) its acceleration at these instants

$a = \frac{dv}{dt} = \frac{d}{dt}(12t^2 - 36t + 24)$

$= 24t - 36$

When $t = 1s$,

$a = 24 - 36 = -12ms^2$

When $t = 2s$,

$a = 48 - 36 = 12ms^2$

(iii) its velocity when its acceleration is zero.

Acceleration is zero when $\frac{dv}{dt} = 0$

$24t - 36 = 0$

$t = \frac{36}{24} = \frac{3}{2}s$

Velocity v

$= 12\left(\frac{3}{2}\right)^2 - 36\left(\frac{3}{2}\right) = 24$

$= -3ms^{-1}$

i.e. the particle is moving in opposite direction.

(c) A particle of mass 5kg moves such that

$s = \begin{pmatrix} 2 - cos3t \\ 6sin2t \end{pmatrix}$

(i) Show that the particle never crosses the y-axis

For any point on the y-axis , x= 0

$2 - cos3t = 0$

cos3t = 2

$3t = cos^{-1}(2)$

$t = \frac{1}{3}cos^{-1}(2)$

Since $cos^{-1}(2)$ has no value, the particle does not cross y-axis

(ii) Find the velocity of the particle when $t = \frac{\pi}{6}$

$v = \frac{dx}{dt} = \frac{d}{dx}\begin{pmatrix} 2 - cos3t \\ 6sin2t \end{pmatrix}$

$= \begin{pmatrix} 3sin3t \\ 12cos2t \end{pmatrix}$

$At\ t = \frac{\pi}{6}$

$v = \begin{pmatrix} 3sin\frac{3\pi}{6} \\ 12cos\frac{2\pi}{6} \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} ms^{-1}$

(d) The acceleration of a car t s after starting from rest is $\frac{75-10t-t^2}{20}ms^{-2}$ until the instant when this expression vanishes. After this instant, the speed of this car remains constant. Find the maximum acceleration.

Solution

A is maximum when $\frac{d(a)}{dt} = 0$

$\frac{d}{dt}\left(\frac{75+10t-t^2}{20}\right) = \frac{10-2t}{20}$

A is maximum when $\frac{10-2t}{20} = 0$

t = 5s

$a_{max} = \frac{75+10(5)-(5)^2}{20} = \frac{100}{20} = 5ms^{-2}$

(e) The distance s m of a particle from a fixed point is given by

$s = t^2(t^2+6))$ where t is the time. Find the velocity and acceleration of the particles when t = 1s

Solution

$s = t^2(t^2+6)$

$= t^4 + 6t^2$

$v = \frac{d(s)}{dt} = \frac{d}{dt}(t^4 + 6t^2)$

$= 4t^3 + 12t$

$at\ t = 1s$

$v = 4(1)^3 + 12(1) = 16ms^{-1}$

$a = \frac{d(v)}{dt} = \frac{d}{dt}(4t^3 + 12t)$

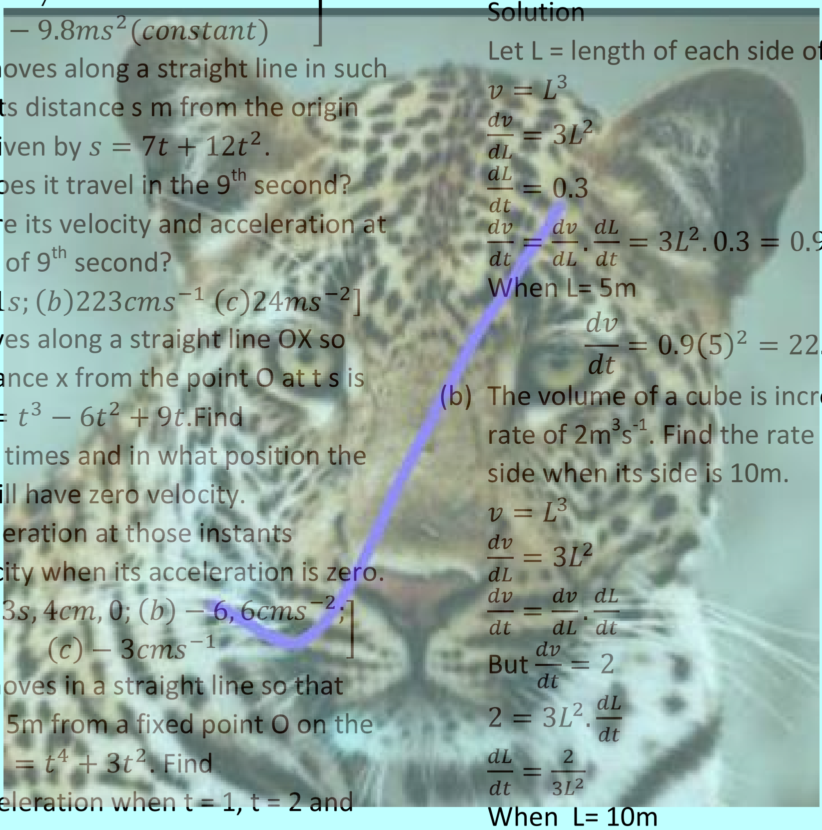
$= 12t^2 + 12$

$at\ t = 1s$

$a = 12(1)^2 + 12 = 24ms^{-2}$

Revision exercise 1

1. A ball is thrown vertically upwards and its height after t seconds is h m where
   $h = 25.2t - 4.9t^2$
   Find
   (a) its height and velocity after 3s
   (b) when it is momentarily at rest
   (c) the greatest height reached
   (d) the distance moved in the 3^{rd} second
   (e) the acceleration when $t = 2\frac{4}{7}$
   $\begin{bmatrix} (a) \quad 31.5m, -4.2ms^{-1}; \\ (b)\ t = 2\frac{4}{7}; (c) 32.4m; (d) 2.5m; \\ (e) - 9.8ms^2 (constant) \end{bmatrix}$



2. A particle moves along a straight line in such a way that its distance s m from the origin after t s is given by $s = 7t + 12t^2$.
   (a) What does it travel in the 9^{th} second?
   (b) What are its velocity and acceleration at the end of 9^{th} second?
   $[(a) 211s; (b) 223cms^{-1}\ (c) 24ms^{-2}]$
3. A point moves along a straight line OX so that its distance x from the point O at t s is given by $s = t^3 - 6t^2 + 9t$. Find
   (a) at what times and in what position the point will have zero velocity.
   (b) its acceleration at those instants
   (c) its velocity when its acceleration is zero.
   $\begin{bmatrix} (a) 1s, 3s, 4cm, 0; (b) - 6, 6cms^{-2}; \\ (c) - 3cms^{-1} \end{bmatrix}$
4. A particle moves in a straight line so that after t s it is 5m from a fixed point O on the line where $s = t^4 + 3t^2$. Find
   (a) The acceleration when t = 1, t = 2 and t = 3s.
   (b) The average acceleration between t = 1 and t =3s
   $[(a) 18, 54, 114ms^{-1}; (b) 58ms^{-2}]$
5. A particle moves a long a straight line so that after t s, its distance from a fixed point O on the line is 5m where $s = t^3 - 3t^2 + 2t$
   (a) When is the particle is at O?
   (b) What is the velocity and acceleration at these times?
   (c) What is the average acceleration between t = 0 and t = 2s.
   $\begin{bmatrix} (a) after\ 0, 1, 2s; (b) 2, -1, 2ms^{-1}; \\ -6, 0, 6ms^{-2}; (c) 0ms^{-1}; (d) 0ms^{-1} \end{bmatrix}$

Rates of change/measurement

This deals with aspects that vary with others.

Example 2

(a) The side of a cube is increasing at the rate of $0.3ms^{-1}$.
Find the rate of volume when the length is 5m.
Solution
Let L = length of each side of the cube.
$v = L^3$
$\frac{dv}{dL} = 3L^2$
$\frac{dL}{dt} = 0.3$
$\frac{dv}{dt} = \frac{dv}{dL} \cdot \frac{dL}{dt} = 3L^2 . 0.3 = 0.9L^2$
When L= 5m
$$\frac{dv}{dt} = 0.9(5)^2 = 22.5m^3s^{-1}$$

(b) The volume of a cube is increasing at the rate of $2m^3s^{-1}$. Find the rate of change of the side when its side is 10m.
$v = L^3$
$\frac{dv}{dL} = 3L^2$
$\frac{dv}{dt} = \frac{dv}{dL} \cdot \frac{dL}{dt}$
But $\frac{dv}{dt} = 2$
$2 = 3L^2 . \frac{dL}{dt}$
$\frac{dL}{dt} = \frac{2}{3L^2}$
When L= 10m
$\frac{dL}{dt} = \frac{2}{3(10)^2} = 0.007ms^{-1}$

(c) The volume of a cube increases uniformly at $a^3m^3s^{-1}$. Find an expression for the rate of increase of the surface area when the area of a face is b^2m^2.
Solution
Let L = side of the cube
A = surface area
V = volume of the cube
$v = L^3$
$\frac{dv}{dL} = 3L^2$

$\frac{dv}{dt} = \frac{dv}{dL}.\frac{dL}{dt}$

But $\frac{dv}{dt} = a^2$

$2 = 3L^2.\frac{dL}{dt}$

$\frac{dL}{dt} = \frac{a^3}{3L^2}$

For face area = $b^2 = L^2$ since L=b

Surface area of a cube = $6b^2$

$A = 6L^2$

$\frac{dA}{dL} = 12L$

$But\ \frac{dA}{dt} = \frac{dA}{dL}.\frac{dL}{dt}$

$= 12L.\frac{a^3}{3L^2} = \frac{4a^3}{L} = \frac{4a^3}{b}$

(d) A spherical balloon is inflated at a rate of $5m^3s^{-1}$. Find the rate of increase of radius when the radius is 3m.

Solution

If v and r are the volume and radius of the sphere at time t, then

$v = \frac{4}{3}\pi r^3$

$\frac{dv}{dr} = 4\pi r^2$

$\frac{dv}{dr} = 5$

But $\frac{dv}{dt} = \frac{dv}{dr}.\frac{dr}{dt}$

$5 = 4\pi r^2.\frac{dr}{dt}$

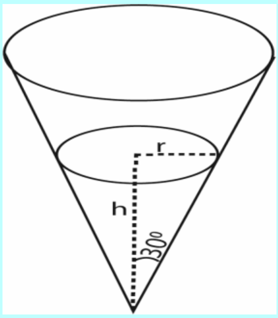
$\frac{dr}{dt} = \frac{5}{4\pi r^2}$

When r=3

$\frac{dr}{dt} = \frac{5}{4\pi(3)^2} = \frac{5}{36\pi}ms^{-1}$

(e) A hollow can of semi-vertical angle 30^0 is held with its vertex downwards. Water is poured into the cone at the rate of $3m^3s^{-1}$. Find the rate at which the depth of water in the cone is increasing when the depth is 5m.

Let the depth of water in the cone be h m



From the diagram above

$\tan 30 = \frac{r}{h}$

$\frac{1}{\sqrt{3}} = \frac{r}{h}$

$r = \frac{h}{\sqrt{3}}$

The volume v m^3 of water in the cone is given by $v = \frac{1}{3}\pi r^2 h$

Substituting for r

$v = \frac{1}{3}\pi\left(\frac{h}{\sqrt{3}}\right)^2 h = \frac{\pi h^3}{9}$

$\frac{dv}{dh} = \frac{\pi h^2}{3}$

But $\frac{dv}{dt} = \frac{dv}{dh}.\frac{dh}{dt}$

$3 = \frac{\pi h^2}{3}.\frac{dh}{dt}$

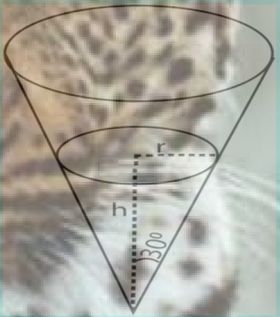
$\frac{dh}{dt} = \frac{9}{\pi h^2}$

When h = 5

$\frac{dh}{dt} = \frac{9}{\pi(5)^2} = \frac{9}{25\pi}ms^{-1}$

∴the rate of change of height is $\frac{9}{25\pi}ms^{-1}$

(f) An inverted cone with a vertical angle of 60^0 is collecting water leaking from a tap at a rate of $2m^3s^{-1}$. If the height of water collected in the cone is 10m, find the rate at which the surface are of eater is increasing.



$\tan 30 = \frac{r}{h}$

$\frac{1}{\sqrt{3}} = \frac{r}{h}$

$r = \frac{h}{\sqrt{3}}$

The volume v m^3 of water in the cone is given by $v = \frac{1}{3}\pi r^2 h$

Substituting for r

$v = \frac{1}{3}\pi\left(\frac{h}{\sqrt{3}}\right)^2 h = \frac{\pi h^3}{9}$

$\frac{dv}{dh} = \frac{\pi h^2}{3}$

Let A = surface area

$A = \pi r^2$

Substituting for r

$A = \pi\left(\frac{h}{\sqrt{3}}\right)^2 = \frac{\pi h^2}{3}$

$\frac{dA}{dh} = \frac{2}{3}\pi h$

But $\frac{dv}{dt} = \frac{dv}{dh}.\frac{dh}{dt}$

$2 = \frac{\pi h^2}{3}.\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{6}{\pi h^2}$

Also, $\frac{dA}{dt} = \frac{dA}{dh}.\frac{dh}{dt} = \frac{2}{3}\pi h \cdot \frac{6}{\pi h^2} = \frac{4}{h}$

Substituting for h =10

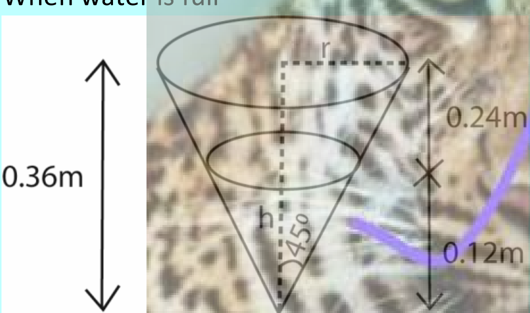
$\frac{dA}{dt} = \frac{4}{10} = 0.4m^2s^{-1}$

∴the rate at which the surface area is changing is $0.4m^2s^{-1}$.

(g) A hollow circular cone with vertical angle 90^0 and height 0.36m is inverted and filled with water. This water begins to leak away through a small hole in the vertex. If the level of the water begins to sink at a rate of 0.01m in 120s, and the water continues to leak a way at the same rate, at what rate is the level sinking when the water is 0.24m from the top?

Solution

When water is full



$\tan 45^0 = \frac{r}{h}$

$1 = \frac{r}{h}$

$r = h$

The volume v m^3 of water in the cone is given by $v = \frac{1}{3}\pi r^2 h$

Substituting for r

$v = \frac{1}{3}\pi(1)^2 h = \frac{\pi h^3}{3}$

$\frac{dv}{dh} = \pi h^2$

$\frac{dh}{dt} = \frac{0.01}{120}$

But $\frac{dv}{dt} = \frac{dv}{dh}.\frac{dh}{dt}$

$= \pi h^2.\frac{0.01}{120}$

Substituting for h

$\frac{dv}{dt} = \pi(0.36)^2.\frac{0.01}{120}$..................(i)

When water level is 0.36 - 0.24 = 0.12m

$\frac{dv}{dt} = \pi(0.12)^2.\frac{dh}{dt}$..................(ii)

Equating (i)and (ii)

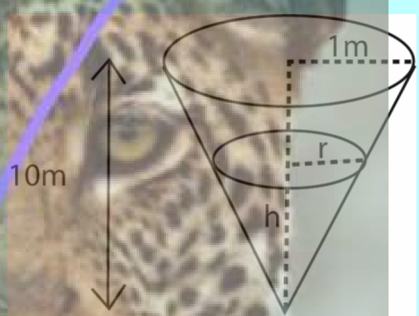
$\pi(0.36)^2.\frac{0.01}{120} = \pi(0.12)^2.\frac{dh}{dt}$

$$\frac{dh}{dt} = \frac{(0.36)^2}{(0.12)^2}.\frac{0.01}{120} = 7.5x10^{-4}ms^{-1}$$

(h) A hollow right circular cone of height 10m and base radius 1m is catching the drips from a tap leaking at a rate $0.002m^3s^{-1}$. Find the rate at which the surface area of water is increasing when water is half way up the cone

Solution

Let h and r be the height and radius of water level at time t



Expressing r in term of h, from similarity of figures,

$\frac{h}{10} = \frac{r}{1}$

$r = \frac{h}{10}$

Volume, v of a cone $= \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3}\pi\left(\frac{h}{10}\right)^2 h$$

$$= \frac{\pi h^3}{300}$$

$\frac{dv}{dh} = \frac{\pi h^2}{100}$

But $\frac{dv}{dt} = \frac{dv}{dh}.\frac{dh}{dt}$

$0.002 = \frac{\pi h^2}{100}.\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{0.2}{\pi h^2}$

Surface area, A = πr^2

Substituting for r

$$A = \pi\left(\frac{h}{10}\right)^2 = \frac{\pi h^2}{100}$$

$$\frac{dA}{dh} = \frac{\pi h}{50}$$

Now $\frac{dA}{dt} = \frac{dA}{dh}.\frac{dh}{dt} = \frac{\pi h}{50}.\frac{0.2}{\pi h^2} = \frac{0.004}{h}$

When water is half way up, h=5m

$$\frac{dA}{dt} = \frac{0.004}{(5)} = 0.0008m^2s^{-1}$$

Revision exercise 2

1. The side of a square is increasing at the rate of $5cms^{-1}$. Find the rate of increase of the area when the length of the side is 10cm. [$100cm^2s^{-1}$]
2. The volume of a cube is increasing at the rate of $18cm^3s^{-1}$. Find the rate of increase of the length of a side when the volume is $125cm^3$. $\left[\frac{6}{25}cms^{-1}\right]$
3. The radius of a circle is increasing at the rate of $\frac{1}{3}cms^{-1}$. Find the rate of increase of the area when the radius is 5cm. $\left[\frac{10\pi}{3}cm^2s^{-1}\right]$
4. The volume of a sphere is increasing at a rate of $(12\pi)cm^3s^{-1}$. Find the rate of increase of the radius when the radius is 6cm. $\left[\frac{1}{12}cms^{-1}\right]$
5. The area of a square is increasing at the rate of $7cm^2s^{-1}$. Find the rate of increase of the length of a side when this area is $100cm^2$. $\left[\frac{7}{10}cms^{-1}\right]$
6. The area of a circle is increasing at the rate of $(4\pi)cm^2s^{-1}$. Find the rate of increase of the radius when this radius is $\frac{1}{2}$cm. [$4cms^{-1}$]
7. The surface area of a sphere is increasing at a rate of $2cm^2s^{-1}$. Find the rate of increase of the radius when the surface area is (100π) cm^2? $\left[\frac{1}{20\pi}cms^{-1}\right]$
8. A boy is inflating a spherical balloon at the rat of $10cm^3s^{-1}$. Find the rate of increase of the surface area of the balloon when the radius is 5m.[$4cm^2s^{-1}$]
9. A hollow cone of semi-vertical angle 45^0 is held with its vertex pointing downwards. It receives water at a rate of $3cm^3$ per minute. Find the rate at which the depth of water in the cone is increasing when the depth is 2cm. $\left[\frac{3}{4\pi}cmmin^{-1}\right]$

Small changes

Suppose a function y = f(x) and δy and δx are increments in y and x respectively

Then as δx → 0, $\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$

$\Rightarrow \delta y = \frac{dy}{dx}.\delta x$

The above expression is used to find small changes in the variable x.

Example 3

(a) If $y = x^5$, find the approximate percentage increase in y due to increase of 0.1 percent in x.

$y = x^5$

$\frac{dy}{dx} = 5x^4$

But $\delta y = \frac{dy}{dx}.\delta x = 5x^4.\delta x$

$\frac{\delta y}{y} = \frac{5x^4.\delta x}{x^5} = 5\frac{\delta x}{x}$

But $\frac{\delta x}{x} = 0.1\%$

$\frac{\delta y}{y} = 5x0.1\%$ =0.5%

(b) An error of 2 ½ % is made in the measurement of the area of a circle. What percentage error results in

(i) The radius

A= πr^2

$\frac{dA}{dr} = 2\pi r$

$\delta A = \frac{dA}{dr}.\delta r$

$= 2\pi r\,.\delta r$

$\frac{\delta A}{A} = \frac{2\pi r}{\pi r^2}.\delta r = 2\frac{\delta r}{r}$

$\frac{1}{2}.\frac{\delta A}{A} = \frac{\delta r}{r}$

$\frac{\delta r}{r} = \frac{1}{2}.\frac{5}{2} = \frac{5}{4} = 1\frac{1}{4}\%$

(ii) The circumference

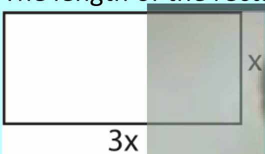
$$c = 2\pi r$$
$$\frac{dc}{dr} = 2\pi$$
$$\delta c = \frac{dc}{dr}.\delta r = 2\pi\delta r$$
$$\frac{\delta c}{c} = \frac{2\pi\delta r}{2\pi r} = \frac{\delta r}{r}$$
$$\frac{\delta c}{c} = 1\frac{1}{4}\%$$

(c) One side of a rectangle is three times the other. If the perimeter increases by 2%. What is the percentage increase in area?

Solution

Let the width of the rectangle = x

The length of the rectangle = 3x



P = 2(3x +x) = 8x

$$\frac{dP}{dx} = 8$$
$$\delta P = \frac{dP}{dx}.\delta x = 8\delta x$$
$$\frac{\delta P}{P} = \frac{8\delta x}{8x} = \frac{\delta x}{x} = 2\%$$
$$A = 3x^2$$
$$\frac{dA}{dx} = 6x$$
$$\delta A = \frac{dA}{dx}\delta x = 6x\delta x$$
$$\frac{\delta A}{A} = \frac{6x\delta x}{3x^2} = \frac{2\delta x}{x}$$
$$\frac{\delta A}{A} = 2x2\% = 4\%$$

(d) Find an approximate for $\sqrt{25.01}$

Let y = $\sqrt{x} = x^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$
$$\delta y = \frac{dy}{dx}\delta x = \frac{1}{2\sqrt{x}}\delta x$$

Taking x= 25 and δx=0.01

$$\delta y = \frac{1}{2\sqrt{25}}x0.01 = \frac{1}{10}x0.01 = 0.001$$
$$Now\ (x + \delta x)^{\frac{1}{2}} = y + \delta y$$
$$\sqrt{25.01} = 5 + 0.001 = 5.001$$

(e) Find an approximate of $\sqrt{101}$

Solution

Let y = $\sqrt{x} = x^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$
$$\delta y = \frac{dy}{dx}\delta x = \frac{1}{2\sqrt{x}}\delta x$$

Taking x= 100 and δx=1

$$\delta y = \frac{1}{2\sqrt{100}}x0.01 = \frac{1}{20}x1 = 0.05$$
$$Now\ (x + \delta x)^{\frac{1}{2}} = y + \delta y$$
$$\sqrt{101} = 10 + 0.05 = 10.05$$

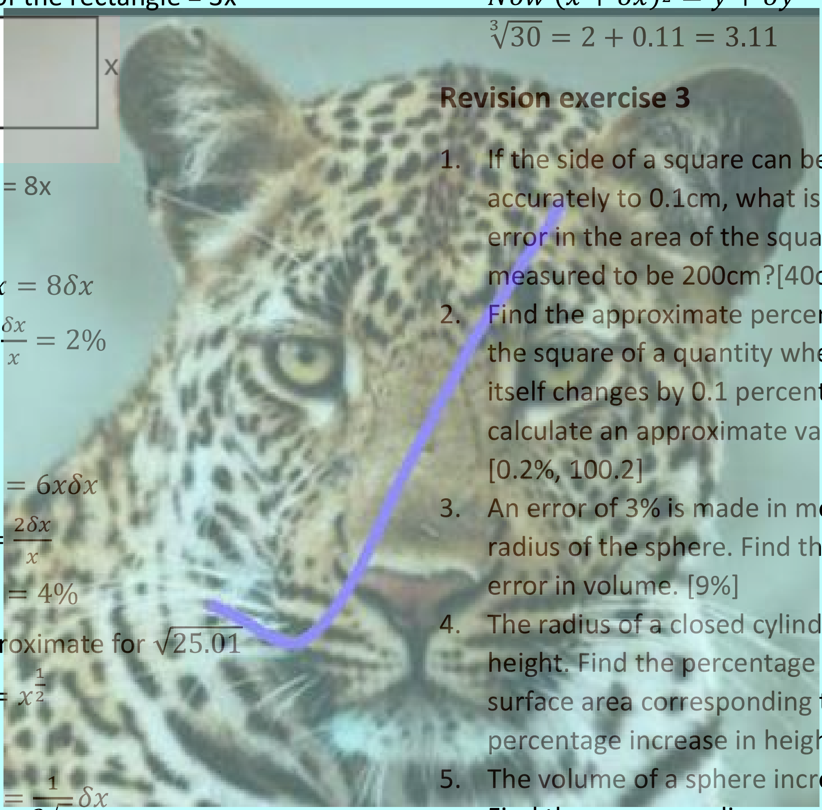
(f) Find $\sqrt[3]{30}$

Let y = $\sqrt[3]{x} = x^{\frac{1}{3}}$

$$\frac{dy}{dx} = \frac{1}{3(\sqrt[3]{x})^2}$$
$$\delta y = \frac{dy}{dx}\delta x = \frac{1}{3(\sqrt[3]{x})^2}\delta x$$

Taking x= 27 and δx= 3

$$\delta y = \frac{1}{3\left(\sqrt[3]{27}\right)^2}x3 = \frac{1}{9} = 0.11$$
$$Now\ (x + \delta x)^{\frac{1}{2}} = y + \delta y$$
$$\sqrt[3]{30} = 2 + 0.11 = 3.11$$



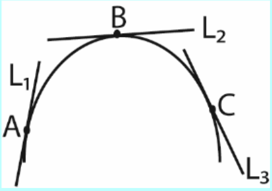
Revision exercise 3

1. If the side of a square can be measured accurately to 0.1cm, what is the possible error in the area of the square whose side measured to be 200cm?[40cm^2]
2. Find the approximate percentage change on the square of a quantity when the quantity itself changes by 0.1 percent. Hence calculate an approximate value of $(10.01)^2$. [0.2%, 100.2]
3. An error of 3% is made in measuring the radius of the sphere. Find the percentage error in volume. [9%]
4. The radius of a closed cylinder is equal to its height. Find the percentage increase in total surface area corresponding to unit percentage increase in height. [2%]
5. The volume of a sphere increases by 2%. Find the corresponding percentage increase surface area.$\left[1\frac{1}{3}\%\right]$
6. If $y = x + \frac{1}{x}$. Find the approximate increase in y when x increases from 2 to 2.4. [0.03]
7. Find the percentage increase in the volume of a cube when all the edges of the cube are increased in length of 2%. [6%]
8. The time period, T, of a pendulum of length L is given by $T = 2\pi\sqrt{\frac{L}{g}}$; where π and g are constants. Find the approximate percentage increase in T when the length of the pendulum increases by 4% [2%]

9. Find an approximate value for $\sqrt[3]{64.96}$. [4.02]
10. Find an approximate value for $(5.02)^3$. [126.5]

Tangents and Normals to curves

A tangent is a straight line drawn that touches a curve at only one point.



Lines 1, 2 and 3 above touch y= f(x) at point A, B and C respectively, hence they are tangents to the curve.

Gradient of a curve

A curve has varying gradients. A gradient at a point on a curve is obtained by finding a gradient of a tangent to the curve.

For a curve y = f(x), the gradient of a curve at any particular point is given by

$\frac{dy}{dx} = f'(x)$

Example 4

(a) Find the gradient of a curve $f(x) = x^2 + \frac{1}{x}$ at the point (1, 2)

Solution

$f(x) = x^2 + \frac{1}{x} = x^2 + x^{-1}$

$f'(x) = 2x - x^{-2} = 2x - \frac{1}{x^2}$

At point (1, 2), x = 1

$f'^{(x)} = 2(1) - \frac{1}{(1)^2} = 2 - 1 = 1$

∴the gradient of the curve at point (1,) is 1

(b) Find the coordinates of points for the curve $y = x^3$ whose gradient is 12

$y = x^3$

$\frac{dy}{dx} = 3x^2$

⇨ $12 = 3x^2$

$x^2 = 4$

$x = \pm 2$

When x = 2, $y = 2^3 = 8$

Hence the point is (2,8)

When x = -2, $y = -2^3 = -8$

Hence the point is (2,-8)

(c) The curve is defined by $y = ax^2 + b$ where a and b are constants. Given that the gradient of the curve at the point (2, -2) is 3. Find the values of a and b.

Solution

$y = ax^2 + b$

$\frac{dy}{dx} = 2ax$

⇨ $2ax = 3$

At point (2, -2), x = 2 and y = -2

⇨ 4a = 3

$a = \frac{3}{4}$

Also $2x\frac{3}{4}x(2)^2 + b = -2$

6 + b =-2

b = -8

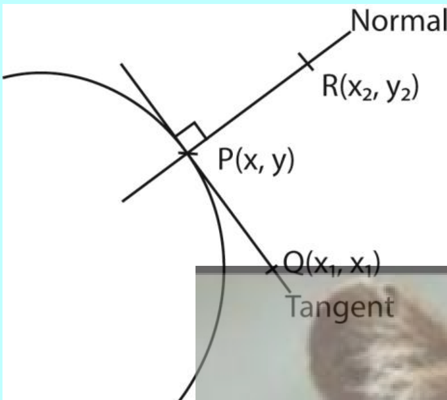
Hence $a = \frac{3}{4}$ and b = -8

Revision exercise 4

1. Find the gradient at the given point of the following curves.
   (a) $y = 2x^3 + 4$ at(3, 58) [54]
   (b) $y = \frac{x+5}{x}$ at (-1, -4) [-5]
   (c) $y = 6\sqrt{x} + \frac{1}{2\sqrt{x}}$ at $\left(\frac{1}{9}, \frac{7}{2}\right)$ $\left[\frac{9}{4}\right]$
   (d) $y = \frac{4-x^3}{x^2}$ at (-2, 3) [0]
2. Find the coordinates of the points on each of the following curves where the gradient is as stated
   (a) $y = 3x^2$, gradient -6 (-1, 3)
   (b) $y = x^3 - x^2 + 3$, grad 0 $\left(\frac{2}{3}, \frac{77}{27}\right)$
   (c) $y = \frac{x^2+3}{2x^2}$, grad 3 (-1, 2)
   (d) $y = 4\sqrt{x} - x$, grad5 $\left(\frac{1}{9}, \frac{11}{9}\right)$
3. A curve is given by $y = ax^2 + b\sqrt{x}$, where a and b are constants. If the gradient of the curve at (1, 1) is 5; find a and b. [a= 3 and b =-2]
4. Given that the curve $y = px^2 + qx$ has a gradient 7 at the point (6,8), find the values of the constants A and B. $\left[p = \frac{17}{18}; q = \frac{-13}{3}\right]$

5. A curve $y = \frac{p}{x} + q$ passes through the point (3, 9) with gradient 5. Find the value of constant p and q. [p= -45; q=24]

Equation of a tangent and the normal to the curve



Equation of the tangent

The gradient of the curve at any point P(x, y) is $\frac{dy}{dx} = m$.

If Q (x_1, y_1) is another point on the tangent then;

Gradient of the tangent $\overline{PQ} = m$

$\frac{y_1 - y}{x_1 - x} = m$

By cross multiplication, the equation of the tangent is obtained

Equation of the normal

The normal to a curve at any point say P is a straight line through P which is perpendicular to the tangent at P

When the gradient of the tangent to the curve is m, then the gradient of the normal is $\frac{-1}{m}$.

If R (x_2, y_2) is another point on the normal then

The gradient of the normal $\overline{PR}$ = $\frac{-1}{m}$.

Then$\frac{y_2 - y}{x_2 - x} = \frac{-1}{m}$

By cross multiplication, the equation of the normal is obtained

Example 5

(a) Find the equations of the equation of the tangent and the normal to the curve $y = x^3$ at P(2,8)

Solution

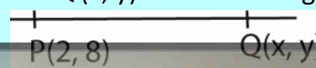
$y = x^3$

$\frac{dy}{dx} = 3x^2$

At x = 2

The gradient of the tangent = $3(2)^2$ = 12

Let Q (x, y) lie on the tangent



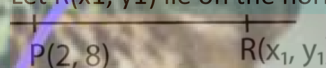
Grad $\overline{PQ} = 12$

$\frac{y-8}{x-2} = 12$

$y - 8 = 12(x - 2)$

$y = 12x - 16$

Let R(x1, y1) lie on the normal



Grad $\overline{PR} = \frac{-1}{12}$

$\frac{y-8}{x-2} = \frac{-1}{12}$

$12(y - 8) = -(x - 2)$

$y = \frac{2-x}{12} + 8$

$y = \frac{98-x}{12}$

(b) The equation of the curve at the point$P\left(-2, \frac{1}{4}\right)$is given by $f(x) = \frac{1}{x^2}$. Find

(i) equation of the tangent

let y = f(x)

$y = \frac{1}{x^2} = x^{-2}$

$\frac{dy}{dx} = -2x^{-3} = \frac{-2}{x^3}$

At x= -2, gradient =$\frac{-2}{(-2)^2} = \frac{1}{4}$

Let Q(x, y) lie on the tangent, then

$\frac{y - \frac{1}{4}}{x-(-2)} = \frac{1}{4}$

$4\left(y - \frac{1}{4}\right) = x + 2$

4y – 1 = x+2

$y = \frac{x+3}{4}$

(ii) the equation of the normal at P

Gradient of the normal = $-1 \div \frac{1}{4} = -4$

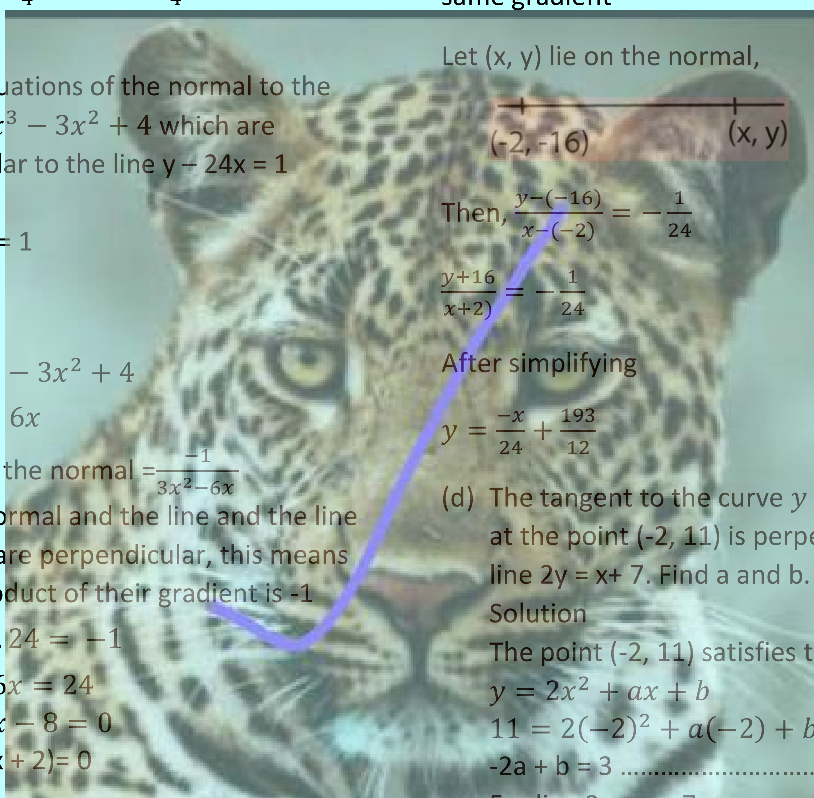
Let R(x, y) lie on the normal, then

$\frac{y-\frac{1}{4}}{x-(-2)} = -4$

$y - \frac{1}{4} = -4(x+2)$

$4y - 1 = -16(x+2)$

$y = \frac{-16(x+2)+1}{4} = \frac{-16x-15}{4}$



(c) Find the equations of the normal to the curve $y = x^3 - 3x^2 + 4$ which are perpendicular to the line y – 24x = 1

Solution

For y – 24x = 1

y = 24x+1

$\frac{dy}{dx} = 24$

$For\ y = x^3 - 3x^2 + 4$

$\frac{dy}{dx} = 3x^2 - 6x$

Gradient of the normal =$\frac{-1}{3x^2-6x}$

Since the normal and the line and the line y = 24x + 1 are perpendicular, this means that the product of their gradient is -1

⇨ $\frac{-1}{3x^2-6x}.24 = -1$

$3x^2 - 6x = 24$

$x^2 - 2x - 8 = 0$

(x – 4)(x + 2)= 0

Either x – 4 = 0 and x = 4

Or(x + 2) = 0 and x = -2

When x = 4

$y = (4)^3 - 3(4)^2 + 4 = 20$

(x, y) = (4, 20)

Gradient of the normal =$\frac{-1}{3(4)^2-6(4)} - \frac{-1}{24}$

Let (x, y) lie on the normal,

Then $\frac{y-20}{x-4} = -\frac{1}{24}$

Simplifying

$y = \frac{-x}{24} + \frac{121}{6}$

When x = -2

$y = (-2)^3 - 3(-2)^2 + 4 = -16$

(x, y) = (-2, -16)

Gradient of the normal =$\frac{-1}{3(-2)^2-6(-2)} - \frac{-1}{24}$ or since the two normal are parallel, they have the same gradient

Let (x, y) lie on the normal,

(-2, -16) (x, y)

Then, $\frac{y-(-16)}{x-(-2)} = -\frac{1}{24}$

$\frac{y+16}{x+2} = -\frac{1}{24}$

After simplifying

$y = \frac{-x}{24} + \frac{193}{12}$

(d) The tangent to the curve $y = 2x^2 + ax + b$ at the point (-2, 11) is perpendicular to the line 2y = x+ 7. Find a and b.

Solution

The point (-2, 11) satisfies the equation

$y = 2x^2 + ax + b$

$11 = 2(-2)^2 + a(-2) + b$

-2a + b = 3 ………………………………………(i)

For line 2y = x + 7

$y = \frac{x}{2} + \frac{7}{2}$

Gradient = $\frac{1}{2}$

For line $y = 2x^2 + ax + b$

$\frac{dy}{dx} = 4x + a$

Since the tangent to the curve and given line are perpendicular.

$\frac{1}{2}.(4x + a\ ) = -1$

$4x + a = -2$

At x = -2

$4(-2) + a = -2$

$a = 6$

From (i)

-2(6) + a = 3

b = 15

Hence a = 6 and b = 15

(e) The curve is given by parametrically by $x = \frac{2}{t}$ and $y = 3t^2 - 1$, at the point (2, 2). Find

(i) Equation of the tangent

Solution

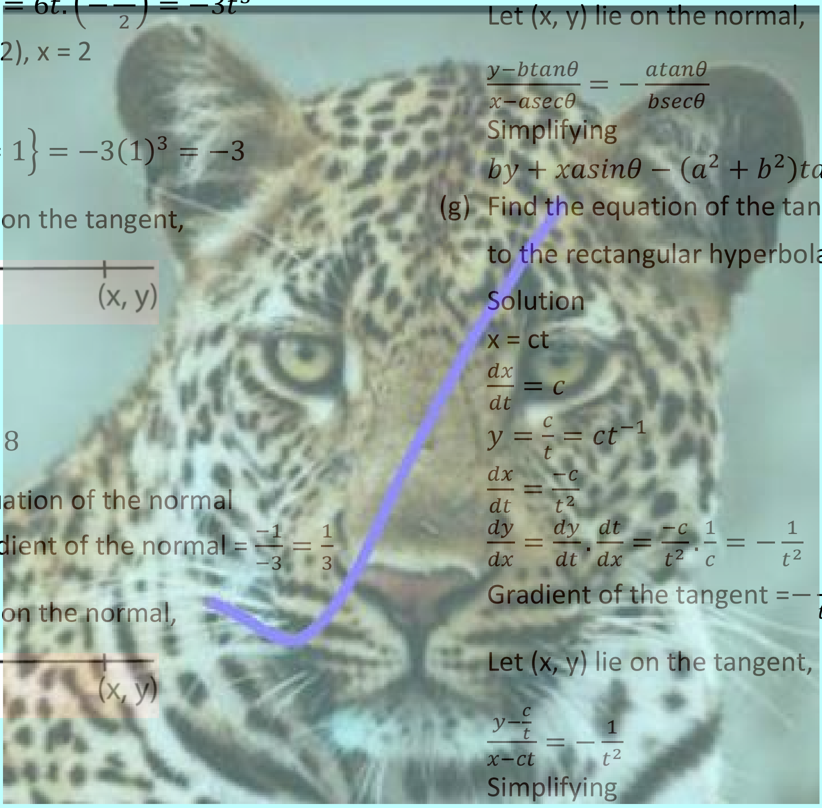
$x = \frac{2}{t} = 2t^{-1}$

$\frac{dx}{dt} = -2t^{-1} = \frac{-2}{t^2}$

$y = 3t^2 - 1,$

$\frac{dx}{dt} = 6t$

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = 6t.\left(-\frac{t^2}{2}\right) = -3t^3$



At point (2, 2), x = 2

$2 = \frac{2}{t}$; t = 1

⇨ $\left\{\frac{dx}{dt}\middle| t = 1\right\} = -3(1)^3 = -3$

Let (x, y) lie on the tangent,

(2, 2) (x, y)

$\frac{y-2}{x-2} = -3$

$y = -3x + 8$

(ii) The equation of the normal

The gradient of the normal $= \frac{-1}{-3} = \frac{1}{3}$

Let (x, y) lie on the normal,

(2, 2) (x, y)

$\frac{y-2}{x-2} = \frac{1}{3}$

3(y – 2) = x – 2

3y = x + 4

$y = \frac{x+4}{3}$

(f) Find the equation of tangent and normal of hyperbola with parametric equations x = asecθ and y = btanθ

Solution

$\frac{dx}{d\theta} = a\sec\theta\tan\theta$

$\frac{dy}{d\theta} = b\sec^2\theta$

$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b\sec\theta}{a\tan\theta}$

Gradient of the tangent $= \frac{b\sec\theta}{a\tan\theta}$

Let (x, y) lie on the tangent,

(asecθ, btanθ) (x, y)

$\frac{y - b\tan\theta}{x - a\sec\theta} = \frac{b\sec\theta}{a\tan\theta}$

Simplifying

$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta - 1 = 0$

Gradient of the normal $= -\frac{a\tan\theta}{b\sec\theta}$

Let (x, y) lie on the normal,

$\frac{y - b\tan\theta}{x - a\sec\theta} = -\frac{a\tan\theta}{b\sec\theta}$

Simplifying

$by + xa\sin\theta - (a^2 + b^2)\tan\theta = 0$

(g) Find the equation of the tangent and normal to the rectangular hyperbola at a $P\left(ct, \frac{c}{t}\right)$

Solution

x = ct

$\frac{dx}{dt} = c$

$y = \frac{c}{t} = ct^{-1}$

$\frac{dx}{dt} = \frac{-c}{t^2}$

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{-c}{t^2} \cdot \frac{1}{c} = -\frac{1}{t^2}$

Gradient of the tangent $= -\frac{1}{t^2}$

Let (x, y) lie on the tangent,

$\frac{y - \frac{c}{t}}{x - ct} = -\frac{1}{t^2}$

Simplifying

$yt^2 + x = 2ct$

Gradient of the normal $= t^2$

Let (x, y) lie on the normal,

$\frac{y - \frac{c}{t}}{x - ct} = t^2$

Simplifying

$yt - t^3x = (1 - t^4)$c

Revision exercise 5

1. Find the equation of the normal to the curve $y = 3x^2 + 7x - 2$ at a point P where x= -1 $[y = x -]$

2. The normal to the curve $y = x^2 - 4x$ at point (3, -3) cuts x-axis at A and y-axis at B. find the equation of the normal and the coordinates of A and B. $\left[y = -\frac{1}{2}(x-3); A(-3,0); B\left(0, \frac{-3}{2}\right)\right]$
3. Find the equation of the tangent to the curve $y = x^2 - 3x + 1$ at a point where the cuts the y-axis [y = -3x+1]
4. Find the equation of the tangent and the normal to the curve x = 6t^2 and y = t^3 – 4t, at the point where t = -1. $\left[y = \frac{1}{12}x + \frac{5}{2}; y = -12x + 75\right]$
5. Find the equation of the normal to the curve $y = x^3 - 8$ at the point where the curve cuts the x-axis [12y +x =3]
6. The two tangents to the curve y = x2 at the point where y=9, intersect at point P, find the coordinates of P. [(0.-9)]
7. Find the coordinates of the point of intersection of two normal to the curve $y = x^2 + 3x + 5$ which make an angle of 45^0 with the x-axis. $\left[\frac{-3}{2}, \frac{7}{2}\right]$
8. (a) Find the equation of the normal at the point (2, 3) on the curve $y = 2x^3 - 12x^2 + 23x - 11$ [y=x+1]
   (b) Find also the coordinate of points where the normal meets the curve again
   [(1, 2),(3, 4)]
9. The tangent of the curve $y = ax^2 + 1$ at the point (1, b) has gradient 6. Find the values of a and b. [a = 3, b = 4]
10. Find the equation of the tangents to the curve $x = t^2 and\ y = 6t -$ at the points where x = 1 [y = 3x – 4; y + 3x + 10 = 0]
11. Find the equations of the tangents to the curve $x = \frac{4}{t} and\ y = t^2 - 3t + 2$ at the point where the curve crosses the x-axis. [y + x = 2, 4y – x + 4 =0]
12. A curve is given by $x = t^3, y = 4t$. The tangent at the point t =2meets the tangent at the point t =-1 at point Q. find the coordinates of Q. [(-2, 2)]

Turning/ stationary points

A point on a curve such that its gradient is zero, $\frac{dy}{dx} = 0$, is called a stationary point.

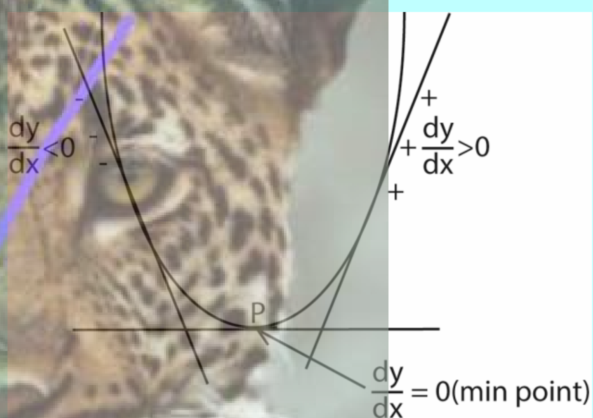
At this, the tangent to the curve is horizontal and the curve is 'flat'

There are three types of stationary points;

- Minimum point
- Maximum point
- Point of inflexion

Minimum point

This is obtained at the lowest point of the curve (valley like). In this, the gradient of the curve is negative to the left of the left of turning point and positive to the right



In summary we have

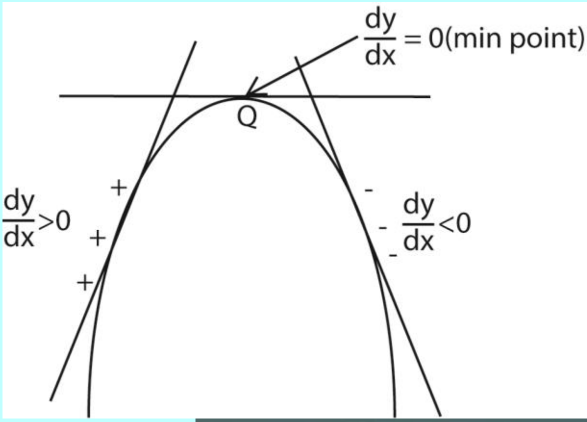
To the left of P	At point P	To the right of P
$\frac{dy}{dx} < 0(-)$	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} > 0(+)$

When using second derivative to investigate where the turning point is minimum. In this case, considering point P, the gradient of the derived function is positive.

i.e. at point P, $\frac{d^2y}{dx^2} > 0$ (positive value)

Maximum point

This is obtained at the highest point of curve (mountain like). In this case, the gradient of the curve is positive to the left and negative to the right of the turning point



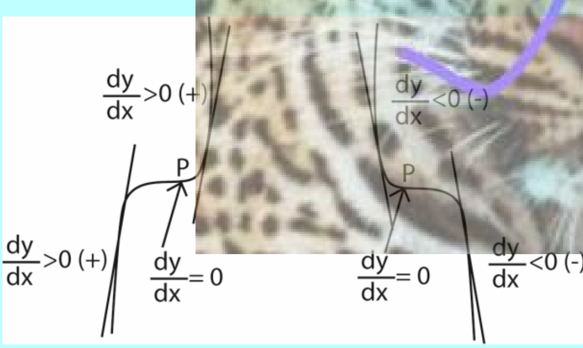
In summary, we have

To the left of Q	At point Q	To the right of Q
$\frac{dy}{dx} > 0(+)$	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} < 0(-)$

When using second derivative to investigate where the turning point is minimum. In this case, considering point Q, the gradient of the derived function is negative.

Point of inflexion

In this case, the gradient has the same sign on each side of the stationary point



In summary

To the left of P	At point P	To the right of P
$\frac{dy}{dx} > 0(+)$	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} > 0(+)$

Or

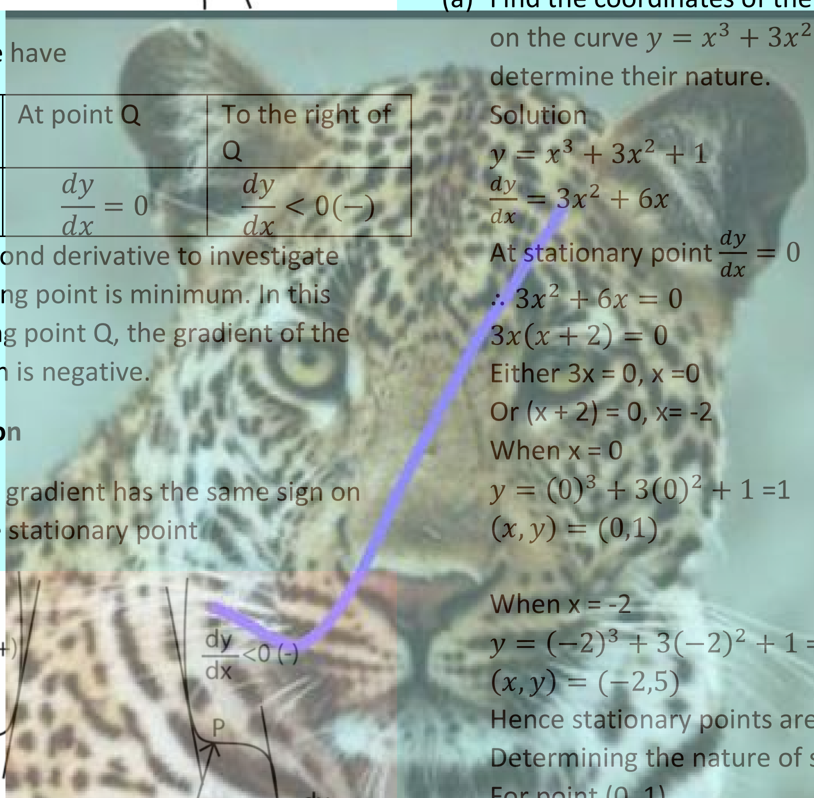
To the left of P	At point P	To the right of P
$\frac{dy}{dx} < 0(-)$	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} < 0(-)$

When using second derivative to investigate whether the turning point is a point of inflexion. In this case, considering point P, the gradient of the derived function is zero. I.e. at point P, $\frac{d^2y}{dx^2} = 0$

However, the second derivatives can also be zero at a maximum or minimum point. For this reason, therefore, we must examine the sign of $\frac{dy}{dx}$ at each side of the point.

Example 6

(a) Find the coordinates of the stationary points on the curve $y = x^3 + 3x^2 + 1$ and determine their nature.



Solution

$y = x^3 + 3x^2 + 1$

$\frac{dy}{dx} = 3x^2 + 6x$

At stationary point $\frac{dy}{dx} = 0$

$\therefore 3x^2 + 6x = 0$

$3x(x + 2) = 0$

Either 3x = 0, x =0

Or (x + 2) = 0, x= -2

When x = 0

$y = (0)^3 + 3(0)^2 + 1$ =1

$(x, y) = (0,1)$

When x = -2

$y = (-2)^3 + 3(-2)^2 + 1$ =5

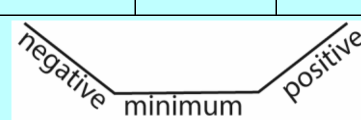
$(x, y) = (-2,5)$

Hence stationary points are (0, 1) and (-2, 5)

Determining the nature of stationary points.

For point (0, 1)

x	-1	0	1
$\frac{dy}{dx}$	-3	0	9



The stationary point (0, 1) is minimum

Alternatively

$\frac{dy}{dx} = 3x^2 + 6x$

$\frac{d^2y}{dx^2} = 6x + 6$

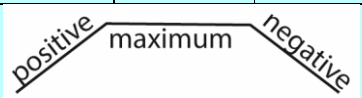
At point (0, 1)

$\frac{d^2y}{dx^2} = 6(0) + 6 = 6$

Hence the stationary point (0, 1) is minimum

For point (-2, 5)

x	-2.5	-2	1
$\frac{dy}{dx}$	3.75	0	-2.25



The stationary point (-2, 5) is maximum

Alternatively

$\frac{dy}{dx} = 3x^2 + 6x$

$\frac{d^2y}{dx^2} = 6x + 6$

At point (-2, 5)

$\frac{d^2y}{dx^2} = 6(-2) + 6 = -6$

Hence the stationary point (-2, 5) is maximum

(b) Find and distinguish between the nature of the turning points of the curves

(i) $y = x^3 - x^2 - 5x + 6$

Solution

$y = x^3 - x^2 - 5x + 6$

$\frac{dy}{dx} = 3x^2 - 2x - 5$

At stationary point $\frac{dy}{dx} = 0$

$\therefore 3x^2 - 2x - 5 = 0$

$(x + 1)(3x - 5) = 0$

Either $(x + 1)$ = 0, x = -1

Or $(3x - 5)$ = = 0, x= $\frac{5}{3}$

When x = -1

$y = (-1)^3 - (-1)^2 - 5(-1) + 6 = 9$

$(x, y) = (-1,9)$

When x = $\frac{5}{3}$

$y = \left(\frac{5}{3}\right)^3 - \left(\frac{5}{3}\right)^2 - 5\left(\frac{5}{3}\right) + 6 = \frac{-13}{27}$

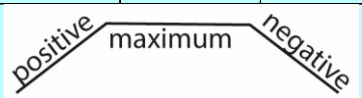
$(x, y) = (\frac{5}{3}, \frac{-13}{27})$

Hence stationary points are $(-1,9)$and $(\frac{5}{3}, \frac{-13}{27})$

Determining the nature of stationary points.

For point (-1, 9)

x	-2	-1	0
$\frac{dy}{dx}$	11	0	-1



The stationary point (-1, 9) is maximum

Alternatively

$\frac{dy}{dx} = 3x^2 - 2x - 5$

$\frac{d^2y}{dx^2} = 6x - 2$

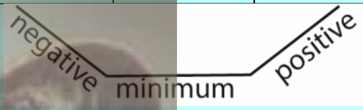
At point (-1, 9)

$\frac{d^2y}{dx^2} = 6(-1) - 2 = -8$ (< 0)

Hence the stationary point (-1, 9) is maximum

For point $(\frac{5}{3}, \frac{-13}{27})$

x	1	$\frac{5}{3}$	2
$\frac{dy}{dx}$	-4	0	3



The stationary point $(\frac{5}{3}, \frac{-13}{27})$is a minimum.

Alternatively

$\frac{dy}{dx} = 3x^2 - 2x - 5$

$\frac{d^2y}{dx^2} = 6x - 2$

At point $(\frac{5}{3}, \frac{-13}{27})$

$\frac{d^2y}{dx^2} = 6\left(\frac{5}{3}\right) - 2 = 8$ (> 0)

Hence the stationary point $(\frac{5}{3}, \frac{-13}{27})$ is maximum

(ii) $y = x^4 + 2x^3$

Solution

$y = x^4 + 2x^3$

$\frac{dy}{dx} = 4x^3 + 6x^2$

At stationary point $\frac{dy}{dx} = 0$

$\therefore 4x^3 + 6x^2 = 0$

$2x^2(2x + 3) = 0$

Either $2x^2$ = 0, x = 0

Or $(2x + 3)$ = = 0, x= $-\frac{3}{2}$

When x = 0

$y = (0)^4 + 2(0)^3 = 0$

$(x, y) = (0,0)$

When x = $-\frac{3}{2}$

$y = \left(-\frac{3}{2}\right)^4 + 2\left(-\frac{3}{2}\right)^3 = \frac{27}{16}$

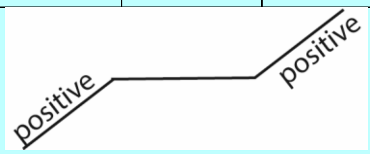
$(x, y) = \left(-\frac{3}{2}, \frac{27}{16}\right)$

Hence stationary points are (0,0)and $\left(-\frac{3}{2}, \frac{27}{16}\right)$

Determining the nature of stationary points.

For point (0, 0)

x	-1	0	1
$\frac{dy}{dx}$	2	0	10



The stationary point (0, 0) is inflexion

Alternatively

$\frac{dy}{dx} = 4x^3 + 6x^2$

$\frac{d^2y}{dx^2} = 12x^2 + 12x$

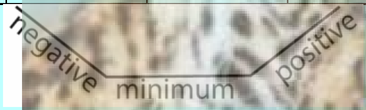
For (0, 0)

$\frac{d^2y}{dx^2} = 12(0)^2 + 12(0) = 0$

Hence the stationary point (0, 0) is inflexion

For point $\left(-\frac{3}{2}, \frac{27}{16}\right)$

x	-2	$-\frac{3}{2}$	-1
$\frac{dy}{dx}$	-8	0	2



The stationary point $\left(-\frac{3}{2}, \frac{27}{16}\right)$ is minimum

Alternatively

$\frac{dy}{dx} = 4x^3 + 6x^2$

$\frac{d^2y}{dx^2} = 12x^2 + 12x$

At point $(\frac{5}{3}, \frac{-13}{27})$

$\frac{d^2y}{dx^2} = 12\left(\frac{5}{3}\right)^2 + 12\left(\frac{5}{3}\right)$ =9 (> 0)

Hence the stationary point $(\frac{5}{3}, \frac{-13}{27})$ is maximum

Revision exercise 6

1. Find the coordinates and the nature of the stationary points of the curves
   (a) $y = x^2 + \frac{16}{x}$ [(2, 12), minimum]
   (b) $x = 4 - t^3$ and $y = t^2 - 2t$ [(3, -1), minimum]
   (c) $y = \frac{2-x^3}{x^4}\left[\left(2, \frac{-3}{8}\right), mim\right]$
   (d) $y = \frac{2}{x^3} - \frac{1}{x^2}\left[\left(3, \frac{-1}{27}\right), mim\right]$
   (e) $y = \frac{1}{x} - \frac{3}{x^2}\left[\left(6, \frac{1}{12}\right), max\right]$
2. Find the maximum and minimum values of the function 2sint+ cos2t $\left[\left(\frac{\pi}{2}, 1\right), min; \left(\frac{\pi}{6}, \frac{3}{2}\right), max; \left(\frac{5\pi}{6}, \frac{3}{2}\right), min\right]$
3. If $p = 4x^2 - 10x + 7$, find the minimum of p and the corresponding value of x at which it occurs. $\left[\frac{3}{4}, \frac{5}{4}\right]$
4. If $v = 30x - 6x^2$, find the maximum of v and the corresponding value of x at which it occurs. $\left[37\frac{1}{2}, 2\frac{1}{2}\right]$

Application of maxima and minima to problem solving

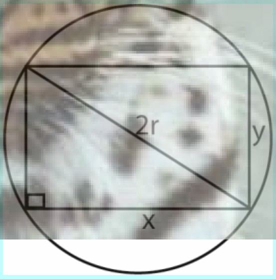
In all case of maximum or minimum values of functions, their derivatives are equal to zero.

Example 7

(a) Find the dimensions of a rectangle with maximum area that can be inscribed in a circle of radius r.

Solution

Let x and y be the dimensions of the rectangle



Area of a rectangle $A = xy$

From the figure, $x^2 + y^2 = (2r)^2$

$x^2 + y^2 = 4r^2$

$y = \sqrt{4r^2 - x^2}$

$\Rightarrow A = x\sqrt{4r^2 - x^2} = x(4r^2 - x^2)^{\frac{1}{2}}$

$\frac{dA}{dx} = \left(\sqrt{4r^2 - x^2}\right)(1) + \frac{x(-2x)}{2\sqrt{4r^2 - x^2}}$

$= \frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}}$

Area is maximum when $\frac{dA}{dx} = 0$

$\Rightarrow \frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}} = 0$

$4r^2 - 2x^2 = 0$

$4r^2 = 2x^2$

$x = r\sqrt{2}$

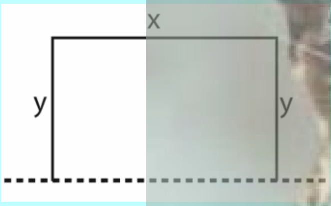
$y = \sqrt{2r^2} = r\sqrt{2}$

Hence the figure is a square of side $r\sqrt{2}$

(b) A figure wishes to enclose a rectangular piece of land of area 1250cm2 whose one side is bound by a straight bank of river. Find the least possible length of barbed wire required to fence the other three sides of land

Solution

Let x and y be the dimensions of the rectangular land



Type equation here.

Area = xy

1250 = xy

y= $\frac{1250}{x}$

Perimeter, P(length of the wire) = 2y + x

$P = 2\left(\frac{1250}{x}\right) + x = \frac{2500}{x} + x$

$\frac{dP}{dx} = 1 - \frac{2500}{x^2}$

P is minimum when $\frac{dP}{dx} = 0$

$\Rightarrow 1 - \frac{2500}{x^2} = 0$

$x^2 = 2500$

$x = 50m$

$y = \frac{1250}{50} = 25m$

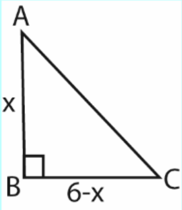
∴the minimum possible length

P = 25 x 2 + 50 = 100m

(c) In a right angled triangle ABC where <ABC = 90^0, the length AB and BC vary such that their sum is 6. Find the maximum area of the rectangle

Solution

Let AB = x , then BC = 6-x



$A = \frac{1}{2}(x)(6-x) = 3x - \frac{x^2}{2}$

$\frac{dA}{dx} = 3 - x$

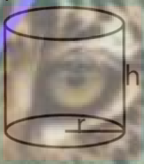
Area is maximum when $\frac{dA}{dx} = 0$

⇨ 3 – x =0

x = 3

Hence maximum area = $\frac{1}{2}(3)(6-3) = 4.5$

(d) A company that manufactures animal feed wishes to pack the feed in enclosed cylindrical tins. What should be the dimensions of each tin if each is to have a volume of 250πcm^3 and the minimum possible surface area?



Volume, $v = \pi r^2 h$

$\pi r^2 h = 250\pi$

$h = \frac{250}{r^2}$

Surface area A = $2\,\pi r^2 + 2\pi rh$

By substitution

A = $2\,\pi r^2 + 2\pi r\left(\frac{250}{r^2}\right) = 2\,\pi r^2 + \frac{500\pi}{r}$

$\frac{dA}{dr} = 4\pi r - \frac{500\pi}{r^2}$

Surface area is minimum when $\frac{dA}{dr} = 0$

$\Rightarrow 4\pi r - \frac{500\pi}{r^2} = 0$

$r^3 = 125$

$r = 5cm$

$h = \frac{250}{(5)^2} = 10cm$

minimum surface area

= $2\,\pi(5)^2 + 2\pi(5)(10)$

=150πcm^2

(e) An enclosed, right circular base radius rcm and height hm has a volume 54πcm^3, show that the total surface area $A = \frac{108\pi}{r} + 2\pi r^2$. Hence find the radius and height corresponding to the minimum surface area.

Volume, $v = \pi r^2 h$

$\pi r^2 h = 54\pi$

$h = \frac{54}{r^2}$

Surface area A = 2 $\pi r^2 + 2\pi rh$

By substitution

A = 2 $\pi r^2 + 2\pi r\left(\frac{54}{r^2}\right)$

$= \frac{108\pi}{r} + 2\pi r^2$(as required)

$\frac{dA}{dr} = 4\pi r - \frac{108\pi}{r^2}$

Surface area is minimum when $\frac{dA}{dr} = 0$

$\Rightarrow 4\pi r - \frac{108\pi}{r^2} = 0$

$r^3 = 27$

$r = 3cm$

$h = \frac{54}{(3)^2} = 6cm$

Hence the surface area is minimum when radius = 3cm and height = 6cm

(f) Write down an expression of the volume v and surface area s of a closed cylinder of radius r and height h. if the surface area is kept constant, show that the volume of the cylinder will be maximum when h = 2r.

Solution

$v = \pi r^2 h$

$s = 2\pi rh + 2\pi r^2$

From $s = 2\pi rh + 2\pi r^2$

$h = \frac{s - 2\pi r^2}{2\pi r}$

Substituting h in $v = \pi r^2 h$

$v = \pi r^2\left(\frac{s-2\pi r^2}{2\pi r}\right)$

$= r\left(\frac{s-2\pi r^2}{2}\right)$

$= \frac{sr}{2} - \pi r^3$

$\frac{dv}{dr} = \frac{s}{2} - 3\pi r^2$

v is maximum when $\frac{dv}{dr} = 0$

$\Rightarrow \frac{s}{2} - 3\pi r^2 = 0$

$\frac{s}{2} = 3\pi r^2$

$s = 6\pi r^2$

Substituting s in h

$h = \frac{6\pi r^2 - 2\pi r^2}{2\pi r} = \frac{4\pi r^2}{2\pi r} = 2r$

$\therefore h = 2r$

Revision exercise 7

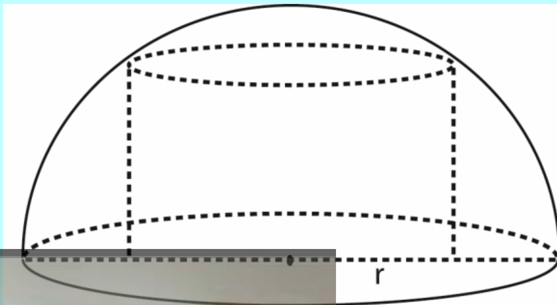
1. A rectangular enclosure is formed by using 1200m fencing. Find the greatest possible area that can be enclosed in this way and the corresponding dimensions of the rectangular enclosure.
[90,000m^2, 300m square]
2. An open tank with a square base is made from 12m^2 of metal sheet. Find the length of the side of the base for the volume of the tank to be maximum and find this volume.
[2m, 4m^2]
3. A cylindrical tin without a lid is made of sheet of metal. If s the area of the sheet used, without waste v, the volume of the tin and r the radius of the cross-section, prove that 2v = sr – πr^3. If s is given, prove that the volume of the tin is greatest when the ratio of the height to the diameter is 1:2.
4. A strip of wire of length 150cm is cut into two pieces. One piece is bent to form a square of x cm and the other piece is bend to form a rectangle which is twice long as wide.
Find the expression, interms of x, for the
(i) Width of the rectangle $\left[25 - \frac{2}{3}x\right]$
(ii) Length of rectangle$\left[50 - \frac{4}{3}x\right]$
(iii) Area of rectangle $\left[1250 - \frac{200}{3}x + \frac{8}{9}x^2\right]$

(iv) Given also that the sum of the two areas enclosed is a minimum, calculate the value of x. $\left[\frac{300}{17}\right]$

5. A closed cuboid plastic box is to be made with an external surface area of 216cm^2. The base is to be such that its length is four times its breadth. Find the length of the base of the box if the volume of the box is to be maximum and find this maximum volume. [12m, 172.8cm^3]
6. A cylindrical can, with no lid, has a circular base of radius r cm, the total surface area is 300π cm^2.
   (a) *Show that the volume* $v\ cm^3$ of the can is given by $v = \frac{\pi r}{2}(300 - r^2)$
   (b) Given that r may vary, find the positive value of r for which $\frac{dv}{dr} = 0$ [10]
   (c) Show that this value of r gives a maximum value of v. [v''(10)=-30π <0]
7. A cylindrical tank, open at the top and of height hm and radius rm, has a capacity of 1m^3. Show that $h - \frac{1}{\pi r^2}$
   If its total internal surface area is sm^2. Show that $s = \frac{2}{r} + \pi r^2$
   Determine the value of r which makes surface area s as small as possible $\left[\sqrt[3]{\frac{1}{11}}\right]$

Revision exercise 8 (topical)

1. The distance s m of a particle from a fixed $s = t^2(t^2 + 6) - 4t(t-1)(t+1)$ where t is time. Find the velocity and acceleration of the particle when t = 1s [8ms^{-1}, 0ms^{-2}]
2. Find the equation of the tangent to the curve $x^2 + y^2 - 2xy = 4$ at (1, -1)[y =-1]
3. A curve is defined by parametric equation $x = t^2 - t, y = 3t + 4$, find the equation of the tangent at (2, 10) [y =x+8]
4. A hemisphere bowl of radius a cm is initially full of water. The water runs out through a small hole at the bottom of the bowl at a rate$a^2\{36x(2a - x\}^{-1}cms^{-1}$. Find how long it will take for the depth of the water to be $\frac{1}{3}a\ cm$ and the rate at which the depth is decreasing at this instant.$\left[20.4s, \frac{a}{20}cms^{-1}\right]$
5. An inverted cone with vertical angle of 60^0 is collecting water leaking from a tap at a rate of 0.2cm^{3}s^{-1}. If the height of water in the cone is 10cm, find the rate the surface are of water is increasing [12cm^{2}s^{-1}]
6. A cylinder is inscribed in a semi-hemisphere of radius r as shown in the figure below



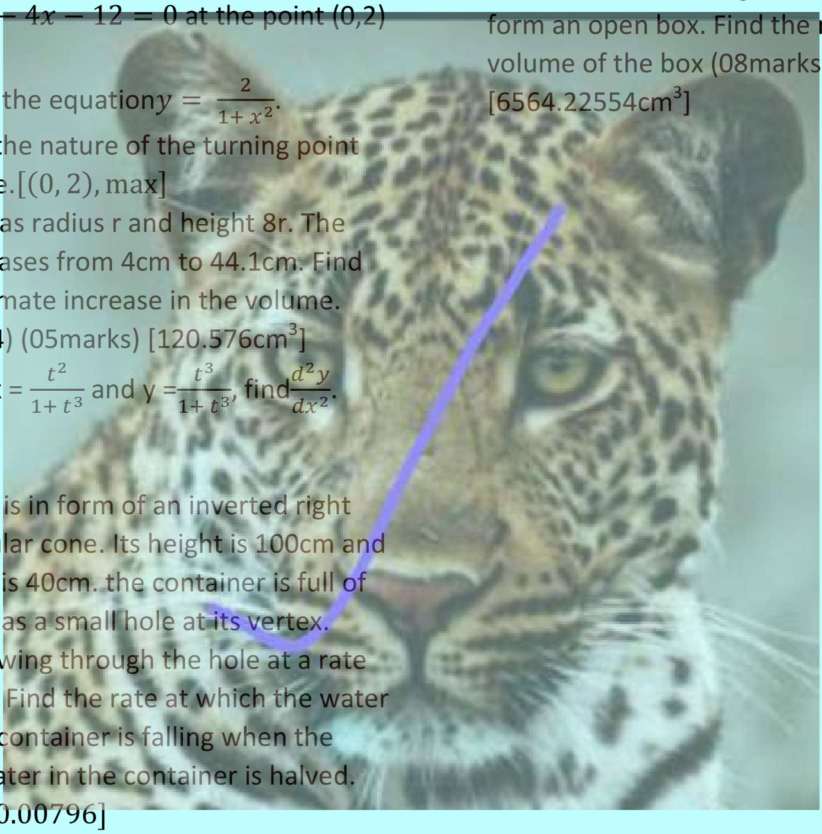
   Find the maximum volume of the cylinder in terms of r. $\left[\frac{2\pi r^3}{3\sqrt{3}}\right]$
7. Using calculus of small increments or otherwise find $\sqrt{98}$ correct to 1 decimal place. [9.9]
8. If $y = \sqrt{(5x^2 + 3)}$, show that
   $$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 5$$
9. A right circular cone of radius r has a minimum volume, the sum of vertical height h, and the circumference is 15cm. if the radius varies, show that the maximum volume of the cone is $\frac{625}{\pi}cm^3$.
10. The distance of the particle moving in a straight line from a fixed point after time t is given by $x = e^{-1}sint$. Show that the particle is instantaneously at rest at $t = \frac{\pi}{4}$. Find the acceleration at $t = \frac{\pi}{4}$s.[-0.6447]
11. A spherical balloon is inflated such that the rate at which its radius is increasing is 0.5cms-1. Find the rate at which
    (a) The volume is increasing at this instant [157.08cm^{3}s^{-1}]
    (b) The surface area is increasing when r= 8.5cm [106.814cm^{2}s-1]
12. A curve is represented by parametric equations $x = 3t\ and\ y = \frac{4}{t^2+1}$. Find the general equation for the tangent to the curve in terms of x, y and t. hence determine the equation of the tangent at the point (3,2).

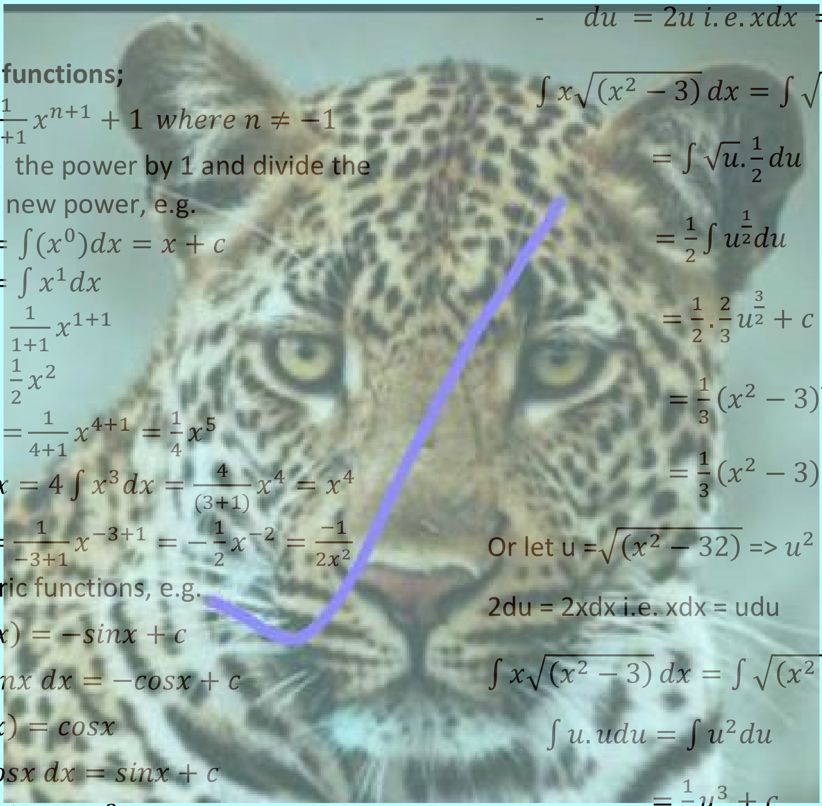
$[3y(t^2+1)^2 + 8tx = 24t^2 + 12(t^2+1; 12]$

13. Given that $R = q\sqrt{(1000-q^2)}$, find
    (a) $\frac{dR}{dq}$
    (b) The value of q when R is maximum$\left[\frac{1000-2q^2}{\sqrt{(1000-q^2)}}, (b)\sqrt{500}\right]$
14. The base radius of a right circular cone increase and the volume changes by 2%. If the height of the cone remains constant, find the percentage increase in the circumference of the base [1%]
15. Find the equation of the normal to the curve $x^2y + 3y^2 - 4x - 12 = 0$ at the point (0,2) [y= -3x +2]
16. A curve has the equation$y = \frac{2}{1+x^2}$. Determine the nature of the turning point on the curve.$[(0,2), \text{max}]$
17. A cylinder has radius r and height 8r. The radius increases from 4cm to 44.1cm. Find the approximate increase in the volume. (use π= 3.14) (05marks) [120.576cm^3]
18. Given that x = $\frac{t^2}{1+t^3}$ and y =$\frac{t^3}{1+t^3}$, find$\frac{d^2y}{dx^2}$.
    $\left[\frac{6}{t}\left(\frac{1+t^3}{2-t^3}\right)^3\right]$
19. A container is in form of an inverted right angled circular cone. Its height is 100cm and base radius is 40cm. the container is full of water and has a small hole at its vertex. Water is flowing through the hole at a rate of 10cm^{3}s^{-1}. Find the rate at which the water level in the container is falling when the height of water in the container is halved. (05marks) [0.00796]
20. A curve whose equation is x^{2}y + y^2 -3x =3 passes through points A(1, 2)and B(-1, 0). The tangent at A and the normal at the curve at B intersect at point C. Determine;
    (a) equation of the tangent. (06marks)
    $\left[y = \frac{1}{5}x + \frac{11}{5}\right]$
    (b) coordinates of C. (06marks)$[C(-19, 6)]$

Determine the equation of the tangent to the curve y^3 + y^2 –x^4 = 1 at the point (1, 1) (05marks) [5y = 4x + 1]

21. Find the equation of the tangent to the curve y = $\frac{a^3}{x^2}$ at the point ($\frac{a}{t}, at^2$). (05marks)
22. given that y = $In\left\{e^x\left(\frac{x-2}{x+2}\right)^{\frac{3}{4}}\right\}$, show that $\frac{dy}{dx} = \frac{x^2-1}{x^2-4}$ (05marks)
23. A rectangular sheet is 50cm long and 40cm wide. A square of x cm is cut off from each corner. The remaining sheet is folded to form an open box. Find the maximum volume of the box (08marks) [6564.22554cm^3]





Integration (A-level)

It the reverse of differentiation.

During integration the following concepts should be considered.

(a) Polynomial functions;

$\int x^n dx = \frac{1}{n+1} x^{n+1} + 1 \; where \; n \neq -1$

i.e. increase the power by 1 and divide the term by the new power, e.g.

(i) $\int 1dx = \int (x^0)dx = x + c$

(ii) $\int xdx = \int x^1 dx$

$= \frac{1}{1+1} x^{1+1}$

$= \frac{1}{2} x^2$

(iii) $\int x^4 dx = \frac{1}{4+1} x^{4+1} = \frac{1}{4} x^5$

(iv) $\int 4x^3 dx = 4 \int x^3 dx = \frac{4}{(3+1)} x^4 = x^4$

(v) $\int x^{-3} = \frac{1}{-3+1} x^{-3+1} = -\frac{1}{2} x^{-2} = \frac{-1}{2x^2}$

(b) Trigonometric functions, e.g.

(i) $\frac{d}{dx}(cosx) = -sinx + c$

- $\int sinx \, dx = -cosx + c$

(ii) $\frac{d}{dx}(sinx) = cosx$

- $\int cosx \, dx = sinx + c$

(iii) $\frac{d}{dx}(tanx) = sec^2 + c$

- $\int sec^2 \, dx = tanx + c$

(iv) $\frac{d}{dx}(cotx) = -cosec^2 x$

- $\int cosec^2 x = -cotx + c$

Methods of integration

The choice of the method depends on judgement. Below are some of the methods:

Integration by change of variable where a derivative exist/integration by recognition or inspection

Example 1

(i) $\int x\sqrt{(x^2 - 2)}$dx

Solution

Let u = x^2-2

- $du = 2u \; i.e. xdx = \frac{1}{2} du$

$\int x\sqrt{(x^2 - 3)}\, dx = \int \sqrt{(x^2 - 3)}\, xdx$

$= \int \sqrt{u}. \frac{1}{2} du$

$= \frac{1}{2} \int u^{\frac{1}{2}} du$

$= \frac{1}{2}. \frac{2}{3} u^{\frac{3}{2}} + c$

$= \frac{1}{3} (x^2 - 3)^{\frac{3}{2}} + c$

$= \frac{1}{3} (x^2 - 3)\sqrt{(x^2 - 3)} + c$

Or let u = $\sqrt{(x^2 - 32)}$ => $u^2 = x^2 - 3$

2du = 2xdx i.e. xdx = udu

$\int x\sqrt{(x^2 - 3)}\, dx = \int \sqrt{(x^2 - 3)}\, xdx$

$\int u.udu = \int u^2 du$

$= \frac{1}{3} u^3 + c$

$= \frac{1}{3} (x^2 - 3)\sqrt{(x^2 - 3)} + c$

(ii) $\int xcosec^2(x^2)dx$

Solution

Let u = x^2 => du =2xdx i.e. xdx = $\frac{1}{2} du$

$\int xcosec^2(x^2)dx = \int cosec^2(x^2)xdx$

$= \frac{1}{2} \int cosec^2 udu$

$= -cotu + c$

$= -cotx^2 + c$

(iii) $\int_0^1 \frac{x^2 - 1}{\sqrt{(x^3 - 3x + 5)}} dx$

Solution

Let u = $x^3 - 3x + 5$ => $du = (3x^2 - 3)dx$

i.e. $(3x2 - 3)dx = \frac{1}{3}du$

$\therefore \int_0^1 \frac{x^2-1}{\sqrt{(x^3-3x+5)}}dx = \frac{1}{3}\int_0^1 \frac{1}{\sqrt{u}}dx$

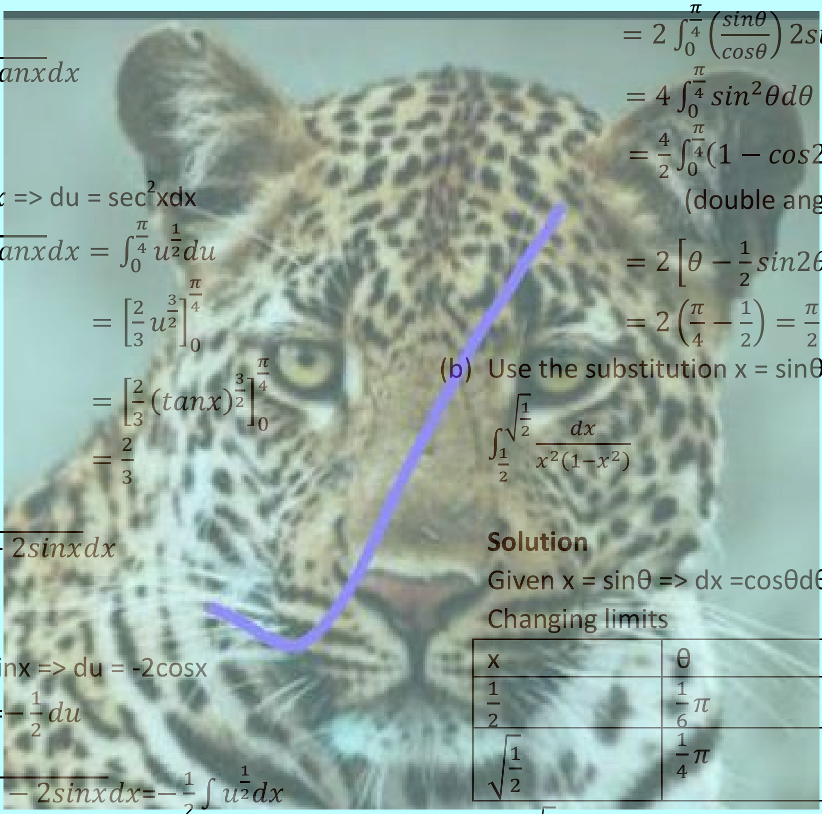
$= \frac{1}{3}u^{\frac{1}{2}}(2) + c$

$= \frac{2}{3}\sqrt{(x^3 - 3x + 5)}$

- $\int_0^1 \frac{x^2-1}{\sqrt{(x^3-3x+5)}}dx = \frac{2}{3}\left[\sqrt{(x^3 - 3x + 5)}\right]_0^1$

$= \frac{2}{3}\left(\sqrt{1 - 3 + 5} - \sqrt{5}\right)$

$= \frac{2}{3}\left(\sqrt{3} - \sqrt{5}\right) = 0.336$



(iv) $\int_0^{\frac{\pi}{4}} sec^2x\sqrt{tanx}dx$

Solution

Let u = $tanx$ => du = sec^2xdx

$\int_0^{\frac{\pi}{4}} sec^2x\sqrt{tanx}dx = \int_0^{\frac{\pi}{4}} u^{\frac{1}{2}}du$

$= \left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^{\frac{\pi}{4}}$

$= \left[\frac{2}{3}(tanx)^{\frac{3}{2}}\right]_0^{\frac{\pi}{4}}$

$= \frac{2}{3}$

(v) $\int cosx\sqrt{1 - 2sinx}dx$

Solution

Let u = 1-2sinx => du = -2cosx

i.e. cosxdx = $-\frac{1}{2}du$

$\therefore \int cosx\sqrt{1 - 2sinx}dx = -\frac{1}{2}\int u^{\frac{1}{2}}dx$

$= \frac{1}{2}.\frac{2}{3}u^{\frac{3}{2}} + c$

$= \frac{1}{3}(1 - 2sinx)^{\frac{3}{2}} + c$

$= \frac{1}{3}(1 - 2sinx)\sqrt{(1 - 2sinx)} + c$

Integration by change of variable where a derivative is given

Example 2

(a) Using the substation x =cos2θ or otherwise, prove that

$\int_0^1 \sqrt{\left(\frac{1-x}{1+x}\right)}dx = \frac{\pi}{2} - 1$

Solution

Given x= cos2θ => dx = -2sin2θdθ

Changing limits

x	θ
0	$\frac{1}{4}\pi$
1	0

$\int_0^1 \sqrt{\left(\frac{1-x}{1+x}\right)}dx = -2\int_{\frac{\pi}{4}}^0 \sqrt{\left(\frac{1-cos2\theta}{1+cos\theta}\right)}sin2\theta\, d\theta$

$= 2\int_0^{\frac{\pi}{4}} \sqrt{\left(\frac{2sin^2\theta}{2cos^2\theta}\right)}sin2\theta\, d\theta$

$= 2\int_0^{\frac{\pi}{4}} \left(\frac{sin\theta}{cos\theta}\right)2sin\theta cos\theta\, d\theta$

$= 4\int_0^{\frac{\pi}{4}} sin^2\theta d\theta$

$= \frac{4}{2}\int_0^{\frac{\pi}{4}}(1 - cos2\theta)d\theta$

(double angle form

$= 2\left[\theta - \frac{1}{2}sin2\theta\right]_0^{\frac{\pi}{4}}$

$= 2\left(\frac{\pi}{4} - \frac{1}{2}\right) = \frac{\pi}{2} - 1$

(b) Use the substitution x = sinθ to evaluate

$\int_{\frac{1}{2}}^{\sqrt{\frac{1}{2}}} \frac{dx}{x^2(1-x^2)}$

Solution

Given x = sinθ => dx =cosθdθ

Changing limits

x	θ
$\frac{1}{2}$	$\frac{1}{6}\pi$
$\sqrt{\frac{1}{2}}$	$\frac{1}{4}\pi$

$\therefore \int_{\frac{1}{2}}^{\sqrt{\frac{1}{2}}} \frac{dx}{x^2(1-x^2)} = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{cos\theta d\theta}{sin^2\theta(1-sin^2\theta)}$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{d\theta}{sin^2\theta} = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} cosec^2\theta d\theta$

$= [-cot\theta]_{\frac{\pi}{6}}^{\frac{\pi}{4}} = [cot\theta]_{\frac{\pi}{4}}^{\frac{\pi}{6}}$

$= cot\frac{\pi}{6} - cot\frac{\pi}{4} = \sqrt{3} - 1$

Revision exercise 1

Integrate the following using the suggested substitution in each case.

1. $\int x(x+4)^3 dx$, u = x + 4
$\left[\frac{1}{5}(x-1)(x+4)^4 + c\right]$
2. $\int (x-4)(x-1)^3 dx$, u = x – 1
$\left[\frac{1}{4}(4x-19)(x-1)^4 + c\right]$
3. $\int x(2x-3)^2 dx$, u = 2x – 3

$\left[\frac{1}{16}(2x+1)(2x-3)^3 + c\right]$

4. $\int (3x+1)\,(2x-5)^2 dx$, u = 2x – 5
$\left[\frac{1}{48}(18x+23)(2x-5)^3 + c\right]$

5. $\int \frac{x}{x+3} dx$, u= x+3 $[x - 3In(x+3) + c]$
6. $\int \frac{x}{(x+1)^2} dx$, u= x+1 $\left[\frac{1}{x+1} + In(x+1) + c\right]$
7. $\int \frac{x+1}{(2x-3)^3} dx$, u= 2x – 1 $\left[-\frac{4x+1}{8(2x-3)^2}\right]$
8. $\int \sqrt{(x+1)}\, dx$, u = x + 1
$\left[\frac{2}{15}(3x-2)\sqrt{(x+1)^2} + c\right]$
9. $\int x\sqrt{(x-1)}\, dx, u = \sqrt{(x-1)}$

$\left[\frac{2}{15}(3x+2)\sqrt{(x-1)^2} + c\right]$

10. $\int (x-4)\sqrt{(x+5)}$, u = x + 5
$\left[\frac{2}{5}(x-10)\sqrt{(x+1)^3} + c\right]$
11. $\int (3x-2)\sqrt{(1-2x)}dx, u = \sqrt{(1-2x)}$
$\left[\frac{1}{15}(7-9x)\sqrt{(1-2x)^2} + c\right]$
12. $\int \frac{x}{\sqrt{x+1}} dx$, u= x+1
$\left[\frac{2}{3}(x-2)\sqrt{x+1} + c\right]$
13. $\int \frac{x}{\sqrt{x-3}} dx$, u= $\sqrt{x-3}$
$\left[\frac{2}{3}(x+6)\sqrt{x-3} + c\right]$
14. $\int \frac{x-2}{\sqrt{x-4}} dx$, u= $x-4$
$\left[\frac{2}{3}(x+2)\sqrt{x-4} + c\right]$
15. $\int \frac{x+3}{\sqrt{5-x}} dx$, u= $\sqrt{5-x}$

$\left[-\frac{2}{3}(x+19)\sqrt{5-x} + c\right]$

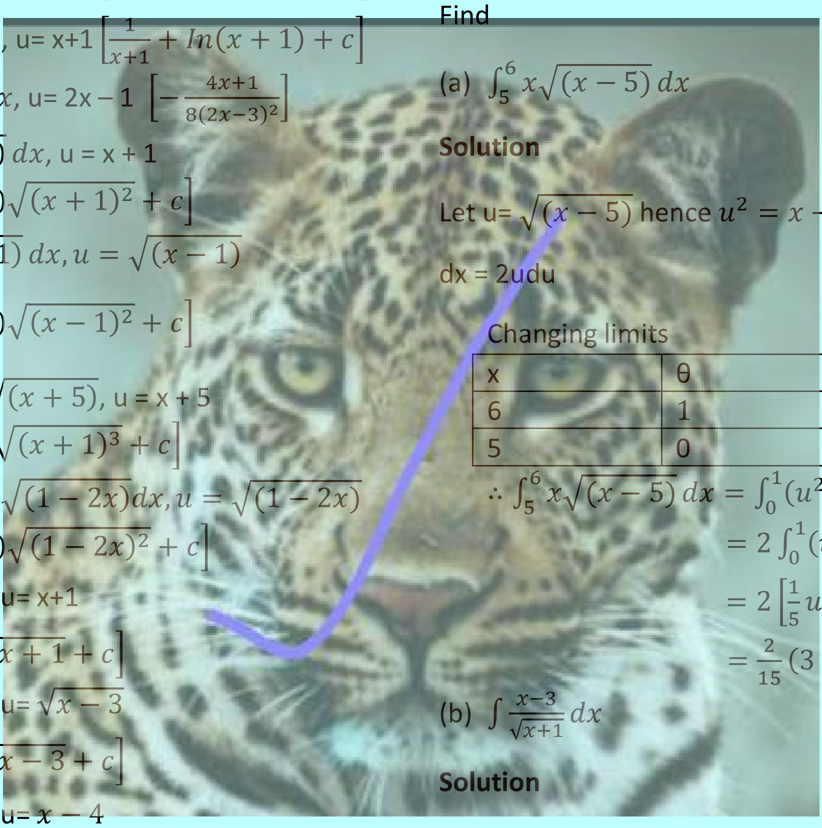
16. Use the substitution $x = \frac{1}{u}$ to evaluate $\int_1^2 \frac{dx}{x\sqrt{x^2-1}}$ $\left[\frac{\pi}{3}\right]$
17. Use the substitution u = $\sqrt{x-2}$ to evaluate $\int_3^4 \frac{3x}{\sqrt{x-2}} dx$ $[16\sqrt{2} - 4]$
18. Evaluate
(a) $\int_3^5 x(x-3)^2 dx$ [12]
(b) $\int_4^7 \frac{5-x}{\sqrt{x-3}}$dx $\left[-\frac{2}{3}\right]$
(c) $\int_1^3 (3x+1)(2-x)^4\, dx$ $\left[2\frac{4}{5}\right]$
19. By using the substitution u = $\sqrt{1+x^2}$, show that $\int_0^{\sqrt{3}} x^2\sqrt{(1+x^2)}\, dx = 3\frac{13}{15}$

Integration by change of variable where a derivative not exist

Here a term is solved by changing it to another variable

Example 3

Find



(a) $\int_5^6 x\sqrt{(x-5)}\, dx$

Solution

Let u= $\sqrt{(x-5)}$ hence $u^2 = x - 5$ => x =u^2 + 5

dx = 2udu

Changing limits

x	θ	
6	1	
5	0	

$$\therefore \int_5^6 x\sqrt{(x-5)}\, dx = \int_0^1 (u^2+5)u.2udu$$
$$= 2\int_0^1 (u^4+5u^2)\, du$$
$$= 2\left[\frac{1}{5}u^5 + \frac{5}{3}u^3\right]_0^1$$
$$= \frac{2}{15}(3+25) = \frac{56}{15}$$

(b) $\int \frac{x-3}{\sqrt{x+1}} dx$

Solution

Let u= $\sqrt{x+1}$ i.e.$u^2 = x+1$

- x = u^2 – 1 and dx = 2udu

$$\therefore \int \frac{x-3}{\sqrt{x+1}} dx = \int \frac{(u^2-1)}{u}.2udu$$
$$= 2\int (u^2-4)du$$
$$= 2\left(\frac{1}{3}u^3 - 4u\right) + c$$
$$= \frac{2}{3}u(u^2-12) + c$$
$$= \frac{2}{3}\sqrt{(x+1)}(x+1-12) + c$$
$$= \frac{2}{3}(x-11)\sqrt{(x+1)} + c$$

(c) $\int (2x-1)(x+2)^3\, dx$

Solution

Let u = x+2 => x = u-2 and dx = du

$\therefore \int (2x-1)(x+2)^3\, dx$

$= \int [2(u-2) - 1u^3 du]$

$= \int (2u^4 - 5u^3)\, du$

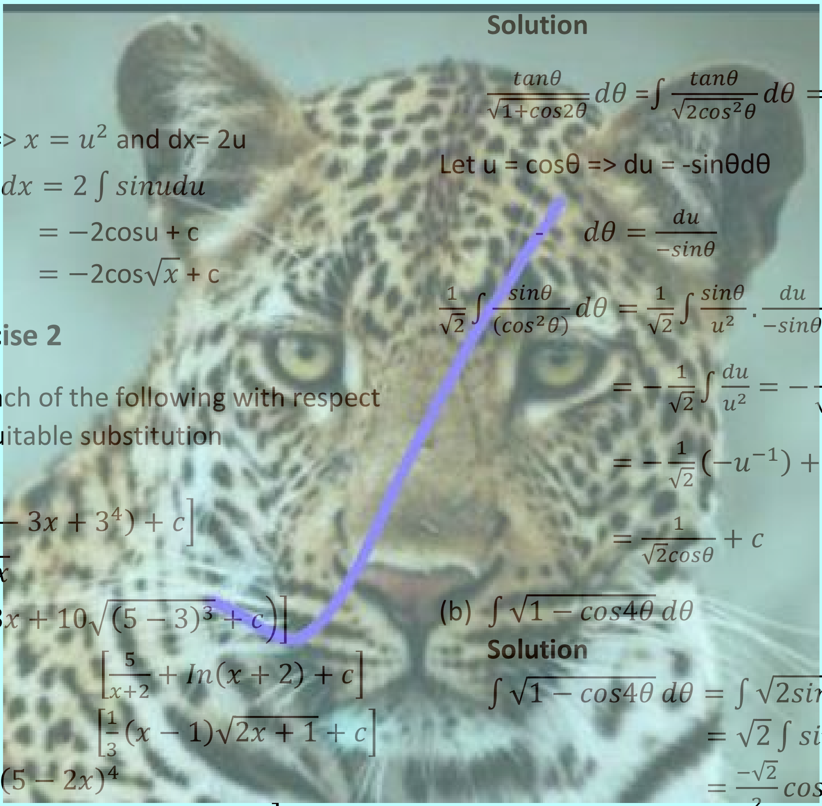
$= \frac{2}{5}u^5 - \frac{5}{4}u^4 + c$

$= \frac{1}{20}u^4(8u - 25) + c$

$= \frac{1}{20}(x+2)^4(8(x+2+-25) + c$

$= \frac{1}{20}(8x-9)(x+2)^4 + c$

(d) $\int \frac{sin\sqrt{x}}{\sqrt{x}} dx$



Solution

Let u = $\sqrt{x}$ => $x = u^2$ and dx= 2u

$\therefore \int \frac{sinu}{u}.2udx = 2\int sinudu$

$= -2cosu + c$

$= -2cos\sqrt{x} + c$

Revision exercise 2

1. Integrate each of the following with respect to x using suitable substitution
   (a) $x(x+3)^3$
   $\left[\frac{1}{20}(4x - 3x + 3^4) + c\right]$
   (b) $x\sqrt{5-x}$
   $\left[-\frac{2}{15}\left(3x + 10\sqrt{(5-3)^3} + c\right)\right]$
   (c) $\frac{x-3}{(x+2)^2}$ $\left[\frac{5}{x+2} + In(x+2) + c\right]$
   (d) $\frac{x}{\sqrt{2x+1}}$ $\left[\frac{1}{3}(x-1)\sqrt{2x+1} + c\right]$
   (e) $(x-3)(5-2x)^4$
   $\left[\frac{1}{120}(31-10x)(5-2x)^5 + c\right]$
   (f) $\frac{x}{\sqrt{(x+1)^3}}$ $\left[\frac{2(x+2)}{\sqrt{x+1}} + c\right]$
   (g) $\frac{x+3}{(3-x)^2}$ $\left[\frac{6}{3-x} + In(3-x) + c\right]$
   (h) $x^2(x-1)^4$
   $\left[\frac{1}{105}(15x^2 + 5x + 1)(x-1)^5 + c\right]$
   (i) $x\sqrt{(1-x)^3}$
   $\left[-\frac{2}{35}(5x+2)\sqrt{(1-x)^5}\right]$

Integrations involving trigonometric functions

A. The double formulae, i.e.

- $Cos2x = 2cos^2x - 1 = 1 - 2sin^2x$
- $1 + cos2x = 2cos^2x$
- $1 - cos2x = 2sin^2x$
- Sin2x =2sinxcos2

Example 4

Find the following integrals

(a) $\frac{tan\theta}{\sqrt{1+cos2\theta}} d\theta$

Solution

$\frac{tan\theta}{\sqrt{1+cos2\theta}} d\theta = \int \frac{tan\theta}{\sqrt{2cos^2\theta}} d\theta = \int \frac{sin\theta}{\sqrt{2}(cos^2\theta)} d\theta$

Let u = cosθ => du = -sinθdθ

$d\theta = \frac{du}{-sin\theta}$

$\frac{1}{\sqrt{2}}\int \frac{sin\theta}{(cos^2\theta)} d\theta = \frac{1}{\sqrt{2}}\int \frac{sin\theta}{u^2} . \frac{du}{-sin\theta}$

$= -\frac{1}{\sqrt{2}}\int \frac{du}{u^2} = -\frac{1}{\sqrt{2}}\int u^{-2} du$

$= -\frac{1}{\sqrt{2}}(-u^{-1}) + c$

$= \frac{1}{\sqrt{2}cos\theta} + c$

(b) $\int \sqrt{1 - cos4\theta}\, d\theta$

Solution

$\int \sqrt{1 - cos4\theta}\, d\theta = \int \sqrt{2sin^2 2\theta} d\theta$

$= \sqrt{2}\int sin2\theta d\theta$

$= \frac{-\sqrt{2}}{2} cos2\theta + c$

(c) $\int sin3\theta cos3\theta d\theta = \frac{1}{2}\int 2sin3\theta cos3\theta d\theta$

$= \frac{1}{2}\int sin6\theta$

$= -\frac{1}{12} cos6\theta + c$

B. The factor formulae, i.e.

- $cos\alpha + cos\beta = 2cos\left(\frac{\alpha+\beta}{2}\right) cos\left(\frac{\alpha-\beta}{2}\right)$
- $cos\alpha - cos\beta = -2sin\left(\frac{\alpha+\beta}{2}\right) sin\left(\frac{\alpha-\beta}{2}\right)$
- $sin\alpha + sin\beta = 2sin\left(\frac{\alpha+\beta}{2}\right) cos\left(\frac{\alpha-\beta}{2}\right)$
- $sin\alpha - sin\beta = -2cos\left(\frac{\alpha+\beta}{2}\right) sin\left(\frac{\alpha-\beta}{2}\right)$

Example 5

Find the following integrals

(a) $\int cos2\theta cos\theta d\theta$

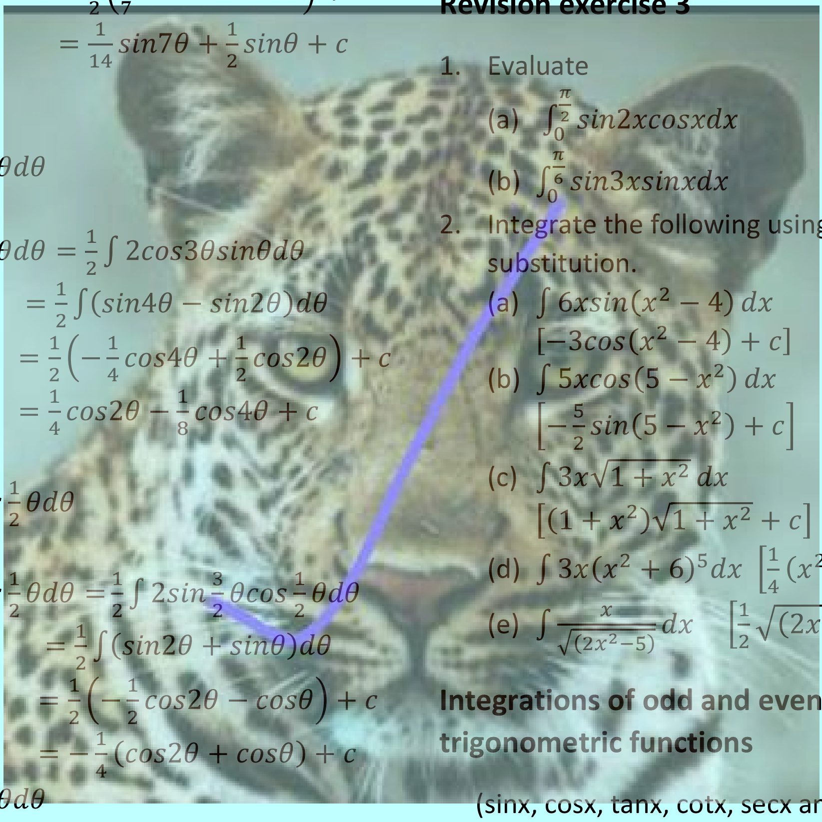
Solution

$$\int cos2\theta cos\theta d\theta = \frac{1}{2}\int 2cos2\theta cos\theta d\theta$$
$$= \frac{1}{2}\int (cos3\theta + cos\theta)d\theta$$
$$= \frac{1}{2}\left(\frac{1}{3}sin3\theta + sin\theta\right) + c$$
$$= \frac{1}{6}sin3\theta + \frac{1}{2}sin\theta + c$$

(b) $\int sin4\theta sin3\theta d\theta$

Solution

$$\int sin4\theta sin3\theta d\theta = \frac{1}{2}\int 2sin4\theta sin3\theta d\theta$$
$$= \frac{1}{2}\int (cos7\theta - cos\theta)d\theta$$
$$= \frac{1}{2}\left(\frac{1}{7}sin7\theta - sin\theta\right) + c$$
$$= \frac{1}{14}sin7\theta + \frac{1}{2}sin\theta + c$$



(c) $\int cos3\theta sin\theta d\theta$

Solution

$$\int cos3\theta sin\theta d\theta = \frac{1}{2}\int 2cos3\theta sin\theta d\theta$$
$$= \frac{1}{2}\int (sin4\theta - sin2\theta)d\theta$$
$$= \frac{1}{2}\left(-\frac{1}{4}cos4\theta + \frac{1}{2}cos2\theta\right) + c$$
$$= \frac{1}{4}cos2\theta - \frac{1}{8}cos4\theta + c$$

(d) $\int sin\frac{3}{2}\theta cos\frac{1}{2}\theta d\theta$

Solution

$$\int sin\frac{3}{2}\theta cos\frac{1}{2}\theta d\theta = \frac{1}{2}\int 2sin\frac{3}{2}\theta cos\frac{1}{2}\theta d\theta$$
$$= \frac{1}{2}\int (sin2\theta + sin\theta)d\theta$$
$$= \frac{1}{2}\left(-\frac{1}{2}cos2\theta - cos\theta\right) + c$$
$$= -\frac{1}{4}(cos2\theta + cos\theta) + c$$

(e) $\int sin2\theta cos\theta d\theta$

Solution

$$\int sin2\theta cos\theta d\theta = \frac{1}{2}\int 2sin2\theta cos\theta d\theta$$
$$= \frac{1}{2}\int (sin3\theta + sin\theta)d\theta$$
$$= \frac{1}{2}\left(-\frac{1}{3}cos3\theta - cos\theta\right) + c$$
$$= -\frac{1}{6}(cos2\theta + 3cos\theta) + c$$

Note

(i) The integral $\int sin2\theta cos2\theta d\theta$, where the angles are the same can be solved in two ways.

Method I: double angle formula

$$\int sin2\theta cos2\theta d\theta = \frac{1}{2}\int sin4\theta d\theta$$
$$= -\frac{1}{8}cos4\theta + c$$

Method II: the factor formula

$$\int sin2\theta cos2\theta d\theta = \frac{1}{2}\int (sin4\theta + sin0)d\theta$$
$$= -\frac{1}{8}cos4\theta + c$$

(ii) The integral of $\int sin4\theta cos2\theta$ where the angles are different, use method I because method II is inapplicable.

Revision exercise 3

1. Evaluate
   (a) $\int_0^{\frac{\pi}{2}} sin2xcosxdx$ $\left[\frac{2}{3}\right]$
   (b) $\int_0^{\frac{\pi}{6}} sin3xsinxdx$ [0.1083]
2. Integrate the following using appropriate substitution.
   (a) $\int 6xsin(x^2 - 4)\,dx$ $[-3cos(x^2 - 4) + c]$
   (b) $\int 5xcos(5 - x^2)\,dx$ $\left[-\frac{5}{2}sin(5 - x^2) + c\right]$
   (c) $\int 3x\sqrt{1 + x^2}\,dx$ $\left[(1 + x^2)\sqrt{1 + x^2} + c\right]$
   (d) $\int 3x(x^2 + 6)^5 dx$ $\left[\frac{1}{4}(x^2 + 6)^6 + c\right]$
   (e) $\int \frac{x}{\sqrt{(2x^2-5)}}dx$ $\left[\frac{1}{2}\sqrt{(2x^2 - 5)}\right] + c$

Integrations of odd and even powers of trigonometric functions

(sinx, cosx, tanx, cotx, secx and cosecx)

Integration of trigonometric functions rose to odd powers

The Pythagoras theorem in trigonometry is handy namely

- $\cos^2 x + \sin^2 x = 1$
- $1 + \tan^2 x = \sec^2 x$
- $1 + \cot^2 x = \mathrm{cosec}^2 x$

Example 6

Integrate the following

(a) $\int cos^5 xdx$

Solution

$\int cos^5xdx$=$\int cos^4xcosxdx$

$= \int(1 - sin^2x)^2cosxdx$

$= \int(cosx - 2sin^2xcosx + sin^4xcosx)dx$

$= sinx - \frac{2}{3}sin^3x + \frac{1}{5}sin^5x + c$

(b) $\int cos^3 2xdx$

Solution

$\int cos^3 2xdx = \int sin^2 2xsin2xdx$

$= \int(1 - cos^2 2x)sin2xdx$

$= \int sin2x - cos^2 2csin2xdx$

$= -\frac{1}{2}cos2x + \frac{1}{2}\left(\frac{1}{3}\right) cos^3 2x + c$

$= \frac{1}{6}cos^3 2x - \frac{1}{2}cos2x + c$

(c) $\int_0^{\frac{\pi}{12}} cos^3 6xdx$

Solution

$\int_0^{\frac{\pi}{12}} cos^3 6xdx = \int_0^{\frac{\pi}{12}}(1 - sin^2 6x)cos6xdx$

$= \int_0^{\frac{\pi}{12}}(cos6x - sin^2 6xcos6x)\, dx$

$= \left[\frac{1}{6}sin6x - \frac{1}{6}\left(\frac{1}{3}\right) sin^3 6x\right]_0^{\frac{\pi}{12}}$

$= \frac{1}{6}sin\frac{\pi}{2} - \frac{1}{18}\left(sin\frac{\pi}{2}\right)^3 = \frac{1}{9}$

(d) $\int sin^3xdx$

Solution

$\int sin^3xdx = \int sinx(1 - cos^2x)dx$

$= \int(sinx - sixcos^2x)dx$

$= -cosx - \left(-\frac{1}{3}cos^3x\right) + c$

$= \frac{1}{3}cos^3x - cosx$

(e) $\int tan^3xdx$

Solution

$\int tan^3xdx = \int tanxtan^2x\, dx$

$= \int tanx(sec^2x - 1)dx$

$= \int tanxsec^2xdx - \int tanx\, dx$

By inspection

$\frac{d}{dx}(In(cosx) = -tanx$

$=> \int -tanxdx = In(cosx)$

Also

$\frac{d}{dx}(tan^2x) = 2tanxsec^2x$

$=> \int tanxsec^2xdx = \frac{1}{2}tan^2x$

$\therefore \int tan^3xdx = \frac{1}{2}tan^2x + In(cosx) + c$

Or

$\frac{d}{dx}In(secx) = tanx$

$=> \int tanxdx = In(secx)$

Also $\frac{d}{dx}(sec^2x) = 2tanxsec^2x$

$=> \int tanxsec^2xdx = \frac{1}{2}sec^2x$

$\therefore \int tan^3xdx = \frac{1}{2}sec^2x + In(secx) + c$

(f) $\int tan^5 2xdx$

Solution

$\int tan^5 2xdx = \int tan2xtan^4 2xdx$

$= \int tan2x(sec^2 2x - 1)^2dx$

$= \int tan2x(sec^4 2x - 2sec^2 2x) - 1)dx$

$= \int t(an2xsec^4 2x - 2tan2xsec^2 2x - tan2x)dx$

$= \frac{1}{8}sec^4 2x - \frac{1}{2}sec^2 2x + \frac{1}{2}In(sec2x) + c$

Or

$\int tan^5 2xdx = \int tan^3 2xtan^2 2xdx$

$= \int tan^3 2x(sec^2 2x - 1)dx$

$= \int tan^3 2xsec^2 2xdx - \int tan^3 2xdx$

$= \frac{1}{8}tan^4 2x - \int tan2x(sec^2 2x - 1)dx$

$= \frac{1}{8}tan^4 2x - \int tan2xsec^2 2xdx + \int tan2x\, dx$

$= \frac{1}{8}tan^4 2x - \frac{1}{4}tan^2 2x + \frac{1}{2}In(sec2x) + c$

(g) $\int cot^3xdx$

Solution

$\int cot^3xdx = \int cotxcot^2xdx$

$= \int cotx(cosec^2x - 1)dx$

$= \int cotxcosec^2xdx - \int cotxdx$

$= \frac{1}{2}cot2x - In(sinx) + c$

Note: the integration of odd powers of secx and cosec x are done using integration by parts.

Integration of trigonometric functions rose to even powers

These are worked out using double angle formulae.

Example 7

Find the integrals of the following

(a) $\int cos^2xdx$

Solution

$\int cos^2xdx = \int \frac{1}{2}(1 + cos2x)dx$

$$= \int\left(\frac{1}{2}+\frac{1}{2}cos2x\right)dx$$
$$= \frac{x}{2}+\frac{1}{4}sin2x+c$$

(b) $\int cos^4 xdx$

Solution

$$\int cos^4xdx = \int(cos^2x)^2dx$$
$$= \int\frac{1}{4}\left((1+cos2x)\right)^2dx$$
$$= \int\frac{1}{4}(1+2cos2x+cos^22x)dx$$
$$= \int\left(\frac{1}{4}+\frac{1}{2}cos2x\right)dx+\frac{1}{4}\int\frac{1}{2}(1+cos4x)dx$$
$$= \frac{1}{4}x+\frac{1}{4}sin2x+\frac{1}{8}x+\frac{1}{32}sin4x+c$$

(c) $\int sin^2 3xdx$

Solution

$$\int sin^23xdx = \frac{1}{2}\int(1-cos6x)dx$$
$$= \frac{1}{2}x-\frac{1}{12}sin6x+c$$

(d) $\int tan^4xdx$

Solution

$$\int tan^4xdx = \int tan^2x.tan^2xdx$$
$$= \int tan^2x(sec^2x-1)dx$$
$$= \int tan^2xsec^2xdx - \int tan^2x\,dx$$
$$= \frac{1}{3}tan^3x - \int(sec^2-1)\,dx$$
$$= \frac{1}{3}tan^3x - tanx + x + c$$

(e) $\int sec^4xdx$

Solution

$$\int sec^4xdx = \int sec^2x.sec^2xdx$$
$$= \int sec^2x(tan^2x+1)dx$$
$$= \int(sec^2tan^2x+sec^2x)dx$$
$$= \frac{1}{3}tan^3x+tanx+c$$

(f) $\int cosec^2\left(\frac{1}{2}x\right)dx$

Solution

$$\int cosec^2\left(\frac{1}{2}x\right)dx = -2cot\frac{1}{2}x+c$$

(g) $\int cot^4xdx$

Solution

$$\int cot^4xdx = \int cot^2x.cot^2xdx$$
$$= \int cot^2x(cosec^2x-1)dx$$
$$= \int cot^2xcosec^2x - \int cot^2x\,dx$$
$$= \frac{1}{3}cot^3x - \int(cosec^2x-1)dx$$
$$= \frac{1}{3}cot^3x+cotx+x+c$$

Exercise 4

1. Integrate each of the following
   (a) $sinxcos^5x$ $\left[-\frac{1}{6}cos^6x+c\right]$
   (b) cos^34x $\left[\frac{1}{4}sin4x-\frac{1}{2}sin^34x+c\right]$
   (c) sin^3xcos^2x $\left[-\frac{1}{3}cos^3x+\frac{1}{5}cos^5x+c\right]$
   (d) cos^3xsin^4x $\left[\frac{1}{5}sin^5x-\frac{1}{7}sin^7x+c\right]$
   (e) cos^32xsin^22x $\left[\frac{1}{6}sin^32x-\frac{1}{10}sin^52x+c\right]$
   (f) $sin2xsin^2x$ $\left[\frac{1}{2}sin^4+c\right]$
   (g) cos^3xsin^3x $\left[\frac{1}{4}sin^4x-\frac{1}{6}sin^6x+c\right]$
2. Integrate each of the following
   (a) cot^22x $\left[\frac{1}{2}x+\frac{1}{8}sin4x+c\right]$
   (b) cos^23x $\left[\frac{1}{2}x+\frac{1}{8}sin6x+c\right]$
   (c) sin^3xcos^2x $\left[\frac{1}{2}x+\frac{1}{16}sin8x+c\right]$
   (d) cos^26x $\left[\frac{1}{2}x+\frac{1}{24}sin12x+c\right]$
   (e) $sin^2\left(\frac{1}{2}x\right)$ $\left[\frac{1}{2}x-\frac{1}{2}sinx+c\right]$
   (f) cos^4x $\left[\frac{3}{8}x+\frac{1}{4}sin2x+\frac{1}{32}sin4x+c\right]$
   (g) sin^42x $\left[\frac{3}{8}x-\frac{1}{4}sin4x+\frac{1}{64}sin8x+c\right]$
3. Integrate each of the following
   (a) cot^2x $[-cotx-x+c]$
   (b) tan^22x $\left[\frac{1}{2}tan2x+c\right]$
   (c) sec^2xtan^3x $\left[\frac{1}{4}tan^4x+c\right]$
   (d) $cosec^22xcot^42x$ $\left[-\frac{1}{10}cot^52x\right]$
   (e) tan^3x $\left[\frac{1}{2}tan^2x+In(cosx)\right]$
   (f) cot^43x $\left[-\frac{1}{9}cot^33x+\frac{1}{3}cot3x+c\right]$
   (g) tan^45x $\left[\frac{1}{15}tan^35x-\frac{1}{5}tan5x+x+c\right]$
   (h) tan^52x
   $\left[\frac{1}{8}tan^42x-\frac{1}{4}tan^22x+\frac{1}{2}In(sec2x)+c\right]$
   (i) $cosecxcot^3x$ $\left[cosecx-\frac{1}{3}cosec^3x+c\right]$
   (j) tan^5xsecx
   $\left[secx-\frac{2}{3}sec^3x+\frac{1}{5}sec^5c+c\right]$
4. Find the Integral of each of the following
   (a) $cosec^2x$ $[-cotx+c]$
   (b) sec^23x $\left[\frac{1}{3}tan3x+c\right]$
   (c) $cosec^2\left(\frac{1}{3}x\right)$ $\left[-3cot\left(\frac{1}{3}x\right)+c\right]$
   (d) sec^4x $\left[\frac{1}{3}tan^3x+tanx+c\right]$

(e) $\text{cosec}^4 5x$ $\quad \left[-\frac{1}{15}cot^3 5x - \frac{1}{5}cot5x + c\right]$

(f) $\sec^4 3x$ $\quad \left[\frac{1}{9}tan^3 3x + \frac{1}{3}tan3x + c\right]$

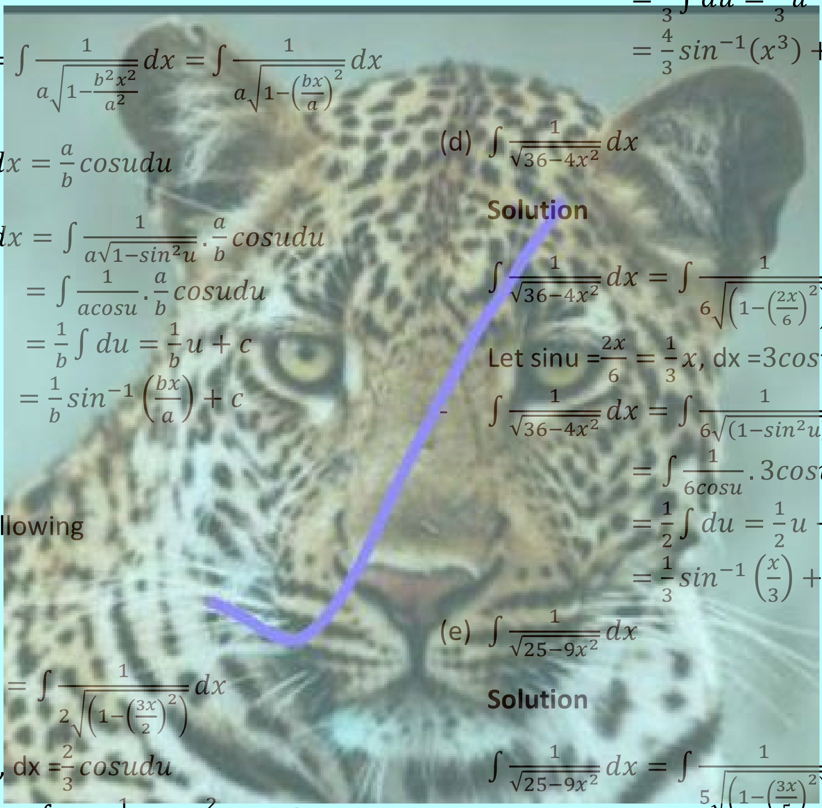
(g) $\sec^6 x$ $\left[\frac{1}{5}tan^5 x + \frac{2}{3}tan^3 x + tanx + c\right]$

Integration involving inverse trigonometric functions

A. From $\frac{d}{dx}\left(\sin^{-1}x\right) = \frac{1}{\sqrt{1-x^2}}$

- $\int \frac{1}{\sqrt{1-x^2}} = \sin^{-1}x + c$

This result enables the integration of the form $\int \frac{1}{\sqrt{a^2-b^2x^2}}dx$ to be workout, i.e.



$$\int \frac{1}{\sqrt{a^2-b^2x^2}}dx = \int \frac{1}{a\sqrt{1-\frac{b^2x^2}{a^2}}}dx = \int \frac{1}{a\sqrt{1-\left(\frac{bx}{a}\right)^2}}dx$$

Let $\frac{bx}{a} = sinu$; $dx = \frac{a}{b}cosudu$

- $\int \frac{1}{\sqrt{a^2-b^2x^2}}dx = \int \frac{1}{a\sqrt{1-sin^2u}}.\frac{a}{b}cosudu$

$$= \int \frac{1}{acosu}.\frac{a}{b}cosudu$$

$$= \frac{1}{b}\int du = \frac{1}{b}u + c$$

$$= \frac{1}{b}sin^{-1}\left(\frac{bx}{a}\right) + c$$

Example 8

Integrate the following

(a) $\int \frac{1}{\sqrt{4-9x^2}}dx$

Solution

$\int \frac{1}{\sqrt{4-9x^2}}dx = \int \frac{1}{2\sqrt{\left(1-\left(\frac{3x}{2}\right)^2\right)}}dx$

Let sinu $=\frac{3x}{2}$, dx $=\frac{2}{3}cosudu$

- $\int \frac{1}{\sqrt{4-9x^2}}dx = \int \frac{1}{2\sqrt{(1-sin^2u)}}.\frac{2}{3}cosudu$

$$= \int \frac{1}{2cosu}.\frac{2}{3}cosudu$$

$$= \frac{1}{3}\int du = \frac{1}{3}u + c$$

$$= \frac{1}{3}sin^{-1}\left(\frac{3x}{2}\right) + c$$

(b) $\int \frac{1}{\sqrt{1-16x^2)}}dx$

Solution

$\int \frac{1}{\sqrt{1-16x^2}}dx = \int \frac{1}{\sqrt{(1-(4x)^2)}}dx$

Let sinu $=4x$, dx $=\frac{1}{4}cosudu$

- $\int \frac{1}{\sqrt{(1-(4x)^2)}}dx = \int \frac{1}{\sqrt{(1-sin^2u)}}.\frac{1}{4}cosudu$

$$= \int \frac{1}{cosu}.\frac{1}{4}cosudu$$

$$= \frac{1}{4}\int du = \frac{1}{4}u + c$$

$$= \frac{1}{4}sin^{-1}(4x) + c$$

(c) $\int \frac{4x^2}{\sqrt{1-x^6)}}dx$

Solution

$\int \frac{4x^2}{\sqrt{1-x^6)}}dx = \int \frac{4x^2}{\sqrt{(1-(x^3)^2)}}dx$

Let sinu $=x^3$, dx $=\frac{1}{3x^2}cosudu$

- $\int \frac{4x^2}{\sqrt{(1-(x^3)^2)}}dx = \int \frac{4x^3}{\sqrt{(1-sin^2u)}}.\frac{1}{3x^2}cosudu$

$$= \int \frac{1}{cosu}.\frac{4}{3}cosudu$$

$$= \frac{4}{3}\int du = \frac{4}{3}u + c$$

$$= \frac{4}{3}sin^{-1}(x^3) + c$$

(d) $\int \frac{1}{\sqrt{36-4x^2}}dx$

Solution

$\int \frac{1}{\sqrt{36-4x^2}}dx = \int \frac{1}{6\sqrt{\left(1-\left(\frac{2x}{6}\right)^2\right)}}dx$

Let sinu $=\frac{2x}{6} = \frac{1}{3}x$, dx $=3cosudu$

- $\int \frac{1}{\sqrt{36-4x^2}}dx = \int \frac{1}{6\sqrt{(1-sin^2u)}}.3cosudu$

$$= \int \frac{1}{6cosu}.3cosudu$$

$$= \frac{1}{2}\int du = \frac{1}{2}u + c$$

$$= \frac{1}{3}sin^{-1}\left(\frac{x}{3}\right) + c$$

(e) $\int \frac{1}{\sqrt{25-9x^2}}dx$

Solution

$\int \frac{1}{\sqrt{25-9x^2}}dx = \int \frac{1}{5\sqrt{\left(1-\left(\frac{3x}{5}\right)^2\right)}}dx$

Let sinu $=\frac{3x}{5}$, dx $=\frac{5}{3}cosudu$

- $\int \frac{1}{\sqrt{25-9x^2}}dx = \int \frac{1}{5\sqrt{(1-sin^2u)}}.\frac{5}{3}cosudu$

$$= \int \frac{1}{5cosu}.\frac{5}{3}cosudu$$

$$= \frac{1}{3}\int du = \frac{1}{3}u + c$$

$$= \frac{1}{3}sin^{-1}\left(\frac{3x}{5}\right) + c$$

(f) $\int \frac{1}{\sqrt{3-2x-x^2}}dx$

Solution

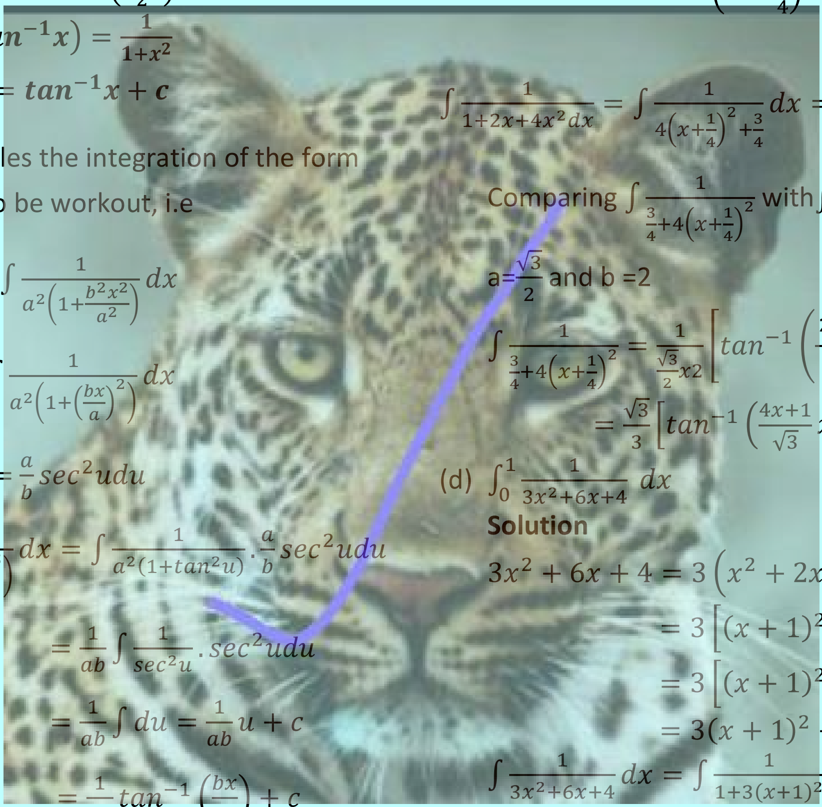
$3 - 2x - x^2 = -(x^2 + 2x - 3)$

By completing squares

$$-(x^2+2x-3) = -[(x+1)^2-3-1]$$
$$= 4-(x+1)^2$$
$$\int \frac{1}{\sqrt{3-2x-x^2}}dx = \int \frac{1}{\sqrt{4-(x+1)^2}}dx$$
$$\int \frac{1}{\sqrt{4-(x+1)^2}}dx = \int \frac{1}{2\sqrt{1-\left(\frac{x+1}{2}\right)^2}}$$

Let sinu $=\frac{x+1}{2}$, dx $=2cosudu$

$$- \quad \int \frac{1}{\sqrt{4-(x+1)^2}}dx = \int \frac{1}{2\sqrt{(1-sin^2u)}}.2cosudu$$
$$= \int \frac{1}{2cosu}.2cosudu$$
$$= \int du = u + c$$
$$= sin^{-1}\left(\frac{x+1}{2}\right) + c$$



B. From $\frac{d}{dx}(tan^{-1}x) = \frac{1}{1+x^2}$

$$- \quad \int \frac{1}{1+x^2}dx = tan^{-1}x + c$$

This result enables the integration of the form $\int \frac{1}{\sqrt{a^2+b^2x^2}}dx$ to be workout, i.e

$$\int \frac{1}{a^2+b^2x^2}dx = \int \frac{1}{a^2\left(1+\frac{b^2x^2}{a^2}\right)}dx$$
$$= \int \frac{1}{a^2\left(1+\left(\frac{bx}{a}\right)^2\right)}dx$$

Let $\frac{bx}{a}$= tanu, dx= $\frac{a}{b}sec^2udu$

$$=> \int \frac{1}{a^2\left(1+\left(\frac{bx}{a}\right)^2\right)}dx = \int \frac{1}{a^2(1+tan^2u)}.\frac{a}{b}sec^2udu$$
$$= \frac{1}{ab}\int \frac{1}{sec^2u}.sec^2udu$$
$$= \frac{1}{ab}\int du = \frac{1}{ab}u + c$$
$$= \frac{1}{ab}tan^{-1}\left(\frac{bx}{a}\right) + c$$

Example 9

Find

(a) $\int \frac{1}{9+25x^2}dx$

Solution

Comparing $\int \frac{1}{9+25x^2}$ with $\int \frac{1}{a^2+b^2x^2}dx$

a=3 and b =5

$$\int \frac{1}{9+25x^2} = \frac{1}{3x5}\left[tan^{-1}\left(\frac{5}{3}x\right)\right] + c$$
$$= \frac{1}{15}\left[tan^{-1}\left(\frac{5}{3}x\right)\right] + c$$

(b) $\int \frac{1}{5+9x^2}dx$

Solution

Comparing $\int \frac{1}{5+9x^2}$ with $\int \frac{1}{a^2+b^2x^2}dx$

a=$\sqrt{5}$ and b =3

$$\int \frac{1}{9+25x^2} = \frac{1}{\sqrt{5}x3}\left[tan^{-1}\left(\frac{3}{\sqrt{5}}x\right)\right] + c$$
$$= \frac{1}{3\sqrt{5}}\left[tan^{-1}\left(\frac{3\sqrt{5}}{5}x\right)\right] + c$$

(c) $\int \frac{1}{1+2x+4x^2}dx$

Solution

$$1+2x+4x^2 = 4x^2+2x+1$$
$$= 4\left(x^2+\frac{1}{2}x+\frac{1}{4}\right)$$
$$= 4\left[\left(x+\frac{1}{4}\right)^2+\frac{1}{4}-\frac{1}{16}\right]$$
$$= 4\left(x+\frac{1}{4}\right)^2+\frac{3}{4}$$
$$\int \frac{1}{1+2x+4x^2dx} = \int \frac{1}{4\left(x+\frac{1}{4}\right)^2+\frac{3}{4}}dx = \int \frac{1}{\frac{3}{4}+4\left(x+\frac{1}{4}\right)^2}dx$$

Comparing $\int \frac{1}{\frac{3}{4}+4\left(x+\frac{1}{4}\right)^2}$ with $\int \frac{1}{a^2+b^2x^2}dx$

a=$\frac{\sqrt{3}}{2}$ and b =2

$$\int \frac{1}{\frac{3}{4}+4\left(x+\frac{1}{4}\right)^2} = \frac{1}{\frac{\sqrt{3}}{2}x2}\left[tan^{-1}\left(\frac{2\left(x+\frac{1}{4}\right)}{\frac{\sqrt{3}}{2}}x\right)\right] + c$$
$$= \frac{\sqrt{3}}{3}\left[tan^{-1}\left(\frac{4x+1}{\sqrt{3}}x\right)\right] + c$$

(d) $\int_0^1 \frac{1}{3x^2+6x+4}\ dx$

Solution

$$3x^2+6x+4 = 3\left(x^2+2x+\frac{4}{3}\right)$$
$$= 3\left[(x+1)^2+\frac{4}{3}-1\right]$$
$$= 3\left[(x+1)^2+\frac{1}{3}\right]$$
$$= 3(x+1)^2+1$$
$$\int \frac{1}{3x^2+6x+4}dx = \int \frac{1}{1+3(x+1)^2}$$

Comparing $\int \frac{1}{1+3(x+1)^2}$ with $\int \frac{1}{a^2+b^2x^2}dx$

a=1 and b =$\sqrt{3}$

$$\int \frac{1}{1+3(x+1)^2} = \frac{1}{\sqrt{3}}\left[tan^{-1}\left(\frac{\sqrt{3}(x+1)}{1}x\right)\right] + c$$
$$= \frac{1}{\sqrt{3}}\left[tan^{-1}(\sqrt{3}(x+1))\right] + c$$
$$\therefore \int_0^1 \frac{1}{3x^2+6x+4}\ dx = \left[\frac{1}{\sqrt{3}}\left[tan^{-1}(\sqrt{3}(x+1))\right]\right]_0^1$$
$$= \frac{1}{\sqrt{3}}\left[tan^{-1}(2\sqrt{3}) - tan^{-1}(\sqrt{3})\right]$$
$$= 0.44$$

Revision exercise 5

Find

(a) $\int \frac{1}{9+x^2}dx$ $\left[\frac{1}{3}tan^{-1}\left(\frac{1}{3}x\right)+c\right]$

(b) $\int_{-2}^{2}\frac{1}{4+x^2}dx$ [0.7854]

(c) $\int_{-\sqrt{3}}^{3}\frac{1}{x^2+3}dx$ [1.833]

(d) $\int \frac{1}{4-x^2}dx$ $[sin^{-1}x+c]$

(e) $\int \frac{1}{\sqrt{16-x^2}}dx$ $\left[sin^{-1}\left(\frac{x}{4}\right)+c\right]$

(f) $\int \frac{1}{\sqrt{49-x^2}}dx$ $\left[sin^{-1}\left(\frac{x}{7}\right)+c\right]$

(g) $\int \frac{1}{\sqrt{25-4x^2}}dx$ $\left[\frac{1}{2}sin^{-1}\left(\frac{2}{4}x\right)+c\right]$

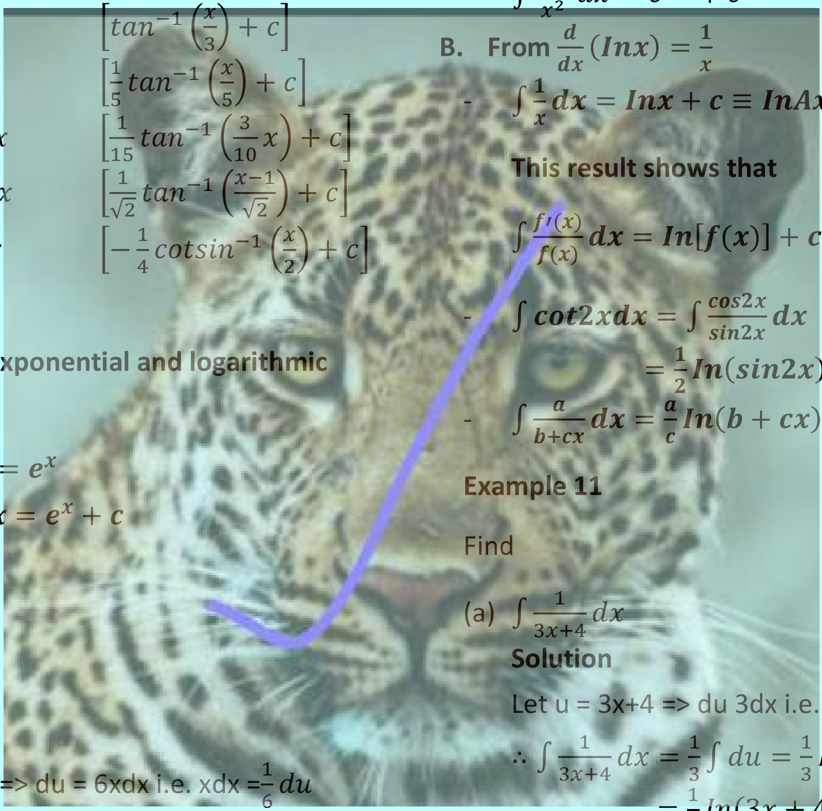
(h) $\int \frac{3}{9+x^2}dx$ $\left[tan^{-1}\left(\frac{x}{3}\right)+c\right]$

(i) $\int \frac{1}{25+x^2}dx$ $\left[\frac{1}{5}tan^{-1}\left(\frac{x}{5}\right)+c\right]$

(j) $\int \frac{2}{100+9x^2}dx$ $\left[\frac{1}{15}tan^{-1}\left(\frac{3}{10}x\right)+c\right]$

(k) $\int \frac{1}{3-2x+x^2}dx$ $\left[\frac{1}{\sqrt{2}}tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right)+c\right]$

(l) $\int \frac{1}{x^2\sqrt{4-x^2}}dx$ $\left[-\frac{1}{4}cotsin^{-1}\left(\frac{x}{2}\right)+c\right]$



Integration of exponential and logarithmic functions.

