COURSE OUTLINE

Algebra (2A and 2B)
- Simultaneous equations
- Inequalities
- Indices
- Surds
- Logarithms
- Quadratics and Polynomials
- Series
- Permutations and combinations
- Binomial theorem
- Complex numbers

Trigonometry (1A and 1B)
- Trigonometry
- Further topics in trigonometry

Vectors (1A and 1B)
- Introduction to three dimensional vectors
- Location of lines in space
- Location of planes in space
- Ratio theorem and polygons

Calculus (3A and 3B)
- Differentiation 1
- Application of differentiation 1
- Integration and its application
- Derivative of trigonometric functions
- Integration of trigonometric functions
- Derivatives of exponential and logarithmic functions
- Curve sketching for rational functions
- Maclaurin's theorem
- Further integration
- Partial fractions
- Differential equations
- Applications of differential equations

Geometry (1A and 1B)
- Coordinate geometry 1
- Loci
- Circles
- Parabolas
- Ellipse

ALGEBRA

Algebra is a branch of mathematics that deals with symbols and the arithmetic operations across these symbols. It is the study of variables and the rules for manipulating these variables in ***formulae.*** *These symbols do not have any fixed values and are called* ***variables****. In our real-life problems, we often see certain values that keep on changing. But there is a constant need to represent these changing values. Here in algebra, these values are often represented with symbols such as x, y, z, p, or q, and these symbols are called variables. Further, these symbols are manipulated through various arithmetic operations of* ***addition, subtraction, multiplication****, and* ***division****, with the objective to find the values.*

Algebra of Surds.

Surds are irrational numbers that can't be represented accurately as a fraction or recurring decimal, so they are left as a square root $e.g\ \sqrt{2},\ \ \sqrt{3},\ \ \sqrt{17}, \ldots$.. An irrational number is a number that cannot be expressed as a ratio of two integers. *Surds are used in real life to make sure that important calculations are precise for example by engineers building bridges. The diagonal of a 1 × 1 square is root two. This is a sure. Any application of Pythagoras's theorem you may use in interior design or garden design uses surds. Make sure you convert them into 'normal' numbers before you order the timber from the DIY store!*

Rules of Surds

(i). $a\sqrt{c} \pm b\sqrt{c} = (a \pm b)\sqrt{c}$ *(ii).* $a\sqrt{b} \times c\sqrt{d} = ac\sqrt{bd}$ *(iii)* $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$

When surds appear in the denominator, they are always removed. The process of removing surds from the denominator is called ***rationalizing the denominator***. To rationalize the denominator of a fraction, use any method that can remove the surd from the denominator, including multiplying both the numerator and denominator by the denominator. i.e. $\frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{a}\times\sqrt{b}}{\sqrt{b}\times\sqrt{b}}$ or multiply the denominator and numerator by a denominator conjugate $\frac{a+\sqrt{b}}{c-\sqrt{d}} \times \frac{(c+\sqrt{d})}{(c+\sqrt{d})}$

Examples

1. Simplify the following

(i). $\sqrt{8}$ (ii). $\sqrt{48}$ (iii). $\sqrt{125y^2}$

(iv). $\sqrt{50} + 4\sqrt{27} + 2\sqrt{18} - \sqrt{32}$ (v). $(\sqrt{3}+7)(\sqrt{3}-7)$ (vi). $(1-\sqrt{5})(\sqrt{2}+\sqrt{5})$

2. Express the following with rational denominators.

(i). $\frac{2+\sqrt{3}}{1-\sqrt{3}}$ (ii). $\frac{(2+\sqrt{5})(3+\sqrt{5})(\sqrt{5}-2)}{(\sqrt{5}-1)(1+\sqrt{5})}$ (iii). $\frac{2x}{x-\sqrt{x^2-2}}$

3. *Find the square root of* $6 + 2\sqrt{5}$

Solution

Let $\sqrt{6 + 2\sqrt{5}} = \sqrt{x_1} + \sqrt{x_2}$

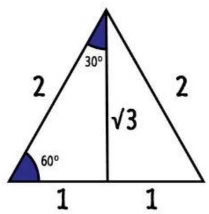
$x_1 + 2\sqrt{x_1 x_2} + x_2 = 6{+}2\sqrt{5}$

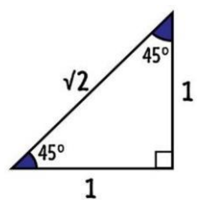
$x_1 + x_2 = 6 \; and \; x_1 x_2 = 5$

$\sqrt{6 + 2\sqrt{5}} = \pm(\mathbf{1} + \sqrt{\mathbf{5}})$

Relationships between surds and angles of $\mathbf{30^\circ, 45^\circ}$ ***and*** $\mathbf{60^\circ}$

Angles of 30° and 60° using an **equilateral triangle** *and 45° using* **a right angled isosceles triangle**.





$\sin 60^\circ = \frac{\sqrt{3}}{2}$ $\cos 60^\circ = \frac{1}{2}$ $\tan 60^\circ = \sqrt{3}$

$\sin 30^\circ = \frac{1}{2}$ $\cos 30^\circ = \frac{\sqrt{3}}{2}$ $\tan 30^\circ = \frac{1}{\sqrt{3}}$

$\sin 45^\circ = \frac{1}{\sqrt{2}}$ $\cos 45^\circ = \frac{1}{\sqrt{2}}$

$\tan 45^\circ = 1$

Exercise

1. Simplify the following

(a). $\frac{6\sqrt{10}+2\sqrt{40}}{\sqrt{2}\times\sqrt{20}}$ (b). $\frac{1}{3-\sqrt{2}} + \frac{1}{3+\sqrt{7}}$ (c). $\frac{1}{3\sqrt{5}-\sqrt{3}} + \frac{1}{\sqrt{5}+\sqrt{3}}$

2. Show that $\frac{\sqrt{2}}{\sqrt{2}+\sqrt{3}-\sqrt{5}} = \frac{3+\sqrt{6}+\sqrt{15}}{6}$ 3. Express $\sqrt{1.08}$ in the form $\frac{a}{b}\sqrt{c}$

4. If $t = \frac{1}{2}(1 + \sqrt{5})$, show that $t^2 = 1 + t$ 5. If $= b\sqrt{2}$, Show that $\frac{a^3-2a^2b+b^3}{ab(a+3b)} = \frac{2+11\sqrt{2}}{14}$

6. Given that $\sqrt{52 - 30\sqrt{3}} = p + q\sqrt{3}$, find the values of $p \; and \; q$.

7. Find the square root of $8 - 2\sqrt{11}$ $[\pm(\sqrt{5} - \sqrt{3})]$

8. Given that $\sqrt{3} = 1.732 \; and \; \sqrt{2} = 1.414$, by rationalizing the denominators, express the following as fractions with rational denominators.

(i). $\frac{\sqrt{3}-2}{2-\sqrt{3}}$ (ii). $\frac{\sqrt{3}-2}{2\sqrt{3}+\sqrt{2}}$ (iii). $\frac{3\sqrt{2}-2\sqrt{3}}{3\sqrt{2}+2\sqrt{3}}$ (iv). $\frac{1+\sqrt{2}+\sqrt{3}}{\sqrt{2}+\sqrt{3}}$ (v). $\frac{(\sqrt{5}+2)^2-(\sqrt{5}-2)^2}{8\sqrt{5}}$

(vi). $\frac{\sqrt{x}}{\sqrt{x}+\sqrt{x+2}}$ (vii). $\frac{(1+\sqrt{2})}{\left(2+\frac{5}{\sqrt{3}}\right)}$

10. Show that $\frac{4}{3}\sqrt{\frac{300}{4}}+\frac{10}{\sqrt{3}}$ can be written as $k\sqrt{a}$, where k and a are integers.

11.Express in surd form and rationalize the denominators:

a) $\frac{1}{1+cos45°}$ b) $\frac{2}{1-cos30°}$ c) $\frac{1+tan60°}{1-tan60°}$ d) $\frac{1}{(1-sin45°)^2}$

INDICES AND LOGARITHMS

Indices

An index is the number which expresses the power to which a given quantity (the base) is raised e.g. for y^x, x is the index to which the quantity y is raised.

Laws of Indices

(i) $a^m \times a^n = a^{m+n}$

(ii). $a^m \div a^n = a^{m-n}$

(iii). $(a^m)^n = a^{mn}$

(iv). $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

(v). $a^m \times b^m = (ab)^m$

(vi). $a^m \div b^m = \left(\frac{a}{b}\right)^m$

(vii). $a^0 = 1$ provided $a \neq 0$

(viii). $a^{-m} = \frac{1}{a^m}$ Provided $a \neq 0$

Proofs

(i). $\boldsymbol{a^m} \times \boldsymbol{a^n} = \underbrace{(a \times a \times \ldots \times a)}_{m\ times} \times \underbrace{(a \times a \times \ldots \times a)}_{n\ times}$

$= \underbrace{(a \times a \times \ldots \times a)}_{(m+n)\ times}$

$= a^{m+n}$

$\boldsymbol{a^m \times a^n = a^{(m+n)}}$

($\boldsymbol{i}$). $\boldsymbol{a^m} \div \boldsymbol{a^n} = \underbrace{(a \times a \times \ldots \times a)}_{m\ times} \div \underbrace{(a \times a \times \ldots \times a)}_{n\ times}$

$= \underbrace{(a \times a \times \ldots \times a)}_{(m-n) times\ since\ n\ of\ m's\ cancel\ out}$

$= a^{m-n}$

$\boldsymbol{a^m \div a^n = a^{(m-n)}}$

(iii). $\boldsymbol{(a^m)^n} = \underbrace{(a \times a \times \ldots \times a)}_{m\ times} \times \underbrace{(a \times a \times \ldots \times a)}_{m\ times} \times \underbrace{(a \times a \times \ldots \times a)}_{m\ times}$

$= \underbrace{(a \times a \times \ldots \times a)}_{(m \times n)\ times}$

$= a^{mn}$ ***therefore,*** $(a^m)^n = a^{mn}$

(iv). $\boldsymbol{a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times \ldots \times a^{\frac{m}{n}} = a^{\left(\frac{m}{n}+\frac{m}{n}+\cdots+\frac{m}{n}\right)n}}$

(v). $\boldsymbol{a^m} \times \boldsymbol{b^m} = \underbrace{(a \times a \times \ldots \times a)}_{m\ times} \times \underbrace{(b \times b \times \ldots \times b)}_{m\ times}$

