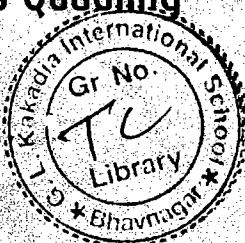


Advanced Level Mathematics
Pure Mathematics 1

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Introduction

Cambridge International Examinations (CIE) Advanced Level Mathematics has been created especially for the new CIE mathematics syllabus. There is one book corresponding to each syllabus unit, except that units P2 and P3 are contained in a single book. This book covers the first Pure Mathematics unit, P1.

The syllabus content is arranged by chapters which are ordered so as to provide a viable teaching course. The early chapters develop the foundations of the syllabus; students may already be familiar with some of these topics. Later chapters, however, are largely independent of each other, and teachers may wish to vary the order in which they are used.

Some chapters, particularly Chapters 2, 3 and the first four sections of Chapter 8, contain material which is not in the examination syllabus for P1, and which therefore cannot be the direct focus of examination questions. Some of this is necessary background material, such as indices and surds; some is useful knowledge, such as graphs of powers of x , the use and meaning of modulus, and work on sequences.

A few sections include important results which are difficult to prove or outside the syllabus. These sections are marked with an asterisk (*) in the section heading, and there is usually a sentence early on explaining precisely what it is that the student needs to know.

Occasionally within the text paragraphs appear in *this type style*. These paragraphs are usually outside the main stream of the mathematical argument, but may help to give insight, or suggest extra work or different approaches.

Graphic calculators are not permitted in the examination, but they are useful aids in learning mathematics. In the book the authors have noted where access to a graphic calculator would be especially helpful but have not assumed that they are available to all students.

Numerical work is presented in a form intended to discourage premature approximation. In ongoing calculations inexact numbers appear in decimal form like 3.456..., signifying that the number is held in a calculator to more places than are given. Numbers are not rounded at this stage; the full display could be, for example, 3.456 123 or 3.456 789. Final answers are then stated with some indication that they are approximate, for example '1.23 correct to 3 significant figures'.

There are plenty of exercises, and each chapter ends with a Miscellaneous exercise which includes some questions of examination standard. Three Revision exercises consolidate work in preceding chapters. The book concludes with two Practice examination papers.

In some exercises a few of the later questions may go beyond the likely requirements of the P1 examination, either in difficulty or in length or both. Some questions are marked with an asterisk, which indicates that they require knowledge of results outside the syllabus.

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1 Coordinates, points and lines

This chapter uses coordinates to describe points and lines in two dimensions. When you have completed it, you should be able to

- find the distance between two points
- find the mid-point of a line segment, given the coordinates of its end points
- find the gradient of a line segment, given the coordinates of its end points
- find the equation of a line through a given point with a given gradient
- find the equation of the line joining two points
- recognise lines from different forms of their equations
- find the point of intersection of two lines
- tell from their gradients whether two lines are parallel or perpendicular.

1.1 The distance between two points

When you choose an origin, draw an x -axis to the right on the page and a y -axis up the page and choose scales along the axes, you are setting up a coordinate system.

The coordinates of this system are called **cartesian coordinates** after the French mathematician René Descartes, who lived in the 17th century.

In Fig. 1.1, two points A and B have cartesian coordinates $(4, 3)$ and $(10, 7)$. The part of the line AB which lies between A and B is called a **line segment**. The length of the line segment is the distance between the points.

A third point C has been added to Fig. 1.1 to form a right-angled triangle. You can see that C has the same x -coordinate as B and the same y -coordinate as A ; that is, C has coordinates $(10, 3)$.

It is easy to see that AC has length $10 - 4 = 6$, and CB has length $7 - 3 = 4$. Using Pythagoras' theorem in triangle ABC shows that the length of the line segment AB is

$$\sqrt{(10-4)^2 + (7-3)^2} = \sqrt{6^2 + 4^2} = \sqrt{36+16} = \sqrt{52}.$$

You can use your calculator to give this as $7.21\dots$, if you need to, but often it is better to leave the answer as $\sqrt{52}$.

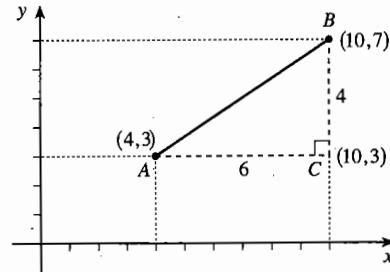


Fig. 1.1

The idea of coordinate geometry is to use algebra so that you can do calculations like this when A and B are any points, and not just the particular points in Fig. 1.1. It often helps to use a notation which shows at a glance which point a coordinate refers to. One way of doing this is with suffixes, calling the coordinates of the first point (x_1, y_1) , and

the coordinates of the second point (x_2, y_2) . Thus, for example, x_1 stands for 'the x -coordinate of the first point'.

Fig. 1.2 shows this general triangle. You can see that C now has coordinates (x_2, y_1) , and that $AC = x_2 - x_1$ and $CB = y_2 - y_1$. Pythagoras' theorem now gives

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

An advantage of using algebra is that this formula works whatever the shape and position of the triangle. In Fig. 1.3, the coordinates of A are negative, and in Fig. 1.4 the line slopes downhill rather than uphill as you move from left to right. Use Figs. 1.3 and 1.4 to work out for yourself the length of AB in each case. You can then use the formula to check your answers.

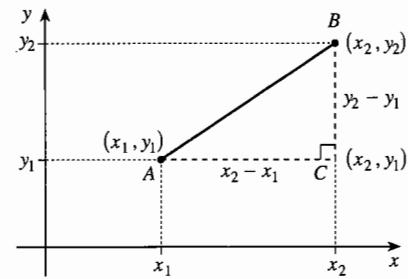


Fig. 1.2

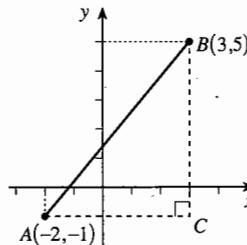


Fig. 1.3

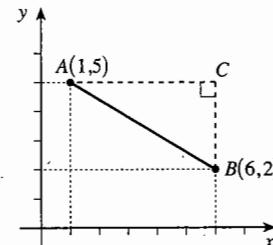


Fig. 1.4

In Fig. 1.3,

$$\begin{aligned} x_2 - x_1 &= 3 - (-2) = 3 + 2 = 5 \quad \text{and} \quad y_2 - y_1 = 5 - (-1) = 5 + 1 = 6, \\ \text{so } AB &= \sqrt{(3 - (-2))^2 + (5 - (-1))^2} = \sqrt{5^2 + 6^2} = \sqrt{25 + 36} = \sqrt{61}. \end{aligned}$$

And in Fig. 1.4,

$$\begin{aligned} x_2 - x_1 &= 6 - 1 = 5 \quad \text{and} \quad y_2 - y_1 = 2 - 5 = -3, \\ \text{so } AB &= \sqrt{(6 - 1)^2 + (2 - 5)^2} = \sqrt{5^2 + (-3)^2} = \sqrt{25 + 9} = \sqrt{34}. \end{aligned}$$

Also, it doesn't matter which way round you label the points A and B . If you think of B as 'the first point' (x_1, y_1) and A as 'the second point' (x_2, y_2) , the formula doesn't change. For Fig. 1.1, it would give

$$BA = \sqrt{(4 - 10)^2 + (3 - 7)^2} = \sqrt{(-6)^2 + (-4)^2} = \sqrt{36 + 16} = \sqrt{52}, \text{ as before.}$$

The distance between the points (x_1, y_1) and (x_2, y_2) (or the length of the line segment joining them) is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

1.2 The mid-point of a line segment

You can also use coordinates to find the mid-point of a line segment.

Fig. 1.5 shows the same line segment as in Fig. 1.1, but with the mid-point M added. The line through M parallel to the y -axis meets AC at D . Then the lengths of the sides of the triangle ADM are half of those of triangle ACB , so that

$$AD = \frac{1}{2}AC = \frac{1}{2}(10 - 4) = \frac{1}{2}(6) = 3,$$

$$DM = \frac{1}{2}CB = \frac{1}{2}(7 - 3) = \frac{1}{2}(4) = 2.$$

The x -coordinate of M is the same as the x -coordinate of D , which is

$$4 + AD = 4 + \frac{1}{2}(10 - 4) = 4 + 3 = 7.$$

The y -coordinate of M is

$$3 + DM = 3 + \frac{1}{2}(7 - 3) = 3 + 2 = 5.$$

So the mid-point M has coordinates $(7, 5)$.

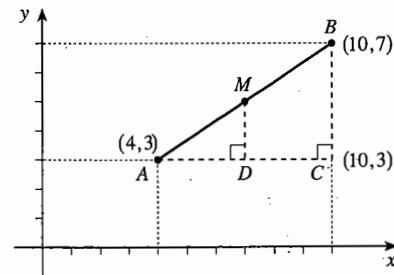


Fig. 1.5

In Fig. 1.6 points M and D have been added in the same way to Fig. 1.2. Exactly as before,

$$AD = \frac{1}{2}AC = \frac{1}{2}(x_2 - x_1), \quad DM = \frac{1}{2}CB = \frac{1}{2}(y_2 - y_1).$$

So the x -coordinate of M is

$$\begin{aligned} x_1 + AD &= x_1 + \frac{1}{2}(x_2 - x_1) = x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_1 \\ &= \frac{1}{2}x_1 + \frac{1}{2}x_2 = \frac{1}{2}(x_1 + x_2). \end{aligned}$$

The y -coordinate of M is

$$\begin{aligned} y_1 + DM &= y_1 + \frac{1}{2}(y_2 - y_1) = y_1 + \frac{1}{2}y_2 - \frac{1}{2}y_1 \\ &= \frac{1}{2}y_1 + \frac{1}{2}y_2 = \frac{1}{2}(y_1 + y_2). \end{aligned}$$

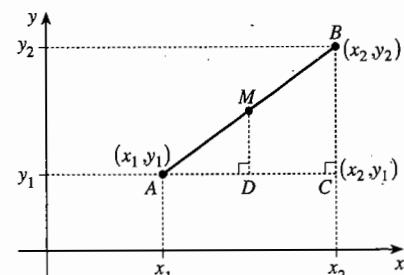


Fig. 1.6

The mid-point of the line segment joining (x_1, y_1) and (x_2, y_2) has coordinates

$$\left(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2) \right).$$

Now that you have an algebraic form for the coordinates of the mid-point M you can use it for any two points. For example, for Fig. 1.3 the mid-point of AB is

$$\left(\frac{1}{2}((-2)+3), \frac{1}{2}((-1)+5)\right) = \left(\frac{1}{2}(1), \frac{1}{2}(4)\right) = \left(\frac{1}{2}, 2\right).$$

And for Fig. 1.4 it is $\left(\frac{1}{2}(1+6), \frac{1}{2}(5+2)\right) = \left(\frac{1}{2}(7), \frac{1}{2}(7)\right) = \left(3\frac{1}{2}, 3\frac{1}{2}\right)$.

Again, it doesn't matter which you call the first point and which the second. In Fig. 1.5, if you take (x_1, y_1) as $(10, 7)$ and (x_2, y_2) as $(4, 3)$, you find that the mid-point is

$$\left(\frac{1}{2}(10+4), \frac{1}{2}(7+3)\right) = (7, 5), \text{ as before.}$$

1.3 The gradient of a line segment

The gradient of a line is a measure of its steepness. The steeper the line, the larger the gradient.

Unlike the distance and the mid-point, the gradient is a property of the whole line, not just of a particular line segment. If you take any two points on the line and find the increases in the x - and y -coordinates as you go from one to the other, as in Fig. 1.7, then the value of the fraction

$$\frac{\text{y-step}}{\text{x-step}}$$

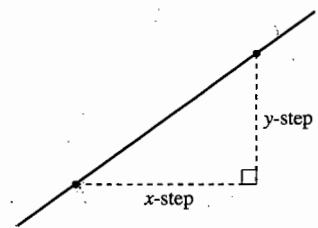


Fig. 1.7

is the same whichever points you choose. This is the **gradient** of the line.

In Fig. 1.2 on page 2 the x -step and y -step are $x_2 - x_1$ and $y_2 - y_1$, so that:

The gradient of the line joining (x_1, y_1) to (x_2, y_2) is $\frac{y_2 - y_1}{x_2 - x_1}$.

This formula applies whether the coordinates are positive or negative. In Fig. 1.3, for example, the gradient of AB is $\frac{5 - (-1)}{3 - (-2)} = \frac{5 + 1}{3 + 2} = \frac{6}{5}$.

But notice that in Fig. 1.4 the gradient is $\frac{2 - 5}{6 - 1} = \frac{-3}{5} = -\frac{3}{5}$; the negative gradient tells you that the line slopes downhill as you move from left to right.

As with the other formulae, it doesn't matter which point has the suffix 1 and which has the suffix 2. In Fig. 1.1, you can calculate the gradient as either $\frac{7 - 3}{10 - 4} = \frac{4}{6} = \frac{2}{3}$, or $\frac{3 - 7}{4 - 10} = \frac{-4}{-6} = \frac{2}{3}$.

Two lines are **parallel** if they have the same gradient.

Example 1.3.1

The ends of a line segment are $(p-q, p+q)$ and $(p+q, p-q)$. Find the length of the line segment, its gradient and the coordinates of its mid-point.

For the length and gradient you have to calculate

$$x_2 - x_1 = (p+q) - (p-q) = p+q-p+q = 2q$$

$$\text{and } y_2 - y_1 = (p-q) - (p+q) = p-q-p-q = -2q.$$

$$\text{The length is } \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(2q)^2 + (-2q)^2} = \sqrt{4q^2 + 4q^2} = \sqrt{8q^2}.$$

$$\text{The gradient is } \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2q}{2q} = -1.$$

For the mid-point you have to calculate

$$x_1 + x_2 = (p-q) + (p+q) = p-q+p+q = 2p$$

$$\text{and } y_1 + y_2 = (p+q) + (p-q) = p+q+p-q = 2p.$$

$$\text{The mid-point is } \left(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2) \right) = \left(\frac{1}{2}(2p), \frac{1}{2}(2p) \right) = (p, p).$$

Try drawing your own figure to illustrate the results in this example.

Example 1.3.2

Prove that the points $A(1,1)$, $B(5,3)$, $C(3,0)$ and $D(-1,-2)$ form a parallelogram.

You can approach this problem in a number of ways, but whichever method you use, it is worth drawing a sketch. This is shown in Fig. 1.8.

Method 1 (using distances) In this method, find the lengths of the opposite sides. If both pairs of opposite sides are equal, then $ABCD$ is a parallelogram.

$$AB = \sqrt{(5-1)^2 + (3-1)^2} = \sqrt{20}.$$

$$DC = \sqrt{(3-(-1))^2 + (0-(-2))^2} = \sqrt{20}.$$

$$CB = \sqrt{(5-3)^2 + (3-0)^2} = \sqrt{13}.$$

$$DA = \sqrt{(-1-1)^2 + (1-(-2))^2} = \sqrt{13}.$$

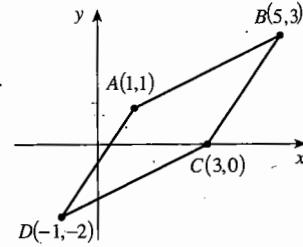


Fig. 1.8

Therefore $AB = DC$ and $CB = DA$, so $ABCD$ is a parallelogram.

Method 2 (using mid-points) In this method, begin by finding the mid-points of the diagonals AC and BD . If these points are the same, then the diagonals bisect each other, so the quadrilateral is a parallelogram.

The mid-point of AC is $\left(\frac{1}{2}(1+3), \frac{1}{2}(1+0) \right)$, which is $\left(2, \frac{1}{2} \right)$. The mid-point of BD is $\left(\frac{1}{2}(5+(-1)), \frac{1}{2}(3+(-2)) \right)$, which is also $\left(2, \frac{1}{2} \right)$. So $ABCD$ is a parallelogram.

Method 3 (using gradients) In this method, find the gradients of the opposite sides. If both pairs of opposite sides are parallel, then $ABCD$ is a parallelogram.

The gradients of AB and DC are $\frac{3-1}{5-1} = \frac{2}{4} = \frac{1}{2}$ and $\frac{0-(-2)}{3-(-1)} = \frac{2}{4} = \frac{1}{2}$ respectively, so AB is parallel to DC . The gradients of DA and CB are both $\frac{3}{2}$, so DA is parallel to CB . As the opposite sides are parallel, $ABCD$ is a parallelogram.

Exercise 1A

Do not use a calculator. Where appropriate, leave square roots in your answers.

- 1 Find the lengths of the line segments joining these pairs of points. In parts (e) and (h) assume that $a > 0$; in parts (i) and (j) assume that $p > q > 0$.

(a) $(2, 5)$ and $(7, 17)$	(b) $(-3, 2)$ and $(1, -1)$
(c) $(4, -5)$ and $(-1, 0)$	(d) $(-3, -3)$ and $(-7, 3)$
(e) $(2a, a)$ and $(10a, -14a)$	(f) $(a+1, 2a+3)$ and $(a-1, 2a-1)$
(g) $(2, 9)$ and $(2, -14)$	(h) $(12a, 5b)$ and $(3a, 5b)$
(i) (p, q) and (q, p)	(j) $(p+4q, p-q)$ and $(p-3q, p)$
- 2 Show that the points $(1, -2)$, $(6, -1)$, $(9, 3)$ and $(4, 2)$ are vertices of a parallelogram.
- 3 Show that the triangle formed by the points $(-3, -2)$, $(2, -7)$ and $(-2, 5)$ is isosceles.
- 4 Show that the points $(7, 12)$, $(-3, -12)$ and $(14, -5)$ lie on a circle with centre $(2, 0)$.
- 5 Find the coordinates of the mid-points of the line segments joining these pairs of points.

(a) $(2, 11), (6, 15)$	(b) $(5, 7), (-3, 9)$
(c) $(-2, -3), (1, -6)$	(d) $(-3, 4), (-8, 5)$
(e) $(p+2, 3p-1), (3p+4, p-5)$	(f) $(p+3, q-7), (p+5, 3-q)$
(g) $(p+2q, 2p+13q), (5p-2q, -2p-7q)$	(h) $(a+3, b-5), (a+3, b+7)$
- 6 A $(-2, 1)$ and B $(6, 5)$ are the opposite ends of the diameter of a circle. Find the coordinates of its centre.
- 7 $M(5, 7)$ is the mid-point of the line segment joining A $(3, 4)$ to B. Find the coordinates of B.
- 8 A $(1, -2)$, B $(6, -1)$, C $(9, 3)$ and D $(4, 2)$ are the vertices of a parallelogram. Verify that the mid-points of the diagonals AC and BD coincide.
- 9 Which one of the points A $(5, 2)$, B $(6, -3)$ and C $(4, 7)$ is the mid-point of the other two? Check your answer by calculating two distances.
- 10 Find the gradients of the lines joining the following pairs of points.

(a) $(3, 8), (5, 12)$	(b) $(1, -3), (-2, 6)$
(c) $(-4, -3), (0, -1)$	(d) $(-5, -3), (3, -9)$
(e) $(p+3, p-3), (2p+4, -p-5)$	(f) $(p+3, q-5), (q-5, p+3)$
(g) $(p+q-1, q+p-3), (p-q+1, q-p+3)$	(h) $(7, p), (11, p)$

- 11** Find the gradients of the lines AB and BC where A is $(3, 4)$, B is $(7, 6)$ and C is $(-3, 1)$. What can you deduce about the points A , B and C ?
- 12** The point $P(x, y)$ lies on the straight line joining $A(3, 0)$ and $B(5, 6)$. Find expressions for the gradients of AP and PB . Hence show that $y = 3x - 9$.
- 13** A line joining a vertex of a triangle to the mid-point of the opposite side is called a median. Find the length of the median AM in the triangle $A(-1, 1)$, $B(0, 3)$, $C(4, 7)$.
- 14** A triangle has vertices $A(-2, 1)$, $B(3, -4)$ and $C(5, 7)$.
- Find the coordinates of M , the mid-point of AB , and N , the mid-point of AC .
 - Show that MN is parallel to BC .
- 15** The points $A(2, 1)$, $B(2, 7)$ and $C(-4, -1)$ form a triangle. M is the mid-point of AB and N is the mid-point of AC .
- Find the lengths of MN and BC .
 - Show that $BC = 2MN$.
- 16** The vertices of a quadrilateral $ABCD$ are $A(1, 1)$, $B(7, 3)$, $C(9, -7)$ and $D(-3, -3)$. The points P , Q , R and S are the mid-points of AB , BC , CD and DA respectively.
- Find the gradient of each side of $PQRS$.
 - What type of quadrilateral is $PQRS$?
- 17** The origin O and the points $P(4, 1)$, $Q(5, 5)$ and $R(1, 4)$ form a quadrilateral.
- Show that OR is parallel to PQ .
 - Show that OP is parallel to RQ .
 - Show that $OP = OR$.
 - What shape is $OPQR$?
- 18** The origin O and the points $L(-2, 3)$, $M(4, 7)$ and $N(6, 4)$ form a quadrilateral.
- Show that $ON = LM$.
 - Show that ON is parallel to LM .
 - Show that $OM = LN$.
 - What shape is $OLMN$?
- 19** The vertices of a quadrilateral $PQRS$ are $P(1, 2)$, $Q(7, 0)$, $R(6, -4)$ and $S(-3, -1)$.
- Find the gradient of each side of the quadrilateral.
 - What type of quadrilateral is $PQRS$? *Trapezium*
- 20** The vertices of a quadrilateral are $T(3, 2)$, $U(2, 5)$, $V(8, 7)$ and $W(6, 1)$. The mid-points of UV and VW are M and N respectively. Show that the triangle TMN is isosceles.
- 21** The vertices of a quadrilateral $DEFG$ are $D(3, -2)$, $E(0, -3)$, $F(-2, 3)$ and $G(4, 1)$.
- Find the length of each side of the quadrilateral.
 - What type of quadrilateral is $DEFG$? *Kite*
- 22** The points $A(2, 1)$, $B(6, 10)$ and $C(10, 1)$ form an isosceles triangle with AB and BC of equal length. The point G is $(6, 4)$.
- Write down the coordinates of M , the mid-point of AC .
 - Show that $BG = 2GM$ and that BGM is a straight line.
 - Write down the coordinates of N , the mid-point of BC .
 - Show that AGN is a straight line and that $AG = 2GN$.

1.4 What is meant by the equation of a straight line or of a curve?

How can you tell whether or not the points $(3, 7)$ and $(1, 5)$ lie on the curve $y = 3x^2 + 2$? The answer is to substitute the coordinates of the points into the equation and see whether they fit; that is, whether the equation is **satisfied** by the coordinates of the point.

For $(3, 7)$: the right side is $3 \times 3^2 + 2 = 29$ and the left side is 7, so the equation is not satisfied. The point $(3, 7)$ does not lie on the curve $y = 3x^2 + 2$.

For $(1, 5)$: the right side is $3 \times 1^2 + 2 = 5$ and the left side is 5, so the equation is satisfied. The point $(1, 5)$ lies on the curve $y = 3x^2 + 2$.

The equation of a line or curve is a rule for determining whether or not the point with coordinates (x, y) lies on the line or curve.

This is an important way of thinking about the equation of a line or curve.

1.5 The equation of a line

Example 1.5.1

Find the equation of the line with gradient 2 which passes through the point $(2, 1)$.

Fig. 1.9 shows the line of gradient 2 through $A(2, 1)$, with another point $P(x, y)$ lying on it. P lies on the line if (and only if) the gradient of AP is 2.

The gradient of AP is $\frac{y-1}{x-2}$. Equating this to 2 gives $\frac{y-1}{x-2} = 2$, which is $y-1=2x-4$, or $y=2x-3$.

In the general case, you need to find the equation of the line with gradient m through the point A with coordinates (x_1, y_1) . Fig. 1.10 shows this line and another point P with coordinates (x, y) on it. The gradient of AP is $\frac{y-y_1}{x-x_1}$.

Equating to m gives $\frac{y-y_1}{x-x_1} = m$, or $y-y_1 = m(x-x_1)$.

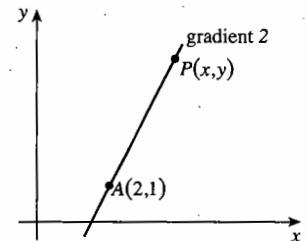


Fig. 1.9

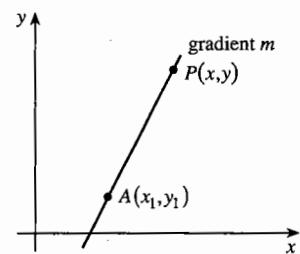


Fig. 1.10

The equation of the line through (x_1, y_1) with gradient m is $y - y_1 = m(x - x_1)$.

Notice that the coordinates of $A(x_1, y_1)$ satisfy this equation.

Example 1.5.2

Find the equation of the line through the point $(-2, 3)$ with gradient -1 .

Using the equation $y - y_1 = m(x - x_1)$ gives the equation $y - 3 = -1(x - (-2))$, which is $y - 3 = -x - 2$ or $y = -x + 1$. As a check, substitute the coordinates $(-2, 3)$ into both sides of the equation, to make sure that the given point does actually lie on the line.

Example 1.5.3

Find the equation of the line joining the points $(3, 4)$ and $(-1, 2)$.

To find this equation, first find the gradient of the line joining $(3, 4)$ to $(-1, 2)$.

Then you can use the equation $y - y_1 = m(x - x_1)$.

The gradient of the line joining $(3, 4)$ to $(-1, 2)$ is $\frac{2-4}{(-1)-3} = \frac{-2}{-4} = \frac{1}{2}$.

The equation of the line through $(3, 4)$ with gradient $\frac{1}{2}$ is $y - 4 = \frac{1}{2}(x - 3)$. After multiplying out and simplifying you get $2y - 8 = x - 3$, or $2y = x + 5$.

Check this equation mentally by substituting the coordinates of the other point.

1.6 Recognising the equation of a line

The answers to Examples 1.5.1–1.5.3 can all be written in the form $y = mx + c$, where m and c are numbers.

It is easy to show that any equation of this form is the equation of a straight line. If $y = mx + c$, then $y - c = m(x - 0)$, or

$$\frac{y - c}{x - 0} = m \quad (\text{except when } x = 0).$$

This equation tells you that, for all points (x, y) whose coordinates satisfy the equation, the line joining $(0, c)$ to (x, y) has gradient m . That is, (x, y) lies on the line through $(0, c)$ with gradient m .

The point $(0, c)$ lies on the y -axis. The number c is called the **y -intercept** of the line.

To find the x -intercept, put $y = 0$ in the equation, which gives $x = -\frac{c}{m}$. But notice that

you can't do this division if $m = 0$. In that case the line is parallel to the x -axis, so there is no x -intercept.

When $m = 0$, all the points on the line have coordinates of the form $(\text{something}, c)$. Thus the points $(1, 2)$, $(-1, 2)$, $(5, 2)$, ... all lie on the straight line $y = 2$, shown in Fig. 1.11. As a special case, the x -axis has equation $y = 0$.

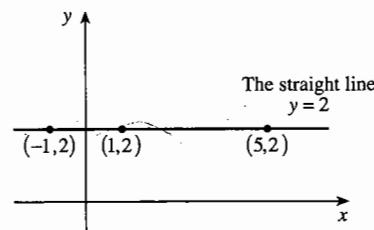


Fig. 1.11

Similarly, a straight line parallel to the y -axis has an equation of the form $x = k$. All points on it have coordinates $(k, \text{something})$. Thus the points $(3,0)$, $(3,2)$, $(3,4)$, ... all lie on the line $x = 3$, shown in Fig. 1.12. The y -axis itself has equation $x = 0$.

The line $x = k$ does not have a gradient; its gradient is undefined. Its equation cannot be written in the form $y = mx + c$.

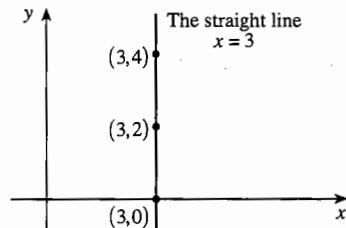


Fig. 1.12

1.7 The equation $ax + by + c = 0$

Suppose you have the equation $y = \frac{2}{3}x + \frac{4}{3}$. It is natural to multiply by 3 to get $3y = 2x + 4$, which can be rearranged to get $2x - 3y + 4 = 0$. This equation is in the form $ax + by + c = 0$ where a , b and c are constants.

Notice that the straight lines $y = mx + c$ and $ax + by + c = 0$ both contain the letter c , but it doesn't have the same meaning. For $y = mx + c$, c is the y -intercept, but there is no similar meaning for the c in $ax + by + c = 0$.

A simple way to find the gradient of $ax + by + c = 0$ is to rearrange it into the form $y = \dots$. Here are some examples.

Example 1.7.1

Find the gradient of the line $2x + 3y - 4 = 0$.

Write this equation in the form $y = \dots$, and then use the fact that the straight line $y = mx + c$ has gradient m .

From $2x + 3y - 4 = 0$ you find that $3y = -2x + 4$ and $y = -\frac{2}{3}x + \frac{4}{3}$. Therefore, comparing this equation with $y = mx + c$, the gradient is $-\frac{2}{3}$.

Example 1.7.2

One side of a parallelogram lies along the straight line with equation $3x - 4y - 7 = 0$. The point $(2,3)$ is a vertex of the parallelogram. Find the equation of one other side.

The line $3x - 4y - 7 = 0$ is the same as $y = \frac{3}{4}x - \frac{7}{4}$, so its gradient is $\frac{3}{4}$. The line through $(2,3)$ with gradient $\frac{3}{4}$ is $y - 3 = \frac{3}{4}(x - 2)$, or $3x - 4y + 6 = 0$.

1.8 The point of intersection of two lines

Suppose that you have two lines with equations $2x - y = 4$ and $3x + 2y = -1$. How do you find the coordinates of the point of intersection of these lines?

You want the point (x,y) which lies on both lines, so the coordinates (x,y) satisfy both equations. Therefore you need to solve the equations simultaneously.

From these two equations, you find $x = 1$, $y = -2$, so the point of intersection is $(1, -2)$.

This argument applies to straight lines with any equations provided they are not parallel. To find points of intersection, solve the equations simultaneously. The method can also be used to find the points of intersection of two curves.

Exercise 1B

1 Test whether the given point lies on the straight line (or curve) with the given equation.

- | | |
|---|---|
| (a) $(1, 2)$ on $y = 5x - 3$ | (b) $(3, -2)$ on $y = 3x - 7$ |
| (c) $(3, -4)$ on $x^2 + y^2 = 25$ | (d) $(2, 2)$ on $3x^2 + y^2 = 40$ |
| (e) $(1, 1\frac{1}{2})$ on $y = \frac{x+2}{3x-1}$ | (f) $\left(5p, \frac{5}{p}\right)$ on $y = \frac{5}{x}$ |
| (g) $(p, (p-1)^2 + 1)$ on $y = x^2 - 2x + 2$ | (h) $(t^2, 2t)$ on $y^2 = 4x$ |

2 Find the equations of the straight lines through the given points with the gradients shown. Your final answers should not contain any fractions.

- | | |
|--|---|
| (a) $(2, 3)$, gradient 5 | (b) $(1, -2)$, gradient -3 |
| (c) $(0, 4)$, gradient $\frac{1}{2}$ | (d) $(-2, 1)$, gradient $-\frac{3}{8}$ |
| (e) $(0, 0)$, gradient -3 | (f) $(3, 8)$, gradient 0 |
| (g) $(-5, -1)$, gradient $-\frac{3}{4}$ | (h) $(-3, 0)$, gradient $\frac{1}{2}$ |
| (i) $(-3, -1)$, gradient $\frac{3}{8}$ | (j) $(3, 4)$, gradient $-\frac{1}{2}$ |
| (k) $(2, -1)$, gradient -2 | (l) $(-2, -5)$, gradient 3 |
| (m) $(0, -4)$, gradient 7 | (n) $(0, 2)$, gradient -1 |
| (o) $(3, -2)$, gradient $-\frac{5}{8}$ | (p) $(3, 0)$, gradient $-\frac{3}{5}$ |
| (q) $(d, 0)$, gradient 7 | (r) $(0, 4)$, gradient m |
| (s) $(0, c)$, gradient 3 | (t) $(c, 0)$, gradient t |

3 Find the equations of the lines joining the following pairs of points. Give your final answer without fractions and in one of the forms $y = mx + c$ or $ax + by + c = 0$.

- | | |
|--------------------------------|------------------------------|
| (a) $(1, 4)$ and $(3, 10)$ | (b) $(4, 5)$ and $(-2, -7)$ |
| (c) $(3, 2)$ and $(0, 4)$ | (d) $(3, 7)$ and $(3, 12)$ |
| (e) $(10, -3)$ and $(-5, -12)$ | (f) $(3, -1)$ and $(-4, 20)$ |
| (g) $(2, -3)$ and $(11, -3)$ | (h) $(2, 0)$ and $(5, -1)$ |
| (i) $(-4, 2)$ and $(-1, -3)$ | (j) $(-2, -1)$ and $(5, -3)$ |
| (k) $(-3, 4)$ and $(-3, 9)$ | (l) $(-1, 0)$ and $(0, -1)$ |
| (m) $(2, 7)$ and $(3, 10)$ | (n) $(-5, 4)$ and $(-2, -1)$ |
| (o) $(0, 0)$ and $(5, -3)$ | (p) $(0, 0)$ and (p, q) |
| (q) (p, q) and $(p+3, q-1)$ | (r) $(p, -q)$ and (p, q) |
| (s) (p, q) and $(p+2, q+2)$ | (t) $(p, 0)$ and $(0, q)$ |

4 Find the gradients of the following lines.

- | | | |
|---------------------|--------------------|--------------------|
| (a) $2x + y = 7$ | (b) $3x - 4y = 8$ | (c) $5x + 2y = -3$ |
| (d) $y = 5$ | (e) $3x - 2y = -4$ | (f) $5x = 7$ |
| (g) $x + y = -3$ | (h) $y = 3(x + 4)$ | (i) $7 - x = 2y$ |
| (j) $3(y - 4) = 7x$ | (k) $y = m(x - d)$ | (l) $px + qy = pq$ |

5 Find the equation of the line through $(-2, 1)$ parallel to $y = \frac{1}{2}x - 3$.

6 Find the equation of the line through $(4, -3)$ parallel to $y + 2x = 7$.

7 Find the equation of the line through $(1, 2)$ parallel to the line joining $(3, -1)$ and $(-5, 2)$.

8 Find the equation of the line through $(3, 9)$ parallel to the line joining $(-3, 2)$ and $(2, -3)$.

9 Find the equation of the line through $(1, 7)$ parallel to the x -axis.

10 Find the equation of the line through $(d, 0)$ parallel to $y = mx + c$.

11 Find the points of intersection of the following pairs of straight lines.

- | | |
|------------------------------------|-------------------------------------|
| (a) $3x + 4y = 33$, $2y = x - 1$ | (b) $y = 3x + 1$, $y = 4x - 1$ |
| (c) $2y = 7x$, $3x - 2y = 1$ | (d) $y = 3x + 8$, $y = -2x - 7$ |
| (e) $x + 5y = 22$, $3x + 2y = 14$ | (f) $2x + 7y = 47$, $5x + 4y = 50$ |
| (g) $2x + 3y = 7$, $6x + 9y = 11$ | (h) $3x + y = 5$, $x + 3y = -1$ |
| (i) $y = 2x + 3$, $4x - 2y = -6$ | (j) $ax + by = c$, $y = 2ax$ |
| (k) $y = mx + c$, $y = -mx + d$ | (l) $ax - by = 1$, $y = x$ |

12 Let P , with coordinates (p, q) , be a fixed point on the 'curve' with equation $y = mx + c$ and let Q , with coordinates (r, s) , be any other point on $y = mx + c$. Use the fact that the coordinates of P and Q satisfy the equation $y = mx + c$ to show that the gradient of PQ is m for all positions of Q .

13 There are some values of a , b and c for which the equation $ax + by + c = 0$ does not represent a straight line. Give an example of such values.

1.9 The gradients of perpendicular lines

In Section 1.3 it is stated that two lines are parallel if they have the same gradient. But what can you say about the gradients of two lines which are perpendicular?

Firstly, if a line has a positive gradient, then the perpendicular line has a negative gradient, and vice versa. But you can be more exact than this.

In Fig. 1.13, if the gradient of PB is m , you can draw a 'gradient triangle' PAB in which PA is one unit and AB is m units.

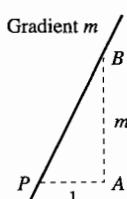


Fig. 1.13

In Fig 1.14, the gradient triangle PAB has been rotated through a right-angle to $PA'B'$, so that PB' is perpendicular to PB . The y -step for $PA'B'$ is 1 and the x -step is $-m$, so

$$\text{gradient of } PB' = \frac{\text{y-step}}{\text{x-step}} = \frac{1}{-m} = -\frac{1}{m}.$$

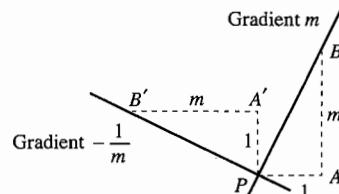


Fig. 1.14

Therefore the gradient of the line perpendicular to PB is $-\frac{1}{m}$.

Thus if the gradients of the two perpendicular lines are m_1 and m_2 , then $m_1m_2 = -1$. It is also true that if two lines have gradients m_1 and m_2 , and if $m_1m_2 = -1$, then the lines are perpendicular. To prove this, see Miscellaneous exercise 1 Question 22.

Two lines with gradients m_1 and m_2 are perpendicular if

$$m_1m_2 = -1, \quad \text{or} \quad m_1 = -\frac{1}{m_2}, \quad \text{or} \quad m_2 = -\frac{1}{m_1}.$$

Notice that the condition does not work if the lines are parallel to the axes. However, you can see that a line $x = \text{constant}$ is perpendicular to one of the form $y = \text{constant}$.

Example 1.9.1

Show that the points $(0, -5)$, $(-1, 2)$, $(4, 7)$ and $(5, 0)$ form a rhombus.

You could tackle this question in several ways. This solution shows that the points form a parallelogram, and then that its diagonals are perpendicular.

The mid-points of the diagonals are $\left(\frac{1}{2}(0+4), \frac{1}{2}(-5+7)\right)$, or $(2, 1)$, and $\left(\frac{1}{2}((-1)+5), \frac{1}{2}(2+0)\right)$, or $(2, 1)$. As these are the same point, the quadrilateral is a parallelogram.

The gradients of the diagonals are $\frac{7-(-5)}{4-0} = \frac{12}{4} = 3$ and $\frac{0-2}{5-(-1)} = \frac{-2}{6} = -\frac{1}{3}$. As the product of the gradients is -1 , the diagonals are perpendicular. Therefore the parallelogram is a rhombus.

Example 1.9.2

Find the coordinates of the foot of the perpendicular from $A(-2, -4)$ to the line joining $B(0, 2)$ and $C(-1, 4)$.

Always draw a diagram, like Fig. 1.15; it need not be to scale. The foot of the perpendicular is the point of intersection, P , of BC and the line through A perpendicular to BC . First find the gradient of BC and its equation.

The gradient of BC is $\frac{4-2}{-1-0} = \frac{2}{-1} = -2$.

The equation of BC is $y - 2 = -2(x - 0)$,
which simplifies to $2x + y = 2$.

The gradient of the line through A
perpendicular to BC is $-\frac{1}{-2} = \frac{1}{2}$.

The equation of this line is

$$y - (-4) = \frac{1}{2}(x - (-2)), \\ \text{or } x - 2y = 6.$$

These lines meet at the point P , whose
coordinates satisfy the simultaneous
equations $2x + y = 2$ and $x - 2y = 6$.
This is the point $(2, -2)$.

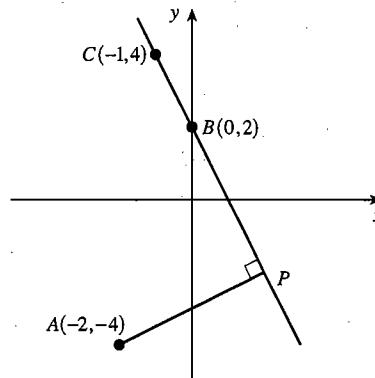


Fig. 1.15

Exercise 1C

- 1 In each part write down the gradient of a line which is perpendicular to one with the given gradient.

(a) 2	(b) -3	(c) $\frac{3}{4}$	(d) $-\frac{5}{6}$
(e) -1	(f) $1\frac{3}{4}$	(g) $-\frac{1}{m}$	(h) m
(i) $\frac{p}{q}$	(j) 0	(k) $-m$	(l) $\frac{a}{b-c}$
- 2 In each part find the equation of the line through the given point which is perpendicular to the given line. Write your final answer so that it doesn't contain fractions.

(a) $(2, 3)$, $y = 4x + 3$	(b) $(-3, 1)$, $y = -\frac{1}{2}x + 3$
(c) $(2, -5)$, $y = -5x - 2$	(d) $(7, -4)$, $y = 2\frac{1}{2}$
(e) $(-1, 4)$, $2x + 3y = 8$	(f) $(4, 3)$, $3x - 5y = 8$
(g) $(5, -3)$, $2x = 3$	(h) $(0, 3)$, $y = 2x - 1$
(i) $(0, 0)$, $y = mx + c$	(j) (a, b) , $y = mx + c$
(k) (c, d) , $ny - x = p$	(l) $(-1, -2)$, $ax + by = c$
- 3 Find the equation of the line through the point $(-2, 5)$ which is perpendicular to the line $y = 3x + 1$. Find also the point of intersection of the two lines.
- 4 Find the equation of the line through the point $(1, 1)$ which is perpendicular to the line $2x - 3y = 12$. Find also the point of intersection of the two lines.
- 5 A line through a vertex of a triangle which is perpendicular to the opposite side is called an altitude. Find the equation of the altitude through the vertex A of the triangle ABC where A is the point $(2, 3)$, B is $(1, -7)$ and C is $(4, -1)$.

- 6** $P(2,5)$, $Q(12,5)$ and $R(8,-7)$ form a triangle.
- Find the equations of the altitudes (see Question 5) through (i) R and (ii) Q .
 - Find the point of intersection of these altitudes.
 - Show that the altitude through P also passes through this point.

Miscellaneous exercise 1

- Show that the triangle formed by the points $(-2,5)$, $(1,3)$ and $(5,9)$ is right-angled.
- Find the coordinates of the point where the lines $2x + y = 3$ and $3x + 5y - 1 = 0$ meet.
- A triangle is formed by the points $A(-1,3)$, $B(5,7)$ and $C(0,8)$.
 - Show that the angle ACB is a right angle.
 - Find the coordinates of the point where the line through B parallel to AC cuts the x -axis.
- $A(7,2)$ and $C(1,4)$ are two vertices of a square $ABCD$.
 - Find the equation of the diagonal BD .
 - Find the coordinates of B and of D .
- A quadrilateral $ABCD$ is formed by the points $A(-3,2)$, $B(4,3)$, $C(9,-2)$ and $D(2,-3)$.
 - Show that all four sides are equal in length.
 - Show that $ABCD$ is not a square.
- P is the point $(7,5)$ and l_1 is the line with equation $3x + 4y = 16$.
 - Find the equation of the line l_2 which passes through P and is perpendicular to l_1 .
 - Find the point of intersection of the lines l_1 and l_2 .
 - Find the perpendicular distance of P from the line l_1 .
- Prove that the triangle with vertices $(-2,8)$, $(3,20)$ and $(11,8)$ is isosceles. Find its area.
- The three straight lines $y = x$, $7y = 2x$ and $4x + y = 60$ form a triangle. Find the coordinates of its vertices.
- Find the equation of the line through $(1,3)$ which is parallel to $2x + 7y = 5$. Give your answer in the form $ax + by = c$.
- Find the equation of the perpendicular bisector of the line joining $(2,-5)$ and $(-4,3)$.
- The points $A(1,2)$, $B(3,5)$, $C(6,6)$ and D form a parallelogram. Find the coordinates of the mid-point of AC . Use your answer to find the coordinates of D .
- The point P is the foot of the perpendicular from the point $A(0,3)$ to the line $y = 3x$.
 - Find the equation of the line AP .
 - Find the coordinates of the point P .
 - Find the perpendicular distance of A from the line $y = 3x$.

- 13** Points which lie on the same straight line are called collinear. Show that the points $(-1, 3)$, $(4, 7)$ and $(-11, -5)$ are collinear.
- 14** Find the equation of the straight line that passes through the points $(3, -1)$ and $(-2, 2)$, giving your answer in the form $ax + by + c = 0$. Hence find the coordinates of the point of intersection of the line and the x -axis. (OCR)
- 15** The coordinates of the points A and B are $(3, 2)$ and $(4, -5)$ respectively. Find the coordinates of the mid-point of AB , and the gradient of AB .
Hence find the equation of the perpendicular bisector of AB , giving your answer in the form $ax + by + c = 0$, where a , b and c are integers. (OCR)
- 16** The curve $y = 1 + \frac{1}{2+x}$ crosses the x -axis at the point A and the y -axis at the point B .
 (a) Calculate the coordinates of A and of B .
 (b) Find the equation of the line AB .
 (c) Calculate the coordinates of the point of intersection of the line AB and the line with equation $3y = 4x$. (OCR)
- 17** The straight line p passes through the point $(10, 1)$ and is perpendicular to the line r with equation $2x + y = 1$. Find the equation of p .
Find also the coordinates of the point of intersection of p and r , and deduce the perpendicular distance from the point $(10, 1)$ to the line r . (OCR)
- 18** Show by calculation that the points $P(0, 7)$, $Q(6, 5)$, $R(5, 2)$ and $S(-1, 4)$ are the vertices of a rectangle.
- 19** The line $3x - 4y = 8$ meets the y -axis at A . The point C has coordinates $(-2, 9)$. The line through C perpendicular to $3x - 4y = 8$ meets it at B . Calculate the area of the triangle ABC .
- 20** The points $A(-3, -4)$ and $C(5, 4)$ are the ends of the diagonal of a rhombus $ABCD$.
 (a) Find the equation of the diagonal BD .
 (b) Given that the side BC has gradient $\frac{5}{3}$, find the coordinates of B and hence of D .
- 21** Find the equations of the medians (see Exercise 1A Question 13) of the triangle with vertices $(0, 2)$, $(6, 0)$ and $(4, 4)$. Show that the medians are concurrent (all pass through the same point).
- 22** Two lines have equations $y = m_1x + c_1$ and $y = m_2x + c_2$, and $m_1m_2 = -1$. Prove that the lines are perpendicular.

2 Surds and indices

The first part of this chapter is about expressions involving square and cube roots. The second part is about index notation. When you have completed it, you should

- be able to simplify expressions involving square, cube and other roots
- know the rules of indices
- know the meaning of negative, zero and fractional indices
- be able to simplify expressions involving indices.

2.1 Different kinds of number

At first numbers were used only for counting, and 1, 2, 3, ... were all that was needed. These are **natural numbers**, or **positive integers**.

Then it was found that numbers could also be useful for measurement and in commerce. For these purposes fractions were also needed. Integers and fractions together make up the **rational numbers**. These are numbers which can be expressed in the form $\frac{p}{q}$ where p and q are integers, and q is not 0.

One of the most remarkable discoveries of the ancient Greek mathematicians was that there are numbers which cannot be expressed in this way. These are called **irrational numbers**. The first such number to be found was $\sqrt{2}$, which is the length of the diagonal of a square with side 1 unit, by Pythagoras' theorem. The argument that the Greeks used to prove that $\sqrt{2}$ cannot be expressed as a fraction can be adapted to show that the square root, cube root, ... of any positive integer is either an integer or an irrational number. Many other numbers are now known to be irrational, of which the most well known is π .

Rational and irrational numbers together make up the **real numbers**. Integers, rational and irrational numbers, and real numbers can be either positive, negative or zero.

When rational numbers are written as decimals, they either come to a stop after a number of places, or the sequence of decimal digits eventually starts repeating in a regular pattern. For example,

$$\begin{aligned}\frac{7}{10} &= 0.7, & \frac{7}{11} &= 0.6363\dots, & \frac{7}{12} &= 0.5833\dots, & \frac{7}{13} &= 0.538\ 461\ 538\ 461\ 53\dots, \\ \frac{7}{14} &= 0.5, & \frac{7}{15} &= 0.466\dots, & \frac{7}{16} &= 0.4375, & \frac{7}{17} &= 0.411\ 764\ 705\ 882\ 352\ 941\ 176\dots\end{aligned}$$

The reverse is also true: if a decimal number stops or repeats indefinitely then it is a rational number. So if an irrational number is written as a decimal, the pattern of the decimal digits never repeats however long you continue the calculation.

2.2 Surds and their properties

When you met expressions such as $\sqrt{2}$, $\sqrt{8}$ and $\sqrt{12}$ before, it is likely that you used a calculator to express them in decimal form. You might have written

$$\sqrt{2} = 1.414\dots \text{ or } \sqrt{2} = 1.414 \text{ correct to 3 decimal places or } \sqrt{2} \approx 1.414.$$

Why is the statement ' $\sqrt{2} = 1.414$ ' incorrect?

Expressions like $\sqrt{2}$ or $\sqrt[3]{9}$ are called **surds**. This section is about calculating with surds. You need to remember that \sqrt{x} always means the *positive* square root of x (or zero when $x = 0$).

The main properties of surds that you will use are:

$$\sqrt{xy} = \sqrt{x} \times \sqrt{y} \quad \text{and} \quad \sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}.$$

You can see that as $(\sqrt{x} \times \sqrt{y}) \times (\sqrt{x} \times \sqrt{y}) = (\sqrt{x} \times \sqrt{x}) \times (\sqrt{y} \times \sqrt{y}) = x \times y = xy$, and as $\sqrt{x} \times \sqrt{y}$ is positive, it is the square root of xy . Therefore $\sqrt{xy} = \sqrt{x} \times \sqrt{y}$. Similar reasoning will convince you that $\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$.

The following examples illustrate these properties:

$$\sqrt{8} = \sqrt{4 \times 2} = \sqrt{4} \times \sqrt{2} = 2\sqrt{2}; \quad \sqrt{12} = \sqrt{4 \times 3} = \sqrt{4} \times \sqrt{3} = 2\sqrt{3};$$

$$\sqrt{18} \times \sqrt{2} = \sqrt{18 \times 2} = \sqrt{36} = 6; \quad \frac{\sqrt{27}}{\sqrt{3}} = \sqrt{\frac{27}{3}} = \sqrt{9} = 3.$$

It is well worth checking some or all of the calculations above on your calculator.

Example 2.2.1

Simplify (a) $\sqrt{28} + \sqrt{63}$, (b) $\sqrt{5} \times \sqrt{10}$.

Alternative methods of solution may be possible, as in part (b).

$$(a) \quad \sqrt{28} + \sqrt{63} = (\sqrt{4} \times \sqrt{7}) + (\sqrt{9} \times \sqrt{7}) = 2\sqrt{7} + 3\sqrt{7} = 5\sqrt{7}.$$

$$(b) \quad \textbf{Method 1} \quad \sqrt{5} \times \sqrt{10} = \sqrt{5 \times 10} = \sqrt{50} = \sqrt{25 \times 2} = 5\sqrt{2}.$$

$$\textbf{Method 2} \quad \sqrt{5} \times \sqrt{10} = \sqrt{5} \times (\sqrt{5} \times \sqrt{2}) = (\sqrt{5} \times \sqrt{5}) \times \sqrt{2} = 5\sqrt{2}.$$

It is sometimes useful to be able to remove a surd from the denominator of a fraction such as $\frac{1}{\sqrt{2}}$. You can do this by multiplying top and bottom by $\sqrt{2}$: $\frac{1 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{\sqrt{2}}{2}$.

Results which it is often helpful to use are:

$$\frac{x}{\sqrt{x}} = \sqrt{x}, \text{ and its reciprocal } \frac{1}{\sqrt{x}} = \frac{\sqrt{x}}{x}.$$

Removing the surd from the denominator is called **rationalising the denominator**.

Example 2.2.2

Rationalise the denominator in the expressions (a) $\frac{6}{\sqrt{2}}$, (b) $\frac{3\sqrt{2}}{\sqrt{10}}$.

$$(a) \frac{6}{\sqrt{2}} = \frac{3 \times 2}{\sqrt{2}} = 3 \times \frac{2}{\sqrt{2}} = 3\sqrt{2}.$$

$$(b) \frac{3\sqrt{2}}{\sqrt{10}} = \frac{3\sqrt{2}}{\sqrt{5} \times \sqrt{2}} = \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5}.$$

Similar rules to those for square roots also apply to cube roots and higher roots.

Example 2.2.3

Simplify (a) $\sqrt[3]{16}$, (b) $\sqrt[3]{12} \times \sqrt[3]{18}$.

$$(a) \sqrt[3]{16} = \sqrt[3]{8 \times 2} = \sqrt[3]{8} \times \sqrt[3]{2} = 2 \times \sqrt[3]{2}.$$

$$(b) \sqrt[3]{12} \times \sqrt[3]{18} = \sqrt[3]{12 \times 18} = \sqrt[3]{216} = 6.$$

Example 2.2.4

Fig. 2.1 shows the vertical cross-section of a roof of a building as a right-angled triangle ABC , with $AB = 15$ m. The height of the roof, BD , is 10 m. Calculate x and y .

Starting with triangle ADB , by Pythagoras' theorem $z^2 + 10^2 = 15^2$, so $z^2 = 225 - 100 = 125$ and

$$z = \sqrt{125} = \sqrt{25 \times 5} = 5\sqrt{5}.$$

Now notice that the triangles ADB and ABC are similar. You can see the similarity more clearly by flipping triangle ADB over to make the side AB horizontal, as in Fig. 2.2. The sides of triangles ADB and ABC must therefore be in the same proportion, so

$$\frac{x}{15} = \frac{y}{10} = \frac{15}{z}. \text{ Since } \frac{15}{z} = \frac{15}{5\sqrt{5}} = \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5},$$

$$x = 15 \times \frac{3\sqrt{5}}{5} = 9\sqrt{5} \quad \text{and} \quad y = 10 \times \frac{3\sqrt{5}}{5} = 6\sqrt{5}.$$

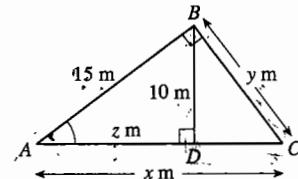


Fig. 2.1

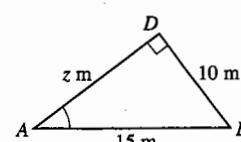


Fig. 2.2

Use Pythagoras' theorem in triangle ABC to check that $x^2 = 15^2 + y^2$.

Exercise 2A

1 Simplify the following without using a calculator.

(a) $\sqrt{3} \times \sqrt{3}$ (b) $\sqrt{10} \times \sqrt{10}$ (c) $\sqrt{16} \times \sqrt{16}$ (d) $\sqrt{8} \times \sqrt{2}$
 (e) $\sqrt{32} \times \sqrt{2}$ (f) $\sqrt{3} \times \sqrt{12}$ (g) $5\sqrt{3} \times \sqrt{3}$ (h) $2\sqrt{5} \times 3\sqrt{5}$
 (i) $3\sqrt{6} \times 4\sqrt{6}$ (j) $2\sqrt{20} \times 3\sqrt{5}$ (k) $(2\sqrt{7})^2$ (l) $(3\sqrt{3})^2$
 (m) $\sqrt[3]{5} \times \sqrt[3]{5} \times \sqrt[3]{5}$ (n) $(2\sqrt[4]{3})^4$ (o) $(2\sqrt[3]{2})^6$ (p) $\sqrt[4]{125} \times \sqrt[4]{5}$

2 Simplify the following without using a calculator.

(a) $\sqrt{18}$	(b) $\sqrt{20}$	(c) $\sqrt{24}$	(d) $\sqrt{32}$
(e) $\sqrt{40}$	(f) $\sqrt{45}$	(g) $\sqrt{48}$	(h) $\sqrt{50}$
(i) $\sqrt{54}$	(j) $\sqrt{72}$	(k) $\sqrt{135}$	(l) $\sqrt{675}$

3 Simplify the following without using a calculator.

(a) $\sqrt{8} + \sqrt{18}$ (b) $\sqrt{3} + \sqrt{12}$ (c) $\sqrt{20} - \sqrt{5}$
 (d) $\sqrt{32} - \sqrt{8}$ (e) $\sqrt{50} - \sqrt{18} - \sqrt{8}$ (f) $\sqrt{27} + \sqrt{27}$
 (g) $\sqrt{99} + \sqrt{44} + \sqrt{11}$ (h) $8\sqrt{2} + 2\sqrt{8}$ (i) $2\sqrt{20} + 3\sqrt{45}$
 (j) $\sqrt{52} - \sqrt{13}$ (k) $20\sqrt{5} + 5\sqrt{20}$ (l) $\sqrt{48} + \sqrt{24} - \sqrt{75} + \sqrt{96}$

4 Simplify the following without using a calculator.

(a) $\frac{\sqrt{8}}{\sqrt{2}}$ (b) $\frac{\sqrt{27}}{\sqrt{3}}$ (c) $\frac{\sqrt{40}}{\sqrt{10}}$ (d) $\frac{\sqrt{50}}{\sqrt{2}}$
 (e) $\frac{\sqrt{125}}{\sqrt{5}}$ (f) $\frac{\sqrt{54}}{\sqrt{6}}$ (g) $\frac{\sqrt{3}}{\sqrt{48}}$ (h) $\frac{\sqrt{50}}{\sqrt{200}}$

5 Rationalise the denominator in each of the following expressions, and simplify them.

(a) $\frac{1}{\sqrt{3}}$ (b) $\frac{1}{\sqrt{5}}$ (c) $\frac{4}{\sqrt{2}}$ (d) $\frac{6}{\sqrt{6}}$
 (e) $\frac{11}{\sqrt{11}}$ (f) $\frac{2}{\sqrt{8}}$ (g) $\frac{12}{\sqrt{3}}$ (h) $\frac{14}{\sqrt{7}}$
 (i) $\frac{\sqrt{6}}{\sqrt{2}}$ (j) $\frac{\sqrt{2}}{\sqrt{6}}$ (k) $\frac{3\sqrt{5}}{\sqrt{3}}$ (l) $\frac{4\sqrt{6}}{\sqrt{5}}$
 (m) $\frac{7\sqrt{2}}{2\sqrt{3}}$ (n) $\frac{4\sqrt{2}}{\sqrt{12}}$ (o) $\frac{9\sqrt{12}}{2\sqrt{18}}$ (p) $\frac{2\sqrt{18}}{9\sqrt{12}}$

6 Simplify the following, giving each answer in the form $k\sqrt{3}$.

(a) $\sqrt{75} + \sqrt{12}$ (b) $6 + \sqrt{3}(4 - 2\sqrt{3})$ (c) $\frac{12}{\sqrt{3}} - \sqrt{27}$
 (d) $\frac{2}{\sqrt{3}} + \frac{\sqrt{2}}{\sqrt{6}}$ (e) $\sqrt{2} \times \sqrt{8} \times \sqrt{27}$ (f) $(3 - \sqrt{3})(2 - \sqrt{3}) - \sqrt{3} \times \sqrt{27}$

7 ABCD is a rectangle in which $AB = 4\sqrt{5}$ cm and $BC = \sqrt{10}$ cm. Giving each answer in simplified surd form, find

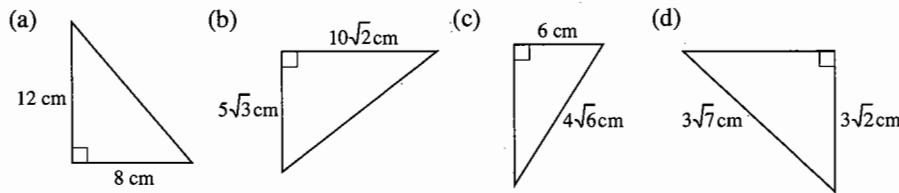
8 Solve the following equations, giving each answer in the form $k\sqrt{2}$.

(a) $x\sqrt{2} = 10$ (b) $2y\sqrt{2} - 3 = \frac{5y}{\sqrt{2}} + 1$ (c) $z\sqrt{32} - 16 = z\sqrt{8} - 4$

9 Express in the form $k\sqrt[3]{3}$

(a) $\sqrt[3]{24}$, (b) $\sqrt[3]{81} + \sqrt[3]{3}$, (c) $(\sqrt[3]{3})^4$, (d) $\sqrt[3]{3000} - \sqrt[3]{375}$.

10 Find the length of the third side in each of the following right-angled triangles, giving each answer in simplified surd form.



11 You are given that, correct to 12 decimal places, $\sqrt{26} = 5.099\ 019\ 513\ 593$.

(a) Find the value of $\sqrt{104}$ correct to 10 decimal places.

(b) Find the value of $\sqrt{650}$ correct to 10 decimal places.

(c) Find the value of $\frac{13}{\sqrt{26}}$ correct to 10 decimal places.

12 Solve the simultaneous equations $7x - (3\sqrt{5})y = 9\sqrt{5}$ and $(2\sqrt{5})x + y = 34$.

13 Simplify the following.

(a) $(\sqrt{2} - 1)(\sqrt{2} + 1)$

(b) $(2 - \sqrt{3})(2 + \sqrt{3})$

(c) $(\sqrt{7} + \sqrt{3})(\sqrt{7} - \sqrt{3})$

(d) $(2\sqrt{2} + 1)(2\sqrt{2} - 1)$

(e) $(4\sqrt{3} - \sqrt{2})(4\sqrt{3} + \sqrt{2})$

(f) $(\sqrt{10} + \sqrt{5})(\sqrt{10} - \sqrt{5})$

(g) $(4\sqrt{7} - \sqrt{5})(4\sqrt{7} + \sqrt{5})$

(h) $(2\sqrt{6} - 3\sqrt{3})(2\sqrt{6} + 3\sqrt{3})$

14 In Question 13, every answer is an integer. Copy and complete each of the following.

(a) $(\sqrt{3} - 1)(\quad) = 2$

(b) $(\sqrt{5} + 1)(\quad) = 4$

(c) $(\sqrt{6} - \sqrt{2})(\quad) = 4$

(d) $(2\sqrt{7} + \sqrt{3})(\quad) = 25$

(e) $(\sqrt{11} + \sqrt{10})(\quad) = 1$

(f) $(3\sqrt{5} - 2\sqrt{6})(\quad) = 21$

The examples in Questions 15 and 16 indicate a method for rationalising the denominator in cases which are more complicated than those in Question 5.

15 (a) Explain why $\frac{1}{\sqrt{3}-1} = \frac{1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}$ and hence show that $\frac{1}{\sqrt{3}-1} = \frac{\sqrt{3}+1}{2}$.

* (b) Show that $\frac{1}{2\sqrt{2}+\sqrt{3}} = \frac{2\sqrt{2}-\sqrt{3}}{5}$.

16 Rationalise the denominators and simplify these fractions.

(a) $\frac{1}{2-\sqrt{3}}$

(b) $\frac{1}{3\sqrt{5}-5}$

(c) $\frac{4\sqrt{3}}{2\sqrt{6}+3\sqrt{2}}$

2.3 Working with indices

In the 16th century, when mathematics books began to be printed, mathematicians were finding how to solve cubic and quartic equations. They found it was more economical to write and to print the products xxx and $xxxx$ as x^3 and x^4 .

This is how index notation started. But it turned out to be much more than a convenient shorthand. The new notation led to important mathematical discoveries, and mathematics as it is today would be inconceivable without index notation.

You will already have used simple examples of this notation. In general, the symbol a^m stands for the result of multiplying m a s together:

$$a^m = \overbrace{a \times a \times a \times \dots \times a}^{m \text{ of these}}.$$

The number a is called the **base**, and the number m is the **index** (plural ‘indices’). Notice that, although a can be any kind of number, m must be a positive integer. Another way of describing this is ‘ a raised to the m th power’, or more shortly ‘ a to the power m ’. Expressions in index notation can often be simplified by using a few simple rules.

One of these is the **multiplication rule**,

$$a^m \times a^n = \overbrace{a \times a \times \dots \times a}^{m \text{ of these}} \times \overbrace{a \times a \times \dots \times a}^{n \text{ of these}} = \overbrace{a \times a \times \dots \times a}^{m+n \text{ of these}} = a^{m+n}.$$

This is used, for example, in finding the volume of a cube of side a :

$$\text{volume} = \text{base area} \times \text{height} = a^2 \times a = a^2 \times a^1 = a^{2+1} = a^3.$$

Closely linked with this is the **division rule**,

$$\begin{aligned} a^m \div a^n &= \overbrace{(a \times a \times \dots \times a)}^{m \text{ of these}} \div \overbrace{(a \times a \times \dots \times a)}^{n \text{ of these}} \\ &= \overbrace{a \times a \times \dots \times a}^{m-n \text{ of these}} \quad (\text{since } n \text{ of the } a\text{s cancel out}) \\ &= a^{m-n}, \quad \text{provided that } m > n. \end{aligned}$$

Another rule is the **power-on-power rule**,

$$\begin{aligned} (a^m)^n &= \overbrace{a \times a \times \dots \times a}^{m \text{ of these}} \times \overbrace{a \times a \times \dots \times a}^{m \text{ of these}} \times \dots \times \overbrace{a \times a \times \dots \times a}^{m \text{ of these}} \\ &= \overbrace{a \times a \times \dots \times a}^{m \times n \text{ of these}} = a^{m \times n}. \end{aligned}$$

One further rule, the **factor rule**, has two bases but just one index:

$$(a \times b)^m = \overbrace{(a \times b) \times (a \times b) \times \dots \times (a \times b)}^{m \text{ of these brackets}} = \overbrace{a \times a \times \dots \times a}^{m \text{ of these}} \times \overbrace{b \times b \times \dots \times b}^{m \text{ of these}} = a^m \times b^m.$$

In explaining these rules multiplication signs have been used. But, as in other parts of algebra, they are usually omitted if there is no ambiguity. For completeness, here are the rules again.

Multiplication rule:	$a^m \times a^n = a^{m+n}$
Division rule:	$a^m \div a^n = a^{m-n}$, provided that $m > n$
Power-on-power rule:	$(a^m)^n = a^{m \times n}$
Factor rule:	$(a \times b)^m = a^m \times b^m$

Example 2.3.1

Simplify $(2a^2b)^3 \div (4a^4b)$.

$$\begin{aligned}
 (2a^2b)^3 \div (4a^4b) &= (2^3(a^2)^3b^3) \div (4a^4b) && \text{factor rule} \\
 &= (8a^{2 \times 3}b^3) \div (4a^4b) && \text{power-on-power rule} \\
 &= (8+4) \times (a^6 + a^4) \times (b^3 + b^1) && \text{rearranging} \\
 &= 2a^{6-4}b^{3-1} && \text{division rule} \\
 &= 2a^2b^2.
 \end{aligned}$$

2.4 Zero and negative indices

The definition of a^m in Section 2.3, as the result of multiplying m a s together, makes no sense if m is zero or a negative integer. You can't multiply -3 a s or 0 a s together. But extending the meaning of a^m when the index is zero or negative is possible, and useful, since it turns out that the rules still work with such index values.

Look at this sequence: $2^5 = 32$, $2^4 = 16$, $2^3 = 8$, $2^2 = 4$,

On the left sides, the base is always 2, and the indices go down by 1 at each step. On the right, the numbers are halved at each step. So you might continue the process

$$\dots, 2^2 = 4, 2^1 = 2, 2^0 = 1, 2^{-1} = \frac{1}{2}, 2^{-2} = \frac{1}{4}, 2^{-3} = \frac{1}{8}, \dots$$

and you can go on like this indefinitely. Now compare

$$2^1 = 2 \text{ with } 2^{-1} = \frac{1}{2}, \quad 2^2 = 4 \text{ with } 2^{-2} = \frac{1}{4}, \quad 2^3 = 8 \text{ with } 2^{-3} = \frac{1}{8}.$$

It looks as if 2^{-m} should be defined as $\frac{1}{2^m}$, with the special value in the middle $2^0 = 1$.

This observation, extended to any base a (except 0), and any positive integer m , gives the **negative power rule**.

$$\text{Negative power rule: } a^{-m} = \frac{1}{a^m} \quad \text{and} \quad a^0 = 1.$$

Here are some examples to show that, with these definitions, the rules established in Section 2.3 for positive indices still work with negative indices. Try making up other examples for yourself.

$$\begin{aligned}
 \text{Multiplication rule: } & a^3 \times a^{-7} = a^3 \times \frac{1}{a^7} = \frac{1}{a^7 \div a^3} \\
 & = \frac{1}{a^{7-3}} \quad \text{using the division rule for positive indices} \\
 & = \frac{1}{a^4} = a^{-4} = a^{3+(-7)} \\
 \text{Power-on-power rule: } & (a^{-2})^{-3} = \left(\frac{1}{a^2}\right)^{-3} = \frac{1}{(1/a^2)^3} = \frac{1}{1/(a^2)^3} \\
 & = \frac{1}{1/a^6} \quad \text{using the power-on-power rule for positive indices} \\
 & = a^6 = a^{(-2) \times (-3)} \\
 \text{Factor rule: } & (ab)^{-3} = \frac{1}{(ab)^3} = \frac{1}{a^3 b^3} \quad \text{using the factor rule for positive indices} \\
 & = \frac{1}{a^3} \times \frac{1}{b^3} = a^{-3} b^{-3}.
 \end{aligned}$$

Example 2.4.1

If $a = 5$, find the value of $4a^{-2}$.

The important thing to notice is that the index -2 goes only with the a and not with the 4 . So $4a^{-2}$ means $4 \times \frac{1}{a^2}$. When $a = 5$, $4a^{-2} = 4 \times \frac{1}{25} = 0.16$.

Example 2.4.2

Simplify (a) $4a^2b \times (3ab^{-1})^{-2}$, (b) $\left(\frac{MLT^{-2}}{L^2}\right) \div \left(\frac{LT^{-1}}{L}\right)$.

(a) **Method 1** Turn everything into positive indices.

$$\begin{aligned}
 4a^2b \times (3ab^{-1})^{-2} &= 4a^2b \times \frac{1}{(3a \times 1/b)^2} = 4a^2b \times \frac{1}{9a^2 \times 1/b^2} = 4a^2b \times \frac{b^2}{9a^2} \\
 &= \frac{4}{9}b^{1+2} = \frac{4}{9}b^3.
 \end{aligned}$$

Method 2 Use the rules directly with positive and negative indices.

$$\begin{aligned}
 4a^2b \times (3ab^{-1})^{-2} &= 4a^2b \times (3^{-2}a^{-2}(b^{-1})^{-2}) \quad \text{factor rule} \\
 &= 4a^2b \times (3^{-2}a^{-2}b^2) \quad \text{power-on-power rule} \\
 &= \left(4 \times \frac{1}{3^2}\right) \times (a^2a^{-2}) \times (bb^2) = \frac{4}{9}a^0b^3 = \frac{4}{9}b^3.
 \end{aligned}$$

(b) This is an application in mechanics: M, L, T stand for dimensions of mass, length and time in the measurement of viscosity. Taking the brackets separately,

$$\left(\frac{MLT^{-2}}{L^2}\right) = ML^{1-2}T^{-2} = ML^{-1}T^{-2} \quad \text{and} \quad \left(\frac{LT^{-1}}{L}\right) = L^{1-1}T^{-1} = L^0T^{-1} = T^{-1},$$

$$\text{so } \left(\frac{MLT^{-2}}{L^2} \right) \div \left(\frac{LT^{-1}}{L} \right) = (ML^{-1}T^{-2}) \div T^{-1} = ML^{-1}T^{-2-(-1)} = ML^{-1}T^{-1}$$

One application of negative indices is in writing down very small numbers. You probably know how to write very large numbers in standard form, or scientific notation. For example, it is easier to write the speed of light as $3.00 \times 10^8 \text{ m s}^{-1}$ than as $300\,000\,000 \text{ m s}^{-1}$. Similarly, the wavelength of red light, about 0.000 000 75 metres, is more easily appreciated written as $7.5 \times 10^{-7} \text{ metres}$.