A. From $\frac{d}{dx}e^x = e^x$

- $\int e^x dx = e^x + c$

Example 10

Find

(a) $\int xe^{x^2}dx$

Solution

Let u = $3x^2$ => du = 6xdx i.e. xdx =$\frac{1}{6}du$

$\int xe^{x^2}dx = \frac{1}{6}\int e^u du = \frac{1}{6}e^u + c$

$\therefore \int xe^{x^2}dx = \frac{1}{6}e^{x^2} + c$

(b) $\int secxtanxe^{secx}dx$

Solution

Let u = secx => du=secxtanxdx

$\int secxtanxe^{secx}dx = \int e^u du = e^u + c$

$\therefore \int secxtanxe^{secx}dx = e^{secx} + c$

(c) $\int \frac{e^{cotx}}{sin^2x}dx$

Solution

Let u = cotx

- Du = -cosec^{2}x = $-\frac{1}{sin^2x}$

$\int \frac{e^{cotx}}{sin^2x}dx = -\int e^u du = -e^u + c$

$\therefore \int \frac{e^{cotx}}{sin^2x}dx = -e^{cotx} + c$

(d) $\int \frac{e^{-\frac{1}{x}}}{x^2}dx$

Solution

Let u = $-\frac{1}{x}$ => du = $\frac{1}{x^2}dx$

$\int \frac{e^{-\frac{1}{x}}}{x^2}dx = \int e^u du = -e^u + c$

$\therefore \int \frac{e^{-\frac{1}{x}}}{x^2}dx = e^{-\frac{1}{x}} + c$

B. From $\frac{d}{dx}(Inx) = \frac{1}{x}$

- $\int \frac{1}{x}dx = Inx + c \equiv InAx$

This result shows that

$\int \frac{f'(x)}{f(x)}dx = In[f(x)] + c$ **i.e.**

- $\int cot2xdx = \int \frac{cos2x}{sin2x}dx$

$= \frac{1}{2}In(sin2x) + c$

- $\int \frac{a}{b+cx}dx = \frac{a}{c}In(b+cx) + k$

Example 11

Find

(a) $\int \frac{1}{3x+4}dx$

Solution

Let u = 3x+4 => du 3dx i.e. dx=$\frac{1}{3}du$

$\therefore \int \frac{1}{3x+4}dx = \frac{1}{3}\int du = \frac{1}{3}Inu + c$

$= \frac{1}{3}In(3x+4) + c$

(b) $\int \frac{x}{1-5x^2}dx$

Solution

Let u = $1-5x^2$

=> du -10xdx i.e. dx=$-\frac{1}{10x}du$

$\therefore \int \frac{x}{1-5x^2}dx = \frac{1}{10}\int \frac{1}{u}du = \frac{1}{10}Inu + c$

$= \frac{1}{3}In(1-5x^2) + c$

(c) $\int tan^3xdx$

Solution

$\int tan^3xdx = \int tan^2xtanxdx$ (an odd power)

$= \int(sec^2x-1)tanxdx$

$= \int sec^2xtanxdx - \int tanxdx$

For $\int sec^2xtanxdx$

Let u = tanx, => du = sec^{2}xdx

$\therefore \int sec^2xtanxdx = \int udu = \frac{1}{2}tan^2x + c$

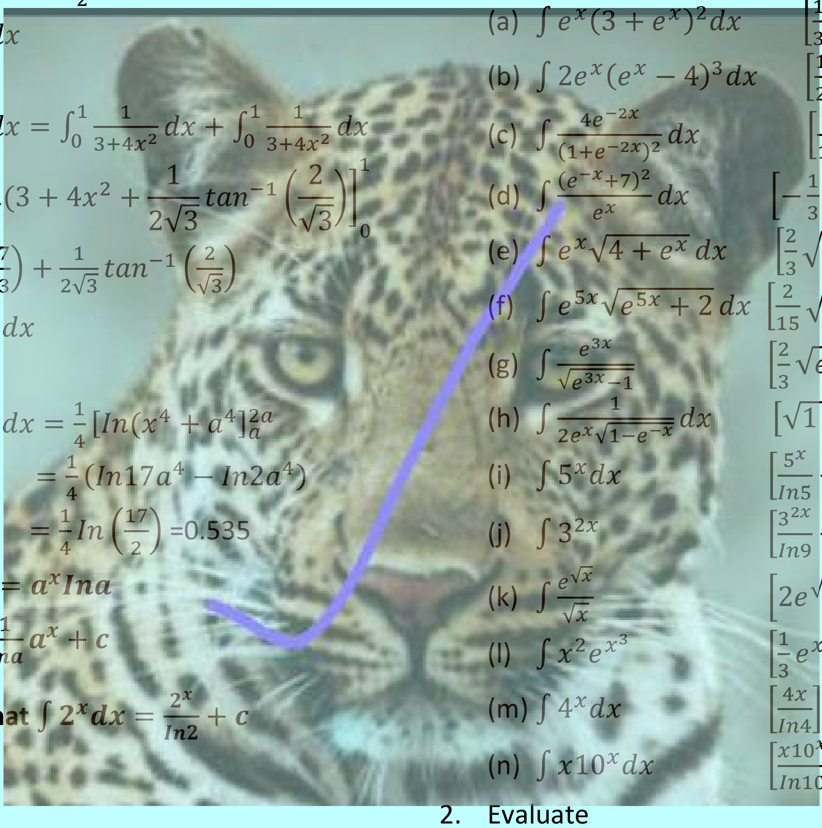
For$\int tanxdx = \int \frac{sinx}{cosx}dx$

Let u =cosx, => du = -sinxdx

$\therefore \int tanxdx = -\int \frac{1}{u}du = -Inu + c$

= -In(cosx)+c =In(secx)+c

$\therefore \int tan^3xdx = \frac{1}{2}tan^2x + In(secx) + c$



(d) $\int_0^1 \frac{x+1}{3+4x^2}dx$

Solution

$\int_0^1 \frac{x+1}{3+4x^2}dx = \int_0^1 \frac{1}{3+4x^2}dx + \int_0^1 \frac{1}{3+4x^2}dx$

$= \left[\frac{1}{8}In(3 + 4x^2 + \frac{1}{2\sqrt{3}}tan^{-1}\left(\frac{2}{\sqrt{3}}\right)\right]_0^1$

$= \frac{1}{8}In\left(\frac{7}{3}\right) + \frac{1}{2\sqrt{3}}tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$

(e) $\int_a^{2a} \frac{x^3}{x^4+a^4}dx$

Solution

$\int_a^{2a} \frac{x^3}{x^4+a^4}dx = \frac{1}{4}[In(x^4 + a^4]_a^{2a}$

$= \frac{1}{4}(In17a^4 - In2a^4)$

$= \frac{1}{4}In\left(\frac{17}{2}\right)$ =0.535

C. From $\frac{d}{dx}a^x = a^xIna$

⇨ $\int a^xdx = \frac{1}{Ina}a^x + c$

It follows that $\int 2^xdx = \frac{2^x}{In2} + c$

Example 12

Integrate

(a) $\int x^2 2^{3x^2}dx$

Solution

Let u = 3x^3, => du = $9x^2$ i.e. $x^2dx = \frac{1}{9}du$

$\int x^2 2^{3x^2}dx = \frac{1}{9}\int 2^u du = \frac{1}{9}\frac{2^u}{In2} + c$

$= \frac{1}{9}\frac{2^{3x^2}}{In2} + c$

(b) $\int cosx.5^{sinx}dx$

Solution

Let u = sinx, => du = cosxdx

$\int cosx.5^{sinx}dx = \int 5^u du = \frac{5^u}{In5} + c$

$= \frac{5^{sinx}}{In5} + c$

(c) $\int \frac{3^{cotx}}{sin^2x}dx$

Solution

Let u = cotx, => du=-cosec^{2}x

$\int \frac{3^{cotx}}{sin^2x}dx = -\int 3^{cotx}cosec^2xdx$

$= \int 3^u du = \frac{3^u}{In3} + c$

$= \frac{3^{cotx}}{In3} + c$

Revision exercise 6

1. Find the following integrals
   - (a) $\int e^x(3 + e^x)^2dx$ $\left[\frac{1}{3}(3 + e^x)^3 + c\right]$
   - (b) $\int 2e^x(e^x - 4)^3dx$ $\left[\frac{1}{2}(e^x - 4)^4 + c\right]$
   - (c) $\int \frac{4e^{-2x}}{(1+e^{-2x})^2}dx$ $\left[\frac{2}{1+e^{-2x}} + c\right]$
   - (d) $\int \frac{(e^{-x}+7)^2}{e^x}dx$ $\left[-\frac{1}{3}(e^{-x} + 7)^3 + c\right]$
   - (e) $\int e^x\sqrt{4 + e^x}\,dx$ $\left[\frac{2}{3}\sqrt{(4 + e^x)^3 + c}\right]$
   - (f) $\int e^{5x}\sqrt{e^{5x} + 2}\,dx$ $\left[\frac{2}{15}\sqrt{(e^{5x} + 2)^3} + c\right]$
   - (g) $\int \frac{e^{3x}}{\sqrt{e^{3x}-1}}$ $\left[\frac{2}{3}\sqrt{e^{3x} - 1} + c\right]$
   - (h) $\int \frac{1}{2e^x\sqrt{1-e^{-x}}}dx$ $\left[\sqrt{1 - e^{-x}} + c\right]$
   - (i) $\int 5^xdx$ $\left[\frac{5^x}{In5} + c\right]$
   - (j) $\int 3^{2x}$ $\left[\frac{3^{2x}}{In9} + c\right]$
   - (k) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}$ $\left[2e^{\sqrt{x}} + c\right]$
   - (l) $\int x^2e^{x^3}$ $\left[\frac{1}{3}e^{x^3}\right]$
   - (m) $\int 4^xdx$ $\left[\frac{4x}{In4}\right]$
   - (n) $\int x10^xdx$ $\left[\frac{x10^x}{In10} - \frac{10^x}{(In10)^2} + c\right]$
2. Evaluate
   - (a) $\int_1^3 e^xdx$ $[e(e^2 - 1)]$
   - (b) $\int_0^3 e^{-x}dx$ $\left[1 - \frac{1}{e^3}\right]$
   - (c) $\int_1^2 2e^{(2x+1)}$ $[e^3(e^2 - 1)]$
   - (d) $\int_{-1}^1 2e^{(1-2x)}$ $\left[e^3 - \frac{1}{e}\right]$
   - (e) $\int_0^1(4xe^{x^2} + 1)dx$ $[2e - 1]$

Integration involving partial fractions

There are three established types of partial fractions depending on the nature of the denominator.

A. Denominators with linear factors e.g. 3x – 1, x + 2 and 3x – 4.

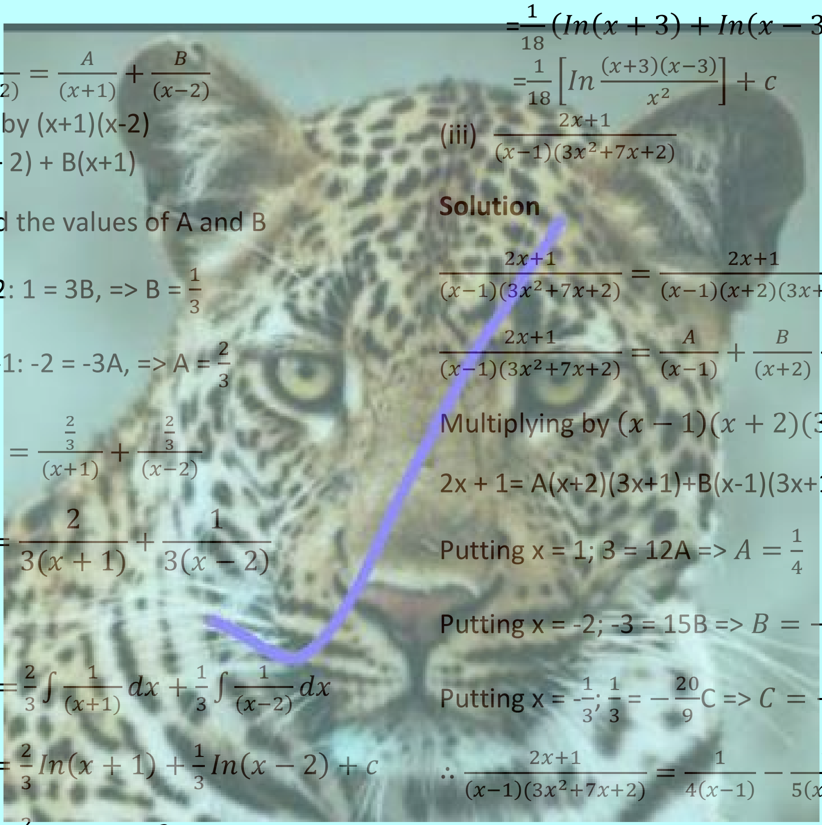
Each linear factor (ax + b) in the denominator has a corresponding partial fraction of the form $\frac{c}{(ax+b)}$' where a, b and c are constants.

Example 13

(a) Express each of the following in partial fraction. Hence find the integral of each with respect to x.

(i) $\frac{x-1}{(x+1)(x-2)}$

Solution



Let $\frac{x-1}{(x+1)(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x-2)}$

Multiplying by (x+1)(x-2)

⇨ x – 1 = A(x – 2) + B(x+1)

then we find the values of A and B

Putting x = 2: 1 = 3B, => B = $\frac{1}{3}$

Putting x = -1: -2 = -3A, => A = $\frac{2}{3}$

$$\therefore \frac{x-1}{(x+1)(x-2)} = \frac{\frac{2}{3}}{(x+1)} + \frac{\frac{2}{3}}{(x-2)}$$

$$= \frac{2}{3(x+1)} + \frac{1}{3(x-2)}$$

Hence,

$$\int \frac{x-1}{(x+1)(x-2)} dx = \frac{2}{3}\int \frac{1}{(x+1)} dx + \frac{1}{3}\int \frac{1}{(x-2)} dx$$

$$= \frac{2}{3} In(x+1) + \frac{1}{3} In(x-2) + c$$

$$= \frac{2}{3} In(x+1)^2(x-2) + c$$

(ii) $\frac{1}{x^3-9x}$

Solution

$$\frac{1}{x^3-9x} = \frac{1}{x(x^2-9)} = \frac{1}{x(x-3)(x+3)}$$

⇨ $\frac{1}{x^3-9x} = \frac{A}{x} + \frac{B}{(x-3)} + \frac{C}{(x+3)}$

Multiplying through with $x(x-3)(x+3)$

1= $A(x^2-9) + B(x^2+3x) + C(x^2-3x)$

Putting x = 0; 1 = -9A => $A = -\frac{1}{9}$

Putting x = 3; 1 = 18B => $B = \frac{1}{18}$

Putting x = -3; 1 = 18C => $C = \frac{1}{18}$

⇨ $\frac{1}{x^3-9x} = -\frac{1}{9x} + \frac{1}{18(x-3)} + \frac{1}{18(x+3)}$

Hence,

$$\int \frac{1}{x^3-9x} dx$$

$$= -\frac{1}{9}\int \frac{1}{x} dx + \frac{1}{18}\int \frac{1}{(x-3)} dx + \frac{1}{18}\int \frac{1}{(x+3)} dx$$

$$=-\frac{1}{9} Inx + \frac{1}{18} In(x+3) + \frac{1}{18} In(x-3) + c$$

$$=\frac{1}{18}(In(x+3) + In(x-3) - 2Inx) + c$$

$$=\frac{1}{18}\left[In\frac{(x+3)(x-3)}{x^2}\right] + c$$

(iii) $\frac{2x+1}{(x-1)(3x^2+7x+2)}$

Solution

$$\frac{2x+1}{(x-1)(3x^2+7x+2)} = \frac{2x+1}{(x-1)(x+2)(3x+1)}$$

$$\frac{2x+1}{(x-1)(3x^2+7x+2)} = \frac{A}{(x-1)} + \frac{B}{(x+2)} + \frac{1}{(3x+1)}$$

Multiplying by $(x-1)(x+2)(3x+1)$

2x + 1= A(x+2)(3x+1)+B(x-1)(3x+1)+C(x-1)(x+2)

Putting x = 1; 3 = 12A => $A = \frac{1}{4}$

Putting x = -2; -3 = 15B => $B = -\frac{1}{5}$

Putting x = -$\frac{1}{3}$; $\frac{1}{3} = -\frac{20}{9}$C => $C = -\frac{3}{20}$

$$\therefore \frac{2x+1}{(x-1)(3x^2+7x+2)} = \frac{1}{4(x-1)} - \frac{1}{5(x+2)} - \frac{3}{20(3x+1)}$$

Hence,

$$\int \frac{2x+1}{(x-1)(3x^2+7x+2)} dx$$

$$= \frac{1}{4}\int \frac{1}{(x-1)} dx - \frac{1}{5}\int \frac{1}{(x+2)} dx - \frac{3}{20}\int \frac{1}{(3x+1)} dx$$

$$= \frac{1}{4} In(x-1) - \frac{1}{5} In(x+2) - \frac{3}{20} In(3x+1)$$

$$= \frac{1}{20} In\frac{(x-1)^5}{(x+2)^4(3x+1)^3}$$

(iv) $\frac{2x^2-x+1}{(x^2-1)(x+2)}$

Solution

$$\frac{2x^2-x+1}{(x^2-1)(x+2)} = \frac{2x^2-x+1}{(x+1)(x-1)(x+2)}$$

$\Rightarrow \quad \frac{2x^2-x+1}{(x^2-1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{C}{(x+2)}$

Multiplying through by (x+1)(x-1)(x+2)

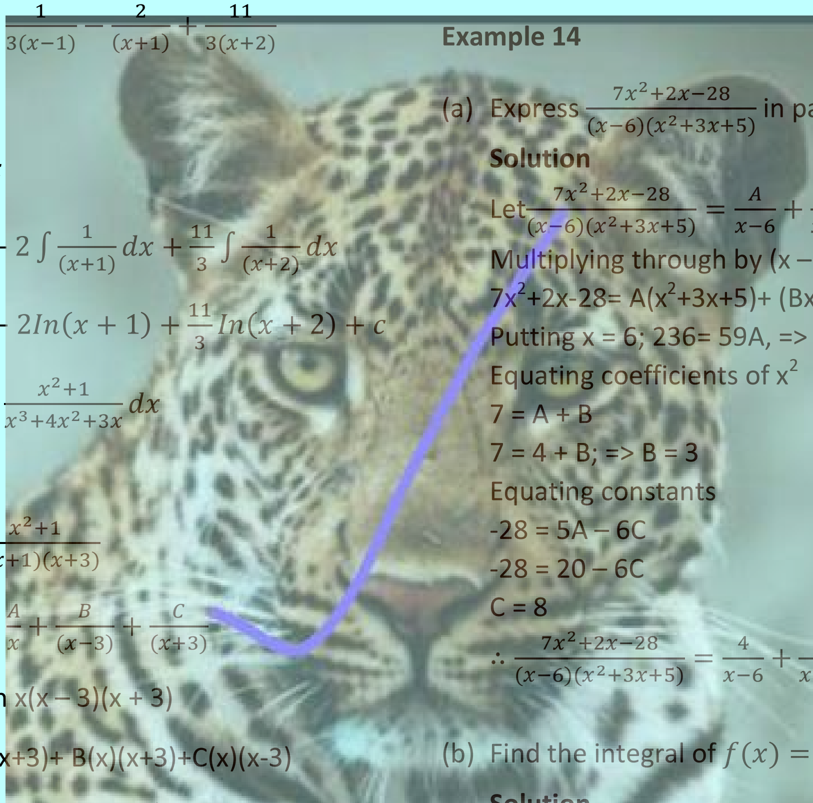
$2x^2 - x+1= A(x-1)(x+2)+B(x+1)(x+2)+C(x+1)(x-1)$

Putting x = -1; 4 = -2A => $A = -2$

Putting x = 1; 2 = 6B => $B = \frac{1}{3}$

Putting x = -2; 11 = 3C => $C = \frac{11}{3}$

$\therefore \frac{2x^2-x+1}{(x^2-1)(x+2)} = \frac{1}{3(x-1)} - \frac{2}{(x+1)} + \frac{11}{3(x+2)}$



Hence,

$\int \frac{2x^2-x+1}{(x^2-1)(x+2)} dx$

$= \frac{1}{3}\int \frac{1}{(x-1)} dx - 2\int \frac{1}{(x+1)} dx + \frac{11}{3}\int \frac{1}{(x+2)} dx$

$= \frac{1}{3} In(x-1) - 2In(x+1) + \frac{11}{3} In(x+2) + c$

(b) Evaluate $\int_1^3 \frac{x^2+1}{x^3+4x^2+3x} dx$

Solution

$\frac{x^2+1}{x^3+4x^2+3x} = \frac{x^2+1}{x(x+1)(x+3)}$

Let $\frac{x^2+1}{x^3+4x^2+3x} = \frac{A}{x} + \frac{B}{(x-3)} + \frac{C}{(x+3)}$

Multiplying with x(x – 3)(x + 3)

x^2+ 1 = A(x – 3)(x+3)+ B(x)(x+3)+C(x)(x-3)

Putting x = 0; 1 = 3A => $A = \frac{1}{3}$

Putting x = -1; 2 = -2B => $B = -1$

Putting x = -3; 10 = 6C => $C = \frac{5}{3}$

$\therefore \frac{x^2+1}{x^3+4x^2+3x} = \frac{1}{3x} - \frac{1}{(x+1)} + \frac{5}{3(x+3)}$

Hence

$\int_1^3 \frac{x^2+1}{x^3+4x^2+3x} dx$

$= \frac{1}{3}\int_1^3 \frac{1}{x} dx - \int_1^3 \frac{1}{(x+1)} dx + \frac{5}{3}\int_1^3 \frac{1}{(x+3)} dx$

$= \left[\frac{1}{3} Inx - In(x+1) + \frac{5}{3} In(x+3)\right]_1^3$

$= \left\{\frac{1}{3} In3 - In4 + \frac{5}{3} In6)\right\} - \left\{\frac{1}{3} In1 - In2 + \frac{5}{3} In4)\right\}$

$= 0.3488$

B. Denominators with linear factors Quadratic factors

Each quadratic factors (ax^2+bx +c) has a corresponding partial fraction of the form $\frac{Ax+B}{(ax^2+bx+c)}$ where a, b, c and A and B are constants.

Example 14

(a) Express $\frac{7x^2+2x-28}{(x-6)(x^2+3x+5)}$ in partial fraction.

Solution

Let $\frac{7x^2+2x-28}{(x-6)(x^2+3x+5)} = \frac{A}{x-6} + \frac{Bx+c}{x^2+3x+5}$

Multiplying through by (x – 6)(x^2+3x+5)

$7x^2$+2x-28= A(x^2+3x+5)+ (Bx +C)(x – 6)

Putting x = 6; 236= 59A, => A = 4

Equating coefficients of x^2

7 = A + B

7 = 4 + B; => B = 3

Equating constants

-28 = 5A – 6C

-28 = 20 – 6C

C = 8

$\therefore \frac{7x^2+2x-28}{(x-6)(x^2+3x+5)} = \frac{4}{x-6} + \frac{3x+8}{x^2+3x+5}$

(b) Find the integral of $f(x) = \frac{2x-1}{(x-1)(x^2+1)}$

Solution

Let $\frac{2x-1}{(x-1)(x^2+1)} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+1)}$

Multiplying through by $(x-1)(x^2+1)$

2x – 1 = A(x^2+1)+ (Bx + C)(x-1)

Putting x = 1; 1 = 2A => A =$\frac{1}{2}$

Putting x= 0; -1 = A –C => C = $\frac{3}{2}$

Putting x= -1; 2A + 2B – 2C => B = $-\frac{1}{2}$

$\therefore \frac{2x-1}{(x-1)(x^2+1)} = \frac{1}{2(x-1)} + \frac{-\frac{1}{2}x+\frac{3}{2}}{(x^2+1)}$

$\frac{2x-1}{(x-1)(x^2+1)} = \frac{1}{2(x-1)} + \frac{3-x}{2(x^2+1)}$

Note the values of x = 0 and x = -1 are conveniently chosen, but the constants B and C by expansion of the expression and equating constants, i.e.

$-1 = A - C \Rightarrow C = \frac{3}{2}$

$2 = C - B$

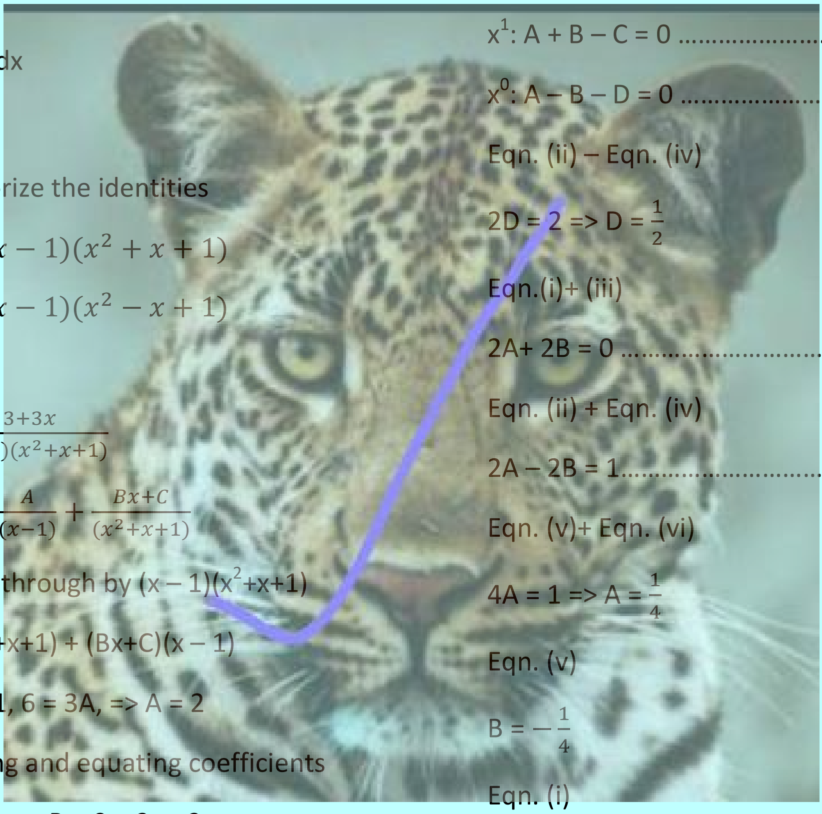
$B = \frac{3}{2} - 2 = -\frac{1}{2}$

Thus,

$$\int \frac{2x-1}{(x-1)(x^2+1)}dx$$

$$= \frac{1}{2}\int \frac{1}{(x-1)}dx + \frac{3}{2}\int \frac{1}{(x^2+1)}dx - \frac{1}{2}\int \frac{x}{(x^2+1)}dx$$

$$= \frac{1}{2}In(x-1) + \frac{3}{2}tan^{-1}x - \frac{1}{4}In(x^2+1) + c$$



(c) Evaluate

(i) $\int_2^3 \frac{3+3x}{x^3-1}dx$

Solution

Note memorize the identities

$x^3 - 1 = (x-1)(x^2+x+1)$

$x^3 + 1 = (x-1)(x^2-x+1)$

Then

$$\frac{3+3x}{x^3-1} = \frac{3+3x}{(x-1)(x^2+x+1)}$$

$$Let\ \frac{3+3x}{x^3-1} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+x+1)}$$

Multiplying through by $(x-1)(x^2+x+1)$

$3+3x = A(x^2+x+1) + (Bx+C)(x-1)$

Putting $x = 1$, $6 = 3A$, $\Rightarrow A = 2$

By expanding and equating coefficients

x^2: $A + B = 0$, $\Rightarrow B = 0 - 2 = -2$

x^0: $A - C = 3$, $\Rightarrow C = 2 - 3 = -1$

$$\therefore \frac{3+3x}{x^3-1} = \frac{2}{(x-1)} - \frac{2x+1}{(x^2+x+1)}$$

$$\int_2^3 \frac{3+3x}{x^3-1}dx = 2\int_2^3 \frac{1}{(x-1)}dx - \int_2^3 \frac{2x+1}{(x^2+x+1)}$$

$$= [2In(x-1) - In(x^2+x+1)]_2^3$$

$$= 2In(2) + In\left(\frac{7}{13}\right)$$

$$= 0.7673$$

(ii) $\int_2^3 \frac{x^2}{x^4-1}dx$

Solution

$$\frac{x^2}{x^4-1} = \frac{x^2}{(x-1)(x+1)(x^2+1)}$$

$$\text{Let} \frac{x^2}{(x-1)(x+1)(x^2+1)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{Cx+D}{(x^2+1)}$$

By multiplying through by $(x-1)(x+1)(x^2+1)$

$x^2 = A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x^2-1)$

By equating coefficients

x^3: $A + B + C = 0$ ……………………. (i)

x^2: $A - B + D = 1$ ……………………..(ii)

x^1: $A + B - C = 0$ ……………………….(iii)

x^0: $A - B - D = 0$ ……………………….(iv)

Eqn. (ii) – Eqn. (iv)

$2D = 2 \Rightarrow D = \frac{1}{2}$

Eqn.(i)+ (iii)

$2A + 2B = 0$ ………………………….. (v)

Eqn. (ii) + Eqn. (iv)

$2A - 2B = 1$…………………………..(vi)

Eqn. (v)+ Eqn. (vi)

$4A = 1 \Rightarrow A = \frac{1}{4}$

Eqn. (v)

$B = -\frac{1}{4}$

Eqn. (i)

$C = 0$

$$\therefore \frac{x^2}{x^4-1} = \frac{1}{4(x-1)} - \frac{1}{4(x+1)} + \frac{1}{2(x^2+1)}$$

$$\int \frac{x^2}{x^4-1}dx$$

$$= \frac{1}{4}\int \frac{1}{(x-1)}dx - \frac{1}{4}\int \frac{1}{(x+1)}dx + \frac{1}{2}\int \frac{1}{(x^2+1)}dx$$

$$= \frac{1}{4}In(x-1) - \frac{1}{4}In(x+1) + \frac{1}{2}tan^{-1}x + c$$

$$\int_2^3 \frac{x^2}{x^4-1}dx$$

$$= \left[\frac{1}{4}In(x-1) - \frac{1}{4}In(x+1) + \frac{1}{2}tan^{-1}x\right]_2^3$$

$$= \frac{1}{4}\left[In\frac{x-1}{x+1} + 2tan^{-1}x\right]_2^3$$

$$= \frac{1}{4}\left\{\left[In\frac{2}{4} + 2tan^{-1}3\right] - \left[In\frac{1}{3} + 2tan^{-1}2\right]\right\}$$

$$= \frac{1}{4}\left[In\frac{1}{2} - In\frac{1}{3} + 2(tan^{-1}3 - tan^{-1}2)\right]$$

$$= \frac{1}{4}(0.405 + 0.1\pi)$$

$$= 0.18$$

(iii) $\int_0^1 \frac{x^2+6}{(x^2+4)(x^2+9)}dx$

Solution

Let $\frac{x^2+6}{(x^2+4)(x^2+9)} = \frac{Ax+B}{(x^2+4)} + \frac{Cx+D}{(x^2+9)}$

Multiplying by $(x^2+4)(x^2+9)$

$x^2+6 = (Ax+B)(x^2+9) + (Cx+D)(x^2+4)$

$x^2+6 = (A+C)x^3+(B+D)x^2+(9A+4C)x+9B+4D$

Equating coefficients

x^3: A + C = 0 ....(i)

x^2: B+D = 1....................(ii)

x^1: 9A + 4C = 0 ....(iii)

x^0: 9B + 4D = 6 ....(iv)

Solving simultaneously

A= C= 0; B=$\frac{2}{5}$ and D =$\frac{3}{5}$

$$\therefore \frac{x^2+6}{(x^2+4)(x^2+9)} = \frac{2}{5(x^2+4)} + \frac{3}{5(x^2+9)}$$

$$\int_0^1 \frac{x^2+6}{(x^2+4)(x^2+9)}dx$$

$$= \frac{2}{5}\int_0^1 \frac{1}{(x^2+4)}dx + \frac{3}{5}\int_0^1 \frac{1}{(x^2+9)}dx$$

$$= \frac{1}{5}\left[tan^{-1}\frac{1}{2}x + tan^{-1}\frac{1}{3}x\right]_0^1$$

$$= \frac{1}{5}tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}\right) = 0.1571$$

C. Repeated factors

Each repeated factor $(ax^2+b)^n$ in the denominator has corresponding partial fraction of the form: $\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$, where a, b, A_i are constants (i = 1, 2, ...n)

Example 15

Express each of the follow in partial fraction and hence find their integrals.

(a) $\frac{4x-9}{(x-3)^2}$

Solution

Let $\frac{4x-9}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2}$

Multiplying through by $(x-3)^2$

4x – 9 = A(x-3) + B = Ax – 3A + B

Equating coefficients

x^1: x = 4

x^0: -3A + B = 4; B = 3

$\therefore \frac{4x-9}{(x-3)^2} = \frac{4}{x-3} + \frac{3}{(x-3)^2}$

Hence

$$\int \frac{4x-9}{(x-3)^2}dx = 4\int \frac{1}{x-3}dx + 3\int (x-3)^{-2}\,dx$$

$$= 4In(x-3) - \frac{3}{x-3} + c$$

(b) $\frac{3x-14}{x^2-8x+16}$

Solution

$\frac{3x-14}{x^2-8x+16} = \frac{3x-14}{(x-4)^2}$

Let $\frac{3x-14}{(x-4)^2} = \frac{A}{x-4} + \frac{B}{(x-4)^2}$

Multiplying through by $(x-4)^2$

3x – 14 = A(x-4) + B = Ax – 4A + B

Equating coefficients

x^1: x = 3

x^0: -4A + B = -14; B = -2

$\therefore \frac{3x-14}{(x-3)^2} = \frac{3}{x-3} - \frac{2}{(x-4)^2}$

Hence

$$\int \frac{3x-14}{(x-4)^2}dx = 3\int \frac{1}{x-4}dx - 2\int (x-4)^{-2}\,dx$$

$$= 3In(x-4) + \frac{2}{x-4} + c$$

(c) $\frac{2x^2-5x+7}{(x-2)(x-1)^2}$

Solution

Let $\frac{2x^2-5x+7}{(x-2)(x-1)^2} = \frac{A}{x-2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$

Multiplying through by $(x-2)(x-1)^2$

$2x^2-5x+7 = A(x-1)^2+B(x-2)(x-1)+C(x-2)$

Putting x = 1: 4 = -C, => C = -4

Putting x = 2: A = 5

Putting x = 0, 7 =A – 2B – 2C; B = -2

$\therefore \frac{2x^2-5x+7}{(x-2)(x-1)^2} = \frac{5}{x-2} - \frac{2}{x-1} - \frac{4}{(x-1)^2}$

Hence

$\int \frac{2x^2-5x+7}{(x-2)(x-1)^2} dx$

$= 5\int \frac{1}{x-2} dx - 2\int \frac{1}{x-1} dx - 4\int (x-1)^{-2} dx$

$= 5In(x-2) - 2In(x-1) - \frac{4}{x-1} + c$

(d) $\frac{7x+2}{3x^3+x^2}$

Solution

$\frac{7x+2}{3x^3+x^2} = \frac{7x+2}{x^2(3x+1)}$

Let $\frac{7x+2}{x^2(3x+1)} = \frac{A}{(3x+1)} + \frac{B}{x} + \frac{C}{x^2}$

Multiplying through by $x^2(3x+1)$

$7x + 2 = Ax^2 + Bx(3x+1) + C(3x+1)$

Putting x = 0; c = 2

Putting x = $-\frac{1}{3}$; $\frac{A}{9} = 2 - \frac{7}{3}$ => A = -3

Putting x= -1; -5 = A + 2B- 2C, => B = 1

$\therefore \frac{7x+2}{x^2(3x+1)} = \frac{-3}{(3x+1)} + \frac{1}{x} + \frac{2}{x^2}$

Hence

$\int \frac{7x+2}{x^2(3x+1)} dx$

$= -\int \frac{3}{(3x+1)} dx + \int \frac{1}{x} dx + 2\int x^{-2} dx$

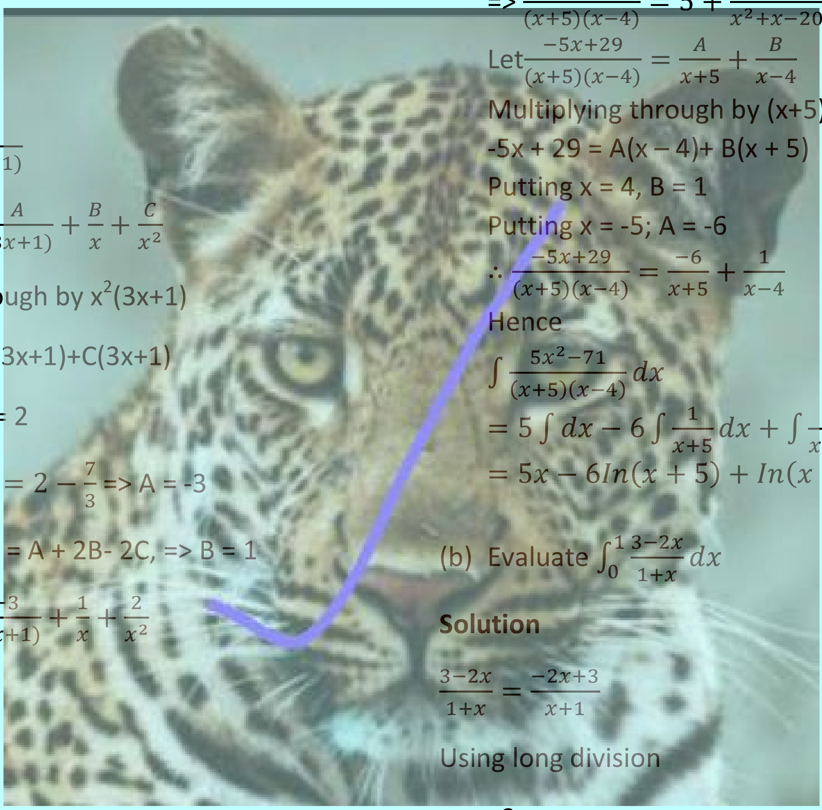
$= -In(3x+1) + Inx - \frac{2}{x} + c$

$= In\frac{x}{3x+3} - \frac{2}{x} + c$In

Integration of improper fractions

Improper fractions are those whose index of the numerator is equal to or greater than that of the denominators.

They are first changed to proper fraction by long division or otherwise, before being integrated.



Example 16

(a) Express $\frac{5x^2-71}{(x+5)(x-4)}$ in partial fractions.

Hence find $\int \frac{5x^2-71}{(x+5)(x-5)} dx$

Solution

$\frac{5x^2-71}{(x+5)(x-4)} = \frac{5x^2-71}{x^2+x-20}$

Using long division

$$\begin{array}{r} 5 \\ x^2+x-20 \overline{)\,5x^2+0x-71} \\ -\ \underline{5x^2+5x-100} \\ -5x+29 \end{array}$$

=> $\frac{5x^2-71}{(x+5)(x-4)} = 5 + \frac{-5x+29}{x^2+x-20}$

Let $\frac{-5x+29}{(x+5)(x-4)} = \frac{A}{x+5} + \frac{B}{x-4}$

Multiplying through by (x+5)(x-4)

-5x + 29 = A(x – 4)+ B(x + 5)

Putting x = 4, B = 1

Putting x = -5; A = -6

$\therefore \frac{-5x+29}{(x+5)(x-4)} = \frac{-6}{x+5} + \frac{1}{x-4}$

Hence

$\int \frac{5x^2-71}{(x+5)(x-4)} dx$

$= 5\int dx - 6\int \frac{1}{x+5} dx + \int \frac{1}{x-4} dx$

$= 5x - 6In(x+5) + In(x-4) + c$

(b) Evaluate $\int_0^1 \frac{3-2x}{1+x} dx$

Solution

$\frac{3-2x}{1+x} = \frac{-2x+3}{x+1}$

Using long division

$$\begin{array}{r} -2 \\ x+1 \overline{)\,-2x+3} \\ \underline{-2x-2} \\ 5 \end{array}$$

$\therefore \frac{3-2x}{1+x} = -2 + \frac{5}{x+1}$

Hence

$\int_0^1 \frac{3-2x}{1+x} dx = -2\int_0^1 dx + 5\int_0^1 \frac{1}{x+1} dx$

$= [-2x + 5In(x+1)]_0^1$

= -2+ 5In2

$= 1.4657$

Revision exercise 7

1. Express the following into partial fraction

(a) $\frac{8x}{x^2-4x-12}$ $\left[\frac{6}{x-6}+\frac{2}{x+2}\right]$

(b) $\frac{x^4-x^3+x^2+1}{x^3+x}$ $\left[x-1+\frac{1}{x}+\frac{x-1}{x^2+1}\right]$

(c) $\frac{5x-1}{2x^2+x}-10$ $\left[\frac{3}{2x+5}+\frac{1}{x-2}\right]$

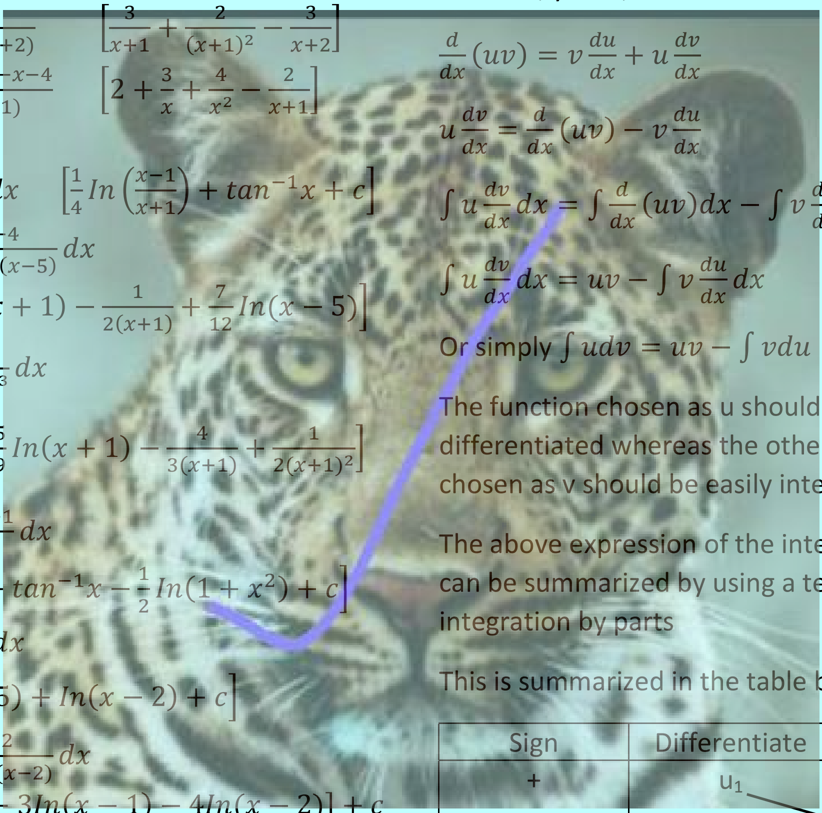
(d) $\frac{2x^2-7x+1}{(2x+1)(2x-1)(x-2)}$

$\left[\frac{1}{2x+1}+\frac{2}{3(2x-1)}-\frac{1}{3(x-2)}\right]$

(e) $\frac{6x+7}{(x^2+2)(x+3)}$ $\left[\frac{x+3}{x^2+2}-\frac{1}{x+3}\right]$

(f) $\frac{5x+7}{(x+1)^2(x+2)}$ $\left[\frac{3}{x+1}+\frac{2}{(x+1)^2}-\frac{3}{x+2}\right]$

(g) $\frac{2x^3+3x^2-x-4}{x^2(x+1)}$ $\left[2+\frac{3}{x}+\frac{4}{x^2}-\frac{2}{x+1}\right]$



2. Find

(a) $\int\frac{x^2}{x^4-1}dx$ $\left[\frac{1}{4}In\left(\frac{x-1}{x+1}\right)+tan^{-1}x+c\right]$

(b) $\int\frac{x^2-4}{(x+1)^2(x-5)}dx$

$\left[\frac{5}{12}In(x+1)-\frac{1}{2(x+1)}+\frac{7}{12}In(x-5)\right]$

(c) $\int\frac{3x^2+x+1}{(x-2)(x+1)^3}dx$

$\left[\frac{5}{9}In(x-2)-\frac{5}{9}In(x+1)-\frac{4}{3(x+1)}+\frac{1}{2(x+1)^2}\right]$

(d) $\int\frac{x^4-x^3+x^2+1}{x^3+x}dx$

$\left[\frac{x^2}{2}-x+Inx+tan^{-1}x-\frac{1}{2}In(1+x^2)+c\right]$

(e) $\int\frac{5x-1}{2x^2+x-10}dx$

$\left[\frac{3}{2}In(2x+5)+In(x-2)+c\right]$

(f) $\int\frac{x^2-9x+2}{(x11)(x-1)(x-2)}dx$

$[2In(x+1)+3In(x-1)-4In(x-2)]+c$

(g) $\int\frac{9x+7}{(2x^2+3)(x+2)}dx$

$\left[\frac{1}{2}In(2x^2+3)+\frac{5}{\sqrt{10}}tan^{-1}\left(\sqrt{\frac{2}{3}}x\right)-In(x+2)+c\right]$

(h) $\int\frac{7+5x-6x^2}{(2x+1)^2(x+2)}dx$

$\left[\frac{3}{2}In(2x+1)-\frac{1}{2x+1}-3In(x+2)+c\right]$

(i) $\int\frac{x^2+7x-14}{(x+5)(x-3)}dx$

$[x+3In(x+5)+2In(x-3)+c]$

3. Evaluate

(a) $\int_0^2\frac{3x^4+7x^3+8x^2+53-186}{(x+4)(x^2+9)}dx$ [-4.5489]

(b) $\int_2^3\frac{x^2}{x^4-1}dx$ [0.18]

(c) $\int_1^3\frac{x^2+1}{x^3+4x^2+3x}dx$ [0.3489]

(d) $\int_0^1\frac{x^3}{x^2+1}dx$ [0.1535]

(e) $\int_6^7\frac{x^2-4}{(x+1)^2(x-5)}dx$ [0.4689]

(f) $\int_3^4\frac{3x^2+x+1}{(x-2)(x+1)^3}dx$ [0.3165]

(g) $\int_0^2\frac{8x}{x^2-4x-12}dx$ [1.05]

Integration by parts

This stems from differentiating the product of a function, y = uv,

$$\frac{d}{dx}(uv)=v\frac{du}{dx}+u\frac{dv}{dx}$$

$$u\frac{dv}{dx}=\frac{d}{dx}(uv)-v\frac{du}{dx}$$

$$\int u\frac{dv}{dx}dx=\int\frac{d}{dx}(uv)dx-\int v\frac{du}{dx}dx$$

$$\int u\frac{dv}{dx}dx=uv-\int v\frac{du}{dx}dx$$

Or simply $\int udv=uv-\int vdu$

The function chosen as u should be easily differentiated whereas the other function chosen as v should be easily integrated.

The above expression of the integration by parts can be summarized by using a technique of integration by parts

This is summarized in the table below

Sign	Differentiate	Integrates
+	u_1	$\frac{dv}{dx}$
-	u_2	v_1
+	u_3	v_2
-	u_4	v_3

NB: the signs change as +, -, + etc.

The u function is differentiated until a zero value is obtained otherwise we continue with differentiation.

The integral of the function is equal to the sum of result shown in the table above.

Integration by parts is applied in the following areas:

A. Integration products of polynomials by parts

Example 17

(a) Find

(i) $\int x(x+2)^3dx$

Solution

Let u = x and $\frac{dv}{dx} = (x+2)^3$

$\frac{du}{dx} = 1$; v = $\frac{1}{4}(x+2)^4$

From $\int udv = uv - \int vdu$

$\int x(x+2)^3dx$

$=\frac{1}{4}x(x+2)^4 - \int 1.\frac{1}{4}(x+2)^4dx$

$=\frac{1}{4}x(x+2)^4 - \frac{1}{4}\int(x+2)^4dx$

$=\frac{1}{4}x(x+2)^4 - \frac{1}{20}(x+2)^5 + c$

$=\frac{1}{2o}(x+2)^4(5x-x-2) + c$

$=\frac{1}{2o}(x+2)^4(4x-2) + c$

$=\frac{1}{1o}(x+2)^4(x-1) + c$

Or by using basic techniques

Sign	Differentiate	Integrates
+	x	$(x+2)^3$
-	1	$\frac{1}{4}(x+2)^4$
+	0	$\frac{1}{20}(x+2)^5$

$\int x(x+2)^3dx = \frac{1}{4}x(x+2)^4 - \frac{1}{20}(x+2)^5 + c$

$= \frac{1}{1o}(x+2)^4(x-1) + c$

$\therefore \int x(x+2)^3dx = \frac{1}{1o}(x+2)^4(x-1) + c$

(ii) $\int(x+3)(x-4)^5dx$

Solution

Let u = (x+3) and $\frac{dv}{dx} = (x-4)^5$

$\frac{du}{dx} = 1$; v = $\frac{1}{6}(x-4)^6$

$\int(x+3)(x-4)^5dx$

$=\frac{1}{6}(x+3)(x-4)^6 - \frac{1}{6}\int 1.(x-4)^6$dx

$=\frac{1}{6}(x+3)(x-4)^6 - \frac{1}{6}\int(x-4)^6dx$

$=\frac{1}{6}(x+3)(x-4)^6 - \frac{1}{42}(x-4)^7 + c$

$=\frac{1}{42}(x-4)^6((7(x+3) - x + 4) + c$

$=\frac{1}{42}(x-4)^6(6x+25) + c$

Sign	Differentiate	Integrates
+	x+3	$(x-4)^5$
-	1	$\frac{1}{6}(x+2)^6$
+	0	$\frac{1}{42}(x+2)^7$

$\int(x+3)(x-4)^5dx$

$=(x+3)(x-4)^5 - \frac{1}{42}x(x+2)^7 + c$

$\therefore \int(x+3)(x-4)^5dx$

$=\frac{1}{42}(x-4)^6(6x+25) + c$

(iii) $\int\frac{3x-4}{(x+2)^4}dx$

Solution

$\int\frac{3x-4}{(x+2)^4}dx = \int(3x-4)(x+2)^{-4}$dx

Let u =(3x – 4) and $\frac{dv}{dx} = (x+2)^{-4}$

$\frac{du}{dx} = 3$; v = $-\frac{1}{3}(x+2)^{-3}$

$\int\frac{3x-4}{(x+2)^4}dx$

$= -\frac{1}{3}(3x-4)(x+2)^{-3} - \int 3.-\frac{1}{3}(x+2)^{-3}dx$

$= -\frac{1}{3}(3x-4)(x+2)^{-3} + \int(x+2)^{-3}dx$

$= -\frac{1}{3}(3x-4)(x+2)^{-3} - \frac{1}{2}(x+2)^{-2} + c$

$= \frac{4-3x}{3(x+2)^3} - \frac{1}{2(x+2)^2} + c$

$= \frac{2(4-3x)-3(x+2)}{6(x+2)^3} + c = \frac{2-9x}{6(x+2)^3} + c$

$\therefore \int\frac{3x-4}{(x+2)^4}dx = \frac{2(4-3x)-3(x+2)}{6(x+2)^3} + c = \frac{2-9x}{6(x+2)^3} + c$

Sign	Differentiate	Integrates
+	3x-4	$(x-4)^{-4}$
-	3	$\frac{1}{3}(x-4)^{-3}$
+	0	$\frac{1}{6}(x-4)^{-2}$

$\int\frac{3x-4}{(x+2)^4}dx$

$= -\frac{1}{3}(3x-4)(x-4)^{-3} - \frac{1}{2}(x-4)^{-2} + c$

$= \frac{2-9x}{6(x+2)^3} + c$

(b) Evaluate

(i) $\int_0^2 x(x-3)^2dx$

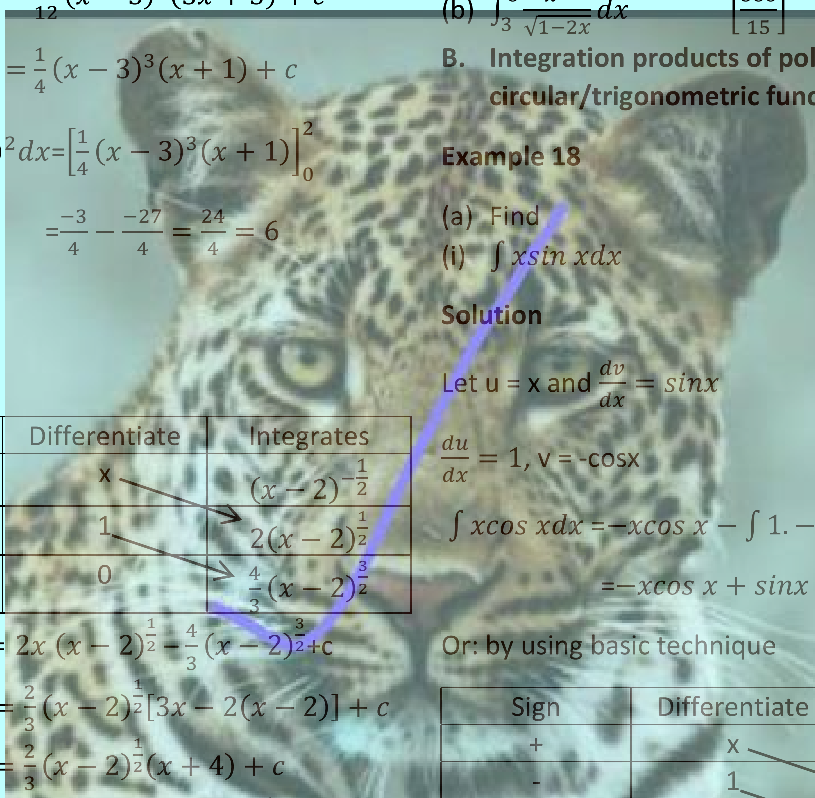
Solution

Sign	Differentiate	Integrates
+	x	$(x-3)^2$
-	1	$\frac{1}{3}(x-3)^3$
+	0	$\frac{1}{12}(x-3)^4$

$\int x(x-3)^2dx = \frac{1}{3}x(x-3)^3 - \frac{1}{12}(x-3)^4+c$

$= \frac{1}{12}(x-3)^3(4x - x + 3) + c$

$= \frac{1}{12}(x-3)^3(3x+3) + c$



$= \frac{1}{4}(x-3)^3(x+1) + c$

⇨ $\int_0^2 x(x-3)^2dx = \left[\frac{1}{4}(x-3)^3(x+1)\right]_0^2$

$= \frac{-3}{4} - \frac{-27}{4} = \frac{24}{4} = 6$

(i) $\int_3^6 \frac{x}{\sqrt{x-2}}dx$

Solution

Sign	Differentiate	Integrates
+	x	$(x-2)^{-\frac{1}{2}}$
-	1	$2(x-2)^{\frac{1}{2}}$
+	0	$\frac{4}{3}(x-2)^{\frac{3}{2}}$

$\int \frac{x}{\sqrt{x-2}}dx = 2x(x-2)^{\frac{1}{2}} - \frac{4}{3}(x-2)^{\frac{3}{2}}+c$

$= \frac{2}{3}(x-2)^{\frac{1}{2}}[3x - 2(x-2)] + c$

$= \frac{2}{3}(x-2)^{\frac{1}{2}}(x+4) + c$

⇨ $\int_3^6 \frac{x}{\sqrt{x-2}}dx = \left[\frac{2}{3}(x-2)^{\frac{1}{2}}(x+4)\right]_3^6$

$= \left[\frac{2}{3}(6-2)^{\frac{1}{2}}(6+4)\right] - \left[\frac{2}{3}(3-2)^{\frac{1}{2}}(3+4)\right]$

$= \frac{2}{3}(20-7) = \frac{26}{3} = 8\frac{2}{3}$

Revision exercise 8

1. Integrate

(a) $\int(x-1)(x+2)^2$dx

$\left[\frac{1}{4}(x-2)(x+2)^3 + c\right]$

(b) $\int(3x-1)(2x+3)^2$dx

$\left[\frac{1}{48}(18x-17)(2x+3)^3 + c\right]$

(c) $\int(2-5x)(4-x)^4$dx

$\left[\frac{1}{30}(25x+8)(4-x)^5\right]$

(d) $\int \frac{x-2}{(2x-3)^2}$dx

$\left[\frac{1}{4}In(2x-3) + \frac{1}{4(2x-3)} + c\right]$

(e) $\int \frac{x+4}{\sqrt{3x-2}}$dx

$\left[\frac{2}{27}(3x+40)\sqrt{3x-2} + c\right]$

(f) $\int \frac{3x+1}{\sqrt{1-2x}}$dx

$In\left(\frac{2-x}{5-x}\right) + \frac{1}{5-x} + c$

2. Evaluate

(a) $\int_{-1}^1 x^2(x+3)^3dx$ $\left[\frac{108}{5}\right]$

(b) $\int_3^6 \frac{x^2}{\sqrt{1-2x}}dx$ $\left[\frac{586}{15}\right]$

B. Integration products of polynomials and circular/trigonometric functions by parts

Example 18

(a) Find

(i) $\int xsin\ xdx$

Solution

Let u = x and $\frac{dv}{dx} = sinx$

$\frac{du}{dx} = 1$, v = -cosx

$\int xcos\ xdx = -xcos\ x - \int 1.-cosx$

$= -xcos\ x + sinx + c$

Or: by using basic technique

Sign	Differentiate	Integrates
+	x	$sinx$
-	1	$-cosx$
+	0	$-sinx$

$\int xcos\ xdx = -xcos\ x + sinx + c$

(ii) $\int x^2cosxdx$

Solution

Let u = x^2 and $\frac{dv}{dx} = cosx$

$\frac{du}{dx} = 2x$, v = sinx

$\int x^2cosxdx = x^2sinx - 2\int xsinxdx$+c

Let u = x and $\frac{dv}{dx} = sinx$

$\frac{du}{dx} = 1$, v = -cosx

$\int x^2cosxdx$
$= x^2sinx - 2[-xcosx - \int -cosxdx] + c$
$= x^2sinx - 2[-xcosx + \int cosxdx] + c$
$= x^2sinx - 2[-xcosx + sinx] + c$
$= x^2sinx + 2xcosx - 2sinx + c$

Or using basic technique

Sign	Differentiate	Integrates
+	x^2	$cosx$
-	2x	$sinx$
+	2	$-cosx$
-	0	$-sinx$

$\int x^2cosxdx$ =x^2sins+ 2xcosx – 2sinx + c

(iii) $\int x^2sin^2xdx$

Solution

Let u = x^2 and $\frac{dv}{dx} = sin^2x = \frac{1}{2}(1 - cos2x)$

$\frac{du}{dx} = 2x$, v = $\frac{1}{2}\left(x - \frac{1}{2}sin2x\right)$

$\int x^2sin^2xdx$
=$\frac{1}{2}x^2\left(x - \frac{1}{2}sin2x\right) - \int 2x.\frac{1}{2}\left(x - \frac{1}{2}sin2x\right)dx$
=$\frac{1}{2}x^2\left(x - \frac{1}{2}sin2x\right) - \int x^2dx + \frac{1}{2}\int xsin2xdx$
=$\frac{1}{2}x^2\left(x - \frac{1}{2}sin2x\right) - \frac{1}{3}x^3 + \frac{1}{2}\int xsin2xdx$

Let u = x and $\frac{dv}{dx} = sin2x$
$\frac{du}{dx} = 1$and v = -$\frac{1}{2}$cos2x
$\int xsin2xdx$ =-$\frac{1}{2}$xcos2x+$\frac{1}{4}$sin2x+c

Substituting for $\int xsin2xdx$
$\int x^2sin^2xdx$
=$\frac{1}{2}x^2\left(x - \frac{1}{2}sin2x\right) - \frac{1}{3}x^3 + \frac{1}{2}\left[-\frac{1}{2}xcos2x + \frac{1}{4}sin2x\right]$
=$\frac{1}{2}x^2\left(x - \frac{1}{2}sin2x\right) - \frac{1}{3}x^3 - \frac{1}{4}xcos2x + \frac{1}{8}sin2x + c$

$$= \frac{1}{6}x^3 - \frac{1}{4}x^2sin2x - \frac{1}{4}xcos2x + \frac{1}{8}sin2x + c$$

Or by using basic technique

Sign	Differentiate	Integrates
+	x^2	$sin^2x = \frac{1}{2}(1 - cos2x)$
-	2x	$\frac{1}{2}x - \frac{1}{4}sins2x$
+	2	$\frac{1}{4}x^2 + \frac{1}{8}cos2x$
-	0	$\frac{1}{12}x^3 + \frac{1}{16}sin2x$

$\int x^2sin^2xdx$

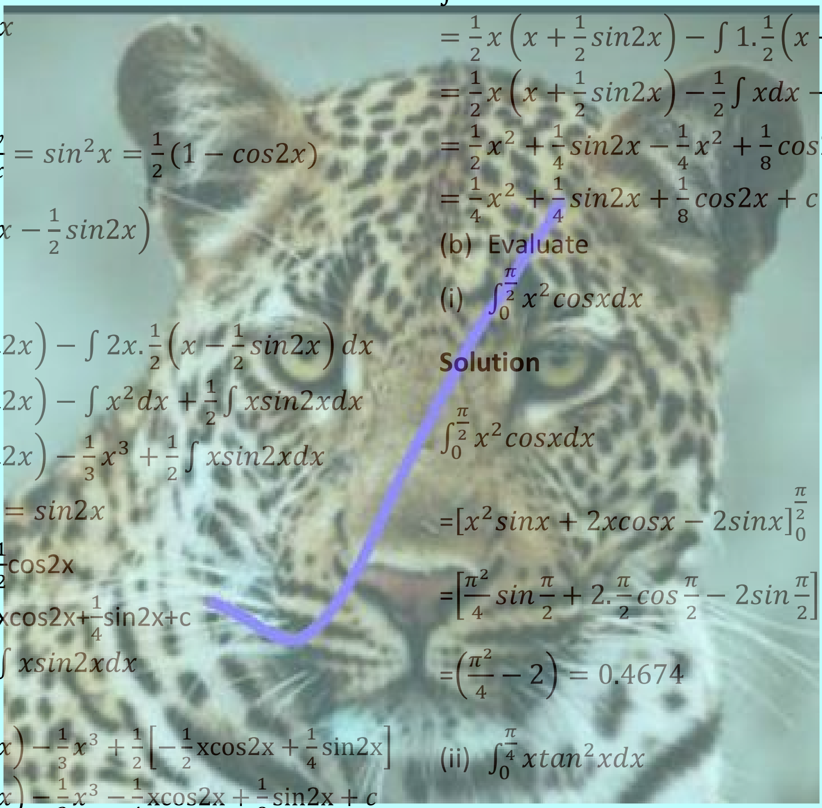
$= \frac{1}{2}x^2\left(x - \frac{1}{2}sin2x\right) - 2x\left(\frac{1}{4}x^2 + \frac{1}{8}cos2x\right) +$
$2\left(\frac{1}{12}x^3 + \frac{1}{16}sin2x\right)$

$$= \frac{1}{6}x^3 - \frac{1}{4}x^2sin2x - \frac{1}{4}xcos2x + \frac{1}{8}sin2x + c$$

(iv) $\int xcos^2xdx$

Solution

$\int xcos^2xdx$
Let u = x and $\frac{dv}{dx} = cos^2x$= $\frac{1}{2} + \frac{1}{2}cos2x$
$\frac{du}{dx} = 1$, v =$\frac{1}{2}\left(x + \frac{1}{2}sin2x\right)$
$\int xcos^2xdx$
$= \frac{1}{2}x\left(x + \frac{1}{2}sin2x\right) - \int 1.\frac{1}{2}\left(x + \frac{1}{2}sin2x\right)$ dx
$= \frac{1}{2}x\left(x + \frac{1}{2}sin2x\right) - \frac{1}{2}\int xdx - \frac{1}{4}\int sin2x$dx
$= \frac{1}{2}x^2 + \frac{1}{4}sin2x - \frac{1}{4}x^2 + \frac{1}{8}cos2x + c$
$= \frac{1}{4}x^2 + \frac{1}{4}sin2x + \frac{1}{8}cos2x + c$



(b) Evaluate

(i) $\int_0^{\frac{\pi}{2}} x^2cosxdx$

Solution

$\int_0^{\frac{\pi}{2}} x^2cosxdx$

=$[x^2sinx + 2xcosx - 2sinx]_0^{\frac{\pi}{2}}$

=$\left[\frac{\pi^2}{4}sin\frac{\pi}{2} + 2.\frac{\pi}{2}cos\frac{\pi}{2} - 2sin\frac{\pi}{2}\right] - 0$

=$\left(\frac{\pi^2}{4} - 2\right) = 0.4674$

(ii) $\int_0^{\frac{\pi}{4}} xtan^2xdx$

Solution

Let u = x and $\frac{dv}{dx} = tan^2x = sec^2x - 1$

$\frac{du}{dx} = 1$; v = tanx - x

$\int xtan^2xdx$=xtanx - x^2-$\int(tanx - x)dx$
=xtanx - x^2+lncosx+$\frac{1}{2}x^2 + c$

=xtanx +lncosx$-\frac{1}{2}x^2 + c$

Or by using basic technique

Sign	Differentiate	Integrates
+	x	$tan^2x = sec^2x - 1$
-	1	tanx – x
+	0	-Incosx$-\frac{1}{2}x^2 +$

$\int xtan^2xdx$=xtanx - x^2+Incosx+$\frac{1}{2}x^2 + c$

=xtanx +Incosx$-\frac{1}{2}x^2 + c$

Hence;

$$\int_0^{\frac{\pi}{4}} xtan^2xdx = \left[\text{xtanx } + \text{Incosx} - \frac{1}{2}x^2\right]_0^{\frac{\pi}{4}}$$

$$= \left[\frac{\pi}{4}\tan\frac{\pi}{4} + \text{Incos}\frac{\pi}{4} - \frac{1}{2}\left(\frac{\pi}{4}\right)^2\right] - 0$$

$$= 0.1304$$

Revision Exercise 9

1. Integrate each of the following
   (a) $\int xsin2xdx$
   $\left[-\frac{x}{2}cos2x + \frac{1}{4}sin2x + c\right]$
   (b) $\int x^2sinxdx$
   $[-x^2cosx + 2xsinx + 2cosx + c]$
   (c) $\int (x+1)^2sinxdx$
   $[(1 - 2x - x^2)cosx = 2(x+1)sinx + c]$
   (d) $\int x^2sinxcosxdx$
   $\left[\frac{1}{8}cos2x(1 - 2x^2) + \frac{1}{4}xsin2x + c\right]$
   (e) $\int x^3 \cos x^2\, dx$
   $\left[\frac{1}{2}x^2 \sin x^2 + \frac{1}{2}\cos x^2 + c\right]$
   (f) $\int (xcosx)^2dx$
   $\left[\frac{1}{6}x^3 + \frac{1}{8}(2x^2 - 1) + \frac{1}{4}xsin2x + c\right]$
2. Evaluate
   (a) $\int_0^{\pi} x^2sinxdx$ $[5.8696]$
   (b) $\int_0^{\pi} x^2cos2xdx$ $[0.0584]$
   (c) $\int_0^{\frac{\pi}{4}} xtan^2xdx$ $[0.1304]$

C. Integration products of polynomials and exponential functions by parts

Examples 19

(a) Find

(i) $\int xe^xdx$

Solution

Let u = x and $\frac{dv}{dx} = e^x$

$\frac{du}{dx} = 1$; v =e^x

$\int xe^xdx = xe^x - \int 1.e^xdx$

$= xe^x - e^x + c$

Or by using basic technique

Sign	Differentiate	Integrates
+	x	e^x
-	1	e^x
+	0	e^x

$\int xe^xdx = xe^x - e^x + c$

(ii) $\int xe^{-x}dx$

Solution

Let u = x and $\frac{dv}{dx} = e^{-x}$

$\frac{du}{dx} = 1$; v =$-e^{-x}$

$\int xe^xdx = -xe^{-x} - \int 1.-e^{-x}dx$

$= -xe^{-x} + \int e^{-x}dx$

$= -xe^x - e^x + c$

Or by using basic technique

Sign	Differentiate	Integrates
+	x	e^{-x}
-	1	$-e^{-x}$
+	0	e^x

$\int xe^xdx = -xe^x - e^x + c$

(iii) $\int xe^{3x}dx$

Solution

Let u = x and $\frac{dv}{dx} = e^{3x}$

$\frac{du}{dx} = 1$; v =$\frac{1}{3}e^{3x}$

$\int xe^{3x}dx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + c$

Or by using basic technique

Sign	Differentiate	Integrates
+	x	e^{3x}
-	1	$\frac{1}{3}e^{3x}$
+	0	$\frac{1}{9}e^{3x}$

$\int xe^xdx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + c$

(b) Find

(i) $\int x.2^x dx$

Solution

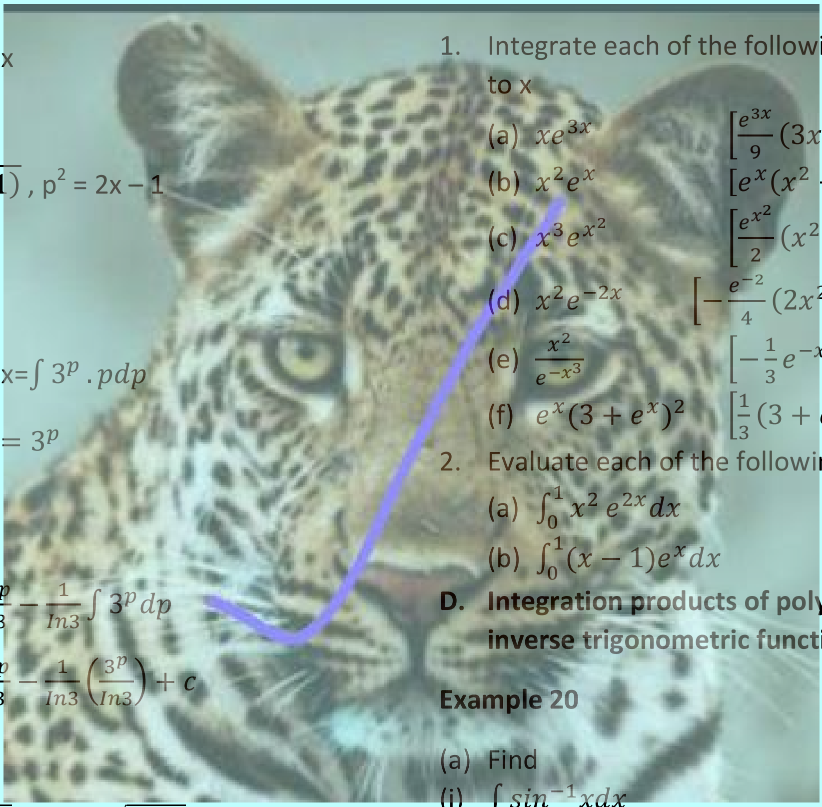
Let u = x and $\frac{dv}{dx} = 2^x$

$\frac{du}{dx} = 1$; v = $\frac{2^x}{In2}$

$\int x.2^x dx = \frac{x.2^x}{In2} - \frac{1}{In2}\int 2^x dx$

$= \frac{x.2^x}{In2} - \frac{1}{In2}\left(\frac{2^x}{In2}\right) + c$

$= \frac{2x}{In2}(x-1) + c$



(ii) $\int 3^{\sqrt{(2x-1)}} dx$

Solution

Let p = $\sqrt{(2x-1)}$, $p^2 = 2x - 1$

2pdp = 2dx

pdp = dx

⇨ $\int 3^{\sqrt{(2x-1)}} dx = \int 3^p . pdp$

Let u = p and $\frac{dv}{dp} = 3^p$

$\frac{du}{dp} = 1$, v = $\frac{3^p}{In3}$

$\int 3^p . pdp = \frac{3^p.p}{In3} - \frac{1}{In3}\int 3^p dp$

$= \frac{3^p.p}{In3} - \frac{1}{In3}\left(\frac{3^p}{In3}\right) + c$

∴ $\int 3^{\sqrt{(2x-1)}} dx$

$= \frac{\sqrt{(2x-1)}3^{\sqrt{(2x-1)}}}{In3} - \frac{1}{In3}\left(\frac{3^{\sqrt{(2x-1)}}}{In3}\right) + c$

$= \frac{3^{\sqrt{(2x-1)}}}{In3}\left(\sqrt{(2x-1)} - \frac{1}{In3}\right) + c$

(c) Evaluate

(i) $\int_0^1 xe^{-x} dx$

Solution

$\int_0^1 xe^{-x} dx = [-xe^{-x} - e^{-x}]_0^1$

$= (-e^{-1} - e^{-1}) - (0 - e^0)$

$= -2e^{-1} + 1$

$= 1 - \frac{2}{e}$

$= 0.2642$

(ii) $\int_0^1 xe^{3x} dx$

Solution

$\int_0^1 xe^{3x} dx = \left[\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x}\right]_0^1$

$= \left[\frac{1}{3}e^3 - \frac{1}{9}e^3\right] - \left[0 - \frac{1}{9}e^0\right]$

$= \frac{2}{9}e^3 + \frac{1}{9} = 4.5746$

Revision exercise 10

1. Integrate each of the following with respect to x

(a) xe^{3x} $\left[\frac{e^{3x}}{9}(3x-1) + c\right]$

(b) x^2e^x $[e^x(x^2 - 2x + 2) + c]$

(c) $x^3e^{x^2}$ $\left[\frac{e^{x^2}}{2}(x^2 - 1) + c\right]$

(d) x^2e^{-2x} $\left[-\frac{e^{-2}}{4}(2x^2 + 2x + 1) + c\right]$

(e) $\frac{x^2}{e^{-x^3}}$ $\left[-\frac{1}{3}e^{-x^3} + c\right]$

(f) $e^x(3 + e^x)^2$ $\left[\frac{1}{3}(3 + e^x)^3 + c\right]$

2. Evaluate each of the following

(a) $\int_0^1 x^2 e^{2x} dx$ [1.5973]

(b) $\int_0^1 (x-1)e^x dx$ [2]

D. Integration products of polynomials and inverse trigonometric functions by parts

Example 20

(a) Find

(i) $\int sin^{-1}xdx$

Solution

$\int sin^{-1}xdx = \int 1.sin^{-1}xdx$

Let u=$sin^{-1}x$ and $\frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$; v = x

$\int sin^{-1}xdx = xsin^{-1}x - \int \frac{x}{\sqrt{1-x^2}} dx$

For $\int \frac{x}{\sqrt{1-x^2}} dx$

Let u = 1 – x^2

Du= -2x

$-\frac{1}{2x}du = dx$

$\int \frac{x}{\sqrt{1-x^2}}dx = \int \frac{x}{u^{\frac{1}{2}}}.-\frac{1}{2x}du$

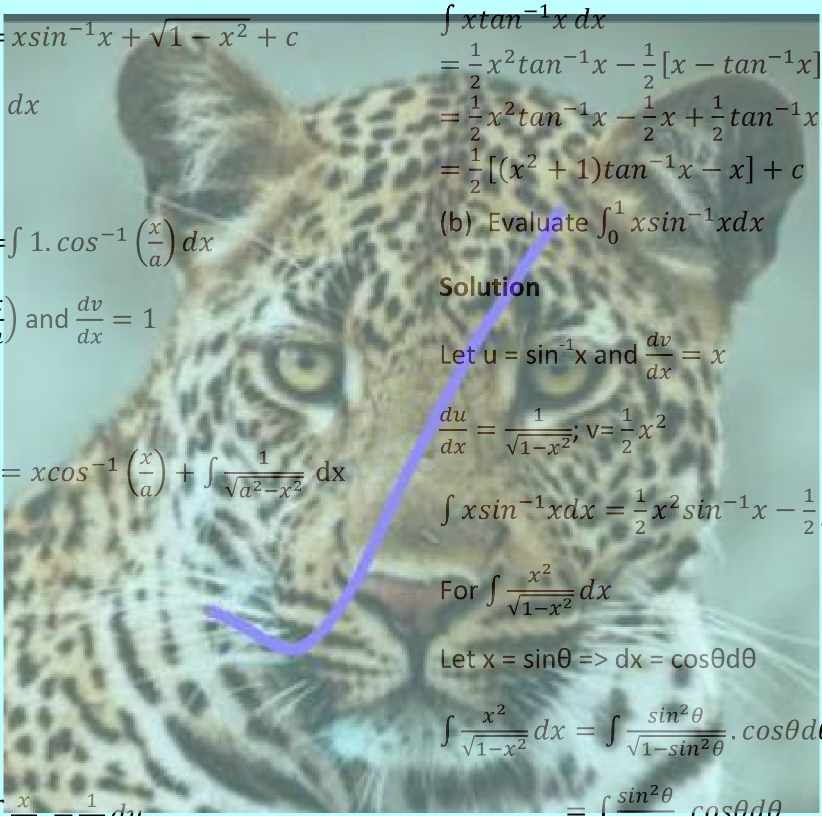
$= -\frac{1}{2}\int u^{-\frac{1}{2}}du$

$= -\frac{1}{2}\left[2u^{\frac{1}{2}} + c\right] = -u^{\frac{1}{2}} + c$

By substitution

$\int sin^{-1}xdx = xsin^{-1}x + u^{\frac{1}{2}} + c$

$\therefore \int sin^{-1}xdx = xsin^{-1}x + \sqrt{1-x^2} + c$



(ii) $\int cos^{-1}\left(\frac{x}{a}\right)dx$

Solution

$\int cos^{-1}\left(\frac{x}{a}\right)dx = \int 1.cos^{-1}\left(\frac{x}{a}\right)dx$

Let u = $cos^{-1}\left(\frac{x}{a}\right)$ and $\frac{dv}{dx} = 1$

$\frac{du}{dx} = -\frac{1}{\sqrt{a^2-x^2}}$

$\int cos^{-1}\left(\frac{x}{a}\right)dx = xcos^{-1}\left(\frac{x}{a}\right) + \int \frac{1}{\sqrt{a^2-x^2}}$ dx

For $\int \frac{x}{\sqrt{a^2-x^2}}dx$

Let u = $a^2 - x^2$

Du= -2x

$-\frac{1}{2x}du = dx$

$\int \frac{x}{\sqrt{a^2-x^2}}dx = \int \frac{x}{u^{\frac{1}{2}}}.-\frac{1}{2x}du$

$= -\frac{1}{2}\int u^{-\frac{1}{2}}du$

$= -\frac{1}{2}\left[2u^{\frac{1}{2}} + c\right] = -u^{\frac{1}{2}} + c$

By substitution

$\int cos^{-1}\left(\frac{x}{a}\right)dx = xcos^{-1}x + u^{\frac{1}{2}} + c$

$\therefore \int cos^{-1}\left(\frac{x}{a}\right)dx = xcos^{-1}x + \sqrt{a^2-x^2} + c$

(iii) $\int xtan^{-1}x\,dx$

Solution

Let u = $tan^{-1}x$ and $\frac{dv}{dx} = x$

$\frac{du}{dx} = \frac{1}{1+x^2}$; v = $\frac{1}{2}x^2$

$\int xtan^{-1}x\,dx = \frac{1}{2}x^2tan^{-1}x - \frac{1}{2}\int \frac{x^2}{1+x^2}$dx

For $\int \frac{x^2}{1+x^2}$dx = $\int \left(1 - \frac{1}{1+x^2}\right)dx$

$= \int dx - \int \frac{1}{1+x^2}dx$

$= x - tan^{-1}x + c$

By substitution

$\int xtan^{-1}x\,dx$

$= \frac{1}{2}x^2tan^{-1}x - \frac{1}{2}[x - tan^{-1}x] + c$

$= \frac{1}{2}x^2tan^{-1}x - \frac{1}{2}x + \frac{1}{2}tan^{-1}x + c$

$= \frac{1}{2}[(x^2+1)tan^{-1}x - x] + c$

(b) Evaluate $\int_0^1 xsin^{-1}xdx$

Solution

Let u = $sin^{-1}x$ and $\frac{dv}{dx} = x$

$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$; v= $\frac{1}{2}x^2$

$\int xsin^{-1}xdx = \frac{1}{2}x^2sin^{-1}x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}dx$

For $\int \frac{x^2}{\sqrt{1-x^2}}dx$

Let x = sinθ => dx = cosθdθ

$\int \frac{x^2}{\sqrt{1-x^2}}dx = \int \frac{sin^2\theta}{\sqrt{1-sin^2\theta}}.cos\theta d\theta$

$= \int \frac{sin^2\theta}{cos\theta}.cos\theta d\theta$

$= \int sin^2\theta\,d\theta$

$= \frac{1}{2}\int(1 - cos2\theta)\,d\theta$

$= \frac{1}{2}\theta - \frac{1}{4}sin2\theta + c$

$= \frac{1}{2}\theta - \frac{1}{4}(2sin\theta cos\theta) + c$

$= \frac{1}{2}\theta - \frac{1}{2}(sin\theta cos\theta) + c$

$= \frac{1}{2}sin^{-1}x - \frac{1}{2}x\sqrt{1-x^2} + c$

$\therefore \int xsin^{-1}xdx$

$= \frac{1}{2}x^2 sin^{-1}x - \frac{1}{4}sin^{-1}x + \frac{1}{4}x\sqrt{1-x^2}$

$\int_0^1 xsin^{-1}xdx$

$= \left[\frac{1}{2}x^2 sin^{-1}x - \frac{1}{4}sin^{-1}x + \frac{1}{4}x\sqrt{1-x^2}\right]_0^1$

$= \left[\frac{1}{2}.1.sin^{-1}1 - \frac{1}{4}sin^{-1}(1)\right] - (0)$

$= \frac{\pi}{4} - \frac{\pi}{8} = \frac{\pi}{8}$

Revision exercise 11

1. Find the following integrals

(a) $\int tan^{-1}3xdx$

$\left[xtan^{-1}x - \frac{1}{6}In(1+9x^2) + c\right]$

(b) $\int \frac{xsin^{-1}x}{\sqrt{1-x^2}}dx$

$\left[xsin^{-1}x + \sqrt{1-x^{2+c}}\right]$

(c) $\int sec^{-1}x\,dx$

$\left[xsec^{-1}x - In\left(x+\sqrt{x^2-1}\right) + c\right]$

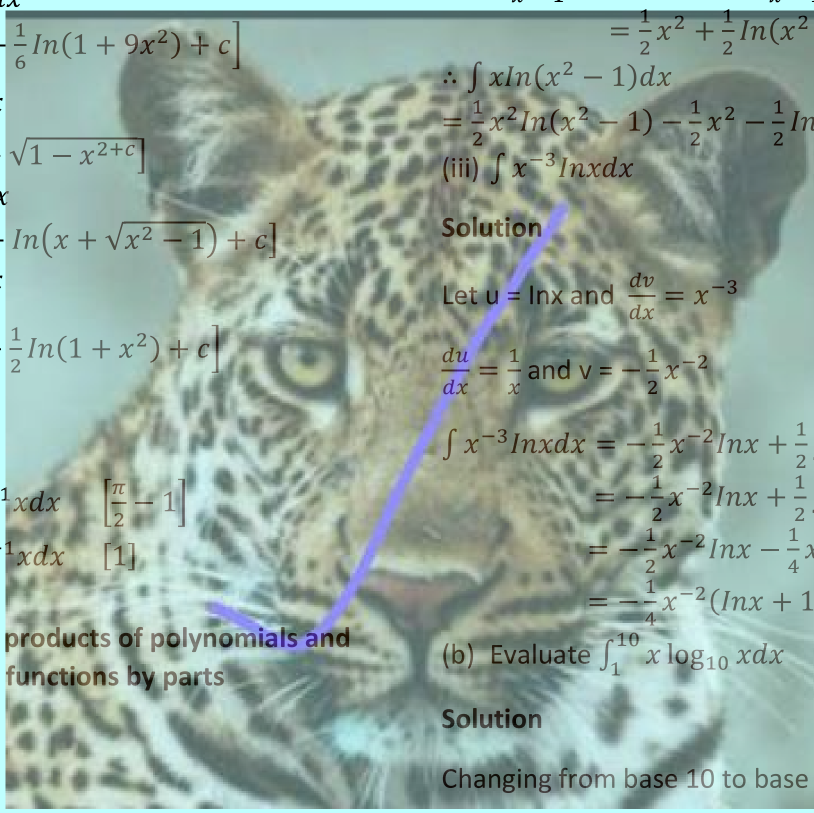
(d) $\int cot^{-1}xdx$

$\left[xcot^{-1}x + \frac{1}{2}In(1+x^2) + c\right]$

2. Evaluate

(a) $\int_0^1 sin^{-1}xdx$ $\left[\frac{\pi}{2} - 1\right]$

(b) $\int_0^1 cos^{-1}xdx$ $[1]$



E. Integration products of polynomials and logarithmic functions by parts

Example 21

(a) Integrate

(i) $\int Inx^2dx$

Solution

$\int Inx^2dx = \int 1.Inx^2dx$

Let u = Inx^2, $\frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{2x}{x^2} = \frac{2}{x}$; v = x

$\int Inx^2dx = xInx^2 - 2\int x.\frac{1}{x}dx$

$= xInx^2 - 2x + c$

$= 2xInx - 2x + c$

$\therefore \int Inx^2dx = 2xInx - 2x + c$

(ii) $\int xIn(x^2-1)dx$

Solution

Let u = $In(x^2-1)$ and $\frac{dv}{dx} = x$

$\frac{du}{dx} = \frac{2x}{x^2-1}$; $v = \frac{1}{2}x^2$

$\int xIn(x^2-1)dx$

$= \frac{1}{2}x^2In(x^2-1) - \int \frac{1}{2}x^2.\frac{2x}{x^2-1}dx$

$= \frac{1}{2}x^2In(x^2-1) - \int \frac{x^3}{x^2-1}dx$

For $\int \frac{x^3}{x^2-1}dx$

By using long division

$\frac{x^3}{x^2-1} = x + \frac{x}{x^2-1}$

$\Rightarrow \int \frac{x^3}{x^2-1}dx = \int xdx + \int \frac{x}{x^2-1}dx$

$= \frac{1}{2}x^2 + \frac{1}{2}In(x^2-1) + c$

$\therefore \int xIn(x^2-1)dx$

$= \frac{1}{2}x^2In(x^2-1) - \frac{1}{2}x^2 - \frac{1}{2}In(x^2-1) + c$

(iii) $\int x^{-3}Inxdx$

Solution

Let u = Inx and $\frac{dv}{dx} = x^{-3}$

$\frac{du}{dx} = \frac{1}{x}$ and v = $-\frac{1}{2}x^{-2}$

$\int x^{-3}Inxdx = -\frac{1}{2}x^{-2}Inx + \frac{1}{2}\int \frac{1}{x}.x^{-2}$ dx

$= -\frac{1}{2}x^{-2}Inx + \frac{1}{2}\int x^{-3}dx$

$= -\frac{1}{2}x^{-2}Inx - \frac{1}{4}x^{-2} + c$

$= -\frac{1}{4}x^{-2}(Inx + 1) + c$

(b) Evaluate $\int_1^{10} x\log_{10}xdx$

Solution

Changing from base 10 to base e

$\log_{10}x = \frac{Ine}{In10}$

$\int_1^{10} x\log_{10}xdx = \frac{1}{In10}\int_1^{10} xInxdx$

Let u= Inx; $\frac{dv}{dx} = x$

$\frac{du}{dx} = \frac{1}{x}$; $v = \frac{1}{2}x^2$

$$\frac{1}{In10}\int_1^{10} xInxdx = \frac{1}{In10}\left[\frac{1}{2}x^2Inx - \frac{1}{4}x^2\right]_1^{10}$$

$= \frac{1}{In10}\left[(50In10 - 25) - \frac{1}{4}\right]$

$= \frac{1}{In10}\left[50In10 - \frac{99}{4}\right]$ $= 50 - \frac{99}{4In10}$

Revision exercise 12

1. Integrate each of the following

   (a) $xInx$ $\left[\frac{x^2}{4}(2Inx-1)+c\right]$

   (b) x^2Inx $\left[\frac{x^2}{9}(3Inx-1)+c\right]$

   (c) $\sqrt{x}Inx$ $\left[\frac{2}{9}\sqrt{x^3}(3Inx-2)+c\right]$

   (d) $(Inx)^2$ $[x(2-2Inx+(Inx)^2)+c]$

   (e) $\frac{Inx}{x^2}$ $\left[-\frac{1}{x}(Inx+1)+c\right]$

   (f) $3^x x$ $\left[\frac{3^x}{(In3)^2}(xIn3-1)+c\right]$

   (g) $x(Inx)^2$

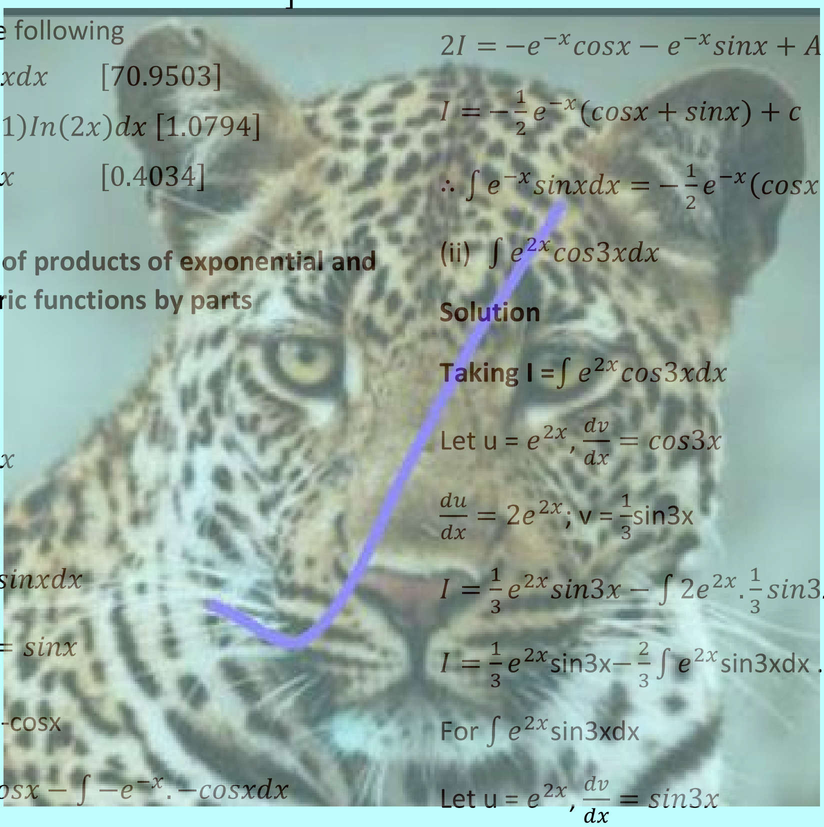
   $\left[\frac{1}{4}x^2(1-2Inx+2(Inx)^2)+c\right]$

2. Evaluate the following

   (a) $\int_2^4 x^3 Inx dx$ $[70.9503]$

   (b) $\int_2^4 (x-1)In(2x)dx$ $[1.0794]$

   (c) $\int_1^4 \frac{Inx}{x^2}dx$ $[0.4034]$



F. Integration of products of exponential and trigonometric functions by parts

Example 22

(a) Find

(i) $\int e^{-x}sinxdx$

Solution

Taking I = $\int e^{-x}sinxdx$

Let u = e^{-x}, $\frac{dv}{dx} = sinx$

$\frac{du}{dx} = -e^{-x}$; v = -cosx

$=> I = -e^{-x}cosx - \int -e^{-x}.-cosxdx$

$I = -e^{-x}cosx - \int e^{-x}.cosxdx$ …(*)

For $\int e^{-x}.cosxdx$

Let u = e^{-x}, $\frac{dv}{dx} = cosx$

$\frac{du}{dx} = -e^{-x}$; v = sinx

$\int e^{-x}.cosxdx = e^{-x}sinx - \int -e^{-x}sinx$

$= e^{-x}sinx + I$ ………(**)

Substituting for (**) in equation (*)

$I = -e^{-x}cosx - e^{-x}sinx - I$

$2I = -e^{-x}cosx - e^{-x}sinx - I + A$

$I = -\frac{1}{2}e^{-x}(cosx + sinx) + c$

$\therefore \int e^{-x}sinxdx = -\frac{1}{2}e^{-x}(cosx + sinx) + c$

Or by using basic technique

sign	Differentiate	integrate
+	e^{-x}	sinx
-	$-e^{-x}$	-cosx
+	e^{-x}	-sinx

$I = -e^{-x}cosx - e^{-x}sinx - \int e^{-x}sinx$

$I = -e^{-x}cosx - e^{-x}sinx - I$

$2I = -e^{-x}cosx - e^{-x}sinx + A$

$I = -\frac{1}{2}e^{-x}(cosx + sinx) + c$

$\therefore \int e^{-x}sinxdx = -\frac{1}{2}e^{-x}(cosx + sinx) + c$

(ii) $\int e^{2x}cos3xdx$

Solution

Taking I = $\int e^{2x}cos3xdx$

Let u = e^{2x}, $\frac{dv}{dx} = cos3x$

$\frac{du}{dx} = 2e^{2x}$; v = $\frac{1}{3}$sin3x

$I = \frac{1}{3}e^{2x}sin3x - \int 2e^{2x}.\frac{1}{3}sin3xdx$

$I = \frac{1}{3}e^{2x}\text{sin3x} - \frac{2}{3}\int e^{2x}\text{sin3xdx}$ … (*)

For $\int e^{2x}$sin3xdx

Let u = e^{2x}, $\frac{dv}{dx} = sin3x$

$\frac{du}{dx} = 2e^{2x}$; v =- $\frac{1}{3}$cos3x

$\int e^{2x}$ sin3xdx

$= -\frac{1}{3}e^{2x}\text{cos3x} - \int 2e^{2x}.-\frac{1}{3}\text{cos3xdx}$

$= -\frac{1}{3}e^{2x}\text{cos3x} + \frac{2}{3}\int e^{2x}\text{cos3xdx}$

$= -\frac{1}{3}e^{2x}\text{cos3x} + \frac{2}{3}I$ ………………….(**)

Substituting (**) into (*)

$I = \frac{1}{3}e^{2x}\text{sin3x} - \frac{2}{3}\left(-\frac{1}{3}e^{2x}\text{cos3x} + \frac{2}{3}I\right)$

$$= \frac{1}{3}e^{2x}\text{sin3x} + \frac{2}{9}e^{2x}\text{cos3x} + \frac{4}{9}I + \text{c}$$

$$\frac{13}{9}I = \frac{1}{3}e^{2x}\text{sin3x} + \frac{2}{9}e^{2x}\text{cos3x} + \text{A}$$

$$I = \frac{3}{13}e^{2x}\text{sin3x} + \frac{2}{13}e^{2x}\text{cos3x} + \text{c}$$

$$I = \frac{1}{13}e^{2x}(3sin3x + 2cos3x) + c$$

$$\therefore \int e^{2x}cos3xdx = \frac{1}{13}e^{2x}(3sin3x + 2cos3x) + c$$

(iii) $\int e^{3x}sin2xdx$

Solution

Taking I = $\int e^{3x}sin2xdx$

Let u = e^{3x}, $\frac{dv}{dx} = sin2x$

$\frac{du}{dx} = 3e^{3x}$; v = $-\frac{1}{2}$cos2x

$$I = -\frac{1}{2}e^{3x}cos2x - \int 3e^{3x}.-\frac{1}{2}cos2xdx$$

$$I = -\frac{1}{2}e^{3x}\text{cos2x} + \frac{3}{2}\int e^{3x}\text{cos2xdx} \ldots (*)$$

For $\int e^{3x}$cos2xdx

Let u = e^{3x}, $\frac{dv}{dx} = cos2x$

$\frac{du}{dx} = 3e^{3x}$; v = $\frac{1}{2}$sin2x

$\int e^{3x}$ cos2xdx

$$= \frac{1}{2}e^{3x}\text{sin2x} - \int 3e^{3x}.\frac{1}{2}\text{sin2xdx}$$

$$= \frac{1}{2}e^{3x}\text{sin2x} - \frac{3}{2}\int e^{3x}\text{sin2xxdx}$$

$$= \frac{1}{2}e^{3x}\text{sin2x} - \frac{3}{2}I \text{ ....(**)}$$

Substituting (**) into (*)

$$I = -\frac{1}{2}e^{3x}cos2x + \frac{3}{2}\left(\frac{1}{2}e^{3x}\text{sin2x} - \frac{3}{2}I\right)$$

$$= -\frac{1}{2}e^{3x}cos2x + \frac{3}{4}e^{3x}\text{sin2x} - \frac{9}{4}I + \text{c}$$

$$\frac{13}{4}I = -\frac{1}{2}e^{3x}cos2x + \frac{3}{4}e^{3x}\text{sin2x} + \text{A}$$

$$I = -\frac{2}{13}e^{3x}\text{cos2x} + \frac{3}{13}e^{3x}\text{sin2x} + \text{c}$$

$$I = \frac{1}{13}e^{3x}(3sin2x - 2cos2x) + c$$

$$\therefore \int e^{3x}sin2xdx = \frac{1}{13}e^{2x}(3sin2x - 2cos3x) + c$$

Or using basic technique

sign	Differentiate	integrate
+	e^{3x}	sin2x
-	$3e^{3x}$	$-\frac{1}{2}$cos2x
+	$9e^{3x}$	$-\frac{1}{4}$sinx

$$I = -\frac{1}{2}e^{3x}cos2x + \frac{3}{4}e^{3x}\text{sin2x} - \frac{9}{4}I + \text{c}$$

$$\frac{13}{4}I = -\frac{1}{2}e^{3x}cos2x + \frac{3}{4}e^{3x}\text{sin2x} + \text{A}$$

$$I = -\frac{2}{13}e^{3x}\text{cos2x} + \frac{3}{13}e^{3x}\text{sin2x} + \text{c}$$

$$I = \frac{1}{13}e^{3x}(3sin2x - 2cos2x) + c$$

$$\therefore \int e^{3x}sin2xdx = \frac{1}{13}e^{2x}(3sin2x - 2cos3x) + c$$

(b) Evaluate $\int_0^{\infty} e^{-2x}sin3xdx$

Solution

Taking I =$e^{-2x}sin3xdx$

Let u = e^{-2x}, $\frac{dv}{dx} = sin3x$

$\frac{du}{dx} = -2e^{-2x}$; v = $-\frac{1}{3}$cos3x

$$I = -\frac{1}{3}e^{-2x}\text{cos3x} - \int 2e^{2x}.\frac{1}{3}\text{sin3xdx}$$

$$I = \frac{1}{3}e^{3x}\text{sin3x} - \frac{2}{3}\int e^{-2x}\text{cos3xdx} \text{ ...... (*)}$$

For $\int e^{-2x}$cos3xdx

Let u = e^{-2x}, $\frac{dv}{dx} = cosx$

$\frac{du}{dx} = -2e^{2x}$; v = $\frac{1}{3}$sin3x

$\int e^{-2x}$cos3xdx

$$= -\frac{1}{3}e^{-2x}\text{sin3x} + \frac{2}{3}\int e^{2x}\text{sins3xdx}$$

$$= -\frac{1}{3}e^{-2x}\text{sin3x} + \frac{2}{3}I \text{ ....(**)}$$

Substituting (**) into (*)

$$I = -\frac{1}{3}e^{-2x}\text{cos3x} - \frac{2}{3}\left(\frac{1}{3}e^{-2x}\text{cos3x} + \frac{2}{3}I\right)$$

$$= -\frac{1}{3}e^{-2x}\text{cos3x} - \frac{2}{9}e^{-2x}\text{sin3x} - \frac{4}{9}I + \text{c}$$

$$\frac{13}{9}I = -\frac{1}{3}e^{-2x}\text{scos3x} - \frac{2}{9}e^{-2x}\text{sin3x} + \text{c}$$

$I = -\frac{1}{13}e^{-2x}(3cos3x + 2sin3x) + c$

$\therefore \int e^{-2x}sin3xdx$

$= -\frac{1}{13}e^{-2x}(3cos3x + 2sin3x) + c$

⇨ $\int_0^{\infty} e^{-2x}sin3xdx$

$=\left[-\frac{1}{13}e^{-2x}(3cos3x + 2sin3x)\right]_0^{\infty}$

$= \frac{3}{13}$ since $e^{\infty} = 0$

Revision exercise 13

Integrate each of the following with respect to x

(a) $e^x cosx$ $\quad\left[\frac{1}{2}e^x(sinx + cosx) + c\right]$

(b) $e^x sinxx$ $\quad\left[\frac{1}{2}e^x(sinx - cosx) + c\right]$

(c) $e^{ax}cosbx$ $\quad\left[\frac{e^{ax}}{b^2+a^2}(acosbx + bsinbx) + c\right]$

(d) $e^{3x}sin2x$ $\quad\left[\frac{1}{13}e^{3x}(3sin2x - 2cos2x) + c\right]$

G. Integration of products of trigonometric functions by parts

A student should take note of the following

(i) $\int tanxdx = In(\sec x)$+ c

Proof

$\frac{d}{dx}In(sex)dx = \frac{secxtanx}{secx} = tanx$

Hence $\int tanxdx = In(\sec x)$+ c

(ii) $\int cosecxdx = -In(cosecx + cotx) + c$

Proof

$\frac{d}{dx}In(cosecx + cotx)dx = \frac{-cosecxcotx - cosec^2x}{cosecx+cotx}$

$= \frac{-cosecx(cotx - cosecx)}{cosecx+cotx}$

$= -$cosecx

$\therefore \int cosecxdx = -In(cosecx + cotx) + c$

(iii) $\int secxdx = In(secx + tanx) + c$

Proof

$\frac{d}{dx}In(secx + tanx)dx = \frac{secxtanx + sec^2x}{secx+tanx}$

$= \frac{secx(In(secx+tanx))}{secx+tanx}$

=secx

$\therefore \int secxdx = In(secx + tanx) + c$

(iv) $\int cotxdx = In(sinx) + c$

Proof

$\frac{d}{dx}In(sinx) = \frac{cosx}{sinx} = cotx$

$\therefore \int cotxdx = In(sinx) + c$

Example 22

(a) Find

(i) $\int sec^3xdx$

Solution

Taking $I = \int sec^3xdx = \int secxsec^2xdx$

Let u = secx and $\frac{dv}{dx} = sec^2x$

$\frac{du}{dx} = secxtanx$; v = tanx

$I = secxtanx - \int(secxtanx)tanxdx$

$= secxtanx - \int secxtan^2xdx$

$= secxtanx - \int secx(sec^2x - 1)dx$

$= secxtanx - \int sec^3xdx + \int secx\,dx$

$= secxtanx$ − I + In(secx+tanx)+c

2I = $secxtanx$ + In(secx+tanx)+c

$I = \frac{1}{2}[secxtanx + \text{In}(\text{secx} + \text{tanx})] + c$

$\therefore \int sec^3xdx$

$= \frac{1}{2}[secxtanx + \text{In}(\text{secx} + \text{tanx})] + c$

(ii) $\int cosec^3xdx$

Taking $I = \int cosec^3xdx = \int cosecxcosec^2xdx$

Let u = cosecx and $\frac{dv}{dx} = cosec^2x$

$\frac{du}{dx} = cosecxcotx$; v = -cotx

$I = -cotxcosecx - \int(secxcotx)cotxdx$

$= -cosecxcotx - \int secxcot^2xdx$

$= -cosecxcotx - \int cosecx(cosec^2x - 1)dx$

$= -cosecxtanx - \int cosec^3xdx + \int cosecx\,dx$

$= -cotxcosecx - I + \int cosecx\,dx$+A

$2I = -cotxcosex) - In(cosecx + cotx)$+c

$I = -\frac{1}{2}[cotxcosecx + In(cosecx + cotx)] + c$

$$\therefore \int cosec^3 x dx$$

$$= -\frac{1}{2}[cotxcosecx + In(cosecx + cotx)] + c$$

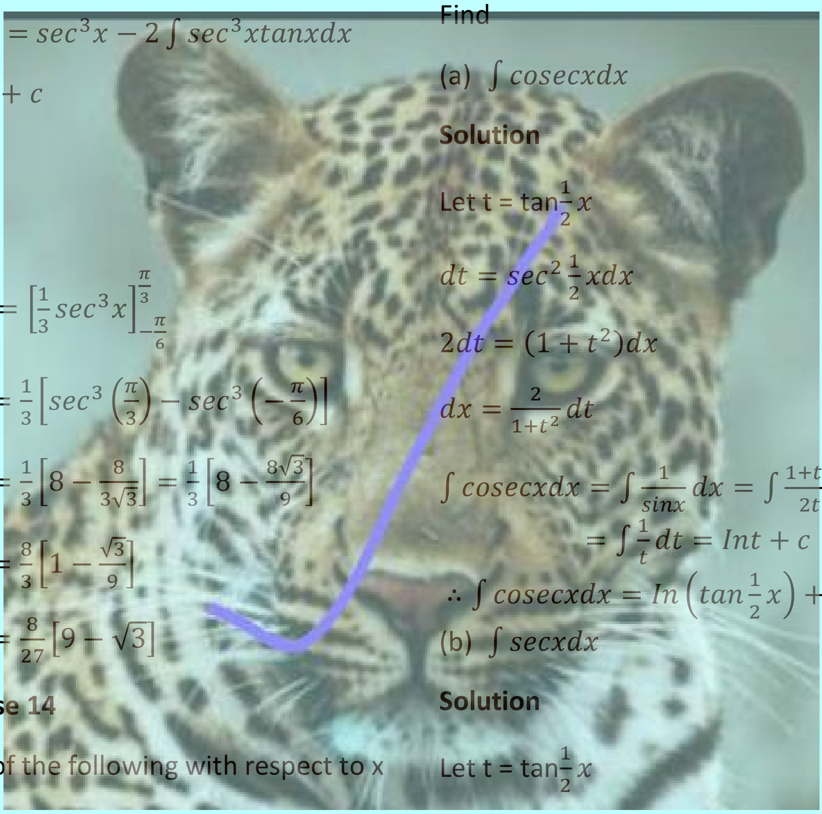
(b) Show that $\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} sec^3 x tanx = \frac{8}{27}(9 - \sqrt{3})$

Solution

$\int sec^3 xtanxdx = \int sec^2 xsecxtanx$

Let u $=sec^2 x$ and $\frac{dv}{dx} = secxtanx$

$\frac{du}{dx} = 2sec^2 x; v = secx$



$\int sec^3 xtanxdx = sec^3 x - 2\int sec^3 xtanxdx$

$I - sec^3 x - 2I + c$

$3I = sec^3 x$

$I = \frac{1}{3} sec^3 x + c$

$$\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} sec^3 xtanx = \left[\frac{1}{3} sec^3 x\right]_{-\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{1}{3}\left[sec^3\left(\frac{\pi}{3}\right) - sec^3\left(-\frac{\pi}{6}\right)\right]$$

$$= \frac{1}{3}\left[8 - \frac{8}{3\sqrt{3}}\right] = \frac{1}{3}\left[8 - \frac{8\sqrt{3}}{9}\right]$$

$$= \frac{8}{3}\left[1 - \frac{\sqrt{3}}{9}\right]$$

$$= \frac{8}{27}\left[9 - \sqrt{3}\right]$$

Revision exercise 14

Integrate each of the following with respect to x

1. $sec^3 x$

   $\left[\frac{1}{2}[secxtanx + \text{In}(\text{secx} + \text{tanx})] + c\right]$
2. $cosec^3 x$

   $\left[-\frac{1}{2}[cotxcosecx + In(cosecx + cotx)] + c\right]$
3. $sec^3 xtanx$

   $\left[\frac{1}{3} sec^3 x\right]$

Integration using t- substitution

Case 1

We know that if t = $tan\frac{1}{2}\theta$, then

$\sin\theta = \frac{2t}{1+t^2}$ and

$cos\theta = \frac{1-t^2}{1+t^2}$

Generally

If t = $tan\frac{1}{2}k\theta$, then

$\sin k\theta = \frac{2t}{1+t^2}$ and

$cosk\theta = \frac{1-t^2}{1+t^2}$

Example 23

Find

(a) $\int cosecxdx$

Solution

Let t = $\tan\frac{1}{2}x$

$dt = sec^2\frac{1}{2}xdx$

$2dt = (1 + t^2)dx$

$dx = \frac{2}{1+t^2}dt$

$$\int cosecxdx = \int \frac{1}{sinx} dx = \int \frac{1+t^2}{2t} . \frac{2}{1+t^2} dt$$
$$= \int \frac{1}{t} dt = Int + c$$

$$\therefore \int cosecxdx = In\left(tan\frac{1}{2}x\right) + c$$

(b) $\int secxdx$

Solution

Let t = $\tan\frac{1}{2}x$

$dt = sec^2\frac{1}{2}xdx$

$2dt = (1 + t^2)dx$

$dx = \frac{2}{1+t^2}dt$

$$\int secxdx = \int \frac{1}{cosx} dx = \int \frac{1+t^2}{1-t^2} . \frac{2}{1+t^2} dt$$
$$= \int \frac{2}{1-t^2} dt = \int \frac{2}{(1+t)(1-t)} dt$$

Let $\frac{2}{(1+t)(1-t)} = \frac{A}{1+t} + \frac{B}{1-t}$

2= A(1 – t) + B(1 + t)

Putting t = 1, B = 1

Putting t = -1; A= 1

=>$\frac{2}{(1+t)(1-t)} = \frac{1}{1+t} + \frac{1}{1-t}$

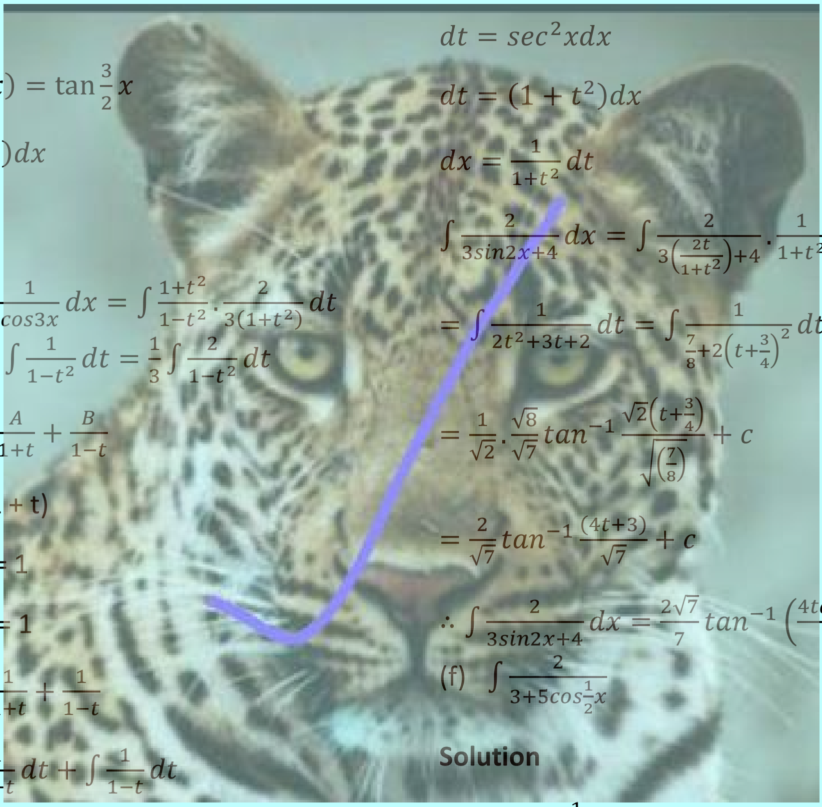
$\int \frac{2}{1-t^2} dt = \int \frac{1}{1+t} dt + \int \frac{1}{1-t} dt$

$= In(1+t) - In(1-t) + c = In\left(\frac{1+t}{1-t}\right) + c$

$\therefore \int secxdx = In\left(\frac{1+tan\frac{1}{2}x}{1-tan\frac{1}{2}x}\right) + c$

(c) $\int sec3xdx$

Solution



Let t = $tan\frac{1}{2}(3x) = \tan\frac{3}{2}x$

$2dt = 3(1+t^2)dx$

$dx = \frac{2}{3(1+)} dt$

$\int sec3xdx = \int \frac{1}{cos3x} dx = \int \frac{1+t^2}{1-t^2} \cdot \frac{2}{3(1+t^2)} dt$

$= \frac{2}{3} \int \frac{1}{1-t^2} dt = \frac{1}{3} \int \frac{2}{1-t^2} dt$

Let$\frac{2}{(1+t)(1-t)} = \frac{A}{1+t} + \frac{B}{1-t}$

2= A(1 – t) + B(1 + t)

Putting t = 1, B = 1

Putting t = -1; A= 1

=>$\frac{2}{(1+t)(1-t)} = \frac{1}{1+t} + \frac{1}{1-t}$

$\int \frac{2}{1-t^2} dt = \int \frac{1}{1+t} dt + \int \frac{1}{1-t} dt$

$= In(1+t) - In(1-t) + c = In\left(\frac{1+t}{1-t}\right) + c$

$\therefore \int sec3xdx = \frac{1}{3} In\left(\frac{1+tan\frac{1}{2}x}{1-tan\frac{1}{2}x}\right) + c$

(d) $\int \frac{1}{3-2cosx} dx$

Solution

Let t = $tan\frac{1}{2}x$

$dt = sec^2\frac{1}{2}xdx$

$2dt = (1+t^2)dx$

$dx = \frac{2}{1+t^2} dt$

$\int \frac{1}{3-2cosx} dx = \int \frac{1}{3-2\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2}{1+t^2} dt = 2\int \frac{1}{1+5t^2} dt$

$= \frac{2}{\sqrt{5}} tan^{-1}(\sqrt{5}t) + c = \frac{2\sqrt{5}}{5} tan^{-1}(\sqrt{5}t) + c$

$\therefore \int \frac{1}{3-2cosx} dx = \frac{2\sqrt{5}}{5} tan^{-1}\left(\sqrt{5}tan\frac{1}{2}x\right) + c$

(e) $\int \frac{2}{3sin2x+4} dx$

Solution

Let t = $tanx$

$dt = sec^2xdx$

$dt = (1+t^2)dx$

$dx = \frac{1}{1+t^2} dt$

$\int \frac{2}{3sin2x+4} dx = \int \frac{2}{3\left(\frac{2t}{1+t^2}\right)+4} \cdot \frac{1}{1+t^2} dt$

$= \int \frac{1}{2t^2+3t+2} dt = \int \frac{1}{\frac{7}{8}+2\left(t+\frac{3}{4}\right)^2} dt$

$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{8}}{\sqrt{7}} tan^{-1} \frac{\sqrt{2}\left(t+\frac{3}{4}\right)}{\sqrt{\left(\frac{7}{8}\right)}} + c$

$= \frac{2}{\sqrt{7}} tan^{-1} \frac{(4t+3)}{\sqrt{7}} + c$

$\therefore \int \frac{2}{3sin2x+4} dx = \frac{2\sqrt{7}}{7} tan^{-1}\left(\frac{4tanx+3}{\sqrt{7}}\right) + c$

(f) $\int \frac{2}{3+5cos\frac{1}{2}x}$

Solution

Let t = $tan\frac{1}{4}x$

$dt = \frac{1}{4} sec^2\frac{1}{4}xdx$

$dt = \frac{1}{4}(1+t^2)dx$

$dx = \frac{4}{1+t^2} dt$

(g) $\int \frac{2}{3+5cos\frac{1}{2}x} = \int \frac{2}{3+5\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{4}{1+t^2} dt$

$= \int \frac{2}{4-t^2} dt = \int \frac{2}{(2+t)(2-t)} dt$

Let $\frac{2}{(2+t)(2-t)} = \frac{A}{(2+t)} + \frac{B}{2-t}$

2= A(2 – t)+ B(2 + t)

Putting t = 2; B= $\frac{1}{2}$

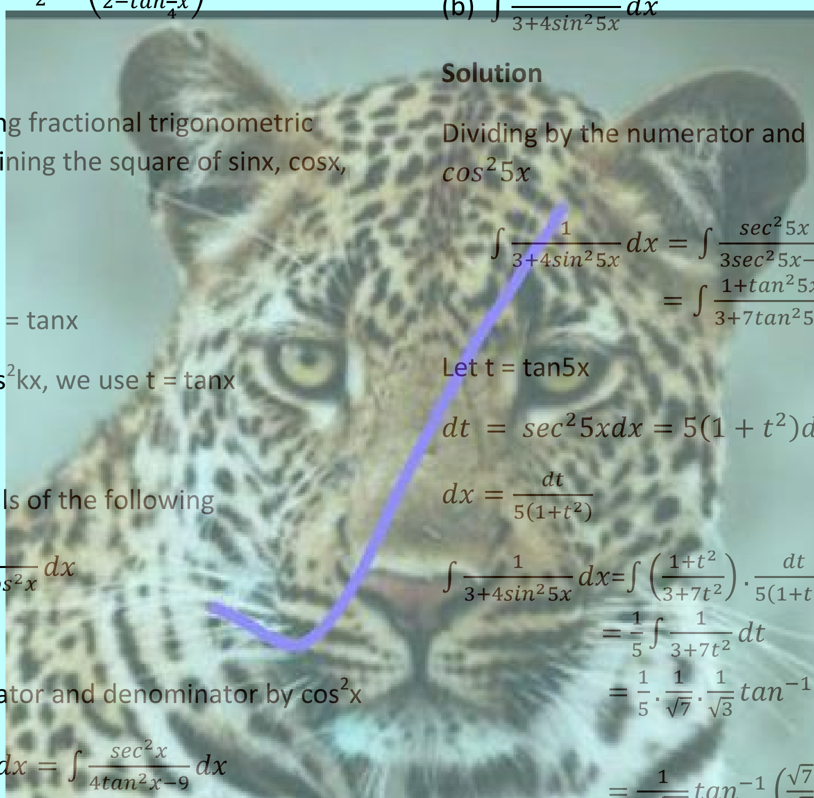
Putting t = -2; A= $\frac{1}{2}$

$$\int \frac{2}{3+5cos\frac{1}{2}x}dx = \frac{1}{2}\int\frac{1}{2+t}dt + \frac{1}{2}\int\frac{1}{2-t}dt$$

$$= \frac{1}{2}In(2+t) - \frac{1}{2}In(2-t) + c$$

$$= \frac{1}{2}In\left(\frac{2+t}{2-t}\right) + c$$

$$\therefore \int \frac{2}{3+5cos\frac{1}{2}x}dx = \frac{1}{2}In\left(\frac{2+tan\frac{1}{4}x}{2-tan\frac{1}{4}x}\right) + c$$



Case II

When integrating fractional trigonometric functions containing the square of sinx, cosx, etc.

