## 9.1 Equation of a line in three dimensions

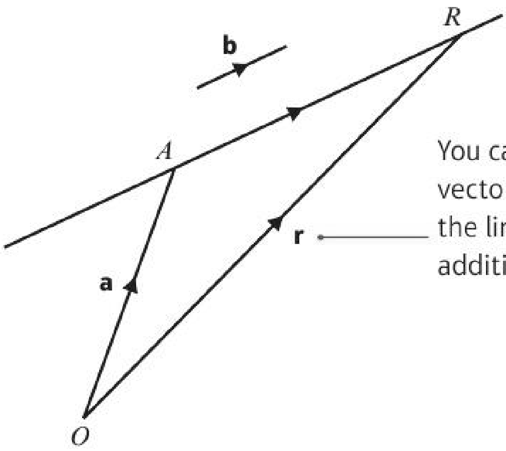
You need to know how to write the equation of a straight line in vector form.

Suppose a straight line passes through a given point $A$, with position vector **a**, and is parallel to the given vector **b**. Only one such line is possible. Let $R$ be an arbitrary point on the line, with position vector **r**.

Since $\overrightarrow{AR}$ is parallel to **b**, $\overrightarrow{AR} = \lambda\mathbf{b}$, where $\lambda$ is a scalar.

The vector **b** is called the **direction vector** of the line.

So the position vector **r** can be written as $\mathbf{a} + \lambda\mathbf{b}$.




You can find the position vector of any point $R$ on the line by using vector addition ($\triangle OAR$):

$\mathbf{r} = \mathbf{a} + \overrightarrow{AR}$

- **A vector equation of a straight line passing through the point $A$ with position vector a, and parallel to the vector b is**

  $$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$$

  **where $\lambda$ is a scalar parameter.**

**Notation** **r** is the position vector of a general point on the line. Scalar parameters in vector equations are often given Greek letters such as $\lambda$ (lambda) and $\mu$ (mu).

By taking different values of the parameter $\lambda$, you can find the position vectors of different points that lie on the straight line.

**Online** Explore the vector equation of a line using GeoGebra.

### Example 1

Find a vector equation of the straight line which passes through the point $A$, with position vector $3\mathbf{i} - 5\mathbf{j} + 4\mathbf{k}$, and is parallel to the vector $7\mathbf{i} - 3\mathbf{k}$.

Here $\mathbf{a} = \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 7 \\ 0 \\ -3 \end{pmatrix}$.

**b** is the direction vector.

An equation of the line is

$$\mathbf{r} = \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} + \lambda\begin{pmatrix} 7 \\ 0 \\ -3 \end{pmatrix}$$

or $\mathbf{r} = (3\mathbf{i} - 5\mathbf{j} + 4\mathbf{k}) + \lambda(7\mathbf{i} - 3\mathbf{k})$

or $\mathbf{r} = (3 + 7\lambda)\mathbf{i} + (-5)\mathbf{j} + (4 - 3\lambda)\mathbf{k}$

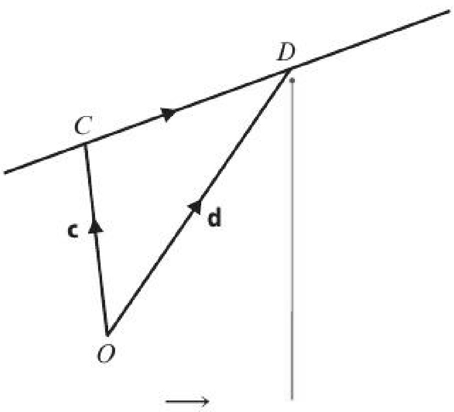
or $\mathbf{r} = \begin{pmatrix} 3 + 7\lambda \\ -5 \\ 4 - 3\lambda \end{pmatrix}$

You sometimes need to show the separate $x$, $y$, $z$ components in terms of $\lambda$.

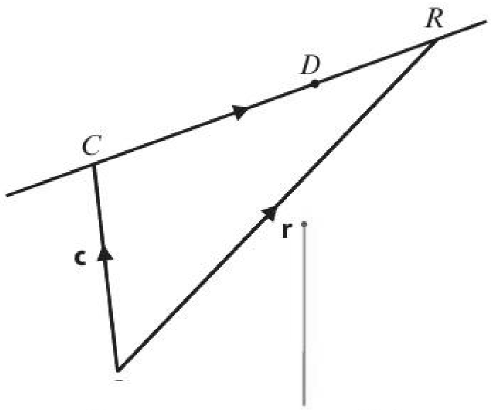
You can represent a 3D vector using column notation, $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, or using **ijk**-notation, $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

← **Pure Year 2, Chapter 12**

Now suppose a straight line passes through two given points $C$ and $D$, with position vectors **c** and **d** respectively. Again, only one such line is possible.




You can use $\overrightarrow{CD}$ as a direction vector for the line: $\overrightarrow{CD} = \mathbf{d} - \mathbf{c}$.




You can now use one of the two given points and the direction vector to form an equation for the straight line.

- **A vector equation of a straight line passing through the points $C$ and $D$, with position vectors c and d respectively, is**

  $$\mathbf{r} = \mathbf{c} + \lambda(\mathbf{d} - \mathbf{c})$$

  **where $\lambda$ is a scalar parameter.**

**Note** You can use any point on the straight line as the initial point in the vector equation. An alternative vector equation for this line would be $\mathbf{r} = \mathbf{d} + \lambda(\mathbf{d} - \mathbf{c})$.

## Example 2

Find a vector equation of the straight line which passes through the points $A$ and $B$, with coordinates $(4, 5, -1)$ and $(6, 3, 2)$ respectively.

$$\mathbf{a} = \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$$

Write down the position vectors of $A$ and $B$.

$$\mathbf{b} - \mathbf{a} = \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$$

Find a direction vector for the line.

Use one of the given points to form the equation.

$$\mathbf{r} = \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix} + t\begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$$

You don't have to use λ for the parameter. In this example, the parameter is represented by the letter $t$.

or $\mathbf{r} = (4\mathbf{i} + 5\mathbf{j} - \mathbf{k}) + t(2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$

or $\mathbf{r} = (4 + 2t)\mathbf{i} + (5 - 2t)\mathbf{j} + (-1 + 3t)\mathbf{k}$

or $\mathbf{r} = \begin{pmatrix} 4 + 2t \\ 5 - 2t \\ -1 + 3t \end{pmatrix}$

You can give your answer in any of these forms.

Example 3

The straight line $l$ has vector equation $\mathbf{r} = (3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}) + t(\mathbf{i} - 6\mathbf{j} - 2\mathbf{k})$.
Given that the point $(a, b, 0)$ lies on $l$, find the value of $a$ and the value of $b$.

$\mathbf{r} = \begin{pmatrix} 3 + t \\ 2 - 6t \\ -5 - 2t \end{pmatrix}$

You can write the equation in this form.

$-5 - 2t = 0$

$t = -\frac{5}{2}$

Use the $z$-coordinate (which is equal to zero) to find the value of $t$.

$a = 3 + t = \frac{1}{2}$

$b = 2 - 6t = 17$

$a = \frac{1}{2}$ and $b = 17$

Find $a$ and $b$ using the value of $t$.

Example 4

The straight line $l$ has vector equation $\mathbf{r} = (2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}) + \lambda(6\mathbf{i} - 2\mathbf{j} + 4\mathbf{k})$.
Show that another vector equation of $l$ is $\mathbf{r} = (8\mathbf{i} + 3\mathbf{j} + \mathbf{k}) + \mu(3\mathbf{i} - \mathbf{j} + 2\mathbf{k})$.

Use the equation $\mathbf{r} = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix} + \lambda\begin{pmatrix} 6 \\ -2 \\ 4 \end{pmatrix}$.

When $\lambda = 1$, $\mathbf{r} = \begin{pmatrix} 8 \\ 3 \\ 1 \end{pmatrix}$, so the point $(8, 3, 1)$ lies on $l$.

To show that $(8\mathbf{i} + 3\mathbf{j} + \mathbf{k})$ lies on $l$, find a value of $\lambda$ that gives this point. It is often easier to work in column vectors.

$\begin{pmatrix} 6 \\ -2 \\ 4 \end{pmatrix} = 2\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ so these two vectors are parallel.

If one vector is a scalar multiple of another then the vectors are parallel.

So an alternative form of the equation is

$\mathbf{r} = \begin{pmatrix} 8 \\ 3 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

**Watch out** Using the same value of the parameter in each equation will give **different** points on the line. You should use a different letter for the parameter of the second equation.

You also need to be able to write the equation of a line in three dimensions in **Cartesian form**. This means that the equation is given in terms of coordinates relative to the $x$-, $y$- and $z$-axes.

- **If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ the equation of the line with vector equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ can be given in Cartesian form as**

$$\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}$$

Each of the three expressions is equal to $\lambda$.

## Example 5

With respect to the fixed origin $O$, the line $l$ is given by the equation

$$\mathbf{r} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

**a** Prove that a Cartesian form of the equation of $l$ is

$$\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}$$

**b** Hence find a Cartesian equation of the line with equation $\mathbf{r} = \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}$.

a $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 + \lambda b_1 \\ a_2 + \lambda b_2 \\ a_3 + \lambda b_3 \end{pmatrix}$

Write the position vector of the general point on the line as $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

$x = a_1 + \lambda b_1$, $y = a_2 + \lambda b_2$, $z = a_3 + \lambda b_3$

Use the vector equation of the line to write expressions for $x$, $y$ and $z$ in terms of $\lambda$.

Rearranging,

$$\lambda = \frac{x - a_1}{b_1},\ \lambda = \frac{y - a_2}{b_2},\ \lambda = \frac{z - a_3}{b_3}$$

Make $\lambda$ the subject of each equation.

So $\dfrac{x - a_1}{b_1} = \dfrac{y - a_2}{b_2} = \dfrac{z - a_3}{b_3}$

For any point on the line, the value of $\lambda$ is a constant, so equate the three different expressions for $\lambda$.

b $\dfrac{x - 4}{-1} = \dfrac{y - 3}{2} = \dfrac{z + 2}{5}$

If you need to convert between vector and Cartesian forms you can quote this result without proof in your exam. Be careful with the signs on the top of each fraction.

## Example 6

The line $l$ has equation $\mathbf{r} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, and the point $P$ has position vector $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$.

**a** Show that $P$ does not lie on $l$.

Given that a circle, centre $P$, intersects $l$ at points $A$ and $B$, and that $A$ has position vector $\begin{pmatrix} 0 \\ -3 \\ 6 \end{pmatrix}$,

**b** find the position vector of $B$.

**a** $\mathbf{r} = \begin{pmatrix} -2 + \lambda \\ 1 - 2\lambda \\ 4 + \lambda \end{pmatrix}$

> **Problem-solving**
>
> It is often useful to write the general point on a line as a single vector. You can write each component in the form $a + \lambda b$.

If $P(2, 1, 3)$ lies on the line then

$2 = -2 + \lambda \Rightarrow \lambda = 4$

$1 = 1 - 2\lambda \Rightarrow \lambda = 0$

$3 = 4 + \lambda \Rightarrow \lambda = -1$

so $P$ does not lie on $l$.

> If $P$ lies on $l$, there is one value of $\lambda$ that satisfies all 3 equations. You only need to show that two of these equations are not consistent to show that $P$ does not lie on $l$.

**b** $\overrightarrow{AP} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ -3 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix}$

$|\overrightarrow{AP}| = \sqrt{2^2 + 4^2 + (-3)^2} = \sqrt{29}$

> The distance between the points with position vectors $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is $\sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}$. As $P$ is the centre of the circle and $A$ lies on the circle, the radius of the circle is $\sqrt{29}$.
>
> ← **Pure Year 2, Chapter 12**

The position vector of $B$ is $\begin{pmatrix} -2 + \lambda \\ 1 - 2\lambda \\ 4 + \lambda \end{pmatrix}$.

> Use the general point on the line to represent the position vector of $B$.

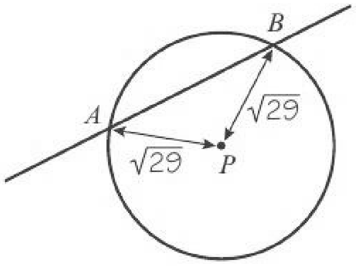
$\overrightarrow{BP} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} -2 + \lambda \\ 1 - 2\lambda \\ 4 + \lambda \end{pmatrix} = \begin{pmatrix} 4 - \lambda \\ 2\lambda \\ -1 - \lambda \end{pmatrix}$

$(4 - \lambda)^2 + 4\lambda^2 + (-1 - \lambda)^2 = 29$

> $B$ lies on the circle so the length $|\overrightarrow{BP}| = \sqrt{29}$.

$16 - 8\lambda + \lambda^2 + 4\lambda^2 + 1 + 2\lambda + \lambda^2 = 29$

> Solve the resulting quadratic equation to find two possible values of $\lambda$. One will correspond to point $A$, and the other will correspond to point $B$.




$$6\lambda^2 - 6\lambda + 17 = 29$$
$$6\lambda^2 - 6\lambda - 12 = 0$$
$$\lambda^2 - \lambda - 2 = 0$$
$$(\lambda - 2)(\lambda + 1) = 0$$

So $\lambda = 2$ or $\lambda = -1$

$\lambda = 2$ gives $\begin{pmatrix} 0 \\ -3 \\ 6 \end{pmatrix}$. This is the position vector of point $A$.

$\lambda = -1$ gives $\begin{pmatrix} -3 \\ 3 \\ 3 \end{pmatrix}$. This is the position vector of point $B$.

> Substitute values of $\lambda$ into $\begin{pmatrix} -2 + \lambda \\ 1 - 2\lambda \\ 4 + \lambda \end{pmatrix}$. Check that one of the values gives the position vector of $A$. The other value must give the position vector for $B$.

## Exercise 9A

**1** For the following pairs of vectors, find a vector equation of the straight line which passes through the point, with position vector $\mathbf{a}$, and is parallel to the vector $\mathbf{b}$:

**a** $\mathbf{a} = 6\mathbf{i} + 5\mathbf{j} - \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$

**b** $\mathbf{a} = 2\mathbf{i} + 5\mathbf{j}$, $\mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

**c** $\mathbf{a} = -7\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

**d** $\mathbf{a} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$

**e** $\mathbf{a} = \begin{pmatrix} 6 \\ -11 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ 5 \\ -2 \end{pmatrix}$

**2** For the points $P$ and $Q$ with position vectors $\mathbf{p}$ and $\mathbf{q}$ respectively, find:

**i** the vector $\overrightarrow{PQ}$

**ii** a vector equation of the straight line that passes through $P$ and $Q$

**a** $\mathbf{p} = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$, $\mathbf{q} = 5\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

**b** $\mathbf{p} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$, $\mathbf{q} = 4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

**c** $\mathbf{p} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, $\mathbf{q} = -2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$

**d** $\mathbf{p} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$

**e** $\mathbf{p} = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix}$

**3** Find a vector equation of the line which is parallel to the $z$-axis and passes through the point $(4, -3, 8)$.

**4** **a** Find a vector equation of the line which passes through the points:

**i** $(2, 1, 9)$ and $(4, -1, 8)$

**ii** $(-3, 5, 0)$ and $(7, 2, 2)$

**iii** $(1, 11, -4)$ and $(5, 9, 2)$

**iv** $(-2, -3, -7)$ and $(12, 4, -3)$

**b** Write down a Cartesian equation in the form $\dfrac{x - a_1}{b_1} = \dfrac{y - a_2}{b_2} = \dfrac{z - a_3}{b_3}$ for each line in part **a**.

