# PREFACE

I feel extremely happy to come out with this revised edition of complete pure mathematics which now contains numerous sample questions.

It has been planned in such away that each topic begins with precise introductory notes, followed by worked examples, sample questions with their solutions where by much emphasis has been put on common examination questions in order to equip students with special techniques of handling them, lastly exercise for students to try.

The users are warned that the topics have been arranged such that you complete a full major topic before you continue to the next topic except chapter six and eleven. Therefore the arrangement is:-

- Chapter 1 to 4 – algebra 1.
- Chapter 5 – Trigonometry.
- Chapter 6 – Geometry 1
- Chapter 7 – Algebra 2.
- Chapter 8 – Analysis 1.
- Chapter 9 – Analysis 2.
- Chapter 10 – Vector.
- Chapter 11 – Geometry 2.

This book has reached the final stage because of many individual contributions. First and foremost I acknowledge Mr. Mwaka Charles, the Headmaster of St. Mary's College Kisubi, for all his assistance and for providing conducive environment in order for me to complete my task.

I would like to take this opportunity to send my appreciation to Mr. Okello Alfred D H/Master of St. Mary's College, Mr. Obot Keith, Head of Chemistry Department and Mr. Otigo Evan the former Head of Mathematics Department, Mr. Otti Walter Nyeko, Mr. Mungawu Peter, Mr. Akena S.P. and other members of Maths Department of the College for their advices, encouragement and material support given to me in order to make sure this edition comes out successfully.

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My special thanks goes to my dear wife, Mary Opio and other family members for their encouragement and moral support during the trying moments.

I would like to request all the users that for any mistake identified or suggestions for improvement of this book write to me at the following address

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## Books by the Same Author

1. Complete Advanced Level Pure Mathematics
2. A' Level Physics Practical
3. O' Level Physics Practical
4. Enjoy O' Level Mathematics

This book is dedicated to my late parents; Noah and Edida Econga.

I wish the users double successes in this double columns book.

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# CHAPTER 1

## INDICES, SURDS AND LOGARITHMS:

Index is the power to which number is raised to and the number is called a base.

For example $2^3$, 3 is index of 2 and 2 is a base.

The fundamental rules of indices

1. $a^m \times a^n = a^{m+n}$ indicates $\Rightarrow$ addition of indices

E.g. $5^3 \times 5^4 = 5^{3+4} = 78125$

2. $a^m \div a^n = a^{m-n}, (m \rangle 0)$ $\Rightarrow$ Subtraction of indices

E.g. $2^3 \div 2^2 = 2^{3-2} = 2^1 = 2$

3. $(a^m)^n = a^{mn}; (m \rangle 0)$ $\Rightarrow$ Multiplication of indices

E.g. $(2^3)^4 = 2^{3 \cdot 4} = 2^{12} = 4096$

4. $a^{-m} = \dfrac{1}{a^m}, m \rangle 0$

$\Rightarrow$ Negative index

E.g. $4^{-3} = \dfrac{1}{4^3} = \dfrac{1}{64} = 0.15625$

5. $a^{m/n} = \sqrt[n]{a^m}$ $n \rangle 0$ & $m \rangle 0$

$= \left(\sqrt[n]{a}\right)^m$

$\Rightarrow$ *Rational indices*

E.g

$$(64)^{2/3} = \sqrt[3]{64^2} = \left(\sqrt[3]{64}\right)^2$$
$$= \sqrt[3]{4096} = (4)^2$$
$$= 16 \qquad = 16$$

6. $a^0 = 1$ $\Rightarrow$ index Zero

E.g. $100^0 = 1$

7. $a^{1/m} = \sqrt[m]{a}$ ; m$\rangle$ 0

$\Rightarrow$ simple fractional index

$81^{1/4} = \sqrt[4]{81} = 3$

8. $a^{-p/q} = \dfrac{1}{a^{p/q}} = \dfrac{1}{\sqrt[q]{a^p}} = \dfrac{1}{\left(\sqrt[q]{a}\right)^p}$;

$\Rightarrow$ Negative rational indices

E.g

$$9^{-3/2} = \frac{1}{9^{3/2}} = \frac{1}{\sqrt[2]{9^3}} = \frac{1}{\left(\sqrt[2]{9}\right)^3} = \frac{1}{27}$$

9. $(ab)^m = a^m b^m$

$\Rightarrow$ distribution of index

E.g. $(2 \times 5)^2 = 2^2 \times 5^2 = 4 \times 25 = 100$

**Example** 1. (a) Express with positive integers

(i) $\sqrt[4]{t^3} \times \sqrt{t^{1/2}}$

(ii) $\dfrac{2a^{-3}b^2}{7c^{-4}d^3}$

(b) Evaluate

(i) $\left(\dfrac{8}{27}\right)^{-1/3}$

(ii) $\left(\dfrac{81}{16}\right)^{3/4}$

***Solution***

(a)(i) $\sqrt[4]{t^3} \times \sqrt{t^{1/2}}$

$= t^{3/4} \times \left(t^{1/2}\right)^{1/2}$

$= t^{3/4} \times t^{1/4}$

$= t^{(3/4 + 1/4)}$

$= t^1$

$= t$

(ii) $\dfrac{2a^{-3}b^2}{7c^{-4}d^3}$

$= \dfrac{2\frac{1}{a^3}b^2}{7\frac{1}{c^4}d^3}$

$= \dfrac{2c^4b^2}{7a^3d^3}$

(b) (i) $\left(\frac{8}{27}\right)^{-1/3} = \frac{1}{\left(\frac{8}{27}\right)^{1/3}}$

$$= \left(\frac{27}{8}\right)^{1/3} = \frac{27^{1/3}}{8^{1/3}} = \frac{(3^3)^{1/3}}{(2^3)^{1/3}} = \frac{3^{3\times 1/3}}{2^{3\times 1/3}} = \frac{3}{2}$$

(ii) $\left(\frac{81}{16}\right)^{3/4} = \frac{81^{3/4}}{16^{3/4}}$

$$= \frac{(3^4)^{3/4}}{(2^4)^{3/4}} = \frac{3^{4\times 3/4}}{2^{4\times 3/4}} = \frac{3^3}{2^3} = \frac{27}{8}$$

(2) Simplify :(i) $(x^4yz^{-3})^2 \times \sqrt{(x^{-5}yz)} \div (xz)^{7/2}$

(ii) $\frac{3(2^{2n+1}) - 4(2^{n-1})}{2^{n+1} - 2^n}$

***Solution:***

(i) $(x^4yz^{-3})^2 \times \sqrt{(x^{-5}yz)} \div (xz)^{7/2}$

$$= \frac{(x^4yz^{-3})^2 \times (x^{-5}yz)^{1/2}}{(xz)^{7/2}}$$

$$= \frac{x^8y^2z^{-6} \times x^{-5/2}y^{1/2}z^{1/2}}{x^{7/2}z^{7/2}}$$

$$= \frac{x^{8-5/2}y^{2+1/2}z^{-6+1/2}}{x^{7/2}z^{7/2}}$$

$$= \left(x^{8-5/2}y^{2+1/2}z^{-6+1/2}\right) \times \left(x^{-7/2} \times z^{-7/2}\right)$$

$$= \left(x^{8-5/2-7/2}\right)\left(y^{5/2}\right)\left(z^{-6+1/2-7/2}\right)$$

$$= x^2y^{5/2}z^{-9}$$

(ii) $\frac{3(2^{n+1}) - 4(2^{n-1})}{2^{n+1} - 2^n} = \frac{3\times 2\times 2^n - 4\times \frac{1}{2}\times 2^n}{2\times 2^n - 2^n}$

$$= \frac{6\times 2^n - 2\times 2^n}{2\times 2^n - 2^n}$$

$$= \frac{2^n(6-2)}{2^n(2-1)} = 4$$

(3) Simplify: $\frac{\sqrt[3]{x^2} + x^{-1/3}}{\sqrt{x}}$

***Solution:***

$$\frac{\sqrt[3]{x^2} + x^{-1/3}}{\sqrt{x}} = \frac{x^{2/3} + x^{-1/3}}{x^{1/2}} = \frac{x^{2/3} + \frac{1}{x^{1/3}}}{x^{1/2}}$$

$$= \frac{x^{2/3+1/3} + 1}{x^{1/3} \times x^{1/2}} = \frac{x+1}{(x)^{(2+3)/6}} = \frac{x+1}{x^{5/6}}$$

$$= x^{1/6} + x^{-5/6}$$

(4) Find the value of $x$ in each of the following

(a) (i) $\frac{6^5 \times 6^x}{36} = 6^9$

(ii) $2^{3x+1} = 3^{x+2}$

(b) Find the value of $x$ and $y$ if

$5^x \times 25^{2x} = 1$ and $3^{5x} \times 9^y = \frac{1}{9}$

***Solution:***

(a) (i) $\frac{6^5 \times 6^x}{36} = \frac{6^{5+x}}{6^2} = 6^{5+x-2} = 6^{3+x}$

$$\frac{6^5 \times 6^x}{36} = 6^{3+x} = 6^9$$

Comparing the powers;

$$3 + x = 9$$

$$x = 6$$

(ii) $2^{3x+1} = 2(2^{3x}) = 2(2^3)^x$

and $3^{x+2} = 3^2(3)^x$

$$2(2^3)^x = 3^2(3)^x$$

$$\frac{2^{3x}}{3^x} = \frac{3^2}{2}$$

$$\log_{10}\left(\frac{2^{3x}}{3^x}\right) = \log_{10}\left(\frac{3^2}{2}\right)$$

$$3x\log_{10}2 - x\log_{10}3 = 2\log_{10}3 - \log_{10}2$$

$$3x(0.3010) - x(0.4771) = 2(0.4771) - 0.3010$$
$$0.9030x - 0.4771x = 0.9542 - 0.3010$$
$$0.4260x = 0.6532$$
$$\Rightarrow x = 1.5333$$

OR $\quad 2^{3x+1} = 3^{x+2}$

$$\log_{10} 2^{3x+1} = \log_{10} 3^{x+2}$$
$$(3x+1)\log_{10} 2 = (x+2)\log_{10} 3$$
$$(3x+1)\log_{10} 2 - (x+2)\log_{10} 3 = 0$$
$$(3x+1)(0.3010) - (x+2)(0.4771) = 0$$
$$0.9030x + 0.3010 - 0.4771x - 0.9542 = 0$$
$$0.426x = 0.6532$$
$$\Rightarrow x = 1.5333$$

(b) $\quad 5^{x} \times 25^{2y} = 1$ and $\quad 3^{5x} \times 9^{y} = \frac{1}{9}$

$5^{x} \times (5^{2})^{2y} = 5^{0}$ $\qquad$ $3^{5x} \times 3^{2y} = 9^{-1}$

$5^{x+4y} = 5^{0}$ $\qquad$ $3^{5x+2y} = 3^{-2}$

Comparing powers with both cases we have;

$x + 4y = 0$ .....(i) and

$5x + 2y = -2$ .....(ii)

2(ii) – (i)

$$10x + 4x = -4$$
$$x + 4y = 0 \; -$$
$$9x \quad = -4$$
$$x = -\frac{4}{9}$$

From equation (i) if we replace $x$ we have

$$-\frac{4}{9} + 4y = 0$$
$$4y = \frac{4}{9}$$
$$y = \frac{1}{9}$$

Hence $x = -\frac{4}{9}$ and $y = \frac{1}{9}$

Or from $5^{x+4y} = 5^{0}$ taking $\log_5$ on both sides

We have $(x+4y)\log_5 5 = 0\log_5 5$

$x + 4y = 0$ .....(i)