Computers and calculators often give users the option to work in scientific notation, and if numbers become too large (or too small) to be displayed in ordinary numerical form they will switch into standard form, for example 3.00E8 or 7.5E-7. The symbol E stands for ‘exponent’, which is yet another word for ‘index’. You can write this in scientific notation by simply replacing the symbol Em by $\times 10^m$, for any integer m.

Example 2.4.3

Calculate the universal constant of gravitation, G, from $G = \frac{gR^2}{M}$ where, in SI units, $g \approx 9.81$, $R \approx 6.37 \times 10^6$ and $M \approx 5.97 \times 10^{24}$. (R and M are the earth’s radius and mass, and g is the acceleration due to gravity at the earth’s surface.)

$$\begin{aligned} G &\approx \frac{9.81 \times (6.37 \times 10^6)^2}{5.97 \times 10^{24}} = \frac{9.81 \times (6.37)^2}{5.97} \times \frac{(10^6)^2}{10^{24}} \\ &\approx 66.7 \times \frac{10^{12}}{10^{24}} = 6.67 \times 10^1 \times 10^{-12} = 6.67 \times 10^{1-12} = 6.67 \times 10^{-11}. \end{aligned}$$

Exercise 2B

1 Simplify the following expressions.

- | | | |
|------------------------------------|----------------------------------|------------------------------------|
| (a) $a^2 \times a^3 \times a^7$ | (b) $(b^4)^2$ | (c) $c^7 \div c^3$ |
| (d) $d^5 \times d^4$ | (e) $(e^5)^4$ | (f) $(x^3y^2)^2$ |
| (g) $5g^5 \times 3g^3$ | (h) $12h^{12} \div 4h^4$ | (i) $(2a^2)^3 \times (3a)^2$ |
| (j) $(p^2q^3)^2 \times (pq^3)^3$ | (k) $(4x^2y)^2 \times (2xy^3)^3$ | (l) $(6ac^3)^2 \div (9a^2c^5)$ |
| (m) $(3m^4n^2)^3 \times (2mn^2)^2$ | (n) $(49r^3s^2)^2 \div (7rs)^3$ | (o) $(2xy^2z^3)^2 \div (2xy^2z^3)$ |

2 Simplify the following, giving each answer in the form 2^n .

- | | | | |
|-------------------------------------|--------------------------------------|--------------------|-----------------------------|
| (a) $2^{11} \times (2^5)^3$ | (b) $(2^3)^2 \times (2^2)^3$ | (c) 4^3 | (d) 8^2 |
| (e) $\frac{2^7 \times 2^8}{2^{13}}$ | (f) $\frac{2^2 \times 2^3}{(2^2)^2}$ | (g) $4^2 \div 2^4$ | (h) $2 \times 4^4 \div 8^3$ |

3 Express each of the following as an integer or a fraction.

- | | | | |
|--------------------------------------|--------------|-------------------------------------|--------------------------------------|
| (a) 2^{-3} | (b) 4^{-2} | (c) 5^{-1} | (d) 3^{-2} |
| (e) 10^{-4} | (f) 1^{-7} | (g) $\left(\frac{1}{2}\right)^{-1}$ | (h) $\left(\frac{1}{3}\right)^{-3}$ |
| (i) $\left(2\frac{1}{2}\right)^{-1}$ | (j) 2^{-7} | (k) 6^{-3} | (l) $\left(1\frac{1}{3}\right)^{-3}$ |

2.5 Fractional indices

In Section 2.4, you saw that the four index rules still work when m and n are integers, but not necessarily positive. What happens if m and n are not necessarily integers?

If you put $m = \frac{1}{2}$ and $n = 2$ in the power-on-power rule, you find that

$$(x^{\frac{1}{2}})^2 = x^{\frac{1}{2} \times 2} = x^1 = x.$$

Putting $x^{\frac{1}{2}} = y$, this equation becomes $y^2 = x$, so $y = \sqrt{x}$ or $y = -\sqrt{x}$, which is $x^{\frac{1}{2}} = \sqrt{x}$ or $-\sqrt{x}$. Defining $x^{\frac{1}{2}}$ to be the positive square root of x , you get $x^{\frac{1}{2}} = \sqrt{x}$.

Similarly, if you put $m = \frac{1}{3}$ and $n = 3$, you can show that $x^{\frac{1}{3}} = \sqrt[3]{x}$. More generally, by putting $m = \frac{1}{n}$, you find that $(x^{\frac{1}{n}})^n = x^{\frac{1}{n} \times n} = x$, which leads to the result

$$x^{\frac{1}{n}} = \sqrt[n]{x}.$$

Notice that for the case $x^{\frac{1}{2}} = \sqrt{x}$, you must have $x \geq 0$, but for the case $x^{\frac{1}{3}} = \sqrt[3]{x}$ you do not need $x \geq 0$, because you can take the cube root of a negative number.

A slight extension of the $x^{\frac{1}{n}} = \sqrt[n]{x}$ rule can show you how to deal with expressions of the form $x^{\frac{2}{3}}$. There are two alternatives:

$$x^{\frac{2}{3}} = x^{\frac{1}{3} \times 2} = (x^{\frac{1}{3}})^2 = (\sqrt[3]{x})^2 \quad \text{and} \quad x^{\frac{2}{3}} = x^{2 \times \frac{1}{3}} = (x^2)^{\frac{1}{3}} = \sqrt[3]{x^2}.$$

(If x has an exact cube root it is usually best to use the first form; otherwise the second form is better.) In general, similar reasoning leads to the **fractional power rule**.

$$\boxed{\text{Fractional power rule: } x^{\frac{m}{n}} = (\sqrt[n]{x})^m = \sqrt[n]{x^m}.}$$

Fractional powers can also be written as $x^{1/2}$, $x^{m/n}$ and so on.

Example 2.5.1

Simplify (a) $9^{\frac{1}{2}}$, (b) $3^{\frac{1}{2}} \times 3^{\frac{3}{2}}$, (c) $16^{-\frac{3}{4}}$.

$$(a) 9^{\frac{1}{2}} = \sqrt{9} = 3. \quad (b) 3^{\frac{1}{2}} \times 3^{\frac{3}{2}} = 3^{\frac{1}{2} + \frac{3}{2}} = 3^2 = 9.$$

$$(c) \text{Method 1} \quad 16^{-\frac{3}{4}} = (2^4)^{-\frac{3}{4}} = 2^{-3} = \frac{1}{8}.$$

$$\text{Method 2} \quad 16^{-\frac{3}{4}} = \frac{1}{16^{\frac{3}{4}}} = \frac{1}{(\sqrt[4]{16})^3} = \frac{1}{2^3} = \frac{1}{8}.$$

There are often good alternative ways for solving problems involving indices, and you should try experimenting with them. Many people prefer to think with positive indices rather than negative ones; if you are one of them, writing $16^{-\frac{3}{4}} = \frac{1}{16^{\frac{3}{4}}}$, as in method 2 of Example 2.5.1(c), makes good sense as a first step.

Example 2.5.2

Simplify (a) $(2\frac{1}{4})^{-\frac{1}{2}}$, (b) $2x^{\frac{1}{2}} \times 3x^{-\frac{5}{2}}$, (c) $\frac{(2x^2y^2)^{-\frac{1}{2}}}{(2xy^{-2})^{\frac{3}{2}}}$.

$$(a) (2\frac{1}{4})^{-\frac{1}{2}} = \left(\frac{9}{4}\right)^{-\frac{1}{2}} = \left(\frac{4}{9}\right)^{\frac{1}{2}} = \sqrt{\frac{4}{9}} = \frac{2}{3}.$$

$$(b) 2x^{\frac{1}{2}} \times 3x^{-\frac{5}{2}} = 6x^{\frac{1}{2}-\frac{5}{2}} = 6x^{-2} = \frac{6}{x^2}.$$

(c) **Method 1** The numerator is $(2x^2y^2)^{-\frac{1}{2}} = \frac{1}{(2x^2y^2)^{\frac{1}{2}}} = \frac{1}{2^{\frac{1}{2}}xy}$, so

$$\frac{(2x^2y^2)^{-\frac{1}{2}}}{(2xy^{-2})^{\frac{3}{2}}} = \frac{1}{2^{\frac{1}{2}}xy} \times \frac{1}{2^{\frac{3}{2}}x^{\frac{3}{2}}y^{-3}} = \frac{1}{2^2x^{\frac{5}{2}}y^{-2}} = \frac{y^2}{4x^{\frac{5}{2}}}.$$

Method 2 Dividing by $(2xy^{-2})^{\frac{3}{2}}$ is equivalent to multiplying by $(2xy^{-2})^{-\frac{3}{2}}$, so

$$\frac{(2x^2y^2)^{-\frac{1}{2}}}{(2xy^{-2})^{\frac{3}{2}}} = (2x^2y^2)^{-\frac{1}{2}}(2xy^{-2})^{-\frac{3}{2}} = (2^{-\frac{1}{2}}x^{-1}y^{-1})(2^{-\frac{3}{2}}x^{-\frac{3}{2}}y^3) = 2^{-2}x^{-\frac{5}{2}}y^2.$$

Notice that in part (c) the final answer has been given in different forms for the two methods. Which is ‘simpler’ is a matter of taste.

Exercise 2C

1 Evaluate the following without using a calculator.

- | | | | |
|---------------------------|---------------------------|-------------------------|-----------------------------|
| (a) $25^{\frac{1}{2}}$ | (b) $8^{\frac{1}{3}}$ | (c) $36^{\frac{1}{2}}$ | (d) $32^{\frac{1}{3}}$ |
| (e) $81^{\frac{1}{4}}$ | (f) $9^{-\frac{1}{2}}$ | (g) $16^{-\frac{1}{4}}$ | (h) $49^{-\frac{1}{2}}$ |
| (i) $1000^{-\frac{1}{3}}$ | (j) $(-27)^{\frac{1}{3}}$ | (k) $64^{\frac{2}{3}}$ | (l) $(-125)^{-\frac{4}{3}}$ |

2 Evaluate the following without using a calculator.

- | | | | |
|------------------------------------|-----------------------------------|--------------------------|--------------------------------------|
| (a) $4^{\frac{1}{2}}$ | (b) $(\frac{1}{4})^2$ | (c) $(\frac{1}{4})^{-2}$ | (d) $4^{-\frac{1}{2}}$ |
| (e) $(\frac{1}{4})^{-\frac{1}{2}}$ | (f) $(\frac{1}{4})^{\frac{1}{2}}$ | (g) 4^2 | (h) $(\frac{1}{4})^{\frac{1}{4}})^2$ |

3 Evaluate the following without using a calculator.

- | | | | |
|------------------------------|--------------------------------------|------------------------------------|-------------------------------------|
| (a) $8^{\frac{2}{3}}$ | (b) $4^{\frac{3}{2}}$ | (c) $9^{-\frac{3}{2}}$ | (d) $27^{\frac{4}{3}}$ |
| (e) $32^{\frac{2}{5}}$ | (f) $32^{\frac{3}{5}}$ | (g) $64^{-\frac{5}{6}}$ | (h) $4^{2\frac{1}{2}}$ |
| (i) $10\ 000^{-\frac{3}{4}}$ | (j) $(\frac{1}{125})^{-\frac{4}{3}}$ | (k) $(3\frac{3}{8})^{\frac{2}{3}}$ | (l) $(2\frac{1}{4})^{-\frac{1}{2}}$ |

4 Simplify the following expressions.

- | | | |
|--|---|---|
| (a) $a^{\frac{1}{3}} \times a^{\frac{5}{3}}$ | (b) $3b^{\frac{1}{2}} \times 4b^{-\frac{3}{2}}$ | (c) $(6c^{\frac{1}{4}}) \times (4c)^{\frac{1}{2}}$ |
| (d) $(d^2)^{\frac{1}{3}} \div (d^{\frac{1}{3}})^2$ | (e) $(2x^{\frac{1}{2}} y^{\frac{1}{3}})^6 \times (\frac{1}{2} x^{\frac{1}{4}} y^{\frac{3}{4}})^4$ | (f) $(24e)^{\frac{1}{3}} \div (3e)^{\frac{1}{3}}$ |
| (g) $\frac{(5p^2 q^4)^{\frac{1}{3}}}{(25pq^2)^{-\frac{1}{3}}}$ | (h) $(4m^3 n)^{\frac{1}{4}} \times (8mn^3)^{\frac{1}{2}}$ | (i) $\frac{(2x^2 y^{-1})^{-\frac{1}{4}}}{(8x^{-1} y^2)^{-\frac{1}{2}}}$ |

5 Solve the following equations.

- | | | | |
|----------------------------|----------------------------|-----------------------------------|-----------------------------------|
| (a) $x^{\frac{1}{2}} = 8$ | (b) $x^{\frac{1}{3}} = 3$ | (c) $x^{\frac{2}{3}} = 4$ | (d) $x^{\frac{3}{2}} = 27$ |
| (e) $x^{-\frac{1}{2}} = 8$ | (f) $x^{-\frac{2}{3}} = 9$ | (g) $x^{\frac{3}{2}} = x\sqrt{2}$ | (h) $x^{\frac{3}{2}} = 2\sqrt{x}$ |

6 The time, T seconds, taken by a pendulum of length l metres to complete one swing is given by $T = 2\pi l^{\frac{1}{2}} g^{-\frac{1}{2}}$ where $g \approx 9.81 \text{ m s}^{-2}$.

- (a) Find the value of T for a pendulum of length 0.9 metres.
 (b) Find the length of a pendulum which takes 3 seconds for a complete swing.

7 The radius, r cm, of a sphere of volume V cm³ is given by $r = \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}}$. Find the radius of a sphere of volume 1150 cm³.

8 Solve the following equations.

- | | | | |
|----------------|---------------------------|-----------------------------------|----------------------------------|
| (a) $4^x = 32$ | (b) $9^y = \frac{1}{27}$ | (c) $16^z = 2$ | (d) $100^x = 1000$ |
| (e) $8^y = 16$ | (f) $8^z = \frac{1}{128}$ | (g) $(2^t)^3 \times 4^{t-1} = 16$ | (h) $\frac{9^y}{27^{2y+1}} = 81$ |

Miscellaneous exercise 2

1 Simplify

- | | |
|---|--|
| (a) $5(\sqrt{2} + 1) - \sqrt{2}(4 - 3\sqrt{2})$, | (b) $(\sqrt{2})^4 + (\sqrt{3})^4 + (\sqrt{4})^4$, |
| (c) $(\sqrt{5} - 2)^2 + (\sqrt{5} - 2)(\sqrt{5} + 2)$, | (d) $(2\sqrt{2})^5$. |

2 Simplify

- | | |
|---|------------------------------------|
| (a) $\sqrt{27} + \sqrt{12} - \sqrt{3}$, | (b) $\sqrt{63} - \sqrt{28}$, |
| (c) $\sqrt{100000} + \sqrt{1000} + \sqrt{10}$, | (d) $\sqrt[3]{2} + \sqrt[3]{16}$. |

3 Rationalise the denominators of the following.

- | | | | |
|---------------------------|---------------------------|------------------------------------|----------------------------------|
| (a) $\frac{9}{2\sqrt{3}}$ | (b) $\frac{1}{5\sqrt{5}}$ | (c) $\frac{2\sqrt{5}}{3\sqrt{10}}$ | (d) $\frac{\sqrt{8}}{\sqrt{15}}$ |
|---------------------------|---------------------------|------------------------------------|----------------------------------|

4 Simplify

- | | |
|--|--|
| (a) $\frac{4}{\sqrt{2}} - \frac{4}{\sqrt{8}}$, | (b) $\frac{10}{\sqrt{5}} + \sqrt{20}$, |
| (c) $\frac{1}{\sqrt{2}}(2\sqrt{2} - 1) + \sqrt{2}(1 - \sqrt{8})$, | (d) $\frac{\sqrt{6}}{\sqrt{2}} + \frac{3}{\sqrt{3}} + \frac{\sqrt{15}}{\sqrt{5}} + \frac{\sqrt{18}}{\sqrt{6}}$. |

5 Express $\frac{5}{\sqrt{7}}$ in the form $k\sqrt{7}$ where k is a rational number.

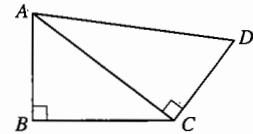
(OCR)

6 Justify the result $\sqrt{12} \times \sqrt{75} = 30$

- (a) using surds, (b) using fractional indices.

7 In the diagram, angles ABC and ACD are right angles.

Given that $AB = CD = 2\sqrt{6}$ cm and $BC = 7$ cm, show that the length of AD is between $4\sqrt{6}$ cm and $7\sqrt{2}$ cm.



8 In the triangle PQR , Q is a right angle, $PQ = (6 - 2\sqrt{2})$ cm and $QR = (6 + 2\sqrt{2})$ cm.

- (a) Find the area of the triangle. (b) Show that the length of PR is $2\sqrt{22}$ cm.

9 Simplify $\sqrt[3]{36} \times \sqrt[6]{\frac{4}{3}} \times \sqrt{27}$ by writing each factor in index notation.

10 In the triangle ABC , $AB = 4\sqrt{3}$ cm, $BC = 5\sqrt{3}$ cm and angle B is 60° . Use the cosine rule to find, in simplified surd form, the length of AC .

11 Solve the simultaneous equations $5x - 3y = 41$ and $(7\sqrt{2})x + (4\sqrt{2})y = 82$.

12 Use the 'raise to power' key on your calculator to find, correct to 5 significant figures,

(a) $\frac{1}{3.7^5}$, (b) $\sqrt[5]{3.7}$.

13 The coordinates of the points A and B are $(2, 3)$ and $(4, -3)$ respectively. Find the length of AB and the coordinates of the midpoint of AB . (OCR)

14 (a) Find the equation of the line l through the point $A(2, 3)$ with gradient $-\frac{1}{2}$.

(b) Show that the point P with coordinates $(2 + 2t, 3 - t)$ will always lie on l whatever the value of t .

(c) Find the values of t such that the length AP is 5 units.

(d) Find the value of t such that OP is perpendicular to l (where O is the origin).

Hence find the length of the perpendicular from O to l .

15 P and Q are the points of intersection of the line

$$\frac{x}{a} + \frac{y}{b} = 1 \quad (a > 0, b > 0)$$

with the x - and y -axes respectively. The distance PQ is 20 and the gradient of PQ is -3 . Find the values of a and b .

16 The sides of a parallelogram lie along the lines $y = 2x - 4$, $y = 2x - 13$, $x + y = 5$ and $x + y = -4$. Find the length of one side, and the perpendicular distance between this and the parallel side. Hence find the area of the parallelogram.

17 Evaluate the following without using a calculator.

(a) $(\frac{1}{2})^{-1} + (\frac{1}{2})^{-2}$ (b) $32^{-\frac{4}{3}}$ (c) $(4^{\frac{3}{2}})^{-\frac{1}{3}}$ (d) $(1\frac{7}{9})^{1\frac{1}{2}}$

18 Express $(9a^4)^{-\frac{1}{2}}$ as an algebraic fraction in simplified form. (OCR)

19 By letting $y = x^{\frac{1}{3}}$, or otherwise, find the values of x for which $x^{\frac{1}{3}} - 2x^{-\frac{1}{3}} = 1$. (OCR)

20 Solve the equation $4^{2x} \times 8^{x-1} = 32$.

21 Express $\frac{1}{(\sqrt{a})^{\frac{4}{3}}}$ in the form a^n , stating the value of n . (OCR)

22 Simplify

(a) $(4p^{\frac{1}{4}}q^{-3})^{\frac{1}{2}}$,

(b) $\frac{(5b)^{-1}}{(8b^6)^{\frac{1}{3}}}$,

(c) $(2x^6y^8)^{\frac{1}{4}} \times (8x^{-2})^{\frac{1}{4}}$,

(d) $(m^{\frac{1}{3}}n^{\frac{1}{2}})^2 \times (m^{\frac{1}{6}}n^{\frac{1}{3}})^4 \times (mn)^{-2}$.

23 Given that, in standard form, $3^{236} \approx 4 \times 10^{112}$, and $3^{-376} \approx 4 \times 10^{-180}$, find approximations, also in standard form, for

(a) 3^{376} ,

(b) 3^{612} ,

(c) $(\sqrt{3})^{236}$,

(d) $(3^{-376})^{\frac{5}{2}}$.

24 The table below shows, for three of the planets in the solar system, details of their mean distance from the sun and the time taken for one orbit round the sun.

Planet	Mean radius of orbit r metres	Period of revolution T seconds
Mercury	5.8×10^{10}	7.6×10^6
Jupiter	7.8×10^{11}	3.7×10^8
Pluto	5.9×10^{12}	7.8×10^9

(a) Show that r^3T^{-2} has approximately the same value for each planet in the table.

(b) The earth takes one year for one orbit of the sun. Find the mean radius of the earth's orbit around the sun.

25 Simplify

(a) $2^{-\frac{3}{2}} + 2^{-\frac{1}{2}} + 2^{\frac{1}{2}} + 2^{\frac{3}{2}}$, giving your answer in the form $k\sqrt{2}$,

(b) $(\sqrt{3})^{-3} + (\sqrt{3})^{-2} + (\sqrt{3})^{-1} + (\sqrt{3})^0 + (\sqrt{3})^1 + (\sqrt{3})^2 + (\sqrt{3})^3$, giving your answer in the form $a + b\sqrt{3}$.

26 Express each of the following in the form 2^n .

(a) $2^{70} + 2^{70}$

(b) $2^{-400} + 2^{-400}$

(c) $2^{\frac{1}{3}} + 2^{\frac{1}{3}} + 2^{\frac{1}{3}} + 2^{\frac{1}{3}}$

(d) $2^{100} - 2^{99}$

(e) $8^{0.1} + 8^{0.1} + 8^{0.1} + 8^{0.1} + 8^{0.1} + 8^{0.1} + 8^{0.1}$

27 Solve the equation $\frac{125^{3x}}{5^{x+4}} = \frac{25^{x-2}}{3125}$.

28 The formulae for the volume and the surface area of a sphere are $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$ respectively, where r is the sphere's radius. Find expressions for

(a) S in terms of V , (b) V in terms of S ,

giving your answer in the form $(S \text{ or } V) = 2^m 3^n \pi^p (V \text{ or } S)^q$.

29 The kinetic energy, K joules, possessed by an object of mass m kg moving with speed $v \text{ m s}^{-1}$ is given by the formula $K = \frac{1}{2}mv^2$. Find the kinetic energy possessed by a bullet of mass 2.5×10^{-2} kg moving with speed $8 \times 10^2 \text{ m s}^{-1}$.

3 Functions and graphs

This chapter introduces the idea of a function and investigates the graphs representing functions of various kinds. When you have completed it, you should

- understand function notation, and the terms ‘domain’ and ‘range’
- know the shapes of graphs of powers of x
- know the shapes of graphs of functions of the form $f(x) = ax^2 + bx + c$
- be able to suggest possible equations of such functions from their graphs
- know how to use factors to sketch graphs
- be able to find the point(s) of intersection of two graphs.

If you have access to a graphic calculator or a computer with graph-plotting software, you can use it to check for yourself the graphs which accompany the text, and to carry out further research along similar lines.

3.1 The idea of a function

You are already familiar with formulae which summarise calculations that need to be performed frequently, such as:

the area of a circle with radius x metres is πx^2 square metres;

the volume of a cube of side x metres is x^3 cubic metres;

the time to travel k kilometres at x kilometres per hour is $\frac{k}{x}$ hours.

You will often have used different letters from x in these formulae, such as r for radius or s for speed, but in this chapter x will always be used for the letter in the formula, and y for the quantity you want to calculate. Notice that some formulae also involve other letters, called **constants**; these might be either a number like π , which is irrational and cannot be written out in full, or a quantity like the distance k , which you choose for yourself depending on the distance you intend to travel.

Expressions such as πx^2 , x^3 and $\frac{k}{x}$ are examples of **functions** of x . Having chosen a value for x , you can get a unique value of y from it.

It is often useful to have a way of writing functions in general, rather than always having to refer to particular functions. The notation which is used for this is $f(x)$ (read ‘ f of x ’, or sometimes just ‘ f x ’). The letter f stands for the function itself, and x for the number for which you choose to evaluate it.

If you want to refer to the value of the function when x has a particular value, say $x = 2$, then you write the value as $f(2)$. For example, if $f(x)$ stands for the function x^3 , then $f(2) = 2^3 = 8$.

If a problem involves more than one function, you can use a different letter for each function. Two functions can, for example, be written as $f(x)$ and $g(x)$.

Functions are not always defined by algebraic formulae. Sometimes it is easier to describe them in words, or to define them using a flow chart or a computer program. All that matters is that each value of x chosen leads to a unique value of $y = f(x)$.

3.2 Graphs, domain and range

You know how to draw graphs. You set up a coordinate system for cartesian coordinates using x - and y -axes, and choose a scale on each axis.

The axes divide the plane of the paper or screen into four quadrants, numbered as shown in Fig. 3.1. The first quadrant is in the top right corner, where x and y are both positive. The other quadrants then follow in order going anticlockwise round the origin.

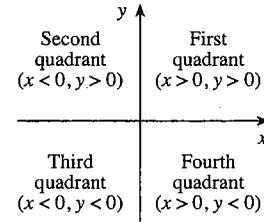


Fig. 3.1

Example 3.2.1

In which quadrants is $xy > 0$?

If the product of two numbers is positive, either both are positive or both are negative. So either $x > 0$ and $y > 0$, or $x < 0$ and $y < 0$. The point (x, y) therefore lies in either the first or the third quadrant.

It is often convenient to describe the direction of the y -axis as 'vertical', and the x -axis as 'horizontal'. But of course if you are drawing the graph on a horizontal surface like a table, these descriptions are not strictly accurate.

The graph of a function $f(x)$ is made up of all the points whose coordinates (x, y) satisfy the equation $y = f(x)$. When you draw such a graph by hand, you choose a few values of x and work out $y = f(x)$ for these. You then plot the points with coordinates (x, y) , and join up these points by eye, usually with a smooth curve. If you have done this accurately, the coordinates of other points on the curve will also satisfy the equation $y = f(x)$. Calculators and computers make graphs in much the same way, but they can plot many more points much more quickly.

When you produce a graph of $y = mx + c$ or $y = x^2$ you cannot show the whole graph. However small the scale, and however large the screen or the paper, the graph will eventually spill over the edge. This is because x can be any real number, as large as you like in both positive and negative directions. When you have to draw a graph like this, the skill is to choose the values of x between which to draw the graph so that you include all the important features.

You have met some functions which can't be defined for all real numbers. Examples are $\frac{1}{x}$, which has no meaning when x is 0; and \sqrt{x} , which has no meaning when x is negative.

Here is another example for which there is a restriction on the values of x you can choose.

Example 3.2.2

Draw the complete graph of $f(x) = \sqrt{4 - x^2}$.

You can only calculate the values of the function $f(x) = \sqrt{4 - x^2}$ if x is between -2 and 2 inclusive.

If $x > 2$ or $x < -2$, the value of $4 - x^2$ is negative, and a negative number does not have a square root.

Also $y = f(x)$ cannot be negative (recall that square roots are positive or zero by definition) and it can't be greater than $\sqrt{4} = 2$. So the graph of $f(x) = \sqrt{4 - x^2}$, shown in Fig. 3.2, lies between -2 and 2 inclusive in the x -direction, and between 0 and 2 inclusive in the y -direction.

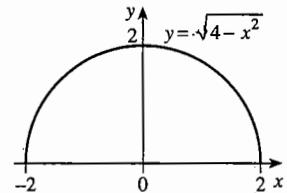


Fig. 3.2

Even when you use a function which has a meaning for all real numbers x , you may be interested in it only when x is restricted in some way. For example, the formula for the volume of a cube is $V = x^3$. Although you can calculate x^3 for any real number x , you would only use this formula for $x > 0$.

Here is an example in which x is restricted to a finite interval.

Example 3.2.3

A wire of length 4 metres is cut into two pieces, and each piece is bent into a square. How should this be done so that the two squares together have

- (a) the smallest area, (b) the largest area?

Let the two pieces have lengths x metres and $(4 - x)$ metres. The areas of the squares in Fig. 3.3 are then $(\frac{1}{4}x)^2$ and $(\frac{1}{4}(4 - x))^2$ square metres. So the total area, y square metres, is given by

$$y = \frac{1}{16}(x^2 + (16 - 8x + x^2)) = \frac{1}{8}(x^2 - 4x + 8).$$

Notice that, since $(x - 2)^2 = x^2 - 4x + 4$, this can be written as

$$y = \frac{1}{8}((x - 2)^2 + 4).$$

You can evaluate this expression for any real number x , but the problem only has meaning if $0 < x < 4$. Fig. 3.4 shows the graph of the area function for this interval. As $(x - 2)^2 \geq 0$, the area is least when $x = 2$, when it is 0.5 m^2 .

From the graph it looks as if the largest area is 1 m^2 , when $x = 0$ and $x = 4$; but these values of x are excluded, since they do not produce two pieces of wire. You can get areas as near to 1 m^2 as you like, but you cannot achieve this target. There is therefore no largest area.

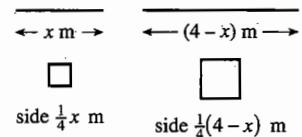


Fig. 3.3

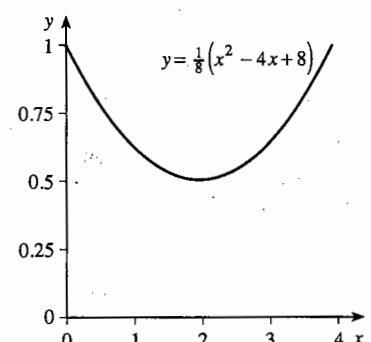


Fig. 3.4

So two possible reasons why a function $f(x)$ might not be defined for all real numbers x are

- the algebraic expression for $f(x)$ may have meaning only for some x
- only some x are relevant in the context in which the function is being used.

The set of numbers x for which a function $f(x)$ is defined is called the **domain** of the function. For example, the domains of the functions in Examples 3.2.2 and 3.2.3 are the intervals $-2 \leq x \leq 2$ and $0 < x < 4$. The largest possible domain of the function $\frac{1}{x}$ is all the real numbers except 0, but if the function is used in a practical problem you may choose a smaller domain, such as all positive real numbers.

Once you have decided the domain of a function $f(x)$, you can ask what values $f(x)$ can take. This set of values is called the **range** of the function.

In Example 3.2.2 the range is $0 \leq y \leq 2$. In Example 3.2.3 the graph shows that the range is $\frac{1}{2} \leq y < 1$. Note that the value $y = \frac{1}{2}$ is attained when $x = 2$, but the value $y = 1$ is not attained if $0 < x < 4$.

The function $f(x) = \frac{1}{x}$, with domain all real numbers except 0, takes all values except 0.

Exercise 3A

- 1 Given $f(x) = 2x + 5$, find the values of
 (a) $f(3)$, (b) $f(0)$, (c) $f(-4)$, (d) $f\left(-2\frac{1}{2}\right)$.
- 2 Given $f(x) = 3x^2 + 2$, find the values of
 (a) $f(4)$, (b) $f(\pm 1)$, (c) $f(\pm 3)$, (d) $f(3)$.
- 3 Given $f(x) = x^2 + 4x + 3$, find the values of
 (a) $f(2)$, (b) $f\left(\frac{1}{2}\right)$, (c) $f(\pm 1)$, (d) $f(\pm 3)$.
- 4 Given $g(x) = x^3$ and $h(x) = 4x + 1$,
 (a) find the value of $g(2) + h(2)$; (b) find the value of $3g(-1) - 4h(-1)$;
 (c) show that $g(5) = h(31)$; (d) find the value of $h(g(2))$.
- 5 Given $f(x) = x^n$ and $f(3) = 81$, determine the value of n .
- 6 Given that $f(x) = ax + b$ and that $f(2) = 7$ and $f(3) = 12$, find a and b .
- 7 Find the largest possible domain of each of the following functions.
 (a) \sqrt{x} (b) $\sqrt{-x}$ (c) $\sqrt{x-4}$ (d) $\sqrt{4-x}$
 (e) $\sqrt{x(x-4)}$ (f) $\sqrt{2x(x-4)}$ (g) $\sqrt{x^2 - 7x + 12}$ (h) $\sqrt{x^3 - 8}$
 (i) $\frac{1}{x-2}$ (j) $\frac{1}{\sqrt{x-2}}$ (k) $\frac{1}{1+\sqrt{x}}$ (l) $\frac{1}{(x-1)(x-2)}$
- 8 The domains of these functions are the set of all positive real numbers. Find their ranges.
 (a) $f(x) = 2x + 7$ (b) $f(x) = -5x$ (c) $f(x) = 3x - 1$
 (d) $f(x) = x^2 - 1$ (e) $f(x) = (x+2)^2 - 1$ (f) $f(x) = (x-1)^2 + 2$

9 Find the range of each of the following functions. All the functions are defined for all real values of x .

(a) $f(x) = x^2 + 4$

(b) $f(x) = 2(x^2 + 5)$

(c) $f(x) = (x - 1)^2 + 6$

(d) $f(x) = -(1 - x)^2 + 7$

(e) $f(x) = 3(x + 5)^2 + 2$

(f) $f(x) = 2(x + 2)^4 - 1$

10 These functions are each defined for the given domain. Find their ranges.

(a) $f(x) = 2x$ for $0 \leq x \leq 8$

(b) $f(x) = 3 - 2x$ for $-2 \leq x \leq 2$

(c) $f(x) = x^2$ for $-1 \leq x \leq 4$

(d) $f(x) = x^2$ for $-5 \leq x \leq -2$

11 Find the range of each of the following functions. All the functions are defined for the largest possible domain of values of x .

(a) $f(x) = x^8$

(b) $f(x) = x^{11}$

(c) $f(x) = \frac{1}{x^3}$

(d) $f(x) = \frac{1}{x^4}$

(e) $f(x) = x^4 + 5$

(f) $f(x) = \frac{1}{4}x + \frac{1}{8}$

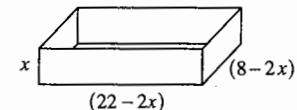
(g) $f(x) = \sqrt{4 - x^2}$

(h) $f(x) = \sqrt{4 - x}$

12 A piece of wire 24 cm long has the shape of a rectangle. Given that the width is w cm, show that the area, A cm², of the rectangle is given by the function $A = 36 - (6 - w)^2$. Find the greatest possible domain and the corresponding range of this function in this context.

13 Sketch the graph of $y = x(8 - 2x)(22 - 2x)$.

Given that y cm³ is the volume of a cuboid with height x cm, length $(22 - 2x)$ cm and width $(8 - 2x)$ cm, state an appropriate domain for the function given above.



3.3 Graphs of powers of x

(i) Positive integer powers

Consider first the graphs of functions of the form $f(x) = x^n$, where n is a positive integer. Notice that $(0,0)$ and $(1,1)$ satisfy the equation $y = x^n$ for all these values of n , so that all the graphs include the points $(0,0)$ and $(1,1)$.

First look at the graphs when x is positive. Then x^n is also positive, so that the graphs lie entirely in the first quadrant. Fig. 3.5 shows the graphs for $n = 1, 2, 3$ and 4 for values of x from 0 to somewhere beyond 1.

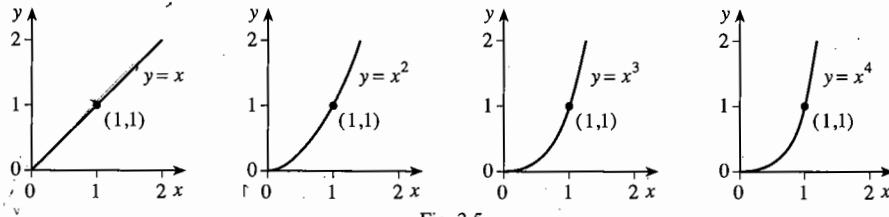


Fig. 3.5

Points to notice are:

- $n = 1$ is a special case: it gives the straight line $y = x$ through the origin, which makes an angle of 45° with each axis.

- For $n > 1$ the x -axis is a tangent to the graphs at the origin. This is because, when x is small, x^n is very small. For example, $0.1^2 = 0.01$, $0.1^3 = 0.001$, $0.1^4 = 0.0001$.
- For each increase in the index n , the graph stays closer to the x -axis between $x = 0$ and $x = 1$, but then climbs more steeply beyond $x = 1$. This is because $x^{n+1} = x \times x^n$, so that $x^{n+1} < x^n$ when $0 < x < 1$ and $x^{n+1} > x^n$ when $x > 1$.

What happens when x is negative depends on whether n is odd or even. To see this, suppose $x = -a$, where a is a positive number.

If n is even, $f(-a) = (-a)^n = a^n = f(a)$. So the value of y on the graph is the same for $x = -a$ and $x = a$. This means that the graph is symmetrical about the y -axis. This is illustrated in Fig. 3.6 for the graphs of $y = x^2$ and $y = x^4$. Functions with the property that $f(-a) = f(a)$ for all values of a are called **even functions**.

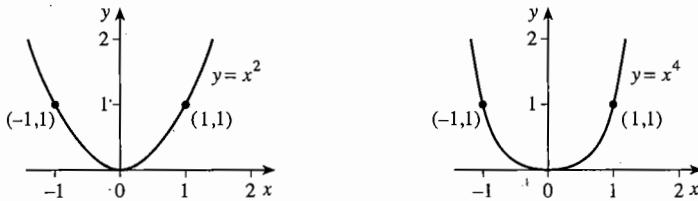


Fig. 3.6

If n is odd, $f(-a) = (-a)^n = -a^n = -f(a)$. The value of y for $x = -a$ is minus the value for $x = a$. Note that the points with coordinates (a, a^n) and $(-a, -a^n)$ are symmetrically placed on either side of the origin. This means that the whole graph is symmetrical about the origin. This is illustrated in Fig. 3.7 for the graphs of $y = x$ and $y = x^3$. Functions with the property that $f(-a) = -f(a)$ for all values of a are called **odd functions**.

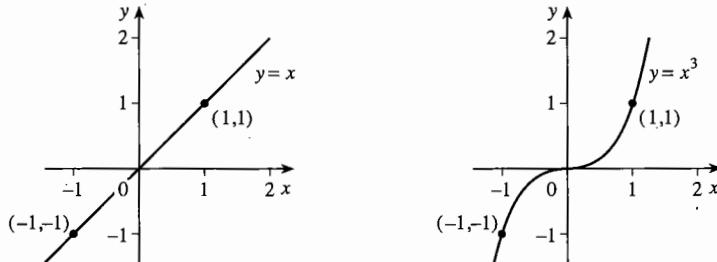


Fig. 3.7

(ii) Negative integer powers

You can write a negative integer n as $-m$, where m is a positive integer. Then x^n becomes x^{-m} , or $\frac{1}{x^m}$.

It is again simplest to begin with the part of the graph for which x is positive. Then $\frac{1}{x^m}$ is also positive, so the graph lies in the first quadrant. Just as when n is positive, the point $(1, 1)$ lies on the graph. But there is an important difference when $x = 0$, since then $x^m = 0$ and $\frac{1}{x^m}$ is not defined. So there is no point on the graph for which $x = 0$.

To look at this more closely, take a value of x close to 0, say 0.01. Then for $n = -1$ the corresponding value of y is $0.01^{-1} = \frac{1}{0.01^1} = \frac{1}{0.01} = 100$; and for $n = -2$ it is $0.01^{-2} = \frac{1}{0.01^2} = \frac{1}{0.0001} = 10\ 000$. Even if you use a very small scale, the graph of x^n will disappear off the top of the page or screen as x is reduced towards zero.

What happens if x is large? For example, take $x = 100$. Then for $n = -1$ the corresponding value of y is $100^{-1} = \frac{1}{100^1} = \frac{1}{100} = 0.01$; and for $n = -2$ it is $100^{-2} = \frac{1}{100^2} = \frac{1}{10\ 000} = 0.0001$. So x^n becomes very small, and the graph comes very close to the x -axis.

Now consider the part of the graph for which x is negative. You found in (i) above that, for positive n , this depends on whether n is odd or even. The same is true when n is negative, and for the same reason. If n is even, x^n is an even function and its graph is symmetrical about the y -axis. If n is odd, x^n is an odd function and its graph is symmetrical about the origin.

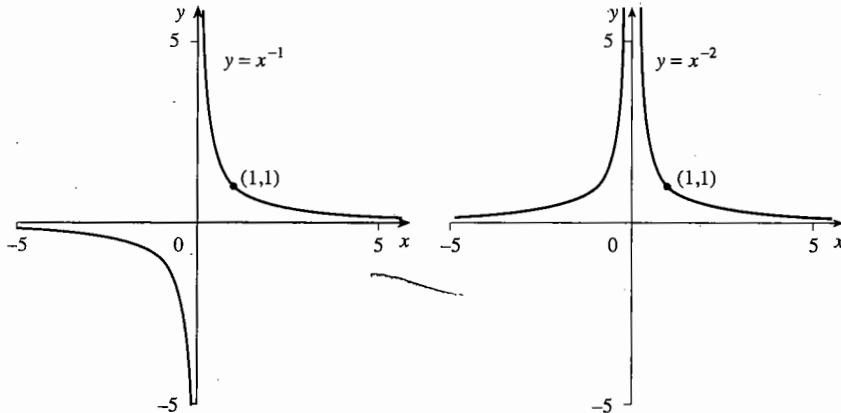


Fig. 3.8

All these properties are shown by the graphs of $y = x^n$ for $n = -1$ and $n = -2$ in Fig. 3.8.

(iii) Fractional powers

When n is a fraction, the function x^n may or may not be defined for negative values of x . For example, $x^{\frac{1}{2}}$ (the cube root of x) and $x^{-\frac{1}{2}}$ have values when $x < 0$, but $x^{\frac{1}{3}}$ (the square root of x) and $x^{-\frac{3}{4}}$ do not. Even when x^n is defined for negative x , some calculators and computers are not programmed to do the calculation. So it is simplest to restrict the discussion to values of $x \geq 0$.

It is easy to sketch the graphs of many of these functions by comparing them with graphs of integer powers of x . Here are two examples.

The graph of $y = x^{\frac{1}{2}}$ must lie between the graphs of $y = x^2$ and $y = x^3$.

The graph of $y = x^{-\frac{1}{2}}$ is not defined when $x = 0$; its graph resembles the graph of $y = x^{-1}$ (see Fig. 3.8), but lies below it when $x < 1$ and above it when $x > 1$.

If you are able to experiment for yourself with other fractional powers using a calculator or a computer, you will find that:

- It is still true that the graph of $y = x^n$ contains the point $(1,1)$.
- If n is positive the graph also contains the point $(0,0)$.
- If $n > 1$ the x -axis is a tangent to the graph; if $0 < n < 1$ the y -axis is a tangent. (To show this convincingly you may need to zoom in on a section of the graph close to the origin.)

Much the most important of these graphs is that of $y = x^{\frac{1}{2}}$, or $y = \sqrt{x}$. The clue to finding the shape of this graph is to note that if $y = x^{\frac{1}{2}}$, then $x = y^2$.

The graph can therefore be obtained from that of $y = x^2$ by swapping the x - and y -axes. This has the effect of tipping the graph on its side, so that instead of facing upwards it faces to the right.

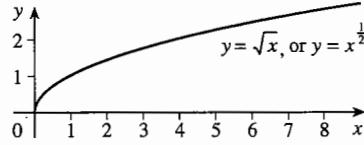


Fig. 3.9

But this is not quite the whole story. If $x = y^2$, then either $y = +\sqrt{x}$ or $y = -\sqrt{x}$. Since you want only the first of these possibilities, you must remove the part of the graph of $x = y^2$ below the x -axis, leaving only the part shown in Fig. 3.9 as the graph of $y = x^{\frac{1}{2}}$, or $y = \sqrt{x}$.

3.4 The modulus of a number

Suppose that you want to find the difference between the heights of two people. With numerical information, the answer is straightforward: if their heights are 90 cm and 100 cm, you would answer 10 cm; and if their heights were 100 cm and 90 cm, you would still answer 10 cm. But how would you answer the question if their heights were H cm and h cm? The answer is, it depends which is bigger: if $H > h$, you would answer $(H - h)$ cm; if $h > H$ you would answer $(h - H)$ cm; and if $h = H$ you would answer 0 cm, which is either $(H - h)$ cm or $(h - H)$ cm.

Questions like this, in which you want an answer that is always positive or zero, lead to the idea of the modulus.

The **modulus** of x , written $|x|$ and pronounced ‘mod x ’, is defined by

$$\begin{aligned} |x| &= x && \text{if } x \geq 0, \\ |x| &= -x && \text{if } x < 0. \end{aligned}$$

Using the modulus notation, you can now write the difference in heights as $|H - h|$ whether $H > h$, $h > H$ or $h = H$.

Another situation in which the modulus is useful is when you want to talk about numbers which are large numerically, but which are negative, such as -1000 or $-1000\,000$. You can describe these as ‘negative numbers with large modulus’.

For example, for large positive values of x , the value of $\frac{1}{x}$ approaches 0. The same is true for negative values of x with large modulus. So you can say that, when $|x|$ is large, $\left|\frac{1}{x}\right|$ is close to zero; or in a numerical example, when $|x| > 1000$,

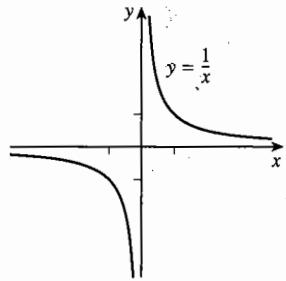
$$\left|\frac{1}{x}\right| < 0.001. \text{ (See Fig. 3.10.)}$$


Fig. 3.10

Some calculators have a key which converts any number in the display to its modulus. This key is often labelled [ABS], which stands for ‘absolute value’.

Exercise 3B

If you have access to a graphic calculator, use it to check your answers to Questions 4, 5 and 6.

1 Sketch the graphs of

(a) $y = x^5$, (b) $y = x^{6\frac{1}{2}}$, (c) $y = x^{10}$, (d) $y = x^{15}$.

2 Three graphs have equations (p) $y = x^{-2}$, (q) $y = x^{-3}$, (r) $y = x^{-4}$.

A line $x = k$ meets the three graphs at points P , Q and R , respectively. Give the order of the points P , Q and R on the line (from the bottom up) when k takes the following values.

(a) 2 (b) $\frac{1}{2}$ (c) $-\frac{1}{2}$ (d) -2

3 For what values of x are these inequalities satisfied? Sketch graphs illustrating your answers.

(a) $0 < x^{-3} < 0.001$ (b) $x^{-2} < 0.0004$ (c) $x^{-4} \geq 100$ (d) $8x^{-4} < 0.000\,05$

4 Sketch the graphs of these equations for $x > 0$.

(a) $y = x^{\frac{3}{2}}$	(b) $y = x^{\frac{1}{3}}$	(c) $y = -2x^{\frac{1}{2}}$
(d) $y = 4x^{-\frac{1}{4}}$	(e) $y = x^{-\frac{4}{3}}$	(f) $y = x^{\frac{2}{3}} - x^{-\frac{2}{3}}$

5 Sketch these graphs, including negative values of x where appropriate.

(a) $y = x^{\frac{2}{3}}$	(b) $y = x^{\frac{3}{4}}$	(c) $y = x^{\frac{4}{3}}$
(d) $y = x^{-\frac{1}{3}}$	(e) $y = x^{\frac{4}{5}}$	(f) $y = x^{-\frac{2}{3}}$

6 Draw sketch graphs with these equations.

(a) $y = x^2 + x^{-1}$	(b) $y = x + x^{-2}$	(c) $y = x^2 - x^{-1}$
(d) $y = x^{-2} - x^{-1}$	(e) $y = x^{-2} - x^{-3}$	(f) $y = x^{-2} - x^{-4}$

7 Of the following functions, one is even and two are odd. Determine which is which.

(a) $y = x^7$ (b) $y = x^4 + 3x^2$ (c) $y = x(x^2 - 1)$

8 State the values of the following.

- | | | |
|-------------|-------------------------------------|---------------|
| (a) $ -7 $ | (b) $\left -\frac{1}{200} \right $ | (c) $ 9-4 $ |
| (d) $ 4-9 $ | (e) $ \pi-3 $ | (f) $ \pi-4 $ |

9 Find the values of $|x-x^2|$ when x takes the values

- | | | |
|---------|---------------------|--------|
| (a) 2, | (b) $\frac{1}{2}$, | (c) 1, |
| (d) -1, | (e) 0. | |

10 You are given that $y = \frac{1}{x^2}$. What can you say about y if

- | | |
|-------------------|--------------------|
| (a) $ x > 100$, | (b) $ x < 0.01$? |
|-------------------|--------------------|

11 You are given that $y = \frac{1}{x^3}$.

- | |
|--|
| (a) What can you say about y if $ x < 1000$? |
| (b) What can you say about x if $ y > 1000$? |

12 The number, N , of people at a football match was reported as '37 000 to the nearest thousand'. Write this statement as an inequality using the modulus sign.

13 The mathematics marks, m and n , of two twins never differ by more than 5. Write this statement as an inequality using the modulus sign.

14 A line has length x cm. You are given that $|x-5.23| < 0.005$. How would you explain this in words?

3.5 Graphs of the form $y = ax^2 + bx + c$

In Chapter 1, you found out how to sketch graphs of straight lines, and what the constants m and c mean in the equation $y = mx + c$.

Exercise 3C gives you experience of plotting the graphs of functions with equations of the form $y = ax^2 + bx + c$.

If you have access to a graphic calculator, use it and do all the questions; if not, work in a group and share out the tasks.

A summary of the main points appears after the exercise.

Exercise 3C

1 Draw, on the same set of axes, the graphs of

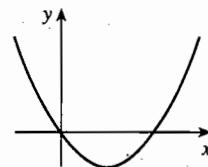
- | | | |
|--------------------------|--------------------------|----------------------|
| (a) $y = x^2 - 2x + 5$, | (b) $y = x^2 - 2x + 1$, | (c) $y = x^2 - 2x$. |
|--------------------------|--------------------------|----------------------|

2 Draw, on the same set of axes, the graphs of

- | | | |
|-------------------------|-------------------------|-------------------------|
| (a) $y = x^2 + x - 4$, | (b) $y = x^2 + x - 1$, | (c) $y = x^2 + x + 2$. |
|-------------------------|-------------------------|-------------------------|

- 3 The diagram shows the graph of $y = ax^2 - bx$. On a copy of the diagram, sketch the graphs of

- (a) $y = ax^2 - bx + 4$,
 (b) $y = ax^2 - bx - 6$.



- 4 What is the effect on the graph of $y = ax^2 + bx + c$ of changing the value of c ?

- 5 Draw the graphs of

- (a) $y = x^2 - 4x + 1$, (b) $y = x^2 - 2x + 1$,
 (c) $y = x^2 + 1$, (d) $y = x^2 + 2x + 1$.

- 6 Draw the graph of $y = 2x^2 + bx + 4$ for different values of b . How does changing b affect the curve $y = ax^2 + bx + c$?

- 7 Draw the graphs of

- (a) $y = x^2 + 1$, (b) $y = 3x^2 + 1$,
 (c) $y = -3x^2 + 1$, (d) $y = -x^2 + 1$.

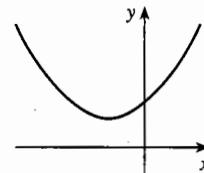
- 8 Draw the graphs of

- (a) $y = -4x^2 + 3x + 1$, (b) $y = -x^2 + 3x + 1$,
 (c) $y = x^2 + 3x + 1$, (d) $y = 4x^2 + 3x + 1$.

- 10 How does changing a affect the shape of the graph of $y = ax^2 + bx + c$?

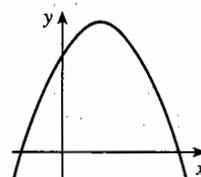
- 11 Which of the following could be the equation of the curve shown in the diagram?

- (a) $y = x^2 - 2x + 5$
 (b) $y = -x^2 - 2x + 5$
 (c) $y = x^2 + 2x + 5$
 (d) $y = -x^2 + 2x + 5$



- 12 Which of the following could be the equation of the curve shown in the diagram?

- (a) $y = -x^2 + 3x + 4$
 (b) $y = x^2 - 3x + 4$
 (c) $y = x^2 + 3x + 4$
 (d) $y = -x^2 - 3x + 4$



3.6 The shapes of graphs of the form $y = ax^2 + bx + c$

In Exercise 3C, you should have discovered a number of results, which are summarised in the box below.

All the graphs have the same general shape, which is called a **parabola**. These parabolas have a vertical **axis of symmetry**. The point where a parabola meets its axis of symmetry is called the **vertex**.

Changing c moves the graph up and down in the y -direction.

Changing b moves the axis of symmetry of the graph in the x -direction. If a and b have the same sign the axis of symmetry is to the left of the y -axis; if a and b have opposite signs the axis of symmetry is to the right of the y -axis.

If a is positive the vertex is at the lowest point of the graph; if a is negative the vertex is at the highest point. The larger the size of $|a|$ the more the graph is elongated, that is, lengthened in the y -direction.

3.7 The point of intersection of two graphs

The principle for finding the point of intersection of two curves is the same as that for finding the point of intersection of two graphs which are straight lines.

Suppose that you have two graphs, with equations $y = f(x)$ and $y = g(x)$. You want the point (x, y) which lies on both graphs, so the coordinates (x, y) satisfy both equations. Therefore x must satisfy the equation $f(x) = g(x)$.

Example 3.7.1

Find the points of intersection of the line $y = 2$ with the graph $y = x^2 - 3x + 4$ (see Fig. 3.11).

Solving these equations simultaneously gives
 $x^2 - 3x + 4 = 2$, which reduces to $x^2 - 3x + 2 = 0$.

Factorising gives $(x - 1)(x - 2) = 0$, so

$$x = 1 \text{ or } x = 2.$$

Substituting these values in either equation ($y = 2$ is obviously easier!) to find y , the points of intersection are $(1, 2)$ and $(2, 2)$.

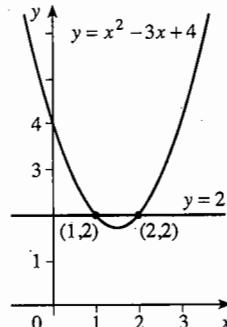


Fig. 3.11

Example 3.7.2

Find the point of intersection of the line $y = 2x - 1$ with the graph $y = x^2$ (see Fig. 3.12).

Solving these equations gives $2x - 1 = x^2$, which is $x^2 - 2x + 1 = 0$. This factorises as $(x - 1)^2 = 0$, giving $x = 1$.

Substituting these values in either equation to find y gives the point of intersection as $(1, 1)$.

The reason that there is only one point of intersection is that this line is a tangent to the graph. If you have access to a graphic calculator, draw the graph to check this statement.

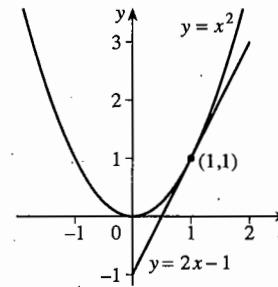


Fig. 3.12

Example 3.7.3

Find the points of intersection of the graphs $y = x^2 - 2x - 6$ and $y = 12 + x - 2x^2$.

Solving these equations simultaneously gives $x^2 - 2x - 6 = 12 + x - 2x^2$, which is $3x^2 - 3x - 18 = 0$. Dividing by 3 gives $x^2 - x - 6 = 0$, which factorises as $(x + 2)(x - 3) = 0$, giving $x = -2$ or $x = 3$.

Substituting these values in either equation to find y gives the points of intersection as $(-2, 2)$ and $(3, -3)$.

Exercise 3D

1 Find the point or points of intersection for the following lines and curves.

- | | |
|------------------------------------|---|
| (a) $x = 3$ and $y = x^2 + 4x - 7$ | (b) $y = 3$ and $y = x^2 - 5x + 7$ |
| (c) $y = 8$ and $y = x^2 + 2x$ | (d) $y + 3 = 0$ and $y = 2x^2 + 5x - 6$ |

2 Find the points of intersection for the following lines and curves.

- | | |
|---|--|
| (a) $y = x + 1$ and $y = x^2 - 3x + 4$ | (b) $y = 2x + 3$ and $y = x^2 + 3x - 9$ |
| (c) $y = 3x + 11$ and $y = 2x^2 + 2x + 5$ | (d) $y = 4x + 1$ and $y = 9 + 4x - 2x^2$ |
| (e) $3x + y - 1 = 0$ and $y = 6 + 10x - 6x^2$ | |

3 In both the following, show that the line and curve meet only once and find the point of intersection.

- | | |
|---|--|
| (a) $y = 2x + 2$ and $y = x^2 - 2x + 6$ | (b) $y = -2x - 7$ and $y = x^2 + 4x + 2$ |
|---|--|

4 Find the points of intersection between the curve $y = x^2 - x$ and the line

- | | |
|---------------|-------------------|
| (a) $y = x$, | (b) $y = x - 1$. |
|---------------|-------------------|

If you have access to a graphic calculator, use it to see how the curve and lines are related.

5 Find the points of intersection between the curve $y = x^2 + 5x + 18$ and the lines

- | | |
|---------------------|---------------------|
| (a) $y = -3x + 2$, | (b) $y = -3x + 6$. |
|---------------------|---------------------|

If you have access to a graphic calculator, use it to see how the curve and lines are related.

6 Find the points of intersection between the line $y = x + 5$ and the curves

(a) $y = 2x^2 - 3x - 1$, (b) $y = 2x^2 - 3x + 7$.

If you have access to a graphic calculator, use it to see how the line and curves are related.

7 Find the points of intersection of the following curves.

(a) $y = x^2 + 5x + 1$ and $y = x^2 + 3x + 11$

(b) $y = x^2 - 3x - 7$ and $y = x^2 + x + 1$

(c) $y = 7x^2 + 4x + 1$ and $y = 7x^2 - 4x + 1$

8 Find the points of intersection of the following curves.

(a) $y = \frac{1}{2}x^2$ and $y = 1 - \frac{1}{2}x^2$ (b) $y = 2x^2 + 3x + 4$ and $y = x^2 + 6x + 2$

(c) $y = x^2 + 7x + 13$ and $y = 1 - 3x - x^2$ (d) $y = 6x^2 + 2x - 9$ and $y = x^2 + 7x + 1$

(e) $y = (x-2)(6x+5)$ and $y = (x-5)^2 + 1$ (f) $y = 2x(x-3)$ and $y = x(x+2)$

9 Find the point or points of intersection of these pairs of graphs. Illustrate your answers with sketch graphs.

(a) $y = 8x^2$, $y = 8x^{-1}$ (b) $y = x^{-1}$, $y = 3x^{-2}$ (c) $y = x$, $y = 4x^{-3}$
 (d) $y = 8x^{-2}$, $y = 2x^{-4}$ (e) $y = 9x^{-3}$, $y = x^{-5}$ (f) $y = \frac{1}{4}x^4$, $y = 16x^{-2}$

3.8 Using factors to sketch graphs

The graphs of some functions of the form $f(x) = ax^2 + bx + c$ which factorise can also be drawn in another way. For example, take the functions

$$f(x) = x^2 - 6x + 5 = (x-1)(x-5),$$

$$g(x) = 12x - 4x^2 = -4x(x-3).$$

In the first case, $f(1) = 0$ and $f(5) = 0$, so that the points $(1,0)$ and $(5,0)$ lie on the graph of $f(x)$. This is shown in Fig. 3.13.

Similarly $g(0) = g(3) = 0$, so that $(0,0)$ and $(3,0)$ lie on the graph of $g(x)$. This is shown in Fig. 3.14.

You can draw the graph of any function of this type which can be factorised as

$$a(x-r)(x-s),$$

by first noting that it cuts the x -axis at the points $(r,0)$ and $(s,0)$. The sign of the constant a tells you whether it ‘bends upwards’ (like $y = x^2$) or ‘bends downwards’.

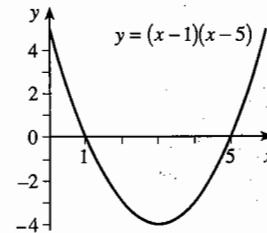


Fig. 3.13

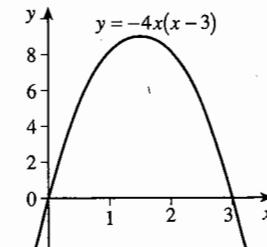


Fig. 3.14

In Figs. 3.13 and 3.14 different scales have been used on the two axes. If equal scales had been used the elongation in both figures would have been more obvious.

Example 3.8.1

Sketch the graph of $f(x) = 3x^2 - 2x - 1$.

You can factorise the expression as $f(x) = (3x + 1)(x - 1)$, but to apply the factor method you need to write it as

$$f(x) = 3\left(x + \frac{1}{3}\right)(x - 1).$$

So the graph passes through $(-\frac{1}{3}, 0)$ and $(1, 0)$. The constant 3 tells you that the graph faces upwards and is elongated.

This is enough information to give a good idea of the shape of the graph, from which you can draw a sketch like Fig. 3.15. It is also worth noting that $f(0) = -1$, so that the graph cuts the y -axis at the point $(0, -1)$.

Note that the sketch does not have marks against the axes, except to say where the graph cuts them.

You can extend the factor method to drawing graphs of functions with more than two factors. For example,

$$f(x) = a(x - r)(x - s)(x - t)$$

defines a function whose equation, when multiplied out, starts with $f(x) = ax^3 - \dots$.

The graph passes through the points $(r, 0)$, $(s, 0)$ and $(t, 0)$. The constant a tells you whether, for large values of x , the graph lies in the first or the fourth quadrant.

This is shown in Figs. 3.16 and 3.17, which show the graphs of $y = 2x(x - 1)(x - 4)$ and $y = -(x + 2)(x - 1)^2$.

Notice that, in Fig. 3.17, the factor $(x - 1)$ is squared, so that there are only two points of the graph on the x -axis.

At $(1, 0)$ the x -axis is a tangent to the graph.

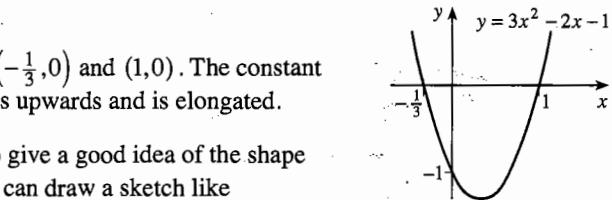


Fig. 3.15

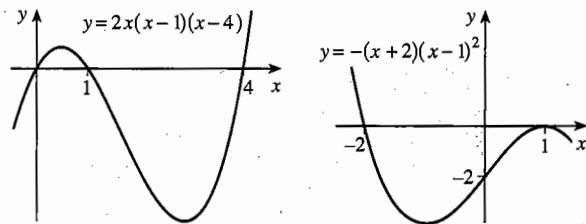


Fig. 3.16

Fig. 3.17

3.9 Predicting functions from their graphs

You can also use the factor form to predict the equation of a function of the type $f(x) = ax^2 + bx + c$, if you know the points where its graph crosses the x -axis and the coordinates of one other point on the graph.

Example 3.9.1

Find the equation of the graph of the type $y = ax^2 + bx + c$ which crosses the x -axis at the points $(1,0)$ and $(4,0)$ and also passes through the point $(3,-4)$.

Since the curve cuts the axes at $(1,0)$ and $(4,0)$, as in Fig. 3.18, the equation has the form

$$y = a(x - 1)(x - 4).$$

Since the point $(3,-4)$ lies on this curve,
 $-4 = a(3-1)(3-4)$, giving $-4 = -2a$, so $a = 2$.

The equation of the curve is therefore

$$y = 2(x - 1)(x - 4), \quad \text{or} \quad y = 2x^2 - 10x + 8.$$

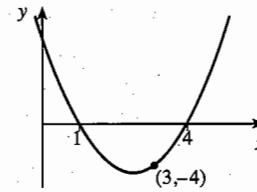


Fig. 3.18

Exercise 3E

1 Sketch the following graphs.

- | | | |
|--------------------------|--------------------------|---------------------------|
| (a) $y = (x - 2)(x - 4)$ | (b) $y = (x + 3)(x - 1)$ | (c) $y = x(x - 2)$ |
| (d) $y = (x + 5)(x + 1)$ | (e) $y = x(x + 3)$ | (f) $y = 2(x + 1)(x - 1)$ |

2 Sketch the following graphs.

- | | | |
|--|----------------------------|---|
| (a) $y = 3(x + 1)(x - 5)$ | (b) $y = -2(x - 1)(x - 3)$ | (c) $y = -(x + 3)(x + 5)$ |
| (d) $y = 2\left(x + \frac{1}{2}\right)(x - 3)$ | (e) $y = -3(x - 4)^2$ | (f) $y = -5(x - 1)\left(x + \frac{4}{5}\right)$ |

3 By first factorising the function, sketch the following graphs.

- | | | |
|-------------------------|----------------------------|---------------------------|
| (a) $y = x^2 - 2x - 8$ | (b) $y = x^2 - 2x$ | (c) $y = x^2 + 6x + 9$ |
| (d) $y = 2x^2 - 7x + 3$ | (e) $y = 4x^2 - 1$ | (f) $y = -(x^2 - x - 12)$ |
| (g) $y = -x^2 - 4x - 4$ | (h) $y = -(x^2 - 7x + 12)$ | (i) $y = 11x - 4x^2 - 6$ |

4 Find the equation, in the form $y = x^2 + bx + c$, of the parabola which

- (a) crosses the x -axis at the points $(2,0)$ and $(5,0)$,
- (b) crosses the x -axis at the points $(-7,0)$ and $(-10,0)$,
- (c) passes through the points $(-5,0)$ and $(3,0)$,
- (d) passes through the points $(-3,0)$ and $(1,-16)$.

5 Sketch the following graphs.

- | | |
|----------------------------------|------------------------------|
| (a) $y = (x + 3)(x - 2)(x - 3)$ | (b) $y = x(x - 4)(x - 6)$ |
| (c) $y = x^2(x - 4)$ | (d) $y = x(x - 4)^2$ |
| (e) $y = -(x + 6)(x + 4)(x + 2)$ | (f) $y = -3(x + 1)(x - 3)^2$ |

6 Find the equation, in the form $y = ax^2 + bx + c$, of the parabola which

- (a) crosses the x -axis at $(1,0)$ and $(5,0)$ and crosses the y -axis at $(0,15)$,
- (b) crosses the x -axis at $(-2,0)$ and $(7,0)$ and crosses the y -axis at $(0,-56)$,
- (c) passes through the points $(-6,0)$, $(-2,0)$ and $(0,-6)$,
- (d) crosses the x -axis at $(-3,0)$ and $(2,0)$ and also passes through $(1,16)$,
- (e) passes through the points $(-10,0)$, $(7,0)$ and $(8,90)$.

7 Sketch the following graphs.

(a) $y = x^2 - 4x - 5$	(b) $y = 4x^2 - 4x + 1$	(c) $y = -x^2 - 3x + 18$
(d) $y = 2x^2 - 9x + 10$	(e) $y = -(x^2 - 4x + 9)$	(f) $y = 3x^2 + 9x$

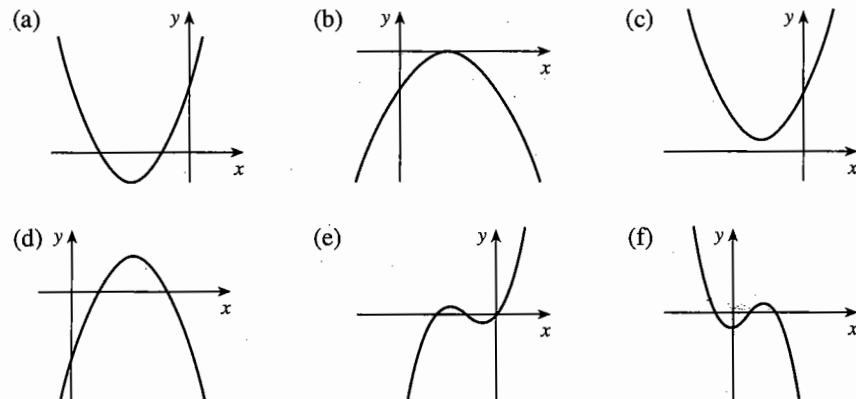
8 Here are the equations of nine parabolas.

A $y = (x - 3)(x - 8)$	B $y = 14 + 5x - x^2$	C $y = 6x^2 - x - 70$
D $y = x(3 - x)$	E $y = (x + 2)(x - 7)$	F $y = -3(x + 3)(x + 7)$
G $y = x^2 + 2x + 1$	H $y = x^2 + 8x + 12$	I $y = x^2 - 25$

Answer the following questions without drawing the graphs of these parabolas.

- Which of the parabolas cross the y -axis at a positive value of y ?
- For which of the parabolas is the vertex at the highest point of the graph?
- For which of the parabolas is the vertex to the left of the y -axis?
- Which of the parabolas pass through the origin?
- Which of the parabolas do not cross the x -axis at two separate points?
- Which of the parabolas have the y -axis as their axis of symmetry?
- Which two of the parabolas have the same axis of symmetry?
- Which of the parabolas have the vertex in the fourth quadrant?

9 Suggest a possible equation for each of the graphs shown below.



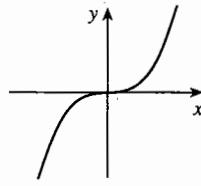
Miscellaneous exercise 3

1 The function f is defined by $f(x) = 7x - 4$.

- Find the values of $f(7)$, $f(\frac{1}{2})$ and $f(-5)$.
- Find the value of x such that $f(x) = 10$.
- Find the value of x such that $f(x) = x$.
- Find the value of x such that $f(x) = f(37)$.

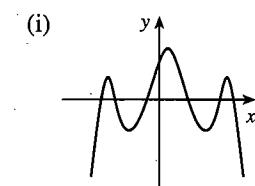
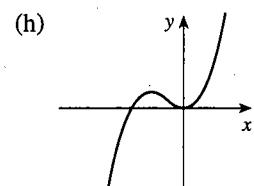
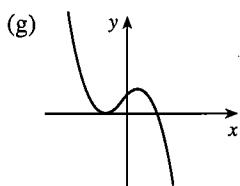
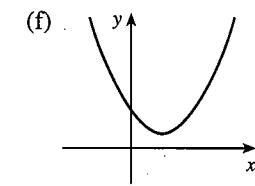
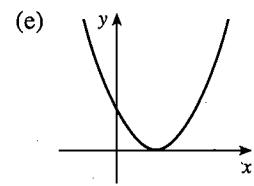
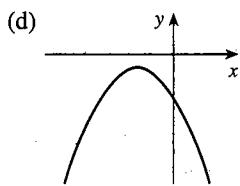
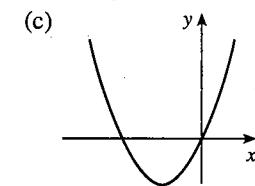
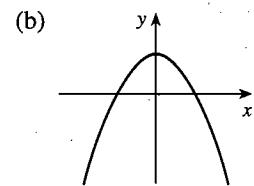
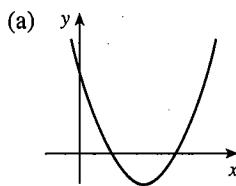
2 The function f is defined by $f(x) = x^2 - 3x + 5$. Find the two values of x for which $f(x) = f(4)$.

- 3 The diagram shows the graph of $y = x^n$, where n is an integer. Given that the curve passes between the points $(2, 200)$ and $(2, 2000)$, determine the value of n .