(P) **5** The point $(1, p, q)$ lies on the line $l$. Find the values of $p$ and $q$, given that the equation of $l$ is:

**a** $\mathbf{r} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -4 \\ -9 \end{pmatrix}$

**b** $\mathbf{r} = \begin{pmatrix} -4 \\ 6 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -5 \\ -8 \end{pmatrix}$

**c** $\mathbf{r} = \begin{pmatrix} 16 \\ -9 \\ -10 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

(P) **6** The line $l_1$ has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}$. The line $l_2$ has equation $\dfrac{x - 4}{2} = \dfrac{y + 1}{-4} = \dfrac{z - 3}{-8}$

Show that $l_1$ and $l_2$ are parallel.

(P) **7** Show that the line $l_1$ with equation $\mathbf{r} = (3 + 2\lambda)\mathbf{i} + (2 - 3\lambda)\mathbf{j} + (-1 + 4\lambda)\mathbf{k}$ is parallel to the line $l_2$ which passes through the points $A(5, 4, -1)$ and $B(3, 7, -5)$.

**8** Show that the points $A(-3, -4, 5)$, $B(3, -1, 2)$ and $C(9, 2, -1)$ are collinear.

**Hint** Points are said to be **collinear** if they all lie on the same straight line.

**9** Show that the points with position vectors $\begin{pmatrix} 1 \\ 7 \\ -2 \end{pmatrix}$, $\begin{pmatrix} 3 \\ -1 \\ 8 \end{pmatrix}$ and $\begin{pmatrix} 10 \\ 4 \\ 0 \end{pmatrix}$ do not lie on the same straight line.

(E/P) **10** The points $P(2, 0, 4)$, $Q(a, 5, 1)$ and $R(3, 10, b)$, where $a$ and $b$ are constants, are collinear. Find the values of $a$ and $b$. **(5 marks)**

(E) **11** The line $l_1$ has equation

$$\mathbf{r} = (8\mathbf{i} - 5\mathbf{j} + 4\mathbf{k}) + \lambda(3\mathbf{i} + \mathbf{j} - 6\mathbf{k})$$

$A$ is the point on $l_1$ such that $\lambda = -2$.

The line $l_2$ passes through $A$ and is parallel to the line with equation

$$\mathbf{r} = (10\mathbf{i} + 3\mathbf{j} - 9\mathbf{k}) + \lambda(2\mathbf{i} - 4\mathbf{j} + \mathbf{k})$$

Find an equation for $l_2$. **(6 marks)**

(E/P) **12** The point $A$ with coordinates $(4, a, 0)$ lies on the line $L$ with vector equation

$$\mathbf{r} = (10\mathbf{i} + 8\mathbf{j} - 12\mathbf{k}) + \lambda(\mathbf{i} - \mathbf{j} + b\mathbf{k})$$

where $a$ and $b$ are constants.

**a** Find the values of $a$ and $b$. **(3 marks)**

The point $X$ lies on $L$ where $\lambda = -1$.

**b** Find the coordinates of $X$. **(1 mark)**

(E) **13** The line $l$ has equation $\mathbf{r} = \begin{pmatrix} 3 \\ -5 \\ 9 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$.

$A$ and $B$ are the points on $l$ with $\lambda = 5$ and $\lambda = 2$ respectively.
Find the distance $AB$. **(4 marks)**

(E) **14** The line $l$ has equation $\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$.

$C$ and $A$ are the points on $l$ with $\lambda = 4$ and $\lambda = 3$ respectively.
A circle has centre $C$ and intersects $l$ at the points $A$ and $B$.
Find the position vector of $B$. **(3 marks)**

(E/P) **15** The line $l$ has equation $x - 5 = \dfrac{y + 1}{3} = \dfrac{z - 6}{-2}$

A circle $C$ has centre $(4, -1, 2)$ and radius $3\sqrt{5}$.
Given that $C$ intersects $l$ at two distinct points, $A$ and $B$, find the coordinates of $A$ and $B$. **(7 marks)**

**Problem-solving**

Write $x - 5$ as $\dfrac{x - 5}{1}$ and convert the equation of the line into vector form.

(E/P) **16** The line $l_1$ has equation $\mathbf{r} = \begin{pmatrix} -4 \\ 6 \\ 5 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. $A$ and $B$ are the points on $l_1$ with $\lambda = 2$ and $\lambda = 5$ respectively.

**a** Find the position vectors of $A$ and $B$. **(2 marks)**

The point $P$ has position vector $\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$.

The line $l_2$ passes through the point $P$ and is parallel to the line $l_1$.

**b** Find a vector equation of the line $l_2$. **(2 marks)**

The points $C$ and $D$ both lie on line $l_2$ such that $AB = AC = AD$.

**c** Show that $P$ is the midpoint of $CD$. **(7 marks)**

(E/P) **17** A tightrope is modelled as a line segment between points with coordinates (2, 3, 8) and (22, 18, 8), relative to a fixed origin $O$, where the units of distance are metres. Two support cables are anchored to a fixed point $A$ on the wire. The other ends of the cables are anchored
to points with coordinates (14, 1, 0) and (6, 17, 0) respectively.

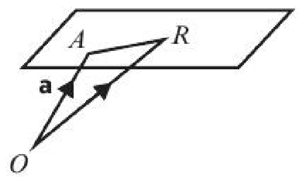
**a** Given that the support cables are both 12 m long, find the coordinates of $A$. **(8 marks)**

**b** Give one criticism of this model. **(1 mark)**

## 9.2 Equation of a plane in three dimensions

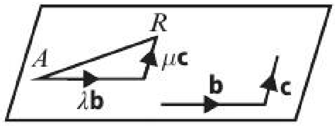
The equation of a plane can be written in vector form.

Suppose a plane passes through a given point $A$, with position vector **a**. Let $R$ be an arbitrary point on the plane, with position vector **r**.




Then, using the triangle law, $\mathbf{r} = \mathbf{a} + \overrightarrow{AR}$.

Since $\overrightarrow{AR}$ lies in the plane, it can be written as $\lambda\mathbf{b} + \mu\mathbf{c}$, where **b** and **c** are non-parallel vectors in the plane and where $\lambda$ and $\mu$ are scalars.




So the position vector **r** can be written as $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$.

- **The vector equation of a plane is $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, where:**
  - **r is the position vector of a general point in the plane**
  - **a is the position vector of a point in the plane**
  - **b and c are non-parallel, non-zero vectors in the plane**
  - **$\lambda$ and $\mu$ are scalars**

### Example 7

Find, in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, an equation of the plane that passes through the points $A(2, 2, -1)$, $B(3, 2, -1)$ and $C(4, 3, 5)$.

$\overrightarrow{AB}$ and $\overrightarrow{AC}$ are vectors which lie in the plane.

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{i}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = 2\mathbf{i} + \mathbf{j} + 6\mathbf{k}$

So an equation of the plane is

$\mathbf{r} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k} + \lambda\mathbf{i} + \mu(2\mathbf{i} + \mathbf{j} + 6\mathbf{k})$

There are many other forms of this answer which are also correct. You could use $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ or $4\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$ instead of $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ in the equation.

You could write this equation as

$$\mathbf{r} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mu\begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix}$$

Example 8

Verify that the point $P$ with position vector $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ lies in the plane with vector equation $\mathbf{r} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$$\mathbf{r} = \begin{pmatrix} 3 + 2\lambda + \mu \\ 4 + \lambda - \mu \\ -2 + \lambda + 2\mu \end{pmatrix}$$

The position vector of any point on the plane can be written in this form.

If $P$ lies on the plane,

$$\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 + 2\lambda + \mu \\ 4 + \lambda - \mu \\ -2 + \lambda + 2\mu \end{pmatrix}$$

$2 = 3 + 2\lambda + \mu$ so $2\lambda + \mu = -1$ (1)

$2 = 4 + \lambda - \mu$ so $\lambda - \mu = -2$ (2)

$-1 = -2 + \lambda + 2\mu$ so $\lambda + 2\mu = 1$ (3)

Solving equations (2) and (3) simultaneously,

(3) − (2): $3\mu = 3$ so $\mu = 1$

Sub in (2): $\lambda - 1 = -2$ so $\lambda = -1$

Check in equation (1):

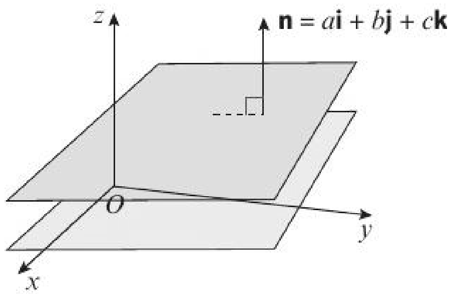
$2\lambda + \mu = -2 + 1 = -1$ so $P$ lies in the plane.

**Problem-solving**

If the point $P$ lies on the plane then there will be values of $\lambda$ and $\mu$ that satisfy **all three** of these equations simultaneously. Solve one pair of equations simultaneously, then check that the solutions satisfy the third equation.

The direction of a plane can be described by giving a **normal vector, n**. This is a vector that is perpendicular to the plane.

One normal vector can describe an infinite number of parallel planes, so the normal vector on its own is not enough information to define a plane uniquely.




- **A Cartesian equation of a plane in three dimensions can be written in the form $ax + by + cz = d$ where $a$, $b$, $c$ and $d$ are constants, and $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is the normal vector to the plane.**

**Note** Compare this equation to the Cartesian equation of a line in two dimensions: $ax + by = c$.

**Online** Explore the vector and Cartesian equations of a plane using GeoGebra.

You can derive this result using the **scalar product**, which you will learn about later in this chapter.

## Example 9

The plane $\Pi$ is perpendicular to the normal vector $\mathbf{n} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and passes through the point $P$ with position vector $8\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}$.
Find a Cartesian equation of $\Pi$.

**Notation** Planes are often represented by the capital Greek letter pi, $\Pi$.

$3x - 2y + z = d$

The general equation is $ax + by + cz = d$ where the normal vector is $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$.

$3 \times 8 - 2 \times 4 + 1 \times (-7) = 9$

Substitute the values of $x$, $y$ and $z$ for point $P$ into this question to find the value of $d$.

So $d = 9$ and the Cartesian equation of $\Pi$ is $3x - 2y + z = 9$.

## Exercise 9B

1 Find, in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, an equation of the plane that passes through the points:
 **a** (1, 2, 0), (3, 1, −1) and (4, 3, 2)
 **b** (3, 4, 1), (−1, −2, 0) and (2, 1, 4)
 **c** (2, −1, −1), (3, 1, 2) and (4, 0, 1)
 **d** (−1, 1, 3), (−1, 2, 5) and (0, 4, 4).

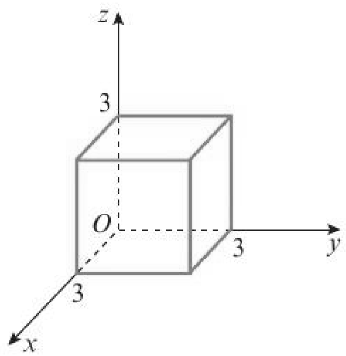
2 The plane $\Pi$ is perpendicular to the normal vector $\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$ and passes through the point with position vector $\begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix}$. Find a Cartesian equation of $\Pi$.

(P) 3 Find the value of $k$, given that the plane $\Pi$ with vector equation $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ passes through the points:
 **a** $(7, -1, k)$
 **b** $(1, k, 11)$
 **c** $(k, -4, 10)$
 **d** $(10, k, -k)$

4 A Cartesian equation of the plane $\Pi$ is $2x - 3y + 5z = 1$.
 **a** Verify that the plane passes through the point: **i** (1, 2, 1) **ii** (2, −4, −3)
 **b** Write down an equation of a normal vector to the plane.

5 The line $l$ is normal to the plane $\Pi$ with Cartesian equation $5x - 3y - 4z = 9$ and passes through the point (2, 3, −2). Find:
 **a** a vector equation of $l$
 **b** a Cartesian equation of $l$

6 The diagram shows a cube with a vertex at the origin and sides of length 3.
Find a Cartesian equation for each face of the cube.



E/P **7** Show that the points (2, 2, 3), (1, 5, 3), (4, 3, −1) and (3, 6, −1) are coplanar. **(6 marks)**

**Notation** Points are said to be **coplanar** if they all lie on the same plane.

E/P **8** Show that the points (2, 3, 4), (2, −1, 3), (5, 3, −2) and (−1, −9, 8) are **not** coplanar. **(6 marks)**

E/P **9** The plane $\Pi$ has vector equation $\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k} + \lambda(-2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}) + \mu(4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$.
The point $A$ lies on $\Pi$ such that $\lambda = 1$ and $\mu = 2$.

**a** Find the position vector of $A$. **(2 marks)**

Point $B$ has position vector (1, −7, 2).

**b** Show that $B$ lies on $\Pi$. **(2 marks)**

The line $l$ passes through points $A$ and $B$.

**c** Find a vector equation of $l$. **(3 marks)**

The point $C$ lies on $l$ such that $|\overrightarrow{OA}| = |\overrightarrow{OC}|$.

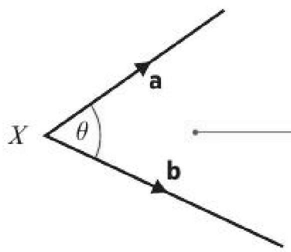
**d** Find the position vector of $C$. **(3 marks)**

**Challenge**

A plane has vector equation $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + \lambda(\mathbf{i} - 2\mathbf{j} + \mathbf{k}) + \mu(2\mathbf{i} - \mathbf{j} + 3\mathbf{k})$.
A line has vector equation $\mathbf{r} = (2\mathbf{i} + 6\mathbf{j} + \mathbf{k}) + t(5\mathbf{i} - 7\mathbf{j} + 6\mathbf{k})$.
Show that the line lies entirely within the plane.

## 9.3 Scalar product

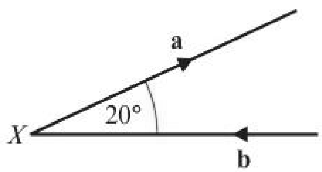
You need to know the definition of the scalar product of two vectors in either two or three dimensions, and how it can be used to find the angle between two vectors. To define the scalar product you need to know how to find the **angle between two vectors**.

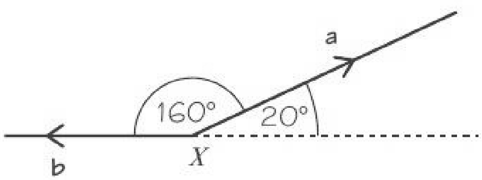



On the diagram, the angle between the vectors **a** and **b** is $\theta$.
Notice that **a** and **b** are both directed away from the point $X$.

**Example 10**

Find the angle between the vectors **a** and **b** on the diagram.