And from $3^{5x+2y} = 3^{-2}$ taking $\log_3$ on both sides

$(5x+2y)\log_3 3 = -2\log_3 3$

$5x + 2y = -2$ .....(ii)

Then solve as before so that we have

$x = -\frac{4}{9}$ and $y = \frac{1}{9}$

**Exercise 1 (a)**

1. Solve the equation:

   i) $y^{\frac{3}{2}} = 64$, ii $9^{x}(3^{x+1}) = 81$

2. Find the values of x and y in the equations.

   $3 = 2^{2y}x$ $\quad$ *and* $\quad \frac{3}{64} = 2^{-2y}x$

3. simplify the following expressions

   *i)* $\dfrac{(x)^{\frac{1}{t}} \div x^{-t}}{(x)^{t+\frac{1}{t}}}$

   *ii)* $\dfrac{(xy)^{\frac{1}{2}}}{(x^{2})(\sqrt{xy})^{3}}$

   *iii)* $\dfrac{(9^{n+1})(6^{n-1})}{(3^{2n-1})(2^{n})}$

4. a) Simplify $1 + \log\left(\frac{64}{x^{4}}\right)^{\frac{1}{2}} + 2\log_8 x$

   b) Solve for x in $2 + 3\log x = \log(0.1)$

5. Simplify the following

   *i)* $\dfrac{3(2^{n+1}) - 4(2^{n-1})}{2^{n+1} - 2^{n}}$

   *ii)* $\dfrac{(8^{t+2})(4^{2t-1})}{(2^{t}) + (4)^{\frac{t}{2}}}$

   *iii)* $(x-2)^{\frac{5}{2}} + 2(x-2)^{\frac{3}{2}}$

**Surds:** These are numbers which cannot be evaluated exactly hence they are irrational e.g.

$7^{1/2}, 61^{1/2}, 20^{1/5}$ .....In short they are numbers which when converted to decimal cannot be exact but they are approximated e.g. $7^{1/2} = 2.6458$ (4dpts)

**Rule set upon surds**

(1) $t^{1/2} \times n^{1/2} = (tn)^{1/2}$ e.g $(2)^{1/2} \times (3)^{1/2} = (6)^{1/2}$

(2) $\dfrac{t^{1/2}}{n^{1/2}} = \left(\dfrac{t}{n}\right)^{1/2}$

(3) $\left(t^{1/2} + n^{1/2}\right)^2 = t + n + 2(tn)^{1/2}$

(4) $\left(t^{1/2} - n^{1/2}\right)^2 = t + n - 2(tn)^{1/2}$

E.g. $\left(5^{1/2} - 3^{1/2}\right)^2 = 5 + 3 - 2(5 \times 3)^{1/2}$

5 $\left(a \pm b^{1/2}\right)^2 = a^2 + b \pm 2a(b)^{1/2}$

E.g $\left(2 \pm 3^{1/2}\right)^2 = 4 + 3 \pm 4(3)^{1/2}$

(6) $\left(a^{1/2} + b^{1/2}\right)\left(a^{1/2} - b^{1/2}\right) = a - b$

(7) $\left(a + b^{1/2}\right)\left(a - b^{1/2}\right) = a^2 - b$

1. Simplify

(a) $\dfrac{1}{3(5)^{1/2} - (3)^{1/2}} + \dfrac{1}{(5)^{1/2} + 3}$

(b) $\dfrac{(3)^{1/2} - 2^{1/2}}{3^{1/2} + 2^{1/2}}$

***Solution:***

(a) $\dfrac{3(5)^{1/2} + (3)^{1/2}}{(3(5)^{1/2} + (3)^{1/2})(3(5)^{1/2} - (3)^{1/2})} + \dfrac{(5)^{1/2} - 3}{((5)^{1/2} + 3)((5)^{1/2} - 3)}$

$\dfrac{3(5)^{1/2} + (3)^{1/2}}{9(5) - 3} + \dfrac{5^{1/2} - 3}{5 - 9}$

$\dfrac{3(5)^{1/2} + (3)^{1/2}}{42} + \dfrac{5^{1/2} - 3}{-4}$

$\dfrac{6(5)^{1/2} + 2(3)^{1/2} - 21(5)^{1/2} + 63}{84}$

$\dfrac{63 + 2(3)^{1/2} - 15(5)^{1/2}}{84}$

$= \dfrac{3}{4} + \dfrac{(3)^{1/2}}{42} - \dfrac{5(5)^{1/2}}{28}$

(b) $\dfrac{\left(3^{1/2} - 2^{1/2}\right)\left(3^{1/2} - 2^{1/2}\right)}{\left(3^{1/2} + 2^{1/2}\right)\left(3^{1/2} - 2^{1/2}\right)} = \dfrac{3 + 2 - 2(6)^{1/2}}{3 - 2} = 5 - 2(6)^{1/2}$

Hence $\dfrac{3^{1/2} - 2^{1/2}}{3^{1/2} - 2^{1/2}} = 5 - 2(6)^{1/2}$

2. Simplify (a) $\dfrac{3(2)^{1/2} + 2(3)^{1/2}}{3(2)^{1/2} - 2(3)^{1/2}}$

(b) $\left(a + 2^{1/2}\right)^4 + \left(a - 2^{1/2}\right)^4$

***Solution:***

(a) $\dfrac{\left(3(2)^{1/2} + 2(3)^{1/2}\right)\left(3(2)^{1/2} + 2(3)^{1/2}\right)}{\left(3(2)^{1/2} - 2(3)^{1/2}\right)\left(3(2)^{1/2} + 2(3)^{1/2}\right)}$

$\dfrac{9(2) + 6(6)^{1/2} + 6(6)^{1/2} + 4(3)}{9(2) - 4(3)}$

$\dfrac{30 + 12(6)^{1/2}}{18 - 12} = \dfrac{30 + 12(6)^{1/2}}{6} = 5 + 2(6)^{1/2}$

(b) $\left(a + (2)^{1/2}\right)^4 + \left(a - (2)^{1/2}\right)^4$

From $\left(a + 2^{\frac{1}{2}}\right)^4 = \left[\left(a + 2^{\frac{1}{2}}\right)^2\right]^2$

$= \left(a^2 + 2 + 2a\, 2^{\frac{1}{2}}\right)^2$

$= (a^2 + 2)^2 + 8a^2 + 2(a^2 + 2)\left(2a\, 2^{\frac{1}{2}}\right)$

$\left(a + (2)^{1/2}\right)^4 = \left[\left(a + 2^{\frac{1}{2}}\right)^2\right]^2$

$= \left((a^2 + 2) - 2a\, 2^{\frac{1}{2}}\right)^2$

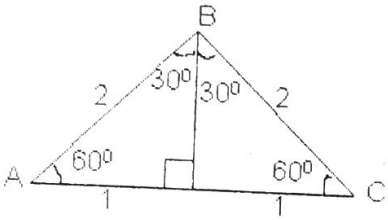
$$\Longrightarrow \left(a+(2)^{1/2}\right)^4+\left(a-(2)^{1/2}\right)^4$$

3. Express in surd form and rationalize the denominator

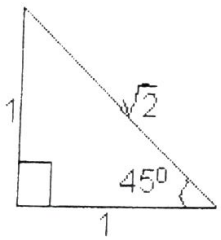
(a) $\dfrac{1+\tan 60^0}{1-\tan 60^0}$ (b) $\dfrac{1}{1+\cos 45^0}$ (c) $\dfrac{2}{1-\cos 30^0}$

$5-2(6)^{1/2}$

**Solution:**




$\cos 30^0=\dfrac{3}{2}$, $\tan 60^0=\dfrac{\sqrt{3}}{1}=\sqrt{3}$




$\sin 45^0=\cos 45^0=\dfrac{1}{\sqrt{2}}=\dfrac{\sqrt{2}}{2}$, $\tan 45^0=1$

$(6)^{1/2}$

Hence (a) $\dfrac{1+\tan 60^0}{1-\tan 60^0}=\dfrac{1+\sqrt{3}}{1-\sqrt{3}}$

$$=\frac{(1+\sqrt{3})(1+\sqrt{3})}{(1-\sqrt{3})(1+\sqrt{3})}=\frac{1+\sqrt{3}+3+\sqrt{3}}{1+\sqrt{3}-\sqrt{3}-3}$$

$$=\frac{4+2\sqrt{3}}{1-3}=\frac{4+2\sqrt{3}}{-2}$$

$$=-2-\sqrt{3}$$

(b) $\dfrac{1}{1+\cos 45^0}$

$$=\frac{1}{1+\frac{\sqrt{2}}{2}}$$

$$=\frac{2}{2+\sqrt{2}}=\frac{2\times(2-\sqrt{2})}{(2+\sqrt{2})(2-\sqrt{2})}$$

$$=\frac{4-2\sqrt{2}}{4-2}=\frac{4-2\sqrt{2}}{2}$$

$$=2-\sqrt{2}$$

(c) $\dfrac{2}{1-\cos 30^0}$

$$=\frac{2}{1-\frac{\sqrt{3}}{2}}=\frac{4}{2-\sqrt{3}}=\frac{4(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})}$$

$$=\frac{8+4\sqrt{3}}{4+2\sqrt{3}-2\sqrt{3}-3}$$

$$=\frac{8+4\sqrt{3}}{4-3}$$

$$=8+4\sqrt{3}$$

$$=4(2+\sqrt{3})$$

4. Simplify (i) $\dfrac{1+\sqrt{2}+\sqrt{5}}{\sqrt{2}+\sqrt{3}}$

(ii) $\dfrac{\sqrt[3]{x}+x^{-1/3}}{\sqrt{x}}$

***Solution:***

(i) $\dfrac{1+\sqrt{2}+\sqrt{5}}{\sqrt{2}+\sqrt{5}}$

$$=\frac{(1+\sqrt{2}+\sqrt{5})(\sqrt{2}-\sqrt{3})}{(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})}$$

$$=\frac{\sqrt{2}-\sqrt{3}+2-\sqrt{6}+\sqrt{10}-\sqrt{15}}{2-\sqrt{6}+\sqrt{6}-3}$$

$$=\frac{(\sqrt{2}+\sqrt{10})-(\sqrt{3}+\sqrt{15})+2-\sqrt{6}}{2-3}$$

$$= \frac{\sqrt{2}(1+\sqrt{5}) - \sqrt{3}(1+\sqrt{5}) + 2 - \sqrt{6}}{-1}$$

$$= (\sqrt{3}-\sqrt{2})(1+\sqrt{5}) + \sqrt{6} - 2$$

(ii) $$\frac{\sqrt[3]{x} + x^{-1/3}}{\sqrt{x}} = \frac{x^{1/3} + \frac{1}{x^{1/3}}}{x^{1/2}}$$

$$= \frac{\left(x^{\frac{2}{3}} + 1\right)}{x^{1/2 + 1/3}}$$

$$\frac{x^{\frac{2}{3}} + 1}{x^{5/6}}$$

$$= \frac{\left(x^{\frac{2}{3}} + 1\right)^{6/5}}{\left(x^{\frac{5}{6}}\right)^{\frac{6}{5}}}$$

$$= \frac{\left(x^{2/3} + 1\right)^{\frac{6}{5}}}{x}$$

5. (a) Express $\frac{1+\sqrt{3}}{(\sqrt{3}-1)^3}$ in the form $a + b\sqrt{c}$, where $a, b$ *and* $c$ are rational.