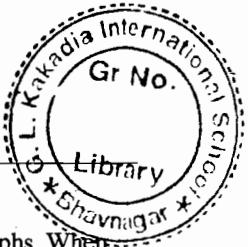


- 4 Find the points of intersection of the curves $y = x^2 - 7x + 5$ and $y = 1 + 2x - x^2$.
- 5 Find the points of intersection of the line $y = 2x + 3$ and the curve $y = 2x^2 + 3x - 7$.
- 6 Find the coordinates of the point at which the line $3x + y - 2 = 0$ meets the curve $y = (4x - 3)(x - 2)$.
- 7 Find the coordinates of any points of intersection of the curves $y = (x - 2)(x - 4)$ and $y = x(2 - x)$. Sketch the two curves to show the relationship between them.
- 8 Given that k is a positive constant, sketch the graphs of
- (a) $y = (x + k)(x - 2k)$, (b) $y = (x + 4k)(x + 2k)$,
 (c) $y = x(x - k)(x - 5k)$, (d) $y = (x + k)(x - 2k)^2$.
- 9 The function f is defined by $f(x) = ax^2 + bx + c$. Given that $f(0) = 6$, $f(-1) = 15$ and $f(1) = 1$, find the values of a , b and c .
- 10 Find the point where the line $y = 3 - 4x$ meets the curve $y = 4(4x^2 + 5x + 3)$.
- * 11 Sketch the graphs of
- (a) $y = (x + 4)(x + 2) + (x + 4)(x - 5)$, (b) $y = (x + 4)(x + 2) + (x + 4)(5 - x)$.
- 12 A function f is defined by $f(x) = ax + b$. Given that $f(-2) = 27$ and $f(1) = 15$, find the value of x such that $f(x) = -5$.
- 13 A curve with equation $y = ax^2 + bx + c$ crosses the x -axis at $(-4, 0)$ and $(9, 0)$ and also passes through the point $(1, 120)$. Where does the curve cross the y -axis?
- 14 The curve $y = ax^2 + bx + c$ passes through the points $(-1, 22)$, $(1, 8)$, $(3, 10)$, $(-2, p)$ and $(q, 17)$. Find the value of p and the possible values of q .
- 15 Show that the curves $y = 2x^2 + 5x$, $y = x^2 + 4x + 12$ and $y = 3x^2 + 4x - 6$ have one point in common and find its coordinates.
- 16 Given that the curves $y = x^2 - 3x + c$ and $y = k - x - x^2$ meet at the point $(-2, 12)$, find the values of c and k . Hence find the other point where the two curves meet.
- 17 Find the value of the constant p if the three curves $y = x^2 + 3x + 14$, $y = x^2 + 2x + 11$ and $y = px^2 + px + p$ have one point in common.
- 18 The straight line $y = x - 1$ meets the curve $y = x^2 - 5x - 8$ at the points A and B . The curve $y = p + qx - 2x^2$ also passes through the points A and B . Find the values of p and q .
- 19 The line $y = 10x - 9$ meets the curve $y = x^2$. Find the coordinates of the points of intersection.

20 Suggest a possible equation for each of the graphs shown below.



- 21 Find, in surd form, the points of intersection of the curves $y = x^2 - 5x - 3$ and $y = 3 - 5x - x^2$.
- 22 The line $y = 6x + 1$ meets the curve $y = x^2 + 2x + 3$ at two points. Show that the coordinates of one of the points are $(2 - \sqrt{2}, 13 - 6\sqrt{2})$, and find the coordinates of the other point.
- 23 Show that the curves $y = 2x^2 - 7x + 14$ and $y = 2 + 5x - x^2$ meet at only one point. Without further calculation or sketching, deduce the number of points of intersection of
- (a) $y = 2x^2 - 7x + 12$ and $y = 2 + 5x - x^2$, (b) $y = 2x^2 - 7x + 14$ and $y = 1 + 5x - x^2$,
- (c) $y = 2x^2 - 7x + 34$ and $y = 22 + 5x - x^2$.
- 24 What can you say about $\frac{|x|}{x}$ if (a) $x > 0$, (b) $x < 0$?



4 Quadratics

This chapter is about quadratic expressions of the form $ax^2 + bx + c$ and their graphs. When you have completed it, you should

- know how to complete the square in a quadratic expression
- know how to locate the vertex and the axis of symmetry of the quadratic graph $y = ax^2 + bx + c$
- be able to solve quadratic equations
- know that the discriminant of the quadratic expression $ax^2 + bx + c$ is the value of $b^2 - 4ac$, and know how to use it
- be able to solve a pair of simultaneous equations involving a quadratic equation and a linear equation
- be able to recognise and solve equations which can be reduced to quadratic equations by a substitution.

4.1 Quadratic expressions

You know that the equation $y = bx + c$ has a graph which is a straight line. The expression $y = bx + c$ is called a linear equation. You know from Chapter 3 that if you add on a term ax^2 , giving $y = ax^2 + bx + c$, the graph is a parabola. The expression $ax^2 + bx + c$, where a , b and c are constants, is called a **quadratic**. Thus x^2 , $x^2 - 6x + 8$, $2x^2 - 3x + 4$ and $-3x^2 - 5$ are all examples of quadratics.

You can write any quadratic as $ax^2 + bx + c$, where a , b and c are constants. The values of b and c can be any numbers you please, including 0, but a cannot be 0 (the expression would not then be a quadratic). The numbers a , b and c are called **coefficients**: a is the coefficient of x^2 , b is the coefficient of x and c is often called the constant term.

The coefficients of x^2 and x in $2x^2 - x + 4$ are 2 and -1 , and the constant term is 4.

4.2 Completed square form

You can write a quadratic expression such as $x^2 - 6x + 8$ in a number of ways. These include the factor form $(x - 4)(x - 2)$, useful for finding where the parabola $y = x^2 - 6x + 8$, shown in Fig. 4.1, cuts the x -axis; and the form $(x - 3)^2 - 1$, useful for locating the vertex of the parabola and also for finding the range of the function $f(x) = x^2 - 6x + 8$, as shown in Example 3.2.3.

Note that you cannot always write a quadratic expression in factor form. For instance, try $x^2 + 1$ or $x^2 + 2x + 3$.

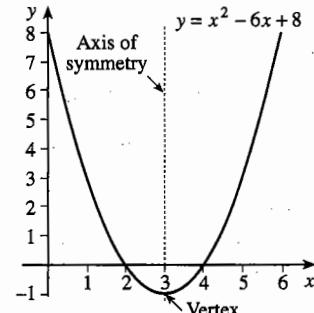


Fig. 4.1

If you write the equation of the graph $y = x^2 - 6x + 8$ in the form $y = (x - 3)^2 - 1$, you can locate the axis of symmetry and the vertex quite easily. Since $(x - 3)^2$ is a perfect square its value is always greater than or equal to 0, and is 0 only when $x = 3$. That is, $(x - 3)^2 \geq 0$, and since $y = (x - 3)^2 - 1$, it follows that $y \geq -1$. Since $(x - 3)^2 = 0$ when $x = 3$, the vertex is at the point $(3, -1)$. The axis of symmetry is the line $x = 3$.

The form $(x - 3)^2 - 1$ is called the **completed square form**. Here are some more examples of its use.

Example 4.2.1

Locate the vertex and the axis of symmetry of the quadratic graph $y = 3 - 2(x + 2)^2$.

Since $2(x + 2)^2 \geq 0$, and $2(x + 2)^2 = 3 - y$, it follows that $3 - y \geq 0$, so $y \leq 3$. As $(x + 2)^2 = 0$ when $x = -2$, the vertex of the graph is the point with coordinates $(-2, 3)$, the greatest value of y is 3 and the axis of symmetry is $x = -2$.

Example 4.2.2

Solve the equation $3(x - 2)^2 - 2 = 0$.

As $3(x - 2)^2 - 2 = 0$, $3(x - 2)^2 = 2$ and $(x - 2)^2 = \frac{2}{3}$.

Therefore $(x - 2) = \pm\sqrt{\frac{2}{3}}$, so $x = 2 \pm \sqrt{\frac{2}{3}}$.

4.3 Completing the square

When you try to write the quadratic expression $x^2 + bx + c$ in completed square form, the key point is to note that when you square $x + \frac{1}{2}b$ you get

$$\left(x + \frac{1}{2}b\right)^2 = x^2 + bx + \frac{1}{4}b^2, \text{ so } x^2 + bx = \left(x + \frac{1}{2}b\right)^2 - \frac{1}{4}b^2.$$

Now add c to both sides:

$$x^2 + bx + c = \left(x^2 + bx\right) + c = \left\{ \left(x + \frac{1}{2}b\right)^2 - \frac{1}{4}b^2 \right\} + c.$$

Example 4.3.1

Write $x^2 + 10x + 32$ in completed square form.

$$x^2 + 10x + 32 = (x^2 + 10x) + 32 = \{(x + 5)^2 - 25\} + 32 = (x + 5)^2 + 7.$$

Don't try to memorise the form $x^2 + bx + c = (x + \frac{1}{2}b)^2 - \frac{1}{4}b^2 + c$. Learn that you halve the coefficient of x , and write $x^2 + bx = (x + \frac{1}{2}b)^2 - \frac{1}{4}b^2$. Then add c to both sides.

If you need to write $ax^2 + bx + c$ in completed square form, but the coefficient a of x^2 is not 1, you can rewrite $ax^2 + bx + c$ by taking out the factor a from the first two terms:

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x\right) + c.$$

Then complete the square of the quadratic expression $x^2 + \frac{b}{a}x$ inside the bracket.

Example 4.3.2

Express $2x^2 + 10x + 7$ in completed square form.

Start by taking out the factor 2 from the terms which involve x :

$$2x^2 + 10x + 7 = 2(x^2 + 5x) + 7.$$

Dealing with the term inside the bracket,

$$x^2 + 5x = \left(x + \frac{5}{2}\right)^2 - \frac{25}{4},$$

$$\begin{aligned} \text{so } 2x^2 + 10x + 7 &= 2\left(x^2 + 5x\right) + 7 = 2\left(\left(x + \frac{5}{2}\right)^2 - \frac{25}{4}\right) + 7 \\ &= 2\left(x + \frac{5}{2}\right)^2 - \frac{25}{2} + 7 = 2\left(x + \frac{5}{2}\right)^2 - \frac{11}{2}. \end{aligned}$$

It's worth checking your result mentally at this stage.

If the coefficient of x^2 is negative, the technique is similar, as shown in Example 4.3.3.

Example 4.3.3

Express $3 - 4x - 2x^2$ in completed square form.

Start by taking out the factor -2 from the terms which involve x :

$$3 - 4x - 2x^2 = 3 - 2(x^2 + 2x).$$

Dealing with the term inside the bracket, $x^2 + 2x = (x + 1)^2 - 1$,

$$\begin{aligned} \text{so } 3 - 4x - 2x^2 &= 3 - 2(x^2 + 2x) = 3 - 2\{(x + 1)^2 - 1\} \\ &= 3 - 2(x + 1)^2 + 2 = 5 - 2(x + 1)^2. \end{aligned}$$

Example 4.3.4

Express $12x^2 - 7x - 12$ in completed square form, and use your result to find the factors of $12x^2 - 7x - 12$.

$$\begin{aligned} 12x^2 - 7x - 12 &= 12\left(x^2 - \frac{7}{12}x\right) - 12 = 12\left\{\left(x - \frac{7}{24}\right)^2 - \frac{49}{576}\right\} - 12 \\ &= 12\left\{\left(x - \frac{7}{24}\right)^2 - \frac{625}{576}\right\} = 12\left\{\left(x - \frac{7}{24}\right)^2 - \left(\frac{25}{24}\right)^2\right\}. \end{aligned}$$

You can now use the formula $a^2 - b^2 = (a - b)(a + b)$ with a as $x - \frac{7}{24}$ and b as $\frac{25}{24}$ to factorise the expression inside the brackets as the difference of two squares:

$$\begin{aligned} 12\left\{\left(x - \frac{7}{24}\right)^2 - \left(\frac{25}{24}\right)^2\right\} &= 12\left(x - \frac{7}{24} - \frac{25}{24}\right)\left(x - \frac{7}{24} + \frac{25}{24}\right) \\ &= 12\left(x - \frac{4}{3}\right)\left(x + \frac{3}{4}\right) = 3\left(x - \frac{4}{3}\right) \times 4\left(x + \frac{3}{4}\right) \\ &= (3x - 4)(4x + 3). \end{aligned}$$

Example 4.3.5

Express $x^2 - 8x + 12$ in completed square form. Use your result to find the range of the function $f(x) = x^2 - 8x + 12$, which is defined for all real values of x .

$$x^2 - 8x + 12 = (x - 4)^2 - 4.$$

As $(x - 4)^2 \geq 0$ for all values of x ,

$$x^2 - 8x + 12 = (x - 4)^2 - 4 \geq -4, \text{ so } f(x) \geq -4.$$

Writing $f(x)$ as y , the range is $y \geq -4$.

Exercise 4A

- 1** Find (i) the vertex and (ii) the equation of the line of symmetry of each of the following quadratic graphs.

(a) $y = (x - 2)^2 + 3$	(b) $y = (x - 5)^2 - 4$	(c) $y = (x + 3)^2 - 7$
(d) $y = (2x - 3)^2 + 1$	(e) $y = (5x + 3)^2 + 2$	(f) $y = (3x + 7)^2 - 4$
(g) $y = (x - 3)^2 + c$	(h) $y = (x - p)^2 + q$	(i) $y = (ax + b)^2 + c$

- 2** Find (i) the least (or, if appropriate, the greatest) value of each of the following quadratic expressions and (ii) the value of x for which this occurs.

(a) $(x + 2)^2 - 1$	(b) $(x - 1)^2 + 2$	(c) $5 - (x + 3)^2$
(d) $(2x + 1)^2 - 7$	(e) $3 - 2(x - 4)^2$	(f) $(x + p)^2 + q$
(g) $(x - p)^2 - q$	(h) $r - (x - t)^2$	(i) $c - (ax + b)^2$

- 3** Solve the following quadratic equations. Leave surds in your answer.

(a) $(x - 3)^2 - 3 = 0$	(b) $(x + 2)^2 - 4 = 0$	(c) $2(x + 3)^2 = 5$
(d) $(3x - 7)^2 = 8$	(e) $(x + p)^2 - q = 0$	(f) $a(x + b)^2 - c = 0$

- 4** Express the following in completed square form.

(a) $x^2 + 2x + 2$	(b) $x^2 - 8x - 3$	(c) $x^2 + 3x - 7$
(d) $5 - 6x + x^2$	(e) $x^2 + 14x + 49$	(f) $2x^2 + 12x - 5$
(g) $3x^2 - 12x + 3$	(h) $7 - 8x - 4x^2$	(i) $2x^2 + 5x - 3$

- 5** Use the completed square form to factorise the following expressions.

(a) $x^2 - 2x - 35$	(b) $x^2 - 14x - 176$	(c) $x^2 + 6x - 432$
(d) $6x^2 - 5x - 6$	(e) $14 + 45x - 14x^2$	(f) $12x^2 + x - 6$

- 6** Use the completed square form to find as appropriate the least or greatest value of each of the following expressions, and the value of x for which this occurs.

(a) $x^2 - 4x + 7$	(b) $x^2 - 3x + 5$	(c) $4 + 6x - x^2$
(d) $2x^2 - 5x + 2$	(e) $3x^2 + 2x - 4$	(f) $3 - 7x - 3x^2$

- 7 Each of the following functions is defined for all real values of x . By completing the square write $f(x)$ as $(x - p)^2 + q$, and hence find their ranges.
- (a) $f(x) = x^2 - 6x + 10$ (b) $f(x) = x^2 + 7x + 1$ (c) $f(x) = x^2 - 3x + 4$
- 8 By completing the square find (i) the vertex, and (ii) the equation of the line of symmetry, of each of the following parabolas.
- (a) $y = x^2 - 4x + 6$ (b) $y = x^2 + 6x - 2$ (c) $y = 7 - 10x - x^2$
 (d) $y = x^2 + 3x + 1$ (e) $y = 2x^2 - 7x + 2$ (f) $y = 3x^2 - 12x + 5$
- 9 The domain of each of the following functions is the set of all positive real numbers. Find the range of each function.
- (a) $f(x) = (x + 2)(x + 1)$ (b) $f(x) = (x - 1)(x - 2)$ (c) $f(x) = (2x - 1)(x - 2)$

4.4 Solving quadratic equations

You will be familiar with solving quadratic equations of the form $x^2 - 6x + 8 = 0$ by factorising $x^2 - 6x + 8$ into the form $(x - 2)(x - 4)$, and then using the argument:

if $(x - 2)(x - 4) = 0$
 then either $x - 2 = 0$ or $x - 4 = 0$
 so $x = 2$ or $x = 4$.

The **solution** of the equation $x^2 - 6x + 8 = 0$ is $x = 2$ or $x = 4$. The numbers 2 and 4 are the **roots** of the equation.

If the quadratic expression has factors which you can find easily, then this is certainly the quickest way to solve the equation. However, the expression may not have factors, or they may be hard to find: try finding the factors of $30x^2 - 11x - 30$.

If you cannot factorise a quadratic expression easily to solve an equation, then use the quadratic formula:

The solution of $ax^2 + bx + c = 0$, where $a \neq 0$, is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

It is useful to know how this formula is derived by expressing $ax^2 + bx + c$ in completed square form. Start by dividing both sides of the equation by a (which cannot be zero, otherwise the equation would not be a quadratic equation):

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Completing the square of the expression on the left side, you find that

$$x^2 + \frac{b}{a}x + \frac{c}{a} = \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = \left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}.$$

So you can continue with the equation

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2} = 0, \text{ which is } \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

There are two possibilities,

$$x + \frac{b}{2a} = +\sqrt{\frac{b^2 - 4ac}{4a^2}} \text{ or } -\sqrt{\frac{b^2 - 4ac}{4a^2}},$$

giving

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This shows that if $ax^2 + bx + c = 0$ and $a \neq 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Example 4.4.1

Use the quadratic formula to solve the equations

- (a) $2x^2 - 3x - 4 = 0$, (b) $2x^2 - 3x + 4 = 0$, (c) $30x^2 - 11x - 30 = 0$.

(a) Comparing this with $ax^2 + bx + c = 0$, put $a = 2$, $b = -3$ and $c = -4$. Then

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \times 2 \times (-4)}}{2 \times 2} = \frac{3 \pm \sqrt{9 + 32}}{4} = \frac{3 \pm \sqrt{41}}{4}.$$

Sometimes you will be expected to leave the roots in surd form. At other times you may be required to give the roots in the form $\frac{3 + \sqrt{41}}{4} \approx 2.35$ and $\frac{3 - \sqrt{41}}{4} \approx -0.85$. Try substituting these numbers in the equation and see what happens.

(b) Putting $a = 2$, $b = -3$ and $c = 4$,

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \times 2 \times 4}}{2 \times 2} = \frac{3 \pm \sqrt{9 - 32}}{4} = \frac{3 \pm \sqrt{-23}}{4}.$$

But -23 does not have a square root. This means that the equation $2x^2 - 3x + 4 = 0$ has no roots.

Try putting $2x^2 - 3x + 4$ in completed square form; what can you deduce about the graph of $y = 2x^2 - 3x + 4$?

(c) Putting $a = 30$, $b = -11$ and $c = -30$,

$$\begin{aligned}x &= \frac{-(-11) \pm \sqrt{(-11)^2 - 4 \times 30 \times (-30)}}{2 \times 30} = \frac{11 \pm \sqrt{121 + 3600}}{60} \\&= \frac{11 \pm \sqrt{3721}}{60} = \frac{11 \pm 61}{60} \\ \text{So } x &= \frac{72}{60} = \frac{6}{5} \text{ or } x = -\frac{50}{60} = -\frac{5}{6}.\end{aligned}$$

This third example factorises, but the factors are difficult to find. But once you know the roots of the equation you can deduce that $30x^2 - 11x - 30 = (6x + 5)(5x - 6)$.

4.5 The discriminant $b^2 - 4ac$

If you look back at Example 4.4.1 you will see that in part (a) the roots of the equation involved surds, in part (b) there were no roots, and in part (c) the roots were fractions.

You can predict which case will arise by calculating the value of the expression under the square root sign, $b^2 - 4ac$, and thinking about the effect that

this value has in the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

- If $b^2 - 4ac$ is a perfect square, the roots will be integers or fractions.
- If $b^2 - 4ac > 0$, the equation $ax^2 + bx + c = 0$ will have two roots.
- If $b^2 - 4ac < 0$, there will be no roots.
- If $b^2 - 4ac = 0$, the roots are given by $x = \frac{-b \pm 0}{2a} = -\frac{b}{2a}$, and there is one root only. Sometimes it is said that there are two coincident roots, or a **repeated root**, because the root values $\frac{-b+0}{2a}$ and $\frac{-b-0}{2a}$ are equal.

The expression $b^2 - 4ac$ is called the **discriminant** of the quadratic expression $ax^2 + bx + c$ because, by its value, it discriminates between the types of solution of the equation $ax^2 + bx + c = 0$.

Example 4.5.1

What can you deduce from the values of the discriminants of the quadratics in the following equations?

- | | |
|-------------------------|-------------------------|
| (a) $2x^2 - 3x - 4 = 0$ | (b) $2x^2 - 3x - 5 = 0$ |
| (c) $2x^2 - 4x + 5 = 0$ | (d) $2x^2 - 4x + 2 = 0$ |

(a) As $a = 2$, $b = -3$ and $c = -4$, $b^2 - 4ac = (-3)^2 - 4 \times 2 \times (-4) = 9 + 32 = 41$.

The discriminant is positive, so the equation $2x^2 - 3x - 4 = 0$ has two roots. Also, as 41 is not a perfect square, the roots are irrational.

(b) As $a = 2$, $b = -3$ and $c = -5$, $b^2 - 4ac = (-3)^2 - 4 \times 2 \times (-5) = 9 + 40 = 49$.

The discriminant is positive, so the equation $2x^2 - 3x - 5 = 0$ has two roots. Also, as 49 is a perfect square, the roots are rational.

(c) $b^2 - 4ac = (-4)^2 - 4 \times 2 \times 5 = 16 - 40 = -24$. As the discriminant is negative, the equation $2x^2 - 4x + 5 = 0$ has no roots.

(d) $b^2 - 4ac = (-4)^2 - 4 \times 2 \times 2 = 16 - 16 = 0$. As the discriminant is zero, the equation $2x^2 - 4x + 2 = 0$ has only one (repeated) root.

Example 4.5.2

The equation $kx^2 - 2x - 7 = 0$ has two real roots. What can you deduce about the value of the constant k ?

The discriminant is $(-2)^2 - 4 \times k \times (-7) = 4 + 28k$. As the equation has two real roots, the value of the discriminant is positive, so $4 + 28k > 0$, and $k > -\frac{1}{7}$.

Example 4.5.3

The equation $3x^2 + 2x + k = 0$ has a repeated root. Find the value of k .

The equation has repeated roots if $b^2 - 4ac = 0$; that is, if $2^2 - 4 \times 3 \times k = 0$. This gives $k = \frac{1}{3}$.

Notice how, in the above examples, there is no need to solve the quadratic equation. You can find all you need to know from the discriminant.

Exercise 4B

- 1 Use the quadratic formula to solve the following equations. Leave irrational answers in surd form. If there is no solution, say so. Keep your answers for use in Question 8.

(a) $x^2 + 3x - 5 = 0$	(b) $x^2 - 4x - 7 = 0$	(c) $x^2 + 6x + 9 = 0$
(d) $x^2 + 5x + 2 = 0$	(e) $x^2 + x + 1 = 0$	(f) $3x^2 - 5x - 6 = 0$
(g) $2x^2 + 7x + 3 = 0$	(h) $8 - 3x - x^2 = 0$	(i) $5 + 4x - 6x^2 = 0$

- 2 Use the value of the discriminant $b^2 - 4ac$ to determine whether the following equations have two roots, one root or no roots.

(a) $x^2 - 3x - 5 = 0$	(b) $x^2 + 2x + 1 = 0$	(c) $x^2 - 3x + 4 = 0$
(d) $3x^2 - 6x + 5 = 0$	(e) $2x^2 - 7x + 3 = 0$	(f) $5x^2 + 9x + 4 = 0$
(g) $3x^2 + 42x + 147 = 0$	(h) $3 - 7x - 4x^2 = 0$	

In parts (i) and (j), the values of p and q are positive.

(i) $x^2 + px - q = 0$	(j) $x^2 - px - q = 0$
------------------------	------------------------

- 3 The following equations have a repeated root. Find the value of k in each case. Leave your answers as integers, exact fractions or surds.

(a) $x^2 + 3x - k = 0$	(b) $kx^2 + 5x - 8 = 0$	(c) $x^2 - 18x + k = 0$
(d) $-3 + kx - 2x^2 = 0$	(e) $4x^2 - kx + 6 = 0$	(f) $kx^2 - px + q = 0$

- 4 The following equations have the number of roots shown in brackets. Deduce as much as you can about the value of k .

(a) $x^2 + 3x + k = 0$ (2)	(b) $x^2 - 7x + k = 0$ (1)	(c) $kx^2 - 3x + 5 = 0$ (0)
(d) $3x^2 + 5x - k = 0$ (2)	(e) $x^2 - 4x + 3k = 0$ (1)	(f) $kx^2 - 5x + 7 = 0$ (0)
(g) $x^2 - kx + 4 = 0$ (2)	(h) $x^2 + kx + 9 = 0$ (0)	

5 Use the value of the discriminant to determine the number of points of intersection of the following graphs with the x -axis.

- | | | |
|-------------------------|--------------------------|---------------------------|
| (a) $y = x^2 - 5x - 5$ | (b) $y = x^2 + x + 1$ | (c) $y = x^2 - 6x + 9$ |
| (d) $y = x^2 + 4$ | (e) $y = x^2 - 10$ | (f) $y = 3 - 4x - 2x^2$ |
| (g) $y = 3x^2 - 5x + 7$ | (h) $y = x^2 + bx + b^2$ | (i) $y = x^2 - 2qx + q^2$ |

6 If a and c are both positive, what can be said about the graph of $y = ax^2 + bx - c$?

7 If a is negative and c is positive, what can be said about the graph of $y = ax^2 + bx + c$?

8 You will need your answers to Question 1, in rational or surd form, not decimals.

(A) For Question 1(a), (b) and (d), find (i) the sum and (ii) the product of the roots. What do you notice? What happens if there is only one (repeated) root?

(B) If α and β are the roots of the quadratic equation $x^2 + bx + c = 0$ then these arise from factors $(x - \alpha)$ and $(x - \beta)$ of $x^2 + bx + c$. Show that the equation $x^2 + bx + c = 0$ has roots which have sum $-b$ and product c .

(C) Extend (B) to find expressions in terms of a , b and c for (i) the sum and (ii) the product of the roots of the equation $ax^2 + bx + c = 0$.

4.6 Simultaneous equations

This section shows how to solve a pair of simultaneous equations such as $y = x^2$ and $5x + 4y = 21$. This takes forward the ideas in Section 3.7.

Example 4.6.1

Solve the simultaneous equations $y = x^2$, $x + y = 6$.

These equations are usually best solved by finding an expression for x or y from one equation and substituting it into the other. In this case, it is easier to substitute for y from the first equation into the second, giving $x + x^2 = 6$. This rearranges to $x^2 + x - 6 = 0$, so $(x + 3)(x - 2) = 0$, giving $x = 2$ or $x = -3$.

You can find the corresponding y -values from the equation $y = x^2$. They are $y = 4$ and $y = 9$ respectively.

The solution is therefore $x = 2, y = 4$ or $x = -3, y = 9$.

Check that, for each pair of values, $x + y = 6$.

Note that the answers go together in pairs. It would be wrong to give the answer in the form $x = 2$ or $x = -3$ and $y = 4$ or $y = 9$, because the pairs of values $x = 2, y = 9$ and $x = -3, y = 4$ do not satisfy the original equations. You can see this if you interpret the question as finding the points of intersection of the graphs $y = x^2$ and $x + y = 6$, as in Fig. 4.2.

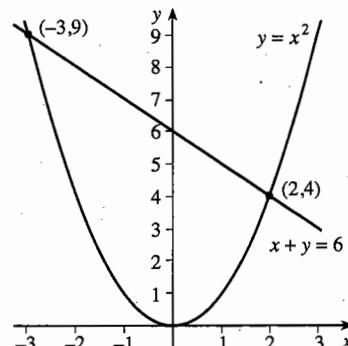


Fig. 4.2

Example 4.6.2

Solve the simultaneous equations $x^2 - 2xy + 3y^2 = 6$ and $2x - 3y = 3$.

It is not easy to find expressions for either x or y from the first equation, so begin with the second equation. You are less likely to make mistakes if you avoid fractions. From the second equation, $2x = 3 + 3y$, so, squaring this equation,

$$4x^2 = (3 + 3y)^2 = 9 + 18y + 9y^2.$$

You now have expressions for $4x^2$ and $2x$, so you would like to substitute for them in the first equation. It is helpful to multiply the first equation by 4. Then

$$4x^2 - 8xy + 12y^2 = 24, \text{ or } 4x^2 - 4y \times 2x + 12y^2 = 24;$$

$$\text{so } (9 + 18y + 9y^2) - 4y(3 + 3y) + 12y^2 = 24.$$

This reduces to $9y^2 + 6y - 15 = 0$, and, dividing by 3, to $3y^2 + 2y - 5 = 0$.

Solving this equation gives $(y - 1)(3y + 5) = 0$, so $y = 1$ or $y = -\frac{5}{3}$.

Substituting in the second equation to find x , you obtain $x = 3$ and $x = -1$ respectively. Therefore the solution is $x = 3, y = 1$ and $x = -1, y = -\frac{5}{3}$.

Example 4.6.3

At how many points does the line $x + 2y = 3$ meet the curve $2x^2 + y^2 = 4$?

From $x + 2y = 3$, $x = 3 - 2y$. Substituting for x in $2x^2 + y^2 = 4$, $2(3 - 2y)^2 + y^2 = 4$, so $2(9 - 12y + 4y^2) + y^2 = 4$, which reduces to $9y^2 - 24y + 14 = 0$.

The discriminant of this equation is $24^2 - 4 \times 9 \times 14 = 576 - 504 = 72$. As this is positive, the equation has two solutions, so the line meets the curve at two points.

4.7 Equations which reduce to quadratic equations

Sometimes you will come across equations which are not quadratic, but which can be changed into quadratic equations, usually by making the right substitution.

Example 4.7.1

Solve the equation $t^4 - 13t^2 + 36 = 0$.

This is called a quartic equation because it has a t^4 term, but if you let x stand for t^2 , the equation becomes $x^2 - 13x + 36 = 0$, which is a quadratic equation in x .

Then $(x - 4)(x - 9) = 0$, so $x = 4$ or $x = 9$.

Now recall that $x = t^2$, so $t^2 = 4$ or $t^2 = 9$, giving $t = \pm 2$ or $t = \pm 3$.

Example 4.7.2

Solve the equation $\sqrt{x} = 6 - x$

(a) by letting y stand for \sqrt{x} (b) by squaring both sides of the equation.

(a) Letting $\sqrt{x} = y$, the equation becomes $y = 6 - y^2$ or $y^2 + y - 6 = 0$. Therefore $(y+3)(y-2) = 0$, so $y = 2$ or $y = -3$. But, as $y = \sqrt{x}$, and \sqrt{x} is never negative, the only solution is $y = 2$, giving $x = 4$.

(b) Squaring both sides gives $x = (6 - x)^2 = 36 - 12x + x^2$ or $x^2 - 13x + 36 = 0$. Therefore $(x - 4)(x - 9) = 0$, so $x = 4$ or $x = 9$. Checking the answers shows that when $x = 4$, the equation $\sqrt{x} = 6 - x$ is satisfied, but when $x = 9$, $\sqrt{x} = 3$ and $6 - x = -3$, so $x = 9$ is not a root. Therefore $x = 4$ is the only root.

This is important. If you square you will find the root or roots of the equation $\sqrt{x} = -(6 - x)$ as well as the roots that you are actually looking for. Notice that $x = 9$ does satisfy this last equation, but $x = 4$ doesn't! The moral is that, when you square an equation in the process of solving it, it is essential to check your answers.

Exercise 4C

1 Solve the following pairs of simultaneous equations.

- | | | | |
|---------------------|---------------------------|---------------------|---------------------|
| (a) $y = x + 1$, | $x^2 + y^2 = 25$ | (b) $x + y = 7$, | $x^2 + y^2 = 25$ |
| (c) $y = x - 3$, | $y = x^2 - 3x - 8$ | (d) $y = 2 - x$, | $x^2 - y^2 = 8$ |
| (e) $2x + y = 5$, | $x^2 + y^2 = 25$ | (f) $y = 1 - x$, | $y^2 - xy = 0$ |
| (g) $7y - x = 49$, | $x^2 + y^2 - 2x - 49 = 0$ | (h) $y = 3x - 11$, | $x^2 + 2xy + 3 = 0$ |

2 Find the coordinates of the points of intersection of the given straight lines with the given curves.

- | | | | |
|----------------------|--------------------------|---------------------|-----------------------|
| (a) $y = 2x + 1$, | $y = x^2 - x + 3$ | (b) $y = 3x + 2$, | $x^2 + y^2 = 26$ |
| (c) $y = 2x - 2$, | $y = x^2 - 5$ | (d) $x + 2y = 3$, | $x^2 + xy = 2$ |
| (e) $3y + 4x = 25$, | $x^2 + y^2 = 25$ | (f) $y + 2x = 3$, | $2x^2 - 3xy = 14$ |
| (g) $y = 2x - 12$, | $x^2 + 4xy - 3y^2 = -27$ | (h) $2x - 5y = 6$, | $2xy - 4x^2 - 3y = 1$ |

3 In each case find the number of points of intersection of the straight line with the curve.

- | | | | |
|---------------------|--------------------|------------------------------|------------|
| (a) $y = 1 - 2x$, | $x^2 + y^2 = 1$ | (b) $y = \frac{1}{2}x - 1$, | $y = 4x^2$ |
| (c) $y = 3x - 1$, | $xy = 12$ | (d) $4y - x = 16$, | $y^2 = 4x$ |
| (e) $3y - x = 15$, | $4x^2 + 9y^2 = 36$ | (f) $4y = 12 - x$, | $xy = 9$ |

4 Solve the following equations; give irrational answers in terms of surds.

- | | | |
|--------------------------|---------------------------|--------------------------|
| (a) $x^4 - 5x^2 + 4 = 0$ | (b) $x^4 - 10x^2 + 9 = 0$ | (c) $x^4 - 3x^2 - 4 = 0$ |
| (d) $x^4 - 5x^2 - 6 = 0$ | (e) $x^6 - 7x^3 - 8 = 0$ | (f) $x^6 + x^3 - 12 = 0$ |

5 Solve the following equations. (In most cases, multiplication by an appropriate expression will turn the equation into a form you should recognise.)

- | | | |
|---|---|---|
| (a) $x = 3 + \frac{10}{x}$ | (b) $x + 5 = \frac{6}{x}$ | (c) $2t + 5 = \frac{3}{t}$ |
| (d) $x = \frac{12}{x+1}$ | (e) $\sqrt{t} = 4 + \frac{12}{\sqrt{t}}$ | (f) $\sqrt{t}(\sqrt{t} - 6) = -9$ |
| (g) $x - \frac{2}{x+2} = \frac{1}{3}$ | (h) $\frac{20}{x+2} - 1 = \frac{20}{x+3}$ | (i) $\frac{12}{x+1} - \frac{10}{x-3} = -3$ |
| (j) $\frac{15}{2x+1} + \frac{10}{x} = \frac{55}{2}$ | (k) $y^4 - 3y^2 = 4$ | (l) $\frac{1}{y^2} - \frac{1}{y^2+1} = \frac{1}{2}$ |

6 Solve the following equations.

- | | | |
|--------------------------|---|--|
| (a) $x - 8 = 2\sqrt{x}$ | (b) $x + 15 = 8\sqrt{x}$ | (c) $t - 5\sqrt{t} - 14 = 0$ |
| (d) $t = 3\sqrt{t} + 10$ | (e) $\sqrt[3]{x^2} - \sqrt[3]{x} - 6 = 0$ | (f) $\sqrt[3]{t^2} - 3\sqrt[3]{t} = 4$ |

Miscellaneous exercise 4

- 1** Solve the simultaneous equations $x + y = 2$ and $x^2 + 2y^2 = 11$. (OCR)
- 2** The quadratic polynomial $x^2 - 10x + 17$ is denoted by $f(x)$. Express $f(x)$ in the form $(x - a)^2 + b$ stating the values of a and b .
Hence find the least possible value that $f(x)$ can take and the corresponding value of x . (OCR)
- 3** Solve the simultaneous equations $2x + y = 3$ and $2x^2 - xy = 10$. (OCR)
- 4** For what values of k does the equation $2x^2 - kx + 8 = 0$ have a repeated root?
- 5** By expressing the function $f(x) = (2x + 3)(x - 4)$ in completed square form, find the range of the function $f(x)$.
- 6** (a) Solve the equation $x^2 - (6\sqrt{3})x + 24 = 0$, giving your answer in terms of surds, simplified as far as possible.
(b) Find all four solutions of the equation $x^4 - (6\sqrt{3})x^2 + 24 = 0$ giving your answers correct to 2 decimal places. (OCR)
- 7** Show that the line $y = 3x - 3$ and the curve $y = (3x + 1)(x + 2)$ do not meet.
- 8** Express $9x^2 - 36x + 52$ in the form $(Ax - B)^2 + C$, where A , B and C are integers.
Hence, or otherwise, find the set of values taken by $9x^2 - 36x + 52$ for real x . (OCR)
- 9** Find the points of intersection of the curves $y = 6x^2 + 4x - 3$ and $y = x^2 - 3x - 1$, giving the coordinates correct to 2 decimal places.
- 10** (a) Express $9x^2 + 12x + 7$ in the form $(ax + b)^2 + c$ where a , b , c are constants whose values are to be found.
(b) Find the set of values taken by $\frac{1}{9x^2 + 12x + 7}$ for real values of x . (OCR)
- 11** Find, correct to 3 significant figures, all the roots of the equation $8x^4 - 8x^2 + 1 = \frac{1}{2}\sqrt{3}$. (OCR)
- 12** Find constants a , b and c such that, for all values of x ,
 $3x^2 - 5x + 1 = a(x + b)^2 + c$.
Hence find the coordinates of the minimum point on the graph of $y = 3x^2 - 5x + 1$.
(Note: the minimum point or maximum point is the vertex.) (OCR, adapted)
- 13** Find the points of intersection of the curve $xy = 6$ and the line $y = 9 - 3x$. (OCR)

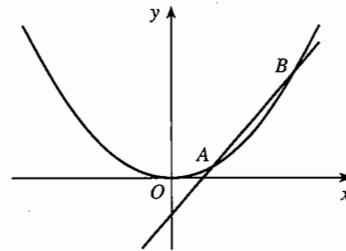
- 14** The equation of a curve is $y = ax^2 - 2bx + c$, where a , b and c are constants with $a > 0$.

- (a) Find, in terms of a , b and c , the coordinates of the vertex of the curve.
 (b) Given that the vertex of the curve lies on the line $y = x$, find an expression for c in

terms of a and b . Show that in this case, whatever the value of b , $c \geqslant -\frac{1}{4a}$.

(OCR, adapted)

- 15** (a) The diagram shows the graphs of $y = x - 1$ and $y = kx^2$, where k is a positive constant. The graphs intersect at two distinct points A and B . Write down the quadratic equation satisfied by the x -coordinates of A and B , and hence show that $k < \frac{1}{4}$.



- (b) Describe briefly the relationship between the graphs of $y = x - 1$ and $y = kx^2$ in each of the cases (i) $k = \frac{1}{4}$, (ii) $k > \frac{1}{4}$.
 (c) Show, by using a graphical argument or otherwise, that when k is a negative constant, the equation $x - 1 = kx^2$ has two real roots, one of which lies between 0 and 1.

- 16** Use the following procedure to find the least (perpendicular) distance of the point $(1, 2)$ from the line $y = 3x + 5$, *without* having to find the equation of a line perpendicular to $y = 3x + 5$ (as you did in Chapter 1).

- (a) Let (x, y) be a general point on the line. Show that its distance, d , from $(1, 2)$ is given by $d^2 = (x - 1)^2 + (y - 2)^2$.
 (b) Use the equation of the line to show that $d^2 = (x - 1)^2 + (3x + 3)^2$.
 (c) Show that $d^2 = 10x^2 + 16x + 10$.
 (d) By completing the square, show that the minimum distance required is $\frac{3}{5}\sqrt{10}$.

- 17** Using the technique of Question 16,

- (a) find the perpendicular distance of $(2, 3)$ from $y = 2x + 1$,
 (b) find the perpendicular distance of $(-1, 3)$ from $y = -2x + 5$,
 (c) find the perpendicular distance of $(2, -1)$ from $3x + 4y - 7 = 0$.

- 18** Point O is the intersection of two roads which cross at right angles; one road runs from north to south, the other from east to west. Car A is 100 metres due west of O and travelling east at a speed of 20 m s^{-1} , and Car B is 80 metres due north of O and travelling south at 20 m s^{-1} .

- (a) Show that after t seconds their distance apart, d metres, is given by

$$d^2 = (100 - 20t)^2 + (80 - 20t)^2.$$

 (b) Show that this simplifies to $d^2 = 400((5-t)^2 + (4-t)^2)$.
 (c) Show that the minimum distance apart of the two cars is $10\sqrt{2}$ metres.

- 19** Point O is the intersection of two roads which cross at right angles; one road runs from north to south, the other from east to west. Find the least distance apart of two motorbikes A and B which are initially approaching O on different roads in the following cases.
- Both motorbikes are 10 metres from O . A is travelling at 20 m s^{-1} , B at 10 m s^{-1} .
 - A is 120 metres from O travelling at 20 m s^{-1} , B is 80 metres from O travelling at 10 m s^{-1} .
 - A is 120 metres from O travelling at 20 m s^{-1} , B is 60 metres from O travelling at 10 m s^{-1} .
- 20** (a) Express $2 - 4x - x^2$ and $24 + 8x + x^2$ in their completed square forms.
(b) Show that the graphs with equations $y = 2 - 4x - x^2$ and $y = 24 + 8x + x^2$ do not intersect.
(c) By giving an example, show that it is possible to find graphs with equations of the forms $y = A - (x - a)^2$ and $y = B + (x - b)^2$ with $A > B$ which do not intersect.
- 21** A recycling firm collects aluminium cans from a number of sites. It crushes them and then sells the aluminium back to a manufacturer.

The profit from processing t tonnes of cans each week is $\$p$, where

$$p = 100t - \frac{1}{2}t^2 - 200.$$

By completing the square, find the greatest profit the firm can make each week, and how many tonnes of cans it has to collect and crush each week to achieve this profit.

5 Inequalities

This chapter is about inequality relationships, and how to solve inequalities. When you have completed it, you should

- know the rules for working with inequality symbols
- be able to solve linear inequalities
- be able to solve quadratic inequalities.

5.1 Notation for inequalities

You often want to compare one number with another and say which is the bigger. This comparison is expressed by using the inequality symbols $>$, $<$, \leq and \geq . You have already met inequalities in Chapters 3 and 4.

The symbol $a > b$ means that a is greater than b . You can visualise this geometrically as in Fig. 5.1, which shows three number lines, with a to the right of b .

Notice that it does not matter whether a and b are positive or negative. The position of a and b in relation to zero on the number line is irrelevant. In all three lines, $a > b$. As an example, in the bottom line, $-4 > -7$.

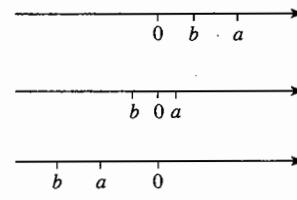


Fig. 5.1

Similarly, the symbol $a < b$ means that a is less than b . You can visualise this geometrically on a number line, with a to the left of b .

These four expressions are equivalent.

$$\begin{array}{ll} a > b & a \text{ is greater than } b \\ b < a & b \text{ is less than } a \end{array}$$

The symbol $a \geq b$ means ‘either $a > b$ or $a = b$ '; that is, a is greater than or equal to, but not less than, b . Similarly, the symbol $a \leq b$ means ‘either $a < b$ or $a = b$ '; that is, a is less than or equal to, but not greater than, b .

These expressions are equivalent.

$$\begin{array}{ll} a \geq b & a \text{ is greater than or equal to } b \\ b \leq a & b \text{ is not greater than } a \end{array}$$

The symbols $<$ and $>$ are called **strict** inequalities, and the symbols \leq and \geq are called **weak** inequalities.

5.2 Solving linear inequalities

When you solve an inequality such as $3x + 10 > 10x - 11$, you have to write a simpler statement with precisely the same meaning. In this case the simpler statement turns out to be $x < 3$. But how do you get from the complicated statement to the simple one?

Adding or subtracting the same number on both sides

You can add or subtract the same number on both sides of an inequality. For instance you can add the number 11 to both sides. In the example you would get

$$(3x + 10) + 11 > (10x - 11) + 11,$$

$$3x + 21 > 10x.$$

Justifying such a step involves showing that, for any number c , 'if $a > b$ then $a + c > b + c$ '.

This is saying that if a is to the right of b on the number line, then $a + c$ is to the right of $b + c$. Fig. 5.2 shows that this is true whether c is positive or negative.

Since subtracting c is the same as adding $-c$, you can also subtract the same number from both sides.

In the example, if you subtract $3x$ from both sides you get

$$(3x + 21) - 3x > 10x - 3x,$$

$$21 > 7x.$$

Multiplying both sides by a positive number

You can multiply (or divide) both sides of an inequality by a positive number. In the example above, you can divide both sides by the positive number 7 (or multiply both sides by $\frac{1}{7}$), and get:

$$21 \times \frac{1}{7} > 7x \times \frac{1}{7},$$

$$3 > x.$$

Here is a justification of the step, 'if $c > 0$ and $a > b$, then $ca > cb$ '.

As $a > b$, a is to the right of b on the number line.

As $c > 0$, ca and cb are enlargements of the positions of a and b relative to the number 0.

Fig. 5.3 shows that, whether a and b are positive or negative, ca is to the right of cb , so $ca > cb$.

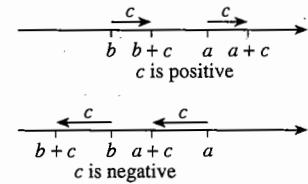


Fig. 5.2

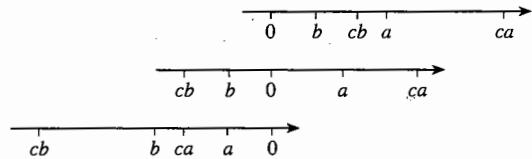


Fig. 5.3

Multiplying both sides by a negative number

If $a > b$, and you subtract $a + b$ from both sides, then you get $-b > -a$, which is the same as $-a < -b$. This shows that if you multiply both sides of an inequality by -1 , then you change the direction of the inequality. Suppose that you wish to multiply the inequality $a > b$ by -2 . This is the same as multiplying $-a < -b$ by 2 , so $-2a < -2b$.

You can also think of multiplying by -2 as reflecting the points corresponding to a and b in the origin, and then multiplying by 2 as an enlargement.

You can summarise this by saying that if you multiply (or divide) both sides of an inequality by a negative number, you must change the direction of the inequality. Thus if $c < 0$ and $a > b$, then $ca < cb$ (see Fig. 5.4).

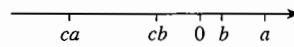


Fig. 5.4

Summary of operations on inequalities

- You can add or subtract a number on both sides of an inequality.
- You can multiply or divide an inequality by a positive number.
- You can multiply or divide an inequality by a negative number, but you must change the direction of the inequality.

Solving inequalities is simply a matter of exploiting these three rules.

Example 5.2.1

Solve the inequality $-3x < 21$.

In this example you need to divide both sides by -3 . Remembering to change the direction of the inequality, $-3x < 21$ becomes $x > -7$.

Example 5.2.2

Solve the inequality $\frac{1}{3}(4x + 3) - 3(2x - 4) \geq 20$.

Use the rule about multiplying by a positive number to multiply both sides by 3, in order to clear the fractions. In the solution, a reason is given only when an operation is carried out which affects the inequality.

$$\begin{aligned} \frac{1}{3}(4x + 3) - 3(2x - 4) &\geq 20, \\ (4x + 3) - 9(2x - 4) &\geq 60, && \text{multiply both sides by 3} \\ 4x + 3 - 18x + 36 &\geq 60, \\ -14x + 39 &\geq 60, \\ -14x &\geq 21, && \text{subtract 39 from both sides} \\ x &\leq -\frac{3}{2}. && \text{divide both sides by } -14, \text{ change } \geq \text{ to } \leq \end{aligned}$$

Solving inequalities of this type is similar to solving equations. However, when you multiply or divide by a number, remember to reverse the inequality if that number is negative.

Exercise 5A

Solve the following inequalities.

- | | | | | |
|----------|---|---|-----------------------------------|------------------------------|
| 1 | (a) $x - 3 > 11$ | (b) $x + 7 < 11$ | (c) $2x + 3 \leq 8$ | (d) $3x - 5 \geq 16$ |
| | (e) $3x + 7 > -5$ | (f) $5x + 6 \leq -10$ | (g) $2x + 3 < -4$ | (h) $3x - 1 \leq -13$ |
| 2 | (a) $\frac{x+3}{2} > 5$ | (b) $\frac{x-4}{6} \leq 3$ | (c) $\frac{2x+3}{4} < -5$ | (d) $\frac{3x+2}{5} \leq 4$ |
| | (e) $\frac{4x-3}{2} \geq -7$ | (f) $\frac{5x+1}{3} > -3$ | (g) $\frac{3x-2}{8} < 1$ | (h) $\frac{4x-2}{3} \geq -6$ |
| 3 | (a) $-5x \leq 20$ | (b) $-3x \geq -12$ | (c) $5-x < -4$ | (d) $4-3x \leq 10$ |
| | (e) $2-6x \leq 0$ | (f) $6-5x > 1$ | (g) $6-5x > -1$ | (h) $3-7x < -11$ |
| 4 | (a) $\frac{3-x}{5} < 2$ | (b) $\frac{5-x}{3} \geq 1$ | (c) $\frac{3-2x}{5} > 3$ | (d) $\frac{7-3x}{2} < -1$ |
| | (e) $\frac{5-4x}{2} \leq -3$ | (f) $\frac{3-2x}{5} > -7$ | (g) $\frac{3+2x}{4} < 5$ | (h) $\frac{7-3x}{4} \leq -5$ |
| 5 | (a) $x - 4 \leq 5 + 2x$ | (b) $x - 3 \geq 5 - x$ | (c) $2x + 5 < 4x - 7$ | |
| | (d) $3x - 4 > 5 - x$ | (e) $4x \leq 3(2 - x)$ | (f) $3x \geq 5 - 2(3 - x)$ | |
| | (g) $6x < 8 - 2(7 + x)$ | (h) $5x - 3 > x - 3(2 - x)$ | (i) $6 - 2(x + 1) \leq 3(1 - 2x)$ | |
| 6 | (a) $\frac{1}{3}(8x+1) - 2(x-3) > 10$ | (b) $\frac{5}{2}(x+1) - 2(x-3) < 7$ | | |
| | (c) $\frac{2x+1}{3} - \frac{4x+5}{2} \leq 0$ | (d) $\frac{3x-2}{2} - \frac{x-4}{3} < x$ | | |
| | (e) $\frac{x+1}{4} + \frac{1}{6} \geq \frac{2x-5}{3}$ | (f) $\frac{x}{2} - \frac{3-2x}{5} \leq 1$ | | |
| | (g) $\frac{x-1}{3} - \frac{x+1}{4} > \frac{x}{2}$ | (h) $\frac{x}{3} \geq 5 - \frac{3x}{4}$ | | |

5.3 Quadratic inequalities

In Chapter 4, you saw that a quadratic function might take one of three forms:

$$f(x) = ax^2 + bx + c \quad \text{the usual form}$$

$$f(x) = a(x-p)(x-q) \quad \text{the factor form}$$

$$f(x) = a(x-r)^2 + s \quad \text{the completed square form.}$$

If you need to solve a quadratic inequality of the form $f(x) < 0$, $f(x) > 0$, $f(x) \leq 0$ or $f(x) \geq 0$, by far the easiest form to use is the factor form.

Here are some examples which show ways of solving quadratic inequalities.

Example 5.3.1

Solve the inequality $(x - 2)(x - 4) < 0$.

Method 1 Sketch the graph of $y = (x - 2)(x - 4)$.

The graph cuts the x -axis at $x = 2$ and $x = 4$. As the coefficient of x^2 is positive, the parabola bends upwards, as shown in Fig. 5.5.

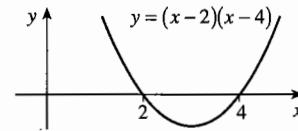


Fig. 5.5

You need to find the values of x such that $y < 0$.

From the graph you can see that this happens when x lies between 2 and 4, that is $x > 2$ and $x < 4$.

Remembering that $x > 2$ is the same as $2 < x$, you can write this as $2 < x < 4$, meaning that x is greater than 2 and less than 4.

When you write an inequality of the kind $r < x$ and $x < s$ in the form $r < x < s$, it is essential that $r < s$. It makes no sense to write $7 < x < 3$; how can x be both greater than 7 and less than 3?

An inequality of the type $r < x < s$ (or $r < x \leq s$ or $r \leq x < s$ or $r \leq x \leq s$) is called an **interval**.

Method 2 Find the values of x for which $(x - 2)(x - 4) = 0$. These values, $x = 2$ and $x = 4$, are called the **critical values** for the inequality.

Make a table showing the signs of the factors in the product $(x - 2)(x - 4)$.

	$x < 2$	$x = 2$	$2 < x < 4$	$x = 4$	$x > 4$
$x - 2$	-	0	+	+	+
$x - 4$	-	-	-	0	+
$(x - 2)(x - 4)$	+	0	-	0	+

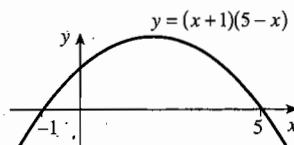
Table 5.6

From Table 5.6 you can see that $(x - 2)(x - 4) < 0$ when $2 < x < 4$.

Example 5.3.2

Solve the inequality $(x + 1)(5 - x) \leq 0$.

Fig. 5.7 shows the graph of $y = (x + 1)(5 - x)$. As the coefficient of x^2 is negative, the parabola has its vertex at the top. So $y \leq 0$ when either $x \leq -1$ or $x \geq 5$.



Note that in this case the inequality is also satisfied by the critical values -1 and 5 .

Fig. 5.7

Example 5.3.3

Solve the inequality $x^2 \leq a^2$, where $a > 0$.

This is the same as $x^2 - a^2 \leq 0$ or $(x+a)(x-a) \leq 0$. The critical values are $x = -a$ and $x = a$.

	$x < -a$	$x = -a$	$-a < x < a$	$x = a$	$x > a$
$x+a$	-	0	+	+	+
$x-a$	-	-	-	0	+
$(x+a)(x-a)$	+	0	-	0	+

Table 5.8

Table 5.8 shows that, if $x^2 \leq a^2$, then $-a \leq x \leq a$. It also shows that, if $-a \leq x \leq a$, then $x^2 \leq a^2$.

The result of Example 5.3.3 is important. You can write it more shortly as:

If $a > 0$, then these statements are equivalent.

$$x^2 \leq a^2 \quad -a \leq x \leq a$$

It is usually easiest to solve inequalities by using graphical or tabular methods. If you have access to a graphic calculator, you can use it to obtain the sketch, which makes the whole process even easier.

Example 5.3.4 shows how inequality arguments can be expressed in a more algebraic form.

Example 5.3.4

Solve the inequalities (a) $(2x+1)(x-3) < 0$, (b) $(2x+1)(x-3) > 0$.

(a) If the product of two factors is negative, one of them must be negative, and the other positive. So there are two possibilities to consider.

If $2x+1$ is negative and $x-3$ is positive, then $x < -\frac{1}{2}$ and $x > 3$. This is obviously impossible.

But if $2x+1$ is positive and $x-3$ is negative, then $x > -\frac{1}{2}$ and $x < 3$, which happens if $-\frac{1}{2} < x < 3$.

(b) If the product of two factors is positive, either both are positive or both are negative.

If both $2x+1$ and $x-3$ are positive, then $x > -\frac{1}{2}$ and $x > 3$, which happens if $x > 3$.

If both $2x+1$ and $x-3$ are negative, then $x < -\frac{1}{2}$ and $x < 3$, which happens if $x < -\frac{1}{2}$.

So $(2x+1)(x-3) > 0$ if $x > 3$ or $x < -\frac{1}{2}$.

You could solve both parts at once by constructing a table as in Example 5.3.3, and reading off the sign from the last line.

There may be times when you don't have access to a graphic calculator, or when factorising the given expression is difficult or impossible. In those cases, you should complete the square, as described in Section 4.3.

Example 5.3.5

Solve algebraically the inequalities (a) $2x^2 - 8x + 11 \leq 0$, (b) $2x^2 - 8x + 5 \leq 0$.

$$(a) \text{ By completing the square, } 2x^2 - 8x + 11 = 2(x-2)^2 + 3.$$

The smallest value of $2(x-2)^2 + 3$ is 3, and it occurs when $x = 2$. So there are no values of x for which $2x^2 - 8x + 11 \leq 0$.

$$(b) \quad 2x^2 - 8x + 5 = 2(x-2)^2 - 3,$$

$$\text{so} \quad 2(x-2)^2 - 3 \leq 0,$$

$$(x-2)^2 - \frac{3}{2} \leq 0,$$

$$(x-2)^2 \leq \frac{3}{2}.$$

Using the result in the box on page 70,

$$-\sqrt{\frac{3}{2}} \leq x-2 \leq \sqrt{\frac{3}{2}}, \quad \text{or} \quad 2 - \sqrt{\frac{3}{2}} \leq x \leq 2 + \sqrt{\frac{3}{2}}.$$

Exercise 5B

1 Use sketch graphs to solve the following inequalities.

- | | | |
|--------------------------|------------------------|---------------------------|
| (a) $(x-2)(x-3) < 0$ | (b) $(x-4)(x-7) > 0$ | (c) $(x-1)(x-3) < 0$ |
| (d) $(x-4)(x+1) \geq 0$ | (e) $(2x-1)(x+3) > 0$ | (f) $(3x-2)(2x+5) \leq 0$ |
| (g) $(x+2)(4x+5) \geq 0$ | (h) $(1-x)(3+x) < 0$ | (i) $(3-2x)(5-x) > 0$ |
| (j) $(x-5)(x+5) < 0$ | (k) $(3-4x)(3x+4) > 0$ | (l) $(2+3x)(2-3x) \leq 0$ |

2 Use a table based on critical values to solve the following inequalities.

- | | | |
|---------------------------|------------------------|---------------------------|
| (a) $(x-3)(x-6) < 0$ | (b) $(x-2)(x-8) > 0$ | (c) $(x-2)(x+5) \leq 0$ |
| (d) $(x-3)(x+1) \geq 0$ | (e) $(2x+3)(x-2) > 0$ | (f) $(3x-2)(x+5) \leq 0$ |
| (g) $(x+3)(5x+4) \geq 0$ | (h) $(2-x)(5+x) < 0$ | (i) $(5-2x)(3-x) > 0$ |
| (j) $(3x+1)(3x-1) \geq 0$ | (k) $(2-7x)(3x+4) < 0$ | (l) $(5+3x)(1-3x) \leq 0$ |

3 Use an algebraic method to solve the following inequalities. Leave irrational numbers in terms of surds. Some inequalities may be true for all values of x , others for no values of x .

- | | | |
|--------------------------|------------------------|----------------------------|
| (a) $x^2 + 3x - 5 > 0$ | (b) $x^2 + 6x + 9 < 0$ | (c) $x^2 - 5x + 2 < 0$ |
| (d) $x^2 - x + 1 \geq 0$ | (e) $x^2 - 9 < 0$ | (f) $x^2 + 2x + 1 \leq 0$ |
| (g) $2x^2 - 3x - 1 < 0$ | (h) $8 - 3x - x^2 > 0$ | (i) $2x^2 + 7x + 1 \geq 0$ |

4 Use any method you like to solve the following inequalities.

- | | | |
|-------------------------|----------------------------|----------------------------|
| (a) $x^2 + 5x + 6 > 0$ | (b) $x^2 - 7x + 12 < 0$ | (c) $x^2 - 2x - 15 \leq 0$ |
| (d) $2x^2 - 18 \geq 0$ | (e) $2x^2 - 5x + 3 \geq 0$ | (f) $6x^2 - 5x - 6 < 0$ |
| (g) $x^2 + 5x + 2 > 0$ | (h) $7 - 3x^2 < 0$ | (i) $x^2 + x + 1 < 0$ |
| (j) $2x^2 - 5x + 5 > 0$ | (k) $12x^2 + 5x - 3 > 0$ | (l) $3x^2 - 7x + 1 \leq 0$ |

Miscellaneous exercise 5

1 Solve the inequality $x^2 - x - 42 \leq 0$.

2 Solve the inequality $(x+1)^2 < 9$.

3 Solve the inequality $x(x+1) < 12$.

(OCR)

4 Solve the inequality $x - x^3 < 0$.

5 Solve the inequality $x^3 \geq 6x - x^2$.

Use the discriminant ' $b^2 - 4ac$ ' in answering Questions 6 to 8. You may need to check the value $k = 0$ separately.

6 Find the values of k for which the following equations have two separate roots.

- | | | |
|-------------------------|-------------------------|-------------------------|
| (a) $kx^2 + kx + 2 = 0$ | (b) $kx^2 + 3x + k = 0$ | (c) $x^2 - 2kx + 4 = 0$ |
|-------------------------|-------------------------|-------------------------|

7 Find the values of k for which the following equations have no roots.

- | | | |
|--------------------------|----------------------------|--------------------------|
| (a) $kx^2 - 2kx + 5 = 0$ | (b) $k^2x^2 + 2kx + 1 = 0$ | (c) $x^2 - 5kx - 2k = 0$ |
|--------------------------|----------------------------|--------------------------|

8 Find the range of values of k for which the equation $x^2 + 3kx + k = 0$ has any roots.

9 Find the set of values of x for which $9x^2 + 12x + 7 > 19$.

(OCR)

10 Sketch, on the same diagram, the graphs of $y = \frac{1}{x}$ and $y = x - \frac{3}{2}$. Find the solution set of the inequality $x - \frac{3}{2} > \frac{1}{x}$.

(OCR)

11 Solve each of the following inequalities.

(a) $\frac{x}{x-2} < 5$

(b) $x(x-2) < 5$

(OCR, adapted)

Revision exercise 1

- 1 The line l_1 passes through the points $A(4,8)$ and $B(10,26)$. Show that an equation for l_1 is $y = 3x - 4$.

The line l_1 intersects the line l_2 , which has equation $y = 5x + 4$, at C . Find the coordinates of C .

- 2 Show that any root of the equation $5 + x - \sqrt{3 + 4x} = 0$ is also a root of the equation $x^2 + 6x + 22 = 0$. Hence show that the equation $5 + x - \sqrt{3 + 4x} = 0$ has no solutions.
- 3 Write $x^2 + 10x + 38$ in the form $(x + b)^2 + c$ where the values of b and c are to be found.
(a) State the minimum value of $x^2 + 10x + 38$ and the value of x for which this occurs.
(b) Determine the values of x for which $x^2 + 10x + 38 \geq 22$.
- 4 Simplify $(4x^{\frac{1}{2}}y)^2 \div (2x^{-1}y^2)$.
- 5 Solve the inequalities (a) $2x^2 - 5x + 2 \leq 0$, (b) $(2x - 3)^2 < 16$, (c) $\frac{1}{3}x - \frac{1}{4}(2x - 5) < \frac{1}{5}$.
- 6 Show that the equation $2^{x+1} + 2^{x-1} = 160$ can be written in the form $2.5 \times 2^x = 160$. Hence find the value of x which satisfies the equation.
- 7 Find the values of k such that the straight line $y = 2x + k$ meets the curve with equation $x^2 + 2xy + 2y^2 = 5$ exactly once.
- 8 Display on the same axes the curves with equations $y = x^3$ and $y = \sqrt[3]{x}$, and give the coordinates of their points of intersection.
- 9 A mail-order photographic developing company offers a picture-framing service to its customers. It will enlarge and mount any photograph, under glass and in a rectangular frame. Its charge is based on the size of the enlargement. It charges \$6 per metre of perimeter for the frame and \$15 per square metre for the glass. Write down an expression for the cost of enlarging and mounting a photograph in a frame which is x metres wide and y metres high.
- A photograph was enlarged and mounted in a square frame of side z metres at a cost of \$12. Formulate and solve a quadratic equation for z .
- 10 Find the equation of the straight line through $A(1,4)$ which is perpendicular to the line passing through the points $B(2,-2)$ and $C(4,0)$. Hence find the area of the triangle ABC , giving your answer in the simplest possible form.
- 11 Solve the inequalities
(a) $2(3-x) < 4 - (2-x)$, (b) $(x-3)^2 < x^2$, (c) $(x-2)(x-3) \geq 6$.
- 12 The quadratic equation $(p-1)x^2 + 4x + (p-4) = 0$ has a repeated root. Find the possible values of p .
- 13 Solve the simultaneous equations

$$\begin{aligned}2x + 3y &= 5, \\x^2 + 3xy &= 4.\end{aligned}$$

- 14** Prove that the triangle with vertices at the points $(1,2)$, $(9,8)$ and $(12,4)$ is right-angled, and calculate its area.
- 15** Find where the line $y = 5 - 2x$ meets the curve $y = (3 - x)^2$. What can you deduce from your answer?
- 16** A rhombus has opposite vertices at $(-1,3)$ and $(5,-1)$. Find the equations of its diagonals. One of the other vertices is $(0,-2)$. Find the fourth vertex.
- 17** Points A and B have coordinates $(-1,2)$ and $(7,-4)$ respectively.
- Write down the coordinates of M , the mid-point of AB .
 - Calculate the distance MB .
 - The point P lies on the circle with AB as diameter and has coordinates $(2,y)$ where y is positive. Calculate the value of y , giving your answer in surd form.
- 18** Solve the inequalities (a) $x^2 - x - 2 > 0$, (b) $(x+1)(x-2)(x-3) > 0$.
- 19** Two of the sides of a triangle have lengths 4 cm and 6 cm, and the angle between them is 120° . Calculate the length of the third side, giving your answer in the form $m\sqrt{p}$, where m and p are integers, and p is prime.
- 20** A triangle has vertices $O(0,0)$, $A(2,6)$ and $B(12,6)$. Write down the equation of the perpendicular bisector of AB , and find the perpendicular bisector of OA . Find the coordinates of the point C where these lines meet, and calculate the distances of C from O , A and B .
Write down the area of triangle OAB . Hence find the length of the perpendicular from A to OB , and deduce that angle AOB is 45° . (MEI, adapted)
- 21** A quadrilateral has vertices $A(-1,1)$, $B(1,2)$, $C(4,1)$ and $D(3,4)$. Find the lengths and the equations of the two diagonals AC and BD . (OCR)
- 22** The quadratic function $f(x) = px^2 + qx + r$ has $f(0) = 35$, $f(1) = 20$ and $f(2) = 11$. Find the values of the constants p , q and r .
Express $f(x)$ in the form $a(x+b)^2 + c$. Use your answer to find the smallest value of $f(x)$. (OCR, adapted)
- 23** Use the substitution $y = 3^x$ to find the values of x which satisfy the equation $3^{2x+2} - 10 \times 3^x + 1 = 0$.
- 24** Show that $\sqrt{N+1} - \sqrt{N} = \frac{1}{\sqrt{N+1} + \sqrt{N}}$. Use this to explain why $\sqrt{101}$ is close to, but slightly less than, 10.05.
Without using a calculator, find the roots of $x^2 + 7x - 13 = 0$, giving your answers correct to 2 decimal places.
- 25** If $ab^{0.4} = c$, express b in terms of a and c
- in index notation,
 - in surd notation.

6 Differentiation

This chapter is about finding the gradient of the tangent at a point on a graph. When you have completed it, you should be able to

- calculate an approximation to the gradient at a point on a curve, given its equation
- calculate the exact gradient at a point on a quadratic curve and certain other curves
- find the equations of the tangent and normal to a curve at a point.

This chapter is divided into two parts. In the first part, Sections 6.1 to 6.5, you will develop results experimentally and use them to solve problems about tangents to graphs. In the second part, Sections 6.6 and 6.7, the experimental results are proved. You may, if you wish, omit the second part of the chapter on a first reading, but you should still tackle Miscellaneous exercise 6 at the end of the chapter.

6.1 Calculating gradients of chords

Think of a simple curve such as the graph of $y = x^2$. As your eye moves along the x -axis, can you describe, in mathematical terms, how the direction of the curve changes?

Just as a straight line has a numerical gradient, so any curve, provided it is reasonably smooth, has a steepness or **gradient** which can be measured at any given point. The difference is that for the curve the gradient will change as you move along it; mathematicians use this gradient to describe the curve's direction.

In Chapter 1 you saw how to find the gradient of a straight line through two points when you know their coordinates. You cannot use this method directly on a curve because it is not a straight line. Instead you find the gradient of the **tangent** to the curve at the point you have chosen, since (as you can see from Fig. 6.1) the tangent has the same steepness as the curve at that point. However, this creates another difficulty; you can only find the gradient of a line if you know the coordinates of two points on it.

Fig. 6.2 shows three chords (straight lines through two points of the curve) which get closer and closer to the tangent line; it turns out that a good way to begin is by finding the gradient of these chords, because for these you *can* use the techniques of Chapter 1.

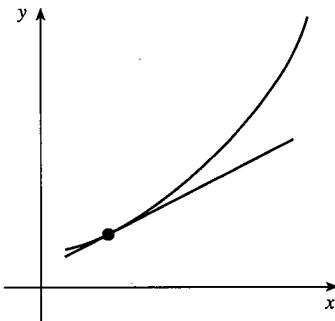


Fig. 6.1

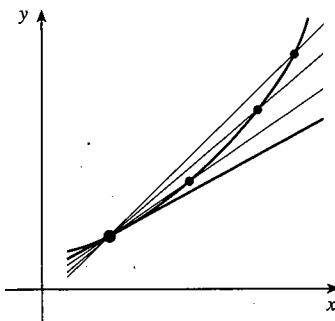


Fig. 6.2

Example 6.1.1

Find the gradient and the equation of the chord joining the points on the curve $y = x^2$ with coordinates $(0.4, 0.16)$ and $(0.7, 0.49)$.

From the formula in Section 1.3, the gradient of the chord is

$$\frac{0.49 - 0.16}{0.7 - 0.4} = \frac{0.33}{0.3} = 1.1.$$

The formula in Section 1.5 then gives the equation of the chord as

$$y - 0.16 = 1.1(x - 0.4), \text{ which is}\\ y = 1.1x - 0.28.$$

Fig. 6.3 shows that this is the equation of the whole line through the two points, not just the line segment between the two points which people often think of as the 'chord'.

At this point it is useful to introduce some new notation. The Greek letter δ (delta) is used as an abbreviation for 'the increase in'. Thus 'the increase in x ' is written as δx , and 'the increase in y ' as δy . These are the quantities called the ' x -step' and ' y -step' in Section 1.3. Thus in Example 6.1.1 from one end of the chord to the other the x -step is $0.7 - 0.4 = 0.3$ and the y -step is $0.49 - 0.16 = 0.33$, so you can write

$$\delta x = 0.3, \quad \delta y = 0.33.$$

With this notation, you can write the gradient of the chord as $\frac{\delta y}{\delta x}$.

Some people use the capital letter Δ rather than δ . Either is acceptable.

Notice that, in the fraction $\frac{\delta y}{\delta x}$, you cannot 'cancel out' the deltas; they do not stand for a number. While you are getting used to the notation it is a good idea to read δ as 'the increase in', so that you are not tempted to treat it as an ordinary algebraic symbol. Remember also that δx or δy could be negative, making the x -step or y -step a decrease.

Using this notation for the gradient of the chord, the first line of Example 6.1.1 would read

$$\frac{\delta y}{\delta x} = \frac{0.49 - 0.16}{0.7 - 0.4} = \frac{0.33}{0.3} = 1.1.$$

Example 6.1.2

Find the gradient of the chord joining the points on the curve $y = x^2$ with x -coordinates 0.4 and 0.41.