$= \underbrace{(a \times b) \times (a \times b) \times \ldots (a \times b)}_{m\ times}$

$= (ab)^m$,

therefore $\boldsymbol{a^m \times b^m = (ab)^m}$

$n\ times$

$$= \left(a^{\frac{m}{n}}\right)^n$$
$$= a^m$$
$$\left(\boldsymbol{a^{\frac{m}{n}}}\right)^{\boldsymbol{n}} = \boldsymbol{a^m} \text{ therefore, } a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

(vi). $a^m \div b^m = \underbrace{(a \times a \times \ldots \times a)}_{m\ times} \div \underbrace{(b \times b * \ldots * b)}_{m\ times}$

$$= \underbrace{\left(\frac{a}{b}\right) \times \left(\frac{a}{b}\right) \ldots \times \left(\frac{a}{b}\right)}_{m\ times}$$

$= \left(\frac{a}{b}\right)^m$, therefore $\boldsymbol{a^m \div b^m = \left(\frac{a}{b}\right)^m}$

(vii). $a^o = a^{(m-m)} = \frac{a^m}{a^m} = 1$, therefore, $\boldsymbol{a^o = 1}$

(viii). $a^{-m} = a^{0-m} = \frac{a^o}{a^m} = {}^1/_{a^m}$, therefore $\boldsymbol{a^{-m} = \frac{1}{a^m}}$

Exercise

1. Simplify the following expressions

(i). $4^{\frac{3}{2}}$

(ii). $\left(\frac{2}{3}\right)^{-3}$

(iii). $\left(\frac{1}{8}\right)^{\frac{4}{3}}$

(iv). $\frac{4^{-1} \times 9^{\frac{1}{2}}}{8^{-2}}$

(v). $\frac{0.064 \times 50}{0.05 \times 4}$

(vi). $\left(\frac{1}{3}\right)^4 \times \frac{3^4}{27} \times 9^3$

(vii). $\frac{(0.125)^2 \times \left(\frac{1}{16}\right)^2}{64^{-3}}$

(viii). $\frac{27^{\frac{1}{2}} \times 243^{\frac{1}{2}}}{243^{\frac{4}{5}}}$

(ix). $\frac{64^{2y} \div 16^y}{128^y \times 4^{2y}}$

(x). $\frac{a^4 b^3 \times a^2 c^4 \times b^{-2} c^3}{a^{-3} b^3 c^3}$

(xi). $\frac{x^{p+\frac{1}{2}q} \times y^{2p-q}}{(xy^2)^p \times \sqrt{x^q}}$

(xii). $\left(\frac{2a^7 b^{-4}}{8a^9 b^{-2}}\right)^{-3} (-6a^{-2} b^c)^{-2}$

(xiii). $7x^{-3} y^{-4} \left(\frac{3x^{-1} y^5}{9x^{-1} y^2}\right)^{-3}$

(xiv). $\frac{(2xy^2 z^3)^2}{2xy^2 z^{-3}}$

(xv). $\frac{(2x^2 y^{-1})^{\frac{-1}{4}}}{(8x^{-1} y^2)^{\frac{-1}{2}}}$

(xvi). $\frac{5^{2x+1} \times 3125}{5^{2(x+3)}}$

Solve each of the following equations;

(i). $x^{\frac{1}{3}} = 3$ (ii). $x^{\frac{4}{3}} = 81$ (iii). $3^{x+2} = 27^{2x-1}$ (iv). $2x^{\frac{3}{4}} = x^{\frac{1}{2}}$

(v). $32^{\frac{3}{4}} \div x^{\frac{1}{2}} = 2$ (vi). $\left(\frac{1}{4}\right)^{x} \times 2^{x+1} = \frac{1}{8}$ (vii). $\left(\frac{1}{81}\right)^{-2y} \times 3^{y} = 243$

(viii). $81^{\frac{1}{4}} \div y^{\frac{1}{3}} = 6$ (ix). $4^{2x} \times 8^{x-1} = 32$ (x). $8^{x} + 2^{3x-1} = 96$

(xi). $4^{x+3}.16^{x} = 8^{3x}$ (xii). $4^{n}.16^{n}.2^{6n} = 4096$ (xiii). $27^{x-3} = 3 \times 9^{x-2}$

(xiv). $\left(\frac{1}{9}\right)^{2y}\left(\frac{1}{3}\right)^{-y} \div \frac{1}{27} = 3^{-5y}$ (xv). $\frac{243\times3^{2y}}{179\times3^{y}\times3^{2y-1}} = 81$ (xvi). $\frac{5^{x+1}}{5} = \frac{5^{2x}}{5^{1-x}}$

(xvii). $\frac{5^{x+1}}{5^{x+4}} = \frac{25^{x-2}}{3125}$ (xviii). $2^{2+2x} + 3 \times 2^{x} - 1 = 0$ (xix). $x^{\frac{1}{3}} - 3 = 28x^{\frac{-1}{3}}$

3. Express $81\sqrt{3}$ in the form 3^{p}.

4. Given that $2^{x} - 3 = 0$, without using tables or calculator, evaluate 8^{x+3}

5. If $5^{x} \times 5^{4y} = 1$ *and* $3^{x} \times 3^{3y} = 3^{-2}$, find x *and* y.

6. Solve the equations $3^{2x+y} = 12$ *and* $2^{x-y} = 4$, simultaneously.

7. Simply; $\frac{(1+x)^{\frac{1}{2}}-\frac{1}{2}x(1+x)^{\frac{-1}{2}}}{1+x}$

Solution

$$\frac{(1+x)^{\frac{1}{2}}-\frac{1}{2}x(1+x)^{\frac{-1}{2}}}{1+x} = \frac{(1+x)^{\frac{1}{2}}}{1+x} - \frac{\frac{1}{2}x(1+x)^{\frac{-1}{2}}}{1+x}$$

$$= \frac{1}{(1+x)^{\frac{1}{2}}} - \frac{x}{2(1+x)^{\frac{3}{2}}}$$

$$= \frac{2(1+x)-x}{2(1+x)^{\frac{3}{2}}}$$

$$= \frac{2+x}{2(1+x)^{\frac{3}{2}}}$$

Alternatively;

$$\frac{(1+x)^{\frac{1}{2}}-\frac{1}{2}x(1+x)^{\frac{-1}{2}}}{1+x} = \left((1+x)\right)^{\frac{-1}{2}}\left[\frac{(1+x)-\frac{1}{2}x}{1+x}\right]$$

OR

$$\frac{(1+x)^{\frac{1}{2}}-\frac{1}{2}x(1+x)^{\frac{-1}{2}}}{1+x} = \left[\frac{(1+x)^{\frac{1}{2}}-\frac{1}{2}x(1+x)^{\frac{-1}{2}}}{1+x}\right]\left[\frac{(1+x)^{\frac{-1}{2}}}{(1+x)^{\frac{-1}{2}}}\right] = \frac{2+x}{2(1+x)^{\frac{3}{2}}}$$

8. Solve the equations.

(i). $x-1=\sqrt{x+1}$ (ii).$\sqrt{x+4}=\sqrt{2x}$
(iii). $2\sqrt{x-1}-\sqrt{x+4}=1$ (iv). $\sqrt{2x+3}-\sqrt{x+1}=\sqrt{x-2}$

Solution

(iii). $2\sqrt{x-1}=1+\sqrt{x+4}$

$$4(x-1)=1+2\sqrt{x+4}+x+4$$
$$4x-4=1+2\sqrt{x+4}+x+4$$
$$3x-9=2\sqrt{x+4}$$
$$9x^2-58x+65=0$$
$$x=5\ or\ 1.444 \text{ (Discard)}$$

Test to remove the extraneous root that results.
From double squaring
$x=5$

Exercise

1. Simplify the following expressions

(a). $(x-2)^{\frac{5}{2}}+2(x-2)^{\frac{3}{2}}$ (b). $\frac{8^{n+2}\times4^{2n-1}}{2^n\times4^{\frac{n}{2}}}$

(c). $\frac{3(2^{n+1})-4(2^{n-1})}{2^{n+1}-2^n}$ (d). $\frac{\frac{1}{2}x(1+x)^{\frac{-1}{2}}-\frac{1}{2}x^{\frac{-1}{2}}(1+x)^{\frac{1}{2}}}{x}$

2. Express (i). $4^{\left(1-\frac{n}{n}\right)}\times2^{n+3}\times16^{\frac{-1}{2}}$ in powers of 2

(ii). $9^{\left(1-\frac{n}{2}\right)}\times3^{n-3}\times81^{\frac{-1}{2}}$ in the form 3^a

3. Solve the equations below.

(a). $5^x=25^{(2x-5)}$ (b). $8^x=4^{(1-x)}$
(c). $\left(\frac{1}{4}\right)^x\times2^{x+1}=\frac{1}{8}$ (d). $5^x=125\sqrt{5}$
(e). $4^{x-1}=3^{x+1}$ (f). $6^x=\left(\frac{2}{3}\right)^{\left(x-\frac{1}{2}\right)}$
(g). $x^3-8-9x^{\frac{3}{2}}=0$ (h). $6x^{\frac{1}{3}}+5+x^{\frac{-1}{3}}=0$
(i). $27(3^x)-3^{2x}=6$ (j). $2^{2(x-1)}-3(2^x)+8=0$
(k). $3^{(2x-1)}+26(3^x)=9$ (l). $3^{(2x+1)}-3^{(x+1)}-3^x+1=0$
(m). $2^{(2x-1)}+\frac{3}{2}=2^{(x-1)}$ (n). $2^x-2^{-x}=3$

4. Solve the equations $2^{x+y}=8$ and $3^{2x-y}=27$ simultaneously.

5. Solve the following equations and in each case justify your answers.

(i). $2\sqrt{x}+\sqrt{2x+1}=7$ (ii). $\sqrt{3x+4}-\sqrt{x+7}=3$
(iii). $\sqrt{2-x}+\sqrt{x+3}=3$ (iv). $\sqrt{3+\sqrt{x}}=\sqrt{x-9}$

LOGARITHMS

The Logarithms of a positive number to a given base is the exponent (power) to which the base must be raised to produce that number i.e. if $a^x = N\ then\ x = \log_a N$. Much of the power of logarithms is their usefulness in solving exponential equations. Some examples of this include sound (*decibel measures*), earthquakes, the brightness of stars, and chemistry (pH balance, a measure of acidity and alkalinity).