We use the

t-substitution, t = tanx

For sin^2kx or cos^2kx, we use t = tanx

Example 24

Find the integrals of the following

(a) $\int \frac{1}{4sin^2x-9cos^2x}dx$

Solution

Dividing numerator and denominator by cos^2x

$$\int \frac{1}{4sin^2x-9cos^2x}dx = \int\frac{sec^2x}{4tan^2x-9}dx$$

$$= \int\frac{1+tan^2x}{4tan^2x-9}dx$$

Let t = tanx

$$dt = sec^2xdx = (1+t^2)dx$$

$$dx = \frac{dt}{(1+t^2)}$$

$$\int\frac{1+tan^2x}{4tan^2x-9}dx = \int\frac{1+t^2}{4t^2-9}.\frac{dt}{(1+t^2)}$$

$$= \int\frac{1}{(2t+3)(2t-3)}dt$$

Let $\frac{1}{(2t+3)(2t-3)} = \frac{A}{(2t+3)} + \frac{B}{(2t-3)}$

1 = A(2t – 3) + B(2t + 3)

Putting t = $\frac{3}{2}$; B = $\frac{1}{6}$

Putting t = $-\frac{3}{2}$; A = $-\frac{1}{6}$

$$⇨ \int\frac{1}{(2t+3)(2t-3)}dt$$
$$= \frac{1}{6}\int\frac{B}{(2t-3)}dt + \frac{1}{6}\int\frac{1}{(2t+3)}dt$$
$$= \frac{1}{6}.\frac{1}{2}In(2t-3) - \frac{1}{6}.\frac{1}{2}In(2t+3) + c$$
$$= \frac{1}{12}In\left(\frac{2t-3}{2t+3}\right) + c$$

$$\int\frac{1}{4sin^2x-9cos^2x}dx = \frac{1}{12}In\left(\frac{2tanx-3}{2tanx+3}\right) + c$$

(b) $\int\frac{1}{3+4sin^25x}dx$

Solution

Dividing by the numerator and denominator cos^25x

$$\int\frac{1}{3+4sin^25x}dx = \int\frac{sec^25x}{3sec^25x-4}dx$$
$$= \int\frac{1+tan^25x}{3+7tan^25x}dx$$

Let t = tan5x

$$dt = sec^25xdx = 5(1+t^2)dx$$

$$dx = \frac{dt}{5(1+t^2)}$$

$$\int\frac{1}{3+4sin^25x}dx = \int\left(\frac{1+t^2}{3+7t^2}\right).\frac{dt}{5(1+t^2)}$$
$$= \frac{1}{5}\int\frac{1}{3+7t^2}dt$$
$$= \frac{1}{5}.\frac{1}{\sqrt{7}}.\frac{1}{\sqrt{3}}tan^{-1}\left(\frac{\sqrt{7}}{\sqrt{3}}t\right) + c$$
$$= \frac{1}{5\sqrt{21}}tan^{-1}\left(\frac{\sqrt{7}}{\sqrt{3}}tan5x\right) + c$$

$$\therefore \int\frac{1}{3+4sin^25x}dx = \frac{\sqrt{21}}{105}tan^{-1}\left(\frac{\sqrt{21}}{3}tan5x\right) + c$$

(c) $\int\frac{sin^23x}{1+cos^23x}dx$

Solution

Dividing numerator and denominator by cos^23x

$$\int\frac{sin^23x}{1+cos^23x}dx = \int\frac{tan^23x}{sec^23x+1}dx = \int\frac{tan^23x}{2+tan^23x}dx$$

Let t = tan3x

$$dt = 3sec^23xdx = 3(1+t^2)dx$$

$$dx = \frac{dt}{3(1+t^2)}$$

$$\int \frac{tan^2 3x}{2+tan^2 3x} dx = \int \frac{t^2}{2+t^2} \cdot \frac{dt}{3(1+t^2)}$$
$$= \frac{1}{3} \int \frac{t^2}{(2+t^2)(1+t^2)} dt$$

Let $\frac{t^2}{(2+t^2)(1+t^2)} = \frac{Ax+B}{(2+t^2)} + \frac{Cx+D}{(1+t^2)}$

By equating coefficients and solving simultaneously

A = 2, C = -1, B = D = 0

$$\int \frac{sin^2 3x}{1+cos^2 3x} dx = \int \frac{2}{(2+t^2)} dt - \int \frac{1}{(1+t^2)} dt$$

$$= \frac{1}{3}\left[\frac{2}{\sqrt{2}} tan^{-1}\left(\frac{t}{\sqrt{2}}\right) - tan^{-1} t\right] + c$$

$$= \frac{1}{3}\left[\frac{2}{\sqrt{2}} tan^{-1}\left(\frac{tan3x}{\sqrt{2}}\right) - tan^{-1}(tan3x)\right] + c$$

$$= \frac{\sqrt{2}}{3} tan^{-1}\left(\frac{tan3x}{\sqrt{2}}\right) - \frac{1}{3} tan^{-1}(tan3x) + c$$

(d) $\int \frac{1}{cos2x - 3sin^2 x} dx$

Solution

$$\int \frac{1}{cos2x - 3sin^2 x} dx = \int \frac{1}{1-5sin^2 x} dx$$

Dividing the numerator and denominator by $cos_2 x$

$$\int \frac{1}{sec^2 - 5tan^2 x} dx$$

Let t = tanx

$$dt = sec^2 x dx = (1+t^2) dx$$

$$dx = \frac{dt}{(1+t^2)}$$

$$\int \frac{1}{1-5sin^2 x} dx = \int \frac{1}{1-4t^2} \cdot \frac{dt}{(1+t^2)}$$
$$= \int \frac{1}{1-4t^2} dt = \int \frac{1}{(1+2t)(1-2t)} dt$$

Let $\frac{1}{(1+2t)(1-2t} = \frac{A}{1+2t} + \frac{B}{1-2t}$

1 = A(1 – 2t) + B (1 + 2t)

Putting t = $\frac{1}{2}$; B = $\frac{1}{2}$

Putting t = $-\frac{1}{2}$; A = $\frac{1}{2}$

$$\int \frac{1}{(1+2t)(1-2t)} dt = \frac{1}{2} \int \frac{dt}{1+2t} + \frac{1}{2} \int \frac{dt}{1-2t}$$

$$= \frac{1}{2}\left[\frac{1}{2} In(1+2t) - \frac{1}{2} In(1-2t)\right] + c$$

$$= \frac{1}{4} In\left(\frac{1+2t}{1-2t}\right) + c$$

$$\therefore \int \frac{1}{cos2x - 3sin^2 x} dx = \frac{1}{4} In\left(\frac{1+2tanx}{1-2tanx}\right) + c$$

Revision exercise 14

1. Integrate the following

(a) $\int \frac{4}{3+5sinx} dx$ $\left[\frac{3tan\frac{1}{2}x+1}{tan\frac{1}{2}x+3} + c\right]$

(b) $\int \frac{1}{4+5cosx} dx$ $\left[\frac{1}{3} In\left(\frac{3+tan\frac{1}{2}x}{3-tan\frac{1}{2}x}\right) + c\right]$

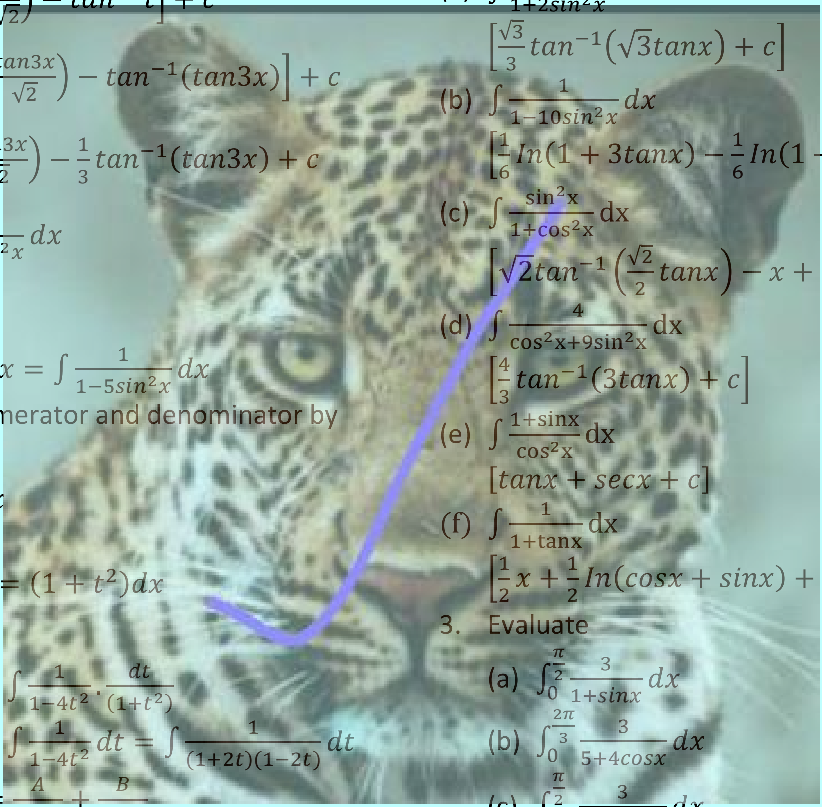
(c) $\int \frac{1}{1+5sin2x} dx$ $\left[-\frac{1}{1+tanx} + c\right]$

(d) $\int \frac{4}{5+3cos\frac{1}{2}x} dx$ $\left[tan^{-1}\left(\frac{1}{2} tan\frac{1}{4}x\right) + c\right]$

(e) $\int \frac{4}{2+sin\frac{1}{2}x} dx$

$\left[\frac{8\sqrt{3}}{9} tan^{-1}\left(\frac{2tan\frac{1}{4}x+1}{\sqrt{3}}\right) + c\right]$

2. Integrate each of the following



(a) $\int \frac{1}{1+2sin^2 x} dx$

$\left[\frac{\sqrt{3}}{3} tan^{-1}(\sqrt{3}tanx) + c\right]$

(b) $\int \frac{1}{1-10sin^2 x} dx$

$\left[\frac{1}{6} In(1+3tanx) - \frac{1}{6} In(1-3tanx)\right] + c$

(c) $\int \frac{\sin^2 x}{1+\cos^2 x} dx$

$\left[\sqrt{2} tan^{-1}\left(\frac{\sqrt{2}}{2} tanx\right) - x + c\right]$

(d) $\int \frac{4}{\cos^2 x + 9\sin^2 x} dx$

$\left[\frac{4}{3} tan^{-1}(3tanx) + c\right]$

(e) $\int \frac{1+\sin x}{\cos^2 x} dx$

$[tanx + secx + c]$

(f) $\int \frac{1}{1+tanx} dx$

$\left[\frac{1}{2}x + \frac{1}{2} In(cosx + sinx) + c\right]$

3. Evaluate

(a) $\int_0^{\frac{\pi}{2}} \frac{3}{1+sinx} dx$ [3]

(b) $\int_0^{\frac{2\pi}{3}} \frac{3}{5+4cosx} dx$ $\left[\frac{1}{3}\pi\right]$

(c) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{3}{4+5cosx} dx$ $[2In2]$

(d) $\int_0^{\frac{\pi}{2}} \frac{5}{3sinx+4cosx} dx$ $[In6]$

Integration of special cases involving splitting the numerator

Case 1

When a fractional integrand with quadratic denominator expressed in the form of $\frac{f(x)}{g(x)}$ is such that g(x) cannot be factorized or written in simple partial fractions, it is normally very useful to express it as a fraction by splitting the numerator.

i.e. Numerator = A(derivative of denominator + B

Example 25

Find the integral of each of the following

(a) $\int \frac{2x-1}{4x^2+3} dx$

Solution

Numerator=$A\left[\frac{d}{dx}(4x^2+3)\right]+B$

2x – 1 =A(8x)+B

Putting x = 0, B= -1

Putting x = 1, A = $\frac{1}{4}$

$\int \frac{2x-1}{4x^2+3} dx = \frac{1}{4}\int \frac{8x}{4x^2+3} dx - \int \frac{1}{4x^2+3} dx$

$= \frac{1}{4} In(4x^2+3) - \frac{\sqrt{3}}{6} tan^{-1}\left(\frac{2\sqrt{3}}{3}x\right) + c$

(b) $\int \frac{2x+3}{x^2+2x+10} dx$

Solution

Numerator=$A\left[\frac{d}{dx}(x^2+2x+10)\right]+B$

2x +3 =A(2x+2)+B

Putting x = -1, B= 1

Putting x = 0, A = 1

$\int \frac{2x+3}{x^2+2x+10} dx$

$= \frac{1}{4}\int \frac{2x+2}{x^2+2x+10} dx + \int \frac{1}{x^2+2x+10} dx$

$= In(x^2+2x+10) + \int \frac{1}{9+(x+1)^2} dx$

$= In(x^2+2x+10) + \frac{1}{3} tan^{-1}\left(\frac{x+1}{3}\right) + c$

(c) $\int \frac{x}{x^2+3x+5} dx$

Solution

Numerator=$A\left[\frac{d}{dx}(x^2+3x+5)\right]+B$

x =A(2x+3)+B

Putting x = $-\frac{3}{2}$, B= $-\frac{3}{2}$

Putting x = 0, A = $-\frac{1}{2}$

$\int \frac{x}{x^2+3x+5} dx$

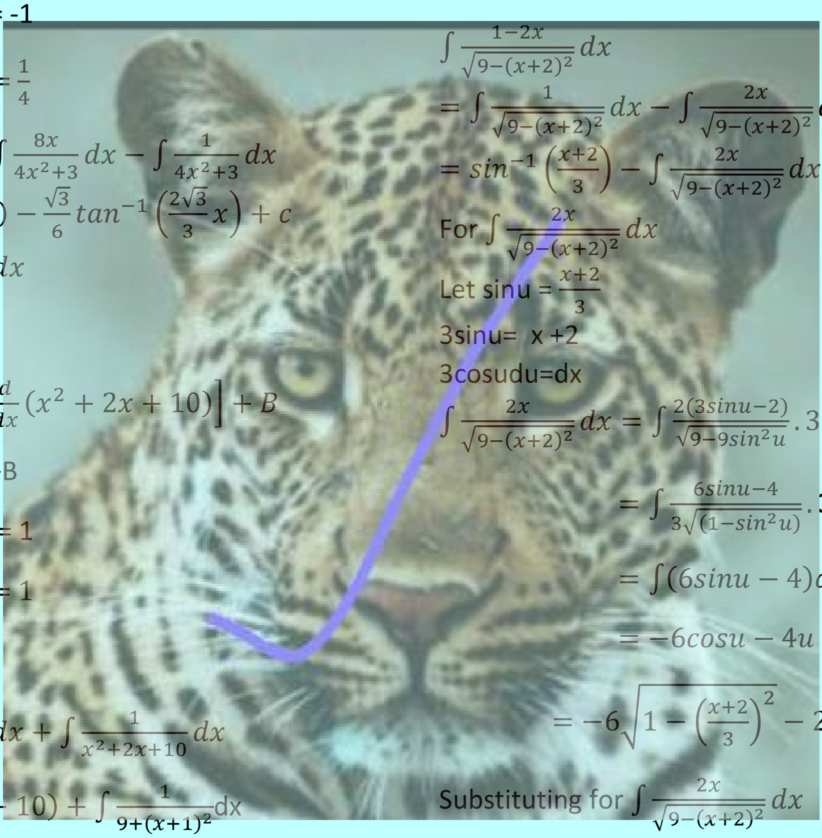
$= \frac{1}{2}\int \frac{2x+3}{x^2+3x+5} dx - \frac{3}{2}\int \frac{1}{x^2+3x+5} dx$

$= \frac{1}{2} In(x^2+3x+5) - \frac{3}{2}\int \frac{1}{\frac{11}{4}+\left(x+\frac{3}{2}\right)^2} dx$

$= \frac{1}{2} In(x^2+3x+5) - \frac{3}{\sqrt{11}} tan^{-1}\left(\frac{2x+3}{\sqrt{11}}\right) + c$

(d) $\int \frac{1-2x}{9-(x+2)^2} dx$

Solution



$\int \frac{1-2x}{\sqrt{9-(x+2)^2}} dx$

$= \int \frac{1}{\sqrt{9-(x+2)^2}} dx - \int \frac{2x}{\sqrt{9-(x+2)^2}} dx$

$= sin^{-1}\left(\frac{x+2}{3}\right) - \int \frac{2x}{\sqrt{9-(x+2)^2}} dx$

For $\int \frac{2x}{\sqrt{9-(x+2)^2}} dx$

Let sinu = $\frac{x+2}{3}$

3sinu= x +2

3cosudu=dx

$\int \frac{2x}{\sqrt{9-(x+2)^2}} dx = \int \frac{2(3sinu-2)}{\sqrt{9-9sin^2u}} . 3cosudu$

$= \int \frac{6sinu-4}{3\sqrt{(1-sin^2u)}} . 3cosudu$

$= \int (6sinu-4) du$

$= -6cosu - 4u + c$

$= -6\sqrt{1-\left(\frac{x+2}{3}\right)^2} - 2sin^{-1}\left(\frac{x+2}{3}\right) + c$

Substituting for $\int \frac{2x}{\sqrt{9-(x+2)^2}} dx$

$\int \frac{1-2x}{\sqrt{9-(x+2)^2}} dx$

$= sin^{-1}\left(\frac{x+2}{3}\right) + -6\sqrt{1-\left(\frac{x+2}{3}\right)^2} - 4sin^{-1}\left(\frac{x+2}{3}\right) + c$

$= 5sin^{-1}\left(\frac{x+2}{3}\right) + 6\sqrt{1-\left(\frac{x+2}{3}\right)^2} + c$

Case II

When finding the integral of fractional trigonometric function expressed in the form $\int \frac{acosx+bsinx}{c\,cosx+dsinx}$, a, b, c and d are constants, we split the numerator as:

Numerator = A(derivative of denominator)+ (denominator)

Example 26

1. Find

(a) $\int \frac{2cosx+9sinx}{3cosx+sinx}dx$

Solution

Let 2cosx+ 9sinx =A$\frac{d}{dx}$(3cosx+sinx)+B(3cosx+sinx)

2cosx+ 9sinx =A(-3sinx+cosx)+B(3cosx+sinx)

2cosx+ 9sinx =(A+3B)cosx +(-3A+B)sinx

Equating coefficients:

For cosx: A+3B = 2 ………………… (i)

For sinx: -3A+B = 9…………………(ii)

Solving Eqn. (i) and Eqn. (ii) simultaneously

A= $-\frac{5}{2}$ and B = $\frac{3}{2}$

⇨ $\int \frac{2cosx+9sinx}{3cosx+sinx}dx$

$=-\frac{5}{2}\int \frac{-3sinx+cosx}{3cosx+sinx}dx + \frac{3}{2}\int \frac{3cosx+sinx}{3cosx+sinx}$

$=-\frac{5}{2}$In(3cosx+sinx)+ $\frac{3}{2}x + c$

(b) $\int \frac{3sinx}{4cosx-sinx}dx$

Solution

Let 3sinx = A$\frac{d}{dx}$(4cosx–sinx) + B(4cosx–sinx)

3sinx = A(-4sinx–cosx) + B (4cosx – sinx)

3sinx = (-A+B)cosx +(-4A-B)sinx

Equating coefficients

For cosx: -A + 4B = 0 ………………… (i)

For sin x: -4A- B = 3 ………………… (ii)

Solving Eqn. (i) and Eqn. (ii) simultaneously

A= $-\frac{12}{17}$ and B= $-\frac{3}{17}$

$\int \frac{3sinx}{4cosx-sinx}dx$

$= -\frac{12}{17}\int \frac{-4sinx-cosx}{4cosx-sinx}dx - \frac{3}{17}\int \frac{4cosx-sinx}{4cosx-sinx}dx$

$= -\frac{12}{17}In(4cosx - sinx) - \frac{3}{17}x$+c

2. Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{cosx-sinx}{3cosx+2sinx}dx$

Solution

Let 3sinx = A$\frac{d}{dx}$(3cosx+2sinx) + B(3cosx+2sinx)

cosx-sinx = A(-3sinx+2cosx) + B (3cosx +2sinx)

cosx- sinx = (2A+3B)cosx +(-3A+2B)sinx

Equating coefficients

For cosx: 2A + 3B = 1 ………………… (i)

For sin x: -3A+2B = -1 ………………… (ii)

Solving Eqn. (i) and Eqn. (ii) simultaneously

A= $\frac{5}{13}$ and B= $-\frac{1}{13}$

$\int \frac{cosx-sinx}{3cosx+2sinx}dx$

$=\frac{5}{13}\int \frac{2cosx-3sinx}{3cosx+2sinx}dx + \frac{1}{13}\int \frac{3cosx+2sinx}{3cosx+2sinx}dx$

$=\frac{5}{13}In(3cosx + 2sinx) + \frac{1}{13}x + c$

$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{cosx-sinx}{3cosx+2sinx}dx$

$=\left[\frac{5}{13}In(3cosx + 2sinx) + \frac{1}{13}x + c\right]_{\frac{1}{6}\pi}^{\frac{1}{2}\pi}$

$=\left[\frac{5}{13}In\left(3cos\frac{\pi}{2} + 2sin\frac{\pi}{2}\right) + \frac{1}{13}.\frac{\pi}{2}\right] - \left[\frac{5}{13}In\left(3cos\frac{\pi}{6} + 2sin\frac{\pi}{6}\right) + \frac{1}{13}.\frac{\pi}{6}\right]$

$=\left[\frac{5}{13}In2 + \frac{\pi}{26}\right] - \left[\frac{5}{13}In\frac{2+\sqrt{3}}{2} + \frac{\pi}{78}\right]$

$=\frac{5}{13}In\left(\frac{4}{2+3\sqrt{3}}\right) + \frac{\pi}{39}$

Revision exercise 15

1. Integrate each of the following

(a) $\int \frac{x+2}{x^2+2x+4}dx$

$\left[\frac{1}{2}In(x^2 + 2x + 4) + \frac{\sqrt{3}}{3}tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + c\right]$

(b) $\int \frac{x}{x^2-x+3}dx$

$\left[\frac{1}{2}In(x^2 - x + 3) + \frac{2}{\sqrt{11}}tan^{-1}\left(\frac{2x-1}{\sqrt{11}}\right) + c\right]$

(c) $\int \frac{2(x+1)}{x^2+4x+8}dx$

$\left[In(x^2+4x+8) - tan^{-1}\left(\frac{x+2}{2}\right) + c\right]$

(d) $\int \frac{5x+7}{x^2+4x+8}dx$

$\left[\frac{5}{2}In(x^2+4x+8) - \frac{3}{2}tan^{-1}\left(\frac{x+2}{2}\right) + c\right]$

2. Integrate the following

(a) $\int \frac{cosx-2sinx}{3cosx+4sinx}dx$

$\left[\frac{2}{5}In(4sinx+3cox) - \frac{1}{5}x + c\right]$

(b) $\int \frac{cosx}{2cosx-sinx}dx$

$\left[-\frac{1}{5}In(2cosx - sinx) + \frac{2}{5}x + c\right]$

(c) $\int \frac{cosx}{cosx-2sinx}dx$

$\left[-\frac{2}{5}In(cosx - 2sinx) - \frac{1}{5}x + c\right]$

(d) $\int \frac{2cosx+sinx}{4cosx+3sinx}dx$

$\left[-\frac{14}{15}In(4cosx+3sinx) - \frac{11}{5}x + c\right]$

Revision exercise 16: general topical revision questions

1. Find

(a) $\int sinxdx$ $\left[xsin^{-1}x + \sqrt{1-x^2} + c\right]$

(b) $\int xsec^2xdx$ $[xtanx + Incosx + c]$

(c) $\int \frac{x^2}{\sqrt{1-x^2}}dx$ $\left[\sqrt{1-x^2}\left(\frac{-2-x^2}{3}\right) + c\right]$

(d) $\int In(x^2-4)dx$

$\left[xIn(x^2-4) - 2x + 2\left(In\frac{x+2}{x-2}\right) + c\right]$

(e) $\int \frac{dx}{3-2cosx}dx$ $\left[\frac{2}{\sqrt{5}}tan^{-1}\left(\sqrt{5}tan\frac{x}{2}\right) + c\right]$

(f) $\int 3^{\sqrt{2x-1}}dx$

$\left[\frac{3^{\sqrt{2x-1}}}{In3}\left(\sqrt{2x-1} - \frac{1}{In3}\right) + c\right]$

(g) $\int sin^2xdx$ $\left[\frac{1}{2}\left(x - \frac{1}{2}sin2x\right) + c\right]$

(h) $\int tan^3xdx$ $\left[\frac{1}{2}tan^2x - Incosx + c\right]$

(i) $\int \frac{4x^2}{\sqrt{1-x^6}}dx$ $\left[\frac{4}{3}sin^{-1}(x^3) + c\right]$

(j) $\int \frac{x^2}{x^4-1}dx$ $\left[\frac{1}{4}In\left(\frac{x-1}{x+1}\right) + \frac{1}{2}tan^{-1}x + c\right]$

(k) $\int \frac{2x}{\sqrt{x^2+4}}dx$ $\left[2\sqrt{x^2+4} + c\right]$

(l) $\int xInxdx$ $\left[\frac{x^2}{2}Inx - \frac{x^2}{4} + c\right]$

(m) $\int x^3e^{x^4}dx$ $\left[\frac{1}{4}e^{x^4} + c\right]$

(n) $\int \frac{1}{1+sin^2x}$ $\left[\frac{\sqrt{2}}{2}tan^{-1}(\sqrt{2}tanx) + c\right]$

(o) $\int Inxdx$ $[x(Inx-1) + c]$

(p) $\int x^2sin2xdx$

$\left[-\frac{1}{2}x^2cos2x + \frac{1}{2}xsin2x + \frac{1}{4}cos2x + c\right]$

(q) $\int Inx^2dx$ $[2x(Inx-1) + c]$

(r) $\int \frac{dx}{e^x-1}$ $[In(1-e^{-x}) + c]$

(s) $\int \frac{x^2}{(1+x^2)^{\frac{1}{2}}}dx$ $\left[\frac{1}{3}(1+x^2)^{\frac{1}{2}}(x^2-2) + c\right]$

(t) $\int \frac{dx}{1-cosx}$ $\left[-cot\left(\frac{x}{2}\right) + c\right]$

(u) $\int \frac{x^4-x^3+x^2+1}{x^3+x}dx$

$\left[\frac{x^2}{2} - x + Inx + tan^{-1}x - \frac{1}{2}In(1+x^2) + c\right]$

(v) $\int \frac{sin^{-1}2x}{\sqrt{1-4x^2}}dx$ $\left[\left(\frac{sin^{-1}2x}{2}\right)^2 + c\right]$

(w) $\int x(1-x^2)^{\frac{1}{2}}dx$ $\left[\frac{1}{3}(1-x^2)^{\frac{3}{2}} + c\right]$

(x) $\int \frac{1+\sqrt{x}}{2\sqrt{x}}dx$ $\left[\sqrt{x} + \frac{x}{2} + c\right]$

(y) $\int x^2e^xdx$ $[x^2e^x - 2xe^x + 2e^x + c]$

(z) $\int \frac{dx}{x^2\sqrt{(25-x^2)}}dx$ $\left[-\frac{1}{25}\left(\frac{5\sqrt{25-x^2}}{x^2}\right) + C\right]$

2. Evaluate

(a) $\int_0^{\frac{\pi}{2}} xcos^2xdx$ [0.3669]

(b) $\int_1^{\sqrt{3}} (x+tanx)dx$ [1.0003]

(c) $\int_0^{\frac{\pi}{2}} sin2xcosxdx$ $\left[\frac{2}{3}\right]$

(d) $\int_1^3 \frac{x^2+1}{x^3+4x^2+3x}dx$ [0.3489]

(e) $\int_0^1 \frac{x}{\sqrt{1+x}}dx$ [0.3905]

(f) $\int_0^{\frac{\pi}{2}} \frac{cosx}{1+sin^2x}dx$ [0.7854]

(g) $\int_0^{\frac{\pi}{6}} sinxsin3xdx$ [0.1083]

(h) $\int_0^1 \frac{x^3}{x^2+1}dx$ [0.15345]

(i) $\int_0^{\sqrt{\frac{\pi}{2}}} 2xcosx^2dx$ [1]

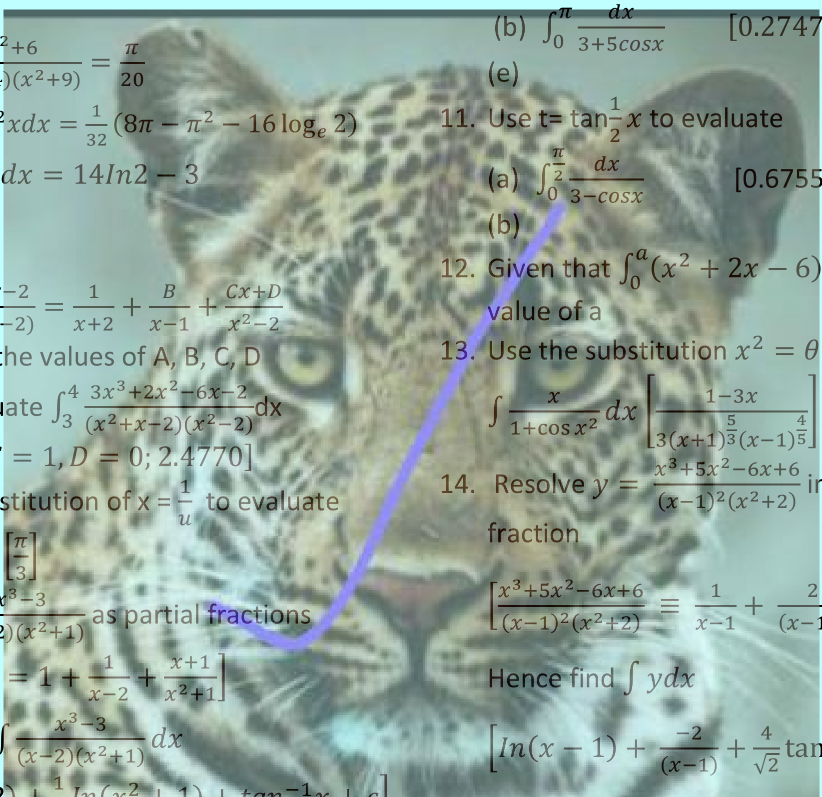
(j) $\int_0^2 \frac{8x}{x^2-4x-12}dx$ [1.05]

(k) $\int_0^{\frac{\pi}{2}} \frac{dx}{1+sinx+cosx}$ [In2]

(l) $\int_0^{\frac{\pi}{2}} sin2xcosxdx$ $\left[\frac{2}{3}\right]$

(m) $\int_4^6 \frac{dx}{x^2-2x-3}$ [0.1905]

(n) $\int_0^{\frac{\pi}{2}} xsin^22xdx$ $\left[\frac{\pi^2}{16}\right]$



(o) $\int_1^{\sqrt{3}} \frac{x^2}{\sqrt{x^4-x^2}} dx$ $[2]$

(p) $\int_1^3 \frac{3x^2+4x+1}{x^3+2x^2+x} dx$ $[In12]$

(q) $\int_0^{\frac{\pi}{2}} x^2 sinxdx$ $[\pi - 2]$

(r) $\int_0^1 xe^{2x} dx$ $[2.0973]$

(s) $\int_{\frac{\pi}{3}}^{\pi} xsinxdx$ $[2.7992]$

(t) $\int_{\frac{1}{2}}^1 10x\sqrt{(1 - x^2)}\, dx$ $[2.165]$

(u) $\int_0^{\frac{\pi}{2}} sin5xcos3xdx$ $\left[\frac{1}{2}\right]$

(v) $\int_0^{\frac{\sqrt{3}}{2}} \frac{dx}{9+4x^2} dx$ $\left[\frac{\pi}{36}\right]$

3. Show that

(a) $\int_0^1 \frac{x^2+6}{(x^2+4)(x^2+9)} = \frac{\pi}{20}$

(b) $\int_0^{\frac{\pi}{2}} xtan^2xdx = \frac{1}{32}(8\pi - \pi^2 - 16\log_e 2)$

(c) $\int_2^4 xInxdx = 14In2 - 3$

(d)

4. Given that
$\frac{3x^3+2x^2-6x-2}{(x^2+x-2)(x^2-2)} = \frac{1}{x+2} + \frac{B}{x-1} + \frac{Cx+D}{x^2-2}$
Determine the values of A, B, C, D
Hence evaluate $\int_3^4 \frac{3x^3+2x^2-6x-2}{(x^2+x-2)(x^2-2)}$dx
$[A = B = C = 1, D = 0; 2.4770]$

5. Use the substitution of x = $\frac{1}{u}$ to evaluate
$\int_1^2 \frac{dx}{x\sqrt{x^2-1}}$ $\left[\frac{\pi}{3}\right]$

6. Express $\frac{x^3-3}{(x-2)(x^2+1)}$ as partial fractions
$\left[\frac{x^3-3}{(x-2)(x^2+1)} = 1 + \frac{1}{x-2} + \frac{x+1}{x^2+1}\right]$
Hence find $\int \frac{x^3-3}{(x-2)(x^2+1)} dx$
$\left[x + In(x-2) + \frac{1}{2}In(x^2+1) + tan^{-1}x + c\right]$

7. Express f(x) = $\frac{2x^2-x+14}{(4x^2-1)(x+3)}$ in partial fraction
$\left[\frac{2x^2-x+14}{(4x^2-1)(x+3)} = \frac{-3}{2x+1} + \frac{2}{2x-1} + \frac{1}{x+3}\right]$
Hence evaluate $\int_1^3 f(x)dx$ $[0.7440]$

8. Using the substitution 2x+1 = u, find
$\int_0^1 \frac{xdx}{(2x+1)^2}$ $\left[\frac{1}{18}\right]$

9. Express

(a) f(x) = $\frac{6x}{(x-2)(x+4)^2}$ in partial fraction
$\left[\frac{6x}{(x-2)(x+4)^2} = \frac{1}{3(x-2)} - \frac{1}{3(x+4)} + \frac{4}{(x+4)^2}\right]$
Hence evaluate $\int f(x)dx$
$\left[\frac{1}{3}In\left(\frac{x-2}{x+4}\right) - \frac{4}{(x+4)} + c\right]$

(b) f(x) = $\frac{3x^2+x+1}{(x-2)(x+1)^2}$ in partial fraction
$\frac{3x^2+x+1}{(x-2)(x+1)^2}$
$= \frac{5}{9(x-2)} - \frac{5}{9(x+2)} + \frac{4}{3(x+1)^2} - \frac{1}{(x+1)^3}$
Hence evaluate
$\int_3^4 \frac{3x^2+x+1}{(x-2)(x+1)^2} dx$ $[0.317]$

(c)

10. Using the substitution x = 3sinθ, evaluate

(a) $\int_0^3 \sqrt{\left(\frac{3+x}{3-x}\right)}\, dx$ $[7.7125]$

(b) $\int_0^{\pi} \frac{dx}{3+5cosx}$ $[0.2747]$

(e)

11. Use t= tan$\frac{1}{2}x$ to evaluate

(a) $\int_0^{\frac{\pi}{2}} \frac{dx}{3-cosx}$ $[0.6755]$

(b)

12. Given that $\int_0^a (x^2 + 2x - 6)\, dx = 0$, find the value of a [a=-6]

13. Use the substitution $x^2 = \theta$ to find
$\int \frac{x}{1+\cos x^2} dx$ $\left[\frac{1-3x}{3(x+1)^{\frac{5}{3}}(x-1)^{\frac{4}{5}}}\right]$

14. Resolve $y = \frac{x^3+5x^2-6x+6}{(x-1)^2(x^2+2)}$ into partial fraction
$\left[\frac{x^3+5x^2-6x+6}{(x-1)^2(x^2+2)} \equiv \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{4}{(x^2+2)}\right]$
Hence find $\int ydx$
$\left[In(x-1) + \frac{-2}{(x-1)} + \frac{4}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + c\right]$

15. Express f(x) = $\frac{1}{x^2(x-1)}$ in partial fraction
$\left[\frac{1}{x^2(x-1)} = -\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-1}\right]$
Hence evaluate $\int_2^3 f(x)dx$ $[0.12102]$

Application of integration

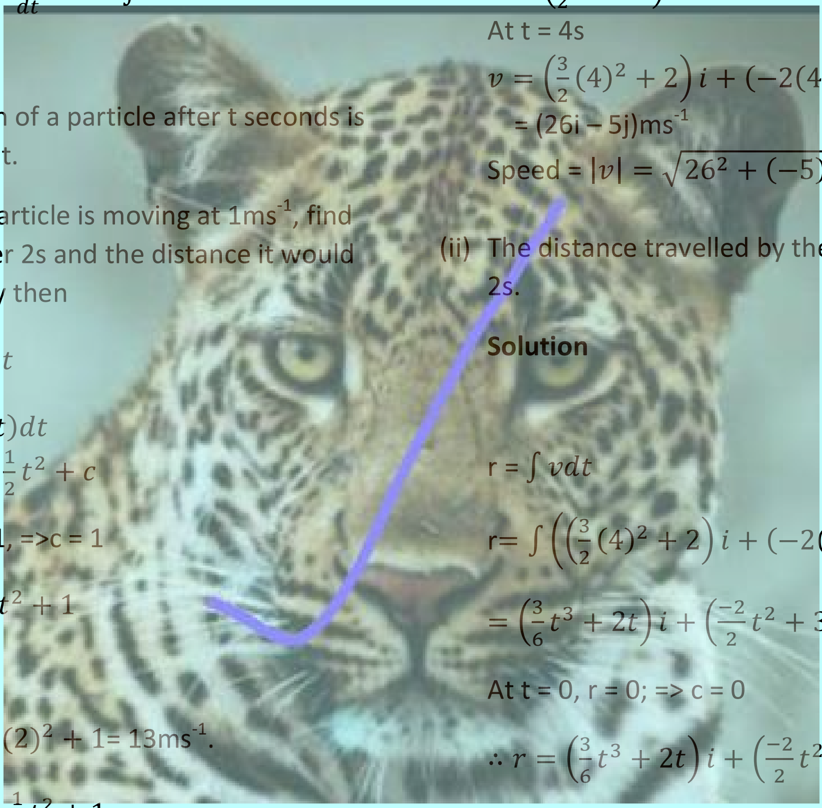
Like differentiation, integration has a wide spectrum of application, some of which are discussed below

Acceleration, velocity, displacement

Given the acceleration, a, of a particle, its velocity, v and displacement, s can be computed as long as the initial values are known.

Acceleration, a = $\frac{dv}{dt}$ => v = $\int adt$

Also, velocity v = $\frac{ds}{dt}$ => s = $\int vdt$



Example 27

The acceleration of a particle after t seconds is given by a = 5 + t.

If initially, the particle is moving at 1ms^{-1}, find the velocity after 2s and the distance it would have covered by then

Given $\frac{dv}{dt} = 5 + t$

⇨ $dv = (5 + t)dt$

$v = 5t + \frac{1}{2}t^2 + c$

Whet t = 0, v = 1, =>c = 1

$\therefore v = 5t + \frac{1}{2}t^2 + 1$

When t = 2s

$v = 5(2) + \frac{1}{2}(2)^2 + 1$= 13ms^{-1}.

And $\frac{ds}{dt} = 5t + \frac{1}{2}t^2 + 1$

$ds = \left(5t + \frac{1}{2}t^2 + 1\right)dt$

s = $\frac{5}{2}t^2 + \frac{1}{6}t^3 + t + c$

when t = 0, s= 0 => c = 0

$\therefore$s = $\frac{5}{2}t^2 + \frac{1}{6}t^3 + t$

At t =2s

s = $\frac{5}{2}(2)^2 + \frac{1}{6}(2)^3 + 2$=13$\frac{1}{3}$m

Example 28

A particle with a velocity (2i+3j)ms^{-1} is accelerated uniformly at the rate of (3ti – 2j)ms^{-1} from the origin. Find

(i) The speed reached by the particle at t = 4s.

Solution

Given a= 3ti – 2j

v = $\int adt$ =$\int(3\text{ti} - 2\text{j})dt$

=$\frac{3}{2}t^2 i - 2tj + c$

At t= 0, 2i+3j

c =2i+3j

By substitution

$v = \left(\frac{3}{2}t^2 + 2\right)i + (-2t + 3)j$

At t = 4s

$v = \left(\frac{3}{2}(4)^2 + 2\right)i + (-2(4) + 3)j$

= (26i – 5j)ms^{-1}

Speed = $|v| = \sqrt{26^2 + (-5)^2}$ =26.5ms^{-1}

(ii) The distance travelled by the particle after 2s.

Solution

r = $\int vdt$

r= $\int\left(\left(\frac{3}{2}(4)^2 + 2\right)i + (-2(4) + 3)j\right)dt$

$= \left(\frac{3}{6}t^3 + 2t\right)i + \left(\frac{-2}{2}t^2 + 3t\right)j$ + c

At t = 0, r = 0; => c = 0

$\therefore r = \left(\frac{3}{6}t^3 + 2t\right)i + \left(\frac{-2}{2}t^2 + 3t\right)j$

At t = 2

$\therefore r = \left(\frac{8}{2} + 4\right)i + (-4 + 6)j$

$|r| = \sqrt{8^2 + 2^2}$ =8.25m

Hence the distance = 8.25m

Example 29

A particle has initial position of (7i+5j)m. the particle moves with constant velocity of (ai+bi)ms-1 and 3s later its position is (10i – j)m. fins the values of a and b.

Solution

Given v = ai +bj

r =$\int vdt = \int(ai + bi)dt + c$

$= ati + btj + c$

at t = 0; r = c =(7i+5j)m

$\therefore r = (at + 7)i + (bt + 5)j$

After 3s

$10i - j = (3a + 7)i + (3b + 5)j$

Equating corresponding vectors

For i: 10 = 3a+7 => a = 1

For j: -1 = 3b +5 => b = -2

$\therefore a = 1 \text{ and } b = -2$

Example 30

A particle of mass 2kg, initially at rest at (0, 0, 0) is acted on by a force $\begin{pmatrix} 2t \\ 2t \\ 4t \end{pmatrix} N$. Find

(i) its acceleration at time t

from F = Ma

$\begin{pmatrix} 2t \\ 2t \\ 4t \end{pmatrix} = 2a \Rightarrow a = \begin{pmatrix} t \\ t \\ 2t \end{pmatrix}$

(ii) its velocity after 3s

velocity v =$\int adt = \int \begin{pmatrix} t \\ t \\ 2t \end{pmatrix} dt$

v=$\begin{pmatrix} \frac{t^2}{2} \\ \frac{t^2}{2} \\ t^2 \end{pmatrix} + c$

at t = 0, v = 0 => c= 0

$\therefore$v=$\begin{pmatrix} \frac{t^2}{2} \\ \frac{t^2}{2} \\ t^2 \end{pmatrix}$

At t = 3s

$v = \frac{9}{2}i + \frac{9}{2}j + 9k$

(iii) the distance of the particle travelled after 3s.

r= $\int vdt$ =$\int \left(\frac{t^2}{2}i + \frac{t^2}{2}j + t^2k\right) dt$

= $\left(\frac{t^3}{6}i + \frac{t^3}{6}j + \frac{1}{3}t^3k\right) + c$

At t = 0, r = 0 => c = 0

$\therefore r = \left(\frac{t^3}{6}i + \frac{t^3}{6}j + \frac{1}{3}t^3k\right)$

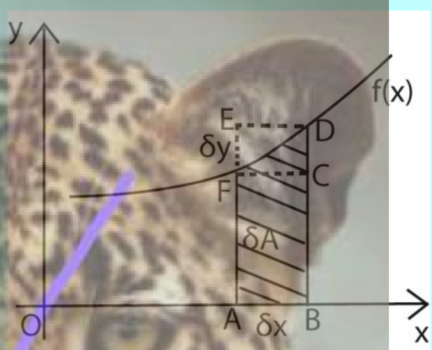
At t = 3

$r = \left(\frac{3^3}{6}i + \frac{3^3}{6}j + \frac{1}{3}.3^3k\right) = \left(\frac{9}{2}i + \frac{9}{2}j + 9k\right)$

$|r| = \sqrt{\left(\frac{9}{2}\right)^2 + \left(\frac{9}{2}\right)^2 + 9^2}$ =11.02m

Area under a curve

If the area under the curve y = f(x) for α≤ x ≤β is required, a small strip can be used for analysis



Suppose the shaded region is δA, the area of the shaded strip lies between areas of the rectangles ABCF and AVDE.

i.e. Area of ABCF ≤ δA ≤ ABDE.

$y\delta x \leq \delta A \leq (y + \delta y)dx$

Dividing by δx

$y \leq \frac{\delta A}{\delta x} \leq (y + \delta y)$

$\lim_{\delta x \to 0} \frac{\delta A}{\delta x} \to \frac{dA}{dx}$ and $\delta y \to 0$

Hence $\frac{dA}{dx} = y$

Integrating both sides with respect to x

$\int \frac{dA}{dx} dx = \int ydx$

Now for the interval $\alpha \leq x \leq \beta$

$A = \int_{\alpha}^{\beta} ydx \ Or \ A = \int_{\alpha}^{\beta} f(x)dx$

Note: when finding the area under the curve, it is advisable that you sketch the curve first in order to establish the required region.

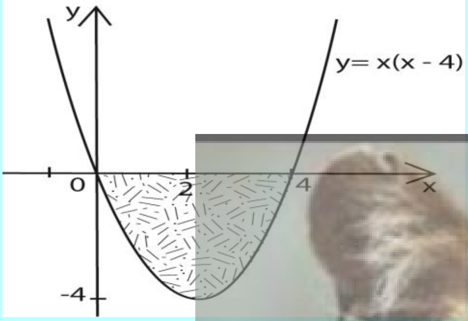
Area between the curve and the x-axis

Example 31

(i) Find the area enclosed by y = x(x – 4) and x-axis

Solution

By sketching the graph y = x(x – 4) with the x-axis we have



Area required = $\int_0^4 x(x-4)dx$

$=\int_0^4 x^2 - 4xdx$

$=\left[\frac{x^3}{3} - 2x^2\right]_0^4$

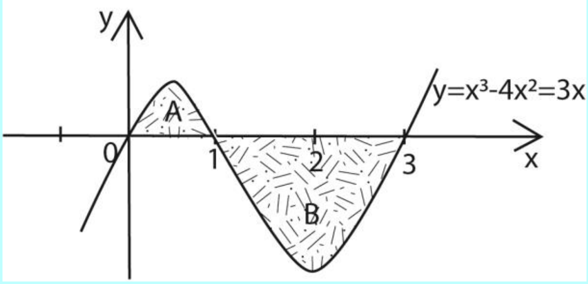
$=\frac{64}{3} - 32 = \frac{-32}{3}$

Hence the area under the curve is $\frac{32}{3}$ $sq.units$ (- sign indicates that the area is below the x-axis).

(ii) Find the area enclosed by the curve $y = x^3 – 4x^2 + 3x$ and the x-axis from x = 0 and x = 3

Solution

By sketching the graph $y = x^3 – 4x^2 + 3x$ with the x-axis we have



Required area = A+ B

Area A $=\int_0^1 (x^3 - 4x^2 + 3x)\, dx$

$=\left[\frac{x^4}{4} - \frac{4x^3}{3} + \frac{3x^2}{2}\right]_0^1$

$=\left(\frac{1}{4} - \frac{4}{3} + \frac{3}{2}\right) - (0) = \frac{5}{12}$

Area B $=\int_1^3 (x^3 - 4x^2 + 3x)\, dx$

$=\left[\frac{x^4}{4} - \frac{4x^3}{3} + \frac{3x^2}{2}\right]_1^3$

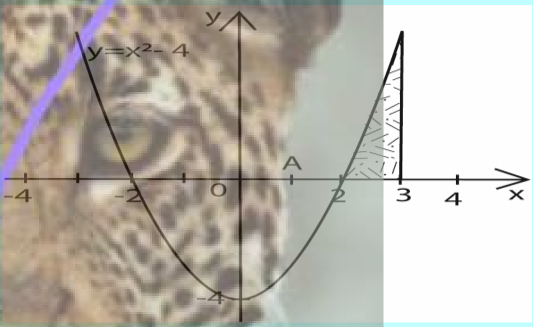
$=\left(\frac{81}{4} - 36 + \frac{27}{2}\right) - \left(\frac{1}{4} - \frac{4}{3} + \frac{3}{2}\right) = -\frac{8}{3}$

Area$=\frac{5}{12} + \frac{8}{3} = \frac{37}{12} sq.units$

(iii) Find the area between $y=x^2 – 4$, the x-axis and line x = 3.

Solution

By sketching the graph of $y=x^2 – 4$ with the x-axis, we have



Required $= \int_2^3 (x^2 - 4)\, dx$

$= \frac{1}{3}\left[x^3 - 4x\right]_2^3$

$= \frac{7}{3}$ sq. units

Area between the curve and the y-axis

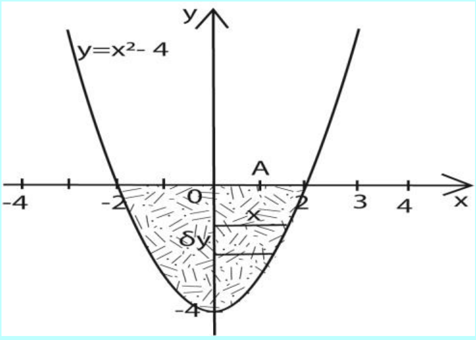
This involves finding the area under the curve with respect to y or by subtracting the area under the curve with the x-axis from the rectangle (s) formed.

Example 32

Find the area enclosed by the curve $y = x^2- 4$ and the y = x2 -4 and y-axis between

(i) y = -4 and y = 0

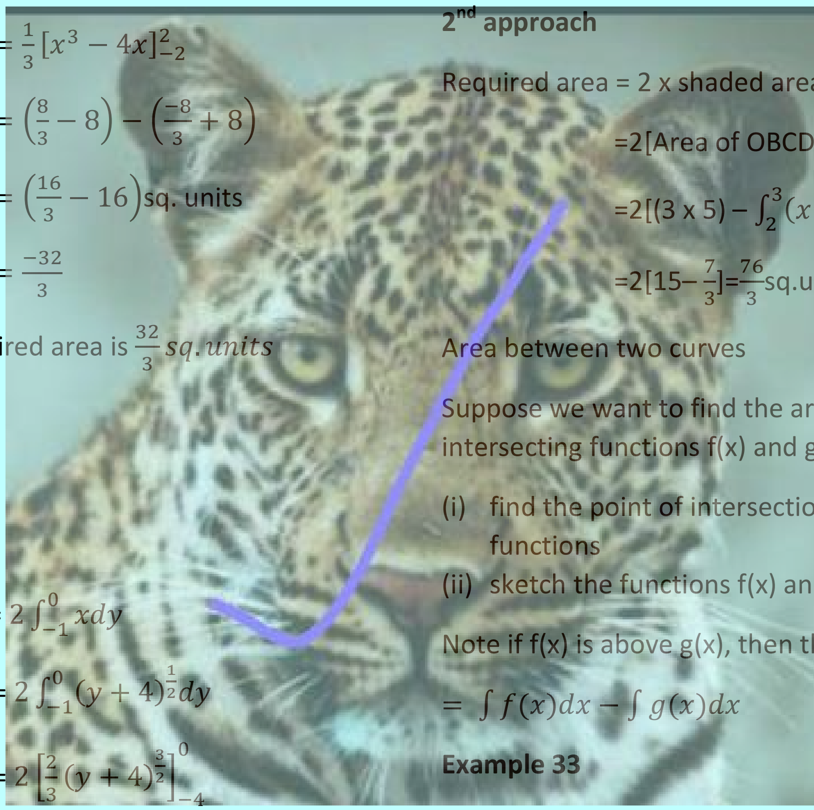
Solution



1st Approach

Integrating with respect to x

Required area $= \int_{-2}^{2}(x^2-4)dx$



$= \frac{1}{3}[x^3-4x]_{-2}^{2}$

$= \left(\frac{8}{3}-8\right)-\left(\frac{-8}{3}+8\right)$

$= \left(\frac{16}{3}-16\right)$sq. units

$= \frac{-32}{3}$

Hence the required area is $\frac{32}{3}\, sq.units$

2nd approach

y = $x^2 - 4$

x = $(y+4)^{\frac{1}{2}}$

Required area = $2\int_{-1}^{0} xdy$

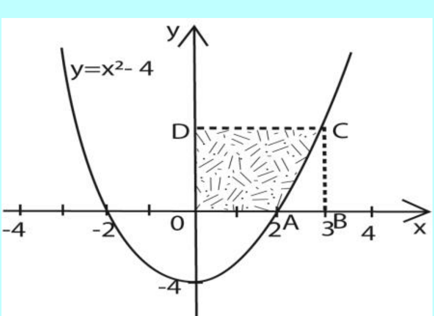
$= 2\int_{-1}^{0}(y+4)^{\frac{1}{2}}dy$

$= 2\left[\frac{2}{3}(y+4)^{\frac{3}{2}}\right]_{-4}^{0}$

$= 2\frac{2}{3}[(8)-(0)]$

$= \frac{32}{3}$ sq. units

(ii) y = 0 and y = 5



1st approach

Required area= 2 x shaded region

Required area = $2\int_{0}^{5} xdy$

$= 2\int_{0}^{5}(y+4)^{\frac{1}{2}}dy$

$= 2\left[\frac{2}{3}(y+4)^{\frac{3}{2}}\right]_{0}^{5}$

$= 2\frac{2}{3}[(2)-(8)]$

$= \frac{76}{3}$ sq. units

2nd approach

Required area = 2 x shaded area

=2[Area of OBCD – area of ABC]

=2[(3 x 5) – $\int_{2}^{3}(x^2-4)dx$]

=2[15– $\frac{7}{3}$]= $\frac{76}{3}$ sq.units

Area between two curves

Suppose we want to find the area between two intersecting functions f(x) and g(x), required it to

(i) find the point of intersection of the functions

(ii) sketch the functions f(x) and g(x)

Note if f(x) is above g(x), then the required area

$= \int f(x)dx - \int g(x)dx$

Example 33

Find the area enclosed between the curves

(a) y = x^2 – 4 and y = 4 – x^2

Solution

Finding the points of intersection

$x^2 - 4 = 4 - x^2$

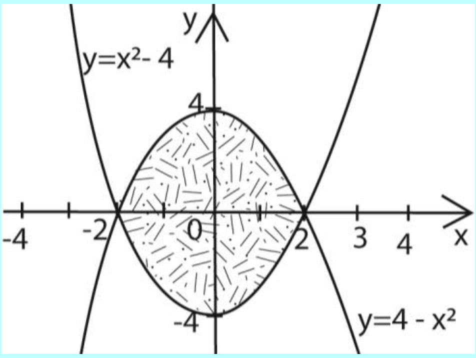
$2x^2 = 8$

x = 2 or x = 2

when x = 2, y = 0

when x = -2, y = 0

The sketch of the functions:



Required area

$= \int_{-2}^{2}[(4-x^2)-(x^2-4)]dx$

$= \int_{-2}^{2}(8-2x^2)dx$

$= \left[8x-\frac{2x^3}{3}\right]_{-2}^{2}$

$= \left(16-\frac{16}{3}\right)-\left(-16+\frac{16}{3}\right)$

$= \frac{32}{3}+\frac{32}{3} = \frac{64}{3}$ sq. units

(b) $y = 2x^2+7x+3$ and $y = 9+4x-x^2$

Solution

Finding the points of intersection

$2x^2+7x+3 = 9+4x-x^2$

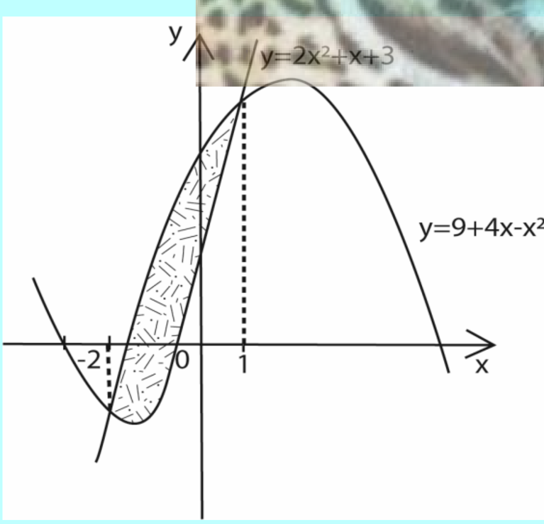
$3x^2+3x-6 = 0$

$(x+2)(x-1) = 0$

x = -2 of x = 1

When x = -2, y = -3

When x = 1, y = 12



Required area

$= \int_{-2}^{1}[(9+4x-x^2)-(2x^2+7x+3)]\,dx$

$= \int_{-2}^{1}(6-3x-3x^2)\,dx$

$= \left[6x-\frac{3x^2}{2}-x^3\right]_{-2}^{1}$

$= \left(6-\frac{3}{2}-1\right)-(-12-6+8)$

$= 13.5\,sq.units$

Example 34

Find the area enclosed between the curve $y = x^2 - x - 3$ and the line $2x + 1$

Solution

Finding the points of intersection

$x^2 - x - 3 = 2x + 1$

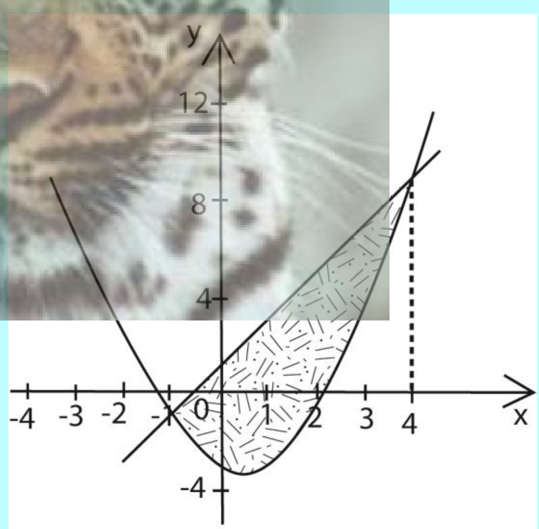
$x^2 - 3x - 4 = 0$

$(x+1)(x-4) = 0$

X = -1 or x = 4

When x = -1, y = -1

When x= 4, y = 9



Area required

$= \int_{-1}^{4}[(2x+1)-(x^2-x-3)]\,dx$

$= \int_{-1}^{4}(4+3x-x^2)dx$

$= \left[4x+\frac{3}{2}x^2-\frac{1}{3}x^3\right]_{-1}^{4}$

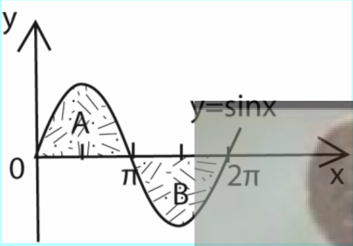
$=\left(16+24-\frac{64}{3}\right)-\left(-4+\frac{3}{3}+\frac{1}{2}\right)$

=20.83 sq.units

Example 35

Find the area enclosed by the curve y = sinx and the x-axis between x = 0 and x = 2π.

Solution



Required area = A + B

$=\int_0^{\pi} sinxdx + \int_{\pi}^{2\pi} sinxdx$

$=[-cosx]_0^{\pi} + [-cosx]_{\pi}^{2\pi}$

$=-(-cos\pi - cos0) - (-cos2\pi - cos\pi)$

=-(-1 – 1)- (-1 – 1)

= 2 +2 = 4sq. units

Volume of a solid of revolution

A solid of revolution is formed when a given area rotates about a fixed axix. Due to the way in which it is formed, it is referred to as solid of revolution.

These bodies have always got axes of symmetry.

The solids formed is subdivided into small cylindrical disks of thickness δx and height y.

Volume of each disk = $\pi y^2 dx$

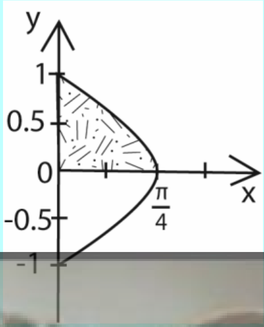
Thereforethe colume of the whole solid of revolution is obtained by rotating through one revolution about the x-axis, the region bounded by the curve y = f(x) and the linex x = a and x =b is given by v = $\int_a^b \pi y^2 dx$

If the rotation is about the y-axis, the volume is given by v =$\int_a^b \pi x^2 dy$

Example 36

(a) Find the volume of revolution when the portion of the curve y = cos2x for 0≤x≤$\frac{\pi}{4}$ is rotated through four right angles about the x-axis.

Solution



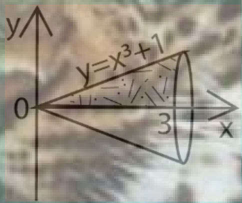
$V=\pi\int_0^{\frac{\pi}{2}} y^2dx = \pi\int_0^{\frac{\pi}{2}} cos^2 2xdx$

$=\pi\int_0^{\frac{\pi}{2}}(1+cos4x)dx$

$=\frac{\pi}{2}\left[x+\frac{1}{4}sin4x\right]_0^{\frac{\pi}{2}}$

$=\frac{1}{8}\pi^2$ cubic units

(b) Find the volume of the area bounded by the curve y = x^3 + 1, the x-axis and limits x = 0 and x = 3 when rotated through four right angles about the x-axis.



$V=\pi\int_0^3 y^2dx = \pi\int_0^3 (x^3+1)^2 dx$

$=\pi\int_0^3 (x^6+2x^3+1)^2 dx$

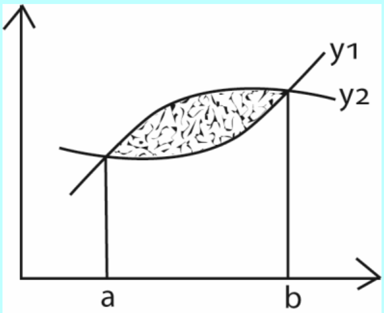
$=\pi\left[\frac{x^7}{7}+\frac{x^4}{2}+x\right]_0^3$

$=\pi\left(\frac{3^7}{7}+\frac{3^4}{2}+3\right)-(0)$

= 1118.25 cubic units.

Rotation the area enclosed between two curves

If we have two curves y_1 and y_2 that enclose some area between a and b as shown below



Now if we rotate this area about the x-axis the volume of the solid formed is given by

$v = \pi \int_a^b [(y_2)^2 - (y_1)^2]\, dx$

Example 36

(a)A cup is madeby rotating the area between $y = x^2$ and $y = x+1$ with $x \geq 0$ about the x-axis. Find the volume of the material needed to make the cup.

Solution

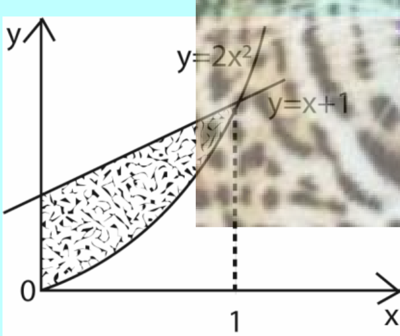
Finding the points of intersection

$2x^2 = x + 1$

$2x^2 – x + 1 = 0$

$(2x+1)(x – 1) = 0$

$x = 1$ since we only need to consider $x \geq 0$.



V = $\pi \int_0^1 [(y+1)^2 - (2x^2)^2] dx$

$= \pi \int_0^1 (x^2 + 2x + 1 - 4x^4) dx$

$= \pi \left[\frac{x^3}{3} + x^2 + x - \frac{4x^5}{5}\right]_0^1$

$= \pi \left(\frac{1}{3} + 1 + 1 - \frac{4}{5}\right) - 0$

$= \frac{23}{15}\pi$ units cubed

Example 37

Find the volume of revolution when the portion of the area between the curves $y = x^2$ and $x = y^2$ is rotated through 360^0 about the x-axis.

Solution

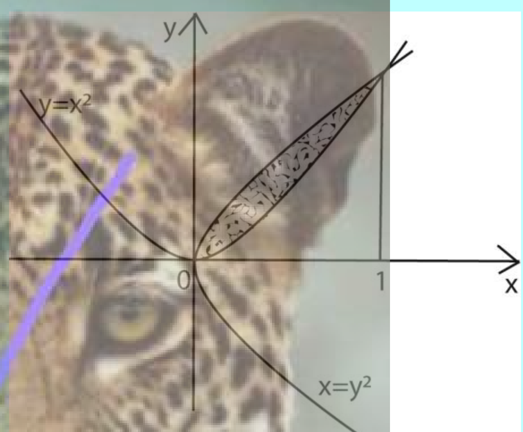
Points of intersection

$x^2 = x^{\frac{1}{2}} => x^4 = x$

$x^4 - x = 0$

$x(x^3 - 1) = 0$

Either x = 0 or x = 1



The volumeof revolution

$= \pi \int_0^1 \left[\left(\sqrt{x}\right)^2 - (x^2)^2\right] dx$

$= \pi \int_0^1 (x - x^4) dx$

$= \pi \left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1$

$= \pi \left[\left(\frac{1}{2} - \frac{1}{5}\right) - 0\right]$

$= \frac{3}{10}\pi$

Example 38

Find the volume generated when the area enclosed by the curve $y = 4 – x^2$ and the line $y = 4 – 2x$ is rotated through 2π.

Solution

Finding the points of intersection

$4 – 2x = 4 – x^2$

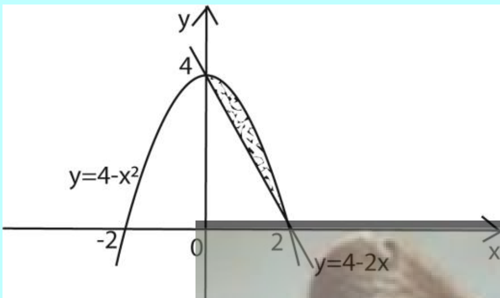
$x^2 - 2x = 0$

$x(x - 2) = 0$

Either x = 0 or x = 2

When x = 0, y = 4

When x = 2, y = 0



Required volume

$=\pi \int_0^2 [(4 - x^2)^2 - (4 - 2x)^2] dx$

$= \pi \int_0^2 [(16 - 8x^2 + x^4) - (16 - 8x - 4x^2)] dx$

$=\pi \int_0^2 (x^4 - 4x^2 + 8x)\, dx$

$=\pi \left[\frac{x^5}{5} - \frac{4x^3}{3} + 4x^2\right]_0^2$

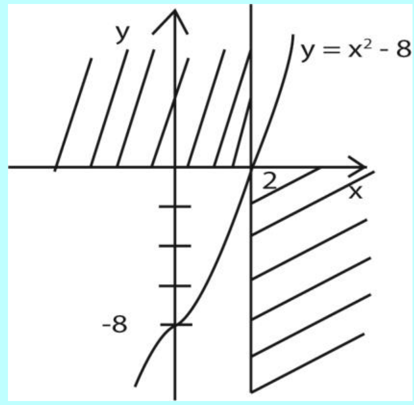
$=\frac{176}{15}\pi = 36.86$ cubic units

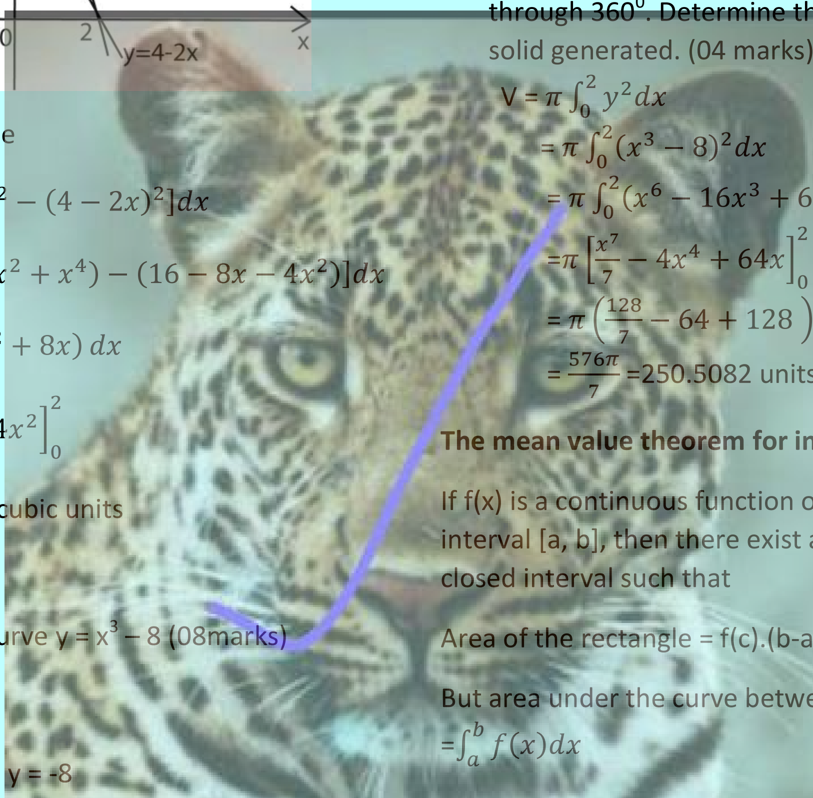
Example 39

(a) Sketch the curve $y = x^3 - 8$ (08marks)

$y = x^3 - 8$
Intercepts
When x = 0, y = -8
When y = 0, x = 2
(x, y) = (2, 0)
Turning point: $\frac{dy}{dx} = 3x^2$
3x2 = 0
x = 0
$\frac{d^2y}{dx^2} = 6x$
$\frac{d^2y}{dx^2} = 0, x = 0$
Point of reflection= (0, 8)

	x < 2	x > 2
y	-	+





(b) The area enclosed by the curve in (a), the y-axis and x-axis is rotate about the line y = 0 through 360^0. Determine the volume of the solid generated. (04 marks)

$V = \pi \int_0^2 y^2 dx$

$= \pi \int_0^2 (x^3 - 8)^2 dx$

$= \pi \int_0^2 (x^6 - 16x^3 + 64) dx$

$=\pi \left[\frac{x^7}{7} - 4x^4 + 64x\right]_0^2$

$= \pi \left(\frac{128}{7} - 64 + 128\right)$

$= \frac{576\pi}{7} = 250.5082$ units^3

The mean value theorem for integrals

If f(x) is a continuous function on the closed interval [a, b], then there exist a number c in the closed interval such that

Area of the rectangle = f(c).(b-a)

But area under the curve between a and b $= \int_a^b f(x)dx$

Equating the two

$\int_a^b f(x)dx - f(c).(b - a)$

Dividing both sides by (b – a)

$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$

Where f(c) is the height of the rectangle

This height is the average value of the function over the interval in the question.

Hence the mean value of f(x) over a closed interval (a, b) is given by

M.V $=\frac{1}{b-a}\int_a^b f(x)dx$

Example 40

Find the mean value of y= x^2 + 2 for x = 1 and x = 4.

Solution

M.V $=\frac{1}{4-1}\int_1^4 (x^2+2)dx$

$=\frac{1}{3}\int_1^4 (x^2+2)dx$

$=\frac{1}{3}\left[\frac{x^3}{3}+2x\right]_1^4$

$=\frac{1}{3}\left[\left(\frac{64}{3}+8\right)-\left(\frac{1}{3}+2\right)\right]=9$

Example 41

Find the mean value of

y = $\frac{1}{1+sin^2\theta}$ for $0\le\theta\le\frac{\pi}{4}$

Solution

M.V $=\frac{1}{\frac{\pi}{4}-0}\int_0^{\frac{\pi}{4}}\frac{1}{1+sin^2\theta}dx$

$=\frac{4}{\pi}\int_0^{\frac{\pi}{4}}\frac{sec^2\theta}{sec^2\theta+tan^2\theta}dx$

$=\frac{4}{\pi}\int_0^{\frac{\pi}{4}}\frac{1+tan^2\theta}{1+2tan^2\theta}dx$

Let t = tan θ => dt = $sec^2\theta d\theta = (1+t^2)d\theta$

$d\theta=\frac{dt}{1+t^2}$

Changing limits

When θ = 0, t = 0 and when $\theta=\frac{\pi}{4}$, t = 1

$\therefore M.V=\frac{\pi}{4}\int_0^1\frac{1+t^2}{1+2t^2}.\frac{dt}{1+t^2}$

$=\frac{\pi}{4}\int_0^1\frac{1}{1+2t^2}dt$

$=\frac{\pi}{4}\left[\frac{1}{\sqrt{2}}tan^{-1}\sqrt{2}t\right]_0^1$

$=\frac{2\sqrt{2}}{\pi}tan^{-1}\sqrt{2}$

= 0.86

Example 42

Find the mean value of y = x(4 – x)in the interval where y ≥ 0.

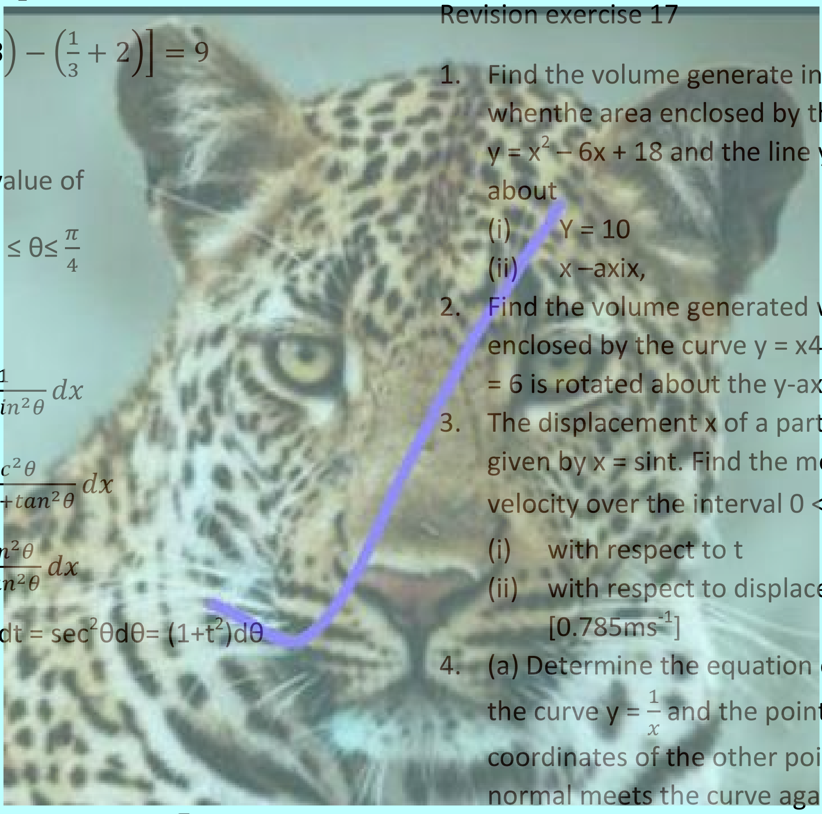
Solution

Given y ≥ 0 => x(4 –x)≥0 (positive)

The solution is 0 ≤ x ≤ 4

M.V $=\frac{1}{4-0}\int_0^4 x(4-x)dx=\frac{1}{4}\int_0^4(4x-x^2)dx$

$=\frac{1}{4}\left[2x^2-\frac{x^3}{3}\right]_0^4=\frac{8}{3}$



Revision exercise 17

1. Find the volume generate in each case whenthe area enclosed by the curve y = x^2 – 6x + 18 and the line y = 10 is rotated about
   (i) Y = 10 [1541π units^3]
   (ii) x –axix, [256π units^3]
2. Find the volume generated when the area enclosed by the curve y = x4 from y = 3 and y = 6 is rotated about the y-axis [6.33π units^3]
3. The displacement x of a particle at time t is given by x = sint. Find the mean value of its velocity over the interval 0 < t < $\frac{\pi}{2}$
   (i) with respect to t [0.637ms^{-1}]
   (ii) with respect to displacement x [0.785ms^{-1}]
4. (a) Determine the equation of the normal to the curve y = $\frac{1}{x}$ and the point x = 2. Find the coordinates of the other point where the normal meets the curve again.
   [2y – 8x+15 =0; $\left(-\frac{1}{8},-8\right)$
   (b) Find the area of the region bounded by the curve y =$\frac{1}{x(2x+1)}$, the x-axis and the lines x = 1 and x = 2. $\left(In\left(\frac{6}{5}\right)\right)$
5. A shell is formed by rotating the portion of the parabola y2= 4x for which 0 ≤ x ≤ 1 through two right angles about its axis. Find
   (i) the volume of the solid formed [2π]
   (ii) the area of the base of the solid formed [4π units^2]

6. Show that the tangents at (-1,3) and (1,5)on the curve $y = 2x^2 + x + 2$ passes through the origin. Find the area enclosed between the curve and these two tangents $\left[\frac{4}{3}\right]$

7. Sketch the curve $y = x - \frac{8}{x^2}$ for $x > 0$, showing any a symptotes. Find the area enclosed by the x-axis, the line x = 4 and the curve $x - \frac{8}{x^2}$. [10 sq. units]
If this area is now rotated about the x-axis through 3600, determine the volume of the solid generated, correct to 3 significant figures. [42.1 cubic units]

8. Show that the tangents to the curve $4 - 2x - 2x^2$ at points 9-1, 4) and $\left(\frac{1}{2}, 2\frac{1}{2}\right)$respectively passes through the point $\left(-\frac{1}{4}, 5\frac{1}{2}\right)$. Calculate the area of the curve enclosed between the curve and the x-axis. [9sq.units]

9. (i) find the Cartesian equation of the curve given parametrically by
$x = \frac{1+t}{1-t}$, $y = \frac{2t^2}{1-t}$ $\left[y = \frac{(x-1)^2}{x+1}\right]$
(ii) sketch the curve
(iii) find the area enclosed between the curve and the line y = 1 [1.955sq.units]

10. Given the curve y = sin3x, find the
(a)(i) the value of $\frac{dy}{dx}$ at the point $\left(\frac{\pi}{2}, 0\right)$
(ii) equation of the tangent to the curve at this point [y= 3x+π)

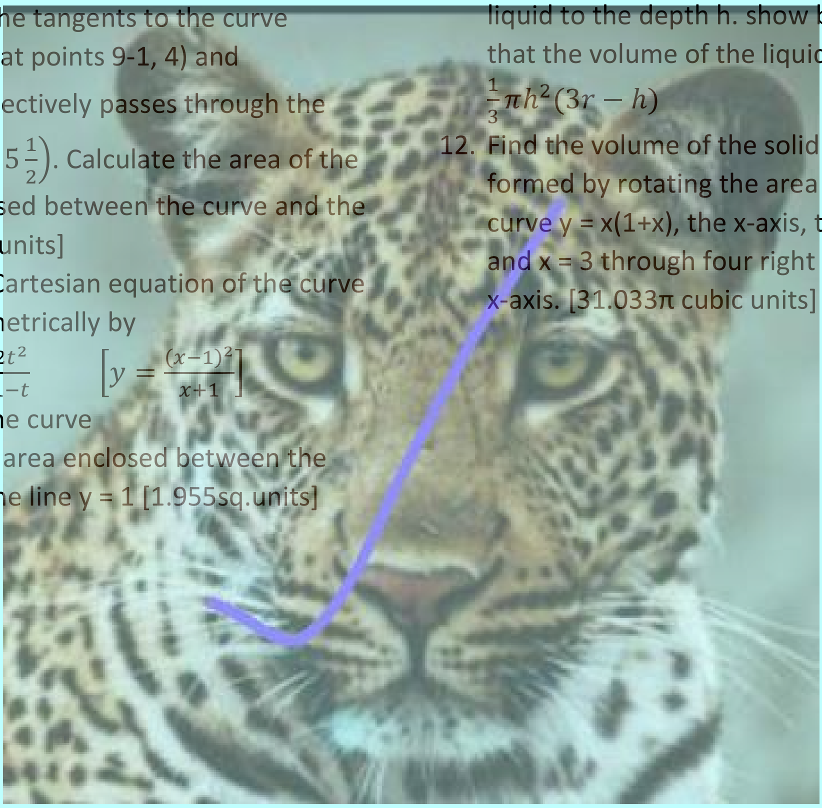
(b) (i) sketch the curve y = sin3x

(ii) Calculate the area bounded by the tangent in (a)(i) above, the curve and y-axis

[0.9783sq. units]

11. A hemisphericalbowl of internal radius r is fixed with its rim horizontal and contains a liquid to the depth h. show by integration that the volume of the liquid in the bowl is $\frac{1}{3}\pi h^2(3r - h)$

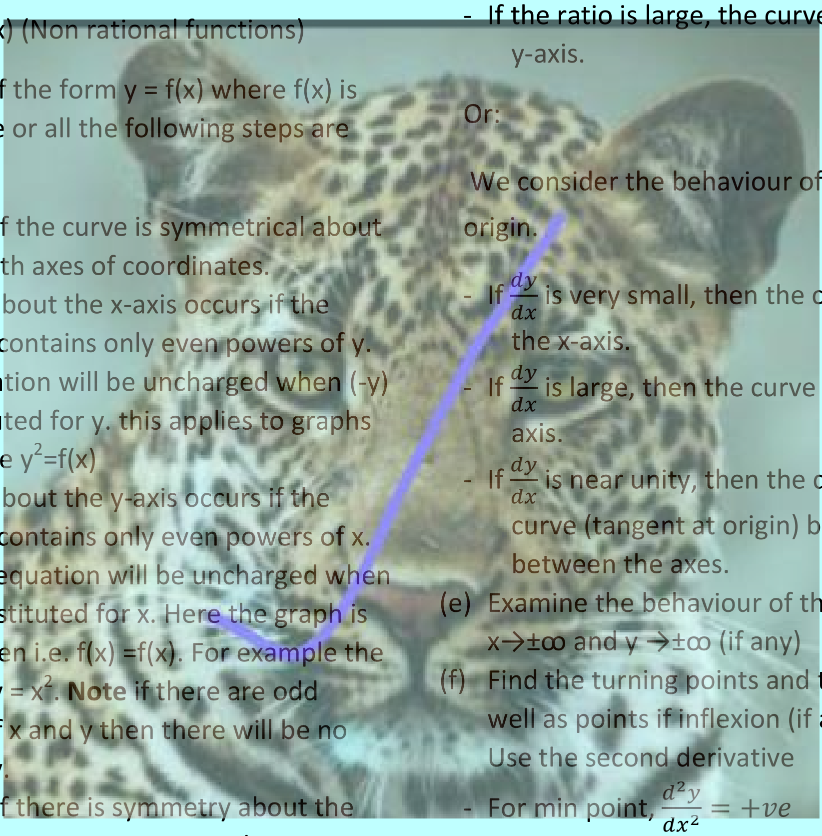
12. Find the volume of the solid of revolution formed by rotating the area enclosed by the curve y = x(1+x), the x-axis, the lines x = 2 and x = 3 through four right anglesabout the x-axis. [31.033π cubic units]



Curve sketching (A-level)

The procedures of curve sketching depend on the nature of the curve to be sketched

Graphs of y = f(x) (Non rational functions)



For any graph of the form y = f(x) where f(x) is not linear, some or all the following steps are followed.

(a) Determine if the curve is symmetrical about either or both axes of coordinates.
  - Symmetry about the x-axis occurs if the equation contains only even powers of y. here equation will be uncharged when (-y) is substituted for y. this applies to graphs of the type $y^2=f(x)$
  - Symmetry about the y-axis occurs if the equation contains only even powers of x. Here the equation will be uncharged when (-x) is substituted for x. Here the graph is said to even i.e. f(x) =f(x). For example the graph of $y = x^2$. **Note** if there are odd powers of x and y then there will be no symmetry.

(b) Determine if there is symmetry about the origin. Here symmetry occurs when a change in the sign of x causes a change in the sign of y without altering its numerical value.

(c) Find the intercepts i.e. the curve cuts the x-axis at a point when y = 0 and cuts the y-axis at the point when x = 0.

(d) The curve passes through the origin if (x, y)= (0, 0)

The behaviour of the neighbour of the curve through the origin is studied by considering the ratio of $\frac{y}{x}$.

- If this ratio is small, the curve keeps close to x-axis near the origin.
- If the ratio is unity, the direction of the curve bisects the angle between the axes.
- If the ratio is large, the curve keeps near the y-axis.

Or:

We consider the behaviour of $\frac{dy}{dx}$ near the origin.

- If $\frac{dy}{dx}$ is very small, then the curve lies near the x-axis.
- If $\frac{dy}{dx}$ is large, then the curve lies near the y-axis.
- If $\frac{dy}{dx}$ is near unity, then the direction of the curve (tangent at origin) bisects the angle between the axes.

(e) Examine the behaviour of the function as x→±∞ and y →±∞ (if any)

(f) Find the turning points and their nature as well as points if inflexion (if any)
Use the second derivative
  - For min point, $\frac{d^2y}{dx^2} = +ve$
  - For max point, $\frac{d^2y}{dx^2} = -ve$
  - Point of inflexion, $\frac{d^2y}{dx^2} = 0$

Example 1

(a) Sketch the graph of y= 5 + 4x – x^2.
Steps taken
  - Finding intercepts
    x – intercept; y = 0
    0 = 5 + 4x – x^2.
    5 + 5x –x – x^2 = 0
    5(1 + x) – x(1 + x) =0
    (5 – x)(1 + x)

Either 5 – x = 0; x = 5

Or 1 + x = 0; x = -1

Hence the curve cuts the x-axis at point (-1, 0) and (0, 5)

y – intercept, when x = 0, y = 5

hence the curve cuts the y-axis at point (0, 5)

- As x→+∞, y →-∞ and x→-∞, y →+∞
- Finding turning point

$\frac{dy}{dx} = 4 - 2x$

At turning point $\frac{dy}{dx} = 0$

2x – 4 = 0; x = 2

When x = 2; y= 5 + 4(2) – $(2)^2$= 9

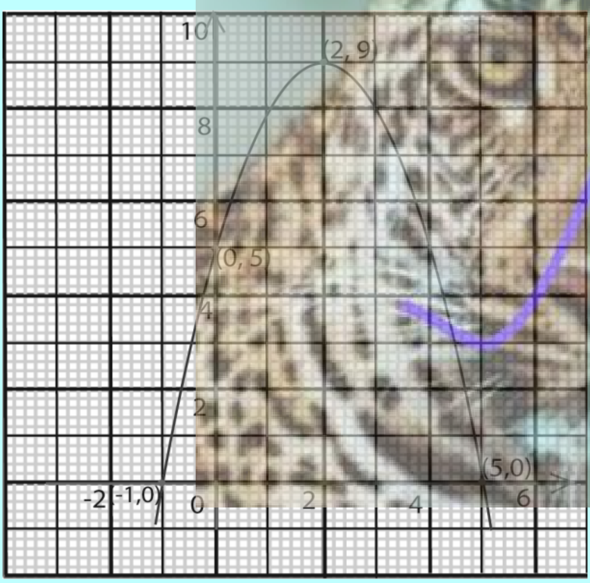
Hence turning point = (2, 9)

Finding the nature of turning point

$\frac{dy}{dx} = 4 - 2x$

$\frac{d^2y}{dx^2} = -2$

Since $\frac{d^2y}{dx^2} < 0$, hence the turning point is maximum.



(b) Sketch the curve y = $x^3 - x^2 - 5x + 6$

Steps taken

For y – intercept; x = 0, y = 0

Hence the y – intercept is (0, 6)

For x – intercept, y = 0

$x^3 - x^2 - 5x + 6 = 0$

error approach is used to find the first factor i.e. (x-2), then other factor is found by long division

$$
\begin{array}{r}
x^2 + x - 3 \\
(x-2)\overline{)\,x^3 - x^2 - 5x + 6} \\
-\underline{X^3 - 2x^2} \\
x^2 - 5x + 6 \\
-\underline{\;x^2 - 2x\;} \\
3x + 6 \\
-\underline{\;3x + 6\;} \\
0 + 0
\end{array}
$$

⇨ $x^3 - x^2 - 5x + 6 = (x - 2)(x^2 + x - 3) = 0$

Solving $x^2 + x - 3 = 0$

$x = \frac{-1\pm\sqrt{1+12}}{2} = \frac{-1\pm3.6}{2}$

x = 1.3 or -2.6

Hence the x- intercepts are (2, 0), (1.3, 0) and (-2.3, 0)

Finding turning points

$y = x^3 - x^2 - 5x + 6$

$\frac{dy}{dx} = 3x^2$-2x-5

At turning point $\frac{dy}{dx} = 0$

$3x^2$-2x-5 = (3x – 5)(x+1) = 0

Either 3x -5 = 0 x = $\frac{5}{3}$

Or x + 1 = 0; x =-1

When x = $\frac{5}{3}$;

$y = \left(\frac{5}{3}\right)^3 - \left(\frac{5}{3}\right)^2 - 5\left(\frac{5}{3}\right) + 6 = \frac{-13}{27}$

When x = -1

$y = (-1)^3 - (-1)^2 - 5(-1) + 6 = 9$

Hence turning points are $\left(\frac{5}{3}; \frac{-13}{27}\right)$ and(-1, 9)

Finding the nature of turning points

$\frac{dy}{dx} = 3x^2$-2x-5

$$\frac{d^2y}{dx^2} = 6x - 2$$

For$\left(\frac{5}{3}; \frac{-13}{27}\right)$

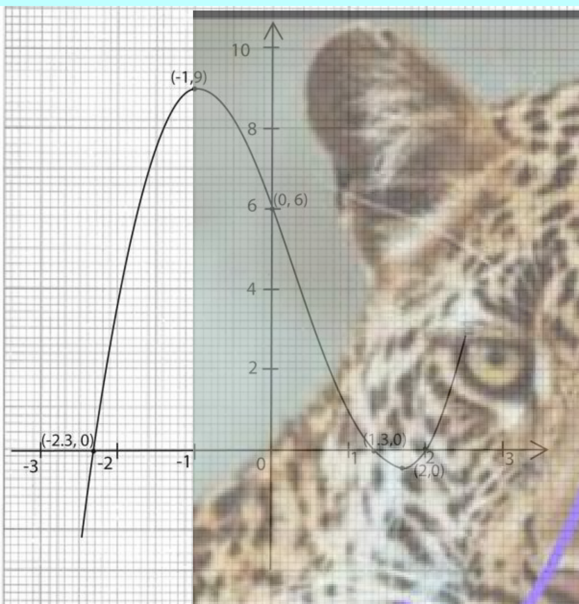
$$\frac{d^2y}{dx^2} = 6\left(\frac{5}{3}\right) - 2 = 8\ (> 0)$$

$\therefore \left(\frac{5}{3}; \frac{-13}{27}\right)$is minimum

For (-1, 9)

$$\frac{d^2y}{dx^2} = 6(-1) - 2 = -8\ (< 0)$$

$\therefore (-1; 9\ )$is maximum



(c) sketch the curve y = $x^3 - 8$

y = $x^3 - 8$

Intercepts

When x = 0, y = -8

When y = 0, x = 2

(x, y) = (2, 0)

Turning point: $\frac{dy}{dx} = 3x^2$

$3x^2 = 0$

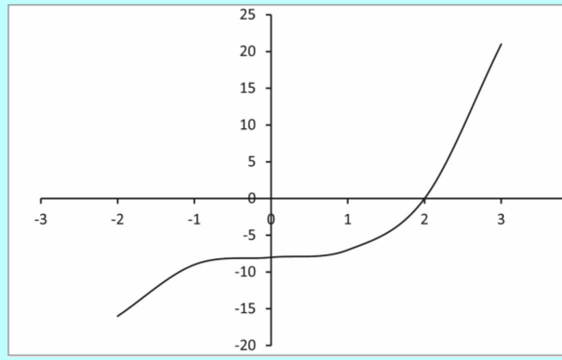
x = 0

$$\frac{d^2y}{dx^2} = 6x$$

$$\frac{d^2y}{dx^2} = 0, x = 0$$

Point of reflection= (0, 8)

	x < 2	x > 2
y	-	+



(d) Sketch the curve $y = x^2(x-4)$

Steps taken

- Finding the intercepts

y – intercept, (0,0)

hence y – intercept is (0, 0)

For x – intercept, y= 0

=>$x^2(x-4) = 0$

Either x = 0 or x = 4

Hence x-intercept are (0, 0) and (4, 0)

- As x→+∞, y →+∞ and x→-∞, y →-∞
- Finding turning point(s)

$$y = x^2(x-4) = x^3 - 4x^2$$

$$\frac{dy}{dx} = 3x^2 - 8x$$

At turning point, $\frac{dy}{dx} = 0$

⇨ $3x^2 - 8x = x(3x-8) = 0$

Either x = 0

Or x = $\frac{8}{3}$

When x = 0; y=0

When x =$\frac{8}{3}$; =3$\left(\frac{8}{3}\right)^2 - 8\left(\frac{8}{3}\right) = \frac{-256}{27}$

Hence turning points are (0,0) and$\left(\frac{8}{3}, \frac{-256}{27}\right)$

- Finding the nature of turning points

$$\frac{dy}{dx} = 3x^2 - 8x$$

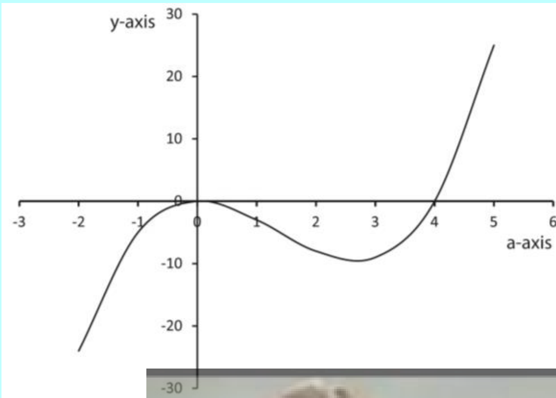
$$\frac{d^2y}{dx^2} = 6x - 8$$

For (0, 0); $\frac{d^2y}{dx^2} = 6(0) - 8 = -8\ (< 0)$

Hence (0, 0) is maximum

For $\left(\frac{8}{3}, \frac{-256}{27}\right)$; $\frac{d^2y}{dx^2} = 6\left(\frac{8}{3}\right) - 8 = 8\ (> 0)$

Hence $\frac{8}{3}$ is minimum



Graphs of rational functions

Rational functions are fractions expressed in the form $y = \frac{f(x)}{g(x)}$.

The basic principles followed when sketching rational curves

(a) Determine if the curve is symmetrical about either or both axes of coordinates.

(b) Find the intercepts on both axes.

(c) Examine the behaviour of the curve as x tends to infinity.

(d) Find the turning points and their nature

(e) Determine the possible asymptotes of the curve

- Vertical asymptote is the value of x which make(s)y tend to infinity. Here we equate the denominator of the function to zero
- Horizontal asymptote is the value of x which make(s)x tend to infinity. Here we divide terms of the numerator and denominators by x with the highest power.

  Alternatively; when finding the horizontal asymptote, we re-arrange the equation and solve for x or make x the subject and then observe the limits, i.e. x→∞, see how y behaves
- Slating asymptotes; this only occurs if horizontal asymptote does not exist and the frictions is improper. Here we divide the terms of the numerator by those of the denominator an y equated to the quotient becomes the asymptote, i.e. asymptote is the y-quotient.

(f) Determine the region where the curve exists/does not exist. This is done by finding a quadratic equation in x such that for real values of x; $b^2>4ac$

Example 2

(a) Sketch the graph of $y = \frac{(x-1)(x+2)}{(x-2)(x+1)}$

Solution

Steps taken

- Finding intercepts

  For y-intercepts; x= 0, $y = \frac{(-1)(2)}{(-2)(+1)} = 1$

  Hence the y-intercept = (0, 1)

  For x-intercept y= 0, $\frac{(x-1)(x+2)}{(x-2)(x+1)} = 0$;

  x=1 or x = -2

  Hence the x-intercept are (1, 0) and (-2, 0)
- Finding turning points

  $y = \frac{(-1)(2)}{(-2)(+1)} = \frac{x^2+x-2}{x^2-x-2}$

  $\frac{dy}{dx} = \frac{(x^2-x-2)(2x+1)-(x^2+x-2)(2x-1)}{(x^2-x-2)^2}$

  At turning point, $\frac{dy}{dx} = 0$

  ⇨ $\frac{(x^2-x-2)(2x+1)-(x^2+x-2)(2x-1)}{(x^2-x-2)^2} = 0$

  $(2x^3 - x^2 - 5x - 2)$

  $-(2x^3 + x2 - 5x + 2) = 0$

  $2x^2 + 4 = 0$

  $x^2 + 2 = 0$

  There is no real value of x, hence there is no turning points.
- Finding asymptotes;

  Vertical asymptote

  $(x - 2)(x + 1) = 0$

  Either (x – 2) = 0; x = 2

  Or (x + 1) = 0; x = -1

  Hence the vertical asymptotes are x = 2 and x = 1

  Horizontal asymptotes

  $y = \frac{x^2+x-2}{x^2-x-2}$

  Dividing terms on the LHS by x^2

  $y = \frac{1+\frac{1}{x}-\frac{2}{x^2}}{1-\frac{1}{x}-\frac{2}{x^2}}$

  As x →∞, y→1

Hence the horizontal asymptote y = 1

- Finding the regions where the curve does not exist

$y = \frac{x^2+x-2}{x^2-x-2}$

$y(x^2 - x - 2) = x^2 + x - 2$

$y(x^2 - x - 2) - x^2 - x + 2 = 0$

$(y-1)x^2 + (-y-1)x + (2-2y) = 0$

For real value of x, $b^2 > 4ac$

⇨ $(-y-1)^2 > 8(y-1)(1-y)$

$(y+1)^2 + 8(y-1)^2 > 0$

$9y^2 - 6y + 9 > 0$

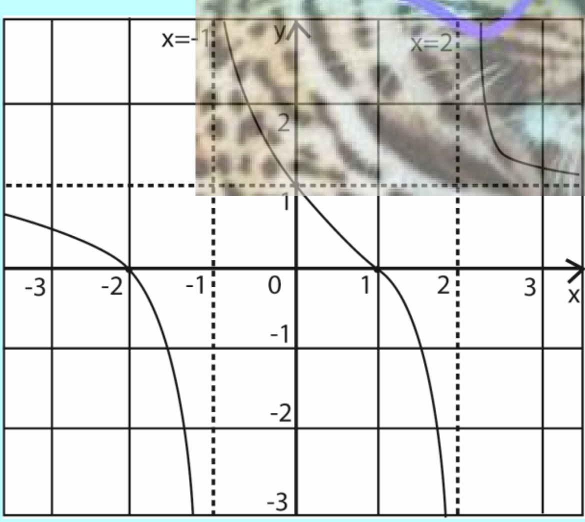
There is no real value of y which means that there is no restriction on y

- Determining the sign of the function throughout its domain. The function will only change sign where the curve cuts the x-axis and vertical asymptotes

The critical value are -, -1, 1, 2

	X<-2	-2<x<-1	-1<x<1	1<x<2	x>2
x-1	-	-	-	+	+
x+2	-	+	+	+	+
x-2	-	-	-	-	+
X+1	-	-	+	+	+
y	+	-	+	-	+

Graph of $y = \frac{(x-1)(x+2)}{(x-2)(x+1)}$



(b) Sketch the graph

Sketch the graph of $y = \frac{x(x-2)}{x+1}$

Steps taken

- Finding the intercepts

For y – intercept; x = 0, and y =0

Hence the y-intercept is (0, 0)

For x- intercept y= 0

⇨ $\frac{x(x-2)}{x+1} = 0$

$x(x-2) = 0$

Either x = 0

Or (x-2) = 0; x = 2

Hence the x-intercepts are(0, 0) and (2, 0)

- Finding turning points

$y = \frac{x^2-2x}{x+1}$

$\frac{dy}{dx} = \frac{x^2-2x(1)-(x+1)(2x-2)}{(x+1)^2} = \frac{x^2+2x-2}{(x+1)^2}$

At turning points, $\frac{dy}{dx} = 0$

⇨ $x^2 + 2x - 2 = 0$

$x = \frac{-2\pm\sqrt{2^2-(4x1x-2)}}{2x1}$

$x = \frac{-2\pm\sqrt{12}}{2} = -1 \pm 1.732$

When x $= -1 + 1.732 = 0.732$

$y = \frac{0.732(0.732-2)}{0.732+1} = -0.54$

When x $= -1 - 1.732 = -2.732$

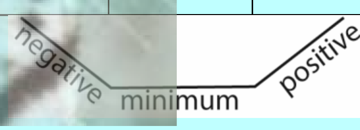
$y = \frac{-2.732(-2.732-2)}{-2.732+1} = -7.46$

Hence the turning points are (0.73, -0.54) and (-2.73, -7.46)

- Finding the nature of the turning points

For (0.73, -0.54)

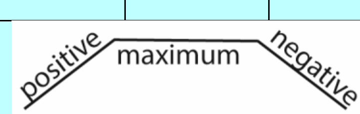
x	0	0.73	3
$\frac{dy}{dx}$	-2	0	0.25



Hence the turning point (0.73, -0.54) is minimum

For (-2.73, -7.46)

x	-3	-2.73	-2
$\frac{dy}{dx}$	0.25	0	-2



Hence the turning point (-2.73, -7.46) is maximum

- Finding asymptotes

For vertical asymptote, the denominator = 0

⇨ x+ 1 = 0; x = -1

since the function is improper fraction, there must be slating asymptote.

Dividing the numerator by denominator;

$$\begin{array}{r|l} & x-3 \\ \hline x+1 & x^2-2x \\ & -\ x^2+x \\ \hline & -\ 3x-3 \\ \hline & 3 \end{array}$$

The slanting asymptote is y = x – 3

X	0	3
y	-3	0

- Finding the region where the curve does not exist.

$y = \frac{x^2-2x}{x+1}$

$y(x+1) = x^2 - 2x$

$x^2 - (2+y)x - y = 0$

For real values of x, $b^2 \geq 4ac$

$(2+y)^2 > 4x\ 1\ x(-y)$

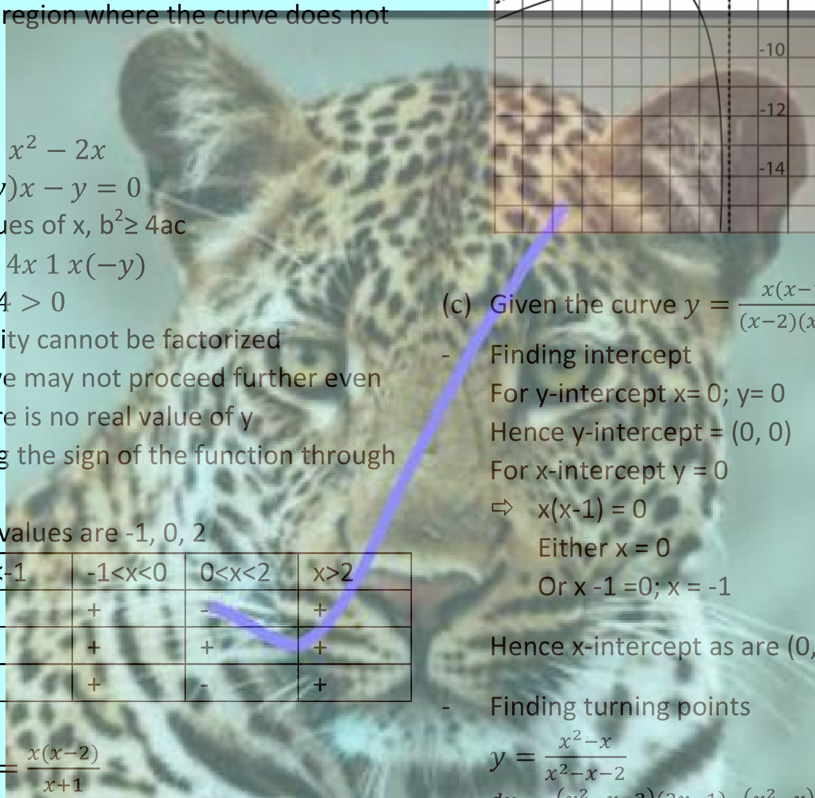
$y^2 + 8y + 4 > 0$

The inequality cannot be factorized therefore we may not proceed further even though there is no real value of y

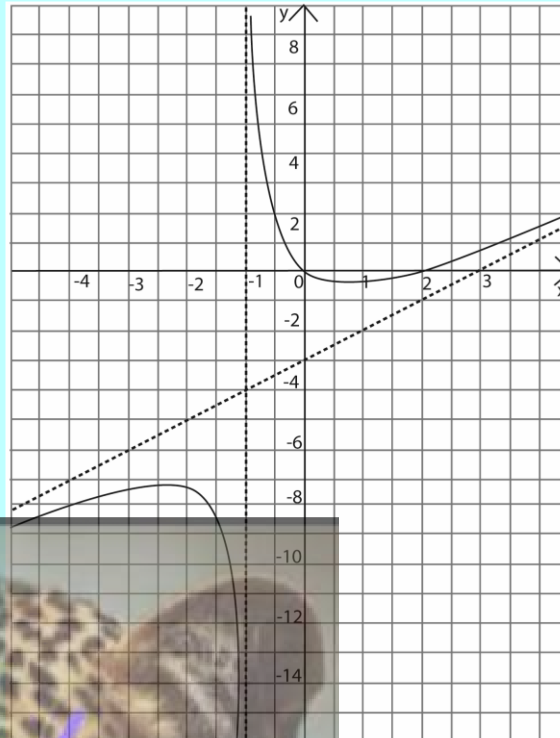
- Determining the sign of the function through it domain

The critical values are -1, 0, 2

	X<-1	-1<x<0	0<x<2	x>2
x(x-1)	+	+	-	+
x+1	-	+	+	+
y	-	+	-	+



Graph of $y = \frac{x(x-2)}{x+1}$



(c) Given the curve $y = \frac{x(x-1)}{(x-2)(x+1)}$

- Finding intercept

For y-intercept x= 0; y= 0

Hence y-intercept = (0, 0)

For x-intercept y = 0

⇨ x(x-1) = 0

Either x = 0

Or x -1 =0; x = -1

Hence x-intercept as are (0, 0) and (1, 0)

- Finding turning points

$y = \frac{x^2-x}{x^2-x-2}$

$\frac{dy}{dx} = \frac{(x^2-x-2)(2x-1)-(x^2-x)(2x-1)}{(x^2-x-2)^2}$

At turning point $\frac{dy}{dx} = 0$

$(2x-1)\{(x^2-x-2)-(x^2-x)\} = 0$

(2x – 1)(-2) = 0

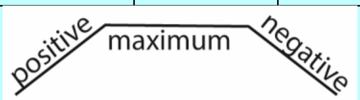
⇨ 2x – 1 = 0; x =$\frac{1}{2}$

When x = $\frac{1}{2}$, $y = \frac{\left(\frac{1}{2}\right)^2 - \frac{1}{2}}{\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right) - 2} = \frac{1}{9}$

Hence turning point is $\left(\frac{1}{2}, \frac{1}{9}\right)$

- Determining nature of turning point

x	0	$\frac{1}{2}$	1
$\frac{dy}{dx}$	2	0	-6



Hence the turning point $\left(\frac{1}{2},\frac{1}{9}\right)$ is maximum.

- Finding the asymptote(s)

For vertical asymptote

(x – 2)(x + 1) = 0

Either (x – 2) = 0; x = 2

Or (x + 1) =0; x = -1

For horizontal asymptote

Dividing the numerator and denominator on the RHS by x^2.

$y = \frac{1+\frac{1}{x}}{1-\frac{1}{x}-\frac{2}{x^2}}$

As x → ∞, y→1

- Finding the region where x does not exist

$y = \frac{x(x-1)}{(x-2)(x+1)}$

$y(x-2)(x+1) = x(x-1)$

$y(x^2-x-2) = x^2-x$

$yx^2-yx-2y-x^2+x = 0$

$(y-1)x^2+(1-y)x-2y = 0$

For real values of x, $b^2 \geq 4ac$

$(1-y)^2 > -8y(y-1)$

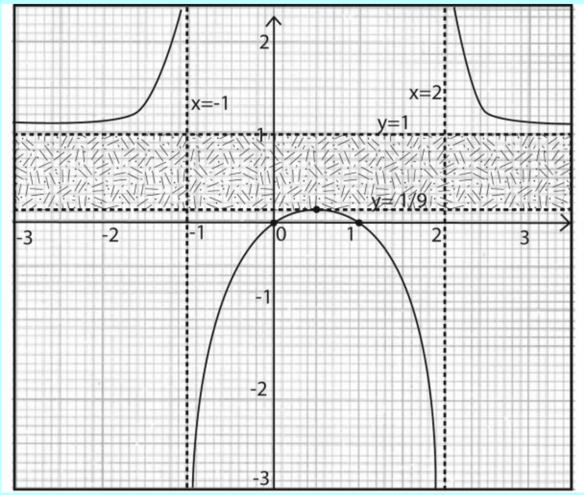
$1-2y+y^2+8y(y-1) > 0$

(9y – 1)(y – 1) = 0

	$y<\frac{1}{9}$	$\frac{1}{9}<y<1$	Y > 1
9y – 1	-	+	+
y – 1	-	-	+
(9y – 1)(y – 1)	+	-	+

Hence the curve does not lie in the range

$\frac{1}{9}<y<1$



(d) Sketch the curve $y = \frac{x^2+4x+3}{x+2}$

- Finding the range of values over which the curve does not exist

$y = \frac{x^2+4x+3}{x+2}$

$y(x+2) = x^2+4x+3$

$x^2+(4-y)x+(3-2y) = 0$

For real values of x, $b^2 \geq 4ac$

$(4-y)^2 > 4(3-2y)$

$16-8y+y^2-12+8y \geq 0$

$y^2+4 \geq 0$

Since there are no real values of y, this means that there is no restriction on y.

- Finding intercepts

For y intercept, x= 0; $y = \frac{3}{2}$

Hence y- intercept is $\left(0,\frac{3}{2}\right)$

For x – intercepts y = 0

⇨ $\frac{x^2+4x+3}{x+2} = 0$

$x^2+4x+3 = 0$

(x + 3)(x+ 1) = 0

Either (x + 3) = 0; x= -3

Or (x+1) = 0, x = -1

Hence x-intercepts are (-1, 0) and (-3, 0)

- Finding turning points

$y = \frac{x^2+4x+3}{x+2}$

$\frac{dy}{dx} = \frac{(x+2)(2x+4)-(x^2+4x+3)(1)}{(x+2)^2}$

At turning points, $\frac{dy}{dx} = 0$

$x^2 + 4x + 5x = 0$

$x = \frac{-4\pm\sqrt{4^2-4(1)(5)}}{2(1)}$

$x = \frac{-4\pm\sqrt{-4}}{2}$

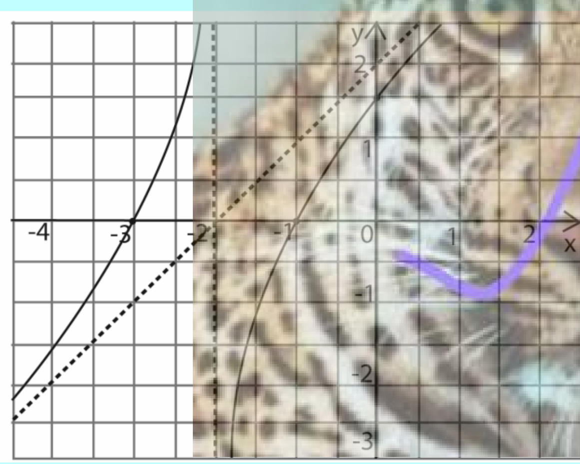
Since there is no real value of x, this means that the curve has no turning point

- Finding vertical asymptote
  (x +2) = 0
  x = -2
  Since the function is improper fraction, there could be a slating asymptote

```
              x + 2
(x+2) ) x^2 + 4x + 3
       - x^2 + 2x
               2x + 3
             - 2x + 4
                   -1
```

Hence the slanting asymptote is y =x + 2

A curve $y = \frac{x^2+4x+3}{x+2}$



(e) A curve is given by $y = \frac{(x-1)}{(2x-1)(x+1)}$

(i) Show that for real values of x, y cannot take on values in the interval $\left(\frac{1}{9}, 1\right)$

$y = \frac{(x-1)}{(2x-1)(x+1)}$

$y(2x-1)(x+1) = x - 1$

$y(2x^2 + x - y) = x - 1$

$2yx^2 + (y-1)x + (1-y) = 0$

For real values of x, $b^2 \geq 0$

$(y-1)^2 \geq 8y(1-y)$

$(y-1)^2 + 8y(y-1) \geq 0$

(9y – 1)(y – 1) = 0

	y<$\frac{1}{9}$	$\frac{1}{9} < y < 1$	Y > 1
9y – 1	-	+	+
y – 1	-	-	+
(9y – 1)(y – 1)	+	-	+

Hence the curve does not lie in the range $\frac{1}{9} < y < 1$

(ii) Determine the turning points of the curve

$y = \frac{(x-1)}{(2x-1)(x+1)} = \frac{(x-1)}{2x^2+x-1}$

$\frac{dy}{dx} = \frac{(2x^2+x-1)(1)-(x-1)(4x+1)}{(2x^2+x-1)^2}$

$= \frac{-2x^2+4x}{(2x^2+x-1)^2}$

At turning point $\frac{dy}{dx} = 0$

⇨ $-2x^2 + 4x = 0$

–2x(x- 2) = 0

Either 2x = 0; x= 0

Or (x - 2) = 0; x= 2

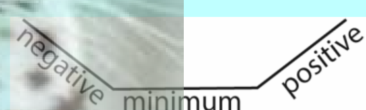
When x = 0; $y = \frac{-1}{-1} = 1$ => (x, y) =(0, 1)

When x = 2, $y = \frac{1}{3x3} = \frac{1}{9}$=> (x, y) =$\left(2, \frac{1}{9}\right)$

Determining the nature of turning points

For (0,1)

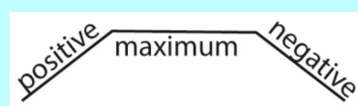
x	-0.5	0	0.5
$\frac{dy}{dx}$	-2.5	0	1.5



Hence (0, 1) is minimum

For$\left(2, \frac{1}{9}\right)$

x	1	2	3
$\frac{dy}{dx}$	+0.025	0	-0.074



Hence $\left(2, \frac{1}{9}\right)$is maximum

(iii) State with reasons the asymptotes of the curve

For vertical asymptote

(2x – 1)(x + 1) = 0

Either 2x – 1 =0; x=$\frac{1}{2}$

Or (x + 1) = 0; x = -1

For horizontal asymptotes

Dividing the numerator and denominator on the RHS by x

$$y = \frac{1-\frac{1}{x}}{2x-1-\frac{1}{x}}$$

As x→ ∞, y → 0

Hence horizontal asymptote is y = 0

(iv) Sketch the curve

Finding intercepts

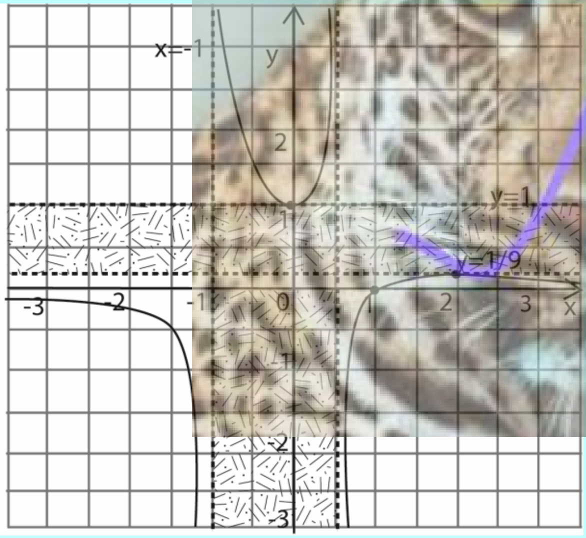
For y-intercept x = 0; y = 1

Hence y-intercept (0,1)

For x-intercept y = 0; x= 1

Hence x – intercept is(1, 0)

A graph of $y = \frac{(x-1)}{(2x-1)(x+1)}$



Sketching graphs of parametric equations

It requires eliminating parameters in the equations given and then following similar steps in above examples.

Example 3

(a) A curve is given by parametric equations

x = t + 2 and y = $\frac{t^2-t}{t+1}$

(a) Find the Cartesian equation of the curve

Solution

x = t – 2; t = x + 2

Substituting t into the equation

$$y = \frac{(x-2)^2-(x-2)}{(x-2)+1} = \frac{(x-2)(x-3)}{(x-1)}$$

Hence Cartesian equation is $y = \frac{(x-2)(x-3)}{(x-1)}$

(b) Sketch the curve

$$y = \frac{(x-2)(x-3)}{(x-1)}$$

- Finding intercepts

For y-intercept, x = 0, y = -6

For x-intercept y = 0

⇨ $(x-2)(x-3) = 0$

Either x – 2 = 0; x = 2

Or (x – 3) = 0; x = 3

Hence x- intercepts are (2, 0) and (3, 0)

- Finding the turning points

$$y = \frac{x^2-5x+6}{(x-1)}$$

$$\frac{dy}{dx} = \frac{(x-1)(2x-5)-(x^2-5x+6)(1)}{(x-1)^2} = \frac{x^2-2x-1}{(x-1)^2}$$

At turning point $\frac{dy}{dx} = 0$

⇨ $x^2 - 2x - 1 = 0$

$$x = \frac{2\pm\sqrt{(-2)^2-4(1)(-1)}}{2(1)}$$

$$x = \frac{2\pm\sqrt{8}}{2} = \frac{2\pm2\sqrt{2}}{2} = 1 \pm \sqrt{2}$$

Either x = 1 = $\sqrt{2}$= 2.4

Or x = 1 $-\sqrt{2}$= -0.4

When x = 2.4

$$y = \frac{(2.4-2)(2.4-3)}{(2.4-1)} = -0.17$$

When x = -0.4

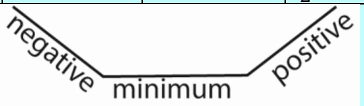
$$y = \frac{(-0.4-2)(-0.4-3)}{(-0.4-1)} = -5.83$$

Hence the turning points are (2.4, -0.17) and (-0.4, - 5.83)

- Finding the nature of turning points

For (2.4, -0.17)

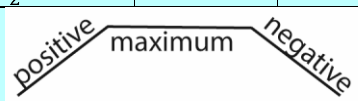
X	2	2.4	3
$\frac{dy}{dx}$	-1	0	$\frac{1}{2}$



Hence the turning point (2.4, -0.17) is minimum.

For (-0.4, -5.83)

X	-1	-0.4	0
$\frac{dy}{dx}$	$\frac{1}{2}$	0	-1



Hence the turning point (-0.4, -5.3) is maximum

- Finding asymptotes

For vertical asymptotes

x – 1 = 0, x= 1

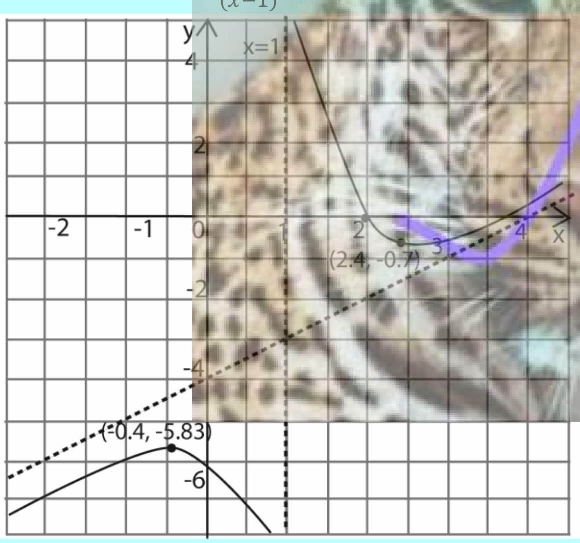
For slanting asymptote

$$\begin{array}{r|l} & x-4 \\ (x-1) & x^2-5x+6 \\ & \underline{-\ x^2-x} \\ & \quad -4x+6 \\ & \quad \underline{-\ -4x+4} \\ & \qquad\quad 2 \end{array}$$

Hence the slanting asymptote is y = x – 4

x	0	4
y	-4	0

Graph $y = \frac{(x-2)(x-3)}{(x-1)}$



(c) A curve is given by parametric equations
x = cos2θ and y =2sinθ.

(i) Find the equation of the normal to the curve at θ=$\frac{5\pi}{6}$

x = cos2θ

$\frac{dx}{d\theta} = -2sin2\theta$

y =2sinθ.

$\frac{dy}{d\theta} = 2cos\theta$

$\frac{dy}{dx} = \frac{dy}{d\theta}.\frac{d\theta}{dx} = \frac{-2cos\theta}{2sin2\theta} = -\frac{cos\theta}{sin2\theta}$

At θ=$\frac{5\pi}{6}$

$\frac{dy}{dx} = \frac{cos\left(\frac{5\pi}{6}\right)}{sin\left(\frac{5\pi}{3}\right)} = -1$

Gradient of the normal =$\frac{-1}{-1} = 1$

x = cos2θ

$x = cos\frac{5\pi}{3}$=$\frac{1}{2}$

$y = 2sin\frac{5\pi}{6} = 1$

Let a point (x, y) lie on the normal

⇨ $\frac{y-1}{x-\frac{1}{2}} = 1$

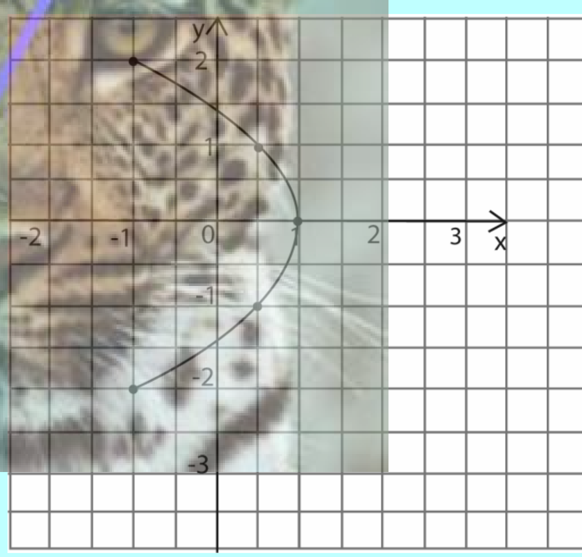
$y = x + \frac{1}{2}$

Hence the equation of the normal to the curve at $\theta = \frac{5\pi}{6}$ is $y = x + \frac{1}{2}$

(ii) Sketch the curve for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
x=cos2θ	-1	-0.5	0.5	1	0.5	-0.5	-1
y=2sinθ	-2	-1.73	-1	0	1.73	1.73	2

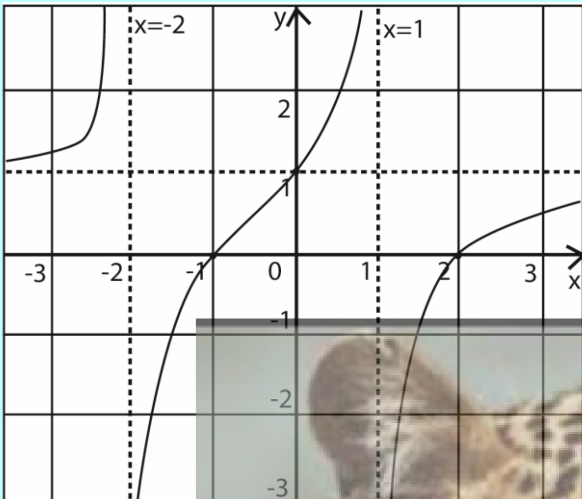
A graph of $x = 1 - \frac{y^2}{2}$



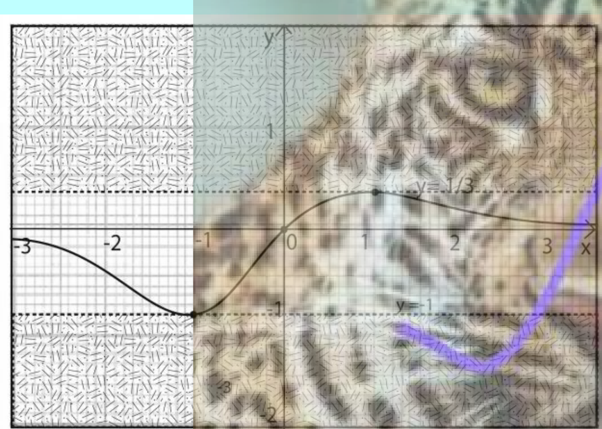
Revision question 1

1. Sketch the graph $y = \frac{(x-2)(x+1)}{(x-1)(x+2)}$

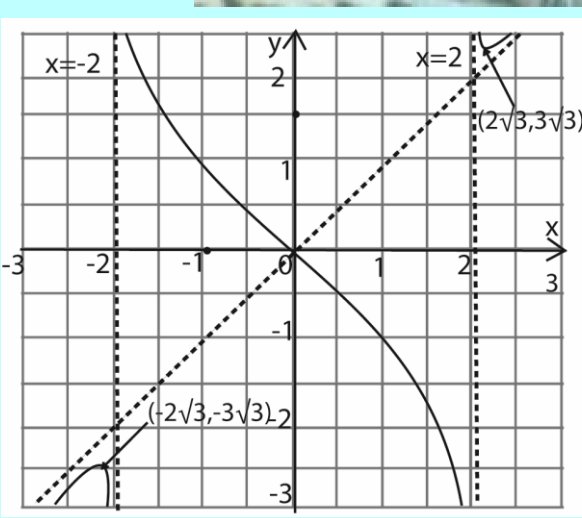
   A graph $y = \frac{(x-2)(x+1)}{(x-1)(x+2)}$



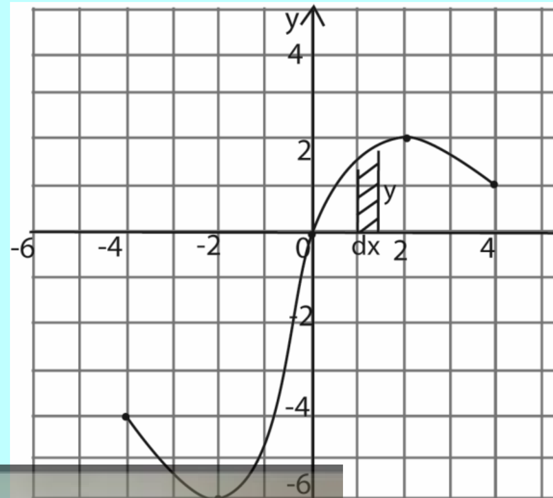
2. Sketch the graph $y = \frac{x}{x^2+x+1}$



3. Sketch the curve $y = \frac{x^2}{x^2-4}$



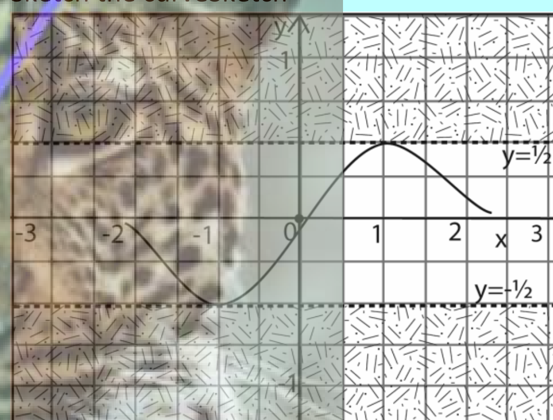
4. (a) Sketch the curve $y = \frac{12}{x^2+2x+4}$



   (b) Find the area enclosed by the curve, x-axis and 0≤ x ≤4 [0.259 to3dp)

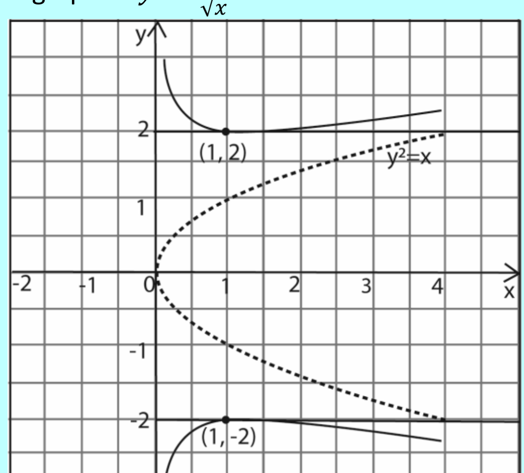
5. Determine the stationary points (including points of inflexion) of the curve $y = \frac{x}{x^2+1}$.