First you need to calculate the y -coordinates of the two points. They are $0.4^2 = 0.16$ and $0.41^2 = 0.1681$.

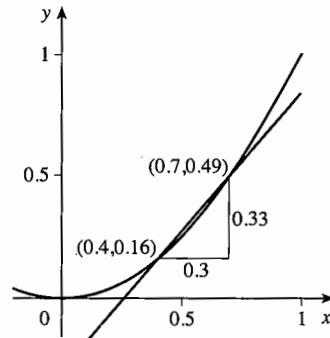


Fig. 6.3

Working in a similar way to Example 6.1.1,

$$\delta x = 0.41 - 0.4 = 0.01 \quad \text{and} \quad \delta y = 0.1681 - 0.16 = 0.0081,$$

so that the gradient of the chord is

$$\frac{\delta y}{\delta x} = \frac{0.0081}{0.01} = 0.81.$$

Fig. 6.4 is not very useful as an illustration, because the two points are so close together. There is a small triangle there, like the triangle in Fig. 6.3, but you could be excused for missing it.

In Fig. 6.4 it has become difficult to distinguish between the chord joining two points close together and the tangent at the point with x -coordinate 0.4.

This shows the way to find the gradient of the tangent at the point on the curve with $x = 0.4$.

In Example 6.1.3, the two points have become even closer together.

Example 6.1.3

Find the gradient of the chord joining the points on the curve $y = x^2$ with x -coordinates 0.4 and 0.400 01.

The coordinates of the two points are $(0.4, 0.4^2)$ and $(0.400\ 01, 0.400\ 01^2)$;

$$\delta x = 0.400\ 01 - 0.4 = 0.000\ 01 \quad \text{and} \quad \delta y = 0.400\ 01^2 - 0.4^2 = 0.000\ 008\ 000\ 1,$$

so that the gradient of the chord is

$$\frac{\delta y}{\delta x} = \frac{0.000\ 008\ 000\ 1}{0.000\ 01} = 0.800\ 01.$$

This result, being so close to 0.8, seems to indicate that the gradient of the tangent to the curve $y = x^2$ at $x = 0.4$ is 0.8. But it does not prove it, because you are still finding the equation of the chord joining two points, no matter how close those points are.

Exercise 6A

In Questions 2 and 3 the parts of questions could be divided, so that groups of students working together have answers to all the parts of each question, and can pool their results.

- 1 Find the equation of the line joining the points with x -values 1 and 2 on the graph $y = x^2$.
- 2 In each part of this question, find the gradient of the chord joining the two points with the given x -coordinates on the graph of $y = x^2$.

<small>(a) 1 and 1.001</small>	<small>(b) 1 and 0.9999</small>	<small>(c) 2 and 2.002</small>
<small>(d) 2 and 1.999</small>	<small>(e) 3 and 3.000 001</small>	<small>(f) 3 and 2.999 99</small>

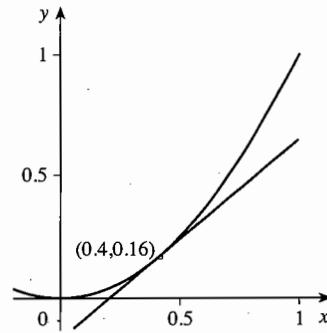


Fig. 6.4

- 3 In each part of this question find the gradient of the chord from the given point on the graph $y = x^2$ to a nearby point. Vary the distance between the point given and the nearby point; make sure that some of the points that you choose are to the left of the given point.
- (a) $(-1, 1)$ (b) $(-2, 4)$ (c) $(10, 100)$
- 4 Use the results of Questions 2 and 3 to make a guess about the gradient of the tangent at any point on the graph of $y = x^2$.
- 5 (a) Use a similar method to that of Questions 2 to 4 to make a guess about the gradients of the tangents at points on the graphs of $y = x^2 + 1$ and $y = x^2 - 2$.
- (b) Use the results from part (a) to make a generalisation about the gradient at any point on the graph of $y = x^2 + c$, where c is any real number.

6.2 The gradient of a tangent to the curve $y = x^2 + c$

If you collect the results from Exercise 6A, you should suspect that the gradient of the tangent to the curve $y = x^2$ at any point is twice the value of the x -coordinate of the point. Another way of saying this is that the gradient formula for the curve $y = x^2$ is $2x$.

For example, at the point $(-3, 9)$ on the curve $y = x^2$, the gradient is $2 \times (-3) = -6$.

This means that the tangent to the curve at this point is the straight line which has gradient -6 and passes through the point $(-3, 9)$.

To find the equation of this tangent, you can use the method in Section 1.5. The equation of the line is

$$\begin{aligned} y - 9 &= -6(x - (-3)), \quad \text{which is} \\ y - 9 &= -6x - 18, \quad \text{or } y = -6x - 9. \end{aligned}$$

You should also see that the gradient formula holds for the curve $y = x^2 + c$ where c is any constant: the gradient at x is also $2x$. After all, the curve $y = x^2 + c$ has the same shape as the curve $y = x^2$, but it is shifted in the y -direction.

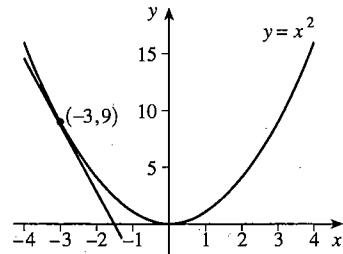


Fig. 6.5

Assume for the moment that these results can be proved. You will find proofs, when you need them, in Section 6.6.

6.3 The normal to a curve at a point

The line passing through the point of contact of the tangent with the curve which is perpendicular to the tangent is called the **normal** to the curve at that point.

Fig. 6.6 shows a curve with equation $y = f(x)$. The tangent and the normal at the point A have been drawn.

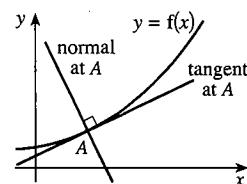


Fig. 6.6

If you know the gradient of the tangent at A , you can calculate the gradient of the normal by using the result of Section 1.9. If the gradient of the tangent is m , then the gradient of the normal is $-\frac{1}{m}$, provided that $m \neq 0$.

Example 6.3.1

Find the equation of the normal to the curve $y = x^2$ at the point for which

- (a) $x = -3$, (b) $x = 0$.

- (a) You found in Section 6.2 that the gradient of the tangent at $(-3, 9)$ is -6 .

The normal has a gradient of $-\frac{1}{-6} = \frac{1}{6}$ and also passes through $(-3, 9)$.

Therefore the equation of the normal is $y - 9 = \frac{1}{6}(x - (-3))$, which simplifies to $6y = x + 57$.

- (b) At $(0, 0)$, the gradient of the tangent is 0 , so the tangent is parallel to the x -axis. The normal is therefore parallel to the y -axis, so its equation is of the form $x = \text{something}$. As the normal passes through $(0, 0)$, its equation is $x = 0$.

If you have access to a graphic calculator, try displaying the curve $y = x^2$, the tangent $y = -6x - 9$ and the normal $6y = x + 57$ on it. You may be surprised by the results.

You should realise that if you draw a curve together with its tangent and normal at a point, the normal will only appear perpendicular in your diagram if the scales are the same on both the x - and y -axes. However, no matter what the scales are, the tangent will always appear as a tangent.

At this stage you should recognise that you need to generalise this result to curves with other equations. In Section 6.2 you saw that the gradient of the tangent at x to the curve $y = x^2 + c$ is $2x$.

Exercise 6B

For Questions 9 to 12 the parts of questions could be divided, so that groups of students working together have answers to all the parts of each question, and can pool their results.

- Find the gradient of the tangent to the graph of $y = x^2$, at each of the points with the given x -coordinate.

(a) 1	(b) 4	(c) 0	(d) -2
(e) -0.2	(f) -3.5	(g) p	(h) $2p$
- Find the gradient of the tangent to the graph of $y = x^2 - 2$, at each of the points with the given x -coordinate.

(a) 1	(b) 4	(c) 0	(d) -2
(e) -0.2	(f) -3.5	(g) p	(h) $2p$
- The y -coordinate of a point P on the graph of $y = x^2 + 5$ is 9. Find the two possible values of the gradient of the tangent to $y = x^2 + 5$ at P .

6.4 The gradient formula for quadratic graphs

Chapter 3 introduced the idea of the general quadratic graph, whose equation can be written $y = ax^2 + bx + c$, where a , b and c are constants. What can you say about the gradient of the tangent to this curve?

From the results of the questions in Exercise 6B you should have found that the gradient formula for $y = ax^2$ is $2ax$. This means, for example, that the graph of $y = 3x^2$ is three times as steep at any given value of x as the graph of $y = x^2$. You should also have found that the gradient formula for $y = x^2 + bx$ is $2x + b$. This means that the gradient formula for $y = x^2 + 4x$ is $2x + 4$, which is the sum of the gradient formulae for x^2 and $4x$.

You already know that $y = x^2$ and $y = x^2 + c$ have the same gradient formula whatever the value of c .

So it seems reasonable to expect that:

The gradient formula for the curve with equation
 $y = ax^2 + bx + c$ is $2ax + b$.

The importance of this result is that you can find the gradient of a function which is the sum of several parts by finding the gradient of each part in turn, and adding the results. You can also find the gradient of a constant multiple of a function by taking the same multiple of the gradient.

Section 6.6 will show how these results can be proved. Meanwhile here are some examples of their use. But first it will help to have some new notation.

Let the equation of a curve be $y = f(x)$. Then the gradient formula is denoted by $f'(x)$. This is pronounced ‘f dashed x’.

The process of finding the gradient of the tangent to a curve is called **differentiation**. When you find the gradient formula, you are **differentiating**.

Just as $f(2)$ stands for the value of the function where $x = 2$, so $f'(2)$ stands for the gradient of $y = f(x)$ at $x = 2$. Thus the dash in $f'(x)$ tells you to differentiate: you then substitute the value of x at which you wish to find the gradient.

The quantity $f'(2)$ is called the **derivative** of $f(x)$ at $x = 2$.

Thus to find the gradient at $x = 2$ on the curve with equation $y = f(x)$, find $f'(x)$, and then substitute $x = 2$ to get $f'(2)$.

Example 6.4.1

- (a) Differentiate $y = 3x^2 - 2x + 5$.
- (b) Find the equations of the tangent and the normal to the graph of $y = 3x^2 - 2x + 5$ at the point for which $x = 1$.

(a) Let $f(x) = 3x^2 - 2x + 5$. For this function $a = 3$, $b = -2$ and $c = 5$. So, differentiating, $f'(x) = 2 \times 3 \times x - 2 = 6x - 2$.

(b) The y -coordinate of the point on the curve for which $x = 1$ is $3 - 2 + 5 = 6$.

When $x = 1$ the gradient of the tangent is $f'(1) = 6 \times 1 - 2 = 4$.

The equation of the tangent is therefore $y - 6 = 4(x - 1)$, or $y = 4x + 2$.

The normal is perpendicular to the tangent, so its gradient is $-\frac{1}{4}$. The equation of the normal is therefore $y - 6 = -\frac{1}{4}(x - 1)$, which simplifies to $x + 4y = 25$.

Example 6.4.2

Differentiate (a) $f(x) = 2(x^2 - 3x - 2)$, (b) $g(x) = (x + 2)(2x - 3)$.

(a) **Method 1** Multiplying out the bracket,

$$f(x) = 2(x^2 - 3x - 2) = 2x^2 - 6x - 4, \text{ so } f'(x) = 4x - 6.$$

Method 2 If a given function is multiplied by a constant, the gradient of that function is multiplied by the same constant. In this case, the multiple is 2, so $f'(x) = 2(2x - 3)$.

(b) For $g(x) = (x + 2)(2x - 3)$, you cannot use method 2 of part (a) because the multiple is not constant. But you can multiply out the brackets to get a quadratic which you can then differentiate.

$$g(x) = (x + 2)(2x - 3) = 2x^2 + x - 6, \text{ so } g'(x) = 4x + 1.$$

If you cannot immediately differentiate a given function using the rules you know, see if you can write the function in a different form which enables you to apply one of the rules.

Example 6.4.3

Find the equation of the tangent to the graph of $y = x^2 - 4x + 2$ which is parallel to the x -axis.

From Section 1.6, a line parallel to the x -axis has gradient 0.

Let $f(x) = x^2 - 4x + 2$. Then $f'(x) = 2x - 4$.

To find when the gradient is 0 you need to solve $2x - 4 = 0$, giving $x = 2$.

When $x = 2$, $y = 2^2 - 4 \times 2 + 2 = -2$.

From Section 1.6, the equation of a line parallel to the x -axis has the form $y = c$. So the equation of the tangent is $y = -2$.

Exercise 6C

For Questions 13 to 16 the parts of questions could be divided, so that groups of students working together have answers to all the parts of each question, and can pool their results.

1 Find the gradient formula for each of the following functions.

- | | | | |
|--------------|--------------------|---------------------|-------------------------------|
| (a) x^2 | (b) $x^2 - x$ | (c) $4x^2$ | (d) $3x^2 - 2x$ |
| (e) $2 - 3x$ | (f) $x - 2 - 2x^2$ | (g) $2 + 4x - 3x^2$ | (h) $\sqrt{2}x - \sqrt{3}x^2$ |

2 For each of the following functions $f(x)$, find $f'(x)$. You may need to rearrange some of the functions before differentiating them.

- | | | | |
|------------------|-----------------------|-----------------|---------------------|
| (a) $3x - 1$ | (b) $2 - 3x^2$ | (c) 4 | (d) $1 + 2x + 3x^2$ |
| (e) $x^2 - 2x^2$ | (f) $3(1 + 2x - x^2)$ | (g) $2x(1 - x)$ | (h) $x(2x + 1) - 1$ |

3 Find the derivative of each of the following functions $f(x)$ at $x = -3$.

- | | | | |
|---------------------|------------------|-----------------|-----------------------|
| (a) $-x^2$ | (b) $3x$ | (c) $x^2 + 3x$ | (d) $2x - x^2$ |
| (e) $2x^2 + 4x - 1$ | (f) $-(3 - x^2)$ | (g) $-x(2 + x)$ | (h) $(x - 2)(2x - 1)$ |

4 For each of the following functions $f(x)$, find x such that $f'(x)$ has the given value.

- | | | | | | |
|--------------------|---|----------------------|----|--------------------|----|
| (a) $2x^2$ | 3 | (b) $x - 2x^2$ | -1 | (c) $2 + 3x + x^2$ | 0 |
| (d) $x^2 + 4x - 1$ | 2 | (e) $(x - 2)(x - 1)$ | 0 | (f) $2x(3x + 2)$ | 10 |

5 Find the equation of the tangent to the curve at the point with the given x -coordinate.

- | | |
|--------------------------------------|-----------------------------------|
| (a) $y = x^2$ where $x = -1$ | (b) $y = 2x^2 - x$ where $x = 0$ |
| (c) $y = x^2 - 2x + 3$ where $x = 2$ | (d) $y = 1 - x^2$ where $x = -3$ |
| (e) $y = x(2 - x)$ where $x = 1$ | (f) $y = (x - 1)^2$ where $x = 1$ |

6 Find the equation of the normal to the curve at the point with the given x -coordinate.

- | | |
|---|--|
| (a) $y = -x^2$ where $x = 1$ | (b) $y = 3x^2 - 2x - 1$ where $x = 1$ |
| (c) $y = 1 - 2x^2$ where $x = -2$ | (d) $y = 1 - x^2$ where $x = 0$ |
| (e) $y = 2(2 + x + x^2)$ where $x = -1$ | (f) $y = (2x - 1)^2$ where $x = \frac{1}{2}$ |

7 Find the equation of the tangent to the curve $y = x^2$ which is parallel to the line $y = x$.

8 Find the equation of the tangent to the curve $y = x^2$ which is parallel to the x -axis.

9 Find the equation of the tangent to the curve $y = x^2 - 2x$ which is perpendicular to the line $2y = x - 1$.

10 Find the equation of the normal to the curve $y = 3x^2 - 2x - 1$ which is parallel to the line $y = x - 3$.

11 Find the equation of the normal to the curve $y = (x - 1)^2$ which is parallel to the y -axis.

12 Find the equation of the normal to the curve $y = 2x^2 + 3x + 4$ which is perpendicular to the line $y = 7x - 5$.

13 Use an exploration method similar to that of Exercise 6B Questions 9 and 10 to make a guess about the gradient formulae for the graphs of $y = x^3$ and $y = x^4$.

14 In each part of this question, find the gradient of the chord joining the two points with the given x -coordinates on the graph of $y = \sqrt{x}$.

- | | | |
|-----------------|------------------------|------------------------|
| (a) 1 and 1.001 | (b) 1 and 0.9999 | (c) 4 and 4.002 |
| (d) 4 and 3.999 | (e) 0.25 and 0.250 001 | (f) 0.25 and 0.249 999 |

15 In each part of this question find the gradient of the chord from the given point to a nearby point for the curve $y = \frac{1}{x}$. Vary the distance between the given point and the nearby point which you choose; make sure that some of the points that you choose are to the left of the given point.

- | | | |
|--------------|----------------|---------------|
| (a) (-1, -1) | (b) (-2, -0.5) | (c) (10, 0.1) |
|--------------|----------------|---------------|

- 16** Use the results of Questions 14 and 15 to make a guess about the gradient of the tangent at any point on the graphs of $y = \sqrt{x}$ and $y = \frac{1}{x}$.

6.5 Some rules for differentiation

You already know the following rules:

If $f(x) = ax^2 + bx + c$, then $f'(x) = 2ax + b$.

If you add two functions, then the derivative of the sum is the sum of the derivatives: if $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$.

If you multiply a function by a constant, you multiply its derivative by the same constant: if $f(x) = ag(x)$, then $f'(x) = ag'(x)$.

You will have found from Exercise 6C Question 13 that the derivative of $f(x) = x^3$ is $f'(x) = 3x^2$, and the derivative of $f(x) = x^4$ is $f'(x) = 4x^3$. You already know that if $f(x) = x^2$, then $f'(x) = 2x$, or $2x^1$. This suggests the rule:

If $f(x) = x^n$, where n is a positive integer, then $f'(x) = nx^{n-1}$.

Example 6.5.1

Find the coordinates of the points on the graph of $y = x^3 - 3x^2 - 4x + 2$ at which the gradient is 5.

Let $f(x) = x^3 - 3x^2 - 4x + 2$. Then $f'(x) = 3x^2 - 6x - 4$. The gradient is 5 when $f'(x) = 5$, that is when $3x^2 - 6x - 4 = 5$. This gives the quadratic equation $3x^2 - 6x - 9 = 0$, which simplifies to $x^2 - 2x - 3 = 0$.

In factor form this is $(x+1)(x-3) = 0$, so $x = -1$ or $x = 3$.

Substituting these values into $y = x^3 - 3x^2 - 4x + 2$ to find the y -coordinates of the points, you find $y = (-1)^3 - 3 \times (-1)^2 - 4 \times (-1) + 2 = -1 - 3 + 4 + 2 = 2$ and $y = 3^3 - 3 \times 3^2 - 4 \times 3 + 2 = 27 - 27 - 12 + 2 = -10$. The coordinates of the required points are therefore $(-1, 2)$ and $(3, -10)$.

The results of Exercise 6C Questions 14 to 16 suggest two more rules:

If $f(x) = \sqrt{x}$, then $f'(x) = \frac{1}{2\sqrt{x}}$.

If $f(x) = \frac{1}{x}$, then $f'(x) = -\frac{1}{x^2}$.

In index notation, these results take the forms:

$$\text{If } f(x) = x^{\frac{1}{2}}, \text{ then } f'(x) = \frac{1}{2}x^{-\frac{1}{2}}.$$

$$\text{If } f(x) = x^{-1}, \text{ then } f'(x) = -x^{-2} = (-1)x^{-2}.$$

This suggests the following rule:

$$\text{If } f(x) = x^n, \text{ where } n \text{ is a rational number, then } f'(x) = nx^{n-1}.$$

Example 6.5.2

Find the equation of the tangent to the graph of $y = 2\sqrt{x}$ at the point where $x = 9$.

$$\text{Let } f(x) = 2\sqrt{x} = 2x^{\frac{1}{2}}.$$

Then, using results in the boxes,

$$f'(x) = 2 \times \frac{1}{2}x^{-\frac{1}{2}} = x^{-\frac{1}{2}}.$$

$$\text{When } x = 9, f'(9) = 9^{-\frac{1}{2}} = \frac{1}{\sqrt{9}} = \frac{1}{3}.$$

The tangent passes through the point $(9, 2\sqrt{9}) = (9, 6)$, so its equation is

$$y - 6 = \frac{1}{3}(x - 9), \quad \text{or} \quad 3y - x = 9.$$

Example 6.5.3

Differentiate each of the functions (a) $x(1+x^2)$, (b) $(1+\sqrt{x})^2$, (c) $\frac{x^2+x+1}{x}$.

$$(a) \text{ Let } f(x) = x(1+x^2).$$

$$\text{Then } f(x) = x + x^3, \text{ so } f'(x) = 1 + 3x^2.$$

$$(b) \text{ Let } f(x) = (1+\sqrt{x})^2.$$

$$\text{Then } f(x) = 1 + 2\sqrt{x} + x = 1 + 2x^{\frac{1}{2}} + x, \text{ so } f'(x) = 2 \times \frac{1}{2}x^{-\frac{1}{2}} + 1 = x^{-\frac{1}{2}} + 1 = \frac{1}{\sqrt{x}} + 1.$$

$$(c) \text{ Let } f(x) = \frac{x^2+x+1}{x}.$$

$$\text{Then, by division, } f(x) = x + 1 + \frac{1}{x} = x + 1 + x^{-1}, \text{ so } f'(x) = 1 + (-1)x^{-2} = 1 - \frac{1}{x^2}.$$

Example 6.5.4

Find the equation of the tangent to $y = \sqrt[3]{x}$ at the point $(8, 2)$.

In index notation $\sqrt[3]{x} = x^{\frac{1}{3}}$. So the rule gives the derivative as $\frac{1}{3}x^{\left(\frac{1}{3}-1\right)}$ or $\frac{1}{3}x^{-\frac{2}{3}}$,

which in surd notation is $\frac{1}{3(\sqrt[3]{x})^2}$. At $(8, 2)$, this is $\frac{1}{3(\sqrt[3]{8})^2} = \frac{1}{12}$.

Thus the equation of the tangent is $y - 2 = \frac{1}{12}(x - 8)$, or $x - 12y + 16 = 0$.

The results stated in this section can be assumed for the remainder of this book. Some of them are proved in Sections 6.6 and 6.7, but if you wish you may omit these final sections and, after working through Exercise 6D, go straight to Miscellaneous exercise 6.

Exercise 6D

- 1** Differentiate the following functions.

(a) $x^3 + 2x^2$	(b) $1 - 2x^3 + 3x^2$	(c) $x^3 - 6x^2 + 11x - 6$
(d) $2x^3 - 3x^2 + x$	(e) $2x^2(1 - 3x^2)$	(f) $(1 - x)(1 + x + x^2)$

- 2** Find $f'(-2)$ for each of the following functions $f(x)$.

(a) $2x - x^3$	(b) $2x - x^2$	(c) $1 - 2x - 3x^2 + 4x^3$
(d) $2 - x$	(e) $x^2(1 + x)$	(f) $(1 + x)(1 - x + x^2)$

- 3** For each of the following functions $f(x)$ find the value(s) of x such that $f'(x)$ is equal to the given number.

(a) x^3	12	(b) $x^3 - x^2$	8	(c) $3x - 3x^2 + x^3$	108
(d) $x^3 - 3x^2 + 2x$	-1	(e) $x(1 + x)^2$	0	(f) $x(1 - x)(1 + x)$	2

- 4** Differentiate the following functions.

(a) $2\sqrt{x}$	(b) $(1 + \sqrt{x})^2$	(c) $y = x - \frac{1}{2}\sqrt{x}$	(d) $x\left(1 - \frac{1}{\sqrt{x}}\right)^2$
(e) $x - \frac{1}{x}$	(f) $\frac{x^3 + x^2 + 1}{x}$	(g) $\frac{(x+1)(x+2)}{x}$	(h) $\left(\frac{\sqrt{x} + x}{\sqrt{x}}\right)^2$

- 5** Find the equation of the tangent to the curve $y = x^3 + x$ at the point for which $x = -1$.

- 6** One of the tangents to the curve with equation $y = 4x - x^3$ is the line with equation $y = x - 2$. Find the equation of the other tangent parallel to $y = x - 2$.

- 7** Find the equation of the tangent at the point $(4, 2)$ to the curve with equation $y = \sqrt{x}$.

- 8** Find the equation of the tangent at the point $(2, \frac{1}{2})$ to the curve with equation $y = \frac{1}{x}$.

- 9** Find the equation of the normal at the point $(1, 2)$ to the graph $y = x + \frac{1}{x}$.

- 10** The graphs of $y = x^2 - 2x$ and $y = x^3 - 3x^2 - 2x$ both pass through the origin. Show that they share the same tangent at the origin.

- 11 Find the equation of the tangent to the curve with equation $y = x^3 - 3x^2 - 2x - 6$ at the point where it crosses the y -axis.
- 12 A curve has equation $y = x(x-a)(x+a)$, where a is a constant. Find the equations of the tangents to the graph at the points where it crosses the x -axis.
- 13 Find the coordinates of the point of intersection of the tangents to the graph of $y = x^2$ at the points at which it meets the line with equation $y = x + 2$.
- 14 Differentiate each of these functions $f(x)$. Give your answers $f'(x)$ in a similar form, without negative or fractional indices.
- | | | | |
|--------------------|------------------------------|------------------------------------|-------------------------------|
| (a) $\frac{1}{4x}$ | (b) $\frac{3}{x^2}$ | (c) x^0 | (d) $\sqrt[4]{x^3}$ |
| (e) $6\sqrt[3]{x}$ | (f) $\frac{4}{\sqrt{x}}$ | (g) $\frac{3}{x} + \frac{1}{3x^3}$ | (h) $\sqrt{16x^5}$ |
| (i) $x\sqrt{x}$ | (j) $\frac{1}{\sqrt[3]{8x}}$ | (k) $\frac{x-2}{x^2}$ | (l) $\frac{1+x}{\sqrt[4]{x}}$ |
- 15 Find the equations of the tangent and the normal to $y = \sqrt[3]{x^2}$ at the point $(8, 4)$.
- 16 The tangent to the curve with equation $y = \frac{1}{x^2}$ at the point $(\frac{1}{2}, 4)$ meets the axes at P and Q . Find the coordinates of P and Q .

6.6* The gradient formula for any quadratic graph

If you wish, you can omit these final sections and go straight to Miscellaneous exercise 6.

The purpose of this section is to show you how to calculate the gradient of a quadratic graph without making any approximations.

Example 6.6.1

Find the gradient of the chord of $y = x^2$ joining the points with x -coordinates p and $p+h$.

The y -coordinates of the points are p^2 and $(p+h)^2$, so for this chord

$$\delta x = h, \quad \delta y = (p+h)^2 - p^2 = p^2 + 2ph + h^2 - p^2 = 2ph + h^2 = h(2p+h),$$

and the gradient is

$$\frac{\delta y}{\delta x} = \frac{h(2p+h)}{h} = 2p+h.$$

Notice that the gradients found in Examples 6.1.1 to 6.1.3 are special cases of this result, as shown in Table 6.7.

	p	$p + h$	h	$\frac{\delta y}{\delta x} = 2p + h$
Example 6.1.1	0.4	0.7	0.3	1.1
Example 6.1.2	0.4	0.41	0.01	0.81
Example 6.1.3	0.4	0.400 01	0.000 01	0.800 01

Table 6.7

The advantage of using algebra is that you don't have to work out the gradients each time from scratch. Table 6.8 shows some more results for $p = 0.4$ with different values of h , some positive and some negative.

Value of h	0.1	0.001	0.000 001	-0.1	-0.001	-0.000 001
Value of $\frac{\delta y}{\delta x}$	0.9	0.801	0.800 001	0.7	0.799	0.799 999

Table 6.8

If you have a graphic calculator, or some computer software for drawing graphs, it is interesting to produce a display showing the graph of $y = x^2$ and the chord through $(0.4, 0.16)$ with each of these gradients in turn. You will find that, when h is very close to 0, so that the two ends of the chord are very close together, it is almost impossible to distinguish the chord from the tangent to the curve. And you can see from Tables 6.7 and 6.8 that, for these chords, the gradient is very close to 0.8.

In fact, by taking h close enough to 0, you can make the gradient of the chord as close to 0.8 as you choose. From Example 6.6.1, the gradient of the chord is $0.8 + h$. So if you want to find a chord through $(0.4, 0.16)$ with a gradient between, say, 0.799 999 and 0.800 001, you can do it by taking h somewhere between -0.000 001 and +0.000 001.

The only value that you cannot take for h is 0 itself. But you can say that

'in the limit, as h tends to 0, the gradient of the chord tends to 0.8'.

The conventional way of writing this is

$$\lim_{h \rightarrow 0} (\text{gradient of chord}) = \lim_{h \rightarrow 0} (0.8 + h) = 0.8.$$

There is nothing special about taking p to be 0.4. You can use the same argument for any other value of p . Example 6.6.1 shows that the gradient of the chord joining (p, p^2) to $(p+h, (p+h)^2)$ is $2p + h$. If you keep p fixed, and let h take different values, then, by the same argument as before,

$$\lim_{h \rightarrow 0} (\text{gradient of chord}) = \lim_{h \rightarrow 0} (2p + h) = 2p.$$

Therefore the gradient at the point (p, p^2) on the curve $y = x^2$ is $2p$.

This shows that:

The gradient formula for the curve $y = x^2$ is $2x$.

A similar approach can be used for any curve if you know its equation.

Fig. 6.9 shows a curve which has an equation of the form $y = f(x)$. Suppose that you want the gradient of the tangent at the point P , with coordinates $(p, f(p))$. The chord joining this point to any other point Q on the curve with coordinates $(p+h, f(p+h))$ has

$$\delta x = h, \quad \delta y = f(p+h) - f(p)$$

so that its gradient is

$$\frac{\delta y}{\delta x} = \frac{f(p+h) - f(p)}{h}$$

Now let the value of h change so that the point Q takes different positions on the curve. Then, if Q is close to P , so that h is close to 0, the gradient of the chord is close to the gradient of the tangent at P , where $x = p$. In the limit, as h tends to 0, this expression tends to $f'(p)$.

If the curve $y = f(x)$ has a tangent at $(p, f(p))$, then its gradient is

$$\lim_{h \rightarrow 0} \frac{f(p+h) - f(p)}{h}$$

This quantity is called the derivative of $f(x)$ at $x = p$; it is denoted by $f'(p)$.

Since p can stand for any value of x , you can write:

The derivative of $f(x)$ for any value of x is $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

Example 6.6.2

Find the derivative of the function $f(x) = 4x - 5$.

From the definition, $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ with $f(x) = 4x - 5$.

The top line is

$$f(x+h) - f(x) = (4(x+h) - 5) - (4x - 5) = 4x + 4h - 5 - 4x + 5 = 4h.$$

Therefore

$$\frac{f(x+h) - f(x)}{h} = \frac{4h}{h} = 4.$$

Then in the limit, as h tends to 0,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} 4 = 4.$$

Of course you could have predicted this result. From the work of Chapter 1, the graph of $f(x) = 4x - 5$ is a straight line with gradient 4. So it should not be a surprise that the derivative of $f(x) = 4x - 5$ is 4.

Similarly, you would expect the gradient of the function $f(x) = mx + c$, whose graph is the straight line $y = mx + c$, to be given by $f'(x) = m$.

Example 6.6.3

Find the derivative of the function $f(x) = 3x^2$.

For this function, $f(x+h) = 3(x+h)^2 = 3x^2 + 6xh + 3h^2$, so

$$f(x+h) - f(x) = (3x^2 + 6xh + 3h^2) - 3x^2 = 6xh + 3h^2 = h(6x + 3h)$$

$$\text{and } \frac{f(x+h) - f(x)}{h} = \frac{h(6x + 3h)}{h} = 6x + 3h.$$

Then in the limit, as h tends to 0,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} (6x + 3h) = 6x.$$

Notice that the derivative of $f(x) = x^2$ is $2x$ and the derivative of $f(x) = 3x^2$ is $6x$, which is $3 \times 2x$. This is an example of a general rule first seen in Section 6.5:

If you multiply a function by a constant, then you multiply its derivative by the same constant.

Example 6.6.4

Find the derivative of the function $f(x) = 3x^2 + 4x - 5$.

For the function $f(x) = 3x^2 + 4x - 5$,

$$f(x+h) = 3(x+h)^2 + 4(x+h) - 5 = 3x^2 + 6xh + 3h^2 + 4x + 4h - 5,$$

$$\begin{aligned} \text{so } f(x+h) - f(x) &= (3x^2 + 6xh + 3h^2 + 4x + 4h - 5) - (3x^2 + 4x - 5) \\ &= 6xh + 3h^2 + 4x + 4h - 5 - 3x^2 - 4x + 5 \\ &= 6xh + 3h^2 + 4h = h(6x + 3h + 4), \end{aligned}$$

$$\text{and } \frac{f(x+h) - f(x)}{h} = \frac{h(6x + 3h + 4)}{h} = 6x + 3h + 4.$$

Then in the limit, as h tends to 0,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} (6x + 3h + 4) = 6x + 4.$$

Examples 6.6.2 to 6.6.4 illustrate another general rule. The function in Example 6.6.4 is the sum of the functions in Examples 6.6.2 and 6.6.3, and the gradient in Example 6.6.4 is the sum of the gradients in Examples 6.6.2 and 6.6.3.

The general rule is:

If you add two functions, then you find the derivative of the resulting function by adding the derivatives of the individual functions.

6.7* The gradient formula for some other functions

For some functions the method used in Section 6.6 lands you in some tricky algebra, and it is easier to use a different notation. Instead of finding the gradient of the chord joining the points with x -coordinates p and $p+h$ (or x and $x+h$), you can take the points to have coordinates $(p, f(p))$ and $(q, f(q))$, so that

$$\delta x = q - p \quad \text{and} \quad \delta y = f(q) - f(p).$$

Then the gradient is $\frac{\delta y}{\delta x} = \frac{f(q) - f(p)}{q - p}$.

To see how this works, here is Example 6.6.3 worked in this notation.

Example 6.7.1

Find the derivative at $x = p$ of the function $f(x) = 3x^2$.

For this function, $f(q) - f(p) = 3q^2 - 3p^2 = 3(q^2 - p^2) = 3(q-p)(q+p)$.

$$\text{So } \frac{\delta y}{\delta x} = \frac{f(q) - f(p)}{q - p} = \frac{3(q - p)(q + p)}{q - p} = 3(q + p).$$

Now, in this method, q has taken the place of $p + h$, so that instead of taking the limit ‘as h tends to 0’ you take it ‘as q tends to p ’. It is easy to see that, as q tends to p , $3(q + p)$ tends to $3(p + p) = 3(2p) = 6p$.

Therefore, if $f(x) = 3x^2$, $f'(p) = 6p$. Since this holds for any value of p , you can write $f'(x) = 6x$.

In this notation, the definition of the derivative takes the form:

The derivative of $f(x)$ at $x = p$ is $f'(p) = \lim_{q \rightarrow p} \frac{f(q) - f(p)}{q - p}$.

Example 6.7.2

Find the derivative of the function $f(x) = x^4$ at $x = p$.

At $x = p$, $f(p) = p^4$, and at $x = q$, $f(q) = q^4$. The chord joining (p, p^4) and (q, q^4) has $\delta x = q - p$, $\delta y = q^4 - p^4$.

Notice that you can write δy as $(q^2)^2 - (p^2)^2$, so you can use the difference of two squares twice to obtain

$$\delta y = (q^2 - p^2)(q^2 + p^2) = (q - p)(q + p)(q^2 + p^2).$$

Therefore

$$\frac{\delta y}{\delta x} = \frac{(q - p)(q + p)(q^2 + p^2)}{q - p} = (q + p)(q^2 + p^2).$$

Then in the limit, as q tends to p ,

$$f'(p) = \lim_{q \rightarrow p} \frac{f(q) - f(p)}{q - p} = \lim_{q \rightarrow p} ((q + p)(q^2 + p^2)) = 2p(2p^2) = 4p^3.$$

Example 6.7.3

Find the derivative of the function $f(x) = \sqrt{x}$ at $x = p$.

At $x = p$, $f(p) = \sqrt{p}$, and at $x = q$, $f(q) = \sqrt{q}$. The chord joining (p, \sqrt{p}) and (q, \sqrt{q}) has $\delta x = q - p$, $\delta y = \sqrt{q} - \sqrt{p}$.

Notice that you can write δx as the difference of two squares in the form

$$\delta x = q - p = (\sqrt{q})^2 - (\sqrt{p})^2 = (\sqrt{q} - \sqrt{p})(\sqrt{q} + \sqrt{p}).$$

$$\text{So } \frac{\delta y}{\delta x} = \frac{\sqrt{q} - \sqrt{p}}{(\sqrt{q} - \sqrt{p})(\sqrt{q} + \sqrt{p})} = \frac{1}{\sqrt{q} + \sqrt{p}}.$$

Then in the limit, as q tends to p ,

$$\begin{aligned} f'(p) &= \lim_{q \rightarrow p} \frac{f(q) - f(p)}{q - p} \\ &= \lim_{q \rightarrow p} \frac{1}{\sqrt{q} + \sqrt{p}} \\ &= \frac{1}{\sqrt{p} + \sqrt{p}} = \frac{1}{2\sqrt{p}} = \frac{1}{2}p^{-\frac{1}{2}}. \end{aligned}$$

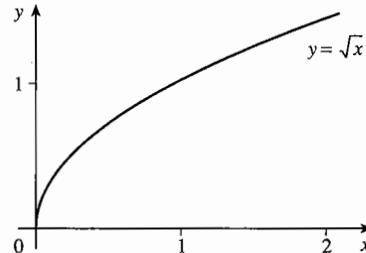


Fig. 6.10

Notice that this does not work when $p = 0$. In this case $\frac{\delta y}{\delta x} = \frac{1}{\sqrt{q}}$, which does not have any limit as $q \rightarrow 0$. You can see from the graph of $y = \sqrt{x}$ in Fig. 6.10 that the tangent at $x = 0$ is the y -axis, which does not have a gradient.

Example 6.7.4

Find the derivative of the function $f(x) = \frac{1}{x}$ at $x = p$.

At $x = p$, $f(p) = \frac{1}{p}$, and at $x = q$, $f(q) = \frac{1}{q}$.

The chord joining $\left(p, \frac{1}{p}\right)$ and $\left(q, \frac{1}{q}\right)$ has $\delta x = q - p$, $\delta y = \frac{1}{q} - \frac{1}{p} = \frac{p-q}{qp} = -\frac{q-p}{qp}$,

$$\text{and } \frac{\delta y}{\delta x} = \frac{-\left(\frac{q-p}{qp}\right)}{q-p} = -\frac{1}{qp}.$$

Then, in the limit as q tends to p ,

$$f'(p) = \lim_{q \rightarrow p} \frac{f(q) - f(p)}{q - p} = \lim_{q \rightarrow p} \left(-\frac{1}{qp} \right) = -\frac{1}{p^2} = -p^{-2}.$$

If you have access to a graphic calculator, display this curve and see why the gradient is always negative.

Exercise 6E

Use the method of Section 6.7 in this exercise.

- 1 Find the derivative of the function $f(x) = x^3$ at $x = p$.

(You will need to use either the expansion $(p+h)^3 = p^3 + 3p^2h + 3ph^2 + h^3$ or the product of factors $(q-p)(q^2 + qp + p^2) = q^3 - p^3$.)

- 2 Find the derivative of the function $f(x) = x^8$ at $x = p$. (Let $p+h = q$ and use the difference of two squares formula on $q^8 - p^8$ as often as you can.)

- 3 Find the derivative of the function $f(x) = \frac{1}{x^2}$ at $x = p$.

Miscellaneous exercise 6

- 1 Find the equation of the tangent to $y = 5x^2 - 7x + 4$ at the point (2,10).
- 2 Given the function $f(x) = x^3 + 5x^2 - x - 4$, find
 - (a) $f'(-2)$,
 - (b) the values of a such that $f'(a) = 56$.
- 3 Find the equation of the normal to $y = x^4 - 4x^3$ at the point for which $x = \frac{1}{2}$.
- 4 Show that the equation of the tangent to $y = \frac{1}{x}$ at the point for which $x = p$ is $p^2y + x = 2p$. At what point on the curve is the equation of the tangent $9y + x + 6 = 0$?
- 5 The tangent to the curve $y = 6\sqrt{x}$ at the point (4,12) meets the axes at A and B . Show that the distance AB may be written in the form $k\sqrt{13}$, and state the value of k .
- 6 Find the coordinates of the two points on the curve $y = 2x^3 - 5x^2 + 9x - 1$ at which the gradient of the tangent is 13.
- 7 Find the equation of the normal to $y = (2x - 1)(3x + 5)$ at the point (1,8). Give your answer in the form $ax + by + c = 0$, where a , b and c are integers.
- 8 The curve $y = x^2 - 3x - 4$ crosses the x -axis at P and Q . The tangents to the curve at P and Q meet at R . The normals to the curve at P and Q meet at S . Find the distance RS .
- 9 The equation of a curve is $y = 2x^2 - 5x + 14$. The normal to the curve at the point (1,11) meets the curve again at the point P . Find the coordinates of P .
- 10 At a particular point of the curve $y = x^2 + k$, the equation of the tangent is $y = 6x - 7$. Find the value of the constant k .
- 11 Show that the curves $y = x^3$ and $y = (x+1)(x^2 + 4)$ have exactly one point in common, and use differentiation to find the gradient of each curve at this point. (OCR)
- 12 At a particular point of the curve $y = 5x^2 - 12x + 1$ the equation of the normal is $x + 18y + c = 0$. Find the value of the constant c .
- 13 The graphs of $y = x^m$ and $y = x^n$ intersect at the point $P(1,1)$. Find the connection between m and n if the tangent at P to each curve is the normal to the other curve.
- 14 The tangents at $x = \frac{1}{4}$ to $y = \sqrt{x}$ and $y = \frac{1}{\sqrt{x}}$ meet at P . Find the coordinates of P .
- 15 The normals at $x = 2$ to $y = \frac{1}{x^2}$ and $y = \frac{1}{x^3}$ meet at Q . Find the coordinates of Q .

7 Applications of differentiation

In the last chapter you learnt what differentiation means and how to differentiate a lot of functions. This chapter shows how you can use differentiation to sketch graphs and apply it to real-world problems. When you have completed it, you should

- understand that the derivative of a function is itself a function
- appreciate the significance of positive, negative and zero derivatives
- be able to locate maximum and minimum points on graphs
- know that you can interpret a derivative as a rate of change of one variable with respect to another
- be familiar with the notation $\frac{dy}{dx}$ for a derivative
- be able to apply these techniques to solve real-world problems.

7.1 Derivatives as functions

In Chapter 6 you were introduced to differentiation by carrying out a number of ‘explorations’. For example, in Exercise 6A Question 5(a) you were asked to make guesses about the gradient of the tangent at various points on the graph of $y = f(x)$, where $f(x) = x^2 - 2$. Table 7.1 shows the results you were expected to get.

p	-2	-1	1	2	3	10
$f(p)$	2	-1	-1	2	7	98
$f'(p)$	-4	-2	2	4	6	20

Table 7.1

What this suggests is that the gradient is also a function of x , given by the formula $2x$. In Chapter 6 this formula was called the derivative. But when you are thinking of it as a function rather than using its value for a particular x , it is sometimes called the **derived function**. It is denoted by $f'(x)$, and in this example $f'(x) = 2x$.

Also, just as you can draw the graph of the function $f(x)$, so it is possible to draw the graph of the derived function $f'(x)$. It is interesting to show these two graphs aligned one above the other on the page, as in Fig. 7.2.

On the left half of the graph, where $x < 0$, the graph of $f'(x)$ is below the x -axis, indicating that the gradient of $f(x)$ is negative. On the right, where the gradient of $f(x)$ is positive, the graph of $f'(x)$ is above the x -axis.

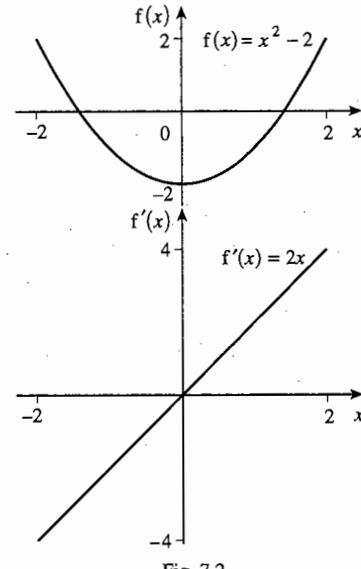


Fig. 7.2

You can now write down what you know about differentiation in terms of the derived function:

If $f(x) = x^n$, where n is a rational number, then $f'(x) = nx^{n-1}$.

The derived function of $f(x) + g(x)$ is $f'(x) + g'(x)$.

The derived function of $cf(x)$, where c is a constant, is $cf'(x)$.

Example 7.1.1

Find the derived function of $f(x) = x^2 - \frac{1}{3}x^3$.

Using the results from the box, the derived function is $f'(x) = 2x - x^2$.

The graphs of $f(x)$ and $f'(x)$ in Example 7.1.1 are drawn in Fig. 7.3. Here are some points to notice.

When $x < 0$ the gradient of the graph of $f(x) = x^2 - \frac{1}{3}x^3$ is negative, and the values of the derived function are also negative.

When $x = 0$, the gradient of $f(x)$ is 0, and the value of $f'(0)$ is 0.

Between $x = 0$ and $x = 2$ the gradient of $f(x)$ is positive, and $f'(x)$ is positive.

When $x = 2$, the gradient of $f(x)$ is 0, and $f'(2) = 0$.

When $x > 2$, the gradient of $f(x)$ is negative, and the values of the derived function are also negative.

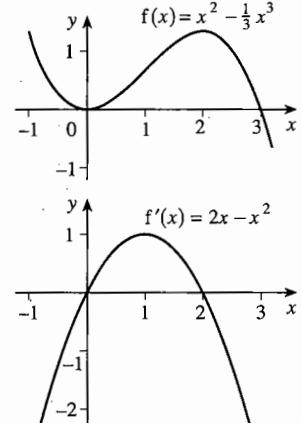


Fig. 7.3

Exercise 7A

1 Draw and compare the graphs of $y = f(x)$ and $y = f'(x)$ in each of the following cases.

(a) $f(x) = 4x$

(b) $f(x) = 3 - 2x$

(c) $f(x) = x^2$

(d) $f(x) = 5 - x^2$

(e) $f(x) = x^2 + 4x$

(f) $f(x) = 3x^2 - 6x$

2 Draw and compare the graphs of $y = f(x)$ and $y = f'(x)$ in each of the following cases.

(a) $f(x) = (2 + x)(4 - x)$

(b) $f(x) = (x + 3)^2$

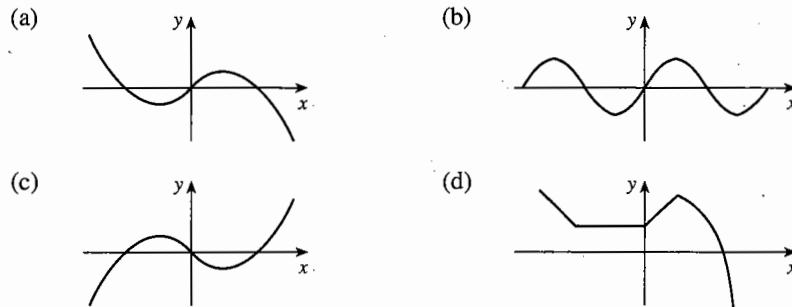
(c) $f(x) = x^4$

(d) $f(x) = x^2(x - 2)$

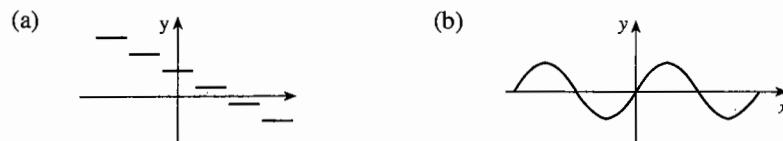
(e) $f(x) = \sqrt{x}$ for $x \geq 0$

(f) $f(x) = \frac{1}{x}$ for $x \neq 0$

- 3 In each part of the question, the diagram shows the graph of $y = f(x)$. Draw a graph of the derived function $y = f'(x)$.



- 4 In each part of the question, the diagram shows the graph of the derived function $y = f'(x)$. Draw a possible graph of $y = f(x)$.



7.2 Increasing and decreasing functions

For simplicity, the word ‘function’ in this chapter will mean functions which are **continuous** within their domains. This includes all the functions you have met so far, but cuts out functions such as ‘the fractional part of x ’, which is defined for all positive real numbers but whose graph (shown in Fig. 7.4) has jumps in it.

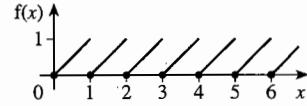


Fig. 7.4

You can use the idea that the derivative of a function is itself a function to investigate the shape of a graph from its equation.

Example 7.2.1

Find the interval in which $f(x) = x^2 - 6x + 4$ is increasing, and the interval in which it is decreasing.

The derivative is $f'(x) = 2x - 6 = 2(x - 3)$. This means that the graph has a positive gradient for $x > 3$. That is, $f(x)$ is increasing for $x > 3$.

For $x < 3$ the gradient is negative, and the values of y are getting smaller as x gets larger. That is, $f(x)$ is decreasing for $x < 3$.

The results are illustrated in Fig. 7.5 on the next page.

What about $x = 3$ itself? At first sight you might think that this has to be left out of both the increasing and the decreasing intervals, but this would be wrong! If you imagine moving along the curve from left to right, then as soon as you have passed through $x = 3$ the gradient becomes positive and the curve starts to climb. However close you are to $x = 3$, the value of y is greater than $f(3) = -5$. So you can say that $f(x)$ is increasing for $x \geq 3$; similarly, it is decreasing for $x \leq 3$.

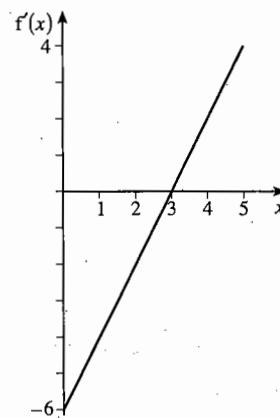
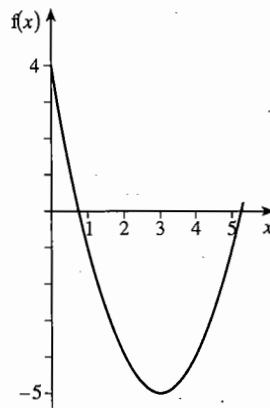


Fig. 7.5

You can use the reasoning in Example 7.2.1 for any function. Fig. 7.6 shows the graph of a function $y = f(x)$ whose derivative $f'(x)$ is positive in an interval $p < x < q$. You can see that larger values of y are associated with larger values of x . More precisely, if x_1 and x_2 are two values of x in the interval $p \leq x \leq q$, and if $x_2 > x_1$, then $f(x_2) > f(x_1)$. A function with this property is said to be **increasing** over the interval $p \leq x \leq q$.

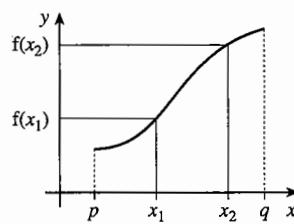


Fig. 7.6

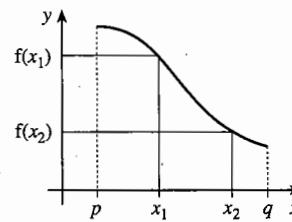


Fig. 7.7

If $f'(x)$ is negative in the interval $p < x < q$, as in Fig. 7.7, the function has the opposite property; if $x_2 > x_1$, then $f(x_2) < f(x_1)$. A function with this property is **decreasing** over the interval $p \leq x \leq q$.

If $f''(x) > 0$ in an interval $p < x < q$, then $f(x)$ is increasing over the interval $p \leq x \leq q$.

If $f''(x) < 0$ in $p < x < q$, then $f(x)$ is decreasing over $p \leq x \leq q$.

Notice that, for $f(x)$ to be increasing for $p \leq x \leq q$, the gradient $f'(x)$ does not have to be positive at the ends of the interval, where $x = p$ or $x = q$. At these points it may be 0, or even undefined. This may seem a minor distinction, but it has important consequences. It is a pay-off from the decision to work only with continuous functions.

The word 'interval' is used not only for values of x between finite end points, but also for values of x satisfying inequalities $x > p$ or $x < q$, which extend indefinitely in either the positive or negative direction.

Example 7.2.2

For the function $f(x) = x^4 - 4x^3$, find the intervals in which $f(x)$ is increasing and those in which it is decreasing.

Begin by expressing $f'(x)$ in factors as

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3).$$

As x^2 is always positive (except at $x = 0$), to find where $f'(x) > 0$ you need only solve the inequality $x - 3 > 0$, giving $x > 3$. So $f(x)$ is increasing over the interval $x \geq 3$, now including the end point.

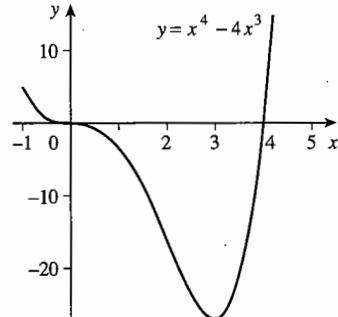


Fig. 7.8

The solution of $x - 3 < 0$ is $x < 3$; but to find where $f'(x) < 0$ you have to exclude $x = 0$, so that $f'(x) < 0$ if $x < 0$ or $0 < x < 3$. Therefore $f(x)$ is decreasing over the intervals $x \leq 0$ and $0 \leq x \leq 3$.

However, these last two intervals have the value $x = 0$ in common, so you can combine them as a single interval $x \leq 3$. It follows that $f(x)$ is decreasing over the interval $x \leq 3$.

Note also that $f'(x) = 0$ when $x = 0$ and $x = 3$. You can check all these properties from the graph of $y = f(x)$ shown in Fig. 7.8.

Example 7.2.2 shows that the rule given above, connecting the sign of $f'(x)$ with the property that $f(x)$ is increasing or decreasing, can be slightly relaxed.

If $f''(x) > 0$ in an interval $p < x < q$ except at isolated points where $f''(x) = 0$, then $f(x)$ is increasing in the interval $p \leq x \leq q$.

If $f''(x) < 0$ in an interval $p < x < q$ except at isolated points where $f''(x) = 0$, then $f(x)$ is decreasing over $p \leq x \leq q$.

The next example is about a function which involves fractional powers of x for $x < 0$. Fractional powers sometimes present problems, because some of them are not defined when x is negative. But the indices in this example involve only cube roots. There is no difficulty in taking the cube root of a negative number.

Example 7.2.3

Find the intervals in which the function $f(x) = x^{\frac{2}{3}}(1-x)$ is increasing and those in which it is decreasing.

To differentiate, write the function as $f(x) = x^{\frac{2}{3}} - x^{\frac{5}{3}}$, so that

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}} - \frac{5}{3}x^{\frac{2}{3}},$$

which you can write as

$$f'(x) = \frac{1}{3}x^{-\frac{1}{3}}(2-5x).$$

In this last expression, $x^{-\frac{1}{3}}$ is positive when $x > 0$ and negative when $x < 0$. The factor $2-5x$ is positive when $x < 0.4$ and negative when $x > 0.4$. Fig. 7.9 shows that

$f(x)$ is increasing in the interval $0 \leq x \leq 0.4$,

$f(x)$ is decreasing in the intervals $x \leq 0$ and $x \geq 0.4$.

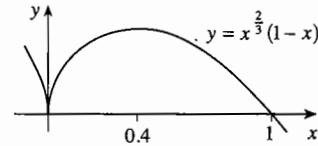


Fig. 7.9

7.3 Maximum and minimum points

Example 7.2.1 showed that, for $f(x) = x^2 - 6x + 4$, $f(x)$ is decreasing for $x \leq 3$ and increasing for $x \geq 3$. It follows from the definition of decreasing and increasing functions that, if $x_1 < 3$, then $f(x_1) > f(3)$; and that, if $x_2 > 3$, then $f(x_2) > f(3)$. That is, for every value of x other than 3, $f(x)$ is greater than $f(3) = -5$. You can say that $f(3)$ is the minimum value of $f(x)$, and that $(3, -5)$ is the minimum point on the graph of $y = f(x)$.

A minimum point does not have to be the lowest point on the whole graph, but is the lowest point in its immediate neighbourhood. In Fig. 7.9, $(0,0)$ is a minimum point; this is shown by the fact that $f(x) > 0$ for every number $x < 1$ except $x = 0$ although $f(x) < 0$ when $x > 1$.

This leads to a definition, which is illustrated by Fig. 7.10.

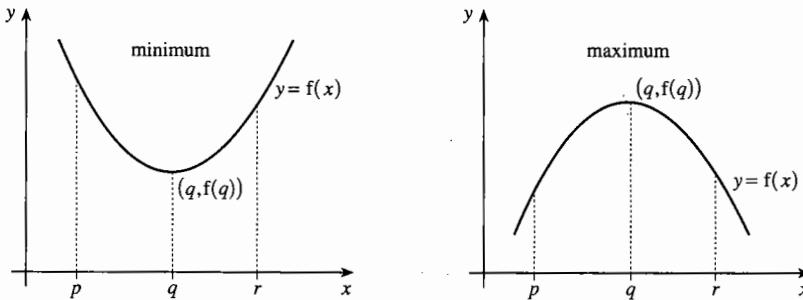


Fig. 7.10

A function $f(x)$ has a **minimum** at $x = q$ if there is an interval $p < x < r$ containing q in which $f(x) > f(q)$ for every value of x except q .

It has a **maximum** if $f(x) < f(q)$ for every value of x in the interval except q .

The point $(q, f(q))$ is called a **minimum point**, or a **maximum point**.

Thus, in Example 7.2.3, $f(x)$ has a minimum at $x = 0$, and a maximum at $x = 0.4$.

Minimum and maximum points are sometimes also called **turning points**.

You will see that at the minimum point in Fig. 7.8 and the maximum point in Fig. 7.9 the graph has gradient 0. But at the minimum point $(0,0)$ in Fig. 7.9 the tangent to the graph is the y -axis, so that the gradient is undefined. These examples illustrate a general rule:

If $(q, f(q))$ is a minimum or maximum point of the graph of $y = f(x)$, then either $f'(q) = 0$ or $f'(q)$ is undefined.

Notice, though, that Fig. 7.8 has another point at which the gradient is 0, which is neither a minimum nor a maximum, namely the point $(0,0)$. A point of a graph where the gradient is 0 is called a **stationary point**. So Figs. 7.8 and 7.9 illustrate the fact that a stationary point may be a minimum or maximum point, but may be neither.

A way to decide between a minimum and a maximum point is to find the sign of the gradient $f'(x)$ on either side of $x = q$. To follow the details you may again find it helpful to refer to the graphs in Fig. 7.10.

If $f'(x) < 0$ in an interval $p < x < q$, and $f'(x) > 0$ in an interval $q < x < r$, then $(q, f(q))$ is a minimum point.

If $f'(x) > 0$ in $p < x < q$, and $f'(x) < 0$ in $q < x < r$, then $(q, f(q))$ is a maximum point.

You may be happy to accept this on the evidence of the graphs, but it can also be argued from statements which you have already met. Consider the minimum case.

Suppose that x_1 is a number in $p < x < q$. Then, since $f'(x) < 0$ in that interval, it follows from Section 7.2 that $f(x_1) > f(q)$.

Now suppose that x_2 is a number in $q < x < r$. Since $f'(x) > 0$ in that interval, $f(q) < f(x_2)$.

This shows that, if x is any number in the interval $p < x < r$ other than q , then $f(x) > f(q)$. From the definition, this means that $f(x)$ has a minimum at $x = q$.

All these results can be summed up as a procedure.

To find the minimum and maximum points on the graph of $y = f(x)$:

Step 1 Decide the domain in which you are interested.

Step 2 Find an expression for $f'(x)$.

Step 3 List the values of x in the domain for which $f'(x)$ is either 0 or undefined.

Step 4 Taking each of these values of x in turn, find the sign of $f'(x)$ in intervals immediately to the left and to the right of that value.

Step 5 If these signs are – and + respectively, the graph has a minimum point. If they are + and – it has a maximum point. If the signs are the same, it has neither.

Step 6 For each value of x which gives a minimum or maximum, calculate $f(x)$.

Example 7.3.1

Find the minimum point on the graph with equation $y = \sqrt{x} + \frac{4}{x}$.

$$\text{Let } f(x) = \sqrt{x} + \frac{4}{x}.$$

Step 1 As \sqrt{x} is defined for $x \geq 0$ but $\frac{1}{x}$ is not defined for $x = 0$, the largest possible domain for $f(x)$ is the positive real numbers.

Step 2 The derivative $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} - 4x^{-2}$ can be written as $f'(x) = \frac{x^{\frac{3}{2}} - 8}{2x^2}$.

Step 3 The derivative is defined for all positive real numbers, and is 0 when $x^{\frac{3}{2}} = 8$. Raising both sides to the power $\frac{2}{3}$ and using the power-on-power rule,

$$x = (x^{\frac{3}{2}})^{\frac{2}{3}} = 8^{\frac{2}{3}} = 4$$
.

Step 4 If $0 < x < 4$, the bottom line, $2x^2$, is positive, and

$$x^{\frac{3}{2}} - 8 < 4^{\frac{3}{2}} - 8 = 8 - 8 = 0, \text{ so that } f'(x) < 0.$$

If $x > 4$, $2x^2$ is still positive, but $x^{\frac{3}{2}} - 8 > 4^{\frac{3}{2}} - 8 = 0$, so that $f'(x) > 0$.

Step 5 The sign of $f'(x)$ is – on the left of 4 and + on the right, so the function has a minimum at $x = 4$.

Step 6 Calculate $f(4) = \sqrt{4} + \frac{4}{4} = 2 + 1 = 3$. The minimum point is $(4, 3)$.

If you have a graphic calculator, it is interesting to use it to display $y = f(x)$ together with $y = \sqrt{x}$ and $y = \frac{4}{x}$, from which it is made up. You will find that $y = f(x)$ is very flat around the minimum; it would be difficult to tell by eye exactly where the minimum occurs.

Notice that this theory gives you another way to find the range of some functions. For the function in Example 7.3.1 with domain $x > 0$, the range is $y \geq 3$.

Exercise 7B

- 1 For each of the following functions $f(x)$, find $f'(x)$ and the interval in which $f(x)$ is increasing.

(a) $x^2 - 5x + 6$	(b) $x^2 + 6x - 4$	(c) $7 - 3x - x^2$
(d) $3x^2 - 5x + 7$	(e) $5x^2 + 3x - 2$	(f) $7 - 4x - 3x^2$

- 2 For each of the following functions $f(x)$, find $f'(x)$ and the interval in which $f(x)$ is decreasing.

(a) $x^2 + 4x - 9$	(b) $x^2 - 3x - 5$	(c) $5 - 3x + x^2$
(d) $2x^2 - 8x + 7$	(e) $4 + 7x - 2x^2$	(f) $3 - 5x - 7x^2$

- 3 For each of the following functions $f(x)$, find $f'(x)$ and any intervals in which $f(x)$ is increasing.

(a) $x^3 - 12x$	(b) $2x^3 - 18x + 5$	(c) $2x^3 - 9x^2 - 24x + 7$
(d) $x^3 - 3x^2 + 3x + 4$	(e) $x^4 - 2x^2$	(f) $x^4 + 4x^3$
(g) $3x - x^3$	(h) $2x^5 - 5x^4 + 10$	(i) $3x + x^3$

- 4 For each of the following functions $f(x)$, find $f'(x)$ and any intervals in which $f(x)$ is decreasing. In part (i), n is an integer.

(a) $x^3 - 27x$ for $x \geq 0$	(b) $x^4 + 4x^2 - 5$ for $x \geq 0$	(c) $x^3 - 3x^2 + 3x - 1$
(d) $12x - 2x^3$	(e) $2x^3 + 3x^2 - 36x - 7$	(f) $3x^4 - 20x^3 + 12$
(g) $36x^2 - 2x^4$	(h) $x^5 - 5x$	(i) $x^n - nx$ ($n > 1$)

- 5 For each of the following functions $f(x)$, find $f'(x)$, the intervals in which $f(x)$ is decreasing, and the intervals in which $f(x)$ is increasing.

(a) $x^{\frac{3}{2}}(x-1)$, for $x > 0$	(b) $x^{\frac{3}{4}} - 2x^{\frac{7}{4}}$, for $x > 0$	(c) $x^{\frac{2}{3}}(x+2)$
(d) $x^{\frac{3}{2}}(x^2 - 13)$	(e) $x + \frac{3}{x}$, for $x \neq 0$	(f) $\sqrt{x} + \frac{1}{\sqrt{x}}$, for $x > 0$

- 6 For the graphs of each of the following functions:

- (i) find the coordinates of the stationary point;
- (ii) say, with reasoning, whether this is a maximum or a minimum point;
- (iii) check your answer by using the method of ‘completing the square’ to find the vertex;
- (iv) state the range of values which the function can take.

(a) $x^2 - 8x + 4$	(b) $3x^2 + 12x + 5$	(c) $5x^2 + 6x + 2$
(d) $4 - 6x - x^2$	(e) $x^2 + 6x + 9$	(f) $1 - 4x - 4x^2$

7 Find the coordinates of the stationary points on the graphs of the following functions, and find whether these points are maxima or minima.

- | | | |
|----------------------------------|-----------------------------------|------------------------------|
| (a) $2x^3 + 3x^2 - 72x + 5$ | (b) $x^3 - 3x^2 - 45x + 7$ | (c) $3x^4 - 8x^3 + 6x^2$ |
| (d) $3x^5 - 20x^3 + 1$ | (e) $2x + x^2 - 4x^3$ | (f) $x^3 + 3x^2 + 3x + 1$ |
| (g) $x + \frac{1}{x}$ | (h) $x^2 + \frac{54}{x}$ | (i) $x - \frac{1}{x}$ |
| (j) $x - \sqrt{x}$, for $x > 0$ | (k) $\frac{1}{x} - \frac{3}{x^2}$ | (l) $x^2 - \frac{16}{x} + 5$ |
| (m) $x^{\frac{1}{3}}(4-x)$ | (n) $x^{\frac{1}{3}}(x+6)$ | (o) $x^4(1-x)$ |

8 Find the ranges of each of these functions $f(x)$, defined over the largest possible domains.

- | | | |
|-------------------|------------------|-----------------------|
| (a) $x^2 + x + 1$ | (b) $x^4 - 8x^2$ | (c) $x + \frac{1}{x}$ |
|-------------------|------------------|-----------------------|

7.4 Derivatives as rates of change

The quantities x and y in a relationship $y = f(x)$ are often called **variables**, because x can stand for any number in the domain and y for any number in the range. When you draw the graph you have a free choice of values of x , and then work out the values of y . So x is called the **independent variable** and y the **dependent variable**.

These variables often stand for physical or economic quantities, and then it is convenient to use other letters which suggest what these quantities are: for example, t for time, V for volume, C for cost, P for population, and so on.

To illustrate this consider a situation familiar to deep sea divers, that pressure increases with depth below sea level. The independent variable is the depth, z metres, below the surface.

It will soon be clear why the letter d was not used for the depth. The letter z is often used for distances in the vertical direction.

The dependent variable is the pressure, p , measured in bars. At the surface the diver experiences only atmospheric pressure, about 1 bar, but the pressure increases as the diver descends.

At off-shore (coastal) depths the variables are connected approximately by the equation

$$p = 1 + 0.1z.$$

The (z, p) graph is a straight line, shown in Fig. 7.11.

The constant 0.1 in the equation is the amount that the pressure goes up for each extra metre of depth. This is the 'rate of change of pressure with respect to depth'.

If the diver descends a further distance of δz metres, the pressure goes up by δp bars; this rate of change is $\frac{\delta p}{\delta z}$.

It is represented by the gradient of the graph.

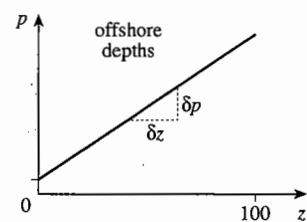


Fig. 7.11

But at ocean depths the (z, p) graph is no longer a straight line: it has the form of Fig. 7.12. The quantity $\frac{\delta p}{\delta z}$ now represents the average rate of change over the extra depth δz . It is represented by the gradient of the chord in Fig. 7.12.

The rate of change of pressure with respect to depth is the limit of $\frac{\delta p}{\delta z}$ as δz tends to 0. The $f'()$ notation, which has so far been used for this limit, is not ideal because it does not mention p ; it is useful to have a notation which includes both of the letters used for the variables. An alternative symbol $\frac{dp}{dz}$ was devised, obtained by replacing the letter δ in the average rate by d in the limit. Formally,

$$\frac{dp}{dz} = \lim_{\delta z \rightarrow 0} \frac{\delta p}{\delta z}.$$

There is no new idea here. It is just a different way of writing the definition of the derivative given in Chapter 6. The advantage is that it can be adapted, using different letters, to express the rate of change whenever there is a function relationship between two variables.

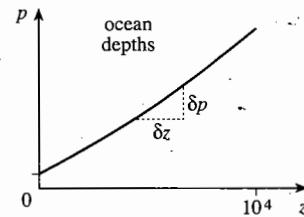


Fig. 7.12

If x and y are the independent and dependent variables respectively in a functional relationship, then the derivative,

$$\frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x},$$

measures the **rate of change of y with respect to x** .

If $y = f(x)$, then $\frac{dy}{dx} = f'(x)$.

Although $\frac{dy}{dx}$ looks like a fraction, for the time being you should treat it as one inseparable symbol made up of four letters and a horizontal line. By themselves, the symbols dx and dy have no meaning. (Later on, though, you will find that in some ways the symbol $\frac{dy}{dx}$ behaves like a fraction. This is another of its advantages over the $f'()$ notation.)

The notation can be used in a wide variety of contexts. For example, if the area of burnt grass, t minutes after a fire has started, is A square metres, then $\frac{dA}{dt}$ measures the rate at which the fire is spreading in square metres per minute. If, at a certain point on the Earth's surface, distances of x metres on the ground are represented by distances of y metres on a map, then $\frac{dy}{dx}$ represents the scale of the map at that point.

Example 7.4.1

A sprinter in a women's 100-metre race reaches her top speed of 12 metres per second after she has run 36 metres. Up to that distance her speed is proportional to the square root of the distance she has run. Show that until she reaches full speed the rate of change of her speed with respect to distance is inversely proportional to her speed.

Suppose that after she has run x metres her speed is S metres per second. You are told that, up to $x = 36$, $S = k\sqrt{x}$, and also that $S = 12$ when $x = 36$. So

$$12 = k\sqrt{36}, \quad \text{giving} \quad k = \frac{12}{6} = 2.$$

The (x, S) relationship is therefore

$$S = 2\sqrt{x} \text{ for } 0 < x < 36.$$

The rate of change of speed with respect to distance is the derivative $\frac{dS}{dx}$, and the derivative of \sqrt{x} (from Section 6.5) is $\frac{1}{2\sqrt{x}}$. Therefore

$$\frac{dS}{dx} = 2 \times \frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{x}}.$$

Since $\sqrt{x} = \frac{S}{2}$, $\frac{dS}{dx}$ can be written as $\frac{2}{S}$.

The rate of change is therefore inversely proportional to her speed.

If she maintains her top speed for the rest of the race, the rate of change of speed with respect to distance drops to 0 for $x > 36$. Fig. 7.13 shows that the gradient, which represents the rate of change, gets smaller as her speed increases, and then becomes zero once she reaches her top speed.