Basic laws of Logarithms

(i). $\log_c c = 1$

(ii). $\log_c 1 = 0\ , provided\ c \neq 0$

(iii). $a^{\log_a b} = b$

(iv). $\log_a b^n = n \log_a b$

(v). $\log_c ab = \log_c a + \log_c b$

(vi).$\log_c \left(\frac{a}{b}\right) = \log_c a - \log_c b$

(vii). $\log_a b = \frac{\log_c b}{\log_c a}$

(i). $\log_c c = 1$

Proof, *let* $\log_c c = x$

$c^x = c,$

By comparison

$c^x = c^1$

$x = 1$

$\log_c c = 1$

(ii). $\log_c 1 = 0$

Proof, *let* $\log_c 1 = x$

$c^x = 1$

$c^x = c^0,$

By comparison

$x = 0$

$\log_c c = 0$

(iii). $a^{\log_a b} = b$

Proof, let $a^{\log_a b} = x$

$\log_a x = \log_a b$

$x = b$

$a^{\log_a b} = b$

(iv). Log$_a$ $b^n = n \log_a b$

Proof, let $\log_a b^n = x$

$a^x = b^n$

$a^{\frac{x}{n}} = b$

$\frac{x}{n} = \log_a b$

$x = n\log_a b$

Therefore,

$\log_a b^n = n\log_a b$

(v). $\log_c ab = \log_c a + \log_c b$

Proof, let $\log_c a = x\ and\ \log_c b = y$

$c^x = a\ and\ c^y = b$

$ab = c^x c^y$

$= c^{(x+y)}$

$\log_c ab = \log_c c^{(x+y)}$

$= (x + y) \log_c c$

$= (x + y)$

$\log_c ab = \log_c a + \log_c b$

(vi). $\mathbf{Log}_c\left(\frac{a}{b}\right) = \mathbf{log}_c\, a + \mathbf{log}_c\, b$

Proof, let $\log_c a = x$ and $\log_c b = y$

$$c^x = a \text{ and } c^y = b$$
$$\frac{a}{b} = \frac{c^x}{c^y}$$
$$\frac{a}{b} = c^{(x-y)}$$

*Introduce **log** to base **c** on both sides*

$\log_c\left(\frac{a}{b}\right) = \log_c c^{(x-y)}$

$\log_c\left(\frac{a}{b}\right) = (x-y)\log_c c = x - y$

$\log_c\left(\frac{a}{b}\right) = \log_c a + \log_c b$

(vii). $\mathbf{log}_a\, b = \frac{\mathbf{log}_c\, b}{\mathbf{log}_c\, a}$

Proof, let $\log_a b = x$

$a^x = b,$

Introduce **log** *to base* **c** *on both sides*

$$\log_c a^x = \log_c b$$
$$x\log_c a = \log_c b$$
$$x = \frac{\log_c b}{\log_c a},$$
$$\text{but } \log_a b = x$$

$therefore, \log_a b = \frac{\log_c b}{\log_c a}$

Examples

1. Solve the equations below

(i). $\log_5 x + \log_x 5 = 2.5$

(ii). $\log_4(6-x) = \log_2 x$

(iii). $\log_4 x^2 - 6\log_x 4 - 1 = 0$

(iv). $\log_{25} 4x^2 = \log_5(3-x^2)$

Solution

(i). $\mathbf{log_5\, x + log_x\, 5 = 2.5}$

$\log_5 x + \frac{\log_5 5}{\log_5 x} = 2.5$

Let $\log_5 x = a$

$a + \frac{1}{a} = 2.5$

$2a^2 + 2 = 5a$

$2a^2 - 5a + 2 = 0$

$2a^2 - 4a - a + 2 = 0$

$2a(a-2) - (a-2) = 0$

$(2a-1)(a-2) = 0$

either $2a - 1 = 0\, ; a = \frac{1}{2}$

or $a - 2 - 0\, ; a = 2$

when $a = 2$, $\log_5 x = 2$

$x = 5^2 = 25$

when $= \frac{1}{2}$, $\log_5 x = \frac{1}{2}\, ;\, x = \sqrt{5}$

Either $x = 25$ *or* $x = \sqrt{5}$

(ii). $\mathbf{log_4(6-x) = log_2\, x}$

$\frac{\log_2(6-x)}{\log_2 4} = \log_2 x$

$\log_2(6-x) = \log_2 x$

$6 - x = x^2$

$x^2 + x - 6 = 0$

$x^2 + 3x - 2x - 6 = 0$

$x(x+3) + 2(x-3) = 0$

$(x-2)(x+3) = 0$

Either $(x+3) = 0\, ; x = -3\ (discard)$

Or $x - 2 = 0\, ; x = 2$

$x = 2$

2. Show that $\boldsymbol{\log_6 x = \frac{\log_3 x}{1+\log_3 2}}$ hence evaluate $\boldsymbol{\log_6 4}$ given that $\boldsymbol{\log_3 2 = 0.631}$

Solution

$$\log_6 x = \frac{\log_3 x}{\log_3 6} = \frac{\log_3 x}{\log_3(2\times 3)} = \frac{\log_3 x}{\log_3 3 + \log_3 2} = \frac{\log_3 x}{1+\log_3 2}$$

$$\text{Hence, } \log_6 4 = \frac{\log_3 4}{1+\log_3 2} = \frac{2\log_3 2}{1+\log_3 2} = \frac{2(0.631)}{1+0.631} = 0.774$$

2. Solve the equations $\boldsymbol{5\log_x y = 1 \; and \; xy = 64}$ simultaneously.

Solution

$5\log_x y = 1$……………….....……..(i)

$xy = 64$ …………………………. (ii)

from (i)

$x = y^5$

Substitute for x in (ii)

$y^6 = 64$

$y^6 = 2^{\,6}$,

by comparison

$y = 2$

$x = 2^5 = 32$

Therefore, $x = 32 \; and \; y = 2$

3. Prove that $\boldsymbol{\log_c(a+b)^2 = 2\log_c a + \log_c\left(1+\frac{2b}{a}+\frac{b^2}{a^2}\right)}$

Solution

$$\begin{aligned}\log_c(a+b)^2 &= \log_c(a^2+2ab+b^2)\\ &= \log_c a^2\left(1+\frac{2b}{a}+\frac{b^2}{a^2}\right)\\ &= \log_c a^2 + \log_c\left(1+\frac{2b}{a}+\frac{b^2}{a^2}\right)\\ &= 2\log_c a + \log_c\left(1+\frac{2b}{a}+\frac{b^2}{a^2}\right)\\ \log_c(a+b)^2 &= 2\log_c a + \log_c\left(1+\frac{2b}{a}+\frac{b^2}{a^2}\right)\end{aligned}$$

4. If $\boldsymbol{\log_a n = x}$ *and* $\boldsymbol{\log_c n = y}$ where $\boldsymbol{n \neq 1}$ prove that $\boldsymbol{\frac{x-y}{x+y} = \frac{\log_b c - \log_c n}{\log_b c + \log_b a}}$

Solution

$$\frac{x-y}{x+y} = \frac{\log_b c - \log_c n}{\log_b c + \log_b a} = \frac{\left(\frac{\log_b n}{\log_b a} - \frac{\log_b n}{\log_b a}\right)}{\left(\frac{\log_b n}{\log_b a} + \frac{\log_b n}{\log_b a}\right)}$$

$$\frac{\log_b n}{\log_b n} = \left[\frac{\frac{1}{\log_b a} - \frac{1}{\log_b c}}{\frac{1}{\log_b a} + \frac{1}{\log_b c}}\right] = \frac{\log_b c - \log_b a}{\log_b c + \log_b a}$$

$$\frac{x-y}{x+y} = \frac{\log_b c - \log_b a}{\log_b c + \log_b a}$$

5. If $\boldsymbol{\log_b c = a}$, $\boldsymbol{\log_c a = b}$ *and* $\boldsymbol{\log_a b = c}$ prove that $\boldsymbol{abc = 1}$

Solution

$a = \log_b c\ ;\ \ b^a = c$ ………………….(i)

$\mathrm{b} = \log_c a\ ;\ \ c^b = a$…………………...(ii)

$\mathrm{c} = \log_a b\ ;\ \ a^c = b$…………………..(iii)

Substitute for $\boldsymbol{c}$ in (ii)

$(b^a)^b = a$

Substitute for $\boldsymbol{a}$ in (iii)

$(b^{ab})^c = b\ ;\ b^{abc} = b^1$

$abc = 1$

Alternatively $abc = \log_b c \times \log_c a \times \log_a b = 1$

6. If $\boldsymbol{x = \log_a bc}$, $\boldsymbol{y = \log_b ac}$ ***and*** $\boldsymbol{z = \log_c ab}$, *prove that*; $\boldsymbol{x + y + z = xyz - 2}$

Solution

$x = \log_a bc\ ;\ a^x = bc$ ……………. (i)

$y = \log_b ac\ ;\ b^y = ac$ ……………. (ii)

$z = \log_c ab\ ;\ c^z = ab$ ……………. (iii)

From (i); $\boldsymbol{a = b^{\frac{1}{x}} c^{\frac{1}{x}}}$

Substitute $\boldsymbol{a}$ in (ii)

$\boldsymbol{b^y = b^{\frac{1}{x}} c^{\frac{1}{x}} c}$; $c = b^{\frac{xy-1}{x+1}}$……………. (iv)

Substitute for $\boldsymbol{a}$ in (iii)

$c^z = b^{\frac{1}{x}} c^{\frac{1}{x}} b$

$c = b^{\frac{1+x}{zx-1}}$……………... (v)

On equating (iv) and (v)

$$\frac{1+x}{zx-1}=\frac{xy-1}{x+1}$$

$1+2x+x^2=x^2yz-xz-xy+1$, ***therefore,*** $\boldsymbol{x+y+z=xyz-2}$

Exercise

1. Without using tables or calculator evaluate $\frac{\log 81}{\log 729}$

2. Simplify the following logarithms

(i). $1+\log_{10}\left(\frac{4}{x^4}\right)-2\log_{10}x$ (ii). $\frac{1}{2}\log_{10}64-3\log_{10}\left(\frac{a}{5}\right)+\log_{10}a^3$

(iii). $\log\sqrt{x^2-1}+\frac{1}{2}\log\left[\frac{x+1}{x-1}\right]$ (iv). $\frac{\log_3 32+\log_3 8-1}{\log_3 243\div\log_3 27}$

