17. Vectors

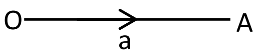
Vectors

A vector is a quantity with both magnitude and direction.

Examples include displacement, velocity, acceleration, force, momentum etc.

Representation of vectors

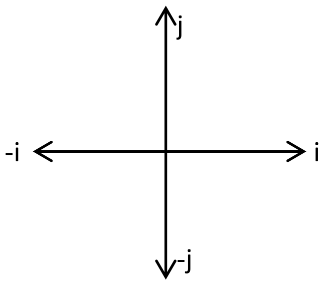
A vector is represented by a line segment or a small letter



The above vector can be represented as OA, a, $\overrightarrow{OA}$, $\underline{a}$ etc. which can be used interchangeably.

Vectors in two dimensions

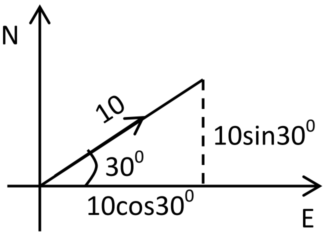
These are the representation of magnitude and directions of quantities in x –y plane.
x – direction is represented by **i** or **–i** while
y – direction is represented by **j** or **–j**.



In the figure above the unit vectors in the x – y plane are $i = \begin{pmatrix}1\\0\end{pmatrix}$ and are $j = \begin{pmatrix}0\\1\end{pmatrix}$

Illustration

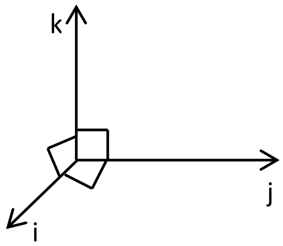
(i) The velocity of a body moving eastward at $5kms^{-1}$ is represented by 5i.
(ii) The velocity of a body moving northwards at x $5kms^{-1}$ is represented by 15j.
(iii) The velocity of a body moving westward at $8kms^{-1}$ is represented by -8i
(iv) The velocity of a body moving southward at $6kms^{-1}$ is represented by -6j.
(v) A body moving at $10ms^{-1}$ in the direction $N60^0E$ is represented as



$= 5\sqrt{3}I + 5j$

Vectors in three dimensions

These represent magnitudes and directions in x, y and z planes and are represented by i, j and k



Where i and j represent direction in x – y plane (or east-north directions on ground) while k represent direction in z- plane (vertical plane)
In summary the unit vectors in the x, y and z planes are

$$i = \begin{pmatrix}1\\0\\0\end{pmatrix},\quad j = \begin{pmatrix}0\\1\\0\end{pmatrix},\quad k = \begin{pmatrix}0\\0\\1\end{pmatrix},$$

For example, the vector of a body that moves 10m due East, 8m due north and 12m vertically is represented as

10i + 8j + 12k or $\begin{pmatrix}10\\8\\12\end{pmatrix}$

Basic concepts

Position vector

If a point P in a two dimensional geometry has Cartesian coordinates (x, y), the position vector of P is given by OP = p = $\begin{pmatrix}x\\y\end{pmatrix}$ or

OP = p = xi + yj

If P has coordinates (x, y, z) in a three dimensional geometry, its position vector is given by

OP = p = $\begin{pmatrix}x\\y\\z\end{pmatrix}$ or xi + yj + zk

Displacement vector

If points P and Q have coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively, the displacement vector PQ is denoted by either PQ, $\underline{PQ}$ or $\overrightarrow{PQ}$ where

PQ = OQ –OP

$$= \begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix} - \begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}$$

$$= \begin{pmatrix}x_2 - x_1\\y_2 - y_1\\z_2 - z_1\end{pmatrix}$$

Example 1

Given the following pair of points, find their respective displacement vectors, P

(i) P (3, 10) and Q (1, 1)

Solution

PQ = OQ –OP

$$= \begin{pmatrix}\mathbf{3}\\\mathbf{10}\end{pmatrix} - \begin{pmatrix}\mathbf{1}\\\mathbf{1}\end{pmatrix} = \begin{pmatrix}\mathbf{2}\\\mathbf{9}\end{pmatrix}$$

(ii) P(4, 0, 2) and Q(2, 4, 1)

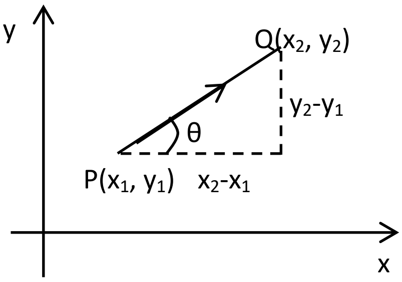
Solution

PQ = OQ –OP

$$= \begin{pmatrix}4\\0\\2\end{pmatrix} - \begin{pmatrix}2\\4\\1\end{pmatrix} = \begin{pmatrix}2\\-4\\1\end{pmatrix}$$

Direction of displacement vector

Direction of displacement vector in 2- D geometry is given by



$$\tan\theta = \frac{y_2 - y_1}{x_2 - x_2} \; or \; \theta = \tan^{-1}\left(\frac{y_2 - y_1}{x_2 - x_2}\right)$$

Example 2

Find the direction of the displacement PQ with the horizontal, given the following points

(i) P(2, 4) and Q (6, 8)

Solution

$\tan\theta = \frac{8-4}{6-2} = 1$

$\theta = \tan^{-1} 1 = 45^0$

(ii) P(1, 1) and Q(3, 5)

Solution

$\tan\theta = \frac{5-1}{3-1} = 2$

$\theta = \tan^{-1} 2 = 63.4^0$

Modulus of a vector

Modulus of a vector is the same as magnitude of a vector.

(i) For P = xi + yj

Modulus of P,= $|P| = \sqrt{x^2 + y^2}$

(ii) For P = xi + yj + zk

Modulus of P,= $|P| = \sqrt{x^2 + y^2 + z^2}$

Example 3

Find the modulus of the following vectors

(i) P = 3i + 4j

Solution

$|P| = \sqrt{3^2 + 4^2} = 5$

(ii) P = 3i + 4j + 5k

Solution

$|P| = \sqrt{3^2 + 4^2 + 5^2}$ = 7.071

Unit vector

This is a vector whose magnitude or length is equal to one.

Example 4

Show that the vector $P = \frac{3}{5}i + \frac{4}{5}$ is a unit vector

$$|P| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{5}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{\frac{25}{25}} = 1$$

The unit vector parallel to a given vector

The unit vector parallel to a vector P or a vector in direction of P is denoted by $\hat{P}$ where

$$\hat{P} = \frac{P}{|P|}$$

Example 5

Find the unit vectors parallel to each of the following vectors.

(i) p = 6i + 8j

Solution

$$\hat{p} = \frac{6i+8j}{|6i+8j|}$$
$$= \frac{6i+8j}{\sqrt{6^2+8^2}}$$
$$= \frac{6i+8j}{\sqrt{100}}$$
$$= \frac{6i+8j}{10} = \frac{3i+4j}{5}$$
$$= \frac{3}{5}i + \frac{4}{5}$$

(ii) q= 3i + 4j + 5k

Solution

$$\hat{q} = \frac{3i+4j+5k}{|3i+4j+5k|}$$
$$= \frac{3i+4j+5k}{\sqrt{3^2+4^2+5^2}}$$
$$= \frac{3i+4j+5k}{\sqrt{50}}$$
$$= \frac{3i+4j+5k}{5\sqrt{2}}$$
$$= \frac{3}{5\sqrt{2}}i + \frac{4}{5\sqrt{2}} + \frac{5}{5\sqrt{2}}k$$
$$= \frac{3\sqrt{2}}{5}i + \frac{4\sqrt{2}}{5} + \frac{5\sqrt{2}}{5}k$$

Revision exercise 1

1. Find the magnitude of each of the following vectors
   (a) 3i + 4j [5]
   (b) 6i + 8j [10]
   (c) $\begin{pmatrix}3\\4\\5\end{pmatrix}$ [$5\sqrt{2}$]
   (d) $\begin{pmatrix}5\\8\\10\end{pmatrix}$ [13.75]
2. Find the value of q in each of the following
   (a) $|3i + qj| = 5$ [4]
   (b) $|2i + qj + 4k| = 6$ [4]
   (c) $|qi + 4j + 4k|$ = $2\sqrt{17}$ [6]
3. Find the direction θ to the horizontal of each of the following vectors.
   (a) $P = \begin{pmatrix}1\\1\end{pmatrix}$ [45]
   (b) $q = \begin{pmatrix}3\\4\end{pmatrix}$ [53.13]
4. Find a unit vector in the direction of each of the following vectors
   (a) p = 8i + 6j $\left[\frac{4}{5}i + \frac{3}{5}j\right]$
   (b) q = 5i + 8j $\left[\frac{5}{\sqrt{89}}i + \frac{8}{\sqrt{89}}j\right]$
   (c) $r = \begin{pmatrix}7\\-9\end{pmatrix}$ $\left[\begin{pmatrix}\frac{5}{\sqrt{89}}\\\frac{8}{\sqrt{89}}\end{pmatrix}\right]$
   (d) 3i – 2j + 5k $\left[\frac{3}{\sqrt{38}}i - \frac{2}{\sqrt{38}} + \frac{5}{\sqrt{38}}k\right]$
   (e) i + 3j + 2k $\left[\frac{1}{\sqrt{14}}i - \frac{3}{\sqrt{14}} + \frac{2}{\sqrt{14}}k\right]$
   (f) $\begin{pmatrix}3\\-12\\4\end{pmatrix}$ $\begin{pmatrix}\frac{3}{13}\\\frac{-12}{13}\\\frac{4}{13}\end{pmatrix}$
5. Find a vector of magnitude $\sqrt{7}$in the direction of the vector $\begin{pmatrix}5\\-3\\1\end{pmatrix}$. $\begin{bmatrix}\sqrt{5}\\\frac{-3\sqrt{3}}{3}\\\frac{\sqrt{5}}{5}\end{bmatrix}$
6. Find $\overrightarrow{PR}$ in each case given that
   (a) $\overrightarrow{PQ}$ = 2i -4j + 5k and $\overrightarrow{QR}$ = 3i +6j – 2k
   $[5i + 2j + 3k]$
   (b) $\overrightarrow{PQ} = \begin{pmatrix}3\\6\\-8\end{pmatrix}$ and $\overrightarrow{QR} = \begin{pmatrix}5\\-5\\0\end{pmatrix}$. $\left[\begin{pmatrix}8\\1\\-8\end{pmatrix}\right]$
7. (a) Given that $\overrightarrow{PQ}$ = 5i – 7j – 2k and

$\overrightarrow{PR}$ = 2i + 3j – 2k, find $\overrightarrow{QR}$? $[-3i + 10j]$

(b) Given that $\overrightarrow{PQ} = \begin{pmatrix}2\\3\\4\end{pmatrix}$ and $\overrightarrow{PR} = \begin{pmatrix}4\\5\\6\end{pmatrix}$,

find $\overrightarrow{QR}$? $\left[\begin{pmatrix}2\\2\\1\end{pmatrix}\right]$

8. Given that $\overrightarrow{PQ}$ = ai + 6j + 4k, $\overrightarrow{QR}$ = 4i + bj + -2k, and $\overrightarrow{PR}$ = -3i + ck, find the possible values of the constants a, b, c.
$[a = -7, b = -6, c = 1]$

Vector algebra

Addition and subtraction of vectors

When adding or subtracting two or more vectors, corresponding elements are added or subtracted.

Example 6

1. Given that p = 2i + 3k and q = 3i + 6j + 5k find
   (i) p + q = (2+3)i + (0+6)j + (3+5)k
   = 5i + 6j + 8k
   (ii) p – q = (2 – 3)i + (0 – 6)j + (3 – 5)k
   = –i – 6j – 2k
2. Given that $p = \begin{pmatrix}5\\7\\9\end{pmatrix}$ and $q = \begin{pmatrix}1\\2\\3\end{pmatrix}$, find
   (i) p + q = $\begin{pmatrix}5\\7\\9\end{pmatrix} + \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}6\\9\\12\end{pmatrix}$
   (ii) p + q = $\begin{pmatrix}5\\7\\9\end{pmatrix} - \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}4\\5\\6\end{pmatrix}$

Multiplication or division of vectors by a scalar

When a vector is multiplied or divided by a scalar the size of the vector changes but the direction remains unchanged

Example 7

Given the vectors $p = \begin{pmatrix}1\\2\\-1\end{pmatrix}$ and $q = \begin{pmatrix}-2\\3\\2\end{pmatrix}$

find

(i) 3p

$$3p = 3\begin{pmatrix}1\\2\\-1\end{pmatrix} = \begin{pmatrix}3\\6\\-3\end{pmatrix}$$

(ii) 3p + 2q

Solution

$$3p + 2q = 3\begin{pmatrix}1\\2\\-1\end{pmatrix} + 2\begin{pmatrix}-2\\3\\2\end{pmatrix}$$
$$= \begin{pmatrix}3\\6\\-3\end{pmatrix} + \begin{pmatrix}-4\\6\\4\end{pmatrix}$$
$$= \begin{pmatrix}-1\\12\\1\end{pmatrix}$$

Coplanar vectors

The vectors p, q and r are said to be coplanar when there exist scalars say α and β such that $r = \alpha p + \beta q$

Example 8

(a) Given $p = \begin{pmatrix}3\\4\end{pmatrix}$, $q = \begin{pmatrix}1\\2\end{pmatrix}$ and $r = \begin{pmatrix}4\\0\end{pmatrix}$, find scalars α and β such that $r = \alpha p + \beta q$

Solution

$\alpha\begin{pmatrix}3\\4\end{pmatrix} + \beta\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}4\\0\end{pmatrix}$

3α + β = 4 ....(i)

4α + 2β = 0 ....(ii)

Solving equation (i) and (ii) simultaneously, we obtain $\alpha = \frac{4}{5}$ and $\beta = \frac{-8}{5}$

(b) Given $p = \begin{pmatrix}2\\3\\1\end{pmatrix}$, $q = \begin{pmatrix}-1\\2\\2\end{pmatrix}$ and $r = \begin{pmatrix}0\\7\\5\end{pmatrix}$, find scalars α, β and γ such that $r = \alpha p + \beta q$

Solution

2α - β = 0 ....(i)

3α + 2β = 7 ....(ii)

α + 2β = 5 ....(iii)

Solving equation (i), (ii) and (iii) simultaneously, we obtain$\alpha = 1$ and $\beta = 2$

Equal vectors

Two or more vectors are said to be equal when they have the same magnitude and direction.

Example 9

Given that vectors p = αi + 2j + (4 – β)k and q= (2 – β)i + 2j + 8k are equal find the values of α and β.

Solution

p and q are equal

αi + 2j + (4 – β)k = q= (2 – β)i + 2j + 8k

⇨ 4 – β = 8
β = -4
α = 2 – β = 2 - (-4) = 6
Hence α = 6 and β = -4

Parallel vectors

Vectors p and q are parallel when one of them is a scalar multiple of another i.e. p = kq where k is a constant.

To show that a given points are collinear

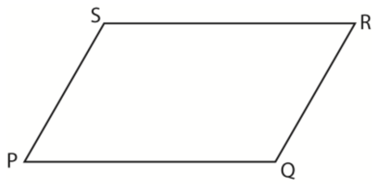
If points P, Q and R are collinear, then

(i) PQ and PR or QR are parallel.
(ii) PQ = kQR where k is a constant and there is a common point on LHS and RHS in this case Q.

Example 10

a. PQRS is a parallelogram with coordinates P(2,4), Q(-1, 5) and R(4, 8). Find the coordinate of S.

Solution



PS = QR
OS – OP = OR – OQ
OS = OR – OQ + OP
$=\binom{4}{8}-\binom{-1}{5}+\binom{2}{4}=\binom{7}{7}$
Hence S(7,7)

b. The position vectors of P, Q and R are $OP=\begin{pmatrix}1\\-2\\2\end{pmatrix}$, $OQ=\begin{pmatrix}3\\2\\-1\end{pmatrix}$, and $OR=\begin{pmatrix}7\\10\\-7\end{pmatrix}$, prove that P, Q and R are collinear.

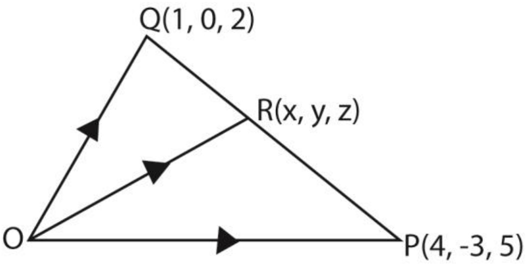
For collinear points PQ = kQR
PQ = OQ – OP
$=\begin{pmatrix}3\\2\\-1\end{pmatrix}-\begin{pmatrix}1\\-2\\2\end{pmatrix}=\begin{pmatrix}2\\4\\-3\end{pmatrix}$
PR = OR- OP
$=\begin{pmatrix}7\\10\\-7\end{pmatrix}-\begin{pmatrix}1\\-2\\2\end{pmatrix}=\begin{pmatrix}6\\12\\-9\end{pmatrix}=3\begin{pmatrix}2\\4\\-3\end{pmatrix}$
Substituting for PQ and QR
⇨ PQ = 2QR

Hence QR is parallel to PQ, since Q is common to both sides of the equation, then P, Q and R are collinear

c. Given points P(2, 1, 0), Q (5, 2, 4) and R(14, 5, 16); show that the points are collinear
Solution
PQ = kQR
PQ = OQ – OP
$=\begin{pmatrix}5\\2\\4\end{pmatrix}-\begin{pmatrix}2\\1\\0\end{pmatrix}=\begin{pmatrix}3\\1\\4\end{pmatrix}$
PR = OR- OP
$=\begin{pmatrix}14\\5\\16\end{pmatrix}-\begin{pmatrix}2\\1\\0\end{pmatrix}=\begin{pmatrix}12\\4\\16\end{pmatrix}=4\begin{pmatrix}3\\1\\4\end{pmatrix}$
Substituting for PQ and QR
⇨ PQ = 3QR

Hence QR is parallel to PQ, since Q is common to both sides of the equation, then P, Q and R are collinear

d. Given that $OP=\begin{pmatrix}4\\-3\\5\end{pmatrix}$ $and\ OQ=\begin{pmatrix}1\\0\\2\end{pmatrix}$ find the coordinates of point R such that PR:PQ = 1:2 and points P, Q and R re collinear.
Solution



$PR = \frac{1}{2}PQ$

$OR - OP = \frac{1}{2}(OQ - OP)$

$OR = \frac{1}{2}(OQ - OP) + OP = \frac{1}{2}(OQ + OP)$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{2}\left[\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix}\right] = \begin{pmatrix} 2.5 \\ -1.5 \\ 3.5 \end{pmatrix}$$

Hence coordinates of R(2.5, -1.5, 3,5)

The ratio theorem (section formula)

Consider the division of a line PR by a point Q as shown below:

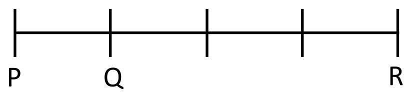


Fig: 1

In the Fig: 1 above, point Q divides line PR internally; PQ:QR = 1:3

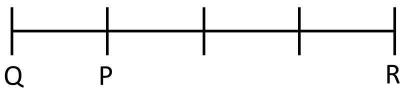


Fig: 2

In Fig: 2, point Q divides PR externally PQ:QR = -1:4 or 1: -4 (depending on the direction considered.

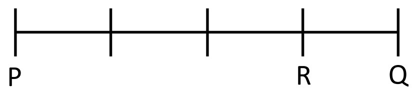


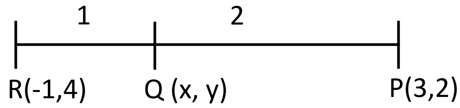
Fig: 3

In fig: 3, Point Q divides PR externally.
PQ :QR = 4: -1 or QP : RQ = 4:1
Note PQ and QR are in opposite direction.