   Sketch the curveSketch

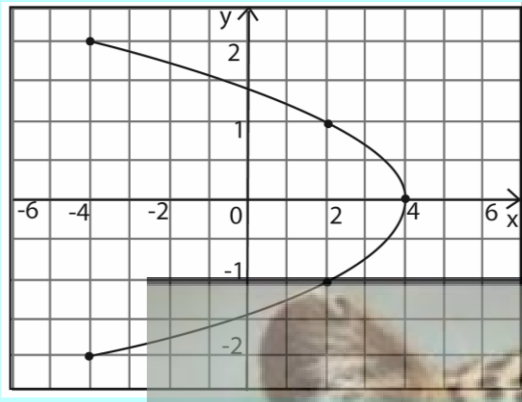


6. Sketch the curve give by the following parametric equations x= t² and $y = t + \frac{1}{t}$.

   A graph of $y = \frac{x+1}{\sqrt{x}}$

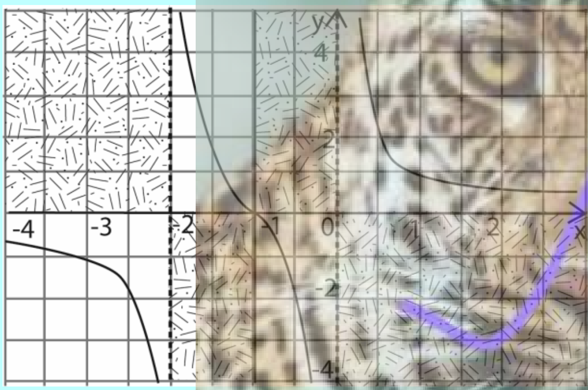


7. A curve is given by the parametric equations x= 4cos2t and y = 2sint
   (i) Find the equation of the normal to the curve at t = $\frac{5\pi}{6}$ [y = 4x – 7]
   (ii) Sketch the curve for $-\frac{\pi}{2} < t < \frac{\pi}{2}$

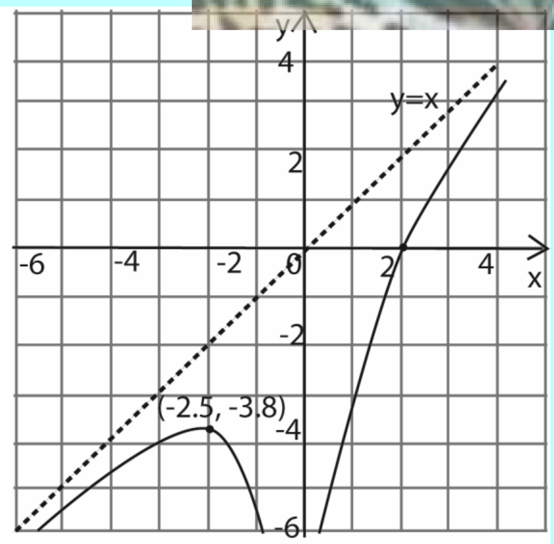


   (iii) Find the area enclosed by the curve and the y-axis [7.543 units (3d,p)]

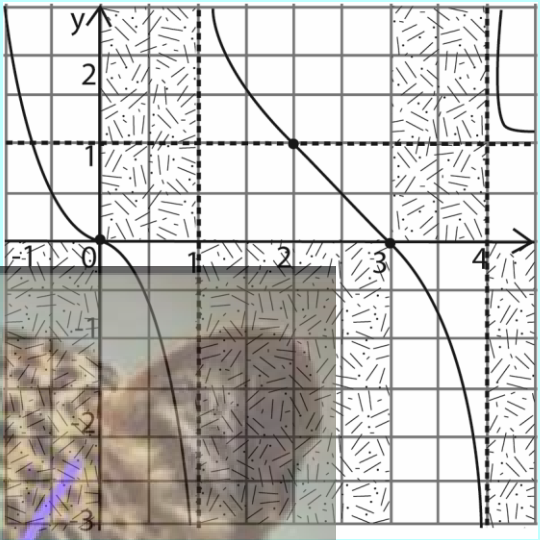
8. Sketch the curve $y = \frac{x+1}{x^2+2x}$



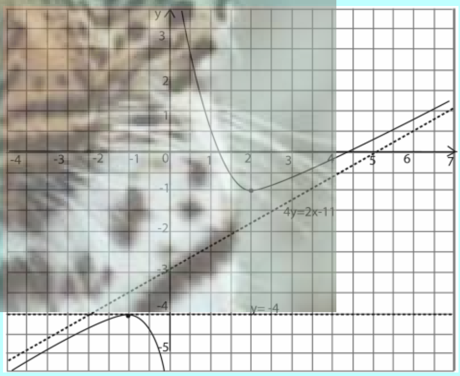
9. Sketch the curve =$y = x - \frac{8}{x^2}$



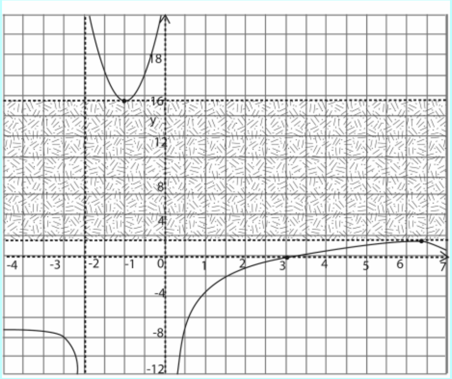
10. Given the curve $y = \frac{x(x-3)}{(x-1)(x-4)}$
   (i) Show that the curve does not have turning points $\left[\frac{dy}{dx} = 0; has\ no\ roots)\right]$
   (ii) Find the equations of asymptotes. Hence sketch the graph

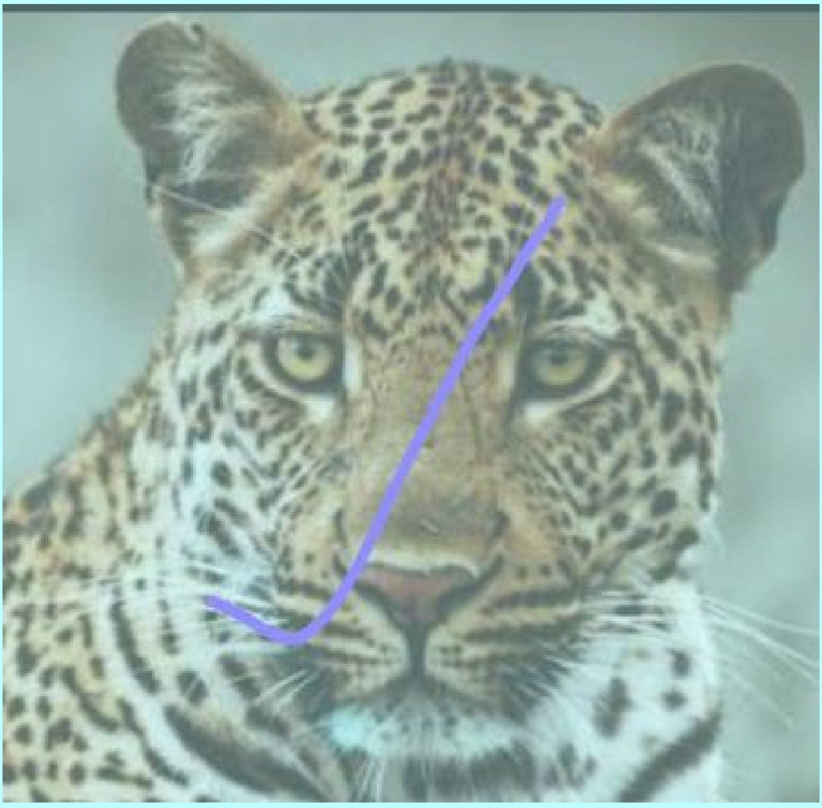


11. Determining the nature of the turning points of the curve $y = \frac{x^2-6x+5}{2x+1}$, sketch the graph of the curve for x= -2 to x=7. Show any asymptotes.



12. Sketch the curve $y = \frac{4(x-3)}{x(x+2)}$

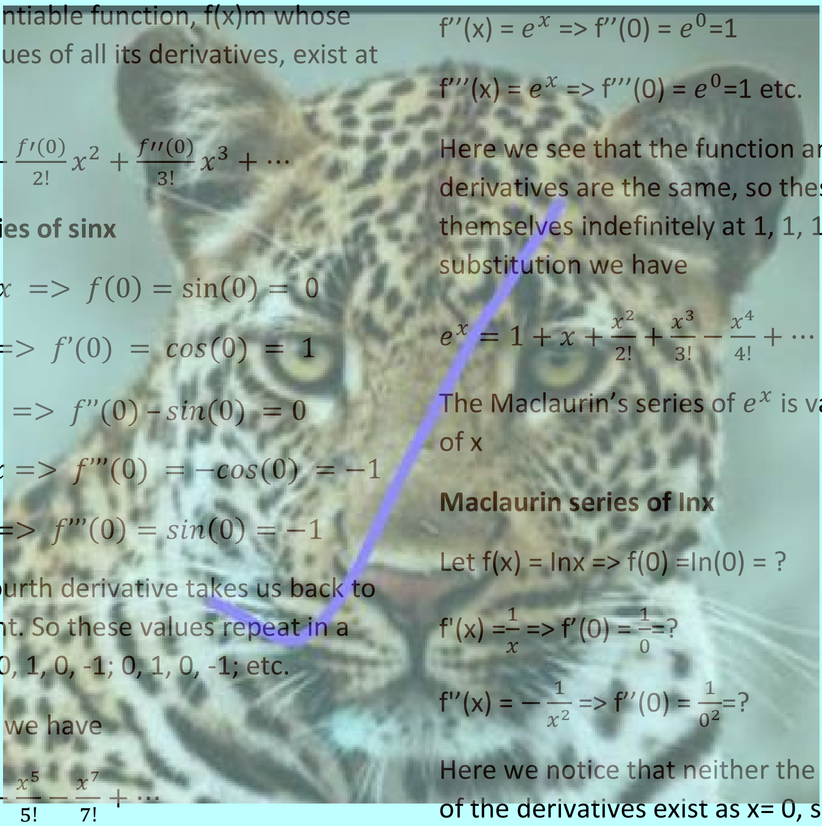




Approximations

The Maclaurin's Theorem

The polynomial of Maclaurin's series of any infinitely differentiable function, f(x)m whose value and all values of all its derivatives, exist at x = 0



$$f(x) = f(0)x + \frac{f'(0)}{2!}x^2 + \frac{f''(0)}{3!}x^3 + \cdots$$

Maclaurin's series of sinx

Let $f(x) = \ sinx \ => \ f(0) = \sin(0) = \ 0$

$f'(x) \ = cosx \ => \ f'(0) \ = \ cos(0) \ = \ 1$

$f''(x) = -sinx \ => \ f''(0) - sin(0) \ = 0$

$f'''(x) = -cosx => \ f'''(0) \ = -cos(0) \ = -1$

$f^{iv}(x) = sinx => \ f'''(0) = sin(0) = -1$

Note that the fourth derivative takes us back to the starting point. So these values repeat in a cycle of four as 0, 1, 0, -1; 0, 1, 0, -1; etc.

By substitution, we have

$$sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

The Maclaurin's series of sin x is valid for all values of x.

Maclaurin series of cos x

$$cosx = \frac{d}{dx}(sinx)$$

$$= \frac{d}{dx}(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots)$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{3!} - \frac{x^6}{6!} + \cdots$$

The Maclaurin's series of cos x is valid for all values of x.

Maclaurin's series of e^x

Let f(x) = e^x => f(0) = e^0=1

f'(x) = e^x => f'(0) = e^0=1

f''(x) = e^x => f''(0) = e^0=1

f'''(x) = e^x => f'''(0) = e^0=1 etc.

Here we see that the function and all its derivatives are the same, so these values repeat themselves indefinitely at 1, 1, 1,1, etc. by substitution we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots$$

The Maclaurin's series of e^x is valid for all values of x

Maclaurin series of Inx

Let f(x) = Inx => f(0) =In(0) = ?

f'(x) =$\frac{1}{x}$ => f'(0) = $\frac{1}{0}$=?

f''(x) = $-\frac{1}{x^2}$ => f''(0) = $\frac{1}{0^2}$=?

Here we notice that neither the function nor any of the derivatives exist as x= 0, so there is no polynomial Maclaurin's expansion of natural logarithm, Inx.

Maclaurin series of In(1+x)

Let f(x) = In(1+x) => f(0) =In(1+0) = 0

f'(x) =$-\frac{1}{1+x}$ => f'(0) = $-\frac{1}{1+0}$= 1

f''(x) = $\frac{-1}{(1+x)^2}$ => f''(0) = $\frac{1}{(1+0)^2}$ =-1

f'''(x) = $\frac{2}{(1+x)^3}$ => f'''(0) = $\frac{2}{(1+0)^3}$ =2

f^{iv}(x) = $-\frac{3x2}{(1+x)^4}$ => f^{iv} (0) = $\frac{-3x2}{(1+0)^4}$ =-3x2 etc by substitution we have

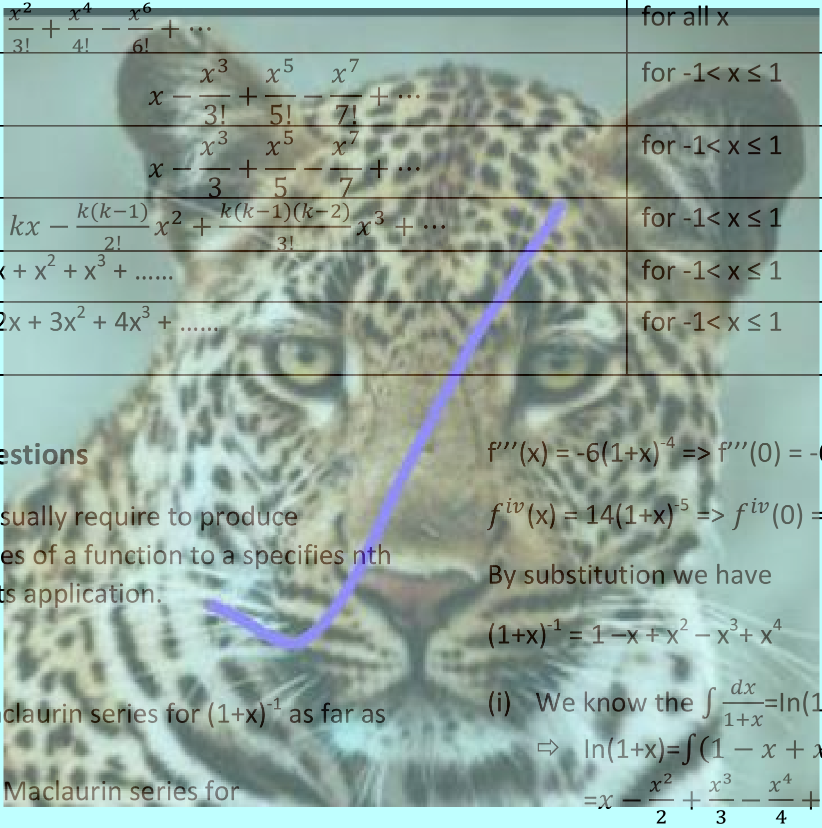
$$In(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$$

The Maclaurin's series of In(1+x) is valid for -1< x ≤ 1

Note the validity of **Maclaurin series** is arrived at by using ratio test theorem whose derivation is outside the scope of our coverage

Summary

f(x)	Expansion	Validity
e^x	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$	for all x
e^{-x}	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$	for all x
sinx	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	for all x
cosx	$1 - \frac{x^2}{3!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	for all x
$tan^{-1}x$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	for -1< x ≤ 1
In(1+x)	$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	for -1< x ≤ 1
$(1+x)^k$	$1 + kx - \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$	for -1< x ≤ 1
$\frac{1}{1-x}$	$1 + x + x^2 + x^3 + \dots\dots$	for -1< x ≤ 1
$\frac{1}{(1-x)^2}$	$1 + 2x + 3x^2 + 4x^3 + \dots\dots$	for -1< x ≤ 1



Answering questions

The questions usually require to produce Maclaurin's series of a function to a specifies nth term and then its application.

Examples

1. Find the Maclaurin series for $(1+x)^{-1}$ as far as x^4.
   Deduce the Maclaurin series for
   (i) In(1+x)
   (ii) In(1-x)
   (iii) $In\left(\frac{1-x}{1+x}\right)$

   Solution

   Maclaurin expansion series is given by

   $$f(x) = f(0) + f'^{(0)}x + \frac{f''(0)}{2!} + \frac{f'''(0)}{3!} + \dots$$

   Let $f(x) = (1+x)^{-1} \Rightarrow f(0) = (1+0)^{-1} = 0$

   $f'(x) = -1(1+x)^{-2} \Rightarrow f'(0) = -1(1+0)^{-2} = -1$

   $f''(x) = 2(1+x)^{-3} \Rightarrow f'(0) = 2(1+0)^{-3} = 2$

   $f'''(x) = -6(1+x)^{-4} \Rightarrow f'''(0) = -6(1+0)^{-4} = -6$

   $f^{iv}(x) = 14(1+x)^{-5} \Rightarrow f^{iv}(0) = 24(1+0)^{-5} = 24$

   By substitution we have

   $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4$

   (i) We know the $\int \frac{dx}{1+x}$=In(1+x)
   ⇨ In(1+x)=$\int(1 - x + x^2 - x^3 + x^4)dx$
   $= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$

   This valid for-1 < x ≤ 1

   (ii) Replacing x by –x in (i)

   In(1-x) $= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5}$

   This valid for-1 < x ≤ 1

   (iii) Subtracting (ii) from (i)
   In(1+x) – In(1-x)= $2x + \frac{2x^3}{3} + \frac{2x^5}{5}$

2. Find the Maclaurin series for (1+x)-1 as far as x^4.
   Deduce the Maclaurin series for

(i) $\frac{1}{1+x^2}$ as far as x^6.

(ii) $tan^{-1}x$ as far as x^7.

Show that $tan^{-1}\left(\frac{1}{2}\right) + tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{4}$

Solution

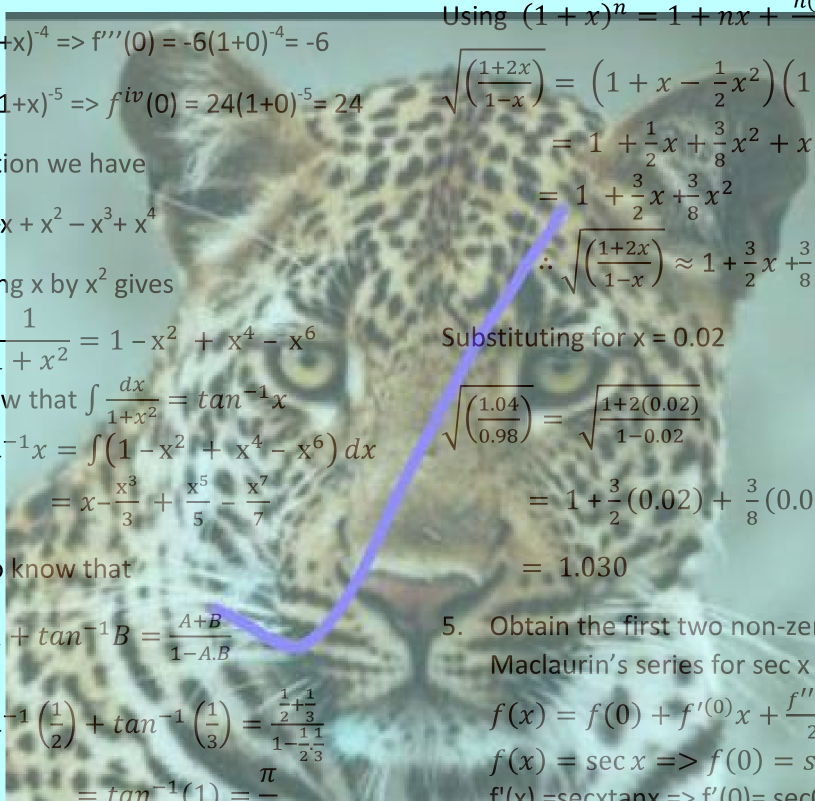
Maclaurin expansion series is given by

$$f(x) = f(0) + f'^{(0)}x + \frac{f''(0)}{2!} + \frac{f'''(0)}{3!} + \ldots$$

Let $f(x) = (1+x)^{-1}$ => $f(0) = (1+0)^{-1} = 0$

$f'(x) = -1(1+x)^{-2}$ => $f'(0) = -1(1+0)^{-2} = -1$

$f''(x) = 2(1+x)^{-3}$ => $f'(0) = 2(1+0)^{-3} = 2$



$f'''(x) = -6(1+x)^{-4}$ => $f'''(0) = -6(1+0)^{-4} = -6$

$f^{iv}(x) = 14(1+x)^{-5}$ => $f^{iv}(0) = 24(1+0)^{-5} = 24$

By substitution we have

$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4$

(i) Replacing x by x^2 gives

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6$$

(ii) We know that $\int \frac{dx}{1+x^2} = tan^{-1}x$

⇨ $tan^{-1}x = \int (1 - x^2 + x^4 - x^6)\, dx$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$$

We also know that

$$tan^{-1}A + tan^{-1}B = \frac{A+B}{1-A.B}$$

⇨ $tan^{-1}\left(\frac{1}{2}\right) + tan^{-1}\left(\frac{1}{3}\right) = \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}$

$$= tan^{-1}(1) = \frac{\pi}{4}$$

3. Use Maclaurin theorem to expand e^x up to the term x^4, use your expansion to evaluate e correct to 4 decimal places.

Let $f(x) = e^x$ => $f(0) = e^0 = 1$

$f'(x) = e^x$ => $f'(0) = e^0 = 1$

$f''(x) = e^x$ => $f''(0) = e^0 = 1$

$f'''(x) = e^x$ => $f'''(0) = e^0 = 1$ etc.

by substitution we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

Evaluating e

$e = e^1$, substituting for x = 1

$$e^1 = 1 + (1) + \frac{(1)^2}{2!} + \frac{(1)^3}{3!} + \frac{(1)^4}{4!}$$

$$= 2 + \frac{1}{6} + \frac{1}{24} = \frac{65}{24} = 2.7083\ (4d.p)$$

4. Expand $\sqrt{\left(\frac{1+2x}{1-x}\right)}$ up to the term x^2. Hence find the value of $\sqrt{\left(\frac{1.04}{0.98}\right)}$ to four significant figures. (12marks)

$$\sqrt{\left(\frac{1+2x}{1-x}\right)} = (1+2x)^{\frac{1}{2}}(1-x)^{\frac{-1}{2}}$$

Using $(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \cdots$

$$\sqrt{\left(\frac{1+2x}{1-x}\right)} = \left(1 + x - \frac{1}{2}x^2\right)\left(1 + \frac{1}{2}x + \frac{3}{8}x^2\right)$$

$$= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + x + \frac{1}{2}x^2 - \frac{1}{2}x^2$$

$$= 1 + \frac{3}{2}x + \frac{3}{8}x^2$$

$$\therefore \sqrt{\left(\frac{1+2x}{1-x}\right)} \approx 1 + \frac{3}{2}x + \frac{3}{8}x^2$$

Substituting for x = 0.02

$$\sqrt{\left(\frac{1.04}{0.98}\right)} = \sqrt{\frac{1+2(0.02)}{1-0.02}}$$

$$= 1 + \frac{3}{2}(0.02) + \frac{3}{8}(0.02)^2$$

$$= 1.030$$

5. Obtain the first two non-zero terms of Maclaurin's series for sec x

$$f(x) = f(0) + f'^{(0)}x + \frac{f''(0)}{2!} + \frac{f'''(0)}{3!} + \ldots$$

$f(x) = \sec x$ => $f(0) = sec0 = 1$

f'(x) =secxtanx => f'(0)= sec0tan0 =0

$f''(x) = \sec x \sec^2 x + \tan x \sec x \tan x$

⇨ $f''(0) = \sec 0 \sec^2 0 + \tan 0 \sec 0 \tan 0 = 1 + 0 = 1$

Hence the first two non-zero terms of Maclaurin series of $sec\ x = 1 + \frac{x^2}{2}$

Revision exercise

1. Use Maclaurin theorem to expand the following up to
   (i) $In\left(\frac{1+x}{1-x}\right)$ up to x^3. Hence, find the approximation of In2 correct to 3 significant figure

   $\left[In\left(\frac{1+x}{1-x}\right) = 2x + \frac{2}{3}x^3; 0.691\right]$

   (ii) $e^{-x}sinx$
   $\left[x - x^2 + \frac{1}{3}x^3\right]$

   (iii) $In\sqrt{\left(\frac{1+sinx}{1-sinx}\right)}$ $\left[2x + \frac{x^3}{6}\right]$

   (iv) $In(1 + sinx)$ $\left[x - \frac{x^2}{2} + \frac{x^3}{6}\right]$

   (v) $In(1 + x)^2$ $\left[2x - x^2 + \frac{2x^3}{3} - \frac{x^4}{2}\right]$

   (vi) $\frac{1}{\sqrt{(1+x)}}$ $\left[1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16}\right]$

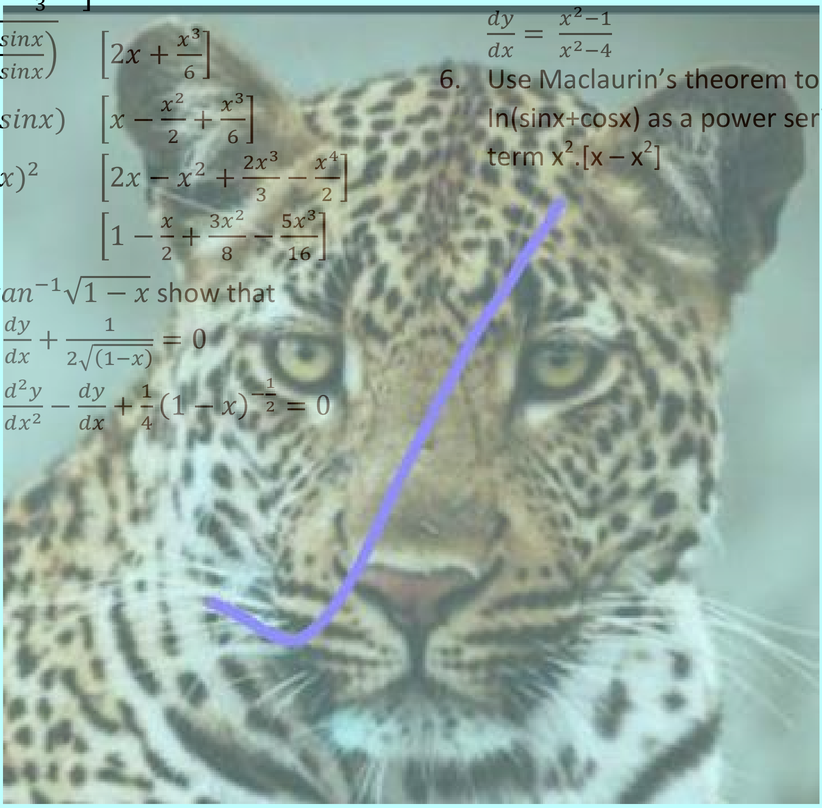
2. Given $y = tan^{-1}\sqrt{1-x}$ show that
   (i) $(2 - x)\frac{dy}{dx} + \frac{1}{2\sqrt{(1-x)}} = 0$
   (ii) $(2 - x)\frac{d^2y}{dx^2} - \frac{dy}{dx} + \frac{1}{4}(1 - x)^{-\frac{1}{2}} = 0$

3. Use Maclaurin theorem to show that
   (i) $\frac{cosx}{1-x^2} = 1 + \frac{1}{2}x^2 + \frac{11}{24}x^4$
   (ii) $e^{-x}sinx = \frac{x}{3}(x^2 - 3x + 3)$. Hence evaluate $e^{-x}sin\frac{\pi}{3}$ to 4d.p [0.3334]

4. Given that $y = e^{tan^{-1}x}$, show that $(1 + x^2)\frac{d^2y}{dx^2} + (2x - 1)\frac{dy}{dx} = 0$. Hence or otherwise, determine the first four non-zero terms of the Maclaurin expansion of y
   $\left[1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right]$

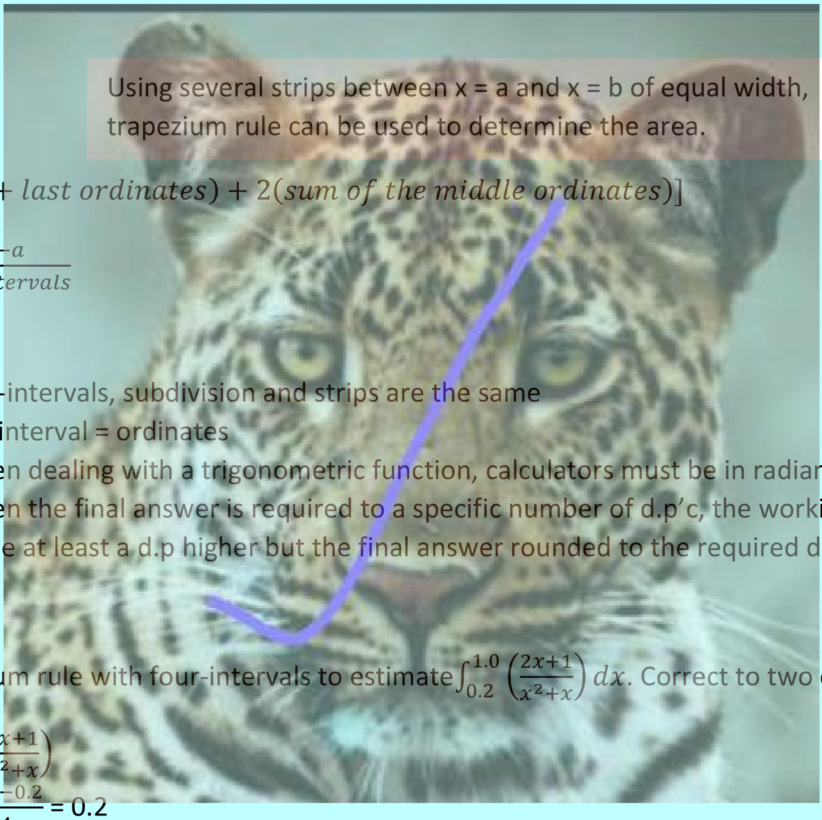
5. Given that y = $In\left\{e^x\left(\frac{x-2}{x+2}\right)^{\frac{3}{4}}\right\}$, show that
   $\frac{dy}{dx} = \frac{x^2-1}{x^2-4}$

6. Use Maclaurin's theorem to express In(sinx+cosx) as a power series up to the term x^2.$[x - x^2]$



Trapezium rule

It is used for estimating an integral area under a curve of continuous function over a given interval [a, b]



if y = f(x)

Using several strips between x = a and x = b of equal width, trapezium rule can be used to determine the area.

A = $\int_a^b y dx$

A ≈ $\frac{1}{2}h[(first + last\ ordinates) + 2(sum\ of\ the\ middle\ ordinates)]$

where h = $\frac{b-a}{subintervals}$

Note

(i) sub-intervals, subdivision and strips are the same

(ii) subinterval = ordinates

(iii) when dealing with a trigonometric function, calculators must be in radian mode

(iv) when the final answer is required to a specific number of d.p'c, the working's should be done at least a d.p higher but the final answer rounded to the required d.p's

Example 1

Use the trapezium rule with four-intervals to estimate $\int_{0.2}^{1.0} \left(\frac{2x+1}{x^2+x}\right) dx$. Correct to two decimal places.

Let y = $\left(\frac{2x+1}{x^2+x}\right)$

h = $\frac{1.0-0.2}{4}$ = 0.2

x	y = $\frac{2x+1}{x^2+x}$	
0.2	5.8333	
0.4		3.2143
0.6		2.2917
0.8		1.8056
1.0	1.5000	
Sum	7.3333	7.3116

$\int_{0.2}^{1.0} \left(\frac{2x+1}{x^2+x}\right) dx = \frac{1}{2}\ x\ 0.2(7.3333 + 7.3116\ )$

= 2.1955

= 2.20 (2D)

Example 2

Use the trapezium rule with seven coordinates to estimate

$\int_0^3 [(1.2)^x - 1]^{\frac{1}{2}}$dx correct to 2 decimal places (05marks)

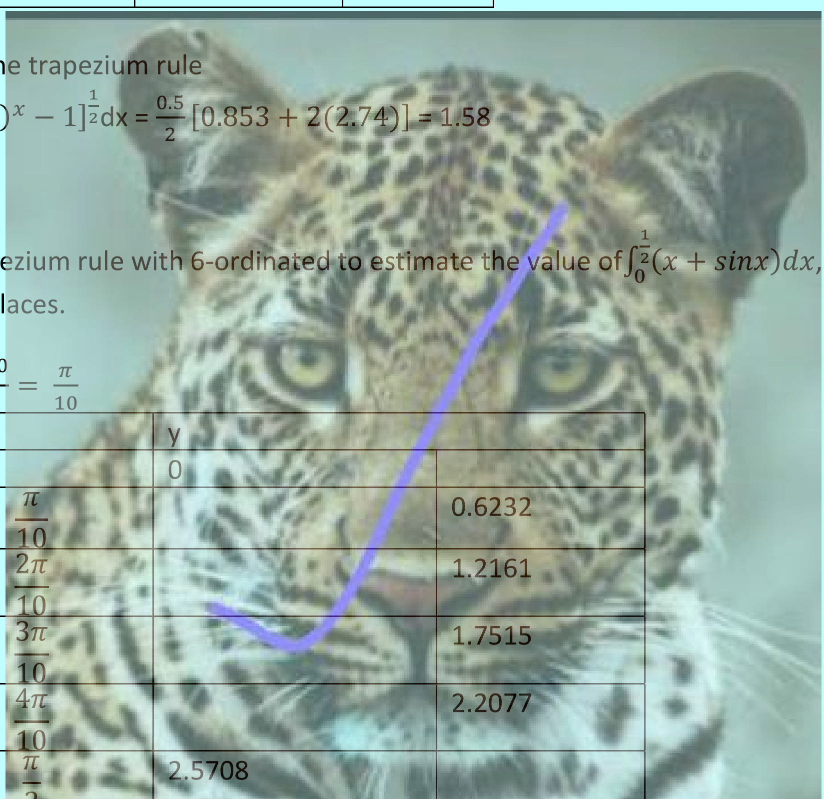
Solution

For 7 ordinates, there are 6 subintervals

Width, h = $\frac{b-a}{subinterval} = \frac{3-0}{6} = 0.5$

Let y = $\sqrt{(1.2)^x - 1}$

x	y	
0	0	
0.5		0.309
1		0.447
1.5		0.561
2		0.663
2.5		0.760
3	0.853	
Sum	0.853	2.74



Using the trapezium rule

$\int_0^3 [(1.2)^x - 1]^{\frac{1}{2}}$dx = $\frac{0.5}{2}[0.853 + 2(2.74)] = 1.58$

Example 3

(a) Use the trapezium rule with 6-ordinated to estimate the value of $\int_0^{\frac{1}{2}}(x + sinx)dx$, correct to three decimal places.

$h = \frac{\frac{1}{2}-0}{5} = \frac{\pi}{10}$

x	y	
0	0	
$\frac{\pi}{10}$		0.6232
$\frac{2\pi}{10}$		1.2161
$\frac{3\pi}{10}$		1.7515
$\frac{4\pi}{10}$		2.2077
$\frac{\pi}{2}$	2.5708	
Sum	2.5708	5.7985

$\int_0^{\frac{1}{2}}(x + sinx)dx = \frac{1}{2}\ x\ \frac{\pi}{10}(2.5708 + 2\ x\ 5.7985)$

= 2.225

(b)(i) Evaluate $\int_0^{\frac{1}{2}}(x + sinx)dx$, correct to three decimal places

$\int_0^{\frac{1}{2}}(x + sinx)dx = \left|\frac{x^2}{2} - \cos x\right|_0^{\frac{1}{2}}$

= $\frac{1}{2}\left(\frac{\pi^2}{4} - 0\right) - (\cos\frac{\pi}{2} - \cos 0)$

= $\frac{\pi^2}{8} + 1$

= 2.234

(ii) Calculate the error in your estimation in (a) above

Error = $|2.234 - 1.225|$ = 0.009

(iii) Suggest how the error may be reduced (06marks)

Increasing on number of strips or subintervals

Example 4

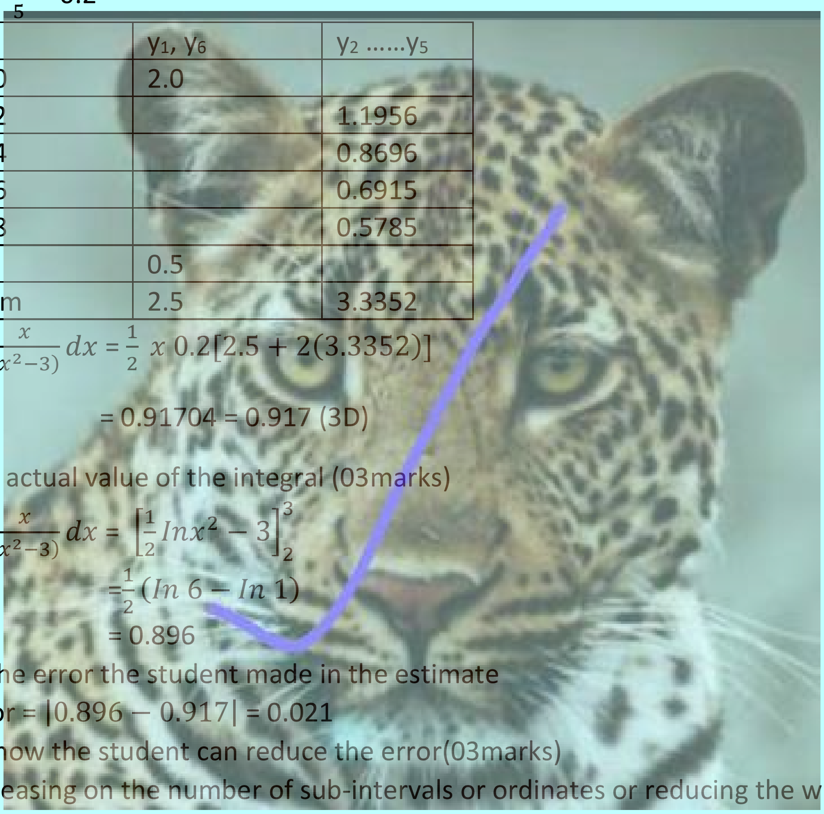
A student used the trapezium rule with five sub-intervals to estimate

$\int_2^3 \frac{x}{(x^2-3)} dx$ correct to **three** decimal places

Determine;

(a) The value the student obtained (06marks)

h = $\frac{3-2}{5}$ =0.2



X	y_1, y_6	y_2 ……y_5
2.0	2.0	
2.2		1.1956
2.4		0.8696
2.6		0.6915
2.8		0.5785
3	0.5	
Sum	2.5	3.3352

$\int_2^3 \frac{x}{(x^2-3)} dx = \frac{1}{2}\ x\ 0.2[2.5 + 2(3.3352)]$

= 0.91704 = 0.917 (3D)

(b) The actual value of the integral (03marks)

$\int_2^3 \frac{x}{(x^2-3)} dx = \left[\frac{1}{2} Inx^2 - 3\right]_2^3$

$= \frac{1}{2}(In\ 6 - In\ 1)$

= 0.896

(c) (i) the error the student made in the estimate

Error = $|0.896 - 0.917|$ = 0.021

(ii) how the student can reduce the error(03marks)

Increasing on the number of sub-intervals or ordinates or reducing the width of h

Example 5

Use trapezium rule with 4 subintervals to estimate to 3 decimal places $\int_0^{\frac{\pi}{2}} \cos x dx$

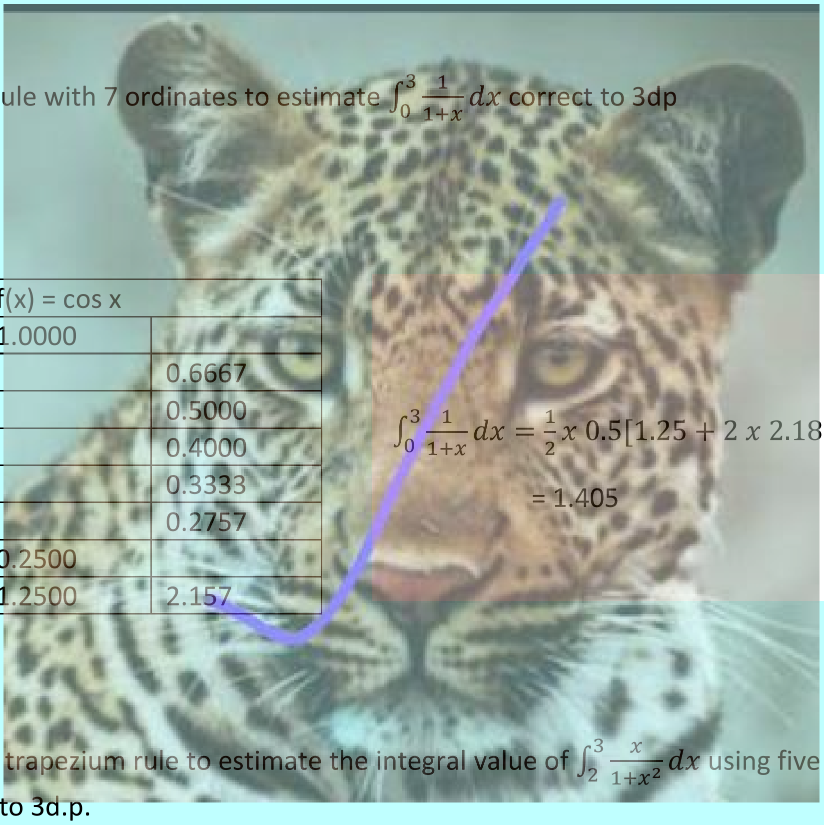
Solution

h = $\frac{\frac{\pi}{2}-0}{4} = \frac{\pi}{8}$

x	f(x) = cos x	
0	1.0000	
$\frac{\pi}{8}$		0.9239
$\frac{2\pi}{8}$		0.7071
$\frac{3\pi}{8}$		0.3827
$\frac{4\pi}{8}$	0.0000	
sum	1.0000	2.0137

$\int_0^{\frac{\pi}{2}} \cos x dx = \frac{1}{2} x \frac{\pi}{8}[1 + 2\ x\ 2.0137]$

=0.987



Example 6

Use trapezium rule with 7 ordinates to estimate $\int_0^3 \frac{1}{1+x} dx$ correct to 3dp

Solution

h = $\frac{3-0}{7-1} = 0.5$

x	f(x) = cos x	
0	1.0000	
0.5		0.6667
1.0		0.5000
1.5		0.4000
2.0		0.3333
2.5		0.2757
3.0	0.2500	
sum	1.2500	2.157

$\int_0^3 \frac{1}{1+x} dx = \frac{1}{2} x\ 0.5[1.25 + 2\ x\ 2.1857]$

= 1.405

Example 7

(a) Use the trapezium rule to estimate the integral value of $\int_2^3 \frac{x}{1+x^2} dx$ using five subinterval and correct to 3d.p.

(b) (i) find the exact value of $\int_2^3 \frac{x}{1+x^2} dx$

(ii) suggest how the error may be reduced.

(a) h = $\frac{3-2}{5} = 0.2$

x	f(x) = $\frac{x}{1+x^2}$	
2.0	0.40000	
2.2		0.37671
2.4		0.35503
2.6		0.33505
2.8		0.31674
3.0	0.30000	
sum	0.70000	1.3353

$\int_2^3 \frac{x}{1+x^2} dx = \frac{1}{2} x\ 0.2[0.7 + 2\ x\ 1.38353]$

= 0.3467

(b)(i) $\int_2^3 \frac{x}{1+x^2}dx = \left[\frac{1}{2}In(1+x^2)\right]_2^3 = \frac{1}{2}(In10\ In5) = 0.3466$

(ii) error = $|exact\ value - approximate\ value| = |0.3466 - 0.3467| = 0.0001$

(iii) the error can be reduced by reducing h or increasing the number of sub-intervals.

Example 8

(a) Use trapezium rule to estimate the integral value of $\int_0^1 x^2e^x\,dx$

(b) (i) find exact value of $\int_0^1 x^2e^x\,dx$

(ii) determine the percentage error in your estimation

(a) h = $\frac{1-0}{5} = 0.2$

x	f(x) = x^2e^x	
0	0	
0.2		0.0489
0.4		0.2387
0.6.		0.6560
0.8		1.4243
1.0	2.7183	
sum	2.7183	2.3679

$\int_0^1 x^2e^x\,dx = \frac{1}{2}x\ 0.2[2.713 + 2\ x\ 2.3679]$

=0.74541≈ 0.745

(b)(i) $\int_0^1 x^2e^x\,dx = [x^2e^x - 2xe^x + 2e^x]_0^1$

= 0.718

(ii) error = |0.718 – 0.745|= 0.027

Percentage error = $\frac{error}{exact\ value}\ x\ 100\%$

$= \frac{0.027}{0.718}\ x\ 100 =$ 3.8%

Revision Exercise

1. (a) Use trapezium rule with six strips to estimate $\int_0^{\pi} xsinxdx$ [3.069]
   (b) Determine the percentage error in your determination. [2.3%]
2. Use the trapezium rule to estimate the approximate value of $\int_0^1 \frac{1}{1+x^2dx}$ using 6 ordinates and correct to 3 decimal places. [0.784]
3. (a) Use trapezium rule with six strips to estimate $\int_2^4 \frac{10}{2x+1}dx$ correct 4dp. [2.9418]
   (b) Determine the percentage error in your estimation and suggest how this error may be reduce. [0.098%]
4. (a) Use trapezium rule to estimate the area of y = 3x between x-axis, x = 1 and x = 2, using five subintervals. Give your answer correct to four significant figures. [5.483]
   (b) Find the exact value of $\int_1^2 3^x dx$ [5.461]
   (c) Find the exact percentage error in calculations (a) and (b) above. [0.4028%]
5. Use trapezium rule with 7 ordinates to estimate $\int_0^3 \frac{1}{1+x}dx$, correct to 3 decimal places [1.405]
6. Use the trapezium rule with 6 ordinates to evaluate $\int_0^1 e^{-x^2}$ correct to 2 decimal place. [0.74]
7. Use the trapezium rule with 6 ordinates to estimate $\int_1^2 \frac{Inx}{x}dx$.Give your answer correct to 3 decimal places [0.237]
8. Find the approximate value to one decimal place of $\int_0^1 \frac{dx}{1+x}$, using the trapezium rule with five strips. [0.7]
9. (a) Use trapezium rule with five subintervals to estimate $\int_0^{\frac{\pi}{3}} \tan x\,dx$ correct to 3dp. [0.704]

(b) (i) Find the exact value of $\int_0^{\frac{\pi}{3}} \tan x \, dx$ to 3 d.p. [0.693]

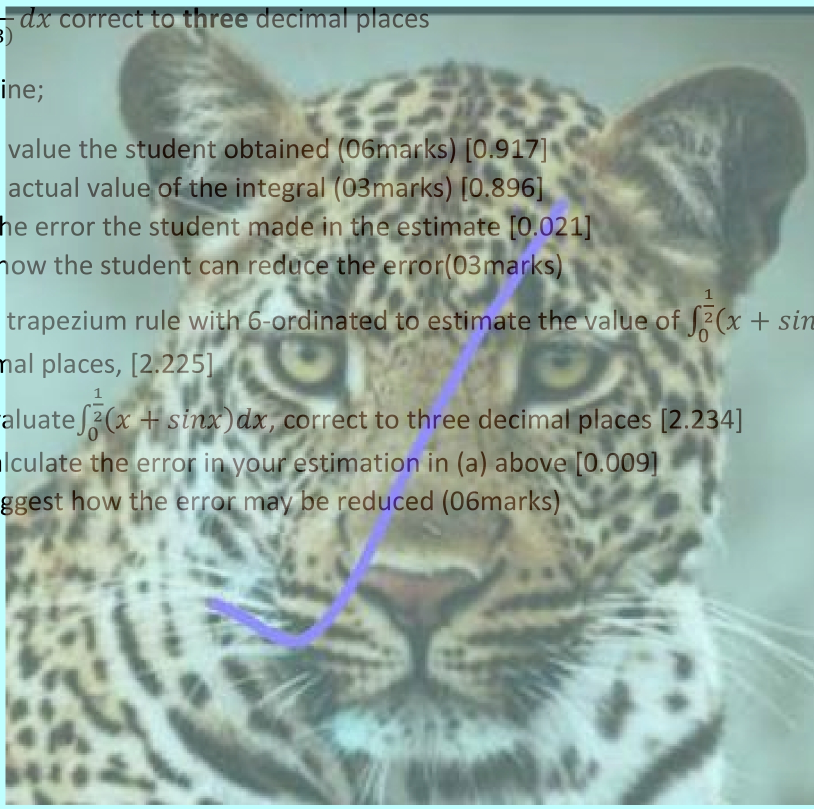
(ii) Calculate the percentage error in your estimation in (a) above [1.587%]

(iii) Suggest how the percentage error in (b)(ii) may be reduced.

10. Use the trapezium rule with four subdivisions to estimate $\int_0^{\frac{\pi}{2}} \frac{1}{1+sinx} dx$. Give your answer correct to three decimal places. [1.013]
11. Find the approximate value of $\int_0^{2} \frac{1}{1+x^2} dx$ using trapezium rule with 6 ordinates. Give your answer to 3 decimal places (05marks)[1.105]
12. Use the trapezium rule with five subintervals to estimate

    $\int_2^{4} \frac{5}{(x+1)} dx$. Give your answer correct to 3 decimal places (05marks)[2.559]
13. A student used the trapezium rule with five sub-intervals to estimate

    $\int_2^{3} \frac{x}{(x^2-3)} dx$ correct to **three** decimal places



    Determine;

    (a) The value the student obtained (06marks) [0.917]

    (b) The actual value of the integral (03marks) [0.896]

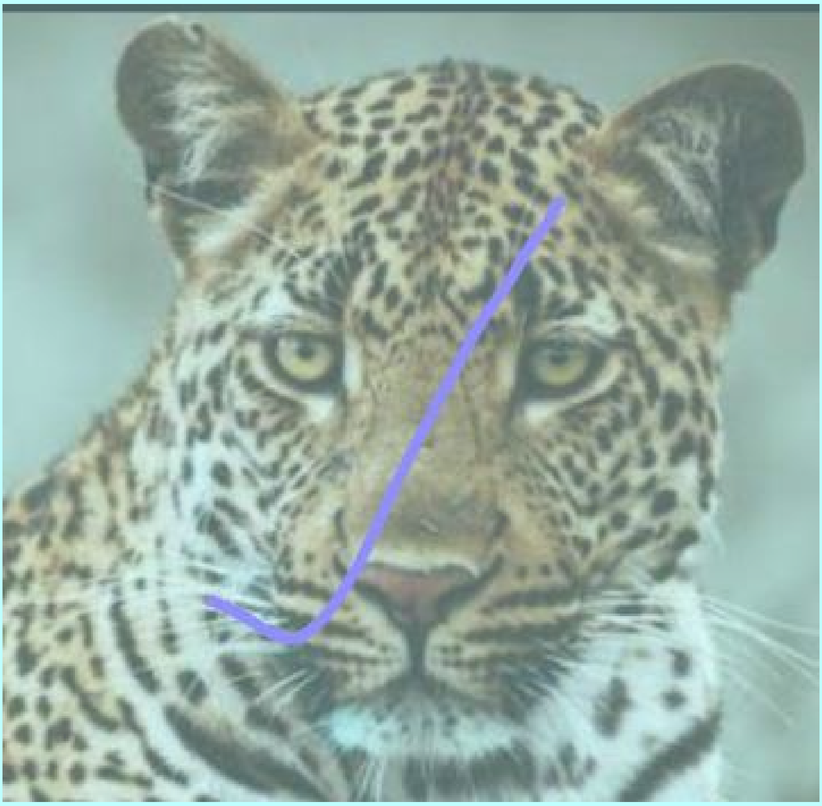
    (c) (i) the error the student made in the estimate [0.021]

    (ii) how the student can reduce the error(03marks)
14. (a) Use the trapezium rule with 6-ordinated to estimate the value of $\int_0^{\frac{1}{2}}(x + sinx)dx$, correct to three decimal places, [2.225]

    (b)(i) Evaluate $\int_0^{\frac{1}{2}}(x + sinx)dx$, correct to three decimal places [2.234]

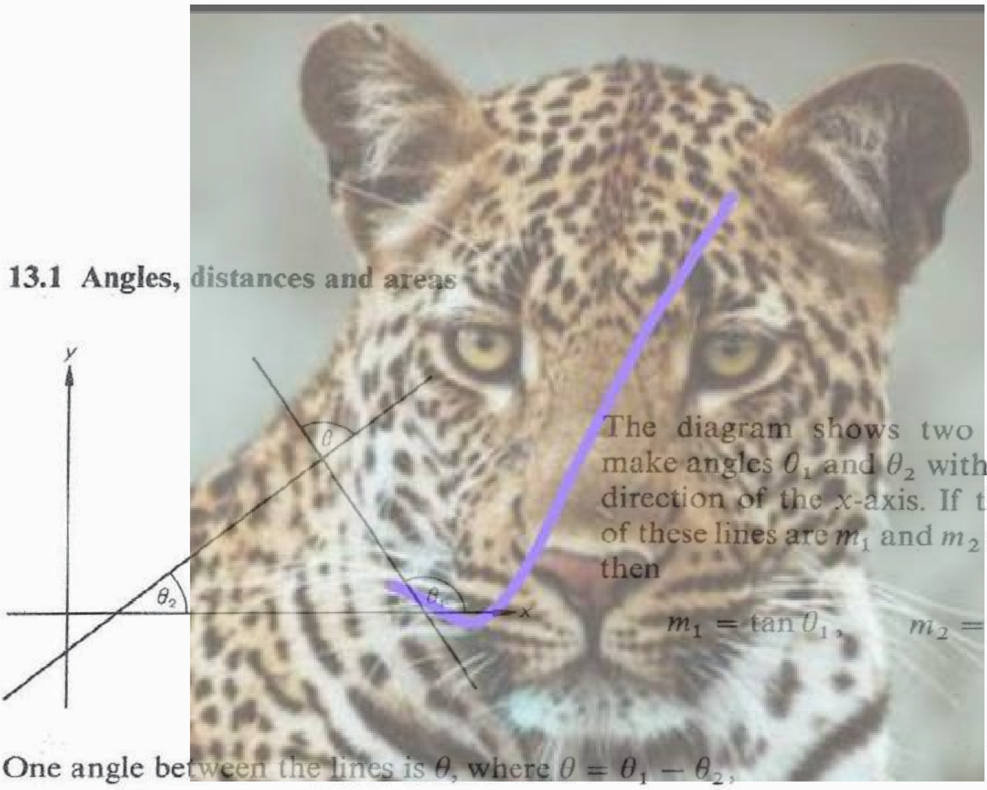
    (ii) Calculate the error in your estimation in (a) above [0.009]

    (iii) suggest how the error may be reduced (06marks)



13 Coordinate geometry

13.1 Angles, distances and areas



The diagram shows two lines which make angles θ_1 and θ_2 with the positive direction of the x-axis. If the gradients of these lines are m_1 and m_2 respectively, then

$$m_1 = \tan\theta_1, \qquad m_2 = \tan\theta_2.$$

One angle between the lines is θ, where $\theta = \theta_1 - \theta_2$,

$$\therefore \qquad \tan\theta = \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}.$$

Hence the angle θ between two lines with gradients m_1 and m_2 is given by

$$\boxed{\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}}$$

Example 1 Find the acute angle between the straight lines $3x - y + 2 = 0$ and $x - 2y - 1 = 0$.

The equations of the lines may be re-written as:

$y = 3x + 2$ and $y = \frac{1}{2}x - \frac{1}{2}$.

Hence the gradients of the lines are 3 and $\frac{1}{2}$.

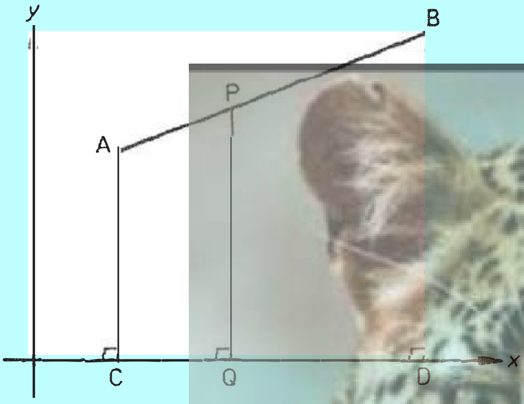
If θ is one angle between the lines, then

$$\tan\theta = \frac{3 - \frac{1}{2}}{1 + 3 \times \frac{1}{2}} = \frac{2\frac{1}{2}}{2\frac{1}{2}} = 1$$

$\therefore$ the acute angle between the lines is 45°.

[Note that the angle between two curves at any point of intersection is defined as the angle between the tangents to the curves at the point.]

Consider now the line joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$. Let P be the point which divides AB in the ratio $\mu:\lambda$. Then if C, D and Q are the feet of the perpendiculars from A, B and P to the x-axis, Q must divide CD in the ratio $\mu:\lambda$. It follows that the x-coordinate of P



$$= x_1 + \frac{\mu}{\lambda + \mu}(x_2 - x_1) = \frac{\lambda x_1 + \mu x_2}{\lambda + \mu}$$

A similar expression is obtained for the y-coordinate of P.

i.e. $\boxed{\text{The point dividing the line } AB \text{ in the ratio } \mu:\lambda \text{ has coordinates } \left(\frac{\lambda x_1 + \mu x_2}{\lambda + \mu}, \frac{\lambda y_1 + \mu y_2}{\lambda + \mu}\right).}$

This formula can be used when P divides AB externally in a given ratio if μ and λ are taken to have opposite signs.
[We recall that the corresponding formula for the position vector of a point which divides a line in the ratio $\mu:\lambda$ was obtained in §7.1.]

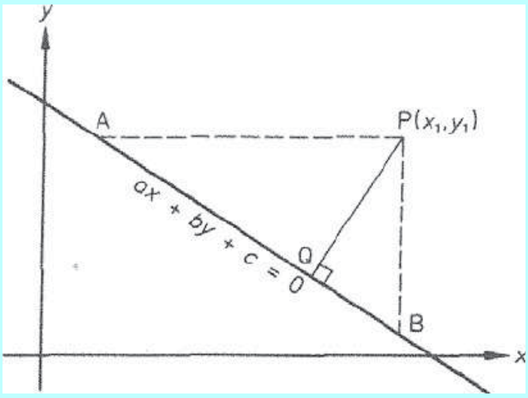
Example 2 If A and B are the points (2, 1) and (−4, 4) respectively, find the coordinates of the point C which divides AB internally in the ratio 2:1 and of the point D which divides AB externally in the ratio 3:2.

The coordinates of C are

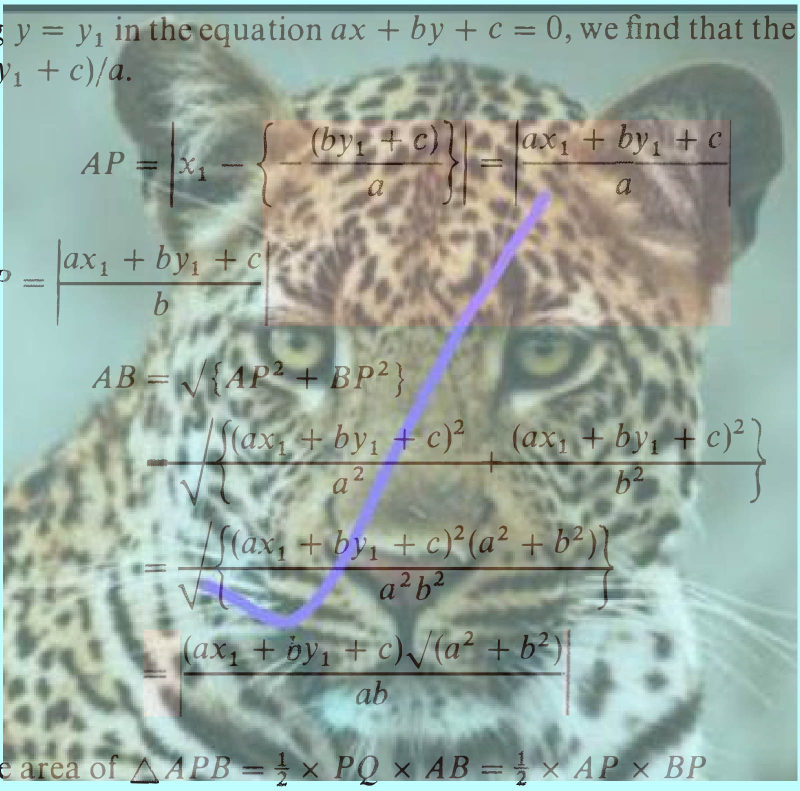
$$\left(\frac{1 \times 2 + 2(-4)}{1 + 2}, \frac{1 \times 1 + 2 \times 4}{1 + 2}\right) \quad \text{i.e.} \quad (-2, 3)$$

Regarding D as the point which divides AB in the ratio 3: −2, the coordinates of D are

$$\left(\frac{(-2)2 + 3(-4)}{-2 + 3}, \frac{(-2)1 + 3 \times 4}{-2 + 3}\right) \quad \text{i.e.} \quad (-16, 10).$$



Next we derive a formula for the perpendicular distance from a point $P(x_1, y_1)$ to a straight line $ax + by + c = 0$. In the diagram A and B are points on the line $ax + by + c = 0$, such that AP is parallel to the x-axis and BP is parallel to the y-axis. The point Q is the foot of the perpendicular from P to the line and so PQ is the required perpendicular distance.



Substituting $y = y_1$ in the equation $ax + by + c = 0$, we find that the x-coordinate of A is $-(by_1 + c)/a$.

$$\therefore \qquad AP = \left| x_1 - \left\{ -\frac{(by_1 + c)}{a} \right\} \right| = \left| \frac{ax_1 + by_1 + c}{a} \right|$$

Similarly $BP = \left| \dfrac{ax_1 + by_1 + c}{b} \right|$

$$\therefore \qquad AB = \sqrt{\{AP^2 + BP^2\}}$$

$$= \sqrt{\left\{ \frac{(ax_1 + by_1 + c)^2}{a^2} + \frac{(ax_1 + by_1 + c)^2}{b^2} \right\}}$$

$$= \sqrt{\left\{ \frac{(ax_1 + by_1 + c)^2 (a^2 + b^2)}{a^2 b^2} \right\}}$$

$$= \left| \frac{(ax_1 + by_1 + c)\sqrt{(a^2 + b^2)}}{ab} \right|$$

However, the area of $\triangle APB = \frac{1}{2} \times PQ \times AB = \frac{1}{2} \times AP \times BP$

$$\therefore \qquad PQ = \frac{AP \times BP}{AB}$$

$$= \left| \frac{(ax_1 + by_1 + c)^2}{ab} \times \frac{ab}{(ax_1 + by_1 + c)\sqrt{(a^2 + b^2)}} \right|$$

$$= \left| \frac{ax_1 + by_1 + c}{\sqrt{(a^2 + b^2)}} \right|$$

i.e. The perpendicular distance from the point (x_1, y_1) to the line

$$ax + by + c = 0 \quad \text{is} \quad \left| \frac{ax_1 + by_1 + c}{\sqrt{(a^2 + b^2)}} \right|$$

Example 3 Find the distance of the point $(3, -5)$ from the line $2x - y = 1$.

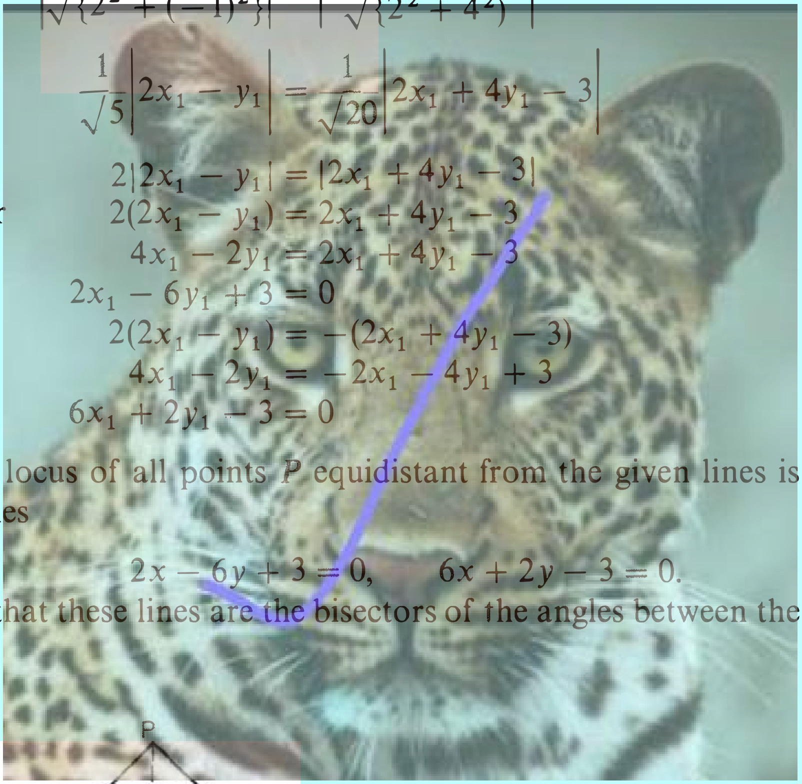
The distance of the point $(3, -5)$ from the line $2x - y - 1 = 0$ is

$$\left|\frac{2 \times 3 - (-5) - 1}{\sqrt{\{2^2 + (-1)^2\}}}\right| = \frac{10}{\sqrt{5}} = 2\sqrt{5}.$$

Example 4 Find the locus of points equidistant from the lines $y = 2x$ and $2x + 4y - 3 = 0$.

Let $P(x_1, y_1)$ be a point equidistant from the lines $2x - y = 0$ and $2x + 4y - 3 = 0$, then

$$\left|\frac{2x_1 - y_1}{\sqrt{\{2^2 + (-1)^2\}}}\right| = \left|\frac{2x_1 + 4y_1 - 3}{\sqrt{\{2^2 + 4^2\}}}\right|$$



$$\therefore \quad \frac{1}{\sqrt{5}}|2x_1 - y_1| = \frac{1}{\sqrt{20}}|2x_1 + 4y_1 - 3|$$

$$\therefore \quad 2|2x_1 - y_1| = |2x_1 + 4y_1 - 3|$$

Thus either

$$2(2x_1 - y_1) = 2x_1 + 4y_1 - 3$$
$$4x_1 - 2y_1 = 2x_1 + 4y_1 - 3$$
$$2x_1 - 6y_1 + 3 = 0$$

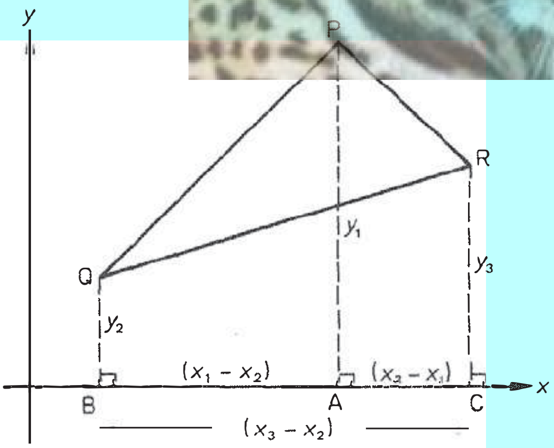
or

$$2(2x_1 - y_1) = -(2x_1 + 4y_1 - 3)$$
$$4x_1 - 2y_1 = -2x_1 - 4y_1 + 3$$
$$6x_1 + 2y_1 - 3 = 0$$

Hence the locus of all points P equidistant from the given lines is the pair of straight lines

$$2x - 6y + 3 = 0, \qquad 6x + 2y - 3 = 0.$$

It follows that these lines are the bisectors of the angles between the given lines.



Finally we consider the area of the triangle whose vertices are $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$.

As shown in the diagram, the area of ΔPQR = area $PQBA$ + area $PACR$ − area $QBCR$.

Since each of these areas is a trapezium,

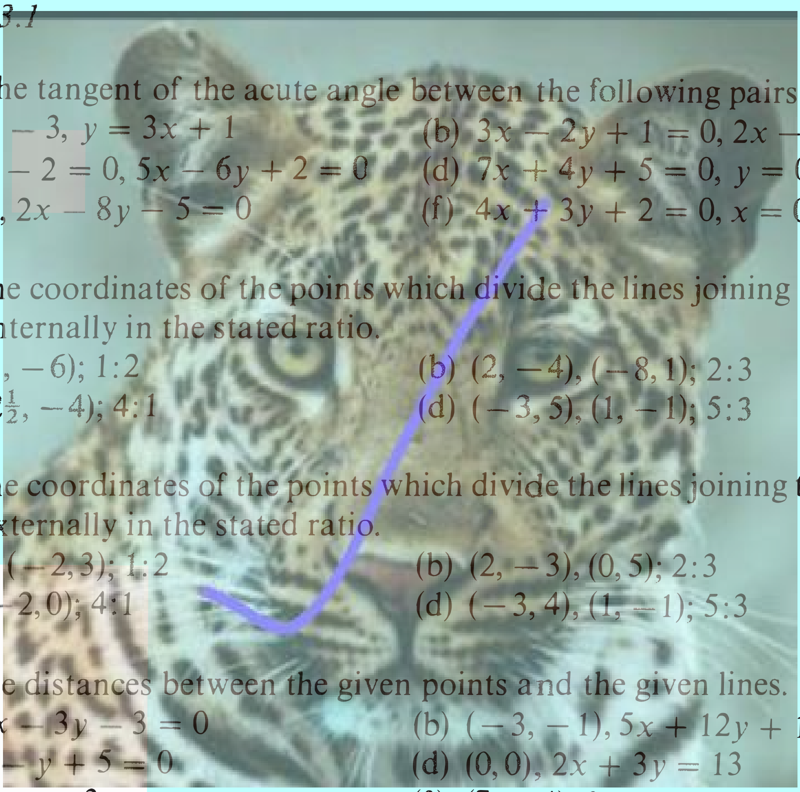
$$\text{area of } \Delta PQR = \tfrac{1}{2}(x_1 - x_2)(y_1 + y_2) + \tfrac{1}{2}(x_3 - x_1)(y_1 + y_3) - \tfrac{1}{2}(x_3 - x_2)(y_2 + y_3)$$
$$= \tfrac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}$$

It is found that, although this result holds for any points P, Q and R, it sometimes gives a negative value for the area. We must therefore include modulus signs in the general formula.

$$\text{Area of triangle} = \tfrac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$$

[Note that this formula is rarely needed in elementary work. Problems on area usually involve right-angled triangles, isosceles triangles or other triangles whose heights and bases are easily calculated.]

Exercise 13.1



1. Find the tangent of the acute angle between the following pairs of lines

(a) $y = 2x - 3$, $y = 3x + 1$
(b) $3x - 2y + 1 = 0$, $2x - y - 7 = 0$
(c) $x + 3y - 2 = 0$, $5x - 6y + 2 = 0$
(d) $7x + 4y + 5 = 0$, $y = 0$
(e) $y = 4x$, $2x - 8y - 5 = 0$
(f) $4x + 3y + 2 = 0$, $x = 0$.

2. Find the coordinates of the points which divide the lines joining the given pairs of points internally in the stated ratio.

(a) $(7, 3)$, $(1, -6)$; $1:2$
(b) $(2, -4)$, $(-8, 1)$; $2:3$
(c) $(0, 6)$, $(2\frac{1}{2}, -4)$; $4:1$
(d) $(-3, 5)$, $(1, -1)$; $5:3$

3. Find the coordinates of the points which divide the lines joining the given pairs of points externally in the stated ratio.

(a) $(5, -1)$, $(-2, 3)$; $1:2$
(b) $(2, -3)$, $(0, 5)$; $2:3$
(c) $(4, 3)$, $(-2, 0)$; $4:1$
(d) $(-3, 4)$, $(1, -1)$; $5:3$

4. Find the distances between the given points and the given lines.

(a) $(1, 2)$, $4x - 3y - 3 = 0$
(b) $(-3, -1)$, $5x + 12y + 1 = 0$
(c) $(2, 5)$, $x - y + 5 = 0$
(d) $(0, 0)$, $2x + 3y = 13$
(e) $(-6, 10)$, $x = 3$
(f) $(7, -4)$, $3y - x - 1 = 0$

5. Find the areas of the triangles with the following vertices:

(a) $(2, 3)$, $(5, -1)$, $(2, -1)$
(b) $(-3, 4)$, $(5, 1)$, $(-3, -2)$
(c) $(-1, 3)$, $(2, -3)$, $(4, 5)$
(d) $(3, -1)$, $(-2, 0)$, $(1, 4)$
(e) $(5, 7)$, $(-4, -11)$, $(0, -3)$
(f) $(-2, 6)$, $(3, -7)$, $(4, 5)$

6. Find the equations of the bisectors of the angles between the given pairs of lines

(a) $3x - y - 3 = 0$, $x + 3y + 1 = 0$
(b) $2x + 3y = 2$, $3x + 2y = 3$
(c) $3x + 6y - 1 = 0$, $x - 2y + 1 = 0$
(d) $y = 7x - 1$, $y = x + 5$

7. Find the angles of the triangle PQR with vertices $P(1, 4)$, $Q(3, -2)$ and $R(5, 2)$.

8. Find the angles of the triangle ABC with vertices $A(1, -3)$, $B(4, 6)$ and $C(-1, 1)$.

9. A triangle has vertices $A(-4, 1)$, $B(3, 0)$ and $C(1, 2)$. If the internal bisector of $\angle B$ meets AC at P, use the fact that $AP:PC = AB:BC$ to find the coordinates of P.

10. A triangle has vertices $P(1, -2)$, $Q(5, 1)$ and $R(6, 10)$. Find the coordinates of the point on QR which is equidistant from the sides PQ and PR.

11. Find the tangent of the acute angle between the parabola $y^2 = 4x$ and the circle $x^2 + y^2 = 5$ at their points of intersection.

12. The curves $y = x^2$ and $y^2 = 8x$ intersect at the origin and at a point A. Find the angle between the curves at A.

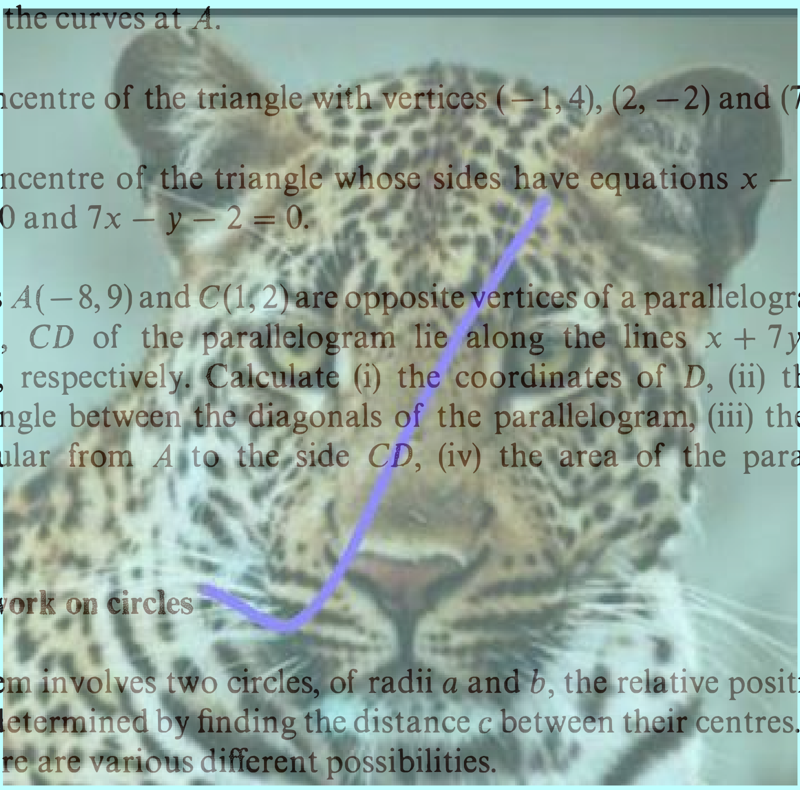
13. Find the incentre of the triangle with vertices $(-1, 4)$, $(2, -2)$ and $(7, 8)$.

14. Find the incentre of the triangle whose sides have equations $x - y + 1 = 0$, $x + 7y + 9 = 0$ and $7x - y - 2 = 0$.

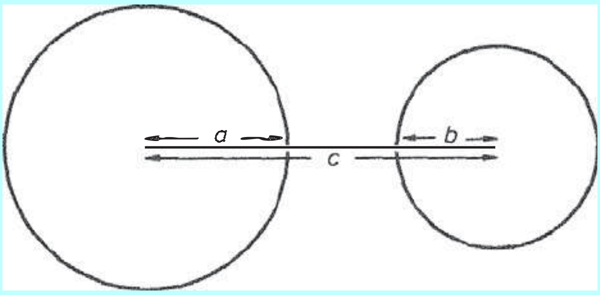
15. The points $A(-8, 9)$ and $C(1, 2)$ are opposite vertices of a parallelogram $ABCD$. The sides BC, CD of the parallelogram lie along the lines $x + 7y - 15 = 0$, $x - y + 1 = 0$, respectively. Calculate (i) the coordinates of D, (ii) the tangent of the acute angle between the diagonals of the parallelogram, (iii) the length of the perpendicular from A to the side CD, (iv) the area of the parallelogram. (JMB)

13.2 Further work on circles

When a problem involves two circles, of radii a and b, the relative positions of the circles can be determined by finding the distance c between their centres. Assuming that $a > b$, there are various different possibilities.

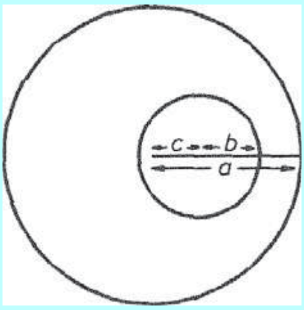


(1)



$c > a + b$

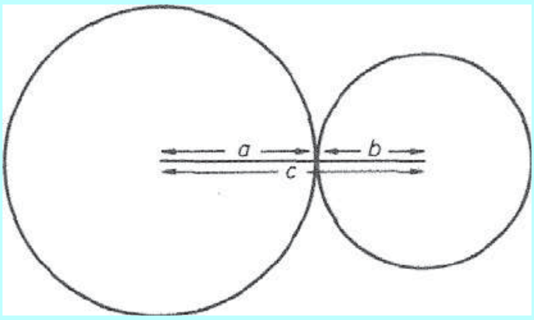
(2)



$c < a - b$

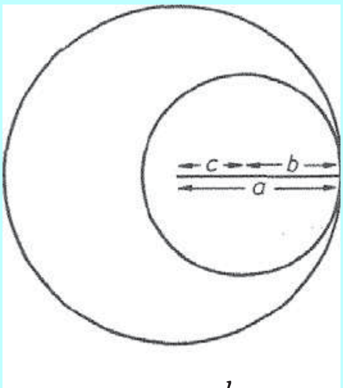
In diagrams (1) and (2) the circles do not intersect.

(3)



$c = a + b$

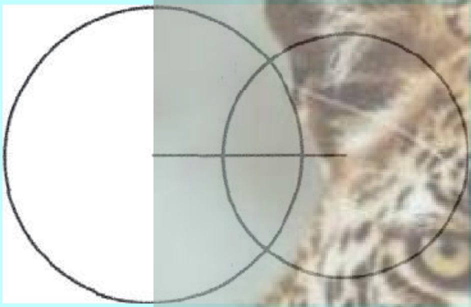
(4)



$c = a - b$

In diagrams (3) and (4) the circles touch either externally or internally.

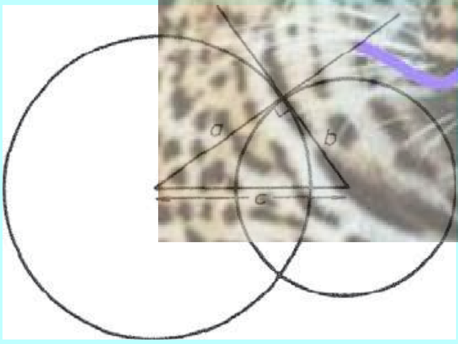
(5)



$a - b < c < a + b$

In diagram (5) the circles have two distinct points of intersection.

(6)



$c^2 = a^2 + b^2$

Two circles which cut at right-angles are called *orthogonal* circles. Diagram (6) shows two such circles.

The *common chord* of two circles is the line joining their points of intersection. The equation of this chord can be obtained without finding the coordinates of the points of intersection.

Example 1 The circles $x^2 + y^2 + 4x - 3y + 1 = 0$ and $x^2 + y^2 + x - y - 2 = 0$ intersect at the points A and B. Find the equation of the common chord AB.

$$x^2 + y^2 + 4x - 3y + 1 = 0 \qquad (1)$$
$$x^2 + y^2 + x - y - 2 = 0 \qquad (2)$$

Subtracting (2) from (1) we obtain:

$$3x - 2y + 3 = 0$$

Since the coordinates of A and B satisfy equations (1) and (2), they must also satisfy this new equation, which represents a straight line.
Hence $3x - 2y + 3 = 0$ is the equation of AB.

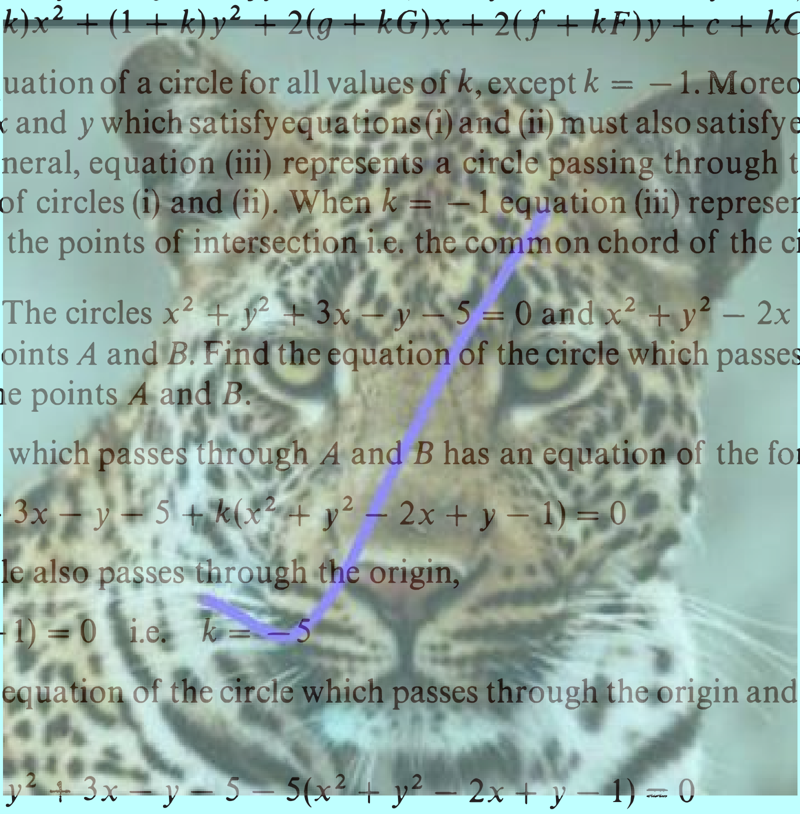
We now extend the method of Example 1 by considering two intersecting circles with equations:

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{(i)}$$
$$x^2 + y^2 + 2Gx + 2Fy + C = 0 \qquad \text{(ii)}$$

From these equations we can form a new equation:

$$x^2 + y^2 + 2gx + 2fy + c + k(x^2 + y^2 + 2Gx + 2Fy + C) = 0 \qquad \text{(iii)}$$

i.e. $\qquad (1 + k)x^2 + (1 + k)y^2 + 2(g + kG)x + 2(f + kF)y + c + kC = 0$



This is the equation of a circle for all values of k, except $k = -1$. Moreover, any pair of values of x and y which satisfy equations (i) and (ii) must also satisfy equation (iii). Hence, in general, equation (iii) represents a circle passing through the points of intersection of circles (i) and (ii). When $k = -1$ equation (iii) represents a straight line through the points of intersection i.e. the common chord of the circles.

Example 2 The circles $x^2 + y^2 + 3x - y - 5 = 0$ and $x^2 + y^2 - 2x + y - 1 = 0$ intersect at points A and B. Find the equation of the circle which passes through the origin and the points A and B.

Any circle which passes through A and B has an equation of the form:

$x^2 + y^2 + 3x - y - 5 + k(x^2 + y^2 - 2x + y - 1) = 0$

If this circle also passes through the origin,

$-5 + k(-1) = 0 \quad$ i.e. $\quad k = -5$

Hence the equation of the circle which passes through the origin and the points A and B is

$$x^2 + y^2 + 3x - y - 5 - 5(x^2 + y^2 - 2x + y - 1) = 0$$

i.e. $$4x^2 + 4y^2 - 13x + 6y = 0.$$

Exercise 13.2

1. Two circles have equations $x^2 + y^2 - 2x - 6y - 54 = 0$ and
$x^2 + y^2 - 8x + 2y + 13 = 0.$

Show that one circle lies entirely inside the other.

2. Prove that the circles whose equations are $x^2 + y^2 - 2x - 2y - 2 = 0$, $x^2 + y^2 - 6x - 10y + 33 = 0$ lie entirely outside one another.

3. A circle C_1 has equation $x^2 + y^2 - 32x - 24y + 300 = 0$ and a second circle C_2 has as diameter the line joining the points $(8, 0)$ and $(0, 6)$. Show that the circles C_1 and C_2 touch externally.

4. A circle C has equation $x^2 + y^2 - 4x + 2y = 40$. Find the equations of the circles with centre $(3, 1)$ which touch circle C. Find also the coordinates of the points of contact.

5. Show that the circles $x^2 + y^2 - 2x - 2y - 2 = 0$ and

$$x^2 - y^2 - 8x - 10y + 32 = 0$$

touch externally and find the coordinates of the point of contact.

6. Find the equations of the two circles with centres $A(1, 1)$ and $B(9, 7)$ which have equal radii and touch each other externally. Find also the equations of the common tangents to these circles.

7. Two circles with centres $(3, -5)$ and $(-6, 7)$ pass through the point $(4, 2)$. Find the equations of these circles. Find also the equation and the length of their common chord.

8. Prove that the circles whose equations are $x^2 + y^2 - 4y - 5 = 0$, $x^2 + y^2 - 8x + 2y + 1 = 0$ cut orthogonally and find the equation of the common chord.

9. Find the equation of the circle with centre $(3, -2)$ and radius 5 units. Find also the equation of the circle with centre $(-7, 3)$ which intersects the original circle at right angles.

10. Show that any circle which passes through the points $(1, 0)$ and $(-1, 0)$ has an equation of the form $x^2 + y^2 - 2\lambda y - 1 = 0$. Prove also that if the circles given by $\lambda = \lambda_1$ and $\lambda = \lambda_2$ cut orthogonally then $\lambda_1 \lambda_2 = -1$.

11. Show that if the circles $x^2 + y^2 + 2gx + 2fy + c = 0$ and

$$x^2 + y^2 + 2Gx + 2Fy + C = 0$$

cut orthogonally, then $2gG + 2fF = c + C$.

12. The circles $x^2 + y^2 - 2x = 0$ and $x^2 + y^2 + 4x - 6y - 3 = 0$ intersect at the points A and B. Find (a) the equation of AB, (b) the equation of the circle which passes through A, B and the point $(1, 2)$.

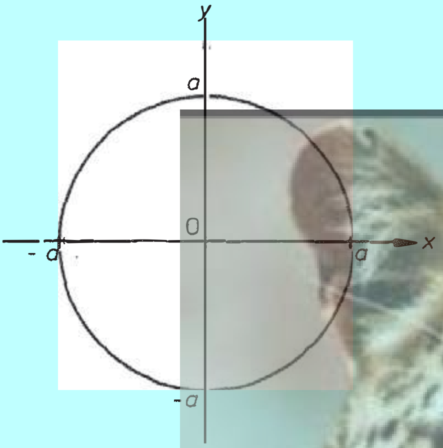
13. Given that the circles $x^2 + y^2 - 3y = 0$ and $x^2 + y^2 + 5x - 8y + 5 = 0$ intersect at P and Q, find the equation of the circle which passes through P, Q and (a) the point $(1, 1)$, (b) the origin, (c) touches the x-axis.

14. The circle $x^2 + y^2 + 3x - 5y - 4 = 0$ and the straight line $y = 2x + 5$ intersect at the points A and B. Find the equation of the circle which passes through A, B and (a) the point $(3, 1)$, (b) the origin, (c) has its centre on the y-axis.

15. Show that the circles $x^2 + y^2 + 8x + 2y + 8 = 0$, $x^2 + y^2 - 2x + 2y - 2 = 0$ have three common tangents, and find their equations. (C)

13.3 Loci using parametric forms

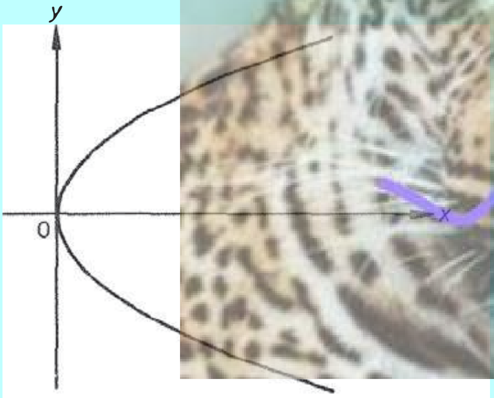
Many problems in coordinate geometry are best solved using the parametric equations of curves. The basic techniques involved were introduced in Book 1 and used to investigate the properties of various curves including the parabola $y^2 = 4ax$. We now consider further applications of these methods. As clear diagrams are often useful in this type of work, we begin by giving sketches of several important curves together with the most commonly used parametric forms of their equations.



The circle $x^2 + y^2 = a^2$.
Parametric equations:

$$x = a\cos\theta,$$
$$y = a\sin\theta.$$

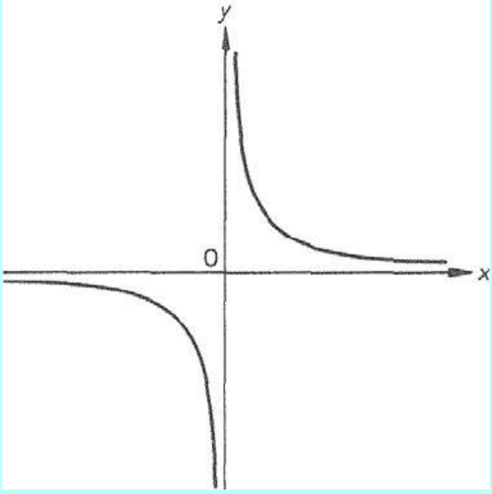
This is the circle with centre O and radius a.



The parabola $y^2 = 4ax$.
Parametric equations:

$$x = at^2,$$
$$y = 2at.$$

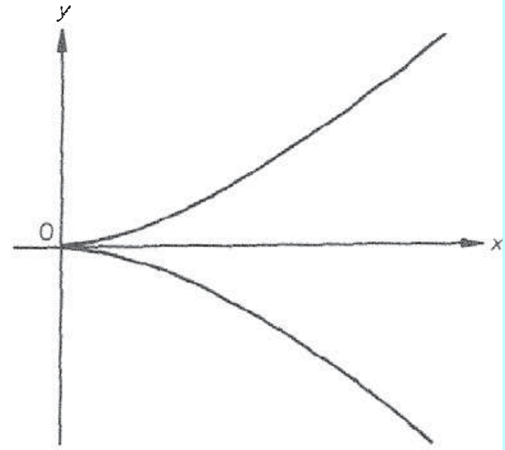
This parabola has vertex O and is symmetrical about the x-axis.



The rectangular hyperbola $xy = c^2$.
Parametric equations:

$$x = ct,$$
$$y = c/t.$$

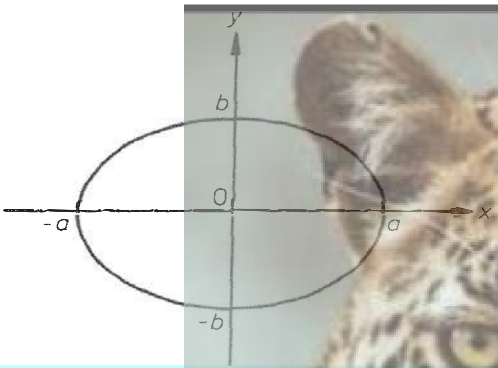
The x- and y-axes are asymptotes to the curve and the curve is symmetrical about the lines $y = \pm x$.



The semi-cubical parabola $ay^2 = x^3$.
Parametric equations:

$$x = at^2,$$
$$y = at^3.$$

The curve has a *cusp* at O and is symmetrical about the x-axis.



The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.
Parametric equations:

$$x = a\cos\theta,$$
$$y = b\sin\theta.$$

The curve is symmetrical about the x- and y-axes.

Example 1 Find the equation of the chord of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ joining the points $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$.

Gradient of $PQ = \dfrac{b\sin\theta - b\sin\phi}{a\cos\theta - a\cos\phi}$

$$= \frac{2b\cos\frac{1}{2}(\theta+\phi)\sin\frac{1}{2}(\theta-\phi)}{-2a\sin\frac{1}{2}(\theta+\phi)\sin\frac{1}{2}(\theta-\phi)} = -\frac{b\cos\frac{1}{2}(\theta+\phi)}{a\sin\frac{1}{2}(\theta+\phi)}$$

Thus the equation of the chord PQ is

$$y - b\sin\theta = -\frac{b\cos\frac{1}{2}(\theta+\phi)}{a\sin\frac{1}{2}(\theta+\phi)}(x - a\cos\theta)$$

i.e. $ay\sin\frac{1}{2}(\theta+\phi) - ab\sin\theta\sin\frac{1}{2}(\theta+\phi)$

$$= -bx\cos\tfrac{1}{2}(\theta+\phi) + ab\cos\theta\cos\tfrac{1}{2}(\theta+\phi)$$

$\therefore\quad bx\cos\frac{1}{2}(\theta+\phi) + ay\sin\frac{1}{2}(\theta+\phi)$

$$= ab\{\cos\theta\cos\tfrac{1}{2}(\theta+\phi) + \sin\theta\sin\tfrac{1}{2}(\theta+\phi)\}$$
$$= ab\cos\{\theta - \tfrac{1}{2}(\theta+\phi)\} = ab\cos\tfrac{1}{2}(\theta-\phi).$$

Hence the equation of the chord PQ may be written:

$$\frac{x\cos\frac{1}{2}(\theta+\phi)}{a} + \frac{y\sin\frac{1}{2}(\theta+\phi)}{b} = \cos\tfrac{1}{2}(\theta-\phi).$$

Example 2 Find the equations of the tangent and the normal to the curve $xy = c^2$ at the point $P(ct, c/t)$. Given that the normal at P meets the curve again at Q, find the coordinates of Q. If the tangent at P meets the y-axis at R, find the equation of the locus of the mid-point M of PR.

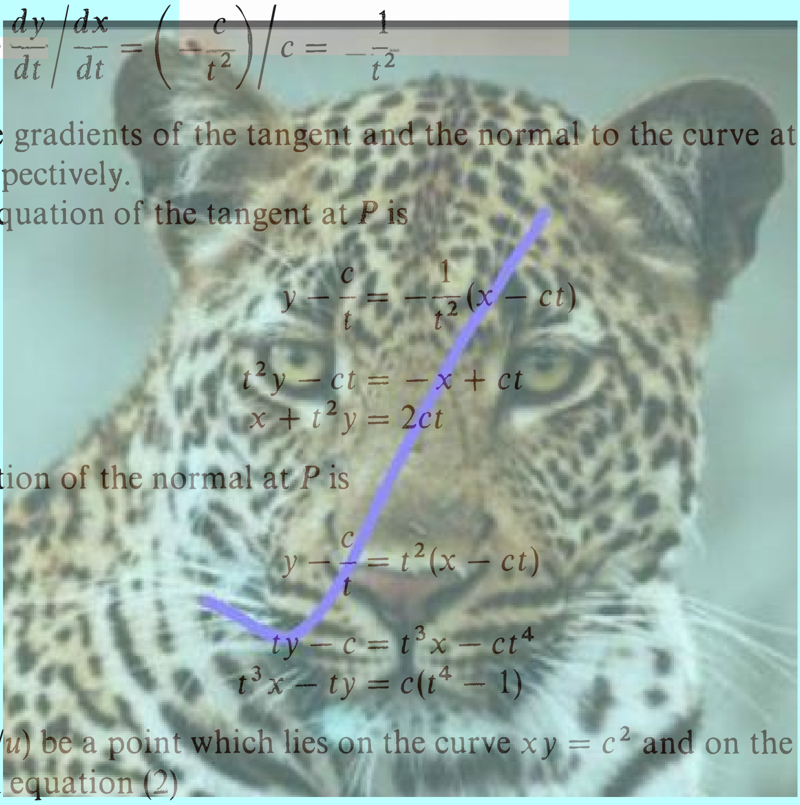
The parametric equations of the curve are

$$x = ct, \qquad y = \frac{c}{t}$$

Differentiating with respect to t we have

$$\frac{dx}{dt} = c, \qquad \frac{dy}{dt} = -\frac{c}{t^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt}\bigg/\frac{dx}{dt} = \left(-\frac{c}{t^2}\right)\bigg/c = -\frac{1}{t^2}$$



Hence the gradients of the tangent and the normal to the curve at P are $-1/t^2$ and t^2 respectively.

$\therefore$ the equation of the tangent at P is

$$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$$

i.e. $\qquad t^2y - ct = -x + ct$

i.e. $\qquad x + t^2y = 2ct \qquad (1)$

The equation of the normal at P is

$$y - \frac{c}{t} = t^2(x - ct)$$

i.e. $\qquad ty - c = t^3x - ct^4$

i.e. $\qquad t^3x - ty = c(t^4 - 1) \qquad (2)$

Let $(cu, c/u)$ be a point which lies on the curve $xy = c^2$ and on the normal at P, then from equation (2)

$$t^3 \times cu - t \times \frac{c}{u} = c(t^4 - 1)$$

$\therefore \quad t^3u^2 + (1 - t^4)u - t = 0$

$\therefore \quad (u - t)(t^3u + 1) = 0$

$\therefore \quad$ either $u = t \quad$ or $\quad u = -1/t^3$

Since the solution $u = t$ gives the point P, the remaining solution $u = -1/t^3$ must give the point Q.
Hence the coordinates of Q are $(-c/t^3, -ct^3)$.
Substituting $x = 0$ in equation (1) we have,

$$t^2y = 2ct \quad \text{i.e.} \quad y = 2c/t$$

$\therefore \quad$ the coordinates of the point R are $(0, 2c/t)$.

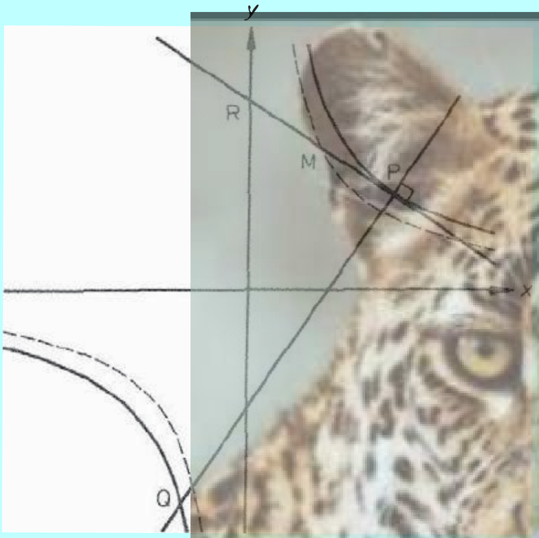
Thus the coordinates of the mid-point M of PR are

$$\left(\frac{1}{2}\{ct+0\},\frac{1}{2}\left\{\frac{c}{t}+\frac{2c}{t}\right\}\right) \quad \text{i.e.} \quad \left(\frac{1}{2}ct,\frac{3c}{2t}\right)$$

Hence as t varies the parametric equations of the locus of M are

$$x=\frac{1}{2}ct, \qquad y=\frac{3c}{2t}$$

Eliminating t from these equations we obtain the Cartesian equation of the locus, namely $xy=\frac{3}{4}c^2$.



As shown in the diagram the locus of M is another rectangular hyperbola.

Exercise 13.3

1. Write down parametric coordinates for a point on each of the following curves and sketch the curves.

(a) $y^2=12x$, (b) $xy=9$, (c) $y^2=x^3$,
(d) $4xy=25$, (e) $4y^2=x^3$, (f) $y^2=10x$.

2. Write down parametric coordinates for a point on each of the following curves and sketch the curves.

(a) $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$, (b) $x^2+y^2=9$, (c) $\dfrac{x^2}{5}+y^2=1$,
(d) $4x^2+9y^2=4$, (e) $9x^2+y^2=9$, (f) $x^2+(y-1)^2=25$.

3. Find the equations of the tangent and normal to the curve $xy=16$ at the point $(4t,4/t)$.

4. Find the equations of the tangent and normal to the curve $x=t$, $y=1/t$ at the point where $t=2$.

5. Find the equations of the tangent and normal to the curve $x = 2\cos\theta$, $y = \sin\theta$ at the point where $\theta = \frac{1}{3}\pi$.

6. P is the point $(5\cos\theta, 4\sin\theta)$ on the curve $x^2/25 + y^2/16 = 1$. Given that S and S' are the points with coordinates $(-3, 0)$ and $(3, 0)$, show that the value of $PS + PS'$ is independent of θ.

7. Find the equations of the tangents to the curve $xy = c^2$ which have gradient -4 and write down the coordinates of their points of contact.

8. The points $P(cp, c/p)$, $Q(cq, c/q)$ and $R(cr, c/r)$ lie on the curve $xy = c^2$. Show that if $\angle RPQ$ is a right angle, then $p^2qr = -1$. Prove further that QR is perpendicular to the tangent at P to the curve.

9. The parametric equations of a curve are $x = 2t$, $y = 2/t$. The tangent to the curve at $P(2t, 2/t)$ meets the x-axis at A; the normal to the curve at P meets the y-axis at B. The point Q divides AB in the ratio 3:1. Find the parametric equations of the locus of Q as P moves on the given curve. (JMB)

10. Let P and Q be points on the rectangular hyperbola $xy = a^2$, with coordinates $(ap, a/p)$ and $(aq, a/q)$. Obtain the equation of the line PQ, and deduce, or obtain otherwise, the equation of the tangent at P to the hyperbola. Show that, if the line PQ meets the coordinate axes at A and B, then the mid-point M of AB is the mid-point of PQ also. Obtain the coordinates of the point T where the tangents at P and Q meet, and show that the line MT passes through the origin. (W)

11. Prove that the normal to the hyperbola $xy = c^2$ at the point $P(ct, c/t)$ has equation $y = t^2x + \dfrac{c}{t} - ct^3$. If the normal at P meets the line $y = x$ at N, and O is the origin, show that $OP = PN$ provided that $t^2 \neq 1$. The tangent to the hyperbola at P meets the line $y = x$ at T. Prove that $OT \,.\, ON = 4c^2$. (O&C)

12. The normal at the point $P(ct, c/t)$ on the rectangular hyperbola $xy = c^2$ meets the curve again at Q. If the normal meets the x-axis at A and the y-axis at B, show that the mid-point of AB is also the mid-point of PQ.

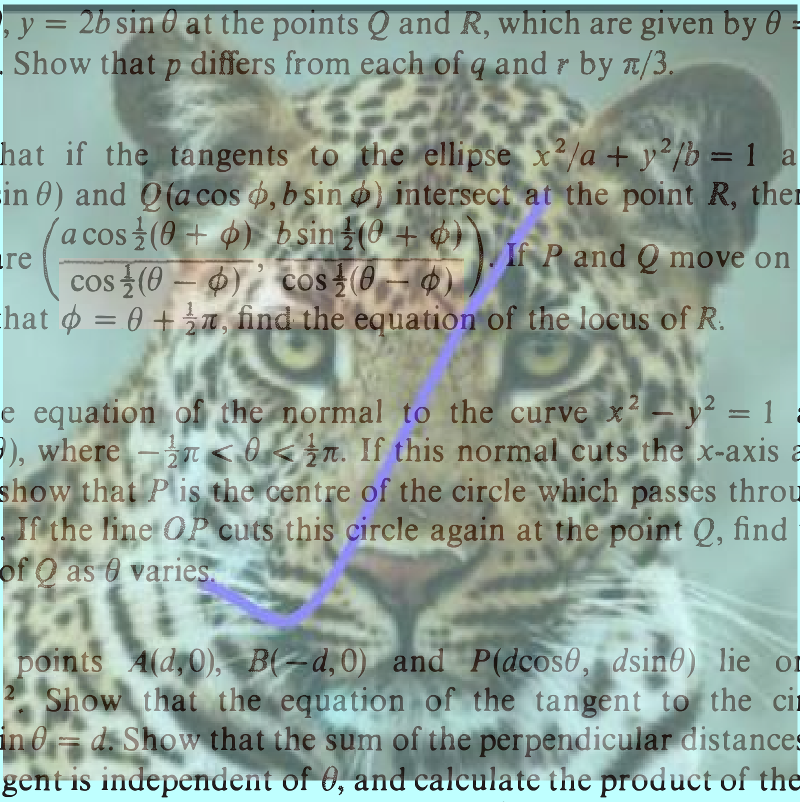
13. Show that the tangent at $P(cp, c/p)$ to the rectangular hyperbola $xy = c^2$ has the equation $p^2y + x = 2cp$. The perpendicular from the origin to this tangent meets it at N, and meets the hyperbola again at Q and R. Prove that (i) the angle QPR is a right angle, (ii) as p varies, the point N lies on the curve whose equation is $(x^2 + y^2)^2 = 4c^2xy$. (C)

14. The tangents at the points $P(cp, c/p)$ and $Q(cq, c/q)$ on the rectangular hyperbolar $xy = c^2$ intersect at the point R. Given that R lies on the rectangular hyperbola $xy = \frac{1}{2}c^2$, find the equation of the locus of the mid-point M of PQ as p and q vary.

15. Find the equation of the tangent to the curve $ay^2 = x^3$ at the point (at^2, at^3) and prove that, apart from one exceptional case, the tangent meets the curve again. Find the coordinates of the point of intersection. What is the exceptional case? (O&C)

16. The tangents to the curve $9y^2 = x^3$ at the points $P(9p^2, 9p^3)$ and $Q(9q^2, 9q^3)$ intersect at the point R. Find the coordinates of R. If p and q vary in such a way that $\angle PRQ$ is always a right angle, find the equation of the locus of R.

17. Show that the equation of the tangent to the curve $x = a\cos t$, $y = b\sin t$ at the point $P(a\cos p, b\sin p)$ is $\dfrac{x}{a}\cos p + \dfrac{y}{b}\sin p = 1$. This tangent meets the curve $x = 2a\cos\theta$, $y = 2b\sin\theta$ at the points Q and R, which are given by $\theta = q$ and $\theta = r$ respectively. Show that p differs from each of q and r by $\pi/3$. (JMB)



18. Show that if the tangents to the ellipse $x^2/a + y^2/b = 1$ at the points $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$ intersect at the point R, then the coordinates of R are $\left(\dfrac{a\cos\frac{1}{2}(\theta+\phi)}{\cos\frac{1}{2}(\theta-\phi)}, \dfrac{b\sin\frac{1}{2}(\theta+\phi)}{\cos\frac{1}{2}(\theta-\phi)}\right)$. If P and Q move on the ellipse in such a way that $\phi = \theta + \frac{1}{2}\pi$, find the equation of the locus of R.

19. Find the equation of the normal to the curve $x^2 - y^2 = 1$ at the point $P(\sec\theta, \tan\theta)$, where $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$. If this normal cuts the x-axis at A and the y-axis at B, show that P is the centre of the circle which passes through A, B and the origin O. If the line OP cuts this circle again at the point Q, find the equation of the locus of Q as θ varies.

20. (a) The points $A(d, 0)$, $B(-d, 0)$ and $P(d\cos\theta, d\sin\theta)$ lie on the circle $x^2 + y^2 = d^2$. Show that the equation of the tangent to the circle at P is $x\cos\theta + y\sin\theta = d$. Show that the sum of the perpendicular distances from A and B to this tangent is independent of θ, and calculate the product of these distances.
(b) The point $P(p, p^3)$ lies on the curve $y = x^3$. Show that the equation of the tangent to the curve at P is $y - 3p^2x + 2p^3 = 0$. Show that when $p = 3$ this tangent passes through the point $A(7/3, 9)$, and find the other two values of p for which the tangent passes through A. (W)

21. A curve has the parametric equations $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, where $0 \leqslant \theta \leqslant \pi$. A point P of the curve has parameter ϕ, where $\phi \neq 0$.

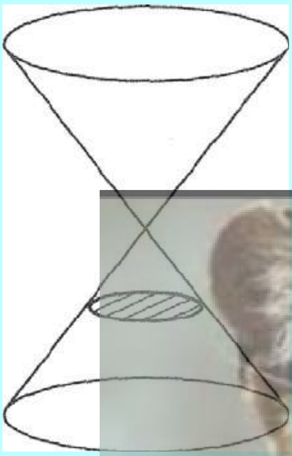
(i) Show that, at P, $\dfrac{dy}{dx} = \dfrac{\sin\phi}{1 - \cos\phi}$.

(ii) The normal to the curve at P meets the x-axis at G, and O is the origin. Show that $OG = a\phi$.

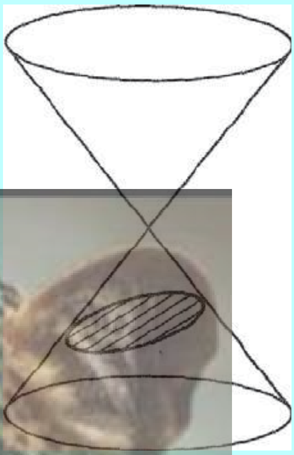
(iii) The tangent to the curve at P meets at K the line through G parallel to the y-axis. Show that $GK = 2a$. (C)

13.4 Conic sections

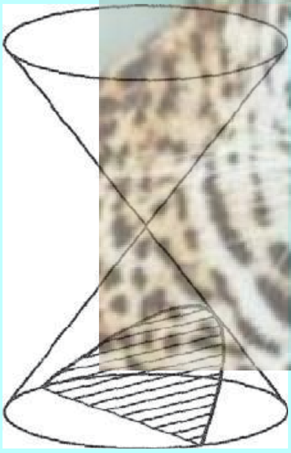
Some of the curves we have been studying arise naturally as cross-sections of a circular cone. The following diagrams show the sections produced when a double cone of semi-vertical angle θ is cut by a plane which does not pass through the vertex. The nature of the curve obtained in each case is determined by the value of the angle α between the plane and the axis of the cone, where $0 \leqslant \alpha \leqslant \frac{1}{2}\pi$.



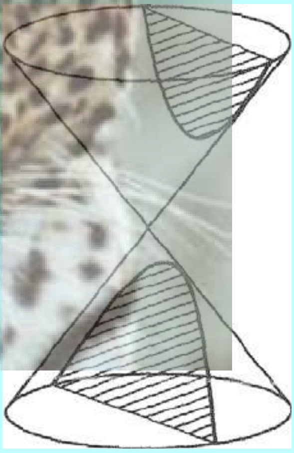
Circle, $\alpha = \frac{1}{2}\pi$



Ellipse, $\alpha > \theta$



Parabola, $\alpha = \theta$

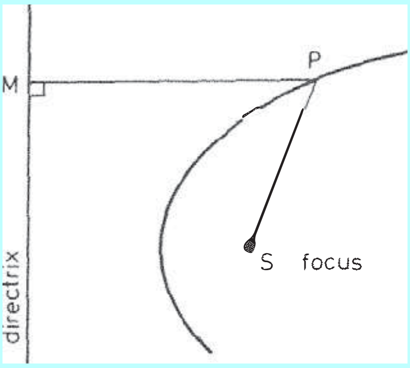


Hyperbola, $\alpha < \theta$

The complete set of conics includes the sections produced by planes through the vertex of the double cone, namely:

for $\alpha < \theta$, a pair of straight lines;
for $\alpha = \theta$, a single straight line;
for $\alpha > \theta$, a single point.

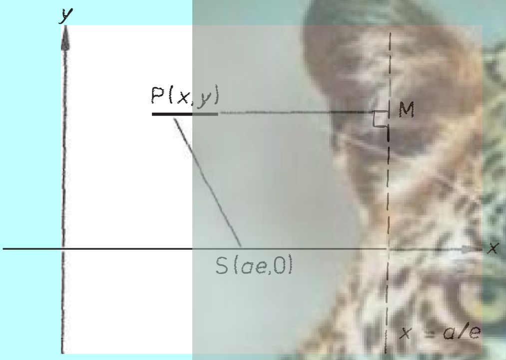
It can be shown that the definition of a parabola as a conic section is consistent with the locus definition used in Book 1. We now consider the corresponding locus definitions of the ellipse and hyperbola.



Suppose that P is a point which moves so that its distance from a fixed point S, the *focus*, is a constant multiple of its distance from a fixed line called the *directrix*. If M is the foot of the perpendicular from P to the directrix then

$$PS = ePM$$

where e is a positive constant called the *eccentricity*. The nature of the locus of P depends on the value taken by e, as follows:



for an ellipse, $0 < e < 1$;
for a parabola, $e = 1$;
for a hyperbola, $e > 1$.

The standard form of the equation of the ellipse is obtained by taking the point $(ae, 0)$ as focus and the line $x = a/e$ as directrix.

$$PS^2 = e^2PM^2$$

$$\therefore \quad (x - ae)^2 + y^2 = e^2\left(\frac{a}{e} - x\right)^2$$

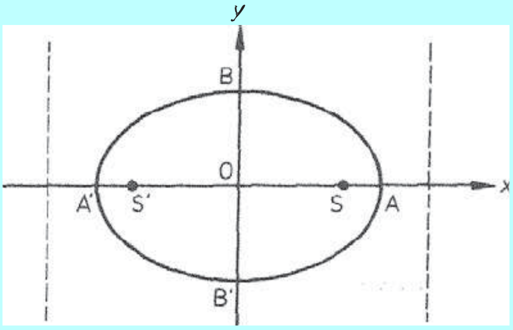
$$x^2 - 2aex + a^2e^2 + y^2 = a^2 - 2aex + e^2x^2$$

$$x^2(1 - e^2) + y^2 = a^2(1 - e^2)$$

i.e.
$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$$

Hence the equation of the ellipse is:

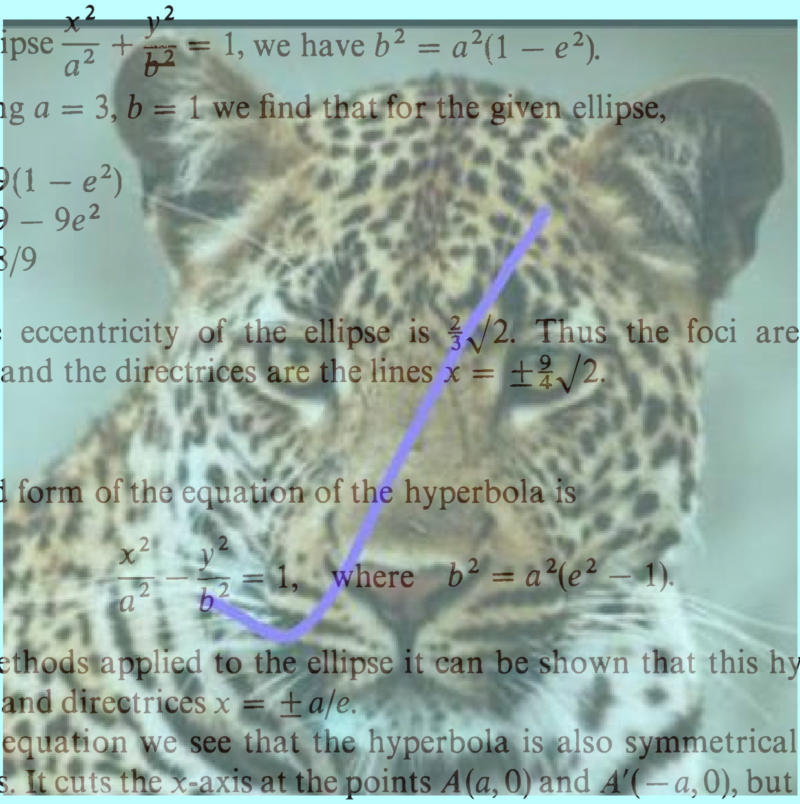
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{where} \quad b^2 = a^2(1 - e^2).$$



From this equation we see that the ellipse is symmetrical about both x- and y-axes. It cuts the x-axis at the points $A(a, 0)$ and $A'(-a, 0)$. It cuts the y-axis at the points $B(b, 0)$ and $B'(-b, 0)$. AA' is called the *major axis* and BB' the *minor axis*. The origin O is the *centre* of the ellipse and any chord through O is a *diameter*. From symmetry of the curve it is clear that the same locus would have been produced using the point $S'(-ae, 0)$ as focus and the line $x = -a/e$ as directrix. Thus the ellipse is said to have foci $(\pm ae, 0)$ and directrices $x = \pm ae$.

[Note that in terms of a and b the coordinates of the foci are $(\pm\sqrt{\{a^2 - b^2\}}, 0)$ and the equations of the directrices are $x = \pm a^2/\sqrt{(a^2 - b^2)}$.]

Example 1 Find the eccentricity, the foci and the directrices of the ellipse $\dfrac{x^2}{9} + y^2 = 1$.



For the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, we have $b^2 = a^2(1 - e^2)$.

Substituting $a = 3$, $b = 1$ we find that for the given ellipse,

$$
\begin{aligned}
1 &= 9(1 - e^2) \\
\therefore \quad 1 &= 9 - 9e^2 \\
\therefore \quad e^2 &= 8/9
\end{aligned}
$$

Hence the eccentricity of the ellipse is $\frac{2}{3}\sqrt{2}$. Thus the foci are the points $(\pm 2\sqrt{2}, 0)$ and the directrices are the lines $x = \pm\frac{9}{4}\sqrt{2}$.

The standard form of the equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad \text{where} \quad b^2 = a^2(e^2 - 1).$$

Using the methods applied to the ellipse it can be shown that this hyperbola has foci $(\pm ae, 0)$ and directrices $x = \pm a/e$.

From the equation we see that the hyperbola is also symmetrical about both x- and y-axes. It cuts the x-axis at the points $A(a, 0)$ and $A'(-a, 0)$, but does not cut the y-axis.

Rearranging the equation we have:

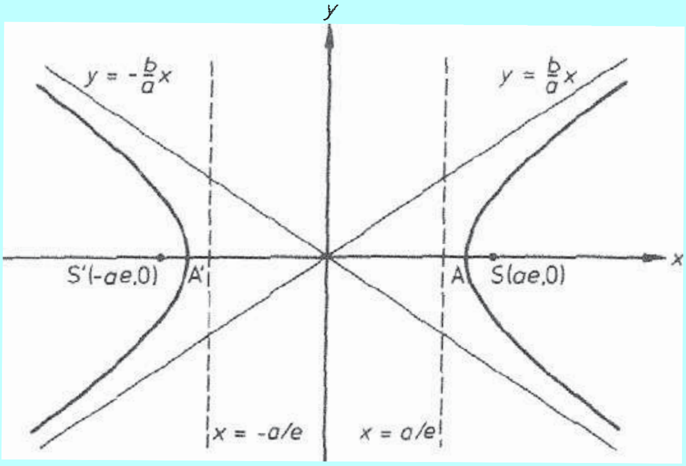
$$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1$$

$$\therefore \quad \frac{y^2}{x^2} = \frac{b^2}{a^2} - \frac{b^2}{x^2}$$

$$\therefore \quad \text{as } |x| \to \infty, \qquad \frac{y^2}{x^2} \to \frac{b^2}{a^2} \quad \text{and} \quad \frac{y}{x} \to \pm\frac{b}{a}$$

i.e. as $|x|$ increases the curve approaches the lines $y = \pm\dfrac{b}{a}x$.

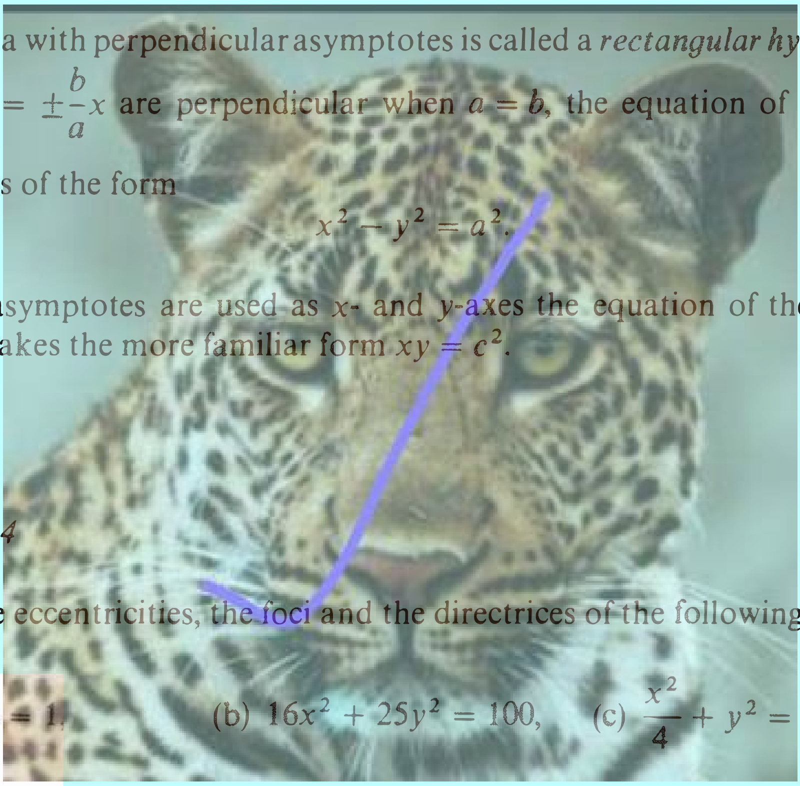
Hence the lines $y = \pm\dfrac{b}{a}x$ are asymptotes to the hyperbola.



A hyperbola with perpendicular asymptotes is called a *rectangular hyperbola*. Since the lines $y = \pm\frac{b}{a}x$ are perpendicular when $a = b$, the equation of a rectangular hyperbola is of the form

$$x^2 - y^2 = a^2.$$

When the asymptotes are used as x- and y-axes the equation of the rectangular hyperbola takes the more familiar form $xy = c^2$.



Exercise 13.4

1. Find the eccentricities, the foci and the directrices of the following ellipses.

(a) $\frac{x^2}{25} + \frac{y^2}{9} = 1$, (b) $16x^2 + 25y^2 = 100$, (c) $\frac{x^2}{4} + y^2 = 1$,

(d) $\frac{x^2}{9} + \frac{y^2}{5} = 1$, (e) $\frac{x^2}{4} + \frac{y^2}{9} = 1$, (f) $4x^2 + y^2 = 1$.

2. Find the eccentricities, the foci, the directrices and the asymptotes of the following hyperbolas.

(a) $\frac{x^2}{16} - \frac{y^2}{9} = 1$, (b) $x^2 - y^2 = 4$, (c) $x^2 - 4y^2 = 4$

3. Find the eccentricities of the following ellipses

(a) $x = 2\cos\theta$, $y = \sin\theta$, (b) $x = 5\cos\theta$, $y = 3\sin\theta$,
(c) $x = 2\cos\theta$, $y = 3\sin\theta$, (d) $x = 5\sin\theta$, $y = 3\cos\theta$.

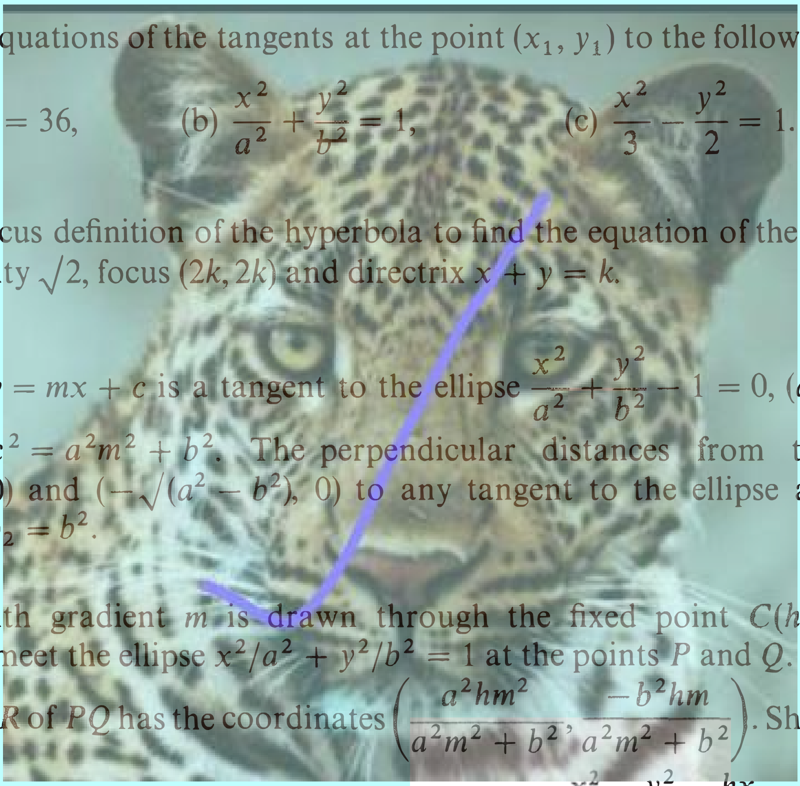
4. Find in the form $x^2/a^2 + y^2/b^2 = 1$ the equations of the ellipses with (a) eccentricity $1/2$, foci $(\pm 2, 0)$, (b) eccentricity $3/5$, foci $(\pm 9, 0)$.