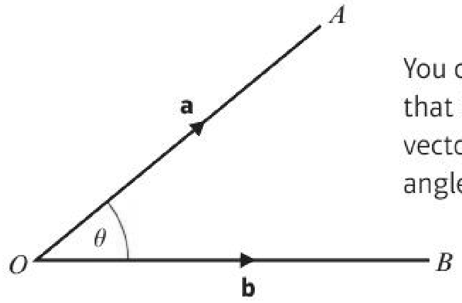

The angle between **a** and **b** is 180° − 20° = 160°.

For the correct angle, **a** and **b** must both be pointing away from $X$, so re-draw to show this.

■ **The scalar product of two vectors a and b is written as a.b, and defined as**

**a.b = |a||b| cos** $\theta$

**where** $\theta$ **is the angle between a and b.**




You can see from this diagram that if **a** and **b** are the position vectors of $A$ and $B$, then the angle between **a** and **b** is $\angle AOB$.

**Notation** The scalar product is often called the **dot product**. You say 'a dot b'.

**Online** Use GeoGebra to consider the scalar product as the component of one vector in the direction of another.

■ **If a and b are the position vectors of the points** $A$ **and** $B$**, then** $\cos(\angle AOB) = \dfrac{\mathbf{a.b}}{|\mathbf{a}||\mathbf{b}|}$

If two vectors **a** and **b** are perpendicular, the angle between them is 90°.

Since cos 90° = 0, **a.b** = |**a**||**b**| cos 90° = 0.

■ **The non-zero vectors a and b are perpendicular if and only if a.b = 0.**

If **a** and **b** are parallel, the angle between them is 0°.

■ **If a and b are parallel, a.b = |a||b|. In particular, a.a = |a|².**

## Example 11

Find the values of

**a** **i.j** **b** **k.k** **c** (4**j**).**k** + (3**i**).(3**i**)

a **i.j** = 1 × 1 × cos 90° = 0

**i** and **j** are unit vectors (magnitude 1), and are perpendicular.

b **k.k** = 1 × 1 × cos 0° = 1

**k** is a unit vector (magnitude 1) and the angle between **k** and itself is 0°.

c (4**j**).**k** + (3**i**).(3**i**)
= (4 × 1 × cos 90°) + (3 × 3 × cos 0°)
= 0 + 9 = 9

## Example 12

Given that $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, prove that $\mathbf{a.b} = a_1b_1 + a_2b_2 + a_3b_3$.

$$\begin{aligned}\mathbf{a}.\mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}).(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})\\
&= a_1\mathbf{i}.(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})\\
&\quad + a_2\mathbf{j}.(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})\\
&\quad + a_3\mathbf{k}.(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})\\
&= (a_1\mathbf{i}).(b_1\mathbf{i}) + (a_1\mathbf{i}).(b_2\mathbf{j}) + (a_1\mathbf{i}).(b_3\mathbf{k})\\
&\quad + (a_2\mathbf{j}).(b_1\mathbf{i}) + (a_2\mathbf{j}).(b_2\mathbf{j}) + (a_2\mathbf{j}).(b_3\mathbf{k})\\
&\quad + (a_3\mathbf{k}).(b_1\mathbf{i}) + (a_3\mathbf{k}).(b_2\mathbf{j}) + (a_3\mathbf{k}).(b_3\mathbf{k})\\
&= (a_1b_1)\mathbf{i}.\mathbf{i} + (a_1b_2)\mathbf{i}.\mathbf{j} + (a_1b_3)\mathbf{i}.\mathbf{k}\\
&\quad + (a_2b_1)\mathbf{j}.\mathbf{i} + (a_2b_2)\mathbf{j}.\mathbf{j} + (a_2b_3)\mathbf{j}.\mathbf{k}\\
&\quad + (a_3b_1)\mathbf{k}.\mathbf{i} + (a_3b_2)\mathbf{k}.\mathbf{j} + (a_3b_3)\mathbf{k}.\mathbf{k}\\
&= a_1b_1 + a_2b_2 + a_3b_3\end{aligned}$$

Use the results for parallel and perpendicular unit vectors:

$\mathbf{i}.\mathbf{i} = \mathbf{j}.\mathbf{j} = \mathbf{k}.\mathbf{k} = 1$

$\mathbf{i}.\mathbf{j} = \mathbf{i}.\mathbf{k} = \mathbf{j}.\mathbf{i} = \mathbf{j}.\mathbf{k} = \mathbf{k}.\mathbf{i} = \mathbf{k}.\mathbf{j} = 0$

The above example leads to a simple formula for finding the scalar product of two vectors given in Cartesian component form:

- **If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$,**

$$\mathbf{a}.\mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$$

You can use this result without proof in your exam.

## Example 13

Given that $\mathbf{a} = 8\mathbf{i} - 5\mathbf{j} - 4\mathbf{k}$ and $\mathbf{b} = 5\mathbf{i} + 4\mathbf{j} - \mathbf{k}$,

**a** Find $\mathbf{a}.\mathbf{b}$.

**b** Find the angle between $\mathbf{a}$ and $\mathbf{b}$, giving your answer in degrees to 1 decimal place.

a $\mathbf{a}.\mathbf{b} = \begin{pmatrix} 8 \\ -5 \\ -4 \end{pmatrix}.\begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix}$

Write in column vector form.

$= (8 \times 5) + (-5 \times 4) + (-4 \times -1)$

Use $\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$.

$= 40 - 20 + 4$

$= 24$

b $\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$

Use the scalar product definition.

$|\mathbf{a}| = \sqrt{8^2 + (-5)^2 + (-4)^2} = \sqrt{105}$

$|\mathbf{b}| = \sqrt{5^2 + 4^2 + (-1)^2} = \sqrt{42}$

Find the modulus of **a** and of **b**.

$\sqrt{105}\sqrt{42} \cos\theta = 24$

Use $\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$.

$$\cos\theta = \frac{24}{\sqrt{105}\sqrt{42}}$$

$\theta = 68.8°$ (1 d.p.)

## Example 14

Given that $\mathbf{a} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 7\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$, find the angle between $\mathbf{a}$ and $\mathbf{b}$, giving your answer in degrees to 1 decimal place.

$$\mathbf{a.b} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 7 \\ -2 \\ 2 \end{pmatrix} = -7 - 2 + 6 = -3$$

$$|\mathbf{a}| = \sqrt{(-1)^2 + 1^2 + 3^2} = \sqrt{11}$$

$$|\mathbf{b}| = \sqrt{7^2 + (-2)^2 + 2^2} = \sqrt{57}$$

For the scalar product formula, you need to find $\mathbf{a.b}$, $|\mathbf{a}|$ and $|\mathbf{b}|$.

$$\sqrt{11}\sqrt{57}\cos\theta = -3$$

Use $\mathbf{a.b} = |\mathbf{a}||\mathbf{b}|\cos\theta$.

$$\cos\theta = \frac{-3}{\sqrt{11}\sqrt{57}}$$

The cosine is negative, so the angle is obtuse.

$$\theta = 96.9° \text{ (1 d.p.)}$$

## Example 15

Given that the vectors $\mathbf{a} = 2\mathbf{i} - 6\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 5\mathbf{i} + 2\mathbf{j} + \lambda\mathbf{k}$ are perpendicular, find the value of $\lambda$.

$$\mathbf{a.b} = \begin{pmatrix} 2 \\ -6 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 2 \\ \lambda \end{pmatrix}$$
$$= 10 - 12 + \lambda$$
$$= -2 + \lambda$$

Find the scalar product.

$$-2 + \lambda = 0$$

For perpendicular vectors, the scalar product is zero.

$$\lambda = 2$$

## Example 16

Given that $\mathbf{a} = -2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} - 8\mathbf{j} + 5\mathbf{k}$, find a vector which is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

$$\mathbf{a}.\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \text{ and } \mathbf{b}.\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

Both scalar products are zero.

$$\begin{pmatrix} -2 \\ 5 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \text{ and } \begin{pmatrix} 4 \\ -8 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$-2x + 5y - 4z = 0 \quad (1)$$

$$4x - 8y + 5z = 0 \quad (2)$$

Let $z = 1$

$-2x + 5y = 4$ (from 1)

$4x - 8y = -5$ (from 2)

So $x = \frac{7}{4}$, $y = \frac{3}{2}$ and $z = 1$

A possible vector is $\frac{7}{4}\mathbf{i} + \frac{3}{2}\mathbf{j} + \mathbf{k}$.

Another possible vector is
$4(\frac{7}{4}\mathbf{i} + \frac{3}{2}\mathbf{j} + \mathbf{k}) = 7\mathbf{i} + 6\mathbf{j} + 4\mathbf{k}$

Choose a (non-zero) value for $z$ (or for $x$, or for $y$).

Solving simutaneously gives
$x = \frac{7}{4}$ and $y = \frac{3}{2}$

You can multiply by a scalar constant to find another vector which is also perpendicular to both **a** and **b**.

**Example 17**

The points $A$, $B$ and $C$ have coordinates (2, −1, 1), (5, 1, 7) and (6, −3, 1) respectively.

**a** Find $\overrightarrow{AB}.\overrightarrow{AC}$

**b** Hence, or otherwise, find the area of triangle $ABC$.

**a** $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix}$

$\overrightarrow{AB}.\overrightarrow{AC} = 3 \times 4 + 2 \times (-2) + 6 \times 0 = 8$

**b** $|\overrightarrow{AB}| = \sqrt{3^2 + 2^2 + 6^2} = 7$

$|\overrightarrow{AC}| = \sqrt{4^2 + (-2)^2 + 0^2} = 2\sqrt{5}$

$\cos(\angle BAC) = \dfrac{\overrightarrow{AB}.\overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|}$

$= \dfrac{8}{7 \times 2\sqrt{5}}$

$= 0.2555\ldots$

$\angle BAC = 75.1937\ldots°$

Area $= \frac{1}{2}|\overrightarrow{AB}||\overrightarrow{AC}|\sin(\angle BAC)$

$= \frac{1}{2} \times 7 \times 2\sqrt{5}\sin(75.1937\ldots°)$

$= 15.13$ (2 d.p.)

Use the scalar product to find the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Then use area $= \frac{1}{2}ab\sin\theta$ to find the area of the triangle.

**Problem-solving**

You could find $\angle BAC$ by finding the lengths $AB$, $BC$ and $AC$ and using the cosine rule, but it is quicker to use a vector method.

## Exercise 9C

**1** The vectors **a** and **b** each have magnitude 3, and the angle between **a** and **b** is 60°. Find **a.b**.

**2** For each pair of vectors, find **a.b**:

**a** $\mathbf{a} = 5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$

**b** $\mathbf{a} = 10\mathbf{i} - 7\mathbf{j} + 4\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} - 5\mathbf{j} - 12\mathbf{k}$

**c** $\mathbf{a} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{b} = -\mathbf{i} - \mathbf{j} + 4\mathbf{k}$

**d** $\mathbf{a} = 2\mathbf{i} - \mathbf{k}$, $\mathbf{b} = 6\mathbf{i} - 5\mathbf{j} - 8\mathbf{k}$

**e** $\mathbf{a} = 3\mathbf{j} + 9\mathbf{k}$, $\mathbf{b} = \mathbf{i} - 12\mathbf{j} + 4\mathbf{k}$

**3** In each part, find the angle between **a** and **b**, giving your answer in degrees to 1 decimal place:

**a** $\mathbf{a} = 3\mathbf{i} + 7\mathbf{j}$, $\mathbf{b} = 5\mathbf{i} + \mathbf{j}$

**b** $\mathbf{a} = 2\mathbf{i} - 5\mathbf{j}$, $\mathbf{b} = 6\mathbf{i} + 3\mathbf{j}$

**c** $\mathbf{a} = \mathbf{i} - 7\mathbf{j} + 8\mathbf{k}$, $\mathbf{b} = 12\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

**d** $\mathbf{a} = -\mathbf{i} - \mathbf{j} + 5\mathbf{k}$, $\mathbf{b} = 11\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$

**e** $\mathbf{a} = 6\mathbf{i} - 7\mathbf{j} + 12\mathbf{k}$, $\mathbf{b} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$

**f** $\mathbf{a} = 4\mathbf{i} + 5\mathbf{k}$, $\mathbf{b} = 6\mathbf{i} - 2\mathbf{j}$

**g** $\mathbf{a} = -5\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - 2\mathbf{j} - 11\mathbf{k}$

**h** $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$

**4** Find the value, or values, of $\lambda$ for which the given vectors are perpendicular:

**a** $3\mathbf{i} + 5\mathbf{j}$ and $\lambda\mathbf{i} + 6\mathbf{j}$

**b** $2\mathbf{i} + 6\mathbf{j} - \mathbf{k}$ and $\lambda\mathbf{i} - 4\mathbf{j} - 14\mathbf{k}$

**c** $3\mathbf{i} + \lambda\mathbf{j} - 8\mathbf{k}$ and $7\mathbf{i} - 5\mathbf{j} + \mathbf{k}$

**d** $9\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$ and $\lambda\mathbf{i} + \lambda\mathbf{j} + 3\mathbf{k}$

**e** $\lambda\mathbf{j} + 3\mathbf{j} - 2\mathbf{k}$ and $\lambda\mathbf{i} + \lambda\mathbf{j} + 5\mathbf{k}$

**5** Find, to the nearest tenth of a degree, the angle that the vector $9\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$ makes with:

**a** the positive $x$-axis

**b** the positive $y$-axis

**6** Find, to the nearest tenth of a degree, the angle that the vector $\mathbf{i} + 11\mathbf{j} - 4\mathbf{k}$ makes with:

**a** the positive $y$-axis

**b** the positive $z$-axis

**7** The angle between the vectors $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $2\mathbf{i} + \mathbf{j} + \mathbf{k}$ is $\theta$. Calculate the exact value of $\cos\theta$.

**8** The angle between the vectors $\mathbf{i} + 3\mathbf{j}$ and $\mathbf{j} + \lambda\mathbf{k}$ is 60°. Show that $\lambda = \pm\sqrt{\frac{13}{5}}$

**9** Find a vector which is perpendicular to both **a** and **b**, where:

**a** $\mathbf{a} = \mathbf{i} + \mathbf{j} - 3\mathbf{k}$, $\mathbf{b} = 5\mathbf{i} - 2\mathbf{j} - \mathbf{k}$

**b** $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$, $\mathbf{b} = \mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$

**c** $\mathbf{a} = 4\mathbf{i} - 4\mathbf{j} - \mathbf{k}$, $\mathbf{b} = -2\mathbf{i} - 9\mathbf{j} + 6\mathbf{k}$

**10** The points $A$ and $B$ have position vectors $2\mathbf{i} + 5\mathbf{j} + \mathbf{k}$ and $6\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ respectively, and $O$ is the origin. Calculate each of the angles in $\triangle OAB$, giving your answers in degrees to 1 decimal place.