Example 11

(a) P(3, 2) and R(-1, 4) are two points on the line . A point Q divides PR in the ratio (i) 2:1, (ii) 4: -1, (iii) 1: -4. Find the coordinates of Q in each case

Solution



Here Q divides the line internally

PQ: QR = 2:1

$\frac{PQ}{QR} = \frac{2}{1}$

PQ =2QR

OQ – OP =2(OR – OQ)

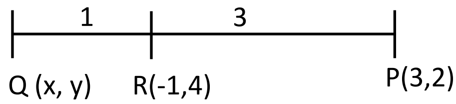
3OQ = 2OR + OP

$=2\begin{pmatrix} -1 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 10 \end{pmatrix}$

$= \begin{pmatrix} \frac{1}{3} \\ \frac{10}{3} \end{pmatrix}$

Hence $Q\left(\frac{1}{3}, \frac{10}{3}\right)$

(ii)



Here Q divides the line externally

PQ: QR = 4: -1 ($\overrightarrow{PQ}$ as positive)

$\frac{PQ}{QR} = \frac{4}{-1}$

-PQ =4QR

-(OQ – OP) =4(OR – OQ)

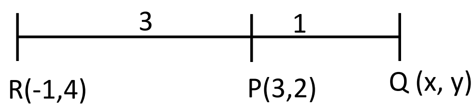
3OQ = 4OR-OP

$=4\begin{pmatrix} -1 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -7 \\ 14 \end{pmatrix}$

OQ $=\begin{pmatrix} -7/3 \\ 14/3 \end{pmatrix}$

Hence $Q(-7/3, 14/3)$

(iii)



Here Q divides the line externally

PQ: QR = 1:-4 (taking PQ positive)

$\frac{PQ}{QR} = \frac{1}{-4}$

-4PQ =PR

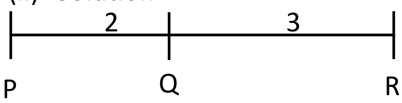
-4(OQ – OP) =(OR – OQ)
3OQ =4OP – OR

$=4\begin{pmatrix}3\\2\end{pmatrix} - \begin{pmatrix}-1\\4\end{pmatrix} = \begin{pmatrix}13\\4\end{pmatrix}$

OQ $=\begin{pmatrix}{}^{13}/_{3}\\ {}^{4}/_{3}\end{pmatrix}$

Hence Q$\left({}^{13}/_{3}, {}^{4}/_{3}\right)$

(b) Two points P and Q are such that P(0, 1, 4) and Q(2, 6, 0). A point Q divides a line PQ in ratio 2:3. Find the position vector of Q if it divides PQ
(i) Internally
(ii) Solution



Here Q divides the line internally

PQ: QR = 2:3

$\frac{PQ}{QR} = \frac{2}{3}$

3PQ =2QR

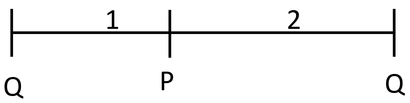
3(OQ – OP) =2(OR – OQ)
3OQ – 3OP =2OR – 2OQ
5OQ=2OR + 3OP

$=2\begin{pmatrix}2\\6\\0\end{pmatrix} + 3\begin{pmatrix}0\\1\\4\end{pmatrix} = \begin{pmatrix}4\\15\\12\end{pmatrix}$

OQ $=\begin{pmatrix}{}^{4}/_{5}\\ 3\\ {}^{12}/_{5}\end{pmatrix}$

Hence $Q\left({}^{4}/_{5}, 3, {}^{12}/_{5}\right)$

(ii) Externally



Here Q divides the line externally

PQ: QR = -2:3

$\frac{PQ}{QR} = \frac{-2}{3}$

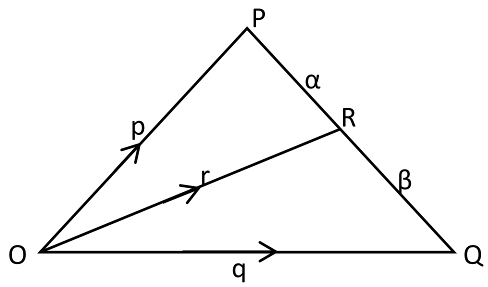
3PQ =-2QR

3(OQ – OP) =-2(OR – OQ)
3OQ – 3OP =-2OR + 2OQ
OQ =3OP + 3OR

$= 3\begin{pmatrix}0\\1\\4\end{pmatrix} - 2\begin{pmatrix}2\\6\\0\end{pmatrix} = \begin{pmatrix}-4\\-9\\12\end{pmatrix}$

Hence Q(-4, -9, 12)

Note: in general, given that Q divides PR in ratio α:β



OR= OP + PR

$= p + \frac{\alpha}{\alpha+\beta}PQ$

$= p + \frac{\alpha}{\alpha+\beta}(-PO + OQ)$

$= p + \frac{\alpha}{\alpha+\beta}(-p + q)$

$= \frac{p(\alpha+\beta)+\alpha(-p+q)}{\alpha+\beta}$

$= \frac{p(\alpha+\beta)-\alpha p+\alpha q}{\alpha+\beta}$

$= \frac{\beta}{\alpha+\beta}p + \frac{\alpha}{\alpha+\beta}q$

Finding the constants of equality

Suppose that r = λa + kb and r = ma+ nb

⇨ λa + kb = ma+ nb

Equating corresponding unit vectors

λ = m and k = n

Example 12

(a) Find the position vector of Q if it divides PR in the ration (i) 1:5 and (ii) 1:-4, given that OR =r, OP = p and OQ = q.

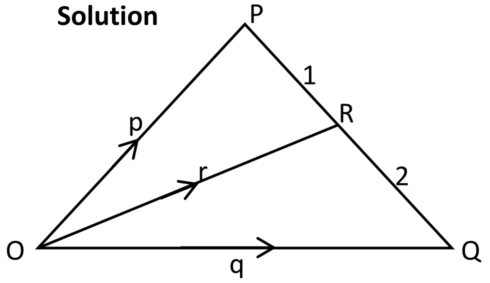
Solution

(i) $r = \frac{5}{1+5}p + \frac{1}{1+5}q = \frac{5}{6}p + \frac{1}{6}q$

(ii) $r = \frac{-4}{1+-4}p + \frac{1}{1+-4}q = \frac{4}{3}p - \frac{1}{3}q$

(b) OPQ is a triangle with vector OP = p, OQ= q. Express in terms of p and q the position vector of OR, where R divides $\overline{PQ}$ in ratio 1:2.

Solution

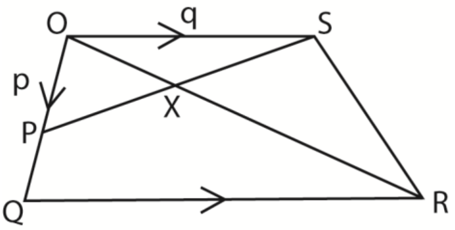


$OR = \frac{2}{1+2}p + \frac{1}{1+2}q = \frac{2}{3}p + \frac{1}{3}q$

(c) The diagram below shows a quadrilateral OSRQ, OS = q, OP = p and SX = kSP.

(i) Express vectors SP and OX in terms of p and q

Solution



SP = SO + OP

= -q + p

= p - q

OX = OS + kSP

= q + k(p – q)

= kp + q(1-k)

(ii) OQ = 3p and QR = 2OS and OX =λOR

Find k and λ.

Solution

OX = λOR

= λ(OQ +QR)

= λ(3p + 2q)

= 3λp + 2λq

Equating corresponding unit vectors

For p

k = 3λ ....(1)

For Q

(1 – k) = 2λ

k = 1 - 2λ ....(2)

Equations (1) and (2)

3λ = 1 – 2λ

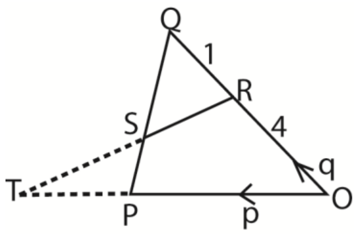
5λ = 1

λ = $\frac{1}{5}$

From eqn. (1)

k = $3x\frac{1}{5} = \frac{3}{5}$

(d) Given that OP = p and OQ = q, point R is on OQ such that $\overline{OR}$: $\overline{RQ}$ = 4:1. Point S is on QP such that QP: SA = 2:3 and RS and OP are both produced, they to meet at point T.



Find

(i) OR and OS in terms of p and q.

Solution

OR = $\frac{4}{5}OQ = \frac{4}{5}q$

OS = OQ + QS

= $q + \frac{2}{5}QP$

= $q + \frac{2}{5}(p - q)$

= $\frac{1}{5}(2p + 3q)$

(ii) OT in terms of p.

Solution

Let OT = αOP and RT = βRS

From ΔOTR

OT = OR + RT

αOP + OR+ βRS

RS = RO + OS

= $\frac{-4}{5}q + \frac{1}{5}(2p + 3q)$

RS = $\frac{1}{5}(2p - q)$

αOP = OR + RT

αp = $\frac{4}{5}q$ + βRS

= $\frac{4}{5}q + \frac{\beta}{5}(2p - q)$

$\alpha p = \frac{2\beta}{5}p + \left(\frac{4}{5} - \frac{\beta}{5}\right)q$

Comparing coefficients

For q: $\frac{4}{5} - \frac{\beta}{5} = 0$

β =4

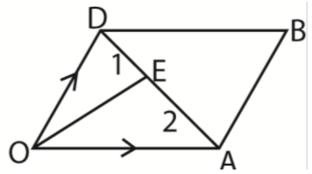
For p, $OT = \frac{2\beta}{5} = \frac{2}{5}x\ 4 = \frac{8}{5}$

$\therefore OT = \frac{8}{5}p$

(e) OABCD is a parallelogram, find the position vectors of E and F such that E divides DA in the ration 1:2 and F divides it externally in ratio 1:2.

Given that OA =$\begin{pmatrix}1\\-2\\2\end{pmatrix}$ and OB=$\begin{pmatrix}3\\2\\-1\end{pmatrix}$

Solution



(i) OA = DB

$\begin{pmatrix}1\\-2\\2\end{pmatrix} = OB - OD$

$\begin{pmatrix}1\\-2\\2\end{pmatrix} = \begin{pmatrix}3\\2\\-1\end{pmatrix} - OD$

$OD = \begin{pmatrix}3\\2\\-1\end{pmatrix} - \begin{pmatrix}1\\-2\\2\end{pmatrix} = \begin{pmatrix}2\\4\\-3\end{pmatrix}$

$DE = OD + DE$

$= \begin{pmatrix}2\\4\\-3\end{pmatrix} + \frac{1}{3}(DA)$

$= \begin{pmatrix}2\\4\\-3\end{pmatrix} + \frac{1}{3}(OA - OD)$

$= \begin{pmatrix}2\\4\\-3\end{pmatrix} + \frac{1}{3}\left[\begin{pmatrix}1\\-2\\2\end{pmatrix} - \begin{pmatrix}2\\4\\-3\end{pmatrix}\right]$

$= \begin{pmatrix}2\\4\\-3\end{pmatrix} + \frac{1}{3}\begin{pmatrix}-1\\-6\\5\end{pmatrix}$

$= \begin{pmatrix}\frac{5}{3}\\2\\\frac{-4}{3}\end{pmatrix}$

(ii) When F divides DA externally, ether 1 or 2 must be negative but not both. From the ratio 1: -2.

⇨ DF : FA = 1 : -2

$\frac{DF}{FA} = \frac{1}{-2}$

-2DF = FA

-2(OF – OD) = OA – OF

OF = 2OD – OA

$= 2\begin{pmatrix}2\\4\\-3\end{pmatrix} - \begin{pmatrix}1\\-2\\2\end{pmatrix}$

OF =$\begin{pmatrix}3\\10\\-8\end{pmatrix}$

Exercise 2

1. Points P, Q, R have respective position vectors $\begin{pmatrix}5\\4\\1\end{pmatrix}, \begin{pmatrix}7\\5\\4\end{pmatrix}$ and $\begin{pmatrix}11\\7\\10\end{pmatrix}$.
   (a) Vectors PQ and QR $\left[\begin{pmatrix}2\\1\\3\end{pmatrix}; \begin{pmatrix}4\\2\\0\end{pmatrix}\right]$
   (b) Deduce that P, Q and R are collinear and find the ration PQ:QR. [1:2]
2. The A, B and C have coordinates (1, -5, 6), (3, -2, 10) and (7, 4,18 respectively. Show that A, B, C are collinear.
3. Show that the points P(5, 4, -5), Q(3, 8, -1) and R(0, 14, 2) are collinear.
4. Given A(2, 13, -5), B(3, x, -3) and C(6, -7, y) are collinear, find the values of x and y [8,3]
5. OABC is a parallelogram with $\overrightarrow{OA}$ = a and $\overrightarrow{OC}$ = c. Sis the point on AB such that AS: SB = 3:1 and T is a point on BC such that BT:TC =1:3
   (a) Express each of the following in terms of a and c.
   (i) $\overrightarrow{AC}$ $[c - a]$
   (ii) $\overrightarrow{SB}$
   (iii) $\overrightarrow{BT}$ $\left[-\frac{1}{4}a\right]$-
   (iv) $\overrightarrow{ST}$ $[1: 4]$
   (b) State the value of the ratio ST:AC

6. Triangle OAB has OA = a and OB = b. C is a point on OA such that OC = $\frac{2}{3}a$. D is the midpoint of AB. When CD is produced it meets OB at E, such that DE = nCD and BE = kb. Express DE in terms of
   (a) n, a and b $\left[\frac{5}{6}na - \frac{1}{2}nb\right]$
   (b) k, a and b $\left[\frac{1}{2}a + \frac{(2k-1)}{2}b\right]$
   (c) hence find the values of n and k. $\left[n = \frac{3}{5}\ and\ k = \frac{1}{5}\right]$
7. Three non-collinear points A, B, and C have position vectors a, b, and c respectively with respect to an origin O. The points M on AC is such that AM:MC = 2:1 and point N on AB is such that AN: NB = 2:1.
   (a) Find in terms of a, b, c the vectors
      (i) BM $\left[\frac{1}{3}a - b + \frac{2}{3}c\right]$
      (ii) CN $\left[\frac{1}{3}a + \frac{2}{3}b - c\right]$
   (b) The lines BM and CN intersect at L. Given that BL = rBM and CL = tCN, where r and t are scalars; express in terms of a, b, c, r and t;
      (i) BL $\left[\frac{1}{3}ra - rb + \frac{2rc}{3}\right]$
      (ii) CL $\left[\frac{1}{3}ta + \frac{2}{3}tb - tc\right]$
   (c) Hence by using triangle BLC, or otherwise, find r and t $\left[r = \frac{3}{5}\ and\ t = \frac{3}{5}\right]$

8. In the rectangle OABC, OA = a and OC = c. R is a point on AB such that AR : RB = 1:2 and S is a point on BC such that BS:SC = 3:1. AS meets OR at P.
   (i) Find an expression of OP in terms a and c $\left[\frac{4}{5}a + \frac{4}{15}c\right]$
   (ii) Show that OP:PR = 4:1.
   (iii) Find the value of the ratio AP:PS $[4:1]$
9. Ina triangle OAB, OA = a and OB = b, M is the midpoint of AB and N is a point on OB such that ON:NB = 1:4. OM meets AN at P.
   (a) Find n expression ofOp in terms of a and b. $\left[\frac{1}{6}(a+b)\right]$
   (b) Find the ratio of AP:PN [5:1]
10. In a trapezium OABC, OA = a, OC = c and CB = 3a. T is a point on BC such that BC: TC =1:2. OT meets AC at P
   (a) Find an expression for OP in terms of a and c. $\left[\frac{2}{3}a + \frac{1}{3}c\right]$
   (b) Deduce that P is a point of trisection of both AC and OT
11. In a rectangle OABC, M is a midpoint of OA and N is a midpoint of AB. OB meets MC at P and NC at Q. show that OP = PQ =QB.
12. In the parallel gram OABC, P is a point on OA such that OP:PA = 1:2 and Q is a point on AB such that AQ:QB = 1:3, OB meets PC at K and QC at M show that OK:KM:MB = 7:9:12

The scalar or dot products

The dot product of vectors p and q inclined at an angle θ to each other is defined as

$p.q\ = |p|.|q|cos\theta, 0 \le \theta \le \pi$

Properties of scalar product.

(a) $i.i = |i|.|i|cos0^0 = 1$ (the angle between i and I is zero)
(b) $i.j = |i|.|j|cos90^0$ =0 (i and j are perpendicular

Thus i.i = j.j = k.k = 1 and i.j = i.k = j.k =0

Hence the dot product of two vectors is a scalar quantity.

Note that a dot product is used to show that the two vectors are perpendicular.

(c) $|p.p| = |p|^2$
(d) $p.(q+r) = p.q + p.r$ (distribution law)
(e) $p.(kq) = (kp).q = k(p.q)$ where k is constant.

Example 13

(a) Given that p = i – 2kand q = 3i – 3j +k; find
   (i) p.q
      Solution
      p.q = (i – 2k).(3i – 3j + k)
      = 3 – 2 = 1
   (ii) the angle between p and q corrected to the nearest degree.
      Solution
      p.q = $\sqrt{1^2+(-2)^2}.\sqrt{3^2+(-3)^2+1^2}$cosθ
      1 = $\sqrt{5}.\sqrt{19}cos\theta$

$\theta = \cos^{-1}\left(\frac{1}{\sqrt{95}}\right) = 84.11^0$

∴the angle between p and q corrected to the nearest degree is 84^0.

(b) Show that the following vectors are perpendicular.

(i) p = 2i +6j + 4k and q = (-2i – 2j + 4k)

Solution

p.q = (2i +6j + 4k). (-2i – 2j + 4k)

= -4 -12 + 14 = 0

(hence perpendicular)

(ii) a = 3i – 4j + k and b = 2i + 3j + 6k

Solution

a.b = (3i – 4j + k).(2i + 3j + 6k)

= 6 - 12 + 6 = 0

(hence perpendicular)

(c) (i) Find the values of the scalar x if the vectors p = 2xi + 7j – k and q = 3xi + xj + 3k

Solution

p.q = $|p|.|q|cos\theta$

If p and q are perpendicular then p.q =0

p.q = (2xi + 7j – k).(3xi + xj + 3k)

$6x^2 + 21x - 3 = 0$

(3x – 1)(2x + 3) = 0

Either x = $\frac{1}{3}$ or x =- $\frac{3}{2}$

(ii) If the angle between the vector p = xi + 2j and q = 3i + j is 45^0, find two possible values of x.

Solution

(xi + 2j)(3i+j)= $\left(\sqrt{x^2+2^2}\right).\left(\sqrt{3^2+1^2}\right)cos45^0$

3x + 2 = $\frac{\sqrt{2}}{2}\left(\sqrt{x^2+4}\right).\left(\sqrt{10}\right)$

⇨ $x^2 + 3x - 4 = 0$

By solving the equation = -4 or x = 1

(d) Find the angle between the vectors

$p = \begin{pmatrix}2\\3\\7\end{pmatrix}$ and $p = \begin{pmatrix}-2\\3\\0\end{pmatrix}$

Solution

$$\begin{pmatrix}2\\3\\7\end{pmatrix}.\begin{pmatrix}-2\\3\\0\end{pmatrix} = \left|\begin{pmatrix}2\\3\\7\end{pmatrix}\right|.\left|\begin{pmatrix}-2\\3\\0\end{pmatrix}\right|cos\theta$$

-4 + 9 = $\sqrt{2^2+3^2+7^2}.\sqrt{(-2)^2+3^2+0^2}\cos\theta$

5 = $\left(\sqrt{62x13}\right)cos\theta$

$\theta = \cos^{-1}\left(\frac{5}{\sqrt{806}}\right) = 79.86^0$

The vector (cross) product

Given two non – zero vectors p and q, their vector (cross) product is denoted by p x q or p∧q is defined p x q = $[p]|q|sin\theta.\mu$ where θ is the angle between p and q and μ is the opposite unit vector to the given vectors. And $0 \le \theta \le \pi$

The cross product is synonymous to determinant of a 3 x 3 matrix.

Properties of vector (cross) product

(a) i x j = $|i||j|sin90^0.k = k$

= 1 x1x k = k

i x k = $|i||j|sin90^0.j =$

= 1 x1 x j = j

j x k = $j|k|sin90^0.i = i$

= 1 x1xi = i

(b) i x i = $|i||i|sin0^0 = 0$

j x j = $|j||j|sin0^0 = 0$

k x k = $|k||k|sin0^0 = 0$

= 1 x1xi = i

Hence the cross product of two vectors is a vector quantity.

Note we use the cross product to show that two vectors are parallel.

(c) p x q = -(p x q)

(d) for any three vectors p, q, and r

p(q x r) = p x q + p x r

(e) The cross product is perpendicular to either of the two vectors crossed.

Suppose we have vectors p = (p_1i + p_2j + p_3k) and q = (q_1i + q_2j + q_3k), the cross product of p and q is

p x q = $\begin{bmatrix} i & -j & k \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{bmatrix}$

= $\begin{vmatrix} p_2 & p_3 \\ q_2 & q_3 \end{vmatrix} i - \begin{vmatrix} p_1 & p_3 \\ q_1 & q_3 \end{vmatrix} j + \begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} k$

=(p_2q_3-p_3q_2)i - (p_1q_3-p_3q_1)j + (p_1q_2-p_2q_1)k

Example 14

(a) Given p = 3i – 2j+k and q = 4i +3j -2k, find p x q and q x p.

Solution

$$p\ x\ q = \begin{vmatrix} i & -j & k \\ 3 & -2 & 1 \\ 4 & 3 & -2 \end{vmatrix}$$

$$= \begin{vmatrix} -2 & 1 \\ 3 & -2 \end{vmatrix} i - \begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} j + \begin{vmatrix} 3 & -2 \\ 4 & 3 \end{vmatrix} \text{k}$$

= (-2 x -2 – 1 x3)i – (3x-2 – 1 x 4)j + (3x3 - -2x4)k

=(4 – 3)i – (-6 – 4)j + (9+8)k

= i +10j + 17k

Or using matrix approach

$$p\ x\ q = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} x \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} -2\ x -2 - 3x1 \\ -(3x - 2 - 4x1) \\ 3x3 - 4x - 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 - 3 \\ -(-6 - 4) \\ 9 + 8 \end{pmatrix} = \begin{pmatrix} 1 \\ 10 \\ 17 \end{pmatrix}$$

q x p = -(p x q)

= -(i +10j + 17k)

= –i – 10j – 17k

Or $p\ x\ q\ = -\begin{pmatrix} 1 \\ 10 \\ 17 \end{pmatrix} = \begin{pmatrix} -1 \\ -10 \\ -17 \end{pmatrix}$

(b) Show that the cross product of vectors $p\ = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$ and $q = \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix}$ is perpendicular to the vectors.

$$p\ x\ q = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} x \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 3x - 2 - 0\ x\ 5 \\ 2\ x - 2 -\ -1x5 \\ 2x0 -\ -1x3 \end{pmatrix}$$

$$= \begin{pmatrix} -6 - 0 \\ -(-4 + 5) \\ 0 + 3 \end{pmatrix} = \begin{pmatrix} -6 \\ -1 \\ 3 \end{pmatrix}$$

Now $\begin{pmatrix} -6 \\ -1 \\ 3 \end{pmatrix} . \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} = -12\ -3 + 15 = 0$

and

and $\begin{pmatrix} -6 \\ -1 \\ 3 \end{pmatrix} . \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix} = 6 - 6 = 0$

Hence the product is perpendicular

(c) Find the vector perpendicular to p= $\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$ and q= $\begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}$

Solution

Approach 1

$$p\ x\ q = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} x \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} -10 \\ 5 \\ 10 \end{pmatrix} = 5 \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

Hence the perpendicular vector is $\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$

Note the $\begin{pmatrix} -10 \\ 5 \\ 10 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$ are parallel.