5. Use the locus definition of the ellipse to find the equation of the ellipse with eccentricity $\frac{2}{3}$, focus $(2, 1)$ and directrix $x = -\frac{1}{2}$.

6. Find the equations of the tangent and the normal to each of the following ellipses at the given point:

(a) $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1; (3, 0)$, (b) $\dfrac{x^2}{8} + \dfrac{y^2}{2} = 1; (-2, 1)$,

(c) $9x^2 + 16y^2 = 40; (2, \frac{1}{2})$, (d) $4x^2 + 5y^2 = 120; (-5, -2)$.



7. Find the equations of the tangents at the point (x_1, y_1) to the following curves:

(a) $4x^2 + 9y^2 = 36$, (b) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, (c) $\dfrac{x^2}{3} - \dfrac{y^2}{2} = 1$.

8. Use the locus definition of the hyperbola to find the equation of the hyperbola with eccentricity $\sqrt{2}$, focus $(2k, 2k)$ and directrix $x + y = k$.

9. The line $y = mx + c$ is a tangent to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 = 0$, $(a > b > 0)$. Show that $c^2 = a^2m^2 + b^2$. The perpendicular distances from the points $(\sqrt{(a^2 - b^2)}, 0)$ and $(-\sqrt{(a^2 - b^2)}, 0)$ to any tangent to the ellipse are p_1, p_2. Show that $p_1p_2 = b^2$. (C)

10. A line with gradient m is drawn through the fixed point $C(h, 0)$, where $0 < h < a$, to meet the ellipse $x^2/a^2 + y^2/b^2 = 1$ at the points P and Q. Prove that the mid-point R of PQ has the coordinates $\left(\dfrac{a^2hm^2}{a^2m^2 + b^2}, \dfrac{-b^2hm}{a^2m^2 + b^2}\right)$. Show that, as m varies, R always lies on the curve whose equation is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{hx}{a^2}$. (C)

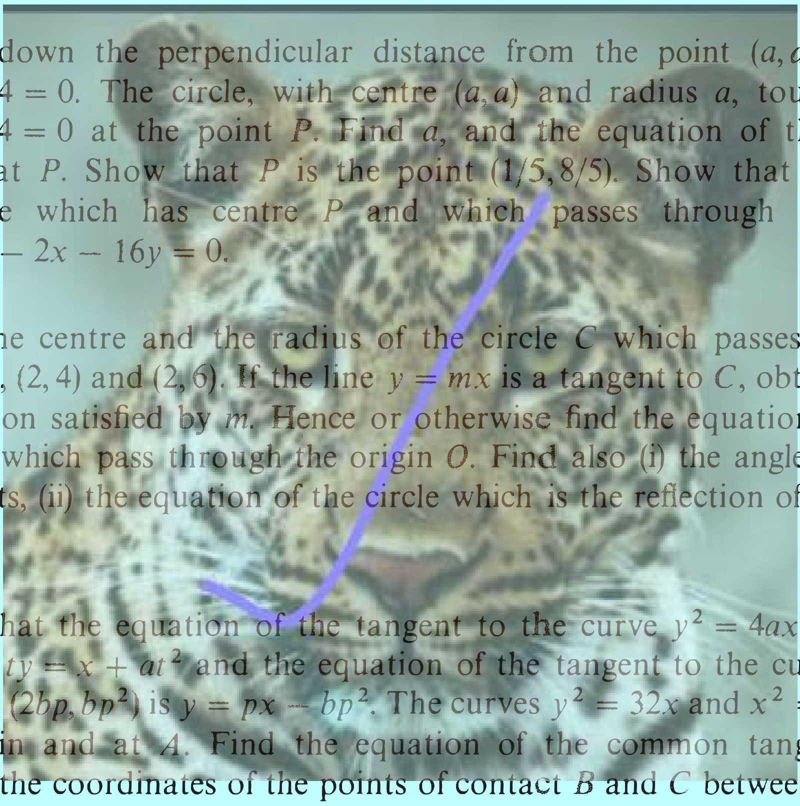
11. The ellipse $x^2/a^2 + y^2/b^2 = 1$ intersects the positive x-axis at A and the positive y-axis at B. Determine the equation of the perpendicular bisector of AB. (i) Given that this line intersects the x-axis at P and that M is the mid-point of AB, prove that the area of triangle PMA is $b(a^2 + b^2)/8a$. (ii) If $a^2 = 3b^2$, find, in terms of b, the coordinates of the points where the perpendicular bisector of AB intersects the ellipse. (JMB)

12. Prove that the hyperbola H_1, with equation $x^2 - y^2 = a^2$, cuts the hyperbola H_2, with equation $xy = c^2$, at right angles at two points P and Q. If the distance between the tangents to H_1 at P and Q is equal to the distance between the tangents to H_2 at P and Q, find the relation between a and c. (O)

Exercise 13.5 (miscellaneous)

1. A straight line parallel to the line $2x + y = 0$ intersects the x-axis at A and the y-axis at B. The perpendicular bisector of AB cuts the y-axis at C. Prove that the gradient of the line AC is $-\frac{3}{4}$. Find also the tangent of the acute angle between the line AC and the bisector of the angle AOB, where O is the origin. (JMB)

2. The triangle ABC has vertices $A(0, 12)$, $B(-9, 0)$, $C(16, 0)$. Find the equations of the internal bisectors of the angles ABC and ACB. Hence, or otherwise, find the equation of the inscribed circle of the triangle ABC. Find also the equation of the circle passing through A, B and C. (C)



3. Write down the perpendicular distance from the point (a, a) to the line $4x - 3y + 4 = 0$. The circle, with centre (a, a) and radius a, touches the line $4x - 3y + 4 = 0$ at the point P. Find a, and the equation of the normal to the circle at P. Show that P is the point $(1/5, 8/5)$. Show that the equation of the circle which has centre P and which passes through the origin is $5(x^2 + y^2) - 2x - 16y = 0$. (L)

4. Find the centre and the radius of the circle C which passes through the points $(4, 2)$, $(2, 4)$ and $(2, 6)$. If the line $y = mx$ is a tangent to C, obtain the quadratic equation satisfied by m. Hence or otherwise find the equations of the tangents to C which pass through the origin O. Find also (i) the angle between the two tangents, (ii) the equation of the circle which is the reflection of C in the line $y = 3x$. (AEB 1977)

5. Show that the equation of the tangent to the curve $y^2 = 4ax$ at the point $(at^2, 2at)$ is $ty = x + at^2$ and the equation of the tangent to the curve $x^2 = 4by$ at the point $(2bp, bp^2)$ is $y = px - bp^2$. The curves $y^2 = 32x$ and $x^2 = 4y$ intersect at the origin and at A. Find the equation of the common tangent to these curves and the coordinates of the points of contact B and C between the tangent and the curves. Calculate the area of the triangle ABC. (AEB 1976)

6. On the same diagram sketch the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 10x = 0$. The line $ax + by + 1 = 0$ is a tangent to both these circles. State the distances of the centres of the circles from this tangent. Hence, or otherwise, find the possible values of a and b and show that, if 2ϕ is the angle between the common tangents, then $\tan\phi = \frac{3}{4}$. (L)

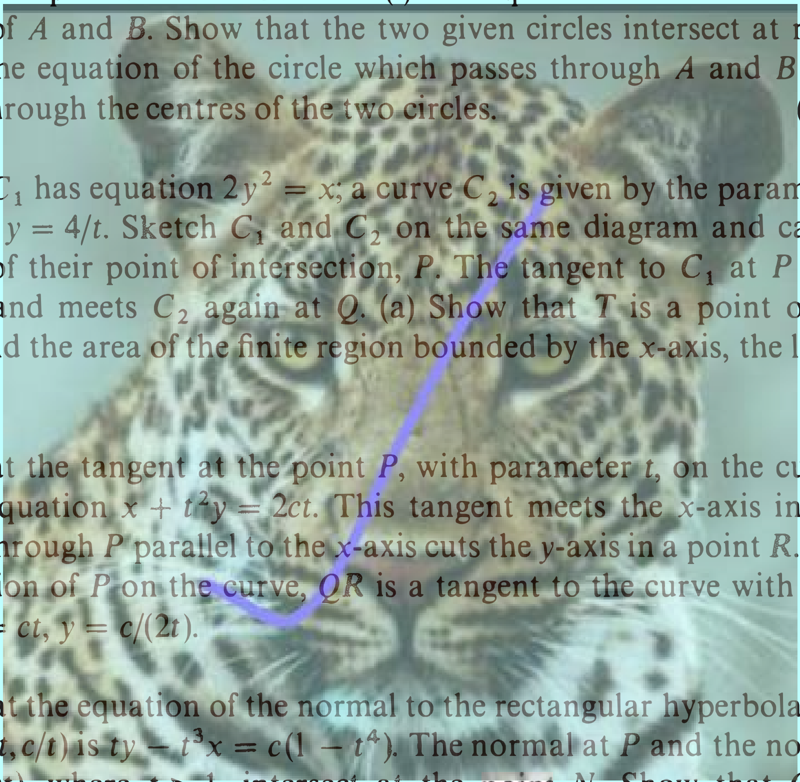
7. Find the centre and radius of each of the circles C_1 and C_2 whose equations are $x^2 + y^2 - 16y + 32 = 0$ and $x^2 + y^2 - 18x + 2y + 32 = 0$ respectively and show that the circles touch externally. Find the coordinates of their point of contact and show that the common tangent at that point passes through the origin. The other tangents from the origin, one to each circle, are drawn. Find, correct to the nearest degree, the angle between these tangents. (SU)

8. Two circles, C_1 and C_2, have equations $x^2 + y^2 - 4x - 8y - 5 = 0$ and $x^2 + y^2 - 6x - 10y + 9 = 0$, respectively. Find the x-coordinates of the points P and Q at which the line $y = 0$ cuts C_1, and show that this line touches C_2. Find the tangent of the acute angle made by the line $y = 0$ with the tangents to C_1 at P and Q. Show that, for all values of the constant λ, the circle C_3 whose equation is $\lambda(x^2 + y^2 - 4x - 8y - 5) + x^2 + y^2 - 6x - 10y + 9 = 0$ passes through the points of intersection of C_1 and C_2. Find the two possible values of λ for which the line $y = 0$ is a tangent to C_3. (JMB)

9. The circles whose equations are

$$x^2 + y^2 - x + 6y + 7 = 0$$
$$\text{and} \quad x^2 + y^2 + 2x + 2y - 2 = 0$$

intersect at the points A and B. Find (i) the equation of the line AB, (ii) the coordinates of A and B. Show that the two given circles intersect at right angles and obtain the equation of the circle which passes through A and B and which also passes through the centres of the two circles. (AEB 1978)



10. A curve C_1 has equation $2y^2 = x$; a curve C_2 is given by the parametric equations $x = 4t$, $y = 4/t$. Sketch C_1 and C_2 on the same diagram and calculate the coordinates of their point of intersection, P. The tangent to C_1 at P crosses the x-axis at T and meets C_2 again at Q. (a) Show that T is a point of trisection of PQ. (b) Find the area of the finite region bounded by the x-axis, the line PT and the curve C_1. (L)

11. Show that the tangent at the point P, with parameter t, on the curve $x = ct$, $y = c/t$ has equation $x + t^2y = 2ct$. This tangent meets the x-axis in a point Q and the line through P parallel to the x-axis cuts the y-axis in a point R. Show that, for any position of P on the curve, QR is a tangent to the curve with parametric equations $x = ct$, $y = c/(2t)$. (L)

12. Prove that the equation of the normal to the rectangular hyperbola $xy = c^2$ at the point $P(ct, c/t)$ is $ty - t^3x = c(1 - t^4)$. The normal at P and the normal at the point $Q(c/t, ct)$, where $t > 1$, intersect at the point N. Show that $OPNQ$ is a rhombus, where O is the origin. Hence, or otherwise, find the coordinates of N. If the tangents to the hyperbola at P and Q intersect at T, prove that the product of the lengths of OT and ON is independent of t. (JMB)

13. The line of gradient $m(\neq 0)$ through the point $A(a, O)$ is a tangent to the rectangular hyperbola $xy = c^2$ at the point P. Find m in terms of a and c, and show that the coordinates of P are $(\frac{1}{2}a, 2c^2/a)$. The line through A parallel to the y-axis meets the hyperbola at Q, and the line joining Q to the origin O intersects AP at R. Given that OQ and AP are perpendicular to each other, find the numerical value of c^2/a^2 and the numerical value of the ratio $AR:RP$. (JMB)

14. The point P in the first quadrant lies on the curve with parametric equations $x = t^2$, $y = t^3$. The tangent to the curve at P meets the curve again at Q and is normal to the curve at Q. Find the coordinates of P and of Q. (L)

15. Sketch the parabola whose parametric equations are $x = t^2$, $y = 2t$ and on the same diagram sketch the curve with parametric equations $x = 10(1 + \cos\theta)$, $y = 10\sin\theta$. These curves touch at the origin and meet again at two other points A and B. The normals at A and B to the parabola meet at P and the tangents to the other curve at A and B meet at Q. Calculate the length of PQ. (L)

16. Obtain an equation of the tangent, at the point with parameter t, to the curve $\mathscr{C}$ whose parametric equations are given by

$$x = 2\sin^3 t, \quad y = 2\cos^3 t, \quad 0 \leqslant t \leqslant \pi/2.$$

Show that, if the tangent meets the coordinate axes in points R and S, then RS is of constant length. Sketch the curve $\mathscr{C}$. Find the area of the finite region enclosed by the curve $\mathscr{C}$ and the coordinate axes. (L)

17. Sketch the curve whose parametric equations are $x = a\cos\phi$, $y = b\sin\phi$, $0 \leqslant \phi \leqslant 2\pi$, where a and b are positive constants. The point P is given by $\phi = \pi/4$. Find (a) the equation of the tangent to the curve at P, (b) the equation of the normal to the curve at P. By evaluating a suitable integral, calculate the area of the region in the first quadrant between the curve and the coordinate axes. Hence deduce the area of the region enclosed by the curve. (L)

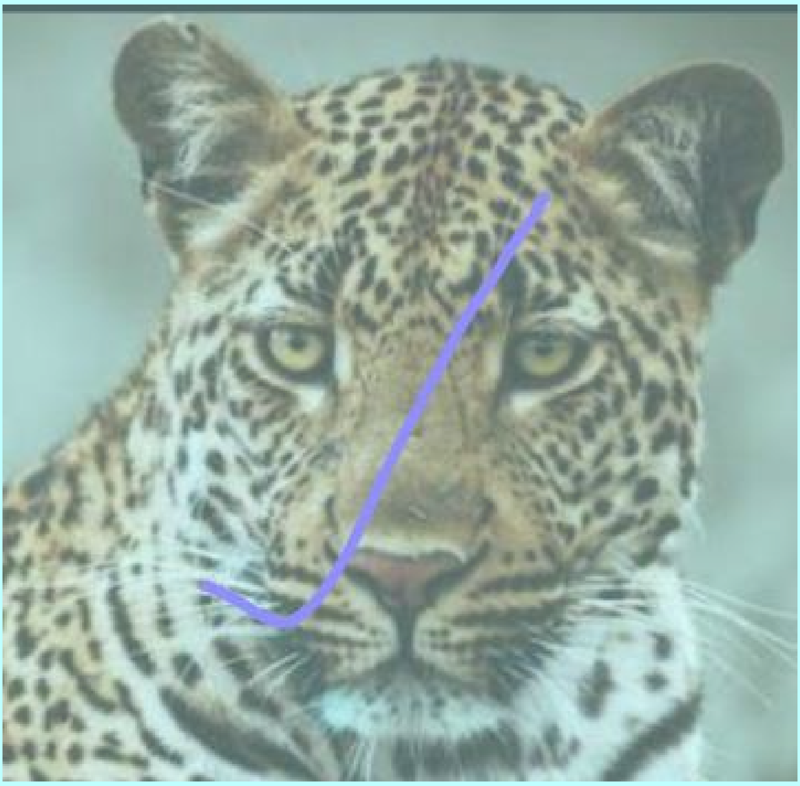
18. Prove that the equation of the tangent at the point (x_1, y_1) on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is $\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1$. The tangent at the point $(2\cos\theta, \sqrt{3}\sin\theta)$ on the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$ passes through the point $P(2, 1)$. Show that $\sqrt{3}\cos\theta + \sin\theta = \sqrt{3}$. Without using tables, calculator or slide rule, find all the solutions of this equation which are in the range $0^\circ \leqslant \theta < 360^\circ$. Hence obtain the coordinates of the points of contact, Q and R, of the tangents to the ellipse from P. Verify that the line through the origin and the point P passes through the mid-point of the line QR. (JMB)

19. Prove that the equation of the normal at $(\alpha\cos\phi, \beta\sin\phi)$ to the ellipse $\dfrac{x^2}{\alpha^2} + \dfrac{y^2}{\beta^2} = 1$ is $\alpha x\sec\phi - \beta y\operatorname{cosec}\phi = \alpha^2 - \beta^2$. P is the point $(a\cos\theta, b\sin\theta)$ on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. M and N are the feet of the perpendiculars from P to the axes. Find the equation of MN. Prove that, for variable θ, MN is always normal to a fixed concentric ellipse and find the equation of this ellipse. (O&C)

20. Prove that the equation of the tangent to the curve $xy^2 = c^3$ $(c > 0)$ at the point $P(ct^2, c/t)$ is $2t^3y + x = 3ct^2$. Prove that the parameters of the points of contact of the tangents which pass through the point $Q(h, k)$ $(k \neq 0)$ satisfy a cubic equation and, by considering the turning values of this cubic, or otherwise, prove that there are three distinct tangents to the curve which pass through Q if $hk^2 < c^3$

and $h > 0$. State a necessary and sufficient condition on the parameters t_1, t_2, t_3 for the tangents at the corresponding points to be concurrent.

When Q lies on the curve, the tangent at Q cuts the curve again at R; if the parameter of Q is t, determine the parameter of R. Prove that, if the tangents Q_1, Q_2, Q_3 are concurrent and cut the curve again at R_1, R_2, R_3 then the tangents at R_1, R_2, R_3 are concurrent. (O&C)



Equation of a line in three dimensions

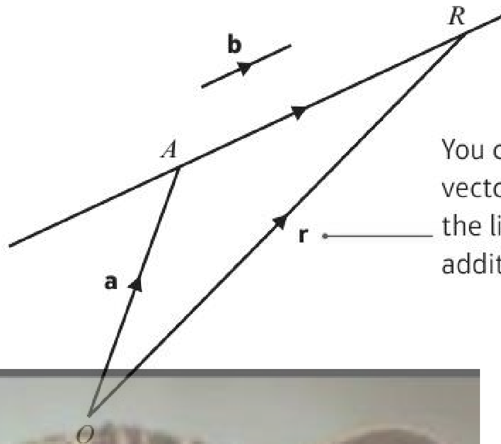
need to know how to write the equation of a straight line in vector form.

pose a straight line passes through a n point A, with position vector **a**, and rallel to the given vector **b**. Only one line is possible. Let R be an arbitrary t on the line, with position vector **r**.

e $\overrightarrow{AR}$ is parallel to **b**, $\overrightarrow{AR} = \lambda\mathbf{b}$, where a scalar.

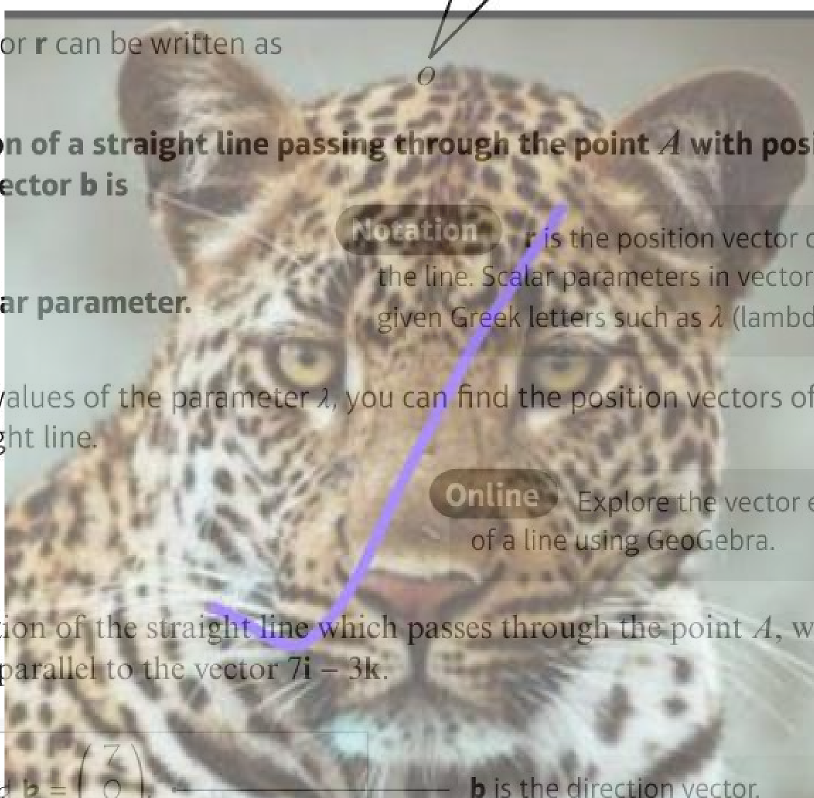
vector **b** is called the **direction vector** ne line.

ne position vector **r** can be written as λ**b**.



You can find the posit vector of any point R the line by using vect addition ($\triangle OAR$):

$\mathbf{r} = \mathbf{a} + \overrightarrow{AR}$



vector equation of a straight line passing through the point A with position vector a, ar arallel to the vector b is

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$$

here λ is a scalar parameter.

Notation **r** is the position vector of a general point o the line. Scalar parameters in vector equations are ofte given Greek letters such as λ (lambda) and μ (mu).

aking different values of the parameter λ, you can find the position vectors of different points lie on the straight line.

Online Explore the vector equation of a line using GeoGebra.

mple 1

d a vector equation of the straight line which passes through the point A, with position vecto $5\mathbf{j} + 4\mathbf{k}$, and is parallel to the vector $7\mathbf{i} - 3\mathbf{k}$.

re $\mathbf{a} = \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 7 \\ 0 \\ -3 \end{pmatrix}$.

b is the direction vector.

equation of the line is

$$\mathbf{r} = \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} + \lambda\begin{pmatrix} 7 \\ 0 \\ -3 \end{pmatrix}$$

$$\mathbf{r} = (3\mathbf{i} - 5\mathbf{j} + 4\mathbf{k}) + \lambda(7\mathbf{i} - 3\mathbf{k})$$

$$\mathbf{r} = (3 + 7\lambda)\mathbf{i} + (-5)\mathbf{j} + (4 - 3\lambda)\mathbf{k}$$

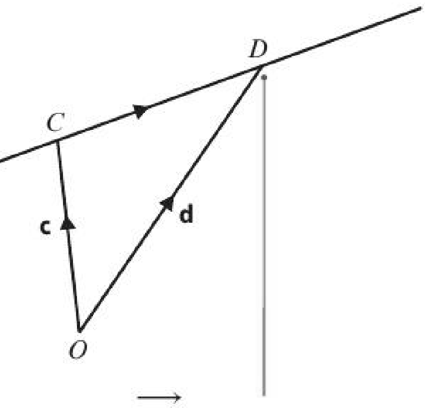
$$\mathbf{r} = \begin{pmatrix} 3 + 7\lambda \\ -5 \\ 4 - 3\lambda \end{pmatrix}$$

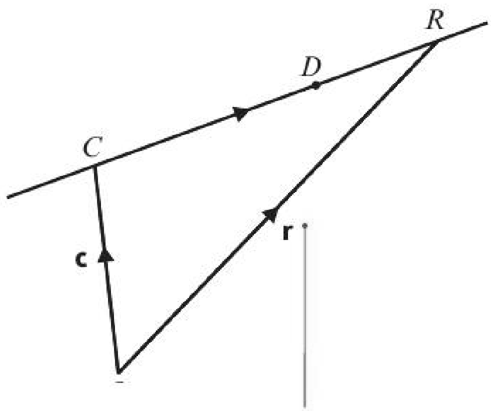
You sometimes need to show the separate x, y, z components in terms of λ.

You can represent a 3D vector using column notation, $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, or using **ijk**-notation, $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

← **Pure Year 2, Chapter**

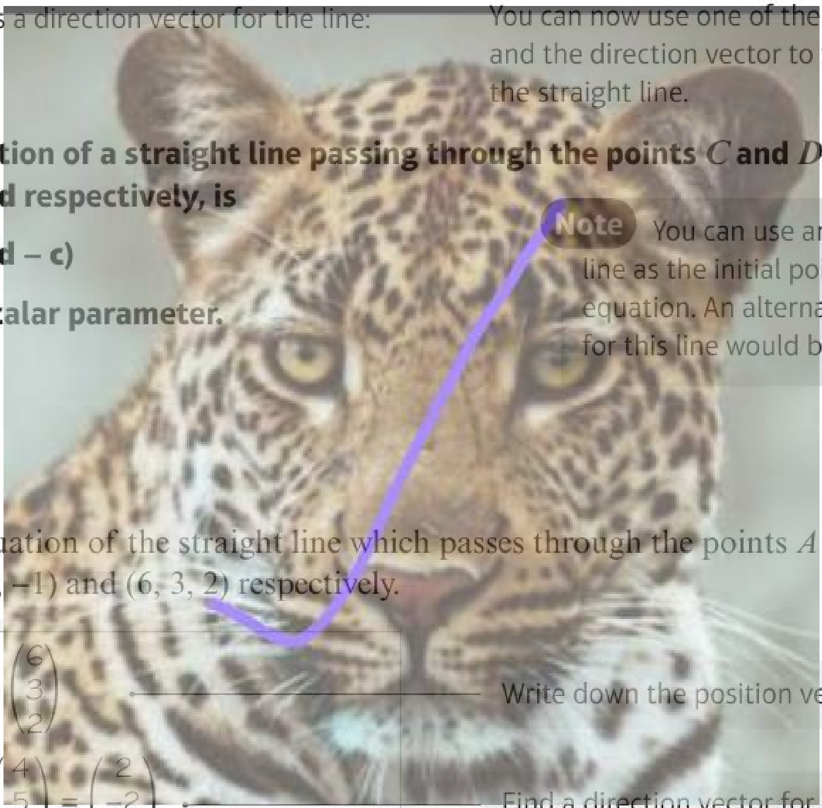
ow suppose a straight line passes through two given points C and D, with position vectors **c** a
spectively. Again, only one such line is possible.





ou can use $\overrightarrow{CD}$ as a direction vector for the line:
$\overrightarrow{CD} = \mathbf{d} - \mathbf{c}$.

You can now use one of the two given points and the direction vector to form an equation f the straight line.



A vector equation of a straight line passing through the points C and D, with position vectors c and d respectively, is

$$\mathbf{r} = \mathbf{c} + \lambda(\mathbf{d} - \mathbf{c})$$

where λ is a scalar parameter.

Note You can use any point on the str line as the initial point in the vector equation. An alternative vector equati for this line would be $\mathbf{r} = \mathbf{d} + \lambda(\mathbf{d} - \mathbf{c})$.

xample 2

ind a vector equation of the straight line which passes through the points A and B, with oordinates $(4, 5, -1)$ and $(6, 3, 2)$ respectively.

$\mathbf{a} = \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix}$ $\quad \mathbf{b} = \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$ — Write down the position vectors of A and B.

$\mathbf{b} - \mathbf{a} = \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$ — Find a direction vector for the line.

Use one of the given points to form the equa

$$\mathbf{r} = \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix} + t\begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$$

You don't have to use λ for the parameter. In example, the parameter is represented by th letter t.

or $\mathbf{r} = (4\mathbf{i} + 5\mathbf{j} - \mathbf{k}) + t(2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$

or $\mathbf{r} = (4 + 2t)\mathbf{i} + (5 - 2t)\mathbf{j} + (-1 + 3t)\mathbf{k}$

or $\mathbf{r} = \begin{pmatrix} 4 + 2t \\ 5 - 2t \\ -1 + 3t \end{pmatrix}$

You can give your answer in any of these forr

mple 3

straight line l has vector equation $\mathbf{r} = (3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}) + t(\mathbf{i} - 6\mathbf{j} - 2\mathbf{k})$.
en that the point $(a, b, 0)$ lies on l, find the value of a and the value of b.

$= \begin{pmatrix} 3 + t \\ 2 - 6t \\ -5 - 2t \end{pmatrix}$ — You can write the equation in this form.

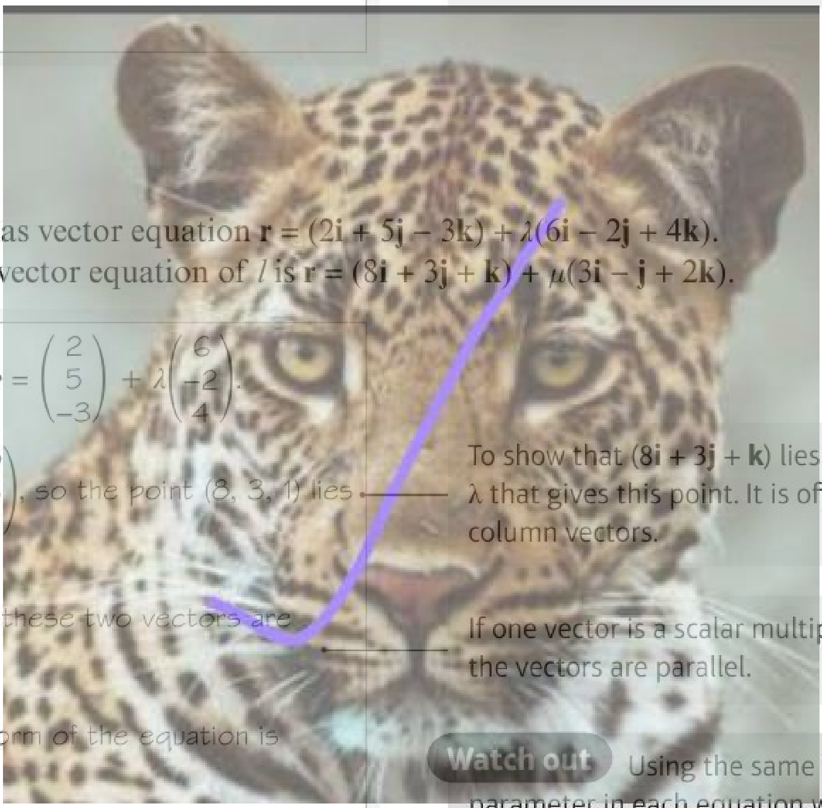
$- 2t = 0$ — Use the z-coordinate (which is equal to zero) to find the value of t.

$= -\frac{5}{2}$

$= 3 + t = \frac{1}{2}$

$= 2 - 6t = 17$ — Find a and b using the value of t.

$= \frac{1}{2}$ and $b = 17$



mple 4

straight line l has vector equation $\mathbf{r} = (2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}) + \lambda(6\mathbf{i} - 2\mathbf{j} + 4\mathbf{k})$.
w that another vector equation of l is $\mathbf{r} = (8\mathbf{i} + 3\mathbf{j} + \mathbf{k}) + \mu(3\mathbf{i} - \mathbf{j} + 2\mathbf{k})$.

e the equation $\mathbf{r} = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix} + \lambda\begin{pmatrix} 6 \\ -2 \\ 4 \end{pmatrix}$.

en $\lambda = 1$, $\mathbf{r} = \begin{pmatrix} 8 \\ 3 \\ 1 \end{pmatrix}$, so the point $(8, 3, 1)$ lies l. — To show that $(8\mathbf{i} + 3\mathbf{j} + \mathbf{k})$ lies on l, find a value λ that gives this point. It is often easier to work column vectors.

$\begin{pmatrix} 6 \\ -2 \\ 4 \end{pmatrix} = 2\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ so these two vectors are rallel. — If one vector is a scalar multiple of another ther the vectors are parallel.

an alternative form of the equation is

$= \begin{pmatrix} 8 \\ 3 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

Watch out Using the same value of the parameter in each equation will give **different** points on the line. You should use a different letter for the parameter of the second equatior

also need to be able to write the equation of a line in three dimensions in **Cartesian form**.
means that the equation is given in terms of coordinates relative to the x-, y- and z-axes.

$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ the equation of the line with vector equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ can be gi Cartesian form as

$$\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}$$

Each of the three expressions is equal to λ.

xample 5

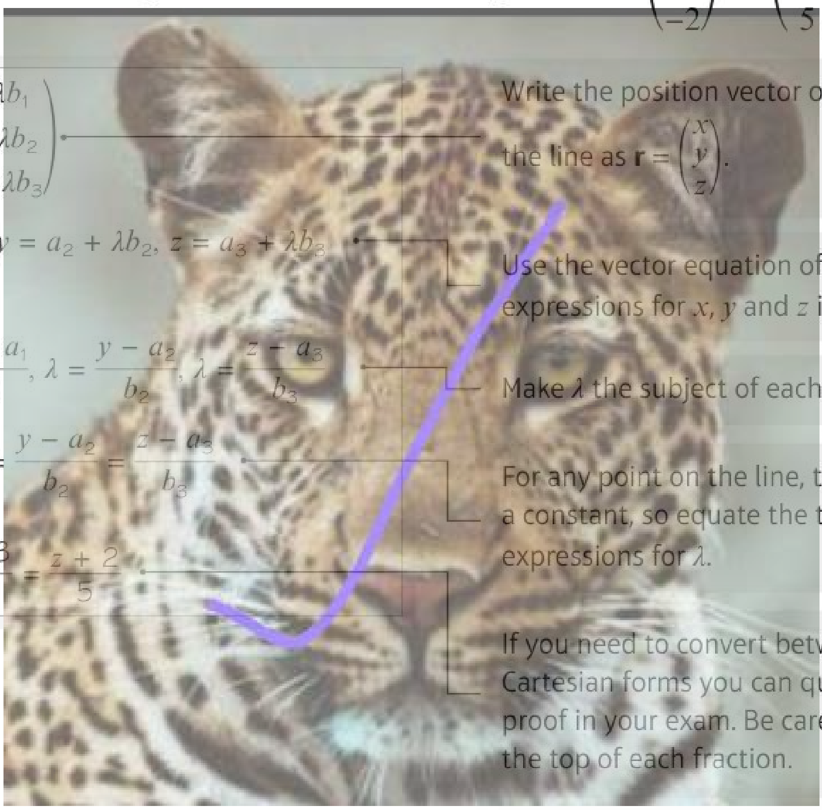
ith respect to the fixed origin O, the line l is given by the equation

$$\mathbf{r} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Prove that a Cartesian form of the equation of l is

$$\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}$$

Hence find a Cartesian equation of the line with equation $\mathbf{r} = \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}$.



a $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 + \lambda b_1 \\ a_2 + \lambda b_2 \\ a_3 + \lambda b_3 \end{pmatrix}$

Write the position vector of the general poin the line as $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

$x = a_1 + \lambda b_1$, $y = a_2 + \lambda b_2$, $z = a_3 + \lambda b_3$

Use the vector equation of the line to write expressions for x, y and z in terms of λ.

Rearranging,

$$\lambda = \frac{x - a_1}{b_1}, \lambda = \frac{y - a_2}{b_2}, \lambda = \frac{z - a_3}{b_3}$$

Make λ the subject of each equation.

So $\quad \dfrac{x - a_1}{b_1} = \dfrac{y - a_2}{b_2} = \dfrac{z - a_3}{b_3}$

For any point on the line, the value of λ is a constant, so equate the three different expressions for λ.

b $\dfrac{x - 4}{-1} = \dfrac{y - 3}{2} = \dfrac{z + 2}{5}$

If you need to convert between vector and Cartesian forms you can quote this result wit proof in your exam. Be careful with the signs the top of each fraction.

xample 6

he line l has equation $\mathbf{r} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, and the point P has position vector $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$.

Show that P does not lie on l.

iven that a circle, centre P, intersects l at points A and B, and that A has position vector $\begin{pmatrix} 0 \\ - \\ 6 \end{pmatrix}$

find the position vector of B.

$$\mathbf{r} = \begin{pmatrix} -2 + \lambda \\ 1 - 2\lambda \\ 4 + \lambda \end{pmatrix}$$

If $P(2, 1, 3)$ lies on the line then

$2 = -2 + \lambda \Rightarrow \lambda = 4$

$1 = 1 - 2\lambda \Rightarrow \lambda = 0$

$3 = 4 + \lambda \Rightarrow \lambda = -1$

so P does not lie on l.

$$\overrightarrow{AP} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ -3 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix}$$

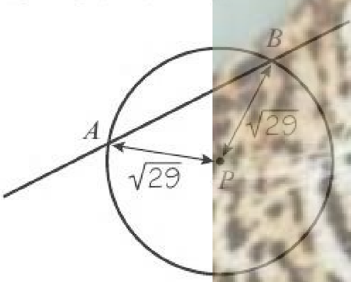
$$|\overrightarrow{AP}| = \sqrt{2^2 + 4^2 + (-3)^2} = \sqrt{29}$$

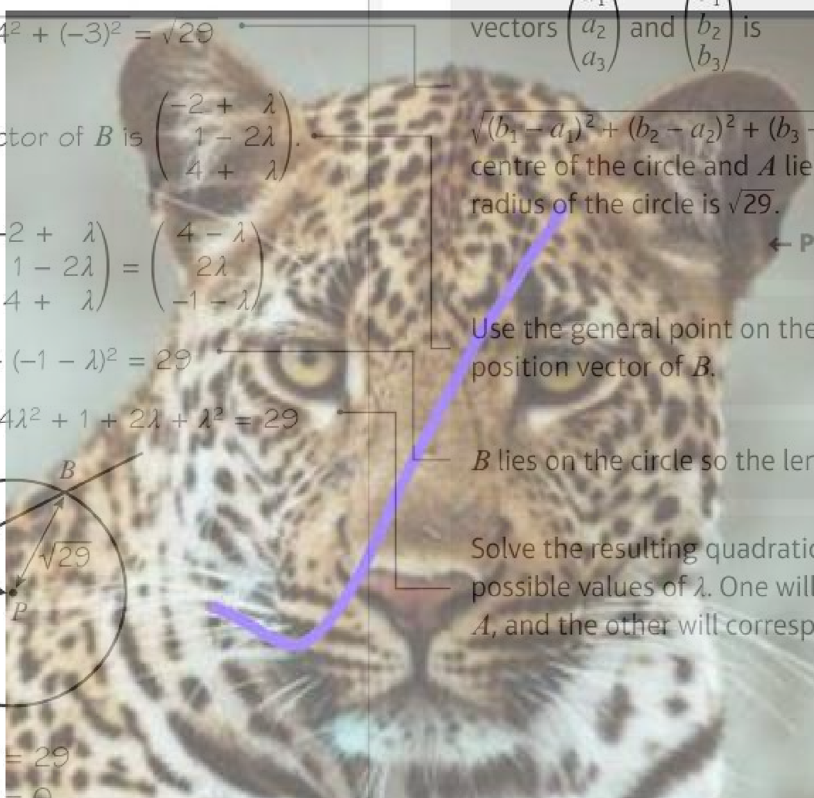
The position vector of B is $\begin{pmatrix} -2 + \lambda \\ 1 - 2\lambda \\ 4 + \lambda \end{pmatrix}$.

$$\overrightarrow{BP} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} -2 + \lambda \\ 1 - 2\lambda \\ 4 + \lambda \end{pmatrix} = \begin{pmatrix} 4 - \lambda \\ 2\lambda \\ -1 - \lambda \end{pmatrix}$$

$$(4 - \lambda)^2 + 4\lambda^2 + (-1 - \lambda)^2 = 29$$

$$16 - 8\lambda + \lambda^2 + 4\lambda^2 + 1 + 2\lambda + \lambda^2 = 29$$





$$6\lambda^2 - 6\lambda + 17 = 29$$

$$6\lambda^2 - 6\lambda - 12 = 0$$

$$\lambda^2 - \lambda - 2 = 0$$

$$(\lambda - 2)(\lambda + 1) = 0$$

So $\lambda = 2$ or $\lambda = -1$

$\lambda = 2$ gives $\begin{pmatrix} 0 \\ -3 \\ 6 \end{pmatrix}$. This is the position vector of point A.

$\lambda = -1$ gives $\begin{pmatrix} -3 \\ 3 \\ 3 \end{pmatrix}$. This is the position vector of point B.

Problem-solving

It is often useful to write the general point on a line as a single vector. You can write each component in the form $a + \lambda b$.

If P lies on l, there is one value of λ that satisfie all 3 equations. You only need to show that two these equations are not consistent to show tha P does not lie on l.

The distance between the points with position vectors $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is $\sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}$. As P is the centre of the circle and A lies on the circle, the radius of the circle is $\sqrt{29}$.

← Pure Year 2, Chapter

Use the general point on the line to represent t position vector of B.

B lies on the circle so the length $|\overrightarrow{BP}| = \sqrt{29}$.

Solve the resulting quadratic equation to find t possible values of λ. One will correspond to poi A, and the other will correspond to point B.

Substitute values of λ into $\begin{pmatrix} -2 + \lambda \\ 1 - 2\lambda \\ 4 + \lambda \end{pmatrix}$. Check tha one of the values gives the position vector of A The other value must give the position vector for B.

xercise 9A

1 For the following pairs of vectors, find a vector equation of the straight line which passes through the point, with position vector $\mathbf{a}$, and is parallel to the vector $\mathbf{b}$:

a $\mathbf{a} = 6\mathbf{i} + 5\mathbf{j} - \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$

b $\mathbf{a} = 2\mathbf{i} + 5\mathbf{j}$, $\mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

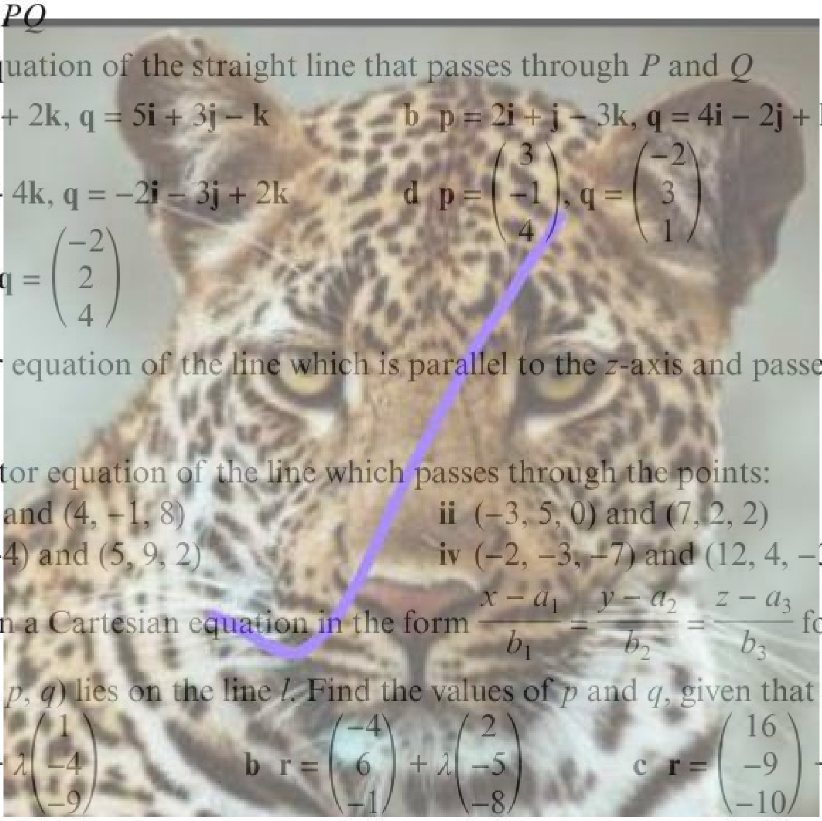
c $\mathbf{a} = -7\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

d $\mathbf{a} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$

e $\mathbf{a} = \begin{pmatrix} 6 \\ -11 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ 5 \\ -2 \end{pmatrix}$

2 For the points P and Q with position vectors $\mathbf{p}$ and $\mathbf{q}$ respectively, find:

i the vector $\overrightarrow{PQ}$



ii a vector equation of the straight line that passes through P and Q

a $\mathbf{p} = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$, $\mathbf{q} = 5\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

b $\mathbf{p} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$, $\mathbf{q} = 4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

c $\mathbf{p} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, $\mathbf{q} = -2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$

d $\mathbf{p} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$

e $\mathbf{p} = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix}$

3 Find a vector equation of the line which is parallel to the z-axis and passes through the po
$(4, -3, 8)$.

4 **a** Find a vector equation of the line which passes through the points:

i $(2, 1, 9)$ and $(4, -1, 8)$

ii $(-3, 5, 0)$ and $(7, 2, 2)$

iii $(1, 11, -4)$ and $(5, 9, 2)$

iv $(-2, -3, -7)$ and $(12, 4, -3)$

b Write down a Cartesian equation in the form $\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}$ for each line in pa

5 The point $(1, p, q)$ lies on the line l. Find the values of p and q, given that the equation of

a $\mathbf{r} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -4 \\ -9 \end{pmatrix}$

b $\mathbf{r} = \begin{pmatrix} -4 \\ 6 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -5 \\ -8 \end{pmatrix}$

c $\mathbf{r} = \begin{pmatrix} 16 \\ -9 \\ -10 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

6 The line l_1 has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}$. The line l_2 has equation $\frac{x - 4}{2} = \frac{y + 1}{-4} = \frac{z - 3}{-8}$

Show that l_1 and l_2 are parallel.

7 Show that the line l_1 with equation $\mathbf{r} = (3 + 2\lambda)\mathbf{i} + (2 - 3\lambda)\mathbf{j} + (-1 + 4\lambda)\mathbf{k}$ is parallel to the li
which passes through the points $A(5, 4, -1)$ and $B(3, 7, -5)$.

8 Show that the points
$A(-3, -4, 5)$, $B(3, -1, 2)$ and
$C(9, 2, -1)$ are collinear.

Hint Points are said to be **colline**
they all lie on the same straight li

9 Show that the points with position vectors $\begin{pmatrix} 1 \\ 7 \\ -2 \end{pmatrix}$, $\begin{pmatrix} 3 \\ -1 \\ 8 \end{pmatrix}$ and $\begin{pmatrix} 10 \\ 4 \\ 0 \end{pmatrix}$ do not lie on the same

The points $P(2, 0, 4)$, $Q(a, 5, 1)$ and $R(3, 10, b)$, where a and b are constants, are collinear. Find the values of a and b. **(5 ma**

The line l_1 has equation

$$\mathbf{r} = (8\mathbf{i} - 5\mathbf{j} + 4\mathbf{k}) + \lambda(3\mathbf{i} + \mathbf{j} - 6\mathbf{k})$$

A is the point on l_1 such that $\lambda = -2$.

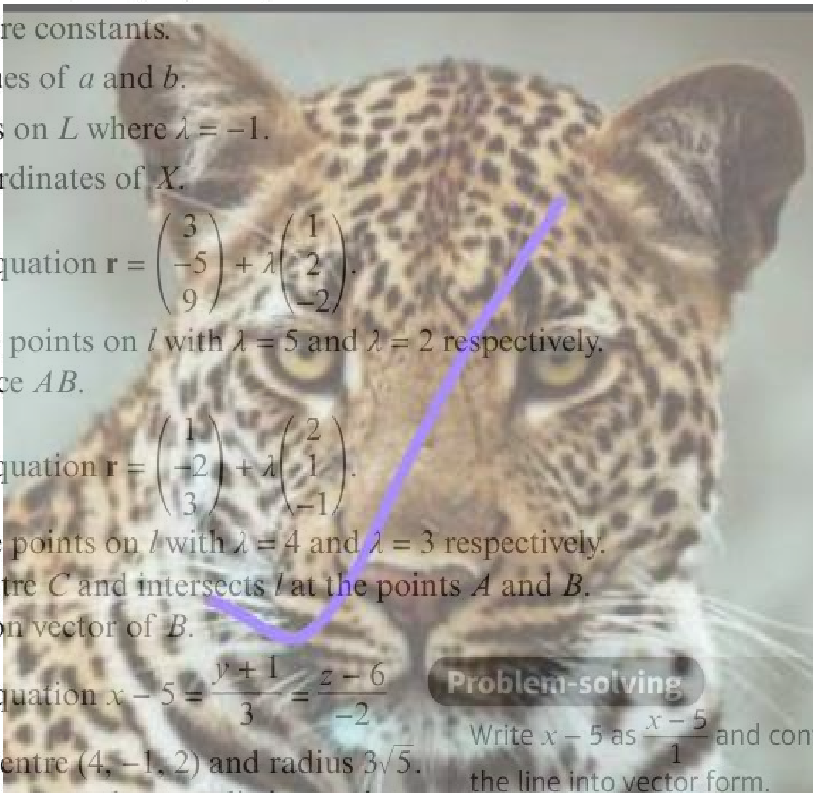
The line l_2 passes through A and is parallel to the line with equation

$$\mathbf{r} = (10\mathbf{i} + 3\mathbf{j} - 9\mathbf{k}) + \lambda(2\mathbf{i} - 4\mathbf{j} + \mathbf{k})$$

Find an equation for l_2. **(6 ma**

The point A with coordinates $(4, a, 0)$ lies on the line L with vector equation

$$\mathbf{r} = (10\mathbf{i} + 8\mathbf{j} - 12\mathbf{k}) + \lambda(\mathbf{i} - \mathbf{j} + b\mathbf{k})$$



where a and b are constants.

a Find the values of a and b. **(3 ma**

The point X lies on L where $\lambda = -1$.

b Find the coordinates of X. **(1 m**

The line l has equation $\mathbf{r} = \begin{pmatrix} 3 \\ -5 \\ 9 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$.

A and B are the points on l with $\lambda = 5$ and $\lambda = 2$ respectively.

Find the distance AB. **(4 ma**

The line l has equation $\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$.

C and A are the points on l with $\lambda = 4$ and $\lambda = 3$ respectively.

A circle has centre C and intersects l at the points A and B.

Find the position vector of B. **(3 ma**

The line l has equation $x - 5 = \frac{y+1}{3} = \frac{z-6}{-2}$

Problem-solving

Write $x - 5$ as $\frac{x-5}{1}$ and convert the equation o the line into vector form.

A circle C has centre $(4, -1, 2)$ and radius $3\sqrt{5}$.

Given that C intersects l at two distinct points, A and B, find the coordinates of A and B. **(7 ma**

The line l_1 has equation $\mathbf{r} = \begin{pmatrix} -4 \\ 6 \\ 5 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. A and B are the points on l_1 with $\lambda = 2$ and $\lambda = 5$ respectively.

a Find the position vectors of A and B. **(2 ma**

The point P has position vector $\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$.

The line l_2 passes through the point P and is parallel to the line l_1.

b Find a vector equation of the line l_2. **(2 ma**

The points C and D both lie on line l_2 such that $AB = AC = AD$.

c Show that P is the midpoint of CD. **(7 ma**

7 A tightrope is modelled as a line segment between points with coordinates (2, 3, 8) and (22, 18, 8), relative to a fixed origin O, where the units of distance are metres. Two suppor cables are anchored to a fixed point A on the wire. The other ends of the cables are anchored
to points with coordinates (14, 1, 0) and (6, 17, 0) respectively.

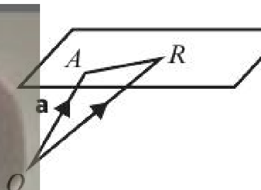
a Given that the support cables are both 12 m long, find the coordinates of A. **(8 mark**

b Give one criticism of this model. **(1 mar**

.2 Equation of a plane in three dimensions

he equation of a plane can be written in vector form.

uppose a plane passes through a given point A, with position vector **a**.
et R be an arbitrary point on the plane, with position vector **r**.



hen, using the triangle law, $\mathbf{r} = \mathbf{a} + \overrightarrow{AR}$.

nce $\overrightarrow{AR}$ lies in the plane, it can be ritten as $\lambda\mathbf{b} + \mu\mathbf{c}$, where **b** and **c** are on-parallel vectors in the plane and here λ and μ are scalars.

o the position vector **r** can be written as $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$.

The vector equation of a plane is $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, where:

- **r is the position vector of a general point in the plane**
- **a is the position vector of a point in the plane**
- **b and c are non-parallel, non-zero vectors in the plane**
- **λ and μ are scalars**

xample 7

ind, in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, an equation of the plane that passes through the points (2, 2, −1), B(3, 2, −1) and C(4, 3, 5).

$\overrightarrow{AB}$ and $\overrightarrow{AC}$ are vectors which lie in the plane.

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{i}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = 2\mathbf{i} + \mathbf{j} + 6\mathbf{k}$

So an equation of the plane is

$\mathbf{r} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k} + \lambda\mathbf{i} + \mu(2\mathbf{i} + \mathbf{j} + 6\mathbf{k})$

There are many other forms of this answer which are also correct. You could use $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ or $4\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$ instead of $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ in the equation.

You could write this equation as

$$\mathbf{r} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mu\begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix}$$

mple 8

fy that the point P with position vector $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ lies in the plane with vector equation

$$\begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 + 2\lambda + \mu \\ 4 + \lambda - \mu \\ -2 + \lambda + 2\mu \end{pmatrix}$$

The position vector of any point on the plane ca be written in this form.

P lies on the plane,

$$\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 + 2\lambda + \mu \\ 4 + \lambda - \mu \\ -2 + \lambda + 2\mu \end{pmatrix}$$

$= 3 + 2\lambda + \mu$ so $2\lambda + \mu = -1$ (1)

$= 4 + \lambda - \mu$ so $\lambda - \mu = -2$ (2)

$= -2 + \lambda + 2\mu$ so $\lambda + 2\mu = 1$ (3)

olving equations (2) and (3) simultaneously:

$-$ (2): $3\mu = 3$ so $\mu = 1$

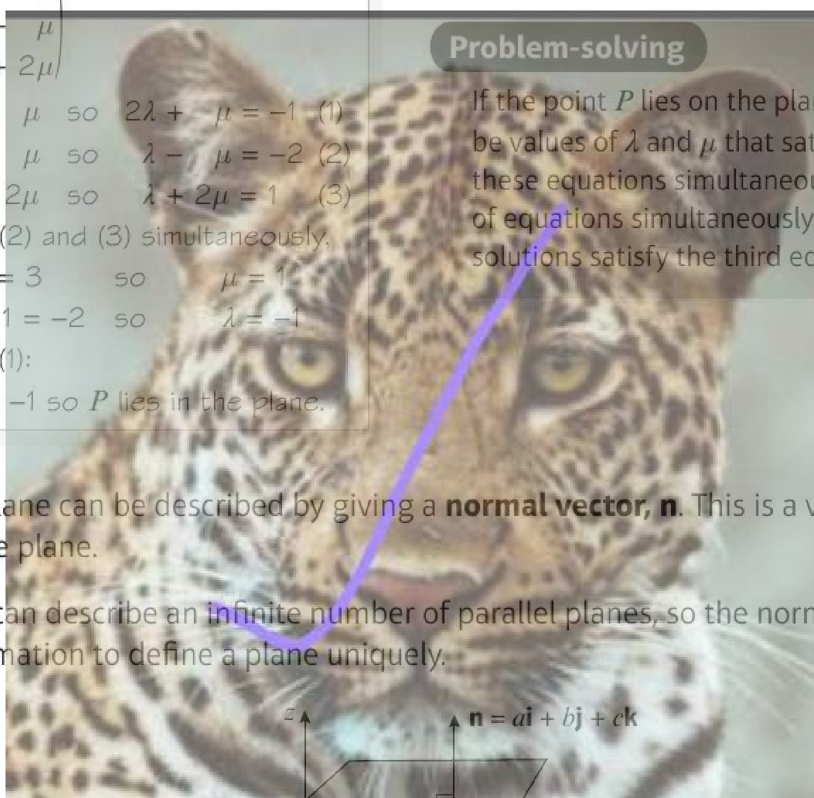
b in (2): $\lambda - 1 = -2$ so $\lambda = -1$

eck in equation (1):

$+ \mu = -2 + 1 = -1$ so P lies in the plane.

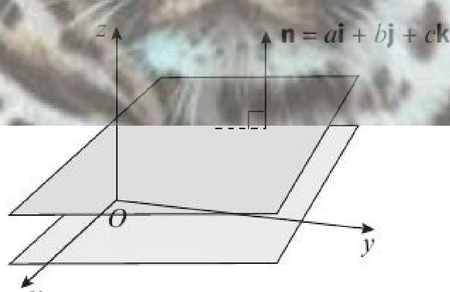
Problem-solving

If the point P lies on the plane then there will be values of λ and μ that satisfy **all three** of these equations simultaneously. Solve one pair of equations simultaneously, then check that th solutions satisfy the third equation.



direction of a plane can be described by giving a **normal vector, n**. This is a vector that is pendicular to the plane.

normal vector can describe an infinite number of parallel planes, so the normal vector on its o t enough information to define a plane uniquely.



Cartesian equation of a plane in three imensions can be written in the form $ax + by + cz = d$ where a, b, c and d are onstants, and $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is the normal vector o the plane.

Note Compare this equation to the Cartesian equation of a line in two dimensions: $ax + by =$

Online Explore the vector and Cartesian equations of a plane using GeoGebra.

can derive this result using the **scalar product**, which you will learn about later in this chapter

xample 9

he plane Π is perpendicular to the normal
ector $\mathbf{n} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and passes through the
oint P with position vector $8\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}$.
ind a Cartesian equation of Π.

Notation Planes are often represented by
capital Greek letter pi, Π.

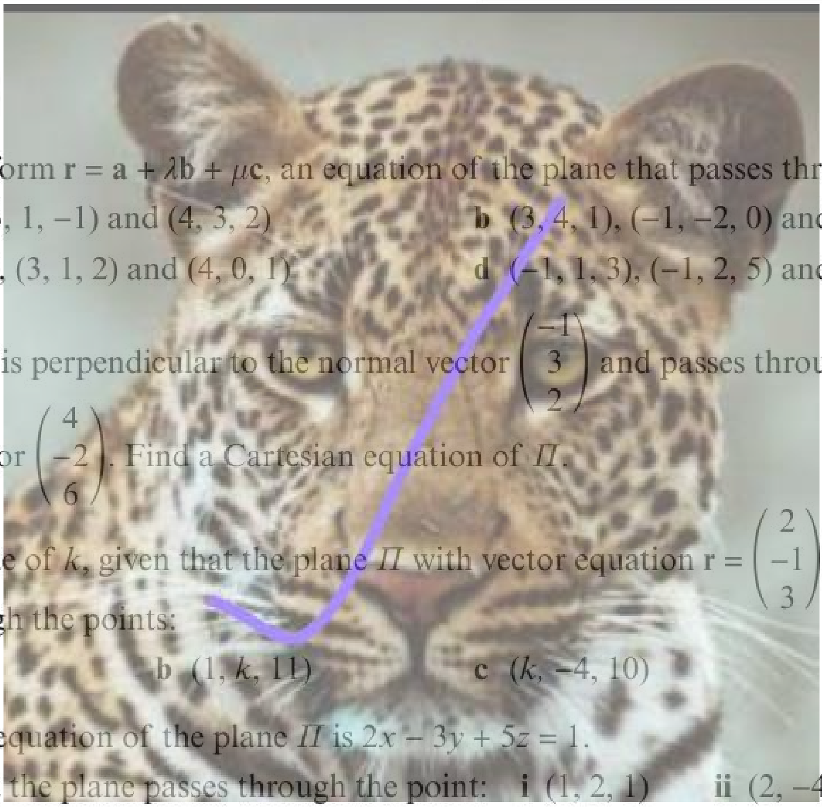
$3x - 2y + z = d$

The general equation is $ax + by + cz = d$ whe
the normal vector is $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$.

$3 \times 8 - 2 \times 4 + 1 \times (-7) = 9$

Substitute the values of x, y and z for point
this question to find the value of d.

So $d = 9$ and the Cartesian equation of Π is
$3x - 2y + z = 9$.



xercise 9B

1 Find, in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, an equation of the plane that passes through the points:

a (1, 2, 0), (3, 1, −1) and (4, 3, 2) **b** (3, 4, 1), (−1, −2, 0) and (2, 1, 4)

c (2, −1, −1), (3, 1, 2) and (4, 0, 1) **d** (−1, 1, 3), (−1, 2, 5) and (0, 4, 4).

2 The plane Π is perpendicular to the normal vector $\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$ and passes through the point wit
position vector $\begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix}$. Find a Cartesian equation of Π.

3 Find the value of k, given that the plane Π with vector equation $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ - \\ 3 \end{pmatrix}$
passes through the points:

a $(7, -1, k)$ **b** $(1, k, 11)$ **c** $(k, -4, 10)$ **d** $(10, k, -k)$

4 A Cartesian equation of the plane Π is $2x - 3y + 5z = 1$.

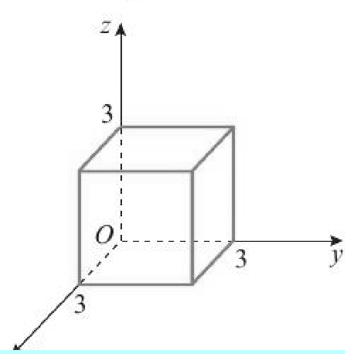
a Verify that the plane passes through the point: **i** (1, 2, 1) **ii** (2, −4, −3)

b Write down an equation of a normal vector to the plane.

5 The line l is normal to the plane Π with Cartesian equation $5x - 3y - 4z = 9$ and passes through the point $(2, 3, -2)$. Find:

a a vector equation of l **b** a Cartesian equation of l

6 The diagram shows a cube with a vertex at the origin and sides of length 3.
Find a Cartesian equation for each face of the cube.



Show that the points (2, 2, 3), (1, 5, 3), (4, 3, −1) and (3, 6, −1) are coplanar. **(6 ma**

Notation Points are said to be **coplanar** if th all lie on the same plane.

Show that the points (2, 3, 4), (2, −1, 3), (5, 3, −2) and (−1, −9, 8) are **not** coplanar. **(6 ma**

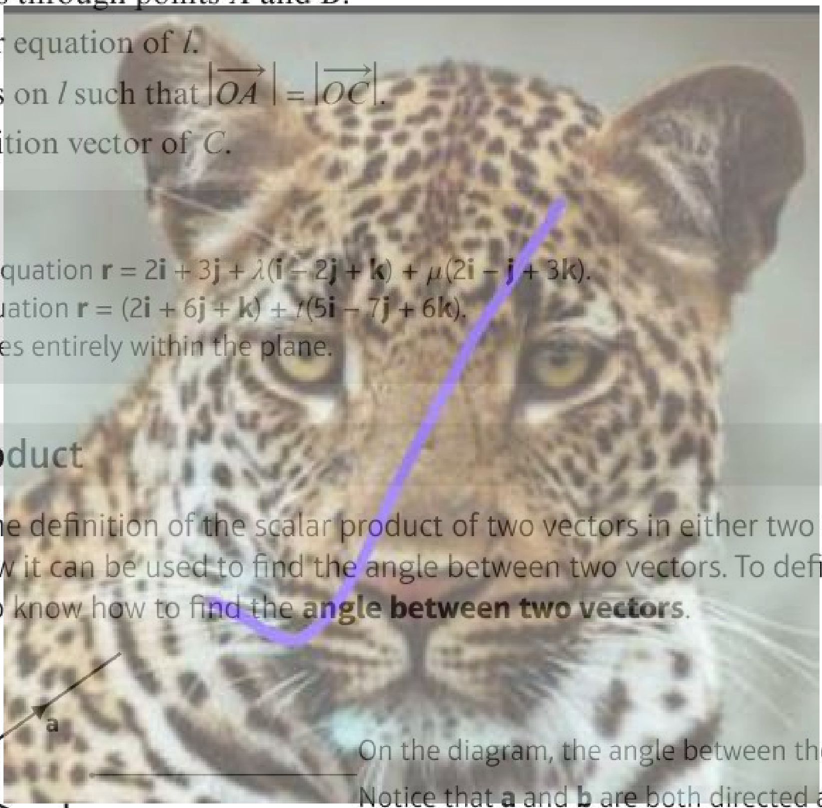
The plane Π has vector equation $\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k} + \lambda(-2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}) + \mu(4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$.
The point A lies on Π such that $\lambda = 1$ and $\mu = 2$.

a Find the position vector of A. **(2 ma**

Point B has position vector (1, −7, 2).

b Show that B lies on Π. **(2 ma**

The line l passes through points A and B.



c Find a vector equation of l. **(3 ma**

The point C lies on l such that $|\overrightarrow{OA}| = |\overrightarrow{OC}|$.

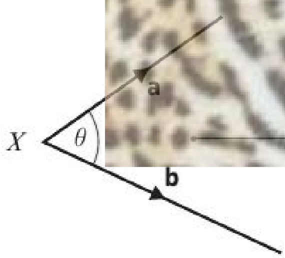
d Find the position vector of C. **(3 ma**

llenge

plane has vector equation $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + \lambda(\mathbf{i} - 2\mathbf{j} + \mathbf{k}) + \mu(2\mathbf{i} - \mathbf{j} + 3\mathbf{k})$.
ne has vector equation $\mathbf{r} = (2\mathbf{i} + 6\mathbf{j} + \mathbf{k}) + t(5\mathbf{i} - 7\mathbf{j} + 6\mathbf{k})$.
ow that the line lies entirely within the plane.

Scalar product

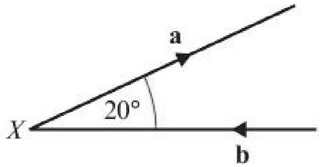
need to know the definition of the scalar product of two vectors in either two or three ensions, and how it can be used to find the angle between two vectors. To define the scalar luct you need to know how to find the **angle between two vectors**.

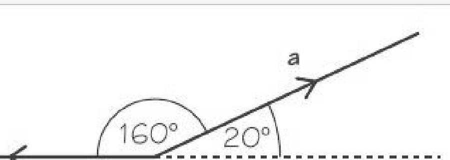


On the diagram, the angle between the vectors **a** and **b** i
Notice that **a** and **b** are both directed away from the poi

mple 10

l the angle between the vectors **a** and **b** on the diagram.





For the correct angle, **a** and **b** must both be pointing away from X, so re-draw to show th

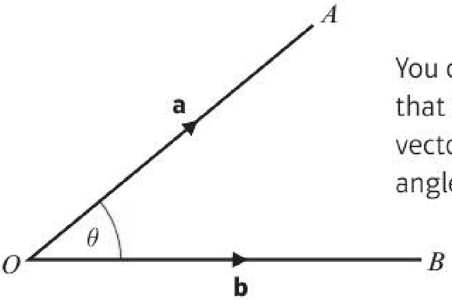
e angle between **a** and **b** is 180° − 20° = 160°.

The scalar product of two vectors a and b is written as a.b, and defined as

$$\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

where θ is the angle between a and b.

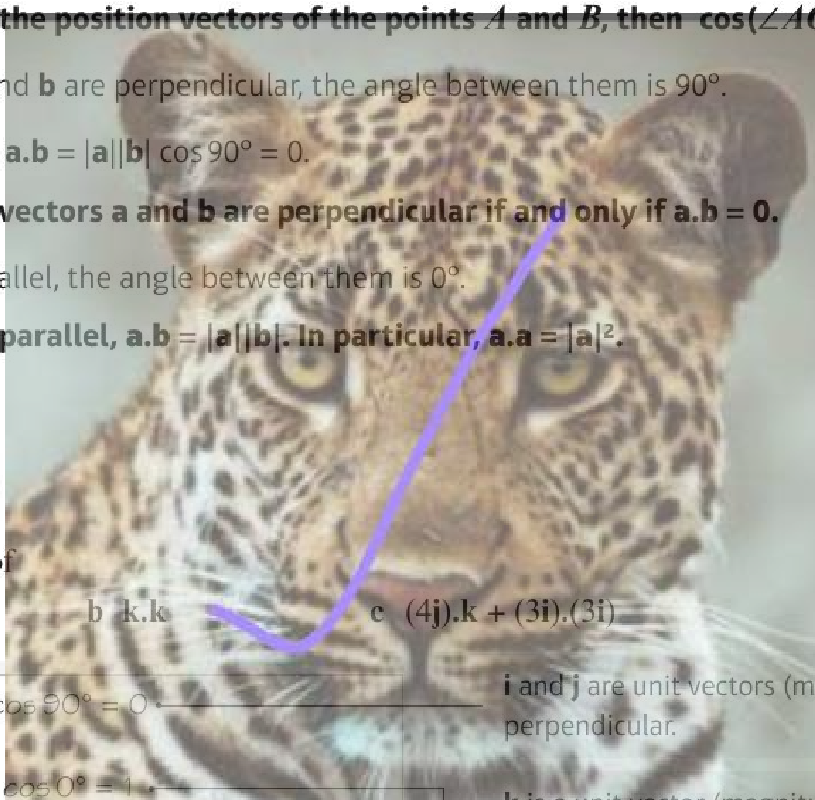
Notation The scalar pro
is often called the **dot pro**
You say 'a dot b'.



You can see from this diagram that if **a** and **b** are the position vectors of A and B, then the angle between **a** and **b** is $\angle AOB$.

Online Use GeoGebra to consider the scalar prod as the component of one v in the direction of another.

If a and b are the position vectors of the points A and B, then $\cos(\angle AOB) = \dfrac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$



two vectors **a** and **b** are perpendicular, the angle between them is 90°.

nce $\cos 90° = 0$, $\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos 90° = 0$.

The non-zero vectors a and b are perpendicular if and only if a.b = 0.

a and **b** are parallel, the angle between them is 0°.

If a and b are parallel, $\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|$. In particular, $\mathbf{a}.\mathbf{a} = |\mathbf{a}|^2$.

xample 11

ind the values of

a $\mathbf{i}.\mathbf{j}$ **b** $\mathbf{k}.\mathbf{k}$ **c** $(4\mathbf{j}).\mathbf{k} + (3\mathbf{i}).(3\mathbf{i})$

a $\mathbf{i}.\mathbf{j} = 1 \times 1 \times \cos 90° = 0$

i and **j** are unit vectors (magnitude 1), and ar perpendicular.

b $\mathbf{k}.\mathbf{k} = 1 \times 1 \times \cos 0° = 1$

k is a unit vector (magnitude 1) and the angl between **k** and itself is 0°.

c $(4\mathbf{j}).\mathbf{k} + (3\mathbf{i}).(3\mathbf{i})$
$= (4 \times 1 \times \cos 90°) + (3 \times 3 \times \cos 0°)$
$= 0 + 9 = 9$

xample 12

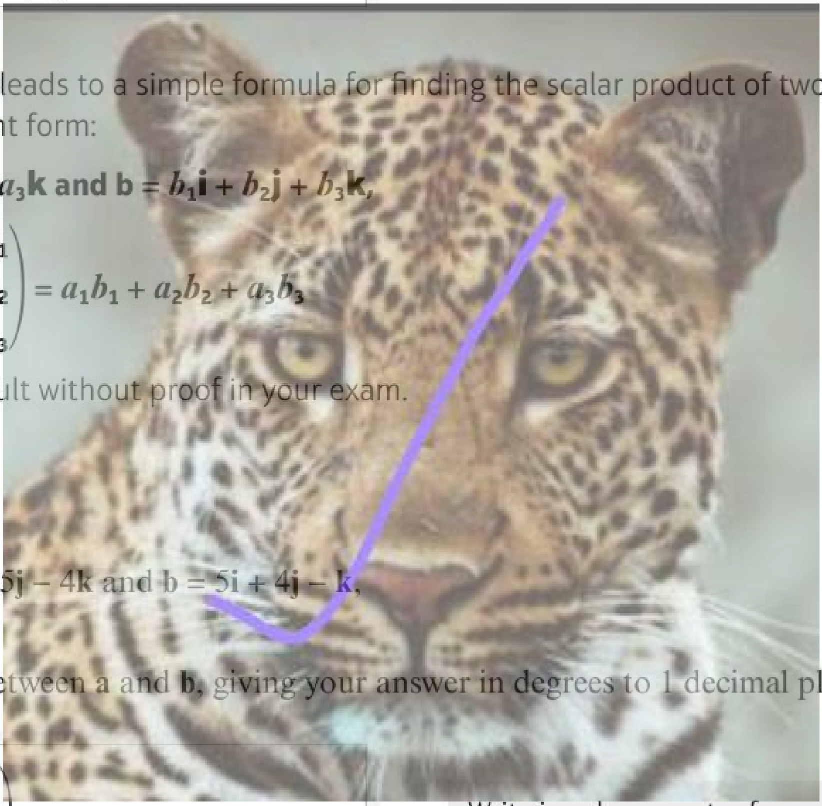
iven that $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, prove that $\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$.

$$
\begin{aligned}
\ldots &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}).(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
&= a_1\mathbf{i}.(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
&\quad + a_2\mathbf{j}.(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
&\quad + a_3\mathbf{k}.(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
&= (a_1\mathbf{i}).(b_1\mathbf{i}) + (a_1\mathbf{i}).(b_2\mathbf{j}) + (a_1\mathbf{i}).(b_3\mathbf{k}) \\
&\quad + (a_2\mathbf{j}).(b_1\mathbf{i}) + (a_2\mathbf{j}).(b_2\mathbf{j}) + (a_2\mathbf{j}).(b_3\mathbf{k}) \\
&\quad + (a_3\mathbf{k}).(b_1\mathbf{i}) + (a_3\mathbf{k}).(b_2\mathbf{j}) + (a_3\mathbf{k}).(b_3\mathbf{k}) \\
&= (a_1b_1)\mathbf{i}.\mathbf{i} + (a_1b_2)\mathbf{i}.\mathbf{j} + (a_1b_3)\mathbf{i}.\mathbf{k} \\
&\quad + (a_2b_1)\mathbf{j}.\mathbf{i} + (a_2b_2)\mathbf{j}.\mathbf{j} + (a_2b_3)\mathbf{j}.\mathbf{k} \\
&\quad + (a_3b_1)\mathbf{k}.\mathbf{i} + (a_3b_2)\mathbf{k}.\mathbf{j} + (a_3b_3)\mathbf{k}.\mathbf{k} \\
&= a_1b_1 + a_2b_2 + a_3b_3
\end{aligned}
$$

Use the results for parallel and perpendicular u vectors:

$\mathbf{i}.\mathbf{i} = \mathbf{j}.\mathbf{j} = \mathbf{k}.\mathbf{k} = 1$

$\mathbf{i}.\mathbf{j} = \mathbf{i}.\mathbf{k} = \mathbf{j}.\mathbf{i} = \mathbf{j}.\mathbf{k} = \mathbf{k}.\mathbf{i} = \mathbf{k}.\mathbf{j} = 0$



above example leads to a simple formula for finding the scalar product of two vectors given in esian component form:

$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$,

$$\mathbf{a}.\mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$$

can use this result without proof in your exam.

mple 13

en that $\mathbf{a} = 8\mathbf{i} - 5\mathbf{j} - 4\mathbf{k}$ and $\mathbf{b} = 5\mathbf{i} + 4\mathbf{j} - \mathbf{k}$,

ind $\mathbf{a}.\mathbf{b}$.

ind the angle between $\mathbf{a}$ and $\mathbf{b}$, giving your answer in degrees to 1 decimal place.

$$\mathbf{a}.\mathbf{b} = \begin{pmatrix} 8 \\ -5 \\ -4 \end{pmatrix}.\begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix}$$

Write in column vector form.

$$
\begin{aligned}
&= (8 \times 5) + (-5 \times 4) + (-4 \times -1) \\
&= 40 - 20 + 4 \\
&= 24
\end{aligned}
$$

Use $\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$.

$\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$

Use the scalar product definition.

$|\mathbf{a}| = \sqrt{8^2 + (-5)^2 + (-4)^2} = \sqrt{105}$

$|\mathbf{b}| = \sqrt{5^2 + 4^2 + (-1)^2} = \sqrt{42}$

Find the modulus of $\mathbf{a}$ and of $\mathbf{b}$.

$\sqrt{105}\sqrt{42} \cos\theta = 24$

Use $\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$.

$$\cos\theta = \frac{24}{\sqrt{105}\sqrt{42}}$$

$$\theta = 68.8° \text{ (1 d.p.)}$$

xample 14

iven that $\mathbf{a} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 7\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$, find the angle between $\mathbf{a}$ and $\mathbf{b}$, giving your an
degrees to 1 decimal place.

$$\mathbf{a}.\mathbf{b} = \begin{pmatrix}-1\\1\\3\end{pmatrix}.\begin{pmatrix}7\\-2\\2\end{pmatrix} = -7 - 2 + 6 = -3$$

$$|\mathbf{a}| = \sqrt{(-1)^2 + 1^2 + 3^2} = \sqrt{11}$$

$$|\mathbf{b}| = \sqrt{7^2 + (-2)^2 + 2^2} = \sqrt{57}$$

For the scalar product formula, you need to f
a.b, |**a**| and |**b**|.

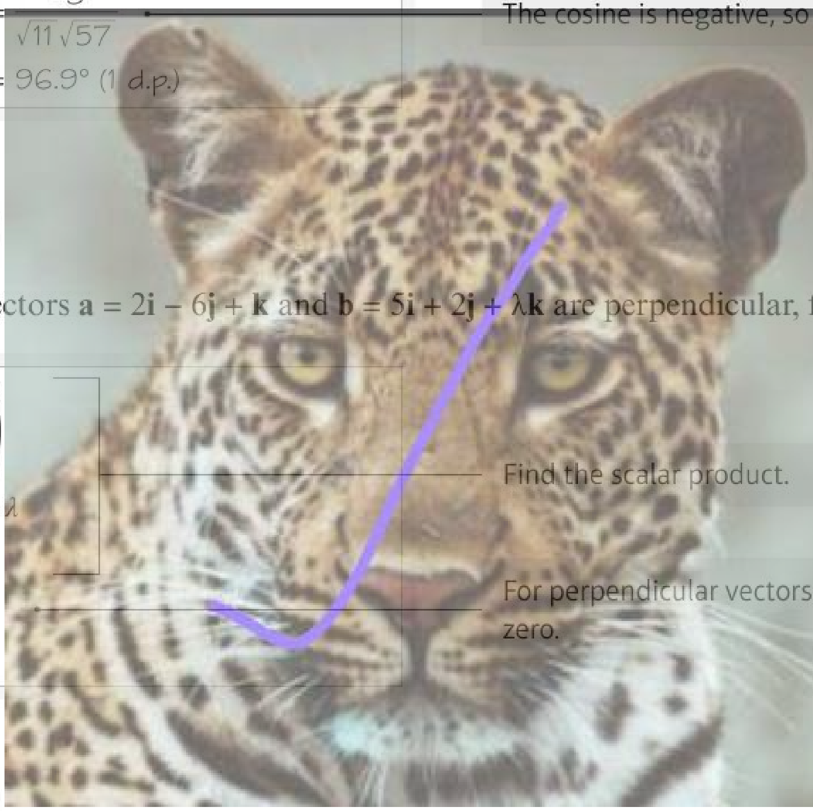
$$\sqrt{11}\sqrt{57}\cos\theta = -3$$

Use $\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$.

$$\cos\theta = \frac{-3}{\sqrt{11}\sqrt{57}}$$

The cosine is negative, so the angle is obtuse

$$\theta = 96.9° \text{ (1 d.p.)}$$



xample 15

iven that the vectors $\mathbf{a} = 2\mathbf{i} - 6\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 5\mathbf{i} + 2\mathbf{j} + \lambda\mathbf{k}$ are perpendicular, find the value of

$$\mathbf{a}.\mathbf{b} = \begin{pmatrix}2\\-6\\1\end{pmatrix}.\begin{pmatrix}5\\2\\\lambda\end{pmatrix}$$

$$= 10 - 12 + \lambda$$

$$= -2 + \lambda$$

Find the scalar product.

$$-2 + \lambda = 0$$

For perpendicular vectors, the scalar product
zero.

$$\lambda = 2$$

xample 16

iven that $\mathbf{a} = -2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} - 8\mathbf{j} + 5\mathbf{k}$, find a vector which is perpendicular to both
and $\mathbf{b}$.

$$\mathbf{a}.\begin{pmatrix}x\\y\\z\end{pmatrix} = 0 \text{ and } \mathbf{b}.\begin{pmatrix}x\\y\\z\end{pmatrix} = 0$$

Both scalar products are zero.

$$\begin{pmatrix}-2\\5\\-4\end{pmatrix}.\begin{pmatrix}x\\y\\z\end{pmatrix} = 0 \text{ and } \begin{pmatrix}4\\-8\\5\end{pmatrix}.\begin{pmatrix}x\\y\\z\end{pmatrix} = 0$$

$$-2x + 5y - 4z = 0 \quad (1)$$

$$4x - 8y + 5z = 0 \quad (2)$$

t $z = 1$

$-2x + 5y = 4$ (from 1)

$4x - 8y = -5$ (from 2)

$x = \frac{7}{4}$, $y = \frac{3}{2}$ and $z = 1$

possible vector is $\frac{7}{4}\mathbf{i} + \frac{3}{2}\mathbf{j} + \mathbf{k}$.

other possible vector is

$\frac{7}{4}\mathbf{i} + \frac{3}{2}\mathbf{j} + \mathbf{k}) = 7\mathbf{i} + 6\mathbf{j} + 4\mathbf{k}$

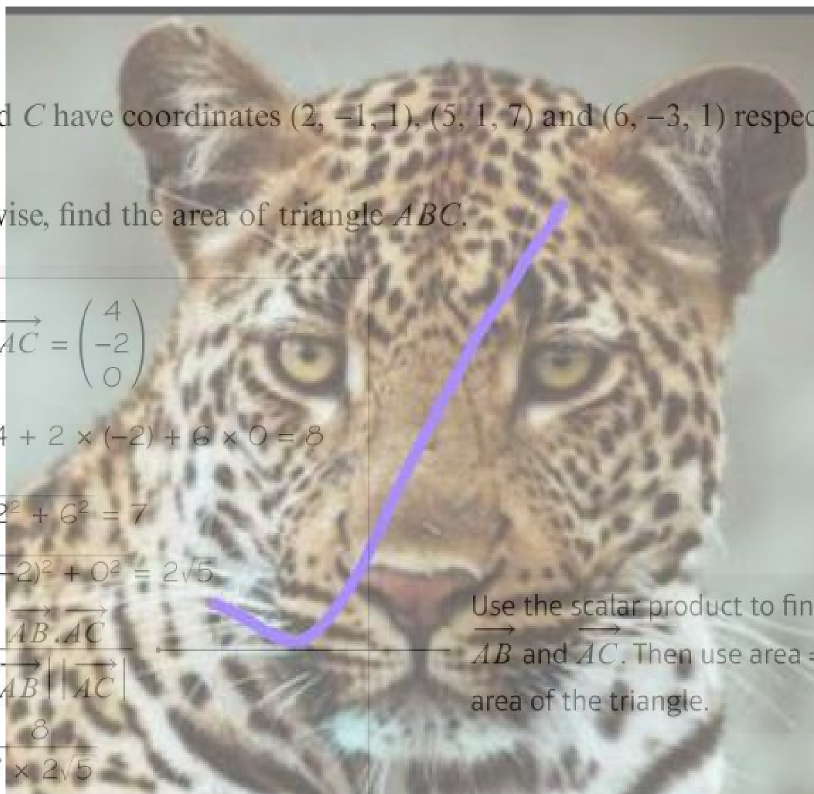
Choose a (non-zero) value for z (or for x, or for y

Solving simutaneously gives
$x = \frac{7}{4}$ and $y = \frac{3}{2}$

You can multiply by a scalar constant to find another vector which is also perpendicular to both **a** and **b**.

mple 17

points A, B and C have coordinates $(2, -1, 1)$, $(5, 1, 7)$ and $(6, -3, 1)$ respectively.
ind $\overrightarrow{AB}.\overrightarrow{AC}$
lence, or otherwise, find the area of triangle ABC.



$\overrightarrow{AB} = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix}$

$\overrightarrow{AB}.\overrightarrow{AC} = 3 \times 4 + 2 \times (-2) + 6 \times 0 = 8$

$|\overrightarrow{AB}| = \sqrt{3^2 + 2^2 + 6^2} = 7$

$|\overrightarrow{AC}| = \sqrt{4^2 + (-2)^2 + 0^2} = 2\sqrt{5}$

$$\cos(\angle BAC) = \frac{\overrightarrow{AB}.\overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|}$$

$$= \frac{8}{7 \times 2\sqrt{5}}$$

$$= 0.2555\ldots$$

$\angle BAC = 75.1937\ldots°$

$$\text{Area} = \frac{1}{2}|\overrightarrow{AB}||\overrightarrow{AC}|\sin(\angle BAC)$$

$$= \frac{1}{2} \times 7 \times 2\sqrt{5}\sin(75.1937\ldots°)$$

$$= 15.13 \text{ (2 d.p.)}$$

Use the scalar product to find the angle betwee $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Then use area $= \frac{1}{2}ab\sin\theta$ to find th area of the triangle.

Problem-solving

You could find $\angle BAC$ by finding the lengths A BC and AC and using the cosine rule, but it is quicker to use a vector method.

xercise 9C

1 The vectors $\mathbf{a}$ and $\mathbf{b}$ each have magnitude 3, and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $60°$. Find $\mathbf{a.b}$

2 For each pair of vectors, find $\mathbf{a.b}$:

a $\mathbf{a} = 5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$

b $\mathbf{a} = 10\mathbf{i} - 7\mathbf{j} + 4\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} - 5\mathbf{j} - 12\mathbf{k}$

c $\mathbf{a} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{b} = -\mathbf{i} - \mathbf{j} + 4\mathbf{k}$

d $\mathbf{a} = 2\mathbf{i} - \mathbf{k}$, $\mathbf{b} = 6\mathbf{i} - 5\mathbf{j} - 8\mathbf{k}$

e $\mathbf{a} = 3\mathbf{j} + 9\mathbf{k}$, $\mathbf{b} = \mathbf{i} - 12\mathbf{j} + 4\mathbf{k}$

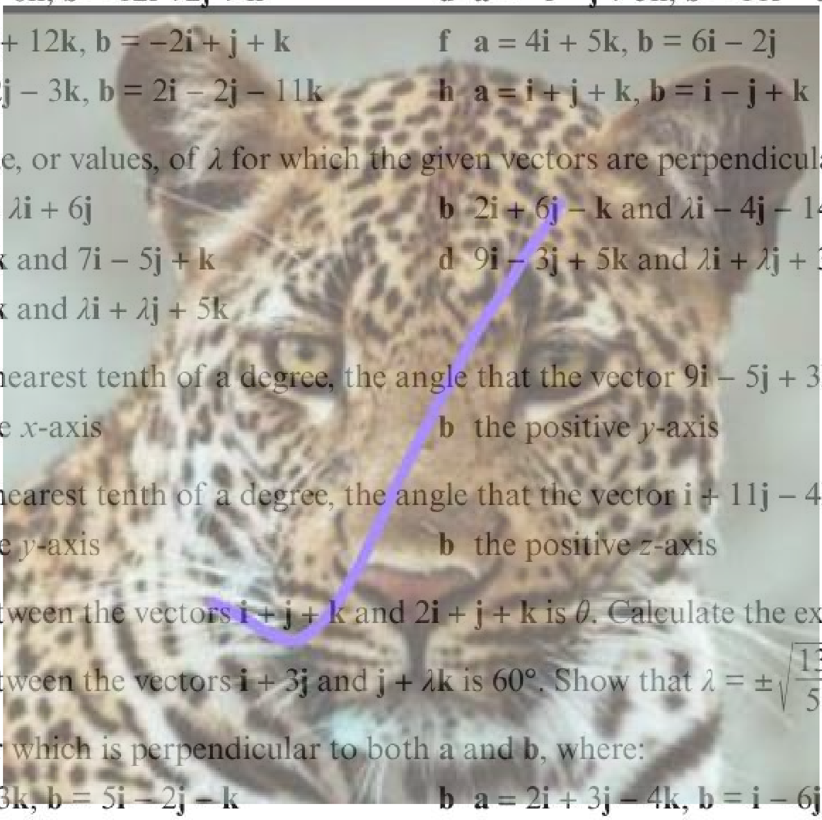
3 In each part, find the angle between $\mathbf{a}$ and $\mathbf{b}$, giving your answer in degrees to 1 decimal pl

a $\mathbf{a} = 3\mathbf{i} + 7\mathbf{j}$, $\mathbf{b} = 5\mathbf{i} + \mathbf{j}$

b $\mathbf{a} = 2\mathbf{i} - 5\mathbf{j}$, $\mathbf{b} = 6\mathbf{i} + 3\mathbf{j}$

c $\mathbf{a} = \mathbf{i} - 7\mathbf{j} + 8\mathbf{k}$, $\mathbf{b} = 12\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

d $\mathbf{a} = -\mathbf{i} - \mathbf{j} + 5\mathbf{k}$, $\mathbf{b} = 11\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$



e $\mathbf{a} = 6\mathbf{i} - 7\mathbf{j} + 12\mathbf{k}$, $\mathbf{b} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$

f $\mathbf{a} = 4\mathbf{i} + 5\mathbf{k}$, $\mathbf{b} = 6\mathbf{i} - 2\mathbf{j}$

g $\mathbf{a} = -5\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - 2\mathbf{j} - 11\mathbf{k}$

h $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$

4 Find the value, or values, of λ for which the given vectors are perpendicular:

a $3\mathbf{i} + 5\mathbf{j}$ and $\lambda\mathbf{i} + 6\mathbf{j}$

b $2\mathbf{i} + 6\mathbf{j} - \mathbf{k}$ and $\lambda\mathbf{i} - 4\mathbf{j} - 14\mathbf{k}$

c $3\mathbf{i} + \lambda\mathbf{j} - 8\mathbf{k}$ and $7\mathbf{i} - 5\mathbf{j} + \mathbf{k}$

d $9\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$ and $\lambda\mathbf{i} + \lambda\mathbf{j} + 3\mathbf{k}$

e $\lambda\mathbf{j} + 3\mathbf{j} - 2\mathbf{k}$ and $\lambda\mathbf{i} + \lambda\mathbf{j} + 5\mathbf{k}$

5 Find, to the nearest tenth of a degree, the angle that the vector $9\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$ makes with:

a the positive x-axis

b the positive y-axis

6 Find, to the nearest tenth of a degree, the angle that the vector $\mathbf{i} + 11\mathbf{j} - 4\mathbf{k}$ makes with:

a the positive y-axis

b the positive z-axis

7 The angle between the vectors $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $2\mathbf{i} + \mathbf{j} + \mathbf{k}$ is θ. Calculate the exact value of cos

8 The angle between the vectors $\mathbf{i} + 3\mathbf{j}$ and $\mathbf{j} + \lambda\mathbf{k}$ is $60°$. Show that $\lambda = \pm\sqrt{\frac{13}{5}}$

9 Find a vector which is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, where:

a $\mathbf{a} = \mathbf{i} + \mathbf{j} - 3\mathbf{k}$, $\mathbf{b} = 5\mathbf{i} - 2\mathbf{j} - \mathbf{k}$

b $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$, $\mathbf{b} = \mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$

c $\mathbf{a} = 4\mathbf{i} - 4\mathbf{j} - \mathbf{k}$, $\mathbf{b} = -2\mathbf{i} - 9\mathbf{j} + 6\mathbf{k}$

0 The points A and B have position vectors $2\mathbf{i} + 5\mathbf{j} + \mathbf{k}$ and $6\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ respectively, and O is the origin. Calculate each of the angles in $\triangle OAB$, giving your answers in degrees to 1 decimal place.

1 The points A, B and C have coordinates $(1, 3, 1)$, $(2, 7, -3)$ and $(4, -5, 2)$ respectively.

a Find the exact lengths of AB and BC.

b Calculate, to one decimal place, the size of $\angle ABC$.

2 Given that the points A and B have coordinates $(7, 4, 4)$ and $(2, 2, 1)$ respectively,

a find the value of $\cos \angle AOB$, where O is the origin **(4 r**

b show that the area of $\triangle AOB$ is $\frac{\sqrt{53}}{2}$ **(3 r**

AB is a diameter of a circle centred at the origin O, and P is a point on the circumference of the circle. By considering the position vectors of A, B and P, prove that AP is perpendicular to BP.

Problem-solving

This is a vector proof of the fact that the angle a semi-circle is 90°.

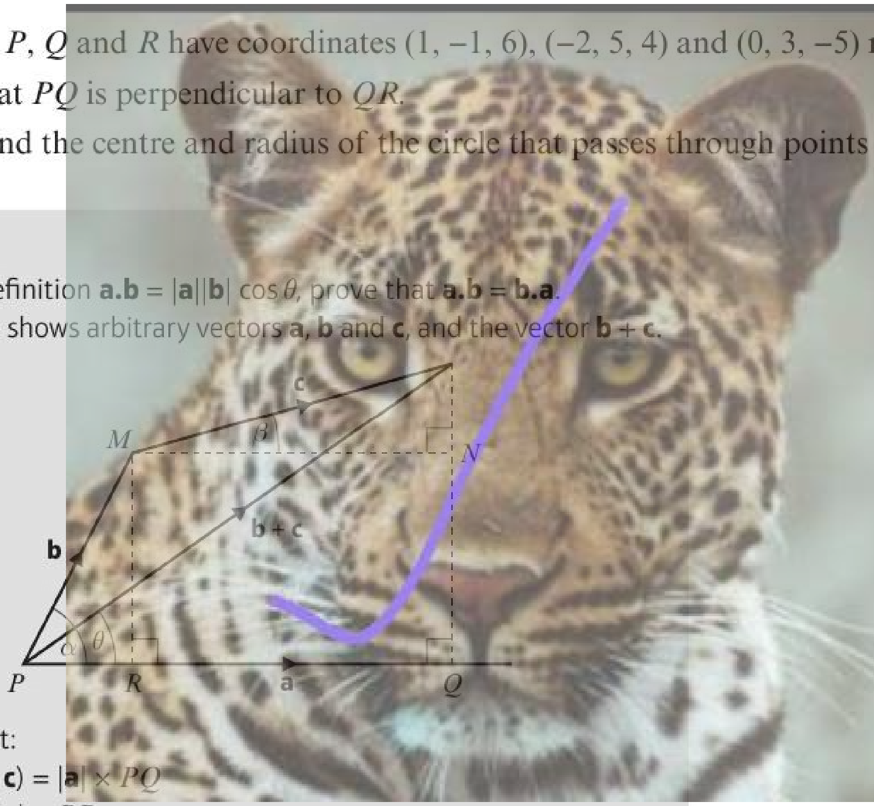
Points A, B and C have coordinates $(5, -1, 0)$, $(2, 4, 10)$ and $(6, -1, 4)$ respectively.

a Find the vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$. **(2 ma**

b Find the area of the triangle ABC. **(4 ma**

c Point D is such that A, B, C and D are the vertices of a parallelogram. Find the coordinat of three possible positions of D. **(3 ma**

d Write down the area of the parallelogram. **(1 m**



The points P, Q and R have coordinates $(1, -1, 6)$, $(-2, 5, 4)$ and $(0, 3, -5)$ respectively.

a Show that PQ is perpendicular to QR. **(3 ma**

b Hence find the centre and radius of the circle that passes through points P, Q and R. **(3 ma**

llenge

Using the definition $\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$, prove that $\mathbf{a}.\mathbf{b} = \mathbf{b}.\mathbf{a}$.

The diagram shows arbitrary vectors **a**, **b** and **c**, and the vector **b** + **c**.

a Show that:

i $\mathbf{a}.(\mathbf{b} + \mathbf{c}) = |\mathbf{a}| \times PQ$

ii $\mathbf{a}.\mathbf{b} = |\mathbf{a}| \times PR$

iii $\mathbf{a}.\mathbf{c} = |\mathbf{a}| \times RQ$

b Hence prove that $\mathbf{a}.(\mathbf{b} + \mathbf{c}) = \mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c}$.

Calculating angles between lines and planes

o straight lines in three dimensions intersect, then you can calculate the size of the angle veen them using the scalar product.

he acute angle θ between two intersecting traight lines is given by

$$\cos\theta = \left|\frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right|$$

Watch out The modulus signs around the whole expression ensure you get an acute ang If you need to work out the size of an **obtuse** angle between two lines, use the formula the subtract the resulting acute angle from 180°.

xample 18

he lines l_1 and l_2 have vector equations $\mathbf{r} = (2\mathbf{i} + \mathbf{j} + \mathbf{k}) + t(3\mathbf{i} - 8\mathbf{j} - \mathbf{k})$ and
$= (7\mathbf{i} + 4\mathbf{j} + \mathbf{k}) + s(2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$ respectively.

iven that l_1 and l_2 intersect, find the size of the acute angle between the lines to one decimal

$\mathbf{a} = \begin{pmatrix} 3 \\ -8 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$ — Use the direction vectors.

$$\cos\theta = \left|\frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right|$$

$$\mathbf{a}.\mathbf{b} = \begin{pmatrix} 3 \\ -8 \\ -1 \end{pmatrix}.\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$$

$$= 6 - 16 - 3 = -13$$

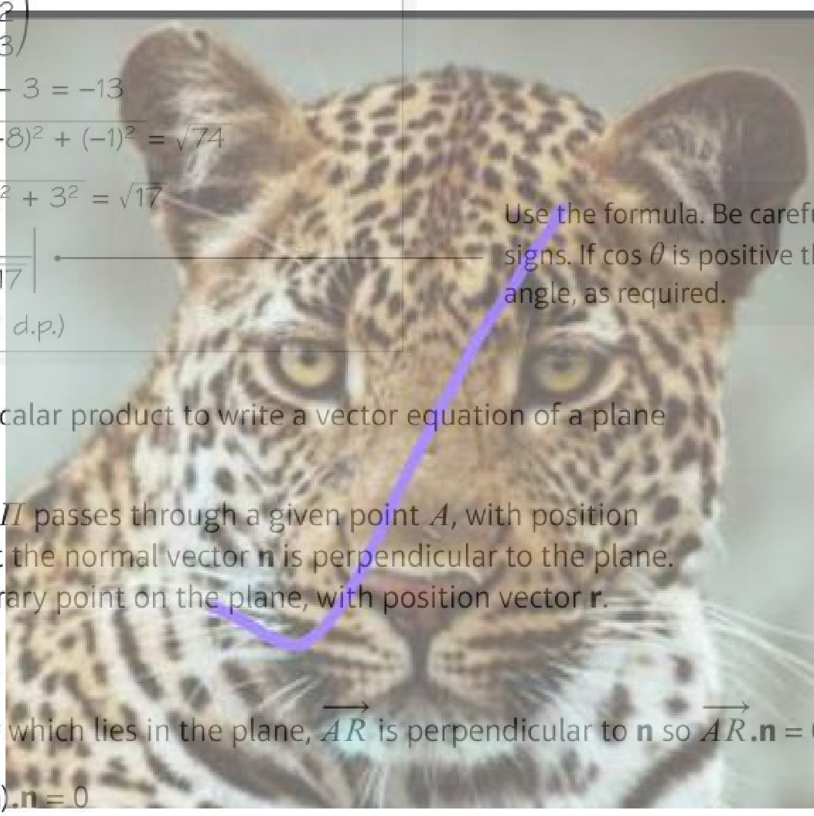
$$|\mathbf{a}| = \sqrt{3^2 + (-8)^2 + (-1)^2} = \sqrt{74}$$

$$|\mathbf{b}| = \sqrt{2^2 + 2^2 + 3^2} = \sqrt{17}$$

$$\cos\theta = \left|\frac{-13}{\sqrt{74}\ \sqrt{17}}\right|$$

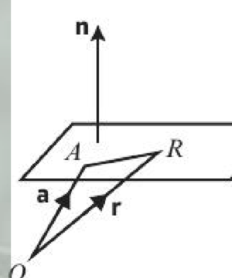
Use the formula. Be careful with the modulus signs. If $\cos\theta$ is positive then θ will be an acu angle, as required.

$$\theta = 68.5° \text{ (1 d.p.)}$$



ou can use the scalar product to write a vector equation of a plane fficiently.

uppose a plane Π passes through a given point A, with position ector $\mathbf{a}$, and that the normal vector $\mathbf{n}$ is perpendicular to the plane. et R be an arbitrary point on the plane, with position vector $\mathbf{r}$.



hen, $\overrightarrow{AR} = \mathbf{r} - \mathbf{a}$

s $\overrightarrow{AR}$ is a vector which lies in the plane, $\overrightarrow{AR}$ is perpendicular to $\mathbf{n}$ so $\overrightarrow{AR}.\mathbf{n} = 0$.

his means $(\mathbf{r} - \mathbf{a}).\mathbf{n} = 0$

ou can rewrite this as $\mathbf{r}.\mathbf{n} = \mathbf{a}.\mathbf{n}$

nce $\mathbf{a}$ is a fixed point, $\mathbf{a}.\mathbf{n}$ is a scalar constant, k, and the equation of the plane Π is $\mathbf{r}.\mathbf{n} = k$.

The scalar product form of the equation of a plane is $\mathbf{r}.\mathbf{n} = k$ where $k = \mathbf{a}.\mathbf{n}$ for any point the plane with position vector a.

xample 19

he plane Π passes through the point A and is perpendicular to the vector $\mathbf{n}$.

iven that $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$ and $\mathbf{n} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$ where O is the origin, find an equation of the plane:

in scalar product form **b** in Cartesian form

$\mathbf{r.n} = k$, where $k = \mathbf{a.n}$

$\mathbf{a.n} = \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix} . \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$

$= 2 \times 3 + 3 \times 1 + (-5) \times (-1)$

$= 6 + 3 + 5 = 14$

So a scalar product form of the equation of Π is $\mathbf{r}.\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = 14$.

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} . \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = 14$

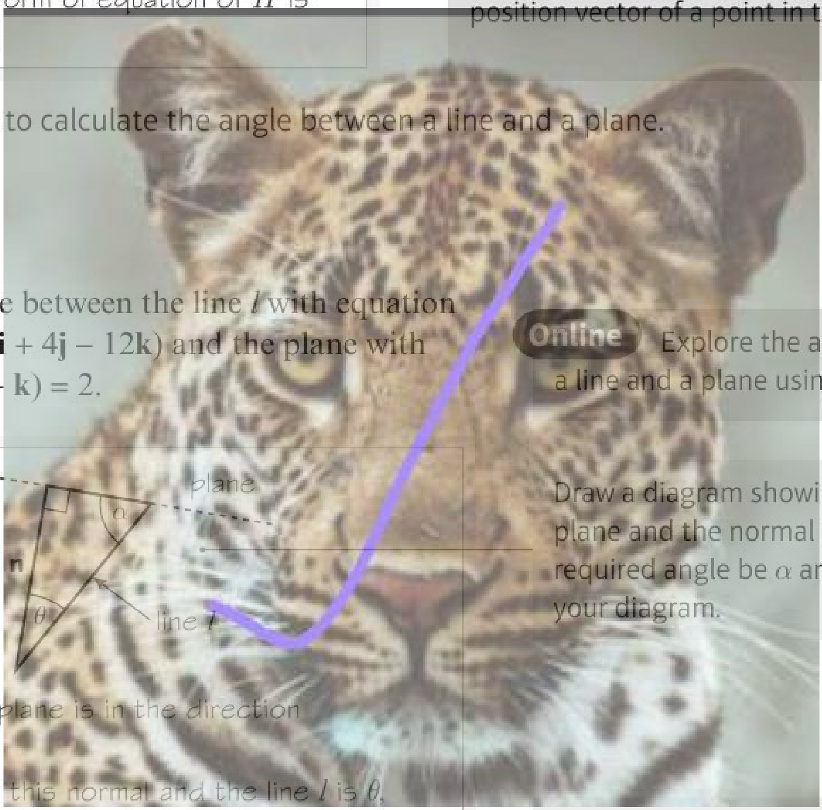
So a Cartesian form of equation of Π is

$3x + y - z = 14$

Use $\mathbf{r.n} = k$ where $k = \mathbf{a.n}$ for any point in the plane with position vector $\mathbf{a}$.

Problem-solving

You can convert between scalar product form a… Cartesian form quickly by writing the general position vector of a point in the plane as $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$



…need to be able to calculate the angle between a line and a plane.

…mple 20

…d the acute angle between the line l with equation $2\mathbf{i} + \mathbf{j} - 5\mathbf{k} + \lambda(3\mathbf{i} + 4\mathbf{j} - 12\mathbf{k})$ and the plane with …ation $\mathbf{r}.(2\mathbf{i} - 2\mathbf{j} - \mathbf{k}) = 2$.

Online Explore the angle between a line and a plane using GeoGebra.

Draw a diagram showing the line, the plane and the normal to the plane. Let t… required angle be α and show α and θ in your diagram.

…e normal to the plane is in the direction

…$= 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$.

…e angle between this normal and the line l is θ.

…ere $\cos\theta = \dfrac{(3\mathbf{i} + 4\mathbf{j} - 12\mathbf{k}).(2\mathbf{i} - 2\mathbf{j} - \mathbf{k})}{\sqrt{3^2 + 4^2 + (-12)^2}\ \sqrt{2^2 + (-2)^2 + (-1)^2}}$

$= \dfrac{10}{13 \times 3} = \dfrac{10}{39}$

First find the angle between the given li… and the normal to the plane.

…the angle between the plane and the line l is α …ere $\alpha + \theta = 90°$.

…$\sin\alpha = \dfrac{10}{39}$ and $\alpha = 14.9°$.

Subtract the angle θ from 90°, to give angl… α, or use the trigonometric connection tha… $\cos\theta = \sin\alpha$.

…he acute angle θ between the line with equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and the plane with equation …$\mathbf{n} = k$ is given by the formula

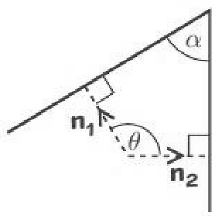
…n $\theta = \left|\dfrac{\mathbf{b.n}}{|\mathbf{b}||\mathbf{n}|}\right|$

ou need to be able to calculate the angle between two planes.

xample 21

ind the acute angle between the planes with equations (4**i** + 4**j** – 7**k**) = 13 and **r**.(7**i** – 4**j** + 4**k**) = 6 respectively.

Online Visualise the angle between two planes using GeoGebra.



Draw a diagram showing the planes and the normals to the planes. Let the required angl α and show α and θ in your diagram.

The normals to the planes are in the directions $\mathbf{n}_1 = 4\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}$ and $\mathbf{n}_2 = 7\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$.

The angle between these normals is θ, where

$$\cos\theta = \frac{(4\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}).(7\mathbf{i} - 4\mathbf{j} + 4\mathbf{k})}{\sqrt{4^2 + 4^2 + (-7)^2}\sqrt{7^2 + (-4)^2 + 4^2}}$$

$$= \frac{28 - 16 - 28}{\sqrt{16 + 16 + 49}\sqrt{49 + 16 + 16}}$$

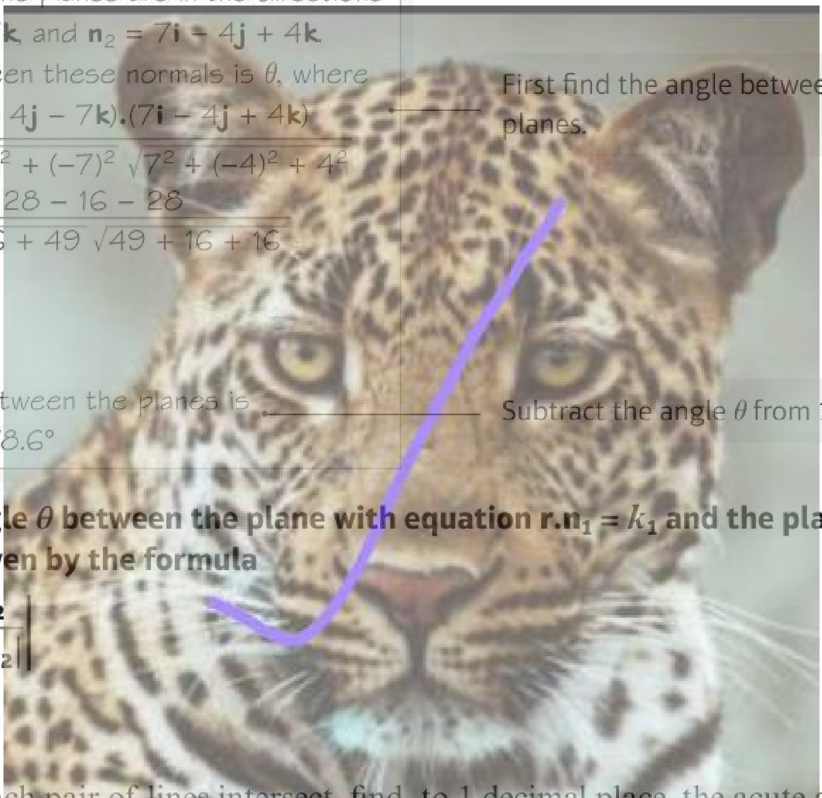
$$= -\frac{16}{81}$$

So $\theta = 101.4°$

First find the angle between the normals to t planes.

So the angle between the planes is $180 - 101.4 = 78.6°$

Subtract the angle θ from 180°, to give angle



The acute angle θ between the plane with equation $\mathbf{r}.\mathbf{n}_1 = k_1$ and the plane with equatio $\mathbf{r}.\mathbf{n}_2 = k_2$ is given by the formula

$$\cos\theta = \left|\frac{\mathbf{n}_1.\mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right|$$

xercise 9D

1 Given that each pair of lines intersect, find, to 1 decimal place, the acute angle between th lines with vector equations:

a $\mathbf{r} = (2\mathbf{i} + \mathbf{j} + \mathbf{k}) + \lambda(3\mathbf{i} - 5\mathbf{j} - \mathbf{k})$ and $\mathbf{r} = (7\mathbf{i} + 4\mathbf{j} + \mathbf{k}) + \mu(2\mathbf{i} + \mathbf{j} - 9\mathbf{k})$

b $\mathbf{r} = (\mathbf{i} - \mathbf{j} + 7\mathbf{k}) + \lambda(-2\mathbf{i} - \mathbf{j} + 3\mathbf{k})$ and $\mathbf{r} = (8\mathbf{i} + 5\mathbf{j} - \mathbf{k}) + \mu(-4\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

c $\mathbf{r} = (3\mathbf{i} + 5\mathbf{j} - \mathbf{k}) + \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k})$ and $\mathbf{r} = (-\mathbf{i} + 11\mathbf{j} + 5\mathbf{k}) + \mu(2\mathbf{i} - 7\mathbf{j} + 3\mathbf{k})$

d $\mathbf{r} = (\mathbf{i} + 6\mathbf{j} - \mathbf{k}) + \lambda(8\mathbf{i} - \mathbf{j} - 2\mathbf{k})$ and $\mathbf{r} = (6\mathbf{i} + 9\mathbf{j}) + \mu(\mathbf{i} + 3\mathbf{j} - 7\mathbf{k})$

e $\mathbf{r} = (2\mathbf{i} + \mathbf{k}) + \lambda(11\mathbf{i} + 5\mathbf{j} - 3\mathbf{k})$ and $\mathbf{r} = (\mathbf{i} + \mathbf{j}) + \mu(-3\mathbf{i} + 5\mathbf{j} + 4\mathbf{k})$

2 Find, in the form $\mathbf{r}.\mathbf{n} = k$, an equation of the plane that passes through the point with pos vector **a** and is perpendicular to the vector **n** where:

a $\mathbf{a} = \mathbf{i} - \mathbf{j} - \mathbf{k}$ and $\mathbf{n} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$

b $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{n} = 5\mathbf{i} - \mathbf{j} - 3\mathbf{k}$

c $\mathbf{a} = 2\mathbf{i} - 3\mathbf{k}$ and $\mathbf{n} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$

d $\mathbf{a} = 4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{n} = 4\mathbf{i} + \mathbf{j} - 5\mathbf{k}$

3 Find a Cartesian equation for each of the planes in question 2

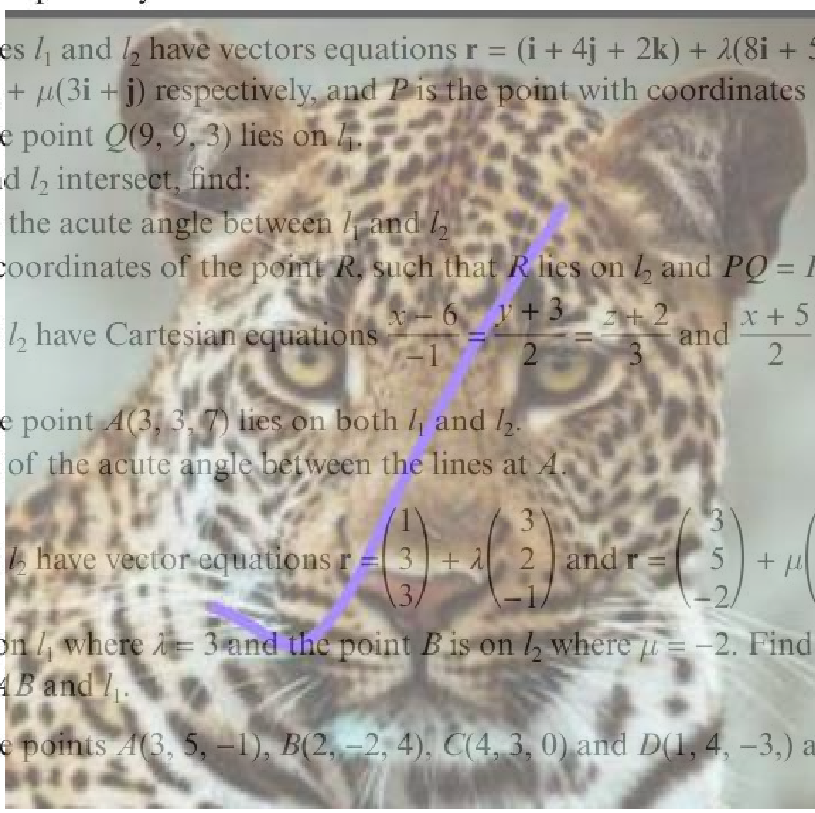
A plane has equation $\mathbf{r.n} = k$, where $\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$. Find a Cartesian equation of the plane.

Find the acute angle between the line with equation $\mathbf{r} = 2\mathbf{i} + \mathbf{j} - 5\mathbf{k} + \lambda(4\mathbf{i} + 4\mathbf{j} + 7\mathbf{k})$ and the plane with equation $\mathbf{r}.(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 13$.

Find the acute angle between the line with equation $\mathbf{r} = -\mathbf{i} - 7\mathbf{j} + 13\mathbf{k} + \lambda(3\mathbf{i} + 4\mathbf{j} - 12\mathbf{k})$ and t
plane with equation $\mathbf{r}.(4\mathbf{i} - 4\mathbf{j} - 7\mathbf{k}) = 9$.

Find the acute angle between the planes with equations $\mathbf{r}.(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = 1$ and $\mathbf{r}.(-4\mathbf{i} + 4\mathbf{j} + 7\mathbf{k}) = 7$ respectively.

Find the acute angle between the planes with equations $\mathbf{r}.(3\mathbf{i} - 4\mathbf{j} + 12\mathbf{k}) = 9$ and $\mathbf{r}.(5\mathbf{i} - 12\mathbf{k}) = 7$ respectively.



The straight lines l_1 and l_2 have vectors equations $\mathbf{r} = (\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}) + \lambda(8\mathbf{i} + 5\mathbf{j} + \mathbf{k})$ and $\mathbf{r} = (\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}) + \mu(3\mathbf{i} + \mathbf{j})$ respectively, and P is the point with coordinates $(1, 4, 2)$.

a Show that the point $Q(9, 9, 3)$ lies on l_1.

Given that l_1 and l_2 intersect, find:

b the cosine of the acute angle between l_1 and l_2

c the possible coordinates of the point R, such that R lies on l_2 and $PQ = PR$.

The lines l_1 and l_2 have Cartesian equations $\frac{x-6}{-1} = \frac{y+3}{2} = \frac{z+2}{3}$ and $\frac{x+5}{2} = \frac{y-15}{-3} = \frac{z-3}{1}$
respectively.

a Show that the point $A(3, 3, 7)$ lies on both l_1 and l_2. **(3 ma**

b Find the size of the acute angle between the lines at A. **(4 ma**

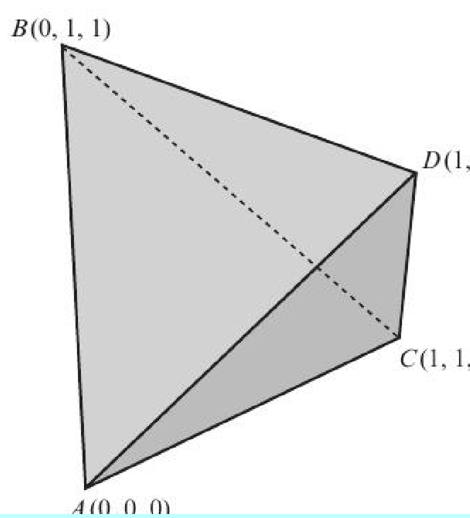
The lines l_1 and l_2 have vector equations $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} + \mu\begin{pmatrix} -4 \\ -3 \\ 1 \end{pmatrix}$.

The point A is on l_1 where $\lambda = 3$ and the point B is on l_2 where $\mu = -2$. Find the size of the ac
angle between AB and l_1. **(6 ma**

a Show that the points $A(3, 5, -1)$, $B(2, -2, 4)$, $C(4, 3, 0)$ and $D(1, 4, -3,)$ are not coplanar. **(6 ma**

b Find the angle between the plane containing A, B and C and the line segment AD. **(4 ma**

A regular tetrahedron has vertices A, B, C and D, with coordinates $(0, 0, 0)$, $(0, 1, 1)$, $(1, 1, 0)$ and $(1, 0, 1)$ respectively. Show that the angle between any two adjacent faces of the tetrahedron is $\arccos\left(\frac{1}{3}\right)$.

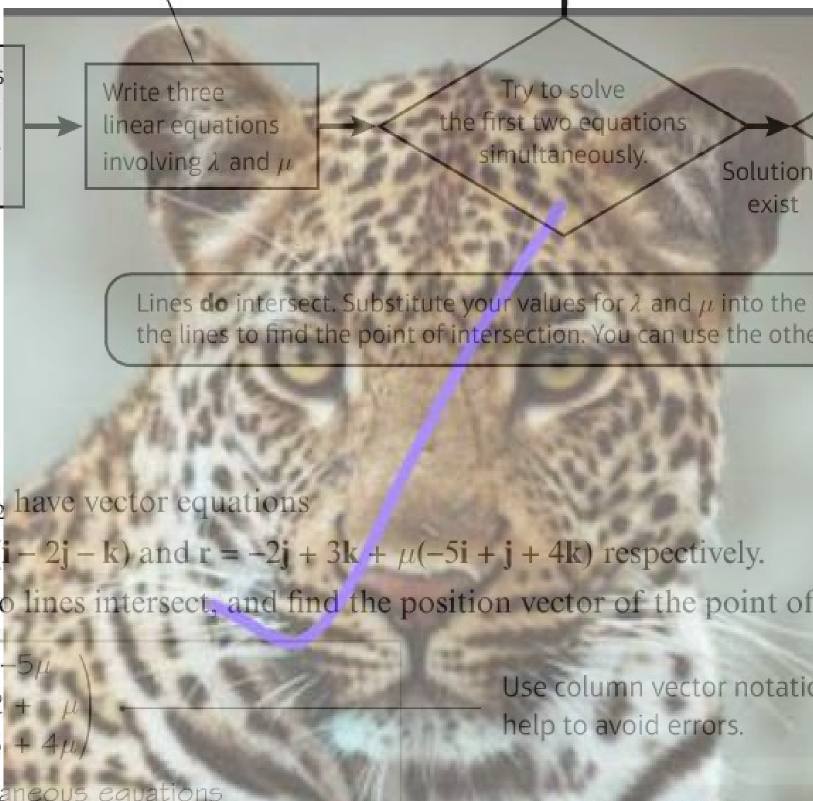


(7 marks)

4 A flagpole is supported by 3 guide ropes which are attached at a point 20 m above the bas the pole. The ends of the ropes are secured at points with position vectors (0, 8, 2), (12, −5, 3) and (−2, 6, 5) relative to the base of the pole, where the units are metres. The flagpole will be stable if the angles between adjacent guide ropes are all greater than 1 Determine whether the flagpole will be stable, showing your working clearly. **(7 r**

.5 Points of intersection

ou need to be able to determine whether two lines meet and, if so, to determine their point of tersection.



λ and μ are the **parameters** in the vector equations of the lines.

Write the equations in column notation and set them equal to each other. → Write three linear equations involving λ and μ → Try to solve the first two equations simultaneously.

No solutions → Lines **do not** intersect

Solutions exist → Do these solutions satisfy t third equation?

No

Yes → Lines **do** intersect. Substitute your values for λ and μ into the equation of one of the lines to find the point of intersection. You can use the other equation to check.

xample 22

he lines l_1 and l_2 have vector equations

$= 3\mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} - \mathbf{k})$ and $\mathbf{r} = -2\mathbf{j} + 3\mathbf{k} + \mu(-5\mathbf{i} + \mathbf{j} + 4\mathbf{k})$ respectively.

how that the two lines intersect, and find the position vector of the point of intersection.

$$\begin{pmatrix} 3 + \lambda \\ 1 - 2\lambda \\ 1 - \lambda \end{pmatrix} = \begin{pmatrix} -5\mu \\ -2 + \mu \\ 3 + 4\mu \end{pmatrix}$$

Use column vector notation for clarity, and to help to avoid errors.

Solve the simultaneous equations

$3 + \lambda = -5\mu$ (1)

and $1 - \lambda = 3 + 4\mu$ (2)

Choose two of the three equations obtained equating x-, y- and z-components and solve resulting simultaneous equations.

Adding gives $4 = 3 - \mu$

and so $\mu = -1$.

Substituting back into equation (1) gives $\lambda = 2$.

Check $\mu = -1$, $\lambda = 2$ also satisfy the third equation.

If the lines intersect there is a pair of values and μ that satisfy the 3 equations simultane

$1 - 2\lambda = -2 + \mu$ gives $-3 = -3$

So the lines do intersect.

Substituting $\lambda = 2$ into $\begin{pmatrix} 3 + \lambda \\ 1 - 2\lambda \\ 1 - \lambda \end{pmatrix}$ gives $\begin{pmatrix} 5 \\ -3 \\ -1 \end{pmatrix}$.

The point where the lines meet is (5, −3, −1).

Check that the point which you obtain after substitution lies on both straight lines.

also need to be able to find the coordinates of the point of intersection of a line with a plane.

mple 23

d the coordinates of the point of intersection of the line l and the plane Π where l has equat
$-\mathbf{i} + \mathbf{j} - 5\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$ and Π has equation $\mathbf{r}.(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = 4$.

e line meets the plane when

$$\begin{pmatrix} 1 + \lambda \\ 1 + \lambda \\ 5 + 2\lambda \end{pmatrix} . \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 4$$

$+ \lambda + 2(1 + \lambda) + 3(-5 + 2\lambda) = 4$

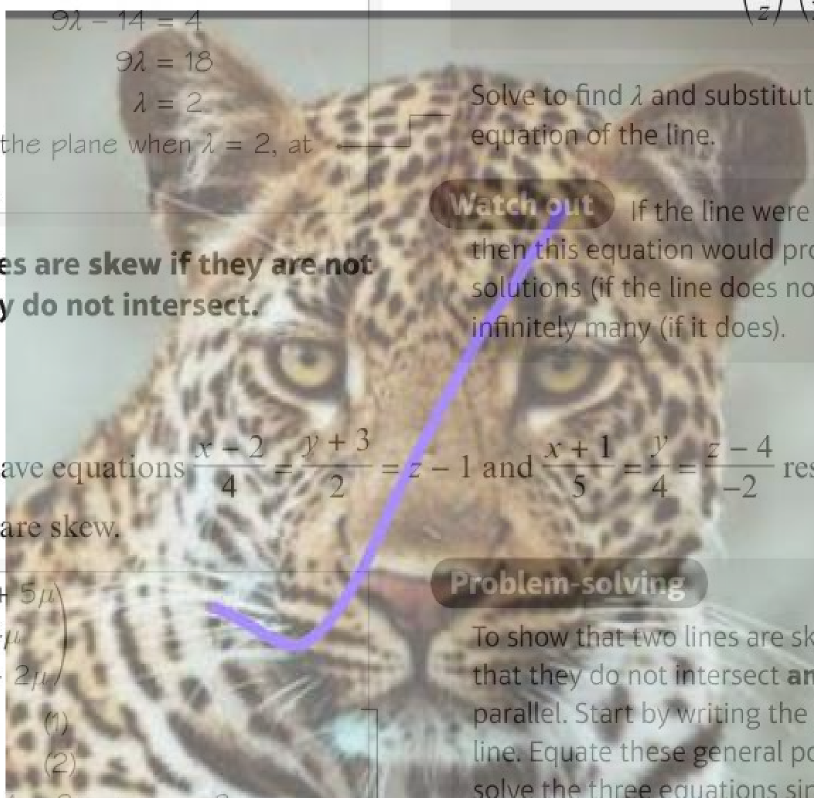
$9\lambda - 14 = 4$

$9\lambda = 18$

$\lambda = 2$

the line meets the plane when $\lambda = 2$, at
e point $(1, 3, -1)$.

Write the equation of the line in column vector form as $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 + \lambda \\ 1 + \lambda \\ -5 + 2\lambda \end{pmatrix}$ and substitute into th equation of the plane $\begin{pmatrix} x \\ y \\ z \end{pmatrix} . \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 4$.



Solve to find λ and substitute its value into the equation of the line.

Watch out If the line were parallel to the plan then this equation would produce either no solutions (if the line does not lie in the plane), o infinitely many (if it does).

wo straight lines are skew if they are not arallel and they do not intersect.

mple 24

lines l_1 and l_2 have equations $\frac{x-2}{4} = \frac{y+3}{2} = z - 1$ and $\frac{x+1}{5} = \frac{y}{4} = \frac{z-4}{-2}$ respectively.
ve that l_1 and l_2 are skew.

$$\begin{pmatrix} 2 + 4\lambda \\ 3 + 2\lambda \\ 1 + \lambda \end{pmatrix} = \begin{pmatrix} -1 + 5\mu \\ 4\mu \\ 4 - 2\mu \end{pmatrix}$$

$+ 4\lambda = -1 + 5\mu$ (1)

$+ 2\lambda = 4\mu$ (2)

$- 2 \times (2)$: $8 = -1 - 3\mu \Rightarrow \mu = -3$

bstituting into (2): $-3 + 2\lambda = -12 \Rightarrow \lambda = -\frac{9}{2}$

eck for consistency:

$+ \lambda = -\frac{7}{2}$ and $4 - 2\mu = 10$.

$+ \lambda \neq 4 - 2\mu$ so equations not consistent
d lines do not intersect.

ection of l_1 is $\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 8 \\ 4 \\ 2 \end{pmatrix}$.

ection of l_2 is $\begin{pmatrix} 5 \\ 4 \\ -2 \end{pmatrix}$.

ection vectors are not scalar multiples of
ch other so lines are not parallel.
nce l_1 and l_2 are skew.

Problem-solving

To show that two lines are skew you need to sho that they do not intersect **and** that they are not parallel. Start by writing the general point on ea line. Equate these general points and attempt to solve the three equations simultaneously.

Solve the first two equations simultaneously, then check to see whether the answer is consistent with the third equation.

If the lines are parallel the direction vectors will be scalar multiples of each other. Multiply the direction vector of l_1 by a scalar to make one component match the direction vector of l_2, the compare the other components.