(b) Simplify $\sqrt{(14 - 4\sqrt{6})}$

(c) $\frac{\sqrt{3}(2-\sqrt{3})}{(\sqrt{5}-\sqrt{3})}$ make it in form $a + b\sqrt{c}$

***Solution:***

(a) $$(\sqrt{3}-1)^3 = \binom{3}{0}\sqrt{3}^3 + \binom{3}{1}\sqrt{3}^2(-1) + \binom{3}{2}\sqrt{3}(-1)^2 + \binom{3}{3}(-1)^3$$

$$= 3\sqrt{3} + 3(3)(-1) + 3\sqrt{3}(1) - 1$$

$$= 3\sqrt{3} + 3\sqrt{3} - 9 - 1$$

$$= 6\sqrt{3} - 10$$

$$\frac{1+\sqrt{3}}{6\sqrt{3}-10} = \frac{(1+\sqrt{3})(6\sqrt{3}+10)}{(6\sqrt{3}-10)(6\sqrt{3}+10)}$$

$$= \frac{6\sqrt{3} + 10 + 6\cdot 3 + 10\sqrt{3}}{6^2 \cdot 3 - 10^2}$$

$$= \frac{16\sqrt{3} + 28}{108 - 100}$$

$$= \frac{16\sqrt{3} + 28}{8}$$

$$= \frac{7}{2} + 2\sqrt{3}.$$

(b) Suppose $14 - 4\sqrt{6} = (\sqrt{a} - 2\sqrt{b})^2$

$14 - 4\sqrt{6} = (a + (4)b - 4\sqrt{ab})$

By comparing R.H.S and L.H.S.

$14 = (a + 4b)$ .....(i)

$ab = 6$ .....(ii)

$a = 14 - 4b$ Substituting for $a$

$(14 - 4b)b = 6$

$7b - 2b^2 = 3$

$2b^2 - 7b + 3 = 0$

$$b = \frac{7 \pm \sqrt{7^2 - 4\times 3\times 2}}{4}$$

$$= \frac{7 \pm 5}{4} = 3 \text{ or } \frac{1}{2}$$

When $b = 3$

$a = 14 - 4(3) = 2$

When $b = \frac{1}{2}$

$a = 14 - 4 \times \frac{1}{2} = 12$

When $a = 2$ $b = 3$

$14 - 4\sqrt{6} = (\sqrt{2} - 2\sqrt{3})^2$

$\therefore \sqrt{14 - 4\sqrt{6}} = \sqrt{2} - 2\sqrt{3}$ or $2\sqrt{3} - \sqrt{2}$

When $a = 12$ and $b = \frac{1}{2}$

$$14-4\sqrt{6}=\left(\sqrt{12}-2\sqrt{\frac{1}{2}}\right)^2$$

$$=2\sqrt{3}-\sqrt{2}$$

But not equal to $\sqrt{2}-2\sqrt{3}$

(c) $\dfrac{\sqrt{3}\left(2-\sqrt{3}\right)}{\left(\sqrt{5}-\sqrt{3}\right)}$

$$=\frac{\sqrt{3}\left(2-\sqrt{3}\right)\left(\sqrt{5}+\sqrt{3}\right)}{\left(\sqrt{5}-\sqrt{3}\right)\left(\sqrt{5}+\sqrt{3}\right)}$$

$$=\frac{\left(2\sqrt{3}-3\right)\left(\sqrt{5}+\sqrt{3}\right)}{\left(\sqrt{5}\right)^2+\sqrt{15}-\sqrt{15}+\left(\sqrt{3}\right)^2}$$

$$=\frac{2\sqrt{15}+2(3)-3\sqrt{5}-3\sqrt{3}}{5-3}$$

$$=\frac{6+2\sqrt{15}-3\sqrt{5}-3\sqrt{3}}{2}$$

$$=3+\sqrt{15}-\frac{3}{2}\left(\sqrt{5}-\sqrt{3}\right)$$

6. Solve for $x$ in the following equation

(i) $\sqrt{x+1}=3$

(ii) $\sqrt{2x+5}-1=x$

(iii) $\sqrt{3x+5}+x=3$

(iv) $3\sqrt{x}-\sqrt{x+5}=3$

***Solution:***

(i) $\sqrt{x+1}=3$

$\left(\sqrt{x+1}\right)^2=3^2$

$x+1=9$

$x=8$

(ii) $\sqrt{2x+5}-1=x$

$\left(\sqrt{2x+5}\right)^2=(x+1)^2$

$2x+5=x^2+2x+1$

$5=x^2+1$

$x^2=4$

$x=\pm 2$

When $x=2$

$\sqrt{4+5}-1=2$

$3-1=2$. Satisfy

When $x=-2$

$\sqrt{-4+5}-1=-2$

$1-1\neq -2$

Hence $x=2$

(iii) $\sqrt{3x+5}+x=3$

$\sqrt{3x+5}=3-x$

$\left(\sqrt{3x+5}\right)^2=(3-x)^2$

$x^2-9x+4=0$

$$x=\frac{9\pm\sqrt{9^2-4\times1\times4}}{2}$$

$$=\frac{9\pm8.0623}{2}$$

$x=8.5311$ or $0.4689$

When $x=8.5311$

$\sqrt{3\times8.5311+5}+8.5311\neq 3$

Does not satisfy.

When $x=0.4689$

$\sqrt{3\times0.4689+5}+0.4689=3$

$3=3$

Satisfy

Hence $x=0.4689$ but not $8.5311$

(iv) $3\sqrt{x}-\sqrt{x+5}=3$

$\left(3\sqrt{x}-\sqrt{x+5}\right)^2=3^2$

$9x-6\sqrt{x}\sqrt{x+5}+(x+5)=9$

$10x+5-6\sqrt{x(x+5)}=9$

$6\sqrt{x(x+5)}=(10x-4)$

$\left(6\sqrt{x(x+5)}\right)^2=(10x-4)^2$

$36x(x+5)=100x^2-80x+16$

$9x^2-45x=25x^2-20x+4$

$16x^2-65x+4=0$

$$x = \frac{65 \pm \sqrt{65^2 - 4 \times 16 \times 4}}{32}$$

$$x = \frac{65 \pm 63}{32}$$

$= 4$ or $0.0625$

When $x = 4$

$$3\sqrt{4} - \sqrt{4+5} = 3$$

$$6 - \sqrt{9} = 3$$

$$3 = 3$$

Since the LHS = RHS.

Hence it satisfies the equation.

When $x = 0.0625$

$$3\sqrt{0.0625} - \sqrt{0.0625 + 5} = 3$$

$$0.75 - 2.25 \neq 3$$

Hence $x = 4$.

7. Solve the following equation, check your solutions in each case whether it or they satisfy the equation.

(i) $\sqrt{2x+1} - \sqrt{x} = \sqrt{x-3}$

(ii) $\sqrt{x+6} + \sqrt{4-x} = \sqrt{1-3x}$

***Solution:***

(i) $\sqrt{2x+1} - \sqrt{x} = \sqrt{x-3}$

$$\left(\sqrt{2x+1} - \sqrt{x}\right)^2 = \left(\sqrt{x-3}\right)^2$$

$$2x + 1 + x - 2\sqrt{x}\sqrt{2x+1} = (x-3)$$

$$(3x+1) - (x-3) = 2\sqrt{x(2x+1)}$$

$$(2x+4)^2 = \left(2\sqrt{x(2x+1)}\right)^2$$

$$4x^2 + 16x + 16 = 4x(2x+1)$$

$$4x^2 + 16x + 16 = 8x^2 + 4x$$

$$4x^2 - 12x - 16 = 0$$

$$x = \frac{12 \pm \sqrt{12^2 + 4 \times 4 \times 16}}{8}$$

$$x = \frac{12 \pm 20}{8}$$

$x = 4$ or $-1$

When $x = 4$

$$\sqrt{8+1} - \sqrt{4} = \sqrt{4-3}$$

$$3 - 2 = 1$$

$x = 4$ Satisfies the equation.

(ii) $\sqrt{x+6} + \sqrt{4-x} = \sqrt{1-3x}$

$$\left(\sqrt{x+6} + \sqrt{4-x}\right)^2 = \left(\sqrt{1-3x}\right)^2$$

$$(x+6) + (4-x) + 2\sqrt{(x+6)(4-x)} = (1-3x)$$

$$2\sqrt{(x+6)(4-x)} + 10 = (1-3x)$$

$$2\sqrt{(x+6)(4-x)} = -(3x+9)$$

$$\left(2\sqrt{(x+6)(4-x)}\right)^2 = \left(-(3x+9)\right)^2$$

$$4(x+6)(4-x) = 9x^2 + 81 + 54x$$

$$4(4x - x^2 + 24 - 6x) = 9x^2 + 54x + 81$$

$$-8x - 4x^2 + 96 = 9x^2 + 54x + 81$$

$$13x^2 + 62x + 15 = 0$$

$$x = \frac{-62 \pm \sqrt{62^2 + 13 \times 4 \times 15}}{2 \times 13}$$

$$= \frac{-62 \pm 68}{26}$$

$$= \frac{6}{26} \text{ or } -5$$

$$= \frac{3}{13} \text{ or } -5$$

When $x = -5$ we will have

$$(-5+6)^{1/2} + (4+5)^{1/2} = (1+15)^{1/2}$$

$$(1)^{1/2} + (9)^{1/2} = (16)^{1/2}$$

$\pm 1 \pm 3 = \pm 4$ it Satisfies

When $x = \frac{3}{13}$ we will have

$$\left(\frac{3}{13} + 6\right)^{1/2} + \left(4 - \frac{3}{13}\right)^{1/2} \neq \left(1 - \frac{3}{13}\right)^{1/2}$$

$$\left(\frac{81}{13}\right)^{1/2} + \left(\frac{49}{13}\right)^{1/2} \neq \left(\frac{4}{13}\right)^{1/2}$$

$$\frac{9}{(13)^{1/2}} + \frac{7}{(13)^{1/2}} \neq \frac{2}{(13)^{1/2}}$$

$$\frac{16}{(13)^{1/2}} \neq \frac{2}{(13)^{1/2}}$$

Hence $x$ = -5 only.