3. Solve the equations below

(i). $\log_n 4+\log_4 n^3=3$ (ii). $\log_4 4x=2\log_x 4$ $[x=4]$

(iii). $3\log_2 p-6\log_p 2+7=0$ (iv). $\log_2 x+\log_4 x+\log_{16} x=\frac{21}{16}$

(v). $\log_{9x}64=\log_x 4$ (vi). $\log_2 x+\log_2 x^2+\log_2 x^3=24$

(vii). $2+3\log x=\log 0.1$ (viii). $5(2^{x+1})=7^x$

(ix). $2^x\times 3^{2x}=5$ (x). $3^x\times 2^{x-1}=108$

4. Solve the following pairs of simultaneous equations.

(i). $\log_{10}x-\log_{10}y=\log_{10}2.5$ *and* $\log_{10}x+\log_{10}y=1$

(ii). $\log_b a+2\log_a b=3$ *and* $\log_a a+\log_a b=3$ *where* $a\neq b$

(iii). $6\log_8 x+6\log_{27}y=7$ *and* $4\log_2 x-4\log_3 y=9$

5. Solve the inequality

(i). $0.8^x>4.0$ (ii). $\log_4 x\times\log_2 x>2$

6. If $\log_3 x=p$ *and* $\log_{18}x=q$, show that $\log_6 3=\frac{q}{p-q}$

7. If $a=\log_5 35$ *and* $b=\log_a 35$, show that $\log_5 21=\frac{1}{2b}(2ab-2b+a)$.

8. Given that $2\log_8 N=p$ *and* $\log_2 2N=q$ $q-p=4$ Find the value of N. $(N=512)$

9. Show that $p=\log_a(a^3y^{-2})$ *and* $q=\log_a(ay^2)$, find $p+q$.

10. Show that $\log_8 x=\frac{2}{3}\log_4 x$ hence without using tables or calculator evaluate $\log_8 6$ *if* $\log_4 3=0.7925$: (0.862)

11. if $\log_a\left(1+\frac{1}{24}\right)=n,\log_a\left(1+\frac{1}{8}\right)=l$ *and* $\log_a\left(1+\frac{1}{25}\right)=m$ *show that*;

$\boldsymbol{l - m - n = \log_a\left(1 + \frac{1}{80}\right)}$.

12. If $2\log n - \log(8n - 24) = \log 2$, show that $\boldsymbol{n^2 - 16n + 48 = 0}$ hence find n.

13. The variable x satisfies the equation $3^x \times 4^{2x+1} = 6^{x+2}$. By taking logarithms on both sides, show that $\boldsymbol{x = \frac{\log 9}{\log 8}}$

14. If $\log_b xy^3 = p$ *and* $\log_b x^3y^2 = q$, *express* $\log_b \sqrt{xy}$ *in terms of* p *and* q.

15. If $x^2 + y^2 = 47xy$, *prove that* $\boldsymbol{\log_a x + \log_b x = 2\log_a\left(\frac{x+y}{7}\right)}$

ROOTS OF QUADRATIC EQUATIONS

Quadratic equations *are used in many fields and in everyday activities for example; Astrology, Engineering, Agriculture, Sciences, Military, and Sports are some of the fields that use quadratic equations. Quadratic equations are used in many real-life situations such as calculating the areas of an enclosed space, the speed of an object, the profit and loss of a product, or curving a piece of equipment for designing.*

One such real-life example is that if an object is projected, then the place where the object will reach the ground, the distance traveled by the object, and the time taken by the object to reach the peak height can all be determined using quadratic equations.

The roots of the quadratic equation $\boldsymbol{ax^2 + bx + c = 0}$ are the values of x that fully satisfy the equation. The roots of a quadratic equation can be determined by applying either the factorization method, graphical method, or by completing squares.

Graphical method

To solve the equation $ax^2 + bx + c = 0$ *graphically, plot a graph of* $y = ax^2 + bx + c$.
The roots of the equation are the x *- coordinates of the points where the graph cuts the x-axis (the line* $y = 0$*). Alternatively, plot graphs of* $y = ax^2$ *and* $y = -(bx + c)$ *on the same axes. The roots of the equation are the* x *coordinates of the points of intersection of the two graphs.* ***(covered in 'O' level)***

Factorization method

To solve the equation $x^2 + bx + c = 0$ *using factorization method, express* $ax^2 + bx + c$ *as a product of its linear factors, equate each factor to zero and then solve each linear equation formed.* ***(covered in 'O' level)***

Method of completing squares

To solve the equation $\boldsymbol{ax^2+bx+c=0}$ *using the method of completing squares, express the equation* $\boldsymbol{ax^2+bx+c=0}$ *in the form* $(\boldsymbol{x}+\boldsymbol{p})^2=\boldsymbol{q}$ *where* $\boldsymbol{p}=\frac{\boldsymbol{b}}{\boldsymbol{2a}}$ *and* $\boldsymbol{q}=\left[\left(\frac{\boldsymbol{b}}{\boldsymbol{2a}}\right)^{\boldsymbol{2}}-\frac{\boldsymbol{c}}{\boldsymbol{a}}\right]$*, take the square roots on both sides of the equation and solve for x.*

The quadratic formula

The quadratic formula is an extension of the method of completing squares.
To solve the equation $ax^2+bx+c=0$ *using the quadratic formula, substitute the values of* a, b *and* c *in the formula* $x=\frac{-b\ \sqrt[2]{b^2-4ac}}{2a}$

Derivation of the quadratic formula

$$\boldsymbol{ax^2+bx+c=0}$$

Divide through the equation by a

$$\boldsymbol{x^2+\frac{b}{a}x+\frac{c}{a}=0}$$

$$\boldsymbol{x^2+\frac{b}{a}x=-\frac{c}{a}}$$

$$\boldsymbol{x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\left(\frac{b}{2a}\right)^2-\frac{c}{a}}$$

$$\boldsymbol{\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}=\frac{b^2-4ac}{4a^2}}$$

$$\boldsymbol{x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}}$$

$$\boldsymbol{x=\frac{-b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}}$$

$$\boldsymbol{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}$$

NB:

The expression $\boldsymbol{b^2-4ac}$ *is the* ***DISCRIMINANT*** *of the quadratic equation. This is because it determines the nature of the roots of the quadratic equation.*

If $\boldsymbol{b^2-4ac\geq 0}$*, the quadratic equation will have two different real roots.*

If $\boldsymbol{b^2-4ac=0}$*, the expression* $\boldsymbol{ax^2+bx+c}$ *is a perfect square hence the equation has a repeated root which is equal to* $\frac{\boldsymbol{-b}}{\boldsymbol{(2a)}}$

If $\boldsymbol{b^2-4ac\leq 0}$*, the quadratic equation will have no real roots i.e. the roots will be complex numbers.*

Example One: Solve the equations (a) $2x^2+5x-12=0$, b) $x^2+11=7x$

Example two: Find the value of k so that the equation $\boldsymbol{4x^2 - 8x + k = 0}$ shall have equal roots.
Here, $a = 4, b = -8 \; and \; c = k$
The condition for equal roots, $b^2 = 4ac$
This gives $(-8)^2 = 4(4k)$
$64 = 16k$
$k = 4$

Example 3. Prove that the roots of the equation $(p - q - r)x^2 + px + q + r = 0$ are real if $p, q \; and \; r$ are real.
The condition for real roots is $b^2 \geq 4ac$
Therefore $p^2 \geq 4(p - q - r)(q + r)$
That is, $p^2 - 4p(q + r) + 4(q + r)^2 \geq 0$
$\{p - 2(q + r)\}^2 \geq 0$

This is always true for the left hand side is the square of a real quantity and therefore cannot be negative

Example 4: If *x* is real, show that the expression $y = \frac{(x^2+x+1)}{(x+1)}$ can have no real value between -3 and 1.

Rearranging as a quadratic in *x*,
$(x + 1)y = x^2 + x + 1$
$x^2 + (1 - y)x + 1 - y = 0.$
For *x* to be real
$(1 - y)^2 \geq 4(1 - y)$
or, (1 - y)(-3 - y) ≥ 0
Changing the signs, for *x* to be real
(y - l)(y + 3) ≥0.
If y lies between - 3 and 1, y + 3 > 0,
and y - 1 < 0 giving (y - l)(y + 3) < 0, the above inequality is not satisfied.
Hence there is no real value between -3 and 1.

SUM AND PRODUCT OF ROOTS OF QUADRATIC EQUATIONS

From**,** $\boldsymbol{x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}}$**,** $\boldsymbol{x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}}$ **and** $\boldsymbol{x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}}$

Sum of the roots, $\boldsymbol{x_1 + x_2 = \frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a} = \frac{-b+\sqrt{b^2-4ac}-b-\sqrt{(b^2-4ac)}}{2a} = \frac{-2b}{2a}}$

Therefore, **Sum of the roots,** $\boldsymbol{x_1 + x_2 = \frac{-b}{a}}$ **--(1)**

Product of the roots, $\boldsymbol{x_1 x_2 = \left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)}$

$\boldsymbol{x_1 x_2 = \frac{b^2-b^2+4ac}{4a^2}}$ Therefore, product of the roots, $\boldsymbol{x_1 x_2 = \frac{c}{a}}$ **--------------(2)**

Suppose α and β are the roots or the quadratic equation $ax^2 + bx + c = 0$, then $x - \alpha = 0$ and $x - \beta = 0$. It follows that $(x - \alpha)(x - \beta) = 0$
On expanding,

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

NB: $(x - \alpha)$ and $(x - \beta)$ are the factors of the expression $ax^2 + bx + c$,
$\boldsymbol{\alpha + \beta}$ is the sum and $\boldsymbol{\alpha\beta}$ is the product of the roots of the equation $x^2 - (\alpha + \beta)x + \alpha\beta = 0$.

The quadratic equation is in the form $\boldsymbol{x^2 - (sum\ of\ roots)x + product\ of\ roots = 0.}$
$\therefore x^2 - (\alpha + \beta)x + \alpha\beta = 0$ is the general equation of a quadratic equation whose roots are α and β. $\alpha + \beta$ is the sum and $\alpha\beta$ is the product of the roots only if the equation is in its standard form i.e. if $\boldsymbol{a}$, the coefficient of x^2 is **1**. It follows that if the quadratic equation whose roots are α and β is $ax^2 + bx + c = 0$ then from equations (1) and (2) above,
$\boldsymbol{\alpha + \beta = -\frac{b}{a}}$ and $\boldsymbol{\alpha\beta = \frac{c}{a}}$

Examples 1
State the product and sum of the roots of the equation $3x^2 + 9x + 7 = 0$

Solution

$$3x^2 + 9x + 7 = 0 \quad \textit{(first divide through by 3)}$$
$$x^2 + 3x + {}^7\!/_3 = 0$$
$$\therefore Sum\ of\ roots = -3$$
$$Product\ of\ roots = {}^7\!/_3$$

Examples 2
Deduce the equation whose roots are 3 *and* -2.