**11** The points $A$, $B$ and $C$ have coordinates $(1, 3, 1)$, $(2, 7, -3)$ and $(4, -5, 2)$ respectively.

**a** Find the exact lengths of $AB$ and $BC$.

**b** Calculate, to one decimal place, the size of $\angle ABC$.

E/P **12** Given that the points $A$ and $B$ have coordinates $(7, 4, 4)$ and $(2, 2, 1)$ respectively,

**a** find the value of $\cos\angle AOB$, where $O$ is the origin **(4 marks)**

**b** show that the area of $\triangle AOB$ is $\frac{\sqrt{53}}{2}$ **(3 marks)**

(P) 13 $AB$ is a diameter of a circle centred at the origin $O$, and $P$ is a point on the circumference of the circle. By considering the position vectors of $A$, $B$ and $P$, prove that $AP$ is perpendicular to $BP$.

**Problem-solving**

This is a vector proof of the fact that the angle in a semi-circle is 90°.

(E/P) 14 Points $A$, $B$ and $C$ have coordinates $(5, -1, 0)$, $(2, 4, 10)$ and $(6, -1, 4)$ respectively.

**a** Find the vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$. **(2 marks)**

**b** Find the area of the triangle $ABC$. **(4 marks)**

**c** Point $D$ is such that $A$, $B$, $C$ and $D$ are the vertices of a parallelogram. Find the coordinates of three possible positions of $D$. **(3 marks)**

**d** Write down the area of the parallelogram. **(1 mark)**

(E/P) 15 The points $P$, $Q$ and $R$ have coordinates $(1, -1, 6)$, $(-2, 5, 4)$ and $(0, 3, -5)$ respectively.

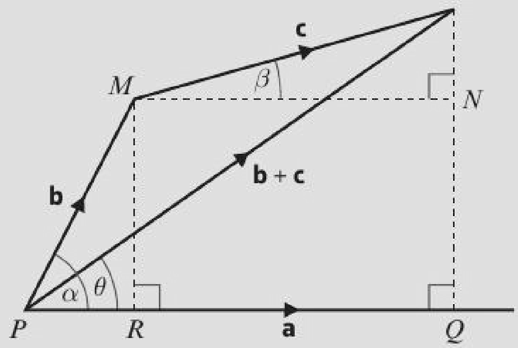
**a** Show that $PQ$ is perpendicular to $QR$. **(3 marks)**

**b** Hence find the centre and radius of the circle that passes through points $P$, $Q$ and $R$. **(3 marks)**

**Challenge**

**1** Using the definition $\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$, prove that $\mathbf{a}.\mathbf{b} = \mathbf{b}.\mathbf{a}$.

**2** The diagram shows arbitrary vectors **a**, **b** and **c**, and the vector **b** + **c**.




**a** Show that:

**i** $\mathbf{a}.(\mathbf{b}+\mathbf{c}) = |\mathbf{a}| \times PQ$

**ii** $\mathbf{a}.\mathbf{b} = |\mathbf{a}| \times PR$

**iii** $\mathbf{a}.\mathbf{c} = |\mathbf{a}| \times RQ$

**b** Hence prove that $\mathbf{a}.(\mathbf{b}+\mathbf{c}) = \mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c}$.

## 9.4 Calculating angles between lines and planes

If two straight lines in three dimensions intersect, then you can calculate the size of the angle between them using the scalar product.

- **The acute angle $\theta$ between two intersecting straight lines is given by**

$$\cos\theta = \left|\frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right|$$

**where a and b are direction vectors of the lines.**

**Watch out** The modulus signs around the whole expression ensure you get an acute angle. If you need to work out the size of an **obtuse** angle between two lines, use the formula then subtract the resulting acute angle from 180°.

## Example 18

The lines $l_1$ and $l_2$ have vector equations $\mathbf{r} = (2\mathbf{i} + \mathbf{j} + \mathbf{k}) + t(3\mathbf{i} - 8\mathbf{j} - \mathbf{k})$ and $\mathbf{r} = (7\mathbf{i} + 4\mathbf{j} + \mathbf{k}) + s(2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$ respectively.

Given that $l_1$ and $l_2$ intersect, find the size of the acute angle between the lines to one decimal place.

$\mathbf{a} = \begin{pmatrix} 3 \\ -8 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$ — Use the direction vectors.

$$\cos\theta = \left| \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \right|$$

$$\mathbf{a}.\mathbf{b} = \begin{pmatrix} 3 \\ -8 \\ -1 \end{pmatrix} . \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$$

$$= 6 - 16 - 3 = -13$$

$$|\mathbf{a}| = \sqrt{3^2 + (-8)^2 + (-1)^2} = \sqrt{74}$$

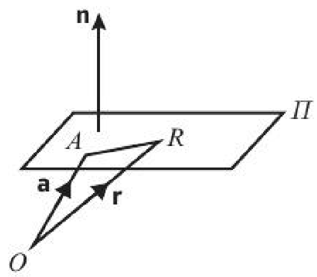
$$|\mathbf{b}| = \sqrt{2^2 + 2^2 + 3^2} = \sqrt{17}$$

$$\cos\theta = \left| \frac{-13}{\sqrt{74}\sqrt{17}} \right|$$

Use the formula. Be careful with the modulus signs. If $\cos\theta$ is positive then $\theta$ will be an acute angle, as required.

$$\theta = 68.5° \text{ (1 d.p.)}$$

You can use the scalar product to write a vector equation of a plane efficiently.




Suppose a plane $\Pi$ passes through a given point $A$, with position vector $\mathbf{a}$, and that the normal vector $\mathbf{n}$ is perpendicular to the plane. Let $R$ be an arbitrary point on the plane, with position vector $\mathbf{r}$.

Then, $\overrightarrow{AR} = \mathbf{r} - \mathbf{a}$

As $\overrightarrow{AR}$ is a vector which lies in the plane, $\overrightarrow{AR}$ is perpendicular to $\mathbf{n}$ so $\overrightarrow{AR}.\mathbf{n} = 0$.

This means $(\mathbf{r} - \mathbf{a}).\mathbf{n} = 0$

You can rewrite this as $\mathbf{r}.\mathbf{n} = \mathbf{a}.\mathbf{n}$

Since $\mathbf{a}$ is a fixed point, $\mathbf{a}.\mathbf{n}$ is a scalar constant, $k$, and the equation of the plane $\Pi$ is $\mathbf{r}.\mathbf{n} = k$.

- **The scalar product form of the equation of a plane is $\mathbf{r}.\mathbf{n} = k$ where $k = \mathbf{a}.\mathbf{n}$ for any point in the plane with position vector $\mathbf{a}$.**

## Example 19

The plane $\Pi$ passes through the point $A$ and is perpendicular to the vector $\mathbf{n}$.

Given that $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$ and $\mathbf{n} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$ where $O$ is the origin, find an equation of the plane:

**a** in scalar product form **b** in Cartesian form a $\mathbf{r.n} = k$, where $k = \mathbf{a.n}$

$$\mathbf{a.n} = \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix} . \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$$

$$= 2 \times 3 + 3 \times 1 + (-5) \times (-1)$$

$$= 6 + 3 + 5 = 14$$

So a scalar product form of the equation of $\Pi$ is $\mathbf{r}.\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = 14$.

b $\begin{pmatrix} x \\ y \\ z \end{pmatrix} . \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = 14$

So a Cartesian form of equation of $\Pi$ is $3x + y - z = 14$

Use $\mathbf{r.n} = k$ where $k = \mathbf{a.n}$ for any point in the plane with position vector $\mathbf{a}$.

**Problem-solving**

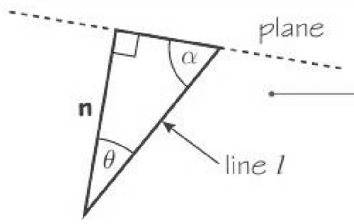
You can convert between scalar product form and Cartesian form quickly by writing the general position vector of a point in the plane as $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

You need to be able to calculate the angle between a line and a plane.

## Example 20

Find the acute angle between the line $l$ with equation $\mathbf{r} = 2\mathbf{i} + \mathbf{j} - 5\mathbf{k} + \lambda(3\mathbf{i} + 4\mathbf{j} - 12\mathbf{k})$ and the plane with equation $\mathbf{r}.(2\mathbf{i} - 2\mathbf{j} - \mathbf{k}) = 2$.

**Online** Explore the angle between a line and a plane using GeoGebra.




Draw a diagram showing the line, the plane and the normal to the plane. Let the required angle be $\alpha$ and show $\alpha$ and $\theta$ in your diagram.

The normal to the plane is in the direction $\mathbf{n} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$.

The angle between this normal and the line $l$ is $\theta$,

where $\cos\theta = \dfrac{(3\mathbf{i} + 4\mathbf{j} - 12\mathbf{k}).(2\mathbf{i} - 2\mathbf{j} - \mathbf{k})}{\sqrt{3^2 + 4^2 + (-12)^2}\,\sqrt{2^2 + (-2)^2 + (-1)^2}}$

$$= \frac{10}{13 \times 3} = \frac{10}{39}$$

First find the angle between the given line and the normal to the plane.

So the angle between the plane and the line $l$ is $\alpha$ where $\alpha + \theta = 90°$.

So $\sin\alpha = \dfrac{10}{39}$ and $\alpha = 14.9°$.

Subtract the angle $\theta$ from 90°, to give angle $\alpha$, or use the trigonometric connection that $\cos\theta = \sin\alpha$.

- **The acute angle $\theta$ between the line with equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and the plane with equation $\mathbf{r.n} = k$ is given by the formula**

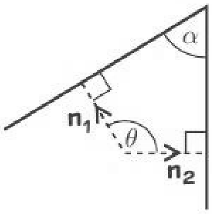
$$\sin\theta = \left|\frac{\mathbf{b.n}}{|\mathbf{b}||\mathbf{n}|}\right|$$

You need to be able to calculate the angle between two planes.

**Example 21**

Find the acute angle between the planes with equations $\mathbf{r}.(4\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}) = 13$ and $\mathbf{r}.(7\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}) = 6$ respectively.

**Online** Visualise the angle between two planes using GeoGebra.




Draw a diagram showing the planes and the normals to the planes. Let the required angle be $\alpha$ and show $\alpha$ and $\theta$ in your diagram.

The normals to the planes are in the directions $\mathbf{n}_1 = 4\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}$, and $\mathbf{n}_2 = 7\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$.

The angle between these normals is $\theta$, where

$$\cos\theta = \frac{(4\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}).(7\mathbf{i} - 4\mathbf{j} + 4\mathbf{k})}{\sqrt{4^2 + 4^2 + (-7)^2}\sqrt{7^2 + (-4)^2 + 4^2}}$$

$$= \frac{28 - 16 - 28}{\sqrt{16 + 16 + 49}\sqrt{49 + 16 + 16}}$$

$$= -\frac{16}{81}$$

First find the angle between the normals to the planes.

So $\theta = 101.4°$

So the angle between the planes is $180 - 101.4 = 78.6°$

Subtract the angle $\theta$ from 180°, to give angle $\alpha$.

- **The acute angle $\theta$ between the plane with equation $\mathbf{r}.\mathbf{n}_1 = k_1$ and the plane with equation $\mathbf{r}.\mathbf{n}_2 = k_2$ is given by the formula**

$$\cos\theta = \left|\frac{\mathbf{n}_1.\mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right|$$

**Exercise 9D**

**1** Given that each pair of lines intersect, find, to 1 decimal place, the acute angle between the lines with vector equations:

**a** $\mathbf{r} = (2\mathbf{i} + \mathbf{j} + \mathbf{k}) + \lambda(3\mathbf{i} - 5\mathbf{j} - \mathbf{k})$ and $\mathbf{r} = (7\mathbf{i} + 4\mathbf{j} + \mathbf{k}) + \mu(2\mathbf{i} + \mathbf{j} - 9\mathbf{k})$

**b** $\mathbf{r} = (\mathbf{i} - \mathbf{j} + 7\mathbf{k}) + \lambda(-2\mathbf{i} - \mathbf{j} + 3\mathbf{k})$ and $\mathbf{r} = (8\mathbf{i} + 5\mathbf{j} - \mathbf{k}) + \mu(-4\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

**c** $\mathbf{r} = (3\mathbf{i} + 5\mathbf{j} - \mathbf{k}) + \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k})$ and $\mathbf{r} = (-\mathbf{i} + 11\mathbf{j} + 5\mathbf{k}) + \mu(2\mathbf{i} - 7\mathbf{j} + 3\mathbf{k})$

**d** $\mathbf{r} = (\mathbf{i} + 6\mathbf{j} - \mathbf{k}) + \lambda(8\mathbf{i} - \mathbf{j} - 2\mathbf{k})$ and $\mathbf{r} = (6\mathbf{i} + 9\mathbf{j}) + \mu(\mathbf{i} + 3\mathbf{j} - 7\mathbf{k})$

**e** $\mathbf{r} = (2\mathbf{i} + \mathbf{k}) + \lambda(11\mathbf{i} + 5\mathbf{j} - 3\mathbf{k})$ and $\mathbf{r} = (\mathbf{i} + \mathbf{j}) + \mu(-3\mathbf{i} + 5\mathbf{j} + 4\mathbf{k})$

**2** Find, in the form $\mathbf{r}.\mathbf{n} = k$, an equation of the plane that passes through the point with position vector $\mathbf{a}$ and is perpendicular to the vector $\mathbf{n}$ where:

**a** $\mathbf{a} = \mathbf{i} - \mathbf{j} - \mathbf{k}$ and $\mathbf{n} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$

**b** $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{n} = 5\mathbf{i} - \mathbf{j} - 3\mathbf{k}$

**c** $\mathbf{a} = 2\mathbf{i} - 3\mathbf{k}$ and $\mathbf{n} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$

**d** $\mathbf{a} = 4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{n} = 4\mathbf{i} + \mathbf{j} - 5\mathbf{k}$

**3** Find a Cartesian equation for each of the planes in question 2.

**4** A plane has equation $\mathbf{r}.\mathbf{n} = k$, where $\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$. Find a Cartesian equation of the plane.

**5** Find the acute angle between the line with equation $\mathbf{r} = 2\mathbf{i} + \mathbf{j} - 5\mathbf{k} + \lambda(4\mathbf{i} + 4\mathbf{j} + 7\mathbf{k})$ and the plane with equation $\mathbf{r}.(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 13$.

**6** Find the acute angle between the line with equation $\mathbf{r} = -\mathbf{i} - 7\mathbf{j} + 13\mathbf{k} + \lambda(3\mathbf{i} + 4\mathbf{j} - 12\mathbf{k})$ and the plane with equation $\mathbf{r}.(4\mathbf{i} - 4\mathbf{j} - 7\mathbf{k}) = 9$.

**7** Find the acute angle between the planes with equations $\mathbf{r}.(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = 1$ and $\mathbf{r}.(-4\mathbf{i} + 4\mathbf{j} + 7\mathbf{k}) = 7$ respectively.

**8** Find the acute angle between the planes with equations $\mathbf{r}.(3\mathbf{i} - 4\mathbf{j} + 12\mathbf{k}) = 9$ and $\mathbf{r}.(5\mathbf{i} - 12\mathbf{k}) = 7$ respectively.

P **9** The straight lines $l_1$ and $l_2$ have vectors equations $\mathbf{r} = (\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}) + \lambda(8\mathbf{i} + 5\mathbf{j} + \mathbf{k})$ and $\mathbf{r} = (\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}) + \mu(3\mathbf{i} + \mathbf{j})$ respectively, and $P$ is the point with coordinates (1, 4, 2).