8. (a) Solve the following equation

$$(3-x)^{1/2} - (7+x)^{1/2} = (16+2x)^{1/2}$$

***Solution:***

Squaring both sides we have

$$(3-x) - 2(3-x)^{1/2}(7+x)^{1/2} + (7+x) = 16 + 2x$$

$$-2(3-x)^{1/2}(7+x)^{1/2} + 10 = 16 + 2x$$

$$-2(3-x)^{1/2}(7+x)^{1/2} = 2x + 6$$

Squaring both sides we have:

$$4(3-x)(7+x) = 36 + 24x + 4x^2$$

$$2x^2 + 10x - 12 = 0$$

$$x^2 + 5x - 6 = 0$$

$$x = -6 \text{ or } 1$$

When $x = -6$ is substituted back in the above equation we have

$$(3+6)^{1/2} - (7-6)^{1/2} = (16-12)^{1/2}$$

$$3 - 1 = 2$$

hence $x$ = -6 satisfies equation

When $x$ = 1 is substituted back in the above equation we have

$$(3-1)^{1/2} - (7--1)^{1/2} = (16-2(-1))^{1/2}$$

$$(2)^{1/2} - (8)^{1/2} = (18)^{1/2}$$

$$1(2)^{1/2} - 2(2)^{1/2} = 3(2)^{1/2}$$

$$-1(2)^{1/2} = 3(2)^{1/2}$$

Dividing through by $(2)^{1/2}$

$-1 \neq 3$ does not satisfies the equation

Hence the root is $x$ = -6

9. Find the square root of

(i) $5 + 2(6)^{1/2}$

(ii) $18 - 12(2)^{1/2}$

***Solution:***

(i) Let the square root of $5 + 2(6)^{1/2}$ be

$$a^{1/2} + b^{1/2}$$

$$\left(5 + 2(6)^{1/2}\right)^{1/2} = a^{1/2} + b^{1/2}$$

Squaring both sides we have

$$5 + 2(6)^{1/2} = \left(a^{1/2} + b^{1/2}\right)^2 = a + b + 2a^{1/2}b^{1/2}$$

$$5 + 2(6)^{1/2} = a + b + 2a(ab)^{1/2}$$

Comparing parts we have

$5 = a + b$ .....(i) and

$6 = ab$ .....(ii)

From equation $b = 5 - a$ and substituting b in (ii)

We have $6 = a(5-a) = 5a - a^2$

$$a^2 - 5a + 6 = 0$$

$$a = 2 \text{ or } 3$$

When $b$ = 3 then $a$ = 2 and

When $a$ = 3 then $b$ = 2

Hence square root of $5 + 2(6)^{1/2}$ is $3^{1/2} + 2^{1/2}$

(ii) Let square root of $18 - 12(2)^{1/2}$ be $x^{1/2} - y^{1/2}$

$$\left(18 - 12(2)^{1/2}\right)^{1/2} = x^{1/2} - y^{1/2}$$

Squaring both sides

$$18 - 12(2)^{1/2} = \left(x^{1/2} - y^{1/2}\right)^2 = x + y - 2(xy)^{1/2}$$

$$18 - 2(6^2 \times 2)^{1/2} = x + y - 2(xy)^{1/2}$$

Comparing parts we have

$18 = x + y$ .....(i) and

$72 = xy$ .....(ii)

From (i) $y = 18 - x$ and substituting for y in (ii)

We have $72 = x(18-x) = 18x - x^2$

$$x^2 - 18x + 72 = 0$$

$$x = \frac{18 \pm \sqrt{18^2 - 4 \times 1 \times 72}}{2}$$

$$x = \frac{18 \pm 6}{2}$$

$$= \frac{18+6}{2} \quad \text{or} \quad \frac{18-6}{2}$$

$$= 12 \quad \text{or} \quad 6$$

Hence square root of

$18 - 12(2)^{1/2}$ is $(12)^{1/2} - (6)^{1/2}$

10. Simplify (a) $\dfrac{1}{1+x^{1/2}} + \dfrac{1}{1-x^{1/2}} + \dfrac{1}{1-x}$

(b) $\left(x^{1/2} + \dfrac{1}{x^{1/2}}\right)\left(x^{1/2} - 1 - \dfrac{1}{x^{1/2}}\right)$

***Solution:***

(a) $$\frac{1\left(1-x^{1/2}\right)}{\left(1+x^{1/2}\right)\left(1-x^{1/2}\right)} + \frac{1\left(1+x^{1/2}\right)}{\left(1-x^{1/2}\right)\left(1+x^{1/2}\right)} + \frac{1}{1-x}$$

$$= \frac{1-x^{1/2}}{1-x} + \frac{1+x^{1/2}}{1-x} + \frac{1}{1-x}$$

$$= \frac{-\left(1-x^{1/2}+1+x^{1/2}+1\right)}{(x-1)}$$

$$= \frac{-3}{x-1} = \frac{3}{1-x}$$

(b) $$\left(x^{1/2} + \frac{1}{x^{1/2}}\right)\left(x^{1/2} - 1 - \frac{1}{x^{1/2}}\right)$$

$$\left(\frac{x+1}{x^{1/2}}\right)\left(\frac{x - x^{1/2} - 1}{x^{1/2}}\right)$$

$$= \frac{x^2 - x(x)^{1/2} - x + x - (x)^{1/2} - 1}{x}$$

$$= \frac{x^2 + x(x)^{1/2} - x^{1/2} - 1}{x}$$

$$= x - x^{1/2} - \frac{1}{x^{1/2}} - \frac{1}{x}$$

$$= x - x^{1/2} - \frac{(x)^{1/2}}{x} - \frac{1}{x}$$

**Exercise 1 (b)**

1. Find the square root of:

   i) $14 + 6\sqrt{5}$, ii) $13 - 2\sqrt{42}$

2. express (i) $\dfrac{3+\sqrt{2}}{3-\sqrt{2}}$ and ii) $\left(\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}\right)$ form $p + q\sqrt{r}$

3. Express the following without a surd in denominator.

   *i)* $\dfrac{2}{1+\sqrt{3}} - \dfrac{3}{3-2\sqrt{3}}$

   *ii)* $\dfrac{1}{1+\sqrt{(x-1)}} - \dfrac{1}{-1+\sqrt{(x+1)}}$

4. Rationalize the denominators and simplif as far as possible the following.

   *i)* $\dfrac{1}{2\sqrt{5}+\sqrt{5}-1}$

   *ii)* $\dfrac{3\sqrt{2}-4}{3-2\sqrt{2}}$

5. Given that to six significant figures. $\sqrt{6} = 2.44949$, evaluate as accurately as possible $\dfrac{1}{(2\sqrt{3}-\sqrt{6})}$, justify the accuracy you have given.

6. Solve for x in the following equations.

   *i)* $\sqrt{(x+6)} + \sqrt{(4-x)} = \sqrt{(1-3x)}$

   *ii)* $\sqrt{(3x+4)} = 2 + \sqrt{(x+2)}$

7. Rationalize the surd $\dfrac{1}{3\sqrt{5}-\sqrt{3}} + \dfrac{1}{3+\sqrt{5}}$

8. Solve:

   *i)* $2\sqrt{(x-1)} - \sqrt{(x+4)} = 1$

   *ii)* $2\sqrt{x-4} = 4 - \sqrt{(x-1)}$

   *iii)* $\sqrt{(4x-2)} + \sqrt{(x+1)} - \sqrt{(7-5x)} = 0$

   *iv)* $\sqrt{(x+2)} + \sqrt{(3x+4)} = 2$

solve the following:

i) $\sqrt{(6-x)}-\sqrt{(1-x)}=\sqrt{(5-3x)}$

ii) $2\sqrt{(x+1)}-3\sqrt{(2x-5)}=\sqrt{(x-2)}$

iii) $\sqrt{(4x+13)}-\sqrt{(x+1)}=\sqrt{(12-x)}$

Leave the answers in simplified surds.

$\cos 15^0$, (ii) $\cos 165^0$ and (iii) $\cos 75^0$

prove that:

$$\tan^2\left(\frac{\pi}{8}\right)=3-2\sqrt{2}$$

$$\tan 15^0=2-\sqrt{3}$$

$$\tan\frac{5}{12}\pi=2+\sqrt{3}$$

Express $\frac{1+\sqrt{2}}{\sqrt{5}+\sqrt{3}}-\frac{1-\sqrt{2}}{\sqrt{5}-\sqrt{3}}$ in the form $a\sqrt{b}+c\sqrt{d}$, hence state values of a, b, c and.

**GARITHM:**

***finition***: Logarithm is the power or index to ch a number (base) is raised to produce a given nber.

mple 1: Logarithm of 100 to base 10 is 2 and thematically can be written as

$_{10}100=2$. $= 2\log_{10}10$

ich means $100=10^2=10\times 10$

mple 2: $\log_2 8=3$

$8=2^3=2\times 2\times 2$

mple 3: $\log_5 625=4$

$625=5^4=5\times 5\times 5\times 5$

mple 4: $\log_{10}1000=3$

$1000=10^3=10\times 10\times 10$

mple 5: $\log_{10}2=0.3010$

$2=10^{0.3010}$

**nple Elementary Rules for Logarithms**

$\log_x PQ=\log_x P+\log_x Q$

$\log_x\left(\frac{P}{Q}\right)=\log_x P-\log_x Q$

3. $\log_x P^n=n\log_x P$

4. $\log_x x=1$

5. $\log_x 1=0$

6. $\log_x(-y)\neq$ does not exist

7. $\log_x\sqrt[n]{m}=\frac{1}{n}\log_x m$

8. $\log_x(P)^{1/t}=-t\log_x P$

9. $\log_x P=\frac{\log_c P}{\log_c x}$

10. $\log_x a=\frac{1}{\log_a x}$

1. To prove that $\log_c(a\ b)=x+y$ if $\log_c a=x$ and $\log_c b=y$;

***Solution;***

Given that $\log_c a=x \Rightarrow a=c^x$

And given that $\log_c b=y \Rightarrow b=c^y$

Then $a\times b=c^x\times c^y$

Using the standard rules of indices we have

$ab=c^{x+y}$

And hence

$\log_c ab=\log_c c^{x+y}$

Using the standard rules of logarithms

$\log_c(ab)=x+y\log_c c$

$=(x+y)\cdot 1$

$\log_c(ab)=x+y$ But $x=\log_c a$ and $y=\log_c b$ $\log_c(ab)=\log(ab)=\log_c a+\log_c b$

2.Prove that $\log_c\left(\frac{a}{b}\right)=\log_c a-\log_c b$.

Given that $c^x=a$ then $\log_c a=x$ and given that $c^y=b$ then $\log_c b=y$

Using the standard rules of indices we have

$$\frac{a}{b}=\frac{c^x}{c^y}=c^{x-y}$$

$\log_c\left(\frac{a}{b}\right) = \log_c c^{x-y} = (x-y)\cdot 1 = x - y$ But

$x = \log_c a$, $y = \log_c b$

$\log_c\left(\frac{a}{b}\right) = \log_c a - \log_c b$

3. Prove that $\log_c a^n = n\log_c a$

Let $a = c^x \quad a^n = c^{nx}$;

Then $\log_c a = x$

Taking $\log_c an = \log_c nx$

$\log_c a^n = nx\log_c c = nx$

But $x = \log_c a$

$\log_c a^n = n\log_c a$

4. Prove that $\log_x P = \frac{\log_c P}{\log_c x}$

Let $\log_x P = t$

$P = x^t$

$\log_c P = t\log_c x$

$t = \frac{\log_c P}{\log_c x}$

But $t = \log_x P$

Hence $\log_x P = \frac{\log_c P}{\log_c x}$

**APPLICATION OF LOGARITHMS:**

1. Solve $\log_5 x + \log_x 5 = 2.5$

***Solution:***

Let $\log_x 5 = n$

$5 = x^n$

$\log_5 5 = n\log_5 x$

$n = \frac{\log_5 5}{\log_5 x} = \frac{1}{\log_5 x}$

Then $\log_5 x + \log_x 5 = 2.5$ becomes

$\log_5 x + \frac{1}{\log_5 x} = \frac{5}{2}$

$\frac{\log_5 x^2 + 1}{\log_5 x} = \frac{5}{2}$

$\log_5 x^2 + 1 = \frac{5}{2}\log_5 x$

$2(\log_5 x^2) + 2 = 5\log_5 x$

Let $\log_5 x = y$

$2y^2 + 2 = 5y$

$2y^2 - 5y + 2 = 0$

$y = \frac{5 \pm \sqrt{-5^2 - 4\times 2\times 2}}{2\times 2} = \frac{5 \pm \sqrt{25-16}}{4}$

$y = \frac{5 \pm \sqrt{9}}{4} = \frac{5+3}{4}$ or $\frac{5-3}{4} = 2$ or $\frac{1}{2}$