Exercise 9E

1 In each case establish whether lines l_1 and l_2 meet and, if they meet, find the coordinates of their point of intersection:

a l_1 has equation $\mathbf{r} = \mathbf{i} + 3\mathbf{j} + \lambda(\mathbf{i} - \mathbf{j} + 5\mathbf{k})$ and l_2 has equation $\mathbf{r} = -\mathbf{i} - 3\mathbf{j} + 2\mathbf{k} + \mu(\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

b l_1 has equation $\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$ and l_2 has equation $\mathbf{r} = 4\mathbf{i} + 3\mathbf{j} + \mu(-\mathbf{i} + \mathbf{j} - \mathbf{k}$

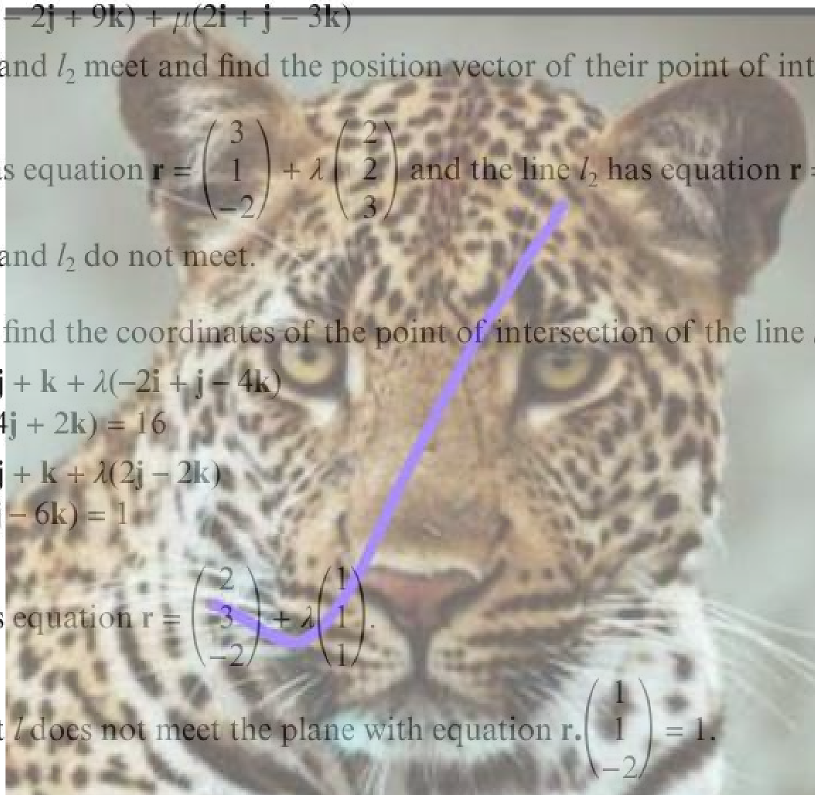
c l_1 has equation $\mathbf{r} = \mathbf{i} + 3\mathbf{j} + 5\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} + \mathbf{k})$ and l_2 has equation $\mathbf{r} = \mathbf{i} + \frac{5}{2}\mathbf{j} + \frac{5}{2}\mathbf{k} + \mu(\mathbf{i} +$

(In each of the above cases λ and μ are scalars.)

2 With respect to a fixed origin O, the lines l_1 and l_2 are given by the equations

l_1: $\mathbf{r} = (-6\mathbf{i} + 11\mathbf{k}) + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k})$

l_2: $\mathbf{r} = (2\mathbf{i} - 2\mathbf{j} + 9\mathbf{k}) + \mu(2\mathbf{i} + \mathbf{j} - 3\mathbf{k})$



Show that l_1 and l_2 meet and find the position vector of their point of intersection. **(6 m**

3 The line l_1 has equation $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$ and the line l_2 has equation $\mathbf{r} = \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$.

Show that l_1 and l_2 do not meet. **(4 m**

4 In each case, find the coordinates of the point of intersection of the line l with the plane Π

a l: $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(-2\mathbf{i} + \mathbf{j} - 4\mathbf{k})$
Π: $\mathbf{r}.(3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = 16$

b l: $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(2\mathbf{j} - 2\mathbf{k})$
Π: $\mathbf{r}.(3\mathbf{i} - \mathbf{j} - 6\mathbf{k}) = 1$

5 The line l has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

a Show that l does not meet the plane with equation $\mathbf{r}.\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = 1$. **(4 m**

b Give a geometrical interpretation to your answer to part **a**. **(1**

6 The line with vector equation $\mathbf{r} = \begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ is perpendicular to the line with vector equation $\mathbf{r} = \begin{pmatrix} 0 \\ 11 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ p \\ p \end{pmatrix}$.

a Find the value of p. **(2 m**

b Show that the two lines meet, and find the coordinates of the point of intersection. **(4 m**

7 The line l_1 has vector equation $\mathbf{r} = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ and the line l_2 has vector equation

$\mathbf{r} = \begin{pmatrix} 4 \\ 1 \\ \ldots \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ \ldots \end{pmatrix}$, where λ and μ are parameters.

The lines l_1 and l_2 intersect at the point A and the acute angle between l_1 and l_2 is θ.

a Find the coordinates of A. **(4 ma**

b Find the value of $\cos\theta$ giving your answer as a simplified fraction. **(4 ma**

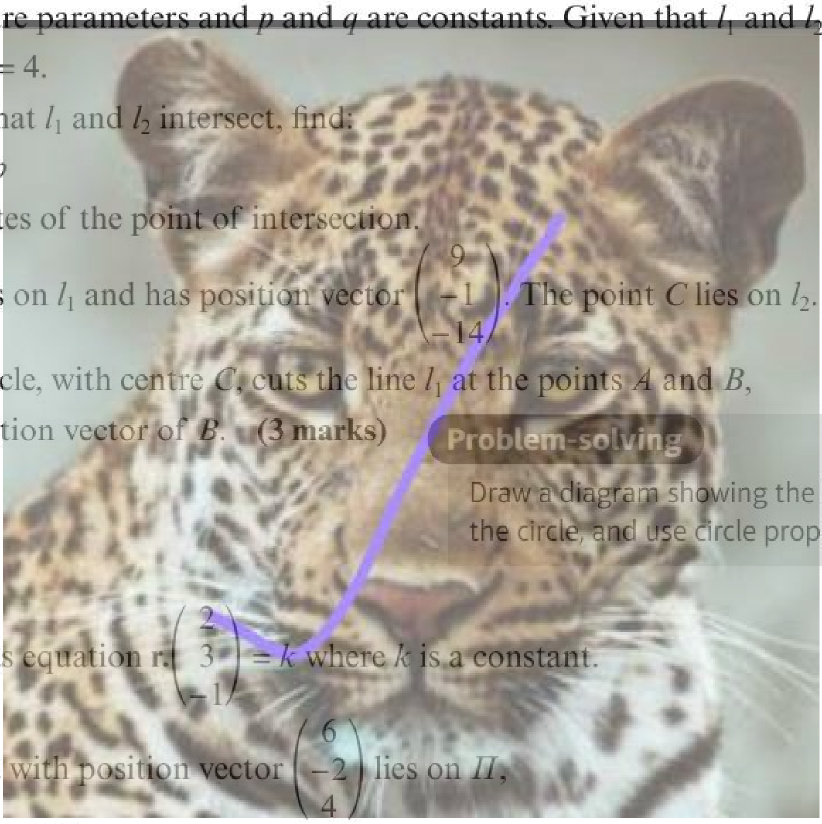
The lines l_1 and l_2 have equations $\frac{x}{-3} = \frac{y+1}{5} = \frac{z-2}{4}$ and $x = \frac{y-1}{-2} = \frac{z+5}{2}$ respectively.

Prove that l_1 and l_2 are skew.

With respect to a fixed origin O the lines l_1 and l_2 are given by the equations

$$l_1\colon \mathbf{r} = \begin{pmatrix} 8 \\ 2 \\ -12 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} \qquad l_2\colon \mathbf{r} = \begin{pmatrix} -4 \\ 10 \\ p \end{pmatrix} + \mu\begin{pmatrix} q \\ 2 \\ -1 \end{pmatrix}$$

where λ and μ are parameters and p and q are constants. Given that l_1 and l_2 are perpendicul



a show that $q = 4$. **(2 ma**

Given further that l_1 and l_2 intersect, find:

b the value of p **(6 ma**

c the coordinates of the point of intersection. **(2 ma**

The point A lies on l_1 and has position vector $\begin{pmatrix} 9 \\ -1 \\ -14 \end{pmatrix}$. The point C lies on l_2.

Given that a circle, with centre C, cuts the line l_1 at the points A and B,

d find the position vector of B. **(3 marks)**

Problem-solving

Draw a diagram showing the lines l_1 and l_2 and the circle, and use circle properties.

The plane Π has equation $\mathbf{r}.\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = k$ where k is a constant.

Given the point with position vector $\begin{pmatrix} 6 \\ -2 \\ 4 \end{pmatrix}$ lies on Π,

a find the value of k **(3 ma**

b find a Cartesian equation for Π. **(2 ma**

The point P has coordinates (6, 4, 8). The line l passes through P and is perpendicular to Π. The line l intersects Π at the point N.

c Find the coordinates of N. **(4 ma**

The line l has a Cartesian equation $\frac{x-3}{5} = \frac{y+2}{3} = \frac{4-z}{1}$

The plane Π has Cartesian equation $4x + 3y - 2z = -10$.

The line intersects the plane at the point P.

a Find the position vector of P. **(5 ma**

b Find the acute angle between the line and the plane at the point of intersection. **(5 ma**

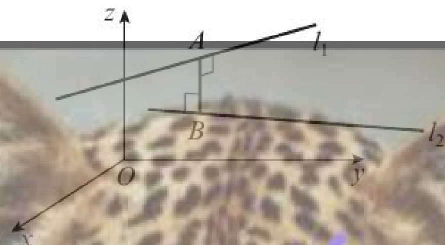
.6 Finding perpendiculars

ou need to be able to calculate the **perpendicular distance** between:

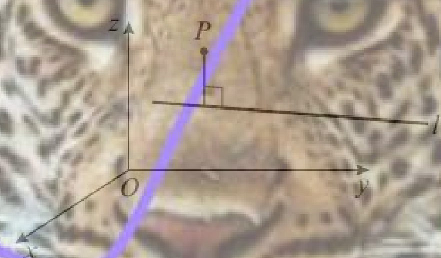
- two lines
- a point and a line
- a point and a plane

each case, the perpendicular distance is the **shortest distance** between them.

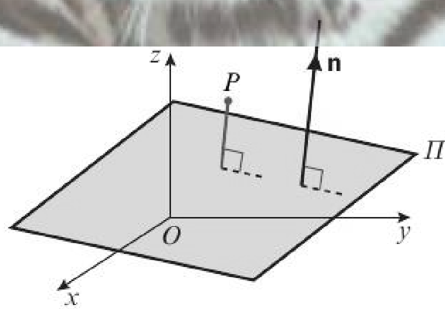
For any two non-intersecting lines l_1 and l_2 there is a unique line segment AB such that A lies on l_1, B lies on l_2 and AB is perpendicular to both lines.



The perpendicular from a point P to a line l is a line through P which meets l at right angles.



The perpendicular from a point P to a plane Π is a line through P which is parallel to t normal vector of the plane, n.



mple 24

w that the shortest distance between the parallel lines with equations

$+ 2\mathbf{j} - \mathbf{k} + \lambda(5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k})$ and $\mathbf{r} = 2\mathbf{i} + \mathbf{k} + \mu(5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k})$,

re λ and μ are scalars, is $\frac{21\sqrt{2}}{10}$

t A be a general point on the first line and B a general point on the second line, then

$\overrightarrow{B} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} + t\begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}$ where $t = \mu - \lambda$.

$\begin{pmatrix} 1 + 5t \\ -2 + 4t \\ 2 + 3t \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} = 0$

$+ 25t - 8 + 16t + 6 + 9t = 0$

$50t = -3$

$t = -\frac{3}{50}$

$\begin{pmatrix} 1 - 5t \\ 2 + 4t \\ 2 + 3t \end{pmatrix} = \begin{pmatrix} 1 - \frac{15}{50} \\ -2 - \frac{12}{50} \\ 2 - \frac{9}{50} \end{pmatrix} = \begin{pmatrix} \frac{35}{50} \\ -\frac{112}{50} \\ \frac{91}{50} \end{pmatrix}$

$|\overrightarrow{AB}| = \sqrt{\frac{35^2 + 112^2 + 91^2}{50^2}}$

$= \frac{21\sqrt{2}}{10}$

the shortest distance between the two es is $\frac{21\sqrt{2}}{10}$

As $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

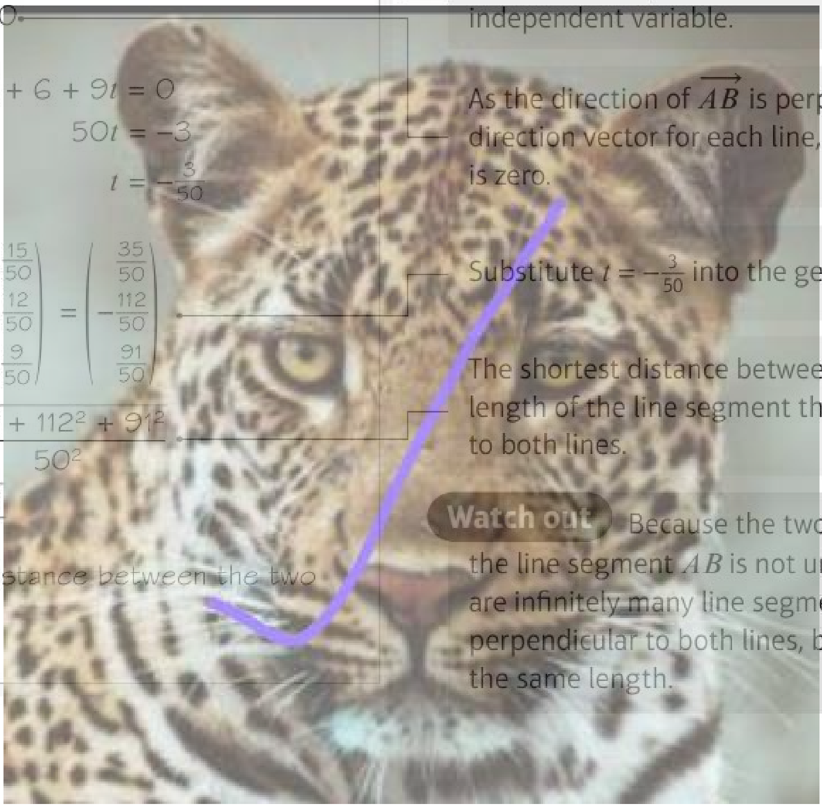
You can set $t = \mu - \lambda$ so that there is only one independent variable.

As the direction of $\overrightarrow{AB}$ is perpendicular to the direction vector for each line, the scalar produc is zero.

Substitute $t = -\frac{3}{50}$ into the general form of $\overrightarrow{AB}$.

The shortest distance between two lines is the length of the line segment that is perpendicula to both lines.

Watch out Because the two lines are parallel, the line segment AB is not unique. There are infinitely many line segments that are perpendicular to both lines, but they will all hav the same length.



mple 25

lines l_1 and l_2 have equations $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix} + \mu\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ respectively,

re λ and μ are scalars.

l the shortest distance between these two lines.

t A be the general point on l_1 with position ctor $\begin{pmatrix} 1 \\ \lambda \\ \lambda \end{pmatrix}$ and let B be the general point on with position vector $\begin{pmatrix} -1 + 2\mu \\ 3 - \mu \\ -1 - \mu \end{pmatrix}$

Online Explore the perpendicular distance between two lines using GeoGebra.

$$\overrightarrow{AB} = \begin{pmatrix} -1 + 2\mu \\ 3 - \mu \\ -1 - \mu \end{pmatrix} - \begin{pmatrix} 1 \\ \lambda \\ \lambda \end{pmatrix} = \begin{pmatrix} -2 + 2\mu \\ 3 - \mu - \lambda \\ -1 - \mu - \lambda \end{pmatrix}$$

Find position vectors of general points on l_1 a[nd l_2] and use these to find $\overrightarrow{AB}$ in terms of μ and λ

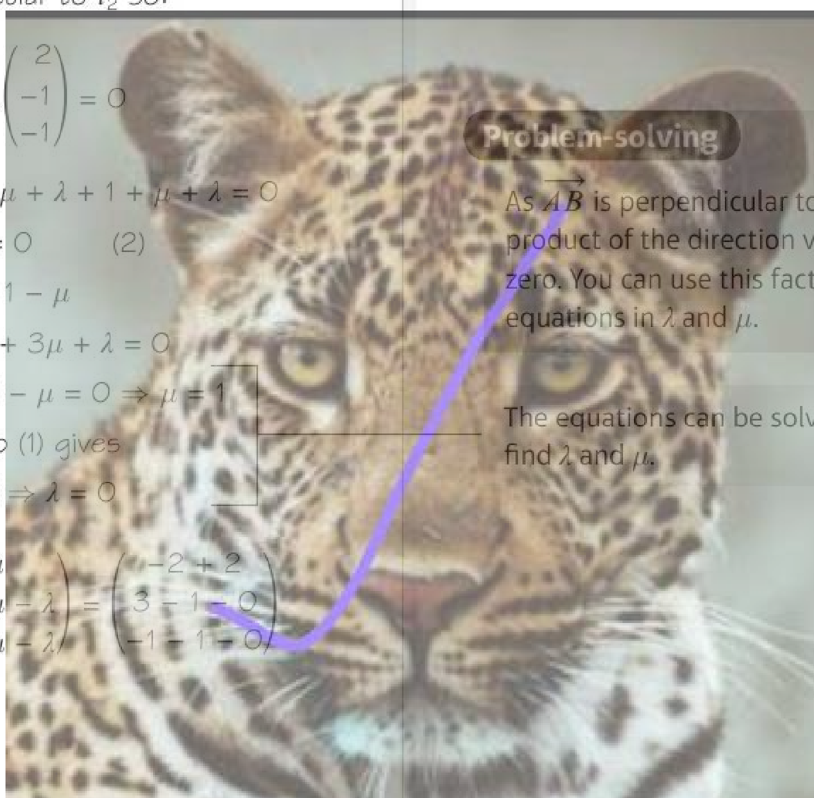
$\overrightarrow{AB}$ is perpendicular to l_1, so:

$$\begin{pmatrix} -2 + 2\mu \\ 3 - \mu - \lambda \\ -1 - \mu - \lambda \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0$$

$3 - \mu - \lambda - 1 - \mu - \lambda = 0$

$2 - 2\mu - 2\lambda = 0 \qquad (1)$

$\overrightarrow{AB}$ is perpendicular to l_2 so:



$$\begin{pmatrix} -2 + 2\mu \\ 3 - \mu - \lambda \\ -1 - \mu - \lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = 0$$

$-4 + 4\mu - 3 + \mu + \lambda + 1 + \mu + \lambda = 0$

$-6 + 6\mu + 2\lambda = 0 \qquad (2)$

From (1), $\quad \lambda = 1 - \mu$

From (2), $\quad -3 + 3\mu + \lambda = 0$

So $-3 + 3\mu + 1 - \mu = 0 \Rightarrow \mu = 1$

Substituting into (1) gives

$2 - 2 - 2\lambda = 0 \Rightarrow \lambda = 0$

Problem-solving

As $\overrightarrow{AB}$ is perpendicular to l_1 and l_2, the scala[r] product of the direction vectors of the lines i[s] zero. You can use this fact to generate two li[near] equations in λ and μ.

The equations can be solved simultaneously [to] find λ and μ.

$$\overrightarrow{AB} = \begin{pmatrix} -2 + 2\mu \\ 3 - \mu - \lambda \\ -1 - \mu - \lambda \end{pmatrix} = \begin{pmatrix} -2 + 2 \\ 3 - 1 - 0 \\ -1 - 1 - 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$$

$\left|\overrightarrow{AB}\right| = \sqrt{0^2 + 2^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$

So the shortest distance between the two lines is $2\sqrt{2}$.

These two lines are **skew**, so in this case the [line] segment AB is unique.

[E]xample 26

[T]he line l has equation $\frac{x - 1}{2} = \frac{y - 1}{-2} = \frac{z + 3}{-1}$, and the point A has coordinates $(1, 2, -1)$.

[a] Find the shortest distance between A and l.

[b] Find a Cartesian equation of the line that is perpendicular to l and passes through A.

Vector equation of l is

$$\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$$

$$\frac{x-1}{2} = \lambda,\ x = 1 + 2\lambda$$
$$\frac{y-1}{-2} = \lambda,\ y = 1 - 2\lambda$$
$$\frac{z+3}{-1} = \lambda,\ z = -3 - \lambda$$

So a general point B, on the line has position vector $\begin{pmatrix} 1 + 2\lambda \\ 1 - 2\lambda \\ -3 - \lambda \end{pmatrix}$.

Let B be the position vector of a general point on

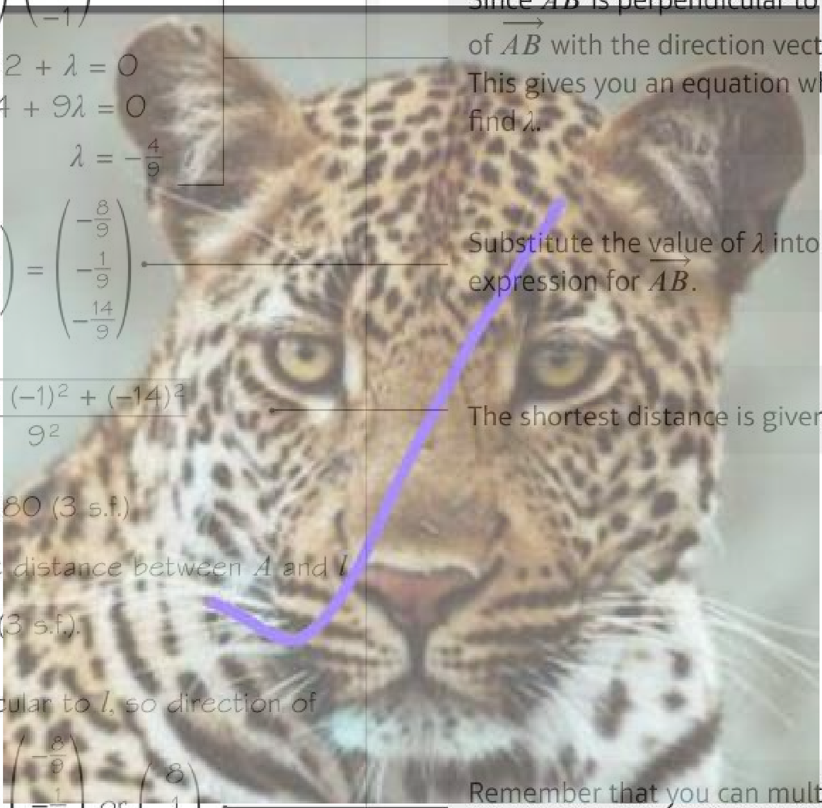
Then $\overrightarrow{AB} = \begin{pmatrix} 1 + 2\lambda \\ 1 - 2\lambda \\ -3 - \lambda \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2\lambda \\ -1 - 2\lambda \\ -2 - \lambda \end{pmatrix}$

Find $\overrightarrow{AB}$ in terms of λ.

$$\begin{pmatrix} 2\lambda \\ -1 - 2\lambda \\ -2 - \lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} = 0$$

$$4\lambda + 2 + 4\lambda + 2 + \lambda = 0$$
$$4 + 9\lambda = 0$$
$$\lambda = -\frac{4}{9}$$

Since $\overrightarrow{AB}$ is perpendicular to l the scalar produc of $\overrightarrow{AB}$ with the direction vector of the line is ze This gives you an equation which you can solve find λ.



$$\overrightarrow{AB} = \begin{pmatrix} 2\lambda \\ -1 - 2\lambda \\ -2 - \lambda \end{pmatrix} = \begin{pmatrix} -\frac{8}{9} \\ -\frac{1}{9} \\ -\frac{14}{9} \end{pmatrix}$$

Substitute the value of λ into your general expression for $\overrightarrow{AB}$.

$$|\overrightarrow{AB}| = \sqrt{\frac{(-8)^2 + (-1)^2 + (-14)^2}{9^2}}$$

$$= \frac{\sqrt{29}}{3} = 1.80 \text{ (3 s.f.)}$$

The shortest distance is given by $|\overrightarrow{AB}|$.

So the shortest distance between A and l is $\frac{\sqrt{29}}{3}$ or 1.80 (3 s.f.).

$\overrightarrow{AB}$ is perpendicular to l, so direction of perpendicular is $\begin{pmatrix} -\frac{8}{9} \\ -\frac{1}{9} \\ -\frac{14}{9} \end{pmatrix}$ or $\begin{pmatrix} 8 \\ 1 \\ 14 \end{pmatrix}$.

Remember that you can multiply the direction vector by a scalar to find a simpler parallel vect

A vector equation of the line through A perpendicular to l is $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 8 \\ 1 \\ 14 \end{pmatrix}$.

A vector equation of the line through the point with position vector **a** with direction **b** is $\mathbf{r} = \mathbf{a} + \mu\mathbf{b}$ where μ is a scalar constant.

So the Cartesian equation of the line is

$$\frac{x-1}{8} = \frac{y-2}{1} = \frac{z+1}{14}$$

ou can use the principles covered above to give meaning to the constant, k, in the scalar produ
rm of the vector equation of a plane.

k is the length of the perpendicular from the origin to a plane Π, where the equation o plane Π is written in the form $\mathbf{r}.\hat{\mathbf{n}} = k$, where $\hat{\mathbf{n}}$ is a unit vector perpendicular to Π.

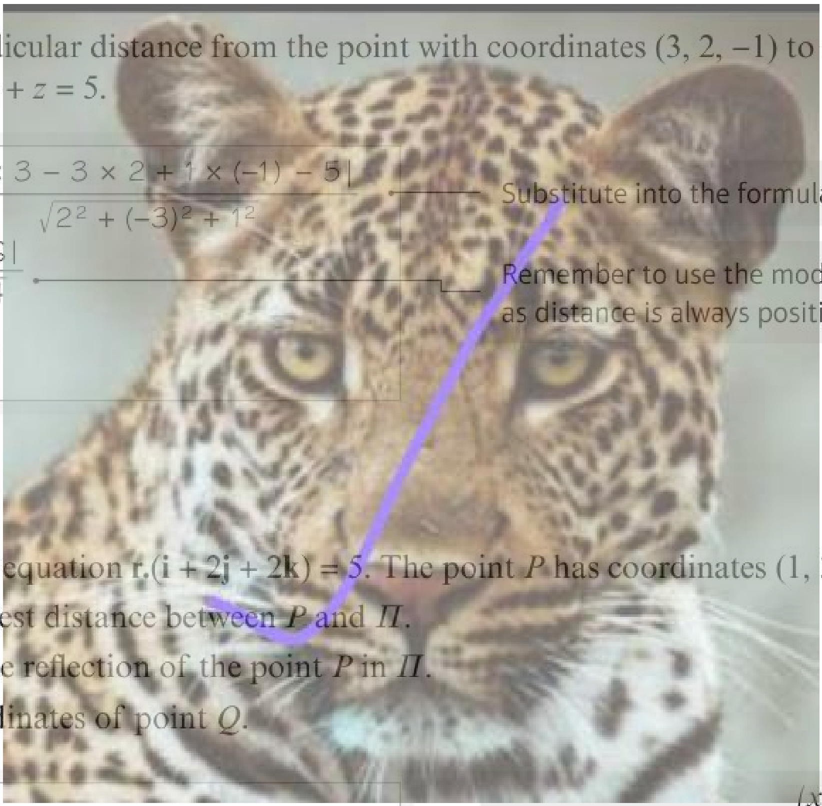
The perpendicular distance from the point with coordinates (α, β, γ) to the plane with equation $ax + by + cz = d$ is

$$\frac{|a\alpha + b\beta + c\gamma - d|}{\sqrt{a^2 + b^2 + c^2}}$$

Note This formula is given in the formulae booklet and you can use it without proof in y exam.

xample 27

ind the perpendicular distance from the point with coordinates (3, 2, −1) to the plane with quation $2x - 3y + z = 5$.



$$\text{Distance} = \frac{|2 \times 3 - 3 \times 2 + 1 \times (-1) - 5|}{\sqrt{2^2 + (-3)^2 + 1^2}}$$

Substitute into the formula.

$$= \frac{|-6|}{\sqrt{14}}$$

Remember to use the modulus of the numer as distance is always positive.

$$= \frac{6}{\sqrt{14}}$$

xample 28

he plane Π has equation $\mathbf{r}.(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) = 5$. The point P has coordinates (1, 3, −2).

Find the shortest distance between P and Π.

he point Q is the reflection of the point P in Π.

Find the coordinates of point Q.

a Cartesian equation of Π is $x + 2y + 2z = 5$.

Use $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\begin{pmatrix} x \\ y \\ z \end{pmatrix} . \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 5$.

$$\text{Distance} = \frac{|1 \times 1 + 2 \times 3 + 2 \times (-2) - 5|}{\sqrt{1^2 + 2^2 + 2^2}}$$

$$= \frac{|-2|}{\sqrt{9}} = \frac{2}{3}$$

Using the formula $\frac{|a\alpha + b\beta + c\gamma - d|}{\sqrt{a^2 + b^2 + c^2}}$ for the distance between a point with coordinates (α, β, γ) and the plane with Cartesian equati $ax + by + cz = d$.

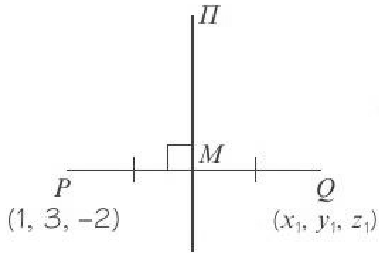
b A perpendicular vector to Π is

$\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

Let Q have coordinates (x_1, y_1, z_1).

Let M be the midpoint of PQ.

A normal vector to the plane $ax + by + cz = a$ $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

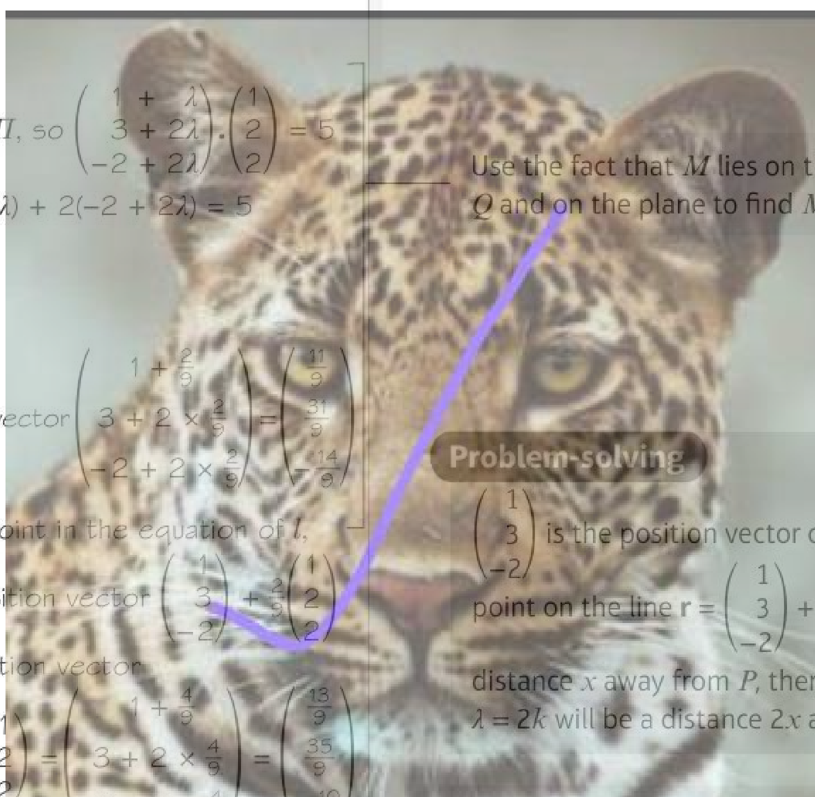


The line joining P to its reflection Q will be perpendicular to the plane, and P and Q will be the same distance from the plane. Draw a diagram showing P, Q, and the midpoint of PQ. Represent the plane Π using a vertical line.

Online Explore reflections in a plane using GeoGebra.

A vector equation of the line through P, M and Q is $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

M lies on this line so has position vector $\begin{pmatrix} 1 + \lambda \\ 3 + 2\lambda \\ -2 + 2\lambda \end{pmatrix}$



M also lies on Π, so $\begin{pmatrix} 1 + \lambda \\ 3 + 2\lambda \\ -2 + 2\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = 5$

$1 + \lambda + 2(3 + 2\lambda) + 2(-2 + 2\lambda) = 5$

$3 + 9\lambda = 5$

$\lambda = \frac{2}{9}$

M has position vector $\begin{pmatrix} 1 + \frac{2}{9} \\ 3 + 2 \times \frac{2}{9} \\ -2 + 2 \times \frac{2}{9} \end{pmatrix} = \begin{pmatrix} \frac{11}{9} \\ \frac{31}{9} \\ -\frac{14}{9} \end{pmatrix}$

Use the fact that M lies on the line joining P ar Q and on the plane to find M.

P is the initial point in the equation of l,

so if M has position vector $\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \frac{2}{9}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

then P has position vector

$\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + 2 \times \frac{2}{9}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 + \frac{4}{9} \\ 3 + 2 \times \frac{4}{9} \\ -2 + 2 \times \frac{4}{9} \end{pmatrix} = \begin{pmatrix} \frac{13}{9} \\ \frac{35}{9} \\ -\frac{10}{9} \end{pmatrix}$

Problem-solving

$\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$ is the position vector of point P, so if the point on the line $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ with $\lambda = k$ is distance x away from P, then the point with $\lambda = 2k$ will be a distance $2x$ away from P.

Point Q has coordinates $\left(\frac{13}{9}, \frac{35}{9}, -\frac{10}{9}\right)$.

You could also use the fact that the midpoint of the line segment joining (x_1, y_1, z_1) to (x_2, y_2, z_2) $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$

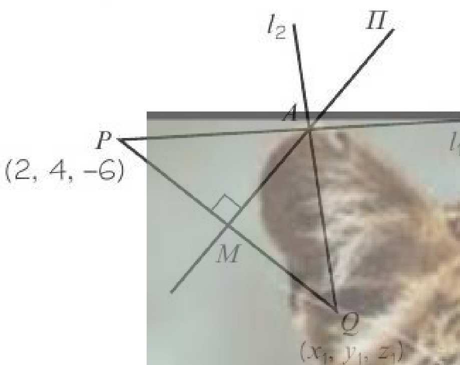
xample 29

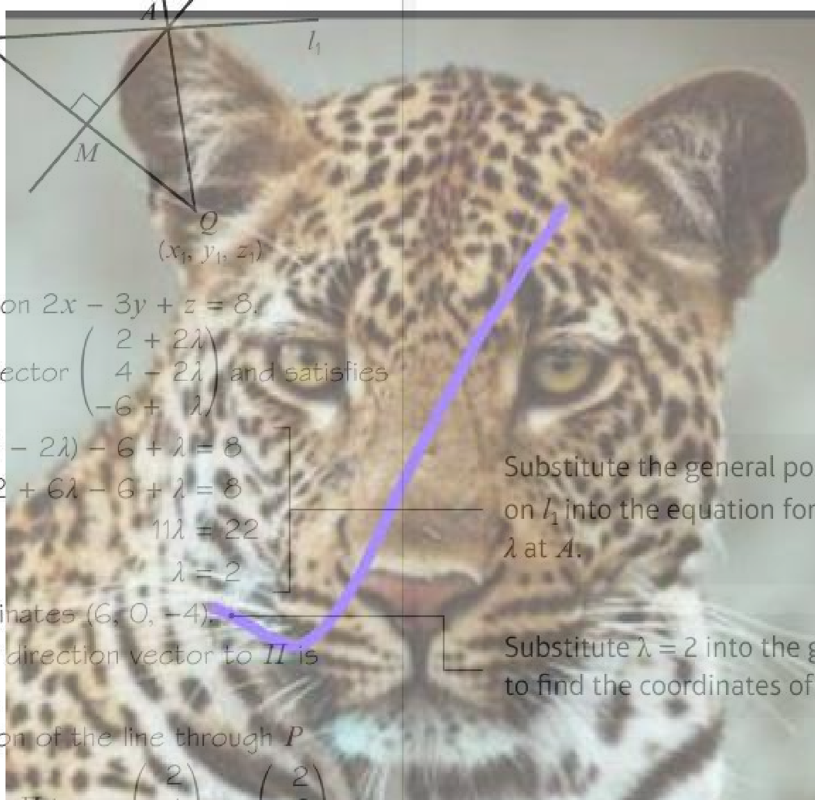
he line l_1 has equation $\dfrac{x-2}{2} = \dfrac{y-4}{-2} = \dfrac{z+6}{1}$. The plane Π has equation $2x - 3y + z = 8$.
he line l_2 is the reflection of line l_1 in the plane Π. Find a vector equation of the line l_2.

A vector equation of l_1 is $\mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

So $P(2, 4, -6)$ is a point on line l_1.

Let A be the point of intersection of l_1 and Π.





Problem-solving

You need to find **two points** on the reflected l_2. One is the point of intersection of l_1 and Π. To find another point on l_2, choose any point and reflect it in the plane.

A lies on l_1 and on $2x - 3y + z = 8$.

A has position vector $\begin{pmatrix} 2+2\lambda \\ 4-2\lambda \\ -6+\lambda \end{pmatrix}$ and satisfies

$2(2 + 2\lambda) - 3(4 - 2\lambda) - 6 + \lambda = 8$
$4 + 4\lambda - 12 + 6\lambda - 6 + \lambda = 8$
$11\lambda = 22$
$\lambda = 2$

Substitute the general position vector of a po on l_1 into the equation for Π to find the valu λ at A.

So A has coordinates $(6, 0, -4)$.

Substitute $\lambda = 2$ into the general position vec to find the coordinates of A.

A perpendicular direction vector to Π is $\mathbf{n} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$.

A vector equation of the line through P perpendicular to Π is $\mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix} + \mu\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$.

Let $Q\ (x_1, y_1, z_1)$ be the point of intersection of this line and l_2.

Q is the reflection of the point (6, 0, −4) in th plane.

Let M be midpoint of PQ.

M lies on line and on $2x - 3y + z = 8$ so satisfies

$2(2 + 2\mu) - 3(4 - 3\mu) - 6 + \mu = 8$

The general point on the line PQ is $\begin{pmatrix} 2+2\mu \\ 4-3\mu \\ -6+\mu \end{pmatrix}$

$4 + 4\mu - 12 + 9\mu - 6 + \mu = 8$
$14\mu = 22$
$\mu = \frac{11}{7}$

So M has position vector $\begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix} + \frac{11}{7}\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

d Q has position vector

$$\begin{pmatrix}2\\4\\6\end{pmatrix} + 2 \times \frac{11}{7}\begin{pmatrix}2\\-3\\1\end{pmatrix} = \begin{pmatrix}\frac{58}{7}\\-\frac{38}{7}\\-\frac{20}{7}\end{pmatrix}$$

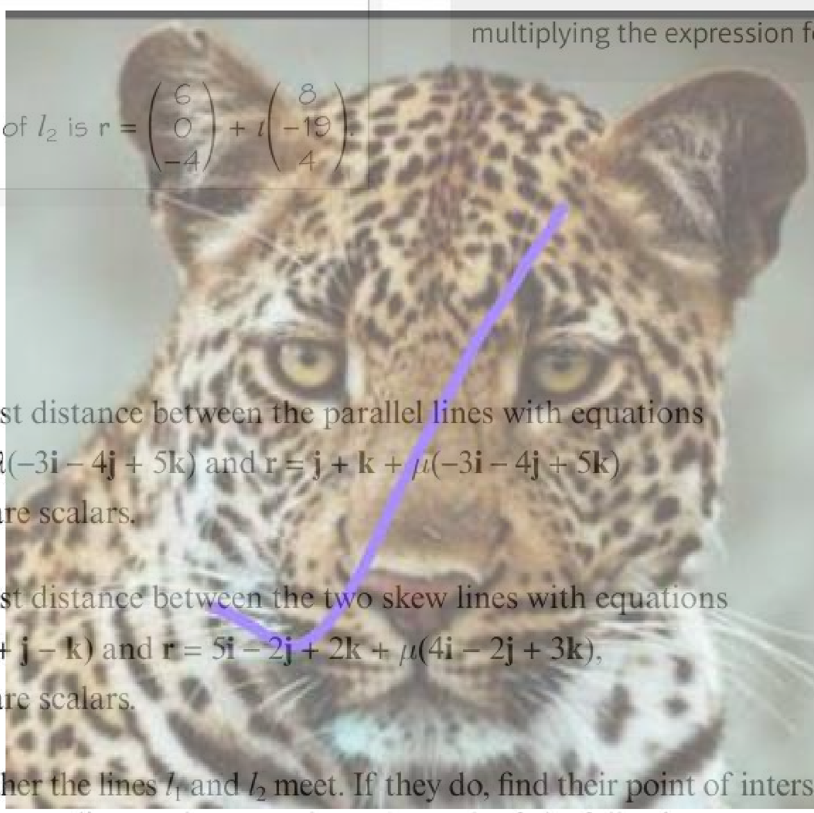
Q is twice as far along the line from P as M.

Q has coordinates $\left(\frac{58}{7}, -\frac{38}{7}, -\frac{20}{7}\right)$.
is the line through Q and A, so has
ection:

$$\overrightarrow{Q} = \begin{pmatrix}\frac{16}{7}\\-\frac{38}{7}\\\frac{8}{7}\end{pmatrix}$$

The direction of l_2 can be simplified to $\begin{pmatrix}8\\-19\\4\end{pmatrix}$ b
multiplying the expression for $\overrightarrow{AQ}$ by $\frac{7}{2}$

vector equation of l_2 is $\mathbf{r} = \begin{pmatrix}6\\0\\-4\end{pmatrix} + t\begin{pmatrix}8\\-19\\4\end{pmatrix}$.



rcise 9F

Find the shortest distance between the parallel lines with equations
$\mathbf{r} = 2\mathbf{i} - \mathbf{j} + \mathbf{k} + \lambda(-3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k})$ and $\mathbf{r} = \mathbf{j} + \mathbf{k} + \mu(-3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k})$
where λ and μ are scalars.

Find the shortest distance between the two skew lines with equations
$\mathbf{r} = \mathbf{j} + \mathbf{k} + \lambda(-\mathbf{i} + \mathbf{j} - \mathbf{k})$ and $\mathbf{r} = 5\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} + \mu(4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$,
where λ and μ are scalars.

Determine whether the lines l_1 and l_2 meet. If they do, find their point of intersection. If they do not, find the shortest distance between them. (In each of the following cases λ and μ are scalars.)

a l_1 has equation $\mathbf{r} = 7\mathbf{i} + 3\mathbf{j} + \mathbf{k} + \lambda(6\mathbf{i} + 2\mathbf{j} - 4\mathbf{k})$ and l_2 has equation $\mathbf{r} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k} - 2\mu\mathbf{j}$

b l_1 has equation $\mathbf{r} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k} + \lambda(2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$ and l_2 has equation $\mathbf{r} = \mathbf{i} - \mathbf{j} + 3\mathbf{k} + \mu(\mathbf{i} - \mathbf{j} +$

c l_1 has equation $\mathbf{r} = \mathbf{i} + \mathbf{j} + 5\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$ and l_2 has equation $\mathbf{r} = -\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \mu(\mathbf{i} + \mathbf{j} + \mathbf{k}$

Find the shortest distance between the point with coordinates $(4, 1, -1)$ and the line with equation $\mathbf{r} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \mu(2\mathbf{i} - \mathbf{j} - \mathbf{k})$, where μ is a scalar.

Find the shortest distance between the parallel planes.

a $\mathbf{r}.(6\mathbf{i} + 6\mathbf{j} - 7\mathbf{k}) = 55$ and $\mathbf{r}.(6\mathbf{i} + 6\mathbf{j} - 7\mathbf{k}) = 22$

b $\mathbf{r} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k} + \lambda(4\mathbf{i} + \mathbf{k}) + \mu(8\mathbf{i} + 3\mathbf{j} + 3\mathbf{k})$ and $\mathbf{r} = 14\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} + \lambda(3\mathbf{j} + \mathbf{k}) + \mu(8\mathbf{i} - 9\mathbf{j} - \mathbf{k})$

6 The plane Π has equation $\mathbf{r}.(10\mathbf{i} + 10\mathbf{j} + 23\mathbf{k}) = 81$.

a Find the perpendicular distance from the origin to plane Π.

b Find the perpendicular distance from the point $(-1, -1, 4)$ to the plane Π.

c Find the perpendicular distance from the point $(2, 1, 3)$ to the plane Π.

d Find the perpendicular distance from the point $(6, 12, -9)$ to the plane Π.

7 The line l has vector equation $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

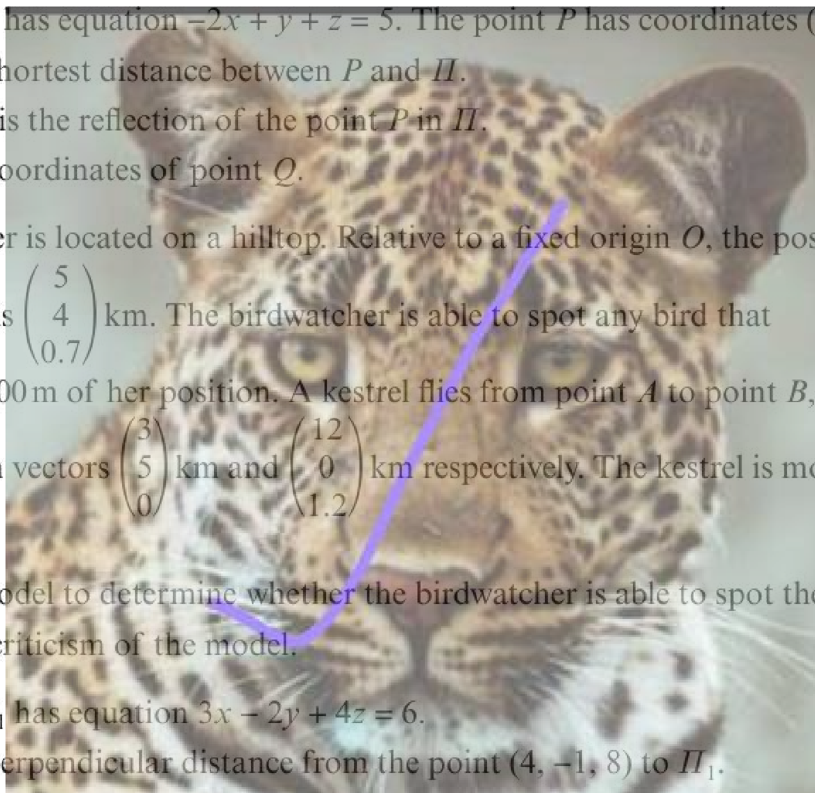
The point P has coordinates $(3, 0, 2)$.

Find the coordinates of the reflection of the point P in the line l. **(5 r**

Problem-solving

Let M be the midpoint of the line segment joining P to its reflection in l. This segment be perpendicular to l and pass through it.

8 The plane Π has equation $-2x + y + z = 5$. The point P has coordinates $(1, 0, 3)$.



a Find the shortest distance between P and Π. **(3 r**

The point Q is the reflection of the point P in Π.

b Find the coordinates of point Q. **(5 r**

9 A birdwatcher is located on a hilltop. Relative to a fixed origin O, the position vector of t

birdwatcher is $\begin{pmatrix} 5 \\ 4 \\ 0.7 \end{pmatrix}$ km. The birdwatcher is able to spot any bird that

flies within 500 m of her position. A kestrel flies from point A to point B, where points A a

have position vectors $\begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}$ km and $\begin{pmatrix} 12 \\ 0 \\ 1.2 \end{pmatrix}$ km respectively. The kestrel is modelled as flying i

straight line.

a Use the model to determine whether the birdwatcher is able to spot the kestrel. **(7 r**

b Give one criticism of the model. **(1**

10 The plane Π_1 has equation $3x - 2y + 4z = 6$.

a Find the perpendicular distance from the point $(4, -1, 8)$ to Π_1. **(3 r**

The plane Π_2 has vector equation $\mathbf{r} = \lambda(2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) + \mu(-3\mathbf{i} + 3\mathbf{k})$ where λ and μ are scala parameters.

b Show that the vector $2\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$ is perpendicular to Π_2. **(2 r**

c Find the acute angle between Π_1 and Π_2. **(3 r**

11 The line l_1 has equation $\dfrac{x+2}{2} = \dfrac{y-2}{-1} = \dfrac{z+1}{-2}$, and the point A has coordinates $(3, -1, 2)$.

a Find the shortest distance between A and l_1. **(5 r**

b Find a Cartesian equation of the line that is perpendicular to l_1 and passes through A. **(3 r**

12 The line l_1 has vector equation $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -3 \\ 4 \end{pmatrix}$. The plane Π has equation $\mathbf{r}.\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = 4$.

The line l_2 is the reflection of line l_1 in the plane Π.

Find a vector equation of the line l_2. **(7 r**

ed exercise 9

The line l passes through the points A and B with position vectors $\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ respectively, relative to a fixed origin O.

a Find a vector equation of the line l. **(4 ma**

b Find the position vector of the point C which lies on the line segment AB such that $AC = 2CB$. **(3 ma**

Find a Cartesian equation of the straight line that passes through the points with coordinates $(7, -1, 2)$ and $(-1, 3, 8)$. **(4 ma**

Find a vector equation of the straight line which passes through the point A with position vector $2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$, and is parallel to the vector $2\mathbf{j} + 3\mathbf{k}$. **(3 ma**

A straight line l has vector equation $\mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$.

a Write down a Cartesian equation for l. **(2 ma**

b Given that the point $(0, a, b)$ lies on l, find the value of a and the value of b. **(3 ma**

A straight line l has vector equation $\mathbf{r} = (\mathbf{i} + 2\mathbf{j} - \mathbf{k}) + \lambda(3\mathbf{i} + \mathbf{j} - 2\mathbf{k})$.
Show that another vector equation of l is $\mathbf{r} = (7\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}) + \lambda(9\mathbf{i} + 3\mathbf{j} - 6\mathbf{k})$.

A straight line l has Cartesian equation $\dfrac{x+2}{1} = \dfrac{y-2}{3} = \dfrac{z+3}{4}$.

a Find a vector form of the equation of l. **b** Verify that the point $(0, 8, 5)$ lies on l.

A plane passes through the points $A(2, -1, 2)$, $B(1, 3, -1)$ and $C(4, 2, 5)$.

a Find a vector form of the equation of the plane.

b Find a Cartesian form of the equation of the plane.

A Cartesian form of the equation of a plane is $3x + 2y - 4z = 18$. Find a vector form of the equation of the plane.

With respect to an origin O, the position vectors of the points L, M and N are $\begin{pmatrix} 4 \\ 7 \\ 7 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ and (respectively.

a Find the vectors $\overrightarrow{ML}$ and $\overrightarrow{MN}$. **(3 ma**

b Prove that $\cos \angle LMN = \frac{9}{10}$ **(3 ma**

Referred to a fixed origin O, the points A, B and C have position vectors $9\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $6\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$ and $3\mathbf{i} + p\mathbf{j} + q\mathbf{k}$ respectively, where p and q are constants.

a Find, in vector form, an equation of the line l which passes through A and B. **(2 ma**

Given that C lies on l,

b find the value of p and the value of q **(2 ma**

c calculate, in degrees, the acute angle between OC and AB. **(3 ma**

The point D lies on AB and is such that OD is perpendicular to AB.

d Find the position vector of D. **(5 ma**

1 Referred to a fixed origin O, the points A and B have position vectors $\begin{pmatrix}1\\2\\-3\end{pmatrix}$ and $\begin{pmatrix}5\\0\\-3\end{pmatrix}$ respe

a Find, in vector form, an equation of the line l_1 which passes through A and B. **(3 m**

The line l_2 has equation $\mathbf{r} = \begin{pmatrix}4\\-4\\3\end{pmatrix} + \mu\begin{pmatrix}1\\-2\\2\end{pmatrix}$, where μ is a scalar parameter.

b Show that A lies on l_2. **(2 m**

c Find, in degrees, the acute angle between the lines l_1 and l_2. **(4 m**

The point C with position vector $\begin{pmatrix}0\\4\\-5\end{pmatrix}$ lies on l_2.

d Find the shortest distance from C to the line l_1. **(4 m**

2 Two submarines are travelling in straight lines through the ocean. Relative to a fixed origi vector equations of the two lines, l_1 and l_2, along which they travel are

l_1: $\mathbf{r} = 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$

l_2: $\mathbf{r} = 9\mathbf{i} + \mathbf{j} - 2\mathbf{k} + \mu(4\mathbf{i} + \mathbf{j} - \mathbf{k})$

where λ and μ are scalars.

a Show that the submarines are moving in perpendicular directions. **(2 m**

b Given that l_1 and l_2 intersect at the point A, find the position vector of A. **(4 m**

The point B has position vector $10\mathbf{j} - 11\mathbf{k}$.

c Show that only one of the submarines passes through the point B. **(3 m**

d Given that 1 unit on each coordinate axis represents 100 m, find, in km, the distance AB. **(2 m**

3 Find the shortest distance between the lines with vector equations

$$\mathbf{r} = 3\mathbf{i} + s\mathbf{j} - \mathbf{k} \quad \text{and} \quad \mathbf{r} = 9\mathbf{i} - 2\mathbf{j} - \mathbf{k} + t(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$

where s and t are scalars. **(4 m**

4 Obtain the shortest distance between the lines with equations

$$\mathbf{r} = (3s - 3)\mathbf{i} - s\mathbf{j} + (s + 1)\mathbf{k} \quad \text{and} \quad \mathbf{r} = (3 + t)\mathbf{i} + (2t - 2)\mathbf{j} + \mathbf{k}$$

where s and t are parameters. **(4 m**

5 Find, in the form $\mathbf{r.n} = p$, an equation of the plane which contains the line l and the point position vector $\mathbf{a}$ where:

a l has equation $\mathbf{r} = \mathbf{i} + \mathbf{j} - 2\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{k})$ and $\mathbf{a} = 4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

b l has equation $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} - 3\mathbf{k})$ and $\mathbf{a} = 3\mathbf{i} + 5\mathbf{j} + \mathbf{k}$

c l has equation $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$ and $\mathbf{a} = 7\mathbf{i} + 8\mathbf{j} + 6\mathbf{k}$

6 Find a Cartesian equation of the plane which passes through the point (1, 1, 1) and conta the line with equation $\dfrac{x-2}{3} = \dfrac{y+4}{1} = \dfrac{z-1}{2}$

A plane passes through the three points A, B and C, whose position vectors, referred to an origin O, are $\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$ and $3\mathbf{i} + \mathbf{j} + 4\mathbf{k}$, $2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$ respectively.

a Find, in the form $l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$, a unit vector normal to this plane. **(4 ma**

b Find also a Cartesian equation of the plane. **(2 ma**

c Find the perpendicular distance from the origin to this plane. **(4 ma**

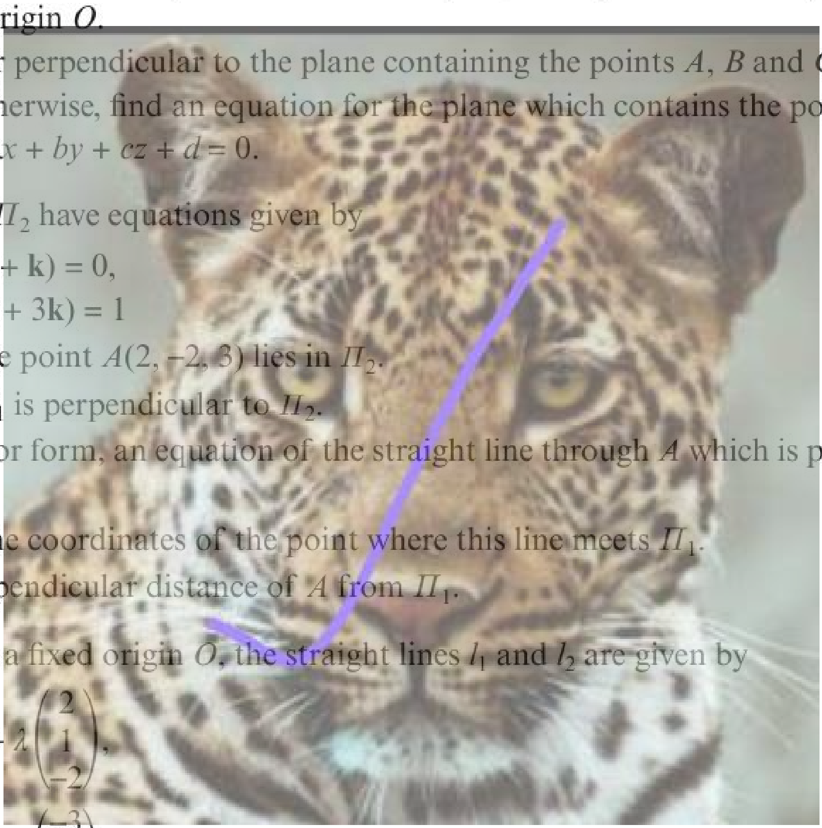
a Show that the vector $\mathbf{i} + \mathbf{k}$ is perpendicular to the plane with vector equation

$\mathbf{r} = \mathbf{i} + s\mathbf{j} + t(\mathbf{i} - \mathbf{k})$ **(3 ma**

b Find the perpendicular distance from the origin to this plane. **(4 ma**

c Hence or otherwise obtain a Cartesian equation of the plane. **(2 ma**

The points A, B and C have position vectors $\mathbf{i} + \mathbf{j} + \mathbf{k}$, $5\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$ respectivel referred to an origin O.



a Find a vector perpendicular to the plane containing the points A, B and C. **(4 ma**

b Hence, or otherwise, find an equation for the plane which contains the points A, B and C, in the form $ax + by + cz + d = 0$. **(2 ma**

Planes Π_1 and Π_2 have equations given by

Π_1: $\mathbf{r}.(2\mathbf{i} - \mathbf{j} + \mathbf{k}) = 0$,

Π_2: $\mathbf{r}.(\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}) = 1$

a Show that the point $A(2, -2, 3)$ lies in Π_2. **(2 ma**

b Show that Π_1 is perpendicular to Π_2. **(4 ma**

c Find, in vector form, an equation of the straight line through A which is perpendicular to **(2 ma**

d Determine the coordinates of the point where this line meets Π_1. **(4 ma**

e Find the perpendicular distance of A from Π_1. **(4 ma**

With respect to a fixed origin O, the straight lines l_1 and l_2 are given by

$$l_1: \mathbf{r} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\-2\end{pmatrix},$$

$$l_2: \mathbf{r} = \begin{pmatrix}1\\2\\2\end{pmatrix} + \mu\begin{pmatrix}-3\\0\\1\end{pmatrix}$$

where λ and μ are scalar parameters.

a Show that the lines intersect. **(3 ma**

b Find the position vector of their point of intersection. **(1 m**

c Find the cosine of the acute angle between the lines. **(4 ma**

The line l_1 has vector equation $\mathbf{r} = 6\mathbf{i} + 8\mathbf{j} + 5\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k})$, where λ is a scalar parameter. The point A has coordinates $(3, a, 2)$, where a is a constant. The point B has coordinates $(8, 6, b)$, where b is a constant. Points A and B lie on the line l_1.

a Find the values of a and b. **(3 ma**

Given that the point O is the origin, and that the point P lies on l_1 such that OP is perpendicular to l_1,

b find the coordinates of P. **(5**

c Hence find the distance OP, giving your answer in surd form. **(2**

3 Relative to a fixed origin O, the point A has position vector $6\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, and the point B h position vector $5\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$. The line l passes through the points A and B.

a Find the vector $\overrightarrow{AB}$. **(2**

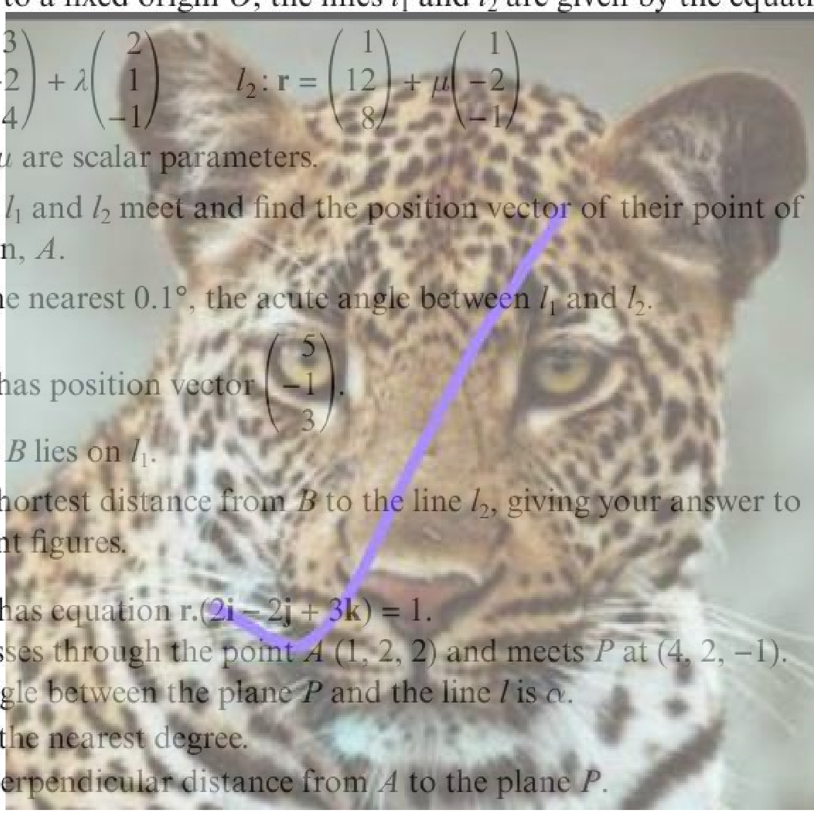
b Find a vector equation for the line l. **(2**

The point C has position vector $4\mathbf{i} + 10\mathbf{j} + 2\mathbf{k}$.

The point P lies on l. Given that the vector $\overrightarrow{CP}$ is perpendicular to l,

c find the position vector of the point P. **(6**

4 With respect to a fixed origin O, the lines l_1 and l_2 are given by the equations



$$l_1: \mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \qquad l_2: \mathbf{r} = \begin{pmatrix} 1 \\ 12 \\ 8 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$$

where λ and μ are scalar parameters.

a Show that l_1 and l_2 meet and find the position vector of their point of intersection, A. **(6**

b Find, to the nearest 0.1°, the acute angle between l_1 and l_2. **(3**

The point B has position vector $\begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$.

c Show that B lies on l_1. **(1**

d Find the shortest distance from B to the line l_2, giving your answer to 3 significant figures. **(4**

5 The plane P has equation $\mathbf{r}.(2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) = 1$.
The line l passes through the point A (1, 2, 2) and meets P at (4, 2, −1).
The acute angle between the plane P and the line l is α.

a Find α to the nearest degree. **(4**

b Find the perpendicular distance from A to the plane P. **(4**

6 Two aeroplanes are modelled as travelling in straight lines. Aeroplane A travels from a poi with position vector $\begin{pmatrix} 120 \\ -80 \\ 13 \end{pmatrix}$ km to a point with position vector $\begin{pmatrix} 200 \\ 20 \\ 5 \end{pmatrix}$ km, relative to a fixe origin O. Aeroplane B starts at a point with position vector $\begin{pmatrix} -20 \\ 35 \\ 5 \end{pmatrix}$ km relative to O, and t the direction of $\begin{pmatrix} 10 \\ -2 \\ 0.1 \end{pmatrix}$.

a Show that the flight paths of the two aeroplanes will intersect, and determine the positi vector of the point of intersection. **(7**

An air traffic controller states that this means that the planes will collide.

b Explain why this conclusion is not necessarily correct. **(2**

llenge

a Show that the equations $4x - 2y + 6z = 10$ and $-2x + y - 3z = -5$ represent the same plane.

b Hence explain why the matrix $\begin{pmatrix} 4 & -2 & 6 \\ -2 & 1 & -3 \\ a & b & c \end{pmatrix}$ is singular for all possible values of a, b and c.

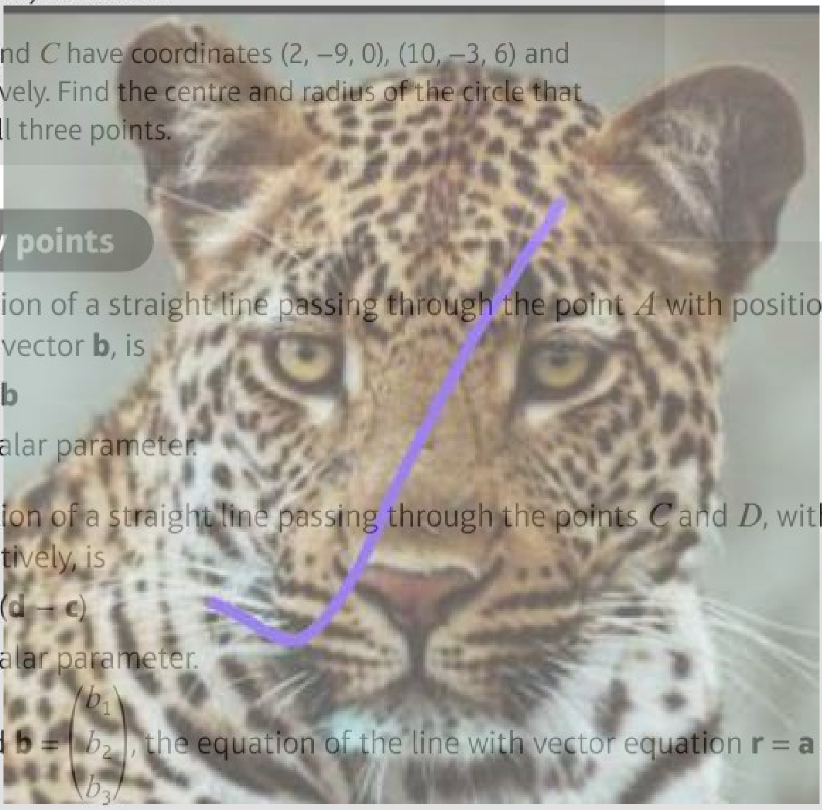
c Find values of a, b and c such that the matrix equation

$$\begin{pmatrix} 4 & -2 & 6 \\ -2 & 1 & -3 \\ a & b & c \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ -5 \\ 15 \end{pmatrix}$$ has

i no solutions

ii infinitely many solutions.

The points A, B and C have coordinates (2, −9, 0), (10, −3, 6) and (8, −1, 2) respectively. Find the centre and radius of the circle that passes through all three points.



mmary of key points

A vector equation of a straight line passing through the point A with position vector **a**, and parallel to the vector **b**, is

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$$

where λ is a scalar parameter.

A vector equation of a straight line passing through the points C and D, with position vector **c** and **d** respectively, is

$$\mathbf{r} = \mathbf{c} + \lambda(\mathbf{d} - \mathbf{c})$$

where λ is a scalar parameter.

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, the equation of the line with vector equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ can be give in Cartesian form as:

$$\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}$$

Each of these three expressions is equal to λ.

The vector equation of a plane is

$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, where:

- **r** is the position vector of a general point in the plane
- **a** is the position vector of a point in the plane
- **b** and **c** are non-parallel, non-zero vectors in the plane
- λ and μ are scalars

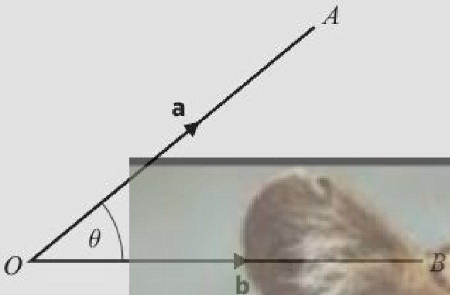
5 A Cartesian equation of a plane in three dimensions can be written in the form $ax + by + c$

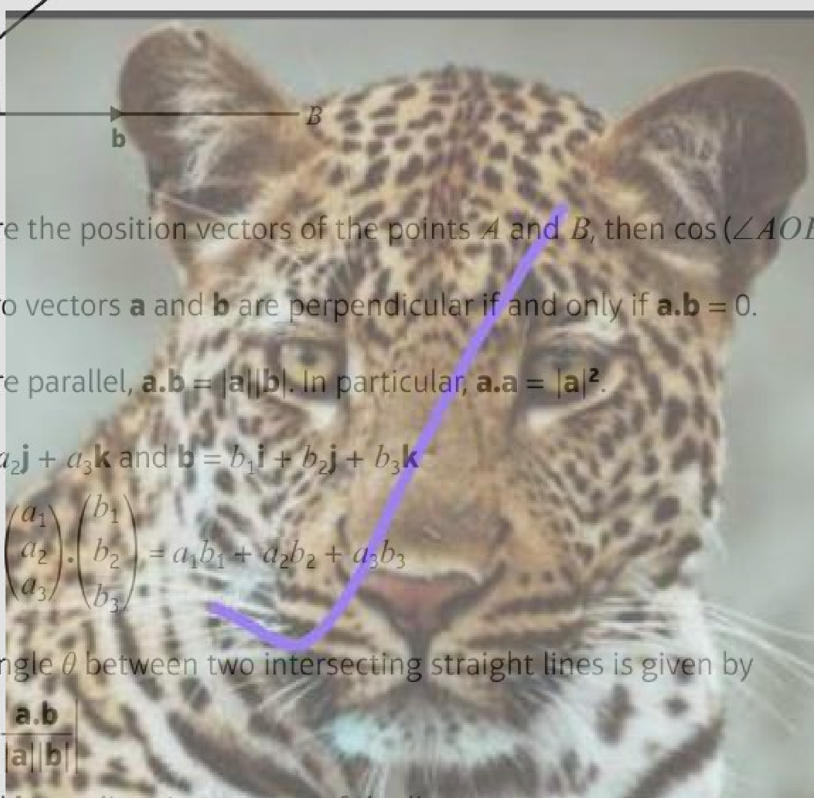
where a, b, c and d are constants, and $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is the normal vector to the plane.

6 The **scalar product** of two vectors **a** and **b** is written as **a.b** (say '**a** dot **b**'), and defined as

$$\mathbf{a.b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

where θ is the angle between **a** and **b**.





7 If **a** and **b** are the position vectors of the points A and B, then $\cos(\angle AOB) = \dfrac{\mathbf{a.b}}{|\mathbf{a}||\mathbf{b}|}$

8 The non-zero vectors **a** and **b** are perpendicular if and only if $\mathbf{a.b} = 0$.

9 If **a** and **b** are parallel, $\mathbf{a.b} = |\mathbf{a}||\mathbf{b}|$. In particular, $\mathbf{a.a} = |\mathbf{a}|^2$.

10 If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

$$\mathbf{a.b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} . \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$$

11 The acute angle θ between two intersecting straight lines is given by

$$\cos\theta = \left|\frac{\mathbf{a.b}}{|\mathbf{a}||\mathbf{b}|}\right|$$

where **a** and **b** are direction vectors of the lines.

12 The scalar product form of the equation of a plane is $\mathbf{r.n} = k$ where $k = \mathbf{a.n}$ for any point in plane with position vector **a**.

13 The acute angle θ between the line with equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and the plane with equation $\mathbf{r.n} = k$ is given by the formula

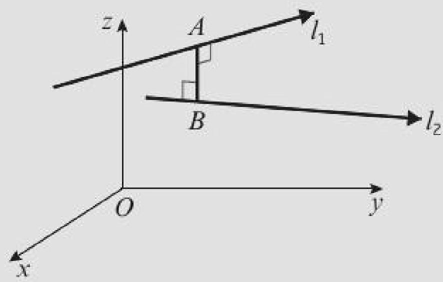
$$\sin\theta = \left|\frac{\mathbf{b.n}}{|\mathbf{b}||\mathbf{n}|}\right|$$

14 The acute angle θ between the plane with equation $\mathbf{r.n}_1 = k_1$ and the plane with equation $\mathbf{r.n}_2 = k_2$ is given by the formula

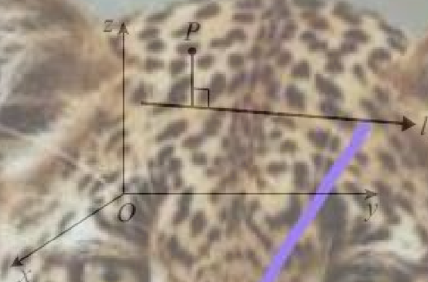
$$\cos\theta = \left|\frac{\mathbf{n}_1.\mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right|$$

Two lines are **skew** if they are not parallel and they do not intersect.

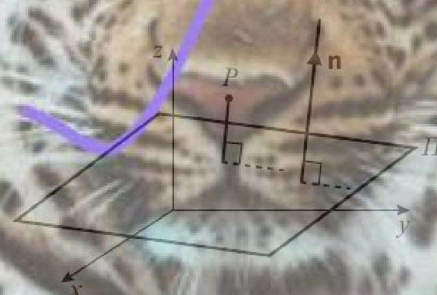
For any two non-intersecting lines l_1 and l_2 there is a unique line segment AB such that A lies on l_1, B lies on l_2 and AB is perpendicular to both lines.



The perpendicular from a point P to a line l is a line drawn from P at right angles to l.



The perpendicular from a point P to a plane Π is a line drawn from P parallel to the normal vector **n**.



k is the length of the perpendicular from the origin to a plane Π, where the equation of plane Π is written in the form $\mathbf{r}.\hat{\mathbf{n}} = k$, where $\hat{\mathbf{n}}$ is a unit vector perpendicular to Π.

The perpendicular distance from the point with coordinates (α, β, γ) and the plane with equation $ax + by + cz = d$ is

$$\frac{|a\alpha + b\beta + c\gamma - d|}{\sqrt{a^2 + b^2 + c^2}}$$

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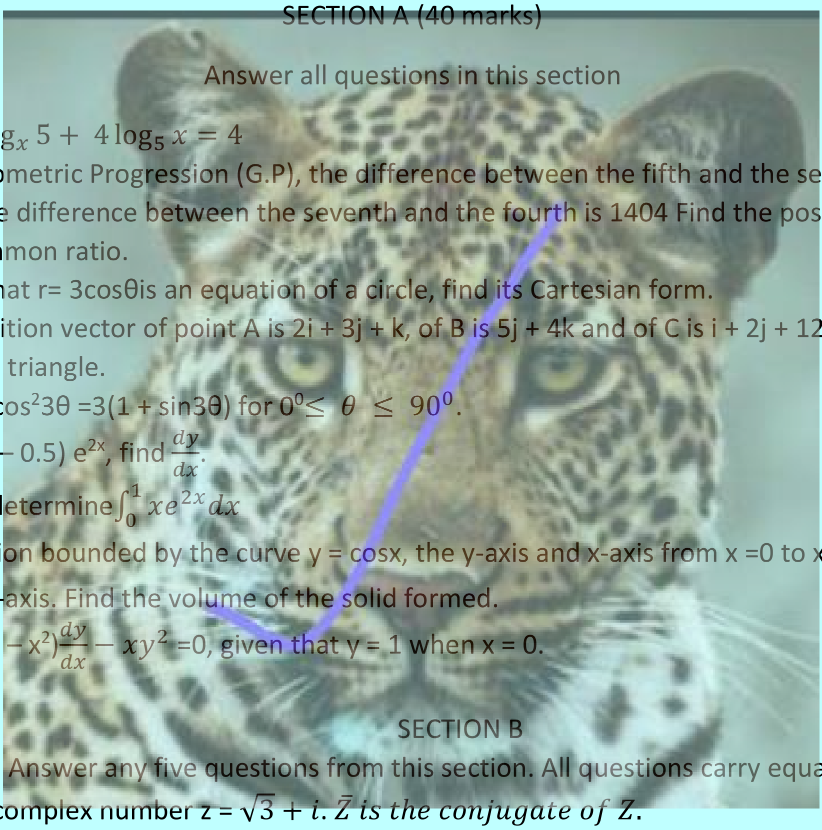


SECTION A (40 marks)

Answer all questions in this section

1. Solve $\log_x 5 + 4\log_5 x = 4$
2. In a Geometric Progression (G.P), the difference between the fifth and the second term is 156. The difference between the seventh and the fourth is 1404 Find the possible values of the common ratio.
3. Given that r= 3cosθis an equation of a circle, find its Cartesian form.
4. The position vector of point A is 2i + 3j + k, of B is 5j + 4k and of C is i + 2j + 12k. Show that ABC is a triangle.
5. Solve $5\cos^2 3\theta = 3(1 + \sin 3\theta)$ for $0^0 \leq \theta \leq 90^0$.
6. If $y = (x - 0.5)e^{2x}$, find $\frac{dy}{dx}$.

   Hence determine $\int_0^1 xe^{2x}dx$
7. The region bounded by the curve y = cosx, the y-axis and x-axis from x =0 to x = $\frac{\pi}{2}$ is rotated about x-axis. Find the volume of the solid formed.
8. Solve $(1 - x^2)\frac{dy}{dx} - xy^2 = 0$, given that y = 1 when x = 0.

SECTION B

Answer any five questions from this section. All questions carry equal marks

9. (a) the complex number $z = \sqrt{3} + i$. $\bar{Z}$ *is the conjugate of* Z.

   (i) Express Z in the modulus argument form

   (ii) On the same Argand diagram plot $\bar{Z}$ and $2\bar{Z} + 3i$

   (b) What are the greatest and least values of $|Z|$ *if* $|Z - 4| \leq 3$?

10. Given the equation $x^3 + x - 10 = 0$.

    (a) Show that x = 2 is a root of the equation

    (b) Deduce the values of α + β and αβ where α and β are roots of the equation. Hence form a quadratic equation whose roots are α^2 and β^2.

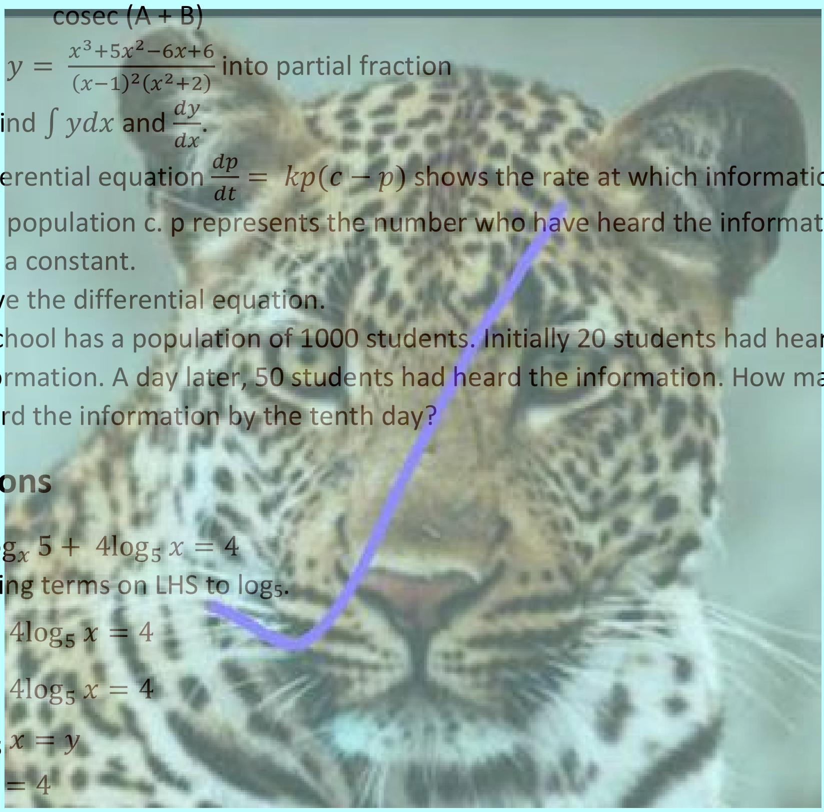
11. (a) Find the point of intersection of the lines $\frac{x-3}{4} = \frac{y-7}{4} = \frac{z+3}{-5}$ and $\frac{x-8}{7} = \frac{y-4}{1} = \frac{z-5}{3}$

    (c) The equations of a line and a plane are $\frac{x-2}{1} = \frac{y-2}{2} = \frac{z+3}{2}$ and 2x + y + 4z = 9resectively. P is a point on the line where x = 3, N is the foot of the perpendicular from P to the plane. Find the coordinates of N.

12. (a) Find the equation of the tangent to the hyperbola whose points are of the parametric form $(2t, \frac{2}{t})$.
(b)(i) Find the equations of the tangents in (a), which are parallel to y + 4x = 0
(ii) Determine the distance between the tangents in (i)

13. A curve has the equation $y = \frac{2}{1+x^2}$.
(a) Determine the nature of the turning point on the curve.
(b) Find the equation of the asymptote. Hence sketch the curve.

14. (a) Prove that $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$
Hence show that $\frac{1-\tan 15^0}{1+\tan 15^0} = \frac{1}{\sqrt{3}}$
(b) Given that $\cos A = \frac{3}{5}$ and $\cos B = \frac{12}{13}$ where A and B are acute, find the values of
(i) tan (A + B)
(ii) cosec (A + B)



15. Resolve $y = \frac{x^3+5x^2-6x+6}{(x-1)^2(x^2+2)}$ into partial fraction
Hence find $\int y dx$ and $\frac{dy}{dx}$.

16. The differential equation $\frac{dp}{dt} = kp(c - p)$ shows the rate at which information flows in a student population c. p represents the number who have heard the information in t days and k is a constant.
(a) Solve the differential equation.
(b) A school has a population of 1000 students. Initially 20 students had heard the information. A day later, 50 students had heard the information. How many students heard the information by the tenth day?

Solutions

1. Solve $\log_x 5 + 4\log_5 x = 4$
Expressing terms on LHS to $\log_5$.
$\frac{\log_5 5}{\log_5 x} + 4\log_5 x = 4$
$\frac{1}{\log_5 x} + 4\log_5 x = 4$
Let $\log_5 x = y$
$\frac{1}{y} + 4y = 4$
$4y^2 - 4y + 1 = 0$
$(2y - 1)(2y - 1) = 0$
$2y = 1$
$y = \frac{1}{2}$
⇨ $\log_5 x = \frac{1}{2}$
$x = 5^{\frac{1}{2}} = \sqrt{5}$

2. In a Geometric Progression (G.P), the difference between the fifth and the second term is 156. The difference between the seventh and the fourth is 1404 Find the possible values of the common ratio.
$U_5 - U_2 = 156$
$ar^4 - ar = 156$
$ar(r^3 - 1) = 156$ ……………. (i)
$U_7 - U_4 = 156$

$Ar^6 - ar^3 = 1404$

$ar^3(r^3 - 1) = 156$ ……………. (ii)

Eqn. (ii) ÷ eqn. (i)

$\frac{ar^3(r^3-1)}{ar(r^3-1)} = \frac{1404}{156}$

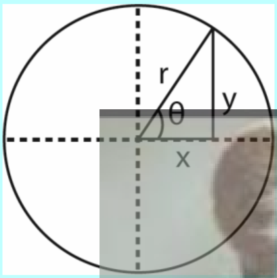
r2 = 9

$r = \pm 3$

∴ r = 3 and r = -3

3. Given that r= 3cosθ is an equation of a circle, find its Cartesian form.

Method I: from the polar coordinates



$x^2 + y^2 = r^2$

x = rcosθ

y = rsinθ

$\cos\theta = \frac{x}{r}$

but r = 3cosθ

$r^2 = 3x$

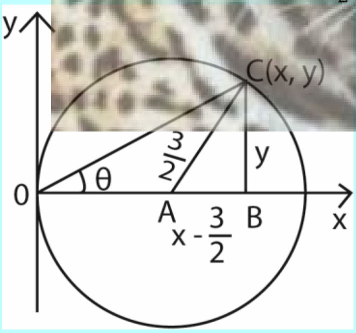
$x^2 + y^2 = 3x$

$\therefore x^2 + y^2 - 3x = 0$

Method II: the Cartesian equation of a circle in polar form of radius a and centre (a, 0) is given by r = 2acosθ

⇨ 2a = 3; $a = \frac{3}{2}$

So the radius of the circle is $\frac{3}{2}$

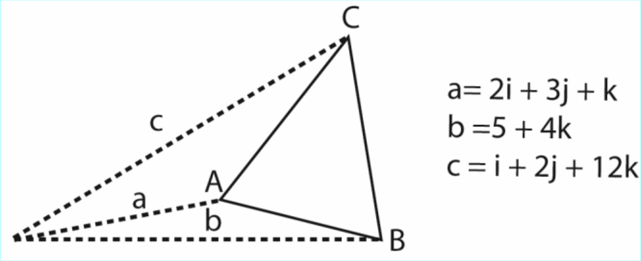


$AC^2 = AD^2 + DC^2$

$\left(\frac{3}{2}\right)^2 = \left(x - \frac{3}{2}\right)^2 + y^2$

This is an equation of a circle in Cartesian form whose centre is $(\frac{3}{2}, 0)$ and radius $\frac{3}{2}$

4. The position vector of point A is 2i + 3j + k, of B is 5j + 4k and of C is i + 2j + 12k. Show that ABC is a triangle.



Two conditions must b fulfilled:

1st condition

For a triangle to be, $\overline{AB} + \overline{BC} + \overline{CA} = 0$

$\overline{AB} + \overline{BC} + \overline{CA}$ = (OB – OA) + (OC –OB) + (OA – OC)

$$= \begin{pmatrix}0\\5\\4\end{pmatrix} - \begin{pmatrix}2\\3\\1\end{pmatrix} + \begin{pmatrix}1\\2\\12\end{pmatrix} - \begin{pmatrix}0\\5\\4\end{pmatrix} + \begin{pmatrix}2\\3\\1\end{pmatrix} - \begin{pmatrix}1\\2\\12\end{pmatrix}$$

$$= \begin{pmatrix}-2\\2\\3\end{pmatrix} + \begin{pmatrix}1\\-3\\8\end{pmatrix} + \begin{pmatrix}1\\1\\-11\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Second condition

We work out for any angle and if it is not 0^0 or 180^0, then we conclude that ABC is a triangle

Now finding angle A

From dot product of vectors

AB.AC = $|AB||AC|cosA$

cosA = $\frac{AB.AC}{|AB||AC|}$

AB.AC = (-2i + 2j +3k).(-I –j + 11k)

= 2 – 2 + 33 = 33

$|AB| = \sqrt{(-2)^2 + 2^2 + 3^2} = \sqrt{4 + 4 + 9} = \sqrt{17}$

$|AC| = \sqrt{(-1)^2 + 1^2 + 11^2} = \sqrt{1 + 1 + 121} = \sqrt{123}$

A = $\cos^{-1}\left(\frac{33}{\sqrt{17\ x\ 123}}\right)$ = 43.8^0

Since A is not 0^0 or 180^0, hence ABC is a triangle

NB. The above two conditions **must** be clearly shown in order for the candidate to get all the marks.

5. Solve $5\cos^2 3\theta = 3(1 + \sin 3\theta)$ for $0^0 \le \theta \le 90^0$.
   $5\cos^2 3\theta = 3(1 + \sin 3\theta)$
   $5(1 - \sin^2 3\theta) = 3(1 + \sin 3\theta)$
   $5 - 5\sin^2 3\theta = 3 + 3\sin 3\theta$
   $5\sin^2 3\theta + 3\sin 3\theta - 0 = 0$
   $(\sin 3\theta + 1)(5\sin 3\theta - 2) = 0$
   $\sin 3\theta + 1 = 0$
   $3\theta = \sin^{-1}(-1) = -90^0, 270^0$

θ = -30^0, 90^0

5sin3θ – 2 = 0

Sin3θ = $\frac{2}{5}$

3θ = $\sin^{-1}(\frac{2}{5})$ = 23.578^0, 156.422^0

θ = 7.859^0, 52.141^0

Hence θ =(7.859^0, 52.141^0, 90^0)

6. If y = (x – 0.5) e^{2x}, find $\frac{dy}{dx}$.

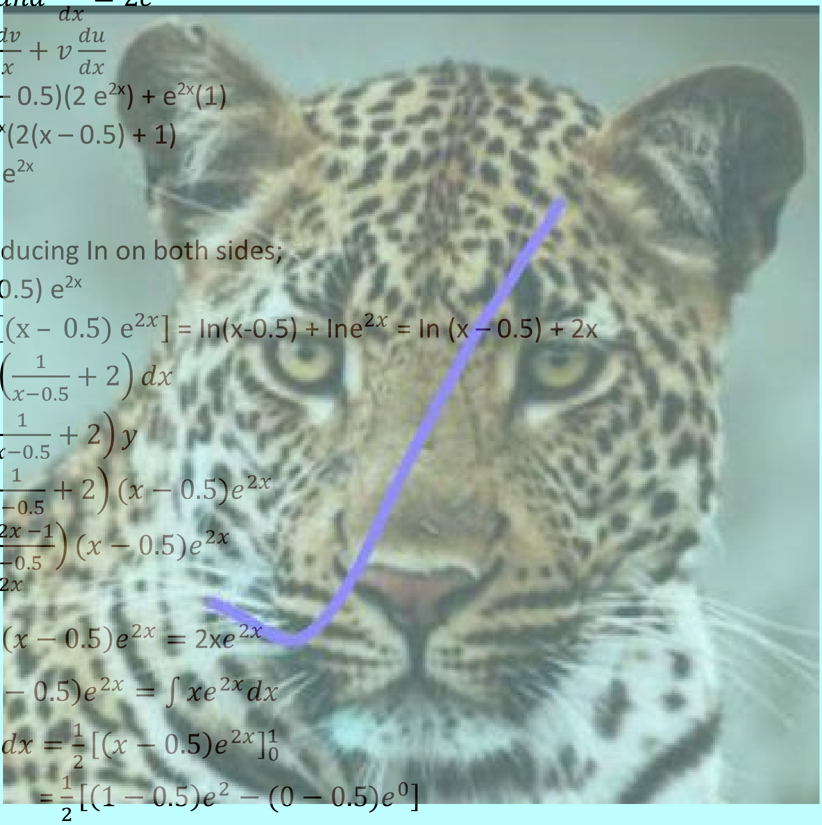
   Hence determine $\int_0^1 xe^{2x}dx$

   Using product rule

   y = (x – 0.5) e^{2x}

   Let u = (x – 0.5) and v = e^{2x}

   $\frac{du}{du} = 1\ and\ \frac{dv}{dx} = 2e^{2x}$



   $\frac{dy}{dx} = u\frac{dv}{x} + v\frac{du}{dx}$

   =(x – 0.5)(2 e^{2x}) + e^{2x}(1)

   = e^{2x}(2(x – 0.5) + 1)

   = 2x e^{2x}

   Or

   By introducing In on both sides;

   y = (x – 0.5) e^{2x}

   Iny = In[(x – 0.5) e^{2x}] = In(x-0.5) + Ine^{2x} = In (x – 0.5) + 2x

   $\frac{1}{y}dy = \left(\frac{1}{x-0.5} + 2\right)dx$

   $\frac{dy}{dx} = \left(\frac{1}{x-0.5} + 2\right)y$

   = $\left(\frac{1}{x-0.5} + 2\right)(x - 0.5)e^{2x}$

   = $\left(\frac{1+2x-1}{x-0.5}\right)(x - 0.5)e^{2x}$

   = 2xe^{2x}

   Now $\frac{d}{dx}(x - 0.5)e^{2x}$ = 2xe^{2x}

   $\frac{1}{2}\int d\,(x - 0.5)e^{2x} = \int xe^{2x}dx$

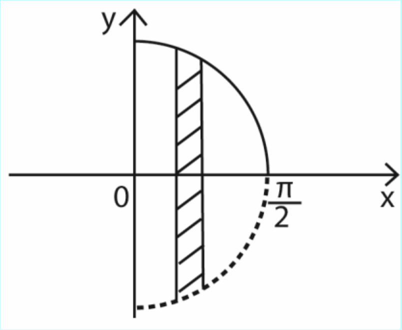
   $\int_0^1 xe^{2x}dx = \frac{1}{2}[(x - 0.5)e^{2x}]_0^1$

   $= \frac{1}{2}[(1 - 0.5)e^2 - (0 - 0.5)e^0]$

   $= \frac{1}{2}\left[\frac{e^2}{2} + \frac{1}{2}\right]$

   $= \frac{e^2}{4} + \frac{1}{4}$ = 2.0973

7. The region bounded by the curve y = cosx, the y-axis and x-axis from x =0 to x = $\frac{\pi}{2}$ is rotated about x-axis. Find the volume of the solid formed.



Volume formed = $\pi \int_0^{\frac{\pi}{2}} y^2 dx$

$= \pi \int_0^{\frac{\pi}{2}} cos^2 x dx$

$= \frac{\pi}{2} \int_0^{\frac{\pi}{2}} (1 + cos2x) dx$

$= \frac{\pi}{2} \left[x + \frac{1}{2} sin2x \right]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2} \left[\left(\frac{\pi}{2} + \frac{1}{2} sin\pi \right) - \left(0 + \frac{1}{2} sin0 \right) \right]$

$= \frac{\pi}{2}\ x\ \frac{\pi}{2} = \frac{\pi^2}{4}$ *cubic units*

8. Solve $(1 - x^2)\frac{dy}{dx} - xy^2 = 0$, given that y = 1 when x = 0.

$(1 - x^2)\frac{dy}{dx} - xy^2 = 0$

$(1 - x^2)\frac{dy}{dx} = xy^2$

By separation of variables

$\frac{dy}{y^2} = \frac{x}{1 - x^2} dx$

$\int y^{-2} dy = \int \frac{x}{1 - x^2} dx$

$\frac{y^{-1}}{-1} = \frac{1}{2} In(1 - x^2) + c$

$-\frac{1}{y} = -\frac{1}{2} In(1 - x^2) + c$

$\frac{1}{y} = In(1 - x^2)^{\frac{1}{2}} + c$

Substituting for y = 1 and x = 0

1 = In(1) + c

c = 1

Hence the equation is $\frac{1}{y} = In(1 - x^2)^{\frac{1}{2}} + 1$ or y = $\frac{1}{In(1 - x^2)^{\frac{1}{2}} + 1}$

SECTION B

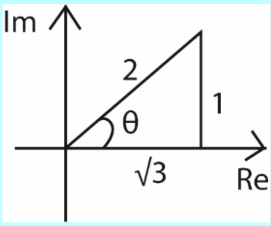
Answer any five questions from this section. All questions carry equal marks

9. (a) The complex number z = $\sqrt{3} + i$. $\bar{Z}$ *is the conjugate of* Z.

(i) Express Z in the modulus argument form

$|z| = \sqrt{(\sqrt{3})^2 + 1^2}$

$= \sqrt{3 + 1} = 2$



$\tan\theta = \frac{1}{\sqrt{3}}$

$\theta = \text{Arg}(z) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

Hence $z = 2\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$ or $z = 2[\cos 30^0 + i\sin 30^0]$

(ii) On the same Argand diagram plot $\overline{Z}$ and $2\overline{Z} + 3i$

$\bar{z} = \sqrt{3} - i$

$|z| = \sqrt{(\sqrt{3})^2 + (-1)^2}$

$= \sqrt{3+1} = 2$

$2\overline{Z} + 3i = 2(\sqrt{3} - i) + 3i = 2\sqrt{3} + i$

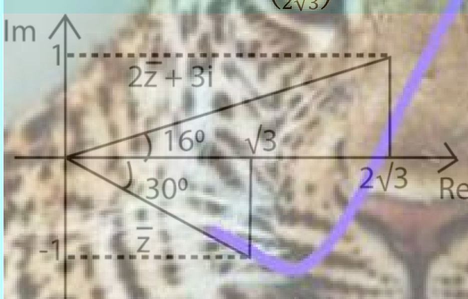
$|2\overline{Z} + 3i| = \sqrt{(2\sqrt{3})^2 + 1^2} = \sqrt{13}$

Finding arg($\bar{z}$):

$\text{Arg}(\bar{z}) = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -30^0$

Finding arg($2\overline{Z} + 3i$):

$\text{Arg}(2\overline{Z} + 3i) = \tan^{-1}\left(\frac{1}{2\sqrt{3}}\right) = 16.2^0$



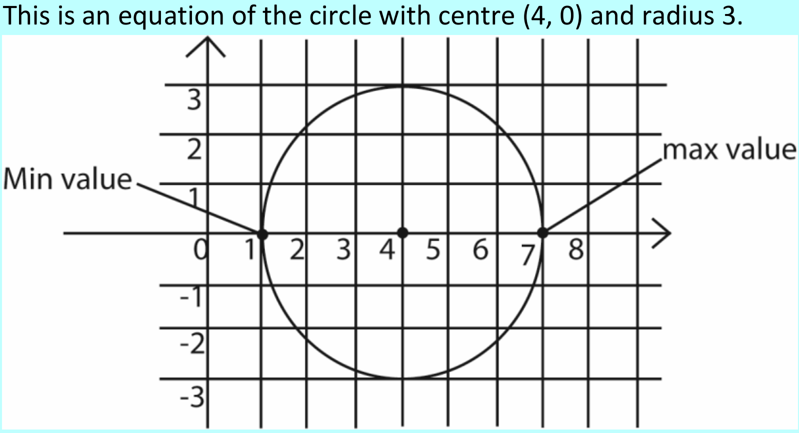
(c) What are the greatest and least values of $|Z|$ *if* $|Z - 4| \le 3$?

$|z - 2| \le 3$

$|x - 4 + iy| \le 3$

$(x - 4)^2 + y^2 \le 3$

This is an equation of the circle with centre (4, 0) and radius 3.



Greatest value of $|z| = 4 + 3 = 7$

Lowest value of $|z| = 4 - 3 = 1$

10. Given the equation x³ + x – 10 = 0.

(a) Show that x = 2 is a root of the equation

Let f(x) = $x^3 + x - 10$

Substituting for x = 2

f(2) = $2^3 + x - 10$

= 8 + 2 – 10

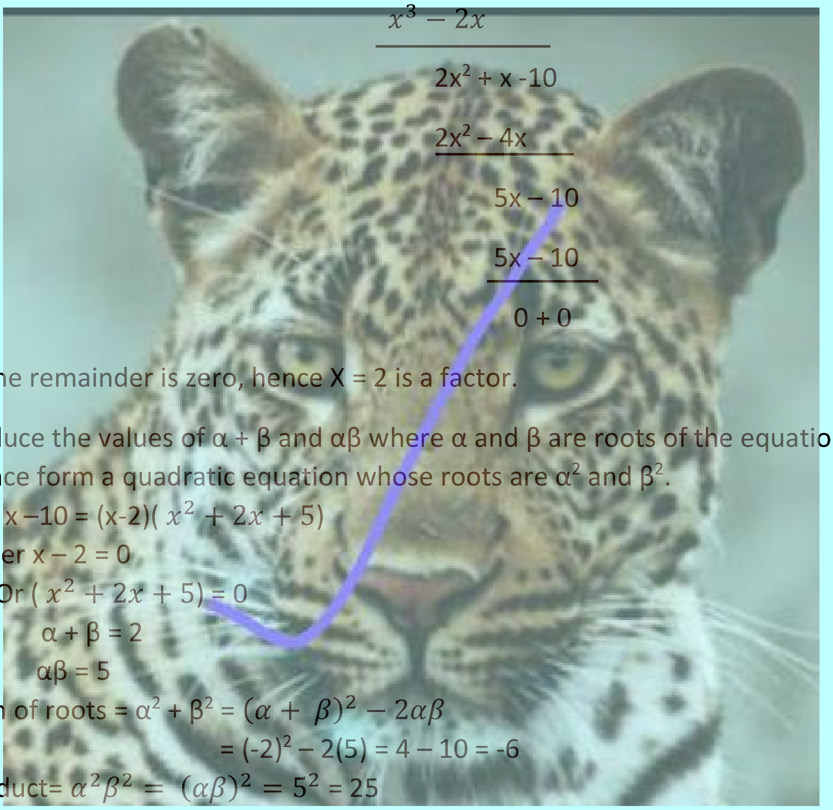
= 10 – 10 = 0

Hence x = 2 is a root of $x^3 + x - 10 = 0$

Or

If x = 2 is a root, then (x-2) should be a factor

$$\begin{array}{r|l} & x^2+2x+5 \\ \hline (x-2) & x^3+2x-10 \\ & x^3-2x \\ \hline & 2x^2+x-10 \\ & 2x^2-4x \\ \hline & 5x-10 \\ & 5x-10 \\ \hline & 0+0 \end{array}$$



Since the remainder is zero, hence X = 2 is a factor.

(b) Deduce the values of α + β and αβ where α and β are roots of the equation. Hence form a quadratic equation whose roots are α^2 and β^2.

⇨ $x^3 + x - 10 = (x-2)(x^2 + 2x + 5)$

Either x – 2 = 0

Or $(x^2 + 2x + 5) = 0$

α + β = 2

αβ = 5

Sum of roots = $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

= $(-2)^2 - 2(5) = 4 - 10 = -6$

Product= $\alpha^2\beta^2 = (\alpha\beta)^2 = 5^2 = 25$

The equation become

$x^2 - (-6)x + 25 = 0$

$x^2 + 6x + 25 = 0$

11. (a) Find the point of intersection of the lines $\frac{x-3}{4} = \frac{y-7}{4} = \frac{z+3}{-5}$ and $\frac{x-8}{7} = \frac{y-4}{1} = \frac{z-5}{3}$

Let $\frac{x-3}{4} = \frac{y-7}{4} = \frac{z+3}{-5} = \mu$

⇨ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 4 \\ -5 \end{pmatrix}$ .... (i)

And

$\frac{x-8}{7} = \frac{y-4}{1} = \frac{z-5}{3}$

⇨ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix}$ .... (ii)

Equating eqn. (i) and eqn. (ii)

$$\begin{pmatrix}5\\7\\-3\end{pmatrix}+\mu\begin{pmatrix}4\\4\\-5\end{pmatrix}=\begin{pmatrix}8\\4\\5\end{pmatrix}+\lambda\begin{pmatrix}7\\1\\3\end{pmatrix}$$

Equating corresponding unit vectors

5 + 4μ = 8 + 7λ

4μ - 7λ = 3.......................... (iii)

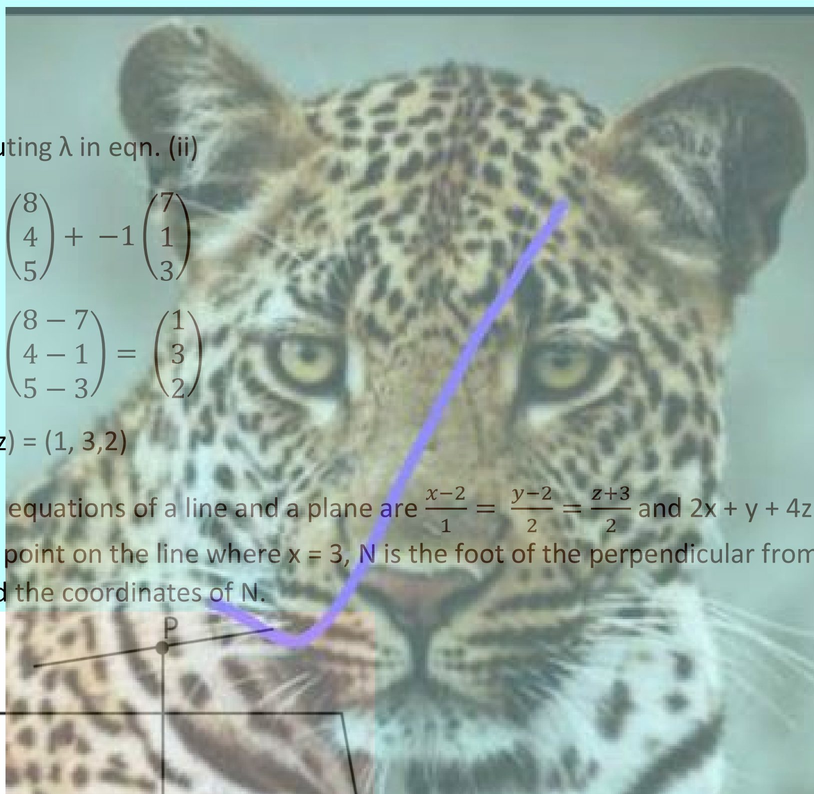
7 + 4μ = 4 + λ

4μ - λ = -3.......................... (iv)

Eqn. (iii) – eqn.(iv)

-6λ = 6

λ = -1



Substituting λ in eqn. (ii)

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}8\\4\\5\end{pmatrix}+-1\begin{pmatrix}7\\1\\3\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}8-7\\4-1\\5-3\end{pmatrix}=\begin{pmatrix}1\\3\\2\end{pmatrix}$$

∴ (x, y, z) = (1, 3,2)

(b) The equations of a line and a plane are $\frac{x-2}{1}=\frac{y-2}{2}=\frac{z+3}{2}$ and 2x + y + 4z = 9resectively. P is a point on the line where x = 3, N is the foot of the perpendicular from P to the plane. Find the coordinates of N.

Line equation

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\2\end{pmatrix}$$

$x = 2 + \lambda$

When x = 3

3 = 2 + λ; λ = 1

⇨ OP = $\begin{pmatrix}2\\2\\3\end{pmatrix}+\begin{pmatrix}1\\2\\2\end{pmatrix}=\begin{pmatrix}3\\4\\5\end{pmatrix}$

∴ P(3, 4, 5)

Plane equation : 2x + y + 4z = 9

$$r\begin{pmatrix}2\\1\\4\end{pmatrix} = 9$$

$$\therefore n = \begin{pmatrix}2\\1\\4\end{pmatrix}$$

NP = n

NP = OP – ON

ON = OP – NP

$$= \begin{pmatrix}3\\4\\5\end{pmatrix} - \begin{pmatrix}2\\1\\4\end{pmatrix} = \begin{pmatrix}1\\3\\1\end{pmatrix}$$

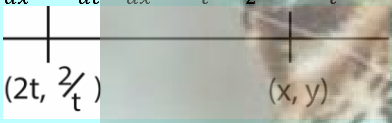
∴ N(1, 3, 1)

12. (a) Find the equation of the tangent to the hyperbola whose points are of the parametric form $(2t, \frac{2}{t})$.

x = 2t, y = $\frac{2}{t} = 2y^{-1}$

$\frac{dx}{dt} = 2, \frac{dy}{dt} = -\frac{2}{t^2}$

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = -\frac{2}{t^2} \cdot \frac{1}{2} = -\frac{1}{t^2}$



Gradient = $\frac{y-\frac{2}{t}}{x-2t}$

But gradient = $-\frac{1}{t^2}$

=> $\frac{y-\frac{2}{t}}{x-2t} = -\frac{1}{t^2}$

$t^2\left(y - \frac{2}{t}\right) = -(x - 2t)$

$t^2y + x - 4t = 0$

(b)(i) Find the equations of the tangents in (a), which are parallel to y + 4x = 0

$t^2y + x - 4t = 0$

$y = -\frac{1}{t^2}x + \frac{4}{t}$

$gradient = -\frac{1}{t^2}$

For y + 4x = 0

y = -4x

gradient = -4

But parallel lines have equal gradient

⇨ $-\frac{1}{t^2} = -4; t^2 = \frac{1}{4}\ and\ t = \pm\frac{1}{2}$

Substituting for t = $\frac{1}{2}$

$y = -\frac{1}{\left(\frac{1}{2}\right)^2}x + \frac{4}{\left(\frac{1}{2}\right)}$

$y = -4x + 8$

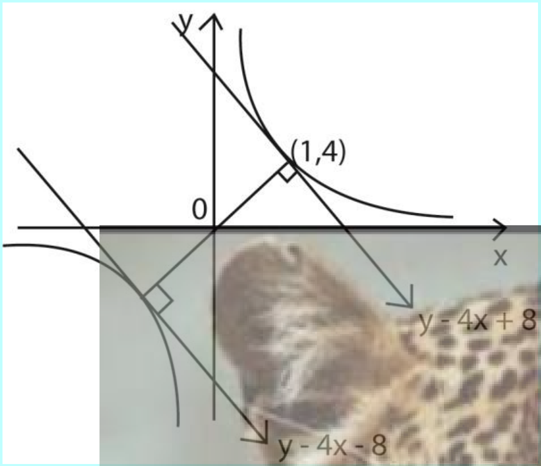
Substituting for t = $-\frac{1}{2}$

$$y = -\frac{1}{\left(-\frac{1}{2}\right)^2}x + \frac{4}{\left(-\frac{1}{2}\right)}$$
$$y = -4x - 8$$

(ii) Determine the distance between the tangents in (i)

By the nature of the parametric points in the form (2t, $\frac{2}{t}$), this is a rectangular hyperbola

Substituting for t = $\pm\frac{1}{2}$, the points become (1, 4) and (-1, -4)



The distance between two tangents = perpendicular distance between them

Using d = $\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$

We use either y = -4x + 8

Considering y = -4x + 8 or y + 4x – 8 = 0; a= 4, b = 1 c = -8

Substituting for (x, y) = (-1, -4)

d = $\left|\frac{4(-1)+1(-4)-8}{\sqrt{1^2 + 4^2}}\right| = \frac{16}{\sqrt{17}}$ = 3.88

NB – distance between the tangents is the perpendicular distance between the tangents

- Candidates or learners should not use the general formula of finding the distance between two points.

13. A curve has the equation $y = \frac{2}{1+x^2}$.

(a) Determine the nature of the turning point on the curve.

$$y = \frac{2}{1+x^2} = 2(1 + x^2)^{-1}$$
$$\frac{dy}{dx} = -2(1 + x^2)^{-2}.2x = \frac{-4x}{(1+x^2)^2}$$

Or

$$y = \frac{2}{1+x^2}$$
$$Iny = In2 - In\,(1 + x^2)$$
$$\frac{1}{y}dy = \left(0 - \frac{2x}{1+x^2}\right)dx$$
$$\frac{dy}{dx} = \frac{-2x}{1+x^2}.\frac{2}{1+x^2} = \frac{-4x}{(1+x^2)^2}$$

At turning points, $\frac{dy}{dx} = 0$

⇨ $\frac{-4x}{(1+x^2)^2} = 0$

$x = 0$

y = $\frac{2}{1+0} = 2$

Hence the turning points is (x, y) = (0, 2)

Finding the nature of the turning point

X	-1	0	1
$\frac{dy}{dx}$	+1	0	-1

Max

Hence the nature of the turning point is maximum

(b) Find the equation of the asymptote. Hence sketch the curve.

$y = \frac{2}{1+x^2}$

For vertical asymptote

$1 + x^2 = 0$

$x^2 = -1$

$x = \sqrt{-1}$ which does not exist

Or

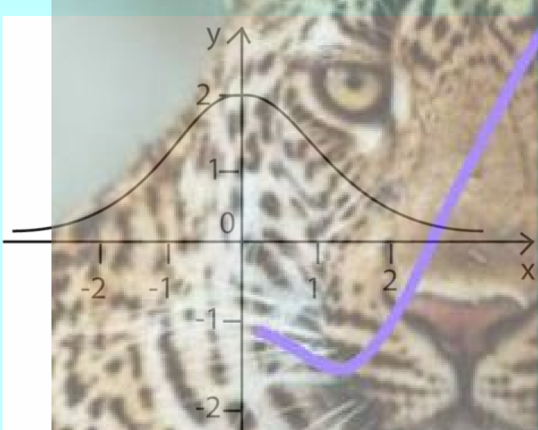
Since $1 + x^2$ cannot be factorized, there is no vertical asymptote

For horizontal asymptote

$y = \frac{2}{1+x^2}$

As $x \rightarrow \infty$, $y \rightarrow 0$

Hence the horizontal asymptote is $y = 0$



NB.

- This is not an open question, therefore candidates should not waster time finding the region where the curve is confined.
- Shading is not very necessary for this question

14. (a) Prove that $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

$\tan(A - B) = \frac{\sin(A-B)}{\cos(A-B)}$

$= \frac{\sin A \cos B - \cos A \sin B}{\cos A \sin B + \sin A \sin B}$

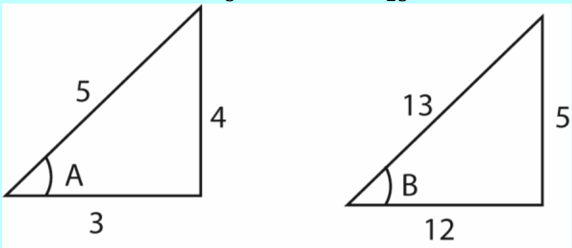
Diving numerator and denominator on the R.H.S by cosAcosB

$\tan(A - B) = \frac{\frac{\sin A \cos B}{\cos A \cos B} - \frac{\cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B} + \frac{\sin A \sin B}{\cos A \cos B}} = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

Hence show that $\frac{1 - \tan 15^0}{1 + \tan 15^0} = \frac{1}{\sqrt{3}}$

$\frac{1 - \tan 15^0}{1 + \tan 15^0} = \frac{\tan 45^0 - \tan 15^0}{1 + \tan 45^0 \tan 15^0} = \tan(45^0 - 15^0)\ \tan 30^0 = \frac{1}{\sqrt{3}}$

(b) Given that cos A = $\frac{3}{5}$ and cos B = $\frac{12}{13}$ where A and B are acute, find the values of



cos A = $\frac{3}{5}$ cos B = $\frac{12}{13}$

sin A = $\frac{4}{5}$ sin B = $\frac{5}{13}$

tan A = $\frac{4}{3}$ tanB = $\frac{5}{12}$

(i) tan (A + B)

tan (A + B) = $\frac{sinA\ cosB + cosA\ sinB}{\cos AsinB - sinAsin\ B}$

$= \frac{\frac{4}{5}.\frac{12}{13}+\frac{3}{5}.\frac{5}{13}}{\frac{3}{5}.\frac{5}{13}-\frac{4}{5}.\frac{5}{13}}$ = 3.9375

(ii) cosec (A + B)

cosec (A + B) = $\frac{1}{\sin(A+B}$ = $\frac{1}{sinAcosB+cosAsinB}$ = $\frac{1}{\frac{4}{5}.\frac{12}{13}+\frac{3}{5}.\frac{5}{13}}$ = 1.0317

15. Resolve $y = \frac{x^3+5x^2-6x+6}{(x-1)^2(x^2+2)}$ into partial fraction

Let $\frac{x^3+5x^2-6x+6}{(x-1)^2(x^2+2)} \equiv \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{(x^2+2)}$

$x^3 + 5x^2 - 6x + 6 \equiv A(x-1)(x^2+2) + B(x^2+2) + (Cx+D)(x-1)^2$

Taking x = 1

1 + 5 – 6 + 6 = 0 + 3B

3B = 6; B = 2

Taking x = 0

6 = -2A + 2B +D

6 = -2A + 4 + D

2 = -2A + D ……………… (i)

Equating coefficient of x^3

1 = A + C ………………… (ii)

Equating coefficients of x^2

5 = -A + B – 2C + D

5 = -A + 2 -2C + D

3 + A + 2C = D

Substituting for D into equation (i)

2 = -2A + 3 + A + 2C

-1= -A + 2C ……………. (iii)

Eqn (ii)+ eqn. (iii)

0 = 3C; C = 0

Substituting for c into eqn. (i)

1 = A + 0

A = 1

Substituting for A into eqn. (i)

2 = -2 + D; D = 4

⇨ $\frac{x^3+5x^2-6x+6}{(x-1)^2(x^2+2)} \equiv \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{4}{(x^2+2)}$

Hence find $\int ydx$ and $\frac{dy}{dx}$.

$\frac{x^3+5x^2-6x+6}{(x-1)^2(x^2+2)} \equiv \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{4}{(x^2+2)}$

$\int ydx = \int \left(\frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{4}{(x^2+2)}\right) dx$

= In(x -1) + $\int \frac{2}{(x-1)^2} + \int \frac{4}{(x^2+2)}$

= In(x – 1) + $\frac{-2}{(x-1)} + \int \frac{4}{(x^2+2)}$

Now $\int \frac{1}{a^2+b^2x^2} = \frac{1}{ab}\tan^{-1}\left(\frac{bx}{a}\right) + c$

Comparing $\frac{1}{2+x^2}$ with $\frac{1}{a^2+b^2x^2}$

a = $\sqrt{2}$ and b = 1

$\int \frac{4}{(x^2+2)} = \frac{4}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)$+ C

$\int ydx = In(x-1) + \frac{-2}{(x-1)} + \frac{4}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)$+ C

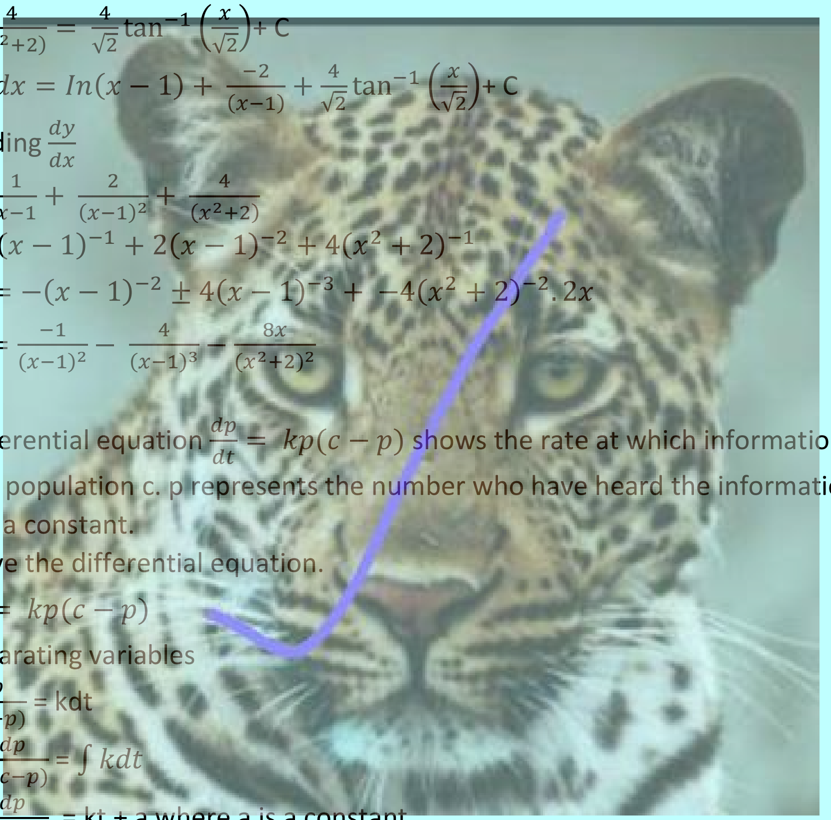
Finding $\frac{dy}{dx}$

y = $\frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{4}{(x^2+2)}$

= $(x-1)^{-1} + 2(x-1)^{-2} + 4(x^2+2)^{-1}$

$\frac{dy}{dx} = -(x-1)^{-2} \pm 4(x-1)^{-3} + -4(x^2+2)^{-2}.2x$

$= \frac{-1}{(x-1)^2} - \frac{4}{(x-1)^3} - \frac{8x}{(x^2+2)^2}$



16. The differential equation $\frac{dp}{dt} = kp(c-p)$ shows the rate at which information flows in a student population c. p represents the number who have heard the information in t days and k is a constant.

(a) Solve the differential equation.

$\frac{dp}{dt} = kp(c-p)$

Separating variables

$\frac{dp}{p(c-p)}$ = kdt

$\int \frac{dp}{p(c-p)} = \int kdt$

$\int \frac{dp}{p(c-p)}$ = kt + a where a is a constant

By partial fractions

$\frac{1}{p(c-p)} \equiv \frac{A}{p} + \frac{B}{c-p}$

$1 \equiv A(c-p) + B(p)$

$1 \equiv Ac - Ap + Bp$

$1 \equiv Ac + (B-A)p$

Equating constants

1 =Ac

A = $\frac{1}{c}$

Equating coefficient of p

0 = B – A

A = B = $\frac{1}{c}$

$\Rightarrow \int \frac{dp}{p(c-p)} = \frac{1}{c}\int \frac{1}{p}dp + \frac{1}{c}\int \frac{1}{c-p}dp$

$$=\frac{1}{c}Inp - \frac{1}{c}In(c-p)$$

$$=\frac{1}{c}In\ \frac{p}{(c-p)}$$

$$\therefore \frac{1}{c}In\ \frac{p}{(c-p)} = kt + a$$

(b) A school has a population of 1000 students. Initially 20 students had heard the information. A day later, 50 students had heard the information. How many students heard the information by the tenth day?