**a** Show that the point $Q(9, 9, 3)$ lies on $l_1$.

Given that $l_1$ and $l_2$ intersect, find:

**b** the cosine of the acute angle between $l_1$ and $l_2$

**c** the possible coordinates of the point $R$, such that $R$ lies on $l_2$ and $PQ = PR$.

E/P **10** The lines $l_1$ and $l_2$ have Cartesian equations $\frac{x-6}{-1} = \frac{y+3}{2} = \frac{z+2}{3}$ and $\frac{x+5}{2} = \frac{y-15}{-3} = \frac{z-3}{1}$ respectively.

**a** Show that the point $A(3, 3, 7)$ lies on both $l_1$ and $l_2$. **(3 marks)**

**b** Find the size of the acute angle between the lines at $A$. **(4 marks)**

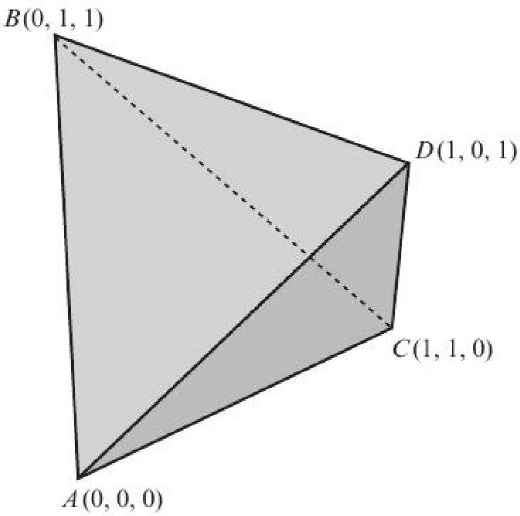
E **11** The lines $l_1$ and $l_2$ have vector equations $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} + \mu\begin{pmatrix} -4 \\ -3 \\ 1 \end{pmatrix}$.

The point $A$ is on $l_1$ where $\lambda = 3$ and the point $B$ is on $l_2$ where $\mu = -2$. Find the size of the acute angle between $AB$ and $l_1$. **(6 marks)**

E/P **12** **a** Show that the points $A(3, 5, -1)$, $B(2, -2, 4)$, $C(4, 3, 0)$ and $D(1, 4, -3,)$ are not coplanar. **(6 marks)**

**b** Find the angle between the plane containing $A$, $B$ and $C$ and the line segment $AD$. **(4 marks)**

E/P **13** A regular tetrahedron has vertices $A$, $B$, $C$ and $D$, with coordinates (0, 0, 0), (0, 1, 1), (1, 1, 0) and (1, 0, 1) respectively. Show that the angle between any two adjacent faces of the tetrahedron is $\arccos\left(\frac{1}{3}\right)$.

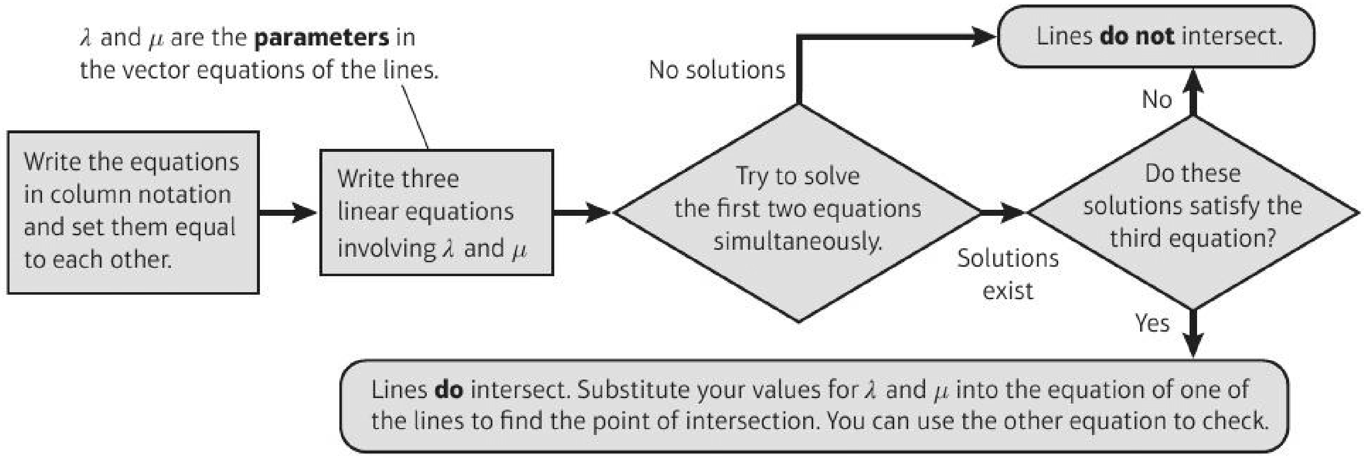



**(7 marks)**

E/P **14** A flagpole is supported by 3 guide ropes which are attached at a point 20 m above the base of the pole. The ends of the ropes are secured at points with position vectors (0, 8, 2), (12, −5, 3) and (−2, 6, 5) relative to the base of the pole, where the units are metres.
The flagpole will be stable if the angles between adjacent guide ropes are all greater than 15°.
Determine whether the flagpole will be stable, showing your working clearly. **(7 marks)**

## 9.5 Points of intersection

You need to be able to determine whether two lines meet and, if so, to determine their point of intersection.




### Example 22

The lines $l_1$ and $l_2$ have vector equations

$\mathbf{r} = 3\mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} - \mathbf{k})$ and $\mathbf{r} = -2\mathbf{j} + 3\mathbf{k} + \mu(-5\mathbf{i} + \mathbf{j} + 4\mathbf{k})$ respectively.

Show that the two lines intersect, and find the position vector of the point of intersection.

$$\begin{pmatrix} 3 + \lambda \\ 1 - 2\lambda \\ 1 - \lambda \end{pmatrix} = \begin{pmatrix} -5\mu \\ -2 + \mu \\ 3 + 4\mu \end{pmatrix}$$

Use column vector notation for clarity, and to help to avoid errors.

Solve the simultaneous equations

$3 + \lambda = -5\mu$ (1)

and $1 - \lambda = 3 + 4\mu$ (2)

Choose two of the three equations obtained by equating $x$-, $y$- and $z$-components and solve the resulting simultaneous equations.

Adding gives $4 = 3 - \mu$

and so $\mu = -1$.

Substituting back into equation (1) gives $\lambda = 2$.

Check $\mu = -1$, $\lambda = 2$ also satisfy the third equation.

If the lines intersect there is a pair of values of $\lambda$ and $\mu$ that satisfy the 3 equations simultaneously.

$1 - 2\lambda = -2 + \mu$ gives $-3 = -3$

So the lines do intersect.

Substituting $\lambda = 2$ into $\begin{pmatrix} 3 + \lambda \\ 1 - 2\lambda \\ 1 - \lambda \end{pmatrix}$ gives $\begin{pmatrix} 5 \\ -3 \\ -1 \end{pmatrix}$.

The point where the lines meet is (5, −3, −1).

Check that the point which you obtain after substitution lies on both straight lines.

You also need to be able to find the coordinates of the point of intersection of a line with a plane.

## Example 23

Find the coordinates of the point of intersection of the line $l$ and the plane $\Pi$ where $l$ has equation $\mathbf{r} = -\mathbf{i} + \mathbf{j} - 5\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$ and $\Pi$ has equation $\mathbf{r}.(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = 4$.

The line meets the plane when

$$\begin{pmatrix} -1 + \lambda \\ 1 + \lambda \\ -5 + 2\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 4$$

Write the equation of the line in column vector form as $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 + \lambda \\ 1 + \lambda \\ -5 + 2\lambda \end{pmatrix}$ and substitute into the equation of the plane $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 4$.

$$-1 + \lambda + 2(1 + \lambda) + 3(-5 + 2\lambda) = 4$$
$$9\lambda - 14 = 4$$
$$9\lambda = 18$$
$$\lambda = 2$$

So the line meets the plane when $\lambda = 2$, at the point $(1, 3, -1)$.

Solve to find $\lambda$ and substitute its value into the equation of the line.

**Watch out** If the line were parallel to the plane then this equation would produce either no solutions (if the line does not lie in the plane), or infinitely many (if it does).

- **Two straight lines are skew if they are not parallel and they do not intersect.**

## Example 24

The lines $l_1$ and $l_2$ have equations $\frac{x-2}{4} = \frac{y+3}{2} = z - 1$ and $\frac{x+1}{5} = \frac{y}{4} = \frac{z-4}{-2}$ respectively.

Prove that $l_1$ and $l_2$ are skew.

**Problem-solving**

To show that two lines are skew you need to show that they do not intersect **and** that they are not parallel. Start by writing the general point on each line. Equate these general points and attempt to solve the three equations simultaneously.

$$\begin{pmatrix} 2 + 4\lambda \\ -3 + 2\lambda \\ 1 + \lambda \end{pmatrix} = \begin{pmatrix} -1 + 5\mu \\ 4\mu \\ 4 - 2\mu \end{pmatrix}$$

$2 + 4\lambda = -1 + 5\mu$ (1)

$-3 + 2\lambda = 4\mu$ (2)

(1) − 2 × (2): $8 = -1 - 3\mu \Rightarrow \mu = -3$

Substituting into (2): $-3 + 2\lambda = -12 \Rightarrow \lambda = -\frac{9}{2}$

Check for consistency:

$1 + \lambda = -\frac{7}{2}$ and $4 - 2\mu = 10$.

Solve the first two equations simultaneously, then check to see whether the answer is consistent with the third equation.

$1 + \lambda \neq 4 - 2\mu$ so equations not consistent and lines do not intersect.

Direction of $l_1$ is $\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 8 \\ 4 \\ 2 \end{pmatrix}$.

Direction of $l_2$ is $\begin{pmatrix} 5 \\ 4 \\ -2 \end{pmatrix}$.

Direction vectors are not scalar multiples of each other so lines are not parallel.

If the lines are parallel the direction vectors will be scalar multiples of each other. Multiply the direction vector of $l_1$ by a scalar to make one component match the direction vector of $l_2$, then compare the other components.

Hence $l_1$ and $l_2$ are skew.

## Exercise 9E

**1** In each case establish whether lines $l_1$ and $l_2$ meet and, if they meet, find the coordinates of their point of intersection:

**a** $l_1$ has equation $\mathbf{r} = \mathbf{i} + 3\mathbf{j} + \lambda(\mathbf{i} - \mathbf{j} + 5\mathbf{k})$ and $l_2$ has equation $\mathbf{r} = -\mathbf{i} - 3\mathbf{j} + 2\mathbf{k} + \mu(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

**b** $l_1$ has equation $\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$ and $l_2$ has equation $\mathbf{r} = 4\mathbf{i} + 3\mathbf{j} + \mu(-\mathbf{i} + \mathbf{j} - \mathbf{k})$

**c** $l_1$ has equation $\mathbf{r} = \mathbf{i} + 3\mathbf{j} + 5\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} + \mathbf{k})$ and $l_2$ has equation $\mathbf{r} = \mathbf{i} + \frac{5}{2}\mathbf{j} + \frac{5}{2}\mathbf{k} + \mu(\mathbf{i} + \mathbf{j} - 2\mathbf{k})$

(In each of the above cases $\lambda$ and $\mu$ are scalars.)

E **2** With respect to a fixed origin $O$, the lines $l_1$ and $l_2$ are given by the equations

$l_1$: $\mathbf{r} = (-6\mathbf{i} + 11\mathbf{k}) + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k})$

$l_2$: $\mathbf{r} = (2\mathbf{i} - 2\mathbf{j} + 9\mathbf{k}) + \mu(2\mathbf{i} + \mathbf{j} - 3\mathbf{k})$

Show that $l_1$ and $l_2$ meet and find the position vector of their point of intersection. **(6 marks)**

E **3** The line $l_1$ has equation $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$ and the line $l_2$ has equation $\mathbf{r} = \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$.

Show that $l_1$ and $l_2$ do not meet. **(4 marks)**

**4** In each case, find the coordinates of the point of intersection of the line $l$ with the plane $\Pi$.

**a** $l$: $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(-2\mathbf{i} + \mathbf{j} - 4\mathbf{k})$

$\Pi$: $\mathbf{r}.(3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = 16$

**b** $l$: $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(2\mathbf{j} - 2\mathbf{k})$

$\Pi$: $\mathbf{r}.(3\mathbf{i} - \mathbf{j} - 6\mathbf{k}) = 1$

E/P **5** The line $l$ has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

**a** Show that $l$ does not meet the plane with equation $\mathbf{r}.\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = 1$. **(4 marks)**

**b** Give a geometrical interpretation to your answer to part **a**. **(1 mark)**

E/P **6** The line with vector equation $\mathbf{r} = \begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ is perpendicular to the line with vector equation $\mathbf{r} = \begin{pmatrix} 0 \\ 11 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ p \\ p \end{pmatrix}$.

**a** Find the value of $p$. **(2 marks)**

**b** Show that the two lines meet, and find the coordinates of the point of intersection. **(4 marks)**

E **7** The line $l_1$ has vector equation $\mathbf{r} = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ and the line $l_2$ has vector equation

$\mathbf{r} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, where $\lambda$ and $\mu$ are parameters.

The lines $l_1$ and $l_2$ intersect at the point $A$ and the acute angle between $l_1$ and $l_2$ is $\theta$.

**a** Find the coordinates of $A$. **(4 marks)**

**b** Find the value of $\cos\theta$ giving your answer as a simplified fraction. **(4 marks)**

E/P **8** The lines $l_1$ and $l_2$ have equations $\frac{x}{-3} = \frac{y+1}{5} = \frac{z-2}{4}$ and $x = \frac{y-1}{-2} = \frac{z+5}{2}$ respectively.

Prove that $l_1$ and $l_2$ are skew.

E/P **9** With respect to a fixed origin $O$ the lines $l_1$ and $l_2$ are given by the equations

$$l_1: \mathbf{r} = \begin{pmatrix} 8 \\ 2 \\ -12 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} \qquad l_2: \mathbf{r} = \begin{pmatrix} -4 \\ 10 \\ p \end{pmatrix} + \mu\begin{pmatrix} q \\ 2 \\ -1 \end{pmatrix}$$

where $\lambda$ and $\mu$ are parameters and $p$ and $q$ are constants. Given that $l_1$ and $l_2$ are perpendicular,

**a** show that $q = 4$. **(2 marks)**

Given further that $l_1$ and $l_2$ intersect, find:

**b** the value of $p$ **(6 marks)**

**c** the coordinates of the point of intersection. **(2 marks)**

The point $A$ lies on $l_1$ and has position vector $\begin{pmatrix} 9 \\ -1 \\ -14 \end{pmatrix}$. The point $C$ lies on $l_2$.

Given that a circle, with centre $C$, cuts the line $l_1$ at the points $A$ and $B$,

**d** find the position vector of $B$. **(3 marks)**

**Problem-solving**

Draw a diagram showing the lines $l_1$ and $l_2$ and the circle, and use circle properties.

E/P **10** The plane $\Pi$ has equation $\mathbf{r}.\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = k$ where $k$ is a constant.