Approach 2

Let the perpendicular vector be $\begin{pmatrix} p \\ q \\ r \end{pmatrix}$

⇨ $\begin{pmatrix} p \\ q \\ r \end{pmatrix} . \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = 0$

3p + 4q + r = 0 ....(ii)

And $\begin{pmatrix} p \\ q \\ r \end{pmatrix} . \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} = 0$

-p + 2q - r = 0 ....(ii)

Eqn. (i) – 2Eqn (ii)

5p + 5r = 0

p + q = 0

Let p = λ, then r = -λ

Substituting for p and r in equation (i)

3λ + 4b – λ = 0

b = $-\frac{1}{2}\lambda$

$$\therefore \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} \lambda \\ -\frac{1}{2}\lambda \\ -\lambda \end{pmatrix} = -\frac{1}{2}\lambda\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

Hence the perpendicular vector is $\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$

(d) Given points P(1, 1, 2), Q(3, 7, 8) and R(4, 10, 11)
Show that PQ is parallel to QR.
Solution
PQ = OQ – OP
= (3i + 7j + 8k) – (i + j + 2k)
= (2i + 6j + 6k)
QR = OR – OQ
= (4i + 10j + 11k) - (3i + 7j + 8k)
= (i + 3j + 3k)

PQ x QR = $\begin{vmatrix} i & -j & k \\ 2 & 6 & 6 \\ 1 & 3 & 3 \end{vmatrix}$

$=\begin{vmatrix} 6 & 6 \\ 3 & 3 \end{vmatrix} i - \begin{vmatrix} 2 & 6 \\ 1 & 3 \end{vmatrix} j + \begin{vmatrix} 2 & 6 \\ 1 & 3 \end{vmatrix}$k

=(6x3 - 6x3)i – (2 x3 – 6x1)j + (2 x 3 – 1x6)k
= (18 – 18)i – (6 – 6)j + (6 – 6)k
0i – 0j – 0k = 0

(e) If vectors p = 2i – 3j + k and q = ai – 6j + bk are parallel, find the values of a and b
Solution

$$pxq = \begin{vmatrix} i & -j & k \\ 2 & -3 & 1 \\ a & -6 & b \end{vmatrix}$$

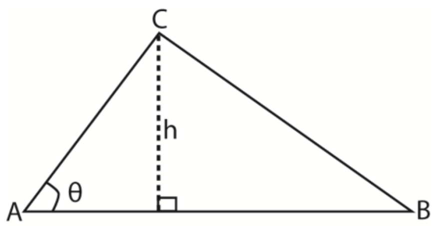
$=\begin{vmatrix} -3 & 1 \\ -6 & b \end{vmatrix} i - \begin{vmatrix} 2 & 1 \\ a & b \end{vmatrix} j + \begin{vmatrix} 2 & -3 \\ a & -6 \end{vmatrix} k$

= (-3b + 6)i –(2b –a)j + (-12+3a)k
⇨ -3b+ 6= 0; b = 2
⇨ 2b –a = 0; a = 2 x 2 = 4
Hence the value of a = 4 and b = 2

Application of dot and cross product of vectors

(1) The triangle

(i) The area of a triangle

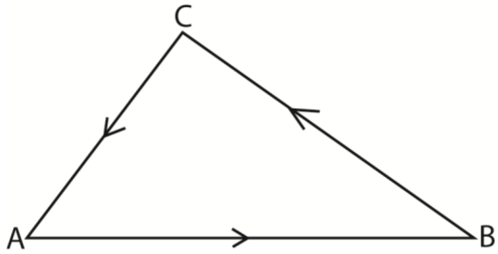


Area of triangle ABC = $\frac{1}{2} x\ \overline{AB}\ x\ h$

$= \frac{1}{2} x\ \overline{AB}\ x\ ACsin\theta$

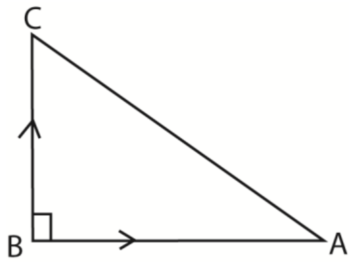
But $|AB\ x\ AC| = |AB||AC|sin\theta$
Hence the area of the triangle $= \frac{1}{2}|AB\ x\ AC|$
In general, the area of a triangle ABC
$= \frac{1}{2}|AB\ x\ AC| = \frac{1}{2}|BA\ x\ CC| = \frac{1}{2}|CB\ x\ CA|$

(ii) To show that given vertices for a triangle



If ABC is a triangle, then it must be a closed polygon.
i.e. $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0$
=> (OB – OA) + (OC – OB) + (OA – OC) =0

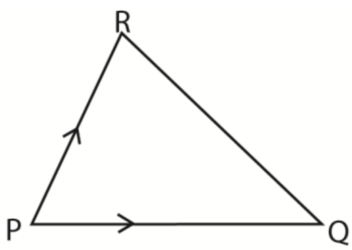
(iii) To show that a given triangle is right angled triangle.
Suppose <ABC = 900



⇨ BA.BC = 0 (dot product of BA and BC)

Example 15

(a) The vertices of a triangle PQR have position vectors p = i + 2j + k, q = i + 3k and r = -1 + 2j –k. Determine the area of the triangle PQR.
Solution



PQ = OQ – OP

$=\begin{pmatrix}1\\0\\3\end{pmatrix}-\begin{pmatrix}1\\2\\1\end{pmatrix}=\begin{pmatrix}0\\-2\\2\end{pmatrix}$

PR = OR – OP

$=\begin{pmatrix}-1\\2\\-1\end{pmatrix}-\begin{pmatrix}1\\2\\1\end{pmatrix}=\begin{pmatrix}-2\\0\\2\end{pmatrix}$

$PQ\ x\ PR=\begin{pmatrix}0\\-2\\2\end{pmatrix}x\begin{pmatrix}-2\\0\\2\end{pmatrix}=\begin{pmatrix}4\\-4\\-4\end{pmatrix}$

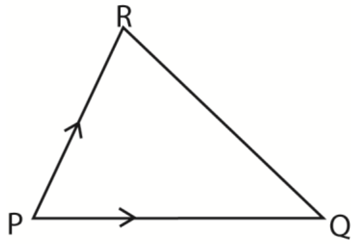
Area of PQR = $\frac{1}{2}|PQ\ x\ PR|$

$=\frac{1}{2}\sqrt{4^2+(-4)^2+(-4)^2}$

$=2\sqrt{3}$ sq. units.

(b) Find the area of a triangle PQR with vertices P(0, 1, 3), Q(1, 5, 7) and R(4, -2, 4)

Solution



PQ = OQ – OP

$=\begin{pmatrix}1\\5\\7\end{pmatrix}-\begin{pmatrix}0\\1\\3\end{pmatrix}=\begin{pmatrix}1\\4\\4\end{pmatrix}$

PR = OR – OP

$=\begin{pmatrix}4\\-2\\4\end{pmatrix}-\begin{pmatrix}0\\1\\3\end{pmatrix}=\begin{pmatrix}4\\-3\\1\end{pmatrix}$

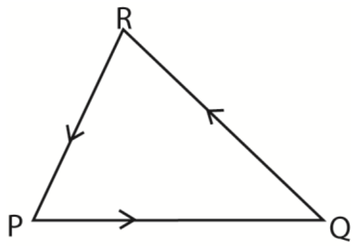
$PQ\ x\ PR=\begin{pmatrix}1\\4\\4\end{pmatrix}x\begin{pmatrix}4\\-3\\1\end{pmatrix}=\begin{pmatrix}16\\15\\-19\end{pmatrix}$

Area of PQR = $\frac{1}{2}|PQ\ x\ PR|$

$=\frac{1}{2}\sqrt{16^2+15^2+(-19)^2}$

=29 sq. units.

(c) Show that the points P(13, -2, 0), Q(7, 1, -3) and R (2, -1, 5) are vertices of a triangle and it is a right angled triangle. Find its area



PQ = OQ – OP

$=\begin{pmatrix}7\\1\\-3\end{pmatrix}-\begin{pmatrix}13\\-2\\0\end{pmatrix}=\begin{pmatrix}-6\\3\\-3\end{pmatrix}$

QR = OR – OQ

$=\begin{pmatrix}2\\-1\\5\end{pmatrix}-\begin{pmatrix}7\\1\\-3\end{pmatrix}=\begin{pmatrix}-5\\-2\\8\end{pmatrix}$

PR = OR – OP

$=\begin{pmatrix}13\\-2\\0\end{pmatrix}-\begin{pmatrix}2\\-1\\5\end{pmatrix}=\begin{pmatrix}11\\-1\\-5\end{pmatrix}$

PQ+QR + RP $=\begin{pmatrix}-6\\3\\-3\end{pmatrix}+\begin{pmatrix}-5\\-2\\8\end{pmatrix}+\begin{pmatrix}11\\-1\\-5\end{pmatrix}$

$=\begin{pmatrix}0\\0\\0\end{pmatrix}$

Hence PQR is a triangle

To show that PQR is a right angled triangle

PQ.QR $=\begin{pmatrix}-6\\3\\-3\end{pmatrix}.\begin{pmatrix}-5\\-2\\8\end{pmatrix}=30-6-20=0$

Hence PQR is a right angled triangle

Area PQR = $\frac{1}{2}(\overline{QP}x\overline{QR})$

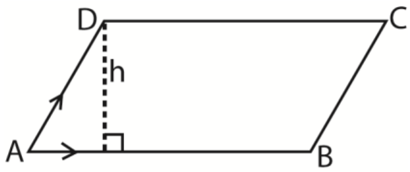
$\overline{QP}=\sqrt{6^2+(-3)^2+3^2}=\sqrt{54}$

$\overline{QR}=\sqrt{(-5)^2+(-2)^2+8^2}=\sqrt{93}$

Area PQR = $\frac{1}{2}(\sqrt{54}x\sqrt{93})$ = 35.4sq. units

(2) The parallelogram

(i) The area of the parallelogram



Taking <DAB = θ

Area of parallelogram = base x height

$= |AB||AD|cos\theta$

$= |ABxAD|$

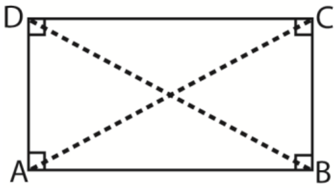
(ii) Properties of parallelogram

- Two sides are parallel and equal, i.e., AB = DC and AD = BC
- The diagonals are not perpendicular and not equal
- Opposite angles are equal i.e.

<DAB = <DCB and <ADC = <ABC

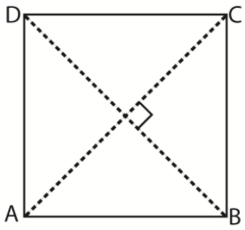
- The sides are not perpendicular i.e., <DAB = <DCB$\neq 90^0$ and <ADC = <ABC$\neq 90^0$

(iii) Properties of a rectangle



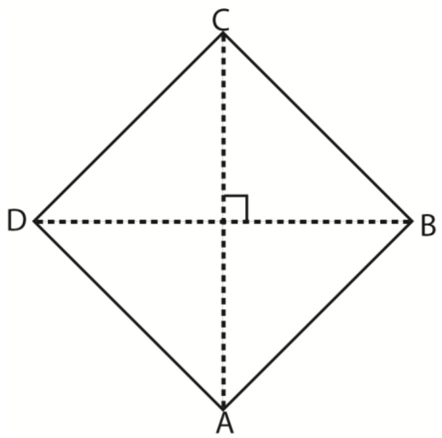
- Two sides are parallel and equal, i.e., AB = DC and AD = BC
- Diagonals are equal and perpendicular
- All angles are equal to 90^0. i.e., <DAB = <ABC =<BCA = <CDA = 90^0.

(iv) Properties of a square



- All sides are parallel and equal, i.e., AB = DC = AD = BC
- Diagonals are equal and perpendicular
- All angles are equal to 90^0. i.e., <DAB = <ABC =<BCA = <CDA = 90^0.

(v) Properties of a rhombus



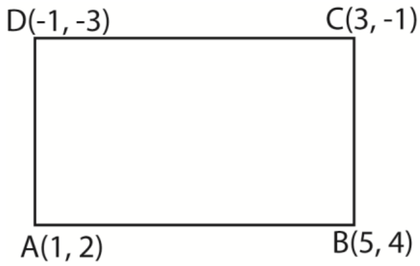
- All sides are parallel and equal, i.e., AB = DC = AD = BC
- Diagonals are equal and perpendicular

Opposite angles are equal but not equal to 90^0. i.e.
<DAB = <DCB$\neq 90^0$ and
<ADC = <ABC$\neq 90^0$

Example 16

(a) A quadrilateral ABCD has coordinates A(1, 2), B(5,4), C(3, -1) and D(-1, -3). Show whether ABCD is a rectangle or parallelogram

Solution



For both a rectangle and parallelogram,
AB = DB and AD = BC
AB = OB – OA

$= \begin{pmatrix}5\\4\end{pmatrix} - \begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}4\\2\end{pmatrix}$

DC = OC – OD

$= \begin{pmatrix}3\\-1\end{pmatrix} - \begin{pmatrix}-1\\-3\end{pmatrix} = \begin{pmatrix}4\\2\end{pmatrix}$

⇨ AB = DC

AD = OD – OA

$= \begin{pmatrix}-1\\-3\end{pmatrix} - \begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}-2\\-5\end{pmatrix}$

DC = OC – OD

$= \begin{pmatrix}3\\-1\end{pmatrix} - \begin{pmatrix}5\\4\end{pmatrix} = \begin{pmatrix}-2\\-5\end{pmatrix}$

⇨ AD = DC

Hence ABCD is either a rectangle or parallelogram.

For a rectangle <DAB = <ABC = 900

⇨ AD.AB = 0

$$\begin{pmatrix}-2\\-5\end{pmatrix}.\begin{pmatrix}4\\2\end{pmatrix} = -8 - 10 = -18$$

Hence ABCD is not a rectangle.

For a parallelogram, <DAB = <BCD and <ABC = <ADC

AD.AB = $|AD||AB|cos\theta$

$-18 \;= \sqrt{4^2 + 2^2}\sqrt{-2^2 + -5^2}cos\theta$

$-18 = \sqrt{20x29}cos\theta$

$\theta = \cos^{-1}\left(\frac{-18}{\sqrt{580}}\right) = 138.4^0$

$CB.CD = |CB||CD|cos\theta$

$\binom{2}{5}.\binom{-4}{-2} = \sqrt{2^2 + 5^2}\sqrt{-4^2 + -2^2}cos\theta$

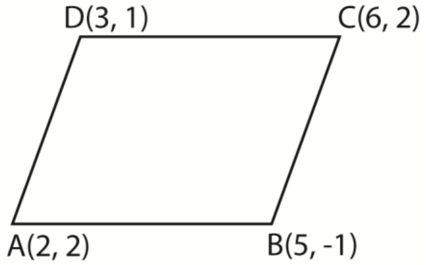
$-8 - 10 = \sqrt{29.20}cos\theta$

$-18 = \sqrt{20x29}cos\theta$

$\theta = \cos^{-1}\left(\frac{-18}{\sqrt{580}}\right) = 138.4^0$

Since <DAB =<BCD; ABCD is a parallelogram.

(b) ABCD is a quadrilateral with A(2, -2), B(5, -1), C(6, 2) and D(3,1). Show whether the quadrilateral is a square or a rhombus.



For square or rhombus

AB = BC = CD = AD

AB = OB – OA

$= \binom{5}{-1} - \binom{2}{-2} = \binom{3}{1}$

$|AB| = \sqrt{3^2 + 1^2} = \sqrt{10}$

BC = OC – OB

$= \binom{6}{2} - \binom{5}{-1} = \binom{1}{3}$

$|BC| = \sqrt{1^2 + 3^2} = \sqrt{10}$

CD = OD – OC

$= \binom{3}{1} - \binom{6}{2} = \binom{-3}{-1}$

$|AB| = \sqrt{-3^2 + -1^2} = \sqrt{10}$

AD = OD – OA

$= \binom{3}{1} - \binom{2}{-2} = \binom{1}{3}$

$|AD| = \sqrt{1^2 + 3^2} = \sqrt{10}$

Since AD is parallel to BC and have the same magnitude; ABCD is either a square or rhombus.

For both a square and rhombus, the diagonals are perpendicular

⇨ AC.BD = 0

AC = OC – OA

$= \binom{6}{2} - \binom{2}{-2} = \binom{4}{4}$

BD = OD – OB

$= \binom{3}{1} - \binom{5}{-1} = \binom{-2}{2}$

AC.BD $= \binom{4}{4}.\binom{-2}{2} = -8 + 8 = 0$

Hence ABCD is either a square of rhombus

For a square <DAB = <ABC = 90^0

⇨ AB.AD = 0

Now $\binom{3}{1}.\binom{1}{3} = 3 + 3 = 6$

Hence ABCD is not a square.

For a rhombus <ADC =<BCD $\neq 90^0$ and <ABC= <ADC $\neq 90^0$

AD.AB = $|AD||AB|cos\theta$

6 $= \sqrt{3^2 + 1^2}\sqrt{1^2 + 3^2}cos\theta$

6 $= \sqrt{10}\sqrt{10}cos\theta$

$\theta = \cos^{-1}\frac{6}{10}$=53.13^0

CB.CD = $|AD||AB|cos\theta$

$\binom{-1}{-3}\binom{-3}{-1} = \sqrt{-1^2 + -31^2}\sqrt{-3^2 + -1^2}cos\theta$

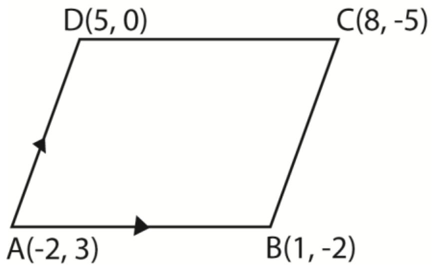
6 $= \sqrt{10}\sqrt{10}cos\theta$

$\theta = \cos^{-1}\frac{6}{10}$=53.13^0

Since <DAB =<BCD $\neq 90^0$; ABCD is a rhombus.

(c) A parallelogram ABCD has vertices A(-2, 3), B(1, -2), C(8, -5) and D(5, 0). Find the area of the parallelogram

Solution



Area of ABCD =$|ABxAD|$

AB = OB – OA

=$\binom{1}{-2} - \binom{-2}{3} = \binom{3}{-5}$

AD = OD – OA

=$\binom{5}{0} - \binom{-2}{3} = \binom{7}{-3}$

AB x AD =$\begin{pmatrix} 3 \\ -5 \\ 0 \end{pmatrix} x \begin{pmatrix} 7 \\ -3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 26 \end{pmatrix}$

$|AB\ x\ AD\ | = \sqrt{26^2} = 26$

The area of ABCD = 26 s. units.

Exercise 3

1. Given that p = 4i + 5j, q = αi - 8j and r = i+βj.
   (a) Find the value of constants α given that p and q are perpendicular. [10]
   (b) Find the value of constant β given that p and r are parallel $\left[\frac{5}{4}\right]$
2. Given that p = 6i – j, q = αi + 2j and r = 2i + βj.
   (a) Find the value of a constant α given that p and q are parallel [-12]
   (b) Find the value of constant β give that p and r are perpendicular. [12]
3. Given that$\begin{pmatrix} \alpha \\ 2+\alpha \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 3 \\ 4-\alpha \end{pmatrix}$ are perpendicular vectors, find the value of α. [18]
4. Find the possible values of the constant α, given the vectors αi + 8j+ (3α + 1)k and (α + 1)I + (α – 1)j – 2k are perpendicular. [2 or -5]
5. Given that the vectors $\begin{pmatrix} t \\ 4 \\ 2t+1 \end{pmatrix}$ and $\begin{pmatrix} t+2 \\ 1-t \\ -1 \end{pmatrix}$ are perpendicular, find the possible values of t [5 or -1]
6. Three point P, Q, and R have position vectors p = 7i + 10j; q = 3i + 12j and r –i + 4j respectively.
   (i) Write down vectors PQ and RQ and show that they are perpendicular. [-4i + 2i; 4i, 8j; PQ.RQ=0]
   (ii) Using a scalar product or otherwise find and RQ [26.6^0]
   (iii) Find the position vectors of S, the mid-point of PR. [-4i, -3j
7. The points A, B, C have position vectors

   A = 2i + j –k, b =3i +4j -2k, c = 5i –j + 2k respectively, relatively to fixed point O.