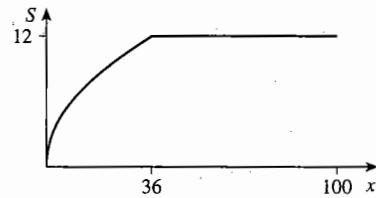


Fig. 7.13

Example 7.4.2

A line of cars, each 5 metres long, is travelling along an open road at a steady speed of S km per hour. There is a recommended separation between each pair of cars given by the formula $(0.18S + 0.006S^2)$ metres. At what speed should the cars travel to maximise the number of cars that the road can accommodate?

It is a good idea to write the separation formula as $(aS + bS^2)$, where $a = 0.18$ and $b = 0.006$. This gives a neater formula, and will also enable you to investigate the effect of changing the coefficients in the formula. But remember when you differentiate that a and b are simply constants, and you can treat them just like numbers.

A 'block', consisting of a car's length and the separation distance in front of it, occupies $5 + aS + bS^2$ metres of road, or $\frac{5 + aS + bS^2}{1000}$ km. For the largest number

of blocks passing a checkpoint in an hour, the time T (in hours) for a single block to pass the checkpoint should be as small as possible. Since the block is moving at speed S km per hour,

$$TS = \frac{5 + aS + bS^2}{1000},$$

$$\text{or } T = \frac{5 + aS + bS^2}{1000S} = 0.001(5S^{-1} + a + bS).$$

Now follow the procedure for finding the minimum value of T .

Step 1 Since the speed must be positive, the domain is $S > 0$.

Step 2 The derivative is $\frac{dT}{dS} = 0.001(-5S^{-2} + b)$.

Step 3 This derivative is defined everywhere in the domain, and is 0 when

$$-\frac{5}{S^2} + b = 0, \text{ which is at } S = \sqrt{\frac{5}{b}}.$$

Step 4 As S increases, $\frac{5}{S^2}$ decreases, so $-\frac{5}{S^2} + b$ increases. Since $\frac{dT}{dS}$ is 0 when $S = \sqrt{\frac{5}{b}}$, the sign of $\frac{dT}{dS}$ is $-$ when $S < \sqrt{\frac{5}{b}}$, and $+$ when $S > \sqrt{\frac{5}{b}}$.

Step 5 Since $\frac{dT}{dS}$ changes from $-$ to $+$, T is a minimum when $S = \sqrt{\frac{5}{b}}$.

Step 6 Substituting $a = 0.18$ and $b = 0.006$ gives $S = \sqrt{\frac{5}{0.006}} \approx 28.87$ and $T \approx 0.000\ 526$ at the minimum point.

This shows that the cars flow best at a speed of just under 29 km h^{-1} . (Each block then takes approximately $0.000\ 526$ hours, or 1.89 seconds, to pass the checkpoint, so that the number of cars which pass in an hour is approximately $\frac{1}{0.000\ 526} \approx 1900$.)

Example 7.4.3

A hollow cone with base radius a cm and height b cm is placed on a table. What is the volume of the largest cylinder that can be hidden underneath it?

The volume of a cylinder of radius r cm and height h cm is V cm 3 , where

$$V = \pi r^2 h.$$

You can obviously make this as large as you like by choosing r and h large enough. But in this problem the variables are restricted by the requirement that the cylinder has to fit under the cone. Before you can follow the procedure for finding a maximum, you need to find how this restriction affects the values of r and h .

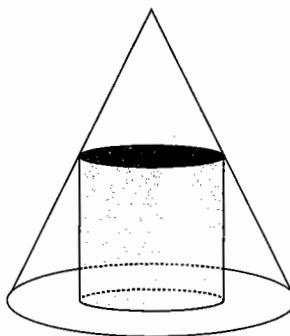


Fig. 7.14

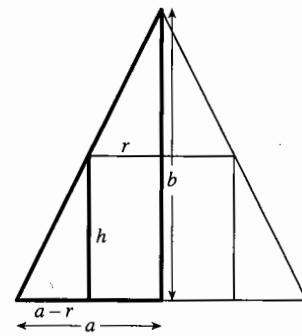


Fig. 7.15

Fig. 7.14 shows the three-dimensional set-up, and Fig. 7.15 is a vertical section through the top of the cone. The similar triangles picked out with heavy lines in Fig. 7.15 show that r and h are connected by the equation

$$\frac{h}{a-r} = \frac{b}{a}, \text{ so that } h = \frac{b(a-r)}{a}.$$

Substituting this expression for h in the formula for V then gives

$$V = \frac{\pi r^2 b(a-r)}{a} = \left(\frac{\pi b}{a}\right)(ar^2 - r^3).$$

Notice that the original expression for V contains two independent variables r and h . The effect of the substitution is to reduce the number of independent variables to one; h has disappeared, and only r remains. This makes it possible to apply the procedure for finding a maximum.

The physical problem only has meaning if $0 < r < a$, so take this interval as the domain of the function. Differentiating by the usual rule (remembering that π , a and b are constants) gives

$$\frac{dV}{dr} = \left(\frac{\pi b}{a}\right)(2ar - 3r^2) = \left(\frac{\pi b}{a}\right)r(2a - 3r).$$

The only value of r in the domain for which $\frac{dV}{dr} = 0$ is $\frac{2}{3}a$. It is easy to check that the sign of $\frac{dV}{dr}$ is + for $0 < r < \frac{2}{3}a$ and - for $\frac{2}{3}a < r < a$.

So the cylinder of maximum volume has radius $\frac{2}{3}a$, height $\frac{1}{3}b$ and volume $\frac{4}{27}\pi a^2 b$. (Since the volume of the cone is $\frac{1}{3}\pi a^2 b$, the cylinder of maximum volume occupies $\frac{4}{9}$ of the space under the cone.)

Exercise 7C

- 1 In each part of this question express each derivative as ‘the rate of change of ... with respect to ...’, and state its physical significance.

 - $\frac{dh}{dx}$, where h is the height above sea level, and x is the horizontal distance travelled along a straight road
 - $\frac{dN}{dt}$, where N is the number of people in a stadium at time t after the gates open
 - $\frac{dM}{dr}$, where M is the magnetic force at a distance r from a magnet
 - $\frac{dv}{dt}$, where v is the velocity of a particle moving in a straight line at time t
 - $\frac{dq}{ds}$, where q is the rate at which petrol is used in a car in litres per km, and S is the speed of the car in km per hour

2 Defining suitable notation and units, express each of the following as a derivative.

 - the rate of change of atmospheric pressure with respect to height above sea level
 - the rate of change of temperature with respect to the time of day
 - the rate at which the tide is rising
 - the rate at which a baby’s weight increases in the first weeks of life

3 (a) Find $\frac{dz}{dt}$ where $z = 3t^2 + 7t - 5$. (b) Find $\frac{d\theta}{dx}$ where $\theta = x - \sqrt{x}$.

(c) Find $\frac{dx}{dy}$ where $x = y + \frac{3}{y^2}$. (d) Find $\frac{dr}{dt}$ where $r = t^2 + \frac{1}{\sqrt{t}}$.

(e) Find $\frac{dm}{dt}$ where $m = (t+3)^2$. (f) Find $\frac{df}{ds}$ where $f = 2s^6 - 3s^2$.

(g) Find $\frac{dw}{dt}$ where $w = 5t$. (h) Find $\frac{dR}{dr}$ where $R = \frac{1-r^3}{r^2}$.

4 A particle moves along the x -axis. Its displacement at time t is $x = 6t - t^2$.

 - What does $\frac{dx}{dt}$ represent?
 - Is x increasing or decreasing when (i) $t = 1$, (ii) $t = 4$?
 - Find the greatest (positive) displacement of the particle. How is this connected to your answer to part (a)?

5 Devise suitable notation to express each of the following in mathematical form.

 - The distance travelled along the motorway is increasing at a constant rate.
 - The rate at which a savings bank deposit grows is proportional to the amount of money deposited.
 - The rate at which the diameter of a tree increases is a function of the air temperature.

- 6 At a speed of S km per hour a car will travel y kilometres on each litre of petrol, where

$$y = 5 + \frac{1}{5}S - \frac{1}{800}S^2.$$

Calculate the speed at which the car should be driven for maximum economy.

- 7 A ball is thrown vertically upwards. At time t seconds its height h metres is given by $h = 20t - 5t^2$. Calculate the ball's maximum height above the ground.
- 8 The sum of two real numbers x and y is 12. Find the maximum value of their product xy .
- 9 The product of two positive real numbers x and y is 20. Find the minimum possible value of their sum.
- 10 The volume of a cylinder is given by the formula $V = \pi r^2 h$. Find the greatest and least values of V if $r + h = 6$.
- 11 A loop of string of length 1 metre is formed into a rectangle with one pair of opposite sides each x cm. Calculate the value of x which will maximise the area enclosed by the string.
- 12 One side of a rectangular sheep pen is formed by a hedge. The other three sides are made using fencing. The length of the rectangle is x metres; 120 metres of fencing is available.
- Show that the area of the rectangle is $\frac{1}{2}x(120 - x)$ m².
 - Calculate the maximum possible area of the sheep pen.
- 13 A rectangular sheet of metal measures 50 cm by 40 cm. Equal squares of side x cm are cut from each corner and discarded. The sheet is then folded up to make a tray of depth x cm. What is the domain of possible values of x ? Find the value of x which maximises the capacity of the tray.
- 14 An open rectangular box is to be made with a square base, and its capacity is to be 4000 cm³. Find the length of the side of the base when the amount of material used to make the box is as small as possible. (Ignore 'flaps').
- 15 An open cylindrical wastepaper bin, of radius r cm and capacity V cm³, is to have a surface area of 5000 cm².
- Show that $V = \frac{1}{2}r(5000 - \pi r^2)$.
 - Calculate the maximum possible capacity of the bin.
- 16 A circular cylinder is to fit inside a sphere of radius 10 cm. Calculate the maximum possible volume of the cylinder. (It is probably best to take as your independent variable the height, or half the height, of the cylinder.)

Miscellaneous exercise 7

- 1 Use differentiation to find the coordinates of the stationary points on the curve

$$y = x + \frac{4}{x}$$

and determine whether each stationary point is a maximum point or a minimum point.

Find the set of values of x for which y increases as x increases.

(OCR)

- 2 The rate at which a radioactive mass decays is known to be proportional to the mass remaining at that time. If, at time t , the mass remaining is m , this means that m and t satisfy the equation

$$\frac{dm}{dt} = -km$$

where k is a positive constant. (The negative sign ensures that $\frac{dm}{dt}$ is negative, which indicates that m is decreasing.)

Write down similar equations which represent the following statements.

- (a) The rate of growth of a population of bacteria is proportional to the number, n , of bacteria present.
- (b) When a bowl of hot soup is put in the freezer, the rate at which its temperature, θ °C, decreases as it cools is proportional to its current temperature.
- (c) The rate at which the temperature, θ °C, of a cup of coffee decreases as it cools is proportional to the excess of its temperature over the room temperature, β °C.

- 3 A car accelerates to overtake a truck. Its initial speed is u , and in a time t after it starts to accelerate it covers a distance x , where $x = ut + kt^2$.

Use differentiation to show that its speed is then $u + 2kt$, and show that its acceleration is constant.

- 4 A car is travelling at 20 m s⁻¹ when the driver applies the brakes. At a time t seconds later the car has travelled a further distance x metres, where $x = 20t - 2t^2$. Use differentiation to find expressions for the speed and the acceleration of the car at this time. For how long do these formulae apply?

- 5 A boy stands on the edge of a cliff of height 60 m. He throws a stone vertically upwards so that its distance, h m, above the cliff top is given by $h = 20t - 5t^2$.

- (a) Calculate the maximum height of the stone above the cliff top.
- (b) Calculate the time which elapses before the stone hits the beach. (It just misses the boy and the cliff on the way down.)
- (c) Calculate the speed with which the stone hits the beach.

- 6 Find the least possible value of $x^2 + y^2$ given that $x + y = 10$.

- 7 The sum of the two shorter sides of a right-angled triangle is 18 cm. Calculate

- (a) the least possible length of the hypotenuse,
- (b) the greatest possible area of the triangle.

- 8 (a) Find the stationary points on the graph of $y = 12x + 3x^2 - 2x^3$ and sketch the graph.

- (b) How does your sketch show that the equation $12x + 3x^2 - 2x^3 = 0$ has exactly three real roots?

- (c) Use your graph to show that the equation $12x + 3x^2 - 2x^3 = -5$ also has exactly three real roots.

- (d) For what range of values of k does the equation $12x + 3x^2 - 2x^3 = k$ have
- (i) exactly three real roots,
 - (ii) only one real root?

- 9** Find the coordinates of the stationary points on the graph of $y = x^3 - 12x - 12$ and sketch the graph.

Find the set of values of k for which the equation $x^3 - 12x - 12 = k$ has more than one real solution. (OCR)

- 10** Find the coordinates of the stationary points on the graph of $y = x^3 + x^2$. Sketch the graph and hence write down the set of values of the constant k for which the equation $x^3 + x^2 = k$ has three distinct real roots.

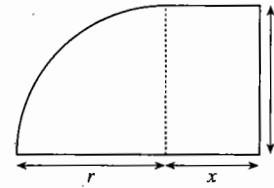
- 11** Find the coordinates of the stationary points on the graph of $y = 3x^4 - 4x^3 - 12x^2 + 10$, and sketch the graph. For what values of k does the equation $3x^4 - 4x^3 - 12x^2 + 10 = k$ have

- (a) exactly four roots, (b) exactly two roots?

- 12** Find the coordinates of the stationary points on the curve with equation $y = x(x-1)^2$. Sketch the curve.

Find the set of real values of k such that the equation $x(x-1)^2 = k^2$ has exactly one real root. (OCR, adapted)

- 13** The cross-section of an object has the shape of a quarter-circle of radius r adjoining a rectangle of width x and height r , as shown in the diagram.



- (a) The perimeter and area of the cross-section are P and A respectively. Express each of P and A in terms of r and x , and hence show that $A = \frac{1}{2}Pr - r^2$.

- (b) Taking the perimeter P of the cross-section as fixed, find x in terms of r for the case when the area A of the cross-section is a maximum, and show that, for this value of x , A is a maximum and not a minimum. (OCR)

- 14** A curve has equation $y = \frac{1}{x} - \frac{1}{x^2}$. Use differentiation to find the coordinates of the stationary point and determine whether the stationary point is a maximum point or a minimum point. Deduce, or obtain otherwise, the coordinates of the stationary point of each of the following curves.

$$(a) \quad y = \frac{1}{x} - \frac{1}{x^2} + 5 \qquad (b) \quad y = \frac{2}{x-1} - \frac{2}{(x-1)^2}$$

- 15** The manager of a supermarket usually adds a mark-up of 20% to the wholesale prices of all the goods he sells. He reckons that he has a loyal core of F customers and that, if he lowers his mark-up to $x\%$ he will attract an extra $k(20-x)$ customers from his rivals. Each week the average shopper buys goods whose wholesale value is £ A . Show that with a mark-up of $x\%$ the supermarket will have an anticipated weekly profit of

$$\text{£} \frac{1}{100} Ax((F + 20k) - kx).$$

Show that the manager can increase his profit by reducing his mark-up below 20% provided that $20k > F$. (OCR)

- 16 The costs of a firm which makes climbing boots are of two kinds:

Fixed costs (plant, rates, office expenses): £2000 per week;

Production costs (materials, labour): £20 for each pair of boots made.

Market research suggests that, if they price the boots at £30 a pair they will sell 500 pairs a week, but that at £55 a pair they will sell none at all; and between these values the graph of sales against price is a straight line.

If they price boots at £ x a pair ($30 \leq x \leq 55$) find expressions for

- (a) the weekly sales, (b) the weekly receipts, (c) the weekly costs
(assuming that just enough boots are made).

Hence show that the weekly profit, £ P , is given by

$$P = -20x^2 + 1500x - 24\,000.$$

Find the price at which the boots should be sold to maximise the profit.

(OCR)

- 17 Sketch the graph of an even function $f(x)$ which has a derivative at every point.

Let P be the point on the graph for which $x = p$ (where $p > 0$). Draw the tangent at P on your sketch. Also draw the tangent at the point P' for which $x = -p$.

- (a) What is the relationship between the gradient at P' and the gradient at P ? What can you deduce about the relationship between $f'(p)$ and $f'(-p)$? What does this tell you about the derivative of an even function?
(b) Show that the derivative of an odd function is even.

8 Sequences

This chapter is about sequences of numbers. When you have completed it, you should

- know that a sequence can be constructed from a formula or an inductive definition
- be familiar with triangle, factorial, Pascal and arithmetic sequences
- know how to find the sum of an arithmetic series.

8.1 Constructing sequences

Here are six rows of numbers, each forming a pattern of some kind. What are the next three numbers in each row?

(a) 1 4 9 16 25 ...

(b) $\frac{1}{2} \frac{2}{3} \frac{3}{4} \frac{4}{5} \frac{5}{6} \dots$

(c) 99 97 95 93 91 ...

(d) 1 1.1 1.21 1.331 1.4641 ...

(e) 2 4 8 14 22 ...

(f) 3 1 4 1 5 ...

Rows of this kind are called **sequences**, and the separate numbers are called **terms**.

The usual notation for the first, second, third, ... terms of a sequence is u_1, u_2, u_3 , and so on. There is nothing special about the choice of the letter u , and other letters such as v, x, t and I are often used instead, especially if the sequence appears in some application. If r is a natural number, then the r th term will be u_r, v_r, x_r, t_r or I_r .

Sometimes it is convenient to number the terms u_0, u_1, u_2, \dots , starting with $r = 0$, but you then have to be careful in referring to 'the first term': do you mean u_0 or u_1 ?

In (a) and (b) you would have no difficulty in writing a formula for the r th term of the sequence. The numbers in (a) could be rewritten as $1^2, 2^2, 3^2, 4^2, 5^2$, and the pattern could be summed up by writing

$$u_r = r^2.$$

The terms of (b) are $\frac{1}{1+1}, \frac{2}{2+1}, \frac{3}{3+1}, \frac{4}{4+1}, \frac{5}{5+1}$, so $u_r = \frac{r}{r+1}$.

In (c), (d) and (e) you probably expect that there is a formula, but it is not so easy to find it. What is more obvious is how to get each term from the one before. For example, in (c) the terms go down by 2 at each step, so that $u_2 = u_1 - 2, u_3 = u_2 - 2, u_4 = u_3 - 2$, and so on. These steps can be summarised by the single equation

$$u_{r+1} = u_r - 2.$$

The terms in (d) are multiplied by 1.1 at each step, so the rule is

$$u_{r+1} = 1.1u_r.$$

Unfortunately, there are many other sequences which satisfy the equation $u_{r+1} = u_r - 2$. Other examples are 10, 8, 6, 4, 2, ... and -2, -4, -6, -8, -10,

The definition is not complete until you know the first term. So to complete the definitions of the sequences (c) and (d) you have to write

$$(c) \quad u_1 = 99 \text{ and } u_{r+1} = u_r - 2,$$

$$(d) \quad u_1 = 1 \text{ and } u_{r+1} = 1.1u_r.$$

Definitions like these are called **inductive definitions**.

Sequence (e) originates from geometry. It gives the greatest number of regions into which a plane can be split by different numbers of circles. (Try drawing your own diagrams with 1, 2, 3, 4, ... circles.) This sequence is developed as $u_2 = u_1 + 2$, $u_3 = u_2 + 4$, $u_4 = u_3 + 6$, and so on. Since the increments 2, 4, 6, ... are themselves given by the formula $2r$, this can be summarised by the inductive definition

$$u_1 = 2 \text{ and } u_{r+1} = u_r + 2r.$$

For (f) you may have given the next three terms as 1, 6, 1 (expecting the even-placed terms all to be 1, and the odd-placed terms to go up by 1 at each step). In fact this sequence had a quite different origin, as the first five digits of π in decimal form! With this meaning, the next three terms would be 9, 2, 6.

This illustrates an important point, that a sequence can never be uniquely defined by giving just the first few terms. Try, for example, working out the first eight terms of the sequence defined by

$$u_r = r^2 + (r-1)(r-2)(r-3)(r-4)(r-5).$$

You will find that the first five terms are the same as those given in (a), but the next three are probably very different from your original guess.

A sequence can only be described unambiguously by giving a formula, an inductive definition in terms of a general natural number r , or some other general rule.

Exercise 8A

1 Write down the first five terms of the sequences with the following definitions.

$$(a) \quad u_1 = 7, \quad u_{r+1} = u_r + 7$$

$$(b) \quad u_1 = 13, \quad u_{r+1} = u_r - 5$$

$$(c) \quad u_1 = 4, \quad u_{r+1} = 3u_r$$

$$(d) \quad u_1 = 6, \quad u_{r+1} = \frac{1}{2}u_r$$

$$(e) \quad u_1 = 2, \quad u_{r+1} = 3u_r + 1$$

$$(f) \quad u_1 = 1, \quad u_{r+1} = u_r^2 + 3$$

2 Suggest inductive definitions which would produce the following sequences.

$$(a) \quad 2 \quad 4 \quad 6 \quad 8 \quad 10 \quad \dots$$

$$(b) \quad 11 \quad 9 \quad 7 \quad 5 \quad 3 \quad \dots$$

$$(c) \quad 2 \quad 6 \quad 10 \quad 14 \quad 18 \quad \dots$$

$$(d) \quad 2 \quad 6 \quad 18 \quad 54 \quad 162 \quad \dots$$

$$(e) \quad \frac{1}{3} \quad \frac{1}{9} \quad \frac{1}{27} \quad \frac{1}{81} \quad \dots$$

$$(f) \quad \frac{1}{2}a \quad \frac{1}{4}a \quad \frac{1}{8}a \quad \frac{1}{16}a \quad \dots$$

$$(g) \quad b-2c \quad b-c \quad b \quad b+c \quad \dots$$

$$(h) \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad \dots$$

$$(i) \quad \frac{p}{q^3} \quad \frac{p}{q^2} \quad \frac{p}{q} \quad \dots$$

$$(j) \quad \frac{a^3}{b^2} \quad \frac{a^2}{b} \quad a \quad b \quad \dots$$

$$(k) \quad x^3 \quad 5x^2 \quad 25x \quad \dots$$

$$(l) \quad 1 \quad 1+x \quad (1+x)^2 \quad (1+x)^3 \quad \dots$$

3 Write down the first five terms of each sequence and give an inductive definition for it.

- | | | |
|-------------------------------------|--------------------------|-------------------------------|
| (a) $u_r = 2r + 3$ | (b) $u_r = r^2$ | (c) $u_r = \frac{1}{2}r(r+1)$ |
| (d) $u_r = \frac{1}{6}r(r+1)(2r+1)$ | (e) $u_r = 2 \times 3^r$ | (f) $u_r = 3 \times 5^{r-1}$ |

4 For each of the following sequences give a possible formula for the r th term.

- | | |
|---|--|
| (a) 9 8 7 6 ... | (b) 6 18 54 162 ... |
| (c) 4 7 12 19 ... | (d) 4 12 24 40 60 ... |
| (e) $\frac{1}{4}, \frac{3}{5}, \frac{5}{6}, \frac{7}{7}, \dots$ | (f) $\frac{2}{2}, \frac{5}{4}, \frac{10}{8}, \frac{17}{16}, \dots$ |

8.2 The triangle number sequence

The numbers of crosses in the triangular patterns in Fig. 8.1 are called triangle numbers. If t_r denotes the r th triangle number, you can see by counting the numbers of crosses in successive rows that

$$t_1 = 1, \quad t_2 = 1 + 2 = 3, \quad t_3 = 1 + 2 + 3 = 6,$$

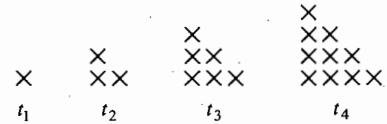


Fig. 8.1

and in general $t_r = 1 + 2 + 3 + \dots + r$, where the dots indicate that all the natural numbers between 3 and r have to be included in the addition.

Fig. 8.2 shows a typical pattern of crosses forming a triangle number t_r . (It is in fact drawn for $r = 9$, but any other value of r could have been chosen.) An easy way of finding a formula for t_r is to make a similar pattern of 'noughts', and then to turn it upside down and place it alongside the pattern of crosses, as in Fig. 8.3. The noughts and crosses together then make a rectangular pattern, $r+1$ objects wide and r objects high. So the total number of objects is $r(r+1)$, half of them crosses and half noughts. The number of crosses alone is therefore

$$t_r = \frac{1}{2}r(r+1).$$

This shows that:

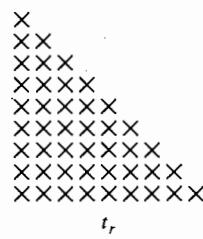


Fig. 8.2

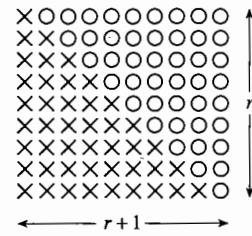


Fig. 8.3

The sum of all the natural numbers from 1 to r is $\frac{1}{2}r(r+1)$.

You can put this argument into algebraic form. If you count the crosses from the top downwards you get

$$t_r = 1 + 2 + 3 + \dots + (r-2) + (r-1) + r,$$

but if you count the noughts from the top downwards you get

$$t_r = r + (r-1) + (r-2) + \dots + 3 + 2 + 1.$$

Counting all the objects in the rectangle is equivalent to adding these two equations:

$$2t_r = (r+1) + (r+1) + (r+1) + \dots + (r+1) + (r+1) + (r+1).$$

with one $(r+1)$ bracket for each of the r rows. It follows that $2t_r = r(r+1)$, so that

$$t_r = \frac{1}{2}r(r+1).$$

It is also possible to give an inductive definition for the sequence t_r . Fig. 8.1 shows that to get from any triangle number to the next you simply add an extra row of crosses underneath. Thus $t_2 = t_1 + 2$, $t_3 = t_2 + 3$, $t_4 = t_3 + 4$, and in general

$$t_{r+1} = t_r + (r+1).$$

You can complete this definition by specifying either $t_1 = 1$ or $t_0 = 0$. If you choose $t_0 = 0$, then you can find t_1 by putting $r=0$ in the general equation, as $t_1 = t_0 + 1 = 0 + 1 = 1$. So you may as well define the triangle number sequence by

$$t_0 = 0 \quad \text{and} \quad t_{r+1} = t_r + (r+1), \quad \text{where } r = 0, 1, 2, 3, \dots .$$

8.3 The factorial sequence

If, in the definition of t_r , you go from one term to the next by multiplication rather than addition, you get the factorial sequence

$$f_{r+1} = f_r \times (r+1), \quad \text{where } r = 0, 1, 2, 3, \dots .$$

There would be little point in defining f_0 to be 0 (think about why this is); instead take f_0 to be 1. (This may seem strange, but you will see the reason in the next chapter.) You then get

$$f_1 = f_0 \times 1 = 1 \times 1 = 1, \quad f_2 = f_1 \times 2 = 1 \times 2 = 2, \quad f_3 = f_2 \times 3 = 2 \times 3 = 6,$$

and if you go on in this way you find that, for any $r \geq 1$,

$$f_r = 1 \times 2 \times 3 \times \dots \times r.$$

This sequence is so important that it has its own special notation, $r!$, read as ‘factorial r ’ or ‘ r factorial’ (or often, colloquially, as ‘ r shriek’).

Factorial r is defined by $0! = 1$ and $(r+1)! = r! \times (r+1)$, where $r = 0, 1, 2, 3, \dots$.

For $r \geq 1$, $r!$ is the product of all the natural numbers from 1 to r .

Many calculators have a special key labelled $[n!]$. For small values of n the display gives the exact value, but the numbers in the sequence increase so rapidly that from about $n = 14$ onwards only an approximate value in standard form can be displayed.

8.4 Pascal sequences

Another important type of sequence based on a multiplication rule is a Pascal sequence. You will find in the next chapter that these sequences feature in the expansion of expressions like $(x + y)^n$. A typical example has an inductive definition

$$p_0 = 1 \quad \text{and} \quad p_{r+1} = \frac{4-r}{r+1} p_r, \quad \text{where } r = 0, 1, 2, 3, \dots$$

Using the inductive definition for $r = 0, 1, 2, \dots$ in turn produces the terms

$$\begin{aligned} p_1 &= \frac{4}{1} p_0 = 4, & p_2 &= \frac{3}{2} p_1 = 6, & p_3 &= \frac{2}{3} p_2 = 4, \\ p_4 &= \frac{1}{4} p_3 = 1, & p_5 &= \frac{0}{5} p_4 = 0, & p_6 &= \frac{(-1)}{6} p_5 = 0, \quad \text{and so on.} \end{aligned}$$

You will see that at a certain stage the sequence has a zero term, and because it is formed by multiplication all the terms after that will be zero. So the complete sequence is

$$1, 4, 6, 4, 1, 0, 0, 0, 0, 0, \dots$$

This is only one of a family of Pascal sequences, and its terms also have a special notation, $\binom{4}{r}$. For example, $\binom{4}{0} = 1$, $\binom{4}{1} = 4$, $\binom{4}{2} = 6$, and so on. Other Pascal sequences have numbers different from 4 in the multiplying factor.

The general definition of a Pascal sequence, whose terms are denoted by $\binom{n}{r}$, is

$$\binom{n}{0} = 1 \quad \text{and} \quad \binom{n}{r+1} = \frac{n-r}{r+1} \binom{n}{r}, \quad \text{where } r = 0, 1, 2, 3, \dots$$

Check for yourself that the Pascal sequences for $n = 0, 1, 2, 3$ are

$$n = 0: 1, 0, 0, 0, 0, \dots$$

$$n = 1: 1, 1, 0, 0, 0, \dots$$

$$n = 2: 1, 2, 1, 0, 0, \dots$$

$$n = 3: 1, 3, 3, 1, 0, \dots$$

The complete pattern of Pascal sequences, without the trailing zeros, is called **Pascal's triangle**. Its earliest recorded use was in China, but Blaise Pascal (a French mathematician of the 17th century, one of the originators of probability theory) was one of the first people in Europe to publish it. It is usually presented in isosceles form (Fig. 8.4), drawing attention to the symmetry of the sequence. But for its algebraic applications the format of Fig. 8.5 is often more convenient, since each column then corresponds to a particular value of r .

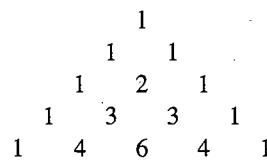


Fig. 8.4

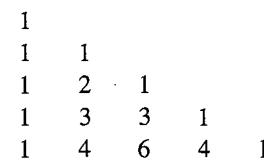


Fig. 8.5

You may be surprised to notice that every number in the pattern in Fig. 8.4 except for the 1s is the sum of the two numbers most closely above it.

You have seen these numbers before: look back at sequence (d) in Section 8.1.

Exercise 8B

1 Using Fig. 8.3 as an example,

- (a) draw a pattern of crosses to represent the r th triangle number t_r ;
- (b) draw another pattern of noughts to represent t_{r-1} ;
- (c) combine these two patterns to show that $t_r + t_{r-1} = r^2$.
- (d) Use the fact that $t_r = \frac{1}{2}r(r+1)$ to show the result in part (c) algebraically.

2 (a) Find an expression in terms of r for $t_r - t_{r-1}$ for all $r \geq 1$.

(b) Use this result and that in Question 1(c) to show that $t_r^2 - t_{r-1}^2 = r^3$.

(c) Use part (b) to write expressions in terms of triangle numbers for $1^3, 2^3, 3^3, \dots, n^3$.
Hence show that $1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{1}{4}n^2(n+1)^2$.

3 Without using a calculator, evaluate the following.

(a) $7!$	(b) $\frac{8!}{3!}$	(c) $\frac{7!}{4! \times 3!}$
----------	---------------------	-------------------------------

4 Write the following in terms of factorials.

- | | | |
|------------------------------------|---------------------------------------|-----------------------|
| (a) $8 \times 7 \times 6 \times 5$ | (b) $9 \times 10 \times 11 \times 12$ | (c) $n(n-1)(n-2)$ |
| (d) $n(n^2 - 1)$ | (e) $n(n+1)(n+2)(n+3)$ | (f) $(n+6)(n+5)(n+4)$ |
| (g) $8 \times 7!$ | (h) $n \times (n-1)!$ | |

5 Simplify the following.

(a) $\frac{12!}{11!}$	(b) $23! - 22!$	(c) $\frac{(n+1)!}{n!}$	(d) $(n+1)! - n!$
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6 Show that $\frac{(2n)!}{n!} = 2^n(1 \times 3 \times 5 \times \dots \times (2n-1))$.

7 Use the inductive definition in Section 8.4 to find the Pascal sequences for

- | | | |
|-------------|-------------|-------------|
| (a) $n=5$, | (b) $n=6$, | (c) $n=8$. |
|-------------|-------------|-------------|

- 8** Use the inductive definition for $\binom{n}{r}$ to show that $\binom{9}{6} = \frac{9 \times 8 \times 7}{1 \times 2 \times 3}$, and show that this can be written as $\frac{9!}{6! \times 3!}$.

Use a similar method to write the following in terms of factorials.

$$(a) \binom{11}{4} \quad (b) \binom{11}{7} \quad (c) \binom{10}{5} \quad (d) \binom{12}{3} \quad (e) \binom{12}{9}$$

- 9** The answers to Question 8 suggest a general result, that $\binom{n}{r} = \frac{n!}{r! \times (n-r)!}$. Assuming this to be true, show that $\binom{n}{r} = \binom{n}{n-r}$.

- 10** Show by direct calculation that

$$(a) \binom{6}{3} + \binom{6}{4} = \binom{7}{4}, \quad (b) \binom{8}{5} + \binom{8}{6} = \binom{9}{6}.$$

Write a general statement, involving n and r , suggested by these results.

- 11** The Pascal sequence for $n = 2$ is 1 2 1.

The sum of the terms in this sequence is $1+2+1=4$.

Investigate the sum of the terms in Pascal sequences for other values of n .

8.5 Arithmetic sequences

An **arithmetic sequence**, or **arithmetic progression**, is a sequence whose terms go up or down by constant steps. Sequence (c) in Section 8.1 is an example. The inductive definition for an arithmetic sequence has the form

$$u_1 = a, \quad u_{r+1} = u_r + d.$$

The number d is called the **common difference**.

Sequence (c) has first term $a = 99$ and common difference $d = -2$.

Example 8.5.1

Senne would like to give a sum of money to a charity each year for 10 years. She decides to give \$100 in the first year, and to increase her contribution by \$20 each year. How much does she give in the last year, and how much does the charity receive from her altogether?

Although she makes 10 contributions, there are only 9 increases. So in the last year she gives $(100 + 9 \times 20) = \$280$.

If the total amount the charity receives is S , then

$$S = 100 + 120 + 140 + \dots + 240 + 260 + 280.$$

With only 10 numbers it is easy enough to add these up, but you can also find the sum by a method similar to that used to find a formula for t_n . If you add up the numbers in reverse order, you get

$$S = 280 + 260 + 240 + \dots + 140 + 120 + 100.$$

Adding the two equations then gives

$$2S = 380 + 380 + 380 + \dots + 380 + 380 + 380,$$

where the number 380 occurs 10 times. So

$$2S = 380 \times 10 = 3800, \text{ giving } S = 1900.$$

Over the 10 years the charity receives \$1900.

This calculation can be illustrated with diagrams similar to Figs. 8.2 and 8.3. Senne's contributions are shown by Fig. 8.6, with the first year in the top row. (Each cross is worth \$20.) In Fig. 8.7 a second copy, with noughts instead of crosses, is put alongside it, but turned upside down. There are then 10 rows, each with 19 crosses or noughts and worth \$380.

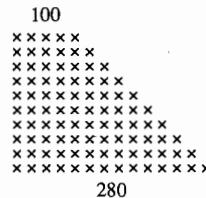


Fig. 8.6

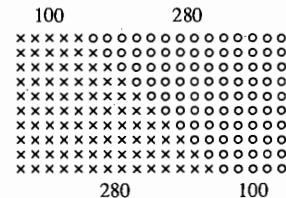


Fig. 8.7

Two features of Example 8.5.1 are typical of arithmetic progressions.

- They usually only continue for a finite number of terms.
- It is often interesting to know the sum of all the terms. In this case, it is usual to describe the sequence as a **series**.

In Example 8.5.1, the annual contributions

$$100, 120, 140, \dots, 240, 260, 280$$

form an arithmetic sequence, but if they are added as

$$100 + 120 + 140 + \dots + 240 + 260 + 280$$

they become an **arithmetic series**.

If the general arithmetic sequence

$$a, a+d, a+2d, a+3d, \dots$$

has n terms in all, then from the first term to the last there are $n-1$ steps of the common difference d . Denote the last term, u_n , by l . Then

$$l = a + (n-1)d.$$

From this equation you can calculate any one of the four quantities a, l, n, d if you know the other three.

Let S be the sum of the arithmetic series formed by adding these terms. Then it is possible to find a formula for S in terms of a, n and either d or l .

Method 1 This generalises the argument used in Example 8.5.1.
The series can be written as

$$S = a + (a+d) + (a+2d) + \dots + (l-2d) + (l-d) + l.$$

Turning this back to front,

$$S = l + (l-d) + (l-2d) + \dots + (a+2d) + (a+d) + a.$$

Adding these,

$$2S = (a+l) + (a+l) + (a+l) + \dots + (a+l) + (a+l) + (a+l),$$

where the bracket $(a+l)$ occurs n times. So

$$2S = n(a+l), \text{ which gives } S = \frac{1}{2}n(a+l).$$

Method 2 This uses the formula for triangle numbers found in Section 8.2.

In the series

$$S = a + (a+d) + (a+2d) + \dots + (a+(n-1)d)$$

you can collect separately the terms involving a and those involving d :

$$S = (a+a+\dots+a) + (1+2+3+\dots+(n-1))d.$$

In the first bracket a occurs n times. The second bracket is the sum of the natural numbers from 1 to $n-1$, or t_{n-1} ; using the formula $t_r = \frac{1}{2}r(r+1)$ with $r = n-1$ gives this sum as

$$t_{n-1} = \frac{1}{2}(n-1)((n-1)+1) = \frac{1}{2}(n-1)n.$$

Therefore

$$S = na + \frac{1}{2}(n-1)nd = \frac{1}{2}n(2a + (n-1)d).$$

Since $l = a + (n-1)d$, this is the same answer as that given by method 1.

Here is a summary of the results about arithmetic series.

An arithmetic series of n terms with first term a and common difference d has last term

$$l = a + (n-1)d$$

and sum

$$S = \frac{1}{2}n(a+l) = \frac{1}{2}n(2a + (n-1)d).$$

Example 8.5.2

Find the sum of the first n odd natural numbers.

Method 1 The odd numbers 1, 3, 5, ... form an arithmetic series with first term $a = 1$ and common difference $d = 2$. So

$$S = \frac{1}{2}n(2a + (n-1)d) = \frac{1}{2}n(2 + (n-1)2) = \frac{1}{2}n(2n) = n^2.$$

Method 2 Take the natural numbers from 1 to $2n$, and remove the n even numbers 2, 4, 6, ..., $2n$. You are left with the first n odd numbers.

The sum of the numbers from 1 to $2n$ is t_r , where $r = 2n$, that is

$$t_{2n} = \frac{1}{2}(2n)(2n+1) = n(2n+1).$$

The sum of the n even numbers is

$$2 + 4 + 6 + \dots + 2n = 2(1 + 2 + 3 + \dots + n) = 2t_n = n(n+1).$$

So the sum of the first n odd numbers is

$$n(2n+1) - n(n+1) = n((2n+1) - (n+1)) = n(n) = n^2.$$

Method 3 Fig. 8.8 shows a square of n rows with n crosses in each row (drawn for $n = 7$). You can count the crosses in the square by adding the numbers in the 'channels' between the dotted L-shaped lines, which gives

$$n^2 = 1 + 3 + 5 + \dots \text{ (to } n \text{ terms).}$$

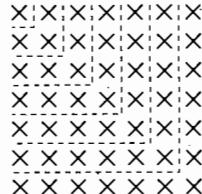


Fig. 8.8

Example 8.5.3

A student reading a 426-page book finds that he reads faster as he gets into the subject. He reads 19 pages on the first day, and his rate of reading then goes up by 3 pages each day. How long does he take to finish the book?

You are given that $a = 19$, $d = 3$ and $S = 426$. Since $S = \frac{1}{2}n(2a + (n-1)d)$,

$$426 = \frac{1}{2}n(38 + (n-1)3),$$

$$852 = n(3n + 35),$$

$$3n^2 + 35n - 852 = 0.$$

Using the quadratic formula,

$$n = \frac{-35 \pm \sqrt{35^2 - 4 \times 3 \times (-852)}}{2 \times 3} = \frac{-35 \pm 107}{6}.$$

Since n must be positive, $n = \frac{-35 + 107}{6} = \frac{72}{6} = 12$. He will finish the book in 12 days.

Exercise 8C

- 1** Which of the following sequences are the first four terms of an arithmetic sequence? For those that are, write down the value of the common difference.
- (a) 7 10 13 16 ... (b) 3 5 9 15 ... (c) 1 0.1 0.01 0.001 ...
 (d) 4 2 0 -2 ... (e) 2 -3 4 -5 ... (f) $p-2q$ $p-q$ p $p+q$...
 (g) $\frac{1}{2}a$ $\frac{1}{3}a$ $\frac{1}{4}a$ $\frac{1}{5}a$... (h) x $2x$ $3x$ $4x$...
- 2** Write down the sixth term, and an expression for the r th term, of the arithmetic sequences which begin as follows.
- (a) 2 4 6 ... (b) 17 20 23 ... (c) 5 2 -1 ...
 (d) 1.3 1.7 2.1 ... (e) 1 $1\frac{1}{2}$ 2 ... (f) 73 67 61 ...
 (g) x $x+2$ $x+4$... (h) $1-x$ 1 $1+x$...
- 3** In the following arithmetic progressions, the first three terms and the last term are given. Find the number of terms.
- (a) 4 5 6 ... 17 (b) 3 9 15 ... 525
 (c) 8 2 -4 ... -202 (d) $2\frac{1}{8}$ $3\frac{1}{4}$ $4\frac{3}{8}$... $13\frac{3}{8}$
 (e) $3x$ $7x$ $11x$... $43x$ (f) -3 $-1\frac{1}{2}$ 0 ... 12
 (g) $\frac{1}{6}$ $\frac{1}{3}$ $\frac{1}{2}$... $2\frac{2}{3}$ (h) $1-2x$ $1-x$ 1 ... $1+25x$
- 4** Find the sum of the given number of terms of the following arithmetic series.
- (a) $2+5+8+\dots$ (20 terms) (b) $4+11+18+\dots$ (15 terms)
 (c) $8+5+2+\dots$ (12 terms) (d) $\frac{1}{2}+1+1\frac{1}{2}+\dots$ (58 terms)
 (e) $7+3+(-1)+\dots$ (25 terms) (f) $1+3+5+\dots$ (999 terms)
 (g) $a+5a+9a+\dots$ (40 terms) (h) $-3p-6p-9p-\dots$ (100 terms)
- 5** Find the number of terms and the sum of each of the following arithmetic series.
- (a) $5+7+9+\dots+111$ (b) $8+12+16+\dots+84$
 (c) $7+13+19+\dots+277$ (d) $8+5+2+\dots+(-73)$
 (e) $-14-10-6-\dots+94$ (f) $157+160+163+\dots+529$
 (g) $10+20+30+\dots+10\,000$ (h) $1.8+1.2+0.6+\dots+(-34.2)$
- 6** In each of the following arithmetic sequences you are given two terms. Find the first term and the common difference.
- (a) 4th term = 15, 9th term = 35 (b) 3rd term = 12, 10th term = 47
 (c) 8th term = 3.5, 13th term = 5.0 (d) 5th term = 2, 11th term = -13
 (e) 12th term = -8, 20th term = -32 (f) 3rd term = -3, 7th term = 5
 (g) 2nd term = $2x$, 11th term = $-7x$ (h) 3rd term = $2p+7$, 7th term = $4p+19$
- 7** Find how many terms of the given arithmetic series must be taken to reach the given sum.
- (a) $3+7+11+\dots$, sum = 820 (b) $8+9+10+\dots$, sum = 162
 (c) $20+23+26+\dots$, sum = 680 (d) $27+23+19+\dots$, sum = -2040
 (e) $1.1+1.3+1.5+\dots$, sum = 1017.6 (f) $-11-4+3+\dots$, sum = 2338

- 8** A squirrel is collecting nuts. It collects 5 nuts on the first day of the month, 8 nuts on the second, 11 on the third and so on in arithmetic progression.
- How many nuts will it collect on the 20th day?
 - After how many days will it have collected more than 1000 nuts?
- 9** Kulsum is given an interest-free loan to buy a car. She repays the loan in unequal monthly instalments; these start at \$30 in the first month and increase by \$2 each month after that. She makes 24 payments.
- Find the amount of her final payment.
 - Find the amount of her loan.
- 10** (a) Find the sum of the natural numbers from 1 to 100 inclusive.
 (b) Find the sum of the natural numbers from 101 to 200 inclusive.
 (c) Find and simplify an expression for the sum of the natural numbers from $n+1$ to $2n$ inclusive.
- 11** An employee starts work on 1 January 2000 on an annual salary of \$30,000. His pay scale will give him an increase of \$800 per annum on the first of January until 1 January 2015 inclusive. He remains on this salary until he retires on 31 December 2040. How much will he earn during his working life?

Miscellaneous exercise 8

- 1** A sequence is defined inductively by $u_{r+1} = 3u_r - 1$ and $u_0 = c$.
- Find the first five terms of the sequence if (i) $c = 1$, (ii) $c = 2$, (iii) $c = 0$, (iv) $c = \frac{1}{2}$.
 - Show that, for each of the values of c in part (a), the terms of the sequence are given by the formula $u_r = \frac{1}{2} + b \times 3^r$ for some value of b .
 - Show that, if $u_r = \frac{1}{2} + b \times 3^r$ for some value of r , then $u_{r+1} = \frac{1}{2} + b \times 3^{r+1}$.
- 2** The sequence u_1, u_2, u_3, \dots is defined by

$$u_1 = 0, \quad u_{r+1} = (2 + u_r)^2.$$
- Find the value of u_4 .
- 3** The sequence u_1, u_2, u_3, \dots , where u_1 is a given real number, is defined by

$$u_{n+1} = \sqrt{(4 - u_n)^2}.$$
- Given that $u_1 = 1$, evaluate u_2 , u_3 and u_4 , and describe the behaviour of the sequence.
 - Given alternatively that $u_1 = 6$, describe the behaviour of the sequence.
 - For what value of u_1 will all the terms of the sequence be equal to each other?
- (OCR, adapted)
- 4** The sequence u_1, u_2, u_3, \dots , where u_1 is a given real number, is defined by

$$u_{n+1} = u_n^2 - 1.$$
- Describe the behaviour of the sequence for each of the cases $u_1 = 0$, $u_1 = 1$ and $u_1 = 2$.
 - Given that $u_2 = u_1$, find exactly the two possible values of u_1 .
 - Given that $u_3 = u_1$, show that $u_1^4 - 2u_1^2 - u_1 = 0$.
- (OCR)

- 5** The r th term of an arithmetic progression is $1+4r$. Find, in terms of n , the sum of the first n terms of the progression. (OCR)
- 6** The sum of the first two terms of an arithmetic progression is 18 and the sum of the first four terms is 52. Find the sum of the first eight terms. (OCR)
- 7** The sum of the first twenty terms of an arithmetic progression is 50, and the sum of the next twenty terms is -50. Find the sum of the first hundred terms of the progression. (OCR)
- 8** An arithmetic progression has first term a and common difference -1. The sum of the first n terms is equal to the sum of the first $3n$ terms. Express a in terms of n . (OCR)
- 9** Find the sum of the arithmetic progression 1, 4, 7, 10, 13, 16, ..., 1000.
Every third term of the above progression is removed, i.e. 7, 16, etc. Find the sum of the remaining terms. (OCR)
- 10** The sum of the first hundred terms of an arithmetic progression with first term a and common difference d is T . The sum of the first 50 odd-numbered terms, i.e. the first, third, fifth, ..., ninety-ninth, is $\frac{1}{2}T - 1000$. Find the value of d . (OCR)
- 11** In the sequence 1.0, 1.1, 1.2, ..., 99.9, 100.0, each number after the first is 0.1 greater than the preceding number. Find
 (a) how many numbers there are in the sequence,
 (b) the sum of all the numbers in the sequence. (OCR)
- 12** The sequence u_1, u_2, u_3, \dots is defined by $u_n = 2n^2$.
 (a) Write down the value of u_3 .
 (b) Express $u_{n+1} - u_n$ in terms of n , simplifying your answer.
 (c) The differences between successive terms of the sequence form an arithmetic progression. For this arithmetic progression, state its first term and its common difference, and find the sum of its first 1000 terms. (OCR)
- 13** A small company producing children's toys plans an increase in output. The number of toys produced is to be increased by 8 each week until the weekly number produced reaches 1000. In week 1, the number to be produced is 280; in week 2, the number is 288; etc. Show that the weekly number produced will be 1000 in week 91.
From week 91 onwards, the number produced each week is to remain at 1000. Find the total number of toys to be produced over the first 104 weeks of the plan. (OCR)
- 14** In 1971 a newly-built flat was sold with a 999-year lease. The terms of the sale included a requirement to pay 'ground rent' yearly. The ground rent was set at £28 per year for the first 21 years of the lease, increasing by £14 to £42 per year for the next 21 years, and then increasing again by £14 at the end of each subsequent period of 21 years.
 (a) Find how many complete 21-year periods there would be if the lease ran for the full 999 years, and how many years there would be left over.
 (b) Find the total amount of ground rent that would be paid in all of the complete 21-year periods of the lease. (OCR)

- 15** An arithmetic progression has first term a and common difference 10. The sum of the first n terms of the progression is 10 000. Express a in terms of n , and show that the n th term of the progression is

$$\frac{10\ 000}{n} + 5(n-1).$$

Given that the n th term is less than 500, show that $n^2 - 101n + 2000 < 0$ and hence find the largest possible value of n . (OCR)

- 16** Three sequences are defined inductively by

- (a) $u_0 = 0$ and $u_{r+1} = u_r + (2r+1)$,
- (b) $u_0 = 0$, $u_1 = 1$ and $u_{r+1} = 2u_r - u_{r-1}$ for $r \geq 1$,
- (c) $u_0 = 1$, $u_1 = 2$ and $u_{r+1} = 3u_r - 2u_{r-1}$ for $r \geq 1$.

For each sequence calculate the first few terms, and suggest a formula for u_r . Check that the formula you have suggested does in fact satisfy all parts of the definition.

- 17** A sequence F_n is constructed from terms of Pascal sequences as follows:

$$F_0 = \binom{0}{0}, F_1 = \binom{1}{0} + \binom{0}{1}, F_2 = \binom{2}{0} + \binom{1}{1} + \binom{0}{2}, \text{ and in general}$$

$$F_n = \binom{n}{0} + \binom{n-1}{1} + \dots + \binom{1}{n-1} + \binom{0}{n}$$

Show that terms of the sequence F_n can be calculated by adding up numbers in Fig. 8.5 along diagonal lines. Verify by calculation that, for small values of n , $F_{n+1} = F_n + F_{n-1}$. (This is called the *Fibonacci sequence*, after the man who introduced algebra from the Arabic world to Italy in about the year 1200.)

Use the Pascal sequence property $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$ (see Exercise 8B Question 10) to explain why $F_3 + F_4 = F_5$ and $F_4 + F_5 = F_6$.

9 The binomial theorem

This chapter is about the expansion of $(x+y)^n$, where n is a positive integer (or zero). When you have completed it, you should

- be able to use Pascal's triangle to find the expansion of $(x+y)^n$ when n is small
- know how to calculate the coefficients in the expansion of $(x+y)^n$ when n is large
- be able to use the notation $\binom{n}{r}$ in the context of the binomial theorem.

9.1 Expanding $(x+y)^n$

The binomial theorem is about calculating $(x+y)^n$ quickly and easily. It is useful to start by looking at $(x+y)^n$ for $n = 2, 3$ and 4 .

The expansions are:

$$(x+y)^2 = x(x+y) + y(x+y) = x^2 + 2xy + y^2,$$

$$\begin{aligned}(x+y)^3 &= (x+y)(x+y)^2 = (x+y)(x^2 + 2xy + y^2) \\&= x(x^2 + 2xy + y^2) + y(x^2 + 2xy + y^2) \\&= x^3 + 2x^2y + xy^2 \\&\quad + x^2y + 2xy^2 + y^3 \\&= \underline{x^3 + 3x^2y + 3xy^2 + y^3},\end{aligned}$$

$$\begin{aligned}(x+y)^4 &= (x+y)(x+y)^3 = (x+y)(x^3 + 3x^2y + 3xy^2 + y^3) \\&= x(x^3 + 3x^2y + 3xy^2 + y^3) + y(x^3 + 3x^2y + 3xy^2 + y^3) \\&= x^4 + 3x^3y + 3x^2y^2 + xy^3 \\&\quad + x^3y + 3x^2y^2 + 3xy^3 + y^4 \\&= \underline{x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4}.\end{aligned}$$

You can summarise these results, including $(x+y)^1$, as follows. The coefficients are in bold type.

$$(x+y)^1 = \mathbf{1}x + \mathbf{1}y$$

$$(x+y)^2 = \mathbf{1}x^2 + \mathbf{2}xy + \mathbf{1}y^2$$

$$(x+y)^3 = \mathbf{1}x^3 + \mathbf{3}x^2y + \mathbf{3}xy^2 + \mathbf{1}y^3$$

$$(x+y)^4 = \mathbf{1}x^4 + \mathbf{4}x^3y + \mathbf{6}x^2y^2 + \mathbf{4}xy^3 + \mathbf{1}y^4$$

Study these expansions carefully. Notice how the powers start from the left with x^n . The powers of x then successively reduce by 1, and the powers of y increase by 1 until reaching the term y^n .

Notice also that the coefficients form the pattern of Pascal's triangle, which you saw in Section 8.4 and which is shown again in Fig. 9.1.

A simple way of building up Pascal's triangle is as follows: start with 1; then add pairs of elements in the row above to get the entry positioned below and between them (as the arrows in Fig. 9.1 show); complete the row with a 1. This is identical to the way in which the two rows are added to give the final result in the expansions of $(x+y)^3$ and $(x+y)^4$ on the previous page.

You should now be able to predict that the coefficients in the fifth row are

$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$

and that

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.$$

Example 9.1.1

Write down the expansion of $(1+y)^6$.

Use the next row of Pascal's triangle, continuing the pattern of powers and replacing x by 1:

$$\begin{aligned}(1+y)^6 &= (1)^6 + 6(1)^5y + 15(1)^4y^2 + 20(1)^3y^3 + 15(1)^2y^4 + 6(1)y^5 + y^6 \\ &= 1 + 6y + 15y^2 + 20y^3 + 15y^4 + 6y^5 + y^6.\end{aligned}$$

Example 9.1.2

Multiply out the brackets in the expression $(2x+3)^4$.

Use the expansion of $(x+y)^4$, replacing x by $(2x)$ and replacing y by 3:

$$\begin{aligned}(2x+3)^4 &= (2x)^4 + 4 \times (2x)^3 \times 3 + 6 \times (2x)^2 \times 3^2 + 4 \times (2x) \times 3^3 + 3^4 \\ &= 16x^4 + 96x^3 + 216x^2 + 216x + 81.\end{aligned}$$

Example 9.1.3

Expand $(x^2+2)^3$.

$$(x^2+2)^3 = (x^2)^3 + 3 \times (x^2)^2 \times 2 + 3 \times x^2 \times 2^2 + 2^3 = x^6 + 6x^4 + 12x^2 + 8.$$

Example 9.1.4

Find the coefficient of x^3 in the expansion of $(3x-4)^5$.

The term in x^3 comes third in the row with coefficients 1, 5, 10, So the term is

$$10 \times (3x)^3 \times (-4)^2 = 10 \times 27 \times 16x^3 = 4320x^3.$$

The required coefficient is therefore 4320.

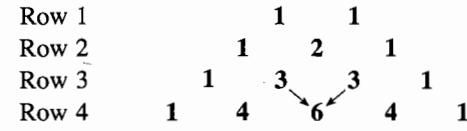


Fig. 9.1

Example 9.1.5

Expand $(1 + 2x + 3x^2)^3$.

To use the binomial expansion, you need to write $1 + 2x + 3x^2$ in a form with two terms rather than three. One way to do this is to consider $(1 + (2x + 3x^2))^3$. Then

$$(1 + (2x + 3x^2))^3 = 1^3 + 3 \times 1^2 \times (2x + 3x^2) + 3 \times 1 \times (2x + 3x^2)^2 + (2x + 3x^2)^3.$$

Now you can use the binomial theorem to expand the bracketed terms:

$$\begin{aligned} (1 + 2x + 3x^2)^3 &= 1 + 3(2x + 3x^2) + 3((2x)^2 + 2 \times (2x) \times (3x^2) + (3x^2)^2) \\ &\quad + ((2x)^3 + 3 \times (2x)^2 \times (3x^2) + 3 \times (2x) \times (3x^2)^2 + (3x^2)^3) \\ &= 1 + (6x + 9x^2) + (12x^2 + 36x^3 + 27x^4) \\ &\quad + (8x^3 + 36x^4 + 54x^5 + 27x^6) \\ &= 1 + 6x + 21x^2 + 44x^3 + 63x^4 + 54x^5 + 27x^6. \end{aligned}$$

In this kind of detailed work, it is useful to check your answers. You could do this by expanding $(1 + 2x + 3x^2)^3$ in the form $((1 + 2x) + 3x^2)^3$ to see if you get the same answer. Rather quicker is to give x a particular value, $x = 1$ for example. Then the left side is $(1 + 2 + 3)^3 = 6^3 = 216$; the right is $1 + 6 + 21 + 44 + 63 + 54 + 27 = 216$. It is important to note that the results are the same; it does not guarantee that the expansion is correct; but if they are different, it is certain that there is a mistake.

Exercise 9A

1 Write down the expansion of each of the following.

- | | | | |
|--------------------|-------------------|---------------------|-----------------------|
| (a) $(2x + y)^2$ | (b) $(5x + 3y)^2$ | (c) $(4 + 7p)^2$ | (d) $(1 - 8t)^2$ |
| (e) $(1 - 5x^2)^2$ | (f) $(2 + x^3)^2$ | (g) $(x^2 + y^3)^3$ | (h) $(3x^2 + 2y^3)^3$ |

2 Write down the expansion of each of the following.

- | | | | |
|-----------------|-------------------|------------------|-------------------|
| (a) $(x + 2)^3$ | (b) $(2p + 3q)^3$ | (c) $(1 - 4x)^3$ | (d) $(1 - x^3)^3$ |
|-----------------|-------------------|------------------|-------------------|

3 Find the coefficient of x in the expansion of

- | | |
|--------------------|--------------------|
| (a) $(3x + 7)^2$, | (b) $(2x + 5)^3$. |
|--------------------|--------------------|

4 Find the coefficient of x^2 in the expansion of

- | | |
|--------------------|--------------------|
| (a) $(4x + 5)^3$, | (b) $(1 - 3x)^4$. |
|--------------------|--------------------|

5 Expand each of the following expressions.

- | | | | |
|------------------|------------------|-------------------|----------------------------|
| (a) $(1 + 2x)^5$ | (b) $(p + 2q)^6$ | (c) $(2m - 3n)^4$ | (d) $(1 + \frac{1}{2}x)^4$ |
|------------------|------------------|-------------------|----------------------------|

6 Find the coefficient of x^3 in the expansion of

- | | |
|--------------------|--------------------|
| (a) $(1 + 3x)^5$, | (b) $(2 - 5x)^4$. |
|--------------------|--------------------|

7 Expand $(1 + x + 2x^2)^2$. Check your answer with a numerical substitution.

8 Write down the expansion of $(x + 4)^3$ and hence expand $(x + 1)(x + 4)^3$.

- 9** Expand $(3x + 2)^2(2x + 3)^3$.
- 10** In the expansion of $(1 + ax)^4$, the coefficient of x^3 is 1372. Find the constant a .
- 11** Expand $(x + y)^{11}$.
- 12** Find the coefficient of x^6y^6 in the expansion of $(2x + y)^{12}$.

9.2 The binomial theorem

The treatment given in Section 9.1 is fine for finding the coefficients in the expansion of $(x + y)^n$ where n is small, but it is hopelessly inefficient for finding the coefficient of $x^{11}y^4$ in the expansion of $(x + y)^{15}$. Just think of all the rows of Pascal's triangle which you would have to write out! What you need is a formula in terms of n and r for the coefficient of $x^{n-r}y^r$ in the expansion of $(x + y)^n$.

Fortunately, the n th row of Pascal's triangle is the n th Pascal sequence given in Section 8.4. It was shown there that

$$\binom{n}{0} = 1 \quad \text{and} \quad \binom{n}{r+1} = \frac{n-r}{r+1} \binom{n}{r}, \quad \text{where } r = 0, 1, 2, \dots$$

In fact, you can write Pascal's triangle as

Row 1		$\binom{1}{0}$	$\binom{1}{1}$
Row 2		$\binom{2}{0}$	$\binom{2}{1}$
Row 3		$\binom{3}{0}$	$\binom{3}{1}$
Row 4	$\binom{4}{0}$	$\binom{4}{1}$	$\binom{4}{2}$
			$\binom{4}{3}$
			$\binom{4}{4}$

and so on.

This enables you to write down a neater form of the expansion of $(x + y)^n$.

The **binomial theorem** states that, if n is a natural number,

$$(x + y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \dots + \binom{n}{n}y^n.$$

To calculate the coefficients, you can use the inductive formula given at the beginning of this section to generate a formula for $\binom{n}{r}$. For example, to calculate $\binom{4}{2}$, start by putting $n = 4$. Then

$$\binom{4}{0} = 1, \text{ so } \binom{4}{1} = \frac{4-0}{0+1} \binom{4}{0} = \frac{4}{1} \times 1 = \frac{4}{1}, \text{ and } \binom{4}{2} = \frac{4-1}{1+1} \binom{4}{1} = \frac{3}{2} \times \frac{4}{1} = \frac{4 \times 3}{1 \times 2}.$$

In the general case,

$$\binom{n}{0} = 1, \quad \binom{n}{1} = \frac{n-0}{0+1} \times 1 = \frac{n}{1}, \quad \binom{n}{2} = \frac{n-1}{1+1} \binom{n}{1} = \frac{n-1}{2} \times \frac{n}{1} = \frac{n(n-1)}{1 \times 2}, \quad \dots$$

Continuing in this way, you find that $\binom{n}{r} = \frac{n(n-1)\dots(n-(r-1))}{1 \times 2 \times \dots \times r}$.

You can also write $\binom{n}{r}$ in the form

$$\binom{n}{r} = \frac{n(n-1)\dots(n-(r-1))}{1 \times 2 \times \dots \times r} \times \frac{(n-r) \times (n-r-1) \times \dots \times 2 \times 1}{(n-r) \times (n-r-1) \times \dots \times 2 \times 1} = \frac{n!}{r!(n-r)!}.$$

Notice that this formula works for $r=0$ and $r=n$ as well as the values in between, since (from Section 8.3) $0!=1$.

The **binomial coefficients** are given by

$$\binom{n}{r} = \frac{n(n-1)\dots(n-(r-1))}{1 \times 2 \times \dots \times r}, \quad \text{or} \quad \binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

When you use the first formula to calculate any particular value of $\binom{n}{r}$, such as $\binom{10}{4}$ or $\binom{12}{7}$, it is helpful to remember that there are as many factors in the top line as there are

in the bottom. So you can start by putting in the denominators, and then count down from 10 and 12 respectively, making sure that you have the same number of factors in the numerator as in the denominator.

$$\binom{10}{4} = \frac{10 \times 9 \times 8 \times 7}{1 \times 2 \times 3 \times 4} = 210, \quad \binom{12}{7} = \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7} = 792.$$

Many calculators give you values of $\binom{n}{r}$, usually with a key labelled $[_n C_r]$. To find $\binom{10}{4}$, you would normally key in the sequence [10, $_n C_r$, 4], but you may need to check your calculator manual for details.

Example 9.2.1

Calculate the coefficient of $x^{11}y^4$ in the expansion of $(x+y)^{15}$.

$$\text{The coefficient is } \binom{15}{4} = \frac{15 \times 14 \times 13 \times 12}{1 \times 2 \times 3 \times 4} = 1365.$$

One other step is required before you can be sure that the values of $\binom{n}{r}$ are the values that you need for the binomial theorem. In Fig. 9.1 you saw that each term of Pascal's triangle, except for the 1s at the end of each row, is obtained by adding the two terms immediately above it. So it should be true that

$$\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1}.$$

For example:

$$\begin{aligned}\binom{6}{3} + \binom{6}{4} &= \frac{6 \times 5 \times 4}{1 \times 2 \times 3} + \frac{6 \times 5 \times 4 \times 3}{1 \times 2 \times 3 \times 4} = \frac{6 \times 5 \times 4 \times 4 + 6 \times 5 \times 4 \times 3}{1 \times 2 \times 3 \times 4} \\ &= \frac{6 \times 5 \times 4}{1 \times 2 \times 3 \times 4} \times (4+3) \\ &= \frac{7 \times 6 \times 5 \times 4}{1 \times 2 \times 3 \times 4} \\ &= \binom{7}{4}.\end{aligned}$$

The proof that $\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1}$ is not easy. You may wish to accept the result and omit the proof, and jump to Example 9.2.2.

To prove this result, start from the right side.

$$\begin{aligned}\binom{n}{r} + \binom{n}{r+1} &= \frac{n(n-1)\dots(n-(r-1))}{1 \times 2 \times \dots \times r} + \frac{n(n-1)\dots(n-r)}{1 \times 2 \times \dots \times r \times (r+1)} \\ &= \frac{n(n-1)\dots(n-(r-1)) \times (r+1) + n(n-1)\dots(n-r)}{1 \times 2 \times \dots \times r \times (r+1)} \\ &= \frac{n(n-1)\dots(n-(r-1))}{1 \times 2 \times \dots \times r \times (r+1)} \times ((r+1) + (n-r)) \\ &= \frac{n(n-1)\dots(n-(r-1))}{1 \times 2 \times \dots \times r \times (r+1)} \times (n+1) \\ &= \frac{(n+1)n(n-1)\dots((n+1)-r)}{1 \times 2 \times \dots \times r \times (r+1)} \\ &= \binom{n+1}{r+1}.\end{aligned}$$

This completes the chain of reasoning which connects Pascal's triangle with the binomial coefficients.

The following example is one in which the value of x is assumed to be small. When this is the case, say for $x = 0.1$, the successive powers of x decrease by a factor of 10 each time and become very small indeed, so higher powers can be neglected in approximations.

In Example 9.2.2 you are asked to put the terms of a binomial expansion in order of **ascending powers of x** . This means that you start with the term with the smallest power of x , then move to the next smallest, and so on.

Example 9.2.2

Find the first four terms in the expansion of $(2 - 3x)^{10}$ in ascending powers of x . By putting $x = \frac{1}{100}$, find an approximation to 1.97^{10} correct to the nearest whole number.

$$\begin{aligned}(2 - 3x)^{10} &= 2^{10} + \binom{10}{1} \times 2^9 \times (-3x) + \binom{10}{2} \times 2^8 \times (-3x)^2 + \binom{10}{3} \times 2^7 \times (-3x)^3 + \dots \\ &= 1024 - 10 \times 512 \times 3x + \frac{10 \times 9}{1 \times 2} \times 256 \times 9x^2 - \frac{10 \times 9 \times 8}{1 \times 2 \times 3} \times 128 \times 27x^3 + \dots \\ &= 1024 - 15\,360x + 103\,680x^2 - 414\,720x^3 + \dots\end{aligned}$$

The first four terms are therefore $1024 - 15\,360x + 103\,680x^2 - 414\,720x^3$.

Putting $x = \frac{1}{100}$ gives

$$\begin{aligned}1.97^{10} &\approx 1024 - 15\,360 \times \frac{1}{100} + 103\,680 \times \left(\frac{1}{100}\right)^2 - 414\,720 \times \left(\frac{1}{100}\right)^3 \\ &= 880.353\,28.\end{aligned}$$

Therefore $1.97^{10} \approx 880$.

The next term is actually $\binom{10}{4} \times 2^6 \times (3x)^4 = 1\,088\,640x^4 = 0.010\,886\,4$ and the rest are very small indeed.

Exercise 9B

- 1 Find the value of each of the following.

(a) $\binom{7}{3}$	(b) $\binom{8}{6}$	(c) $\binom{9}{5}$	(d) $\binom{13}{4}$
(e) $\binom{6}{4}$	(f) $\binom{10}{2}$	(g) $\binom{11}{10}$	(h) $\binom{50}{2}$

- 2 Find the coefficient of x^3 in the expansion of each of the following.

(a) $(1+x)^5$	(b) $(1-x)^8$	(c) $(1+x)^{11}$	(d) $(1-x)^{16}$
---------------	---------------	------------------	------------------

- 3 Find the coefficient of x^5 in the expansion of each of the following.

(a) $(2+x)^7$	(b) $(3-x)^8$	(c) $(1+2x)^9$	(d) $(1-\frac{1}{2}x)^{12}$
---------------	---------------	----------------	-----------------------------

- 4 Find the coefficient of x^6y^8 in the expansion of each of the following.

(a) $(x+y)^{14}$	(b) $(2x+y)^{14}$	(c) $(3x-2y)^{14}$	(d) $(4x+\frac{1}{2}y)^{14}$
------------------	-------------------	--------------------	------------------------------

- 5 Find the first four terms in the expansion in ascending powers of x of the following.

(a) $(1+x)^{13}$	(b) $(1-x)^{15}$	(c) $(1+3x)^{10}$	(d) $(2-5x)^7$
------------------	------------------	-------------------	----------------

6 Find the first three terms in the expansion in ascending powers of x of the following.

(a) $(1+x)^{22}$ (b) $(1-x)^{30}$ (c) $(1-4x)^{18}$ (d) $(1+6x)^{19}$

7 Find the first three terms in the expansion, in ascending powers of x , of $(1+2x)^8$. By substituting $x = 0.01$, find an approximation to 1.02^8 .

8 Find the first three terms in the expansion, in ascending powers of x , of $(2+5x)^{12}$. By substituting a suitable value for x , find an approximation to 2.005^{12} to 2 decimal places.

9 Expand $(1+2x)^{16}$ up to and including the term in x^3 . Deduce the coefficient of x^3 in the expansion of $(1+3x)(1+2x)^{16}$.

10 Expand $(1-3x)^{10}$ up to and including the term in x^2 . Deduce the coefficient of x^2 in the expansion of $(1+3x)^2(1-3x)^{10}$.

11 Given that the coefficient of x in the expansion of $(1+ax)(1+5x)^{40}$ is 207, determine the value of a .

12 Simplify $(1-x)^8 + (1+x)^8$. Substitute a suitable value of x to find the exact value of $0.99^8 + 1.01^8$.

13 Given that the expansion of $(1+ax)^n$ begins $1+36x+576x^2$, find the values of a and n .

Miscellaneous exercise 9

1 Expand $(3+4x)^3$.

2 Find the first three terms in the expansions, in ascending powers of x , of
 (a) $(1+4x)^{10}$, (b) $(1-2x)^{16}$.

3 Find the coefficient of a^3b^5 in the expansions of

(a) $(3a-2b)^8$, (b) $\left(5a+\frac{1}{2}b\right)^8$.

4 Expand $(3+5x)^7$ in ascending powers of x up to and including the term in x^2 . By putting $x = 0.01$, find an approximation, correct to the nearest whole number, to 3.05^7 .