Given the point with position vector $\begin{pmatrix} 6 \\ -2 \\ 4 \end{pmatrix}$ lies on $\Pi$,

**a** find the value of $k$ **(3 marks)**

**b** find a Cartesian equation for $\Pi$. **(2 marks)**

The point $P$ has coordinates (6, 4, 8). The line $l$ passes through $P$ and is perpendicular to $\Pi$. The line $l$ intersects $\Pi$ at the point $N$.

**c** Find the coordinates of $N$. **(4 marks)**

E/P **11** The line $l$ has a Cartesian equation $\frac{x-3}{5} = \frac{y+2}{3} = \frac{4-z}{1}$

The plane $\Pi$ has Cartesian equation $4x + 3y - 2z = -10$.
The line intersects the plane at the point $P$.

**a** Find the position vector of $P$. **(5 marks)**

**b** Find the acute angle between the line and the plane at the point of intersection. **(5 marks)**

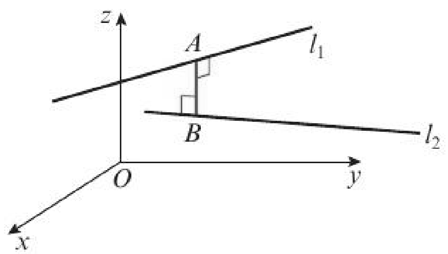
## 9.6 Finding perpendiculars

You need to be able to calculate the **perpendicular distance** between:

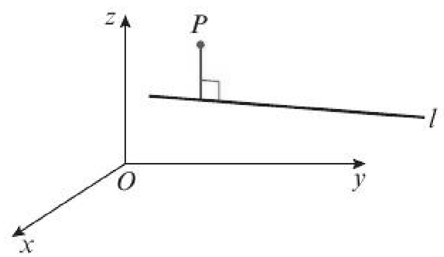
- two lines
- a point and a line
- a point and a plane

In each case, the perpendicular distance is the **shortest distance** between them.

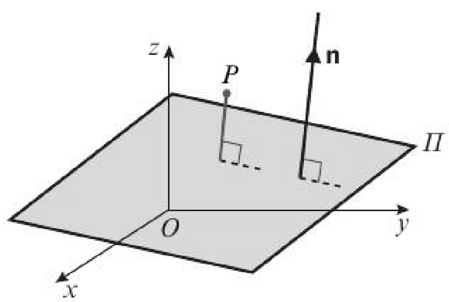
- **For any two non-intersecting lines $l_1$ and $l_2$ there is a unique line segment $AB$ such that $A$ lies on $l_1$, $B$ lies on $l_2$ and $AB$ is perpendicular to both lines.**




- **The perpendicular from a point $P$ to a line $l$ is a line through $P$ which meets $l$ at right angles.**




- **The perpendicular from a point $P$ to a plane $\Pi$ is a line through $P$ which is parallel to the normal vector of the plane, n.**



## Example 24

Show that the shortest distance between the parallel lines with equations
$\mathbf{r} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} + \lambda(5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k})$ and $\mathbf{r} = 2\mathbf{i} + \mathbf{k} + \mu(5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k})$,
where $\lambda$ and $\mu$ are scalars, is $\dfrac{21\sqrt{2}}{10}$

Let $A$ be a general point on the first line and $B$ be a general point on the second line, then

$$\overrightarrow{AB} = \begin{pmatrix}1\\-2\\2\end{pmatrix} + t\begin{pmatrix}5\\4\\3\end{pmatrix} \text{ where } t = \mu - \lambda.$$

$$\begin{pmatrix}1+5t\\-2+4t\\2+3t\end{pmatrix}\cdot\begin{pmatrix}5\\4\\3\end{pmatrix} = 0$$

$$5 + 25t - 8 + 16t + 6 + 9t = 0$$
$$50t = -3$$
$$t = -\tfrac{3}{50}$$

$$\begin{pmatrix}1-5t\\-2+4t\\2+3t\end{pmatrix} = \begin{pmatrix}1-\frac{15}{50}\\-2-\frac{12}{50}\\2-\frac{9}{50}\end{pmatrix} = \begin{pmatrix}\frac{35}{50}\\-\frac{112}{50}\\\frac{91}{50}\end{pmatrix}$$

$$|\overrightarrow{AB}| = \sqrt{\frac{35^2 + 112^2 + 91^2}{50^2}}$$
$$= \frac{21\sqrt{2}}{10}$$

So the shortest distance between the two lines is $\dfrac{21\sqrt{2}}{10}$

As $\begin{pmatrix}1\\-2\\2\end{pmatrix} = \begin{pmatrix}2\\0\\1\end{pmatrix} - \begin{pmatrix}1\\2\\-1\end{pmatrix}$

You can set $t = \mu - \lambda$ so that there is only one independent variable.

As the direction of $\overrightarrow{AB}$ is perpendicular to the direction vector for each line, the scalar product is zero.

Substitute $t = -\frac{3}{50}$ into the general form of $\overrightarrow{AB}$.

The shortest distance between two lines is the length of the line segment that is perpendicular to both lines.

**Watch out** Because the two lines are parallel, the line segment $AB$ is not unique. There are infinitely many line segments that are perpendicular to both lines, but they will all have the same length.

## Example 25

The lines $l_1$ and $l_2$ have equations $\mathbf{r} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}0\\1\\1\end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix}-1\\3\\-1\end{pmatrix} + \mu\begin{pmatrix}2\\-1\\-1\end{pmatrix}$ respectively,
where $\lambda$ and $\mu$ are scalars.
Find the shortest distance between these two lines.

Let $A$ be the general point on $l_1$ with position vector $\begin{pmatrix}1\\\lambda\\\lambda\end{pmatrix}$ and let $B$ be the general point on $l_2$ with position vector $\begin{pmatrix}-1+2\mu\\3-\mu\\-1-\mu\end{pmatrix}$

**Online** Explore the perpendicular distance between two lines using GeoGebra.

$$\overrightarrow{AB} = \begin{pmatrix} -1 + 2\mu \\ 3 - \mu \\ -1 - \mu \end{pmatrix} - \begin{pmatrix} 1 \\ \lambda \\ \lambda \end{pmatrix} = \begin{pmatrix} -2 + 2\mu \\ 3 - \mu - \lambda \\ -1 - \mu - \lambda \end{pmatrix}$$

Find position vectors of general points on $l_1$ and $l_2$ and use these to find $\overrightarrow{AB}$ in terms of $\mu$ and $\lambda$.

$\overrightarrow{AB}$ is perpendicular to $l_1$, so:

$$\begin{pmatrix} -2 + 2\mu \\ 3 - \mu - \lambda \\ -1 - \mu - \lambda \end{pmatrix} . \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0$$

$3 - \mu - \lambda - 1 - \mu - \lambda = 0$

$2 - 2\mu - 2\lambda = 0 \qquad (1)$

$\overrightarrow{AB}$ is perpendicular to $l_2$ so:

$$\begin{pmatrix} -2 + 2\mu \\ 3 - \mu - \lambda \\ -1 - \mu - \lambda \end{pmatrix} . \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = 0$$

$-4 + 4\mu - 3 + \mu + \lambda + 1 + \mu + \lambda = 0$

$-6 + 6\mu + 2\lambda = 0 \qquad (2)$

From (1), $\lambda = 1 - \mu$

From (2), $-3 + 3\mu + \lambda = 0$

So $-3 + 3\mu + 1 - \mu = 0 \Rightarrow \mu = 1$

Substituting into (1) gives

$2 - 2 - 2\lambda = 0 \Rightarrow \lambda = 0$

The equations can be solved simultaneously to find $\lambda$ and $\mu$.

**Problem-solving**

As $\overrightarrow{AB}$ is perpendicular to $l_1$ and $l_2$, the scalar product of the direction vectors of the lines is zero. You can use this fact to generate two linear equations in $\lambda$ and $\mu$.

$$\overrightarrow{AB} = \begin{pmatrix} -2 + 2\mu \\ 3 - \mu - \lambda \\ -1 - \mu - \lambda \end{pmatrix} = \begin{pmatrix} -2 + 2 \\ 3 - 1 - 0 \\ -1 - 1 - 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$$

$|\overrightarrow{AB}| = \sqrt{0^2 + 2^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$

So the shortest distance between the two lines is $2\sqrt{2}$.

These two lines are **skew**, so in this case the line segment $AB$ is unique.

## Example 26

The line $l$ has equation $\frac{x-1}{2} = \frac{y-1}{-2} = \frac{z+3}{-1}$, and the point $A$ has coordinates $(1, 2, -1)$.

**a** Find the shortest distance between $A$ and $l$.

**b** Find a Cartesian equation of the line that is perpendicular to $l$ and passes through $A$.

**a** Vector equation of $l$ is

$$\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$$

$\frac{x-1}{2} = \lambda,\ x = 1 + 2\lambda$

$\frac{y-1}{-2} = \lambda,\ y = 1 - 2\lambda$

$\frac{z+3}{-1} = \lambda,\ z = -3 - \lambda$

So a general point $B$, on the line has position vector $\begin{pmatrix} 1 + 2\lambda \\ 1 - 2\lambda \\ -3 - \lambda \end{pmatrix}$.

Let $B$ be the position vector of a general point on $l$.

Then $\overrightarrow{AB} = \begin{pmatrix} 1 + 2\lambda \\ 1 - 2\lambda \\ -3 - \lambda \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2\lambda \\ -1 - 2\lambda \\ -2 - \lambda \end{pmatrix}$

Find $\overrightarrow{AB}$ in terms of $\lambda$.

$$\begin{pmatrix} 2\lambda \\ -1 - 2\lambda \\ -2 - \lambda \end{pmatrix} . \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} = 0$$

$$4\lambda + 2 + 4\lambda + 2 + \lambda = 0$$

$$4 + 9\lambda = 0$$

$$\lambda = -\tfrac{4}{9}$$

Since $\overrightarrow{AB}$ is perpendicular to $l$ the scalar product of $\overrightarrow{AB}$ with the direction vector of the line is zero. This gives you an equation which you can solve to find $\lambda$.

$$\overrightarrow{AB} = \begin{pmatrix} 2\lambda \\ -1 - 2\lambda \\ -2 - \lambda \end{pmatrix} = \begin{pmatrix} -\frac{8}{9} \\ -\frac{1}{9} \\ -\frac{14}{9} \end{pmatrix}$$

Substitute the value of $\lambda$ into your general expression for $\overrightarrow{AB}$.

$$\left|\overrightarrow{AB}\right| = \sqrt{\frac{(-8)^2 + (-1)^2 + (-14)^2}{9^2}}$$

$$= \frac{\sqrt{29}}{3} = 1.80 \text{ (3 s.f.)}$$

The shortest distance is given by $\left|\overrightarrow{AB}\right|$.

So the shortest distance between $A$ and $l$ is $\frac{\sqrt{29}}{3}$ or 1.80 (3 s.f.).

**b** $\overrightarrow{AB}$ is perpendicular to $l$, so direction of perpendicular is $\begin{pmatrix} -\frac{8}{9} \\ -\frac{1}{9} \\ -\frac{14}{9} \end{pmatrix}$ or $\begin{pmatrix} 8 \\ 1 \\ 14 \end{pmatrix}$.

Remember that you can multiply the direction vector by a scalar to find a simpler parallel vector.

A vector equation of the line through $A$ perpendicular to $l$ is $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 8 \\ 1 \\ 14 \end{pmatrix}$.

A vector equation of the line through the point with position vector $\mathbf{a}$ with direction $\mathbf{b}$ is $\mathbf{r} = \mathbf{a} + \mu\mathbf{b}$ where $\mu$ is a scalar constant.

So the Cartesian equation of the line is

$$\frac{x-1}{8} = \frac{y-2}{1} = \frac{z+1}{14}$$

You can use the principles covered above to give meaning to the constant, $k$, in the scalar product form of the vector equation of a plane.

- **$k$ is the length of the perpendicular from the origin to a plane $\Pi$, where the equation of plane $\Pi$ is written in the form $\mathbf{r}.\hat{\mathbf{n}} = k$, where $\hat{\mathbf{n}}$ is a unit vector perpendicular to $\Pi$.**
- **The perpendicular distance from the point with coordinates $(\alpha, \beta, \gamma)$ to the plane with equation $ax + by + cz = d$ is $\dfrac{|a\alpha + b\beta + c\gamma - d|}{\sqrt{a^2 + b^2 + c^2}}$**

**Note** This formula is given in the formulae booklet and you can use it without proof in your exam.

## Example 27

Find the perpendicular distance from the point with coordinates $(3, 2, -1)$ to the plane with equation $2x - 3y + z = 5$.

$$\text{Distance} = \frac{|2 \times 3 - 3 \times 2 + 1 \times (-1) - 5|}{\sqrt{2^2 + (-3)^2 + 1^2}}$$

Substitute into the formula.

$$= \frac{|-6|}{\sqrt{14}}$$

Remember to use the modulus of the numerator as distance is always positive.

$$= \frac{6}{\sqrt{14}}$$

## Example 28

The plane $\Pi$ has equation $\mathbf{r}.(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) = 5$. The point $P$ has coordinates $(1, 3, -2)$.

**a** Find the shortest distance between $P$ and $\Pi$.

The point $Q$ is the reflection of the point $P$ in $\Pi$.

**b** Find the coordinates of point $Q$.

a Cartesian equation of $\Pi$ is $x + 2y + 2z = 5$.

Use $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\begin{pmatrix} x \\ y \\ z \end{pmatrix} . \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 5$.

$$\text{Distance} = \frac{|1 \times 1 + 2 \times 3 + 2 \times (-2) - 5|}{\sqrt{1^2 + 2^2 + 2^2}}$$

$$= \frac{|-2|}{\sqrt{9}} = \frac{2}{3}$$

Using the formula $\dfrac{|a\alpha + b\beta + c\gamma - d|}{\sqrt{a^2 + b^2 + c^2}}$ for the distance between a point with coordinates $(\alpha, \beta, \gamma)$ and the plane with Cartesian equation $ax + by + cz = d$.

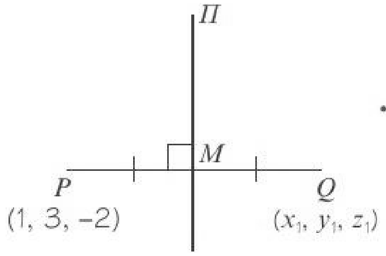
b A perpendicular vector to $\Pi$ is

$\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

A normal vector to the plane $ax + by + cz = d$ is $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

Let $Q$ have coordinates $(x_1, y_1, z_1)$.

Let $M$ be the midpoint of $PQ$.




The line joining $P$ to its reflection $Q$ will be perpendicular to the plane, and $P$ and $Q$ will be the same distance from the plane. Draw a diagram showing $P$, $Q$, and the midpoint of $PQ$. Represent the plane $\Pi$ using a vertical line.