   (a) Evaluate the scalar product (a – b), (c – b). Hence calculate the size of angle ABC. [17. 40.2^0]
   (b) Given that ABCD is parallelogram
      (i) Determine the position vector of D [-4i, -2j +3k]
      (ii) Calculate the area of ABCD,[14.4]
   (c) The point E lies on BA produced so that $\overrightarrow{BE}$ = 3$\overrightarrow{BA}$. Write down the position vector of E. the line CE cuts the line AD at X; find the position vector of X. [-2i + k; $\frac{10}{3}i, \frac{7}{3}$,$\frac{5}{3}k$
8. The point A and B have position vectors i + 2j +2k and 4i +3k respectively, relative an origin O.
   (a) Find the length of OA and OB. [3, 5]
   (b) Find the scalar product of OA and OB. Hence find angle OAB. [48.2^0]
   (c) Find the area of the triangle AOB, giving your answer correct to 2 decimal places. [5.59]
   (d) The point C divides AB in ratio α: 1-α
      (i) Find an expression for OC. [(1+α)I + (2 + α)j + 2(1- α)k]
      (ii) Show that $OC^2 = 14\alpha^2 + 2\alpha + 9$
      (iii) Find the position vectors of the two point on AB whose distance from O is $\sqrt{21}$. [-2i + j +4k; $\frac{25}{7}i + \frac{29}{7}\ \ + \frac{2}{2}k$]
      (iv) Show that the perpendicular distance of O from AB is approximately 2.99

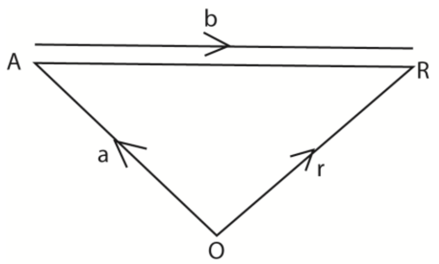
Lines

Equation of a line

An equation of line can be expressed in any of the three forms

(i) Vectors
(ii) Parametric form
(iii) Cartesian form

Finding equations of a line given one point on the line and the vector parallel to the line (direction vector)



In the figure above A is the point on the line with position vector OA = a and b is the vector parallel to the line

Taking R(x, y, z) as general point on the line AR is parallel to b

⇨ AR = λb where λ is a constant
OR – OA = λb
OR = OA + λb
⇨ r = a + λb

This is the vector equation of the line

Suppose that a= $\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ and b = $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$

Substituting for r, a and b into the equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

$x = x_0 + \lambda x_1$

$y = y_0 + \lambda y_1$

$z = z_0 + \lambda z_1$

These are parametric equation of a line

By making λ the subject from the parametric equations,

$$\frac{x-x_0}{x_1} = \frac{y-y_0}{y_1} = \frac{z-z_0}{z_1}$$

This is the Cartesian equation of the line

Different values of λ define positions of R.

If λ < 0, the point R is on the left point A

If λ = 0, the point in question is A

If λ > 0, the point R is on the right of point A

Example 17

(a) Find the vector, parametric and Cartesian equation of the line passing through the point A(1,2,3) and is parallel to the vector 2i – j + k.

Solution

The position vector of A is a = i + 2j + 3k and the parallel vector b = 2i – j +k.

Using r = a + λb

⇨ r = I + 2j +3k + λ(2i –j +k)

Or

r = $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, this a vector equation

Substituting for r = $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$x = 1 + 2\lambda$

$y = 2 - \lambda$

$y = 3 + \lambda$ this parametric equation

From these equation

$$\begin{pmatrix} x-1 \\ y-2 \\ z-3 \end{pmatrix} = \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

=> $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{1}$ is the Cartesian equation

Note

Given the Cartesian equation of a line, the point through which the line passes (1, 2,3) and the vector parallel to this line $\begin{pmatrix}-3\\2\\1\end{pmatrix}$ can be deduced easily, for example the line

$\frac{x-2}{-3} = \frac{y-1}{2} = \frac{z+4}{1}$ pases through

$(-2, 1, 4)$ and is parallel to $\begin{pmatrix}-3\\2\\1\end{pmatrix}$

Hence from the general Cartesian equation of the line, $\frac{x-x_0}{x_1} = \frac{y-y_0}{y_1} = \frac{z-z_0}{z_1}$, the arallel vector is the values of the denominator and for the vector equation, it is the coefficient of the constant.

(b) Find the Cartesian equation of the line that passes the point

(i) P(2, 0, -1) and is parallel to $\begin{pmatrix}-1\\4\\-2\end{pmatrix}$

(ii) M(3, 2,1) and is parallel to 5i + 7k.

Solution

Using general Cartesian equation

$\frac{x-x_0}{x_1} = \frac{y-y_0}{y_1} = \frac{z+1}{z_1}$

(i) $\frac{x-2}{-1} = \frac{y-0}{4} = \frac{z+1}{-2}$

$\frac{x-2}{-1} = \frac{y}{4} = \frac{z+1}{-2}$

(ii) $\frac{x-3}{5} = \frac{y-2}{0} = \frac{z-1}{7}$

$\frac{x-3}{5} = \frac{z-1}{7}$; y = 2

(c) Find the vector equation of the straight line that passes through point (2, 3) and perpendicular to the line r = 3i + 2j + λ(i – 2j).

Solution

Using r = a + λb, substituting for a = 2i + 3j and b = a_1i + b_1j, we have

r = 2i + 3j + λ(a_1i + b_1j)

since the lines are perpendicular, this means that their parallel vectors are also perpendicular

⇨ (a_1i + b_1j)(i – 2j) =0

a_1 -2b_1 = 0

a_1 = 2b_1

⇨ b = 2b_1i + b_1j

b = b_1(2i = + j)

Hence the required line will be parallel to any vector of the form λ(2i +j). So taking 2i + j as one of such vector, the required equation is r = 2i + 3j + λ(2i +j)

(a) Defining a line given two points lying on the line

Finding equations of a line given two points lying on the line

Suppose the line passes through point A and B whose position vectors are a and b.

For a general point R(x, y, z)

AR is parallel to AB

AR = λAB for any value of λ.

OR – OA = λ(OB – OA)

OR = OA + λ(OB – OA)

r = a + λ(b – a)

Example 18

(a) Find the equation of the line passing through the points A(3, 0. -2) and B(4, -2, 1)

Solution

AB = OB – OA = $\begin{pmatrix}3\\0\\-2\end{pmatrix} - \begin{pmatrix}3\\0\\-2\end{pmatrix} = \begin{pmatrix}1\\-2\\3\end{pmatrix}$

Using r = a + λAB

r = $\begin{pmatrix}3\\0\\-2\end{pmatrix} + \lambda\begin{pmatrix}1\\-2\\3\end{pmatrix}$: the vector equation

Substituting for r

$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}3\\0\\-2\end{pmatrix} + \lambda\begin{pmatrix}1\\-2\\3\end{pmatrix}$

x = 3 + λ

y = -2λ

z = -2 + 3λ parametric equation

Making λ the subject

$\frac{x-3}{1} = \frac{y}{-2} = \frac{z+2}{3}$: Cartesian equation

(b) Find the equation of the line passing through points A and B whose position vectors are i +2j – 5k and 2i – 5j +8k

Solution

AB = OB – OA =$\begin{pmatrix}2\\-5\end{pmatrix} - \begin{pmatrix}1\\2\\-5\end{pmatrix} = \begin{pmatrix}1\\-7\\13\end{pmatrix}$

Using r = a + λAB

r =$\begin{pmatrix}1\\2\\-5\end{pmatrix} + \lambda\begin{pmatrix}1\\-7\\13\end{pmatrix}$: the vector equation

Substituting for r

$\begin{pmatrix}x\\y\\z\end{pmatrix}$=$\begin{pmatrix}1\\2\\-5\end{pmatrix} + \lambda\begin{pmatrix}1\\-7\\13\end{pmatrix}$

$x = 1 + \lambda$

$y = 2 - 7\lambda$ Parametric equation

$z = -5 + 13\lambda$

Making λ the subject

$\frac{x-1}{1} = \frac{y-2}{-7} = \frac{z+5}{13}$ Cartesian equation

Note: in a situation where three points lying on a line are given such as ABC, the vector equation of the line is given by

r = a + λ(BC)

(c) Find the equation of the line passing through points A(1, 2, 5), B(2, 1, 0) and C(5, 3, 2).

Solution

BC = OC – OB= $\begin{pmatrix}5\\3\\2\end{pmatrix} - \begin{pmatrix}2\\1\\0\end{pmatrix} = \begin{pmatrix}3\\2\\2\end{pmatrix}$

r = $\begin{pmatrix}1\\2\\5\end{pmatrix} + \lambda\begin{pmatrix}3\\2\\2\end{pmatrix}$: vector equation

Substituting for r

$\begin{pmatrix}x\\y\\z\end{pmatrix}$ = $\begin{pmatrix}1\\2\\5\end{pmatrix} + \lambda\begin{pmatrix}3\\2\\2\end{pmatrix}$:

x = 1 + 3λ

y = 2 +2λ Parametric equation

z = 5 + 2λ

Making λ the subject

$\frac{x-1}{3} = \frac{y-2}{2} = \frac{z-5}{2}$ Cartesian equation

To show that a given point lies on the line

Suppose that a point (a, b, c) lies on

$\frac{x-x_0}{x_1} = \frac{y-y_0}{y_1} = \frac{z+1}{z_1}$. The when we substitute this point into the equation, we obtain a constant for the values of a, b, c.

Example 19

(a) Show that a point with coordinates (4, -1, 12) lies on the line r = 2i + 3j + 4k + λ(i – 2j + 4k)

Solution

$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\3\\4\end{pmatrix} + \lambda\begin{pmatrix}1\\-2\\4\end{pmatrix}$

Substituting (x, y, z) = (4, -1, 12)

$\begin{pmatrix}4\\-1\\12\end{pmatrix} = \begin{pmatrix}2\\3\\4\end{pmatrix} + \lambda\begin{pmatrix}1\\-2\\4\end{pmatrix}$

$\begin{pmatrix}4-2\\-1-3\\12-4\end{pmatrix} = \lambda\begin{pmatrix}1\\-2\\4\end{pmatrix}$ => $\begin{pmatrix}\lambda\\\lambda\\\lambda\end{pmatrix} = \begin{pmatrix}2\\2\\2\end{pmatrix}$

Since the value of λ is constant, the point lies on the line.

(b) Show that a point with position vector i – 9j + k lies on the line r = 3i + 3k – k + λ(i + 6j – k)

Solution

$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}3\\3\\-1\end{pmatrix} + \lambda\begin{pmatrix}1\\6\\-1\end{pmatrix}$

Substituting (x, y, z) =(1, -9, 1)

$\begin{pmatrix}1\\-9\\1\end{pmatrix} = \begin{pmatrix}3\\3\\-1\end{pmatrix} + \lambda\begin{pmatrix}1\\6\\-1\end{pmatrix}$

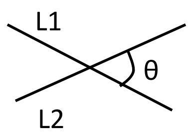
$\begin{pmatrix}1-3\\-9-3\\1+1\end{pmatrix} = \lambda\begin{pmatrix}1\\6\\-1\end{pmatrix}$=> $\begin{pmatrix}\lambda\\\lambda\\\lambda\end{pmatrix} = \begin{pmatrix}-2\\-2\\-2\end{pmatrix}$

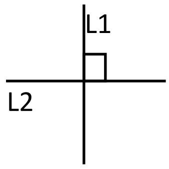
Since the value of λ is constant, the point lies on the line.

Relationship between two lines

There are three types of relationship between lines

(i) Intersection



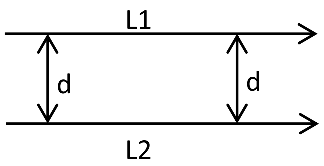


(a)intersecting at θ (b)intersecting at 90^0

If two lines r_1 and r_2 meet, then at the point of intersection, $r_1 = r_2$

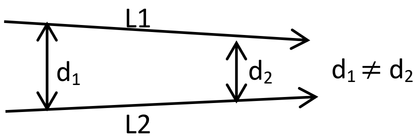
(ii) Parallel lines

These are non-intersecting lines that are equidistant from one another.



(iii) Skew lines

These are non-intersecting lines that are not equidistant from one another (not parallel)



Example 20

(a) Show that lines

$$r_1 = \begin{pmatrix}-2\\8\\-1\end{pmatrix} + \mu\begin{pmatrix}1\\3\\-2\end{pmatrix} \text{ and}$$

$$r_2 = \begin{pmatrix}8\\-1\\3\end{pmatrix} + \lambda\begin{pmatrix}-4\\1\\0\end{pmatrix} \text{ intersect and find the}$$

point of intersection.

Solution

At the point of intersection r1 = r2

$$\begin{pmatrix}-2\\8\\-1\end{pmatrix} + \mu\begin{pmatrix}1\\3\\-2\end{pmatrix} = \begin{pmatrix}8\\-1\\3\end{pmatrix} + \lambda\begin{pmatrix}-4\\1\\0\end{pmatrix}$$

$-2 + \mu = 8 - 4\lambda$ i.e. $\mu + 4\lambda = 10$ ....(i)

$8 + 3\mu = -1 + \lambda$ i.e. $3\mu - \lambda = -9$ ....(ii)

$-1 - 2\mu = 3$ i.e. $\mu = -2$

Substituting for μ in equation (i)

$\lambda = 3$

Checking with eqn. (ii) $3(-2) - 3 = -9$ i.e. there is consistency.

Using $\mu = -2$, the point of intersection is

$$\begin{pmatrix}-2\\8\\-1\end{pmatrix} - 2\begin{pmatrix}1\\3\\-2\end{pmatrix} = \begin{pmatrix}-4\\2\\5\end{pmatrix}$$

Or point of intersection is (-4, 2, 5)

(b) Show that lines

$\frac{x}{1} = \frac{y+2}{2} = \frac{z-5}{-1}$ and $\frac{x-1}{-1} = \frac{y+3}{-3} = \frac{z-4}{1}$

intersect and find the point of intersection.

Solution

$\frac{x}{1} = \frac{y+2}{2}$ i.e $2x - y = 2$ ....(i)

Also

$\frac{x-1}{-1} = \frac{y+3}{-3}$ i.e. $-3x + y = -6$ ....(ii)

Eqn. (i) + Eqn. (ii) $-x = -4$ or $x = 4$

Substituting for x in equation (i)

$2 \times 4 - y = 2$; $y = 6$

Finding the value of z

$\frac{x}{1} = \frac{z-5}{-1}$ => $-x = z - 5$

Substituting for x

$-4 = z - 5$; $z = 1$

Checking for constancy using $\frac{y+3}{-3} = \frac{z-4}{1}$

⇨ $\frac{6+3}{-3} = \frac{1-4}{1} = -3$ (consistent)

Given that $r_1 = a_1 + \lambda_1 b_1$ and $r_2 = a_2 + \lambda_2 b_2$ intersect; then the shortest distance between the lines is zero.

Or $(a_1 - a_2).(b_1 \times b_2) = 0$

(c) Show that the following lines are perpendicular

(i) $\frac{x-1}{2} = \frac{y}{1} = \frac{z-4}{4}$ and $\frac{x}{3} = \frac{y+2}{-2} = \frac{z}{-1}$

The first line is parallel to $b_1 = \begin{pmatrix}2\\1\\4\end{pmatrix}$

The 2nd line is parallel to $b_2 = \begin{pmatrix}3\\-2\\-1\end{pmatrix}$

$b_1 . b_2 = \begin{pmatrix}2\\1\\4\end{pmatrix} . \begin{pmatrix}3\\-2\\-1\end{pmatrix}$ = 6 – 2 – 4 = 0

∴ the lines are perpendicular because $b_1 . b_2 = 0$

(ii) $r_1 = i - j + \lambda_1(i + 2j - k)$ and
$r_2 = 2i + j - k + \lambda_2(-2i - 4j + 2k)$

Solution

The 1st line is parallel to $b_1 = \begin{pmatrix}1\\2\\-1\end{pmatrix}$

The 2nd line is parallel to $b_2 = \begin{pmatrix}-2\\-4\\2\end{pmatrix} = 2\begin{pmatrix}-1\\-2\\1\end{pmatrix}$

Substituting for b_1 into b_2; $b_2 = -2b_1$

Hence the two lines are parallel.

The shortest (perpendicular) distance from a given point to the line.

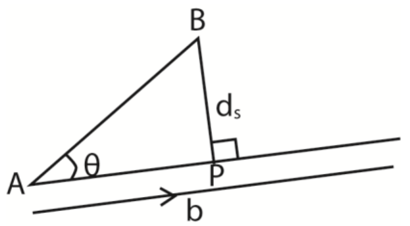
There are several methods of calculating this; here two methods are discussed in examples below

Example 21

(a) Find the perpendicular distance from point, A(1, 1, 3) to the line $\frac{x+4}{2} = \frac{y+1}{3} = \frac{z-1}{3}$.

Solution

Approach 1



In the figure above
A is a point on the line
P is a point at which the perpendicular from B meets the line
B is the vector parallel to the line
θ is the angle between AB and the line (or the parallel vector)
Required is the distance d_s, = $|Bp|$
From the figure, $|BP|$= $|AB|$sinθ .... (i)
But by definition: $|AB\ x\ b|$ =$|AB||b|$sinθ

⇨ $|AB|\sin\theta = \frac{|AB\ x\ b|}{|p|}$ ....(ii)

Combining equations (i) and (ii)

$|PB| = \frac{|AB\ x\ b|}{|b|}$

The shortest distance, $d_s = \frac{|AB\ x\ b|}{|b|}$

From the given line in question A(-4, -1, 1) and $b = \begin{pmatrix}2\\3\\3\end{pmatrix}$

$AB = OB - OA = \begin{pmatrix}1\\1\\3\end{pmatrix} - \begin{pmatrix}-4\\-1\\1\end{pmatrix} = \begin{pmatrix}5\\2\\2\end{pmatrix}$

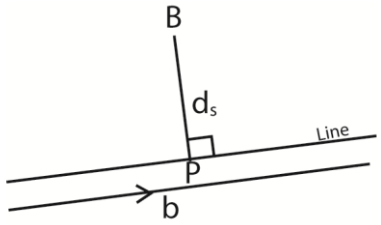
$AB\ x\ b = \begin{pmatrix}5\\2\\2\end{pmatrix} x \begin{pmatrix}2\\3\\3\end{pmatrix} = \begin{pmatrix}0\\-11\\11\end{pmatrix}$

$|AB\ x\ b| = \sqrt{[(-11)^2 + 11^2]} = 11\sqrt{2}$ and

$|b| = \sqrt{2^2 + 3^2 + 3^2} = \sqrt{22}$

$d_s = \frac{|AB\ x\ b|}{|b|} = \frac{11\sqrt{2}}{\sqrt{22}} = \sqrt{11}$ units

Approach 2



P is a point at which the perpendicular line meets the line

The vector equation of the line is

$$r = \begin{pmatrix}-4\\-1\\1\end{pmatrix} + \lambda\begin{pmatrix}2\\3\\3\end{pmatrix}$$

Let λ =λ1 at Q i.e. $p = \begin{pmatrix}-4\\-1\\1\end{pmatrix} + \lambda_1\begin{pmatrix}2\\3\\3\end{pmatrix}$

$$= \begin{pmatrix}-4+2\lambda_1\\-1+3\lambda_1\\1+3\lambda_1\end{pmatrix}$$

Since PB is perpendicular to the line, it is also perpendicular to b.

$BP.\begin{pmatrix}2\\3\\3\end{pmatrix} = 0$ i.e. $(p - B).\begin{pmatrix}2\\3\\3\end{pmatrix} = 0$

$$\left\{\begin{pmatrix}-4+2\lambda_1\\-1+3\lambda_1\\1+3\lambda_1\end{pmatrix} - \begin{pmatrix}1\\1\\3\end{pmatrix}\right\}.\begin{pmatrix}2\\3\\3\end{pmatrix} = 0$$

$-10 + 4\lambda_1 - 6 + 9\lambda_1 - 6 + 9\lambda_1 = 0$

$\lambda_1 = 1$

BP = $\begin{pmatrix} -5+2 \\ -2+3 \\ -2+3 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$

$|BP| = d_s = \sqrt{-3^2 + 1^2 + 1^2} = \sqrt{11}$ units

(b) Find the perpendicular distance from the point A(4, -3, 10) to the line L with vector equation $r = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

Solution

Method 1

The shortest distance $d_s = \frac{|AB\ x\ b|}{|b|}$

From the given line in question, B(1, 2, 3) and b = $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

BA = OA – OB

$= \begin{pmatrix} 4 \\ -3 \\ 10 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ 7 \end{pmatrix}$

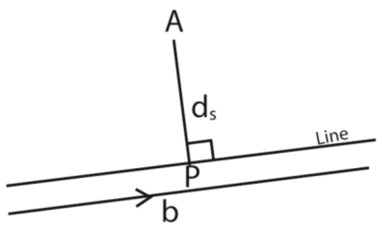
BA x b = $\begin{pmatrix} 3 \\ -5 \\ 7 \end{pmatrix} x \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 15 \\ 12 \end{pmatrix}$

$|BA| = \sqrt{-3^2 + 15^2 + 12^2} = \sqrt{378}$

$|b| = \sqrt{3^2 + -1^2 + 2^2} = \sqrt{14}$

$d_s = \frac{|AB\ x\ b|}{|b|} = \frac{\sqrt{378}}{\sqrt{14}} = \sqrt{27} = 3\sqrt{3}$

Method 2



$r = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

Let $\lambda = \lambda_1$ at P i.e. $p = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 1+3\lambda_1 \\ 2-1\lambda_1 \\ 3+2\lambda_1 \end{pmatrix}$

Since AP is perpendicular to the line, it is also perpendicular to b

AP. $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = 0$

$\left\{\begin{pmatrix} 1+3\lambda_1 \\ 2-1\lambda_1 \\ 3+2\lambda_1 \end{pmatrix} - \begin{pmatrix} 4 \\ -3 \\ 10 \end{pmatrix}\right\} . \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = 0$

$\begin{pmatrix} -3+3\lambda_1 \\ 5-1\lambda_1 \\ -7+2\lambda_1 \end{pmatrix} . \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = 0$

$-9 + 9\lambda_1 - 5 + \lambda_1 - 14 + 4\lambda_1 = 0$

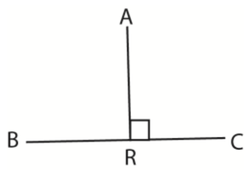
$\lambda_1 = 2$

AP = $\begin{pmatrix} -3+6 \\ 5-2 \\ -7+4 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix}$

$|AP| = \sqrt{3^2 + 3^2 + -3^2} = \sqrt{27} = 3\sqrt{3}$ units

(c) The points A, B, C have position vectors a = 3i – j +4k, b = j – 4k, c = 6i + 4j + 5k respectively. Find the position vector of the point R on BC such that AR is perpendicular to BC. Hence find the perpendicular distance of A from the line BC.