Given c =1000, at t=0, p= 20

By substitution, we have

$$\frac{1}{1000}In\ \frac{20}{1000-20} = 0 + a$$

$$a = \frac{1}{1000}In\ \frac{20}{980} = \frac{1}{1000}In\ \frac{1}{49}$$

$\Rightarrow \frac{1}{1000}In\ \frac{p}{(1000-p)} = kt + \frac{1}{1000}In\ \frac{1}{49}$

After t = 1, p = 50; by substitution, we have

$$\frac{1}{1000}In\ \frac{50}{(1000-50)} = k(1) + \frac{1}{1000}In\ \frac{1}{49}$$

$$k = \frac{1}{1000}In\ \frac{50}{950} - \frac{1}{1000}In\ \frac{1}{49} = \frac{1}{1000}In\ \frac{1}{19} \div \frac{1}{49} = \frac{1}{1000}In\ \frac{49}{19}$$

$\Rightarrow \frac{1}{1000}In\ \frac{p}{(1000-p)} = \left(\frac{1}{1000}In\ \frac{49}{19}\right)t + \frac{1}{1000}In\ \frac{1}{49}$

$$In\ \frac{p}{(1000-p)} = \left[In\left(\frac{49}{19}\right)\right]t + In\ \frac{1}{49}$$

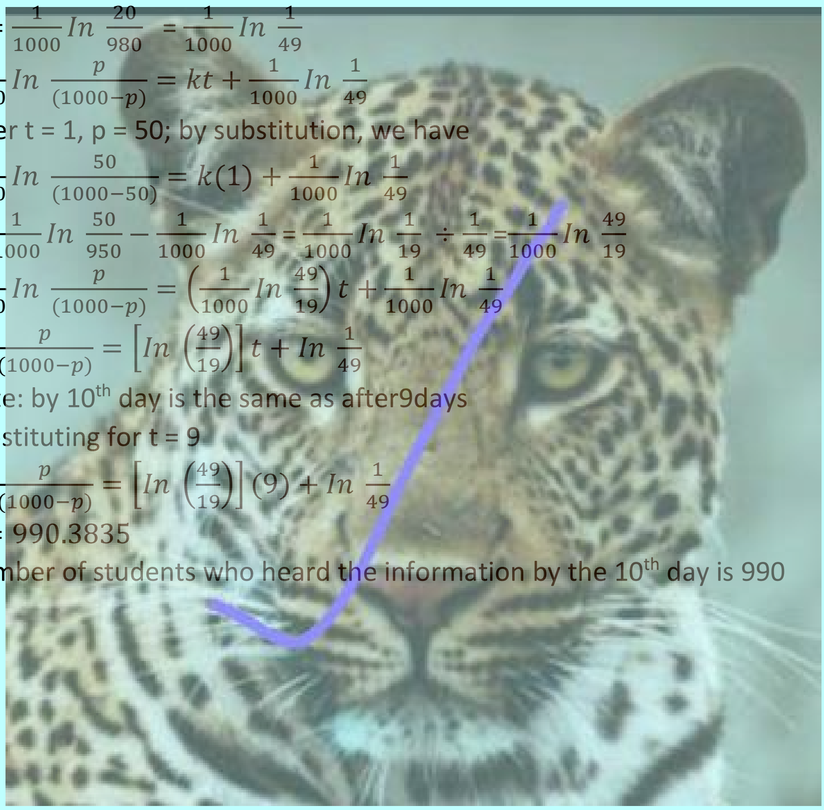
Note: by 10th day is the same as after9days

Substituting for t = 9

$$In\ \frac{p}{(1000-p)} = \left[In\left(\frac{49}{19}\right)\right](9) + In\ \frac{1}{49}$$

$$p = 990.3835$$

Number of students who heard the information by the 10th day is 990



UACE MATHEMATICS PAPER 1 2014

SECTION A (40 marks)

Answer all questions in this section

1. Solve the simultaneous equation
   X – 2y -2z = 0
   2x + 3y + z = 1
   3x –y – 3z = 3(05marks)
2. A focal chord PQ, to the hyperbola $y^2 = 4x$, has a gradient m = 1. Find the coordinates of the mid-point of PQ. (05marks)
3. Given that cos 2A - cos 2B = -p and sin2A – sin2B = q, prove that sec (A + B) = $\frac{1}{q}\sqrt{p^2 + q^2}$ (05marks)
4. Differentiatelog$_5\left(\frac{e^{tanx}}{sin^2x}\right)$.
5. Find the equation of a line through S(1, 0, 2) and T(3, 2, 1) in the form r = a +λb. (05marks)
6. Solve the equation $\sqrt{2x+3} - \sqrt{x+1} = \sqrt{x-2}$ (05marks)
7. Find $\int x(1 - x^2)^{\frac{1}{2}}dx$ (05marks)
8. A cylinder has radius r and height 8r. The radius increases from 4cm to 44.1cm. Find the approximate increase n the volume. (use π= 3.14) (05marks)

SECTION B (60MARKS)

Answer any five questions from this section. All questions carry equal marks)

9. (a) Given that the complex number Z and its conjugate $\bar{Z}$ satisfy the equation
   $Z\bar{Z} + 2iZ = 12 + 6i$. Find Z. (07marks)
   (b)One root of the equation $Z^3 - 3Z^2 + 9Z + 13 = 0\ is\ 2 + 3i$. Determine other roots (05marks)
10. A circle is described by the equation $x^2 + y^2 – 4x – 8y + 16 = 0$. A line given by the equation y = 2(x – 1) cuts the circle at points A and B. A point (x, y) moves such that its distance from the midpoint of AB is equal to its distance from the centre of the circle.
    (a) Calculate the coordinates of A and B. (04 marks)
    (b) Determine the centre and radius of the circle (04marks)
    (c) Find locus of P. (04marks)
11. (a) differentiate y = $\cot^{-1}(Inx)$ with respect to x (06marks)
    (b) Evaluate $\int_{\frac{\pi}{3}}^{\pi} xsinxdx$ (06marks)

12. (a) Find the Cartesian equation of the plane through the points whose position vectors are 2i + 2j + 3k, 3i + j + 2k and -2j + 4k. (06marks)

    (b) Determine the angle between the plane in (a) and the line $\frac{x-2}{2} = \frac{y}{-4} = z - 5$. (06marks)

13. (a) Find the three terms of the expansion $(2 – x)^6$ and use it to find $(1.998)^6$ correct to two decimal places (06 marks)

    (b) Expand $(1 – 3x + 2x^2)^5$ in ascending power of x as far as the x^2 term (06marks)

14. (a) Find the equation of a normal to curve whose parametric equation are x = $b\sec^2\theta$,

    y = $b\tan^2\theta$ (06marks)

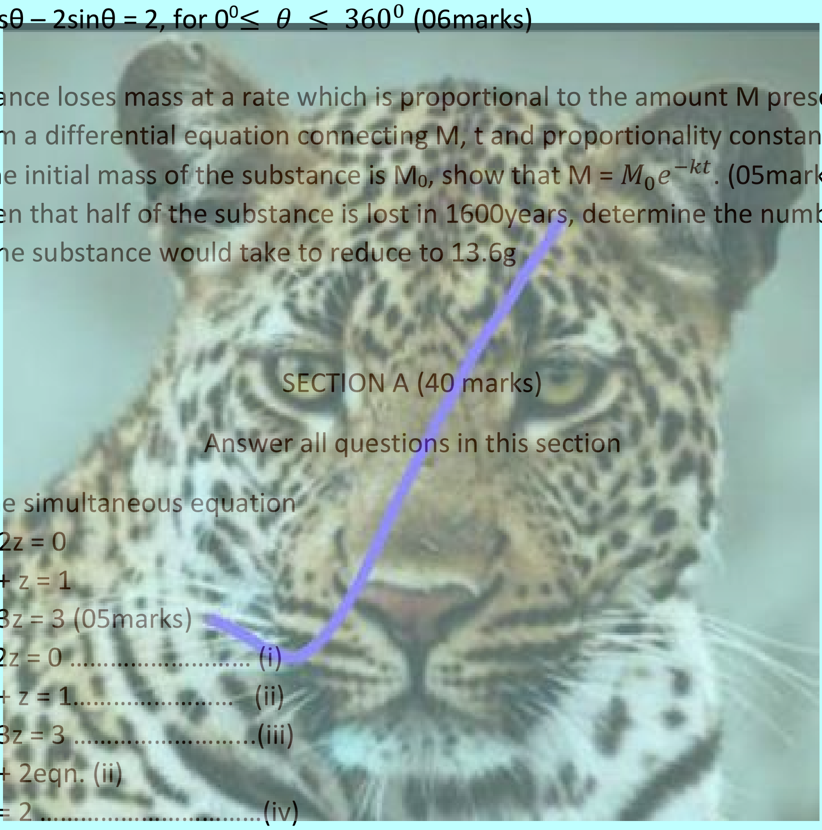
    (b) The area enclosed by the curve $x^2 + y^2 = a^2$, the y-axis and the line y = $\frac{1}{2}$a is rotated through 90^0 about the y-axis. Find the volume of the solid generated. (06marks)

15. Solve

    (a) $4\sin^2\theta – 12\sin2\theta + 35\cos^2\theta = 0$; for $0^0 \leq \theta \leq 90^0$ (06marks)

    (b) $3\cos\theta – 2\sin\theta = 2$, for $0^0 \leq \theta \leq 360^0$ (06marks)

    (c)



16. A substance loses mass at a rate which is proportional to the amount M present at time t.

    (a) Form a differential equation connecting M, t and proportionality constant k. (02marks)

    (b) If the initial mass of the substance is M_0, show that M = M_0e^{-kt}. (05marks)

    (c) Given that half of the substance is lost in 1600years, determine the number of years 15g of the substance would take to reduce to 13.6g

Solutions

SECTION A (40 marks)

Answer all questions in this section

1. Solve the simultaneous equation

   x – 2y - 2z = 0

   2x + 3y + z = 1

   3x –y – 3z = 3 (05marks)

   X – 2y -2z = 0 …………………….. (i)

   2x + 3y + z = 1…………………… (ii)

   3x –y – 3z = 3 …………………….(iii)

   Eqn. (i) + 2eqn. (ii)

   4x + 4y = 2 ………………………….(iv)

   3Eqn. (ii) + eqn. (iii)

   9x + 8y = 6 ………………………. (v)

   2eqn. (iv) – eqn. (v)

   x = -2

   Substituting x = -2 into eqn. (iv)

   5(-2) + 4y = 2

   y = 3

   Substituting x = -2 and y = 3 into eqn. (ii)

   2(-2) + 3(3) + z =2

   z = -4

   ∴ x = -2, y = 3 and z = -4

Alternatively: using row reduction to echelon form

Expressing the equation in matrix form

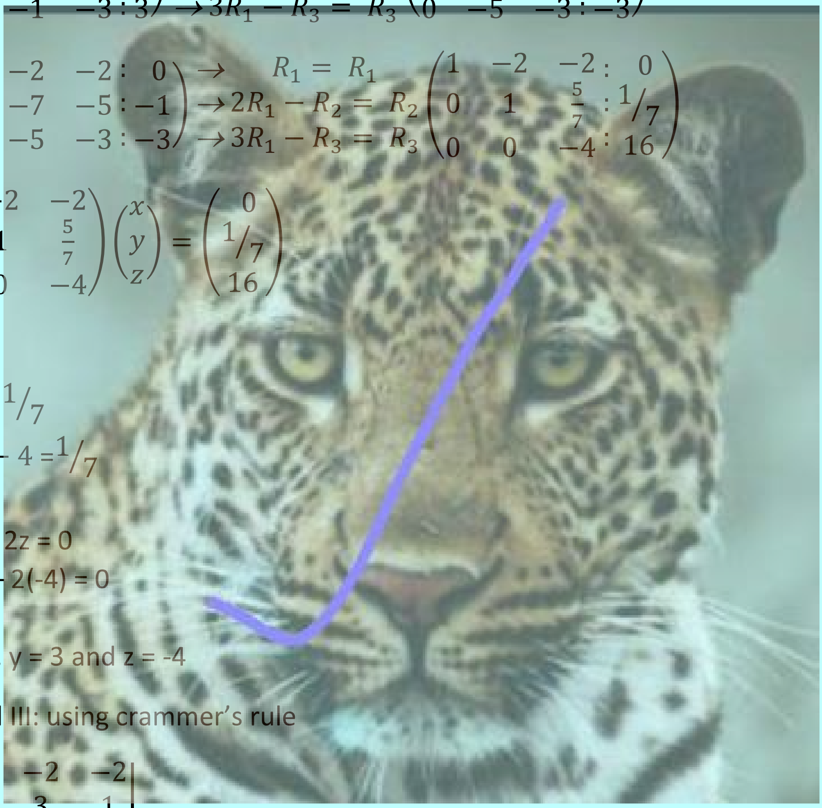
$$\begin{pmatrix} 1 & -2 & -2 \\ 2 & 3 & 1 \\ 3 & -1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

The augmented matrix is

$$\begin{pmatrix} 1 & -2 & -2 : 0 \\ 2 & 3 & 1 : 1 \\ 3 & -1 & -3 : 3 \end{pmatrix}$$

Transforming augmented matrix to a unity triangular matrix

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\begin{pmatrix} 1 & -2 & -2 : 0 \\ 2 & 3 & 1 : 1 \\ 3 & -1 & -3 : 3 \end{pmatrix}\begin{matrix} \rightarrow & R_1 = R_1 \\ \rightarrow & 2R_1 - R_2 = R_2 \\ \rightarrow & 3R_1 - R_3 = R_3 \end{matrix}\begin{pmatrix} 1 & -2 & -2 : & 0 \\ 0 & -7 & -5 : & -1 \\ 0 & -5 & -3 : & -3 \end{pmatrix}$$



$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\begin{pmatrix} 1 & -2 & -2 : & 0 \\ 0 & -7 & -5 : & -1 \\ 0 & -5 & -3 : & -3 \end{pmatrix}\begin{matrix} \rightarrow & R_1 = R_1 \\ \rightarrow & 2R_1 - R_2 = R_2 \\ \rightarrow & 3R_1 - R_3 = R_3 \end{matrix}\begin{pmatrix} 1 & -2 & -2 : & 0 \\ 0 & 1 & \frac{5}{7} : & {}^{1}/_{7} \\ 0 & 0 & -4 : & 16 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & \frac{5}{7} \\ 0 & 0 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ {}^{1}/_{7} \\ 16 \end{pmatrix}$$

-4z = 16

z = -4

$y + \frac{5}{7}z = {}^{1}/_{7}$

$y + \frac{5}{7}x - 4 = {}^{1}/_{7}$

y = 3

x – 2y – 2z = 0

x -2(3) – 2(-4) = 0

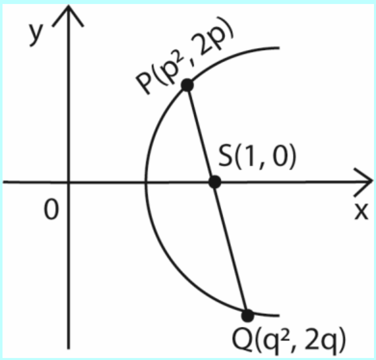
x = -2

∴ x = -2, y = 3 and z = -4

Method III: using crammer's rule

$$D = \begin{vmatrix} 1 & -2 & -2 \\ 2 & 3 & 1 \\ 3 & -1 & -3 \end{vmatrix}$$

2. A focal chord PQ, to the hyperbola $y^2 = 4x$, has a gradient m = 1. Find the coordinates of the mid-point of PQ. (05marks)



Grad of $\overline{PS} = grad\ of\ \overline{SP} = grad\ of\ \overline{PQ} = 1$

$\frac{2p-0}{p^2-1} = 1$

$p^2 - 2p - 1 = 0$

$p = \frac{2\pm\sqrt{2^2-4(1)(-1)}}{2(1)} = \frac{2\pm\sqrt{4+4}}{2} = 1\pm\sqrt{2}$

$=> P\left[\left(1+\ \sqrt{2}\right)^2, \left(2+2\sqrt{2}\right)\right], Q\left[\left(1-\ \sqrt{2}\right)^2, \left(2-2\sqrt{2}\right)\right],$

Let M(x, y) be the coordinates of the mid-point of PQ.

$x = \frac{1}{2}\left[\left(1+\ \sqrt{2}\right)^2, \left(2+2\sqrt{2}\right)\right] = 3$

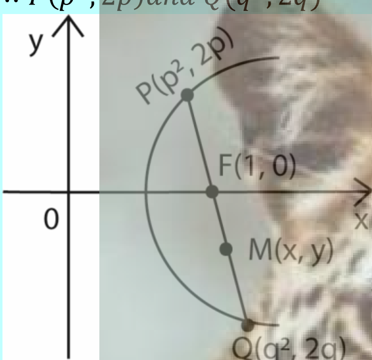
$y = \frac{1}{2}\left[\left(1-\ \sqrt{2}\right)^2, \left(2-2\sqrt{2}\right)\right] = 2$

∴M(3, 2)

Alternatively

From $y^2 = 4x$,; a =1

$\therefore P(p^2, 2p) and\ Q(q^2, 2q)$



From focal chord, pq = 1

Gradient = $\frac{2q-2p}{q^2-p^2} = 1$

$= \frac{2(q-p)}{(q+p)(q-p)} = 1$

⇨ q + p = 2

$M(\frac{p^2+q^2}{2}, p+q)$

x = $\frac{p^2+q^2}{2} = \frac{1}{2}[(p+q)^2 - 2pq] = \frac{1}{2}[(-2)^2 - 2(1)] = \frac{1}{2}(4+2) = 3$

y = p + q = 2

∴M(3, 2)

3. Given that cos 2A - cos 2B = -p and sin2A – sin2B = q, prove that sec (A + B) = $\frac{1}{q}\sqrt{p^2 + q^2}$ (05marks)

cos 2A - cos 2B = -2sin$\frac{1}{2}(2A+2B)sin\frac{1}{2}(2A-2B)$

-p = -2sin (A + B)sin (A-B)

p = 2sin (A + B)sin (A- B) ……………. (i)

sin2A – sin2B = -2cos$\frac{1}{2}(2A+2B)sin\frac{1}{2}(2A-2B)$

q = 2cos (A + B)sin (A- B) ……………. (ii)

Eqn. (i) ÷ eqn. (ii)

$\frac{p}{q} = \tan(A+B)$

$\frac{p^2}{q^2} = tan^2\ (A+B) = sec^2\ (A+B) - 1$

$$sec^2(A+B) = \frac{p^2}{q^2} + 1 = \frac{p^2+q^2}{q^2} = \frac{1}{q^2}(p^2 + q^2)$$

$$\sec(A+B) = \frac{1}{q}\sqrt{(p^2 + q^2)}$$

4. Differentiate$\log_5\left(\frac{e^{tanx}}{sin^2x}\right)$.

Let y =$\log_5\left(\frac{e^{tanx}}{sin^2x}\right)$

$$5^y = \frac{e^{tanx}}{sin^2x}$$

$$yIn5 = Ine^{tanx} - Insin^2x$$

$$yIn5 = tanx - Insin^2x$$

$$\frac{dy}{dx}In5 = sec^2x - \frac{\frac{d}{dx}(sin^2x)}{sin^2x}$$

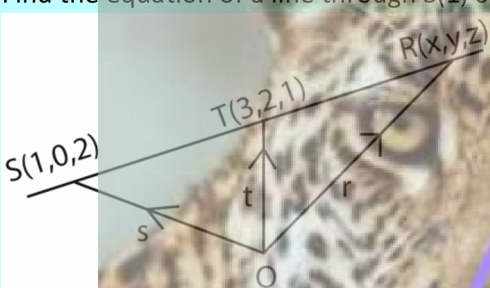
$$= sec^2x - \frac{2sinxcosx}{sin^2x}$$

$$= sec^2x - \frac{2cosx}{\sin x}$$

$$= sec^2x - 2cotx$$

$$\frac{dy}{dx} = \frac{1}{In5}(sec^2x - 2cotx)$$

5. Find the equation of a line through S(1, 0, 2) and T(3, 2, 1) in the form r = a +λb. (05marks)



SR = λST

OR –OS =λ(OT – OS)

$$OR - \begin{pmatrix}1\\0\\2\end{pmatrix} = \lambda\begin{pmatrix}3\\2\\1\end{pmatrix} - \begin{pmatrix}1\\0\\2\end{pmatrix}$$

$$r-\begin{pmatrix}1\\0\\2\end{pmatrix} = \lambda\begin{pmatrix}2\\2\\-1\end{pmatrix}$$

Substituting for r= OR = $\begin{pmatrix}x\\y\\z\end{pmatrix}$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\0\\2\end{pmatrix} + \lambda\begin{pmatrix}2\\2\\-1\end{pmatrix}$$

$$\frac{x-1}{2} = \frac{y}{2} = \frac{z-2}{-1} = \lambda$$

Hence the Cartesian equation is$\frac{x-1}{2} = \frac{y}{2} = \frac{z-2}{-1}$

6. Solve the equation $\sqrt{2x+3} - \sqrt{x+1} = \sqrt{x-2}$ (05marks)

$$\sqrt{2x+3} - \sqrt{x+1} = \sqrt{x-2}$$

Squaring both sides

$\left(\sqrt{2x+3} - \sqrt{x+1}\right)^2 = \left(\sqrt{x-2}\,\right)^2$

$2x+3-2\sqrt{(2x+3)(x+1)}$+x +1 = x – 2

$2x+6-2\sqrt{(2x+3)(x+1)}$ = 0

$x+3 = \sqrt{(2x+3)(x+1)}$

Squaring both side

$(x+3)^2$ =$\left(\sqrt{(2x+3)(x+1)}\right)^2$

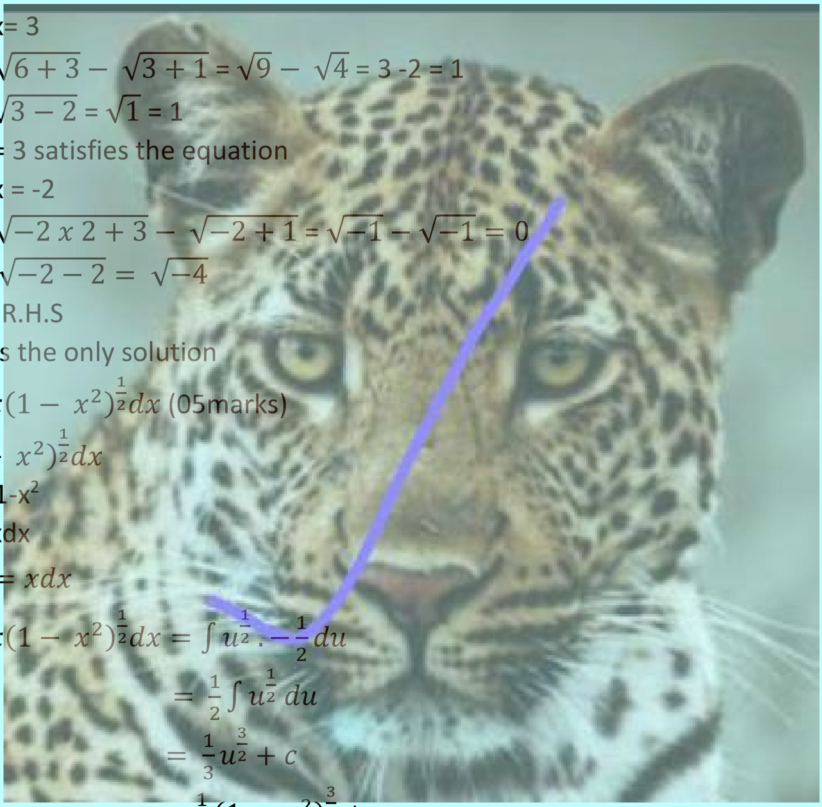
$x^2+6x+9 = 2x^2+5x+3$

$x^2-x-6=0$

(x -3)(x+2) = 0

x = 3 or x = -2

Verification



Taking x= 3

L.H.S = $\sqrt{6+3} - \sqrt{3+1} = \sqrt{9} - \sqrt{4}$ = 3 -2 = 1

R.H.S =$\sqrt{3-2} = \sqrt{1}$ = 1

Here x = 3 satisfies the equation

Taking x = -2

L.H.S = $\sqrt{-2\,x\,2+3} - \sqrt{-2+1} = \sqrt{-1} - \sqrt{-1} = 0$

R.H.S = $\sqrt{-2-2} = \sqrt{-4}$

L.H.S $\neq$ R.H.S

∴ x = 3 is the only solution

7. Find $\int x(1-x^2)^{\frac{1}{2}}dx$ (05marks)

$\int x(1-x^2)^{\frac{1}{2}}dx$

Let u = 1-x^2

du = -2xdx

$-\frac{1}{2}du = xdx$

$=> \int x(1-x^2)^{\frac{1}{2}}dx = \int u^{\frac{1}{2}} \cdot -\frac{1}{2}du$

$= \frac{1}{2}\int u^{\frac{1}{2}}\,du$

$= \frac{1}{3}u^{\frac{3}{2}} + c$

$= \frac{1}{3}(1-x^2)^{\frac{3}{2}} + c$

8. A cylinder has radius r and height 8r. The radius increases from 4cm to 44.1cm. Find the approximate increase n the volume. (use π= 3.14) (05marks)

v =$\pi r^2 h$ = $\pi r^2(8r) = 8\pi r^3$

$\frac{dv}{dr} = 24\pi r^2$

But $\frac{\Delta v}{\Delta r} = \frac{dv}{dr}$

$\Delta v = \frac{dv}{dr}\Delta r$

$\Delta r = 4.1 - 4.0 = 0.1cm$

$=> \Delta v = 24(3.14)(4)^2(0.1)$ = 120.576cm^3

∴ Δv = 120.576cm^3

SECTION B (60MARKS)

Answer any five questions from this section. All questions carry equal marks)

9. (a) Given that the complex number Z and its conjugate $\bar{Z}$ satisfy the equation

$$Z\bar{Z} + 2iZ = 12 + 6i.$$ Find Z. (07marks)

Let z = x + iy then $\bar{z} = x - iy$

(x + iy)(x – iy) +2i(x + iy) = 12 + 6i

$x^2 + y^2 - 2y + 2xi = 12 + 6i$

Equating imaginary parts

2xi = 6i

x = 3

Equating real parts

$x^2 + y^2 - 2y = 12$

By substituting for x

$9 + y^2 - 2y = 12$

$y^2 - 2y - 3 = 0$

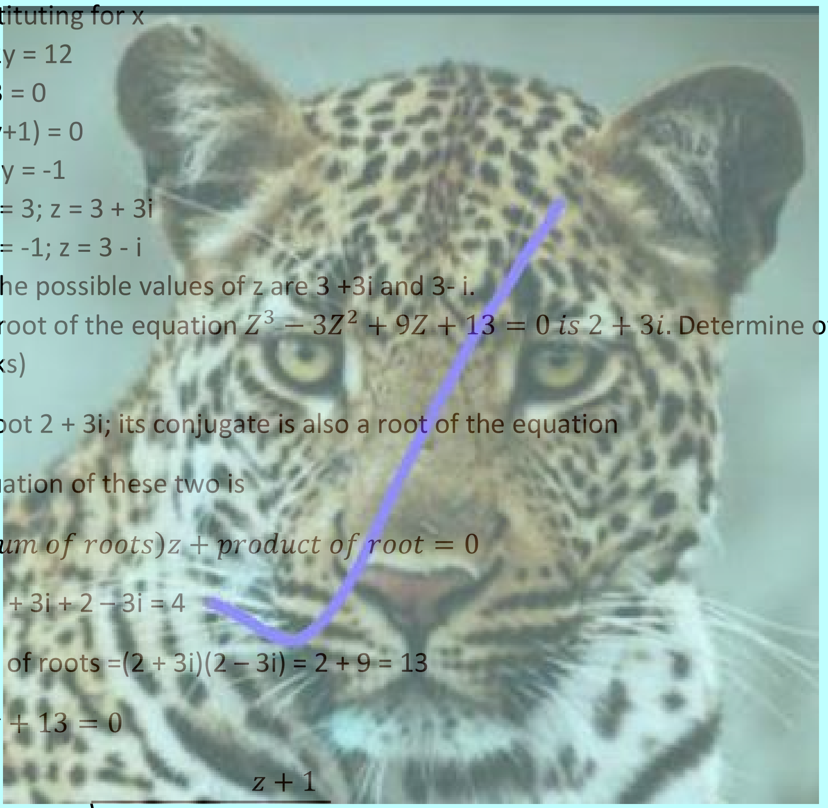
(y – 3)(y+1) = 0

y = 3 or y = -1

when y = 3; z = 3 + 3i

when y = -1; z = 3 - i

Hence the possible values of z are 3 +3i and 3- i.



(b)One root of the equation $Z^3 - 3Z^2 + 9Z + 13 = 0$ *is* $2 + 3i$. Determine other roots (05marks)

Given root 2 + 3i; its conjugate is also a root of the equation

The equation of these two is

$z^2 - (sum\ of\ roots)z + product\ of\ root = 0$

Sum = 2 + 3i + 2 – 3i = 4

Product of roots =(2 + 3i)(2 – 3i) = 2 + 9 = 13

$z^2 - 4z + 13 = 0$

$$\begin{array}{r|l} & z + 1 \\ \hline z^2 - 4z + 13 & z^3 - 3z^2 - 9z + 13 \\ & -z^3 - 4z^2 - 13z \\ \hline & z^2 - 4z + 13 \\ & -z^2 - 4z + 13 \\ \hline & 0 \quad 0 \quad 0 \end{array}$$

⇨ $(z^2 - 4z + 13)(z + 1) = 0$

z + 1= 0

z = -1

So the other roots are 2-3i and -1

10. A circle is described by the equation $x^2 + y^2 - 4x - 8y + 16 = 0$. A line given by the equation $y = 2(x - 1)$ cuts the circle at points A and B. A point (x, y) moves such that its distance from the midpoint of AB is equal to its distance from the centre of the circle.

(a) Calculate the coordinates of A and B. (04 marks)

Substituting for $y = 2(x-1)$ into the equation $x^2 + y^2 - 4x - 8y + 16 = 0$

$x^2 + (2(x-1))^2 - 4x - 8y + 16 = 0$

$5x^2 - 28x + 36 = 0$

$(x - 2)(5x - 18) = 0$

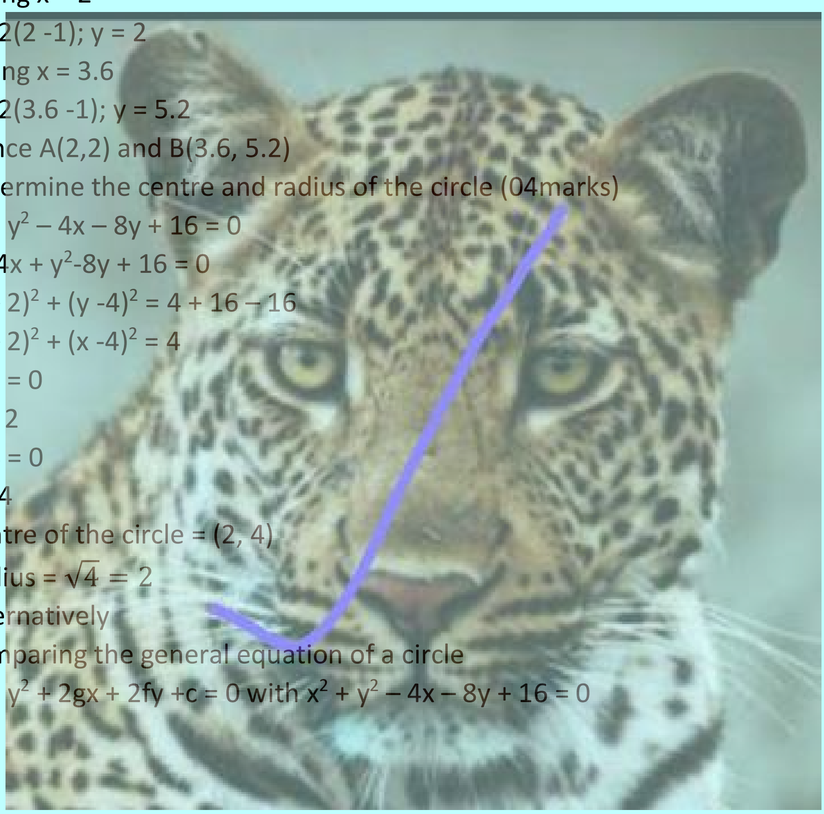
$x - 2 = 0$

$x = 2$

and $5x - 18 = 0$

$x = \frac{18}{5} = 3.6$

taking $x = 2$



$y = 2(2 - 1)$; $y = 2$

taking $x = 3.6$

$y = 2(3.6 - 1)$; $y = 5.2$

Hence A(2,2) and B(3.6, 5.2)

(b) Determine the centre and radius of the circle (04marks)

$x^2 + y^2 - 4x - 8y + 16 = 0$

$x^2 - 4x + y^2 - 8y + 16 = 0$

$(x - 2)^2 + (y - 4)^2 = 4 + 16 - 16$

$(x - 2)^2 + (x - 4)^2 = 4$

$x - 2 = 0$

$x = 2$

$y - 4 = 0$

$y = 4$

Centre of the circle = (2, 4)

Radius = $\sqrt{4} = 2$

Alternatively

Comparing the general equation of a circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ with $x^2 + y^2 - 4x - 8y + 16 = 0$

⇨ $2g = -4$

$g = -2$

$2f = -8$

$f = -4$

Centre of the circle is (-g, -f)

Hence centre of the circle = (2, 4)

Radius = $+\sqrt{g^2 + f^2 - c}$

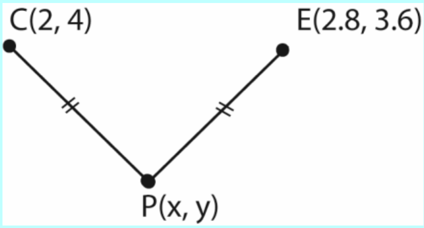
$= +\sqrt{(-2)^2 + (-4)^2 - 16} = \sqrt{4} = 2$

(c) Find locus of P. (04marks)

A(2, 2) and B(3.6, 5.2)

Coordinates of mid-point of AB

$= E\left(\frac{2+3.6}{2}, \frac{2+5.2}{2}\right)$



$\overline{PC} = \overline{PE}$ and $\overline{PC}^2 = \overline{PE}^2$

$(x-2)^2 + (y-4)^2 = (x-2.8)^2 + (y-3.6)^2$

$2x - y = 1$ or y = 2x -1

11. (a) Differentiate y = $\cot^{-1}(Inx)$ with respect to x (06marks)

y = $\cot^{-1}(Inx)$

coty = Inx

$\frac{d}{dx}(coty) = \frac{d}{dx}(Inx)$

$-sec^2y\frac{dy}{x} = \frac{1}{x}$

$\frac{dy}{dx} = \frac{1}{x[1+(In\,x)^2]}$

(b) Evaluate $\int_{\frac{\pi}{3}}^{\pi} xsinxdx$ (06marks)

Let u = x and $\frac{dv}{dx} = sinx$

$\frac{du}{dx} = 1, v = -cosx$

$Using \int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}dx$

$\int xsinxdx = xcosx - \int 1 - cosxdx$

= -xcosx + sinx + c

Or Using basic technique

Sign	Differentiate	Integrate
+	X	sinx
-	1	-cosx
+	0	-sinx

⇨ $\int_{\frac{\pi}{3}}^{\pi} xsinxdx = [-xcosx + snx]_{\frac{\pi}{3}}^{\pi}$

$= (-\pi cos\pi + sin\pi) - \left(-\frac{\pi}{s}cos\frac{\pi}{3} + sin\frac{\pi}{3}\right)$

$= \pi - \left(-\frac{\pi}{6} + \frac{\sqrt{3}}{2}\right) = \frac{7\pi}{6} - \frac{\sqrt{3}}{2} = 2.7992(4D)$

12. (a) Find the Cartesian equation of the plane through the points whose position vectors are 2i + 2j + 3k, 3i + j + 2k and -2j + 4k. (06marks)

Let OA = 2i + 2j + 3k

OB = 3i + j + 2k

OC = -2j + 4k

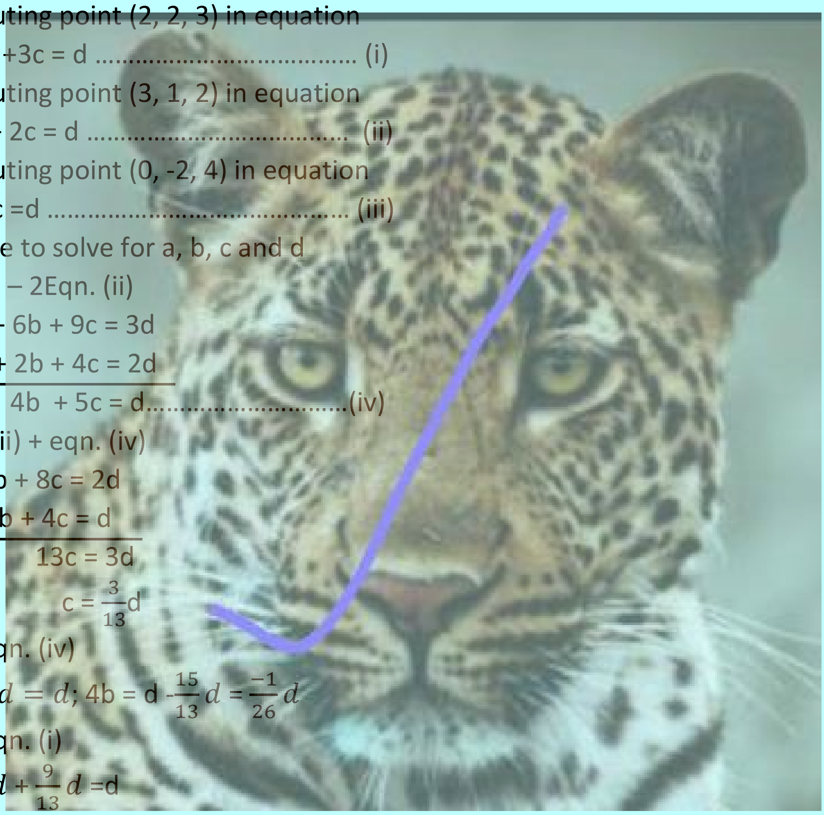
Let R be the general point in the plane

Then AR = μ(AB) + λAC

OR = OA + μ(OB – OA) + λ(OC – OA)

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\2\\3\end{pmatrix} + \mu\left[\begin{pmatrix}3\\1\\2\end{pmatrix} - \begin{pmatrix}2\\2\\3\end{pmatrix} + \lambda\left[\begin{pmatrix}0\\-2\\4\end{pmatrix} - \begin{pmatrix}2\\2\\3\end{pmatrix}\right]\right]$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\2\\3\end{pmatrix} + \mu\begin{pmatrix}1\\-1\\-1\end{pmatrix} + \lambda\begin{pmatrix}-2\\-4\\1\end{pmatrix}$$

$x = 2 + \mu - 2\lambda$ .... (i)
$y = 2 - \mu - 4\lambda$ .... (ii)
$z = 3 - \mu + \lambda$ ....(iii)
$Eqn\ (i) + eqn.(ii)$
$(x + y) = 4 - 6\lambda$ ....(iv)
$Eqn.(i) and\ eqn.(iii)$
x + z = 5 – λ
λ = -x –z + 5
Substituting for λ into eqn. (iv)
x + y =4 -6(-x –z + 5)
5x –y + 6z -26 = 0
Alternatively
Let the equation of the plane be ax + by + cz = d

Substituting point (2, 2, 3) in equation
2a + 2b +3c = d .... (i)
Substituting point (3, 1, 2) in equation
3a + b + 2c = d .... (ii)
Substituting point (0, -2, 4) in equation
-2b + 4c =d .... (iii)
We have to solve for a, b, c and d
3Eqn.(i) – 2Eqn. (ii)

$$\begin{array}{rl} & 6a + 6b + 9c = 3d \\ - & 6a + 2b + 4c = 2d \\ \hline & 4b + 5c = d \text{..............................(iv)} \end{array}$$

2eqn. (iii) + eqn. (iv)

$$\begin{array}{l} -4b + 8c = 2d \\ +4b + 4c = d \\ \hline 13c = 3d \\ c = \frac{3}{13}d \end{array}$$

From eqn. (iv)
4b + $\frac{15}{13}d = d$; 4b = d -$\frac{15}{13}d$ =$\frac{-1}{26}d$
From eqn. (i)
2a -$\frac{2}{26}d + \frac{9}{13}d$ =d
2a = d + $\frac{2}{26}d$ -$\frac{9}{13}d$ = $\frac{10}{26}d$
a = $\frac{5}{26}d$
Substituting for a, b, c in the equation of the plane

$\frac{5}{26}dx - \frac{1}{26}dy + \frac{3}{13}d = d$

$Multiplying\ through\ by\ \frac{26}{d}$

$5x - y + 6z = 26$

(b) Determine the angle between the plane in (a) and the line $\frac{x-2}{2} = \frac{y}{-4} = z - 5$. (06marks)
Let n = normal vector to the plane
b = parallel vector to the plane
⇨ b = 2i – 4j + k
n =5i –j + 6k
Let θ = angle between the line and the plane

b.n = $|b||n|\sin\theta$

$$\begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -1 \\ 6 \end{pmatrix} = (\sqrt{2^2 + (-4)^2 + 1^2} \cdot \sqrt{5^2 + (-1)^2 + 6^2})\sin\theta$$

$$10 + 4 + 6 = (\sqrt{21}.\sqrt{62})sin\theta = \sqrt{1302}sin\theta$$

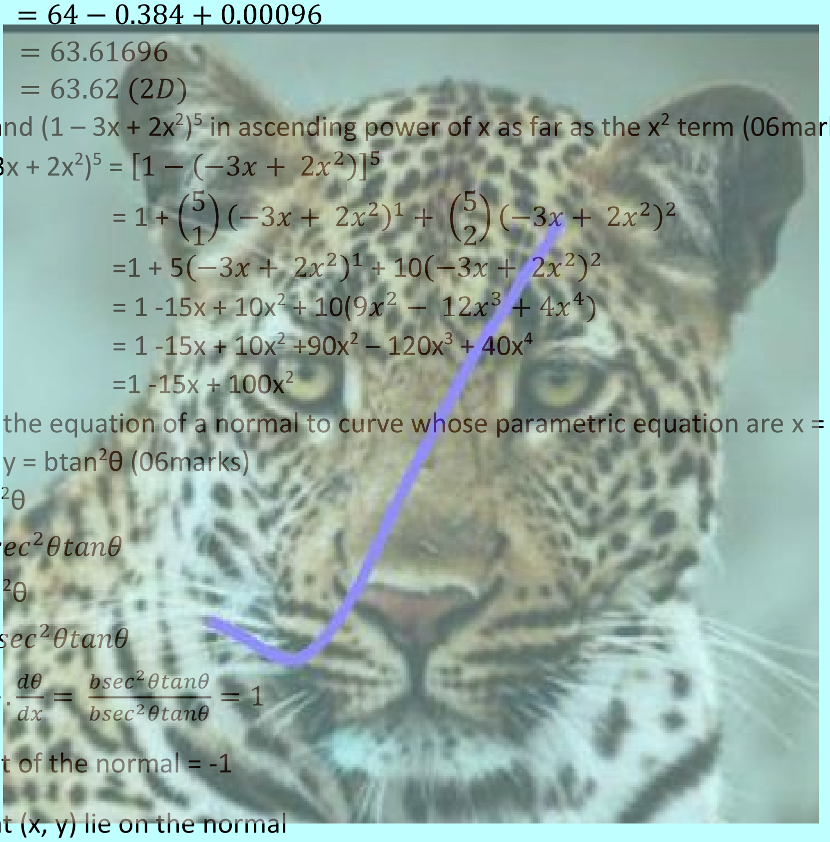
$sin\theta = \frac{20}{\sqrt{1302}}$; θ = 33.66^0 (2D)

13. (a) Find the three terms of the expansion $(2 – x)^6$ and use it to find $(1.998)^6$ correct to two decimal places (06 marks)

$(2 – x)^6 = 2^6 + \binom{6}{1} 2^5(-x)^1 + \binom{6}{2} 4(-x)^2$

$= 64 - 192x + 240x^2$

$$(1.998)^2 = (2 - 0.002)^2$$
$$= 64 - 192(0.002) + 240(0.002)^2$$
$$= 64 - 0.384 + 0.00096$$



$$= 63.61696$$
$$= 63.62\ (2D)$$

(b) Expand $(1 – 3x + 2x^2)^5$ in ascending power of x as far as the x^2 term (06marks)

$(1 – 3x + 2x^2)^5 = [1 - (-3x + 2x^2)]^5$

$= 1 + \binom{5}{1}(-3x + 2x^2)^1 + \binom{5}{2}(-3x + 2x^2)^2$

$=1 + 5(-3x + 2x^2)^1 + 10(-3x + 2x^2)^2$

$= 1 - 15x + 10x^2 + 10(9x^2 - 12x^3 + 4x^4)$

$= 1 - 15x + 10x^2 + 90x^2 – 120x^3 + 40x^4$

$=1 - 15x + 100x^2$

14. (a) Find the equation of a normal to curve whose parametric equation are x = bsec^{2}θ, y = btan^{2}θ (06marks)

x = bsec^{2}θ

$$\frac{dx}{d\theta} = bsec^2\theta tan\theta$$

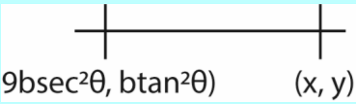
y = btan^{2}θ

$$\frac{dy}{d\theta} = bsec^2\theta tan\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = \frac{bsec^2\theta tan\theta}{bsec^2\theta tan\theta} = 1$$

Gradient of the normal = -1

Let point (x, y) lie on the normal



$$\frac{y - \text{btan}^2\theta}{x - \text{bsec}^2\theta} = -1$$

$$y - \text{btan}^2\theta = -x + \text{bsec}^2\theta$$

(x + y) = b$(\tan^2\theta + \text{bsec}^2\theta)$

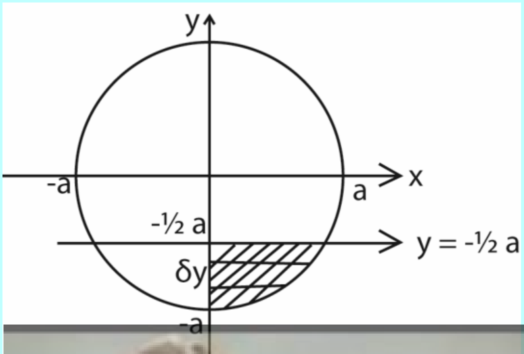
(b) The area enclosed by the curve $x^2 + y^2 = a^2$, the y-axis and the line y = $\frac{1}{2}$a is rotated through 90^0 about the y-axis. Find the volume of the solid generated. (06marks)

Note

(i) The equation $x^2 + y^2 = a^2$ is an equation of the circle with radius a and the centre as its origin

(ii) The required region is of two forms as worked out below. So bot answers are correct even though different.

Case I



$$v = \frac{1}{4}\pi \int_{-a}^{-\frac{a}{2}} x^2 dy$$

$$= \frac{1}{4}\pi \int_{-a}^{-\frac{a}{2}} (a^2 - y^2)dy$$

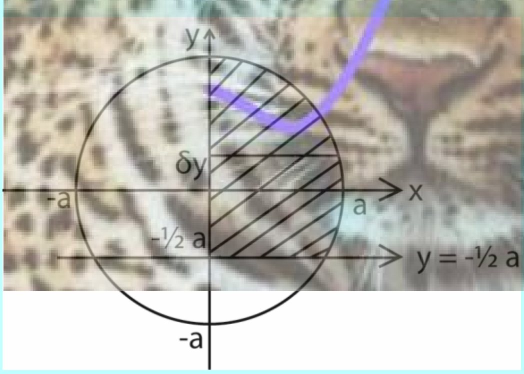
$$= \frac{1}{4}\pi \left[a^2 y - \frac{y^3}{3}\right]_{-a}^{-\frac{a}{2}}$$

$$= \frac{\pi}{4}\left\{\left[a^2\left(-\frac{a}{2}\right) - \frac{\left(-\frac{a}{2}\right)^3}{3}\right] - \left[a^2(-a) - \frac{(-a)^3}{3}\right]\right\}$$

$$= \frac{\pi}{4}\left[\left(-\frac{a^3}{2} + \frac{a^3}{24}\right) - \left(-a^3 + \frac{a^3}{3}\right)\right]$$

$$= \frac{5\pi a^3}{96}$$

Case II



$$y = \frac{\pi}{4}\int_{-\frac{a}{2}}^{a} (a^2 - y^2)dy$$

$$= \frac{1}{4}\pi \left[a^2 y - \frac{y^3}{3}\right]_{-\frac{a}{2}}^{a}$$

$$= \frac{1}{4}\pi \left[\left(a^3 - \frac{a^3}{3}\right) - \left(-\frac{a^3}{2} + \frac{a^3}{24}\right)\right]$$

$$= \frac{\pi}{96}\left[\frac{24a^3 - 8a^3 + 12a^3 - a^3}{24}\right]$$

$$= \frac{27}{96}\pi a^3 = \frac{9}{32}\pi a^3$$

15. Solve

(a) $4\sin^2\theta - 12\sin2\theta + 35\cos^2\theta = 0$; for $0^0 \le \theta \le 90^0$ (06marks)

$4\sin^2\theta - 12\sin2\theta + 35\cos^2\theta = 0$

$4\sin^2\theta - 24\sin\theta\cos\theta + 35\cos^2\theta = 0$

$4\sin^2\theta - 10\sin\theta\cos\theta - 14\sin\theta\cos\theta + 35\cos^2\theta = 0$

$2\sin\theta(2\sin\theta - 5\cos\theta) - 7\cos\theta(2\sin\theta - 5\cos\theta) = 0$

$(2\sin\theta - 7\cos\theta)(2\sin\theta - 5\cos\theta) = 0$

Either $(2\sin\theta - 5\cos\theta) = 0$

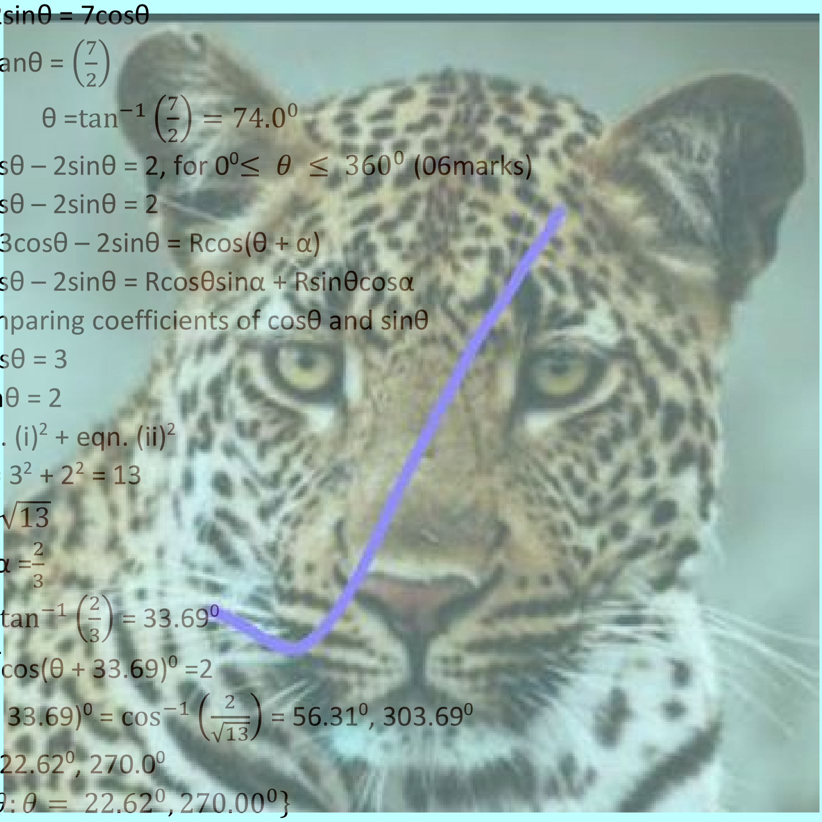
$2\sin\theta = 5\cos\theta$

$\tan\theta = \left(\frac{5}{2}\right)$

$\theta = \tan^{-1}\left(\frac{5}{2}\right)$; $= 68.2^0$

Or $(2\sin\theta - 7\cos\theta)$

$2\sin\theta = 7\cos\theta$



$\tan\theta = \left(\frac{7}{2}\right)$

$\theta = \tan^{-1}\left(\frac{7}{2}\right) = 74.0^0$

(b) $3\cos\theta - 2\sin\theta = 2$, for $0^0 \le \theta \le 360^0$ (06marks)

$3\cos\theta - 2\sin\theta = 2$

Let $3\cos\theta - 2\sin\theta = R\cos(\theta + \alpha)$

$3\cos\theta - 2\sin\theta = R\cos\theta\sin\alpha + R\sin\theta\cos\alpha$

Comparing coefficients of $\cos\theta$ and $\sin\theta$

$R\cos\theta = 3$

$R\sin\theta = 2$

Eqn. $(i)^2$ + eqn. $(ii)^2$

$R^2 = 3^2 + 2^2 = 13$

$R = \sqrt{13}$

$\tan\alpha = \frac{2}{3}$

$\alpha = \tan^{-1}\left(\frac{2}{3}\right) = 33.69^0$

⇨ $\sqrt{13}\cos(\theta + 33.69)^0 = 2$

$(\theta + 33.69)^0 = \cos^{-1}\left(\frac{2}{\sqrt{13}}\right) = 56.31^0, 303.69^0$

$\theta = 22.62^0, 270.0^0$

$\therefore \{\theta : \theta = 22.62^0, 270.00^0\}$

Alternatively

Using the substitution, $t = \tan\frac{1}{2}\theta$

$\cos\theta = \frac{1-t^2}{1+t^2}$

$\sin\theta = \frac{2t}{1+t^2}$

by substitution

$3\frac{1-t^2}{1+t^2} - 2\frac{2t}{1+t^2} = 2$

$3 - 3t^2 - 4t = 2 + 2t^2$

$5t^2 + 4t - 1 = 0$

$(5t - 1)(t + 1) = 0$

Either $(5t - 1) = 0$

$t = \frac{1}{5}$

$\tan\frac{1}{2}\theta = \frac{1}{5}$

$\theta = 2\tan^{-1}\frac{1}{5} = 22.62^0$

Or (t + 1) =0; t = -1

$\tan\frac{1}{2}\theta = -1$

$\theta = 2\tan^{-1}(-1) = 270^0$

$\therefore \{\theta : \theta = 22.62^0, 270.00^0\}$

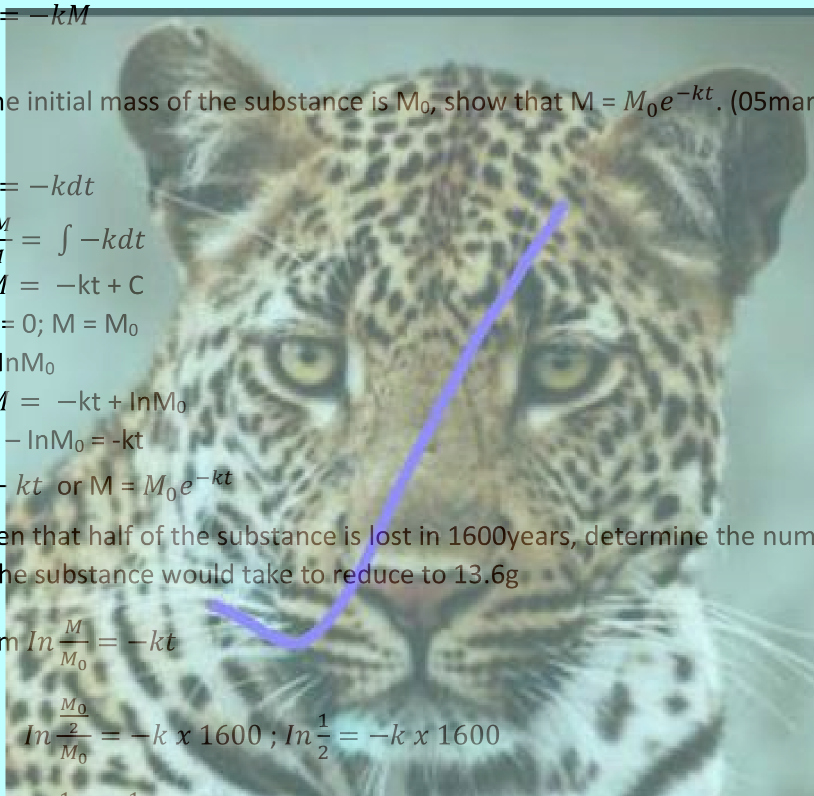
16. A substance loses mass at a rate which is proportional to the amount M present at time t.

(a) Form a differential equation connecting M, t and proportionality constant k. (02marks)

$-\frac{dM}{dt} \propto M$

$-\frac{dM}{dt} = kM$

$\frac{dM}{dt} = -kM$



(b) If the initial mass of the substance is M_0, show that M = M_0e^{-kt}. (05marks)

$\frac{dM}{M} = -kdt$

$\int\frac{dM}{M} = \int -kdt$

$InM = -kt + C$

At t= 0; M = M_0

c = InM_0

⇨ $InM = -kt + InM_0$

$InM - InM_0 = -kt$

$In\frac{M}{M_0} - kt$ or M = M_0e^{-kt}

(c) Given that half of the substance is lost in 1600years, determine the number of years 15g of the substance would take to reduce to 13.6g

From $In\frac{M}{M_0} = -kt$

$In\frac{\frac{M_0}{2}}{M_0} = -k \times 1600$; $In\frac{1}{2} = -k \times 1600$

-k = $\frac{1}{1600}In\frac{1}{2}$

Let the required time be t

$In\frac{13.6}{15} = \frac{1}{1600}In\frac{1}{2}t$

t = 226.17years

UACE MATHEMATICS PAPER 2 2015 guide

SECTION A (40 marks)

Answer all questions in this section

1. The first term of an Arithmetic Progression (A.P) is equal to the first term of a Geometric Progression (G.P) whose common ratio is 1/3 and sum to infinity is 9. If the common difference of he A.P is 2, find the sum of the first ten terms of the A.P. (05marks)
2. Find the equation of a line through the point (5, 3) and perpendicular to the line $2x - y + 4 = 0$ (05marks)
3. Solve for x in : $\log_a(x+3) + \frac{1}{\log_x a} = 2\log_a 2$ (05marks)
4. Given that D(7, 1, 2), E(3, -1, 4) and F(4, -2, 5) are points on a plane, show that ED is perpendicular to EF. (05marks)
5. In a triangle ABC all angles are acute. Angle ABC $= 50^0$, a = 10cm and b = 9cm. Solve the triangle (05marks)
6. Differentiate $e^{-x^2}x^3 sinx$ with respect to x. (05marks)
7. The region enclosed by the curve $y = x^2$, the x-axis and the line x = 2 is rotated through one revolution about x-axis. Find the volume of the solid generated. (05marks)
8. Solve $\frac{dy}{dx} = e^{x+y}$ given that y= 2 when x = 0

SECTION B (60MARKS)

Answer any five questions from this section. All questions carry equal marks

9. (a) Given $f(x) = (x-\alpha)^2 g(x)$, show that $f'(x)$ is divisible by $(x - \alpha)$. (03marks)
   (b) A polynomial $P(x) = x^3 + 4ax^2 + bx + 3$ is divisible by $(x-1)^2$. Use the result in (a) above, to find the values of a and b. Hence solve the equation P(X) = 0 (09 marks)
10. Sketch on the same co-ordinate axes the graphs of the curve $y = 2 + x - x^2$ and $y = x + 1$. Hence determine the area of the region enclosed between the curve and the line. (12marks)
11. (a) Solve $Z\bar{Z} - 5iZ = 5(9-7i)$ where $\bar{Z}$ is the complex conjugate of Z. (06marks)
    (b) (i) Find the Cartesian equation of the curve given as $|Z + 2 - 3i| = 2|Z - 2 + i|$.
    (ii) Show that it represents a circle. Find the centre and radius of the circle. (06marks)
12. (a) Simplify $\frac{cos3\theta + cos5\theta}{sin5\theta - sin3\theta}$ (03marks)
    (b) Show that cot 2θ = $2\frac{1-tan^2\theta}{2tan\theta}$. Hence solve the equation cot 2θ = 4 – tanθ for values of θ between 0^0 and 360^0.

13. Express $\frac{1}{x^2(x-1)}$ as partial fractions. Hence evaluate $\int_2^3 \frac{dx}{x^2(x-1)}$ correct to 3 decimal places. (12 marks)

14. (a) Show that lines $\boldsymbol{a} = \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix} + \alpha \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$ intersect

(06marks)

(b)Find the

(i) point of intersection, P, of the two lines in (a)

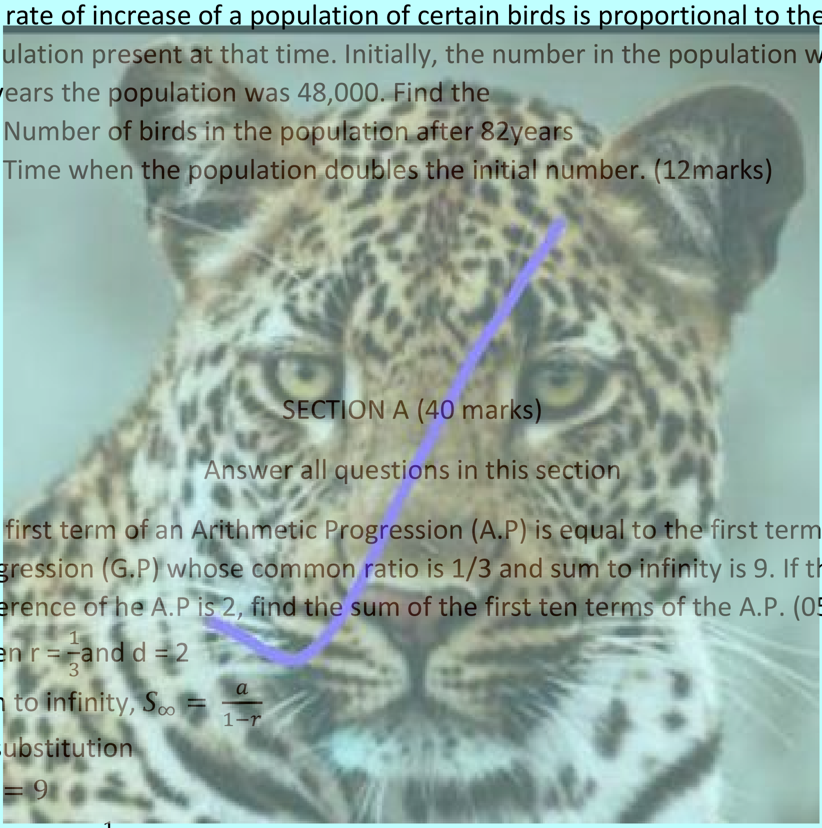
(ii) Cartesian equation of the plane which contains ***a*** and ***b***.

15. The tangents at the points P(cp, $^c/_p$) and Q(cq, $^c/_q$) on the rectangular hyperbola xy = c^2 intersect at R. Given that R lies on the curve xy = $\frac{c^2}{2}$; show that the locus of the mid-point of the PQ is given by xy = $2c^2$. (12marks)

16. The rate of increase of a population of certain birds is proportional to the number in the population present at that time. Initially, the number in the population was32000. After 70 years the population was 48,000. Find the

(a) Number of birds in the population after 82years

(b) Time when the population doubles the initial number. (12marks)



Solutions

SECTION A (40 marks)

Answer all questions in this section

1. The first term of an Arithmetic Progression (A.P) is equal to the first term of a Geometric Progression (G.P) whose common ratio is 1/3 and sum to infinity is 9. If the common difference of he A.P is 2, find the sum of the first ten terms of the A.P. (05marks)

Given r = $\frac{1}{3}$and d = 2

Sum to infinity, $S_\infty = \frac{a}{1-r}$

By substitution

$\frac{a}{1-r} = 9$

$a = 9\,(1 - \frac{1}{3})$ = 6

Using the sum of the first n terms

$$s_n = \frac{n}{2}(2a + (n-1)d\,)$$

$$= \frac{10}{2}[2\,x\,6 + 2(10-1)]$$

= 150

2. Find the equation of a line through the point (5, 3) and perpendicular to the line 2x – y + 4 = 0 (05marks)

Expressing the equation of the line in form y = mx + c

2x – y + 4 = 0

y = 2x + 4

Gradient m_1 = 2

let point (x, y) line on the perpendicular line

Gradient of perpendicular line, $m_2 = \frac{y-3}{x-5}$

But for perpendicular lines, m_1m_2 = -1

⇨ $2\left(\frac{y-3}{x-5}\right) = -1$

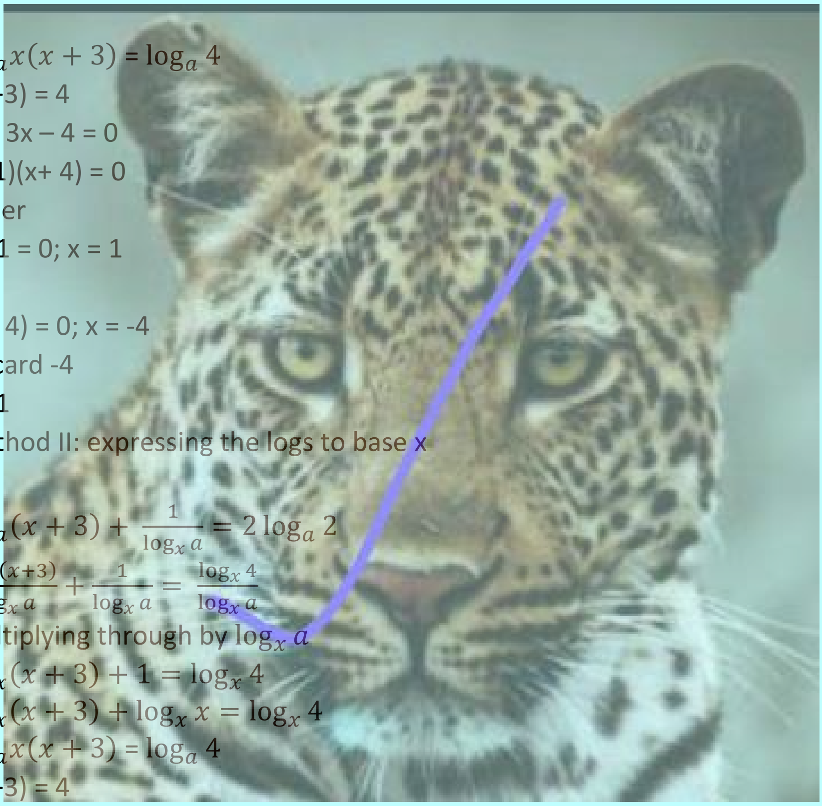
2y + x = 11

3. Solve for x in : $\log_a(x+3) + \frac{1}{\log_x a} = 2\log_a 2$ (05marks)

Method 1: expressing the logs to base a

$\log_a(x+3) + \frac{1}{\log_a a/\log_a x} = \log_a 2^2$

$\log_a(x+3) + \log_a x = \log_a 4$



$\log_a x(x+3) = \log_a 4$

x(x+3) = 4

$x^2 + 3x - 4 = 0$

(x -1)(x+ 4) = 0

Either

x – 1 = 0; x = 1

Or

(x + 4) = 0; x = -4

Discard -4

x = 1

Method II: expressing the logs to base x

$\log_a(x+3) + \frac{1}{\log_x a} = 2\log_a 2$

$\frac{\log_x(x+3)}{\log_x a} + \frac{1}{\log_x a} = \frac{\log_x 4}{\log_x a}$

Multiplying through by $\log_x a$

$\log_x(x+3) + 1 = \log_x 4$

$\log_x(x+3) + \log_x x = \log_x 4$

$\log_a x(x+3) = \log_a 4$

x(x+3) = 4

$x^2 + 3x - 4 = 0$

(x -1)(x+ 4) = 0

Either

x – 1 = 0; x = 1

Or

(x + 4) = 0; x = -4

Discard -4

x = 1

4. Given that D(7, 1, 2), E(3, -1, 4) and F(4, -2, 5) are points on a plane, show that ED is perpendicular to EF. (05marks)

$\overline{ED} = \overline{OD} - \overline{OE}$

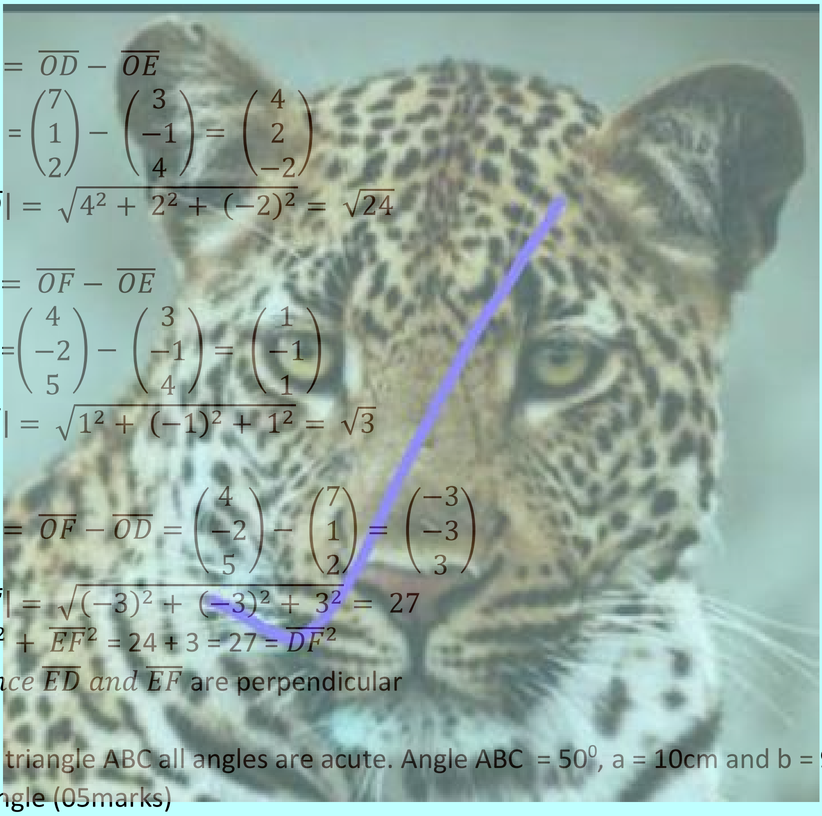
$$= \begin{pmatrix} 7 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}$$

$$\overline{EF} = \overline{OF} - \overline{OE}$$
$$= \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\overline{ED} \,.\, \overline{EF} = \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix} . \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 4 - 2 - 2 = 0$$

Since $\overline{ED} \,.\, \overline{EF} = 0$; $\overline{ED}$ *and* $\overline{EF}$ are perpendicular

Method II



$$\overline{ED} = \overline{OD} - \overline{OE}$$
$$= \begin{pmatrix} 7 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}$$
$$|\overline{ED}| = \sqrt{4^2 + 2^2 + (-2)^2} = \sqrt{24}$$

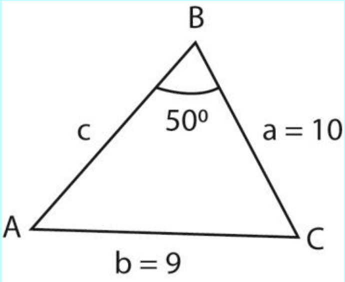
$$\overline{EF} = \overline{OF} - \overline{OE}$$
$$= \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$
$$|\overline{EF}| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$$

$$\overline{DF} = \overline{OF} - \overline{OD} = \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix} - \begin{pmatrix} 7 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \\ 3 \end{pmatrix}$$
$$|\overline{DF}| = \sqrt{(-3)^2 + (-3)^2 + 3^2} = 27$$
$$\overline{ED}^2 + \overline{EF}^2 = 24 + 3 = 27 = \overline{DF}^2$$

Hence $\overline{ED}$ *and* $\overline{EF}$ are perpendicular

5. In a triangle ABC all angles are acute. Angle ABC $= 50^0$, a = 10cm and b = 9cm. Solve the triangle (05marks)

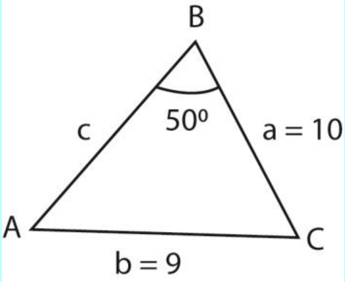
Method I: using sine rule



$$\frac{a}{Sin\ A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$
$$\frac{10}{\sin A} = \frac{9}{\sin 50^0};\ A = 58.34^0$$
$$But\ A + B + C = 180^0$$
$$C = 180^0 - (50^0 + 58.34^0) = 71.66^0$$

$Also, \frac{c}{\sin 71.66^0} = \frac{9}{\sin 50^0}$; c = 11.15cm

Method II: using cosine rule



$9^2 = 10^2 + c^2 - 2 \ x \ 10 \ x \ c \ x \cos 50^0$; c = 11.15cm

$10^2 = 11.15^2 + 9^2 - 2 \ x \ 11.15 \ x \ 9 \ x \cos A^0$; A = 58.34^0

$11.15^2 = 10^2 + 9^2 - 2 \ x \ 10 \ x \ 9 \ x \cos C^0$; C = 71.66^0

6. Differentiate $e^{-x^2}x^3 sinx$ with respect to x. (05marks)

Method I

Let y = $e^{-x^2}x^3 sinx$

Introducing In on both sides

Iny = In$\left(e^{-x^2}x^3 sinx\right)$

Iny = -x^2 + 3Inx + Insinx

Differentiating both sides

$\frac{1}{y}\frac{dy}{dx} = -2x + \frac{3}{x} + \frac{cosx}{sinx}$

$= -2x + \frac{3}{x} + cotx$

$= \frac{2x^2 + 3 + xcotx}{x}$

$\frac{dy}{dx} = \left(\frac{2x^2 + 3 + xcotx}{x}\right)y$

$= \left(\frac{2x^2 + 3 + xcotx}{x}\right)e^{-x^2}x^3 sinx$

$= e^{-x^2}x^2 sinx(-2x^2 + 3 + xcotx)$

Method II

Let p = x^{3}sinx

⇨ y = e^{-x^2}p

$\frac{dy}{dx} = -2xpe^{-x^2} + e^{-x^2}\frac{dp}{dx}$

$\frac{dp}{dx} = 3x^2 sinx + x^3 cosx$

$Substituting \ for \ p \ and \ \frac{dp}{dx}$

$\frac{dy}{dx} = -2xe^{-x^2}(\text{x}^3\text{sinx}) + e^{-x^2}(3x^2 sinx + x^3 cosx)$

$= e^{-x^2}x^2 sinx(-2x^2 + 3 + xcotx)$

Method III

Let y = $e^{-x^2}x^3 sinx$

$e^{x^2}y = x^3 sinx$

$$e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y = 3x^2 sinx + x^3 cosx$$

$$e^{x^2}\frac{dy}{dx} = 3x^2 sinx + x^3 cosx - 2xe^{x^2}y$$

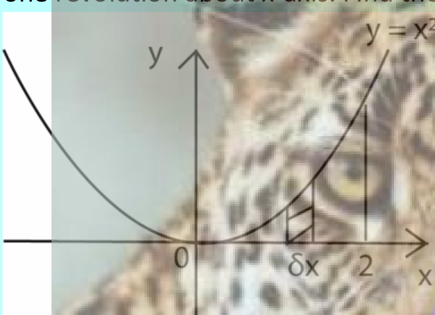
$$\frac{dy}{dx} = e^{-x^2}\left(3x^2 sinx + x^3 cosx - 2xe^{x^2}y\right)$$

$$= e^{-x^2}\left[3x^2 sinx + x^3 cosx - 2xe^{x^2}\left(e^{-x^2}x^3 sinx\right)\right]$$

$$= e^{-x^2}[3x^2 sinx + x^3 cosx - 2x^4 sinx]$$

$$= x^2e^{-x^2}[3sinx + xcosx - 2x^2 sinx]$$

7. The region enclosed by the curve y = x^2, the x-axis and the line x = 2 is rotated through one revolution about x-axis. Find the volume of the solid generated. (05marks)



Volume of element = $\pi y^2 dx$

Volume of solid = $\pi\int_0^2 y^2 dx$

$$= \pi\int_0^2 x^4 dx = \pi\left[\frac{x^5}{5}\right]_0^2 = \pi\left[\frac{32}{5} - \frac{0}{5}\right] = \frac{32\pi}{5} = 20.1062 \text{ unit}^3$$

8. Solve $\frac{dy}{dx} = e^{x+y}$ given that y= 2 when x = 0

$\frac{dy}{dx} = e^{x+y} = e^x.e^y$

By separation of variables

$\int\frac{dy}{e^y} = \int e^x dx$

$-e^{-y} = e^x + c$

Substituting for x = 0 and y= 2

$-e^{-2} = 1 + c$; => c = -1.135

$\therefore -e^{-y} = e^x - 1.135$

SECTION B (60MARKS)

Answer any five questions from this section. All questions carry equal marks

9. (a) Given $f(x) = (x - \alpha)^2 g(x)$, show that $f'(x)$ is divisible by $(x - \alpha)$. (03marks)

$f(x) = (x - \alpha)^2 g(x)$

$f'(x) = 2(x - \alpha)g(x) + (x - \alpha)^2 g'(x)$

$\quad = (x - \alpha)[2g(x)x + (x - \alpha)g'(x)$

Hence $f'(x)$ is divisible by $(x - \alpha)$

Alternatively

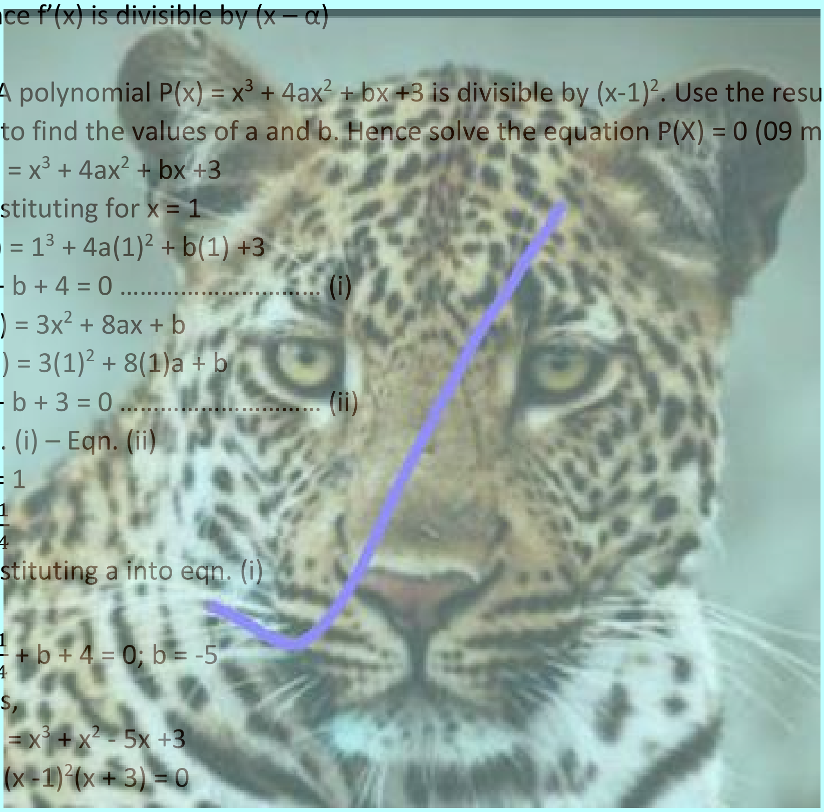
$f'(x) = 2(x - \alpha)g(x) + (x - \alpha)^2 g'(x)$

Substituting $x = \alpha$

$f'(x) = 2(x - \alpha)g(x) + (x - \alpha)^2 g'(x)$

$f'(\alpha) = (\alpha - \alpha)[2g(\alpha) + (\alpha - \alpha)g'(\alpha) = 0$

Hence $f'(x)$ is divisible by $(x - \alpha)$



(b) A polynomial $P(x) = x^3 + 4ax^2 + bx + 3$ is divisible by $(x-1)^2$. Use the result in (a) above, to find the values of a and b. Hence solve the equation $P(X) = 0$ (09 marks)

$P(x) = x^3 + 4ax^2 + bx + 3$

Substituting for $x = 1$

$P(1) = 1^3 + 4a(1)^2 + b(1) + 3$

$4a + b + 4 = 0$ ……………………… (i)

$P'(x) = 3x^2 + 8ax + b$

$P'(1) = 3(1)^2 + 8(1)a + b$

$8a + b + 3 = 0$ ……………………… (ii)

Eqn. (i) – Eqn. (ii)

$4a = 1$

$a = \frac{1}{4}$

Substituting a into eqn. (i)

$4 \times \frac{1}{4} + b + 4 = 0;\ b = -5$

Thus,

$P(x) = x^3 + x^2 - 5x + 3$

⇨ $(x - 1)^2(x + 3) = 0$

Either $(x-1) = 0;\ x = 1$

Or $(x + 3) = 0;\ x = -3$

Hence $x = 1$ and $x = -3$

10. Sketch on the same co-ordinate axes the graphs of the curve $y = 2 + x - x^2$ and $y = x + 1$. Hence determine the area of the region enclosed between the curve and the line. (12marks)

$y = 2 + x - x^2$

Finding intercepts

When $x = 0$; $y = 2$; $(x, y) = (0, 2)$

When $y = 0$

$2 + x - x^2 = 0$

(x + 1)(x – 2) = 0

Either (x + 1) = 0; x = -1; (x, y) = (-1, 0)

Or (x – 2) = 0; x = 2; (x, y) = (2, 0)

Finding turning points

$\frac{dy}{dx} = 1 - 2x = 0$

$x = \frac{1}{2}$

When x = $\frac{1}{2}$; y = $\frac{9}{4}$

x	L		R
Sign $\frac{dy}{dx}$	+	0	-

The curve is maximum at $\left(\frac{1}{2}, \frac{9}{4}\right)$

Finding the intersection of curve and line

x + 1 = 2 + x – x^2

x^2 = 1

x = ± 1

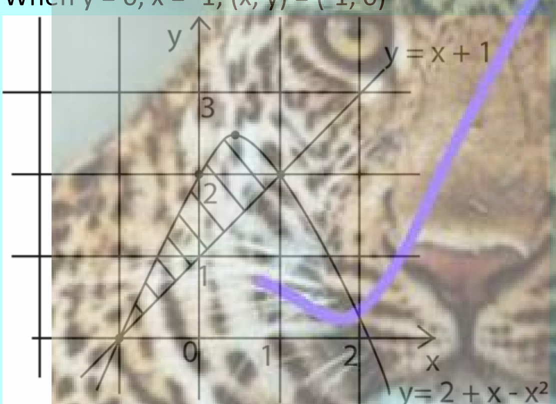
When x = 1, y = 2; (x, y) =(1, 2)

When x = -1, y = 0; (x, y) = (-1, 0)

Finding the points where the line y = x + 1 meets the axes

When x = 0, y = 1; (x, y) = (0, 1)

When y = 0, x = -1; (x, y) = (-1, 0)



y = (2 + x –x^2) – (x + 1) = 1 – x^2

A = $\int_{-1}^{1} 1 - x^2 dx$

= $\left[x + \frac{x^3}{3}\right]_{-1}^{1} = \left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right)$ = 2 - $\frac{2}{3}$ = $\frac{4}{3}$

11. (a) Solve $Z\bar{Z} - 5iZ = 5(9 - 7i)$ where $\bar{Z}$ is the complex conjugate of Z. (06marks)

$Z\bar{Z} - 5iZ = 5(9 - 7i)$

Let Z = x + iy and $\bar{Z} = x - iy$

By substitution

(x + iy)(x - iy) – 5i(x + iy) = 5(9-7i)

$x^2 + y^2$ -5xi + 5y = 45 – 35i

$x^2 + y^2$ + 5y - 5xi = 45 – 35i

Equating real parts

$x^2 + y^2$ + 5x = 45

Equating imaginary parts

-5x = -35

x = 7

Substituting for x into eqn. (i)

49 + y^2 + 5y = 45

y^2 + 5y + 4 = 0

(y + 4)(y +1) = 0

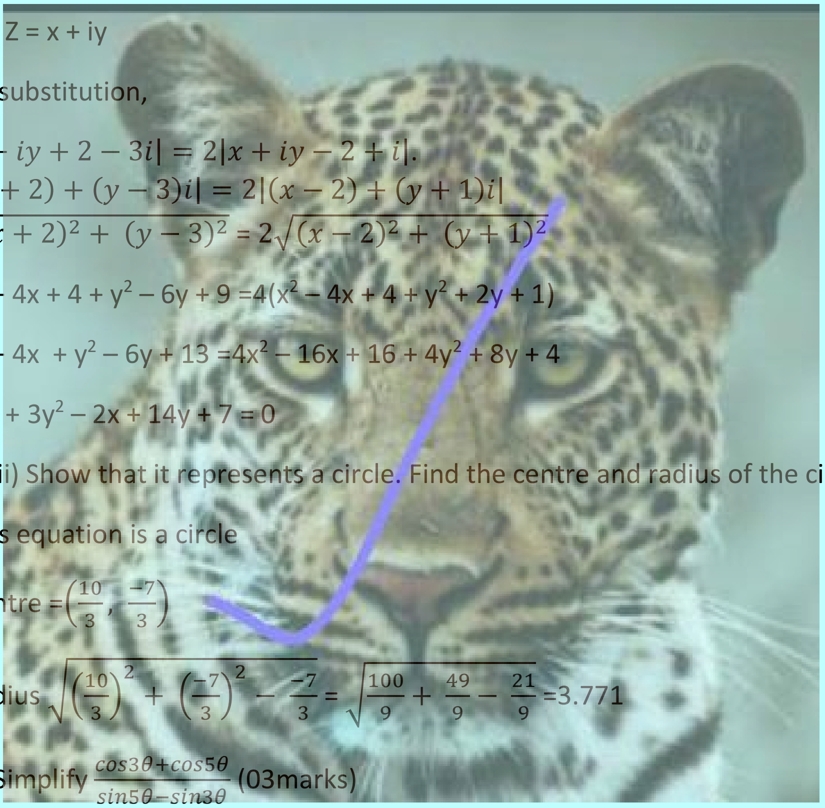
Either y + 4 = 0; y = -4

Or y +1= 0; y = -1

Hence Z = 7 – 4i or Z = 7 – i

(b) (i) Find the Cartesian equation of the curve given as $|Z + 2 - 3i| = 2|Z - 2 + i|$.

$|Z + 2 - 3i| = 2|Z - 2 + i|$.



Let Z = x + iy

By substitution,

$|x + iy + 2 - 3i| = 2|x + iy - 2 + i|$.

$|(x + 2) + (y - 3)i| = 2|(x - 2) + (y + 1)i|$

$\sqrt{(x + 2)^2 + (y - 3)^2} = 2\sqrt{(x - 2)^2 + (y + 1)^2}$

$x^2 + 4x + 4 + y^2 - 6y + 9 = 4(x^2 - 4x + 4 + y^2 + 2y + 1)$

$x^2 + 4x + y^2 - 6y + 13 = 4x^2 - 16x + 16 + 4y^2 + 8y + 4$

$3x^2 + 3y^2 - 2x + 14y + 7 = 0$

(ii) Show that it represents a circle. Find the centre and radius of the circle. (06marks)

This equation is a circle

Centre $= \left(\frac{10}{3}, \frac{-7}{3}\right)$

Radius $\sqrt{\left(\frac{10}{3}\right)^2 + \left(\frac{-7}{3}\right)^2 - \frac{-7}{3}} = \sqrt{\frac{100}{9} + \frac{49}{9} - \frac{21}{9}}$ =3.771

12. (a) Simplify $\frac{cos3\theta + cos5\theta}{sin5\theta - sin3\theta}$ (03marks)

$$\frac{cos3\theta + cos5\theta}{sin5\theta - sin3\theta} = \frac{2\cos 4\theta \cos\theta}{2\cos 4\theta \sin\theta} = \frac{\cos\theta}{\sin\theta} = cot\theta$$

(b) Show that cot 2θ = $\frac{1 - tan^2\theta}{2tan\theta}$. Hence solve the equation cot 2θ = 4 – tanθ for values of θ between 0^0 and 360^0.

$$\cot 2\theta = \frac{\cos 2\theta}{\sin 2\theta} = \frac{cos^2\theta - sin^2\theta}{2sin\theta cos\theta}$$

Dividing the numerator and denominator each by $\cos^2\theta$

$$\cot 2\theta = \frac{\frac{cos^2\theta}{cos^2\theta} - \frac{sin^2\theta}{cos^2\theta}}{\frac{2sin\theta cos\theta}{cos^2\theta}} = \frac{1 - \frac{sin^2\theta}{cos^2\theta}}{\frac{2\sin\theta}{\cos\theta}} = \frac{1 - tan^2\theta}{2\tan\theta}$$

Alternatively

$$\cot 2\theta = \frac{1}{\tan 2\theta} = \frac{1}{2\tan\theta / 1 - tan^2\theta} = \frac{1 - tan^2\theta}{2tan\theta}$$

Or considering the R. H.S

$$\frac{1-tan^2\theta}{2tan\theta} = \frac{1-\frac{sin^2\theta}{cos^2\theta}}{\frac{2\sin\theta}{\cos\theta}} == \frac{cos^2\theta - sin^2\theta}{2sin\theta cos\theta} = \frac{\cos 2\theta}{\sin 2\theta} = \cot 2\theta$$

Alternatively

$$\frac{1-tan^2\theta}{2tan\theta} = \frac{1}{\frac{1-tan^2\theta}{2tan\theta}} = \frac{1}{\tan 2\theta} = \cot 2\theta$$

Solving the equation

$\cot 2\theta = 4 - \tan\theta$

$\frac{1-tan^2\theta}{2tan\theta} = 4 - \tan\theta$

$1 - tan^2\theta = 8\tan\theta - 2tan^2\theta$

$tan^2\theta - 8\tan\theta + 1 = 0$

$\tan\theta = \frac{8 \pm\sqrt{8^2-4(1)(1)}}{2(1)} = \frac{8\pm\sqrt{64-4}}{2}$

= 7.873, 0.127

Taking tanθ =7.873

θ = $\tan^{-1}(7.873)$ = 82.76^0, 262.76^0

Taking tanθ =0.127

θ = $\tan^{-1}(0.127)$ = 7.24^0, 187.24^0

13. Express $\frac{1}{x^2(x-1)}$ as partial fractions. Hence evaluate $\int_2^3 \frac{dx}{x^2(x-1)}$ correct to 3 decimal places. (12 marks)

$\frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$

1 = Ax(x – 1) + B(x -1) + cx^2

Substituting for x = 0

1 = –B, hence B = -1

Substituting for x = 1

c = 1

substituting for x = -1

1 = 2A – 2B + C

1 = 2A + 2 + 1

A = -1

⇨ $\frac{1}{x^2(x-1)} = -\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-1}$

Alternatively

$\frac{1}{x^2(x-1)} = \frac{Ax+B}{x^2} + \frac{C}{x-1}$

Substituting for x = 0

1 = –B, hence B = -1

Substituting for x = 1

c = 1

substituting for x = -1

1 = 2A – 2B + C

1 = 2A + 2 + 1

A = -1

$\Rightarrow \quad \frac{1}{x^2(x-1)} = -\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-1}$

Now $\int_2^3 \frac{1}{x^2(x-1)} = \int_2^3 \left(-\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-1}\right) dx$

$= \left[-Inx + \frac{1}{x} + In(x-1)\right]_2^3$

$= \left[In\left(\frac{x-1}{x}\right) + \frac{1}{x}\right]_2^3$

$= \left(In\,\frac{2}{3} + \frac{1}{3}\right) - \left(In\,\frac{1}{2} + \frac{1}{2}\right)$

$= In\frac{4}{3} - \frac{1}{6}$

$= 0.12102$

14. (a) Show that lines $\boldsymbol{a} = \begin{pmatrix}3\\-4\\2\end{pmatrix} + \alpha\begin{pmatrix}-1\\1\\1\end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix}5\\0\\-2\end{pmatrix} + \beta\begin{pmatrix}-1\\-1\\2\end{pmatrix}$ intersect

(06marks)

For intersection to occur, **a = b**

$$\begin{pmatrix}3\\-4\\2\end{pmatrix} + \alpha\begin{pmatrix}-1\\1\\1\end{pmatrix} = \begin{pmatrix}5\\0\\-2\end{pmatrix} + \beta\begin{pmatrix}-1\\-1\\2\end{pmatrix}$$

3−α = 5 – β .... (i)

−4 + α = 0 – β .... (ii)

2 + 2α = -2 + 2β .... (iii)

Eqn. (i) and (ii)

-2β = -6; β = 3

3 – α = 5-3; α = 1

Substituting for α and β into equation (iii)

2 + 2x 1 = -2 + 2 x 3 =4

Since L.H.S = R.H.S.; the lines intersect

(b)Find the

(i) point of intersection, P, of the two lines in (a)

The parallel vector, n is given by

$n = \begin{pmatrix}-1\\1\\2\end{pmatrix} x \begin{pmatrix}-1\\-1\\2\end{pmatrix} = \begin{vmatrix} i & j & k\\ -1 & 1 & 2\\ -1 & -1 & 2\end{vmatrix}$

$= \big(2-(-2)\big)i - (-2+2)j + (1+1)k$

=4i +0j + 2k

Let R(x, y, z) be a common point in the plane

$$\begin{pmatrix}4\\0\\2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}4\\0\\2\end{pmatrix}\begin{pmatrix}2\\-3\\4\end{pmatrix}$$

$4x + 2z = 8 + 8$

2x + z = 8

Alternatively

The general equation for the plane is given by

r = a + λb + μc

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$

$x = 2 - \lambda - \mu$ ……………… (i)

$y = -3 + \lambda + \mu$ ………..(ii)

$z = 4 + 2\lambda + 2\mu$ …………. (iii)

Eqn. (i) and (ii)

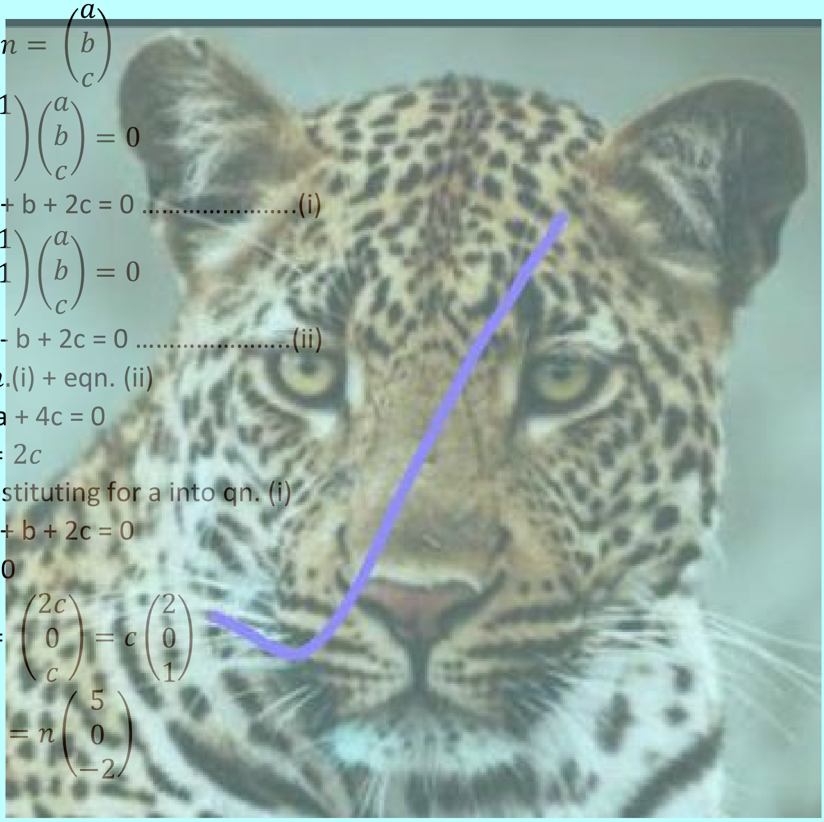
x + y = -1 - 2μ

μ = $\frac{-1-x-y}{2}$

2eqn. (i) + eqn. (iii)

2x + z = 8

Alternatively



Let $n = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$

−a + b + 2c = 0 …………………..(i)

$\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$

−a - b + 2c = 0 …………………..(ii)

eqn.(i) + eqn. (ii)

−2a + 4c = 0

$a = 2c$

Substituting for a into qn. (i)

-2c + b + 2c = 0

b = 0

$n = \begin{pmatrix} 2c \\ 0 \\ c \end{pmatrix} = c \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

$n.r = n \begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix}$

$\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix}$

$2x + z = 10 - 2$

2x + z = 8

(ii) Cartesian equation of the plane which contains ***a*** and ***b***.

15. The tangents at the points P(cp, $^{c}/_{p}$) and Q(cq, $^{c}/_{q}$) on the rectangular hyperbola xy = c^2 intersect at R. Given that R lies on the curve xy = $\frac{c^2}{2}$; show that the locus of the mid-point of the PQ is given by xy = $2c^2$. (12marks)

x= cp, y = $^{c}/_{p}$

$\frac{dx}{dp} = c \text{ and } \frac{dy}{dp} = \frac{-c}{p^2}$

$\frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx} = \frac{-c}{p^2} \cdot \frac{1}{c} = -\frac{1}{p^2}$

Finding the equation of the gradient at P

$$\frac{y-{}^{c}/_{p}}{x-cp} = -\frac{1}{p^2}$$

$$p^2y - cp = -x + cp$$

$p^2y + x - 2cp = 0$ ……… (i)

$$Similarly\ the\ equation\ of\ the\ tangent\ at\ Q$$

$q^2y + x - 2cq = 0$ ……… (ii)

$$Eqn.(i) - eqn.(ii)$$

$$(p^2 - q^2)y = 2c(p-q)$$

$$y = \frac{2c(p-q)}{(p^2-q^2)} = \frac{2c(p-q)}{(p+q)(p-q)} = \frac{2c}{(p+q)}$$

$substituting\ y\ into\ equn.$ (ii)

$$q^2\left(\frac{2c}{(p+q)}\right) + x - 2cq = 0$$

x = $2cq - \frac{2cq^2}{(p+q)} = \frac{2cpq+2cq^2-2cq^2}{(p+q)} = \frac{2cpq}{(p+q)}$

Hence R$\left(\frac{2cpq}{(p+q)}, \frac{2c}{(p+q)}\right)$

Since R lies on xy = $\frac{c^2}{2}$, substitute for the values of x and y

$$\left(\frac{2cpq}{(p+q)}\right)\left(\frac{2c}{(p+q)}\right) = \frac{c^2}{2}$$

$$\frac{(p+q)^2}{pq} = 8$$

$$Finding\ the\ mid-point\ of\ PQ$$

x = $\frac{cp+cq}{2} = \frac{c(p+q)}{2}$

y = $\frac{{}^{c}/_{p}+{}^{c}/_{q}}{2} = \frac{c(p+q)}{2pq}$

xy = $\frac{c(p+q)}{2}.\frac{c(p+q)}{2pq} = \frac{c^2}{4}.\frac{(p+q)^2}{pq} = \frac{c^2}{4}\ x\ 8 = 2c^2$

16. The rate of increase of a population of certain birds is proportional to the number in the population present at that time. Initially, the number in the population was32000. After 70 years the population was 48,000. Find the
    (a) Number of birds in the population after 82years
        Let x be the number of birds at any time t
        $\frac{dx}{dt} \propto x$
        $\frac{dx}{dt} = kx$

        Separating variables

$\frac{1}{x}dx = kdt$

$\int \frac{1}{x}dx = \int kdt$

$Inx = kt + c$

At t = 0; x = 32,000

c = In32,000

⇨ $Inx = kt + \text{In}32{,}000$

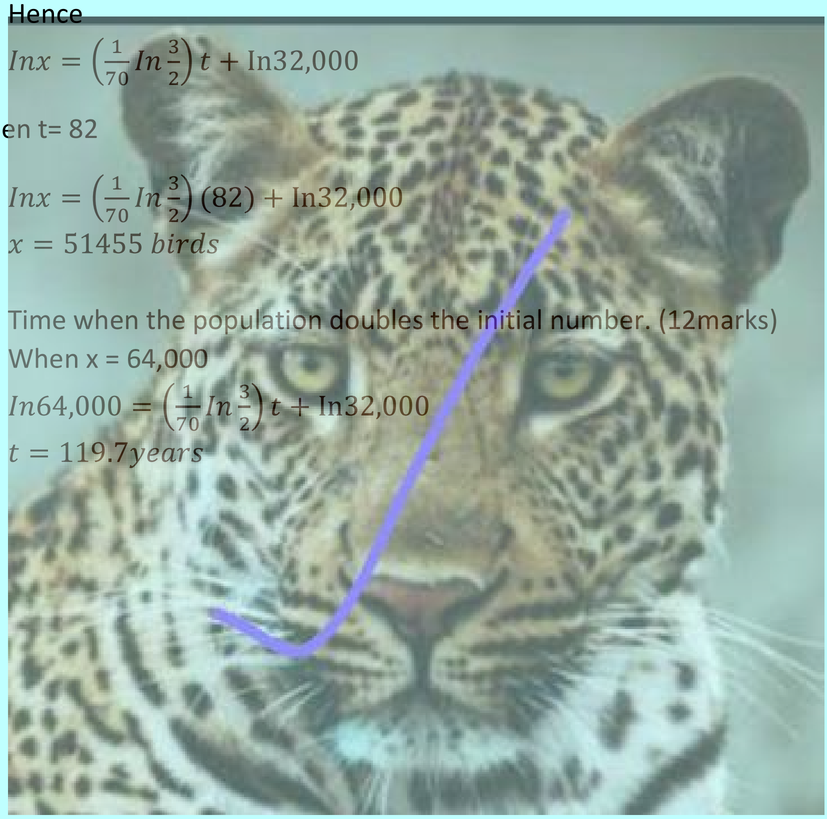
When t = 70, x = 48,000

In48,000 = 70k + In32,000

k = $\frac{1}{70} In \frac{3}{2}$

Hence

⇨ $Inx = \left(\frac{1}{70} In \frac{3}{2}\right) t + \text{In}32{,}000$



When t= 82

⇨ $Inx = \left(\frac{1}{70} In \frac{3}{2}\right)(82) + \text{In}32{,}000$

$x = 51455\ birds$

(b) Time when the population doubles the initial number. (12marks)

When x = 64,000

⇨ $In64{,}000 = \left(\frac{1}{70} In \frac{3}{2}\right) t + \text{In}32{,}000$

$t = 119.7 years$

UACE MATHEMATICS PAPER 1 2016 guide

SECTION A (40 marks)

Answer all questions in this section

1. Without using mathematical tables or calculators, find the value of

$$\frac{(\sqrt{5}+2)^2-(\sqrt{5}-2)^2}{8\sqrt{5}}$$

2. Find the angle between the lines 2x –y = 3 and 11x + 2y = 13
3. Evaluate $\int_{\frac{1}{2}}^{1} 10x\sqrt{(1-x^2)}\,dx$
4. Solve the equation $\frac{dy}{dx} = 1 + y^2$ given that when y = 1 when x = 0.
5. Given that $2x^2 + 7x - 4$, $x^2 + 3x - 4$ and $7x^2 + ax - 8$ have a common factor, find the
   (a) Factors of $2x^2 + 7x - 4$ and $x^2 + 3x - 4$
   (b) Value of a in $7x^2 + ax - 8$.
6. Solve the equation sin2θ + cos2θcos4θ = cos4θcos6θ for $0 \leq \theta \leq \frac{\pi}{4}$.
7. Using small changes; show $(244)^{\frac{1}{5}} = 3\frac{1}{405}$.
8. Three points A(2, -1, 0), B(-2, 5, -4) and C are on a straight line such that 3AB = 2AC. Find the coordinates of C.

SECTION B (60MARKS)

Answer any five questions from this section. All questions carry equal marks

9. (a) If $z_1 = \frac{2i}{1+3i}$ and $z_2 = \frac{3+2i}{5}$, find $|z_1 - z_2|$
   (b) Given the complex number z = x + iy
      (i) Find $\frac{z+i}{z+2}$
      (ii) Show that the locus of $\frac{z+i}{z+2}$ is a straight line when its imaginary part is zero. State the gradient of the line.
10. (a) solve the equation $\cos 2x = 4\cos^2 x - 2\sin^2 x$ for $0 \leq \theta \leq 180^0$
   (b) Show that if sin(x + α) = psin(x – α) then tan x = $\left(\frac{p+1}{p-1}\right)$tanα. Hence solve the equation sin(x + α) = psin(x – α) for p = 2 and α= 20^0.
11. Given that $x = \frac{t^2}{1+t^3}$ and $y = \frac{t^3}{1+t^3}$, find $\frac{d^2y}{dx^2}$.
12. (a) Line A is the intersection of two planes whose equations are
   3x –y + z = 2 and x + 5y +2z = 6. Find the equation of the line.

(b) Given that line B is perpendicular to the plane 3x –y + z = 2and passes through the point C(1, 1, 0), find the

(i) Cartesian equation of line B

(ii) angle between line B and line A in (a) above

13. (a) Find $\int \frac{1+\sqrt{x}}{2\sqrt{x}}$dx

(b) The gradient of the tangent at any point on a curve is x - $\frac{2y}{x}$. The curve passes through the point (2, 4), find the equation of the curve.

14. (a) The point P($at_1^2, 2at_1$) and Q($at_2^2, 2at_2$) are on parabola y^2 = 4ax. OP is perpendicular to OQ, where O is the origin. Show that t_1t_2 + 4 = 0.

(b) The normal to the rectangular hyperbola xy = 8 at point (4, 2) meets the asymptotes at M and N. find the length MN.

15. (a) Prove by induction

1.3 + 2.4 + … + n(n + 2) = $\frac{1}{6}n(n+1)(2n+7)$ for all values of n.

(b) A man deposits Shs. 150,000 at the beginning of every year in a micro finance bank with the understanding that at the end of the seven years he is paid back his money with 5% per annum compound interest. How much does he receive?

16. (a) If $x^2 + 3y^2$ = k, where k is constant, find $\frac{dy}{dx}$at the point (1, 2).

(c) A rectangular field of area 7200m2 is to be fenced using a wire mesh. On one side of the field. Is a straight river. This side is not to be fenced. Find the dimension of the field hat will minimize the amount of wire mesh to be used.

End

Solution

SECTION A (40 marks)

Answer all questions in this section

1. Without using mathematical tables or calculators, find the value of

$\frac{(\sqrt{5}+2)^2-(\sqrt{5}-2)^2}{8\sqrt{5}} = \frac{[(\sqrt{5}+2)+(\sqrt{5}-2)][(\sqrt{5}+2)-(\sqrt{5}-2)]}{8\sqrt{5}} = \frac{(2\sqrt{5})(4)}{8\sqrt{5}} = \frac{8\sqrt{5}}{8\sqrt{5}} = 1$

Or

$\frac{(\sqrt{5}+2)^2-(\sqrt{5}-2)^2}{8\sqrt{5}} = \frac{(5+4\sqrt{5}+4)-(5-4\sqrt{5}+4)}{8\sqrt{2}} = \frac{5+4\sqrt{5}+4-5+4\sqrt{5}-4}{8\sqrt{5}} = \frac{8\sqrt{5}}{8\sqrt{5}} = 1$

2. Find the angle between the lines 2x –y = 3 and 11x +2 y = 13

tanθ = $\frac{m_1 - m_2}{1+m_1m_2}$

For 11x + y = 13; 2y = -11x + 13; y = $\frac{-11}{2}$x + $\frac{13}{2}$ => m_1 = $\frac{-11}{2}$ = -5.5

For 2x –y = 3; y = 2x + 3; => m_2 = 2

tanθ = $\frac{-5.5-2}{1+(-5.5\ x\ 2)} = \frac{-7.5}{-10}$

θ = $\tan^{-1}\left(\frac{7.5}{10}\right)$ = 36.87^0

Alternatively

$n_1 = \binom{2}{-1}, n_2 = \binom{11}{2}$

$n_1 n_2 = \cos\theta |n_1||n_2|$

$n_1 n_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} . \begin{pmatrix} 11 \\ 2 \end{pmatrix}$ = 20

$n_1 = \sqrt{2^2 + (-1)^2} = \sqrt{5}$

$n_2 = \sqrt{11^2 + 2^2} = \sqrt{125}$

$20 = \cos\theta(\sqrt{5})(\sqrt{125})$

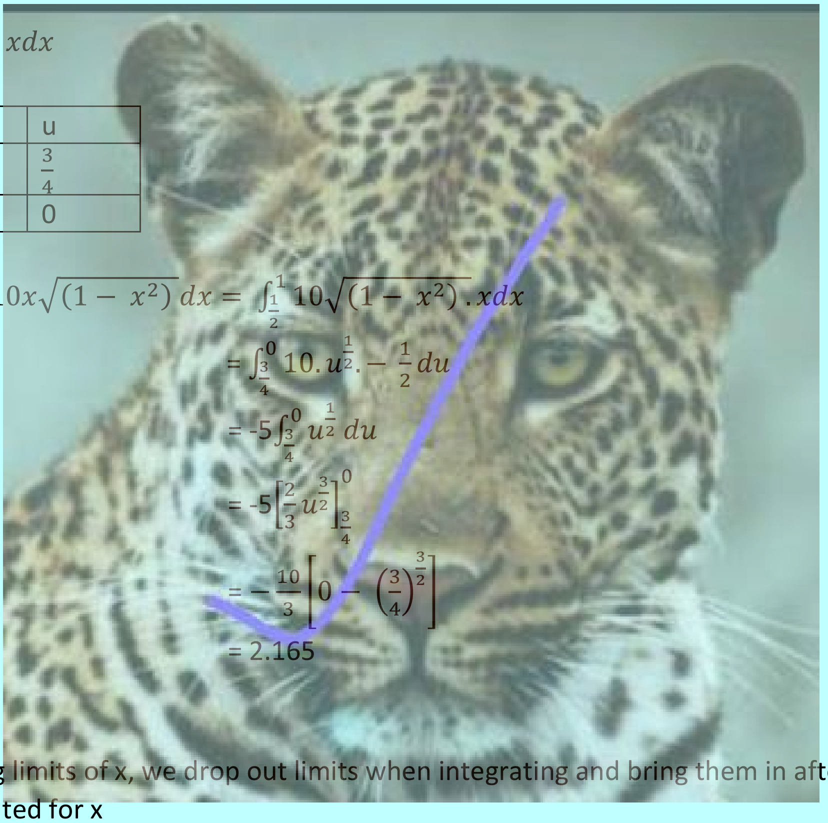
$\cos\theta = \frac{20}{\sqrt{125}}$

$\theta = \cos^{-1} 0.8 = 36.87^0$

3. Evaluate $\int_{\frac{1}{2}}^{1} 10x\sqrt{(1 - x^2)}\, dx$

$\int_{\frac{1}{2}}^{1} 10x\sqrt{(1 - x^2)}\, dx$

Let u = 1- x^2; du = -2xdx



$-\frac{1}{2} du = xdx$

x	u
$\frac{1}{2}$	$\frac{3}{4}$
1	0

⇨ $\int_{\frac{1}{2}}^{1} 10x\sqrt{(1 - x^2)}\, dx = \int_{\frac{1}{2}}^{1} 10\sqrt{(1 - x^2)}\,.xdx$

$= \int_{\frac{3}{4}}^{0} 10.u^{\frac{1}{2}}. -\frac{1}{2} du$

$= -5\int_{\frac{3}{4}}^{0} u^{\frac{1}{2}}\, du$

$= -5\left[\frac{2}{3} u^{\frac{3}{2}}\right]_{\frac{3}{4}}^{0}$

$= -\frac{10}{3}\left[0 - \left(\frac{3}{4}\right)^{\frac{3}{2}}\right]$

= 2.165

Or

By using limits of x, we drop out limits when integrating and bring them in after u has been substituted for x

⇨ $\int 10x\sqrt{(1 - x^2)}\, dx = \int 10\sqrt{(1 - x^2)}\,.xdx$

$= \int 10.u^{\frac{1}{2}}. -\frac{1}{2} du$

$= -5\int u^{\frac{1}{2}}\, du$

$= -5\left(\frac{2}{3} u^{\frac{3}{2}}\right) + c$

$= -\frac{10}{3}(1 - x^2)^{\frac{3}{2}} + c$

Now bringing in limits

$\int_{\frac{1}{2}}^{1} 10x\sqrt{(1 - x^2)}\, dx = -\frac{10}{3}\left[(1 - x^2)^{\frac{3}{2}}\right]_{\frac{1}{2}}^{1}$

$$= -\frac{10}{3}\left[0 - \left(\frac{3}{4}\right)^{\frac{3}{2}}\right]$$

$$= -\frac{10}{3} \cdot - \left(\frac{3}{4}\right)^{\frac{3}{2}}$$

$= 2.165$

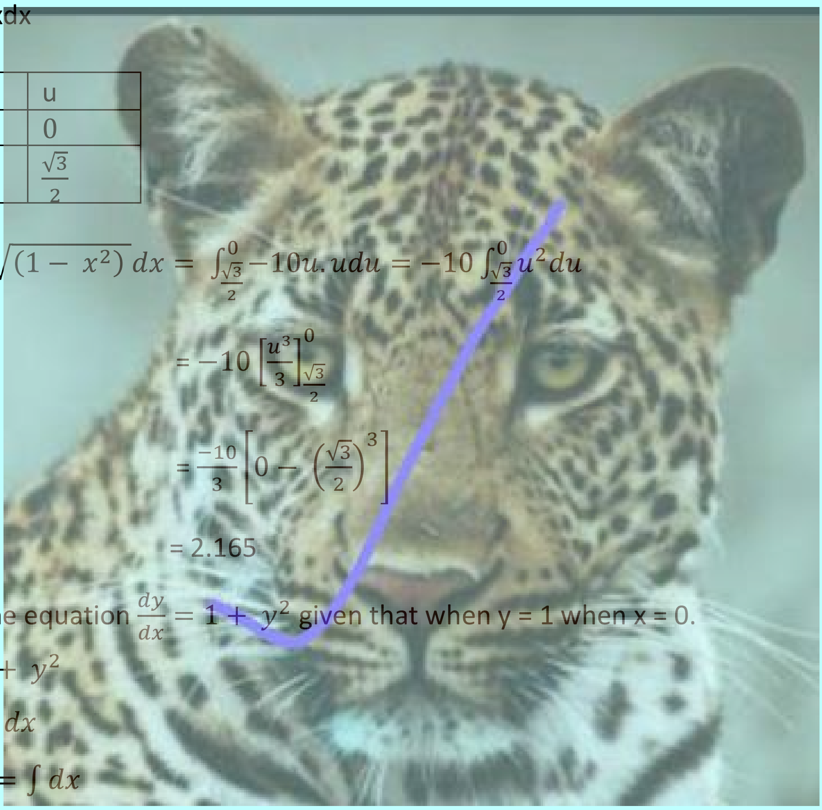
Alternatively

Let $u = \sqrt{1 - x^2}$

$u^2 = 1 - x^2$

$2udu = -2xdx$

$-udu = xdx$



x	u
1	0
$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$

$$\int_{\frac{1}{2}}^{1} 10x\sqrt{(1 - x^2)}\, dx = \int_{\frac{\sqrt{3}}{2}}^{0} -10u.udu = -10\int_{\frac{\sqrt{3}}{2}}^{0} u^2 du$$

$$= -10\left[\frac{u^3}{3}\right]_{\frac{\sqrt{3}}{2}}^{0}$$

$$= \frac{-10}{3}\left[0 - \left(\frac{\sqrt{3}}{2}\right)^3\right]$$

$= 2.165$

4. Solve the equation $\frac{dy}{dx} = 1 + y^2$ given that when y = 1 when x = 0.

$\frac{dy}{dx} = 1 + y^2$

$\frac{dy}{1+y^2} = dx$

$\int \frac{dy}{1+y^2} = \int dx$

$\tan^{-1}(y) = x + C$

Substituting y = 1 when x = 0

$\tan^{-1}(1) = C$

C = $\frac{\pi}{4}$

⇨ $\tan^{-1}(y) = x + \frac{\pi}{4}$

y = tan $\left(x + \frac{\pi}{4}\right)$

5. Given that $2x^2 + 7x - 4$, $x^2 + 3x - 4$ and $7x^2 + ax - 8$ have a common factor, find the
   (a) Factors of $2x^2 + 7x - 4$ and $x^2 + 3x - 4$

$2x^2 + 7x - 4 = 2x^2 + 8x - x - 4$

$= 2x(x + 4) - 1(x + 4)$

$= (2x - 1)(x + 4)$

Hence the factors of $2x^2 + 7x – 4$ are $(2x -1)$ and $(x+4)$

$x^2 + 3x – 4 = x(x + 4) - 1(x + 4)$

$= (x – 1)(x + 4)$

Hence the factors $x^2 + 3x – 4$ are $(x – 1)$ and $(x + 4)$

$\therefore$ the common factor of $2x^2 + 7x – 4$ and $x^2 + 3x – 4$ is $(x + 4)$

(b) Value of a in $7x^2 + ax – 8$.

Since $(x + 4)$ is the common factor of $2x^2 + 7x – 4$ and $x^2 + 3x – 4$; it implies that it a factor of $7x^2 + ax – 8$

Substituting for $x = -4$ in the equation $7x^2 + ax – 8$

$7(-4)^2 - 4a – 8 = 0$

$7 \times 16 – 4a – 8$

$112 – 8 – 4a = 0$

$a = 26$

6. Solve the equation $\sin 2\theta + \cos 2\theta \cos 4\theta = \cos 4\theta \cos 6\theta$ for $0 \leq \theta \leq \frac{\pi}{4}$.

$\sin 2\theta + \cos 2\theta \cos 4\theta = \cos 4\theta \cos 6\theta$

$\sin 2\theta = \cos 4\theta \cos 6\theta - \cos 2\theta \cos 4\theta$

$= \cos 4\theta(\cos 6\theta – \cos 2\theta)$

But $\cos P – \cos Q = -2\sin\frac{(P+Q)}{2}\sin\frac{(P-Q)}{2}$

$\Rightarrow$ $\sin 2\theta = -2\cos 4\theta\left[\sin\frac{(6\theta+2\theta)}{2}\sin\frac{(6\theta-2\theta)}{2}\right]$

$\sin 2\theta = -2\cos 4\theta \sin 4\theta \sin 2\theta$

$\sin 2\theta + 2\cos 4\theta \sin 4\theta \sin 2\theta = 0$

$\sin 2\theta(1 + 2\cos 4\theta \sin 4\theta) = 0$

$\sin 2\theta(1 + \sin 8\theta) = 0$

either

$\sin 2\theta = 0$

$2\theta = \sin^{-1}(0) = 0, \pi, 2\pi$

$\theta = 0, \frac{\pi}{2}, \pi$

or

$(1 + \sin 8\theta) = 0$

$\sin 8\theta = -1$

$8\theta = \sin^{-1}(-1) = 270^0$ or $\frac{3\pi}{2}$

$\theta = \frac{3\pi}{16}$

Hence $\theta = 0, \frac{3\pi}{16}$

7. Using small changes; show $(244)^{\frac{1}{5}} = 3\frac{1}{405}$.

Let $y = x^{\frac{1}{5}}$

$\frac{dy}{dx} = \frac{1}{5}x^{-\frac{4}{5}}$

$= \frac{1}{5x^{\frac{4}{5}}}$

$= \frac{1}{5\left(x^{\frac{1}{5}}\right)^4}$

$\delta y = \frac{1}{5\left(x^{\frac{1}{5}}\right)^4}\delta x$

Taking x = 243 and $\delta x = 1$

$\delta y = \frac{1}{5\left(243^{\frac{1}{5}}\right)^4} . 1 = \frac{1}{405}$

$(x + \delta x) = y + \delta y$b

$= \sqrt[5]{243} + \frac{1}{405}$

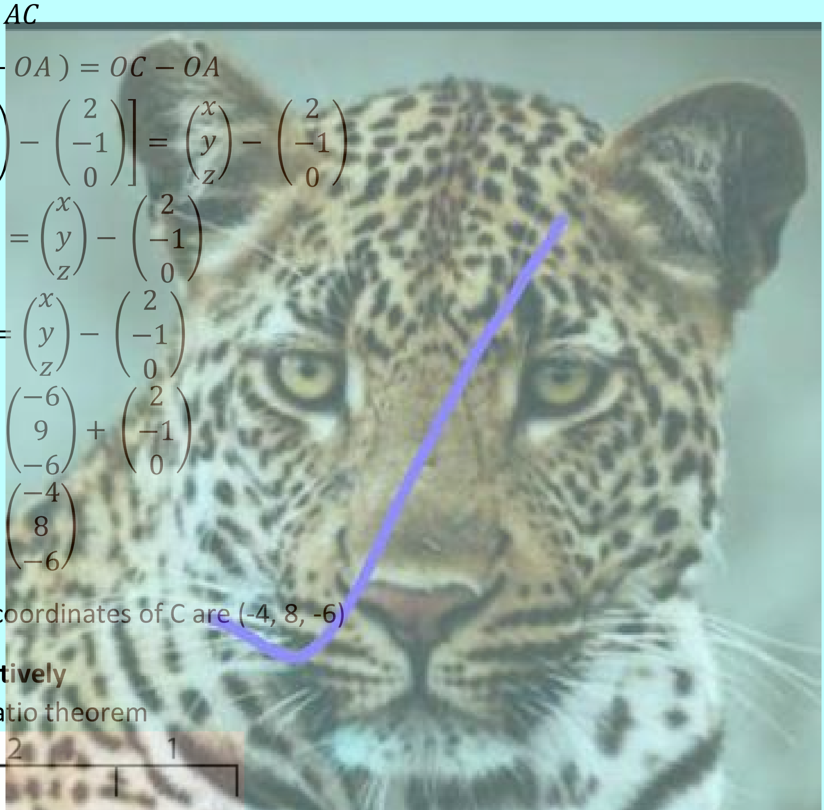
$= 3 + \frac{1}{405}$

$= 3\frac{1}{405}$

8. Three points A(2, -1, 0), B(-2, 5, -4) and C are on a straight line such that 3AB = 2AC. Find the coordinates of C.

3(AB) = 2AC

$\frac{3}{2}AB = AC$



$\frac{3}{2}(OB - OA) = OC - OA$

$\frac{3}{2}\left[\begin{pmatrix}-2\\5\\-4\end{pmatrix} - \begin{pmatrix}2\\-1\\0\end{pmatrix}\right] = \begin{pmatrix}x\\y\\z\end{pmatrix} - \begin{pmatrix}2\\-1\\0\end{pmatrix}$

$\frac{3}{2}\begin{pmatrix}-4\\6\\-4\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix} - \begin{pmatrix}2\\-1\\0\end{pmatrix}$

$\begin{pmatrix}-6\\9\\-6\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix} - \begin{pmatrix}2\\-1\\0\end{pmatrix}$

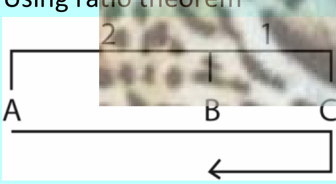
$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-6\\9\\-6\end{pmatrix} + \begin{pmatrix}2\\-1\\0\end{pmatrix}$

$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-4\\8\\-6\end{pmatrix}$

Hence coordinates of C are (-4, 8, -6)

Alternatively

Using ratio theorem



C divides externally in the ratio 3: -1

OC = $\frac{3(OB) - 1(OA)}{3 + (-1)}$

OC = $\frac{1}{2}\left\{3\begin{pmatrix}-2\\5\\-4\end{pmatrix} - \begin{pmatrix}2\\-1\\0\end{pmatrix}\right\}$

$= \frac{1}{2}\begin{pmatrix}-8\\16\\12\end{pmatrix} = \begin{pmatrix}-4\\8\\-6\end{pmatrix}$

Hence C(-4, 8, -6)

Alternatively

B divides AC internally in ration of 2:1

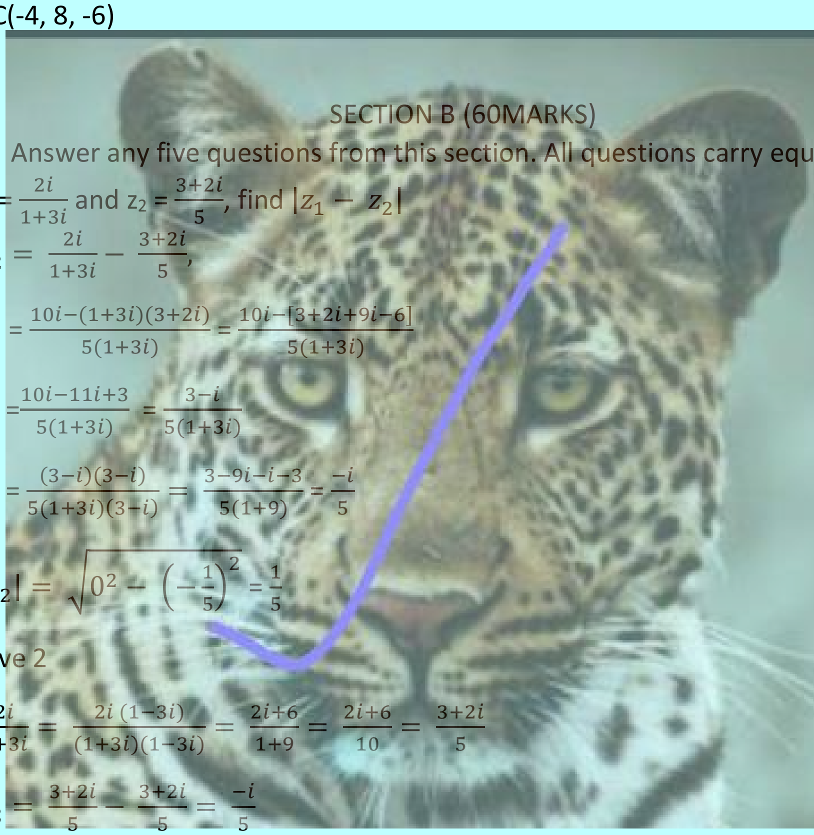
$$\begin{pmatrix}-2\\5\\-4\end{pmatrix} = \frac{1}{3}\left\{2\begin{pmatrix}x\\y\\z\end{pmatrix} + \begin{pmatrix}2\\-1\\0\end{pmatrix}\right\}$$

$$3\begin{pmatrix}-2\\5\\-4\end{pmatrix} = 2\begin{pmatrix}x\\y\\z\end{pmatrix} + \begin{pmatrix}2\\-1\\0\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \frac{1}{2}\left\{\begin{pmatrix}-6\\15\\12\end{pmatrix} - \begin{pmatrix}2\\-1\\0\end{pmatrix}\right\}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-8\\16\\-12\end{pmatrix} = \begin{pmatrix}-4\\8\\-6\end{pmatrix}$$

Hence C(-4, 8, -6)



SECTION B (60MARKS)

Answer any five questions from this section. All questions carry equal marks

9. (a) If $z_1 = \frac{2i}{1+3i}$ and $z_2 = \frac{3+2i}{5}$, find $|z_1 - z_2|$

$$z_1 - z_2 = \frac{2i}{1+3i} - \frac{3+2i}{5},$$

$$= \frac{10i-(1+3i)(3+2i)}{5(1+3i)} = \frac{10i-[3+2i+9i-6]}{5(1+3i)}$$

$$= \frac{10i-11i+3}{5(1+3i)} = \frac{3-i}{5(1+3i)}$$

$$= \frac{(3-i)(3-i)}{5(1+3i)(3-i)} = \frac{3-9i-i-3}{5(1+9)} = \frac{-i}{5}$$

$$|z_1 - z_2| = \sqrt{0^2 - \left(-\frac{1}{5}\right)^2} = \frac{1}{5}$$

Alterative 2

$$z_1 = \frac{2i}{1+3i} = \frac{2i\,(1-3i)}{(1+3i)(1-3i)} = \frac{2i+6}{1+9} = \frac{2i+6}{10} = \frac{3+2i}{5}$$

$$z_1 - z_2 = \frac{3+2i}{5} - \frac{3+2i}{5} = \frac{-i}{5}$$

$$|z_1 - z_2| = \sqrt{0^2 - \left(-\frac{1}{5}\right)^2} = \frac{1}{5}$$

(c) Given the complex number z = x + iy

(i) Find $\frac{z+i}{z+2}$

$$\frac{z+i}{z+2} = \frac{x+i(1+y)}{(x+2)+iy}$$

$$= \frac{[x+i(1+y)][(x+2)-iy]}{[(x+2)+iy][(x+2)-iy]}$$

$$= \frac{x[(x+2)-iy]-i(1+y)[(x+2)-iy]}{(x+2)^2+y^2}$$

$$= \frac{x^2+2x-ixy+i(\,x+2+xy+2y)+y+\,y^2}{(x+2)^2+y^2}$$

$$= \frac{x^2+2x+\,y^2+y+i(2+x+2y)}{(x+2)^2+y^2}$$

(ii) Show that the locus of $\frac{z+i}{z+2}$ is a straight line when its imaginary part is zero. State the gradient of the line.