A vector equation of the line through $P$, $M$ and $Q$ is $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

**Online** Explore reflections in a plane using GeoGebra.

$M$ lies on this line so has position vector $\begin{pmatrix} 1 + \lambda \\ 3 + 2\lambda \\ -2 + 2\lambda \end{pmatrix}$

$M$ also lies on $\Pi$, so $\begin{pmatrix} 1 + \lambda \\ 3 + 2\lambda \\ -2 + 2\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = 5$

$1 + \lambda + 2(3 + 2\lambda) + 2(-2 + 2\lambda) = 5$

$3 + 9\lambda = 5$

$\lambda = \frac{2}{9}$

$M$ has position vector $\begin{pmatrix} 1 + \frac{2}{9} \\ 3 + 2 \times \frac{2}{9} \\ -2 + 2 \times \frac{2}{9} \end{pmatrix} = \begin{pmatrix} \frac{11}{9} \\ \frac{31}{9} \\ -\frac{14}{9} \end{pmatrix}$

Use the fact that $M$ lies on the line joining $P$ and $Q$ and on the plane to find $M$.

$P$ is the initial point in the equation of $l$, so if $M$ has position vector $\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \frac{2}{9}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

then $P$ has position vector

$\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + 2 \times \frac{2}{9}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 + \frac{4}{9} \\ 3 + 2 \times \frac{4}{9} \\ -2 + 2 \times \frac{4}{9} \end{pmatrix} = \begin{pmatrix} \frac{13}{9} \\ \frac{35}{9} \\ -\frac{10}{9} \end{pmatrix}$

**Problem-solving**

$\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$ is the position vector of point $P$, so if the point on the line $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ with $\lambda = k$ is a distance $x$ away from $P$, then the point with $\lambda = 2k$ will be a distance $2x$ away from $P$.

Point $Q$ has coordinates $\left(\frac{13}{9}, \frac{35}{9}, -\frac{10}{9}\right)$.

You could also use the fact that the midpoint of the line segment joining $(x_1, y_1, z_1)$ to $(x_2, y_2, z_2)$ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$

Example 29

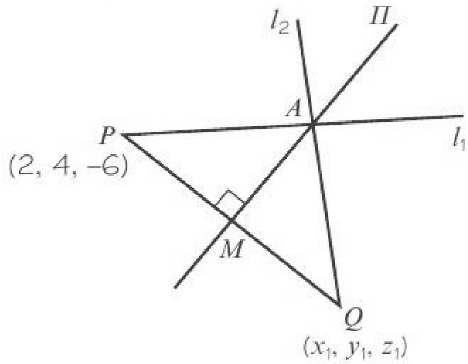
The line $l_1$ has equation $\dfrac{x-2}{2} = \dfrac{y-4}{-2} = \dfrac{z+6}{1}$. The plane $\Pi$ has equation $2x - 3y + z = 8$.

The line $l_2$ is the reflection of line $l_1$ in the plane $\Pi$. Find a vector equation of the line $l_2$.

A vector equation of $l_1$ is $\mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

So $P(2, 4, -6)$ is a point on line $l_1$.

Let $A$ be the point of intersection of $l_1$ and $\Pi$.




$A$ lies on $l_1$ and on $2x - 3y + z = 8$.

$A$ has position vector $\begin{pmatrix} 2 + 2\lambda \\ 4 - 2\lambda \\ -6 + \lambda \end{pmatrix}$ and satisfies

$$2(2 + 2\lambda) - 3(4 - 2\lambda) - 6 + \lambda = 8$$
$$4 + 4\lambda - 12 + 6\lambda - 6 + \lambda = 8$$
$$11\lambda = 22$$
$$\lambda = 2$$

Substitute the general position vector of a point on $l_1$ into the equation for $\Pi$ to find the value of $\lambda$ at $A$.

So $A$ has coordinates $(6, 0, -4)$.

Substitute $\lambda = 2$ into the general position vector to find the coordinates of $A$.

A perpendicular direction vector to $\Pi$ is $\mathbf{n} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$.

A vector equation of the line through $P$ perpendicular to $\Pi$ is $\mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$.

Let $Q$ $(x_1, y_1, z_1)$ be the point of intersection of this line and $l_2$.

$Q$ is the reflection of the point (6, 0, −4) in the plane.

Let $M$ be midpoint of $PQ$.

$M$ lies on line and on $2x - 3y + z = 8$ so satisfies

$$2(2 + 2\mu) - 3(4 - 3\mu) - 6 + \mu = 8$$

The general point on the line $PQ$ is $\begin{pmatrix} 2 + 2\mu \\ 4 - 3\mu \\ -6 + \mu \end{pmatrix}$.

$$4 + 4\mu - 12 + 9\mu - 6 + \mu = 8$$
$$14\mu = 22$$
$$\mu = \tfrac{11}{7}$$

So $M$ has position vector $\begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix} + \frac{11}{7} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

**Problem-solving**

You need to find **two points** on the reflected line, $l_2$. One is the point of intersection of $l_1$ and $\Pi$. To find another point on $l_2$, choose any point on $l_1$ and reflect it in the plane.

and $Q$ has position vector

$$\begin{pmatrix}2\\4\\-6\end{pmatrix} + 2 \times \frac{11}{7}\begin{pmatrix}2\\-3\\1\end{pmatrix} = \begin{pmatrix}\frac{58}{7}\\-\frac{38}{7}\\-\frac{20}{7}\end{pmatrix}$$

$Q$ is twice as far along the line from $P$ as $M$.

So $Q$ has coordinates $\left(\frac{58}{7}, -\frac{38}{7}, -\frac{20}{7}\right)$.

$l_2$ is the line through $Q$ and $A$, so has direction:

$$\overrightarrow{AQ} = \begin{pmatrix}\frac{16}{7}\\-\frac{38}{7}\\\frac{8}{7}\end{pmatrix}$$

The direction of $l_2$ can be simplified to $\begin{pmatrix}8\\-19\\4\end{pmatrix}$ by multiplying the expression for $\overrightarrow{AQ}$ by $\frac{7}{2}$

A vector equation of $l_2$ is $\mathbf{r} = \begin{pmatrix}6\\0\\-4\end{pmatrix} + t\begin{pmatrix}8\\-19\\4\end{pmatrix}$.

## Exercise 9F

1 Find the shortest distance between the parallel lines with equations
$\mathbf{r} = 2\mathbf{i} - \mathbf{j} + \mathbf{k} + \lambda(-3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k})$ and $\mathbf{r} = \mathbf{j} + \mathbf{k} + \mu(-3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k})$
where $\lambda$ and $\mu$ are scalars.

2 Find the shortest distance between the two skew lines with equations
$\mathbf{r} = \mathbf{j} + \mathbf{k} + \lambda(-\mathbf{i} + \mathbf{j} - \mathbf{k})$ and $\mathbf{r} = 5\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} + \mu(4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$,
where $\lambda$ and $\mu$ are scalars.

3 Determine whether the lines $l_1$ and $l_2$ meet. If they do, find their point of intersection. If they do not, find the shortest distance between them. (In each of the following cases $\lambda$ and $\mu$ are scalars.)

**a** $l_1$ has equation $\mathbf{r} = 7\mathbf{i} + 3\mathbf{j} + \mathbf{k} + \lambda(6\mathbf{i} + 2\mathbf{j} - 4\mathbf{k})$ and $l_2$ has equation $\mathbf{r} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k} - 2\mu\mathbf{j}$

**b** $l_1$ has equation $\mathbf{r} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k} + \lambda(2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$ and $l_2$ has equation $\mathbf{r} = \mathbf{i} - \mathbf{j} + 3\mathbf{k} + \mu(\mathbf{i} - \mathbf{j} + \mathbf{k})$

**c** $l_1$ has equation $\mathbf{r} = \mathbf{i} + \mathbf{j} + 5\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$ and $l_2$ has equation $\mathbf{r} = -\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \mu(\mathbf{i} + \mathbf{j} + \mathbf{k})$

4 Find the shortest distance between the point with coordinates (4, 1, −1) and the line with equation $\mathbf{r} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \mu(2\mathbf{i} - \mathbf{j} - \mathbf{k})$, where $\mu$ is a scalar.

5 Find the shortest distance between the parallel planes.

**a** $\mathbf{r}.(6\mathbf{i} + 6\mathbf{j} - 7\mathbf{k}) = 55$ and $\mathbf{r}.(6\mathbf{i} + 6\mathbf{j} - 7\mathbf{k}) = 22$

**b** $\mathbf{r} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k} + \lambda(4\mathbf{i} + \mathbf{k}) + \mu(8\mathbf{i} + 3\mathbf{j} + 3\mathbf{k})$ and $\mathbf{r} = 14\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} + \lambda(3\mathbf{j} + \mathbf{k}) + \mu(8\mathbf{i} - 9\mathbf{j} - \mathbf{k})$

**6** The plane $\Pi$ has equation $\mathbf{r}.(10\mathbf{i} + 10\mathbf{j} + 23\mathbf{k}) = 81$.

**a** Find the perpendicular distance from the origin to plane $\Pi$.

**b** Find the perpendicular distance from the point $(-1, -1, 4)$ to the plane $\Pi$.

**c** Find the perpendicular distance from the point $(2, 1, 3)$ to the plane $\Pi$.

**d** Find the perpendicular distance from the point $(6, 12, -9)$ to the plane $\Pi$.

E/P **7** The line $l$ has vector equation $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

The point $P$ has coordinates $(3, 0, 2)$.

Find the coordinates of the reflection of the point $P$ in the line $l$. **(5 marks)**

**Problem-solving**

Let $M$ be the midpoint of the line segment joining $P$ to its reflection in $l$. This segment must be perpendicular to $l$ and pass through it.

E/P **8** The plane $\Pi$ has equation $-2x + y + z = 5$. The point $P$ has coordinates $(1, 0, 3)$.

**a** Find the shortest distance between $P$ and $\Pi$. **(3 marks)**

The point $Q$ is the reflection of the point $P$ in $\Pi$.

**b** Find the coordinates of point $Q$. **(5 marks)**

E/P **9** A birdwatcher is located on a hilltop. Relative to a fixed origin $O$, the position vector of the birdwatcher is $\begin{pmatrix} 5 \\ 4 \\ 0.7 \end{pmatrix}$ km. The birdwatcher is able to spot any bird that flies within 500 m of her position. A kestrel flies from point $A$ to point $B$, where points $A$ and $B$ have position vectors $\begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}$ km and $\begin{pmatrix} 12 \\ 0 \\ 1.2 \end{pmatrix}$ km respectively. The kestrel is modelled as flying in a straight line.

**a** Use the model to determine whether the birdwatcher is able to spot the kestrel. **(7 marks)**

**b** Give one criticism of the model. **(1 mark)**

E/P **10** The plane $\Pi_1$ has equation $3x - 2y + 4z = 6$.

**a** Find the perpendicular distance from the point $(4, -1, 8)$ to $\Pi_1$. **(3 marks)**

The plane $\Pi_2$ has vector equation $\mathbf{r} = \lambda(2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) + \mu(-3\mathbf{i} + 3\mathbf{k})$ where $\lambda$ and $\mu$ are scalar parameters.

**b** Show that the vector $2\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$ is perpendicular to $\Pi_2$. **(2 marks)**

**c** Find the acute angle between $\Pi_1$ and $\Pi_2$. **(3 marks)**

E/P **11** The line $l_1$ has equation $\frac{x+2}{2} = \frac{y-2}{-1} = \frac{z+1}{-2}$, and the point $A$ has coordinates $(3, -1, 2)$.

**a** Find the shortest distance between $A$ and $l_1$. **(5 marks)**

**b** Find a Cartesian equation of the line that is perpendicular to $l_1$ and passes through $A$. **(3 marks)**

E/P **12** The line $l_1$ has vector equation $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} 4 \\ -3 \\ 4 \end{pmatrix}$. The plane $\Pi$ has equation $\mathbf{r}.\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = 4$.

The line $l_2$ is the reflection of line $l_1$ in the plane $\Pi$.
Find a vector equation of the line $l_2$. **(7 marks)**

## Mixed exercise 9

E **1** The line $l$ passes through the points $A$ and $B$ with position vectors $\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ respectively, relative to a fixed origin $O$.

**a** Find a vector equation of the line $l$. **(4 marks)**

**b** Find the position vector of the point $C$ which lies on the line segment $AB$ such that $AC = 2CB$. **(3 marks)**

E **2** Find a Cartesian equation of the straight line that passes through the points with coordinates $(7, -1, 2)$ and $(-1, 3, 8)$. **(4 marks)**

E **3** Find a vector equation of the straight line which passes through the point $A$ with position vector $2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$, and is parallel to the vector $2\mathbf{j} + 3\mathbf{k}$. **(3 marks)**

E/P **4** A straight line $l$ has vector equation $\mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$.

**a** Write down a Cartesian equation for $l$. **(2 marks)**

**b** Given that the point $(0, a, b)$ lies on $l$, find the value of $a$ and the value of $b$. **(3 marks)**

**5** A straight line $l$ has vector equation $\mathbf{r} = (\mathbf{i} + 2\mathbf{j} - \mathbf{k}) + \lambda(3\mathbf{i} + \mathbf{j} - 2\mathbf{k})$.
Show that another vector equation of $l$ is $\mathbf{r} = (7\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}) + \lambda(9\mathbf{i} + 3\mathbf{j} - 6\mathbf{k})$.

**6** A straight line $l$ has Cartesian equation $\dfrac{x + 2}{1} = \dfrac{y - 2}{3} = \dfrac{z + 3}{4}$.

**a** Find a vector form of the equation of $l$. **b** Verify that the point $(0, 8, 5)$ lies on $l$.

**7** A plane passes through the points $A(2, -1, 2)$, $B(1, 3, -1)$ and $C(4, 2, 5)$.

**a** Find a vector form of the equation of the plane.

**b** Find a Cartesian form of the equation of the plane.

**8** A Cartesian form of the equation of a plane is $3x + 2y - 4z = 18$. Find a vector form of the equation of the plane.

E/P **9** With respect to an origin $O$, the position vectors of the points $L$, $M$ and $N$ are $\begin{pmatrix} 4 \\ 7 \\ 7 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$ respectively.

**a** Find the vectors $\overrightarrow{ML}$ and $\overrightarrow{MN}$. **(3 marks)**

**b** Prove that $\cos\angle LMN = \frac{9}{10}$ **(3 marks)**

E/P **10** Referred to a fixed origin $O$, the points $A$, $B$ and $C$ have position vectors $9\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $6\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$ and $3\mathbf{i} + p\mathbf{j} + q\mathbf{k}$ respectively, where $p$ and $q$ are constants.

**a** Find, in vector form, an equation of the line $l$ which passes through $A$ and $B$. **(2 marks)**

Given that $C$ lies on $l$,

**b** find the value of $p$ and the value of $q$ **(2 marks)**

**c** calculate, in degrees, the acute angle between $OC$ and $AB$. **(3 marks)**

The point $D$ lies on $AB$ and is such that $OD$ is perpendicular to $AB$.

**d** Find the position vector of $D$. **(5 marks)**

(E) **11** Referred to a fixed origin $O$, the points $A$ and $B$ have position vectors $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 0 \\ -3 \end{pmatrix}$ respectively.

**a** Find, in vector form, an equation of the line $l_1$ which passes through $A$ and $B$. **(3 marks)**

The line $l_2$ has equation $\mathbf{r} = \begin{pmatrix} 4 \\ -4 \\ 3 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$, where $\mu$ is a scalar parameter.