Since $y = \log_5 x = 2$ or $\frac{1}{2}$

The $\log_5 x = 2$

$x = 5^2 = 25$

or $\log_5 x = \frac{1}{2}$

$x = 5^{1/2} = \sqrt{5}$

$\therefore x = 25$ or $\sqrt{5}$

2. Solve $3(4^x) - 8(2^x) + 4 = 0$

***Solution:***

$3(2^2)^x - 8(2^x) + 4 = 0$

$3(2)^{2x} - 8(2^x) + 4 = 0$

$3(2^x)^2 - 8(2^x) + 4 = 0$

Let $2^x = t$

$3t^2 - 8t + 4 = 0$

$t = \frac{+8 \pm \sqrt{8^2 - 4\times 3\times 4}}{2\times 3}$

$= \frac{+8 \pm \sqrt{64-48}}{6}$

$$= \frac{+8 \pm 16}{6} = \frac{+8 \pm 4}{6}$$

$$t = \frac{+8+4}{6} \quad \text{or} \quad \frac{+8-4}{6}$$

$$= 2 \quad \text{or} \quad \frac{2}{3}$$

$$2^x = \frac{2}{3} \quad \text{or} \quad 2$$

$$\log_{10} 2^x = \log_{10} \frac{2}{3}$$

$$x \log_{10} 2 = \log_{10} \frac{2}{3}$$

$$x \log_{10} 2 = \log_{10} 2 - \log_{10} 3$$

$$x = \frac{\log_{10} 2 - \log_{10} 3}{\log_{10} 2}$$

$$x = \frac{0.3010 - 0.4771}{0.3010} = \frac{-1761}{0.3010}$$

$$= -0.5851$$

Or $\log_{10} 2^x = \log_{10} 2$

$$x \log_{10} 2 = \log_{10} 2$$

$$x = \frac{\log_{10} 2}{\log_{10} 2} = 1$$

$$x = 1 \quad \text{or} \quad -0.5851$$

3. Solve for $x$ in the equation

$$3^{2x+1} + 26(3^x) = 9$$

***Solution:***

$$3^{2x} \times 3^1 + 26(3^x) = 9$$

$$3(3^{2x}) + 26(3^x) = 9$$

Let $m = 3^x$

$$3m^2 + 26m - 9 = 0$$

$$m = \frac{-26 \pm \sqrt{26^2 + 4 \times 3 \times 9}}{2 \times 3}$$

$$m = \frac{-26 \pm 28}{6}$$

$$m = \frac{1}{3} \quad \text{or} \quad -9$$

$$3^x = \frac{1}{3} = 3^{-1}$$

Comparing powers $x = -1$

Or $3^x = -9$

$$x \log_{10} 3 = \log_{10} -9$$

Since there is no log of a negative number

Hence solution is that $x = -1$

If $\log_a (2 + a) = 2$. Find a

From $\log_a (2 + a) = 2$

$$2 + a = a^2$$

$$a^2 - a - 2 = 0$$

$$a = \frac{1 \pm \pm\sqrt{-1^2 - 4 \times 1 \times 2}}{2 \times 1}$$

$$= \frac{1 \pm \sqrt{9}}{2} = \frac{1 \pm 3}{2}$$

$$a = \frac{4}{2} \quad \text{or} \quad \frac{-2}{2}$$

$$a = 2 \quad \text{or} \quad -1$$

$a = 2$, as -1 is not a base

4. Solve the equations

(a) $\log_8 x + \log_8 \frac{1}{6} = \frac{2}{3}$

(b) $10^3 \log_{10} x = 8$

(c) $3^x - 3^{x^2-6} = 0$

***Solution:***

(a) $\log_8 x + \log_8 \frac{1}{6} = \frac{2}{3}$

$$\log_8 x + \log_8 1 - \log_8 6 = \frac{2}{3}$$

$$\log_8 \frac{x}{6} = \frac{2}{3}$$

$$\frac{x}{6} = (8)^{2/3} = (2^3)^{2/3}$$

$$\frac{x}{6} = (2)^{3 \times 2/3} = 2^2$$

$$x = 2^2 \times 6 = 2^6$$

$$x = 24$$

(b) $10^3 \log_{10} x = 8$

$$\log_{10} x = \frac{8}{10^3} = 0.008$$

$$\Rightarrow x = 10^{0.008} = 1.0186$$

(c) $3^x - 3^{x^2-6} = 0$

$$3^x - 3^{x^2} \times 3^{-6} = 0$$

$$3^x - \frac{3^{x^2}}{3^6} = 0$$

$$3^6(3^x) - 3^{x^2} = 0$$

$$3^6(3^x) - (3^x)^2 = 0$$

Let $3^x = t$

$$3^6(t) - t^2 = 0$$

$$t^2 = 3^6(t)$$

$$t = 3^6 \ OR \ t = 0$$

But $t = 3^x$

$$3^x = 3^6 \ Or \ 3^x \neq 0$$

Comparing powers

$$x = 6$$

Or $\log_3 3^x = \log_3 3^6$

$$x \log_3 3 = 6 \log_3 3$$

$$x = 6$$

Or $\log_{10} 3^x = \log_{10} 3^6$

$$x \log_{10} 3 = 6 \log_{10} 3$$

$$x = \frac{6 \cancel{\log_{10}} 3}{\cancel{\log_{10}} 3}$$

$$x = 6$$

5. Use logarithm to evaluate

(i) $\log_3 18$ and (ii) $(1.75)^{0.67}$

***Solution:***

(i) Let $\log_3 18 = n$

$$18 = 3^n$$

$$\log_{10} 18 = n \log_{10} 3$$

$$n = \frac{\log_{10} 18}{\log_{10} 3}$$

$$= \frac{1.2553}{0.4771}$$

$$= 2.6310$$

$$\Rightarrow \log_3 18 = 2.6310$$

(ii) Let $(1.75)^{0.67} = t$

$$0.67 \log_{10} 1.75 = \log_{10} t$$

$$0.67 \times (0.2430) = \log_{10} t$$

$t = 10^{0.1628}$ or anti - $\log(0.1628)$ t = 1.4548

$$\Rightarrow (1.75)^{0.67} = 1.4548$$

6. Solve the equations

(i) $(0.3)^{x+1} = (0.7)^x$

(ii) $(0.1)^x = (0.2)^5$

(iii) $2^{3x+1} = 3^{x+2}$

(iv) $e^{lnx} + Ine^x = 8$

***Solution:***

(i) $(0.3)^{x+1} = (0.7)^x$

$$(x+1)\log_{10} 0.3 = x \log_{10} 0.7$$

$$(x+1)\log_{10} \frac{3}{10} = x \log_{10} \frac{7}{10}$$

$$(x+1)[\log_{10} 3 - \log_{10} 10] = x[\log_{10} 7 - \log_{10} 10]$$

$$(x+1)[0.4771 - 1] = x[0.8451 - 1]$$

$$(x+1)(-0.5225) = -0.1545x$$

$$-0.5225x - 0.5225 = -0.1545x$$

$$-0.5225 = -0.1545x + 0.5225x$$

$$-0.5225 = 0.3684x$$

$$x = \frac{-0.5225}{0.3684} = -1.4184$$

(ii) $(0.1)^x = (0.2)^5$

$$\left(\frac{1}{10}\right)^x = \left(\frac{2}{10}\right)^5$$

$$x\log_{10}\frac{1}{10} = 5\log_{10}\frac{2}{10}$$

$$x(\log_{10}1 - \log_{10}10) = 5(\log_{10}2 - \log_{10}10)$$

$$x(0-1) = 5(0.3010-1)$$

$$-x = -3.4950$$

$$x = 3.495$$

(iii) $2^{3x+1} = 3^{x+2}$

$$(3x+1)\log_{10}2 = (x+2)\log_{10}3$$

$$(3x+1) = (x+2)\frac{\log_{10}3}{\log_{10}2} = (x+2)\left(\frac{0.4771}{0.3010}\right)$$

$$(3x+1) = (x+2)(1.5850)$$

$$3x+1 = 1.585x + 3.1701$$

$$3x - 1.585x = 3.1701 - 1$$

$$1.415x = 2.1701$$

$$x = \frac{2.1701}{1.415}$$

$$x = 1.5336$$

(iv) $e^{\ln x} + \ln e^x = 8$

But $e^{\ln x} = x$ and

$$\ln e^x = x$$

$$x + x = 8$$

$$2x = 8$$

$$x = 4$$

7(a) If $\log_7 2 = 0.356$, $\log_7 3 = 0.566$

Find the value of

$$2\log_7\left(\frac{7}{15}\right) + \log_7\left(\frac{25}{12}\right) - 2\log_7\left(\frac{7}{3}\right)$$

**Solution**

$$\log_7\left(\frac{7}{15}\right)^2 + \log_7\left(\frac{25}{12}\right) - \log_7\left(\frac{7}{3}\right)^2$$

$$= \log_7\left(\left(\frac{7}{15}\right)^2 \times \left(\frac{25}{12}\right) \Big/ \left(\frac{7}{3}\right)^2\right)$$

$$= \log_7\left(\frac{49}{225} \times \frac{25}{12} \times \frac{9}{49}\right)$$

$$= \log_7\left(\frac{1}{9} \times \frac{1}{12} \times \frac{9}{1}\right)$$

$$= \log_7\frac{1}{12} = \log_7 1 - \log_7 12$$

$$= 0 - \log_7(3 \times 2^2)$$

$$= -\log_7 3 - 2\log_7 2$$

$$= -0.566 - 2(0.356)$$

$$= -1.278$$

There for

$$2\log_7\left(\frac{7}{15}\right) + \log_7\left(\frac{25}{12}\right) - 2\log_7\left(\frac{7}{3}\right) = -1.278$$

7(b) Given $5^x \cdot 25^{2y} = 1$ and $3^{5x} \cdot 9^y = \frac{1}{9}$

Calculate the values of $x$ and $y$

$$5^x \cdot (5^2)^{2y} = 1$$

$$5^x \cdot 5^{4y} = 1$$

$$5^{(x+4y)} = 1$$

$$\log_5 5^{(x+4y)} = \log_5 1$$

$$(x+4y)\log_5 5 = \log_5 1$$

$$(x+4y)\cdot 1 = 0$$

$$x + 4y = 0 \text{ .....(i)}$$

$$3^{5x} \cdot 9^y = \frac{1}{9} = 9^{-1}$$

$$3^{5x} \cdot (3^2)^y = (3^2)^{-1}$$

$$3^{5x} \cdot (3^2)^y = (3^2)^{-1}$$

$$3^{5x} \cdot 3^{2y} = (3)^{-2}$$

$$3^{5x+2y} = (3)^{-2}$$

$$\log_3 3^{(5x+2y)} = \log_3 3^{-2}$$

$$(5x+2y)\log_3 3 = -2\log_3 3$$

$$5x + 2y = -2 \text{ .....(ii)}$$

2(ii) – (i)

$$10x + 4y = -4$$
$$x + 4y = 0 -$$
$$9x \quad = -4$$
$$x = \frac{-4}{9}$$

5(i) – (ii)

$$5x + 20y = 0$$
$$5x + 2y = -2 -$$
$$18y = 2$$
$$y = \frac{2}{18}$$
$$y = \frac{1}{9}$$

Therefore $x = \frac{-4}{9}$ and $y = \frac{1}{9}$.

8. Given that $\log_2 x + 2\log_4 y = 4$,

Show that $xy = 16$. Solve simultaneous equations.