5 Obtain the first four terms in the expansion of $\left(2+\frac{1}{4}x\right)^8$ in ascending powers of x . By substituting an appropriate value of x into this expansion, find the value of 2.0025^8 correct to three decimal places. (OCR)

6 Find, in ascending powers of x , the first three terms in the expansion of $(2-3x)^8$. Use the expansion to find the value of 1.997^8 to the nearest whole number. (OCR)

7 Expand $\left(x^2 + \frac{1}{x}\right)^3$, simplifying each of the terms.

8 Expand $\left(2x - \frac{3}{x^2}\right)^4$.

- 9** Expand and simplify $\left(x + \frac{1}{2x}\right)^6 + \left(x - \frac{1}{2x}\right)^6$. (OCR)
- 10** Find the coefficient of x^2 in the expansion of $\left(x^4 + \frac{4}{x}\right)^3$.
- 11** Find the term independent of x in the expansion of $\left(2x + \frac{5}{x}\right)^6$.
- 12** Find the coefficient of y^4 in the expansion of $(1+y)^{12}$. Deduce the coefficient of
 (a) y^4 in the expansion of $(1+3y)^{12}$,
 (b) y^8 in the expansion of $(1-2y^2)^{12}$,
 (c) x^8y^4 in the expansion of $\left(x + \frac{1}{2}y\right)^{12}$.
- 13** Determine the coefficient of p^4q^7 in the expansion of $(2p-q)(p+q)^{10}$.
- 14** Find the first three terms in the expansion of $(1+2x)^{20}$. By substitution of a suitable value of x in each case, find approximations to
 (a) 1.002^{20} , (b) 0.996^{20} .
- 15** Write down the first three terms in the binomial expansion of $\left(2 - \frac{1}{2x^2}\right)^{10}$ in ascending powers of x . Hence find the value of 1.995^{10} correct to three significant figures. (OCR)
- 16** Two of the following expansions are correct and two are incorrect. Find the two expansions which are incorrect.
 A: $(3+4x)^5 = 243 + 1620x + 4320x^2 + 5760x^3 + 3840x^4 + 1024x^5$
 B: $(1-2x+3x^2)^3 = 1 + 6x - 3x^2 + 28x^3 - 9x^4 + 54x^5 - 27x^6$
 C: $(1-x)(1+4x)^4 = 1 + 15x + 80x^2 + 160x^3 - 256x^5$
 D: $(2x+y)^2(3x+y)^3 = 108x^5 + 216x^4y + 171x^3y^2 + 67x^2y^3 + 13xy^4 + y^6$
- 17** Find and simplify the term independent of x in the expansion of $\left(\frac{1}{2x} + x^3\right)^8$. (OCR)
- 18** Find the term independent of x in the expansion of $\left(2x + \frac{1}{x^2}\right)^9$.
- 19** Evaluate the term which is independent of x in the expansion of $\left(x^2 - \frac{1}{2x^2}\right)^{16}$. (OCR)
- 20** Find the coefficient of x^{-12} in the expansion of $\left(x^3 - \frac{1}{x}\right)^{24}$. (OCR)
- 21** Expand $(1+3x+4x^2)^4$ in ascending powers of x as far as the term in x^2 . By substituting a suitable value of x , find an approximation to 1.0304^4 .
- 22** Expand and simplify $(3x+5)^3 - (3x-5)^3$.
 Hence solve the equation $(3x+5)^3 - (3x-5)^3 = 730$.
- 23** Solve the equation $(7-6x)^3 + (7+6x)^3 = 1736$.

24 Find, in ascending powers of t , the first three terms in the expansions of

(a) $(1+\alpha t)^5$, (b) $(1-\beta t)^8$.

Hence find, in terms of α and β , the coefficient of t^2 in the expansion of $(1+\alpha t)^5(1-\beta t)^8$.

(OCR)

25 (a) Show that

(i) $\binom{6}{4} = \binom{6}{2}$, (ii) $\binom{10}{3} = \binom{10}{7}$, (iii) $\binom{15}{12} = \binom{15}{3}$, (iv) $\binom{13}{6} = \binom{13}{7}$.

(b) State the possible values of x in each of the following.

(i) $\binom{11}{4} = \binom{11}{x}$ (ii) $\binom{16}{3} = \binom{16}{x}$ (iii) $\binom{20}{7} = \binom{20}{x}$ (iv) $\binom{45}{17} = \binom{45}{x}$

(c) Use the definition $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ to prove that $\binom{n}{r} = \binom{n}{n-r}$.

26 The inductive property $\binom{n}{r+1} = \frac{n-r}{r+1} \binom{n}{r}$ was given in Section 8.4. Use this to prove the Pascal triangle property that $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$.

27 (a) Show that

(i) $4 \times \binom{6}{2} = 3 \times \binom{6}{3} = 6 \times \binom{5}{2}$, (ii) $3 \times \binom{7}{4} = 5 \times \binom{7}{5} = 7 \times \binom{6}{4}$.

(b) State numbers a , b and c such that

(i) $a \times \binom{8}{5} = b \times \binom{8}{6} = c \times \binom{7}{5}$, (ii) $a \times \binom{9}{3} = b \times \binom{9}{4} = c \times \binom{8}{3}$.

(c) Prove that $(n-r) \times \binom{n}{r} = (r+1) \times \binom{n}{r+1} = n \times \binom{n-1}{r}$.

28 Prove that $\binom{n}{r-1} + 2\binom{n}{r} + \binom{n}{r+1} = \binom{n+2}{r+1}$.

29 Find the value of 1.0003^{18} correct to 15 decimal places.

30 (a) Expand $(2\sqrt{2} + \sqrt{3})^4$ in the form $a + b\sqrt{6}$, where a and b are integers.

(b) Find the exact value of $(2\sqrt{2} + \sqrt{3})^5$.

31 (a) Expand and simplify $(\sqrt{7} + \sqrt{5})^4 + (\sqrt{7} - \sqrt{5})^4$. By using the fact that $0 < \sqrt{7} - \sqrt{5} < 1$, state the consecutive integers between which $(\sqrt{7} + \sqrt{5})^4$ lies.

(b) Without using a calculator, find the consecutive integers between which the value of $(\sqrt{3} + \sqrt{2})^6$ lies.

32 Find an expression, in terms of n , for the coefficient of x in the expansion

$$(1+4x)+(1+4x)^2+(1+4x)^3+\dots+(1+4x)^n.$$

33 Given that

$$a+b(1+x)^3+c(1+2x)^3+d(1+3x)^3=x^3$$

for all values of x , find the values of the constants a , b , c and d .

10 Trigonometry

This chapter develops work on sines, cosines and tangents. When you have completed it, you should

- know the shapes of the graphs of sine, cosine and tangent for all angles
- know, or be able to find, exact values of the sine, cosine and tangent of certain special angles
- be able to solve simple trigonometric equations
- know and be able to use identities involving $\sin \theta^\circ$, $\cos \theta^\circ$ and $\tan \theta^\circ$.

10.1 The graph of $\cos \theta^\circ$

Letters of the Greek alphabet are often used to denote angles. In this chapter, θ (theta) and ϕ (phi) will usually be used.

You probably first used $\cos \theta^\circ$ in calculations with right-angled triangles, so that $0 < \theta < 90^\circ$. Then you may have used it in any triangle, with $0 < \theta < 180^\circ$. However, if you have a graphic calculator, you will find that it produces a graph of $\cos \theta^\circ$ like that in Fig. 10.3. This section extends the definition of $\cos \theta^\circ$ to angles of any size, positive or negative.

Fig. 10.1 shows a circle of radius 1 unit with centre O ; the circle meets the x -axis at A . Draw a line OP at an angle θ to the x -axis, to meet the circle at P . Draw a perpendicular from P to meet OA at N . Let $ON = x$ units and $NP = y$ units, so that the coordinates of P are (x, y) .

Look at triangle ONP . Using the definition $\cos \theta^\circ = \frac{ON}{OP}$, you find that $\cos \theta^\circ = \frac{x}{1} = x$.

This result, $\cos \theta^\circ = x$, is used as the definition of $\cos \theta^\circ$ for all values of θ .

You can see the consequences of this definition whenever θ is a multiple of 90° .

Example 10.1.1

Find the value of $\cos \theta^\circ$ when (a) $\theta = 180^\circ$, (b) $\theta = 270^\circ$.

(a) When $\theta = 180^\circ$, P is the point $(-1, 0)$. As the x -coordinate of P is -1 ,

$$\cos 180^\circ = -1.$$

(b) When $\theta = 270^\circ$, P is the point $(0, -1)$, so $\cos 270^\circ = 0$.

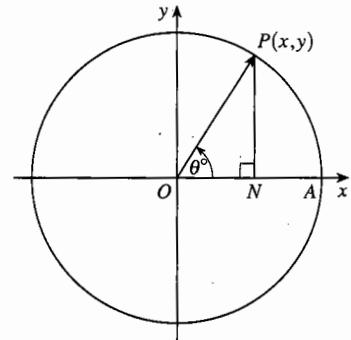


Fig. 10.1

As θ increases, the point P moves round the circle. When $\theta = 360^\circ$, P is once again at A , and as θ becomes greater than 360° , the point P moves round the circle again. It follows immediately that $\cos(\theta - 360^\circ) = \cos\theta^\circ$, and that the values of $\cos\theta^\circ$ repeat themselves every time θ increases by 360° .

If $\theta < 0^\circ$, the angle θ is drawn in the opposite direction, starting once again from A . Fig. 10.2 shows the angle -150° drawn. Thus, if $\theta = -150^\circ$, P is in the third quadrant, and, since the x -coordinate of P is negative, $\cos(-150^\circ)$ is negative.

A calculator will give you values of $\cos\theta^\circ$ for all values of θ . If you have access to a graphic calculator you should use it to display the graph of $\cos\theta^\circ$, shown in Fig. 10.3.

You will have to input the equation of the graph of $\cos\theta^\circ$ as $y = \cos x$ into the calculator, and make sure that the calculator is in degree mode.

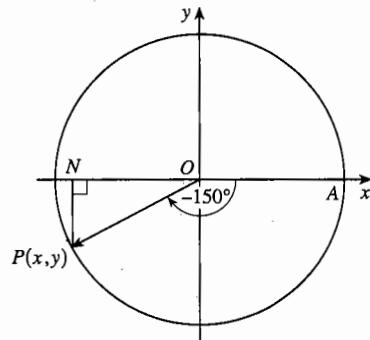


Fig. 10.2

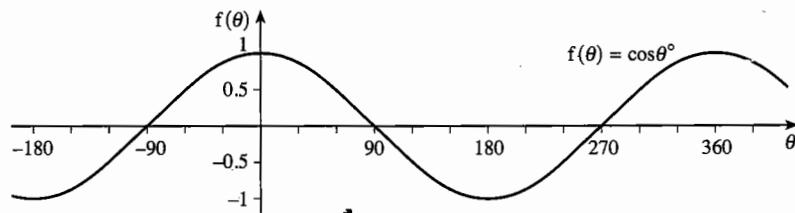


Fig. 10.3

Note that the range of the cosine function is $-1 \leq \cos\theta^\circ \leq 1$. The maximum value of 1 is taken at $\theta = 0, \pm 360, \pm 720, \dots$, and the minimum of -1 at $\theta = \pm 180, \pm 540, \dots$

The graph of the cosine function keeps repeating itself. Functions with this property are called **periodic**; the **period** of such a function is the smallest interval for which the function repeats itself. The period of the cosine function is therefore 360. The property that $\cos(\theta \pm 360^\circ) = \cos\theta^\circ$ is called the **periodic property**. Many natural phenomena have periodic properties, and the cosine is often used in applications involving them.

Example 10.1.2

The height in metres of the water in a harbour is given approximately by the formula $d = 6 + 3 \cos 30t^\circ$ where t is the time in hours from noon. Find (a) the height of the water at 9.45 p.m., and (b) the highest and lowest water levels, and when they occur.

(a) At 9.45 p.m., $t = 9.75$, so $d = 6 + 3 \cos(30 \times 9.75)^\circ = 6 + 3 \cos 292.5^\circ = 7.148\dots$

Therefore the height of the water is 7.15 metres, correct to 3 significant figures.

(b) The maximum value of d occurs when the value of the cosine function is 1, and is therefore $6 + 3 \times 1 = 9$. Similarly, the minimum value is $6 + 3 \times (-1) = 3$. The highest and lowest water levels are 9 metres and 3 metres. The first times that they occur after noon are when $30t = 360$ and $30t = 180$; that is, at midnight and 6.00 p.m.

10.2 The graphs of $\sin \theta^\circ$ and $\tan \theta^\circ$

Using the same construction as for the cosine (see Fig. 10.1), the sine function is defined by

$$\sin \theta^\circ = \frac{NP}{OP} = \frac{y}{1} = y.$$

Like the cosine graph, the sine graph (shown in Fig. 10.4) is periodic, with period 360°. It also lies between -1 and 1 inclusive.

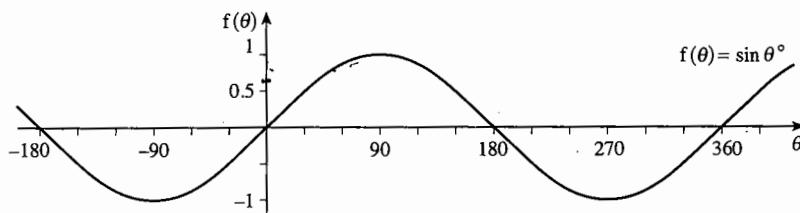


Fig. 10.4

If you return to Fig. 10.1, you will see that $\tan \theta^\circ = \frac{NP}{ON} = \frac{y}{x}$; this is taken as the definition of $\tan \theta^\circ$. The domain of $\tan \theta^\circ$ does not include those angles for which x is zero, namely $\theta = \pm 90^\circ, \pm 270^\circ, \dots$. Fig. 10.5 shows the graph of $\tan \theta^\circ$.

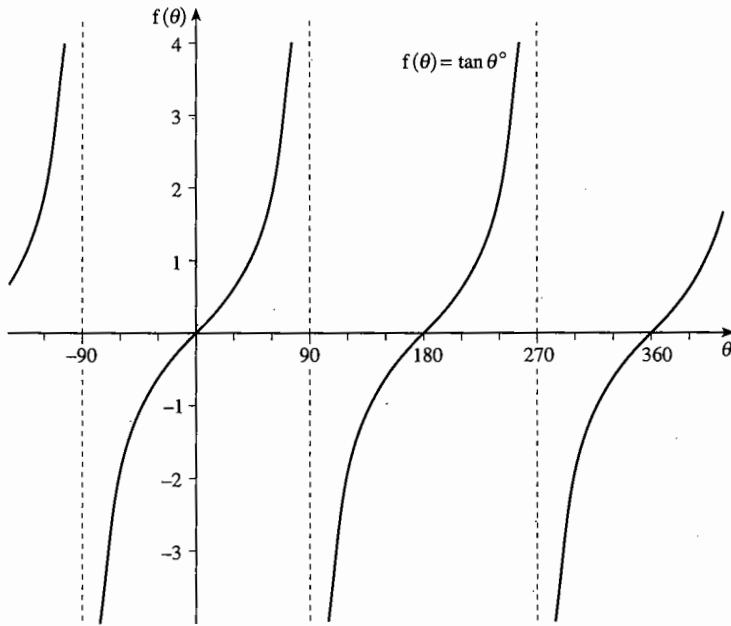


Fig. 10.5

Like the graphs of $\cos \theta^\circ$ and $\sin \theta^\circ$, the graph of $\tan \theta^\circ$ is periodic, but its period is 180°. Thus $\tan(\theta \pm 180^\circ) = \tan \theta^\circ$.

As $\cos \theta^\circ = x$, $\sin \theta^\circ = y$ and $\tan \theta^\circ = \frac{y}{x}$, it follows that $\tan \theta^\circ = \frac{\sin \theta^\circ}{\cos \theta^\circ}$. You could use this as an alternative definition of $\tan \theta^\circ$.

10.3 Exact values of some trigonometric functions

There are a few angles which are a whole number of degrees and whose sines, cosines and tangents you can find exactly. The most important of these are 45° , 60° and 30° .

To find the cosine, sine and tangent of 45° , draw a right-angled isosceles triangle of side 1 unit, as in Fig. 10.6. The length of the hypotenuse is then $\sqrt{2}$ units. Then

$$\cos 45^\circ = \frac{1}{\sqrt{2}}, \quad \sin 45^\circ = \frac{1}{\sqrt{2}}, \quad \tan 45^\circ = 1.$$

If you rationalise the denominators you get

$$\cos 45^\circ = \frac{\sqrt{2}}{2}, \quad \sin 45^\circ = \frac{\sqrt{2}}{2}, \quad \tan 45^\circ = 1.$$

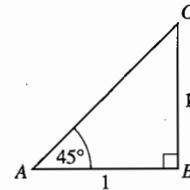


Fig. 10.6

To find the cosine, sine and tangent of 60° and 30° , draw an equilateral triangle of side 2 units, as in Fig. 10.7. Draw a perpendicular from one vertex, bisecting the opposite side. This perpendicular has length $\sqrt{3}$ units, and it makes an angle of 30° with AC . Then

$$\cos 60^\circ = \frac{1}{2}, \quad \sin 60^\circ = \frac{\sqrt{3}}{2}, \quad \tan 60^\circ = \sqrt{3};$$

$$\cos 30^\circ = \frac{\sqrt{3}}{2}, \quad \sin 30^\circ = \frac{1}{2}, \quad \tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

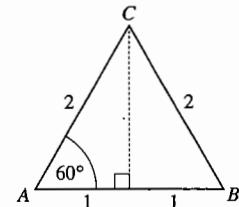


Fig. 10.7

You should learn these results, or be able to reproduce them quickly.

Example 10.3.1

Write down the exact values of (a) $\cos 135^\circ$, (b) $\sin 120^\circ$, (c) $\tan 495^\circ$.

(a) From Fig. 10.3, $\cos 135^\circ = -\cos 45^\circ = -\frac{1}{2}\sqrt{2}$.

(b) From Fig. 10.4, $\sin 120^\circ = \sin 60^\circ = \frac{1}{2}\sqrt{3}$.

(c) From Fig. 10.5, $\tan 495^\circ = \tan(495 - 360)^\circ = \tan 135^\circ = -\tan 45^\circ = -1$.

Exercise 10A

- 1 For each of the following values of θ find, correct to 4 decimal places, the values of
 (i) $\cos \theta^\circ$, (ii) $\sin \theta^\circ$, (iii) $\tan \theta^\circ$.
- | | | | |
|----------|----------|-----------|---------|
| (a) 25 | (b) 125 | (c) 225 | (d) 325 |
| (e) -250 | (f) 67.4 | (g) 124.9 | (h) 554 |

- 2 Find the maximum value and the minimum value of each of the following functions. In each case, give the least positive values of x at which they occur.

(a) $2 + \sin x^\circ$ (b) $7 - 4\cos x^\circ$ (c) $5 + 8\cos 2x^\circ$

(d) $\frac{8}{3 - \sin x^\circ}$ (e) $9 + \sin(4x - 20)^\circ$ (f) $\frac{30}{11 - 5\cos(\frac{1}{2}x - 45)^\circ}$

- 3 (Do not use a calculator for this question.) In each part of the question a trigonometric function of a number is given. Find all the other numbers x , $0 \leq x \leq 360$, such that the same function of x is equal to the given trigonometric ratio. For example, if you are given $\sin 80^\circ$, then $x = 100$, since $\sin 100^\circ = \sin 80^\circ$.

(a) $\sin 20^\circ$ (b) $\cos 40^\circ$ (c) $\tan 60^\circ$ (d) $\sin 130^\circ$

(e) $\cos 140^\circ$ (f) $\tan 160^\circ$ (g) $\sin 400^\circ$ (h) $\cos(-30)^\circ$

(i) $\tan 430^\circ$ (j) $\sin(-260)^\circ$ (k) $\cos(-200)^\circ$ (l) $\tan 1000^\circ$

- 4 (Do not use a calculator for this question.) In each part of the question a trigonometric function of a number is given. Find all the other numbers x , $-180 \leq x \leq 180$, such that the same function of x is equal to the given trigonometric ratio. For example, if you are given $\sin 80^\circ$, then $x = 100$, since $\sin 100^\circ = \sin 80^\circ$.

(a) $\sin 20^\circ$ (b) $\cos 40^\circ$ (c) $\tan 60^\circ$ (d) $\sin 130^\circ$

(e) $\cos 140^\circ$ (f) $\tan 160^\circ$ (g) $\sin 400^\circ$ (h) $\cos(-30)^\circ$

(i) $\tan 430^\circ$ (j) $\sin(-260)^\circ$ (k) $\cos(-200)^\circ$ (l) $\tan 1000^\circ$

- 5 Without using a calculator, write down the exact values of the following.

(a) $\sin 135^\circ$ (b) $\cos 120^\circ$ (c) $\sin(-30)^\circ$ (d) $\tan 240^\circ$

(e) $\cos 225^\circ$ (f) $\tan(-330)^\circ$ (g) $\cos 900^\circ$ (h) $\tan 510^\circ$

(i) $\sin 225^\circ$ (j) $\cos 630^\circ$ (k) $\tan 405^\circ$ (l) $\sin(-315)^\circ$

(m) $\sin 210^\circ$ (n) $\tan 675^\circ$ (o) $\cos(-120)^\circ$ (p) $\sin 1260^\circ$

- 6 Without using a calculator, write down the smallest positive angle which satisfies the following equations.

(a) $\cos \theta^\circ = \frac{1}{2}$ (b) $\sin \phi^\circ = -\frac{1}{2}\sqrt{3}$ (c) $\tan \theta^\circ = -\sqrt{3}$ (d) $\cos \theta^\circ = \frac{1}{2}\sqrt{3}$

(e) $\tan \theta^\circ = \frac{1}{3}\sqrt{3}$ (f) $\tan \phi^\circ = -1$ (g) $\sin \theta^\circ = -\frac{1}{2}$ (h) $\cos \theta^\circ = 0$

- 7 Without using a calculator, write down the angle with the smallest modulus which satisfies the following equations. (If there are two such angles, choose the positive one.)

(a) $\cos \theta^\circ = -\frac{1}{2}$ (b) $\tan \phi^\circ = \sqrt{3}$ (c) $\sin \theta^\circ = -1$ (d) $\cos \theta^\circ = -1$

(e) $\sin \phi^\circ = \frac{1}{2}\sqrt{3}$ (f) $\tan \theta^\circ = -\frac{1}{3}\sqrt{3}$ (g) $\sin \phi^\circ = -\frac{1}{2}\sqrt{2}$ (h) $\tan \phi^\circ = 0$

- 8 The water levels in a dock follow (approximately) a twelve-hour cycle, and are modelled by the equation $D = A + B\sin 30t^\circ$, where D metres is the depth of water in the dock, A and B are positive constants, and t is the time in hours after 8 a.m.

Given that the greatest and least depths of water in the dock are 7.80 m and 2.20 m respectively, find the value of A and the value of B .

Find the depth of water in the dock at noon, giving your answer correct to the nearest cm.

10.4 Symmetry properties of the graphs of $\cos \theta^\circ$, $\sin \theta^\circ$ and $\tan \theta^\circ$

If you examine the graphs of $\cos \theta^\circ$, $\sin \theta^\circ$ and $\tan \theta^\circ$, you can see that they have many symmetry properties. The graph of $\cos \theta^\circ$ is shown in Fig. 10.8.

The graph of $\cos \theta^\circ$ is symmetrical about the vertical axis. This means that if you replace θ by $-\theta$ the graph is unchanged. Therefore

$$\cos(-\theta)^\circ = \cos \theta^\circ.$$

This shows that $\cos \theta^\circ$ is an even function of θ (as defined in Section 3.3).

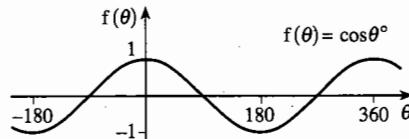


Fig. 10.8

There are other symmetry properties. For example, from Fig. 10.8 you can see that if you decrease (or increase) θ by 180 you change the sign of $f(\theta)$. Therefore

$$\cos(\theta - 180)^\circ = -\cos \theta^\circ.$$

This is called the translation property.

There is one more useful symmetry property. Using the even and translation properties,

$$\cos(180 - \theta)^\circ = \cos(\theta - 180)^\circ = -\cos \theta^\circ.$$

You may have met this property in using the cosine formula for a triangle.

There are similar properties for the graph of $\sin \theta^\circ$, which is shown in Fig. 10.9. You are asked to prove them as part of Exercise 10B. Their proofs are similar to those for the cosine.

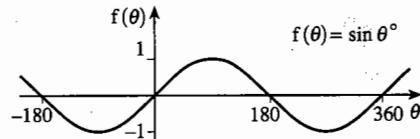


Fig. 10.9

The functions $\cos \theta^\circ$ and $\sin \theta^\circ$ have the following properties.

Periodic property: $\cos(\theta \pm 360)^\circ = \cos \theta^\circ$ $\sin(\theta \pm 360)^\circ = \sin \theta^\circ$

Odd property: $\cos(-\theta)^\circ = \cos \theta^\circ$ $\sin(-\theta)^\circ = -\sin \theta^\circ$

Translation property: $\cos(\theta - 180)^\circ = -\cos \theta^\circ$ $\sin(\theta - 180)^\circ = -\sin \theta^\circ$

$$\cos(180 - \theta)^\circ = -\cos \theta^\circ$$

$$\sin(180 - \theta)^\circ = \sin \theta^\circ$$

If you refer back to the graph of $\tan \theta^\circ$ in Fig. 10.5, and think in the same way as with the cosine and sine graphs, you can obtain similar results:

The function $\tan \theta^\circ$ has the following properties.

$$\text{Periodic property: } \tan(\theta \pm 180)^\circ = \tan \theta^\circ$$

$$\text{Odd property: } \tan(-\theta)^\circ = -\tan \theta^\circ$$

$$\tan(180 - \theta)^\circ = -\tan \theta^\circ$$

Note that the period of the graph of $\tan \theta^\circ$ is 180, and that the translation property of $\tan \theta^\circ$ is the same as the periodic property.

There are also relations between $\cos \theta^\circ$ and $\sin \theta^\circ$. One is shown in Example 10.4.1.

Example 10.4.1

Establish the property that $\cos(90 - \theta)^\circ = \sin \theta^\circ$.

This is easy if $0 < \theta < 90$: consider a right-angled triangle. But it can be shown for any value of θ .

If you translate the graph of $\cos \theta^\circ$ by 90 in the direction of the positive θ -axis, you obtain the graph of $\sin \theta^\circ$, so $\cos(90 - \theta)^\circ = \sin \theta^\circ$. And since the cosine is an even function, $\cos(90 + \theta)^\circ = \cos(90 - \theta)^\circ$. Therefore $\cos(90 - \theta)^\circ = \sin \theta^\circ$.

Another property, which you are asked to prove in Exercise 10B, is $\sin(90 - \theta)^\circ = \cos \theta^\circ$.

Exercise 10B

- 1 Use the symmetric and periodic properties of the sine, cosine and tangent functions to establish the following results.

- | | |
|--|---|
| (a) $\sin(90 - \theta)^\circ = \cos \theta^\circ$ | (b) $\sin(270 + \theta)^\circ = -\cos \theta^\circ$ |
| (c) $\sin(90 + \theta)^\circ = \cos \theta^\circ$ | (d) $\cos(90 + \theta)^\circ = -\sin \theta^\circ$ |
| (e) $\tan(\theta - 180)^\circ = \tan \theta^\circ$ | (f) $\cos(180 - \theta)^\circ = \cos(180 + \theta)^\circ$ |
| (g) $\tan(360 - \theta)^\circ = -\tan(180 + \theta)^\circ$ | (h) $\sin(-\theta - 90)^\circ = -\cos \theta^\circ$ |

- 2 Sketch the graphs of $y = \tan \theta^\circ$ and $y = \frac{1}{\tan \theta^\circ}$ on the same set of axes.

$$\text{Show that } \tan(90 - \theta)^\circ = \frac{1}{\tan \theta^\circ}.$$

- 3 In each of the following cases find the least positive value of α for which

- | | |
|--|---|
| (a) $\cos(\alpha - \theta)^\circ = \sin \theta^\circ$, | (b) $\sin(\alpha - \theta)^\circ = \cos(\alpha + \theta)^\circ$, |
| (c) $\tan \theta^\circ = \tan(\theta + \alpha)^\circ$, | (d) $\sin(\theta + 2\alpha)^\circ = \cos(\alpha - \theta)^\circ$, |
| (e) $\cos(2\alpha - \theta)^\circ = \cos(\theta - \alpha)^\circ$, | (f) $\sin(5\alpha + \theta)^\circ = \cos(\theta - 3\alpha)^\circ$. |

10.5 Solving equations involving the trigonometric functions

Solving the equation $\cos \theta^\circ = k$

To solve the equation $\cos \theta^\circ = k$, you need to assume that $-1 \leq k \leq 1$. If this is not true, there is no solution. In Fig. 10.10, a negative value of k is shown. Note that, in general, there are two roots to the equation $\cos \theta^\circ = k$ in every interval of 360° , the exceptions being when $k = \pm 1$.

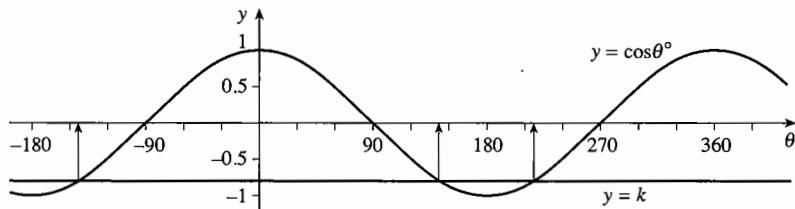


Fig. 10.10

To find an angle θ which satisfies the equation you can use the $[\cos^{-1}]$ key on your calculator (or, on some calculators, the [ARCCOS] key), but unfortunately it will only give you one answer. Usually you want to find all the roots of $\cos \theta^\circ = k$ in a given interval (probably one of width 360°). The problem then is how to find all the other roots in your required interval.

There are three steps in solving the equation of $\cos \theta^\circ = k$.

- Step 1** Find $\cos^{-1} k$.
- Step 2** Use the symmetry property $\cos(-\theta)^\circ = \cos \theta^\circ$ to find another root.
- Step 3** Use the periodic property $\cos(\theta \pm 360)^\circ = \cos \theta^\circ$ to find the roots in the required interval.

Example 10.5.1

Solve the equation $\cos \theta^\circ = \frac{1}{3}$, giving all roots in the interval $0 \leq \theta \leq 360$ correct to 1 decimal place.

Step 1 Use your calculator to find $\cos^{-1} \frac{1}{3} = 70.52\dots$. This is one root in the interval $0 \leq \theta \leq 360$.

Step 2 Use the symmetry property $\cos(-\theta)^\circ = \cos \theta^\circ$ to show that $-70.52\dots$ is another root. Note that $-70.52\dots$ is not in the required interval.

Step 3 Use the periodic property, $\cos(\theta \pm 360)^\circ = \cos \theta^\circ$, to obtain $-70.52\dots + 360 = 289.47\dots$, which is a root in the required interval.

Therefore the roots in the interval $0 \leq \theta \leq 360$ are 70.5 and 289.5, correct to 1 decimal place.

Example 10.5.2

Solve the equation $\cos 3\theta^\circ = -\frac{1}{2}$, giving all the roots in the interval $-180 \leq \theta \leq 180$.

This example is similar to the previous example, except for an important extra step at the beginning, and another at the end.

Let $3\theta = \phi$. Then you have to solve the equation $\cos \phi^\circ = -\frac{1}{2}$. But, as $3\theta = \phi$, if $-180 \leq \theta \leq 180$, then $3 \times (-180) \leq 3\theta \leq 3 \times 180$ so $-540 \leq \phi \leq 540$. So the original problem has become: solve the equation $\cos \phi^\circ = -\frac{1}{2}$ giving all the roots in the interval $-540 \leq \phi \leq 540$. (You should expect six roots of the equation in this interval.)

Step 1 $\cos^{-1}\left(-\frac{1}{2}\right) = 120^\circ$.

Step 2 Another root is -120° .

Step 3 Adding and subtracting multiples of 360° shows that $-120^\circ - 360^\circ = -480^\circ$, $-120^\circ + 360^\circ = 240^\circ$, $120^\circ - 360^\circ = -240^\circ$ and $120^\circ + 360^\circ = 480^\circ$ are also roots.

Therefore the roots of $\cos \phi^\circ = -\frac{1}{2}$ in $-540^\circ \leq \phi \leq 540^\circ$ are -480° , -240° , -120° , 120° , 240° and 480° .

Returning to the original equation, and using the fact that $\theta = \frac{1}{3}\phi$, the roots are -160° , -80° , -40° , 40° , 80° and 160° .

Solving the equation $\sin \theta^\circ = k$

The equation $\sin \theta^\circ = k$, where $-1 \leq k \leq 1$ is solved in a similar way. The only difference is that the symmetry property for $\sin \theta^\circ$ is $\sin(180^\circ - \theta)^\circ = \sin \theta^\circ$.

Step 1 Find $\sin^{-1} k$.

Step 2 Use the symmetry property $\sin(180^\circ - \theta)^\circ = \sin \theta^\circ$ to find another root.

Step 3 Use the periodic property $\sin(\theta \pm 360^\circ)^\circ = \sin \theta^\circ$ to find the roots in the required interval.

Example 10.5.3

Solve the equation $\sin \theta^\circ = -0.7$, giving all the roots in the interval $-180 \leq \theta \leq 180$ correct to 1 decimal place.

Step 1 Use your calculator to find $\sin^{-1}(-0.7) = -44.42\dots$. This is one root in the interval $-180 \leq \theta \leq 180$.

Step 2 Use the symmetry property $\sin(180 - \theta)^\circ = \sin \theta^\circ$ to show that $180 - (-44.42\ldots) = 224.42\ldots$ is another root. Unfortunately it is not in the required interval.

Step 3 Use the periodic property, $\sin(\theta \pm 360)^\circ = \sin \theta^\circ$, to obtain $224.42\ldots - 360 = -135.57\ldots$, which is a root in the required interval.

Therefore the roots in the interval $-180 \leq \theta \leq 180$ are -44.4 and -135.6 , correct to 1 decimal place.

Example 10.5.4

Solve the equation $\sin \frac{1}{3}(\theta - 30)^\circ = \frac{1}{2}\sqrt{3}$, giving all the roots in the interval $0 \leq \theta \leq 360$.

Let $\frac{1}{3}(\theta - 30) = \phi$, so that the equation becomes $\sin \phi^\circ = \frac{1}{2}\sqrt{3}$, with roots required in the interval $-10 \leq \phi \leq 110$.

Step 1 $\sin^{-1}\left(\frac{1}{2}\sqrt{3}\right) = 60$. This is one root in the interval $-10 \leq \phi \leq 110$.

Step 2 Another root is $180 - 60 = 120$, but this is not in the required interval.

Step 3 Adding and subtracting multiples of 360 will not give any more roots in the interval $-10 \leq \phi \leq 110$.

Therefore the only root of $\sin \phi^\circ = \frac{1}{2}\sqrt{3}$ in $-10 \leq \phi \leq 110$ is 60 .

Returning to the original equation, since $\theta = 3\phi + 30$, the root is $\theta = 210$.

Solving the equation $\tan \theta^\circ = k$

The equation $\tan \theta^\circ = k$ is also solved in a similar way. Note that there is generally one root for every interval of 180. Other roots can be found from the periodic property, $\tan(180 + \theta)^\circ = \tan \theta^\circ$.

Step 1 Find $\tan^{-1} k$.

Step 2 Use the periodic property $\tan(180 + \theta)^\circ = \tan \theta^\circ$ to find the roots in the required interval.

Example 10.5.5

Solve the equation $\tan \theta^\circ = -2$, giving all the roots correct to 1 decimal place in the interval $0 \leq \theta \leq 360$.

Step 1 Find $\tan^{-1}(-2) = -63.43\ldots$. Unfortunately, this root is not in the required interval.

Step 2 Add multiples of 180 to get roots in the required interval. This gives $116.56\ldots$ and $296.56\ldots$.

Therefore the roots of $\tan \theta^\circ = -2$ in $0 \leq \theta \leq 360$ are 116.6 and 296.6 , correct to 1 decimal place.

Revisiting Example 10.1.2, here is an application of solving equations of this type.

Example 10.5.6

The height in metres of the water in a harbour is given approximately by the formula $d = 6 + 3 \cos 30t^\circ$ where t is the time measured in hours from noon. Find the time after noon when the height of the water is 7.5 metres for the second time.

To find when the height is 7.5 metres, solve $6 + 3 \cos 30t^\circ = 7.5$. This gives $3 \cos 30t^\circ = 7.5 - 6 = 1.5$, or $\cos 30t^\circ = 0.5$. After substituting $\phi = 30t$, the equation reduces to $\cos \phi^\circ = 0.5$.

Now $\cos^{-1} 0.5 = 60^\circ$, but this gives only the first root, $t = 2$. So, using the symmetry property of the cos function, another root is -60° . Adding 360° gives $\phi = 300^\circ$ as the second root of $\cos \phi^\circ = 0.5$. Thus $30t = 300^\circ$, and $t = 10$.

The water is at height 7.5 metres for the second time at 10.00 p.m.

Exercise 10C

- Find, correct to 1 decimal place, the two smallest positive values of θ which satisfy each of the following equations.

(a) $\sin \theta^\circ = 0.1$	(b) $\sin \theta^\circ = -0.84$	(c) $\sin \theta^\circ = 0.951$
(d) $\cos \theta^\circ = 0.8$	(e) $\cos \theta^\circ = -0.84$	(f) $\cos \theta^\circ = \sqrt{\frac{2}{3}}$
(g) $\tan \theta^\circ = 4$	(h) $\tan \theta^\circ = -0.32$	(i) $\tan \theta^\circ = 0.11$
(j) $\sin(180 + \theta)^\circ = 0.4$	(k) $\cos(90 - \theta)^\circ = -0.571$	(l) $\tan(90 - \theta)^\circ = -3$
(m) $\sin(2\theta + 60)^\circ = 0.3584$	(n) $\sin(30 - \theta)^\circ = 0.5$	(o) $\cos(3\theta - 120)^\circ = 0$
- Find all values of θ in the interval $-180^\circ \leq \theta \leq 180^\circ$ which satisfy each of the following equations, giving your answers correct to 1 decimal place where appropriate.

(a) $\sin \theta^\circ = 0.8$	(b) $\cos \theta^\circ = 0.25$	(c) $\tan \theta^\circ = 2$
(d) $\sin \theta^\circ = -0.67$	(e) $\cos \theta^\circ = -0.12$	(f) $4 \tan \theta^\circ + 3 = 0$
(g) $4 \sin \theta^\circ = 5 \cos \theta^\circ$	(h) $2 \sin \theta^\circ = \frac{1}{\sin \theta^\circ}$	(i) $2 \sin \theta^\circ = \tan \theta^\circ$
- Find all the solutions in the interval $0 < \theta \leq 360^\circ$ of each of the following equations.

(a) $\cos 2\theta^\circ = \frac{1}{3}$	(b) $\tan 3\theta^\circ = 2$	(c) $\sin 2\theta^\circ = -0.6$
(d) $\cos 4\theta^\circ = -\frac{1}{4}$	(e) $\tan 2\theta^\circ = 0.4$	(f) $\sin 3\theta^\circ = -0.42$
- Find the roots in the interval $-180^\circ \leq x \leq 180^\circ$ of each of the following equations.

(a) $\cos 3x^\circ = \frac{2}{3}$	(b) $\tan 2x^\circ = -3$	(c) $\sin 3x^\circ = -0.2$
(d) $\cos 2x^\circ = 0.246$	(e) $\tan 5x^\circ = 0.8$	(f) $\sin 2x^\circ = -0.39$

5 Find the roots (if there are any) in the interval $-180 \leq \theta \leq 180$ of the following equations.

- | | | |
|--|--|---|
| (a) $\cos \frac{1}{2}\theta^\circ = \frac{2}{3}$ | (b) $\tan \frac{2}{3}\theta^\circ = -3$ | (c) $\sin \frac{1}{4}\theta^\circ = -\frac{1}{4}$ |
| (d) $\cos \frac{1}{3}\theta^\circ = \frac{1}{3}$ | (e) $\tan \frac{3}{4}\theta^\circ = 0.5$ | (f) $\sin \frac{2}{5}\theta^\circ = -0.3$ |

6 Without using a calculator, find the exact roots of the following equations, if there are any, giving your answers in the interval $0 < t \leq 360$.

- | | | |
|---|--|--|
| (a) $\sin(2t - 30)^\circ = \frac{1}{2}$ | (b) $\tan(2t - 45)^\circ = 0$ | (c) $\cos(3t + 135)^\circ = \frac{1}{2}\sqrt{3}$ |
| (d) $\tan(\frac{3}{2}t - 45)^\circ = -\sqrt{3}$ | (e) $\cos(2t - 50)^\circ = -\frac{1}{2}$ | (f) $\sin(\frac{1}{2}t + 50)^\circ = 1$ |
| (g) $\cos(\frac{1}{3}t - 50)^\circ = 0$ | (h) $\tan(3t - 180)^\circ = -1$ | (i) $\sin(\frac{1}{4}t - 20)^\circ = 0$ |

7 Find, to 1 decimal place, all values of z in the interval $-180 \leq z \leq 180$ satisfying

- | | | |
|------------------------------|--|--|
| (a) $\sin z^\circ = -0.16$, | (b) $\cos z^\circ(1 + \sin z^\circ) = 0$, | (c) $(1 - \tan z^\circ)\sin z^\circ = 0$, |
| (d) $\sin 2z^\circ = 0.23$, | (e) $\cos(45 - z)^\circ = 0.832$, | (f) $\tan(3z - 17)^\circ = 3$. |

8 Find all values of θ in the interval $0 \leq \theta \leq 360$ for which

- | | | |
|--|--|--|
| (a) $\sin 2\theta^\circ = \cos 36^\circ$, | (b) $\cos 5\theta^\circ = \sin 70^\circ$, | (c) $\tan 3\theta^\circ = \tan 60^\circ$. |
|--|--|--|

9 Find all values of θ in the interval $0 \leq \theta \leq 180$ for which $2\sin\theta^\circ\cos\theta^\circ = \frac{1}{2}\tan\theta^\circ$.

10 For each of the following values, give an example of a trigonometric function involving (i) sine, (ii) cosine and (iii) tangent, with that value as period.

- | | | |
|---------|---------|---------|
| (a) 90 | (b) 20 | (c) 48 |
| (d) 120 | (e) 720 | (f) 600 |

11 Sketch the graphs of each of the following in the interval $0 \leq \phi \leq 360$. In each case, state the period of the function.

- | | | |
|--------------------------------------|--------------------------------------|--|
| (a) $y = \sin 3\phi^\circ$ | (b) $y = \cos 2\phi^\circ$ | (c) $y = \sin 4\phi^\circ$ |
| (d) $y = \tan \frac{1}{3}\phi^\circ$ | (e) $y = \cos \frac{1}{2}\phi^\circ$ | (f) $y = \sin(\frac{1}{2}\phi + 30)^\circ$ |
| (g) $y = \sin(3\phi - 20)^\circ$ | (h) $y = \tan 2\phi^\circ$ | (i) $y = \tan(\frac{1}{2}\phi + 90)^\circ$ |

12 At a certain latitude in the northern hemisphere, the number d of hours of daylight in each day of the year is taken to be $d = A + B\sin kt^\circ$, where A , B , k are positive constants and t is the time in days after the spring equinox.

- Assuming that the number of hours of daylight follows an annual cycle of 365 days, find the value of k , giving your answer correct to 3 decimal places.
- Given also that the shortest and longest days have 6 and 18 hours of daylight respectively, state the values of A and B . Find, in hours and minutes, the amount of daylight on New Year's Day, which is 80 days before the spring equinox.
- A town at this latitude holds a fair twice a year on those days having exactly 10 hours of daylight. Find, in relation to the spring equinox, which two days these are.

10.6 Relations between the trigonometric functions

In algebra you are used to solving equations, which involves finding a value of the unknown, often called x , in an equation such as $2x + 3 - x - 6 = 7$. You are also used to simplifying algebraic expressions like $2x + 3 - x - 6$, which becomes $x - 3$. You may not have realised, however, that these are quite different processes.

When you solve the equation $2x + 3 - x - 6 = 7$, you find that there is one solution, $x = 10$. But the expression $x - 3$ is identical to $2x + 3 - x - 6$ for all values of x . Sometimes it is important to distinguish between these two situations.

If two expressions take the same values for every value of x , they are said to be **identically equal**. This is written with the symbol \equiv , read as ‘is identically equal to’. The statement

$$2x + 3 - x - 6 \equiv x - 3$$

is called an **identity**. Thus an identity in x is an equation which is true for all values of x .

Similar ideas occur in trigonometry. At the end of Section 10.2, it was observed that $\tan \theta^\circ = \frac{\sin \theta^\circ}{\cos \theta^\circ}$, provided that $\cos \theta^\circ \neq 0$. Thus

$$\tan \theta^\circ \equiv \frac{\sin \theta^\circ}{\cos \theta^\circ}.$$

The identity symbol is used even when there are some exceptional values for which neither side is defined. In the example given, neither side is defined when θ is an odd multiple of 90° , but the identity sign is still used.

There is another relationship which comes immediately from the definitions of $\cos \theta^\circ = x$ and $\sin \theta^\circ = y$ in Sections 10.1 and 10.2. As P lies on the circumference of a circle with radius 1 unit, Pythagoras’ theorem gives $x^2 + y^2 = 1$, or $(\cos \theta^\circ)^2 + (\sin \theta^\circ)^2 \equiv 1$.

Conventionally, $(\cos \theta^\circ)^2$ is written as $\cos^2 \theta^\circ$ and $(\sin \theta^\circ)^2$ as $\sin^2 \theta^\circ$, so for all values of θ , $\cos^2 \theta^\circ + \sin^2 \theta^\circ \equiv 1$. This is sometimes called Pythagoras’ theorem in trigonometry.

For all values of θ :

$$\tan \theta^\circ \equiv \frac{\sin \theta^\circ}{\cos \theta^\circ}, \quad \text{provided that } \cos \theta^\circ \neq 0;$$

$$\cos^2 \theta^\circ + \sin^2 \theta^\circ \equiv 1.$$

The convention of using $\cos^n \theta^\circ$ to stand for $(\cos \theta^\circ)^n$ is best restricted to positive powers. In any case, it should never be used with $n = -1$, because of the danger of confusion with $\cos^{-1} x$, which is used to stand for the angle whose cosine is x . If in doubt, you should write $(\cos \theta^\circ)^n$ or $(\cos \theta^\circ)^{-n}$, which could only mean one thing.

You can use the relation $\cos^2 \theta^\circ + \sin^2 \theta^\circ \equiv 1$ in the process of proving the cosine formula for a triangle.

Let ABC be a triangle, with sides $BC = a$, $CA = b$ and $AB = c$. Place the point A at the origin, and let AC lie along the x -axis in the positive x -direction, shown in Fig. 10.11.

The coordinates of C are $(b, 0)$, and those of B are $(c \cos A^\circ, c \sin A^\circ)$, where A stands for the angle BAC . Then, using the distance formula (Section 1.1),

$$\begin{aligned} a^2 &= (b - c \cos A^\circ)^2 + (c \sin A^\circ)^2 \\ &= b^2 - 2bc \cos A^\circ + c^2 \cos^2 A^\circ + c^2 \sin^2 A^\circ \\ &= b^2 - 2bc \cos A^\circ + c^2(\cos^2 A^\circ + \sin^2 A^\circ) \\ &= b^2 + c^2 - 2bc \cos A^\circ, \end{aligned}$$

using $\cos^2 A^\circ + \sin^2 A^\circ = 1$ at the end.

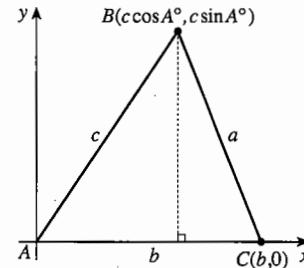


Fig. 10.11

Example 10.6.1

Given that $\sin \theta^\circ = \frac{3}{5}$, and that the angle θ° is obtuse, find, without using a calculator, the values of $\cos \theta^\circ$ and $\tan \theta^\circ$.

Since $\cos^2 \theta^\circ + \sin^2 \theta^\circ = 1$, $\cos^2 \theta^\circ = 1 - \left(\frac{3}{5}\right)^2 = \frac{16}{25}$ giving $\cos \theta^\circ = \pm \frac{4}{5}$. As the angle θ° is obtuse, $90^\circ < \theta < 180^\circ$, so $\cos \theta^\circ$ is negative. Therefore $\cos \theta^\circ = -\frac{4}{5}$.

As $\sin \theta^\circ = \frac{3}{5}$ and $\cos \theta^\circ = -\frac{4}{5}$, $\tan \theta^\circ = \frac{\sin \theta^\circ}{\cos \theta^\circ} = \frac{\frac{3}{5}}{-\frac{4}{5}} = -\frac{3}{4}$.

Example 10.6.2

Solve the equation $3 \cos^2 \theta^\circ + 4 \sin \theta^\circ = 4$, giving all the roots in the interval $-180^\circ < \theta \leq 180^\circ$ correct to 1 decimal place.

As it stands you cannot solve this equation, but if you replace $\cos^2 \theta^\circ$ by $1 - \sin^2 \theta^\circ$ you will obtain the equation $3(1 - \sin^2 \theta^\circ) + 4 \sin \theta^\circ = 4$, which reduces to

$$3 \sin^2 \theta^\circ - 4 \sin \theta^\circ + 1 = 0.$$

This is a quadratic equation in $\sin \theta^\circ$, which you can solve using factors:

$$(3 \sin \theta^\circ - 1)(\sin \theta^\circ - 1) = 0, \text{ giving } \sin \theta^\circ = \frac{1}{3} \text{ or } \sin \theta^\circ = 1.$$

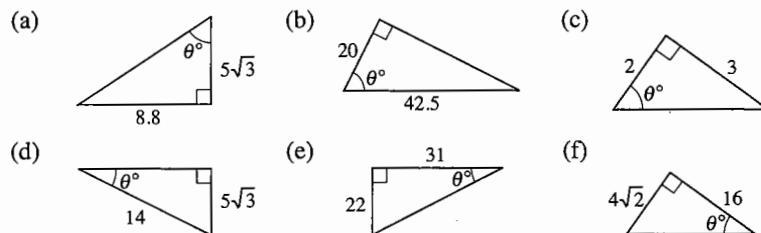
One root is $\sin^{-1} \frac{1}{3} = 19.47\dots$, and the other root, obtained from the symmetry of $\sin \theta^\circ$, is $(180^\circ - 19.47\dots) = 160.52\dots$.

The only root for $\sin \theta^\circ = 1$ is $\theta = 90^\circ$, so the roots are 19.5° , 90° and 160.5° , correct to 1 decimal place.

Exercise 10D

- 1** For each triangle sketched below,

 - use Pythagoras' theorem to find the length of the third side in an exact form;
 - write down the exact values of $\sin \theta^\circ$, $\cos \theta^\circ$ and $\tan \theta^\circ$.



- 2 (a) Given that angle A is obtuse and that $\sin A^\circ = \frac{5}{14}\sqrt{3}$, find the exact value of $\cos A^\circ$.
 (b) Given that $180 < B < 360$ and that $\tan B^\circ = -\frac{21}{20}$, find the exact value of $\cos B^\circ$.
 (c) Find all possible values of $\sin C^\circ$ for which $\cos C^\circ = \frac{1}{2}$.
 (d) Find the values of D for which $-180 < D < 180$ and $\tan D^\circ = 5\sin D^\circ$.

- 3 Use $\tan \theta^\circ \equiv \frac{\sin \theta^\circ}{\cos \theta^\circ}$, $\cos \theta^\circ \neq 0$, and $\cos^2 \theta^\circ + \sin^2 \theta^\circ \equiv 1$ to establish the following.

$$(a) \frac{1}{\sin \theta^\circ} - \frac{1}{\tan \theta^\circ} \equiv \frac{1 - \cos \theta^\circ}{\sin \theta^\circ}$$

$$(b) \frac{\sin^2 \theta^\circ}{1 - \cos \theta^\circ} \equiv 1 + \cos \theta^\circ$$

$$(c) \frac{1}{\cos \theta^\circ} + \tan \theta^\circ \equiv \frac{\cos \theta^\circ}{1 - \sin \theta^\circ}$$

$$(d) \frac{\tan \theta^\circ \sin \theta^\circ}{1 - \cos \theta^\circ} \equiv 1 + \frac{1}{\cos \theta^\circ}$$

- 4 Solve the following equations for θ , giving all the roots in the interval $0^\circ \leq \theta < 360^\circ$ correct to the nearest 0.1° .

$$(a) \quad 4 \sin^2 \theta^\circ - 1 = 0$$

$$(b) \sin^2 \theta^\circ + 2 \cos^2 \theta^\circ = 2$$

$$(c) \quad 10\sin^2\theta^\circ - 5\cos^2\theta^\circ + 2 = 4\sin\theta^\circ$$

$$(d) \quad 4\sin^2 \theta^\circ \cos \theta^\circ = \tan^2 \theta^\circ$$

- 5 Find all values of θ , $-180 < \theta < 180$, for which $2 \tan \theta^\circ - 3 = \frac{2}{\tan \theta^\circ}$.

Miscellaneous exercise 10

- Write down the period of each of the following.
 - $\sin x^\circ$
 - $\tan 2x^\circ$(OCR)
 - By considering the graph of $y = \cos x^\circ$, or otherwise, express the following in terms of $\cos x^\circ$.
 - $\cos(360 - x)^\circ$
 - $\cos(x + 180)^\circ$(OCR)
 - Draw the graph of $y = \cos \frac{1}{2}\theta^\circ$ for θ in the interval $-360 \leq \theta \leq 360$. Mark clearly the coordinates of the points where the graph crosses the θ - and y -axes.
 - Solve the following equations for θ , giving your answers in the interval $0 \leq \theta \leq 360$.
 - $\tan \theta^\circ = 0.4$
 - $\sin 2\theta^\circ = 0.4$(OCR)

- 5** Solve the equation $3\cos 2x^\circ = 2$, giving all the solutions in the interval $0 \leq x \leq 180$ correct to the nearest 0.1. (OCR)
- 6** (a) Give an example of a trigonometric function which has a period of 180.
 (b) Solve for x the equation $\sin 3x^\circ = 0.5$, giving all solutions in the interval $0 < x < 180$. (OCR)
- 7** Find all values of θ° , $0 \leq \theta \leq 360$, for which $2\cos(\theta + 30)^\circ = 1$. (OCR)
- 8** (a) Express $\sin 2x^\circ + \cos(90 - 2x)^\circ$ in terms of a single trigonometric function.
 (b) Hence, or otherwise, find all values of x in the interval $0 \leq x \leq 360$ for which $\sin 2x^\circ + \cos(90 - 2x)^\circ = -1$. (OCR)
- 9** Find the least positive value of the angle A for which
 (a) $\sin A^\circ = 0.2$ and $\cos A^\circ$ is negative; (b) $\tan A^\circ = -0.5$ and $\sin A^\circ$ is negative;
 (c) $\cos A^\circ = \sin A^\circ$ and both are negative; (d) $\sin A^\circ = -0.2275$ and $A > 360$.
- 10** Prove the following identities.
- | | |
|--|--|
| (a) $\frac{1}{\sin \theta^\circ} - \sin \theta^\circ \equiv \frac{\cos \theta^\circ}{\tan \theta^\circ}$ | (b) $\frac{1 - \sin \theta^\circ}{\cos \theta^\circ} \equiv \frac{\cos \theta^\circ}{1 + \sin \theta^\circ}$ |
| (c) $\frac{1}{\tan \theta^\circ} + \tan \theta^\circ \equiv \frac{1}{\sin \theta^\circ \cos \theta^\circ}$ | (d) $\frac{1 - 2 \sin^2 \theta^\circ}{\cos \theta^\circ + \sin \theta^\circ} \equiv \cos \theta^\circ - \sin \theta^\circ$ |
- 11** For each of the following functions, determine the maximum and minimum values of y and the least positive values of x at which these occur.
- | | |
|---------------------------------------|--|
| (a) $y = 1 + \cos 2x^\circ$ | (b) $y = 5 - 4 \sin(x + 30)^\circ$ |
| (c) $y = 29 - 20 \sin(3x - 45)^\circ$ | (d) $y = 8 - 3 \cos^2 x^\circ$ |
| (e) $y = \frac{12}{3 + \cos x^\circ}$ | (f) $y = \frac{60}{1 + \sin^2(2x - 15)^\circ}$ |
- 12** Solve the following equations for θ , giving solutions in the interval $0 \leq \theta \leq 360$.
- | | |
|---|--|
| (a) $\sin \theta^\circ = \tan \theta^\circ$ | (b) $2 - 2 \cos^2 \theta^\circ = \sin \theta^\circ$ |
| (c) $\tan^2 \theta^\circ - 2 \tan \theta^\circ = 1$ | (d) $\sin 2\theta^\circ - \sqrt{3} \cos 2\theta^\circ = 0$ |
- 13** The function t is defined by $t(x) = \tan 3x^\circ$.
- | | |
|---|-------|
| (a) State the period of $t(x)$. | |
| (b) Solve the equation $t(x) = \frac{1}{2}$ for $0 \leq x \leq 180$. | |
| (c) Deduce the smallest positive solution of each of the following equations. | |
| (i) $t(x) = -\frac{1}{2}$ | |
| (ii) $t(x) = 2$ | (OCR) |

- 14** In each of the following, construct a formula involving a trigonometric function which could be used to model the situations described.
- Water depths in a canal vary between a minimum of 3.6 metres and a maximum of 6 metres over 24-hour periods.
 - Petroleum refining at a chemical plant is run on a 10-day cycle, with a minimum production of 15 000 barrels per day and a maximum of 28 000 barrels per day.
 - At a certain town just south of the Arctic circle, the number of hours of daylight varies between 2 and 22 hours during a 360-day year.
- 15** A tuning fork is vibrating. The displacement, y centimetres, of the tip of one of the prongs from its rest position after t seconds is given by
- $$y = 0.1 \sin(100 000t)^\circ.$$
- Find
- the greatest displacement and the first time at which it occurs,
 - the time taken for one complete oscillation of the prong,
 - the number of complete oscillations per second of the tip of the prong,
 - the total time during the first complete oscillation for which the tip of the prong is more than 0.06 centimetres from its rest position.
- 16** One end of a piece of elastic is attached to a point at the top of a door frame and the other end hangs freely. A small ball is attached to the free end of the elastic. When the ball is hanging freely it is pulled down a small distance and then released, so that the ball oscillates up and down on the elastic. The depth d centimetres of the ball from the top of the door frame after t seconds is given by
- $$d = 100 + 10 \cos 500t^\circ.$$
- Find
- the greatest and least depths of the ball,
 - the time at which the ball first reaches its highest position,
 - the time taken for a complete oscillation,
 - the proportion of the time during a complete oscillation for which the depth of the ball is less than 99 centimetres.
- 17** An oscillating particle has displacement y metres, where y is given by $y = a \sin(kt + \alpha)^\circ$, where a is measured in metres, t is measured in seconds and k and α are constants. The time for a complete oscillation is T seconds.
- Find
- k in terms of T ,
 - the number, in terms of k , of complete oscillations per second.

- 18** The population, P , of a certain type of bird on a remote island varies during the course of a year according to feeding, breeding, migratory seasons and predator interactions. An ornithologist doing research into bird numbers for this species attempts to model the population on the island with the annually periodic equation

$$P = N - C \cos \omega t^\circ,$$

where N , C and ω are constants, and t is the time in weeks, with $t=0$ representing midnight on the first of January.

- (a) Taking the period of this function to be 50 weeks, find the value of ω .
- (b) Use the equation to describe, in terms of N and C ,
 - (i) the number of birds of this species on the island at the start of each year;
 - (ii) the maximum number of these birds, and the time of year when this occurs.

- 19** The road to an island close to the shore is sometimes covered by the tide. When the water rises to the level of the road, the road is closed. On a particular day, the water at high tide is a height 4.6 metres above mean sea level. The height, h metres, of the tide is modelled by using the equation $h=4.6\cos kt^\circ$, where t is the time in hours from high tide; it is also assumed that high tides occur every 12 hours.

- (a) Determine the value of k .
- (b) On the same day, a notice says that the road will be closed for 3 hours. Assuming that this notice is correct, find the height of the road above sea level, giving your answer correct to two decimal places.
- (c) In fact, a road repair has raised its level, and it is impassable for only 2 hours 40 minutes. By how many centimetres has the road level been raised? (OCR)

- 20** A simple model of the tides in a harbour on the south coast of Cornwall assumes that they are caused by the attractions of the sun and the moon. The magnitude of the attraction of the moon is assumed to be nine times the magnitude of the attraction of the sun. The period of the sun's effect is taken to be 360 days and that of the moon is 30 days. A model for the height, h metres, of the tide (relative to a mark fixed on the harbour wall), at t days, is

$$h = A \cos \alpha t^\circ + B \cos \beta t^\circ,$$

where the term $A \cos \alpha t^\circ$ is the effect due to the sun, and the term $B \cos \beta t^\circ$ is the effect due to the moon. Given that $h=5$ when $t=0$, determine the values of A , B , α and β .

(OCR, adapted)

11 Combining and inverting functions

This chapter develops the idea of a function, which you first met in Chapter 3. It introduces a kind of algebra of functions, by showing how to find a composite function. When you have completed it, you should

- be able to use correct language and notation associated with functions
- know when functions can be combined by the operation of composition, and be able to form the composite function
- appreciate that a sequence can be regarded as a function whose domain is the natural numbers, or a consecutive subset of the natural numbers
- know the ‘one–one’ condition for a function to have an inverse, and be able to form the inverse function
- know the relationship between the graph of a one–one function and the graph of its inverse function.

The references to calculators in the chapter may not exactly fit your own machine. For example, on some calculators the [=] key is labelled [EXE] (which stands for ‘execute’).

11.1 Function notation

In using a calculator to find values of a function, you carry out three separate steps:

- Step 1** Key in a number (the ‘input’).
Step 2 Key in the function instructions.
Step 3 Read the number in the display (the ‘output’).

Step 2 sometimes involves just a single key, such as ‘square root’, ‘change sign’ or ‘sine’. For example:

Input	Output
4 → [√] → 2	
3 → [+/-] → -3	
30 → [sin] → 0.5	

In this chapter sin, cos and tan stand for these functions as operated by your calculator in degree mode. You enter a number x , and the output is $\sin x^\circ$, $\cos x^\circ$ or $\tan x^\circ$.

Other functions need several keys, such as ‘subtract 3’:

$$7 \rightarrow [-, 3, =] \rightarrow 4.$$

But the principle is the same. The important point is that it is the key sequence inside the square brackets that represents the function. This sequence is the same whatever number you key in as the input in Step 1.

You can think of a function as a kind of machine. Just as you can have a machine which takes fabric and turns it into clothes, so a function takes numbers in the domain and turns them into numbers in the range.

For a general input number x , you can write

$$\begin{aligned}x &\rightarrow [+/\pm] \rightarrow -x, \\x &\rightarrow [-, 3, =] \rightarrow x-3,\end{aligned}$$

and so on. And for a general function,

$$x \rightarrow [f] \rightarrow f(x),$$

where f stands for the key sequence of the function.

This book has often used phrases like ‘the function x^2 ’, ‘the function $\cos x^\circ$ ’, or ‘the function $f(x)$ ’, and you have understood what is meant. Working mathematicians do this all the time. But it is strictly wrong; x^2 , $\cos x^\circ$ and $f(x)$ are symbols for the *output* when the input is x , not for the function itself. When you need to use precise language, you should refer to ‘the function square’, ‘the function cos’ or ‘the function f ’.

Unfortunately only a few functions have convenient names like ‘square’ or ‘cos’. There is no simple name for a function whose output is given by an expression such as $x^2 - 6x + 4$. The way round this is to decide for the time being to call this function f (or any other letter you like). You can then write

$$f : x \mapsto x^2 - 6x + 4.$$

You read this as ‘ f is the function which turns any input number x in the domain into the output number $x^2 - 6x + 4$ ’. Notice the bar at the blunt end of the arrow; it avoids confusion with the arrow which has been used to stand for ‘tends to’ in finding gradients of tangents.

Try to write a key sequence to represent this function. (You may need the memory keys.)

Example 11.1.1

If $f : x \mapsto x(5-x)$, what is $f(3)$?

The symbol $f(3)$ stands for the output when the input is 3. The function f turns the input 3 into the output $3(5-3) = 6$. So $f(3) = 6$.

This idea of using an arrow to show the connection between the input and the output can be linked to the graph of the function. Fig. 11.1 shows the graph of $y = x(5-x)$, with the input number 3 on the x -axis. An arrow which goes up the page from this point and bends through a right angle when it hits the graph takes you to the output number 6 on the y -axis.

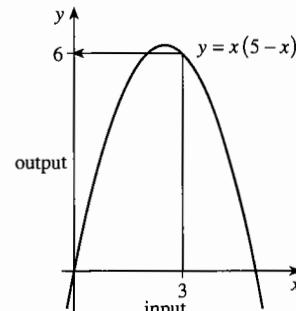


Fig. 11.1

11.2 Forming composite functions

If you want to work out values of $\sqrt{x-3}$, you would probably use the key sequence $[-, 3, =, \sqrt{ }]$ with hardly a thought. But if you look carefully, you will see that three numbers appear in the display during the process. For example, if you use the input 7, the display will show in turn your input number 7, then (after keying [=]) 4, and finally the output 2. In fact, you are really working out two functions, ‘subtract 3’ then ‘square root’, in succession. You could represent the whole calculation:

$$7 \rightarrow [-, 3, =] \rightarrow 4 \rightarrow [\sqrt{ }] \rightarrow 2.$$

The output of the first function becomes the input of the second.

Example 11.2.1

Find the outputs when the functions ‘square’ and ‘sin’ act in succession on the inputs of (a) 30, (b) x .

$$(a) \quad 30 \rightarrow [\text{square}] \rightarrow 900 \rightarrow [\sin] \rightarrow 0.$$

$$(b) \quad x \rightarrow [\text{square}] \rightarrow x^2 \rightarrow [\sin] \rightarrow \sin(x^2)^\circ.$$

Since in (b) the input to the function sin is x^2 , not x , the output is $\sin(x^2)^\circ$, not $\sin x^\circ$.

For a general input, and two general functions f and g , the process would be written:

$$x \rightarrow [f] \rightarrow f(x) \rightarrow [g] \rightarrow g(f(x)).$$

When you work out two functions in succession in this way, you are said to be ‘composing’ them. The result is a third function called the ‘composite function’.

Since the output of the composite function is $g(f(x))$, the composite function itself is denoted by gf . Notice that gf must be read as ‘first f , then g ’. You must get used to reading the symbol gf from right to left. Writing fg means ‘first g , then f ’, which is almost always a different function from gf . For instance, if you change the order of the functions in Example 11.2.1(a), instead of the output 0 you get

$$30 \rightarrow [\sin] \rightarrow 0.5 \rightarrow [\text{square}] \rightarrow 0.25.$$

Example 11.2.2

Let $f: x \mapsto x+3$ and $g: x \mapsto x^2$. Find gf and fg . Show that there is just one number x such that $gf(x) = fg(x)$.

The composite function gf is represented by

$$x \rightarrow [f] \rightarrow x+3 \rightarrow [g] \rightarrow (x+3)^2$$

and fg is represented by

$$x \rightarrow [g] \rightarrow x^2 \rightarrow [f] \rightarrow x^2 + 3.$$

So $gf: x \mapsto (x+3)^2$ and $fg: x \mapsto x^2 + 3$.

If $gf(x) = fg(x)$, $(x+3)^2 = x^2 + 3$, so $x^2 + 6x + 9 = x^2 + 3$, giving $x = -1$.

You can check this with your calculator. If you input -1 and then ‘add 3’ [$+, 3, =$] followed by ‘square’, the display will show in turn $-1, 2, 4$. If you do ‘square’ followed by ‘add 3’, it will show $-1, 1, 4$. When the input is -1 , the outputs are the same although the intermediate displays are different.

Example 11.2.3

If $f : x \mapsto \cos x^\circ$ and $g : x \mapsto \frac{1}{x}$, calculate (a) $gf(60)$, (b) $gf(90)$.

(a) With input 60 , the calculator will show in turn $60, 0.5, 2$, so $gf(60) = 2$.

(b) With input 90 , the calculator will display $90, 0$ and then give an error message! This is because $\cos 90^\circ = 0$ and $\frac{1}{0}$ is not defined.

What has happened in Example 11.2.3(b) is that the number 0 is in the range of the function f , but it is not in the domain of g . You must always be aware that this may happen when you find the composite of two functions. It is time to look again at domains and ranges, so that you can avoid this problem.

11.3 Domain and range

When you see the letters x and y in mathematics, for example in an equation such as $y = 2x - 10$, it is generally understood that they stand for real numbers. But sometimes it is important to be absolutely precise about this. The symbol \mathbb{R} is used to stand for ‘the set of real numbers’, and the symbol \in for ‘belongs to’. With these symbols, you can shorten the statement ‘ x is a real number’, or ‘ x belongs to the set of real numbers’, to $x \in \mathbb{R}$. So you can write

$$f : x \mapsto 2x - 10, \quad x \in \mathbb{R}$$

to indicate that f is the function whose domain is the set of real numbers which turns any input x into the output $2x - 10$.

Strictly, a function is not completely defined unless you state the domain as well as the rule for obtaining the output from the input. For the function above the range is also \mathbb{R} , although you do not need to state this in describing the function.

You know from Chapter 3 that for some functions the domain is only a part of \mathbb{R} , because the expression $f(x)$ only has meaning for some $x \in \mathbb{R}$. (Here \in has to be read as ‘belonging to’ rather than ‘belongs to’.) The set of real numbers for which $f(x)$ has a meaning will be called the ‘natural domain’ of f . With a calculator, if you input a number that is not in the natural domain, the output will be an ‘error’ display.

For the square root function, for example, the natural domain is the set of positive real numbers and zero, so you write

$$\text{square root} : x \mapsto \sqrt{x}, \quad \text{where } x \in \mathbb{R} \text{ and } x \geq 0.$$

If you are given a function described by a formula but no domain is stated, you should assume that the domain is the natural domain.

Example 11.3.1

Find the range of each of the functions

- (a) \sin , with natural domain \mathbb{R} , (b) \sin , with domain $x \in \mathbb{R}$ and $0 < x < 90$.

From the graph of $y = \sin x^\circ$, shown in Fig. 11.2, you can read off the ranges:

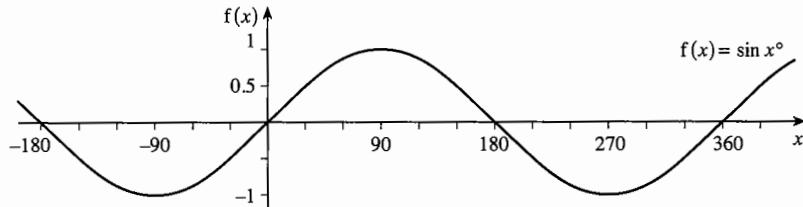


Fig. 11.2

- (a) For $x \in \mathbb{R}$, the range is $y \in \mathbb{R}, -1 \leq y \leq 1$.

- (b) For $x \in \mathbb{R}, 0 < x < 90$, the range is $y \in \mathbb{R}, 0 < y < 1$.

Example 11.3.1 has used the letter x in describing the domain, and y for the range, but other letters would work just as well. The expressions $y \in \mathbb{R}, 0 < y < 1$ and $x \in \mathbb{R}, 0 < x < 1$ and $t \in \mathbb{R}, 0 < t < 1$ all describe the same set of numbers.

It is especially important to understand this when you find composite functions. For example, in Example 11.2.3(a), the number 0.5 appears first as the output for the input 60 to the function $f : x \mapsto \cos x^\circ$, so you might think of it as $y = \cos 60^\circ = 0.5$.