Solution
Sum of roots $= 3 + (-2) = 1$
Product of roots $= 3 \times (-2) = -6$
Recall: $x^2 - (sum\ of\ roots)x + product\ of\ roots = 0$
$\therefore x^2 - x - 6 = 0$ is the required equation

Examples 3
If α *and* β *are the roots of the equation* $3x^2 + 4x - 5 = 0$, *evaluate;*

(a) $\frac{1}{\alpha} + \frac{1}{\beta}$ *(b)* $\alpha^2 + \beta^2$ *(c)* $\alpha - \beta$

(d) $\alpha^3 + \beta^3$ *(e)* $\alpha^3 - \beta^3$ *(f)* $\left(\frac{\alpha^3+1}{\alpha}\right)\left(\frac{\beta^3+1}{\beta}\right)$

Solution
(a) $3x^2 + 4x - 5 = 0$

$x^2 + \frac{4}{3}x - \frac{5}{3} = 0$

$\alpha + \beta = \frac{-4}{3}$ and $\alpha\beta = -\frac{5}{3}$

$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta} = \frac{\left(\frac{-4}{3}\right)}{\left(-\frac{5}{3}\right)}$
$= \frac{4}{5}$

(b) $\boldsymbol{\alpha^2 + \beta^2} = (\boldsymbol{\alpha + \beta})^2 - \boldsymbol{2\alpha\beta} = \left(-\frac{4}{3}\right)^2 - 2\left(-\frac{5}{3}\right) = \frac{16}{9} + \frac{10}{3} = \frac{46}{9}$

(c) $(\boldsymbol{\alpha - \beta})^2 = \boldsymbol{\alpha^2 - 2\alpha\beta + \beta^2} = [(\alpha + \beta)^2 - 2\alpha\beta] - 2\alpha\beta = (\alpha + \beta)^2 - 4\alpha\beta$

$\therefore \alpha - \beta = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta} = \sqrt{\left(-\frac{4}{3}\right)^2 - 4\left(-\frac{5}{3}\right)} = 2.9059$

(d) $\boldsymbol{\alpha^3 + \beta^3} = (\boldsymbol{\alpha + \beta})^3 - \boldsymbol{3\alpha\beta(\alpha + \beta)} = \left(-\frac{4}{3}\right)^2 - 3\left(-\frac{5}{3}\right)\left(-\frac{4}{3}\right) = \frac{64}{27} - \frac{60}{9} = -\frac{244}{27}$

(e) $\boldsymbol{\alpha^3 - \beta^3} = (\boldsymbol{\alpha - \beta})(\boldsymbol{\alpha^2 + \alpha\beta + \beta^2})$

$$= (\propto -\beta)[(\propto +\beta)^2 - 2 \propto \beta + \alpha\beta] = (\alpha - \beta)[(\alpha + \beta)^2 - \alpha\beta]$$

$$= \left[\sqrt{(\alpha + \beta)^2 - 4\alpha\beta}\right][(\alpha + \beta)^2 - \alpha\beta]$$

$$= \left(\sqrt{\left(-\frac{4}{3}\right) - 4\left(-\frac{5}{3}\right)}\right)\left[\left(-\frac{4}{3}\right)^2 - \left(-\frac{5}{3}\right)\right] = 10.00$$

Example 4
The roots of the equation $2x^2 - 7x + 4 = 0$ are α and β. Find the equation whose roots are $^{\alpha}/_{\beta}$ and $^{\beta}/_{\alpha}$.

Solution
$x^2 - \frac{7}{2}x + 2 = 0$

$\alpha + \beta = {}^{7}/_{2}$ and $\alpha\beta = 2$

Sum of new roots $= \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2+\beta^2}{\alpha\beta} = \frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta} = 33$

Product of new roots $= \left(\frac{\alpha}{\beta}\right)\left(\frac{\beta}{\alpha}\right) = 1$

$\therefore$ **$\mathbf{8x^2 - 33x + 8 = 0}$ is the required equation**

Example 5
The roots of the equation $x^2 - px + 8 = 0$ are α and $\alpha + 2$. Find the possible values of α and p.

Solution
$x^2 - px + 8 = 0$
Sum of roots = $\alpha + (\alpha + 2) = p$
$= 2\alpha + 2 = p$
Product of roots = $\alpha(\alpha + 2) = 8$
$= \alpha(\alpha + 2) = 8$
$\alpha^2 + 2\alpha = 8$
From (ii) $\alpha = -4$ or $\alpha = 2$
When $\alpha = 2, p = (2 \times 2) + 2 = 6$
When $\alpha = -4, p = (2x - 4) + 2 = -6$
:. either $\boldsymbol{\alpha = 2}$ and $\boldsymbol{p = 6}$ or $\boldsymbol{\alpha = -4}$ and $\boldsymbol{p = -6}$

Example 6
The roots of the equation $ax^2 + bx + c = 0$ differ by 4. Prove that $b^2 - 4ac = 16a^2$

Solution
Let the roots be α and β
$= \alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = {}^{c}/_{a}$
But $|\alpha - \beta| = 4$
$= (\alpha - \beta)^2 = 16$
$\left(-\frac{b}{a}\right)^2 - 4\left(\frac{c}{a}\right) = 16$
$\frac{b^2}{a^2} - \frac{4c}{a} = 16$
$\therefore \boldsymbol{b^2 - 4ac = 16a^2}$

Example 7
The roots of the equation $x^2 + 2px + q = 0$ differ by 8. Show that $p^2 - 16 = q$.

Solution
Let the roots be α and β.

$\alpha + \beta = 2p$ and $\alpha\beta = q$
But $|\alpha - \beta| = 8$
$(\alpha - \beta)^2 - 4\alpha\beta = 64$
$4^2 - 4q = 64$
$p^2 - q = 16$
$\therefore p^2 - 16 = q$

Example 8
Prove that if the sum of the reciprocals of the roots of the equation $ax^2 + bx + c = 0$ is 1 then $b + c = 0$. If in addition, one root of the equation is twice the other. Find one set of values of a, b and c.

Solution
Let the roots be α and β
$\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = {}^{c}/_{a}$

But $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

$$\frac{\alpha + \beta}{\alpha\beta} = 1$$

$$-\frac{{}^{b}/_{a}}{{}^{c}/_{a}} = 1$$

$c = -b$

In addition, $\alpha = 2\beta$

$$= \frac{\beta + 2\beta}{2\beta^2} = 1$$

$3\beta = 2\beta^2$

$\beta = {}^{3}/_{2}$ and $\alpha = 3$

$$\alpha + \beta = \frac{3}{2} + 3 = {}^{9}/_{2}$$

$$\alpha\beta = \frac{3}{2} \times 3 = {}^{9}/_{2}$$
$2x^2 - 9x + 9 = 0$ is the quadratic equation
$\therefore a = 2, b = -9$ and $c = 9$.

Example 9

If the equation $a^2x^2 + bacx + 8b^2 = 0$ *has equal roots prove that* $ac(x+1)^2 = 4b^2x$ *also has equal roots.*

Solution
Let the roots of $a^2x^2 + bacx + 8b^2 = 0$
Sum of roots $(\alpha + \alpha) = 2\alpha = -\frac{6b}{a}$
Product of roots $\alpha^2 = \frac{ac+8b^2}{a^2}$
From (i), $\alpha = -\frac{3b}{a}$

Substitute for $\propto$ I (ii)
$\dfrac{9b^2}{a^2} = \dfrac{ac + 8b^2}{a^2}$

$b^2 = ac$
Substitute for b^2 in $ac(x+1)^2 = 4b^2x$
$= (x+1)^2 = 4acx$
$(x+1)^2 = 4x$
$x^2 + 2x + 1 = 4x$
$x^2 - 2x + 1 = 0$
$\therefore (x-1)^2 = 0$
Since $(x-1)^2$ is a perfect square, there is only one possible value of x hence $ac(x+1)^2 = 4b^2x$ also has equal roots.

Example 10
Given that α and β are roots of the equation $x^2 + 5x + 6 = 0$.
Form an equation whose roots are
(i) $\frac{1}{1-\alpha}$ and $\frac{1}{1-\beta}$ (ii) $\alpha\beta$ and α
Solution
(i) $x^2 + 5x + 6 = 0$
Sum of roots, $\alpha + \beta = -5$
Product of roots, $\alpha\beta = 6$
Sum of new roots $= \frac{1}{1-\alpha} + \frac{1}{1-\beta} = \frac{(1-\beta)+(1-\alpha)}{(1-\alpha)(1-\beta)}$

$= \dfrac{2 - (\alpha + \beta)}{1 - (\alpha + \beta) + \alpha\beta}$

$= \dfrac{2 - (-5)}{1 -) - 5) + 6}$

$= {}^{7}/_{12}$

Product of new roots $= \left(\frac{1}{1-\alpha}\right)\left(\frac{1}{1-\beta}\right)$

$$= \frac{1}{1-(\alpha+\beta)+\alpha\beta}$$

$$= \frac{1}{12}$$

$$= x^2 - \frac{7}{12}x + \frac{1}{12} = 0$$

$\therefore 12x^2 - 7x + 1 = 0$ is the required equation

Example 11
The roots of the quadratic equation $x^2 - 2x + 2 = 0$ are $\sqrt{\alpha}$ and $\sqrt{\beta}$. Show that one of the equations whose roots are α and β is $x^2 + 4 = 0$.

Solution
$x^2 - 2x + 2 = 0$
Product of roots, $\sqrt{\alpha}.\sqrt{\beta} = 2$
$\left(\sqrt{\alpha} + \sqrt{\beta}\right)^2 = 4$

$\alpha + 4 + \beta = 4$

$\alpha + \beta = 0$
$\therefore x^2 + 4 = 0$

Exercise
1. Express the following in terms of $\alpha + \beta$ and β.
(i) $\alpha^2 - \beta^2$ (ii) $\alpha^4 + \beta^4$ (iii) $\alpha^4 - \beta^4$
Prove that if the difference between the roots of the equation $ax^2 + bx + c = 0$ is 1 then $a^2 = b^2 - 4ac$.