$\log_{10}(x+y) = 1$ And $\log_2 x + 2\log_4 y = 4$

***Solution:***

$$\log_2 x + 2\log_4 y = 4$$
$$\log_2 x + \log_4 y^2 = 4$$

Let $\log_4 y^2 = t$

$$y^2 = 4^t = 2^{2t}$$
$$\log_2 y^2 = \log_2 2^{2t}$$
$$2\log_2 y = 2t\log_2 2$$
$$t = \log_2 y$$
$$\log_2 x + \log_4 y^2 = 4$$

Becomes $\log_2 x + \log_2 y = 4$

$$\log_2 xy = 4$$
$$\Rightarrow xy = 2^4 = 16$$

From $\log_{10}(x+y) = 1$

$$x + y = 10^1$$
$$x + y = 10 \ldots\ldots\ldots\ldots\text{(i)}$$
$$xy = 16 \ldots\ldots\ldots\ldots\text{(ii)}$$

From (i)

$$x = (10 - y)$$

Substituting for $x$ in equation (ii)

$$(10 - y)y = 16$$
$$10y - y^2 = 16$$
$$y^2 - 10y + 16 = 0$$
$$y^2 - 10y + 16 = 0$$
$$y = \frac{10 \pm \sqrt{-10^2 - 4\times1\times16}}{2\times1}$$
$$= \frac{10 \pm \sqrt{100-64}}{2}$$
$$= \frac{10 \pm \sqrt{36}}{2}$$
$$= \frac{10 \pm 6}{2}$$
$$= \frac{16}{2} \text{ or } \frac{4}{2}$$
$$= 8 \text{ or } 2$$

When $y = 8$, $x = 2$ from $x = 10 - y$

$y = 2$, $x = 8$

9. Solve the simultaneous equations

$\log_b a + 2\log_a b = 3$ and $\log_9 a + \log_9 b = 3$

***Solution:***

From $\log_b a + 2\log_a b = 3$

$$\log_b a + \log_a b^2 = 3$$

**********

**Side work;**

Let $\log_a b^2 = t$

$$b^2 = a^t$$
$$\log_b b^2 = t\log_b a$$
$$2\log_b b = t\log_b a$$
$$2 = t\log_b a$$
$$t = \frac{2}{\log_b a}$$

**********

$$\Rightarrow \log_b a + \frac{2}{\log_b a} = 3$$

$$(\log_b a)^2 + 2 = 3\log_b a$$

$$(\log_b a)^2 - 3(\log_b a) + 2 = 0$$

Let $\log_b a = m$

$$m^2 - 3m + 2 = 0$$

$$(m-1)(m-2) = 0$$

$m = 1$ or $2$

$\log_b a = 1$ or $\log_b a = 2$

$a = b^1$ or $a = b^2$

$a = b$ or $a = b^2$ .....(i)

From $\log_9 a + \log_9 b = 3$

$\log_9 ab = 3$

$ab = 9^3 = 729$ .....(ii)

When a = b then $b^2 = 9^3$

$b = \pm\sqrt{9^3} = \pm 27$

Therefore when a = 27 then b = 27 and when a = -27, b = -27

When a = $b^2$ then substituting for a in equation (ii)

$b^3 = 9^3$, b= 9

When b = 9 then a = 81

10. (a) Given that $\log_4 x + 2\log_4 y = 2$

Show that $xy = 16$.

Hence solve for $x$ and $y$ in simultaneous equation

$\log_{10}(x+y) = 1$ And $\log_2 x + 2\log_4 y = 4$

(b) Solve the equation $(4.381)^{2x-5} = (8.032)^x$

***Solution:***

(a) $\log_4 x + 2\log_{16} y = 2$

**********

**Side work;**

Let $\log_{16} y = t$

$$y = 16^t$$

$$\log_4 y = t\log_4 4^2$$

$$\log_2 2^y = 2t\log_4 4$$

$$t = \frac{1}{2}\log_4 y$$

**********

$$\log_4 x + 2\cdot\frac{1}{2}\log_4 y = 2$$

$$\log_4 x + \log_4 y = 2$$

$$\log_4 xy = 2$$

$xy = 4^2 = 16$ As required

If $\log_{10}(x+y) = 1$ and $xy = 16$ .....*

$x + y = 10^1 = x + y = 10$ .....**

From ** $y = 10 - x$ and substituting for y in equation * we have

$$x(10-x) = 16$$

$$x^2 - 10x + 16 = 0$$

$$x = \frac{10 \pm \sqrt{10^2 - 4\times 16\times 1}}{2\times 1}$$

$$= \frac{10 \pm \sqrt{36}}{2}$$

$$= \frac{10 \pm 6}{2}$$

$x = 8$ or $2$

When $x = 2$ and $y = 2$ and when $x = 8$ and $y = 2$

(b) To solve equation $(4.381)^{2x-5} = (8.032)^x$

$$\log_{10}(4.381)^{2x-5} = \log_{10}(8.032)^x$$

$$(2x-5)\log_{10} 4.381 = x\log_{10} 8.032$$

$$(2x-5)0.641573 = x(0.904824)$$

1.28314x-3.207865=0.904824x

$$0.378322x = 3.207865$$

$$x = 8.4779192$$

$$x \approx 8.5$$

11. (a) Evaluate $\dfrac{x^{3/2}+xy}{xy-y^3}-\dfrac{\sqrt{x}}{\sqrt{x}-y}$

$$=\frac{x\sqrt{x}+xy}{xy-y^3}-\frac{\sqrt{x}}{\sqrt{x}-y}$$

$$=\frac{x\sqrt{x}+xy}{y(x-y^2)}-\frac{\sqrt{x}}{(\sqrt{x}-y)}$$

$$=\frac{(x\sqrt{x}+xy)(\sqrt{x}-y)-\sqrt{x}(xy-y^3)}{(y-y^3)(\sqrt{x}-y)}$$

$$=\frac{x^2-xy\sqrt{x}+xy\sqrt{x}-xy^2-xy\sqrt{x}+y^3\sqrt{x}}{xy\sqrt{x}-xy^2-y^3\sqrt{x}+y^4}$$

$$=\frac{\sqrt{x}(x\sqrt{x}-xy-\sqrt{x}y^2+y^3)}{y(x\sqrt{x}-xy-\sqrt{x}y^2+y^3)}$$

$$=\frac{\sqrt{x}}{y}$$

(b) To prove that

$$2\log_a(x+y)=2\log_a x+\log_a\left(1+\frac{2y}{x}+\frac{y^2}{x^2}\right)$$

Let $2\log_a x+\log_a\left(1+\dfrac{2y}{x}+\dfrac{y^2}{x^2}\right)=t$

$$t=\log_a x^2+\log_a\left(\frac{x^2+2xy+y^2}{x^2}\right)$$

$$t=\log_a x^2\cdot\left(\frac{x^2+2xy+y^2}{x^2}\right)$$

From rules of logarithms

$$t=\log_a(x^2+2xy+y^2)=\log_a(x+y)^2$$

$$t=2\log(x+y)$$

Hence

$$2\log_a(x+y)=2\log_a x+\log_a\left(1+\frac{2y}{x}+\frac{y^2}{x^2}\right)$$

(c) If $a=\log_b c,\ b=\log_c a$ and $c=\log_a b$

Prove that $abc=1$

**Solution**

Convert them to common base say base $b$ Let

$\log_c a=t \qquad a=c^t \qquad \log_b a=t\log_b c$

$$t=\frac{\log_b a}{\log_b c}$$

Let $\log_a b=n \qquad b=a^n$

$\log_b b=n\log_b a \qquad n=\dfrac{\log_b b}{\log_b a}$

Then , from $abc=(\log_a c)(\log_b a)(\log_c b)$

Substituting back into right side of equation we have

$$abc=\log_b c\times\frac{\log_b a}{\log_b c}\times\frac{\log_b b}{\log_b a}$$

Hence $abc=1$ as required

12. (a) Given that

$\log_x\left(1+\dfrac{1}{8}\right)=a,\ \log_x\left(1+\dfrac{1}{15}\right)=b$ and

$\log_x\left(1+\dfrac{1}{24}\right)=c$, Show that

$$\log_x\left(1+\frac{1}{80}\right)=a-b-c$$

(b) (i) Express as a single logarithm

$$2\log_x\left(\frac{1}{2}\right)+3\log_x 4-\log_x\left(\frac{4}{5}\right)$$

(ii) Express as a single logarithm

$$\frac{5}{2}\log_n x+\log_n(x+1)-\log_n x$$

(c) Suppose $4\log_{64}N=p$ and $2\log_4 16N=q$ and $q-p=8$. Find $N$.

***Solution:***

(a) From

$$a-b-c=\log\left(1+\frac{1}{8}\right)-\log\left(1+\frac{1}{15}\right)-\log\left(1+\frac{1}{24}\right)$$

$$=\log_x\frac{99}{88}-\log_x\frac{16}{15}-\log_x\frac{25}{24}$$

Using the rules of logarithms

$$a-b-c=\log_x\frac{9}{8}\times\frac{15}{16}\times\frac{24}{25}$$

$$= \log_x \frac{81}{80} = \log_x\left(1+\frac{1}{80}\right)$$

(b) (i) $2\log_x\left(\frac{1}{2}\right) + 3\log_x 4 - \log_x\left(\frac{4}{5}\right)$

$$\log_x\left(\frac{1}{2}\right)^2 + \log_x 4^3 - \log_x\left(\frac{4}{5}\right)$$

Using the rules of logarithm

$$\log_x \frac{\left(\frac{1}{2}\right)^2 \times \left(4^3\right)}{\left(\frac{4}{5}\right)} = \log_x \frac{1}{4} \times 4^3 \times \frac{5}{4} = \log_x 20$$

Therefore

$$\log_x\left(\frac{1}{2}\right) + 3\log_x 4 - \log_x\left(\frac{4}{5}\right) = \log_x 20$$

(ii) $\frac{5}{2}\log_n x + \log_n(x+1) - \log_n x$

$$= \log_n x^{5/2} + \log_n(x+1) - \log_n x$$

$$= \log_n \frac{x^{5/2}(x+1)}{x} = \log_n x^{5/2}(x+1)x^{-1/2}$$

$$= \log_n x^2(x+1)$$

(c) Let $4\log_{64} N$ be $t$

$\log_{64} N^4 = t \qquad N^4 = 64^t$

$\log_4 N^4 = \log_4 4^{3t}$

$\log_4 N^4 = 3t$

$t = \frac{1}{3}\log_4 N^4 = \log_4 N^{4/3}$

$\log_4(16N)^2 - \log_4 N^{4/3} = 8$

$$\log_4\left(\frac{16^2 N^2}{N^{4/3}}\right) = 8$$

$$\frac{16N^2}{N^{4/3}} = 4^8$$

$$\frac{N^2}{N^{4/3}} = 4^4$$

$$N^{2-4/3} = 4^4$$

$$N^{2/3} = 4^4$$

$$N = \left(4^4\right)^{3/2} = 4^6 = 4096$$

13. (a) Given that

$\log_y w = a$ and $\log_x w = b$ where $w \neq 1$

Prove that $\dfrac{\log_n x - \log_n y}{\log_n x + \log_n y} = \dfrac{a-b}{a+b}$

Verify your answer without using a calculator or tables when $y = 4, n = 2, x = 8$ and $w = 4096$

(a) Show that $\log_a x = \dfrac{1}{\log_c a} \times \log_c x$

(b) Prove that $\log_x y = \dfrac{1}{\log_y x}$

***Solution:***

(a)