But when it becomes the input to the function $g : x \mapsto \frac{1}{x}$, it is natural to write $x = 0.5$.

The number 0.5 belongs first to the range of f , then to the domain of g .

This is where Example 11.2.3(b) breaks down. The number 0, which is the output when the input to f is 90, is not in the natural domain of g . So although 90 is in the natural domain of f , it is not in the natural domain of gf .

The general rule is:

To form the **composite function** gf , the domain D of f must be chosen so that the whole of the range of f is included in the domain of g . The function gf is then defined as $gf : x \mapsto g(f(x)), x \in D$.

For the functions in Example 11.2.3, the domain of g is the set \mathbb{R} excluding 0, so the domain of f must be chosen to exclude the numbers x for which $\cos x^\circ = 0$. These are $\dots, -450^\circ, -270^\circ, -90^\circ, +90^\circ, +270^\circ, +450^\circ, \dots$, all of which can be summed up by the formula $90^\circ + 180n^\circ$, where n is an integer.

There is a neat way of writing this, using the standard symbol \mathbb{Z} for the set of integers $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$. The domain of f can then be expressed as $x \in \mathbb{R}, x \neq 90^\circ + 180n^\circ, n \in \mathbb{Z}$.

Example 11.3.2

Find the natural domain and the corresponding range of the function $x \mapsto \sqrt{x(x-3)}$.

You can express the function as gf , where $f: x \mapsto x(x-3)$ and $g: x \mapsto \sqrt{x}$.

The natural domain of g is $x \in \mathbb{R}, x \geq 0$, so you want the range of f to be included in $y \in \mathbb{R}, y \geq 0$. (Switching from x to y is not essential, but you may find it easier.) The solution of the inequality $y = x(x-3) \geq 0$ is $x \geq 3$ or $x \leq 0$.

The natural domain of gf is therefore $x \in \mathbb{R}, x \geq 3$ or $x \leq 0$.

With this domain the range of f is $y \in \mathbb{R}, y \geq 0$, so the numbers input to g are given by $x \in \mathbb{R}, x \geq 0$. With this domain, the range of g is $y \in \mathbb{R}, y \geq 0$. This is therefore the range of the combined function gf .

If you have access to a graphic calculator, try to plot the graph $y = \sqrt{x(x-3)}$, using a window of $-1 \leq x \leq 4$ and $0 \leq y \leq 2$. You should find that no points are plotted for the ‘illegal’ values of x in the interval $0 < x < 3$. If you input a number in this interval, such as 1, you will get as far as $1(1-3) = -2$, but the final $\sqrt{}$ key will give you an error message or some display which does not represent a real number.

11.4 Sequences as functions

Not all functions have for their domain the set of real numbers or a restricted interval of the real numbers. For example, a function might have the set of natural numbers $\{1, 2, 3, \dots\}$ for its domain. This set is denoted by the symbol \mathbb{N} .

Some games (such as chess and Scrabble) are played on a board ruled out in squares. If the board has r squares each way, then the total number of squares is r^2 . So this defines a function

$$f: r \mapsto r^2, \quad \text{where } r \in \mathbb{N}.$$

This is a different function from

$$f: x \mapsto x^2, \quad \text{where } x \in \mathbb{R},$$

because the number of squares each way must be a whole number.

You can make a list of the successive values of $f(r)$:

$$f(1) = 1, f(2) = 4, f(3) = 9, f(4) = 16, f(5) = 25, \dots$$

Notice that these numbers are precisely those in sequence (a) in Section 8.1. This suggests that a sequence can be considered as a function whose domain is \mathbb{N} .

Some sequences have only a finite number of terms. Suppose, for example, that you have 6 identical coins, and $f(r)$ denotes the number of ways of splitting the coins into r piles. Thus $f(2) = 3$, because you can have piles of 1 coin and 5 coins, 2 coins and 4 coins, or 3 coins and 3 coins. Check for yourself that

$$f(1) = 1, f(2) = 3, f(3) = 3, f(4) = 2, f(5) = 1 \text{ and } f(6) = 1;$$

but $f(r)$ has no meaning for $r > 6$. The domain of the function is therefore the set $\{1, 2, 3, 4, 5, 6\}$, which is a subset of consecutive numbers in \mathbb{N} .

A sequence can therefore be defined as a function whose domain is \mathbb{N} or a consecutive subset of \mathbb{N} . For sequences the notation u_r is normally used rather than $f(r)$, but that is simply for convenience.

You saw in Chapter 8 that it is simpler with some sequences to begin with $s = 0$ rather than $r = 1$. For those sequences the domain is $\{0, 1, 2, 3, \dots\}$. The notation for describing this set is $\mathbb{N} \cup \{0\}$.

An important difference between \mathbb{N} and \mathbb{R} is that, for every natural number r , there is a ‘next number’. This is what makes it possible to use an inductive definition to describe a sequence. There is no comparable way of defining a function $f(x)$, where $x \in \mathbb{R}$, because there is no such thing as the ‘next real number’.

Exercise 11A

- 1** Given $f: x \mapsto (3x + 5)^2$, where $x \in \mathbb{R}$, find the values of
 (a) $f(2)$, (b) $f(-1)$, (c) $f(7)$.

2 Given $g: x \mapsto 3x^2 + 5$, where $x \in \mathbb{R}$, find the values of
 (a) $g(2)$, (b) $g(-1)$, (c) $g(7)$.

3 Given $f: x \mapsto \frac{4}{x+5}$, where $x \in \mathbb{R}$ and $x \neq -5$, find the values of
 (a) $f(-1)$, (b) $f(-4)$, (c) $f(3)$.

4 Given $g: x \mapsto \frac{4}{x} + 5$, where $x \in \mathbb{R}$ and $x \neq 0$, find the values of
 (a) $g(-1)$, (b) $g(-4)$, (c) $g(3)$.

5 Find the output if the functions ‘square’ and ‘subtract 4’ act in succession on an input of
 (a) 2, (b) -5, (c) $\frac{1}{2}$, (d) x .

6 Find the output if the functions ‘cos’, ‘add 2’, and ‘cube’ act in succession on an input of
 (a) 0, (b) 90, (c) 120, (d) x .

7 Find the output if the functions ‘square root’, ‘multiply by 2’, ‘subtract 10’ and ‘square’ act in succession on an input of
 (a) 9, (b) 16, (c) $\frac{1}{4}$, (d) x .

8 Determine the key sequence needed to represent each of the following functions.
 (a) $f: x \mapsto 4x + 9$ (b) $f: x \mapsto 4(x + 9)$ (c) $f: x \mapsto 2x^2 - 5$
 (d) $f: x \mapsto 2(x - 5)^2$ (e) $f: x \mapsto (\sqrt{x} - 3)^3$, $x \geq 0$ (f) $f: x \mapsto \sqrt{(x - 2)^2 + 10}$

9 Find the natural domain and corresponding range of each of the following functions.
 (a) $f: x \mapsto x^2$ (b) $f: x \mapsto \cos x^\circ$ (c) $f: x \mapsto \sqrt{x - 3}$
 (d) $f: x \mapsto x^2 + 5$ (e) $f: x \mapsto \frac{1}{\sqrt{x}}$ (f) $f: x \mapsto x(4 - x)$
 (g) $f: x \mapsto \sqrt{x(4 - x)}$ (h) $f: x \mapsto x^2 + 4x + 10$ (i) $f: x \mapsto (1 - \sqrt{x - 3})^2$

- 10** Given that $f: x \mapsto 2x + 1$ and $g: x \mapsto 3x - 5$, where $x \in \mathbb{R}$, find the value of the following.
- (a) $gf(1)$ (b) $gf(-2)$ (c) $fg(0)$ (d) $fg(7)$
 (e) $ff(5)$ (f) $ff(-5)$ (g) $gg(4)$ (h) $gg\left(2\frac{2}{9}\right)$
- 11** Given that $f: x \mapsto x^2$ and $g: x \mapsto 4x - 1$, where $x \in \mathbb{R}$, find the value of the following.
- (a) $fg(2)$ (b) $gg(4)$ (c) $gf(-3)$
 (d) $ff\left(\frac{1}{2}\right)$ (e) $fgf(-1)$ (f) $gfgf(2)$
- 12** Given that $f: x \mapsto 5 - x$ and $g: x \mapsto \frac{4}{x}$, where $x \in \mathbb{R}$ and $x \neq 0$ or 5, find the values of the following.
- (a) $ff(7)$ (b) $ff(-19)$ (c) $gg(1)$ (d) $gg\left(\frac{1}{2}\right)$
 (e) $gggg\left(\frac{1}{2}\right)$ (f) $fffff(6)$ (g) $fgfg(2)$ (h) $fggf(2)$
- 13** Given that $f: x \mapsto 2x + 5$, $g: x \mapsto x^2$ and $h: x \mapsto \frac{1}{x}$, where $x \in \mathbb{R}$ and $x \neq 0$ or $-\frac{5}{2}$, find the following composite functions.
- (a) fg (b) gf (c) fh (d) hf
 (e) ff (f) hh (g) ghf (h) hgf
- 14** Given that $f: x \mapsto \sin x^\circ$, $g: x \mapsto x^3$ and $h: x \mapsto x - 3$, where $x \in \mathbb{R}$, find the following functions.
- (a) hf (b) fh (c) fhg
 (d) fg (e) hhh (f) gf
- 15** Given that $f: x \mapsto x + 4$, $g: x \mapsto 3x$ and $h: x \mapsto x^2$, where $x \in \mathbb{R}$, express each of the following in terms of f , g , h as appropriate.
- (a) $x \mapsto x^2 + 4$ (b) $x \mapsto 3x + 4$ (c) $x \mapsto x^4$
 (d) $x \mapsto 9x^2$ (e) $x \mapsto 3x + 12$ (f) $x \mapsto 3(x^2 + 8)$
 (g) $x \mapsto 9x + 16$ (h) $x \mapsto x^2 + 8x + 16$ (i) $x \mapsto 9x^2 + 48x + 64$
- 16** In each of the following, find the natural domain and the range of the function gf .
- (a) $f: x \mapsto \sqrt{x}$, $g: x \mapsto x - 5$ (b) $f: x \mapsto x + 3$, $g: x \mapsto \sqrt{x}$
 (c) $f: x \mapsto x - 2$, $g: x \mapsto \frac{1}{x}$ (d) $f: x \mapsto \sin x^\circ$, $g: x \mapsto \sqrt{x^2}$
 (e) $f: x \mapsto \sqrt{(x-3)^2}$, $g: x \mapsto \sqrt{x}$ (f) $f: x \mapsto 16 - x^2$, $g: x \mapsto \sqrt[4]{x}$
 (g) $f: x \mapsto x^2 - x - 6$, $g: x \mapsto \sqrt{x}$ (h) $f: x \mapsto x + 2$, $g: x \mapsto \frac{1}{\sqrt{-x}}$
- 17** Given that $f: x \mapsto x^2$ and $g: x \mapsto 3x - 2$, where $x \in \mathbb{R}$, find a , b and c such that
 (a) $fg(a) = 100$, (b) $gg(b) = 55$, (c) $fg(c) = gf(c)$.
- 18** Given that $f: x \mapsto ax + b$ and that $ff: x \mapsto 9x - 28$, find the possible values of a and b .
- 19** For $f: x \mapsto ax + b$, $f(2) = 19$ and $ff(0) = 55$. Find the possible values of a and b .
- 20** The functions $f: x \mapsto 4x + 1$ and $g: x \mapsto ax + b$ are such that $fg = gf$ for all real values of x . Show that $a = 3b + 1$.

- 21** Use function notation to describe
 (a) the triangle number sequence,
 (b) the general arithmetic sequence with first term a and common difference d .
- 22** In prime number theory the following notation is used:

p_r is the r th prime number,

$\pi(r)$ is the number of prime numbers less than or equal to r .

For each of these functions, state

- (a) the domain, (b) the values for $1 \leq r \leq 10$, (c) the range.

(Note: 1 is not a prime number.)

11.5 Reversing functions

If your sister is 2 years older than you, then you are 2 years younger than her. To get her age from yours you use the ‘add 2’ function; to get your age from hers you ‘subtract 2’. The functions ‘add 2’ and ‘subtract 2’ are said to be **inverse functions** of each other. That is, ‘subtract 2’ is the inverse function of ‘add 2’ (and vice versa).

You know many pairs of inverse functions: ‘double’ and ‘halve’, and ‘cube’ and ‘cube root’ are simple examples.

Some functions are their own inverses, such as ‘change sign’; to undo the effect of a change of sign, you just change sign again. Another example is ‘reciprocal’ ($x \mapsto \frac{1}{x}$).

These functions are said to be **self-inverse**.

The inverse of a function f is denoted by the symbol f^{-1} . If f turns an input number x into an output number y , then f^{-1} turns y into x . You can illustrate this graphically by reversing the arrow which symbolises the function, as in Fig. 11.3. The range of f becomes the domain of f^{-1} , and the domain of f becomes the range of f^{-1} .

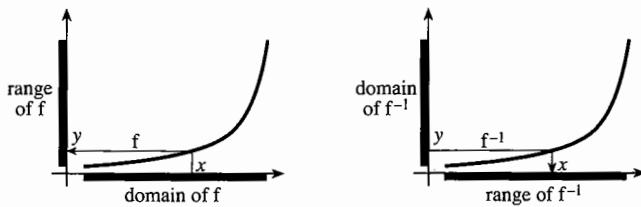


Fig. 11.3

You have already used inverse functions in calculations about triangles. Often you know an angle, and calculate the length of a side by using one of the trigonometric functions such as \tan . But if you know the sides and want to calculate the angle you use the inverse function, which is denoted by \tan^{-1} .

On many calculators you find values of \tan^{-1} by using a sequence of two keys: first an ‘inverse’ key (which on some calculators is labelled ‘shift’ or ‘2nd function’) and then ‘tan’. In the following pages this is referred to as ‘the \tan^{-1} key’, and similarly for the \sin^{-1} and \cos^{-1} functions.

Example 11.5.1

Find the values of $\cos^{-1} y$ when (a) $y = 0.5$, (b) $y = -1$, (c) $y = 1.5$.

Using the $[\cos^{-1}]$ key with inputs $0.5, -1, 1.5$ in turn gives outputs of $60^\circ, 180^\circ$, and an error message!

So, in degree mode, (a) $\cos^{-1} 0.5 = 60^\circ$,
 (b) $\cos^{-1}(-1) = 180^\circ$, but (c) $\cos^{-1} 1.5$ has no meaning.

Fig. 11.4 shows the graph of $y = \cos x^\circ$ with domain $x \in \mathbb{R}, 0^\circ \leq x \leq 180^\circ$. This shows that the range of the function \cos is $-1 \leq y \leq 1$. Since this is the domain of the inverse function, the result in part (c) is explained by the fact that 1.5 lies outside this interval.

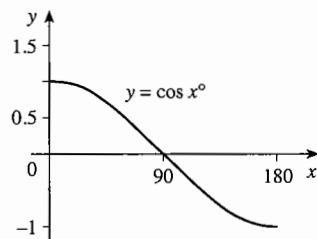


Fig. 11.4

If you try to check by finding the cosines of the answers to this example, you get.

$$\begin{aligned} \text{(a)} \quad 0.5 &\rightarrow [\cos^{-1}] \rightarrow 60 \rightarrow [\cos] \rightarrow 0.5, \\ \text{(b)} \quad -1 &\rightarrow [\cos^{-1}] \rightarrow 180 \rightarrow [\cos] \rightarrow -1. \end{aligned}$$

This is of course what you would expect; the function and its inverse cancel each other out. In general, if $-1 \leq y \leq 1$, then

$$y \rightarrow [\cos^{-1}] \rightarrow [\cos] \rightarrow y.$$

You may therefore be surprised by the result of the next example.

Example 11.5.2

If $f : x \mapsto \sin x^\circ$ and $g : x \mapsto \sin^{-1} x$, evaluate (a) $gf(50)$, (b) $gf(130)$.

Work this example for yourself using the calculator sequence

$$x \rightarrow [\sin] \rightarrow [\sin^{-1}] \rightarrow gf(x).$$

You should get the answers (a) 50 (as you would expect) and (b) 50 .

The answer to part (b) calls for a more careful look at the theory of inverse functions.

11.6 One-one functions

The answers to Example 11.5.2 can be explained by Fig. 11.5, which shows the graph of $y = \sin x^\circ$ over the interval $0^\circ \leq x \leq 180^\circ$. The graph rises from $y = 0$ to $y = 1$ over values for x for which the angle is acute, and then falls symmetrically back to $y = 0$ over values for which the angle is obtuse. This is because the sine of the obtuse angle x° is equal to the sine of the supplementary angle $(180 - x)^\circ$. So $\sin 130^\circ = \sin 50^\circ$, and the calculator gives the value $0.7660\dots$ for both.

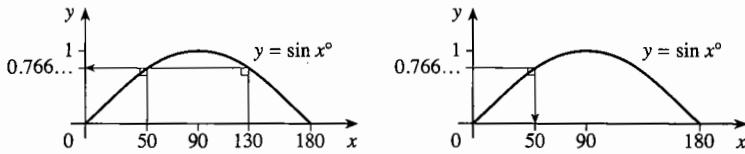


Fig. 11.5

When you use the $[\sin^{-1}]$ key to find $\sin^{-1}(0.766\dots)$, the calculator has to give the same answer in either case. It is programmed to give the answer with the smallest modulus, which in this case is 50.

Exactly the same problem arises whenever you try to reverse a function which has the same output for more than one input. And in mathematics, such ambiguity is not acceptable. The solution adopted is a drastic one, to refuse to define an inverse for any function which has the same output for more than one input. That is, the only functions which have an inverse function are those for which each output in the range comes from only one input. These functions are said to be ‘one-one’.

A function f defined for some domain D is **one-one** if, for each number y in the range R of f there is only one number $x \in D$ such that $y = f(x)$. The function with domain R defined by $f^{-1} : y \mapsto x$, where $y = f(x)$, is the **inverse function** of f .

This definition was illustrated in Fig. 11.3, which was drawn to ensure that the function f was one-one.

In practice, this can be achieved by restricting its domain. For example, the function $x \mapsto \sin x^\circ$, $x \in \mathbb{R}$, whose graph is shown in Fig. 11.2 on page 160, is not one-one, so it does not have an inverse. But the function $x \mapsto \sin x^\circ$, where $x \in \mathbb{R}$ and $-90^\circ \leq x \leq 90^\circ$, shown in Fig. 11.6, is one-one; it is the inverse of this function which is denoted by \sin^{-1} , and activated by the familiar key sequence on the calculator.

Fig. 11.3 suggests that, if you compose a function with its inverse, you get back to the number you started with. That is,

$$f^{-1}f(x) = x, \quad \text{and} \quad ff^{-1}(y) = y.$$

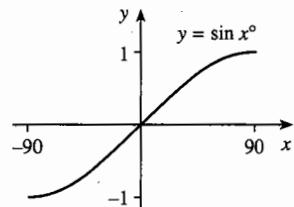


Fig. 11.6

The functions $f^{-1}f$ and ff^{-1} are called **identity functions** because their inputs and outputs are identical. But there is a subtle difference between these two composite functions, since their domains may not be the same; the first has domain D and the second has domain R .

11.7 Finding inverse functions

For very simple functions it is easy to write down an expression for the inverse function. The inverse of ‘add 2’ is ‘subtract 2’, so

$$f : x \mapsto x + 2, x \in \mathbb{R} \quad \text{has inverse} \quad f^{-1} : x \mapsto x - 2, x \in \mathbb{R}.$$

Notice that the inverse could equally well be written as $f^{-1} : y \mapsto y - 2, y \in \mathbb{R}$.

You can sometimes break down more complicated functions into a chain of simple steps. You can then find the inverse by going backwards through each step in reverse order. (This is sometimes called the ‘shoes and socks’ process: you put your socks on before your shoes, but you take off your shoes before your socks. In mathematical notation, $(gf)^{-1} = f^{-1}g^{-1}$, where f denotes putting on your socks and g your shoes.)

However, this method does not always work, particularly if x appears more than once in the expression for the function. Another method is to write $y = f(x)$, and turn the formula round into the form $x = g(y)$. Then g is the inverse of f .

Example 11.7.1

Find the inverse of $f : x \mapsto 2x + 5, x \in \mathbb{R}$.

Note first that f is one-one, and that the range is \mathbb{R} .

Method 1 You can break the function down as

$$x \rightarrow [\text{double}] \rightarrow [\text{add 5}] \rightarrow 2x + 5.$$

To find f^{-1} , go backwards through the chain (read from right to left):

$$\frac{1}{2}(x - 5) \leftarrow [\text{halve}] \leftarrow [\text{subtract 5}] \leftarrow x.$$

So $f^{-1} : x \mapsto \frac{1}{2}(x - 5), x \in \mathbb{R}$.

Method 2 If $y = 2x + 5$,

$$y - 5 = 2x \quad \text{which gives} \quad x = \frac{1}{2}(y - 5).$$

So the inverse function is $f^{-1} : y \mapsto \frac{1}{2}(y - 5), y \in \mathbb{R}$.

The two answers are the same, even though different letters are used.

Example 11.7.2

Restrict the domain of the function $f : x \mapsto x^2 - 2x$, so that an inverse function exists. Find an expression for f^{-1} .

Fig. 11.7 shows the graph of $y = x^2 - 2x, x \in \mathbb{R}$, which is quadratic with its vertex at $(1, -1)$. For $y > -1$ there are two values of x for each y , so the graph does not represent a one-one function. One way of making it one-one is to chop off the part of the graph to the left of its axis of symmetry. This restricts the domain to $x \in \mathbb{R}, x \geq 1$, but the range is still $y \in \mathbb{R}, y \geq -1$.

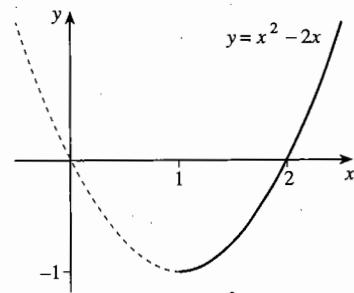


Fig. 11.7

Method 1 Completing the square gives $f(x) = (x - 1)^2 - 1$, so you can break the function down as

$$x \rightarrow [\text{subtract } 1] \rightarrow [\text{square}] \rightarrow [\text{subtract } 1] \rightarrow y.$$

In reverse,

$$1 + \sqrt{y+1} \leftarrow [\text{add } 1] \leftarrow [\sqrt{}] \leftarrow [\text{add } 1] \leftarrow y.$$

So the inverse function is $f^{-1} : y \mapsto 1 + \sqrt{y+1}$, $y \in \mathbb{R}, y \geq -1$.

Notice that the positive square root was chosen to make $x > 1$.

Method 2 If $y = x^2 - 2x$, then $x^2 - 2x - y = 0$.

This is a quadratic equation with roots

$$x = \frac{2 \pm \sqrt{4 + 4y}}{2} = 1 \pm \sqrt{1+y}.$$

Since $x \geq 1$, you must choose the positive sign, giving $x = 1 + \sqrt{1+y}$. So the inverse function is $f^{-1} : y \mapsto 1 + \sqrt{y+1}$, $y \in \mathbb{R}, y \geq -1$.

Example 11.7.3

Find the inverse of the function $f(x) = \frac{x+2}{x-2}$, where $x \in \mathbb{R}$ and $x \neq 2$.

It is not obvious that this function is one-one, or what its range is. However, using the second method and writing $y = \frac{x+2}{x-2}$,

$$\begin{aligned} y(x-2) &= x+2, \\ yx-2y &= x+2, \\ yx-x &= 2y+2, \\ x(y-1) &= 2(y+1), \\ x &= \frac{2(y+1)}{y-1}. \end{aligned}$$

This shows that, unless $y = 1$, there is just one value of x for each value of y . So f must be one-one, the inverse function therefore exists, and

$$f^{-1} : y \mapsto \frac{2(y+1)}{y-1}, \text{ where } y \in \mathbb{R} \text{ and } y \neq -1.$$

11.8 Graphing inverse functions

Fig. 11.8 shows the graph of $y = f(x)$, where f is a one-one function with domain D and range R . Since f^{-1} exists, with domain R and range D , you can also write the equation as $x = f^{-1}(y)$. You can regard Fig. 11.8 as the graph of both f and f^{-1} .

But you sometimes want to draw the graph of f^{-1} in the more conventional form, as $y = f^{-1}(x)$ with the domain along the x -axis. To do this you have to swap the x - and y -axes, which you do by reflecting the graph in Fig. 11.8 in the line $y = x$. (Make sure that you have the same scale on both axes!) Then the x -axis is reflected into the y -axis and vice versa, and the graph of $x = f^{-1}(y)$ is reflected into the graph of $y = f^{-1}(x)$. This is shown in Fig. 11.9.

If f is a one-one function, the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of each other in the line $y = x$.

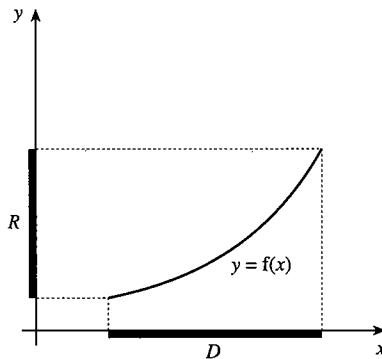


Fig. 11.8

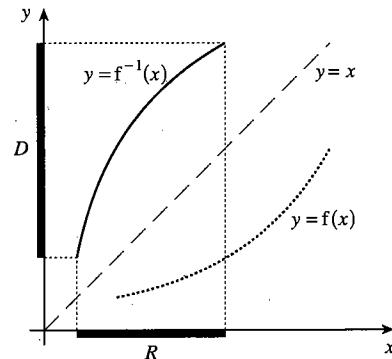


Fig. 11.9

Example 11.8.1

For the function in Example 11.7.2, draw the graphs of $y = f(x)$ and $y = f^{-1}(x)$.

Example 11.7.2 showed that

$$f^{-1}(x) = 1 + \sqrt{x+1}, \quad x \in \mathbb{R}, \quad x \geq -1.$$

Fig. 11.10 shows the graphs of

$$y = f(x) = x^2 - 2x \text{ for } x \geq 1 \text{ and}$$

$$y = f^{-1}(x) = 1 + \sqrt{x+1} \text{ for } x \geq -1.$$
 You can see that these graphs are reflections of each other in the line $y = x$.

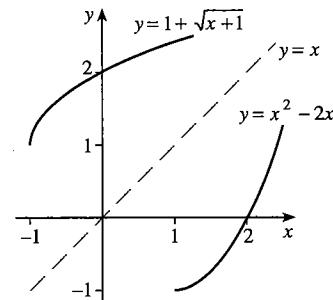


Fig. 11.10

Exercise 11B

- 1 Each of the following functions has domain \mathbb{R} . In each case use a graph to show that the function is one-one, and write down its inverse.
- | | | |
|--|---|---|
| <p>(a) $f : x \mapsto x + 4$</p> | <p>(b) $f : x \mapsto x - 5$</p> | <p>(c) $f : x \mapsto 2x$</p> |
| <p>(d) $f : x \mapsto \frac{1}{4}x$</p> | <p>(e) $f : x \mapsto x^3$</p> | <p>(f) $f : x \mapsto \sqrt[5]{x}$</p> |

- 2** Given the function $f : x \mapsto x - 6, x \in \mathbb{R}$, find the values of
 (a) $f^{-1}(4)$, (b) $f^{-1}(1)$, (c) $f^{-1}(-3)$, (d) $ff^{-1}(5)$, (e) $f^{-1}f(-4)$.
- 3** Given the function $f : x \mapsto 5x, x \in \mathbb{R}$, find the values of
 (a) $f^{-1}(20)$, (b) $f^{-1}(100)$, (c) $f^{-1}(7)$, (d) $ff^{-1}(15)$, (e) $f^{-1}f(-6)$.
- 4** Given the function $f : x \mapsto \sqrt[3]{x}, x \in \mathbb{R}$, find the values of
 (a) $f^{-1}(2)$, (b) $f^{-1}\left(\frac{1}{2}\right)$, (c) $f^{-1}(8)$, (d) $f^{-1}f(-27)$, (e) $ff^{-1}(5)$.
- 5** Each of the following functions has domain \mathbb{R} . Determine which are one-one functions.
 (a) $f : x \mapsto 3x + 4$ (b) $f : x \mapsto x^2 + 1$ (c) $f : x \mapsto x^2 - 3x$
 (d) $f : x \mapsto 5 - x$ (e) $f : x \mapsto \cos x^\circ$ (f) $f : x \mapsto x^3 - 2$
 (g) $f : x \mapsto \frac{1}{2}x - 7$ (h) $f : x \mapsto \sqrt{x^2}$ (i) $f : x \mapsto x(x - 4)$
 (j) $f : x \mapsto x^3 - 3x$ (k) $f : x \mapsto x^9$ (l) $f : x \mapsto \sqrt{x^2 + 1}$
- 6** Determine which of the following functions, with the specified domains, are one-one.
 (a) $f : x \mapsto x^2, x > 0$ (b) $f : x \mapsto \cos x^\circ, -90^\circ \leq x \leq 90^\circ$
 (c) $f : x \mapsto 1 - 2x, x < 0$ (d) $f : x \mapsto x(x - 2), 0 < x < 2$
 (e) $f : x \mapsto x(x - 2), x > 2$ (f) $f : x \mapsto x(x - 2), x < 1$
 (g) $f : x \mapsto \sqrt{x}, x > 0$ (h) $f : x \mapsto x^2 + 6x - 5, x > 0$
 (i) $f : x \mapsto x^2 + 6x - 5, x < 0$ (j) $f : x \mapsto x^2 + 6x - 5, x > -3$
- 7** Each of the following functions has domain $x \geq k$. In each case, find the smallest possible value of k such that the function is one-one.
 (a) $f : x \mapsto x^2 - 4$ (b) $f : x \mapsto (x + 1)^2$ (c) $f : x \mapsto (3x - 2)^2$
 (d) $f : x \mapsto x^2 - 8x + 15$ (e) $f : x \mapsto x^2 + 10x + 1$ (f) $f : x \mapsto (x + 4)(x - 2)$
 (g) $f : x \mapsto x^2 - 3x$ (h) $f : x \mapsto 6 + 2x - x^2$ (i) $f : x \mapsto (x - 4)^4$
- 8** Use method 1 of Example 11.7.1 to find the inverse of each of the following functions.
 (a) $f : x \mapsto 3x - 1, x \in \mathbb{R}$ (b) $f : x \mapsto \frac{1}{2}x + 4, x \in \mathbb{R}$
 (c) $f : x \mapsto x^3 + 5, x \in \mathbb{R}$ (d) $f : x \mapsto \sqrt{x} - 3, x > 0$
 (e) $f : x \mapsto \frac{5x - 3}{2}, x \in \mathbb{R}$ (f) $f : x \mapsto (x - 1)^2 + 6, x \geq 1$
- 9** Use method 2 of Example 11.7.1 to find the inverse of each of the following functions.
 (a) $f : x \mapsto 6x + 5, x \in \mathbb{R}$ (b) $f : x \mapsto \frac{x + 4}{5}, x \in \mathbb{R}$
 (c) $f : x \mapsto 4 - 2x, x \in \mathbb{R}$ (d) $f : x \mapsto \frac{2x + 7}{3}, x \in \mathbb{R}$
 (e) $f : x \mapsto 2x^3 + 5, x \in \mathbb{R}$ (f) $f : x \mapsto \frac{1}{x} + 4, x \in \mathbb{R}$ and $x \neq 0$
 (g) $f : x \mapsto \frac{5}{x - 1}, x \in \mathbb{R}$ and $x \neq 1$ (h) $f : x \mapsto (x + 2)^2 + 7, x \in \mathbb{R}$ and $x \geq -2$
 (i) $f : x \mapsto (2x - 3)^2 - 5, x \in \mathbb{R}$ and $x \geq \frac{3}{2}$ (j) $f : x \mapsto x^2 - 6x, x \in \mathbb{R}$ and $x \geq 3$

10 For each of the following, find the inverse function and sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$.

- | | |
|---|---|
| (a) $f: x \mapsto 4x, x \in \mathbb{R}$ | (b) $f: x \mapsto x + 3, x \in \mathbb{R}$ |
| (c) $f: x \mapsto \sqrt{x}, x \in \mathbb{R}$ and $x \geq 0$ | (d) $f: x \mapsto 2x + 1, x \in \mathbb{R}$ |
| (e) $f: x \mapsto (x - 2)^2, x \in \mathbb{R}$ and $x \geq 2$ | (f) $f: x \mapsto 1 - 3x, x \in \mathbb{R}$ |
| (g) $f: x \mapsto \frac{3}{x}, x \in \mathbb{R}$ and $x \neq 0$ | (h) $f: x \mapsto 7 - x, x \in \mathbb{R}$ |

11 Show that the following functions are self-inverse.

- | | |
|---|---|
| (a) $f: x \mapsto 5 - x, x \in \mathbb{R}$ | (b) $f: x \mapsto -x, x \in \mathbb{R}$ |
| (c) $f: x \mapsto \frac{4}{x}, x \in \mathbb{R}$ and $x \neq 0$ | (d) $f: x \mapsto \frac{6}{5x}, x \in \mathbb{R}$ and $x \neq 0$ |
| (e) $f: x \mapsto \frac{x+5}{x-1}, x \in \mathbb{R}$ and $x \neq 1$ | (f) $f: x \mapsto \frac{3x-1}{2x-3}, x \in \mathbb{R}$ and $x \neq \frac{3}{2}$ |

12 Find the inverse of each of the following functions.

- | | |
|---|--|
| (a) $f: x \mapsto \frac{x}{x-2}, x \in \mathbb{R}$ and $x \neq 2$ | (b) $f: x \mapsto \frac{2x+1}{x-4}, x \in \mathbb{R}$ and $x \neq 4$ |
| (c) $f: x \mapsto \frac{x+2}{x-5}, x \in \mathbb{R}$ and $x \neq 5$ | (d) $f: x \mapsto \frac{3x-11}{4x-3}, x \in \mathbb{R}$ and $x \neq \frac{3}{4}$ |

13 The function $f: x \mapsto x^2 - 4x + 3$ has domain $x \in \mathbb{R}$ and $x > 2$.

- Determine the range of f .
- Find the inverse function f^{-1} and state its domain and range.
- Sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$.

14 The function $f: x \mapsto \sqrt{x-2} + 3$ has domain $x \in \mathbb{R}$ and $x > 2$.

- Determine the range of f .
- Find the inverse function f^{-1} and state its domain and range.
- Sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$.

15 The function $f: x \mapsto x^2 + 2x + 6$ has domain $x \in \mathbb{R}$ and $x \leq k$. Given that f is one-one, determine the greatest possible value of k . When k has this value,

- determine the range of f ,
- find the inverse function f^{-1} and state its domain and range,
- sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$.

16 The inverse of the function $f: x \mapsto ax + b, x \in \mathbb{R}$, is $f^{-1}: x \mapsto 8x - 3$. Find a and b .

17 The function $f: x \mapsto px + q, x \in \mathbb{R}$, is such that $f^{-1}(6) = 3$ and $f^{-1}(-29) = -2$. Find $f^{-1}(27)$.

18 The function $f: x \mapsto x^2 + x + 6$ has domain $x \in \mathbb{R}$ and $x > 0$. Find the inverse function and state its domain and range.

19 The function $f: x \mapsto -2x^2 + 4x - 7$ has domain $x \in \mathbb{R}$ and $x < 1$. Find the inverse function and state its domain and range.

20 For each of the following functions, sketch the graph of $y = f^{-1}(x)$.

- (a) $f: x \mapsto \sin x^\circ, x \in \mathbb{R}$ and $-90^\circ \approx x \approx 90^\circ$
- (b) $f: x \mapsto \cos x^\circ, x \in \mathbb{R}$ and $0^\circ \approx x \approx 180^\circ$
- (c) $f: x \mapsto \tan x^\circ, x \in \mathbb{R}$ and $-90^\circ < x < 90^\circ$

Miscellaneous exercise 11

1 The functions f and g are defined by

$$f: x \mapsto 4x + 9, x \in \mathbb{R}, \quad g: x \mapsto x^2 + 1, x \in \mathbb{R}.$$

Find the value of each of the following.

- | | | |
|--------------|---------------------|-----------------------------------|
| (a) $fg(2)$ | (b) $fg(2\sqrt{3})$ | (c) $gf(-2)$ |
| (d) $ff(-3)$ | (e) $gg(-4)$ | (f) $fgf\left(\frac{1}{2}\right)$ |

2 Find the natural domain and corresponding range of each of the following functions.

- | | | |
|----------------------------|--------------------------------|---------------------------------|
| (a) $f: x \mapsto 4 - x^2$ | (b) $f: x \mapsto (x+3)^2 - 7$ | (c) $f: x \mapsto \sqrt{x+2}$ |
| (d) $f: x \mapsto 5x + 6$ | (e) $f: x \mapsto (2x+3)^2$ | (f) $f: x \mapsto 2 - \sqrt{x}$ |

3 The functions f and g are defined by

$$f: x \mapsto x^3, x \in \mathbb{R}, \quad g: x \mapsto 1 - 2x, x \in \mathbb{R}.$$

Find the functions

- | | | | | |
|------------|------------|-------------|------------|----------------|
| (a) fg , | (b) gf , | (c) gff , | (d) gg , | (e) g^{-1} . |
|------------|------------|-------------|------------|----------------|

4 The function f is defined by $f: x \mapsto 2x^3 - 6, x \in \mathbb{R}$. Find the values of the following.

- | | | | | |
|------------|------------------|------------------|------------------|------------------|
| (a) $f(3)$ | (b) $f^{-1}(48)$ | (c) $f^{-1}(-8)$ | (d) $f^{-1}f(4)$ | (e) $ff^{-1}(4)$ |
|------------|------------------|------------------|------------------|------------------|

5 The function f is defined for all real values of x by $f(x) = x^{\frac{1}{3}} + 10$. Evaluate

- | | | |
|----------------|--------------------|-------|
| (a) $ff(-8)$, | (b) $f^{-1}(13)$. | (OCR) |
|----------------|--------------------|-------|

6 Show that the function $f: x \mapsto (x+3)^2 + 1$, with domain $x \in \mathbb{R}$ and $x > 0$, is one-one and find its inverse.

7 The function f is defined by $f: x \mapsto 4x^3 + 3, x \in \mathbb{R}$. Give the corresponding definition of f^{-1} . State a relationship between the graphs of f and f^{-1} . (OCR)

8 Given that $f(x) = 3x^2 - 4, x > 0$, and $g(x) = x + 4, x \in \mathbb{R}$, find

- | | | |
|---------------------------|-----------------------|-------|
| (a) $f^{-1}(x), x > -4$, | (b) $fg(x), x > -4$. | (OCR) |
|---------------------------|-----------------------|-------|

9 The functions f , g and h are defined by

$$f: x \mapsto 2x + 1, x \in \mathbb{R}, \quad g: x \mapsto x^5, x \in \mathbb{R}, \quad h: x \mapsto \frac{1}{x}, x \in \mathbb{R} \text{ and } x \neq 0.$$

Express each of the following in terms of f , g , h as appropriate.

- | | | | |
|----------------------------------|-------------------------------|---|-------------------------------|
| (a) $x \mapsto (2x+1)^5$ | (b) $x \mapsto 4x+3$ | (c) $x \mapsto x^{\frac{1}{5}}$ | (d) $x \mapsto 2x^{-5} + 1$ |
| (e) $x \mapsto \frac{1}{2x^5+1}$ | (f) $x \mapsto \frac{x-1}{2}$ | (g) $x \mapsto \sqrt[5]{\frac{2}{x^5}+1}$ | (h) $x \mapsto \frac{2}{x-1}$ |

- 10** The function f is defined by $f: x \mapsto x^2 + 1, x \geq 0$. Sketch the graph of the function f and, using your sketch or otherwise, show that f is a one-one function. Obtain an expression in terms of x for $f^{-1}(x)$ and state the domain of f^{-1} .

The function g is defined by $g: x \mapsto x - 3, x \geq 0$. Give an expression in terms of x for $gf(x)$ and state the range of gf . (OCR)

- 11** The functions f and g are defined by

$$f: x \mapsto x^2 + 6x, x \in \mathbb{R}, \quad g: x \mapsto 2x - 1, x \in \mathbb{R}.$$

Find the two values of x such that $fg(x) = gf(x)$, giving each answer in the form $p + q\sqrt{3}$.

- 12** The function f is defined by $f: x \mapsto x^2 - 2x + 7$ with domain $x \leq k$. Given that f is a one-one function, find the greatest possible value of k and find the inverse function f^{-1} .

- 13** Functions f and g are defined by

$$f: x \mapsto x^2 + 2x + 3, x \in \mathbb{R}, \quad g: x \mapsto ax + b, x \in \mathbb{R}.$$

Given that $fg(x) = 4x^2 - 48x + 146$ for all x , find the possible values of a and b .

- 14** The function f is defined by $f: x \mapsto 1 - x^2, x \leq 0$.

(a) Sketch the graph of f .

(b) Find an expression, in terms of x , for $f^{-1}(x)$ and state the domain of f^{-1} .

(c) The function g is defined by $g: x \mapsto 2x, x \leq 0$. Find the value of x for which $fg(x) = 0$. (OCR)

- 15** Functions f and g are defined by $f: x \mapsto 4x + 5, x \in \mathbb{R}$, and $g: x \mapsto 3 - 2x, x \in \mathbb{R}$. Find

$$(a) f^{-1}, \quad (b) g^{-1}, \quad (c) f^{-1}g^{-1}, \quad (d) gf, \quad (e) (gf)^{-1}.$$

- 16** Functions f and g are defined by $f: x \mapsto 2x + 7, x \in \mathbb{R}$, and $g: x \mapsto x^3 - 1, x \in \mathbb{R}$. Find

$$(a) f^{-1}, \quad (b) g^{-1}, \quad (c) g^{-1}f^{-1}, \quad (d) f^{-1}g^{-1}, \\ (e) fg, \quad (f) gf, \quad (g) (fg)^{-1}, \quad (h) (gf)^{-1}.$$

- 17** Given the function $f: x \mapsto 10 - x, x \in \mathbb{R}$, evaluate

$$(a) f(7), \quad (b) f^2(7), \quad (c) f^{15}(7), \quad (d) f^{100}(7).$$

(The notation f^2 represents the composite function ff , f^3 represents fff , and so on.)

- 18** Given the function $f: x \mapsto \frac{x+5}{2x-1}, x \in \mathbb{R}$ and $x \neq \frac{1}{2}$, find

$$(a) f^2(x), \quad (b) f^3(x), \quad (c) f^4(x), \quad (d) f^{10}(x), \quad (e) f^{351}(x).$$

- 19** Given the function $f(x) = \frac{2x-4}{x}, x \in \mathbb{R}$ and $x \neq 0$, find

$$(a) f^2(x), \quad (b) f^{-1}(x), \quad (c) f^3(x), \\ (d) f^4(x), \quad (e) f^{12}(x), \quad (f) f^{82}(x).$$

- 20** Show that a function of the form $x \mapsto \frac{x+a}{x-1}, x \in \mathbb{R}$ and $x \neq 1$, is self-inverse for all values of the constant a .

12 Extending differentiation

This chapter is about differentiating composite functions. When you have completed it, you should

- be able to differentiate composite functions of the form $f(F(x))$
- be able to apply differentiation to rates of change, and to related rates of change.

You may want to leave out Section 12.4 on a first reading; the exercises do not depend on it.

12.1 Differentiating $(ax + b)^n$

To differentiate a function like $(2x + 1)^3$, the only method available to you at present is to use the binomial theorem to multiply out the brackets, and then to differentiate term by term.

Example 12.1.1

Find $\frac{dy}{dx}$ for (a) $y = (2x + 1)^3$, (b) $y = (1 - 3x)^4$.

(a) Expanding by the binomial theorem,

$$y = (2x)^3 + 3 \times (2x)^2 \times 1 + 3 \times (2x) \times 1^2 + 1^3 = 8x^3 + 12x^2 + 6x + 1.$$

$$\text{So } \frac{dy}{dx} = 24x^2 + 24x + 6.$$

It is useful to express the result in factors, as $\frac{dy}{dx} = 6(4x^2 + 4x + 1) = 6(2x + 1)^2$.

(b) Expanding by the binomial theorem,

$$\begin{aligned} y &= 1^4 + 4 \times 1^3 \times (-3x) + 6 \times 1^2 \times (-3x)^2 + 4 \times 1 \times (-3x)^3 + (-3x)^4 \\ &= 1 - 12x + 54x^2 - 108x^3 + 81x^4. \end{aligned}$$

$$\text{So } \frac{dy}{dx} = -12 + 108x - 324x^2 + 324x^3 = -12(1 - 9x + 27x^2 - 27x^3) = -12(1 - 3x)^3.$$

Exercise 12A

In Question 1, see if you can predict what the result of the differentiation will be. If you can predict the result, then check by carrying out the differentiation, and factorising your result. If you can't, differentiate and simplify and look for a pattern in your answers.

- 1 Find $\frac{dy}{dx}$ for each of the following functions. In parts (d) and (e), a and b are constants.
- | | | | |
|------------------|------------------|------------------|------------------|
| (a) $(x + 3)^2$ | (b) $(2x - 3)^2$ | (c) $(1 - 3x)^3$ | (d) $(ax + b)^3$ |
| (e) $(b - ax)^3$ | (f) $(1 - x)^5$ | (g) $(2x - 3)^4$ | (h) $(3 - 2x)^4$ |

- 2 Suppose that $y = (ax + b)^n$, where a and b are constants and n is a positive integer. Guess a formula for $\frac{dy}{dx}$.
- 3 Use the formula you guessed in Question 2, after checking that it is correct, to differentiate each of the following functions, where a and b are constants.
- (a) $(x + 3)^{10}$ (b) $(2x - 1)^5$ (c) $(1 - 4x)^7$ (d) $(3x - 2)^5$
 (e) $(4 - 2x)^6$ (f) $4(2 + 3x)^6$ (g) $(2x + 5)^5$ (h) $(2x - 3)^9$

In Exercise 12A, you were asked to predict how to differentiate a function of the form $(ax + b)^n$, where a and b are constants and n is a positive integer. The result was:

$$\text{If } y = (ax + b)^n, \text{ then } \frac{dy}{dx} = n(ax + b)^{n-1} \times a.$$

Now assume that the same formula works for all n , including fractional and negative values. There is a proof in Section 12.4, but you can skip it on a first reading. The result, however, is important, and you must be able to use it confidently.

$$\boxed{\text{If } a, b \text{ and } n \text{ are constants, and } y = (ax + b)^n, \text{ then } \frac{dy}{dx} = n(ax + b)^{n-1} \times a.}$$

Example 12.1.2

$$\text{Find } \frac{dy}{dx} \text{ when (a) } y = \sqrt{3x+2}, \text{ (b) } y = \frac{1}{1-2x}.$$

(a) Writing $\sqrt{3x+2}$ in index form as $(3x+2)^{\frac{1}{2}}$ and using the result in the box,

$$\frac{dy}{dx} = \frac{1}{2}(3x+2)^{-\frac{1}{2}} \times 3 = \frac{3}{2} \frac{1}{(3x+2)^{\frac{1}{2}}} = \frac{3}{2\sqrt{3x+2}}.$$

$$(b) \text{ In index form } y = (1-2x)^{-1}, \text{ so } \frac{dy}{dx} = -1(1-2x)^{-2} \times (-2) = \frac{2}{(1-2x)^2}.$$

Example 12.1.3

Find any stationary points on the graph $y = \sqrt{2x+1} + \frac{1}{\sqrt{2x+1}}$, and determine whether they are maxima, minima or neither.

$\sqrt{2x+1}$ is defined for $x \geq -\frac{1}{2}$, and $\frac{1}{\sqrt{2x+1}}$ for $x > -\frac{1}{2}$. So the largest possible domain is $x > -\frac{1}{2}$.

$$y = (2x+1)^{\frac{1}{2}} + (2x+1)^{-\frac{1}{2}}, \text{ so}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2}(2x+1)^{-\frac{1}{2}} \times 2 + \left(-\frac{1}{2}\right)(2x+1)^{-\frac{3}{2}} \times 2 = (2x+1)^{-\frac{1}{2}} - (2x+1)^{-\frac{3}{2}} \\ &= \frac{1}{(2x+1)^{\frac{1}{2}}} - \frac{1}{(2x+1)^{\frac{3}{2}}} = \frac{2x+1-1}{(2x+1)^{\frac{3}{2}}} = \frac{2x}{(2x+1)^{\frac{3}{2}}}. \end{aligned}$$

Stationary points are those for which $\frac{dy}{dx} = 0$, which happens when $x = 0$.

To find the nature of the stationary point, notice that the denominator in the expression for $\frac{dy}{dx}$ is positive for all values of x in the domain, and that the numerator is positive for $x > 0$ and negative for $x < 0$. So y has a minimum at $x = 0$.

The result given in the box on page 175 is a special case of a more general result which you can use to differentiate any function of the form $f(ax + b)$.

If a and b are constants, and if $\frac{d}{dx} f(x) = g(x)$,
then $\frac{d}{dx} f(ax + b) = ag(ax + b)$.

For the special case in this section, $f(x) = x^n$ and $g(x) = nx^{n-1}$. Then

$$\frac{d}{dx}(ax + b)^n = \frac{d}{dx}f(ax + b) = ag(ax + b) = an(ax + b)^{n-1}.$$

Exercise 12B

1 Find $\frac{dy}{dx}$ for each of the following.

(a) $y = (4x + 5)^5$ (b) $y = (2x - 7)^8$ (c) $y = (2 - x)^6$ (d) $y = \left(\frac{1}{2}x + 4\right)^4$

2 Find $\frac{dy}{dx}$ for each of the following.

(a) $y = \frac{1}{3x + 5}$ (b) $y = \frac{1}{(4 - x)^2}$ (c) $y = \frac{1}{(2x + 1)^3}$ (d) $y = \frac{4}{(4x - 1)^4}$

3 Find $\frac{dy}{dx}$ for each of the following.

(a) $y = \sqrt{2x + 3}$ (b) $y = \sqrt[3]{6x - 1}$ (c) $y = \frac{1}{\sqrt[4]{4x + 7}}$ (d) $y = 5(3x - 2)^{-\frac{2}{3}}$

4 Given that $y = (2x + 1)^3 + (2x - 1)^3$, find the value of $\frac{dy}{dx}$ when $x = 1$.

5 Find the coordinates of the point on the curve $y = (1 - 4x)^{\frac{3}{2}}$ at which the gradient is -30 .

6 Find the equation of the tangent to the curve $y = \frac{1}{3x + 1}$ at $(-1, -\frac{1}{2})$.

7 Find the equation of the normal to the curve $y = \sqrt{6x + 3}$ at the point for which $x = 13$.

12.2 The chain rule: an informal treatment

When you differentiate $(ax + b)^n$ by using the methods of Section 12.1, you are actually differentiating the composite function

$$x \rightarrow [x, a, +, b, =] \rightarrow ax + b \rightarrow [\text{raise to power } n] \rightarrow (ax + b)^n.$$

That is, you are differentiating $f(F(x))$, where $F: x \mapsto ax + b$ and $f: u \mapsto u^n$.

You will soon see the reason for using different letters, x and u , in describing the two functions. Remember that the function is the same, whatever letter is used. (See Section 11.3.)

The rule at the end of Section 12.1 is a special case in which $F(x) = ax + b$. This section shows how the rule can be generalised to differentiate $f(F(x))$ where F is any function which can be differentiated. As a lead-in, here is a very simple example which suggests the general rule.

Example 12.2.1

Find the derivative of the composite function

$$x \rightarrow [x, a, +, b, =] \rightarrow ax + b \rightarrow [x, c, +, d, =] \rightarrow c(ax + b) + d.$$

If you let $y = c(ax + b) + d$, then

$$\frac{dy}{dx} = \frac{d}{dx}(c(ax + b) + d) = \frac{d}{dx}(cax + cb + d) = ca.$$

However, if you let u stand for the intermediate output $ax + b$, then $y = cu + d$.

$$\text{So } \frac{dy}{du} = c \text{ and } \frac{du}{dx} = a, \text{ and } \frac{dy}{dx} = ca. \text{ That is, } \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

You can also think about this in terms of rates of change.

Recall that:

- $\frac{dy}{dx}$ is the rate at which y changes with respect to x ,
- $\frac{dy}{du}$ is the rate at which y changes with respect to u ,
- $\frac{du}{dx}$ is the rate at which u changes with respect to x .

The equation $\frac{dy}{du} = c$ means that y is changing c times as fast as u ; similarly u is changing a times as fast as x . It is natural to think that if y is changing c times as fast as u , and u is changing a times as fast as x , then y is changing $c \times a$ times as fast as x . Thus, again,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

Example 12.2.2

Find $\frac{dy}{dx}$ when $y = (1+x^2)^3$.

So far there has been no alternative to expanding by the binomial theorem. You have had to write

$$y = (1+x^2)^3 = 1 + 3x^2 + 3x^4 + x^6,$$

$$\begin{aligned}\frac{dy}{dx} &= 6x + 12x^3 + 6x^5 = 6x(1+2x^2+x^4) \\ &= 6x(1+x^2)^2.\end{aligned}$$

But now the relation $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ suggests another approach. If you substitute $u = 1+x^2$, so that $y = u^3$, then

$$\begin{aligned}\frac{dy}{du} &= 3u^2 = 3(1+x^2)^2 \text{ and } \frac{du}{dx} = 2x. \\ \text{So } \frac{dy}{du} \times \frac{du}{dx} &= 3(1+x^2)^2 \times 2x = 6x(1+x^2)^2.\end{aligned}$$

So once again $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

Assume now that this result, known as the chain rule, holds in all cases. It is also sometimes called the ‘composite function’ rule, or the ‘function of a function’ rule.

The chain rule is easy to remember because the term du appears to cancel, but bear in mind that this is simply a helpful feature of the notation. Cancellation has no meaning in this context.

Chain rule

If $y = f(F(x))$, and $u = F(x)$ so that $y = f(u)$, then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

A proof is given in Section 12.4, but you can omit it on a first reading.

Example 12.2.3

Differentiate $y = (2x+1)^{\frac{1}{2}}$ with respect to x .

Substitute $u = 2x+1$, so that $y = u^{\frac{1}{2}}$. Then $\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2}(2x+1)^{-\frac{1}{2}}$ and $\frac{du}{dx} = 2$.

As $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$, $\frac{dy}{dx} = \frac{1}{2}(2x+1)^{-\frac{1}{2}} \times 2 = (2x+1)^{-\frac{1}{2}} = \frac{1}{\sqrt{2x+1}}$.

Example 12.2.4

Find $\frac{dy}{dx}$ when (a) $y = \frac{1}{1+x^2}$, (b) $y = \sqrt{1-x^2}$, (c) $y = \sqrt{1+\sqrt{x}}$.

(a) Substitute $u = 1+x^2$, so $y = \frac{1}{u}$. Then $\frac{dy}{du} = -\frac{1}{u^2} = -\frac{1}{(1+x^2)^2}$ and $\frac{du}{dx} = 2x$.

$$\text{So } \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\frac{1}{(1+x^2)^2} \times 2x = \frac{-2x}{(1+x^2)^2}.$$

(b) Substitute $u = 1-x^2$, so $y = \sqrt{u} = u^{\frac{1}{2}}$.

$$\text{Then } \frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}} = \frac{1}{2\sqrt{1-x^2}} \text{ and } \frac{du}{dx} = -2x.$$

$$\text{So } \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{2\sqrt{1-x^2}} \times (-2x) = \frac{-x}{\sqrt{1-x^2}}.$$

(c) Substitute $u = 1+\sqrt{x}$, so $y = \sqrt{u}$.

$$\text{Then } \frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}} = \frac{1}{2\sqrt{1+\sqrt{x}}} \text{ and } \frac{du}{dx} = \frac{1}{2\sqrt{x}}.$$

$$\text{So } \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{2\sqrt{1+\sqrt{x}}} \times \frac{1}{2\sqrt{x}} = \frac{1}{4\sqrt{x}(1+\sqrt{x})}.$$

Exercise 12C

1 Use the substitution $u = 5x+3$ to differentiate the following with respect to x .

$$(a) y = (5x+3)^6 \quad (b) y = (5x+3)^{\frac{1}{2}} \quad (c) y = \frac{1}{5x+3}$$

2 Use the substitution $u = 1-4x$ to differentiate the following with respect to x .

$$(a) y = (1-4x)^5 \quad (b) y = (1-4x)^{-3} \quad (c) y = \sqrt{1-4x}$$

3 Use the substitution $u = 1+x^3$ to differentiate the following with respect to x .

$$(a) y = (1+x^3)^5 \quad (b) y = (1+x^3)^{-4} \quad (c) y = \sqrt[3]{1+x^3}$$

4 Use the substitution $u = 2x^2+3$ to differentiate the following with respect to x .

$$(a) y = (2x^2+3)^6 \quad (b) y = \frac{1}{2x^2+3} \quad (c) y = \frac{1}{\sqrt{2x^2+3}}$$

5 Differentiate $y = (3x^4+2)^2$ with respect to x by using the chain rule. Confirm your answer by expanding $(3x^4+2)^2$ and then differentiating.

6 Differentiate $y = (2x^3 + 1)^3$ with respect to x

- (a) by using the binomial theorem to expand $y = (2x^3 + 1)^3$ and then differentiating term by term,
 (b) by using the chain rule.

Check that your answers are the same.

7 Use appropriate substitutions to differentiate the following with respect to x .

(a) $y = (x^5 + 1)^4$ (b) $y = (2x^3 - 1)^8$ (c) $y = (\sqrt{x} - 1)^5$

8 Differentiate the following with respect to x ; try to do this without writing down the substitutions.

(a) $y = (x^2 + 6)^4$ (b) $y = (5x^3 + 4)^3$ (c) $y = (x^4 - 8)^7$ (d) $y = (2 - x^9)^5$

9 Differentiate the following with respect to x .

(a) $y = \sqrt{4x + 3}$ (b) $y = (x^2 + 4)^6$ (c) $y = (6x^3 - 5)^{-2}$ (d) $y = (5 - x^3)^{-1}$

10 Given that $f(x) = \frac{1}{1+x^2}$, find (a) $f'(2)$, (b) the value of x such that $f'(x) = 0$.

11 Given that $y = \sqrt[4]{x^3 + 8}$, find the value of $\frac{dy}{dx}$ when $x = 2$.

12 Differentiate the following with respect to x .

(a) $y = (x^2 + 3x + 1)^6$ (b) $y = \frac{1}{(x^2 + 5x)^3}$

13 Find the equation of the tangent to the curve $y = (x^2 - 5)^3$ at the point $(2, -1)$.

14 Find the equation of the tangent to the curve $y = \frac{1}{\sqrt{x-1}}$ at the point $(4, 1)$.

15 Find the equation of the normal to the curve $y = \frac{8}{1-x^3}$ at the point $(-1, 4)$.

16 Use the substitutions $u = x^2 - 1$ and $v = \sqrt{u} + 1$ with the chain rule in the form

$$\frac{dy}{dx} = \frac{dy}{dv} \times \frac{dv}{du} \times \frac{du}{dx}$$

to differentiate $y = (\sqrt{x^2 - 1} + 1)^6$.

17 Use two substitutions to find $\frac{dy}{dx}(\sqrt{1 + \sqrt{4x + 3}})$.

18 A curve has equation $y = (x^2 + 1)^4 + 2(x^2 + 1)^3$. Show that $\frac{dy}{dx} = 4x(x^2 + 1)^2(2x^2 + 5)$ and hence show that the curve has just one stationary point. State the coordinates of the stationary point and, by considering the gradient of the curve on either side of the stationary point, determine its nature.

12.3 Related rates of change

You often need to calculate the rate at which one quantity varies with another when one of them is time. In Section 7.4 it was shown that if r is some quantity, then the rate of change of r with respect to time t is $\frac{dr}{dt}$.

But suppose that r is the radius of a spherical balloon, and you know how fast the volume V of the balloon is increasing. How can you find out how fast the radius is increasing?

Questions like this can be answered by using the chain rule. The situation is best described by a problem.

Example 12.3.1

Suppose that a spherical balloon is being inflated at a constant rate of $5 \text{ m}^3 \text{ s}^{-1}$. At a particular moment, the radius of the balloon is 4 metres. Find how fast the radius of the balloon is increasing at that instant.

First translate the information into a mathematical form.

Let $V \text{ m}^3$ be the volume of the balloon, and let r metres be its radius. Let t seconds be the time for which the balloon has been inflating. Then you are given that $\frac{dV}{dt} = 5$ and $r = 4$, and you are asked to find $\frac{dr}{dt}$ at that moment.

Your other piece of information is that the balloon is spherical, so that $V = \frac{4}{3}\pi r^3$.

The key to solving the problem is to use the chain rule in the form

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}.$$

You can now use $\frac{dV}{dr} = 4\pi r^2$. Substituting the various values into the chain rule formula gives

$$5 = (4\pi \times 4^2) \times \frac{dr}{dt}.$$

Therefore, rearranging this equation, you find that $\frac{dr}{dt} = \frac{5}{64\pi}$, so the radius is increasing at $\frac{5}{64\pi} \text{ m s}^{-1}$.

In practice you do not need to write down so much detail. Here is another example.

Example 12.3.2

The surface area of a cube is increasing at a constant rate of $24 \text{ cm}^2 \text{ s}^{-1}$. Find the rate at which its volume is increasing at the moment when the volume is 216 cm^3 .

Let the side of the cube be x cm at time t seconds, let the surface area be $S \text{ cm}^2$ and let the volume be $V \text{ cm}^3$.

Then $S = 6x^2$, $V = x^3$ and $\frac{dS}{dt} = 24$, and you need to find $\frac{dV}{dt}$ when $V = 216$, which is when $x^3 = 216$, or $x = 6$.

If you know S and want to find V you need to find x first. Similarly, when you know $\frac{dS}{dt}$ and want to find $\frac{dV}{dt}$ you should expect to find $\frac{dx}{dt}$ first.

From the chain rule, $\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt} = 3x^2 \frac{dx}{dt}$, so, when $x = 6$, $\frac{dV}{dt} = 108 \frac{dx}{dt}$.

But $\frac{dS}{dt} = \frac{dS}{dx} \times \frac{dx}{dt} = 12x \frac{dx}{dt}$, so $24 = (12 \times 6) \times \frac{dx}{dt}$, giving $\frac{dx}{dt} = \frac{1}{3}$. Substituting this in the equation $\frac{dV}{dt} = 108 \frac{dx}{dt}$ gives $\frac{dV}{dt} = 108 \times \frac{1}{3} = 36$.

Therefore the volume is increasing at a rate of $36 \text{ cm}^3 \text{ s}^{-1}$.

Exercise 12D

- 1 The number of bacteria present in a culture at time t hours after the beginning of an experiment is denoted by N . The relation between N and t is modelled by

$N = 10\left(1 + \frac{3}{2}t\right)^3$. At what rate per hour will the number of bacteria be increasing when $t = 6$? (OCR, adapted)

- 2 A metal bar is heated to a certain temperature and then the heat source is removed. At time t minutes after the heat source is removed, the temperature, θ degrees Celsius, of the metal bar is given by $\theta = \frac{280}{1 + 0.02t}$. At what rate is the temperature decreasing 100 minutes after the removal of the heat source? (OCR, adapted)

- 3 The length of the side of a square is increasing at a constant rate of 1.2 cm s^{-1} . At the moment when the length of the side is 10 cm , find
 (a) the rate of increase of the perimeter,
 (b) the rate of increase of the area.

- 4 The length of the edge of a cube is increasing at a constant rate of 0.5 mm s^{-1} . At the moment when the length of the edge is 40 mm , find
 (a) the rate of increase of the surface area,
 (b) the rate of increase of the volume.

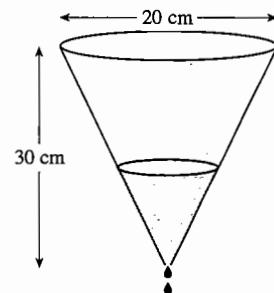
- 5 A circular stain is spreading so that its radius is increasing at a constant rate of 3 mm s^{-1} . Find the rate at which the area is increasing when the radius is 50 mm .

- 6 A water tank has a rectangular base 1.5 m by 1.2 m . The sides are vertical and water is being added to the tank at a constant rate of 0.45 m^3 per minute. At what rate is the depth of water in the tank increasing?

- 7 Air is being lost from a spherical balloon at a constant rate of $0.6 \text{ m}^3 \text{ s}^{-1}$. Find the rate at which the radius is decreasing at the instant when the radius is 2.5 m .

- 8 The volume of a spherical balloon is increasing at a constant rate of $0.25 \text{ m}^3 \text{ s}^{-1}$. Find the rate at which the radius is increasing at the instant when the volume is 10 m^3 .

- 9 A funnel has a circular top of diameter 20 cm and a height of 30 cm. When the depth of liquid in the funnel is 12 cm, the liquid is dripping from the funnel at a rate of $0.2 \text{ cm}^3 \text{ s}^{-1}$. At what rate is the depth of the liquid in the funnel decreasing at this instant?



12.4* Deriving the chain rule

In Section 12.2 the chain rule was shown to work in a number of simple cases, and justified by an informal argument involving rates of change. This section shows how it can be proved, although the argument depends on some assumptions which need to be examined more carefully before the proof is complete. You may omit this section if you wish; there is no exercise depending on it.

In Section 7.4, $\frac{dy}{dx}$ is defined as

$$\frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x}.$$

By changing the letters in the definition,

$$\frac{dy}{du} = \lim_{\delta u \rightarrow 0} \frac{\delta y}{\delta u} \quad \text{and} \quad \frac{du}{dx} = \lim_{\delta x \rightarrow 0} \frac{\delta u}{\delta x}.$$

Now, in these expressions, when y is a function of u , where u is a function of x , then as x changes u changes and so y changes.

Take a particular value of x , and increase x by δx with a corresponding increase of δu in the value of u , which, in turn, increases the value of y by δy . Then

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x}$$

because δy , δu and δx are numbers which you can cancel, assuming that $\delta u \neq 0$.

To find $\frac{dy}{dx}$, you must take the limit as $\delta x \rightarrow 0$, so

$$\frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \left(\frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x} \right).$$

Assuming that as $\delta x \rightarrow 0$, $\delta u \rightarrow 0$ and that $\lim_{\delta x \rightarrow 0} \left(\frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x} \right) = \lim_{\delta x \rightarrow 0} \left(\frac{\delta y}{\delta u} \right) \times \lim_{\delta x \rightarrow 0} \left(\frac{\delta u}{\delta x} \right)$, it follows that

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \left(\frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x} \right) \\ &= \lim_{\delta x \rightarrow 0} \left(\frac{\delta y}{\delta u} \right) \times \lim_{\delta x \rightarrow 0} \left(\frac{\delta u}{\delta x} \right) = \lim_{\delta u \rightarrow 0} \left(\frac{\delta y}{\delta u} \right) \times \lim_{\delta x \rightarrow 0} \left(\frac{\delta u}{\delta x} \right) \\ &= \frac{dy}{du} \times \frac{du}{dx}.\end{aligned}$$

This result, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$, is the chain rule for differentiating composite functions.

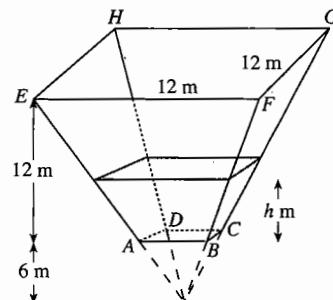
Note that the results in Section 12.1 are particular cases of the chain rule since, if

$$u = ax + b, \quad \frac{du}{dx} = a.$$

Miscellaneous exercise 12

- 1 Differentiate $(4x - 1)^{20}$ with respect to x . (OCR)
- 2 Differentiate $\frac{1}{(3-4x)^2}$ with respect to x . (OCR)
- 3 Differentiate $2(x^4 + 3)^5$ with respect to x .
- 4 Find the equation of the tangent to the curve $y = (x^2 - 5)^6$ at the point $(2, 1)$.
- 5 Given that $y = \sqrt{x^3 + 1}$, show that $\frac{dy}{dx} > 0$ for all $x > -1$.
- 6 Given that $y = \frac{1}{2x-1} + \frac{1}{(2x-1)^2}$, find the exact value of $\frac{dy}{dx}$ when $x = 2$.
- 7 Find the equation of the tangent to the curve $y = (4x + 3)^5$ at the point $(-\frac{1}{2}, 1)$, giving your answer in the form $y = mx + c$. (OCR)
- 8 Find the coordinates of the stationary point of the curve with equation $y = \frac{1}{x^2 + 4}$.
- 9 Find the equation of the normal to the curve $y = \sqrt{2x^2 + 1}$ at the point $(2, 3)$.
- 10 The radius of a circular disc is increasing at a constant rate of 0.003 cm s^{-1} . Find the rate at which the area is increasing when the radius is 20 cm . (OCR)
- 11 A viscous liquid is poured on to a flat surface. It forms a circular patch whose area grows at a steady rate of $5 \text{ cm}^2 \text{ s}^{-1}$. Find, in terms of π ,
 - (a) the radius of the patch 20 seconds after pouring has commenced,
 - (b) the rate of increase of the radius at this instant.

- 12 Find the equation of the tangent to the curve $y = \frac{50}{(2x-1)^2}$ at the point (3,2), giving your answer in the form $ax + by + c = 0$, where a , b and c are integers. (OCR)
- 13 Sketch the graph of $y = (x-2)^2 - 4$ showing clearly on your graph the coordinates of any stationary points and of the intersections with the axes.
Find the coordinates of the stationary points on the graph of $y = (x-2)^3 - 12(x-2)$ and sketch the graph, giving the exact coordinates (in surd form, where appropriate) of the intersections with the axes. (OCR)
- 14 Differentiate $\sqrt{x + \frac{1}{x}}$ with respect to x . (OCR)
- 15 The formulae for the volume of a sphere of radius r and for its surface area are $V = \frac{4}{3}\pi r^3$ and $A = 4\pi r^2$ respectively. Given that, when $r = 5$ m, V is increasing at a rate of 10 m³ s⁻¹, find the rate of increase of A at this instant. (OCR)
- 16 Using differentiation, find the equation of the tangent at the point (2,1) on the curve with equation $y = \sqrt{x^2 - 3}$. (OCR)
- 17 Differentiate $\frac{1}{(3t^2 + 5)^2}$ with respect to t . (OCR)
- 18 (a) Curve C_1 has equation $y = \sqrt{4x - x^2}$. Find $\frac{dy}{dx}$ and hence find the coordinates of the stationary point.
(b) Show that the curve C_2 with equation $y = \sqrt{x^2 - 4x}$ has no stationary point.
- 19 A curve has equation $y = \frac{1}{12}(3x+1)^4 - 8x$.
(a) Show that there is a stationary point where $x = \frac{1}{3}$ and determine whether this stationary point is a maximum or a minimum.
(b) At a particular point of the curve, the equation of the tangent is $48x + 3y + c = 0$. Find the value of the constant c . (OCR)
- 20 If a hemispherical bowl of radius 6 cm contains water to a depth of x cm, the volume of the water is $\frac{1}{3}\pi x^2(18-x)$. Water is poured into the bowl at a rate of 3 cm³ s⁻¹. Find the rate at which the water level is rising when the depth is 2 cm.
- 21 An underground oil storage tank ABCDEFGH is part of an inverted square pyramid, as shown in the diagram. The complete pyramid has a square base of side 12 m and height 18 m. The depth of the tank is 12 m.
When the depth of oil in the tank is h metres, show that the volume V m³ is given by $V = \frac{4}{27}(h+6)^3 - 32$.
Oil is being added to the tank at the constant rate of 4.5 m³ s⁻¹. At the moment when the depth of oil is 8 m, find the rate at which the depth is increasing.