**b** Show that $A$ lies on $l_2$. **(2 marks)**

**c** Find, in degrees, the acute angle between the lines $l_1$ and $l_2$. **(4 marks)**

The point $C$ with position vector $\begin{pmatrix} 0 \\ 4 \\ -5 \end{pmatrix}$ lies on $l_2$.

**d** Find the shortest distance from $C$ to the line $l_1$. **(4 marks)**

(E) **12** Two submarines are travelling in straight lines through the ocean. Relative to a fixed origin, the vector equations of the two lines, $l_1$ and $l_2$, along which they travel are

$l_1$: $\mathbf{r} = 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$
$l_2$: $\mathbf{r} = 9\mathbf{i} + \mathbf{j} - 2\mathbf{k} + \mu(4\mathbf{i} + \mathbf{j} - \mathbf{k})$

where $\lambda$ and $\mu$ are scalars.

**a** Show that the submarines are moving in perpendicular directions. **(2 marks)**

**b** Given that $l_1$ and $l_2$ intersect at the point $A$, find the position vector of $A$. **(4 marks)**

The point $B$ has position vector $10\mathbf{j} - 11\mathbf{k}$.

**c** Show that only one of the submarines passes through the point $B$. **(3 marks)**

**d** Given that 1 unit on each coordinate axis represents 100 m, find, in km, the distance $AB$. **(2 marks)**

(E) **13** Find the shortest distance between the lines with vector equations

$$\mathbf{r} = 3\mathbf{i} + s\mathbf{j} - \mathbf{k} \quad \text{and} \quad \mathbf{r} = 9\mathbf{i} - 2\mathbf{j} - \mathbf{k} + t(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$

where $s$ and $t$ are scalars. **(4 marks)**

(E) **14** Obtain the shortest distance between the lines with equations

$$\mathbf{r} = (3s - 3)\mathbf{i} - s\mathbf{j} + (s + 1)\mathbf{k} \quad \text{and} \quad \mathbf{r} = (3 + t)\mathbf{i} + (2t - 2)\mathbf{j} + \mathbf{k}$$

where $s$ and $t$ are parameters. **(4 marks)**

**15** Find, in the form $\mathbf{r.n} = p$, an equation of the plane which contains the line $l$ and the point with position vector $\mathbf{a}$ where:

**a** $l$ has equation $\mathbf{r} = \mathbf{i} + \mathbf{j} - 2\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{k})$ and $\mathbf{a} = 4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

**b** $l$ has equation $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} - 3\mathbf{k})$ and $\mathbf{a} = 3\mathbf{i} + 5\mathbf{j} + \mathbf{k}$

**c** $l$ has equation $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$ and $\mathbf{a} = 7\mathbf{i} + 8\mathbf{j} + 6\mathbf{k}$

**16** Find a Cartesian equation of the plane which passes through the point (1, 1, 1) and contains the line with equation $\dfrac{x-2}{3} = \dfrac{y+4}{1} = \dfrac{z-1}{2}$

(E) **17** A plane passes through the three points $A$, $B$ and $C$, whose position vectors, referred to an origin $O$, are $\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$ and $3\mathbf{i} + \mathbf{j} + 4\mathbf{k}$, $2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$ respectively.

**a** Find, in the form $l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$, a unit vector normal to this plane. **(4 marks)**

**b** Find also a Cartesian equation of the plane. **(2 marks)**

**c** Find the perpendicular distance from the origin to this plane. **(4 marks)**

(E) **18** **a** Show that the vector $\mathbf{i} + \mathbf{k}$ is perpendicular to the plane with vector equation
$\mathbf{r} = \mathbf{i} + s\mathbf{j} + t(\mathbf{i} - \mathbf{k})$ **(3 marks)**

**b** Find the perpendicular distance from the origin to this plane. **(4 marks)**

**c** Hence or otherwise obtain a Cartesian equation of the plane. **(2 marks)**

(E) **19** The points $A$, $B$ and $C$ have position vectors $\mathbf{i} + \mathbf{j} + \mathbf{k}$, $5\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$ respectively, referred to an origin $O$.

**a** Find a vector perpendicular to the plane containing the points $A$, $B$ and $C$. **(4 marks)**

**b** Hence, or otherwise, find an equation for the plane which contains the points $A$, $B$ and $C$, in the form $ax + by + cz + d = 0$. **(2 marks)**

(E/P) **20** Planes $\Pi_1$ and $\Pi_2$ have equations given by

$\Pi_1$: $\mathbf{r}.(2\mathbf{i} - \mathbf{j} + \mathbf{k}) = 0$,

$\Pi_2$: $\mathbf{r}.(\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}) = 1$

**a** Show that the point $A(2, -2, 3)$ lies in $\Pi_2$. **(2 marks)**

**b** Show that $\Pi_1$ is perpendicular to $\Pi_2$. **(4 marks)**

**c** Find, in vector form, an equation of the straight line through $A$ which is perpendicular to $\Pi_1$. **(2 marks)**

**d** Determine the coordinates of the point where this line meets $\Pi_1$. **(4 marks)**

**e** Find the perpendicular distance of $A$ from $\Pi_1$. **(4 marks)**

(E/P) **21** With respect to a fixed origin $O$, the straight lines $l_1$ and $l_2$ are given by

$$l_1\text{: } \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix},$$

$$l_2\text{: } \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

where $\lambda$ and $\mu$ are scalar parameters.

**a** Show that the lines intersect. **(3 marks)**

**b** Find the position vector of their point of intersection. **(1 mark)**

**c** Find the cosine of the acute angle between the lines. **(4 marks)**

(E/P) **22** The line $l_1$ has vector equation $\mathbf{r} = 6\mathbf{i} + 8\mathbf{j} + 5\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k})$, where $\lambda$ is a scalar parameter. The point $A$ has coordinates $(3, a, 2)$, where $a$ is a constant. The point $B$ has coordinates $(8, 6, b)$, where $b$ is a constant. Points $A$ and $B$ lie on the line $l_1$.

**a** Find the values of $a$ and $b$. **(3 marks)**

Given that the point $O$ is the origin, and that the point $P$ lies on $l_1$ such that $OP$ is perpendicular to $l_1$,

**b** find the coordinates of $P$. **(5 marks)**

**c** Hence find the distance $OP$, giving your answer in surd form. **(2 marks)**

E/P **23** Relative to a fixed origin $O$, the point $A$ has position vector $6\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, and the point $B$ has position vector $5\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$. The line $l$ passes through the points $A$ and $B$.

**a** Find the vector $\overrightarrow{AB}$. **(2 marks)**

**b** Find a vector equation for the line $l$. **(2 marks)**

The point $C$ has position vector $4\mathbf{i} + 10\mathbf{j} + 2\mathbf{k}$.

The point $P$ lies on $l$. Given that the vector $\overrightarrow{CP}$ is perpendicular to $l$,

**c** find the position vector of the point $P$. **(6 marks)**

E/P **24** With respect to a fixed origin $O$, the lines $l_1$ and $l_2$ are given by the equations

$$l_1: \mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \qquad l_2: \mathbf{r} = \begin{pmatrix} 1 \\ 12 \\ 8 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$$

where $\lambda$ and $\mu$ are scalar parameters.

**a** Show that $l_1$ and $l_2$ meet and find the position vector of their point of intersection, $A$. **(6 marks)**

**b** Find, to the nearest 0.1°, the acute angle between $l_1$ and $l_2$. **(3 marks)**

The point $B$ has position vector $\begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$.

**c** Show that $B$ lies on $l_1$. **(1 mark)**

**d** Find the shortest distance from $B$ to the line $l_2$, giving your answer to 3 significant figures. **(4 marks)**

E/P **25** The plane $P$ has equation $\mathbf{r}.(2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) = 1$.
The line $l$ passes through the point $A$ (1, 2, 2) and meets $P$ at (4, 2, −1).
The acute angle between the plane $P$ and the line $l$ is $\alpha$.

**a** Find $\alpha$ to the nearest degree. **(4 marks)**

**b** Find the perpendicular distance from $A$ to the plane $P$. **(4 marks)**

E/P **26** Two aeroplanes are modelled as travelling in straight lines. Aeroplane $A$ travels from a point with position vector $\begin{pmatrix} 120 \\ -80 \\ 13 \end{pmatrix}$ km to a point with position vector $\begin{pmatrix} 200 \\ 20 \\ 5 \end{pmatrix}$ km, relative to a fixed origin $O$. Aeroplane $B$ starts at a point with position vector $\begin{pmatrix} -20 \\ 35 \\ 5 \end{pmatrix}$ km relative to $O$, and flies in the direction of $\begin{pmatrix} 10 \\ -2 \\ 0.1 \end{pmatrix}$.

**a** Show that the flight paths of the two aeroplanes will intersect, and determine the position vector of the point of intersection. **(7 marks)**

An air traffic controller states that this means that the planes will collide.

**b** Explain why this conclusion is not necessarily correct. **(2 marks)**

## Challenge

**1** **a** Show that the equations $4x - 2y + 6z = 10$ and $-2x + y - 3z = -5$ represent the same plane.

**b** Hence explain why the matrix $\begin{pmatrix} 4 & -2 & 6 \\ -2 & 1 & -3 \\ a & b & c \end{pmatrix}$ is singular for all possible values of $a$, $b$ and $c$.

**c** Find values of $a$, $b$ and $c$ such that the matrix equation
$\begin{pmatrix} 4 & -2 & 6 \\ -2 & 1 & -3 \\ a & b & c \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ -5 \\ 15 \end{pmatrix}$ has

**i** no solutions

**ii** infinitely many solutions.

**2** The points $A$, $B$ and $C$ have coordinates (2, −9, 0), (10, −3, 6) and (8, −1, 2) respectively. Find the centre and radius of the circle that passes through all three points.

## Summary of key points

**1** A vector equation of a straight line passing through the point $A$ with position vector **a**, and parallel to the vector **b**, is

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$$

where $\lambda$ is a scalar parameter.

**2** A vector equation of a straight line passing through the points $C$ and $D$, with position vectors **c** and **d** respectively, is

$$\mathbf{r} = \mathbf{c} + \lambda(\mathbf{d} - \mathbf{c})$$

where $\lambda$ is a scalar parameter.

**3** If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, the equation of the line with vector equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ can be given in Cartesian form as:

$$\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}$$

Each of these three expressions is equal to $\lambda$.

**4** The vector equation of a plane is
$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, where:

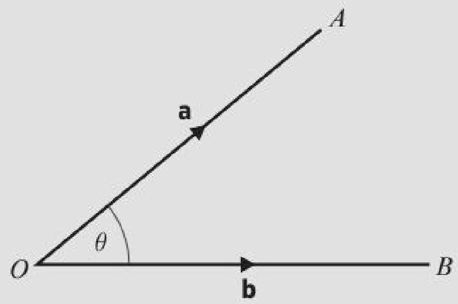
- **r** is the position vector of a general point in the plane
- **a** is the position vector of a point in the plane
- **b** and **c** are non-parallel, non-zero vectors in the plane
- $\lambda$ and $\mu$ are scalars

**5** A Cartesian equation of a plane in three dimensions can be written in the form $ax + by + cz = d$ where $a, b, c$ and $d$ are constants, and $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is the normal vector to the plane.

**6** The **scalar product** of two vectors **a** and **b** is written as **a.b** (say '**a** dot **b**'), and defined as

$$\mathbf{a.b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

where $\theta$ is the angle between **a** and **b**.




**7** If **a** and **b** are the position vectors of the points $A$ and $B$, then $\cos(\angle AOB) = \dfrac{\mathbf{a.b}}{|\mathbf{a}||\mathbf{b}|}$

**8** The non-zero vectors **a** and **b** are perpendicular if and only if **a.b** = 0.

**9** If **a** and **b** are parallel, $\mathbf{a.b} = |\mathbf{a}||\mathbf{b}|$. In particular, $\mathbf{a.a} = |\mathbf{a}|^2$.

**10** If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

$$\mathbf{a.b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} . \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$$

**11** The acute angle $\theta$ between two intersecting straight lines is given by

$$\cos\theta = \left|\frac{\mathbf{a.b}}{|\mathbf{a}||\mathbf{b}|}\right|$$

where **a** and **b** are direction vectors of the lines.

**12** The scalar product form of the equation of a plane is $\mathbf{r.n} = k$ where $k = \mathbf{a.n}$ for any point in the plane with position vector **a**.

**13** The acute angle $\theta$ between the line with equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and the plane with equation $\mathbf{r.n} = k$ is given by the formula

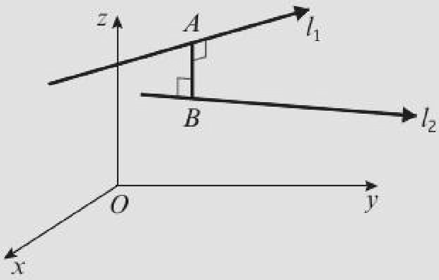
$$\sin\theta = \left|\frac{\mathbf{b.n}}{|\mathbf{b}||\mathbf{n}|}\right|$$

**14** The acute angle $\theta$ between the plane with equation $\mathbf{r.n}_1 = k_1$ and the plane with equation $\mathbf{r.n}_2 = k_2$ is given by the formula

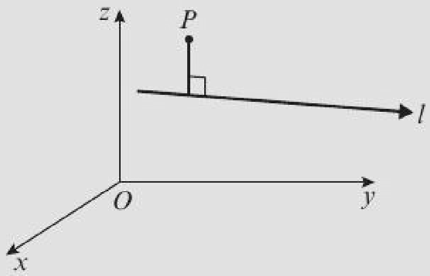
$$\cos\theta = \left|\frac{\mathbf{n}_1.\mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right|$$

**15** Two lines are **skew** if they are not parallel and they do not intersect.

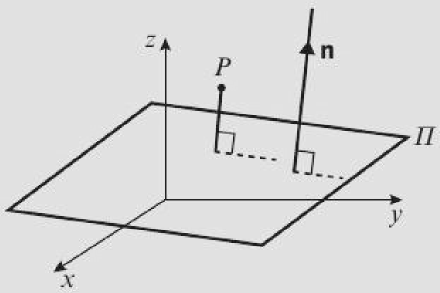
**16** For any two non-intersecting lines $l_1$ and $l_2$ there is a unique line segment $AB$ such that $A$ lies on $l_1$, $B$ lies on $l_2$ and $AB$ is perpendicular to both lines.




**17** The perpendicular from a point $P$ to a line $l$ is a line drawn from $P$ at right angles to $l$.




**18** The perpendicular from a point $P$ to a plane $\Pi$ is a line drawn from $P$ parallel to the normal vector **n**.




**19** $k$ is the length of the perpendicular from the origin to a plane $\Pi$, where the equation of plane $\Pi$ is written in the form $\mathbf{r}.\hat{\mathbf{n}} = k$, where $\hat{\mathbf{n}}$ is a unit vector perpendicular to $\Pi$.

The perpendicular distance from the point with coordinates $(\alpha, \beta, \gamma)$ and the plane with equation $ax + by + cz = d$ is

$$\frac{|a\alpha + b\beta + c\gamma - d|}{\sqrt{a^2 + b^2 + c^2}}$$