$\log_y w = a \qquad \log_x w = b$

$w = y^a \qquad w = x^b$

$\log_n w = a\log_n y \qquad \log_n w = b\log_n x$

$a = \dfrac{\log_n w}{\log_n y} \qquad \dfrac{\log_n w}{\log_n x} = b$

$$\frac{a-b}{a+b} = \frac{\frac{\log_n w}{\log_n y} - \frac{\log_n w}{\log_n x}}{\frac{\log_n w}{\log_n y} + \frac{\log_n w}{\log_n x}}$$

$$= \log w\left(\frac{1}{\log y} - \frac{1}{\log x}\right) \Big/ \log w\left(\frac{+1}{\log y} + \frac{1}{\log x}\right)$$

$$= \frac{\frac{\log_n x - \log_n y}{\log_n y \log_n x}}{\frac{\log_n x + \log_n y}{\log_n y \log_n x}}$$

$$= \frac{\log_n x - \log_n y}{\log_n x + \log_n y}$$

Given $y = 4$, $n = 2$, $x = 8$, and $w = 4096$

$\log_4 4096 = 6^{-a}, \log_8 4096 = 4^{-b}$

$$\frac{6-4}{6+4} = \frac{\log_2 8 - \log_2 4}{\log_2 8 + \log_2 4}$$

$$\frac{2}{10} = \frac{3-2}{3+2}$$

$$\frac{1}{5} = \frac{1}{5}$$

Hence it is verified

(b) Prove that $\log_x y = \dfrac{1}{\log_y x}$

Let $\log_x y = a$

$$y = x^a$$

$$\log_y y = a \log_y x$$

$$1 = a \log_y x$$

$$a = \frac{1}{\log_y x}$$

Hence $\log_x y = \dfrac{1}{\log_y x}$

(c) To find value of $(512)^{-2/9}$

Let $x = (512)^{-2/9}$

Take common log on both sides

$$\log_{10} x = \frac{-2}{9} \log_{10} 512$$

$$\log_{10} x = -0.602060$$

$x = 10^{-0.60206} = \dfrac{1}{4}$ hence $(512)^{-2/9} = \dfrac{1}{4}$

14.(a) Solve the equation

$$x^{2/3} - 5x^{1/3} - 6 = 0$$

(b) Express $\log \dfrac{100a^2}{b^3c}$ in terms of

$\log a, \log b, \log c$

(c) Show that $\log_a \dfrac{1}{m^n} = -n \log_a m$

***Solution:***

(a) Let $x^{1/3}$ be $y$

$$x^{2/3} - 5x^{1/3} - 6 = 0$$

$$y^2 - 5y - 6 = 0$$

$$(y-6)(y+1) = 0$$

$$y = 6 \text{ or } -1$$

Since $x^{1/3} = y$

$x^{1/3} = 6$ or $x^{1/3} \neq -1$

$$\left(x^{1/3}\right)^3 = 6^3$$

$$x = 6^3$$

$$= 216$$

Or $\dfrac{1}{3} \log_{10} x = \log_{10} 6$

$$\log_{10} x = 3 \log_{10} 6$$

$$\log_{10} x = \log_{10} 6^3$$

$$x = 6^3 = 216$$

(b) $\log \dfrac{100a^2}{b^3c^{1/2}}$ since no base is given we assume is common logarithm

$$\log \frac{100a^2}{b^3c^{1/2}} = \log_{10}^{100} + \log_{10}^{a^2} - \log_{10} b^3 - \log_{10} c^{1/2}$$

$$= 2 + 2\log_{10} a - 3\log_{10} b - \frac{1}{2}\log_{10} c$$

(c) To show that $\log_a \dfrac{1}{m^n} = \log_a m^{-n}$

Let $\log_a m = x$

$$m = a^x$$

$$m^{-n} = a^{-nx}$$

$$\log_a m^{-n} = -nx \log_a^a$$

$$\log_a^{\frac{1}{m^n}} = -nx$$

$$= -n \log_a^m$$

$$= \log_a^{m^{-n}}$$

Hence $\log_a \dfrac{1}{m^n} = \log_a m^{-n} = -n \log_a m$

15.a) If $\log_2 x + \log_4 x + \log_{16} x = \dfrac{21}{16}$, Find $x$

(b) Show that $\log_6 x = \dfrac{\log_3 x}{1 + \log_3 2}$. Hence that

$\log_3 2 = 0.631$

Find without using tables or calculator $\log_6 4$ correct to3 significant figures.

(c) Find $x$ if $\log_x 8 - \log_{x^2} 16 = 1$

(d) Simplify;

$$\frac{\log_3\left(\frac{1}{3}\right)^{-3} - \log_3 243 + \log_3\left(27\frac{1}{2}\right)^{8/3} + \log_3\left(a^3+1\right)}{\log_3\left(a^2+2\right)}$$

(a) Let $\log_4 x = t$

$\Rightarrow x = 4^t$

$\log_2 x = \log_2 4^t$

$\log_2 x = \log_2 2^{2t} = 2t\log_2 2 = 2t;$

$2t = \log_2 x$

$t = \frac{1}{2}\log_2 x$

Hence $\log_4 x = \frac{1}{2}\log_2 x$

Let $\log_{16} x = n;\ x = 16^n, x = (2^4)^n = 2^{4n}$

$\log_2 x = \log_2 2^{2n} = 4n\log_2 2$

$= 4n$

$\frac{1}{4}\log_2 x = n$

Therefore since all are been converted to base 2

$\log_2 x + \log_4 x + \log_4 4 = \frac{21}{16}$ Becomes

$\log_2 x + \frac{1}{2}\log_2 x + \frac{1}{4}\log_2 x = \frac{21}{16}$

$\frac{1}{4}(4+2+1)\log_2 x = \frac{21}{16}$

$\frac{7}{4}\log_2 x = \frac{21}{16}$

$\log_2 x = \frac{21}{16}\times\frac{4}{7}$

$\log_2 x = \frac{3}{4}$

$x = 2^{0.75} = 1.6818$

(b) To show that $\log_6 x = \frac{\log_3 x}{1+\log_3 2}$

Let $\log_6 x = t$

$\Rightarrow x = 6^t = (2\times 3)^t$

$\log_3 x = t(\log_3 2 + \log_3 3)$

$\log_3 x = t(\log_3 2 + 1)$

$t = \frac{\log_3 x}{1+\log_3 2}$

Since $t = \log_6 x$

Hence $\log_6 x = \frac{\log_3 x}{1+\log_3 2}$

$\log_6 4 = \frac{\log_3 4}{1+\log_3 2} = \frac{2\log_3 2}{1+\log_3 2}$

$= \frac{2(0.631)}{1+0.631} = 0.774$ (3 sign figures)

(c) From $\log_x 8 - \log_{x^2} 16 = 1$

Let $\log_{x^2} 16 = t$

$16 = (x^2)^t = x^{2t}$

$\log_x 16 = 2t\log_x x = 2t$

$t = \frac{1}{2}\log_x 16$

$\log_x 8 - \frac{1}{2}\log_x 16 = 1$

$\log_x 8 - \log_x 16^{1/2} = 1$

$\log_x 8 - \log_x 4 = 1$

$\log_x\left(\frac{8}{4}\right) = 1$

$\log_x 2 = 1$

$2 = x^1 = x$

$x = 2$

(d) $\dfrac{\log_3\left(\frac{1}{3}\right)^{-3} - \log_3 243 + \log_3\left(27\frac{1}{2}\right)^{8/3} + \log_3 a^3 + 1}{\log_3(a^2)+2}$

$= \dfrac{\log_3(3^{-1})^{-3} - \log_3 3^5 + \log_3\left(3^{3/2}\right)^{8/3} + 3\log_3 a + 1}{2\log_3 a + 2}$

$= \dfrac{\log_3 3^3 - 5\log_3 3 + \log_3 3^{3/2\cdot 8/3} + 3\log_3 3 + 1}{2(\log_3 a + 1)}$

$= \dfrac{3\log_3 3 - 5\log_3 3 + 4\log_3 3 + 3\log_3 a + 1}{2(\log_3 a + 1)}$

$= \dfrac{3 - 5 + 4 + 3\log_3 a + 1}{2(\log_3 a + 1)}$

$= \dfrac{3 + 3\log_3 a}{2(\log_3 a + 1)} = \dfrac{3(\log_3 a + 1)}{2(\log_3 a + 1)} = \dfrac{3}{2}$

## Exercise 1 (c)

1. Solve the simultaneous equations
$\log_x y = \frac{1}{2}$ and $1 + \log_5 x = \log_y 5$

2. solve for x in the equation
$\log_4 6 - x = \log_2 x$

3. prove that $\log_b a = \frac{\log_c a}{\log_c b}$, hence solve for x in equations
(i) $\log_x 5 - 3\log_5 x = 2$.
(ii) $\log_{10} x + \log_x 100 = 3$

4. solve the simultaneous equations:
$\log_{25}(x+y) = \log_5(x-y)$ and $x^2 - y^2 = 8$r

5. Given that $y = \log_a x^3 and z = \log_x a$. Show that zy=3, hence find the values of y and z if $\log a(3\log_a x) - \log a(\log_x a) = \log_a 27$

6. solve the following simultaneous equations:
$\log_2 x + \frac{1}{3}\log_2 y^3 = 2\log_4 6$ *and*
$x\log(128) = y\log_8 2$

7. solve the equation $\log_{25} 4x^2 = \log_5(3 - x^2)$

8. simplify $\log_7 98 - \log_7 30 + \log_7 75$

9. prove that $\log_c ab = \log_c a + \log_c b$. Hence solve the equation
$\log_4 (x-3) + \log_4 (x+3) = 2$

10. $\log_5 x = p$ *and* $\log_{30} x = q$. Show that
$\log_6 5 = \frac{q}{p-q}$. Hence solve:
*i)* $\log_2 x - \log_x 8 = 2$
*ii)* $3\log_2 p - 6\log_p 2 + 7 = 0$
*iii)* $\log_x 4 + \log_4 x^2 = 3$

11. if $a^2 + b^2 = 23ab$, show that
$\log_{10} a + \log_{10} b = 2\log_{10}\left(\frac{a+b}{5}\right)$

12. Prove that $\log_c ab = \log_c b + \log_c a$. Hence solve the equation
$\log_4(x-3) = 3 - \log_4(x+3)$

13. solve
*i)* $\log_{10}\left(e^{\ln(x^2+1)}\right) - \log_{10} e^{\ln x} = \log_{10} 5$
*ii)* $\log_5 x\left(\log_5\left(\frac{2}{x}\right)\right) + 1 = 0$

14. given that $3 + 2\log_2 x = \log_2 y$, show tha $y = 8x^2$, hence, find the root of the equation $3 + 2\log_2 x = \log_2(14x - 3)$

15. solve the simultaneous equations
$\log_9(xy^2) = \frac{1}{2}$ and $(\log_3 x)(\log_3 y) = 3$

16. solve for x if $\log_4(\log_2 x) = \log_2(\log_4 x)$

17. solve for x in the equation $\log_{14} x = \log_7$

18. prove that if $x = \log_a bc$, $y = \log_b ac$ and $z = \log_c ab$ then $x + y + z + 2 = xyz$

19. if $2^x 3^y = 3^x 4^y = 6$, show by taking logarithm to base two and eliminating $\log_2 3$ that $x^2 - 2y^2 = 2x - 3y$

20. solve the simultaneous equations
$\log(x+y) = 1$ and $2\log y - \log(30 - x) = 0$

21. simplify $1 + \log_{10}\left(\frac{4}{x^2}\right)^{\frac{-1}{2}} - 2\log_{10} x$