Solution



BR = λBC

OR – OB = λ(OC- OB)

OR = OB + λ(OC –OB)

r = $\begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} + \lambda \left[\begin{pmatrix} 6 \\ 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix}\right]$

$= \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ 3 \\ 9 \end{pmatrix}$

R = $\begin{pmatrix} 6\lambda \\ 1+3\lambda \\ -4+9\lambda \end{pmatrix}$

Perpendicular distance = $|AR|$

⇨ AR.BC = 0

AR = OR – OA

$= \begin{pmatrix} 6\lambda \\ 1+3\lambda \\ -4+9\lambda \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} -3+6\lambda \\ 2+3\lambda \\ -8+9\lambda \end{pmatrix}$

Hence $\begin{pmatrix} -3+6\lambda \\ 2+3\lambda \\ -8+9\lambda \end{pmatrix} . \begin{pmatrix} 6 \\ 3 \\ 9 \end{pmatrix}$

-18 + 36λ + 6 + 9λ -72 + 81λ = 0

126λ =84

$\lambda = \frac{84}{126} = \frac{2}{3}$

substituting for λ into AR = $\begin{pmatrix} -3+6\lambda \\ 2+3\lambda \\ -8+9\lambda \end{pmatrix}$ we

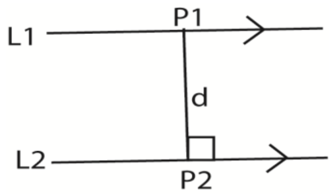
get AR = $\begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$

$|Ar| = \sqrt{1^2 + 4^2 + -2^2} = \sqrt{21}$units

Distance between two lines

Note that:

(i) If r_1 and r_2 are parallel, the distance between the two lines is the length of any line segment P_1P_2 with P_1 on r_1 and P_2 on r_2 perpendicular to both lines



The perpendicular distance, d = $|P_1P_2|$
But $P_1P_2 = OP_2 - OP_1$

⇨ $P_1P_2.b = 0$, where b is a parallel vector to the lines.
This enables us to find the value of either $\mu_1 - \mu_2$ or $\mu_2 - \mu_1$ which is substituted to find d = $|P_1P_2|$

(ii) If r_1 and r_2 are not parallel (skew lines), there are unique points P_1 on r_1 and P_2 on r_2 such that the length of the segment P_1P_2 is the shortest possible distance. The length P_1P_2 is the distance between the two lines which is the common perpendicular to both lines r_1 and r_2.
The distance, d, between skew lines $r_1 = a_1 + \mu_1 b_1$ and $r_2 = a_2 + \mu_2 b_2$ is normally taken to be the shortest distance d = $[(a_1 - a_2).\hat{n}]$, where $\hat{n} = \frac{n}{|n|}$ and n = $b_1 \times b_2$.

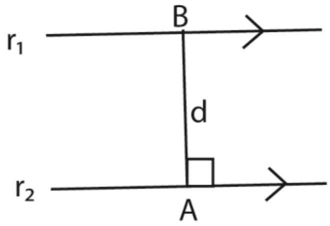
Example 22

Determine the shortest distance between the following pairs of lines

(a) $r_1 = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and

$r_2 = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

Solution



$r_1 = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

⇨ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

So we have A(2+λ, -λ, 3 +2λ)

$r_2 = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

⇨ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

Thus B(1+μ, -1 – μ, 4+2μ)

AB = OB – OA

$= \begin{pmatrix} 1+\mu \\ -1-\mu \\ 4+2\mu \end{pmatrix} - \begin{pmatrix} 2+\lambda \\ -\lambda \\ 3+2\lambda \end{pmatrix}$

$= \begin{pmatrix} -1+\mu-\lambda \\ -1-\mu+\lambda \\ 1+2\mu-2\lambda \end{pmatrix}$

$Now\ AB$.b = 0

$\begin{pmatrix} -1+\mu-\lambda \\ -1-\mu+\lambda \\ 1+2\mu-2\lambda \end{pmatrix} . \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = 0$

$-1 + \mu - \lambda + 1 + \mu - \lambda + 2 + 4\mu - 4\lambda = 0$

$\lambda - \mu = \frac{1}{3}$

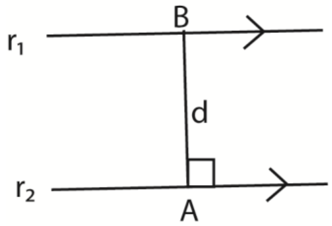
By substitution

AB = $-\frac{4}{3}i - \frac{2}{3} \ + \frac{1}{3}k$

$$d = |AB| = \sqrt{\frac{16}{9} + \frac{4}{9} + \frac{1}{9}} = \frac{\sqrt{21}}{3} units$$

(b) $\frac{x-2}{1} = \frac{y-1}{-1} = \frac{z-3}{2}$ and $\frac{x+1}{1} = \frac{y-3}{-1} = \frac{z-1}{2}$

Solution



Let $\frac{x-2}{1} = \frac{y-1}{-1} = \frac{z-3}{2} = \lambda$ and

$\frac{x+1}{1} = \frac{y-3}{-1} = \frac{z-1}{2} = \mu$

So we have

A(2+λ, 1-λ, 3+2λ) and B(-1-μ, 3+μ, 1+2μ)

AB = OB – OA

=

$$\begin{pmatrix}-1+\mu\\3-\mu\\1+2\mu\end{pmatrix} - \begin{pmatrix}2+\lambda\\1-\lambda\\3+2\lambda\end{pmatrix} = \begin{pmatrix}-3+\mu-\lambda\\2-\mu+\lambda\\-2+2\mu-2\lambda\end{pmatrix}$$

Now AB.b = 0

$$\begin{pmatrix}-3+\mu-\lambda\\2-\mu+\lambda\\-2+2\mu-2\lambda\end{pmatrix}.\begin{pmatrix}1\\-1\\2\end{pmatrix} = 0$$

$$-3+\mu-\lambda-2+\mu-\lambda-4+4\mu-4\lambda = 0$$

$$\mu - \lambda = \frac{9}{6} = \frac{3}{2}$$

By substitution;

AB = $-\frac{3}{2}i - \frac{1}{2} \quad + k$

d = $|AB| = \sqrt{\frac{9}{4} + \frac{1}{4} + 1} = \frac{\sqrt{14}}{2}$ units

(c) $r_1 = \begin{pmatrix}1\\-1\\4\end{pmatrix} + \mu\begin{pmatrix}0\\2\\-2\end{pmatrix}$ and

$r_2 = \begin{pmatrix}2\\3\\2\end{pmatrix} + \mu\begin{pmatrix}-1\\-1\\1\end{pmatrix}$

Solution

$$a_1 - a_2 = \begin{pmatrix}1\\-1\\4\end{pmatrix} - \begin{pmatrix}2\\3\\2\end{pmatrix} = \begin{pmatrix}-1\\-4\\2\end{pmatrix} \text{ and}$$

$$b_1 x b_2 = \begin{pmatrix}0\\2\\-2\end{pmatrix} x \begin{pmatrix}-1\\-1\\1\end{pmatrix} = \begin{pmatrix}4\\2\\2\end{pmatrix}$$

$$|n| = \sqrt{4^2+2^2+2^2} = \sqrt{24}$$

$$\hat{n} = \frac{1}{\sqrt{24}}\begin{pmatrix}4\\2\\2\end{pmatrix}$$

$$(a_1 - a_2).\hat{n} = \frac{1}{\sqrt{24}}\begin{pmatrix}-1\\-4\\2\end{pmatrix}.\begin{pmatrix}4\\2\\2\end{pmatrix}$$

$$= \frac{1}{\sqrt{24}}(-4-8+4) = -\frac{8}{\sqrt{24}}$$

$$d = |(a_1 - a_2).\hat{n}| = \left|-\frac{8}{\sqrt{24}}\right| = \frac{\sqrt{24}}{3}$$

∴ The distance apart is $\frac{\sqrt{24}}{3}$ units

(d) $r_1 = \begin{pmatrix}1\\2\\-1\end{pmatrix} + \mu\begin{pmatrix}1\\0\\1\end{pmatrix}$ and

$r_2 = \begin{pmatrix}2\\-1\\0\end{pmatrix} + \mu\begin{pmatrix}1\\1\\2\end{pmatrix}$

Solution

$$a_1 - a_2 = \begin{pmatrix}1\\2\\-1\end{pmatrix} - \begin{pmatrix}2\\-1\\0\end{pmatrix} = \begin{pmatrix}-1\\3\\-1\end{pmatrix} \text{ and}$$

$$b_1 x b_2 = \begin{pmatrix}1\\0\\1\end{pmatrix} x \begin{pmatrix}1\\1\\2\end{pmatrix} = \begin{pmatrix}-1\\-1\\1\end{pmatrix}$$

$$|n| = \sqrt{-1^2 + -1^2 + 1^2} = \sqrt{3}$$

$$\hat{n} = \frac{1}{\sqrt{3}}(-i - \quad + k)$$

$$(a_1 - a_2).\hat{n} = \frac{1}{\sqrt{3}}\begin{pmatrix}-1\\3\\-1\end{pmatrix}.\begin{pmatrix}-1\\-1\\1\end{pmatrix}$$

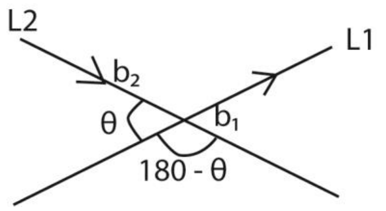
$$= \frac{1}{\sqrt{3}}(1-3-1) = -\frac{3}{\sqrt{3}}$$

$$d = |(a_1 - a_2).\hat{n}| = \left|-\frac{3}{\sqrt{3}}\right| = \sqrt{3}$$

∴ The distance apart is $\sqrt{3}$ units

Angle between two lines

The angle between two lines is equivalent to the angle between their parallel vectors.



It the illustration above, there are two angles: θ and 180 – θ i.e. is obtuse.

Example 23

(a) Determine the acute angle between each of the pairs of the lines

(i) r_1 = 2i + j – k + λ(2i + 3j + 6k) and
r_2 = i + 2j -3k + μ(2i – 2j + k)

Solution

r_1 is parallel to b_1 = 2i + 3j + 6k and

r_2 is parallel to b_2 =2i – 2j + k

Using $b_1b_2 = |b_1||b_2|cos\theta$

(2i + 3j + 6k).(2i – 2j + k)

$=\left(\sqrt{2^2+3^2+6^2}.\sqrt{2^2+-2^2+1^2}\right)cos\theta$

4 – 6 + 6 = (7)(3)cosθ

$\cos^{-1}\left(\frac{4}{21}\right) = 79^0$

(ii) $\frac{x-2}{-4} = \frac{y-3}{3} = \frac{z+1}{-1}$ and $\frac{x-3}{2} = \frac{y-1}{6} = \frac{z+1}{-5}$

Solution

r_1 is parallel to $b_1 = \begin{pmatrix}-4\\3\\-1\end{pmatrix}$ and

r_2 is parallel to $b_2 = \begin{pmatrix}2\\6\\-5\end{pmatrix}$

Using $b_1b_2 = |b_1||b_2|cos\theta$

$\begin{pmatrix}-4\\3\\-1\end{pmatrix}.\begin{pmatrix}2\\6\\-5\end{pmatrix}$

$=\left(\sqrt{-4^2+3^2+-1^2}.\sqrt{2^2+6^2+-5^2}\right)cos\theta$

-8+18+5 = $(\sqrt{26})(\sqrt{65})$cosθ

$\cos^{-1}\left(\frac{15}{\sqrt{1690}}\right) = 68.6^0$

In general, angle θ between the lines

$r_1 = a_1 + \mu b_1$ and $r_2 = a_2 + \lambda b_2$ is

$\theta = arcos\left(\frac{b_1b_2}{|b_1||b_2|}\right)$

(b) Given the equation of two lines are $y = m_1x + c_1$ and $y = m_2x + c_2$. Show that

(i) Their vector equations are respectively $\begin{pmatrix}0\\c_1\end{pmatrix} + \mu\begin{pmatrix}1\\m_1\end{pmatrix}$ and $\begin{pmatrix}0\\c_2\end{pmatrix} + \mu\begin{pmatrix}1\\m_2\end{pmatrix}$ where μ and λ are constant

Solution

⇨ $\frac{x}{1} = \frac{y-c_1}{m_1}$ and $\frac{x}{1} = \frac{y-c_2}{m_2}$

i.e. $\begin{pmatrix}0\\c_1\end{pmatrix} + \mu\begin{pmatrix}1\\m_1\end{pmatrix}$ and $\begin{pmatrix}0\\c_2\end{pmatrix} + \mu\begin{pmatrix}1\\m_2\end{pmatrix}$

(ii) The angle, θ, between them is $\tan^{-1}\left(\frac{m_1-m_2}{1+m_1m_2}\right)$

Solution

$\begin{pmatrix}1\\m_1\end{pmatrix}.\begin{pmatrix}1\\m_2\end{pmatrix}$

$= \sqrt{(1+m_1{}^2)}.\sqrt{(1+m_2{}^2)}$ cosθ

$(1 + m_1m_2)^2=(1 + m_1^2 + m_2^2 + m_1^2m_2^2)\cos^2\theta$

$\sec^2\theta = 1 + \tan^2\theta = \frac{1+m_1{}^2+m_2{}^2+m_1{}^2m_2{}^2}{(1+m_1m_2)^2}$

$\tan^2\theta = \frac{1+m_1{}^2+m_2{}^2+m_1{}^2m_2{}^2}{(1+m_1m_2)^2} - 1 = \frac{(m_1-m_2)^2}{(1+m_1m_2)^2}$

$\tan\theta = \frac{m_1-m_2}{1+m_1m_2}$

$\theta = \tan^{-1}\left(\frac{m_1-m_2}{1+m_1m_2}\right)$

Exercise 4

1. Find the vector equation for the line passing through
   (a) (4, 3) and is parallel to vector i – 2j
   $\left[r = \binom{4}{3} + \lambda\binom{1}{-2}\right]$
   (b) (5, -1, 3) and parallel to vector 4i – 3j +k
   $\left[r = \begin{pmatrix}5\\-1\\3\end{pmatrix} + \lambda\begin{pmatrix}4\\-3\\1\end{pmatrix}\right]$
2. Find a vector equation for the line joining the following point
   (a) (2, 6) and (5, -2) $\left[r = \binom{2}{6} + \mu\binom{3}{-8}\right]$
   (b) (-1, 2, -3) and(6, 3, 0)
   $\left[r = \begin{pmatrix}-1\\2\\-3\end{pmatrix} + \mu\begin{pmatrix}7\\1\\1\end{pmatrix}\right]$
3. (a) Point A and B have coordinates (4, 1) and (2, -5) respectively. Find the vector equation fot the line which passes through the point A and perpendicular to point AB
   $\left[r = \binom{4}{1} + \mu\binom{3}{-1}\right]$
   (b) Point P and Q have coordinates (3, 5) and (-3, -7) respectively. Find vector equation for the line which passes through the point P which is perpendicular to PQ
   $\left[r = \binom{3}{5} + \mu\binom{2}{-1}\right]$
4. Find a vector equation for perpendicular bisector of the points
   (a) (6, 3) and (2, -5) $\left[r = \binom{4}{-1} + \mu\binom{2}{-1}\right]$
   (b) (7, -1) and(3, -3) $\left[r = \binom{3}{5} + \mu\binom{2}{-1}\right]$
5. Points P, Q and R have position vectors 4i – 4j, 2i + 2j and 8i + 6j respectively.

(a) Find a vector equation for L_1 which is perpendicular bisector to points P and Q. $\left[L_1 = \binom{3}{-1} + \mu\binom{3}{1}\right]$

(b) Find a vector equation for L_2 which is perpendicular bisector to points P and Q. $\left[L_2 = \binom{5}{4} + \mu\binom{2}{-3}\right]$

(c) Hence find the position vector of the point $\left[\frac{59}{11}, \frac{4}{11}\right]$

6. Two lines L_1 and L_2 have equations

L_1: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -3 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$ and

L_2: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

(a) Show that L1 and L2 are concurrent (meet at a common point) and find the position vector of their point of intersection. [2i + 5j + 9k]

(b) Find the angle between L_1 and L_2. [15.6^0]

7. Points P, Q, and R have coordinates (-1, 1), (4, 6) and (7, 3) respectively.

(a) Show that the perpendicular distance from the point R to the line PQ is $3\sqrt{2}$.

(b) Deduce the area of the triangle PQR is 15 sq. units

8. Point A, B and C have position vectors –i + 3j + 5k, 5i 6j – 4k and 4i + 7j + 5k respectively. P is the point ON AB such that AP = λAB. Find

(a) AB

(b) CP

(c) The perpendicular distance from the point C to the line AB [$3\sqrt{3}$] {m v

9. Two lines L_1 and L_2 have vector equation

$r_1 = (2 - 3\lambda)i + (1 + \lambda)j + 4\lambda k$ and

$r_2 = (-1 + 3\mu)i + 3j + (4 - \mu)k$. Find

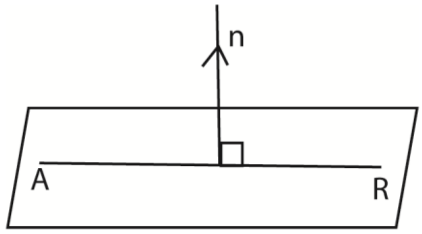
(a) The position vector of their common point of intersection.[r = -4i + 3j +8k]

(b) The angle between the lines [143.7^0]

The Plane

Equation of a plane

(I) Determining the equation of the plane given a vector perpendicular to the plane and one point contained in the plane.



In the figure, A is a point in the plane

n is the perpendicular vector to the plane and R is the general point in the plane

Since n is perpendicular to AR, then

n.AR = 0

i.e. n.(r – a) = 0 ....(i)

Let n = $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, A($x_1$, y_1, z_1) and R (x, y, z)

Substituting these into equation (i)

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}.\begin{pmatrix} x - x_1 \\ y - y_1 \\ z - z_1 \end{pmatrix} = 0$$

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

$ax + by + cz - (ax_1 + by_1 + cz_1) = 0$

let $d = ax_1 + by_1 + cz_1$

⇨ ax + by +cz = d

Hence the Cartesian equation of the plane is ax + by +cz = d where d is a constant and the coefficient of x, y and z form the perpendicular or normal vector.

Note: the above equation may be written as

$$\begin{pmatrix} x \\ y \\ z. \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = d \; i.e. r.n = d$$

This is called the scalar product of the vector equation of the plane

Example 24

(a) Find the vector normal to the plane 3x – 2y + z = 7

Solution

The coefficient of x, y and z is 3i – 2j +k

Hence the normal vector is $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$

(b) Find the equation of the plane that is normal to 5i – j +2k and passes through A(4, 1, -3)

Solution

Either:

$$\begin{pmatrix} x-4 \\ y-1 \\ z+3 \end{pmatrix}.\begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} = 0$$

5x – 20 – y + 1 + 2z + 6 = 0

5x – y + 2z =13

Or:

Using the general equation ax + by + cz = d

5x – y + 2z = (5x 4)- 1 +(2 x -3)

5x – y + 2z = 13

(c) Find the equation of the plane that is normal to 4i + 6j + 5k and passes through the point with position vector i + 3j + k.

Solution

Either:

$$\begin{pmatrix} x-1 \\ y-3 \\ z-1 \end{pmatrix}.\begin{pmatrix} 4 \\ 6 \\ 5 \end{pmatrix} = 0$$

4x – 4 + 6y - 18 + 5z - 5 = 0

4x 6 y + 5z =27

Or:

Using the general equation ax + by + cz = d

4x + 6y + 5z = (4x 1)+ 6 x 3 +(5 x 1)

4x 6 y + 5z =27

(II) Determining the equation of the plane given three non-collinear points.

Several methods are employed including the four outlined in the following example.

Example 25

(a) Find the equation of the plane containing the points A(1, 1, 1), B(5, 0, 0) and C(3, 2, 1)

Solution

Method 1

Let the equation of the plane be

ax + by + cz = d

As the three points lie in the same plane, their coordinates satisfy the above equation

Substituting for A, B and C coordinates in the general equation

For A(1, 1, 1): a + b + c = d ……………………(i)

For B(5, 0, 0): 5a = d ………………………… (ii)

For C(3, 2, 1): 3a + 2b + c …………………… (iii)

Solving for a, b, and c in terms of d:

From Eqn. (ii) a = $\frac{1}{5}d$

Substituting for a into eqn. (i)

$\frac{1}{5}d + b + c = d$ => b + c = $\frac{4}{5}d$ ………… (iv)

Substituting for a into eqn. (iii)

$3\left(\frac{1}{5}d\right) + 2b + c = d$ => 2b + c = $\frac{2}{5}d$……..(v)

Eqn. (v) –eqn. (iv): b = $-\frac{2}{5}d$

Substituting b into eqn. (iv)

$-\frac{2}{5}d + c = \frac{4}{5}d$ => c = $\frac{6}{5}d$

Substituting a, b, and c into the equation

ax + by + cz = d

⇨ $\frac{1}{5}dx - \frac{2}{5}dy + \frac{6}{5}dz = d$

Multiplying through by $\frac{5}{d}$

x – 2y + 6c = 5 is the equation of the plane

Method 2

One of the normal vectors of the plane is

AB.AC = 0

Where AB = OB – OA = $\begin{pmatrix}5\\0\\0\end{pmatrix} - \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}4\\-1\\-1\end{pmatrix}$

And AC = OC – OA = $\begin{pmatrix}3\\2\\1\end{pmatrix} - \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}2\\1\\0\end{pmatrix}$

Now, AB x AC = $\begin{pmatrix}4\\-1\\-1\end{pmatrix} x \begin{pmatrix}2\\1\\0\end{pmatrix} = \begin{pmatrix}1\\-2\\6\end{pmatrix}$

If R(x, y, z) is the general point in the plane, then AR is normal to AB x AC.