(3) One root of the equation $ax^2 + bx + c = 0$ is the square of the other. Show that;
(i)$b^3 = ac(3b - c - a)$ (ii) $c(a-b)^3 = a(c-b)^3$

(4) Prove that if the sum of the squares of the roots of the equation $ax^2 + bx + c = 0$ is 1 then $b^2 = 2ac + a^2$.

(5) If the equations $x^2 - 5x + 2k = 0$ and $x^2 - 2x - 4k = 0$ where $k \neq 0$ have a common root a, find the values of α and k.

(6) The roots of the equation $x^2 - 12x + 2 = 0$ are $\sqrt{\alpha}$ and $\sqrt{\beta}$. Deduce the equation whose roots are α and β.

(7) If $m^2x^2 + 2mnx + n^2 + 1 = 0$ where n and m are constants, show that there isn't any real value of x for any values of m and n.

(8) One root of the quadratic equation $ax^2 + bx + c = 0$ is three times the other. Show that $3b^2 - 16ac = 0$.

(9) Prove that if the difference between the roots of the equation $ax^2 + bx + c = 0$ is 6 then $a^2 = \frac{b^2-4ac}{36}$

(10) If the roots of the equation $x^2 + 3x + 7 = 0$ are $2\alpha + 3$ and $2\beta + 3$, deduce the equation whose roots are $\alpha + 2$ and $\beta + 2$.

(11) Given that the roots of the equation $x^2 + px + q = 0$ are α and β, express $(\alpha - \beta^2)(\beta - \alpha^2)$ in terms of p and q hence deduce that for one root to be the square of the other, $p^2 + 3pq + q = 0$ must hold.

(12) The roots of the quadratic equation $x^2 + px + q = 0$ are α and β. Show that the equation whose roots are $x^2 - q\alpha$ and $\beta^2 - q\beta$ is $x^2 - (x^2 + pq - 2q)x + q^2(q + p + 1) = 0$.

(13) Find the equation whose roots are the squares of the roots of the equation
$3x^2 + 5x - 1 = 0$.

(14) If the roots of the equation $x^2 + bx + c = 0$ are $\sqrt{\propto}$ and $\sqrt{\beta}$, show that;
(i) $\propto + \beta = b^2 - 2c$
(ii) $\propto^2 + \beta^2 = \left(b^2 - 2c - \sqrt{2c}\right)\left(b^2 - 2c + \sqrt{2c}\right)$

(15) Find the values of m for which the equation $x^2 + (m + 3)x + 4m = 0$ has equal roots.
For what values of m is the sum of the roots equal to zero?

(16) The roots of the equation $3x^2 - ax + 6b = 0$ are $\propto$ and β. Find the condition for one root to be;
(i) twice the other
$(81 - b = a^2)$
(ii) the cube of the other
$[a -^2 - 648(b - 1)b^2 - 18(4a^2 + 9)b = 0]$

(17) Given that the roots of the quadratic equation $\frac{1}{x} + \frac{1}{1+x} - \frac{1}{2} = 0$ are $\propto$ and β. Form a quadratic equation whose roots are $(\alpha - \beta)^2$ and $\propto^3 + \beta^3$.
$(x^2 - 62x + 765 = 0)$

(18) Given that $\propto$ and β are the roots of the equation $x^2 + px + q = 0$,
(i) Show that $(\alpha^2 + 1)\,(\beta^2 + 1) = (q - 1)^2 + p^2$
(ii) Find a quadratic equation whose roots are $\frac{\propto}{\propto^2+1}$ and $\frac{\beta}{\beta^2+1}$

$$[((q - 1)^2 + p^2)x^2 + p(1 + q)x + q = 0]$$

(19) Given that the roots of the equation $x^2 + px + q = 0$ are α and β, deduce the condition for one root to be three times the other hence show that $p^2 : q = 16 : 3$.

(20) Prove that the roots of the equation $(\alpha+3)x^2+(6-2\alpha)x+\alpha-1=0$ are real if and only if $\propto$ is not greater than $^3/_2$. Find the value of $\propto$ if one root is six times the other.

(21) Show that if $x^2+bx+q=0$ and $x^2+px+q=0$ have a common root,
then $(c-q)^2=(b-p)(cp-bq)$.

(22) If the roots of the equation $x^2+bx+c=0$ are α and β and the roots of the equation $x^2+bx+\lambda^2c=0$ are and $\alpha+\beta$, and $\alpha\sigma+\beta$
Show that the equation whose roots are is given by

$$x^2+\lambda b^2x+2\lambda^2c(b^2-2c)=0$$

(23) Given that α and β are roots of the equation $ax^2+bx+c=0$, determine the equation whose roots are $\alpha+\beta$ and $\alpha^3+\beta^3$ hence or otherwise solve the equations $\alpha+\beta=3$ and $\alpha^3+\beta^3=26$ simultaneously.

(24) If α and β are the roots of the equation $x^2+px+q=0$, Find the equation whose roots are;

(a) $\alpha+1$ and $\beta+1$

(b) $3\alpha+1$ and $3\beta+1$

(c) $\frac{2}{\alpha}$ and $\frac{2}{\beta}$

(d) $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$

(e) $\frac{1}{(2+\propto)^2}$ and $\frac{1}{(2+\beta)^2}$

(f) $2\alpha+3\beta$ and $3\alpha+2\beta$

(g) $2\alpha-1$ and $2\beta-1$

(h) $\alpha^3\beta$ and $\alpha\beta^3$

(i) $\frac{1}{\alpha^3}$ and $\frac{1}{\beta^3}$

(j) $\frac{\alpha^3-1}{\alpha}$ and $\frac{\beta^3-1}{\beta}$

BIQUADRATIC (QUARTIC) EQUATIONS

A quartic equation is an equation of fourth degree
It is in the form $ax^4+bx^3+cx^2+dx+e=0$ where a, b, c and e are consonants and $a\neq 0$.
A quartic equation can be reduced to a quadratic equation by making the substitution $y=x+\frac{1}{x}$

Examples

If $p=x+\frac{1}{x}$, express $x^2+\frac{1}{x^2}$, $x^3+\frac{1}{x^3}$ and $x^4+\frac{1}{x^4}$ in terms of p.

Solution

$$\boldsymbol{p^2}=\left(\boldsymbol{x}+\frac{\mathbf{1}}{\boldsymbol{x}}\right)^{\mathbf{2}}=x^2+\frac{1}{x^2}+2$$
$$\therefore\ x^2+\frac{1}{x^2}=p^2-2$$

$$\boldsymbol{p^3}=\left(\boldsymbol{x}+\frac{\mathbf{1}}{\boldsymbol{x}}\right)^{\mathbf{3}}=x^3+\frac{1}{x^3}+3x\left(\frac{1}{x}\right)+3x\left(\frac{1}{x}\right)^2$$

$$= x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right)$$
$$= x^3 + \frac{1}{x^3} + 3p$$
$$\therefore \boldsymbol{x^3 + \frac{1}{x^3} = p^3 - 3p = p(p^2 - 3)}$$

$$\boldsymbol{p^4} = \left(\boldsymbol{x + \frac{1}{x}}\right)^{\boldsymbol{4}} = x^4 + \frac{1}{x^4} + 4x^3\left(\frac{1}{x}\right) + 6x\left(\frac{1}{x}\right)^2 + 4x\left(\frac{1}{x}\right)^3$$
$$= x^4 + \frac{1}{x^4} + 4(p^2 - p) + 6$$
$$\therefore x^4 + \frac{1}{x^4} = p^4 - 4p^2 + 2$$

Use the substitution $y = x + \frac{1}{x}$ to solve the equation $2x^4 - 9x^3 + 14x^3 - 9x + 2 = 0$.

Solution
Divide all through by x^2
$2x^2 + \dfrac{2}{x^2} - 9x - \dfrac{9}{x} + 14 = 0$
$2\left(x^2 + \dfrac{1}{x^2}\right) - 9\left(x + \dfrac{1}{x}\right) + 14 = 0$
But $x^2 + \frac{1}{x^2} = y^2 - 2$
$= 2(y^2 - 2) - 9y + 14 = 0$
$y = 2$ or $y = {}^5/_{12}$
When $y = 2, 2 = x + \frac{1}{x}; x = 1$
When $y = {}^5/_2, \frac{5}{2} = x + \frac{1}{x}; x = 2$ or $x = {}^1/_2$
$\therefore \left\{x = x = \dfrac{1}{2}, 1, 2\right\}$

Exercise
(1) Solve the equation;
(i) $(x^2 - 2x)^2 + 24 = 11(x^2 - 2x)$ (ii) $x^2 + 2x + \frac{12}{x^2+2x} = 7$
(2) Use the substitution $y = x + \frac{1}{x}$ to find the real roots of the equation,
(i) $5x^4 - 16x^3 - 42x^2 - 16x + 5 = 0$
(ii) $x^4 + 3x^3 + 2x^2 + 3x + 1 = 0$
(iii) $4x^4 + 17x^3 + 8x^2 + 17x + 4 = 0$
(iv) $x^4 - 4x^3 - 6x^2 - 4x + 1 = 0$

(3) Use the substitution $y = x + \frac{4}{x}$ to find the real roots of the equation

$x^4 - x^3 - 12x^2 + 4x + 16 = 0.$
(4) Use the substitution $y = x^2 - 4x$ to find the real roots of the equation

$$2x^4 - 16x^3 + 77x^2 - 180x + 63 = 0$$

POLYNOMIALS

A polynomial is an algebraic function of the form $a_0x^{n-1} + a_,x^{n-1} + \cdots + a_nx^0$ where $a_0, a_1, \ldots a_{n-1}$ *and* a_n are real numbers and n is a positive integer.The positive integer n is called the degree of the polynomial e.g. the **degree** of the polynomial $x^4 + 2x^3 + 7x^3 + x + 3$ *is* $\mathbf{4}$. Polynomials are everywhere. They are found in roller coasters of amusement parks, the slope of a hill, the curve of a bridge or the continuity of a mountain range. They play a key role in the study of algebra, in analysis and on the whole many mathematical problems involving them.

Algebra of Polynomials

Scalar multiplication
Scalar multiplication of a polynomial is distributive i.e. when a polynomial is multiplied by a scalar; the scalar multiplies all through by each term of the polynomial.

Addition and Subtraction of Polynomials
Polynomials are added or subtracted by adding or subtracting the like terms of the polynomials.