If imaginary part is zero

$(2 + x + 2y) = 0$

$2y = -x - 2$

$y = -\frac{1}{2}x + 1$

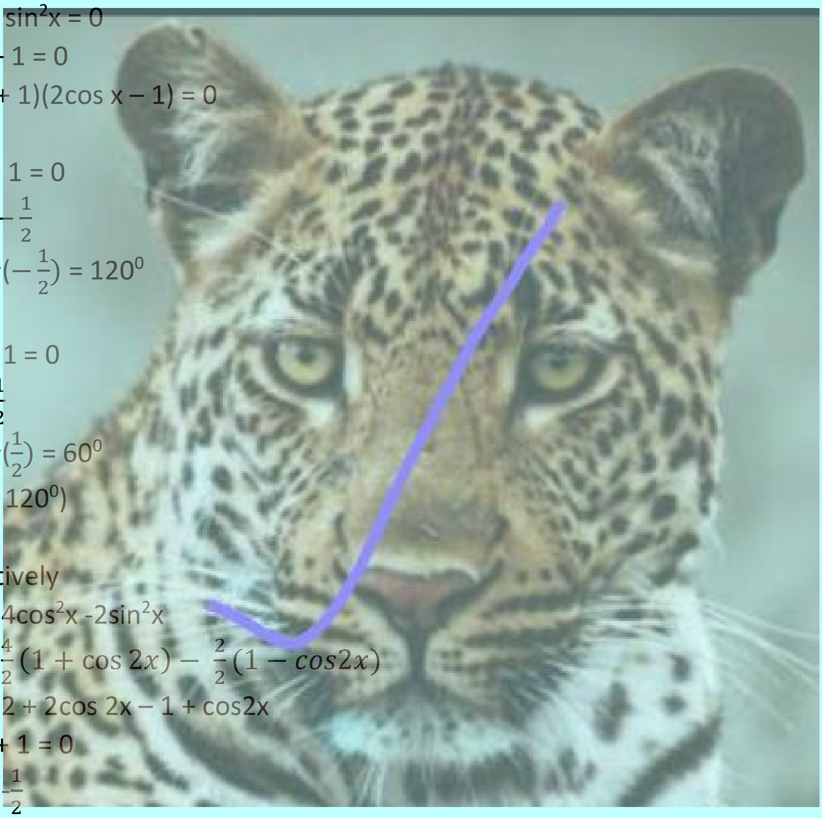
comparing with y = mx + c

the gradient = $-\frac{1}{2}$

10. (a) solve the equation $\cos 2x = 4\cos^2 x - 2\sin^2 x$ for $0 \le \theta \le 180^0$

$\cos 2x = 4\cos^2 x - 2\sin^2 x$

$\cos^2 x - \sin^2 x = 4\cos^2 x - 2\sin^2 x$



$3\cos^2 x - \sin^2 x = 0$

$4\cos^2 x - 1 = 0$

$(2\cos x + 1)(2\cos x - 1) = 0$

Either

$2\cos x + 1 = 0$

$\cos x = -\frac{1}{2}$

$x = \cos^{-1}(-\frac{1}{2}) = 120^0$

Or

$2\cos x - 1 = 0$

$\cos x = \frac{1}{2}$

$x = \cos^{-1}(\frac{1}{2}) = 60^0$

$\therefore x(60^0, 120^0)$

Alternatively

$\cos 2x = 4\cos^2 x - 2\sin^2 x$

$= \frac{4}{2}(1 + \cos 2x) - \frac{2}{2}(1 - \cos 2x)$

$= 2 + 2\cos 2x - 1 + \cos 2x$

$2\cos 2x + 1 = 0$

$\cos 2x = -\frac{1}{2}$

$2x = \cos_{-1}(-\frac{1}{2}) = 120^0, 240^0$

$x = 60^0, 120^0$

Alternatively

$\cos 2x = 4\cos^2 x - 2\sin^2 x$

$\cos^2 x - \sin^2 x = 4\cos^2 x - 2\sin^2 x$

$3\cos^2 x - \sin^2 x = 0$

$\text{Sin}^2 x = 3\cos^2 x$

$\tan^2 x = 3$

$\tan x = \pm\sqrt{3}$

Either

$\tan x = \sqrt{3}$

$x = \tan^{-1}\sqrt{3} = 60^0$

Or

tanx = $-\sqrt{3}$

x = $\tan^{-1} -\sqrt{3}$ = 120^0

Hence x = 60^0, 120^0

Alternatively

cos2x = $4\cos^2$x $-2\sin^2$x

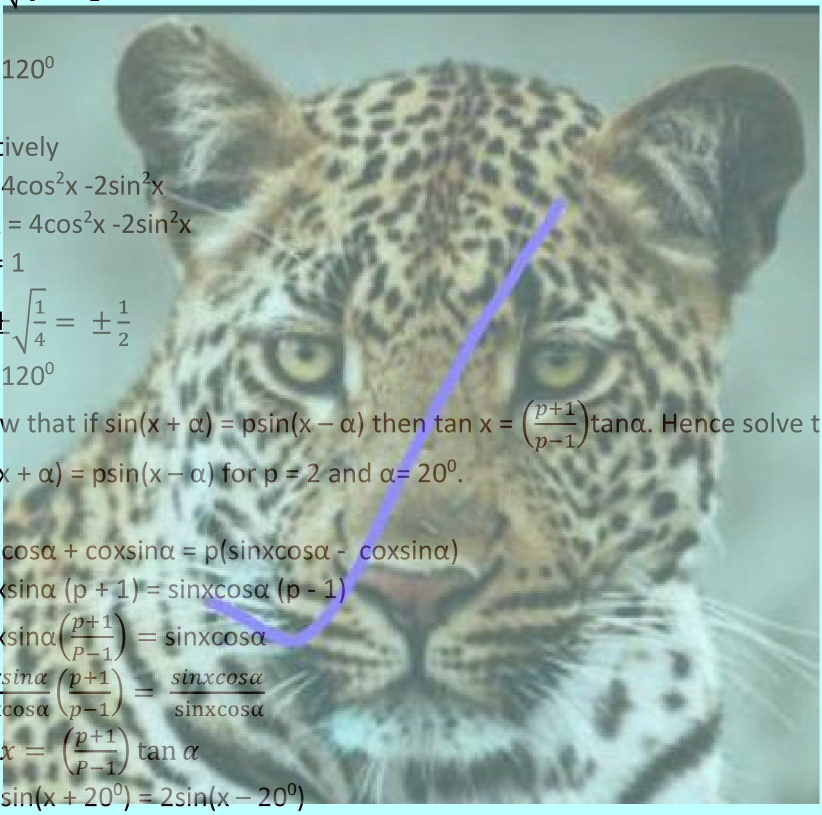
$1-2\sin^2$x = $4(1-\sin^2$x$)-2\sin^2$x

1 = 4 – $4\sin^2$x

$4\sin^2$x = 3

$\sin^2$x = $\frac{3}{4}$

sinx = $\pm\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$



x = 60^0, 120^0

Alternatively

cos2x = $4\cos^2$x $-2\sin^2$x

$1-2\sin^2$x = $4\cos^2$x $-2\sin^2$x

$4\cos^2$x = 1

Cosx = $\pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2}$

x = 60^0, 120^0

(b) Show that if sin(x + α) = psin(x – α) then tan x = $\left(\frac{p+1}{p-1}\right)$tanα. Hence solve the equation sin(x + α) = psin(x – α) for p = 2 and α= 20^0.

sinxcosα + coxsinα = p(sinxcosα - coxsinα)

cosxsinα (p + 1) = sinxcosα (p - 1)

cosxsinα$\left(\frac{p+1}{P-1}\right)$ = sinxcosα

$\frac{cosxsin\alpha}{\text{sinxcos}\alpha}\left(\frac{p+1}{p-1}\right) = \frac{sinxcos\alpha}{\text{sinxcos}\alpha}$

$tanx = \left(\frac{p+1}{P-1}\right)\tan\alpha$

For sin(x + 20^0) = 2sin(x – 20^0)

tanx = $\frac{2+1}{2-1}tan20^0$ = 3tan20^0

x = $\tan^{-1}(3tan20^0)$ = 47.52^0

11. Given that x = $\frac{t^2}{1+t^3}$ and y =$\frac{t^3}{1+t^3}$, find$\frac{d^2y}{dx^2}$.

$$x = \frac{t^2}{1+t^3}$$

$$\frac{dx}{dt} = \frac{2t(1+t^3)-3t^4}{(1+t^3)^2} = \frac{2t+2t^4-3t^4}{(1+t^3)^2} = \frac{2t-t^4}{(1+t^3)^2} = \frac{t(2-t^3)}{(1+t^3)^2}$$

$$y = \frac{t^3}{1+t^3}$$

$$\frac{dy}{dt} = \frac{3t^2(1+t^3)-t^3(3t^2)}{(1+t^3)^2} = \frac{3t^2+3t^5-3t^5}{(1+t^3)^2} = \frac{3t^2}{(1+t^3)^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$= \frac{3t^2}{(1+t^3)^2}\cdot\frac{(1+t^3)^2}{t(2-t^3)} = \frac{3t^2}{t(2-t^3)} = \frac{3t}{2-t^3}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right).\frac{dx}{dt}$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{3(2-t^3)-3t(-3t^3)}{(2-t^3)^2} = \frac{6-3t^3+9t^3}{(2-t^3)^2} = \frac{6+6t^3}{(2-t^3)^2} = \frac{6(1+t^3)}{(2-t^3)^2}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right).\frac{dx}{dt}$$

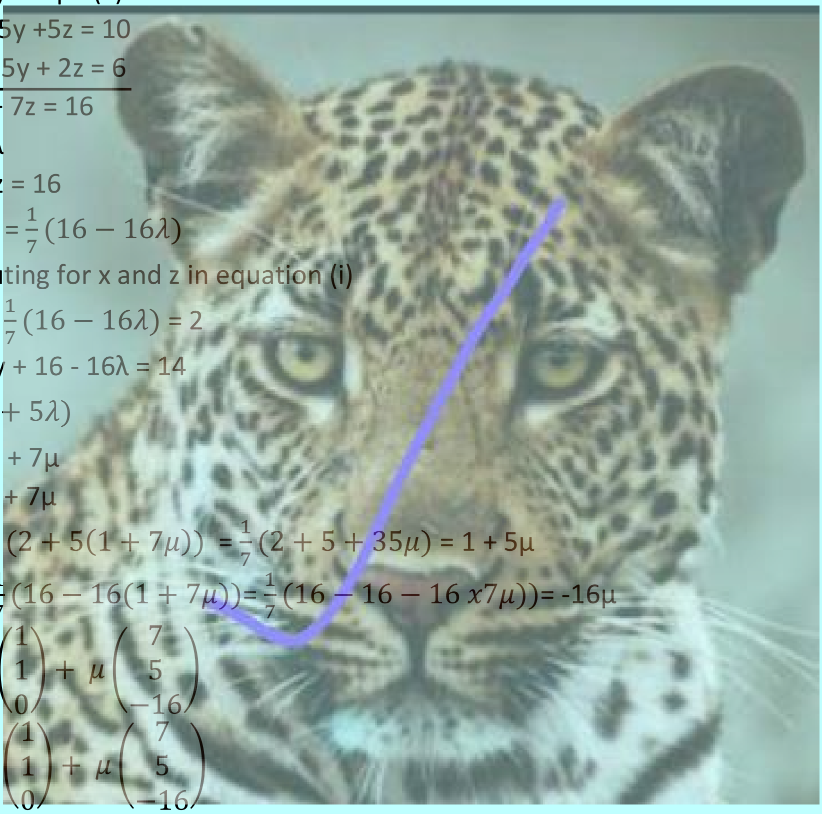
$$= \frac{6(1+t^3)}{(2-t^3)^2}.\frac{(1+t^3)^2}{t(2-t^3)} = \frac{6(1+t^3)^3}{t(2-t^3)^3} = \frac{6}{t}\left(\frac{1+t^3}{2-t^3}\right)^3$$

12. (a) Line A is the intersection of two planes whose equations are $3x - y + z = 2$ and $x + 5y + 2z = 6$. Find the equation of the line.

$3x - y + z = 2$ …………………. (i)

$x + 5y + 2z = 6$ ………………… (ii)

5eqn. (i) + eqn. (ii)



$$\begin{array}{rl} & 15x - 5y + 5z = 10 \\ + & x + 5y + 2z = 6 \\ \hline & 16x + 7z = 16 \end{array}$$

Let $x = \lambda$

$16\lambda + 7z = 16$

$z = \frac{1}{7}(16 - 16\lambda)$

Substituting for x and z in equation (i)

$3\lambda - y + \frac{1}{7}(16 - 16\lambda) = 2$

$21\lambda - 7y + 16 - 16\lambda = 14$

$y = \frac{1}{7}(2 + 5\lambda)$

let $\lambda = 1 + 7\mu$

=> $x = 1 + 7\mu$

$y = \frac{1}{7}(2 + 5(1 + 7\mu)) = \frac{1}{7}(2 + 5 + 35\mu) = 1 + 5\mu$

$z = \frac{1}{7}(16 - 16(1 + 7\mu)) = \frac{1}{7}(16 - 16 - 16\,x7\mu)) = -16\mu$

$$\underline{r} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \mu\begin{pmatrix}7\\5\\-16\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \mu\begin{pmatrix}7\\5\\-16\end{pmatrix}$$

$$\frac{x-1}{7} = \frac{y-1}{5} = \frac{-z}{16}$$

(b) Given that line B is perpendicular to the plane $3x - y + z = 2$and passes through the point C(1, 1, 0), find the

(i) Cartesian equation of line B

Normal to the plane $b = 3i - j + k$

$r = a + \lambda b$

$$= \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}3\\-1\\1\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}3\\-1\\1\end{pmatrix}$$

$$\frac{x-1}{3} = \frac{y-1}{-1} = \frac{z}{1}$$

(ii) angle between line B and line A in (a) above

Let $b_1 = 7i + 5j - 16k$ and $b_2 = 3i - j + k$ and θ = angle between line A and line B

$b_1.b_2 = |b_1||b_2|cos\theta$

$b_1.b_2 = (7i + 5j - 16k).(3i - j + k)$

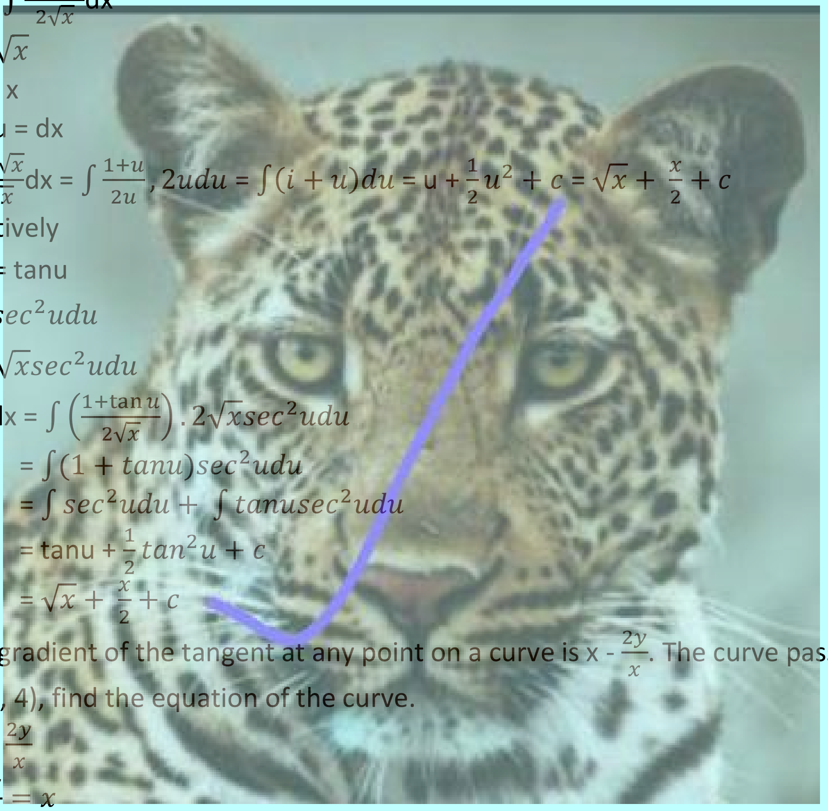
$= 21 - 5 - 16 = 0$

$|b_1||b_2|cos\theta = 0$

cosθ = 0

$\theta = \cos^{-1} 0 = 90^0$

13. (a) Find $\int \frac{1+\sqrt{x}}{2\sqrt{x}}dx$



Let $u = \sqrt{x}$

$u^2 = x$

$2udu = dx$

=> $\int \frac{1+\sqrt{x}}{2\sqrt{x}}dx = \int \frac{1+u}{2u}, 2udu = \int(i+u)du = u + \frac{1}{2}u^2 + c = \sqrt{x} + \frac{x}{2} + c$

Alternatively

Let $\sqrt{x}$ = tanu

$\frac{1}{2\sqrt{x}} = sec^2udu$

$dx = 2\sqrt{x}sec^2udu$

$\int \frac{1+\sqrt{x}}{2\sqrt{x}}dx = \int \left(\frac{1+\tan u}{2\sqrt{x}}\right).2\sqrt{x}sec^2udu$

$= \int(1 + tanu)sec^2udu$

$= \int sec^2udu + \int tanusec^2udu$

$= tanu + \frac{1}{2}tan^2u + c$

$= \sqrt{x} + \frac{x}{2} + c$

(b) The gradient of the tangent at any point on a curve is $x - \frac{2y}{x}$. The curve passes through the point (2, 4), find the equation of the curve.

$\frac{dy}{dx} = x - \frac{2y}{x}$

$\frac{dy}{dx} + \frac{2y}{x} = x$

$the\ integrating\ factor\ \lambda = e^{\int\frac{2}{x}dx} = e^{2Inx} = x^2$

Multiplying through by λ

$x^2\frac{dy}{dx} + 2xy = x^3$

$main\ function = x^2y$

⇨ $\frac{d}{dx}(x^2y) = x^3$

$\int \frac{d}{dx}(x^2y)dx = \int x^3dx$

$x^2y = \frac{1}{4}x^4 + c$

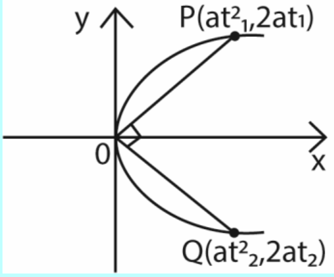
At (2,4)

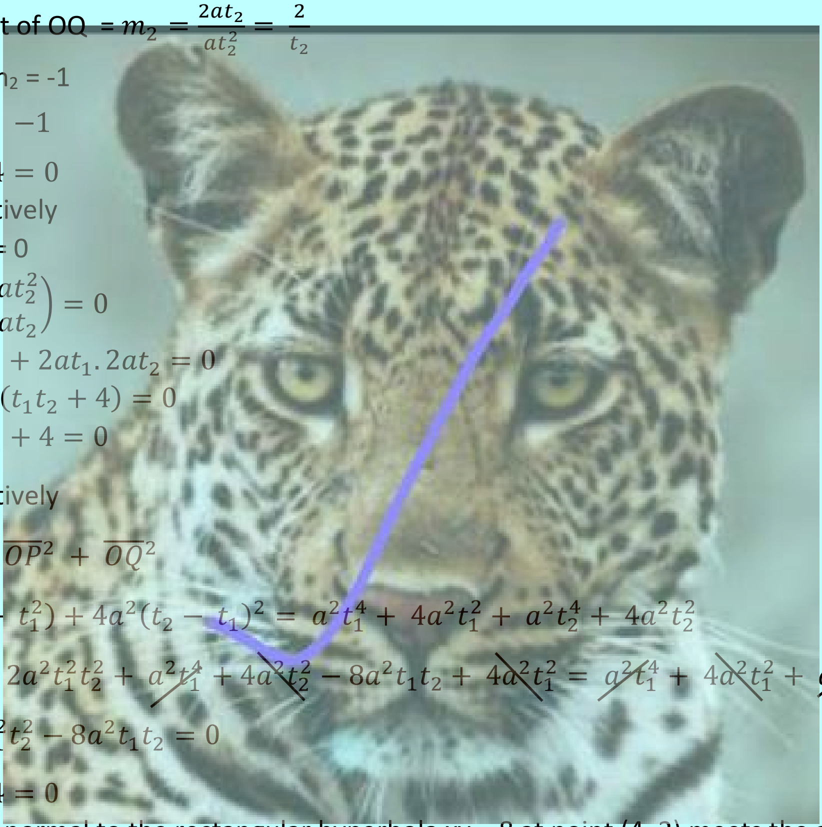
4 x 4 = 4 + c => c = 12

$x^2y = \frac{1}{4}x^4 + 12$ or $y = \frac{1}{4}x^2 + \frac{12}{x^2}$ or $4x^2y = x^4 + 48$

14. (a) The point P($at_1^2, 2at_1$) and Q($at_2^2, 2at_2$) are on parabola $y^2 = 4ax$. OP is perpendicular to OQ, where O is the origin. Show that $t_1t_2 + 4 = 0$.

OP.OQ = 0





Gradient of OP, = $m_1 = \frac{2at_1}{at_1^2} = \frac{2}{t_1}$

Gradient of OQ $= m_2 = \frac{2at_2}{at_2^2} = \frac{2}{t_2}$

But $m_1m_2 = -1$

$\frac{2}{t_1} \cdot \frac{2}{t_2} = -1$

$t_1t_2 + 4 = 0$

Alternatively

OP.OQ = 0

$\begin{pmatrix} at_1^2 \\ at_1 \end{pmatrix} \begin{pmatrix} at_2^2 \\ at_2 \end{pmatrix} = 0$

$at_1^2 . at_2^2 + 2at_1 . 2at_2 = 0$

$aa^2t_1t_2(t_1t_2 + 4) = 0$

⇨ $t_1t_2 + 4 = 0$

Alternatively

$\overline{PQ}^2 = \overline{OP}^2 + \overline{OQ}^2$

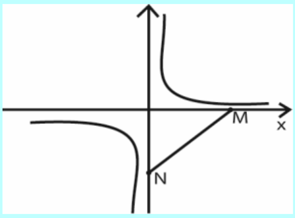
$a^2(t_2^2 - t_1^2) + 4a^2(t_2 - t_1)^2 = a^2t_1^4 + 4a^2t_1^2 + a^2t_2^4 + 4a^2t_2^2$

$\cancel{a^2t_2^4} - 2a^2t_1^2t_2^2 + \cancel{a^2t_1^4} + \cancel{4a^2t_2^2} - 8a^2t_1t_2 + \cancel{4a^2t_1^2} = \cancel{a^2t_1^4} + \cancel{4a^2t_1^2} + \cancel{a^2t_2^4} + \cancel{4a^2t_2^2}$

$-2a^2t_1^2t_2^2 - 8a^2t_1t_2 = 0$

$t_1t_2 + 4 = 0$

(b) The normal to the rectangular hyperbola xy = 8 at point (4, 2) meets the asymptotes at M and N. find the length MN.



The equation of the normal to a rectangular hyperbola $xy = c^2$ at a point ($ct, \frac{c}{t}$) is given by

$t^3x = ty + c(t^4 - 1)$

Comparing $xy = c^2$ with $xy = 8$

⇨ $c^2 = 8$; $c = 2\sqrt{2}$

Also comparing point $\left(ct, {}^{c}/_{t}\right)$ with (4, 2)

⇨ ct = 4

$(2\sqrt{2})t = 4$

t = $\frac{4}{2\sqrt{2}} = \sqrt{2}$

Find the equation of the normal by substituting for c and t.

$(\sqrt{2})^3 = (\sqrt{2})y + 2\sqrt{2}\left[(\sqrt{2})^4 - 1\right]$

$(\sqrt{2})^2 = y + 2\left[(\sqrt{2})^4 - 1\right]$

$2x = y + 6$

y = 2x -6

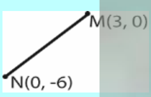
The normal drawn from the curve meets the asymptotes at the x-axis (M) and y-axis N as shown above

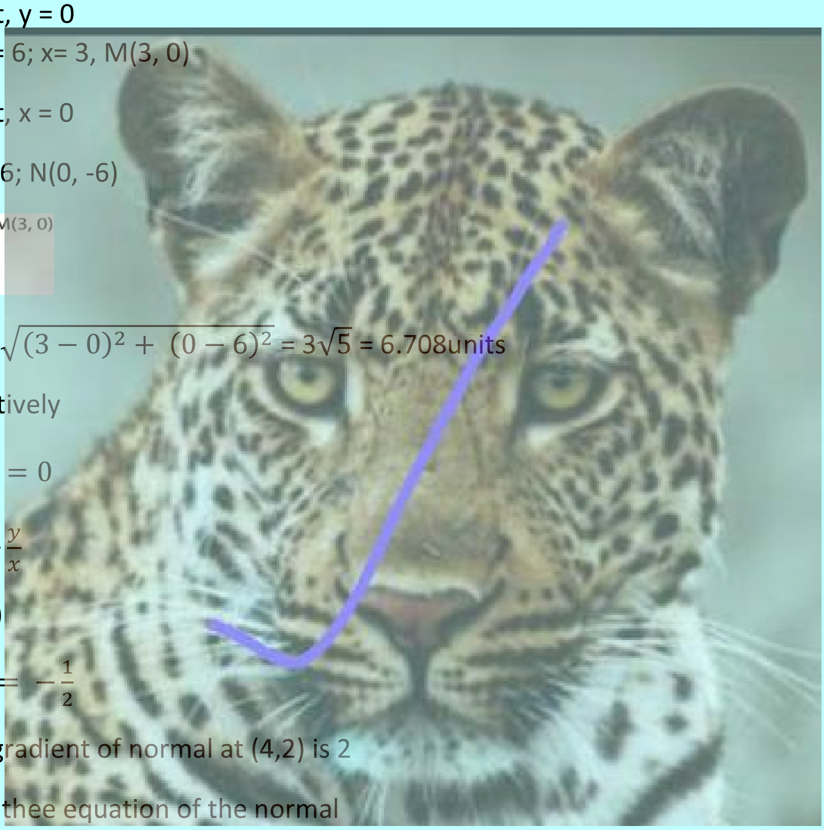
At point, y = 0

⇨ 2x = 6; x= 3, M(3, 0)

At point, x = 0

⇨ y =-6; N(0, -6)





$\overline{NM} = \sqrt{(3-0)^2 + (0-6)^2} = 3\sqrt{5} = 6.708$units

Alternatively

y + $x\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{y}{x}$

At (4, 2)

$\frac{dy}{dx} - \frac{2}{4} = -\frac{1}{2}$

Hence gradient of normal at (4,2) is 2

Finding thee equation of the normal

$\frac{y-2}{x-2} = 2$

y = 2x- 6

Along y-axis at N, x = 0 => y = -6, N(0, -6)

Along y-axis at M, y = 0 => x = 3, M(3, 0)

$\overline{NM} = \sqrt{(3-0)^2 + (0-6)^2} = 3\sqrt{5} = 6.708$units

15. (a) Prove by induction

1.3 + 2.4 + … + n(n + 2) = $\frac{1}{6}n(n+1)(2n+7)$ for all values of n.

Suppose n = 1

L.H.S = 1 x 3 = 3

R.H.S = $\frac{1}{6}x1(1+1)(2+7) = 3$

L.H.S = R.H.S, hence the series holds for n =1

Suppose n = 2

L.H.S= 1 x 3 + 2 x 4 = 11

R.H.S = $\frac{1}{6}\ x\ 2(2+1)(4+7) = 11$

L.H.S = R.H.S, hence the series holds for n =2

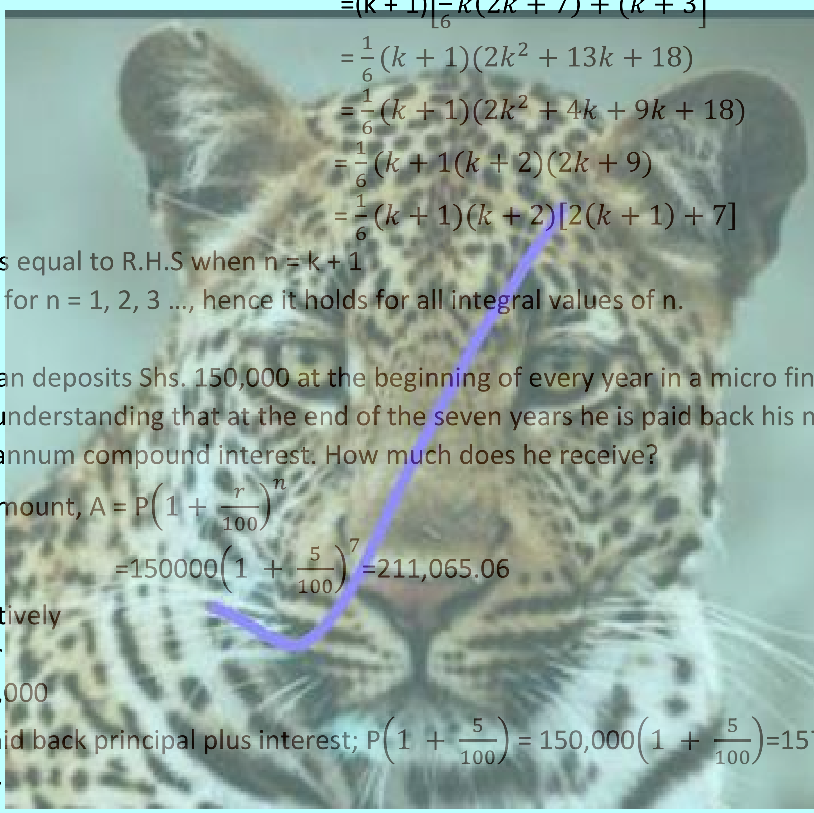
Suppose n = k

1.3 + 2.4 + … + k(k + 2) = $\frac{1}{6}k(k+1)(2k+7)$

For n = k + 1

1.3 + 2.4 + … + k(k + 2), (k + 1)(k + 3) = $\frac{1}{6}k(k+1)(2k+7)$+(k +1)(k + 3)

=(k + 1)$\left[\frac{1}{6}k(2k+7) + (k+3\right]$



= $\frac{1}{6}(k+1)(2k^2+13k+18)$

= $\frac{1}{6}(k+1)(2k^2+4k+9k+18)$

= $\frac{1}{6}(k+1(k+2)(2k+9)$

= $\frac{1}{6}(k+1)(k+2)[2(k+1)+7]$

Which is equal to R.H.S when n = k + 1

It holds for n = 1, 2, 3 …, hence it holds for all integral values of n.

(b) A man deposits Shs. 150,000 at the beginning of every year in a micro finance bank with the understanding that at the end of the seven years he is paid back his money with 5% per annum compound interest. How much does he receive?

Using amount, A = $P\left(1+\frac{r}{100}\right)^n$

$=150000\left(1+\frac{5}{100}\right)^7 = 211{,}065.06$

Alternatively

1st year

P = 150,000

He is paid back principal plus interest; $P\left(1+\frac{5}{100}\right) = 150{,}000\left(1+\frac{5}{100}\right)$=157,500

2nd year

P = 157, 500

He is paid back principal plus interest; $P\left(1+\frac{5}{100}\right) = 157500\left(1+\frac{5}{100}\right)$=165375

3rd year

P = 165375

Interest = $\frac{5}{10}\ x\ 165375 = 8268.75$

He is paid back principal plus interest; $P\left(1+\frac{5}{100}\right) = 165375\left(1+\frac{5}{100}\right)$=173643.75

4th year

P=173643.75

He is paid back principal plus interest; $P\left(1+\frac{5}{100}\right) = 173643.75\left(1+\frac{5}{100}\right)$=182325.94

5th year

P =182325.94

His paid back principal plus interest; $P\left(1+\frac{5}{100}\right) = 182325.94\left(1+\frac{5}{100}\right)$=191442.23

6^{th} year

P = 191442.23

He is paid back principal plus interest; $P\left(1+\frac{5}{100}\right) = 191442.23\left(1+\frac{5}{100}\right)$=201014.35

7^{th} year

P = 201014.35

He is paid back principal plus interest; $P\left(1+\frac{5}{100}\right) = 201014.35\left(1+\frac{5}{100}\right)$=211,065.06

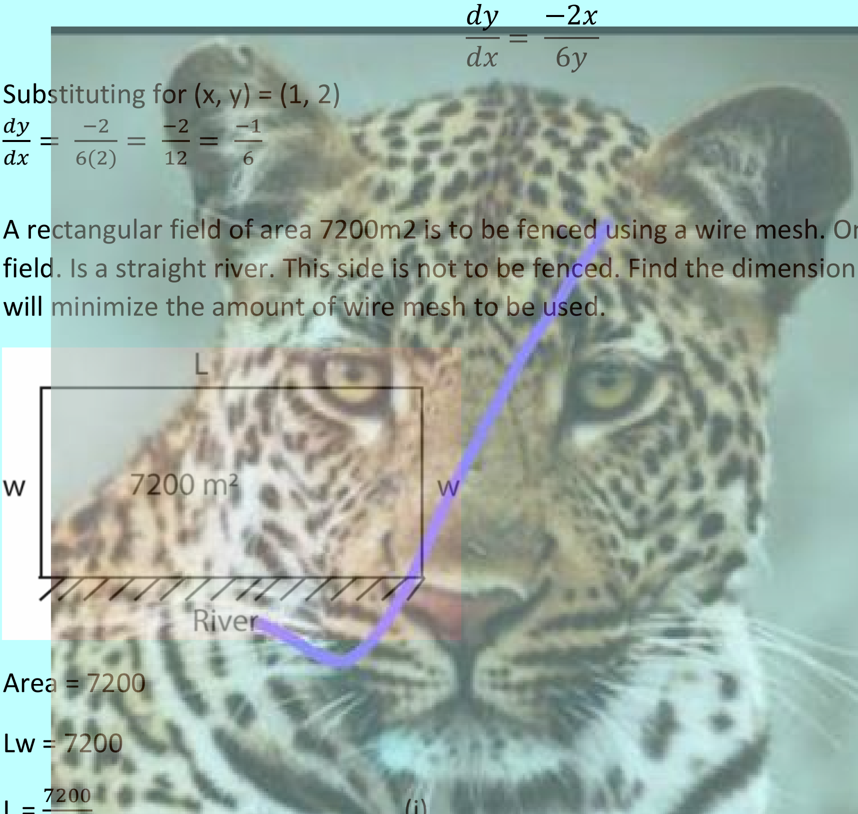
∴ by the 7^{th} year he has accumulated shs. **211,065.06**

16. (a) If $x^2 + 3y^2 = k$, where k is constant, find $\frac{dy}{dx}$ at the point (1, 2).

$x^2 + 3y^2 = k$

$2x + 6y\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{-2x}{6y}$$



Substituting for (x, y) = (1, 2)

$\frac{dy}{dx} = \frac{-2}{6(2)} = \frac{-2}{12} = \frac{-1}{6}$

(b) A rectangular field of area 7200m2 is to be fenced using a wire mesh. On one side of the field. Is a straight river. This side is not to be fenced. Find the dimension of the field hat will minimize the amount of wire mesh to be used.

Area = 7200

Lw = 7200

L = $\frac{7200}{w}$ ………………………………………… (i)

Perimeter, P = 2w + L ………………… (ii)

Substituting eqn. (i) into eqn. (ii)

P = 2w + $\frac{7200}{w}$

$\frac{dp}{dw} = 2 - \frac{7200}{w^2}$

Minimum perimeter occurs when $\frac{dp}{dw} = 0$

⇨ $2 - \frac{7200}{w^2} = 0$

$2w^2 = 7200$

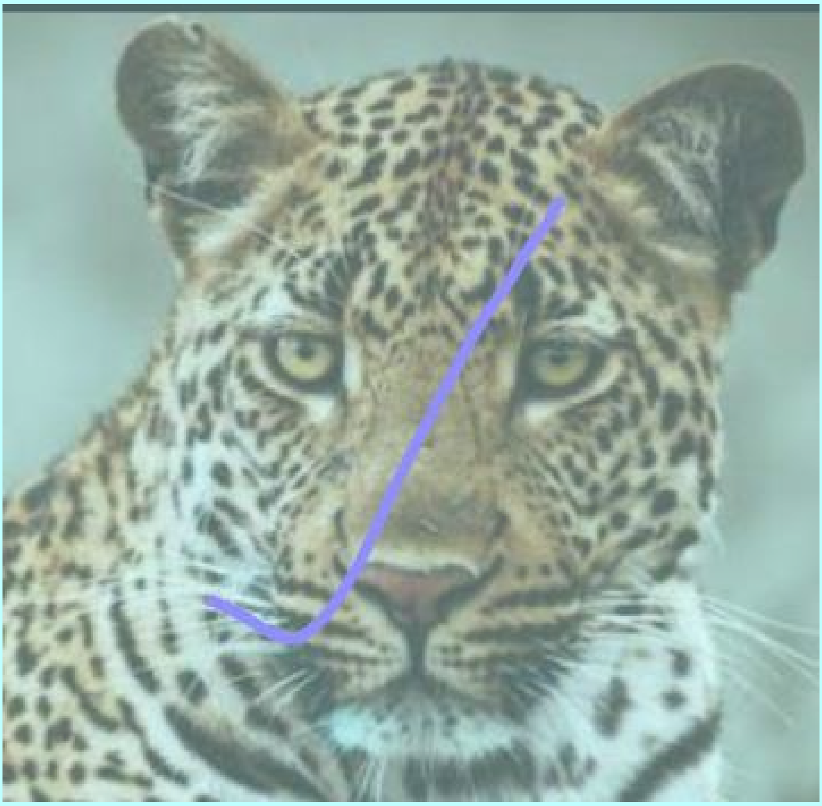
$w^2 = 3600$

w = $\pm$60 or w = 60

from eqn. (ii)

L=$\frac{7200}{60} = 120m$

Hence the dimensions are 60m x 120m



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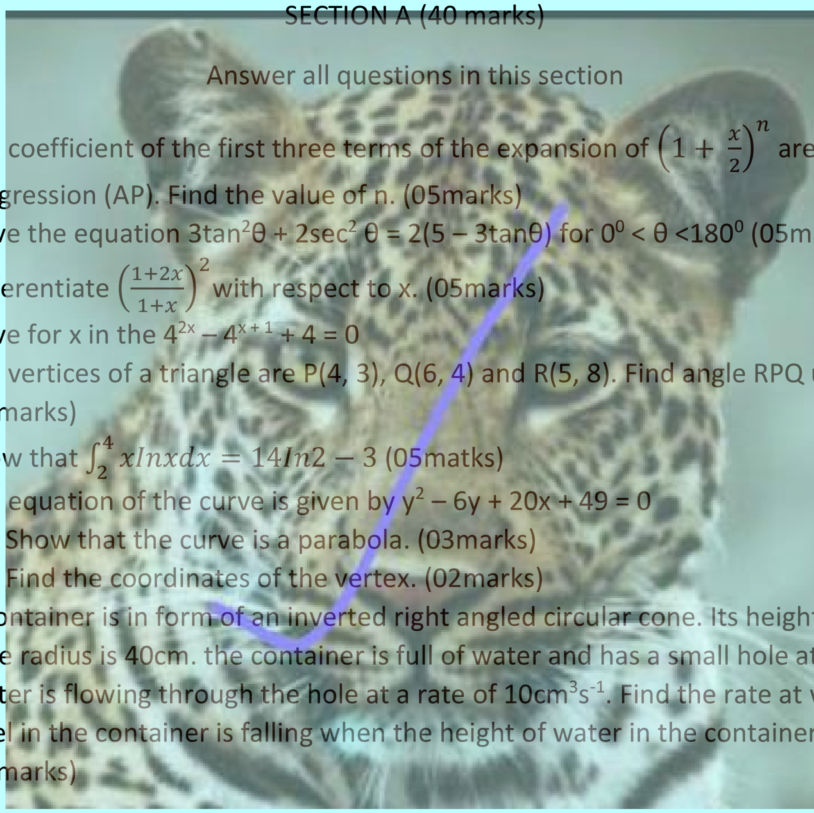


SECTION A (40 marks)

Answer all questions in this section

1. The coefficient of the first three terms of the expansion of $\left(1+\frac{x}{2}\right)^n$ are in arithmetic Progression (AP). Find the value of n. (05marks)
2. Solve the equation $3\tan^2\theta + 2\sec^2\theta = 2(5 - 3\tan\theta)$ for $0^0 < \theta < 180^0$ (05marks)
3. Differentiate $\left(\frac{1+2x}{1+x}\right)^2$ with respect to x. (05marks)
4. Solve for x in the $4^{2x} - 4^{x+1} + 4 = 0$
5. The vertices of a triangle are P(4, 3), Q(6, 4) and R(5, 8). Find angle RPQ using vectors. (05marks)
6. Show that $\int_2^4 xInxdx = 14In2 - 3$ (05matks)
7. The equation of the curve is given by $y^2 - 6y + 20x + 49 = 0$
   (a) Show that the curve is a parabola. (03marks)
   (b) Find the coordinates of the vertex. (02marks)
8. A container is in form of an inverted right angled circular cone. Its height is 100cm and base radius is 40cm. the container is full of water and has a small hole at its vertex. Water is flowing through the hole at a rate of $10cm^3s^{-1}$. Find the rate at which the water level in the container is falling when the height of water in the container is halved. (05marks)

SECTION B (60 MARKS)

Answer any five questions from this section. All questions carry equal marks

9. (a) Given that the complex number Z and its conjugate $\overline{Z}$ satisfy the equation $Z\overline{Z} - 2Z + 2\overline{Z} = 5 - 4i$. Find possible values of Z. (06marks)
   (c) Prove that if $\frac{Z-6i}{Z+8}$ is real, then the locus of the point representing the complex number Z is a straight line. (06marks)
10. A circle whose centre is in the first quadrant touches the x – and y –axes and the line $8x - 15y = 120$. Find the
    (a) equation of the circle (10marks)
    (b) point at which the circle touches the x-axis. (02marks)
11. A curve whose equation is $x^2y + y^2 - 3x = 3$ passes through points A(1, 2)and B(-1, 0). The tangent at A and the normal at the curve at B intersect at point C. Determine;
    (a) equation of the tangent. (06marks)

(b) coordinates of C. (06marks)

12. (a) Express cos $(\theta + 30)^0$ – cos $(\theta + 48)^0$ in the form RsinPsinQ, where R is constant. Hence solve th3 equation
cos $(\theta + 30)^0$ – cos $(\theta + 48)^0 = 0.2$ (06marks)
(b) Prove that in any triangle ABC, $\frac{\sin(A-B)}{\sin(A+B)} = \frac{a^2 - b^2}{c^2}$ (06marks)

13. (a) solve the simultaneous equation
$(x - 4y)^2 = 1$
$3x = 8y = 11$ (06marks)
(b) Solve the inequality
$4x^2 + 2x < 3x + 6$ (06marks)

14. (a) The points A and B have position vectors a and b. A point C with vector position c lies on AB such that $\frac{AC}{AB} = \lambda$. Show that c = $(1 - \lambda)$a + λb. (04marks)
(b) the vector equation of two lines are;
$r_1 = 2i + j + \lambda(i + j + 2k)$ and $r_2 = 2i + 2j + tk + \mu(i + 2j + k)$
where i, j and k are unit vectors and λ, μ and t are constants. Given that the two lines intersect, find
(i) the value of t.
(ii) the coordinates of the point of intersection. (08marks)

15. (a) sketch the curve $y = x^3 - 8$ (08marks)
(b) The area enclosed by the curve in (a), the y-axis and x-axis is rotate about the line y = 0 through 3600. Determine the volume of the solid generated. (04 marks)

16. Solve the differential equation $\frac{dy}{dx} = (xy)^{\frac{1}{2}} In x$, given that y = 1 when x = 1.
Hence find the value of y when x = 4 (12marks)

Solutions

SECTION A (40 marks)

Answer all questions in this section

1. The coefficient of the first three terms of the expansion of $\left(1 + \frac{x}{2}\right)^n$ are in arithmetic Progression (AP). Find the value of n. (05marks)

The expansion of $\left(1 + \frac{x}{2}\right)^n$ is given by pinch

$\left(1 + \frac{x}{2}\right)^n = 1 + \frac{n}{2}x + \frac{\frac{n(n-1)}{4}x^2}{2!} + \cdots$

$= 1 + \frac{n}{2}x + \frac{n(n-1)x^2}{8} + \cdots$

$U_1 = 1$, $U_2 = \frac{n}{2}$; $U_3 = \frac{n(n-1)}{8}$

But 3rd term – 2nd term = 2nd term – 1st term

$\frac{n(n-1)}{8} - \frac{n}{2} = \frac{n}{2} - 1$

$\frac{n(n-1)}{8} = n - 1$

$n(n-1) = 8n - 1$

$n^2 - 9n + 1 = 0$

(n -8)(n -1) = 0

n – 8 = 0

n= 8

2. Solve the equation $3\tan^2\theta + 2\sec^2\theta = 2(5 - 3\tan\theta)$ for $0^0 < \theta < 180^0$ (05marks)

Let t = tan θ

$3t^2 - 2(1 + t^2) = 2(5 - 3t)$

$5t^2 + 6t - 8 = 0$

$t = \frac{-6 \pm \sqrt{6^2 - 4(5)(-8)}}{2(5)} = \frac{-6 \pm 14}{10} = -2 \text{ or } \frac{4}{5}$

Taking t = -2; $\theta = \tan^{-1}(-2) = 116.57^0$

Taking $t = \frac{4}{5}$; $\theta = \tan^{-1}(\frac{4}{5}) = 38.66^0$

Hence $\theta = 38.66^0, 116.57^0$

3. Differentiate $\left(\frac{1+2x}{1+x}\right)^2$ with respect to x. (05marks)

Let $y = \left(\frac{1+2x}{1+x}\right)^2$

$\frac{dy}{dx} = \frac{4(1+2x)(1+x)^2 - 2(1+x)(1+2x)^2}{(1+x)^4}$

$= \frac{2(1+2x)(1+x)[2+2x-1-2x]}{(1+x)^4}$

$= \frac{2(1+2x)(1+x)(1)}{(1+x)^4}$

$\frac{dy}{dx} = \frac{2(1+2x)}{(1+x)^3}$

4. Solve for x in the $4^{2x} - 4^{x+1} + 4 = 0$

$(4^x)^2 - 4(4^x) + 4 = 0$

Let $q = 4^x$

$q^2 - 4q + 4 = 0$

$(q - 2)^2 = 0$

q = 2

⇨ $4^x = 2$

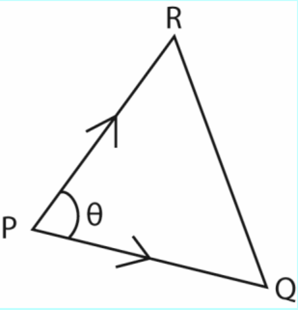
$2^{2x} = 2^1$

2x = 1

$x = \frac{1}{2}$

5. The vertices of a triangle are P(4, 3), Q(6, 4) and R(5, 8). Find angle RPQ using vectors. (05marks)

Let <RPQ = θ



$\overline{PQ} = \binom{6}{4} - \binom{4}{3} = \binom{2}{1}$

$\overline{PR} = \binom{5}{8} - \binom{4}{3} = \binom{1}{5}$

$|\overline{PQ}| = \sqrt{2^2 + 1^2} = \sqrt{5}$

$|\overline{PR}| = \sqrt{1^2 + 5^2} = \sqrt{26}$

$\begin{pmatrix}2\\1\end{pmatrix}\begin{pmatrix}1\\5\end{pmatrix} = \sqrt{5}.\sqrt{26}cos\theta$

$2 + 5 = \sqrt{130}cos\theta$

$\theta = cos^{-1}\left(\frac{7}{\sqrt{130}}\right) = 52.13^0$

6. Show that $\int_2^4 xInxdx = 14In2 - 3$ (05matks)

   Let u = Inx and $v^2 = x$

   ⇨ $u' = \frac{1}{x}$ and $v' = x$

   $\int_2^4 xInxdx = \left[\frac{x^2}{2}Inx\right]_2^4 - \frac{1}{2}\int_2^4 xdx$

   $= \frac{1}{2}(16In4 - 4In2) - \frac{1}{4}[x^2]_2^4$

   $= \frac{1}{2}(16In\,2^2 - 4In2) - \frac{1}{4}(16-4)$

   $= \frac{1}{2}(32In\,2 - 4In2) - \frac{1}{4}(16-4)$

   = 14In2 -3

7. The equation of the curve is given by $y^2 - 6y + 20x + 49 = 0$

   (a) Show that the curve is a parabola. (03marks)

   $y^2 - 6y + 20x + 49 = 0$

   $(y-3)^2 - 9 + 20x + 49 = 0$

   $(y-3)^2 = -20x - 40$

   $(y-3)^2 = -20(x+2)$

   (b) Find the coordinates of the vertex. (02marks)

   V(-2,3)

8. A container is in form of an inverted right angled circular cone. Its height is 100cm and base radius is 40cm. the container is full of water and has a small hole at its vertex. Water is flowing through the hole at a rate of $10cm^3s^{-1}$. Find the rate at which the water level in the container is falling when the height of water in the container is halved. (05marks)

   $\frac{h}{100} = \frac{r}{40}$ =>$r = \frac{2}{5}h$

   $v = \frac{1}{3}\pi\left(\frac{2}{5}h\right)^2 = \frac{4}{75}\pi h^2$

   $\frac{dv}{dt} = -10$

   $\frac{dv}{dh} = \frac{4}{25}\pi h^2$

   $\frac{dh}{dt} = \frac{dh}{dv} x \frac{dv}{dt} = \frac{25}{4\pi h^2}\ x - 10$

   $= \frac{250}{4\pi(50)^2}$

   $= 0.00796$

SECTION B (60 MARKS)

Answer any five questions from this section. All questions carry equal marks

9. (a) Given that the complex number Z and its conjugate $\bar{Z}$ satisfy the equation

   $Z\bar{Z} - 2Z + 2\bar{Z} = 5 - 4i$. Find possible values of Z. (06marks)

   $Z = x + yi$, $\bar{Z} = (x - yi)$

   $Z\bar{Z} - 2Z + 2\bar{Z} = 5 - 4i.$

   $(x + yi)(x - yi) - 2(x + yi) + 2(x - yi) = 5 - 4i$

$x^2 + y^2 - 2x - 2yi + 2x - 2yi = 5 - 4i$

$x^2 + y^2 - 4yi = 5 - 4i$

equating imaginary part

$-4yi = -4$

$y = 1$

equating real parts

$x^2 + y^2 = 5$

$x^2 + 1^2 = 5$

$x = \pm 2$

$\therefore Z = \pm 2 + 1$

(b) Prove that if $\frac{Z-6i}{Z+8}$ is real, then the locus of the point representing the complex number Z is a straight line. (06marks)

$\frac{Z-6i}{Z+8} = \frac{x+yi-6i}{x+yi+8}$

$= \frac{x+(y-6)i}{x+8+yi}$

$= \frac{x+(y-6)i}{x+8+yi} \cdot \frac{x+8-yi}{x+8+yi}$

$= \frac{x^2+8x+y^2-6y+(xy-6x-48-xy)i}{(x+8)^2+y^2}$

$IM\ \frac{z-6i}{z+8} = 0$

$\frac{8y-6x-48}{(x+8)^2+y^2} = 0$

$8y - 6x - 48 = 0$

$4y - 3x - 24 = 0$

Or

$y = \frac{3}{4}x + 6$

10. A circle whose centre is in the first quadrant touches the x – and y –axes and the line $8x - 15y = 120$. Find the

(a) equation of the circle (10marks)

Radius $a = \frac{|8a-15a-120|}{\sqrt{8^2+(-15)^2}}$

$= \frac{|-7a+120|}{17}$

$17a = 7a + 120$

$10a = 120$

$a = 12$

Equation of the circle

$(x - 12)^2 + (y - 12)^2 = 12^2$

$X^2 + y^2 - 24x - 24y + 144 = 0$

(b) point at which the circle touches the x-axis. (02marks)

$y = 0$

$(x - 12)^2 = 0$

$x = 12$

the point (12, 0)

11. A curve whose equation is $x^2y + y^2 - 3x = 3$ passes through points A(1, 2)and B(-1, 0). The tangent at A and the normal at the curve at B intersect at point C. Determine;

(a) equation of the tangent. (06marks)

$x^2y + y^2 - 3x = 3$

$2xy + 2\frac{dy}{dx} - 3 + 2y\frac{dy}{dx} = 0$

At (1,2)

$2(1)(2) + 2\frac{dy}{dx} - 3 + 2(2)\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{1}{5}$

Tangent $y - y_1 = m(x - 1)$

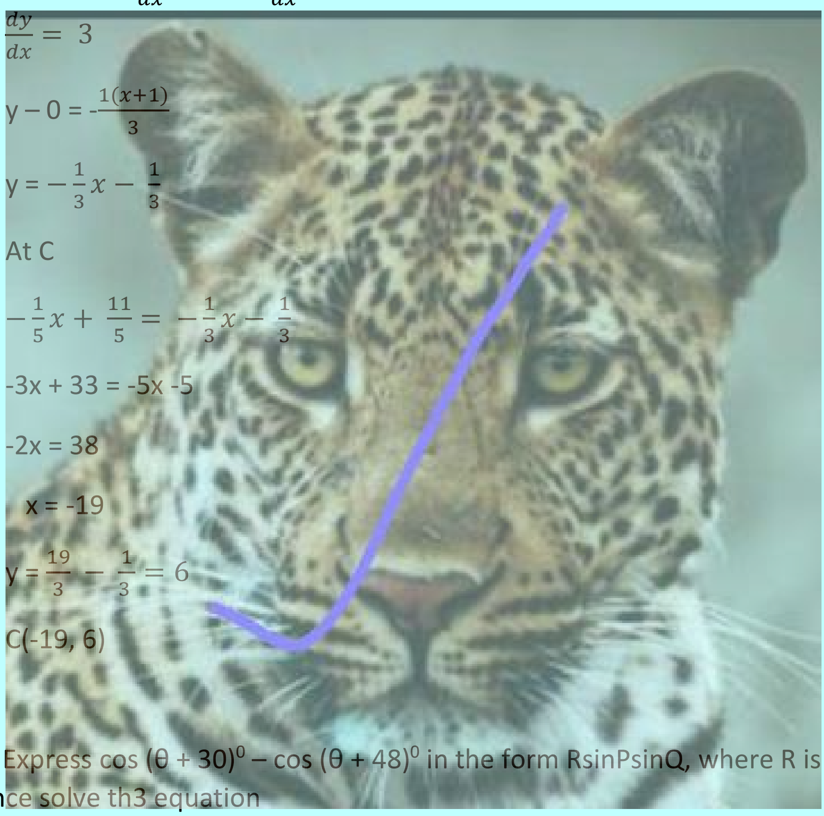
$y - 2 = \frac{1}{5}(x - 1)$

$y = -\frac{1}{5}x + \frac{11}{5}$

(b) coordinates of C. (06marks)

At B(-1, 0)

$2(-1)(0) + 2\frac{dy}{dx} - 3 + 2(0)\frac{dy}{dx} = 0$



$\frac{dy}{dx} = 3$

$y - 0 = -\frac{1(x+1)}{3}$

$y = -\frac{1}{3}x - \frac{1}{3}$

At C

$-\frac{1}{5}x + \frac{11}{5} = -\frac{1}{3}x - \frac{1}{3}$

$-3x + 33 = -5x - 5$

$-2x = 38$

$x = -19$

$y = \frac{19}{3} - \frac{1}{3} = 6$

C(-19, 6)

12. (a) Express $\cos(\theta + 30)^0 - \cos(\theta + 48)^0$ in the form RsinPsinQ, where R is constant. Hence solve th3 equation

$\cos(\theta + 30)^0 - \cos(\theta + 48)^0 = 0.2$ (06marks)

$\cos(\theta + 30)^0 - \cos(\theta + 48)^0$

$= -2\sin\left(\frac{\theta + 30^0 + \theta + 48^0}{2}\right)\sin\left(\frac{\theta + 30^0 - \theta - 48^0}{2}\right)$

$= -2\sin(\theta + 39^0)\sin(-9^0)$

$\cos(\theta + 30)^0 - \cos(\theta + 48)^0 = 0.$

⇨ $-2\sin(\theta + 39^0)\sin(-9^0) = 0.2$

$\sin(\theta + 39^0) = 0.63925$

$\theta + 39^0 = 39.74^0$

$\theta = 0.74^0$

(b) Prove that in any triangle ABC, $\frac{\sin(A-B)}{\sin(A+B)} = \frac{a^2 - b^2}{c^2}$ (06marks)

$$\frac{a^2-b^2}{c^2} = \frac{(2R\sin A)^2-(2R\sin B)^2}{(2R\sin C)^2}$$

$$= \frac{4R^2(\sin^2 A-\sin^2 B}{4R^2\sin^2 C}$$

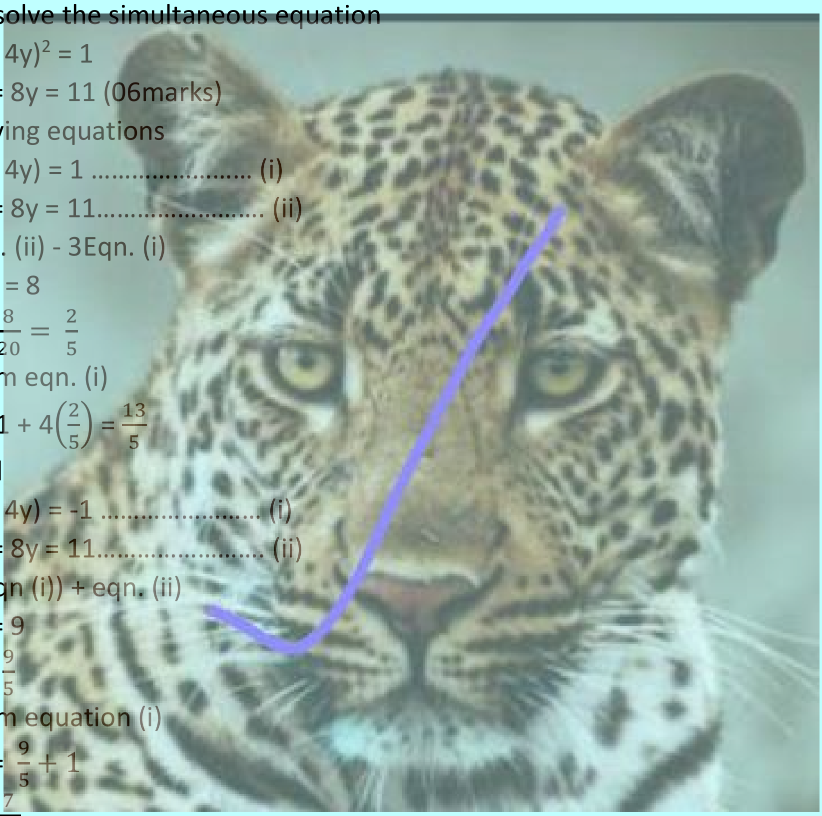
$$= \frac{(\sin A+\sin B)(\sin A-\sin B)}{\sin^2[180^0-(A+B]}$$

$$= \frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right).2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}{\sin^2(A+B)}$$

$$= \frac{\sin(A+B)\sin(A-B)}{\sin^2(A+B)}$$

$$= \frac{\sin(A-B)}{\sin(A+B)}$$

13. (a) solve the simultaneous equation



$(x - 4y)^2 = 1$

$3x = 8y = 11$ (06marks)

Solving equations

$(x - 4y) = 1$ .... (i)

$3x = 8y = 11$........................ (ii)

Eqn. (ii) - 3Eqn. (i)

$20y = 8$

$y = \frac{8}{20} = \frac{2}{5}$

From eqn. (i)

$x = 1 + 4\left(\frac{2}{5}\right) = \frac{13}{5}$

And

$(x - 4y) = -1$ .... (i)

$3x = 8y = 11$........................ (ii)

2(eqn (i)) + eqn. (ii)

$5x = 9$

$x = \frac{9}{5}$

From equation (i)

$4y = \frac{9}{5} + 1$

$y = \frac{7}{10}$

$\therefore (x,y) = \left(\frac{13}{5}, \frac{2}{5}\right), \left(\frac{9}{5}, \frac{7}{10}\right)$

(c) Solve the inequality

$4x^2 + 2x < 3x + 6$ (06marks)

$4x^2 + 5x - 6 < 0$

Critical values

$x = \frac{-5\pm\sqrt{5^2-4(4)(-6)}}{2(4)}$

$= \frac{-5\pm\sqrt{121}}{8}$

$x = -2, \frac{3}{4}$

	$x < -2$	$-2 < x < \frac{3}{4}$	$x > \frac{3}{4}$
$4x^2 + 5x - 6$	+	-	+

$\therefore -2 < x < \frac{3}{4}$

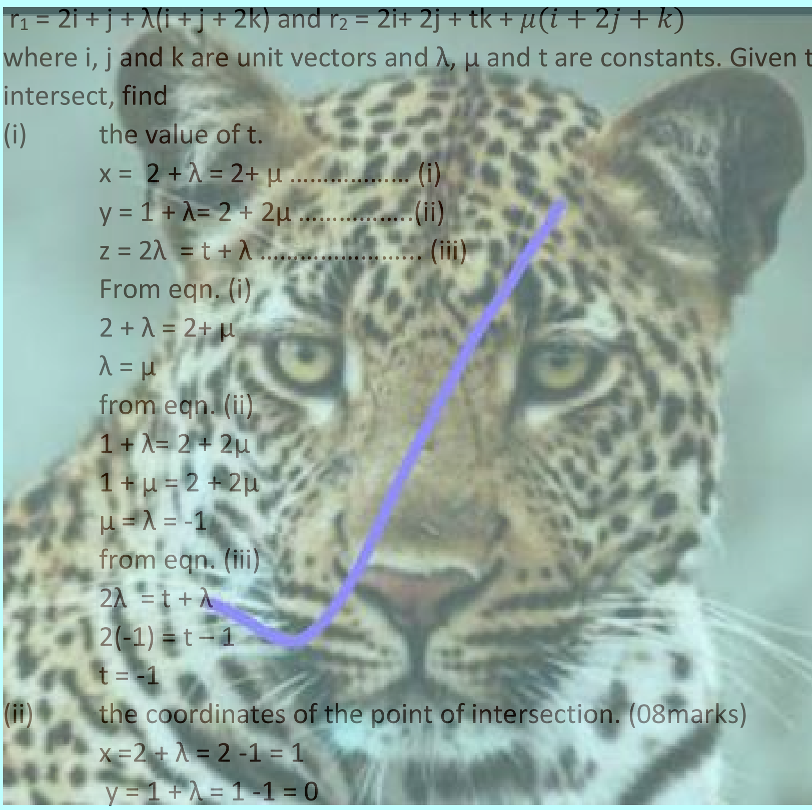
14. (a) The points A and B have position vectors a and b. A point C with vector position c lies on AB such that $\frac{AC}{AB} = \lambda$. Show that c = (1 – λ)a + λb. (04marks)

$$\frac{\overline{AC}}{\overline{AB}} = \lambda$$
$$\overline{AC} = \lambda\overline{AB}$$
$$\overline{OC} - \overline{OA} = \lambda(\overline{OB} - \overline{OA}$$
c – a = λ(b – a)

c = a + λ(b – a)

= (1 – λ)a + λb

(b) the vector equation of two lines are;



r_1 = 2i + j + λ(i + j + 2k) and r_2 = 2i+ 2j + tk + $\mu(i + 2j + k)$

where i, j and k are unit vectors and λ, μ and t are constants. Given that the two lines intersect, find

(i) the value of t.

x = 2 + λ = 2+ μ ……………… (i)

y = 1 + λ= 2 + 2μ ……………..(ii)

z = 2λ = t + λ …………………. (iii)

From eqn. (i)

2 + λ = 2+ μ

λ = μ

from eqn. (ii)

1 + λ= 2 + 2μ

1 + μ = 2 + 2μ

μ = λ = -1

from eqn. (iii)

2λ = t + λ

2(-1) = t – 1

t = -1

(ii) the coordinates of the point of intersection. (08marks)

x =2 + λ = 2 -1 = 1

y = 1 + λ = 1 -1 = 0

z = 2λ = 2(-1) = -2

∴ (x, y, z) = (1, 0, -2)

15. (a) sketch the curve y = x^3 – 8 (08marks)

y = x^3 – 8

Intercepts

When x = 0, y = -8

When y = 0, x = 2

(x, y) = (2, 0)

Turning point: $\frac{dy}{dx} = 3x^2$

$3x^2 = 0$

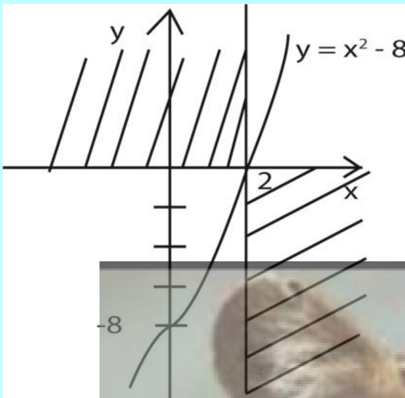
x = 0

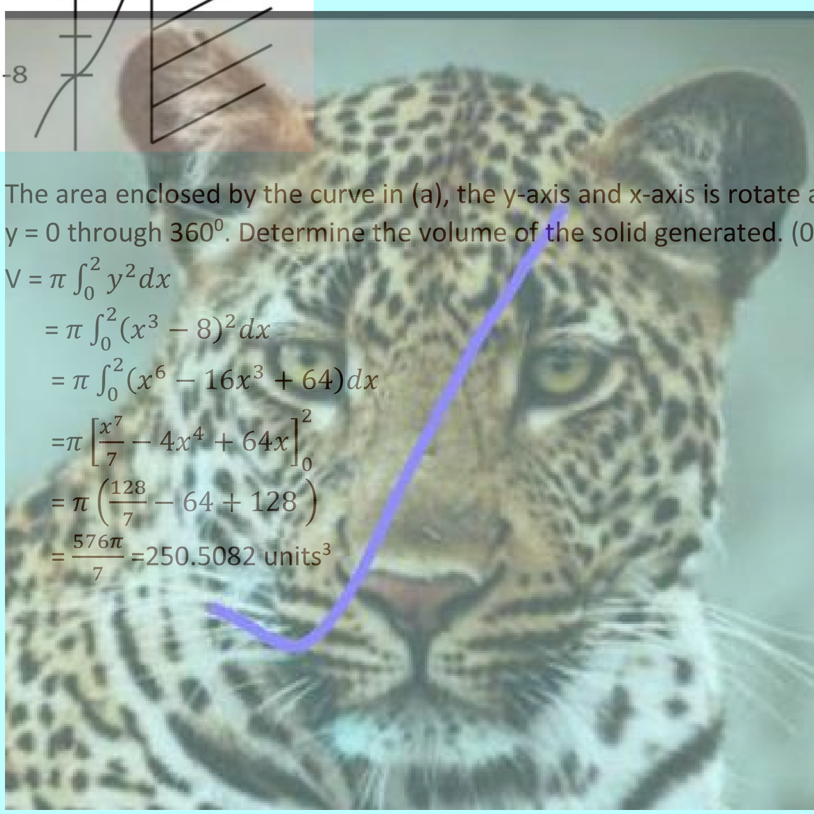
$\frac{d^2y}{dx^2} = 6x$

$$\frac{d^2y}{dx^2} = 0, x = 0$$

Point of reflection= (0, 8)

	x < 2	x > 2
y	-	+





(b) The area enclosed by the curve in (a), the y-axis and x-axis is rotate about the line y = 0 through 360^0. Determine the volume of the solid generated. (04 marks)

V = $\pi \int_0^2 y^2 dx$

$= \pi \int_0^2 (x^3 - 8)^2 dx$

$= \pi \int_0^2 (x^6 - 16x^3 + 64) dx$

$= \pi \left[\frac{x^7}{7} - 4x^4 + 64x\right]_0^2$

$= \pi \left(\frac{128}{7} - 64 + 128\right)$

$= \frac{576\pi}{7}$ =250.5082 units^3

16. Solve the differential equation $\frac{dy}{dx} = (xy)^{\frac{1}{2}}Inx$, given that y = 1 when x = 1. Hence find the value of y when x = 4 (12marks)

$\frac{dy}{dx} = (xy)^{\frac{1}{2}}Inx = \frac{dy}{dx} = y^{\frac{1}{2}}x^{\frac{1}{2}}Inx$

$\int y^{-\frac{1}{2}}\,dy = \int x^{\frac{1}{2}}Inx\,dx$

$2\sqrt{y} = x^{\frac{1}{2}}Inx\,dx$

u = Inx, u' = $\frac{1}{x}$

v' = $x^{\frac{1}{2}}$, v = $\frac{2}{3}x^{\frac{1}{2}}$

$2\sqrt{y} = \frac{2}{3}x\,\sqrt{x}Inx - \frac{4}{9}x\sqrt{x} + c$

$2\sqrt{1} = \frac{2}{3}(1)\,\sqrt{(1)}In(1) - \frac{4}{9}(1)\sqrt{(1)} + c$

c = $2 + \frac{4}{9} = \frac{22}{9}$

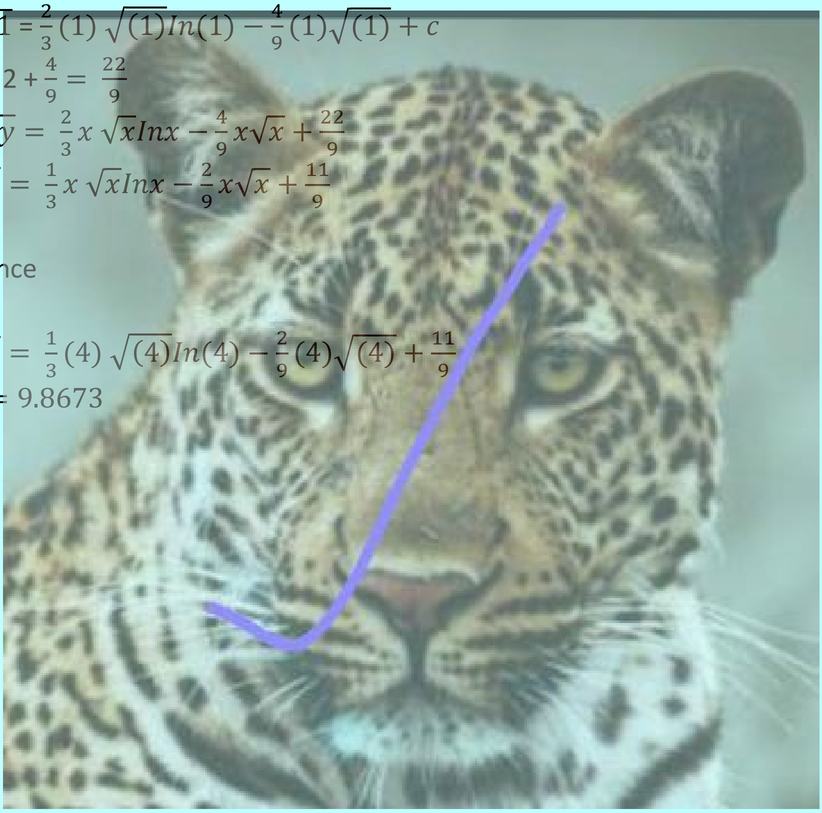
$2\sqrt{y} = \frac{2}{3}x\,\sqrt{x}Inx - \frac{4}{9}x\sqrt{x} + \frac{22}{9}$

$\sqrt{y} = \frac{1}{3}x\,\sqrt{x}Inx - \frac{2}{9}x\sqrt{x} + \frac{11}{9}$

Hence

$\sqrt{y} = \frac{1}{3}(4)\,\sqrt{(4)}In(4) - \frac{2}{9}(4)\sqrt{(4)} + \frac{11}{9}$

$y = 9.8673$



UACE MATHEMATICS PAPER 1 2018 guide

SECTION A (40 marks)

Answer all questions in this section

1. In triangle ABC a= 7cmm b = 4cm and c = 5cm. find the value of
   (a) cos A
   (b) Sin A (05marks)
2. Determine the angle between the lines $\frac{x+4}{8} = \frac{y-2}{2} = \frac{z+1}{-4}$ and the plane 4x + 3y – 3z + 1 = 0 (05marks)
3. Find $\int x^2 e^x dx$ $(05 marks)$
4. Express the function f(x) = x^2 + 12x + 32, in form a$(x + b)^2$ + c.
   Hence find the minimum value of the function f(x) (05mars)
5. A point P moves such that its distance from two points A(-2, 0) and B (8,6) are in ratio AP: PB = 3:2. Show that the locus of P s a circle. (05marks)
6. Determine the equation of the tangent to the curve $y^3 + y^2 - x^4 = 1$ at the point (1, 1) (05marks)
7. Show that 2log4 + $\frac{1}{2}\log 25 - \log 20 = 2\log 2$. (05marks)
8. The region bounded by the curve y = x^2 – 2x and the x-axis from x = 0 to x = 2 is rotated about the x-axis. Calculate the volume of the solid formed. (05marks)

SECTON B: (60 MARKS)

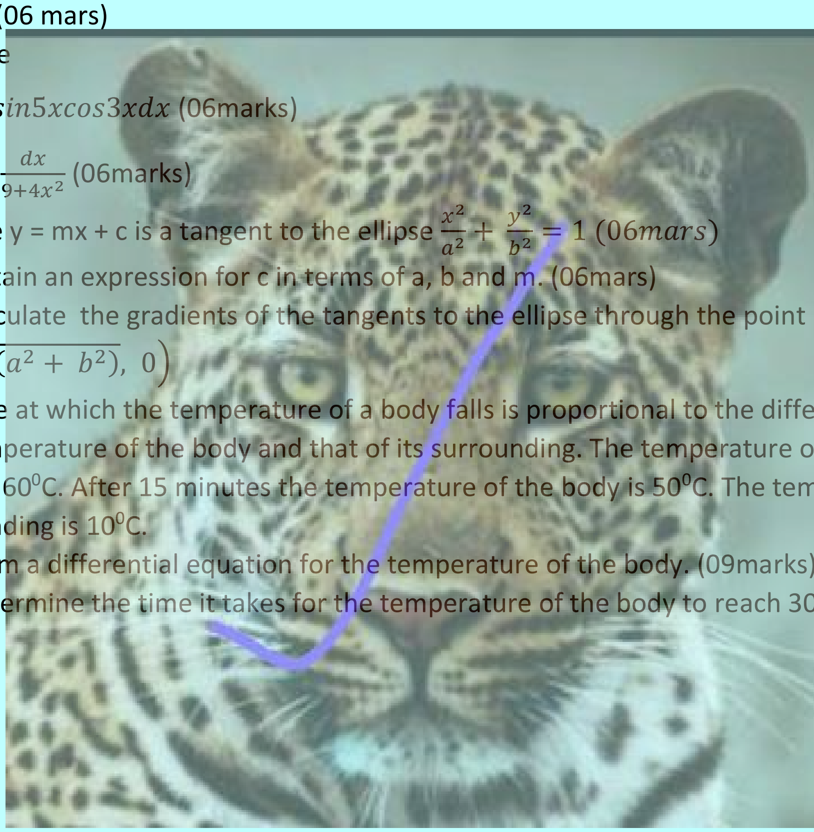
Answer any five questions from this section. All questions carry equal marks

9. The position vectors of the vertices of a triangle are O, r and s, where O is the origin. Show that its area (A) is given by $4A^2 = |r|^2|s|^2 - (r.s)^2$. (06marks)
   Hence, find the area of a triangle when r = $\begin{pmatrix}2\\3\end{pmatrix}$ and s = $\begin{pmatrix}1\\4\end{pmatrix}$ (06marks)

10. Express 5 + 12i in polar form
    Hence, evaluate $\sqrt[3]{5+12i}$, giving your answer in the form a + ib where a and b are corrected to two decimal places. (12marks)
11. (a) Differentiate $\frac{x^3}{\sqrt{(1-2x^2)}}$ with respect to x. (06marks)
    (b) The period T of a swing of a simple pendulum of length of length l is given by
    $T^2 = \frac{4\pi^2 l}{g}$ where g is the acceleration due to gravity.

An error of 2% is made 9n measuring the length, l. determine the resulting percentage error in the period, T. (06marks)

12. (a) Show that tan4θ = $\frac{4t(1-t^2)}{t^4-6t^2+1}$, where t = tanθ (06marks)
    (a) Solve the equation sinx + sin5x = sin2x + sin4x for 0^0 < x < 90^0. (06marks)
13. (a) The first three terms of a Geometric progression (G.P) are 4, 8 and 16. Determine the sum of the first ten terms of the G.P. (04marks)
    (b) An Arithmetic Progression (A.P) has a common difference of 3. A Geometric Progression (G.P) has a common ration of 2. A sequence is formed by subtracting the term of the A.P from the corresponding terms of the G.P. The third term of the sequence is 4. The sixth term of the sequence is 79. Find the first term of the
    (i) A.P (08 marks)
    (ii) G.P (06 mars)
14. Evaluate
    (a) $\int_0^{\frac{\pi}{2}} sin5xcos3xdx$ (06marks)
    (b) $\int_0^{\frac{\sqrt{3}}{2}} \frac{dx}{9+4x^2}$ (06marks)
15. The line y = mx + c is a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\ (06mars)$
    (a) Obtain an expression for c in terms of a, b and m. (06mars)
    (b) Calculate the gradients of the tangents to the ellipse through the point $\left(\sqrt{(a^2 + b^2)},\ 0\right)$
16. The rate at which the temperature of a body falls is proportional to the difference between the temperature of the body and that of its surrounding. The temperature of the body is initially 60^0C. After 15 minutes the temperature of the body is 50^0C. The temperature of the surrounding is 10^0C.
    (a) Form a differential equation for the temperature of the body. (09marks)
    (b) Determine the time it takes for the temperature of the body to reach 30^0C. (03marks)

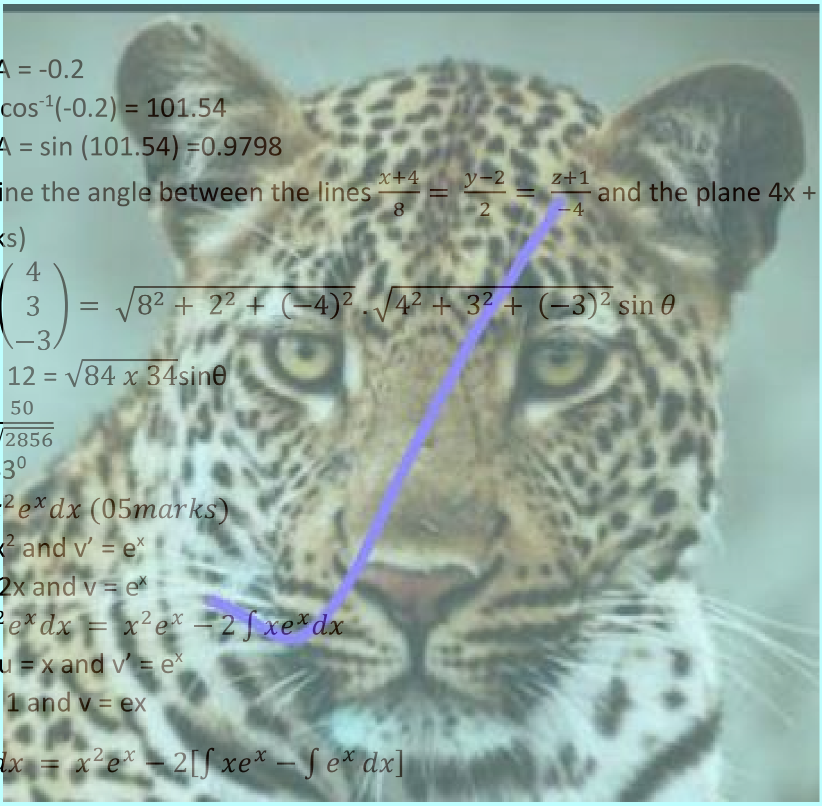


Solutions

SECTION A (40 marks)

Answer all questions in this section

1. In triangle ABC a= 7cmm b = 4cm and c = 5cm. find the value of
   (b) cos A
   $a^2 = b^2 + c^2 - 2bc\cos A$
   $\cos A = \frac{4^2 + 5^2 - 7^2}{2\,(4)(5)} = -0.2$
   (c) Sin A (05marks)
   $\sin^2 A = 1 - \cos^2 A$
   $\sin A = \sqrt{1-(-0.2)^2} = 0.9798$



   or
   cosA = -0.2
   A = $\cos^{-1}(-0.2)$ = 101.54
   sin A = sin (101.54) =0.9798
2. Determine the angle between the lines $\frac{x+4}{8} = \frac{y-2}{2} = \frac{z+1}{-4}$ and the plane 4x + 3y – 3z + 1 = 0 (05marks)
   $$\begin{pmatrix}8\\2\\-4\end{pmatrix}\cdot\begin{pmatrix}4\\3\\-3\end{pmatrix} = \sqrt{8^2+2^2+(-4)^2}\,.\sqrt{4^2+3^2+(-3)^2}\,\sin\theta$$
   $32 + 6 + 12 = \sqrt{84 \ x\ 34}\sin\theta$
   $\sin\theta = \frac{50}{\sqrt{2856}}$
   $\theta = 69.33^0$
3. Find $\int x^2e^x dx$ (05marks)
   Let u = x^2 and v' = e^x
   ⇨ u'= 2x and v = e^x
   $\int x^2e^x dx = x^2e^x - 2\int xe^x dx$
   Again let u = x and v' = e^x
   ⇨ u' = 1 and v = ex
   $\int x^2e^x dx = x^2e^x - 2[\int xe^x - \int e^x dx]$
   $= x^2e^x - 2xe^x + 2e^x + c$
4. Express the function f(x) = $x^2 + 12x + 32$, in form $a(x + b)^2 + c$. Hence find the minimum value of the function f(x) (05mars)
   $f(x) = x^2 + 12x + 32$
   $f(x) = (x + 6)^2 - 6^2 + 32$
   $f(x) = (x + 6)^2 - 4$
   For minimum value to occur, $(x + 6)^2 = 0$
   x = -6
   f'(x) = 2(x + 6)
   Hence the minimum value of f(x) = -4
5. A point P moves such that its distance from two points A(-2, 0) and B (8,6) are in ratio AP: PB = 3:2. Show that the locus of P s a circle. (05marks)
   $\frac{\overline{AP}}{\overline{PB}} = \frac{3}{2}$ => $2\overline{AP} = 3\overline{PB}$

$2\sqrt{(x+2)^2 + y^2} = 3\sqrt{(x-8)^2 + (y-6)^2}$

Squaring both sides

$4(x^2 + 4x + 4 + y^2) = 9(x^2 - 16x + 64 + y^2 - 12y + 36)$

$4x^2 + 16x + 16 + 4y^2 = 9x^2 - 144x + 900 + 9y^2 - 108y$

$5x^2 + 5y^2 - 160x - 108y + 884 = 0$

6. Determine the equation of the tangent to the curve $y^3 + y^2 - x^4 = 1$ at the point (1, 1) (05marks)

$y^3 + y^2 - x^4 = 1$

$3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} + -4x^3 = 0$

At (1, 1)

$3(1)^2\frac{dy}{dx} + 2(1)\frac{dy}{dx} + -4(1)^3 = 0$

$\frac{dy}{dx} = \frac{4}{5}$

$y - 1 = \frac{4}{5}(x-1)$

$5y = 4x + 1$

7. Show that $2\log 4 + \frac{1}{2}\log 25 - \log 20 = 2\log 2$. (05marks)

$2\log 4 + \frac{1}{2}\log 25 - \log 20$

$2\log 2^2 + \frac{1}{2}\log 5^2 - (\log 4 + \log 5)$

$2\log 2^2 + \frac{1}{2}\log 5^2 - \log 4 - \log 5$

$4\log 2 + \log 5 - 2\log 2 - \log 5$

$2\log 2$

8. The region bounded by the curve $y = x^2 - 2x$ and the x-axis from x = 0 to x = 2 is rotated about the x-axis. Calculate the volume of the solid formed. (05marks)

$V = \pi\int_0^2 (x^2 - 2x)^2 dx$

$V = \pi\int_0^2 (x^4 - 4x^3 + 4x^2) dx$

$V = \pi\left[\frac{x^5}{5} - x^4 + \frac{4x^3}{3}\right]_0^2$

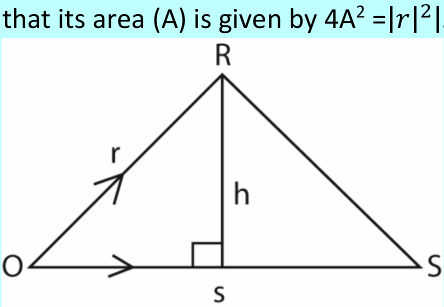
$V = \pi\left(\frac{2^5}{5} - 2^4 + \frac{4(2)^3}{3}\right)$

$= \frac{16\pi}{15}$

SECTON B: (60 MARKS)

Answer any five questions from this section. All questions carry equal marks

9. The position vectors of the vertices of a triangle are O, r and s, where O is the origin. Show that its area (A) is given by $4A^2 = |r|^2|s|^2 - (r.s)^2$. (06marks)



r.s = $|r||s|\cos O$

$(r.s)^2 = |r|^2|s|^2 cos^2\ O$

$sin^2\ O = 1 - \frac{(r.s)^2}{|r|^2|s|^2} = \frac{|r|^2|s|^2 - (r.s)^2}{|r|^2|s|^2}$

$A = \frac{1}{2}|r||s|sinO$

2A = $|r||s|sinO$

4A^2 = $|r|^2|s|^2 sin^2\ O$

4A^2 = $|r|^2|s|^2 . \frac{|r|^2|s|^2 - (r.s)^2}{|r|^2|s|^2}$

4A^2 = $|r|^2|s|^2 - (r.s)^2$

Hence, find the area of a triangle when r = $\begin{pmatrix}2\\3\end{pmatrix}$ and s = $\begin{pmatrix}1\\4\end{pmatrix}$ (06marks)

$|r|^2 = 2^2 + 3^2 = 13$

$|s|^2 = 1^2 + 4^2 = 17$

$r.s = \begin{pmatrix}2\\3\end{pmatrix}.\begin{pmatrix}1\\4\end{pmatrix} = (2\ x\ 1) + (3\ x\ 4) = 14$

∴ 4A^2 = 13 x 17 – 14^2 = 25

$A = \sqrt{\frac{25}{4}}$ = 2.5units

10. Express 5 + 12i in polar form

Hence, evaluate $\sqrt[3]{5 + 12i}$, giving your answer in the form a + ib where a and b are corrected to two decimal places. (12marks)

$|5 + 12i| = \sqrt{5^2 + 12^2} = 13$

Arg(5 + 12i) = tan$^{-1}\left(\frac{12}{5}\right)$ = 67.38^0

5 + 12i = 13(cos 67.38^0 + isin67.38^0)

$\sqrt[3]{5 + 12i} = \sqrt[3]{13}\left[\cos\left(\frac{67.38+2\pi k}{3}\right) + isin\ \left(\frac{67.38+2\pi k}{3}\right)\right]$

$taking\ k = 0, 1,2$

Considering k = 0

$\sqrt[3]{5 + 12i} = \sqrt[3]{13}\left[\cos\left(\frac{67.38+0}{3}\right) + isin\ \left(\frac{67.38+0}{3}\right)\right]$

= 2.17 + 0.90i

Considering k = 1

$\sqrt[3]{5 + 12i} = \sqrt[3]{13}\left[\cos\left(\frac{67.38+2\pi}{3}\right) + isin\ \left(\frac{67.38+2\pi}{3}\right)\right]$

= -1.86 + 1.43i

Considering k = 2

$\sqrt[3]{5 + 12i} = \sqrt[3]{13}\left[\cos\left(\frac{67.38+4\pi}{3}\right) + isin\ \left(\frac{67.38+4\pi}{3}\right)\right]$

= -0.31 – 2.33i

11. (a) Differentiate $\frac{x^3}{\sqrt{(1-2x^2)}}$ with respect to x. (06marks)

$y = \frac{x^3}{\sqrt{(1-2x^2)}}$

$\frac{dy}{dx} = \frac{3x^2\sqrt{1-2x^2} - \frac{-4x(x^3)}{2\sqrt{1-2x^2}}}{\left(\sqrt{1-2x^2}\right)^2}$

$= \frac{3x^2(1-2x^2)+2x^4}{(1-2x^2)\sqrt{1-2x^2}}$

$= \frac{3x^2-6x^4+2x^4}{(1-2x^2)\sqrt{1-2x^2}}$

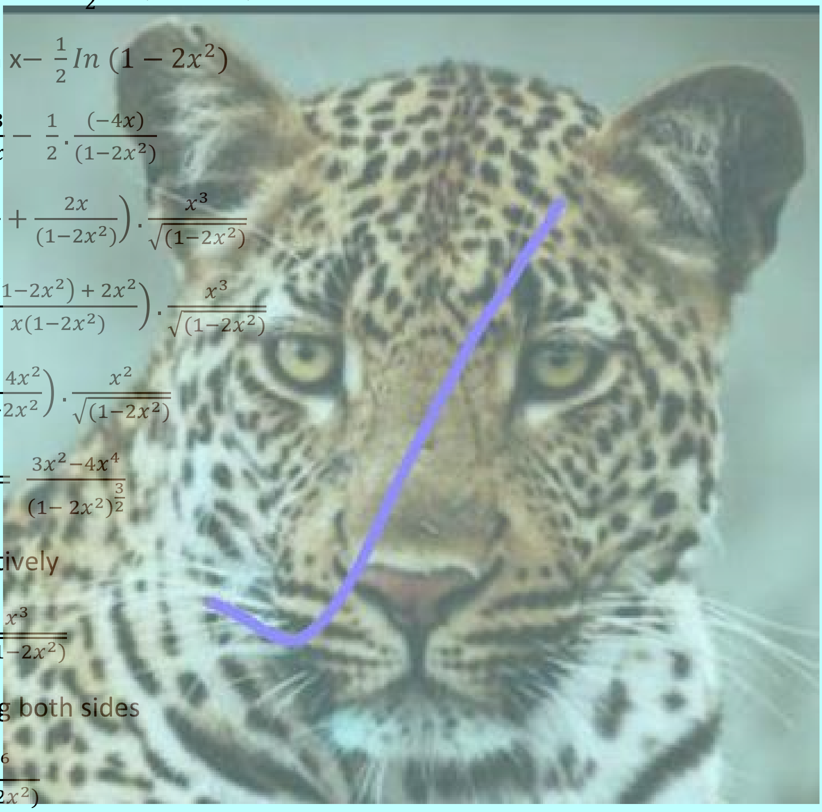
$\frac{dy}{dx} = \frac{3x^2-4x^4}{(1-2x^2)^{\frac{3}{2}}}$

Alternatively

$y = \frac{x^3}{\sqrt{(1-2x^2)}}$

Introducing In on both sides

In y = $In\ x^3 - \frac{1}{2} In\ (1-2x^2)$



=3In x$- \frac{1}{2} In\ (1-2x^2)$

$\frac{1}{y}\frac{dy}{dx} = \frac{3}{x} - \frac{1}{2}.\frac{(-4x)}{(1-2x^2)}$

$\frac{dy}{dx} = \left(\frac{3}{x} + \frac{2x}{(1-2x^2)}\right).\frac{x^3}{\sqrt{(1-2x^2)}}$

$= \left(\frac{3(1-2x^2)+2x^2}{x(1-2x^2)}\right).\frac{x^3}{\sqrt{(1-2x^2)}}$

$=\left(\frac{3-4x^2}{1-2x^2}\right).\frac{x^2}{\sqrt{(1-2x^2)}}$

$=\frac{dy}{dx} = \frac{3x^2-4x^4}{(1-2x^2)^{\frac{3}{2}}}$

Alternatively

$y = \frac{x^3}{\sqrt{(1-2x^2)}}$

Squaring both sides

y^2 = $\frac{x^6}{(1-2x^2)}$

2Iny = 6Inx - In$(1-2x^2)$

$\frac{2}{y}\frac{dy}{dx} = \frac{6}{x} + \frac{4x}{(1-2x^2)}$

$\frac{dy}{dx} = \left(\frac{3}{x} + \frac{2x}{(1-2x^2)}\right).\frac{x^3}{\sqrt{(1-2x^2)}}$

$= \left(\frac{3(1-2x^2)+2x^2}{x(1-2x^2)}\right).\frac{x^3}{\sqrt{(1-2x^2)}}$

$=\left(\frac{3-4x^2}{1-2x^2}\right).\frac{x^2}{\sqrt{(1-2x^2)}}$

$=\frac{dy}{dx} = \frac{3x^2-4x^4}{(1-2x^2)^{\frac{3}{2}}}$

(c) The period T of a swing of a simple pendulum of length of length l is given by $T^2 = \frac{4\pi^2 l}{g}$ where g is the acceleration due to gravity.

An error of 2% is made 9n measuring the length, l. determine the resulting percentage error in the period, T. (06marks)

$T^2 = \frac{4\pi^2 l}{g}$ => $T = 2\pi\sqrt{\frac{l}{g}}$

$2T\frac{dT}{dl} = \frac{4\pi^2}{g}$

$\frac{dT}{dl} = \frac{2\pi^2}{gT} = \frac{2\pi^2}{g\left(2\pi\sqrt{\frac{l}{g}}\right)} = \frac{\pi}{\sqrt{gl}}$

$\frac{\delta T}{\delta l} = \frac{dT}{dl}$ => $\delta T = \frac{dT}{dl}.\delta l$

$\frac{\delta T}{T}\ x\ 100\% = \frac{\frac{\pi\delta l}{\sqrt{gl}}x\ 100\%}{2\pi\sqrt{\frac{l}{g}}}$

$= \frac{\delta l}{2l}x\ 100\%$

$= \frac{1}{2}\ x\ \frac{2}{100}\ x\ 100\% = 1\%$

12. (a) Show that $\tan 4\theta = \frac{4t(1-t^2)}{t^4-6t^2+1}$, where $t = \tan\theta$ (06marks)

$\tan 4\theta = \frac{2\tan 2\theta}{1-\tan^2 2\theta}$ and $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

$= \frac{2\left(\frac{2t}{1-t^2}\right)}{1-\left(\frac{2t}{1-t^2}\right)^2}$

$= \frac{4t(1-t^2)}{t^4-6t^2+1}$

(b) Solve the equation sinx + sin5x = sin2x + sin4x for $0^0 < x < 90^0$. (06marks)

sinx + sin5x = sin2x + sin4x

sin5x –sin 4x-sin2x + sin x = 0

2sin3xcos2x -2sin3xcosx =0

2sin3x[cos2x-cos x] = 0

$-4\sin 3x \sin\frac{3x}{2}\sin\frac{x}{2} = 0$

Either sin3x =0, $\sin\frac{3x}{2} = 0$, $\sin\frac{x}{2} = 0$

$3x, \frac{3x}{2}, \frac{x}{2} = 0^0, 180^0$

$X = 60^0$

13. (a) The first three terms of a Geometric progression (G.P) are 4, 8 and 16. Determine the sum of the first ten terms of the G.P. (04marks)

(b) An Arithmetic Progression (A.P) has a common difference of 3. A Geometric Progression (G.P) has a common ration of 2. A sequence is formed by subtracting the term of the A.P from the corresponding terms of the G.P. The third term of the sequence is 4. The sixth term of the sequence is 79. Find the first term of the

(i) A.P (08 marks)

(a) a = 4, ar = 6

4r = 8

r = 2

$S_{10} = 4\left(\frac{2^{10}-1}{2-1}\right) = 4092$

(ii) G.P (06 mars)

A.P

x, x+3, x +6, x + 9, x + 12, x+ 15, ……

G.P

y, 2y, 4y, 8y, 16y, 32y, ….

4y – (x + 6) = 4

4y – x = 10 ……………..(i)

32y – (x + 15) = 79

32y – x = 94…………. (ii)

Eqn (ii) – Eqn (i)

28y = 84, => y = 3

Substituting for y into eqn (i)

12 – x = 10

x =2

(i) A.P, U_1 = 2

(ii) G.P, U_1 =3

14. Evaluate

(c) $\int_0^{\frac{\pi}{2}} sin5xcos3xdx$ (06marks)

$$\int_0^{\frac{\pi}{2}} sin5xcos3xdx = \frac{1}{2}\int_0^{\frac{\pi}{2}} sin8x + sin2xdx$$

$$= \frac{1}{2}\left[-\frac{1}{8}cos8x - \frac{1}{2}cos2x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[-\frac{1}{8}(cos4\pi - cos0) - \frac{1}{2}(cos\pi - cos0)\right]$$

$$= \frac{1}{2}\left[-\frac{1}{8}(1-1) - \frac{1}{2}(-1-1)\right]$$

$$= \frac{1}{2}$$

(d) $\int_0^{\frac{\sqrt{3}}{2}} \frac{dx}{9+4x^2}$ (06marks)

Let X = $\frac{3}{2}\tan\theta$

dx = $\frac{3}{2}sec^2\theta$

x	θ
0	0
$\frac{\sqrt{3}}{2}$	$\frac{\pi}{6}$

$$\int_0^{\frac{\sqrt{3}}{2}} \frac{dx}{9+4x^2} = \int_0^{\frac{\pi}{6}} \frac{\frac{3}{2}sec^2\theta}{9+4\left(\frac{9}{4}tan^2\theta\right)}$$

$$= \frac{1}{6}\int_0^{\frac{\pi}{6}} \frac{sec^2\theta\, d\theta}{1+tan^2\theta} = \frac{1}{6}\int_0^{\frac{\pi}{6}} d\theta$$

$$= \frac{1}{6}[\theta]_0^{\frac{\pi}{6}} = \frac{1}{6}\left(\frac{\pi}{6} - 0\right)$$

$$= \frac{\pi}{36} = 0.087266$$

15. The line y = mx + c is a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(06mars)$

(c) Obtain an expression for c in terms of a, b and m. (06mars)

$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$

$b^2x^2 + a^2(m^2x^2 + 2mx + c^2) = a^2b^2$

$(b^2 + a^2m^2)x^2 + 2a^2mx + a^2c^2 - a^2b^2$

$(2a^2mc)^2 = 4(b^2 + a^2m^2)(a^2c^2 - a^2b^2)$

$4a^4m^2c^2 = 4a^2(b^2c^2 - b^4 + a^2m^2c^2 - a^2m^2b^2)$

$b^2c^2 = b^4 + a^2m^2b^2$

$c^2 = a^2m^2 + b^2$

(d) Calculate the gradients of the tangents to the ellipse through the point $\left(\sqrt{(a^2 + b^2)}, 0\right)$

y = mx $\pm \sqrt{(a^2m^2 + b^2)}$

0 =m$\sqrt{(a^2 + b^2)} \pm \sqrt{(a^2m^2 + b^2)}$

$\left(\pm\sqrt{(a^2m^2 + b^2)}\right)^2 = \left(\text{m}\sqrt{(a^2 + b^2)}\right)^2$

$a^2m^2 + b^2 = m^2(a^2 + b^2)$

$a^2m^2 + b^2 = m^2a^2 + m^2b^2$

$b^2 = m^2b^2$

$m^2 = 1$

m $= \pm 1$

16. The rate at which the temperature of a body falls is proportional to the difference between the temperature of the body and that of its surrounding. The temperature of the body is initially 60^{0}C. After 15 minutes the temperature of the body is 50^{0}C. The temperature of the surrounding is 10^{0}C.

(c) Form a differential equation for the temperature of the body. (09marks)

$\frac{d\theta}{dt} \propto (\theta - 10)$

$\frac{d\theta}{dt} = k(\theta - 10)$

$\frac{d\theta}{(\theta-10)}$ = -kdt

$\int \frac{d\theta}{(\theta-10)} = -k \int dt$

$In\ (\theta - 10) = -kt + c$

When t = 0, θ = 60

c = $In\ (60 - 10)$ =In50

when t = 15, θ = 50

$In\ 40 = -k\ x\ 15 + In50$

15k = In$\left(\frac{50}{40}\right)$

k = $\frac{1}{15} In \left(\frac{5}{4}\right)$

Hence the differential equation is

$\frac{d\theta}{dt} = \frac{1}{15} In \left(\frac{5}{4}\right)(\theta - 10)$

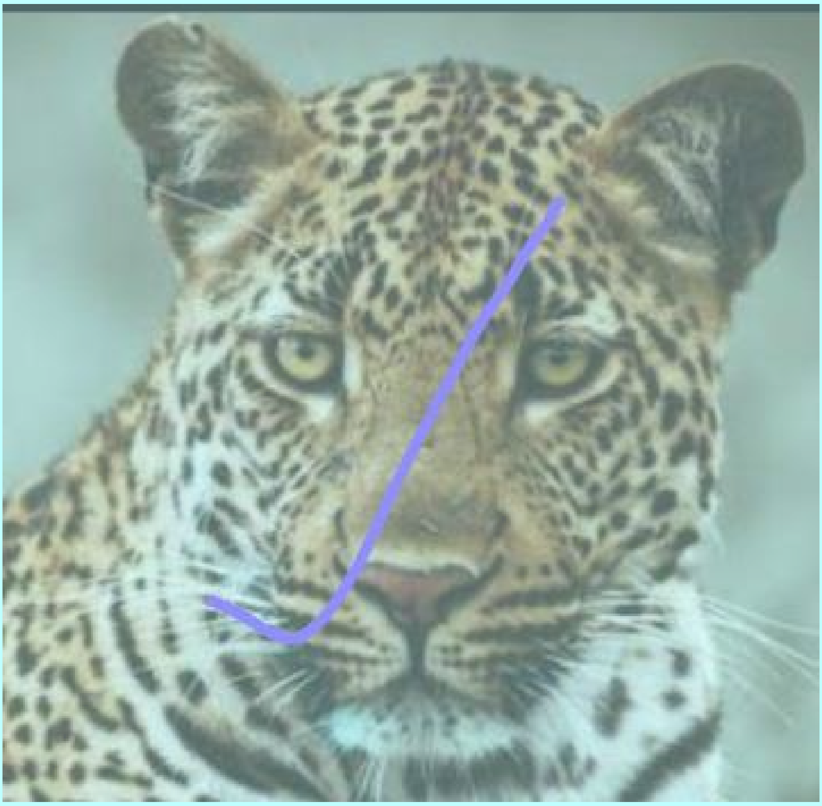
(d) Determine the time it takes for the temperature of the body to reach 30^0C. (03marks)

$In\ (\theta - 10) = -\frac{1}{15} In\left(\frac{5}{4}\right) t + In\ 50$

When t = T and $\theta = 30$

$-\frac{1}{15} In\left(\frac{5}{4}\right) T = In\ 50 - In\ 20$

$T = \frac{15 In\left(\frac{5}{2}\right)}{In\left(\frac{4}{4}\right)} = 61.5943$minutes



UACE MATHEMATICS PAPER 1 2019 guide

SECTION A

1. Show that the modulus of $\frac{(1-i)^6}{(1+i)}$=4$\sqrt{2}$ (05marks)
2. Solve 2cos2θ – 5cos θ = 4 for $0^0 \leq θ \leq 360^0$. (05marks)
3. Using the substitution u = $tan^{-1}x$; show $\int_0^1 \frac{tan^{-1}x}{1+x^2}$dx = $\frac{\pi^2}{32}$ (05marks)
4. Given the plane 4x + 3y – 3z – 4 = 0
   (a) Show that the point A(1,1,1) lies on the plane (02marks)
   (b) Find the perpendicular distance from the plane to the point B(1, 5,1) (03marks)
5. Find the equation of the tangent to the curve y = $\frac{a^3}{x^2}$ at the point ($\frac{a}{t}$, at^2). (05marks)
6. Given that α+ β = $\frac{-1}{3}$ and αβ = $\frac{2}{3}$, form a quadratic equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ (05marks)
7. Find the area enclosed between the curve y = $2x^2$ – 4x and x-axis. (05marks)
8. Given that Q = $\sqrt{80-0.1P}$ and E = $\frac{-dQ}{dP}$. $\frac{P}{Q}$; find E when P = 600

SECTION B

9. (a) Determine the perpendicular distance of the point (4, 6) from the line 2x + 4y -3 = 0 (03marks)
   (b) Show that the angle θ, between two lines with gradient λ_1 and λ_2 is given by θ = $tan^{-1}\left(\frac{\lambda_1-\lambda_2}{1+\lambda_1\lambda_2}\right)$.Hence find the acute angle between the lines x + y + 7 = 0 and $\sqrt{3x} - y + 5 = 0$ (09 marks)
10. (a) Given that 26$\left(1-\frac{1}{26^2}\right)^{1/2}$= a$\sqrt{3}$, find the value of a(05marks)
    (b) Solve the simultaneous equations:

    2x = 3y = 4z

    x^2 - $9y^2$ – 4z + 8 = 0 (07marks)
11. Express 7cos 2θ + 6sin 2θ in form Rcos (2θ – α), where R is a constant and α is an acute angle. (05marks)
    Hence solve 7cos 2θ + 6sin 2θ = 5 for $0^0 \leq \theta \leq 180^0$. (07marks)
12. (a) given that y = $In\left\{e^x\left(\frac{x-2}{x+2}\right)^{\frac{3}{4}}\right\}$, show that $\frac{dy}{dx} = \frac{x^2-1}{x^2-4}$ (05marks)
    (b) Evaluate $\int \frac{dx}{x^2\sqrt{(25-x^2)}}$ (07marks)
13. Four points have coordinates
    A(3, 4, 7), B(13, 9, 2), C(1, 2, 3) and D(10, k,6). The lines AB and CD intersect at P.

Determine the;

(i) Vector equation of the lines AB and CD. (06marks)

(ii) Value of k (04marks)

(iii) Coordinates of P (02marks)

14. Expand $\sqrt{\left(\frac{1+2x}{1-x}\right)}$ up to the term x^2. Hence find the value of $\sqrt{\left(\frac{1.04}{0.98}\right)}$ to four significant figures. (12marks)

15. (a) Differentiate $y = 2x^2 + 3$ from first principles. (04marks)

(b) A rectangular sheet is 50cm long and 40cm wide. A square of x cm is cut off from each corner. The remaining sheet is folded to form an open box. Find the maximum volume of the box (08marks)

16. (a) Find $\int \frac{In\, x}{x^2} dx$ (04marks)

(c) Solve the differential equation $\frac{dy}{dx} + y\cot x = x$, given that y = 1 when $x = \frac{\pi}{2}$. (08marks)

Solutions

SECTION A

1. Show that the modulus of $\frac{(1-i)^6}{(1+i)}=4\sqrt{2}$ (05marks)

Method I

$$\left|\frac{(1-i)^6}{1+i}\right| = \frac{|1-i|^6}{|1+i|} = \frac{\left(\sqrt{1^2+(-1)^2}\right)^6}{\sqrt{1^2+1^2}} = \frac{(\sqrt{2})^6}{\sqrt{2}} = \left(\sqrt{2}\right)^5 = \left(\sqrt{2}\right)^4\sqrt{2} = 4\sqrt{2}$$

Method II

$$\frac{(1-i)^6}{(1+i)} = \frac{1+\ {}^6c_1(-i)^2+\ {}^6c_2(-i)^2+\ {}^6c_3(-i)^2+\ {}^6c_4(-i)^2+\ {}^6c_5(-i)^2+\ {}^6c_6(-i)^2}{1+i}$$

$$= \frac{1-6i-15+20i+15-6i-1}{1+i}$$

$$= \frac{8i}{1+i}\ x\ \frac{1-i}{1-i}$$

$$= \frac{8i+8}{1+1} = 4 + 4i$$

$$|4+4i| = \sqrt{4^2+4^2} = \sqrt{32} = \sqrt{16}.\sqrt{2} = 4\sqrt{2}$$

2. Solve $2\cos 2\theta - 5\cos\theta = 4$ for $0^0 \le \theta \le 360^0$. (05marks)

$2\cos 2\theta - 5\cos\theta = 4$

$2(2\cos^2\theta - 1) - 5\cos\theta = 4$

$4\cos^2\theta - 5\cos\theta - 6 = 0$

$\cos\theta = \frac{5\pm\sqrt{(-5)^2-4(4)(-6)}}{2(4)} = \frac{-3}{4}, 2$

Either $\cos\theta = \frac{-3}{4}$

$\therefore \theta = 138.59^0$ and $\theta = 221.41^0$

3. Using the substitution $u = \tan^{-1}x$; show $\int_0^1 \frac{\tan^{-1}x}{1+x^2}dx = \frac{\pi^2}{32}$ (05marks)

$u = \tan^{-1}x$

$\frac{du}{dx} = \frac{1}{1+x^2}$

$du = \frac{dx}{1+x^2}$

Changing limits

x	U
0	0
1	$\frac{\pi}{4}$

By change of variable;

$$\int_0^1 \frac{tan^{-1} x}{1+x^2}dx = \int_0^{\frac{\pi}{4}} udu = \left[\frac{u^2}{2}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2} x \left(\frac{\pi}{4}\right)^2$$

$$= \frac{\pi^2}{32}$$

4. Given the plane 4x + 3y – 3z-4 = 0
   (a) Show that the point A(1,1,1) lies on the plane (02marks)
   Substitute A(1, 1, 1) into the equation of plane
   4 x 1 + 3 x 1 – 3 x 1 – 4 = 0
   Hence the point lies on the plane
   (b) Find the perpendicular distance from the plane to the point B(1, 5,1) (03marks)
   $d = \frac{|4 \, x \, 1+3 \, x \, 5-3 \, x \, 1-4|}{\sqrt{4^2+3^2+(-3)^2}} = 2.058$

5. Find the equation of the tangent to the curve $y = \frac{a^3}{x^2}$ at the point $(\frac{a}{t}, at^2)$. (05marks)

Given that $y = \frac{a^3}{x^2}, \frac{dy}{dx} = \frac{-2a^3}{x^3}$

Gradient $m = \frac{-2a^3}{\frac{a^3}{t^3}} = -2t^3$

From $y - y_1 = m(x - x_1)$

$y - at^2 = -2t^3\left(x - \frac{a}{t}\right)$

$\therefore y = 3at^2 - 2t^3x$

6. Given that $\alpha + \beta = \frac{-1}{3}$ and $\alpha\beta = \frac{2}{3}$, form a quadratic equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ (05marks)

Sum of the roots $= \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2+\beta^2}{\alpha\beta} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{\left(\frac{-1}{3}\right)^2 - 2 \, x \, \frac{2}{3}}{\frac{2}{3}} = \frac{-11}{6}$

Product of the roots $= \frac{\alpha}{\beta} \, x \, \frac{\beta}{\alpha} = 1$

Equation

x^2 + (sum)x + product = 0

$x^2 - \frac{11}{6}x + 1 = 0$

or

$6x^2 - 11x + 6 = 0$

7. Find the area enclosed between the curve $y = 2x^2 - 4x$ and x-axis. (05marks)

On x-axis, y= 0

$2x^2 - 4x = 0$

Either x = 0 or 2

Area enclosed = $\int_0^2 (2x^2 + 4x)dx$

$= \left[\frac{2x^3}{3} - 2x^2\right]_0^2$

$= \frac{2(2)^3}{3} - 2(2)^2 = \frac{-8}{3}$

$\therefore$ area = $\frac{8}{3}$ square units

8. Given that Q = $\sqrt{80 - 0.1P}$ and E = $\frac{-dQ}{dP} \cdot \frac{P}{Q}$; find E when P = 600

$\frac{dQ}{dP} = \frac{-0.1}{2\sqrt{80-0.1P}}$

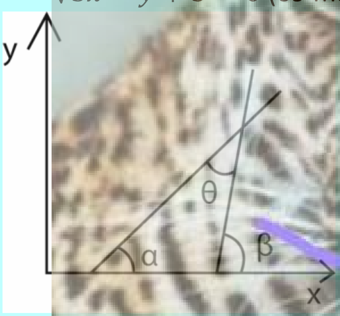
E = $\frac{0.1}{2\sqrt{80-0.1(600)}}$ x $\frac{600}{2\sqrt{80-0.1(600)}}$ = 1.5

SECTION B

9. (a) Determine the perpendicular distance of the point (4, 6) from the line 2x + 4y -3 = 0 (03marks)

Perpendicular distance, d = $\frac{|2(4)+4(6)-3|}{\sqrt{2^2+4^2}} = \frac{29}{\sqrt{20}}$ = 64846

(b) Show that the angle θ, between two lines with gradient λ_1 and λ_2 is given by θ = $\tan^{-1}\left(\frac{\lambda_1 - \lambda_2}{1+\lambda_1\lambda_2}\right)$.Hence find the acute angle between the lines x + y + 7 = 0 and $\sqrt{3x} - y + 5 = 0$ (09 marks)



Tan$\alpha = \lambda_2$, tan$\beta = \lambda_1$

$\alpha + \theta = \beta$; $\theta = \beta - \alpha$

tanθ = tan $(\beta - \alpha)$

$= \frac{\tan\beta - \tan\alpha}{1+\tan\alpha\tan\beta}$

$= \left(\frac{\lambda_1 - \lambda_2}{1+\lambda_1\lambda_2}\right)$

$\therefore \theta = \tan^{-1}\left(\frac{\lambda_1 - \lambda_2}{1+\lambda_1\lambda_2}\right)$

But λ_1 = -1 and $\lambda_2 = \sqrt{3}$

$\theta = \tan^{-1}\left(\frac{-1-\sqrt{3}}{1+(-1\,x\,\sqrt{3})}\right) = 75^0$

10. (a) Given that $26\left(1 - \frac{1}{26^2}\right)^{1/2} = a\sqrt{3}$, find the value of a(05marks)

$26\left(1 - \frac{1}{26^2}\right)^{1/2} = a\sqrt{3}$

$26\left(\frac{26^2-1}{26^2}\right)^{1/2} = a\sqrt{3}$

$(675)^{\frac{1}{2}} = a\sqrt{3}$

a = $\left(\frac{675}{3}\right)^{\frac{1}{2}} = \pm 15$

(b) Solve the simultaneous equations:

$2x = 3y = 4z$

$x^2 - 9y^2 - 4z + 8 = 0$ (07marks)

$2x = 3y = 4z$, substituting $4z = 2x$ and $y = \frac{2x}{3}$ into the equation $x^2 - 9y^2 - 4z + 8 = 0$

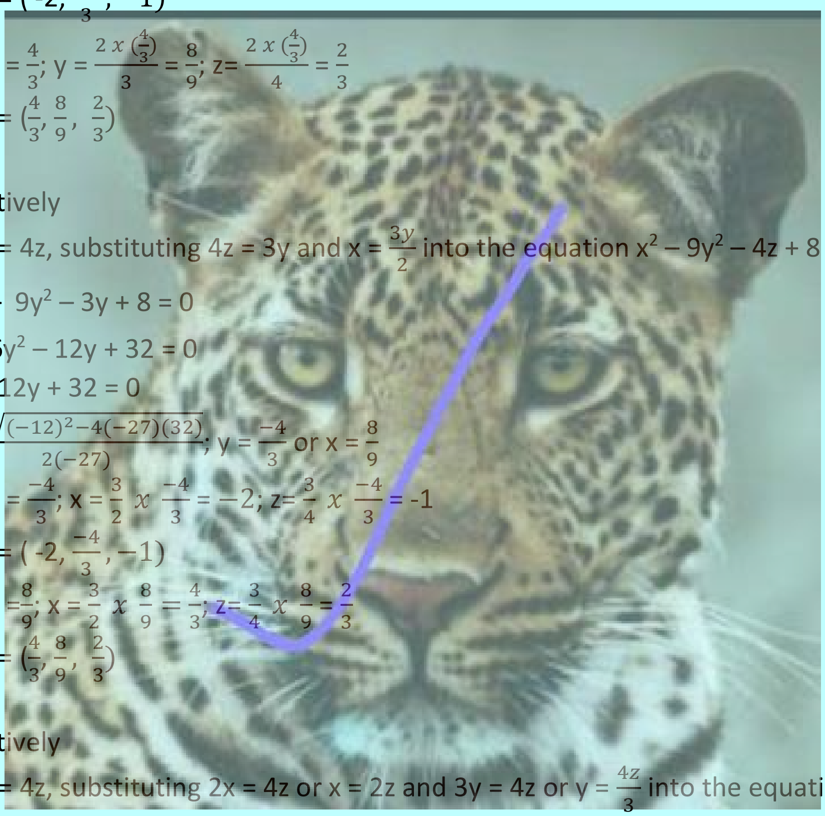
$x^2 - (2x)^2 - 2x + 8 = 0$

$-3x^2 - 2x + 8 = 0$

$x = \frac{2 \pm \sqrt{(-2)^2 - 4(-3)(8)}}{2(-3)}$; $x = -2$ or $x = \frac{4}{3}$

When $x = -2$; $y = \frac{2 \times (-2)}{3} = \frac{-4}{3}$; $z = \frac{2 \times (-2)}{4} = -1$

$(x, y, z) = \left(-2, \frac{-4}{3}, -1\right)$



When $x = \frac{4}{3}$; $y = \frac{2 \times \left(\frac{4}{3}\right)}{3} = \frac{8}{9}$; $z = \frac{2 \times \left(\frac{4}{3}\right)}{4} = \frac{2}{3}$

$(x, y, z) = \left(\frac{4}{3}, \frac{8}{9}, \frac{2}{3}\right)$

Alternatively

$2x = 3y = 4z$, substituting $4z = 3y$ and $x = \frac{3y}{2}$ into the equation $x^2 - 9y^2 - 4z + 8 = 0$

$\left(\frac{3}{2}y\right)^2 - 9y^2 - 3y + 8 = 0$

$9y^2 - 36y^2 - 12y + 32 = 0$

$-27y^2 - 12y + 32 = 0$

$y = \frac{12 \pm \sqrt{(-12)^2 - 4(-27)(32)}}{2(-27)}$; $y = \frac{-4}{3}$ or $x = \frac{8}{9}$

When $y = \frac{-4}{3}$; $x = \frac{3}{2} x \frac{-4}{3} = -2$; $z = \frac{3}{4} x \frac{-4}{3} = -1$

$(x, y, z) = \left(-2, \frac{-4}{3}, -1\right)$

When $y = \frac{8}{9}$; $x = \frac{3}{2} x \frac{8}{9} = \frac{4}{3}$; $z = \frac{3}{4} x \frac{8}{9} = \frac{2}{3}$

$(x, y, z) = \left(\frac{4}{3}, \frac{8}{9}, \frac{2}{3}\right)$

Alternatively

$2x = 3y = 4z$, substituting $2x = 4z$ or $x = 2z$ and $3y = 4z$ or $y = \frac{4z}{3}$ into the equation

$(2z)^2 - (4z)^2 - 4z + 8 = 0$

$4z^2 - 16z^2 - 4z + 8 = 0$

$-12z^2 - 4z + 8 = 0$

$z = \frac{-1 \pm \sqrt{(1)^2 - 4(3)(-2)}}{2(3)}$; $z = -1$ or $z = \frac{2}{3}$

When $z = -1$; $y = \frac{4(-1)}{3} = \frac{-4}{3}$; $x = 2(-1) = -2$

$(x, y, z) = \left(-2, \frac{-4}{3}, -1\right)$

When $z = \frac{2}{3}$; $y = \frac{4}{3} x \frac{2}{3} = \frac{8}{9}$; $x = 2 \times \frac{2}{3} = \frac{4}{3}$

$(x, y, z) = \left(\frac{4}{3}, \frac{8}{9}, \frac{2}{3}\right)$

11. Express 7cos 2θ + 6sin 2θ in form Rcos (2θ – α), where R is a constant and α is an acute angle. (05marks)

7cos 2θ + 6sin 2θ ≡ Rcos (2θ – α)

7cos 2θ + 6sin 2θ ≡ Rcos 2θ cos α + Rsin2θsinα

Comparing both sides

Rcosα = 7 ………………… (i)

Rsinα = 6 ………………… (ii)

(i)2 + (ii)2 gives

R = $\sqrt{7^2 + 6^2}$ = $\sqrt{85}$

From equation (i)

$\sqrt{85}$cosα = 7

$\alpha = cos^{-1}\left(\frac{7}{\sqrt{85}}\right)$ = 40.6^0

Hence solve 7cos 2θ + 6sin 2θ = 5 for $0^0 \leq \theta \leq 180^0$. (07marks)

∴ 7cos 2θ + 6sin 2θ = $\sqrt{85}$ cos(2θ – 40.6^0) =5

$2\theta - 40.6 = cos^{-1}\left(\frac{5}{\sqrt{85}}\right)$ = 57.16^0, 302.84^0

$\theta = 48.88^0, 171.72^0$

12. (a) given that y = $In\left\{e^x\left(\frac{x-2}{x+2}\right)^{\frac{3}{4}}\right\}$, show that $\frac{dy}{dx} = \frac{x^2-1}{x^2-4}$ (05marks)

y = $In\left\{e^x\left(\frac{x-2}{x+2}\right)^{\frac{3}{4}}\right\}$

= $Ine^x + \frac{3}{4}\{In(x-2) + In(x+2)\}$

= $xIne + \frac{3}{4}\{In(x-2) + In(x+2)\}$

$\frac{dy}{dx} = 1 + \frac{3}{4}\left\{\frac{1}{x-2} - \frac{1}{x+2}\right\} = 1 + \frac{3}{4}\left\{\frac{x+2-x+2}{x^2-4}\right\} = 1 + \frac{3}{4}\left\{\frac{4}{x^2-4}\right\} = 1 + \frac{3}{x^2-4} = \frac{x^2-4+3}{x^2-4} = \frac{x^2-1}{x^2-4}$

(b) Evaluate $\int\frac{dx}{x^2\sqrt{(25-x^2)}}$ (07marks)

$\int\frac{dx}{x^2\sqrt{(25-x^2)}}$

Let x = 5sinθ

⇨ dx = 5cosθdθ

$\int\frac{dx}{x^2\sqrt{(25-x^2)}} = \int\frac{5cos\theta d\theta}{25sin^2\theta\sqrt{25-25sin^2\theta}}$

$= \frac{1}{25}\int cosec^2\theta d\theta$

$= -\frac{1}{25}cot\theta + C$

But sinθ = $\frac{x}{5}$, cosθ = $\left(\frac{\sqrt{25-x^2}}{x}\right)$

$\therefore \int\frac{dx}{x^2\sqrt{(25-x^2)}} = -\frac{1}{25}\left(\frac{5\sqrt{25-x^2}}{x^2}\right)$+ C

13. Four points have coordinates

A(3, 4, 7), B(13, 9, 2), C(1, 2, 3) and D(10, k,6). The lines AB and CD intersect at P. Determine the;

(i) Vector equation of the lines AB and CD. (06marks)

$$\underline{r} = \begin{pmatrix}3\\4\\7\end{pmatrix} + \lambda\left[\begin{pmatrix}13\\9\\2\end{pmatrix} - \begin{pmatrix}3\\4\\7\end{pmatrix}\right]$$

$$\underline{r} = \begin{pmatrix}3\\4\\7\end{pmatrix} + \lambda\begin{pmatrix}10\\5\\-5\end{pmatrix}$$

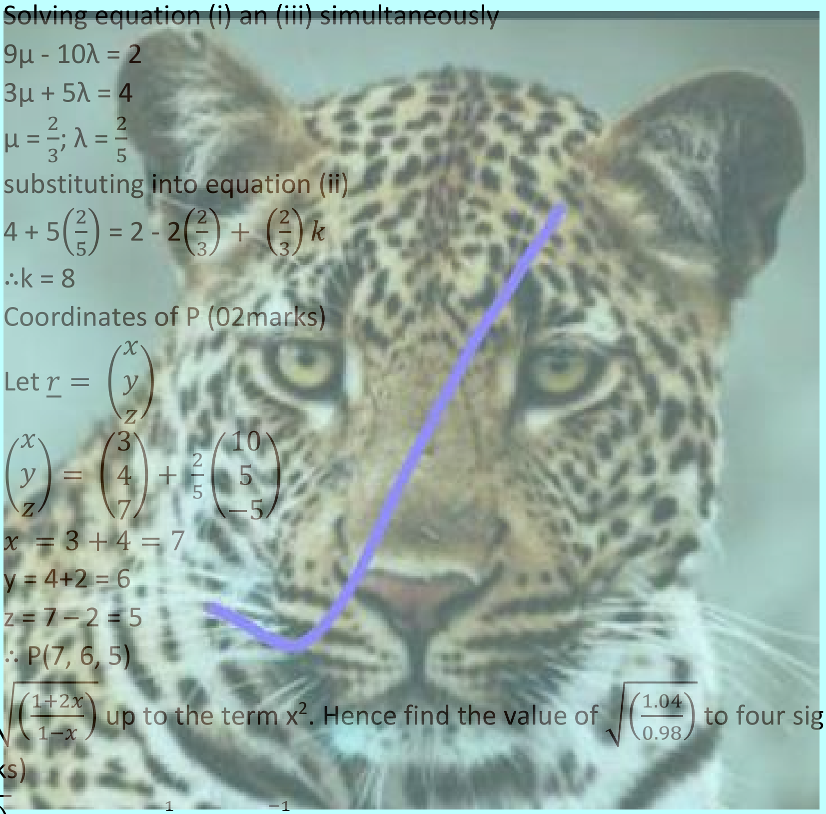
(ii) Value of k (04marks)

Equating corresponding entries

$3 + 10\lambda = 1 + 9\mu$ ………………………… (i)

$4 + 5\lambda = 2 + \mu k - 2\mu$ ……………………….. (ii)

$7 - 5\lambda = 3 + 3\mu$…………………………… (iii)



Solving equation (i) an (iii) simultaneously

$9\mu - 10\lambda = 2$

$3\mu + 5\lambda = 4$

$\mu = \frac{2}{3}$; $\lambda = \frac{2}{5}$

substituting into equation (ii)

$4 + 5\left(\frac{2}{5}\right) = 2 - 2\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)k$

$\therefore k = 8$

(iii) Coordinates of P (02marks)

Let $\underline{r} = \begin{pmatrix}x\\y\\z\end{pmatrix}$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}3\\4\\7\end{pmatrix} + \frac{2}{5}\begin{pmatrix}10\\5\\-5\end{pmatrix}$$

$x = 3 + 4 = 7$

$y = 4+2 = 6$

$z = 7 - 2 = 5$

$\therefore$ P(7, 6, 5)

14. Expand $\sqrt{\left(\frac{1+2x}{1-x}\right)}$ up to the term x^2. Hence find the value of $\sqrt{\left(\frac{1.04}{0.98}\right)}$ to four significant figures. (12marks)

$$\sqrt{\left(\frac{1+2x}{1-x}\right)} = (1+2x)^{\frac{1}{2}}(1-x)^{\frac{-1}{2}}$$

$$using\ (1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \cdots$$

$$\sqrt{\left(\frac{1+2x}{1-x}\right)} = \left(1 + x - \frac{1}{2}x^2\right)\left(1 + \frac{1}{2}x + \frac{3}{8}x^2\right)$$

$$= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + x + \frac{1}{2}x^2 - \frac{1}{2}x^2$$

$$= 1 + \frac{3}{2}x + \frac{3}{8}x^2$$

$$\therefore \sqrt{\left(\frac{1+2x}{1-x}\right)} \approx 1 + \frac{3}{2}x + \frac{3}{8}x^2$$

$$substituting\ for\ x = 0.02$$

$$\sqrt{\left(\frac{1.04}{0.98}\right)} = \sqrt{\frac{1+2(0.02)}{1-0.02}}$$

$$= 1 + \frac{3}{2}(0.02) + \frac{3}{8}(0.02)^2 = 1.030$$

15. (a) Differentiate $y = 2x^2 + 3$ from first principles. (04marks)

$y = 2x^2 + 3$

$y + \delta y = 2(x + \delta x)^2 + 3$

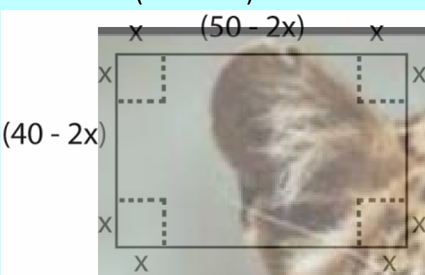
$\delta y = 2x^2 + 4x\delta x + 2(\delta x)^2 + 3 - 2x^2 - 3$

$\delta y = 4x\delta x + 2(\delta x)^2$

$$\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\delta x \to 0}(4x + 2\delta x)$$

$$\frac{\delta y}{\delta x} = 4x$$

(b) A rectangular sheet is 50cm long and 40cm wide. A square of x cm is cut off from each corner. The remaining sheet is folded to form an open box. Find the maximum volume of the box (08marks)



Volume $V = x(50 - 2x)(40 - 2x)$

$V = 2000x - 180x^2 + 4x^3$

$\frac{\delta V}{\delta x} = 2000 - 360x + 12x^2$

For maximum volume, $\frac{\delta V}{\delta x} = 0$

$\therefore 3x^2 - 90x + 500 = 0$

$$x = \frac{90 \pm \sqrt{(-90)^2 - 4(3)(500)}}{2(3)}$$

$x = 7.3624 \; or \; x = 22.6376$

$\therefore x = 7.3624$cm

$$V_{max} = 2000\,x\,(7.3624) - 180(7.3624)^2 + 4(7.3624)^3$$

$= 6564.22554 cm^3$

16. (a) Find $\int \frac{In\,x}{x^2}dx$ (04marks)

Let $u = Inx$, $\frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = \frac{1}{x^2}$, $v = \int x^{-2}dx = \frac{-1}{x}$

Using integration by parts

$$\int u\frac{dv}{dx}dx = u.v - \int v\frac{du}{dx}dx$$

$$\int \frac{In\,x}{x^2}dx = \frac{-1}{x}Inx - \int \frac{-1}{x}.\frac{1}{x}dx$$

$$= \frac{-1}{x}Inx + \int \frac{1}{x^2}dx$$

$$= \frac{-Inx}{x} + \int x^{-2}dx$$

$$= \frac{-Inx}{x} - \frac{1}{x} + C$$

$$\therefore \int \frac{In\,x}{x^2}dx = \frac{-Inx}{x} - \frac{1}{x} + C$$

(c) Solve the differential equation $\frac{dy}{dx} + y\cot x = x$, given that y = 1 when x = $\frac{\pi}{2}$. (08marks)

$\frac{dy}{dx} + y\cot x = x$

Integrating factor,

I.F = $e^{\int \cot x\, dx}$ = $e^{\int \frac{\cos x}{\sin x} dx}$ = $e^{In \sin x}$ = sinx

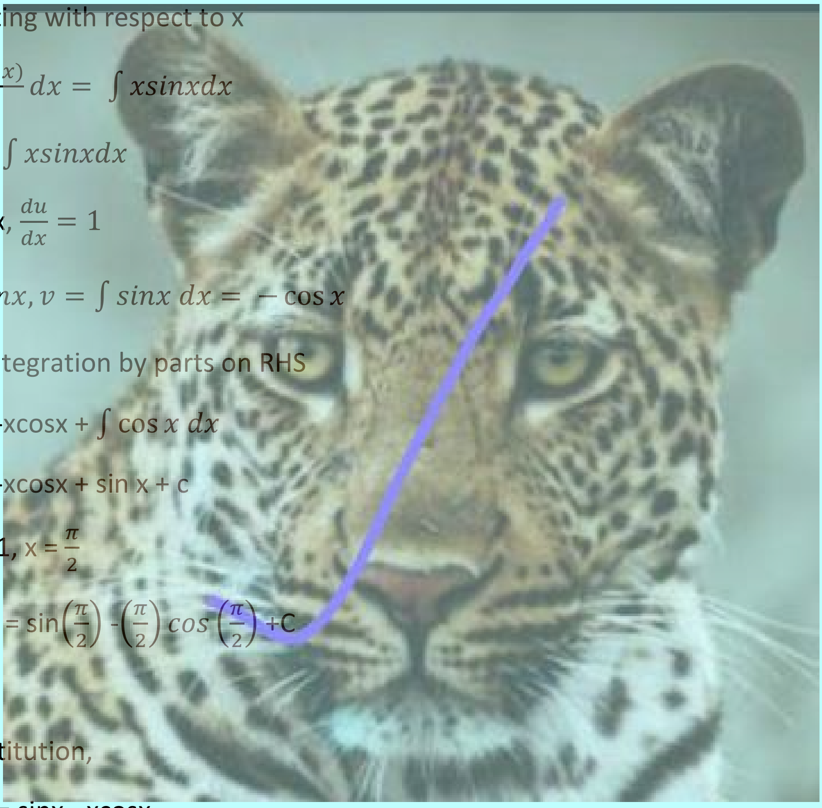
Multiplying all terms by integrating factor

sinx$\frac{dy}{dx}$ + ycotxsinx = xsinx

sinx$\frac{dy}{dx}$ + y$\frac{\cos x}{\sin x}$sinx = xsinx

sinx$\frac{dy}{dx}$ + ycosx = xsinx

$\frac{d(y\sin x)}{dx} dx$ = xsinx



Integrating with respect to x

$\int \frac{d(y\sin x)}{dx} dx = \int xsinxdx$

Ysin x = $\int xsinxdx$

Let u = x, $\frac{du}{dx} = 1$

$\frac{dv}{dx} = sinx, v = \int sinx\, dx = -\cos x$

Using integration by parts on RHS

ysinx = -xcosx + $\int \cos x\, dx$

ysinx = -xcosx + sin x + c

But y = 1, x = $\frac{\pi}{2}$

1sin$\left(\frac{\pi}{2}\right)$ = sin$\left(\frac{\pi}{2}\right)$ -$\left(\frac{\pi}{2}\right)$ $cos\left(\frac{\pi}{2}\right)$ +C

C = 0

By substitution,

∴ ysinx = sinx - xcosx