(r-a).(AB x AC) = 0

$\begin{pmatrix}x-1\\y-1\\z-1\end{pmatrix} . \begin{pmatrix}1\\-2\\6\end{pmatrix} = 0$

x – 1 -2y + 2 + 6z – 6 = 0

x – 2y + 6z = 5

Method 3

Let R be the general point in the plane

The AR = μAB + λAC for scalars μ and λ.

r – a = μAB + λAC

r = a + μAB + λAC

$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \mu\begin{pmatrix}4\\-1\\-1\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\0\end{pmatrix}$

Equating the coefficients of x, y and z

x = 1 + 4μ + λ .... (i)

y = 1 – μ + λ ....(ii)

z = 1 – μ ....(iii)

From eqn. (iii): μ = 1 – z

Substituting for μ

λ = y -1 + (1-z) = y - z

Substituting μ and λ in equation (i)

x = 1 + 4(1-z) + 2(y – z)

= 1 + 4 -4z + 2y-2z

x – 2y + 6z = 5

Note that:

- If the plane passes through the origin, then its equation is r = μb + λc
- The plane r = a + μb + λc passes through point a with position vector a and is parallel to b and c.
- If the vectors a, b and c are coplanar, then the sum of the coefficients of a, b and c must be zero.

Method 4

This involves finding the determinant of a 3 x 3 matrix. Taking A as the principal point, we have

AB x AC = $\begin{pmatrix}x-1 & y-1 & z-1\\4 & -1 & -1\\2 & 1 & 0\end{pmatrix} = 0$

⇨ (x – 1)1 – (y -1)2 + (z – 1)6 = 0
x – 1 – 2y + 2 + 6z – 6 = 0
x – 2y + 6z = 5

(d) Find the equation of the plane containg the points A(1, 2, 5), B(1, 0, 4) and C(5, 2,1)

Solution

Using the determinant method

AB = OB – OA

$= \begin{pmatrix}1\\0\\4\end{pmatrix} - \begin{pmatrix}1\\2\\5\end{pmatrix} = \begin{pmatrix}0\\-2\\-1\end{pmatrix}$

AC = OC - OA

$= \begin{pmatrix}5\\2\\1\end{pmatrix} - \begin{pmatrix}1\\2\\5\end{pmatrix} = \begin{pmatrix}4\\0\\4\end{pmatrix}$

AB x AC = $\begin{pmatrix}x-1 & y-2 & z-5\\0 & -3 & -1\\4 & 0 & -4\end{pmatrix} = 0$

⇨ (x – 1)8 – (y -2)4 + (z – 5)8 = 0

(x – 1)2 – (y -2) + (z – 5)2 = 0

2x –y + 2z = 10

(III) **Determining the equation of the plane given one point and a line in the plane.**

Here more points are obtained from the equation and the problem worked out as in (III).

Example 26

(a) Find the equation of the plane through the point (1, 0, 1) and containing the line $\frac{x}{1} = \frac{y}{-1} = \frac{z}{2}$.

Solution

The vector equation for the line is

r = $\begin{pmatrix}0\\0\\0\end{pmatrix}$+μ$\begin{pmatrix}1\\-1\\2\end{pmatrix}$

Let 1st given point be P(1, 0, -1):

Taking μ = 0: the 2nd point is O(0, 0, 0)

Taking μ = 1; the 3rd point Q(1, -1, 2)

Thus OP = $\begin{pmatrix}1\\0\\1\end{pmatrix}$ and OQ = $\begin{pmatrix}1\\-1\\2\end{pmatrix}$

The normal vector is

Op x OQ= $\begin{pmatrix}1\\0\\1\end{pmatrix} x \begin{pmatrix}1\\-1\\2\end{pmatrix} = \begin{pmatrix}-1\\-3\\-1\end{pmatrix}$

The equation of the plane is

$\begin{pmatrix}x-0\\y-0\\z-0\end{pmatrix}.\begin{pmatrix}-1\\-3\\-1\end{pmatrix} = 0$

x + 3y + z = 0

Using determinant method

Op x OQ= $\begin{pmatrix}x & y & z\\1 & 0 & -1\\1 & -1 & 2\end{pmatrix}$

x(-1) – y(3) + z(-1) = 0

x + 3y + z = 0

x + 3y + z = 0

(IV) **Determining the equation of the plane given two lines in the plane.**

This can be tackled in two ways

Example 27

Find the equation of the plane containing the lines

$\frac{x-3}{5} = \frac{y+1}{2} = \frac{z-3}{1}$ and

$\frac{x-3}{2} = \frac{y+1}{4} = \frac{z-3}{3}$

Solution

Method 1

The corresponding vector equations of the above lines are as follows

$r_1 = \begin{pmatrix}3\\-1\\3\end{pmatrix} + \mu\begin{pmatrix}5\\2\\1\end{pmatrix}$ and

$r_2 = \begin{pmatrix}3\\-1\\3\end{pmatrix} + \mu\begin{pmatrix}2\\4\\3\end{pmatrix}$ respectively

Taking μ = 0, the 1st point is A(3, -1, 3)

Taking μ = 1, the 2nd point is B(8, 1, 4)

Taking λ = 1, the 3rd point is C(5, 3, 6)

So with three points obtained, the above methods can be used.

Now AB = OB – OA =$\begin{pmatrix}8\\1\\4\end{pmatrix} - \begin{pmatrix}3\\-1\\3\end{pmatrix} = \begin{pmatrix}5\\2\\1\end{pmatrix}$

And AC = OC – OA =$\begin{pmatrix}5\\3\\6\end{pmatrix} - \begin{pmatrix}3\\-1\\3\end{pmatrix} = \begin{pmatrix}2\\4\\3\end{pmatrix}$

The normal vector

n = AB x AC= $\begin{pmatrix}5\\2\\1\end{pmatrix} x \begin{pmatrix}2\\4\\3\end{pmatrix} = \begin{pmatrix}2\\-13\\16\end{pmatrix}$

Taking the point (3, -1, 3) which lies on the 1st line: the equation of the plane is

$\begin{pmatrix}x-3\\y+1\\z-3\end{pmatrix}.\begin{pmatrix}2\\-13\\16\end{pmatrix} = 0$

2x – 6 – 13y – 13 + 16z – 48 = 0

2x - 13y + 16z = 67

Method 2

The parallel vectors of the given lines are

$\begin{pmatrix}5\\2\\1\end{pmatrix}$ and $\begin{pmatrix}2\\4\\3\end{pmatrix}$ respectively.

The normal vector, n $\begin{pmatrix}5\\2\\1\end{pmatrix} x \begin{pmatrix}2\\4\\3\end{pmatrix} = \begin{pmatrix}2\\-13\\16\end{pmatrix}$

Taking the point (3, -1, 3) as before

$\begin{pmatrix}x-3\\y+1\\z-3\end{pmatrix}.\begin{pmatrix}2\\-13\\16\end{pmatrix} = 0$

2x - 13y + 16z = 67

(V) **Determining the equation of the plane given one point in the plane and a perpendicular line.**

Example 28

Find the equation of the plane perpendicular to the line $\frac{x-1}{2} = \frac{y}{-1} = \frac{z}{3}$ and passing through point B(1, -3, 2)

Solution

The parallel vector to the line is 2i – j + 3k

This means that this vector is also perpendicular to the plane

The equation of the plane is

$$\begin{pmatrix} x-1 \\ y+3 \\ z-2 \end{pmatrix} . \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = 0$$

2x – 2 – y – 3 + 3z - 6 = 0

2x – y + 3z = 11

(VI) **Determining the equation of the plane given two points in the plane.**

Example 29

Find the equation of the plane containing the points A(1, 2, -1) and B(4, -3, 2)

Solution

a = $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and b = $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

The normal vector is

a x b= $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} x \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -6 \\ -11 \end{pmatrix}$

the equation of the line is thus

$$\begin{pmatrix} x-1 \\ y-2 \\ z+1 \end{pmatrix} . \begin{pmatrix} 1 \\ -6 \\ -11 \end{pmatrix} = 0$$

x – 6y – 11z = 0

(VII) **Determining the equation of the plane given two parallel lines.**

Example 30

Find the equation of the plane passing through (1, 0, -1) and parallel to the line

$r_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ and

$$r_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

Solution

The normal vector,

n = b_1 x b_2 = $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} x \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -6 \\ 7 \end{pmatrix}$

Equation of the plane

-x – 6y +7z = -1(1) – 6(0) + 7(-1)

x + 6y – 7z = 8

(VIII) **Determining the equation of the plane given a line in the plane and a parallel vector.**

Example 31

Find the equation of the plane containing

$r = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ and parallel to $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Solution

The normal vector

n = b_1 x b_2 = $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} x \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -7 \\ -1 \\ 3 \end{pmatrix}$

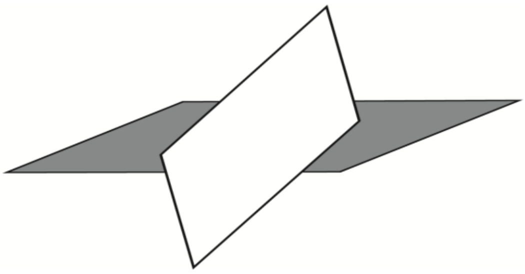
Equation of the plane is

-7x – y + 3z = -7(1) – 2 + 3(1) = -6

7x + y – 3z = 6

Intersection of two planes

Two plane always on a line



Solving the two equations of the lines simultaneously gives equation of this line

Example 32

Find the Cartesian equation of the lines of intersection of the following planes

(a) 3x – 5y + z = 8 and 2x – 3y +z = 3

Solution

Method 1

Note: solving for three unknown from two equations is quite hard, so we express then in terms of a constant say λ

Let 3x – 5y + z = 8 .... (i)

and 2x – 3y +z = 3 .... (ii)

Eqn. (i) – eqn. (ii)

x – 2y = 5

Let x = λ => λ - 26 = 5 i.e. $y = \frac{1}{2}(\lambda - 5)$

Substituting for x and y into eqn. (ii)

$2\lambda - \frac{3}{2}(\lambda - 5) + z = 3 \Rightarrow z = \frac{1}{2}(-9 - \lambda)$

So x = λ, $y = \frac{1}{2}(\lambda - 5), z = \frac{1}{2}(-9 - \lambda)$

To eliminate fractions let λ = 1 + 2μ

x= 1 + 2μ, y = -2 + μ, z = -5 – μ

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$, the vector equation

$\frac{x-1}{2} = \frac{y+2}{1} = \frac{z+5}{-1}$ Cartesian equation

Method 2

The parallel vector

b = n_1 x n_2 = $\begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix} x \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$

$= -\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

The equation of the line is r = a +$\lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

From the equation 3x – 5y + z and 2x – 3y + z = 3, subtracting

⇨ x – 2y = 5

when x = 1, 1 – 2y = 5 i.e. y = -2

substituting in the first equation

3(1) – 5(-2) + z = 8 i.e. z = -5

r = $\begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix}$ +$\lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ vector equation

OR

Substituting for r

$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ = $\begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix}$ +$\lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

$\frac{x-1}{2} = \frac{y+2}{1} = \frac{z+5}{-1}$ Cartesian equation

(b) 3x + 4y + 2z = 3 and 2x – 3y – z = 1

Let 3x + 4y + 2z = 3 ……………(i)

and 2x – 3y – z = 1 …………… (ii)

2en. (i) – eqn. (ii)

17y + 7z = 3

Let y = λ, 17λ + 7z = 3 => z = $\frac{3-17\lambda}{7}$

Substituting for y and z into eqn. (i)

3x + 4λ + $\frac{2}{7}(3 - 17\lambda)$=3 => x = $\frac{1}{7}(5 + 2\lambda)$

x = $\frac{1}{7}(5 + 2\lambda)$, y = λ, z = $\frac{3-17\lambda}{7}$

Let λ = 1+ 7μ (to eliminate fractions)

Then x = 1 +2μ, y = 1+7μ and z = -2 - 17μ

The Cartesian equation is

$\frac{x-1}{2} = \frac{y-1}{7} = \frac{z+2}{-17}$

Method 2

Parallel vector

b = n_1 x n_2 = $\begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} x \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \\ -17 \end{pmatrix}$

The equation of the line is r = a +$\lambda \begin{pmatrix} 2 \\ 7 \\ -17 \end{pmatrix}$

Let 3x + 4y + 2z = 3 ……………(i)

and 2x – 3y – z = 1 …………… (ii)

2en. (i) – eqn. (ii)

17y + 7z = 3

Let y = 1, 17 + 7z = 3 => z = $\frac{3-17}{7} = -2$

Substituting for y and z into eqn. (i)

3x + 4 + 2(-2) = 3 =>x = 1

$r = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 7 \\ -17 \end{pmatrix}$ vector equation

Or

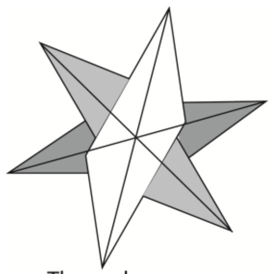
Substituting for r

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 7 \\ -17 \end{pmatrix}$

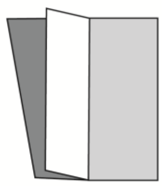
$\frac{x-1}{2} = \frac{y-1}{7} = \frac{z+2}{-17}$, Cartesian equation

Intersection of three planes

Three planes may intersect at a point or on a line (if they meet)



Three planes meeting at a point



Three planes meeting on aline (book pages)

(I) Intersection of three planes at point

Example 33

Find the point of intersection of the three planes

x + 2y – z = 2, 3x – y + z = 3 and 2x + y -3z =3

Solution

Form simultaneous equation

x + 2y – z = 2………….(i)

3x – y + z = 3 …………(ii)

2x + y -3z =3 ………….(iiI)

Solving simultaneously the point of intersection is (1, 1, 1)

(II) Intersection of three planes at point

If a plane and a line meet, they do so at a particular point.

Example 33

(a) Find the point where the line $\frac{x-3}{-1} = \frac{y-1}{2} = \frac{z+3}{4}$ meets the plane
3x – y + 2Z = 8
Solution
Expressing the equation of the line in parametric form
Let $\frac{x-3}{-1} = \frac{y-1}{2} = \frac{z+3}{4} = \lambda$
Then x = 3 – λ, y = 1+2λ and z = -3 + 4λ
Substituting for parametric equations into the equation of the plane
3(3 – λ) - (1+2λ) 2(-3 + 4λ) = 8 => λ = 2
Substituting for λ into parametric equations
x = 3 – 2=1, y = 1+2(2) = 5 and
z = -3 + 4(2) =5
Hence the point of intersection (x, y, z) is (1, 5, 5)

(b) Fins the position vector of a point where the line $r = \begin{pmatrix}2\\-1\\3\end{pmatrix} + \lambda\begin{pmatrix}5\\3\\2\end{pmatrix}$meets the plane r.$\begin{pmatrix}1\\2\\-1\end{pmatrix} = 15$
Solution
$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\-1\\3\end{pmatrix} + \lambda\begin{pmatrix}5\\3\\2\end{pmatrix}$$
The parametric equations of the line are
x =2 + 5λ, y = -1 + 3λ and z = 3 + 2λ
The equation of the plane is x + 2y –z = 15
Substituting parametric equations into the equation of the plane
2 + 5λ + (-1 + 3λ) – (3 + 2λ) = 15
λ = 2
Substituting λ into parametric equations
x = 2 + 5(2) = 12
y = (-1 + 3(2))=5
z = (3 + 2(2)) = 7
Hence the point of intersection (x, y, z) is (12, 5, 7)

Perpendicular distance from a point to the plane

A. Perpendicular distance from the origin to the plane

Rewriting the equation **r.n = d** in the form
r. $\hat{\boldsymbol{n}}$ **=** $\mathbf{d_1}$ where $\hat{\boldsymbol{n}}$ is the unit normal to the plane

Or

By using the general formula, the perpendicular distance $d_p$from a plane ax + by + cz + d = 0 to the point (x_1, y_1, z_1) is given by the expression

$$d_p = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$$

Example 34

(a) Find the distance from the origin to the plane 4x + 8y – z = 18
Solution

The normal vector n =$\begin{pmatrix}4\\8\\-1\end{pmatrix}$ and

$|n| = \sqrt{4^2 + 8^2 + (-1)^2}$=9

Now $\hat{n} = \frac{1}{9}\begin{pmatrix}4\\8\\-1\end{pmatrix}$

⇨ $r.\frac{1}{9}\begin{pmatrix}4\\8\\-1\end{pmatrix} = \frac{18}{9} = 2$

Or

By using the general formula, rewrite the equation of the plane as

4x + 8y – z – 18 = 0, a = 4, b = 8, c = -1 and d= -18

At the origin (x_1, y_1, z_1)= (0, 0, 0)

$$d_p = \frac{|4(0)+8(0)-1(0)-18|}{\sqrt{4^2+8^2+(-1)^2}} = \frac{18}{9} = 2units$$

(b) Find the perpendicular distance from the origin to the plane r.(2i – 14j + 5k) = 10

Solution

The normal vector n $=\begin{pmatrix}2\\-14\\5\end{pmatrix}$ and

$|n| = \sqrt{2^2+(-14)^2+5^2}$=15

Now $\hat{n} = \frac{1}{15}\begin{pmatrix}2\\-14\\5\end{pmatrix}$

⇨ $r.\frac{1}{15}\begin{pmatrix}2\\-14\\5\end{pmatrix} = \frac{10}{15} = \frac{2}{3}$

Or

By using the general formula, rewrite the equation of the plane as

2x – 14y + 5z – 10=0, a = 2, b = -14, c = 5 and d= -10

At the origin (x_1, y_1, z_1)= (0, 0, 0)

$$d_p = \frac{|2(0)-14(0)+5(0)-10|}{\sqrt{2^2+(-14)^2+(-1)^2}} = \frac{10}{15} = \frac{2}{3}units$$

B. **Perpendicular distance for a given point rather than origin to a plane**

Several methods are employed

Example 35

(a) Determine the distance from the lane 12x – 3y – 4z = 39 to the point (5, 3, 1)

Solution

Method1

The perpendicular distance $d_p$from a plane ax + by + cz + d = 0 to the point (x_1, y_1, z_1) is given by the expression

$$d_p = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$$

By substitution

$$d_p = \frac{|12(5)-3(3)-4(1)+39|}{\sqrt{12^2+(-3)^2+(-4)^2}} = 2units$$

Note that the equation of the plane should be rewritten in the form f(x, y, z) = 0 before applying the formula.

Method 2

The parallel plane containing the point given is obtained and the absolute difference of the resulting length of the plane from the origin computed.

Equation of the plane: 12x – 3y – 4z = 39 ..(i)

Equation of parallel plane 12x – 3y – 4z =D for any constant D.

Since this parallel contains the point (5, -3, 1): 12(5)-3(-3)-4(1) = 65 = D

The parallel plane: 12x – 3y – 4z =65 ……(ii)

In both planes, the normal vector

n = 12i – 3j - 4k

$|n| = \sqrt{12^2+(-3)^2+(-4)^2}$ = 13

Dividing equation by 13:

$\frac{12}{13}x - \frac{3}{13}y - \frac{4}{13}z = \frac{65}{13} = 5$

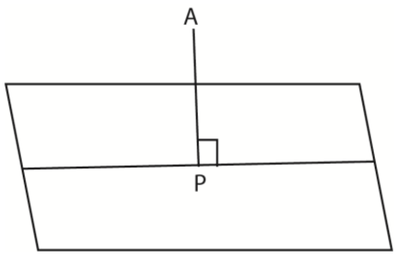
The distance between two planes is

$|5 - 3| = 2$

∴the distance from point (5, -3, 1) to the plane 12x – 3y – 4z = 39 is 2 units.

Method 3

AP= λn (AP is parallel to n)



Given the equation of the plane

12x – 3y – 4z = 39

Let n = $\begin{pmatrix}12\\-3\\-4\end{pmatrix}$ and A(5, -3, 1)

Substitute in AP = λn

p – a = λn

$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}5\\-3\\1\end{pmatrix} + \lambda\begin{pmatrix}12\\-3\\-4\end{pmatrix} = \begin{pmatrix}5-12\lambda\\-3-3\lambda\\1-4\lambda\end{pmatrix}$

$x = 5 - 12\lambda$, y = -3 - 3λ, z = 1 - 4λ

Substitute these in the equation of the plane

12x – 3y – 4z = 39

12(5 - 12λ) – 3(-3 - 3λ) – 4(1 - 4λ) = 39

λ = $-\frac{2}{13}$

AP = $-\frac{2}{13}\begin{pmatrix}12\\-3\\-4\end{pmatrix}$

$|AP| = \frac{2}{13}\sqrt{12^2+(-3)^2+(-4)^2}$= units

Angle between two planes

The angle say θ between the planes $r.n_1 = d_1$ and $r.n_2 = d_2$ is the angle between the normal vectors of the two planes. This is given by

$$\theta = \cos^{-1}\left(\frac{n_1.n_2}{|n_1||n_2|}\right)$$

Example 36

Determine the angle between the planes

4x + 3y + 12z = 10 and 4x – 3y = 7

Solution

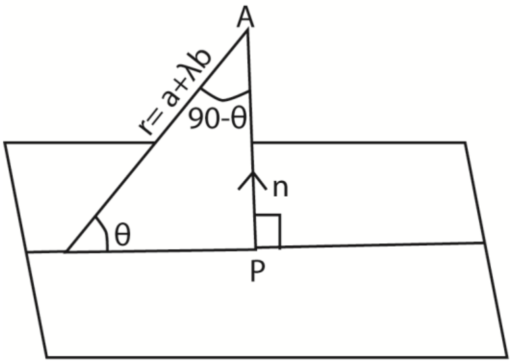
The normal $n_1 = \begin{pmatrix}4\\3\\12\end{pmatrix}$ and $n_2 = \begin{pmatrix}4\\-3\\0\end{pmatrix}$ respectively.

$$\begin{pmatrix}4\\3\\12\end{pmatrix}.\begin{pmatrix}4\\-3\\0\end{pmatrix}=\sqrt{4^2+3^2+12^2}\sqrt{4^2+-3^2}\cos\theta$$

$$16-9=\sqrt{169}\sqrt{25}\cos\theta$$

$$\theta = \cos^{-1}\left(\frac{7}{65}\right) = 83.8^0$$

Angle between a line and a plane



The angle between a line and a plane is the angle between the normal vector to the plane and the parallel vector to the line.