Multiplication of Polynomials
Polynomials are multiplied together by multiplying each term of one polynomial all through by each term of the other polynomial.

Examples
A polynomial $g(x)$ is given by the equation $g(x) = x^4 + x^3 - 3x^2 + 3x - 1$.
Determine $g(2)$.

Given that $f(x) = 3x^2 + 2x - x + 4$ *and* $g(x) = x^3 - x^2 + 7$, find;
(i). $2f(x)$ (ii). $f(x) + g(x)$
(iii). $3f(x) - 2g(x)$ (iv). $g(x) \times f(x)$

solution

(iv). $g(x) \times f(x) = (3x^3 + 2x^2 - x + 4)(x^3 - x^2 + 7)$

$$= 3x^3(x^{\,3} - x + 7) + 2x(x^3 - x^2 + 7) - x\big(x^3 - x^2 + 7) + 4(x^3 - x^2 + 7)\big)$$

$$= 3x^9 - x^5 - 3x^4 + 26x^3 + 10x^2 - 7x + 28$$

Exercise

1. Given that $(x) = x^3 + x^2 - 2x + 4$, find ;

(i). $f(1)$ (ii). $f(-1)$

(iii). $2f(-2)$ (iv). $3f\left(\frac{1}{2}\right)$

2. Given that $f(x) = 2x^3 + 4x^2 - x + 7$ and $(x) = x^3 - 3x^2 + 1$, find the expression for;

(i). $2g(x)$ (ii). $\frac{1}{3}f(x)$

(iii). $g(x) - 2f(x)$ (iv). $\frac{1}{2}f(x) * \frac{1}{3}g(x)$

(v). $2[g(x)]^2 - f(x) * g(x)$ (vi). $2[f(x)]^2 * 4g(x)$

3. $f(x) = 3x^3 + 2x^2 - 4$ and $g(x) = 2x^4 + ax^3 - bx^2 - 3x$. If $f(x) + g(x) = 2x^4 + 5x^3 + 3x^2 - 3x - 4$, find $a \; and \; b$.

4. Simplify the following expressions.

(i). $(3x - 1)(x^3 - 4x^2 + 6x + 8)x^3$

(ii). $(2x^3 + 5x^2 + 7x - 2)(3x^3 - 2x^2 - 4x + 3)$

(iii). $(2x + 1)(x^4 + 3x^2 + x - 2) + x(x^3 + 3x - 4)$

5. Find the coefficient of x^3in the expression $(x^3 + 4x^2 - 7x + 1)(x + 2)$.

6. Given that $p(x) = x^5 + 4x^4 - 6x^2 + 3x + 2 \; and \; q(x) = 2x^6 + 2x^3 + 6x - 1$, find the coefficient of x^3in the expression $p(x) * q(x)$.

REMAINDER AND FACTOR THEOREMS

When a polynomial $f(x)$ is divided by $\alpha x - \beta$, the remainder is $f\left(\frac{\beta}{\alpha}\right)$. This is the statement of the remainder theorem.

Proof of the remainder theorem

Let $Q(x)$ be the quotient and $R(x)$ be the remainder obtained when $f(x)$ is divided by $\alpha x - \beta$

$$\frac{f(x)}{\alpha x - \beta} = Q(x) + \frac{R(x)}{\alpha x - \beta}$$

Multiply all through by $\boldsymbol{\alpha x - \beta}$

$f(x) = Q(x)(\alpha x - \beta) + R(x)$

$R(x) = f(x) - Q(x)(\alpha x - \beta)$

Take $x = \frac{\beta}{\alpha}$

$$R(x) = f\left(\frac{\beta}{\alpha}\right) - Q(x)\left(\alpha\left(\frac{\beta}{\alpha}\right) - \beta\right) = f\left(\frac{\beta}{\alpha}\right) - Q(x)(\beta - \beta) = f\left(\frac{\beta}{\alpha}\right) - 0 = f\left(\frac{\beta}{\alpha}\right)$$

If $\alpha x - \beta$ is ***a factor*** of a polynomial $f(x)$ then the remainder when $f(x)$ is divided by $\alpha x - \beta$ is ***zero*** and if $f\left(\frac{\beta}{\alpha}\right) = 0$ then $\alpha x - \beta$ is a factor of the polynomial $f(x)$. *This is the statement of the factor theorem.*

Proof of the factor theorem
Let $\alpha x + \beta$ be a factor of the polynomial $f(x)$
$f(x) - Q(x)(\alpha x - \beta)$
Take $x = \frac{\beta}{\alpha}$

$$f\left(\frac{\beta}{\alpha}\right) = Q(x)\left[\alpha\left(\frac{\beta}{\alpha}\right) - \beta\right] = 0$$

If $\alpha x - \beta$ is a factor of $f(x)$ then the remainder obtained when $f(x)$ is divided by $\alpha x - \beta$ is zero. The remainder and factor theorems provide a convenient way of finding $f\left(\frac{\beta}{\alpha}\right)$ and $Q(x)$ when $f(x)$ is divided by $\alpha x - \beta$ using a single operation.

NB;
When a polynomial $f(x)$ is divided by a function $g(x)$ of degree n, then the remainder $R(x)$ is a function of degree $n - 1$ e.g. if $g(x)$ is a linear function $R(x)$ is a constant and if $g(x)$ is a quadratic function $R(x)$ is a linear function.

Principle of the undetermined coefficients
If two polynomials of degree n in x are equal for more than n values of x, then they are equal for all values of x. This is the statement of the principle of the undetermined coefficients.

Proof
Consider two polynomials.
$P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$ *and*
$Q(x) = b_0x^n + b_1x^{n-1} + b_2x^{n-2} + \cdots + b_{n-1}x + b_n$
$P(x) - Q(x) = (a_0 - b_0)x^n + (a_1 - b_1)x^{n-1} + (a_2 - b_2)x^{n-2} + \cdots + (a_{n-1} - b_{n-1})x + (a_n - b_n)$

If the polynomials $P(x)$ and $Q(x)$ are equal for more than n values of x, then all the coefficients of $P(x) - Q(x)$ must be zero.
$a_0 - b_0 = 0,\ a_1 - b_1 = 0\ , a_2 - b_2 = 0, \ldots\ , a_{n-1} - b_{n-1} = 0\ \ and\ a_n - b_n = 0$

It follows that $a_0 = b_{0\,,}\ a_1 = b_1, a_2 = b_2\ , \ldots, a_{n-1} = b_{n-1}\ and\ a_n = b_n$

The polynomials P(x) and Q(x) are therefore identical and equal for all values of x.
NB:
The principle holds even when the polynomials are not of the same degree.

Examples

1. Divide $x^3 - 3x^2 + 6x + 5\ by\ x - 2$

Solution

Let $x^3 - 3x^2 + 6x + 5\ by\ x - 2 = f(x) and\ Q(x) = ax^2 + bx + c$

From $f(x) = Q(x) \times g(x) + R(x)$,
$x^3 - 3x^2 + 6x + 5 \equiv (ax^2 + bx + c)(x - 2) + d$
$= ax^3 + bx^2 + cx - 2ax^2 - 2bx - 2c + d$
$= ax^3 + (b - 2a)x^2 + (c - 2b)x + d - 2c$

$a = 1$
$b - 2a = -3\,; b = -3 + 2a = -3 + 2 = 1$
$c - 2b = 6\,; c = 6 + 2b = 6 - 2 = 4$
$d - 2c = 5\,; d = 5 + 2C = 5 + 8 = 13$

$$\frac{x^3 - 3x^2 + 6x + 5}{x - 2} = x^2 - x + 4 + \frac{13}{x - 2}$$

Alternatively, using long division

$$\begin{array}{rl}
 & x^2 - x + 4 \\
x - 2 & \overline{)\,x^3 - 3x^2 + 6x + 5} \\
- & \ \ x^2 - 2x^2 + 0 + 0 \\ \hline
 & \qquad -x^2 + 6x + 5 \\
- & \qquad -x^2 + 2x + 0 \\ \hline
 & \qquad\qquad 4x + 5 \\
- & \qquad\qquad 4x - 8 \\ \hline
 & \qquad\qquad\quad 13
\end{array}$$

$$\frac{x^3 - 3x^2 + 6x + 5}{x - 2} = x^2 - x + 4 + \frac{13}{x - 2}$$

Or, using synthetic division (Detached coefficient) method
$x - 2 = 0\,; x = 2$

$\frac{\beta}{\alpha}$ multiplies all through by each

	x^3	x^2	x^1	x^0	Terms in $f(x)$
2	1	-3	6	5	Coefficients of the terms in $f(x)$
+	0	2	-2	8	
	1	-1	4	13	Sums

Coefficients of the terms in

$$\frac{x^3 - 3x^2 + 6x + 5}{x - 2} = x^2 - x + 4 + \frac{13}{x - 2}$$

2. When a polynomial $P(x)$ is divided by $x-1$, the remainder is 5. When $P(x)$ is divided by $x-2$, the remainder is 7. Find the remainder when $P(x)$ is divided by $(x-1)(x-2)$.