22. solve the following equations
*i)* $5^{2x+1} + 4 = 21(5^x)$
*ii)* $e^{2x} - 4e^x + 3 = 0$
*iii)* $3^{2(x+1)} - 7(2^3 3^x) + 3 = 0$
*iv)* $3^{2(x+1)} - 10(3^x) + 1 = 0$
*v)* $2(3^{2x}) - 5(3^x) + 2 = 0$
*vi)* $2^{2x+1} + \frac{3}{2} = 2^{x+1}$
*vii)* $9^x - 3^{(x+1)} = 10$
*viii)* $3^{2(x+1)} - 28(3^x) + 3 = 0$
*ix)* $e^{2x} - 3e^x - 4 = 0$
*x)* $x^{\frac{2}{3}} - 5x^{\frac{1}{3}} + 6 = 0$

23. if $y = a + bx^n$ is satisfied by the values

| x | 1 | 2 | 4 |
|---|---|---|---|
| y | 7 | 10 | 15 |

show that $n = \log_2\left(\frac{5}{3}\right)$ and deduce the values a and b

24. solve the following simultaneous equations
*i)* $2^x + 4^y = 12$ and $3(2^x) - 2(2^{2y}) = 16$



*ii)* $2^x + 3^y = 5$ a
*iii)* $2^x + 3^y = 44$
iv) $\log_4 x^2 - 6\log$

**Miscellaneous Ex**

1. (a) Rationalizing
(i) $\frac{5}{5-\sqrt{3}} + \frac{5}{5+\sqrt{}}$
(ii) $(1-\sqrt{x}) + \frac{\sqrt{}}{(1-}$
(b) Express the foll denominators
(i) $\frac{\sqrt{6}-\sqrt{3}}{\sqrt{6}-\sqrt{3}+\sqrt{2}}$
(ii) $\frac{1+\sqrt{2}}{\sqrt{(1-x^2)}+\sqrt{(}}$

2. (a) Evaluate (i)
(ii) $2\log_6 3 - 2\log_7$
(b) Solve (i) $4^x -$
(ii) $3^{3x} - 6(3^{2x}) +$

3. Show the result
Using the
$\log_a b$

4. Express
(i) $2\log a + \frac{1}{2}\log$
(ii) $7\log_3 6 + 8\log$

5.(a) Simplify the
(i) $5 \times 4^{3n+1} - 20 \times$
(ii) $\frac{9^{2n+2} \times 6^{2n-3}}{3^{5n} \times (6 \times 2)^n}$

6. (a) If $a = \log_x$
prove that $a + b +$
(b) Show that log
evaluate $\log_5 64 \times$

7. Solve simultane
(a) $5^{x+2} + 7^{y+1} =$
(b) $6\log_3 x + 6\log$

$2^x + 3^y = 5$ and $2^{(x+3)} - 3^{(y+2)} = 23$

) $2^x + 3^y = 44$ and $2^{x+2} + 3^{y+2} = 221$

$\log_4 x^2 - 6\log_x 4 = 1$

**Miscellaneous Exercise 1**

a) Rationalizing each fraction as a separate item

$$\frac{5}{5-\sqrt{3}} + \frac{5}{5+\sqrt{3}}$$

$$\left(1-\sqrt{x}\right) + \frac{\sqrt{x}}{\left(1-\sqrt{x}\right)}$$

Express the following surd number with rational nominators

$$\frac{\sqrt{6}-\sqrt{3}}{\sqrt{6}-\sqrt{3}+\sqrt{2}}$$

$$\frac{1+\sqrt{2}}{\sqrt{\left(1-x^2\right)}+\sqrt{\left(x^2+1\right)}}$$

(a) Evaluate (i) $\dfrac{(\log_3 x)}{(\log_8 x)}$

$2\log_6 3 - 2\log_7 15 + \log_7 0.14 - 10\log_{98} 7$

Solve (i) $4^x - 7\left(2^x\right) + 12 = 0$

$3^{3x} - 6\left(3^{2x}\right) + 11\left(3^x\right) - 6 = 0$

Show the result $\log_a b = \dfrac{1}{\log_b a}$

Using the result to prove

$$\log_a b \times \log_b c = \log_a c$$

Express

$2\log a + \frac{1}{2}\log_{10}(b-3) + \log_2 5^3 + 4$

$7\log_3 6 + 8\log_2 16 + 9\log_{10} 18$

a) Simplify the expression

$5\times 4^{3n+1} - 20\times 8^{2n}$

$\dfrac{9^{2n+2}\times 6^{2n-3}}{3^{5n}\times(6\times 2)^{n-2}}$

(a) If $a = \log_x pq, b = \log_p rq, c = \log_r pq$

ove that $a+b+c = abc-2$

) Show that $\log_a b \times \log_b a = 1$ and hence

aluate $\log_5 64 \times 2\log_4 24$

Solve simultaneous equations

) $5^{x+2} + 7^{y+1} = 3468$ and $7^y = 5^x - 76$

) $6\log_3 x + 6\log_{27} y = 7$ and

$4\log_9 x + 4\log_3 y = 9$.

8.(a) $\log_9 x + \log_9 x^2 + \log_9 x^3 + \log_9 x^4 = 5$

(b) $\log_4 x - \log_4 7 = \dfrac{3}{2}$.

9. (a) Solve the following equations

(i) $4^{2x} + 4^{x+2} - 80 = 0$

(ii) $3^{2y+3} - 3\times 3^y - 4 = 0$

(b) Express

(i) $\frac{5}{2}\log_a x + \log_a(x+1) - \log_a\sqrt{x}$

(ii) $7\log_a 2 - 3\log_a 12 + 5\log_a 3$ as a single logarithm.

(c) Prove that $\log_2 10 = \dfrac{1}{\log_{10} 2}$ and evaluate $\log_e 100$ correct to 3 significant figures taking $e$ as 2.718.

10.(a) Prove that $\log_e a = \dfrac{\log_{10} a}{\log_{10} e}$ ,hence show that $\log_{2.718} 2 = \ln 2$

(b) (i) Prove that $\log\left(\dfrac{a}{b}\right) = \log_c a - \log_c b$

(ii) Find $x$ if $\log_x 8 - \log_x 2^2 = 1$

11.(a) Solve the simultaneous equations

$2^x + 4^y = 12$ and

$3(2)^x - 2(2)^{2y} = 16$

Hence show that $(4)^x + 4(3)^{2y} = 100$.

(b) If $\log_3 x + \log_9 x + \log_{81} x = \dfrac{16}{28}$

find $x$

(c) Simplify $\dfrac{1+\sqrt{3}+\sqrt{5}}{\sqrt{2}+\sqrt{5}}$

12. ( a) (i) Find $\log_8 64\sqrt{4}$ without using calculator or tables.

(ii) Simplify $\left(\log_p q^2\right)\times\left(\log_q p^3\right)$

(b) Find the values of t and m such that

(i) $\log_9 t + \log_9 m = 1$ and

$\log_9 t - \log_9 m = \log_9(1.5)$

(ii) $\log_8(t-2) + \log_8(t+2) = 3$

13.(a) Given that $\dfrac{\log_{10} 7}{\log_{10} 3} = \dfrac{9}{5}$, solve for $x$ and $y$

The simultaneous equations $7^x = 3^{3y+1}$ and

$9^{x-1} = 63^{2y+1}$

(b)(i) Prove that $\log_x q + \log_r p = \log_r pq$

Hence, solve the $\log_e(y+3) + \log_e(3-y) = 2$.

(ii) $\log_5 x + \log_x 5 = 2.5$

14. (a)Rationalize the surd $\dfrac{1}{3\sqrt{3}-3} + \dfrac{1}{\sqrt{5}+3}$

(b) Simplify $1 + \log\dfrac{4}{8x^4} + 2\log_{10} x^2$

(i) Express as a single logarithm and simplify your answer

$$\log(x^2-1) + \frac{1}{2}\log\frac{x+1}{x-1}$$

(ii) Solve the equation

$$3^{2x+1} + 26(3^x) = 9$$

15.(a) Solve the equation $2^{x^2} = 4^{(2x-2)}$

(b) Solve the equation

$$(3-x)-(7+x) = (16+2x)$$

(c) Find the square root of $70 + 14\sqrt{25}$

16.(a)Simplify (i) $(d - d^{-1})(d^{4/3} + d^{-2/3})$

(ii) $\dfrac{3(2^{n+1}) - 4(2^{n-1})}{2^{n+1} - 2^n}$

(b) Using logarithm tables evaluate

(i) $(0.0713)\dfrac{1}{1.2}$ (ii) $\sqrt[5]{0.00237}$

(iii) $\log_5 0.24$

17.(a) Rationalize the denominators of

(i) $\dfrac{\sqrt{2}+2\sqrt{5}}{\sqrt{3}-\sqrt{2}}$ (ii) $\dfrac{\sqrt{11}+3\sqrt{7}}{\sqrt{7}+4\sqrt{11}}$

(b) Express the following in the form $x + y\sqrt{z}$

(i) $(\sqrt{3}+2)(1-2\sqrt{3})$ (ii) $\dfrac{1}{3} + \sqrt{\dfrac{1}{9} + \dfrac{1}{27}}$

(c) Simplify (i) $\sqrt{24} - 3\sqrt{6} - \sqrt{294} + \sqrt{216}$

(ii) $\sqrt{48} + 2\sqrt{12} + \sqrt{27}$

18.(a)Show that (i) $\log_c d \times \log_d a = \log_c a$

(ii) $\log_x(x^a) = a$ (iii) $\log_x\left(\dfrac{1}{x}\right) = -1$

(b) Express as single logarithm

(i) $3\log_a a + \dfrac{3}{2}\log_a a - 2\log_b a$

(ii) $2\log 3 + 3\log 2 - 3\log 6$

19.(a) Solve the pair of equations $27^{x+y} = 9$ and $2x^2 - y^2 = 64$.

(b) Find values of $\dfrac{12^{3/2} \times 16^{1/8}}{27^{1/6} \times 18^{1/2}}$

(c)Express the following in simplest form as possible

(i) $(a+\sqrt{b})(c-\sqrt{b})$ (ii)$(2+3)^3 + (2-3$

20.(a) Solve the simultaneous equation

$\sin(x+y) = \dfrac{1}{2}\sqrt{3}$ and $\cos(2x-y) = \dfrac{1}{2}$

Giving all pairs of values of $x$ and $y$ lyin between $0^0$ and $180^0$.

(b) Express in simplest surd form as possible th equation

$$\frac{\cos(30^0+60^0)}{\sin(60^0+45^0)} \times \tan 60^0$$

21. Solve $2\sqrt{(x-1)} - \sqrt{(x+4)} = 1$

22. Solve the following pair of simultaneous equations.

(i) $\log_2^x + 2\log_4^y = 4$ and $x + 12y = 52$

(ii) $5^{x+2} + 7^{y+1} = 3468$ and $7^y = 5^x - 76$

23. Solve the equations

(i) $5^{2(x+1)} - 28(5^x) + 3 = 0$

(ii) $3^{2(x+1)} - 10(3^x) + 1 = 0$

(iii) $4(7^{2x}) - 9(7^x) + 5 = 0$

(iv) $3^{2x+1} - 26(3^x) = 9$



The equation is
balancing or iden
t is formal stater
of mathematical

QUADRATIC E

Is equation whic
at most e.g. $x^2$
In general form
$ax^2 + bx + c =$
is coefficient c
cannot be zero
from zero.

Methods of sol
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Either $x +$
$x = -4$ or 2
When coeffic
equation is in
Factorise the e
We test by us
and 3, -2 and
of 5, hence we
$\Rightarrow x^2 + 5x +$
Either
$x + 2 = 0$ or
I.e. product
is in form $ax$
Solve $2x^2 -$
$\Rightarrow (2x+3)($
Either $2x +$
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The method
very difficul
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(i) Complet

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