Given a line r = a + λb and the plane r.n = d, the angle θ between them can be computed from the dot product of vectors as

$$b.n = |b||n|\cos(90-\theta)$$

$$=|b||n|\sin\theta$$

$$\theta = \sin^{-1}\theta\left(\frac{b.n}{|b||n|}\right)$$

Example 37

(a) Find the acute angle between the line $\frac{x+1}{4} = \frac{y-2}{1} = \frac{z-3}{-1}$ and the plane 3x – 5y + 4z =5

Solution

The line is parallel to $b = \begin{pmatrix}4\\1\\-1\end{pmatrix}$ and the normal vector to the plane is $n = \begin{pmatrix}3\\-5\\4\end{pmatrix}$

$$\begin{pmatrix}4\\1\\-1\end{pmatrix}.\begin{pmatrix}3\\-5\\4\end{pmatrix}$$

$$=\sqrt{4^2+1^2+-1^2}\sqrt{3^2+-5^2+4^2}\sin\theta$$

$$3=\sqrt{900}\sin\theta$$

$$\theta = \sin^{-1}\frac{3}{30} = 5.7^0$$

(b) Find the angle between the line $r = \begin{pmatrix}-4\\2\\-1\end{pmatrix} + \lambda\begin{pmatrix}2\\-4\end{pmatrix}$ and the plane 4x + 3y – 3z = -1

Solution

The line is parallel to $b = \begin{pmatrix}2\\-4\end{pmatrix}$ and the normal vector to the plane is $n = \begin{pmatrix}4\\3\\-3\end{pmatrix}$

$$\begin{pmatrix}2\\-4\end{pmatrix}.\begin{pmatrix}4\\3\\-3\end{pmatrix}$$

$$=\sqrt{\ ^2+2^2+-4^2}\sqrt{4^2+3^2+-3^2}\sin\theta$$

$$50=\sqrt{2856}\sin\theta$$

$$\theta = \sin^{-1}\frac{50}{\sqrt{2856}} = 69.3^0$$

Exercise 5

1. Find the equation of the plane containing points P(1, 1, 1), Q(1, 2, 0) and R(-1, 2, 1). $[x + 2y + 2z = 5]$
2. Find the equation of the plane containing point (4, -2, 3) and parallel to the plane 3x – 7z = 12

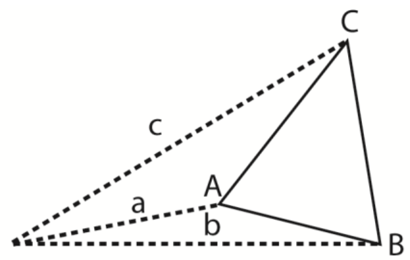
$[3x - 7z = -9]$

3. Show that the point with position vector 7i – 5j – 4k lies in the plane r = 4i + 3j + 2k + λ(i – j –k) +μ(2i + 3j + k). Find the point at which the line x = y – 1=2z intersects the plane 4x – y + 3z = 8 [(2, 3, 1)]
4. Find the parametric equation for the line through the point (0, 1, 2) that is parallel to the plane x + y + z = 2 and perpendicular to the line x = 1 + t, y = 1 – t, z = 2t. $[x = 3t, y = 1 - t, z = 2 - 2t]$
5. Find the distance between parallel planes z= x + 2y + 1 and 3x + 6y – 3z = 4 $\left[\frac{7\sqrt{6}}{18}\right]$
6. Two planes are given by their parametric equation: x = r + s, y = 3s, z = 2r and x = 1+ r + s, y = 2 + r, z = -3 + s. Find the Cartesian equation of the in intersection point. $[6x - 2y - 3z = 0]$
7. The equation of a plane P is given by r.$\begin{pmatrix} 2 \\ 6 \\ 9 \end{pmatrix} = 33$, where r is position vector of P
   Find the perpendicular distance from the origin to the plane$[3\ units]$
8. The line through point (1, -2, 3) and parallel to the line $\frac{x}{3} = \frac{y+1}{-1} = z + 1$ meets the lane x + 2y + 2z = 8 at Q. Find the coordinates of Q. $\left[\left(6, \frac{-11}{3}, \frac{14}{3}\right)\right]$
9. (a) Find the angle between the plane x + 4y – z = 72 and the line r = 9i + 6j + 8k. $[34.5^0]$
10. Obtain the equation of the plane that passes through (1, -2, 2) and perpendicular to the line $\frac{x-9}{4} = \frac{y-6}{-1} = \frac{z-8}{1}$ $[4x - y + z = 8]$
11. Find the parametric equations of the line of intersection of the planes x + y + z = 4 and x – y + 2z + 2= 0
    $[x = 3 + t, y = 2t, z = 1 + 3t]$
12. Find the points of intersection of the three planes 2x – y + 3z = 4, 3x – 2y + 6z = 4 and 7x – 4y + 5z = 11. $[(5, 6, 0)]$
13. Find the Cartesian equation of the plane with parametric vector equation
    r = $\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
    $[x + 2y - 3z = 0]$
14. Find the Cartesian equation of the plane containing the position vector $\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ and parallel to the vectors $\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$.
    $[3y\ \ z = 10]$
15. Find the Cartesian equation of the plane containing the points with position vectors $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}$
    $[3x + 2y + z = 6]$
16. Find the perpendicular distance from the plane r.(2i – 14j +5k) = 10 to the origin $\left[\frac{2}{3}\right]$
17. Find the position vector of the point where the line $r = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$ meets the plane r.$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 15$ $\left[\begin{pmatrix} 12 \\ 5 \\ 5 \end{pmatrix}\right]$
18. Two line have vector equations
    $r = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$and
    $r = \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$. Find the position vector of the point of intersection of the two lines and the Cartesian equation of the plane containing the two lines.
    $\left[\begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix}, 5x - y + 3z = 19\right]$

Example 38 (mixed questions)

1. The position vector of point A is 2i + 3j + k, of B is 5j + 4k and of C is i + 2j + 12k. Show that ABC is a triangle.

Solution



a = 2i + 3k + k

b = 5j + 4k

c = i+2j + 12k

Two conditions must b fulfilled:

1st condition

For a triangle to be, $\overline{AB} + \overline{BC} + \overline{CA} = 0$

$\overline{AB} + \overline{BC} + \overline{CA}$ = 0

= (OB – OA) + (OC –OB) + (OA – OC)

= $\begin{pmatrix}0\\5\\4\end{pmatrix} - \begin{pmatrix}2\\3\\1\end{pmatrix} + \begin{pmatrix}1\\2\\12\end{pmatrix} - \begin{pmatrix}0\\5\\4\end{pmatrix} + \begin{pmatrix}2\\3\\1\end{pmatrix} - \begin{pmatrix}1\\2\\12\end{pmatrix}$

= $\begin{pmatrix}-2\\2\\3\end{pmatrix} + \begin{pmatrix}1\\-3\\8\end{pmatrix} + \begin{pmatrix}1\\1\\-11\end{pmatrix}$ = $\begin{pmatrix}0\\0\\0\end{pmatrix}$

Second condition

We work out for any angle and if it is not 0^0 or 180^0, then we conclude that ABC is a triangle

Now finding angle A

From dot product of vectors

AB.AC = $|AB||AC|cosA$

cosA = $\frac{AB.AC}{|AB||AC|}$

AB.AC = (-2i + 2j +3k).(-I –j + 11k)

= 2 – 2 + 33 = 33

$|AB| = \sqrt{(-2)^2 + 2^2 + 3^2}$

$= \sqrt{4+4+9} = \sqrt{17}$

$|AC| = \sqrt{(-1)^2 + 1^2 + 11^2}$

$= \sqrt{1+1+121} = \sqrt{123}$

A = $\cos^{-1}\left(\frac{33}{\sqrt{17 x 123}}\right)$ = 43.8^0

Since A is not 0^0 or 180^0, hence ABC is a triangle

NB. The above two conditions **must** be clearly shown in order for the candidate to get all the marks.

2. (a) Find the point of intersection of the lines $\frac{x-3}{4} = \frac{y-7}{4} = \frac{z+3}{-5}$ and $\frac{x-8}{7} = \frac{y-4}{1} = \frac{z-5}{3}$

Solution

Let $\frac{x-3}{4} = \frac{y-7}{4} = \frac{z+3}{-5}$ = μ

⇨ $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}5\\7\\-3\end{pmatrix} + \mu\begin{pmatrix}4\\4\\-5\end{pmatrix}$ ………………… (i)

And

$\frac{x-8}{7} = \frac{y-4}{1} = \frac{z-5}{3}$ = λ

⇨ $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}8\\4\\5\end{pmatrix} + \lambda\begin{pmatrix}7\\1\\3\end{pmatrix}$ …………………… (ii)

Equating eqn. (i) and eqn. (ii)

$\begin{pmatrix}5\\7\\-3\end{pmatrix} + \mu\begin{pmatrix}4\\4\\-5\end{pmatrix} = \begin{pmatrix}8\\4\\5\end{pmatrix} + \lambda\begin{pmatrix}7\\1\\3\end{pmatrix}$

Equating corresponding unit vectors

5 + 4μ = 8 + 7λ

4μ - 7λ = 3……………………………… (iii)

7 + 4μ = 4 + λ

4μ - λ = -3…………………………… (iv)

Eqn. (iii) – eqn.(iv)

-6λ = 6

λ = -1

Substituting λ in eqn. (ii)

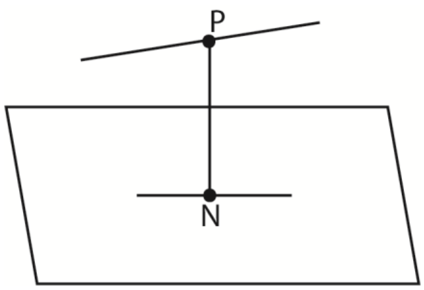
$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}8\\4\\5\end{pmatrix} + -1\begin{pmatrix}7\\1\\3\end{pmatrix}$

$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}8-7\\4-1\\5-3\end{pmatrix} = \begin{pmatrix}1\\3\\2\end{pmatrix}$

∴ (x, y, z) = (1, 3,2)

(b) The equations of a line and a plane are $\frac{x-2}{1} = \frac{y-2}{2} = \frac{z+3}{2}$ and 2x + y + 4z = 9 respectively. P is a point on the line where x = 3, N is the foot of the perpendicular from P to the plane. Find the coordinates of N.

Solution



Line equation

$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\2\end{pmatrix}$

$x = 2 + \lambda$

When x = 3

3 = 2 + λ; λ = 1

⇨ OP = $\begin{pmatrix}2\\2\\3\end{pmatrix} + \begin{pmatrix}1\\2\\2\end{pmatrix} = \begin{pmatrix}3\\4\\5\end{pmatrix}$

∴ P(3, 4, 5)

Plane equation: 2x + y + 4z = 9

$r\begin{pmatrix}2\\1\\4\end{pmatrix} = 9$

$\therefore n = \begin{pmatrix}2\\1\\4\end{pmatrix}$

NP = n

NP = OP – ON

ON = OP – NP

= $\begin{pmatrix}3\\4\\5\end{pmatrix} - \begin{pmatrix}2\\1\\4\end{pmatrix} = \begin{pmatrix}1\\3\\1\end{pmatrix}$

∴ N(1, 3, 1)

3. (a) Find the Cartesian equation of the plane through the points whose position vectors are 2i + 2j + 3k, 3i + j + 2k and -2j + 4k. (06marks)

Solution

Method 1

Let OA = 2i + 2j + 3k

OB = 3i + j + 2k

OC = -2j + 4k

Let R be the general point in the plane

Then AR = μ(AB) + λAC

OR = OA + μ(OB – OA) + λ(OC – OA)

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\2\\3\end{pmatrix} + \mu\left[\begin{pmatrix}3\\1\\2\end{pmatrix} - \begin{pmatrix}2\\2\\3\end{pmatrix} + \lambda\left[\begin{pmatrix}0\\-2\\4\end{pmatrix} - \begin{pmatrix}2\\2\\3\end{pmatrix}\right]\right]$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\2\\3\end{pmatrix} + \mu\begin{pmatrix}1\\-1\\-1\end{pmatrix} + \lambda\begin{pmatrix}-2\\-4\\1\end{pmatrix}$$

$x = 2 + \mu - 2\lambda$ .... (i)

$y = 2 - \mu - 4\lambda$ .... (ii)

$z = 3 - \mu + \lambda$ ....(iii)

$Eqn\ (i) + eqn.(ii)$

$(x + y) = 4 - 6\lambda$ ....(iv)

$Eqn.(i) and\ eqn.(iii)$

x + z = 5 – λ

λ = -x –z + 5

Substituting for λ into eqn. (iv)

x + y =4 -6(-x –z + 5)

5x –y + 6z -26 = 0

Method 2

Let the equation of the plane be

ax + by + cz = d

Substituting point (2, 2, 3) in equation

2a + 2b +3c = d .... (i)

Substituting point (3, 1, 2) in equation

3a + b + 2c = d .... (ii)

Substituting point (0, -2, 4) in equation

-2b + 4c =d .... (iii)

We have to solve for a, b, c and d

3Eqn.(i) – 2Eqn. (ii)

6a + 6b + 9c = 3d

- 6a + 2b + 4c = 2d

4b + 5c = d.............................(iv)

2eqn. (iii) + eqn. (iv)

-4b + 8c = 2d

+4b + 4c = d

13c = 3d

c = $\frac{3}{13}$d

From eqn. (iv)

4b + $\frac{15}{13}d = d$; 4b = d -$\frac{15}{13}d = \frac{-1}{26}d$

From eqn. (i)

2a -$\frac{2}{26}d + \frac{9}{13}d$ =d

2a = d +$\frac{2}{26}d$ -$\frac{9}{13}d = \frac{10}{26}d$

a = $\frac{5}{26}d$

Substituting for a, b, c in the equation of the plane

$\frac{5}{26}dx - \frac{1}{26}dy + \frac{3}{13}d = d$

$Multiplying\ through\ by\ \frac{26}{d}$

$5x - y + 6z = 26$

(b) Determine the angle between the plane in (a) and the line $\frac{x-2}{2} = \frac{y}{-4} = z - 5$. (06marks)

Let n = normal vector to the plane

b = parallel vector to the plane

⇨ b = 2i – 4j + k

n =5i –j + 6k

Let θ = angle between the line and the plane b.n = $|b||n|\sin\theta$

$\begin{pmatrix}2\\-4\\1\end{pmatrix}.\begin{pmatrix}5\\-1\\6\end{pmatrix} = (\sqrt{2^2+(-4)^2+1^2}.$

$\sqrt{5^2+(-1)^2+6^2}$)sinθ

$10+4+6 = (\sqrt{21}.\sqrt{62})sin\theta$

$= \sqrt{1302}sin\theta$

$sin\theta = \frac{20}{\sqrt{1302}}$; θ = 33.66^0 (2D)

4. Three points A(2, -1, 0), B(-2, 5, -4) and C are on a straight line such that 3AB = 2AC. Find the coordinates of C.

Solution

Method 1

3(AB) = 2AC

$\frac{3}{2}AB = AC$

$\frac{3}{2}(OB - OA) = OC - OA$

$\frac{3}{2}\left[\begin{pmatrix}-2\\5\\-4\end{pmatrix} - \begin{pmatrix}2\\-1\\0\end{pmatrix}\right] = \begin{pmatrix}x\\y\\z\end{pmatrix} - \begin{pmatrix}2\\-1\\0\end{pmatrix}$

$\frac{3}{2}\begin{pmatrix}-4\\6\\-4\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix} - \begin{pmatrix}2\\-1\\0\end{pmatrix}$

$\begin{pmatrix}-6\\9\\-6\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix} - \begin{pmatrix}2\\-1\\0\end{pmatrix}$

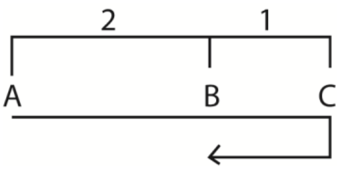
$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-6\\9\\-6\end{pmatrix} + \begin{pmatrix}2\\-1\\0\end{pmatrix}$

$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-4\\8\\-6\end{pmatrix}$

Hence coordinates of C are (-4, 8, -6)

Method 2

Using ratio theorem



C divides externally in the ratio 3: -1

OC = $\frac{3(OB)-1(OA)}{3+(-1)}$

OC = $\frac{1}{2}\left\{3\begin{pmatrix}-2\\5\\-4\end{pmatrix} - \begin{pmatrix}2\\-1\\0\end{pmatrix}\right\}$

$=\frac{1}{2}\begin{pmatrix}-8\\16\\12\end{pmatrix} = \begin{pmatrix}-4\\8\\-6\end{pmatrix}$

Hence C(-4, 8, -6)

Method 3

B divides AC internally in ration of 2:1

$\begin{pmatrix}-2\\5\\-4\end{pmatrix} = \frac{1}{3}\left\{2\begin{pmatrix}x\\y\\z\end{pmatrix} + \begin{pmatrix}2\\-1\\0\end{pmatrix}\right\}$

$3\begin{pmatrix}-2\\5\\-4\end{pmatrix} = 2\begin{pmatrix}x\\y\\z\end{pmatrix} + \begin{pmatrix}2\\-1\\0\end{pmatrix}$

$\begin{pmatrix}x\\y\\z\end{pmatrix} = \frac{1}{2}\left\{\begin{pmatrix}-6\\15\\12\end{pmatrix} - \begin{pmatrix}2\\-1\\0\end{pmatrix}\right\}$

$\begin{pmatrix}x\\y\\z\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-8\\16\\-12\end{pmatrix} = \begin{pmatrix}-4\\8\\-6\end{pmatrix}$

Hence C(-4, 8, -6)

5. (a) Line A is the intersection of two planes whose equations are 3x –y + z = 2 and x + 5y +2z = 6. Find the equation of the line.

3x –y + z = 2 .... (i)

x + 5y +2z = 6 .... (ii)

5eqn. (i) + eqn. (ii)

$$\begin{array}{r} 15x - 5y + 5z = 10 \\ +\quad x + 5y + 2z = 6 \\ \hline 16x + 7z = 16 \end{array}$$

Let x = λ

16λ + 7z = 16

z = $\frac{1}{7}(16-16\lambda)$

Substituting for x and z in equation (i)

3λ – y + $\frac{1}{7}(16-16\lambda)$ = 2

21λ – 7y + 16 - 16λ = 14

y = $\frac{1}{7}(2+5\lambda)$

let λ = 1 + 7μ

=> x = 1 + 7μ

y = $\frac{1}{7}(2+5(1+7\mu))$

= $\frac{1}{7}(2+5+35\mu)$

= 1 + 5μ

z = $\frac{1}{7}(16-16(1+7\mu))$

= $\frac{1}{7}(16-16-16\,x7\mu))$

= -16μ

$\underline{r} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \mu\begin{pmatrix}7\\5\\-16\end{pmatrix}$

$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \mu\begin{pmatrix}7\\5\\-16\end{pmatrix}$

$\frac{x-1}{7} = \frac{y-1}{5} = \frac{-z}{16}$

(b) Given that line B is perpendicular to the plane 3x –y + z = 2and passes through the point C(1, 1, 0), find the

(i) Cartesian equation of line B

Solution

Normal to the plane b = 3i –j + k

r = a + λb

$$= \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}3\\-1\\1\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}3\\-1\\1\end{pmatrix}$$

$$\frac{x-1}{3} = \frac{y-1}{-1} = \frac{z}{1}$$

(ii) angle between line B and line A in (a) above

Solution

Let b_1 = 7i + 5j – 16k and b_2 = 3i –j + k

and θ = angle between line A and line B

$b_1.b_2 = |b_1||b_2|cos\theta$

$b_1.b_2$ = (7i + 5j – 16k).(3i –j + k)

= 21 -5 – 16 = 0

$|b_1||b_2|cos\theta = 0$

cosθ = 0

$\theta = \cos^{-1}0 = 90^0$

6. (a) The points A and B have position vectors a and b. A point C with vector position c lies on AB such that $\frac{AC}{AB} = \lambda$.