Solution

Let the remainder be $ax+b$
$P(x) = (x-1)(x-2)Q(x) + ax + b$
$P(1) = a + b = 5$………………..(i)
$P(2) = 2a + b = 7$………………(ii)

take (ii)-(i)
$a = 2$

substitute for a in (i)
$2 + b = 5$
$b = 5 - 2 = 3$

The remainder is $2x + 3$

3. Factorize $x^3 - 2x^2 - 5x + 6$ completely.
Solution

$$\text{Let } x^3 - 2x^2 - 5x + 6 = f(x)$$
$$f(1) = (1)^3 - 2(1)^2 - 5(1) + 6 = 0$$

$$x - 1 \text{ is a factor of } x^3 - 2x^2 - 5x + 6$$
$$f(-2) = (-2)^3 - 2(-2)^2 - 5(-2) + 6 = 0$$

$$x + 2 \text{ is a factor of } x^3 - 2x^2 - 5x + 6$$
$$f(3) = (3)^3 - 2(3)^2 - 5(3) + 6 = 0$$

$$x - 3 \text{ is a factor of } x^3 - 2x^2 - 5x + 6$$
$$x^3 - 2x^2 - 5x + 6 = (x-1)(x-3)(x+2)$$

Alternatively;

$$\text{Let } x^3 - 2x^2 - 5x + 6 = f(x)$$
$$f(1) = (1)^3 - 2(1)^2 - 5(1) + 6 = 0$$
$$(x-1) \text{ is a factor of } x^3 - 2x^2 - 5x + 6$$

$$\begin{array}{r l}
 & x^2 - x - 6 \\
x - 1 & \overline{)\, x^3 - 2x^2 - 5x + 6} \\
- & x^3 - x^2 + 0 + 0 \\
\hline
 & -x^2 - 5x - 6 \\
- & -x + x + 0 \\
\hline
 & -6x - 6 \\
- & -6x - 6 \\
\hline
\end{array}$$

$$x^3 - 2x^2 - 5x + 6 = (x^2 - x - 6)(x - 1)$$
$$x^2 - x - 6 = x^2 - 3x + 2x - 6$$
$$= x(x - 3) + 2(x - 3) = (x + 2)(x - 3)$$

$$x^3 - 2x^2 - 5x + 6 = (x + 2)(x - 3)(x - 1)$$

4. A polynomial $f(x)$ can be expressed as $f(x) = g(x)Q(x) + R(x)$ where $Q(x)$ is the quotient and $R(x)$ is the remainder when $f(x)$ is divided by $g(x) = (x - \alpha)(x - \beta)$

(i). Show that $R(x) = \frac{(x-\beta)f(\alpha)+(x-\alpha)f(\beta)}{\alpha-\beta}$

(ii). Given that when $f(x)$ is divided by $x - 3$ the remainder is 2 and when divided by $x + 3$ the remainder is -3, use the expression in (i) above to determine the remainder when $f(x)$ is divided by $x^2 - 9$.

Solution

(i). Let $R(x) = ax + b$

$f(x) = Q(x)[(x - \alpha)(x - \beta)] + ax + b$

Take $x = \alpha$

$f(\alpha) = a\alpha + b$……………. (i)

Take $x = \beta$

$f(\beta) = a\beta + b$……………...(ii)

From (i) and (ii), $a = \frac{f(\alpha)-f(\beta)}{\alpha-\beta}$ and $b = \frac{\alpha f(\beta)-\beta f(\alpha)}{\alpha-\beta}$

$$R(x) = \frac{(x - \beta)f(\alpha) + (\alpha - x)f(\beta)}{\alpha - \beta}$$

(ii). $x^2 - 9 = (x - 3)(x + 3)$

$\alpha = 3 \text{ and } \beta = -3$

But $f(\alpha) = 2 \text{ and } f(\beta) = -3$

Hence $R(x) = \frac{(x+3)(2)(3-x)(-3)}{3--3}$

$= \dfrac{5x - 3}{6}$

$= \dfrac{5}{6}x - \dfrac{1}{2}$

5. Factorize $a^2(b - c) + b^2(c - a) + c^2(a - b)$

Solution

Let $a = b$;

$$a^2(b-c) + b^2(c-a) + c^2(a-b) = b^2(b-c) + b^2(c-b) + c^2(b-b) = 0$$
$$a - b\ is\ a\ factor\ of\ a^2(b-c) + b^2(c-a) + c^2(a-b)$$

Similarly $\boldsymbol{b-c}$ and $\boldsymbol{c-a}$ are factors are factors of $a^2(b-c) + b^2(c-a) + c^2(a-b)$

Let $a^2(b-c) + b^2(c-a) + c^2(a-b) = N(a-b)(b-c)(c-a)$

Take $a = 0, b = 1, c = 2$

$$-2 = 2N;$$
$$N = -1$$

$a^2(b-c) + b^2(c-a) + c^2(a-b) = -(a-b)(b-c)(c-a)$

Use the ***principle of the undetermined coefficients*** to find the square root of

$$x^4 + 4x^3 + 8x^2 + 8x + 4$$

Solution

Let $x^4 + 4x^3 + 8x^2 + 8x + 4 = (ax^2 + bx + c)^2$

$$= a^4x^4 + 2ab^3 + (2ab + b)^2 + 2bcx + c^2$$

Comparing coefficients on both sides

$$a = \pm 1\ and\ b = c = \pm 2$$

The square root of $x^4 + 4x^3 + 8x^2 + 8x + 4\ is\ = \pm(x^2 + 2x + 2)$

7. If $2x^2 - 9x + 14 = a(x-1)(x-2) + b(x-1) + c$, Find the values of a, b and c

Solution

$$2x^2 - 9x + 14 = a(x^2 - 3x + 2) + bx - b + c$$
$$= ax^2 - (3a - b)x + 2a - b + c$$
$$a = 2, b = -3\ and\ c = 5$$

7. The polynomials $x^2 + px + q\ and\ 3x^2 + q$ have a common factor $x - b$. If p, q and b are non-zero constants, prove that $3p^2 + 4q = 0$

Solution

Let $x^2 + px + q = f(x) and\ 3x^2 + q =$ g(x)

If $x - b$ is a factor, then

$f(b) = b^2 + bp + q = 0$……………………….(i)

g(b) $= 3b^2 + q = 0$…………………………..(ii)

Take (ii)−3(i)

$$q - 3bp - 3q = 0\ ; b = -\frac{2q}{3p}$$

Substitute for b in (ii)

$$3\left[-\frac{2q}{3p}\right]^2 + q = 0$$
$$4q^2 + 3p^2q = 0$$
$$q(4q + 3p^2) = 0$$

But $q \neq 0$
$$3p^2 + 4q = 0$$

9. Find the relationship between q and r so that $x^3 + 3px^2 + qx + r$ is a perfect cube for all values of x.

Solution

Let $x^3 + 3px^2 + qx + r = (x + a)^3$
$$x^3 + 3px^2 + qx + r = x^3 + 3ax^3 + 3a^2x + a^3$$
Comparing both sides,
$$a = p \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$
$$3a^2 = q \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$
$$a^3 = r \ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}$$
Take $(iii)^2$
$$(a^2)^3 = r^2$$
from (ii) $a^2 = \frac{q}{3}$
$$\left(\frac{q}{3}\right)^3 = r^2$$
$$q^3 = 27r^2$$

10. If $4x^3 + kx^2 + px + 2$ is divided by $x^2 + \lambda^2$, prove that $kp = 8$

Solution

Let $Q(x) = ax + b$
$$4x^3 + kx^2 + px + 2 = (x^2 + \lambda^2)(ax + b)$$
$$4x^3 + kx^2 + px + 2 \ = ax^3 + bx^2 + a\lambda^2x + b\lambda^2$$
Comparing both sides,
$$a = 4, k = b, p = a\lambda^2 \text{ and } b\lambda^2 = 2$$
$$kp = ab\lambda^2 = 4 \times 2 = 8$$

Exercise

1. Divide $x^3 - 2x^2 + 5x + 8\ by\ x - 2$
2. Find the remainder when $x^3 + x - 9$ is divided by $x + 1$.
3. When $ax^4 + bx^3 - x^2 + 2x + 3$ is divided by $x^2 - x - 2$, the remainder is $3x + 5$. Find the values of a and b.
4. Find the values of p and q that make $x^4 + 6x^3 + 13x^2 + px + q$ a perfect square.
5. Factorize $3x^3 - 11x^2 - 19x - 5$ completely.
6. Given that $x + 3$ and $x + 7$ are factors of $ax^2 + 12x + b$, find the values of a and b.
7. Given that $(x + a)(x^2 + bx + 2) = x^3 - 2x^2 - x - 6$, find the values of the constants a and b.
8. Express the polynomial $2x^3 + 5x^2 - 4x - 3$ in the form $(x^2 + x - 2)Q(x) + Ax + B$ where $Q(x)$ is the quotient and $Ax + b$ is the remainder when $2x^3 + 5x^2 - 4x - 3$ is divided by $x^2 + x - 2$.

$$[2x^3 + 5x^2 - 4x - 3 = (x^2 + x - 2)(2x - 3) - 3x + 3]$$

9. The remainders when the polynomial $ap^2 + bp + c$ is divided by $p - 1, p - 2$ and $p + 1$ are $1,$ 1 and 25 respectively. Factorize $ap^2 + bp + c$ completely.

$$[ap^2 + bp + c = (2p - 3)(2p - 3)]$$

10. The remainder obtained when the polynomial $2x^3 + ax^2 - 6x + 1$ is divided by $x + 2$ is twice the remainder obtained when the same polynomial is divided by $x - 1$. Find the value of the constant a.
11. When the polynomial $6x^3 + ax + bx + 4$ is separately divided by $x + 1$ and $x - 3$, the remainders are -15 and 49 respectively. Find the values a and b hence deduce the remainder when the same polynomial is divided by $x^2 - 2x - 3$.
12. The polynomial $x^4 + px^3 - x^2 = qx - 12$ is exactly divisible by both $x + 1$ and $x + 2$. Find the values of p and q hence factorize $x^4 + px^3 - x^2 = qx - 12$ completely.
13. A polynomial P(x) is exactly divisible by $x^2 - 3x + 2$ but has a remainder of 8 when divided by $x + 3$. Find the remainder when P(x) is divided by $x^2 + 2x - 3$.
14. A cubic polynomial P(x) is such that P(0) $= P(1) = P(2) = 1\ and\ P(3) = 13$. Find the expression for P(x).
15. When the polynomial $x^3 - 4x^2 + ax + b$ is divided by $x^2 - 1$, the remainder is $2x + 3$. Determine the values of a and b.
16. The polynomial $x^4 + 4x^3 + ax^2 + bx + c$ is a perfect square. Show that $b + 8 = 2a$ and $16c = b^2$. If the same polynomial leaves a remainder of 4 when divided by $x + 1$, find the values of the constants a, b and c.

17. The polynomial $x^3 + ax^2 + bx + 2$ leaves the same remainder of 4 when separately divided by x-1 and $x + 2$. Find the values a and b.
18. The expression $ax^3 - 8x^2 + bx + 6$ is exactly divisible by $x^2 - 2x - 3$. Find the values of the constants a and b.
19. The polynomial $ax^2 + 2x^2 - 5x + 7$ leaves the same remainder when separately divided by $x + 2$ and $x - 1$. Find the value of the constant a.
20. If $(x + a)(x + 3)(x - 3) \equiv x^3 + bx^2 + cx - 30$, find the values of the constants a, b and c hence solve the equation $x^3 + bx^2 + cx - 30 = 0$.
21. If $(x + a)(x + 3)(x - 3) \equiv 24x^2 - 10x^2 + cx + 1$, Find the values of the constants a, b and c hence solve the equation $24x^2 - 10x^2 + cx + 1 = 0$.

REACHOUT FOR ALL THE REMAINING TOPICS