Show that c = (1 – λ)a + λb. (04marks)

Solution

$\frac{\overline{AC}}{\overline{AB}} = \lambda$

$\overline{AC} = \lambda\overline{AB}$

$\overline{OC} - \overline{OA} = \lambda(\overline{OB} - \overline{OA}$

c – a = λ(b – a)

c = a + λ(b – a)

= (1 – λ)a + λb

(b) The vector equation of two lines are;

r_1 = 2i + j + λ(i + j + 2k) and

r_2 = 2i+ 2j + tk + $\mu(i + 2j + k)$

where i, j and k are unit vectors and λ, μ and t are constants. Given that the two lines intersect, find

(i) the value of t.

x = 2 + λ = 2+ μ ……………… (i)

y = 1 + λ= 2 + 2μ ……………..(ii)

z = 2λ = t + λ …………………… (iii)

From eqn. (i)

2 + λ = 2+ μ

λ = μ

From eqn. (ii)

1 + λ= 2 + 2μ

1 + μ = 2 + 2μ

μ = λ = -1

From eqn. (iii)

2λ = t + λ

2(-1) = t – 1

t = -1

(ii) the coordinates of the point of intersection. (08marks)

x =2 + λ = 2 -1 = 1

y = 1 + λ = 1 -1 = 0

z = 2λ = 2(-1) = -2

∴ (x, y, z) = (1, 0, -2)

7. Determine the angle between the lines $\frac{x+4}{8} = \frac{y-2}{2} = \frac{z+1}{-4}$ and the plane 4x + 3y – 3z + 1 = 0 (05marks)

$$\begin{pmatrix}8\\2\\-4\end{pmatrix}.\begin{pmatrix}4\\3\\-3\end{pmatrix} =$$

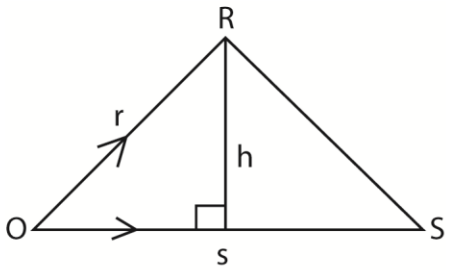
$$\sqrt{8^2 + 2^2 + (-4)^2}.\sqrt{4^2 + 3^2 + (-3)^2}\sin\theta$$

32 + 6 + 12 = $\sqrt{84 \times 34}$sinθ

sin θ = $\frac{50}{\sqrt{2856}}$

θ = 69.33^0

8. The position vectors of the vertices of a triangle are O, r and s, where O is the origin. Show that its area (A) is given by $4A^2 = |r|^2|s|^2 - (r.s)^2$. (06marks)



r.s = $|r||s|\cos\theta$

$(r.s)^2 = |r|^2|s|^2cos^2\theta$

$sin^2\theta = 1 - \frac{(r.s)^2}{|r|^2|s|^2} = \frac{|r|^2|s|^2 - (r.s)^2}{|r|^2|s|^2}$

$A = \frac{1}{2}|r||s|sin\theta$

2A = $|r||s|sin\theta$

$4A^2 = |r|^2|s|^2sin^2\theta$

$4A^2 = |r|^2|s|^2.\frac{|r|^2|s|^2-(r.s)^2}{|r|^2|s|^2}$

$4A^2 = |r|^2|s|^2 - (r.s)^2$

Hence, find the area of a triangle when

r = $\begin{pmatrix}2\\3\end{pmatrix}$ and s = $\begin{pmatrix}1\\4\end{pmatrix}$ (06marks)

$|r|^2 = 2^2 + 3^2 = 13$

$|s|^2 = 1^2 + 4^2 = 17$

$r.s = \begin{pmatrix}2\\3\end{pmatrix}.\begin{pmatrix}1\\4\end{pmatrix} = (2\ x\ 1) + (3\ x\ 4) = 14$

$\therefore 4A^2 = 13 \text{ x } 17 - 14^2 = 25$

$A = \sqrt{\frac{25}{4}}$ = 2.5units

9. Given the plane 4x + 3y – 3z-4 = 0
   (a) Show that the point A(1,1,1) lies on the plane (02marks)

   Solution

   Substitute A(1, 1, 1) into the equation of plane

   4 x 1 + 3 x 1 – 3 x 1 – 4 = 0

   Hence the point lies on the plane

   (b) Find the perpendicular distance from the plane to the point B(1, 5,1) (03marks)

   d = $\frac{|4\ x\ 1+3\ x\ 5-3\ x\ 1-4|}{\sqrt{4^2+3^2+(-3)^2}}$ = 2.058

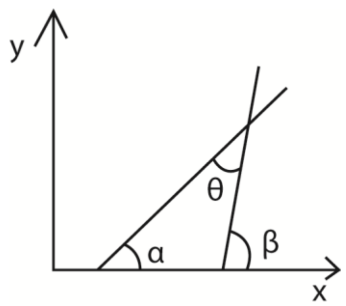
10. (a) Determine the perpendicular distance of the point (4, 6) from the line 2x + 4y -3 = 0 (03marks)

    Solution

    Perpendicular distance, d = $\frac{[2(4)+4(6)-3]}{\sqrt{2^2+4^2}}$ = $\frac{29}{\sqrt{20}}$ = 64846

    (b) Show that the angle θ, between two lines with gradient λ_1 and λ_2 is given by θ = $\tan^{-1}\left(\frac{\lambda_1-\lambda_2}{1+\lambda_1\lambda_2}\right)$.Hence find the acute angle between the lines x + y + 7 = 0 and $\sqrt{3x} - y + 5 = 0$ (09 marks)

    Solution



Tanα = λ_2, tanβ = λ_1

α + θ = β; θ = β – α

tanθ = tan (β – α)

$= \frac{\tan\beta-\tan\alpha}{1+\tan\alpha\tan\beta}$

$= \left(\frac{\lambda_1-\lambda_2}{1+\lambda_1\lambda_2}\right)$

$\therefore$ θ $= \tan^{-1}\left(\frac{\lambda_1-\lambda_2}{1+\lambda_1\lambda_2}\right)$

But λ_1 = -1 and $\lambda_2 = \sqrt{3}$

θ $= \tan^{-1}\left(\frac{-1-\sqrt{3}}{1+(-1\ x\ \sqrt{3})}\right) = 75^0$

Topical revision questions

1. (a) The position vectors of points A, B and C are 2i –j +5k, i – 2j +k and 3i +j – 2k respectively. Give that L and M are the midpoint of AC and CB. Show that LM is parallel to BA.

   (b) Show that the points with position vectors 4i – 8j – 13k, 5i – 2j – 3k and 5i + 4j + 10k are vertices of a triangle
2. (a) The position vector of a body of mass 12 .5kg is $8t^2$i + 6tj meters at a given time t. determine the
   (i) Velocity after 4s.$[64i + 6j]$
   (ii) The force acting on the body [200N horizontally]

   (b) The vector OA is represented by displacement vector a and OB by b. Point R divides AB in the ration λ:μ. Find the position vector of R in terms of vectors a and b and the scalars λ and μ. $\left[r = \frac{\mu}{\mu+\lambda}a + \frac{\lambda}{\mu+\lambda}b\right]$

   (c) If the points P, Q and R have position vector p, q and r respectively, and M is the midpoint of QR, show that the position vector of N is a point on PM that PN:NM =2:1 is $\frac{1}{3}(p+q+r)$.
3. (a) Determine a unit vector perpendicular to the plane containing the points A(0, 2, -4), B(2, 0, 2) and C(-8, 4, 0) $\sqrt{230}$

   (b) Find the equation of the plane $[5x + 14y + 3z = 16]$

   (c) Show that the point (5, -4, 3) lies on the plane [does on lie on the line]

   (d) Write down the equation in form of r = a + μb of the perpendicular through the point P(3, 4, 2) to the plane $[r = 3i + 4j + 2k + \mu(4i + 14j + 3k)]$

(e) If the perpendicular meats the plane at N. determine NP [4.022units]

4. (a) A and B are points whose position vectors are a = 2i + k and b = i – j + 3k respectively. Determine the position vector of the point P that divides AB in the ratio 4:1 $\left[\frac{1}{5}(6i - 4j + 16k)\right]$

(b) Given that a = i – 3j +3k and b = -i -3j + 2k determine

(i) The equation of the plane containing a and b $[-3x + 5y + 6z = 0]$

(ii) The angle the line $\frac{x-4}{4} = \frac{y}{3} = \frac{z-1}{2}$ makes with the plane in (i) above. $[19.446^0]$

5. A vector XY of magnitude a units makes an angle of α with the horizontal. Another vector YZ beginning from the end Y, inclined at an angle β to the same horizontal axis is of magnitude b units. If θ is the angle between the positive directions of the two vectors, where θ = β – α is acute, show that the resultant vector XZ has a magnitude xz equal to $\sqrt{a^2 + b^2 + 2abcos\theta}$ units and is inclined at an angle $\alpha + \sin^{-1}\left(\frac{bsin\theta}{xz}\right)$ to the horizontal. Hence or otherwise calculate the magnitude and direction of the resultant vector of vectors XY and YZ, inclined at 30^0 and 75^0 to the horizontal and magnitude 9 and 6 units respectively. $[47.7^0]$

6. (a) The position vector of points A and B with respect to the origin O are $\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} and \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$ respectively. Determine the equation of the line AB $\left[r = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}\right]$

(b) Find the equation of the plane OPQ where O is the origin and P and Q are points whose position vectors are $\begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ respectively $\left[r.\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = 0 \; or -2x + z = o\right]$

(c) (i) Given that R is a point at which AB meets the plane OPQ, find the coordinates of R $[(7, -7, 14)]$

(ii) Show that the point S(1, -1, 2) lies on OR.

7. The points A, B, and C have position vectors (-2i + 3j), (i – 2j), and (8i -5j) respectively.

(a) Find the vector equation of line AC $\left[r = \binom{-2}{3} + \lambda\binom{10}{-8}\right]$

(b) Determine the coordinates of D if ABCD is a parallelogram [5, 0]

(c) Write down the vector equation of the line through which point B perpendicular to AC and find where it meets AC $\left[\frac{93}{41}, -\frac{17}{41}\right]$

8. (a) In the triangle ABC, P is the point on BC such that BP : PC = λ:μ.
Show that (λ+μ)AP = λAC + μAB

(b) Three non collinear points A, B, and C have position vectors a, b, and c respectively with respect to O. The point M on AC is such that AM:MC = 2:1 and the point N on AB is such that AN:NB = 2:1.

(i) Show that BM = $\frac{1}{3}a - b + \frac{2}{3}c$, and find a similar expression for CN

$$\left[CN = \frac{1}{3}a + \frac{2}{3}b - c\right]$$

(ii) The line BM and CN intersect at L. Given that BL = rBM and CL = sCN, where r and s are scalars, express BL and CL in terms of r, s, a, b, and c.

$$\left[\begin{matrix} BL = \frac{1}{3}sa - rb + \frac{2}{3}rc; \\ CL = \frac{1}{3}sa - \frac{2}{3}sb - sc; \end{matrix}\right]$$

(iii) Hence by using triangle BLC or otherwise find r and s $\left[r = \frac{2}{5}, s = \frac{3}{5}\right]$

9. Find the distance of the point (-2, 0, 6) from the plane 2x – y + 3z = 21 [1.8708 units]

10. ABCD is a quadrilateral with A(2, -2), B(5, -1), C(6, 2) and D(3, 1). Show that the quadrilateral is a rhombus.

11. The points P(4, -6, 1), Q(2, 8, 4)and R(3, 7, 14) lie in the same plane. Find the angle between PQ and QR. $[84.5^0]$

12. (a) Given that OP = $\begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix}$ and OQ = $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, find the coordinates of the point R such that PR:PQ = 1:2 and the points P, Q and R are collinear. [R(2.5, -1.5, 3.5)]
   (b) Show that the vector 5i – 2j +k is perpendicular to the line r = i – 4j + λ(2i + 3j -4k).
   (c) Find the equation of the plane through the point with position vector 5i – 2j +3k perpendicular to the vector 3i + 4j – k. [3x + 4y –z = 4]
13. Calculate the area of a triangle with vertices (-1, 3), (5, 2), (4, -1) [7.6811 sq. units]
14. PQRS is a quadrilateral with vertices P(1, -2), Q(4, -1), R(5,2) and S(2, 1). [show that PQ is parallel to SR and PS is parallel to QR and that $|PQ| = |SP| = |QR| = |PS|$ and PR and QS are perpendicular]
15. The vector equation of lines P and Q are given as $r_p = t(4i + 3j)$and $r_q = 2i + 12 \ + 5(i - j)$ Use the dot product to find the angle between P and Q. [8.13^0]
16. The vector equations of two lines are $r_1 = \binom{5}{-6} + \lambda\binom{3}{1}$ and $r_2 = \binom{4}{1} + \mu\binom{-2}{3}$. Determine the point where r_1 meets r_2. [8, -5]
17. The equation of three planes P, Q and R are 2x – y +3z = 3, 3x + y + 2z = 7 and x +7y – 5z = 13 respectively. Determine where the three planes intersect. [(-2, 5, 4)]
18. (a) Find in Cartesian form the equation of the line passing through the points A(1, 2. 5), B(1, 0, 4) and C(5, 2, 1). [since AB = BC, points A, B and C are not collinear]
   (b) Find the angle between the line $\frac{x+4}{8} = \frac{y-2}{2} = \frac{z+1}{-4}$ and the plane 4x + 3y – 3z + 1. [69.32^0]
19. Show that the equation of the line through points (1, 2, 1) and (4, -2, 2) is given as $\frac{x-1}{3} = \frac{y-2}{-4} = z - 1$
20. (a) Show that the equation of the plane through points with position vector -2i + 4k perpendicular to vector i + 3j – 2k is x + 3y – 2z + 10 = 0
   (b)(i) Show that the vector 2i – 5j + 3.5k is perpendicular to the plane r = 2i –j + λ(4i + 3j + 2k).
   (ii) Calculate the angle between the vectors 3i – 2j + k and the line in b(i) above. [66.6^0]
21. Find the point of intersection of the line $\frac{x}{5} = \frac{y+2}{2} = \frac{z-1}{4}$ with the plane 3x + 4y + 2z – 25 = 0 [(5, 0, 5)]
22. (a) Find the Cartesian equation of the plane through A(0, 3, -4), B(2, -1, 2) and C(7, 4,-1). Show that Q(10, 13, -10) lies in the same plane
   (b) Express the equation of the plane in (a) in the scalar product form.
   $\left[r.\begin{pmatrix} 3 \\ -6 \\ -5 \end{pmatrix} = 2\right]$
   (c) Find the area of ABC in (a) [25.1 Sq. Units]
23. The vertices of a triangle are P(2, -1, 5), Q(7, 1, -3) and R (12, -2, 0). Show that <PQR= 90^0. Find the coordinates of S if PQRS is a rectangle [(8, -4, 8)]
24. (a)Find the equation of the perpendicular line from Pont A =$\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$ onto the line $r = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$. What is the distance of A from r.
   $$\left[OP = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + \mu\begin{pmatrix} -\frac{4}{9} \\ \frac{14}{9} \\ \frac{-8}{9} \end{pmatrix}; 1.795units\right]$$
   (b) Find the angle contained between the line OR and x – y plane, where OR =$\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ [41.81^0]
25. Given that vectors OA = (3, -2, 5) and OB = (9, 1, -1), find the position vector of point C such that C divides AB internally in the ration 5:-3
   $\left[xi + yj + zk = 18i + \frac{11}{2} - 10k\right]$
26. (a) In a triangle ABC, the altitudes from B and C meat the opposite sides at E and F

respectively. BE and CF intersect at O. Taking O as the origin, Use the dot product to prove that AO is perpendicular to BC.

(b) Prove that <ABC = 900 given that A is (0, 5, -3), B(2, 3, -4) and C(1, -1, 2). Find the coordinates of D if ABCD is a rectangle. [(-1, 1, 3)]

27. Find the equation of the plane through the point (1, 2, 3) and perpendicular to the vector r = 4i + 5j + k. [4x 5y +z = 17]

28. (a) Find the equation of the line through A(2, 2, 5) and B(1, 2, 3).
$\left[\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix}\right]$

(b) If the line in (a) above meets the line $\frac{x-1}{1} = \frac{y-2}{0} = \frac{z-1}{3}$ at P, find the
(i) Coordinate of P [3, 2, 7]
(ii) Angle between the lines [171.9^0]

29. Given that the vector ai -2j + k and 2ai + aj – 4k are perpendicular, find the values of a. [-1, 2]

30. (a) Determine the coordinates of the point of intersection of the line $\frac{x+1}{2} = \frac{y-3}{5} = \frac{z+2}{-1}$ and the plane x + y + z = 12 [3, 13,-4]

(b) Find the angle between the line $\frac{x+1}{2} = \frac{y-3}{5} = \frac{z+1}{-1}$ and the plane x + y + z = 12 [50.7685^0 or 39.2515^0]

31. Find the point of intersection of the plane 11x – 3y + 7z = 8 and the line
$r = \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ where μ is a scalar
[-4, -1, 7]

32. (a) Given the vector a = 3i – 2j + k and b = i – 2j +2k, find
(i) the acute angle between the vectors. [36.7^0]
(ii) vector c such that it is perpendicular to both vectors a and b. $\left[c\begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}\right]$

(b) Given that OA = a and OB = b, Point R is on OB such that OR : RB = 4:1. Point P is on BA such that BP:PA = 2:3 and when RP and OA are both produced they meet at point Q. Find
(i) OR and OP in terms of a and b. $\left[OR = \frac{4}{5}b, OP = \frac{1}{5}(2a + 3b)\right]$
(ii) OQ in terms of a. $\left[\frac{8}{5}a\right]$

33. A point P has coordinates (1, -2, 3) and a certain plane has equation x + 2y + 2z = 8. The line through P parallel to the line $\frac{x}{3} = \frac{y+1}{-1} = z + 1$ meets the plane at a point Q. Find the coordinates of Q$\left[\left(6, \frac{-11}{3}, \frac{14}{3}\right)\right]$

34. Given that the position vectors of A, B, and C are $OA = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, OB = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ and
$OC = \begin{pmatrix} 7 \\ -2 \\ 2 \end{pmatrix}$
(a) Prove that A, B and C are collinear
(b) Find the angle between OA and OB [106.1^0]
(c) If OABD is parallelogram, find the position vectors of E and F such that E divides DA in ratio 1:2 and F divides it externally in ratio 1:2.
$\left[E = \begin{pmatrix} \frac{5}{3} \\ 2 \\ \frac{-4}{3} \end{pmatrix}, F = \begin{pmatrix} 3 \\ 10 \\ -8 \end{pmatrix}\right]$

35. Given the vectors a = i -3j + 3k and b = -i – 3j + 2k; find the
(i) Acute angle between vectors a and b [30 .86^0]
(ii) Equation of the plane containing a and b [-3x + 5y + 6z = 0]

36. The position vectors of A and B are OA = 2i – 4j – k and OB = 5i – 2j + 3k respectively. The line AB is produced to meet the plane 2x+ 6y – 3z =-5 at point C. Find the
(a) coordinates of C [(8, 0, 7)]
(b) angle between AB and the plane [9.169^0]

37. The points P(2, 3), Q(-11, 8) and R(-4, -5) are vertices of a parallelogram PQRS which has PR as the diagonal. Find the coordinates of the vertex S. [S(9, -10)]

38. (a) Find the angle between the planes x – 2y + z=0 and x – y = 1 [30^0]
(b) Two lines are given by the parametric equation: -i + 2j + k +t(i – 2j + 3k) and

-3i - + pj + 7k + s(i – j + 2k). If the lines intersect, find

(i) values of t, s and p. [t =10, s = 12, p = -6]

(ii) coordinates of the points of intersection [(9, 18, 31)]

39. given the points A(-3, 3, 4), B(5, 7, 2) and C(1, 1, 4), find the vector equation of a line which joins the mid-point of AB and BC
$\left[\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} + \mu\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}\right]$

40. (a) The equation of the plane R is r.$\begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} = 16$. Where r is the position vector of R. Find the perpendicular distance of the plane from the origin [2.971 units]

(b) Find the Cartesian equation of the plane through the point P(1, 0 -2) and Q(3, -1, 1) and parallel to the line with a vector equation
r = 2i ++ (2α – 1)j + (5 – α)k
[-5x + 2y + 4z + 13 = 0]

41. Find the equation of the line through point (2, 3) and perpendicular to line x + 2y + 5 = 0
[y = 2x – 1]

42. Show that the points A, B and C with position vectors 3i + 3j + k, 8i + 7j + 4k and 11i + 4j + 5k respectively are vertices of a triangle.

43. (a) Find the angle between the lines
$x = \frac{y-1}{2} = \frac{z-2}{3}$ and $\frac{x}{2} = \frac{y+1}{3} = \frac{z+2}{4}$. [8.53^0]

(b) Find in vector form the equation of the line of intersection of two planes
2x + 3y – 2 =4 and x – y + 2z = 5

$$\left[\begin{array}{l} r = \begin{pmatrix} 0 \\ \frac{13}{5} \\ \frac{19}{5} \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \ or \\ r = \begin{pmatrix} \frac{19}{5} \\ \frac{-6}{5} \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \ or \\ r = \begin{pmatrix} \frac{13}{5} \\ 0 \\ \frac{-6}{5} \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \end{array}\right]$$

44. A line passes through the point A(4, 6, 3) and B(1, 3, 3).

(a) Find the vector equation of the line
$\left[r = \begin{pmatrix} 4 \\ 6 \\ 3 \end{pmatrix} + \mu\begin{pmatrix} -3 \\ -3 \\ 0 \end{pmatrix}\right]$

(b) Show that the point C(2, 4. 3) lies on the line in (a) above.

45. Triangle OAB has OA = a and OB = b. C is a point on OA such that OC = $\frac{2}{3}a$. D is the mid-point of AB. When CD is produced it meets OB produced at E., such that DE = nCD and BE = kb. Express DE in terms of

(a) n, a and b $\left[\frac{5n}{6}a + \frac{n}{2}b\right]$

(b) k, a and b $\left[\frac{1}{2}a + \frac{2k-1}{2}b\right]$

Hence find the values of n and k.
$\left[n = \frac{3}{5},\ k = \frac{1}{5}\right]$