- 22 A curve has equation $y = (x^2 - 1)^3 - 3(x^2 - 1)^2$. Find the coordinates of the stationary points and determine whether each is a minimum or a maximum. Sketch the curve.
- 23 Find the coordinates of the stationary point of the curve $y = \frac{1}{2x+1} - \frac{1}{(2x+1)^2}$ and determine whether the stationary point is a maximum or a minimum.
- 24 Find the coordinates of the stationary point of the curve $y = \sqrt{4x-1} + \frac{9}{\sqrt{4x-1}}$ and determine whether the stationary point is a maximum or a minimum.
- 25 (a) Expand $(ax+b)^3$ using the binomial theorem. Differentiate the result with respect to x and show that the derivative is $3a(ax+b)^2$.
- (b) Expand $(ax+b)^4$ using the binomial theorem. Differentiate the result with respect to x and show that the derivative is $4a(ax+b)^3$.
- (c) Write down the expansion of $(ax+b)^n$ where n is a positive integer. Differentiate the result with respect to x . Show that the derivative is $na(ax+b)^{n-1}$.
-

Revision exercise 2

- 1 Find the equation of the tangent at $x = 3$ to the curve with equation $y = 2x^2 - 3x + 2$.
- 2 Find the coordinates of the vertex of the parabola with equation $y = 3x^2 + 6x + 10$
 - (a) by using the completed square form,
 - (b) by using differentiation.
- 3 Find the coordinates of the point on the curve $y = 2x^2 - 3x + 1$ where the tangent has gradient 1.
- 4 The normal to the curve with equation $y = x^2$ at the point for which $x = 2$ meets the curve again at P . Find the coordinates of P .
- 5 A normal to the curve $y = x^2$ has gradient 2. Find where it meets the curve.
- 6 Find the equation of the normal to the curve with equation $y = \sqrt{x}$ at the point $(1,1)$. Calculate the coordinates of the point at which this normal meets the graph of $y = -\sqrt{x}$.
- 7 The height, h centimetres, of a bicycle pedal above the ground at time t seconds is given by the equation
$$h = 30 + 15 \cos 90t^\circ.$$
 - (a) Calculate the height of the pedal when $t = 1\frac{1}{3}$.
 - (b) Calculate the maximum and minimum heights of the pedal.
 - (c) Find the first two positive values of t for which the height of the pedal is 43 cm. Give your answers correct to 2 decimal places.
 - (d) How many revolutions does the pedal make in one minute?
- 8 (a) Calculate the gradient of the graph of $y = 12\sqrt[3]{x}$ at the point where $x = 8$ and hence find the equation of the tangent to the graph of $y = 12\sqrt[3]{x}$ at the point where $x = 8$.
(b) Find the equation of the line passing through the points with coordinates $(15, 30)$ and $(-31, -14)$.
(c) Hence find the coordinates of the point where the tangent in part (a) meets the line in part (b).
- 9 Differentiate $x + \frac{1}{x}$, $2\sqrt{x}$, $\frac{3}{\sqrt{x}}$ and $\frac{(\sqrt{x}+1)^2}{x}$ with respect to x .
- 10 Find the maximum and minimum values of $y = x^3 - 6x^2 + 9x - 8$.
- 11 A curve has equation $y = 2x^3 - 9x^2 + 12x - 5$. Show that one of the stationary points lies on the x -axis, and determine whether this point is a maximum or a minimum.

- 12** Solve the following equations, giving values of θ in the interval $-180 \leq \theta \leq 180$ correct to 1 decimal place.

$$\begin{array}{ll} \text{(a)} & 3\sin^2 \theta^\circ - 2\cos^2 \theta^\circ = 1 \\ \text{(b)} & \cos \theta^\circ \tan \theta^\circ = -\frac{1}{2} \\ \text{(c)} & 3 + 4\tan 2\theta^\circ = 5 \\ \text{(d)} & 4\cos^2 2\theta^\circ = 3 \end{array}$$

- 13** (a) Solve the equation $\tan^2 2x^\circ = \frac{1}{3}$ giving all solutions in the interval $0 \leq x \leq 360$.
 (b) Prove that $\tan^2 \theta^\circ \equiv \frac{1}{\cos^2 \theta^\circ} - 1$.
 (c) Write down the period of the graph of $y = \frac{3}{2 + \cos^2 2x^\circ}$, and also the coordinates of a maximum value of y .

- 14** Differentiate $\sqrt{x} + \frac{1}{\sqrt{x}}$ and $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$ with respect to x .

- 15** By drawing suitable sketch graphs, determine the number of roots of the equation

$$\cos x^\circ = \frac{10}{x}$$

which lie in the interval $-180 < x < 180$.

(OCR, adapted)

- 16** The function $x \mapsto \frac{1-x}{1+2x}$, $x \in \mathbb{R}, x \neq -\frac{1}{2}$, has an inverse. Find the inverse function, giving your answer in similar notation to the original.

- 17** The n th term of a series is $\frac{1}{2}(2n-1)$. Write down the $(n+1)$ th term.
 (a) Prove that the series is an arithmetic progression.
 (b) Find, algebraically, the value of n for which the sum to n terms is 200.

(OCR)

- 18** (a) Determine the first three terms in the binomial expansion of $\left(x - \frac{1}{x}\right)^8$.
 (b) Write down the constant term in the binomial expansion of $\left(2x + \frac{3}{x}\right)^4$.

(OCR)

- 19** Differentiate $y = \frac{1}{\sqrt{2x+3}}$ with respect to x . Draw a sketch of the curve.

- 20** (a) The function f is defined by $f(x) = x^2 - 2x - 1$ for the domain $-2 \leq x \leq 5$. Write $f(x)$ in completed square form. Hence find the range of f . Explain why f does not have an inverse.
 (b) A function g is defined by $g(x) = 2x^2 - 4x - 3$. Write down a domain for g such that g^{-1} exists.

(OCR, adapted)

- 21** The tenth term of an arithmetic progression is 125 and the sum of the first ten terms is 260.

- (a) Show that the first term in the progression is -73.
 (b) Find the common difference.

(OCR)

- 22** The binomial expansion of $(1+ax)^n$, where n is a positive integer, has six terms.
- (a) Write down the value of n .
The coefficient of the x^3 term is $\frac{5}{4}$.
- (b) Find a . (OCR)
- 23** The function f is defined for the domain $x \geq 0$ by $f: x \mapsto 4 - x^2$.
- (a) Sketch the graph of f and state the range of f .
(b) Describe a simple transformation whereby the graph of $y = f(x)$ may be obtained from the graph of $y = x^2$ for $x \geq 0$.
(c) The inverse of f is denoted by f^{-1} . Find an expression for $f^{-1}(x)$ and state the domain of f^{-1} .
(d) Show, by reference to a sketch, or otherwise, that the solution to the equation $f(x) = f^{-1}(x)$ can be obtained from the quadratic equation $x^2 + x - 4 = 0$. Determine the solution of $f(x) = f^{-1}(x)$, giving your value to 2 decimal places.
- 24** Find the coefficient of $\frac{1}{x^4}$ in the binomial expansion of $\left(1 + \frac{3}{x}\right)^6$.
- 25** An arithmetic progression has first term 3 and common difference 0.8. The sum of the first n terms of this arithmetic progression is 231. Find the value of n .
- 26** Differentiate each of the following functions with respect to x .
- (a) $(x^3 + 2x - 1)^3$ (b) $\sqrt{\frac{1}{x^2 + 1}}$
- 27** A spherical star is collapsing in size, while remaining spherical. When its radius is one million kilometres, the radius is decreasing at the rate of 500 km s^{-1} . Find
- (a) the rate of decrease of its volume, (b) the rate of decrease of its surface area.
- 28** Write down the periods of the following trigonometric functions.
- (a) $\cos x^\circ$ (b) $\cos \frac{1}{2}x^\circ$ (c) $\cos \frac{3}{2}x^\circ$
- 29** A woman started a business with a workforce of 50 people. Every two weeks the number of people in the workforce increased by 3 people.
How many people were there in the workforce after 26 weeks?
Each member of the workforce earned \$600 per week. What was the total wage bill for this 26 weeks?
- 30** Sketch the graph of $y = \frac{1}{2}x^2 - 3x + 12$.
The points P and Q on the graph have x -coordinates 0 and 8 respectively. The tangents at P and Q meet at R . Show that the point $(11, 9)$ is equidistant from P , Q and R . (OCR, adapted)
- 31** The origin O and a point $B(p, q)$ are opposite vertices of the square $OABC$. Find the coordinates of the points A and C .
A line l has gradient $\frac{q}{p}$. Find possible values for the gradient of a line at 45° to l .

13 Vectors

This chapter introduces the idea of vectors as a way of doing geometry in two or three dimensions. When you have completed it, you should

- understand the idea of a translation, and how it can be expressed either in column form or in terms of basic unit vectors
- know and be able to use the rules of vector algebra
- understand the idea of displacement and position vectors, and use these to prove geometrical results
- appreciate similarities and differences between geometry in two and three dimensions
- know the definition of the scalar product, and its expression in terms of components
- be able to use the rules of vector algebra which involve scalar products
- be able to use scalar products to solve geometrical problems in two and three dimensions, using general vector algebra or components.

13.1 Translations of a plane

In Section 3.6 you saw that if you translate the graph $y = ax^2 + bx$ through a distance c in the y -direction its new equation is $y = ax^2 + bx + c$. In general, if you translate the graph $y = f(x)$ a distance c units in the y -direction its equation becomes $y = f(x) + c$. A practical way of doing this is to draw the graph on a transparent sheet placed over a coordinate grid, and then to move this sheet up the grid by c units.

The essential feature of a translation is that the sheet moves over the grid without turning. A general translation would move the sheet k units across and l units up the grid. This is shown in Fig. 13.1, where several points move in the same direction through the same distance. Such a translation is called a **vector** and is written $\begin{pmatrix} k \\ l \end{pmatrix}$.

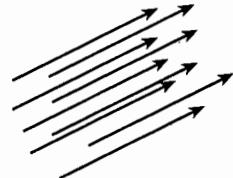


Fig. 13.1

For example, the translation of $y = f(x)$ described above would be performed by the vector $\begin{pmatrix} 0 \\ c \end{pmatrix}$; similarly the vector $\begin{pmatrix} k \\ 0 \end{pmatrix}$ performs a translation of k units across in the x -direction.

In practice, drawing several arrows, as in Fig. 13.1, is not a convenient way of representing a vector. It is usual to draw just a single arrow, as in Fig. 13.2. But you must understand that the position of the arrow in the (x,y) -plane is of no significance. This arrow is just one of infinitely many that could be drawn to represent the vector.

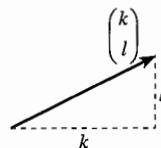


Fig. 13.2

You may meet uses of vectors in other contexts. For example, mechanics uses velocity vectors, acceleration vectors, force vectors, and so on. When you need to make the distinction, the vectors described here are called **translation vectors**. These are the only vectors used in this book.

13.2 Vector algebra

It is often convenient to use a single letter to stand for a vector. In print, bold type is used to distinguish vectors from numbers. For example, in $\mathbf{p} = \begin{pmatrix} k \\ l \end{pmatrix}$, \mathbf{p} is a vector but k and l are numbers, called the **components** of the vector \mathbf{p} in the x - and y -directions.

In handwriting vectors are indicated by a wavy line underneath the letter: $\underline{\mathbf{p}} = \begin{pmatrix} k \\ l \end{pmatrix}$. It is important to get into the habit of writing vectors in this way, so that it is quite clear in your work which letters stand for vectors and which stand for numbers.

If s is any number and \mathbf{p} is any vector, then $s\mathbf{p}$ is another vector. If $s > 0$, the vector $s\mathbf{p}$ is a translation in the same direction as \mathbf{p} but s times as large; if $s < 0$ it is in the opposite direction and $|s|$ times as large. A number such as s is often called a **scalar**, because it usually changes the scale of the vector.

The similar triangles in Fig. 13.3 show that $s\mathbf{p} = \begin{pmatrix} sk \\ sl \end{pmatrix}$. In particular, $(-1)\mathbf{p} = \begin{pmatrix} -k \\ -l \end{pmatrix}$, which is a translation of the same magnitude as \mathbf{p} but in the opposite direction. It is denoted by $-\mathbf{p}$.

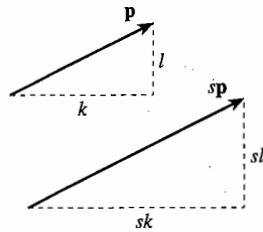


Fig. 13.3

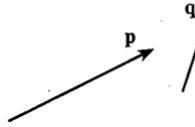


Fig. 13.4

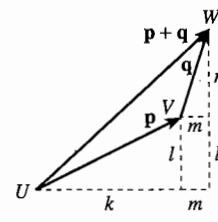


Fig. 13.5

Vectors are added by performing one translation after another. In Fig. 13.4, \mathbf{p} and \mathbf{q} are two vectors. To form their sum, you want to represent them as a pair of arrows by which you can trace the path of a particular point of the moving sheet. In Fig. 13.5, \mathbf{p} is shown by an arrow from U to V , and \mathbf{q} by an arrow from V to W . Then when the translations are combined, the point of the sheet which was originally at U would move first to V and then to W . So the sum $\mathbf{p} + \mathbf{q}$ is represented by an arrow from U to W .

Fig. 13.5 also shows that:

$$\text{If } \mathbf{p} = \begin{pmatrix} k \\ l \end{pmatrix} \text{ and } \mathbf{q} = \begin{pmatrix} m \\ n \end{pmatrix}, \text{ then } \mathbf{p} + \mathbf{q} = \begin{pmatrix} k+m \\ l+n \end{pmatrix}.$$

To form the sum $\mathbf{q} + \mathbf{p}$ the translations are performed in the reverse order. In Fig. 13.6, \mathbf{q} is now represented by the arrow from U to Z ; and since $UVWZ$ is a parallelogram \mathbf{p} is represented by the arrow from Z to W . This shows that

$$\mathbf{p} + \mathbf{q} = \mathbf{q} + \mathbf{p}.$$

This is called the **commutative rule for addition of vectors**.

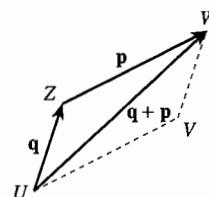


Fig. 13.6

Example 13.2.1

If $\mathbf{p} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$, show that there is a number s such that $\mathbf{p} + s\mathbf{q} = \mathbf{r}$.

You can write $\mathbf{p} + s\mathbf{q}$ in column vector form as

$$\begin{pmatrix} 2 \\ -3 \end{pmatrix} + s \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} + \begin{pmatrix} s \\ 2s \end{pmatrix} = \begin{pmatrix} 2+s \\ -3+2s \end{pmatrix}.$$

If this is equal to \mathbf{r} , then both the x - and y -components of the two vectors must be equal. This gives the two equations

$$2+s=5 \quad \text{and} \quad -3+2s=3.$$

Both these equations are satisfied by $s=3$, so it follows that $\mathbf{p} + 3\mathbf{q} = \mathbf{r}$. You can check this for yourself using squared paper or a screen display, showing arrows representing \mathbf{p} , \mathbf{q} , $\mathbf{p} + 3\mathbf{q}$ and \mathbf{r} .

The idea of addition can be extended to three or more vectors. But when you write $\mathbf{p} + \mathbf{q} + \mathbf{r}$ it is not clear whether you first add \mathbf{p} and \mathbf{q} and then add \mathbf{r} to the result, or whether you add \mathbf{p} to the result of adding \mathbf{q} and \mathbf{r} . Fig. 13.7 shows that it doesn't matter, since the outcome is the same either way. That is,

$$(\mathbf{p} + \mathbf{q}) + \mathbf{r} = \mathbf{p} + (\mathbf{q} + \mathbf{r}).$$

This is called the **associative rule for addition of vectors**.

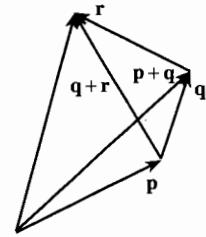


Fig. 13.7

To complete the algebra of vector addition, the symbol $\mathbf{0}$ is needed for the **zero vector**, the 'stay-still' translation, which has the properties that, for any vector \mathbf{p} ,

$$\mathbf{0}\mathbf{p} = \mathbf{0}, \quad \mathbf{p} + \mathbf{0} = \mathbf{p}, \quad \text{and} \quad \mathbf{p} + (-\mathbf{p}) = \mathbf{0}.$$

Vector addition and multiplication by a scalar can be combined according to the two **distributive rules for vectors**:

$$s(\mathbf{p} + \mathbf{q}) = s\mathbf{p} + s\mathbf{q} \quad (\text{from the similar triangles in Fig. 13.8})$$

and $(s+t)\mathbf{p} = s\mathbf{p} + t\mathbf{p}$ (see Fig. 13.9)

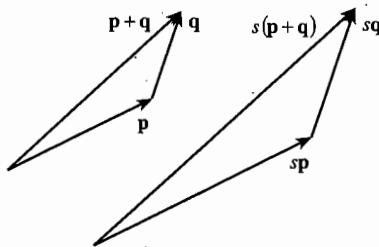


Fig. 13.8

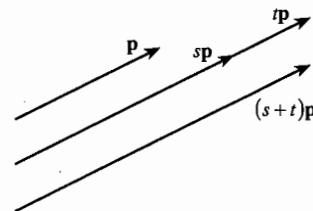


Fig. 13.9

Subtraction of vectors is defined by

$$\mathbf{p} + \mathbf{x} = \mathbf{q} \Leftrightarrow \mathbf{x} = \mathbf{q} - \mathbf{p}.$$

This is illustrated in Fig. 13.10. Notice that to show $\mathbf{q} - \mathbf{p}$ you represent \mathbf{p} and \mathbf{q} by arrows which both start at the same point; this is different from addition, where the arrow representing \mathbf{q} starts where the \mathbf{p} arrow ends.

Comparing Fig. 13.10 with Fig. 13.11 shows that

$$\mathbf{q} - \mathbf{p} = \mathbf{q} + (-\mathbf{p}).$$

In summary, the rules of vector addition, subtraction and multiplication by scalars look very similar to the rules of number addition, subtraction and multiplication. But the diagrams show that the rules for vectors are interpreted differently from the rules for numbers.

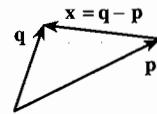


Fig. 13.10

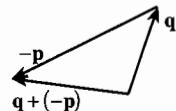


Fig. 13.11

13.3 Basic unit vectors

If you apply the rules of vector algebra to a vector in column form, you can see that

$$\mathbf{p} = \begin{pmatrix} k \\ l \end{pmatrix} = \begin{pmatrix} k+0 \\ 0+l \end{pmatrix} = \begin{pmatrix} k \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ l \end{pmatrix} = k \begin{pmatrix} 1 \\ 0 \end{pmatrix} + l \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ which appear in this last expression are called **basic unit vectors** in the x - and y -directions. They are denoted by the letters \mathbf{i} and \mathbf{j} , so

$$\mathbf{p} = k\mathbf{i} + l\mathbf{j}.$$

This is illustrated by Fig. 13.12. The equation shows that any vector in the plane can be constructed as the sum of multiples of the two basic vectors \mathbf{i} and \mathbf{j} .

The vectors $k\mathbf{i}$ and $l\mathbf{j}$ are called the **component vectors** of \mathbf{p} in the x - and y -directions.

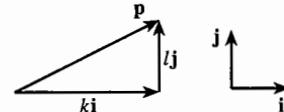


Fig. 13.12

You now have two alternative notations for doing algebra with vectors. For example, if you want to find $3\mathbf{p} - 2\mathbf{q}$, where \mathbf{p} is $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and \mathbf{q} is $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$, you can write either

$$3\begin{pmatrix} 2 \\ 5 \end{pmatrix} - 2\begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 6 \\ 15 \end{pmatrix} - \begin{pmatrix} 2 \\ -6 \end{pmatrix} = \begin{pmatrix} 6-2 \\ 15-(-6) \end{pmatrix} = \begin{pmatrix} 4 \\ 21 \end{pmatrix}$$

$$\text{or } 3(2\mathbf{i} + 5\mathbf{j}) - 2(\mathbf{i} - 3\mathbf{j}) = (6\mathbf{i} + 15\mathbf{j}) - (2\mathbf{i} - 6\mathbf{j}) = 6\mathbf{i} + 15\mathbf{j} - 2\mathbf{i} + 6\mathbf{j} = 4\mathbf{i} + 21\mathbf{j}.$$

You will find that sometimes one of these forms is more convenient than the other, but usually it makes no difference which you use.

Exercise 13A

When you are asked to illustrate a vector equation geometrically, you should show vectors as arrows on a grid of squares, either on paper or on screen.

- 1** Illustrate the following equations geometrically.

(a) $\begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ (b) $3 \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix}$

(c) $\begin{pmatrix} 0 \\ 4 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ (d) $\begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$

(e) $3 \begin{pmatrix} -1 \\ 2 \end{pmatrix} - \begin{pmatrix} -4 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ (f) $4 \begin{pmatrix} 2 \\ 3 \end{pmatrix} - 3 \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$

(g) $\begin{pmatrix} 2 \\ -3 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \end{pmatrix} + \begin{pmatrix} -6 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (h) $2 \begin{pmatrix} 3 \\ -1 \end{pmatrix} + 3 \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ -7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

- 2** Rewrite each of the equations in Question 1 using unit vector notation.

- 3** Express each of the following vectors as column vectors, and illustrate your answers geometrically.

(a) $\mathbf{i} + 2\mathbf{j}$ (b) $3\mathbf{i}$ (c) $\mathbf{j} - \mathbf{i}$ (d) $4\mathbf{i} - 3\mathbf{j}$

- 4** Show that there is a number s such that $s \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$. Illustrate your answer geometrically.

- 5** If $\mathbf{p} = 5\mathbf{i} - 3\mathbf{j}$, $\mathbf{q} = 2\mathbf{j} - \mathbf{i}$ and $\mathbf{r} = \mathbf{i} + 5\mathbf{j}$, show that there is a number s such that $\mathbf{p} + s\mathbf{q} = \mathbf{r}$. Illustrate your answer geometrically.

Rearrange this equation so as to express \mathbf{q} in terms of \mathbf{p} and \mathbf{r} . Illustrate the rearranged equation geometrically.

- 6** Find numbers s and t such that $s \begin{pmatrix} 5 \\ 4 \end{pmatrix} + t \begin{pmatrix} -3 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Illustrate your answer geometrically.

- 7** If $\mathbf{p} = 4\mathbf{i} + \mathbf{j}$, $\mathbf{q} = 6\mathbf{i} - 5\mathbf{j}$ and $\mathbf{r} = 3\mathbf{i} + 4\mathbf{j}$, find numbers s and t such that $s\mathbf{p} + t\mathbf{q} = \mathbf{r}$. Illustrate your answer geometrically.

- 8** Show that it isn't possible to find numbers s and t such that $\begin{pmatrix} 4 \\ -2 \end{pmatrix} + s \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -6 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix} + t \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Give geometrical reasons.

- 9** If $\mathbf{p} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{q} = 4\mathbf{i} - 5\mathbf{j}$ and $\mathbf{r} = \mathbf{i} - 4\mathbf{j}$, find a set of numbers f , g and h such that $f\mathbf{p} + g\mathbf{q} + h\mathbf{r} = \mathbf{0}$. Illustrate your answer geometrically. Give a reason why there is more than one possible answer to this question.

- 10** If $\mathbf{p} = 3\mathbf{i} - \mathbf{j}$, $\mathbf{q} = 4\mathbf{i} + 5\mathbf{j}$ and $\mathbf{r} = 2\mathbf{j} - 6\mathbf{i}$,

- (a) can you find numbers s and t such that $\mathbf{q} = s\mathbf{p} + t\mathbf{r}$,
(b) can you find numbers u and v such that $\mathbf{r} = u\mathbf{p} + v\mathbf{q}$?

Give a geometrical reason for your answers.

13.4 Position vectors

If E and F are two points on a grid, there is a unique translation which takes you from E to F . This translation can be represented by the arrow which starts at E and ends at F , and it is denoted by the symbol \vec{EF} .

Some books use EF in bold type rather than \vec{EF} to emphasise that it is a vector.

However, although this translation is unique, its name is not. If G and H are two other points on the grid such that the lines EF and GH are parallel and equal in length (so that $EFHG$ is a parallelogram, see Fig. 13.13), then the translation \vec{EF} also takes you from G to H , so that it could also be denoted by \vec{GH} . In a vector equation \vec{EF} could be replaced by \vec{GH} without affecting the truth of the statement.

Vectors written like this are sometimes called **displacement vectors**. But they are not a different kind of vector, just translation vectors written in a different way.

There is, however, one especially important displacement vector. This is the translation that starts at the origin O and ends at a point A (\vec{OA} in Fig. 13.13), so that $\vec{OA} = \vec{EF} = \vec{GH}$.

The translation from O to A is called the **position vector** of A .

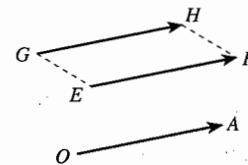


Fig. 13.13

There is a close link between the coordinates of A and the components of its position vector. If A has coordinates (u, v) , then to get from O to A you must move u units in the x -direction and v units in the y -direction, so that the vector \vec{OA} has components u and v .

The position vector of the point A with coordinates (u, v) is

$$\vec{OA} = \begin{pmatrix} u \\ v \end{pmatrix} = u\mathbf{i} + v\mathbf{j}.$$

A useful convention is to use the same letter for a point and its position vector. For example, the position vector of the point A can be denoted by \mathbf{a} . This ‘alphabet convention’ will be used wherever possible in this book. It has the advantages that it economises on letters of the alphabet and avoids the need for repetitive definitions.

13.5 Algebra with position vectors

Multiplication by a scalar has a simple interpretation in terms of position vectors. If the vector $s\mathbf{a}$ is the position vector of a point D , then:

- If $s > 0$, D lies on the directed line OA (produced if necessary) such that $OD = sOA$.
- If $s < 0$, D lies on the directed line AO produced such that $OD = |s|OA$.

This is shown in Fig. 13.14 for $s = \frac{3}{2}$ and $s = -\frac{1}{2}$.

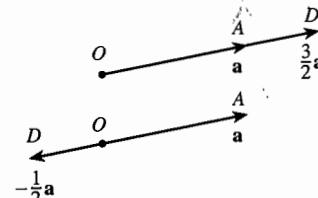


Fig. 13.14

To identify the point with position vector $\mathbf{a} + \mathbf{b}$ is not quite so easy, because the arrows from O to A and from O to B are not related in the way needed for addition (see Fig. 13.5). It is therefore necessary to complete the parallelogram $OACB$, as in Fig. 13.15.

Then

$$\mathbf{a} + \mathbf{b} = \overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{OC} = \overrightarrow{OC}.$$

This is called the **parallelogram rule of addition** for position vectors.

Subtraction can be shown in either of two ways. If you compare Fig. 13.16 with Fig. 13.10, you will see that $\mathbf{b} - \mathbf{a}$ is the displacement vector \overrightarrow{AB} . To interpret this as a position vector, draw a line OE equal and parallel to AB , so that $\overrightarrow{OE} = \overrightarrow{AB}$. Then E is the point with position vector $\mathbf{b} - \mathbf{a}$.

Alternatively, you can write $\mathbf{b} - \mathbf{a}$ as $\mathbf{b} + (-\mathbf{a})$, and then apply the parallelogram rule of addition to the points with position vectors \mathbf{b} and $-\mathbf{a}$. By comparing Figs. 13.16 and 13.17 you can see that this leads to the same point E .

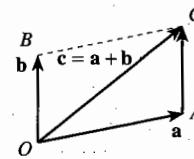


Fig. 13.15

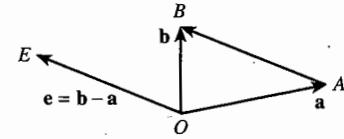


Fig. 13.16

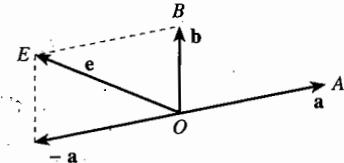


Fig. 13.17

Example 13.5.1

Points A and B have position vectors \mathbf{a} and \mathbf{b} . Find the position vectors of

- the mid-point M of AB ,
- the point of trisection T such that $AT = \frac{2}{3}AB$.

(a) **Method 1** The displacement vector $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$, so $\overrightarrow{AM} = \frac{1}{2}(\mathbf{b} - \mathbf{a})$. Therefore

$$\mathbf{m} = \overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM} = \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a}) = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}.$$

Method 2 If the parallelogram $OACB$ is completed (see Fig. 13.15) then $\mathbf{c} = \mathbf{a} + \mathbf{b}$. Since the diagonals of $OACB$ bisect each other, the mid-point M of AB is also the midpoint of OC . Therefore

$$\mathbf{m} = \frac{1}{2}\mathbf{c} = \frac{1}{2}(\mathbf{a} + \mathbf{b}) = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}.$$

(b) The first method of (a) can be modified. The displacement vector $\overrightarrow{AT} = \frac{2}{3}\overrightarrow{AB} = \frac{2}{3}(\mathbf{b} - \mathbf{a})$, so

$$\mathbf{t} = \overrightarrow{OA} + \overrightarrow{AT} = \mathbf{a} + \frac{2}{3}(\mathbf{b} - \mathbf{a}) = \frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}.$$

The results of this example can be used to prove an important theorem about triangles.

Example 13.5.2

In triangle ABC the mid-points of BC , CA and AB are D , E and F . Prove that the lines AD , BE and CF (called the **medians**) meet at a point G , which is a point of trisection of each of the medians (see Fig. 13.18).

From Example 13.5.1, $\mathbf{d} = \frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{c}$, and the point of trisection on the median AD closer to D has position vector

$$\begin{aligned}\frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{d} &= \frac{1}{3}\mathbf{a} + \frac{2}{3}\left(\frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{c}\right) \\ &= \frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{b} + \frac{1}{3}\mathbf{c}.\end{aligned}$$

This last expression is symmetrical in \mathbf{a} , \mathbf{b} and \mathbf{c} . It therefore also represents the point of trisection on the median BE closer to E , and the point of trisection on CF closer to F .

Therefore the three medians meet each other at a point G , with position vector $\mathbf{g} = \frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$. This point is called the **centroid** of the triangle.

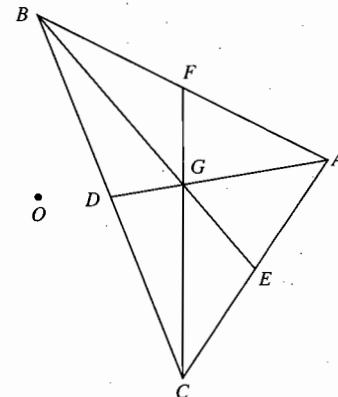


Fig. 13.18

Exercise 13B

In this exercise the alphabet convention is used, that \mathbf{a} stands for the position vector of the point A , and so on.

- 1 The points A and B have coordinates $(3,1)$ and $(1,2)$. Plot on squared paper the points C , D, \dots, H defined by the following vector equations, and state their coordinates.

(a) $\mathbf{c} = 3\mathbf{a}$	(b) $\mathbf{d} = -\mathbf{b}$	(c) $\mathbf{e} = \mathbf{a} - \mathbf{b}$
(d) $\mathbf{f} = \mathbf{b} - 3\mathbf{a}$	(e) $\mathbf{g} = \mathbf{b} + 3\mathbf{a}$	(f) $\mathbf{h} = \frac{1}{2}(\mathbf{b} + 3\mathbf{a})$
- 2 Points A and B have coordinates $(2,7)$ and $(-3,-3)$ respectively. Use a vector method to find the coordinates of C and D , where

(a) C is the point such that $\vec{AC} = 3\vec{AB}$,	(b) D is the point such that $\vec{AD} = \frac{3}{5}\vec{AB}$.
---	---
- 3 C is the point on AB produced such that $\vec{AB} = \vec{BC}$. Express C in terms of \mathbf{a} and \mathbf{b} . Check your answer by using the result of Example 13.5.1(a) to find the position vector of the mid-point of AC .
- 4 C is the point on AB such that $AC:CB = 4:3$. Express \mathbf{c} in terms of \mathbf{a} and \mathbf{b} .
- 5 If C is the point on AB such that $\vec{AC} = t\vec{AB}$, prove that $\mathbf{c} = t\mathbf{b} + (1-t)\mathbf{a}$.
- 6 Write a vector equation connecting \mathbf{a} , \mathbf{b} , \mathbf{c} and \mathbf{d} to express the fact that $\vec{AB} = \vec{DC}$. Deduce from your equation that

(a) $\vec{DA} = \vec{CB}$,
(b) if E is the point such that OAE is a parallelogram, then $OBED$ is a parallelogram.

- 7 ABC is a triangle. D is the mid-point of BC , E is the mid-point of AC , F is the mid-point of AB and G is the mid-point of EF . Express the displacement vectors \vec{AD} and \vec{AG} in terms of \mathbf{a} , \mathbf{b} and \mathbf{c} . What can you deduce about the points A , D and G ?
- 8 $OABC$ is a parallelogram, M is the mid-point of BC , and P is the point of trisection of AC closer to C . Express \mathbf{b} , \mathbf{m} and \mathbf{p} in terms of \mathbf{a} and \mathbf{c} . Deduce that $\mathbf{p} = \frac{2}{3}\mathbf{m}$, and interpret this equation geometrically.
- 9 ABC is a triangle. D is the mid-point of BC , E is the mid-point of AD and F is the point of trisection of AC closer to A . G is the point on FB such that $\vec{FG} = \frac{1}{4}\vec{FB}$. Express \mathbf{d} , \mathbf{e} , \mathbf{f} and \mathbf{g} in terms of \mathbf{a} , \mathbf{b} and \mathbf{c} , and deduce that G is the same point as E . Draw a figure to illustrate this result.
- 10 OAB is a triangle, Q is the point of trisection of AB closer to B and P is the point on OQ such that $\vec{OP} = \frac{2}{5}\vec{OQ}$. AP produced meets OB at R . Express \vec{AP} in terms of \mathbf{a} and \mathbf{b} , and hence find the number k such that $\vec{OA} + k\vec{AP}$ does not depend on \mathbf{a} . Use your answer to express \mathbf{r} in terms of \mathbf{b} , and interpret this geometrically.
Use a similar method to identify the point S where BP produced meets OA .

13.6 Vectors in three dimensions

The power of vector methods is best appreciated when they are used to do geometry in three dimensions. This requires setting up axes in three directions, as in Fig. 13.19. The usual convention is to take x - and y -axes in a horizontal plane (shown shaded), and to add a z -axis pointing vertically upwards.

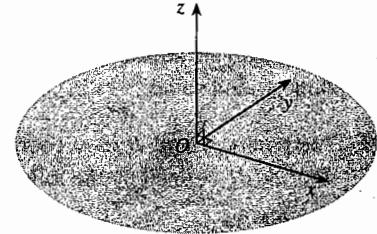


Fig. 13.19

These axes are said to be 'right-handed': if the outstretched index finger of your right hand points in the x -direction, and you bend your middle finger to point in the y -direction, then your thumb can naturally point up in the z -direction.

The position of a point is given by its three coordinates (x, y, z) .

A vector \mathbf{p} in three dimensions is a translation of the whole of space relative to a fixed coordinate framework. (You could imagine Fig. 13.1 as a rainstorm, with the arrows showing the translations of the individual droplets.)

It is written as $\begin{pmatrix} l \\ m \\ n \end{pmatrix}$, which is a translation of l , m and n units in the x -, y - and z -directions. It can also be written in the form $l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$, where $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are basic unit vectors in the x -, y - and z -directions.

Almost everything that you know about coordinates in two dimensions carries over into three dimensions in an obvious way, but you need to notice a few differences:

- The axes can be taken in pairs to define coordinate planes. For example, the x - and y -axes define the horizontal plane, called the xy -plane. All points in this plane have z -coordinate zero, so the equation of the plane is $z = 0$. Similarly the xz -plane and the yz -plane have equations $y = 0$ and $x = 0$; these are both vertical planes.
- The idea of the gradient of a line does not carry over into three dimensions. However, you can still use a vector to describe the direction of a line. This is one of the main reasons why vectors are especially useful in three dimensions.
- In three dimensions lines which are not parallel may or may not meet. Non-parallel lines which do not meet are said to be **skew**.

Example 13.6.1

Points A and B have coordinates $(-5, 3, 4)$ and $(-2, 9, 1)$. Investigate whether or not the point C with coordinates $(-4, 5, 2)$ lies on the line passing through A and B .

The displacement vector \vec{AB} is

$$\mathbf{b} - \mathbf{a} = \begin{pmatrix} -2 \\ 9 \\ 1 \end{pmatrix} - \begin{pmatrix} -5 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ -3 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

The displacement vector \vec{AC} is

$$\mathbf{c} - \mathbf{a} = \begin{pmatrix} -4 \\ 5 \\ 2 \end{pmatrix} - \begin{pmatrix} -5 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}.$$

As $\mathbf{c} - \mathbf{a}$ is not a multiple of $\mathbf{b} - \mathbf{a}$, it is not parallel to $\mathbf{b} - \mathbf{a}$. The points B and C are not in the same direction (or in opposite directions) from A , so C does not lie on the line passing through A and B .

Note that if you change the z -coordinate of C to 3, then C would lie on the line AB .

Example 13.6.2

Points P , Q and R have coordinates $(1, 3, 2)$, $(3, 1, 4)$ and $(4, 1, -5)$ respectively.

- Find the displacement vectors \vec{PQ} and \vec{QR} in terms of the basic vectors \mathbf{i} , \mathbf{j} and \mathbf{k} .
- Find $2\vec{PQ} - \frac{1}{2}\vec{PR} + \frac{1}{2}\vec{QR}$ in terms of the basic vectors \mathbf{i} , \mathbf{j} and \mathbf{k} , and the coordinates of the point reached if you start at R and carry out the translation $2\vec{PQ} - \frac{1}{2}\vec{PR} + \frac{1}{2}\vec{QR}$.

$$(a) \vec{PQ} = \mathbf{q} - \mathbf{p} = (3\mathbf{i} + \mathbf{j} + 4\mathbf{k}) - (\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) = 2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}.$$

$$\vec{QR} = \mathbf{r} - \mathbf{q} = (4\mathbf{i} + \mathbf{j} - 5\mathbf{k}) - (3\mathbf{i} + \mathbf{j} + 4\mathbf{k}) = \mathbf{i} - 9\mathbf{k}.$$

$$(b) \text{ Note first that } \vec{PR} = \mathbf{r} - \mathbf{p} = (4\mathbf{i} + \mathbf{j} - 5\mathbf{k}) - (\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) = 3\mathbf{i} - 2\mathbf{j} - 7\mathbf{k}.$$

$$\begin{aligned} \text{Then } 2\vec{PQ} - \frac{1}{2}\vec{PR} + \frac{1}{2}\vec{QR} &= 2(2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) - \frac{1}{2}(3\mathbf{i} - 2\mathbf{j} - 7\mathbf{k}) + \frac{1}{2}(\mathbf{i} - 9\mathbf{k}) \\ &= 4\mathbf{i} - 4\mathbf{j} + 4\mathbf{k} - \frac{3}{2}\mathbf{i} + \mathbf{j} + \frac{7}{2}\mathbf{k} + \frac{1}{2}\mathbf{i} - \frac{9}{2}\mathbf{k} \\ &= 3\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}. \end{aligned}$$

If you start from R , then the point reached has position vector

$$\mathbf{r} + (3\mathbf{i} - 3\mathbf{j} + \mathbf{k}) = (4\mathbf{i} + \mathbf{j} - 5\mathbf{k}) + (3\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}) = 7\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}.$$

The point is therefore $(7, -2, -2)$.

Exercise 13C

- 1 If $\mathbf{p} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $\mathbf{q} = 5\mathbf{i} + 2\mathbf{j}$ and $\mathbf{r} = 4\mathbf{i} + \mathbf{j} + \mathbf{k}$, calculate the vector $2\mathbf{p} + 3\mathbf{q} - 4\mathbf{r}$ giving your answer as a column vector.
- 2 A and B are points with coordinates $(2, 1, 4)$ and $(5, -5, -2)$. Find the vector \overrightarrow{AB}
 - (a) as a column vector,
 - (b) using \mathbf{i}, \mathbf{j} and \mathbf{k} .
- 3 For each of the following sets of points A , B and C , determine whether the point C lies on the line AB .
 - (a) $A(3, 2, 4)$, $B(-3, -7, -8)$, $C(0, 1, 3)$
 - (b) $A(3, 1, 0)$, $B(-3, 1, 3)$, $C(5, 1, -1)$
 If the answer is yes, draw a diagram showing the relative positions of A , B and C on the line.
- 4 (a) Using the points $A(2, -1, 3)$, $B(3, 1, -4)$ and $C(-1, 1, -1)$, write \overrightarrow{AB} , $2\overrightarrow{AC}$ and $\frac{1}{2}\overrightarrow{BC}$ as column vectors.

 (b) If you start from B and the translation $\overrightarrow{AB} + 2\overrightarrow{AC} + \frac{1}{2}\overrightarrow{BC}$ takes you to D , find the coordinates of D .
- 5 Four points A , B , C and D have position vectors $3\mathbf{i} - \mathbf{j} + 7\mathbf{k}$, $4\mathbf{i} + \mathbf{k}$, $\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $-2\mathbf{j} + 7\mathbf{k}$ respectively. Find the displacement vectors \overrightarrow{AB} and \overrightarrow{DC} . What can you deduce about the quadrilateral $ABCD$?
- 6 Two points A and B have position vectors $\begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$. C is the point on the line segment AB such that $\frac{AC}{CB} = 2$. Find
 - (a) the displacement vector \overrightarrow{AB} ,
 - (b) the displacement vector \overrightarrow{AC} ,
 - (c) the position vector of C .
- 7 Four points A , B , C and D have coordinates $(0, 1, -2)$, $(1, 3, 2)$, $(4, 3, 4)$ and $(5, -1, -2)$ respectively. Find the position vectors of
 - (a) the mid-point E of AC ,
 - (b) the point F on BD such that $\frac{BF}{FD} = \frac{1}{3}$.

Use your answers to draw a sketch showing the relative positions of A , B , C and D .

13.7 The magnitude of a vector

Any translation can be described by giving its magnitude and direction. The notation used for the magnitude of a vector \mathbf{p} , ignoring its direction, is $|\mathbf{p}|$.

If you have two vectors \mathbf{p} and \mathbf{q} which are not equal, but which have equal magnitudes, then you can write $|\mathbf{p}| = |\mathbf{q}|$.

If s is a scalar multiple of \mathbf{p} , then it follows from the definition of $s\mathbf{p}$ (see Section 13.2) that $|s\mathbf{p}| = |s||\mathbf{p}|$. This is true whether s is positive or negative (or zero).

The symbol for the magnitude of a vector is the same as the one for the modulus of a real number, because the concepts are similar. In fact, a real number x behaves just like the vector $x\mathbf{i}$ in one dimension, where \mathbf{i} is a basic unit vector. The vector $x\mathbf{i}$ represents a displacement on the number line, and the modulus $|x|$ then measures the magnitude of the displacement, whether it is in the positive or the negative direction.

A vector of magnitude 1 is called a **unit vector**. The basic unit vectors \mathbf{i} , \mathbf{j} , \mathbf{k} are examples of unit vectors, but there are others: there is a unit vector in every direction.

Unit vectors are sometimes distinguished by a circumflex accent $\hat{\cdot}$ over the letter. For example, a unit vector in the direction of \mathbf{r} may be denoted by $\hat{\mathbf{r}}$. This notation is especially common in mechanics, but it will not generally be used in this chapter.

13.8 Scalar products

So far vectors have been added, subtracted and multiplied by scalars, but they have not been multiplied together. The next step is to define the product of two vectors:

The **scalar product**, or **dot product**, of vectors \mathbf{p} and \mathbf{q} is the number (or scalar) $|\mathbf{p}||\mathbf{q}|\cos\theta$, where θ is the angle between the directions of \mathbf{p} and \mathbf{q} . It is written $\mathbf{p} \cdot \mathbf{q}$ and pronounced ‘ \mathbf{p} dot \mathbf{q} ’.

The angle θ may be acute or obtuse, but it is important that it is the angle between \mathbf{p} and \mathbf{q} , and not (for example) the angle between \mathbf{p} and $-\mathbf{q}$. It is best to show θ in a diagram in which the vectors are represented by arrows with their tails at the same point, as in Fig. 13.20.

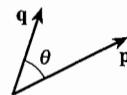


Fig. 13.20

The reason for calling this the ‘scalar product’, rather than simply the product, is that mathematicians also use another product, called the ‘vector product’. But it is important to distinguish the scalar product from ‘multiplication by a scalar’. To avoid confusion, many people prefer to use the alternative name ‘dot product’.

For the same reason, you must always insert the ‘dot’ between \mathbf{p} and \mathbf{q} for the scalar product, but you must *not* insert a dot between s and \mathbf{p} when multiplying by a scalar.

For example, you can never have a scalar product of three vectors, $\mathbf{p}, \mathbf{q}, \mathbf{r}$. You saw in Section 13.2 that the sum of these three vectors can be regarded as $(\mathbf{p} + \mathbf{q}) + \mathbf{r}$ or as $\mathbf{p} + (\mathbf{q} + \mathbf{r})$, and that these expressions are equal. But $(\mathbf{p} \cdot \mathbf{q}) \cdot \mathbf{r}$ has no meaning: $\mathbf{p} \cdot \mathbf{q}$ is a scalar, and you cannot form a dot product of a scalar with the vector \mathbf{r} . Similarly, $\mathbf{p} \cdot (\mathbf{q} \cdot \mathbf{r})$ has no meaning.

However, $s(\mathbf{p} \cdot \mathbf{q})$, where s is scalar, does have a meaning; as you would expect,

$$s(\mathbf{p} \cdot \mathbf{q}) = (s\mathbf{p}) \cdot \mathbf{q}.$$

The proof depends on whether s is positive (see Fig. 13.21) or negative (see Fig. 13.22).

If $s > 0$, then the angle between $s\mathbf{p}$ and \mathbf{q} is θ , so

$$(s\mathbf{p}) \cdot \mathbf{q} = |s\mathbf{p}| |\mathbf{q}| \cos \theta = |s| |\mathbf{p}| |\mathbf{q}| \cos \theta = |s| (|\mathbf{p}| |\mathbf{q}| \cos \theta) = s(\mathbf{p} \cdot \mathbf{q}).$$

If $s < 0$, then the angle between $s\mathbf{p}$ and \mathbf{q} is $\pi - \theta$, and $s = -|s|$, so

$$(s\mathbf{p}) \cdot \mathbf{q} = |s\mathbf{p}| |\mathbf{q}| \cos(\pi - \theta) = |s| |\mathbf{p}| |\mathbf{q}| (-\cos \theta) = -|s| (|\mathbf{p}| |\mathbf{q}| \cos \theta) = s(\mathbf{p} \cdot \mathbf{q}).$$

Another property of the scalar product is that $\mathbf{p} \cdot \mathbf{q} = \mathbf{q} \cdot \mathbf{p}$, which follows immediately from the definition. This is called the **commutative rule for scalar products**.

There are two very important special cases, which you get by taking $\theta = 0$ and putting $\mathbf{p} = \mathbf{q}$, and taking $\theta = \frac{1}{2}\pi$, in the definition of scalar product.

$$\mathbf{p} \cdot \mathbf{p} = |\mathbf{p}|^2 \quad (\mathbf{p} \cdot \mathbf{p} \text{ is sometimes written as } \mathbf{p}^2).$$

If neither \mathbf{p} nor \mathbf{q} is the zero vector,

$$\mathbf{p} \cdot \mathbf{q} = 0 \text{ if and only if } \mathbf{p} \text{ and } \mathbf{q} \text{ are in perpendicular directions.}$$

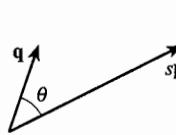


Fig. 13.21

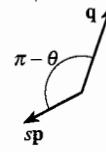


Fig. 13.22

These properties allow you to use vectors to find lengths and to identify right angles.

13.9 Scalar products in component form

The rules in the last section suggest that algebra with scalar products is much like ordinary algebra, except that some expressions (such as the scalar product of three vectors) have no meaning. You need one more rule to be able to use vectors to get geometrical results. This is the **distributive rule** for multiplying out brackets:

$$(\mathbf{p} + \mathbf{q}) \cdot \mathbf{r} = \mathbf{p} \cdot \mathbf{r} + \mathbf{q} \cdot \mathbf{r}.$$

For the present this will be assumed to be true. There is a proof in Section 13.10, but you may if you wish omit it on a first reading.

In the special cases at the end of Section 13.8, take \mathbf{p} and \mathbf{q} to be basic unit vectors.
You then get:

For the basic unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$,

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1 \quad \text{and} \quad \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = \mathbf{i} \cdot \mathbf{j} = 0.$$

It follows that, if vectors \mathbf{p} and \mathbf{q} are written in component form as $\mathbf{p} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$ and $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$, then

$$\begin{aligned}\mathbf{p} \cdot \mathbf{q} &= (l\mathbf{i} + m\mathbf{j} + n\mathbf{k}) \cdot (u\mathbf{i} + v\mathbf{j} + w\mathbf{k}) \\ &= lu\mathbf{i} \cdot \mathbf{i} + lv\mathbf{i} \cdot \mathbf{j} + lw\mathbf{i} \cdot \mathbf{k} + mu\mathbf{j} \cdot \mathbf{i} + mv\mathbf{j} \cdot \mathbf{j} + mw\mathbf{j} \cdot \mathbf{k} \\ &\quad + nu\mathbf{k} \cdot \mathbf{i} + nv\mathbf{k} \cdot \mathbf{j} + nw\mathbf{k} \cdot \mathbf{k} \quad (\text{using the distributive rule}) \\ &= lu \times 1 + lv \times 0 + lw \times 0 + mu \times 0 + mv \times 1 + mw \times 0 \\ &\quad + nu \times 0 + nv \times 0 + nw \times 1 \\ &= lu + mv + nw.\end{aligned}$$

In component form, the scalar product is

$$\begin{pmatrix} l \\ m \\ n \end{pmatrix} \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix} = (l\mathbf{i} + m\mathbf{j} + n\mathbf{k}) \cdot (u\mathbf{i} + v\mathbf{j} + w\mathbf{k}) = lu + mv + nw.$$

This result has many applications. In particular, $\mathbf{p} \cdot \mathbf{p} = l^2 + m^2 + n^2$, giving the length of \mathbf{p} :

$$|\mathbf{p}| = \sqrt{l^2 + m^2 + n^2}.$$

In two dimensions, if $\mathbf{p} = l\mathbf{i} + m\mathbf{j}$ and $\mathbf{q} = u\mathbf{i} + v\mathbf{j}$, then

$$\mathbf{p} \cdot \mathbf{q} = lu + mv$$

so, in component form

$$\begin{pmatrix} l \\ m \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix} = lu + mv.$$

Example 13.9.1

Show that the vectors $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ are perpendicular.

Writing $\mathbf{p} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$, and using $\mathbf{p} \cdot \mathbf{q} = lu + mv$,

$$\mathbf{p} \cdot \mathbf{q} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 3 \end{pmatrix} = 3 \times (-2) + 2 \times 3 = -6 + 6 = 0,$$

Using the result in the box on page 202, since neither \mathbf{p} nor \mathbf{q} is the zero vector and $\mathbf{p} \cdot \mathbf{q} = 0$, the vectors are perpendicular.

Example 13.9.2

Find the angle between the vectors $\mathbf{p} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{q} = 12\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$, giving your answer correct to the nearest tenth of a degree.

The magnitudes of \mathbf{p} and \mathbf{q} are given by

$$|\mathbf{p}| = \sqrt{2^2 + (-2)^2 + 1^2} = \sqrt{4 + 4 + 1} = \sqrt{9} = 3$$

and

$$|\mathbf{q}| = \sqrt{12^2 + 4^2 + (-3)^2} = \sqrt{144 + 16 + 9} = \sqrt{169} = 13.$$

Using $\mathbf{p} \cdot \mathbf{q} = |\mathbf{p}| |\mathbf{q}| \cos \theta = lu + mv + nw$, where θ° is the angle between \mathbf{p} and \mathbf{q} ,

$$3 \times 13 \times \cos \theta^\circ = 2 \times 12 + (-2) \times 4 + 1 \times (-3) = 24 - 8 - 3 = 13,$$

giving

$$\cos \theta^\circ = \frac{13}{39} = \frac{1}{3}, \text{ and thus } \theta = 70.5^\circ.$$

The required angle is 70.5° .

Vectors can give a good method for finding the angle between two straight lines, where it may not be easy or possible to draw a triangle containing the two lines.

Example 13.9.3

A barn (Fig. 13.23) has a rectangular floor $ABCD$ of dimensions 6 m by 12 m. The edges AP , BQ , CR and DS are each vertical and of height 5 m. The ridge UV is symmetrically placed above $PQRS$, and is height 7 m above $ABCD$. Calculate to the nearest tenth of a degree the angle between the lines AS and UR .

Take the unit vectors \mathbf{i} , \mathbf{j} and \mathbf{k} in the directions BC , BA and BQ .

Let $\vec{AS} = \mathbf{e}$ and $\vec{UR} = \mathbf{f}$.

Then

$$\mathbf{e} = 12\mathbf{i} + 5\mathbf{k} \text{ and } \mathbf{f} = 12\mathbf{i} - 3\mathbf{j} - 2\mathbf{k},$$

$$\text{so } |\mathbf{e}| = \sqrt{12^2 + 0^2 + 5^2} = \sqrt{169} = 13$$

$$\text{and } |\mathbf{f}| = \sqrt{12^2 + (-3)^2 + (-2)^2} = \sqrt{157}.$$

Denote the angle between the lines by θ° .

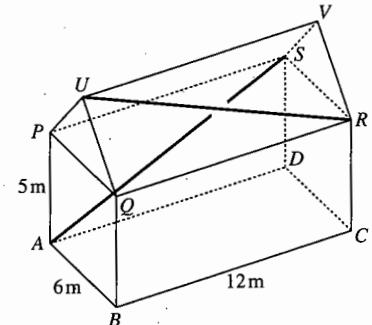


Fig. 13.23

$$\text{Then } 13 \times \sqrt{157} \cos \theta^\circ = 12 \times 12 + 0 \times (-3) + 5 \times (-2) = 134,$$

$$\text{giving } \theta = 34.6, \text{ correct to 1 decimal place.}$$

So the angle between AS and UR is 34.6° .

In this example AS and UR are skew lines. Since AS is parallel to BR , the angle between AS and UR is equal to the angle between AS and BR .

13.10* The distributive rule $(\mathbf{p} + \mathbf{q}) \cdot \mathbf{r} = \mathbf{p} \cdot \mathbf{r} + \mathbf{q} \cdot \mathbf{r}$

The proof of this needs a preliminary result. Fig. 13.24 shows a directed line l and two points A and B (in three dimensions). If lines AD and BE are drawn perpendicular to l , the directed length DE is called the **projection** of the displacement vector \vec{AB} on l .

Here the word ‘directed’ means that a positive direction is selected on l , and that (in this diagram) DE is positive and ED is negative.

Theorem If \mathbf{p} is the displacement vector \vec{AB} , and \mathbf{u} is a unit vector in the direction of l , then the projection of \vec{AB} on l is $\mathbf{p} \cdot \mathbf{u}$.

Proof You will probably find the proof easiest to follow if l is drawn as a vertical line, as in Fig. 13.25. Recall that AD and BE are perpendicular to l , and so are horizontal. The shaded triangles ADM and NEB lie in the horizontal planes through D and E . The point N is such that AN is parallel to l and perpendicular to NB .

Then $DE = AN$, and \mathbf{u} is a unit vector in the direction of AN . If the angle BAN is denoted by θ , then

$$\mathbf{p} \cdot \mathbf{u} = |\mathbf{p}| \times 1 \times \cos \theta = AB \cos \theta = AN = DE,$$

which is the projection of \vec{AB} on l .

Notice that, if B were below A , then the angle between \mathbf{p} and \mathbf{u} would be obtuse, so $\mathbf{p} \cdot \mathbf{u}$ would be negative. On l , E would be below D , so the directed length DE would also be negative.

When you have understood this proof with l vertical, you can try re-drawing Fig. 13.25 with l in some other direction, as in Fig. 13.24. If you then replace ‘horizontal planes’ by ‘planes perpendicular to l ’, the proof will still hold.

Theorem For any vectors \mathbf{p} , \mathbf{q} and \mathbf{r} , $(\mathbf{p} + \mathbf{q}) \cdot \mathbf{r} = \mathbf{p} \cdot \mathbf{r} + \mathbf{q} \cdot \mathbf{r}$.

Proof In Fig. 13.26 the displacement vectors \vec{AB} , \vec{BC} and \vec{AC} represent \mathbf{p} , \mathbf{q} and $\mathbf{p} + \mathbf{q}$. The line l is in the direction of \mathbf{r} ; this is again shown as a vertical line. The horizontal planes through A , B and C cut l at D , E and F respectively, so that AD , BE and CF are perpendicular to l . Let \mathbf{u} be a unit vector in the direction of \mathbf{r} , and denote $|\mathbf{r}|$ by s , so that $\mathbf{r} = s\mathbf{u}$.

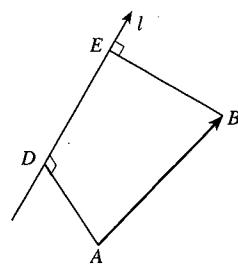


Fig. 13.24

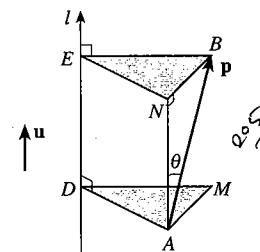


Fig. 13.25

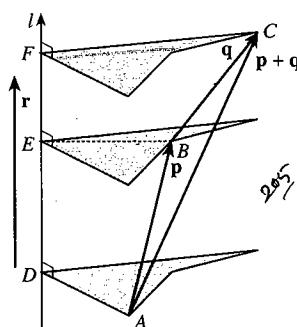


Fig. 13.26

Then

$$\mathbf{p} \cdot \mathbf{r} = \mathbf{p} \cdot (\mathbf{s}\mathbf{u}) = s(\mathbf{p} \cdot \mathbf{u}) = s \times DE,$$

and similarly $\mathbf{q} \cdot \mathbf{r} = s \times EF$ and $(\mathbf{p} + \mathbf{q}) \cdot \mathbf{r} = s \times DF$.

Since DE , EF and DF are directed lengths, it is always true that $DE + EF = DF$, whatever the order of the points D , E and F on l .

Therefore

$$(\mathbf{p} + \mathbf{q}) \cdot \mathbf{r} = s \times DF = s \times (DE + EF) = s \times DE + s \times EF = \mathbf{p} \cdot \mathbf{r} + \mathbf{q} \cdot \mathbf{r}.$$

As before, when you have understood this proof with l vertical, you can adapt it for any other direction of l .

Exercise 13D

- 1 Let $\mathbf{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$. Calculate $\mathbf{a} \cdot \mathbf{b}$, $\mathbf{a} \cdot \mathbf{c}$ and $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$, and verify that $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$.
- 2 Let $\mathbf{a} = 2\mathbf{i} - \mathbf{j}$, $\mathbf{b} = 4\mathbf{i} - 3\mathbf{j}$ and $\mathbf{c} = -2\mathbf{i} - \mathbf{j}$. Calculate $\mathbf{a} \cdot \mathbf{b}$, $\mathbf{a} \cdot \mathbf{c}$ and $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$, and verify that $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$.
- 3 Let $\mathbf{p} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} -1 \\ -9 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 33 \\ -13 \\ -28 \end{pmatrix}$. Calculate $\mathbf{p} \cdot \mathbf{q}$, $\mathbf{p} \cdot \mathbf{r}$ and $\mathbf{q} \cdot \mathbf{r}$. What can you deduce about the vectors \mathbf{p} , \mathbf{q} and \mathbf{r} ?
- 4 Which of the following vectors are perpendicular to each other?
 - (a) $2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$
 - (b) $2\mathbf{i} - 3\mathbf{j} - 6\mathbf{k}$
 - (c) $-3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$
 - (d) $6\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$
- 5 Let $\mathbf{p} = \mathbf{i} - 2\mathbf{k}$, $\mathbf{q} = 3\mathbf{j} + 2\mathbf{k}$ and $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$. Calculate $\mathbf{p} \cdot \mathbf{q}$, $\mathbf{p} \cdot \mathbf{r}$ and $\mathbf{p} \cdot (\mathbf{q} + \mathbf{r})$ and verify that $\mathbf{p} \cdot (\mathbf{q} + \mathbf{r}) = \mathbf{p} \cdot \mathbf{q} + \mathbf{p} \cdot \mathbf{r}$.
- 6 Find the magnitude of each of the following vectors.

(a) $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$	(b) $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$	(c) $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$	(d) $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$
(e) $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$	(f) $\begin{pmatrix} 4 \\ -3 \\ 12 \end{pmatrix}$	(g) $\begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix}$	(h) $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$
(i) $\mathbf{i} - 2\mathbf{k}$	(j) $3\mathbf{j} + 2\mathbf{k}$	(k) $2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$	(l) $2\mathbf{k}$
- 7 Let $\mathbf{a} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$. Find the magnitude of \mathbf{a} , and find a unit vector in the same direction as \mathbf{a} .
- 8 Find unit vectors in the same directions as $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ and $2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

9 Use a vector method to calculate the angles between the following pairs of vectors, giving your answers in degrees to one place of decimals, where appropriate.

(a) $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

(b) $\begin{pmatrix} 4 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ 4 \end{pmatrix}$

(c) $\begin{pmatrix} 4 \\ -6 \end{pmatrix}$ and $\begin{pmatrix} -6 \\ 9 \end{pmatrix}$

(d) $\begin{pmatrix} -1 \\ 4 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}$

(e) $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$

(f) $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ -2 \\ -4 \end{pmatrix}$

10 Let $\mathbf{r}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\mathbf{r}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$. Calculate $|\mathbf{r}_2 - \mathbf{r}_1|$ and interpret your result geometrically.

11 Find the angle between the line joining (1,2) and (3,-5) and the line joining (2,-3) to (1,4).

12 Find the angle between the line joining (1,3,-2) and (2,5,-1) and the line joining (-1,4,3) to (3,2,1).

13 Find the angle between the diagonals of a cube.

14 ABCD is the base of a square pyramid of side 2 units, and V is the vertex. The pyramid is symmetrical, and of height 4 units. Calculate the acute angle between AV and BC, giving your answer in degrees correct to 1 decimal place.

15 Two aeroplanes are flying in directions given by the vectors $300\mathbf{i} + 400\mathbf{j} + 2\mathbf{k}$ and $-100\mathbf{i} + 500\mathbf{j} - \mathbf{k}$. A person from the flight control centre is plotting their paths on a map. Find the acute angle between their paths on the map.

16 The roof of a house has a rectangular base of side 4 metres by 8 metres. The ridge line of the roof is 6 metres long, and centred 1 metre above the base of the roof. Calculate the acute angle between two opposite slanting edges of the roof.

Miscellaneous exercise 13

1 Find which pairs of the following vectors are perpendicular to each other.

$\mathbf{a} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

$\mathbf{b} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

$\mathbf{c} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

$\mathbf{d} = 3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$

2 The vectors \vec{AB} and \vec{AC} are $\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}$ respectively. The vector \vec{AD} is the sum of \vec{AB} and \vec{AC} . Determine the acute angle, in degrees correct to one decimal place, between the diagonals of the parallelogram defined by the points A, B, C and D. (OCR)

3 The vectors \mathbf{AB} and \mathbf{AC} are $\begin{pmatrix} -2 \\ 6 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ -3 \\ 6 \end{pmatrix}$ respectively.

(a) Determine the lengths of the vectors.

(b) Find the scalar product $\mathbf{AB} \cdot \mathbf{AC}$.

(c) Use your result from part (b) to calculate the acute angle between the vectors. Give the angle in degrees correct to one decimal place. (OCR)

- 4 The points A , B and C have position vectors $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -3 \\ 2 \\ 5 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 4 \\ 5 \\ -2 \end{pmatrix}$

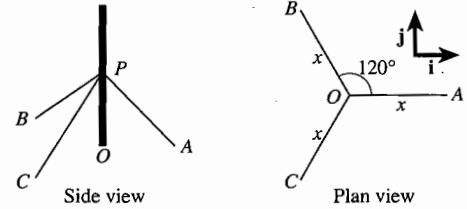
respectively, with respect to a fixed origin. The point D is such that $ABCD$, in that order, is a parallelogram.

- Find the position vector of D .
- Find the position vector of the point at which the diagonals of the parallelogram intersect.
- Calculate the angle BAC , giving your answer to the nearest tenth of a degree.

(OCR)

- 5 A vertical aerial is supported by three straight cables, each attached to the aerial at a point P , 30 metres up the aerial. The cables are attached to the horizontal ground at points A , B and C , each x metres from the foot O of the aerial, and situated symmetrically around it (see the diagrams).

Suppose that \mathbf{i} is the unit vector in the direction \vec{OA} , \mathbf{j} is the unit vector perpendicular to \mathbf{i} in the plane of the ground, as shown in the Plan view, and \mathbf{k} is the unit vector in the direction \vec{OP} .



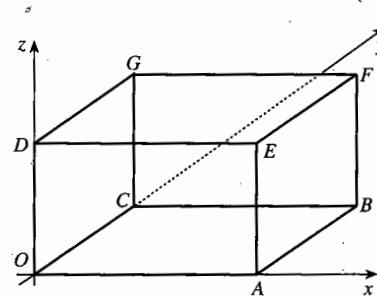
- Write down expressions for the vectors \vec{OA} , \vec{OB} and \vec{OC} in terms of x , \mathbf{i} , \mathbf{j} and \mathbf{k} .
- (i) Write down an expression for the vector \vec{AP} in terms of vectors \vec{OA} and \vec{OP} .
(ii) Hence find expressions for the vectors \vec{AP} and \vec{BP} in terms of x , \mathbf{i} , \mathbf{j} and \mathbf{k} .
- Given that \vec{AP} and \vec{BP} are perpendicular to each other, find the value of x . (OCR)

- 6 The position vectors of three points A , B and C with respect to a fixed origin O are $2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $4\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ respectively. Find unit vectors in the directions of \vec{CA} and \vec{CB} . Calculate angle ACB in degrees, correct to 1 decimal place. (OCR)

- 7 (a) Find the angle between the vectors $2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$ and $3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}$.
(b) The vectors \mathbf{a} and \mathbf{b} are non-zero.
(i) Given that $\mathbf{a} + \mathbf{b}$ is perpendicular to $\mathbf{a} - \mathbf{b}$, prove that $|\mathbf{a}| = |\mathbf{b}|$.
(ii) Given instead that $|\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$, prove that \mathbf{a} and \mathbf{b} are perpendicular.

(OCR)

- 8 $OABCDEFG$, shown in the figure, is a cuboid. The position vectors of A , C and D are $4\mathbf{i}$, $2\mathbf{j}$ and $3\mathbf{k}$ respectively. Calculate
(a) $|AG|$,
(b) the angle between AG and OB . (OCR)



- 9** The three-dimensional vector \mathbf{r} , which has positive components, has magnitude 1 and makes angles of 60° with each of the unit vectors \mathbf{i} and \mathbf{j} .
- Write \mathbf{r} as a column vector.
 - State the angle between \mathbf{r} and the unit vector \mathbf{k} .
- 10** The points A , B and C have position vectors given respectively by $\mathbf{a} = 7\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, $\mathbf{b} = 5\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$, $\mathbf{c} = 6\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}$.
- Find the angle BAC .
 - Find the area of the triangle ABC . (OCR)
- 11** The points A , B and C have position vectors \mathbf{a} , \mathbf{b} and \mathbf{c} respectively relative to the origin O . P is the point on BC such that $\vec{PC} = \frac{1}{10}\vec{BC}$.
- Show that the position vector of P is $\frac{1}{10}(9\mathbf{c} + \mathbf{b})$.
 - Given that the line AP is perpendicular to the line BC , show that $(9\mathbf{c} + \mathbf{b}) \cdot (\mathbf{c} - \mathbf{b}) = 10\mathbf{a} \cdot (\mathbf{c} - \mathbf{b})$.
 - Given also that OA , OB and OC are mutually perpendicular, prove that $OC = \frac{1}{3}OB$. (OCR)
- 12** A mathematical market trader packages fruit in three sizes. An Individual bag holds 1 apple and 2 bananas; a Jumbo bag holds 4 apples and 3 bananas; and a King-size bag holds 8 apples and 7 bananas. She draws two vector arrows \mathbf{a} and \mathbf{b} to represent an apple and a banana respectively, and then represents the three sizes of bag by vectors $\mathbf{I} = \mathbf{a} + 2\mathbf{b}$, $\mathbf{J} = 4\mathbf{a} + 3\mathbf{b}$ and $\mathbf{K} = 8\mathbf{a} + 7\mathbf{b}$. Find numbers s and t such that $\mathbf{K} = s\mathbf{I} + t\mathbf{J}$.
By midday she has sold all her King-size bags, but she has plenty of Individual and Jumbo bags left. She decides to make up some more King-size bags by using the contents of the other bags. How can she do this so that she has no loose fruit left over?
- 13** $ABCD$ is a parallelogram. The coordinates of A , B , D are $(4, 2, 3)$, $(18, 4, 8)$ and $(-1, 12, 13)$ respectively. The origin of coordinates is O .
- Find the vectors \vec{AB} and \vec{AD} . Find the coordinates of C .
 - Show that \vec{OA} can be expressed in the form $\lambda\vec{AB} + \mu\vec{AD}$, stating the values of λ and μ . What does this tell you about the plane $ABCD$? (MEI)
- 14** A balloon flying over flat land reports its position at 7.40 a.m. as $(7.8, 5.4, 1.2)$, the coordinates being given in kilometres relative to a checkpoint on the ground. By 7.50 a.m. its position has changed to $(9.3, 4.4, 0.7)$. Assuming that it continues to descend at the same speed along the same line, find the coordinates of the point where it would be expected to land, and the time when this would occur.
- 15** Prove that, if $(\mathbf{c} - \mathbf{b}) \cdot \mathbf{a} = 0$ and $(\mathbf{c} - \mathbf{a}) \cdot \mathbf{b} = 0$, then $(\mathbf{b} - \mathbf{a}) \cdot \mathbf{c} = 0$. Show that this can be used to prove the following geometrical results.
- The lines through the vertices of a triangle ABC perpendicular to the opposite sides meet in a point.
 - If the tetrahedron $OABC$ has two pairs of perpendicular opposite edges, the third pair of edges is perpendicular.
- Prove also that, in both cases, $OA^2 + BC^2 = OB^2 + CA^2 = OC^2 + AB^2$.

14 Geometric sequences

This chapter introduces another type of sequence. When you have completed it, you should

- recognise geometric sequences and be able to do calculations on them
- know and be able to obtain the formula for the sum of a geometric series
- know the condition for a geometric series to converge, and how to find its limiting sum.

14.1 Geometric sequences

In Chapter 8 you met arithmetic sequences, in which you get from one term to the next by adding a constant. A sequence in which you get from one term to the next by multiplying by a constant is called a geometric sequence.

A **geometric sequence**, or **geometric progression**, is a sequence defined by $u_1 = a$ and $u_{i+1} = ru_i$, where $i \in \mathbb{N}$ and $r \neq 0$ or 1.

The constant r is called the **common ratio** of the sequence.

You should notice two points about this definition. First, since the letter r is conventionally used for the common ratio, a different letter, i , is used for the suffixes.

Secondly, the ratios 0 and 1 are excluded. If you put $r = 0$ in the definition you get the sequence $a, 0, 0, 0, \dots$; if you put $r = 1$ you get a, a, a, a, \dots . Neither is very interesting, and some of the properties of geometric sequences break down if $r = 0$ or 1. However, r can be negative; in that case the terms are alternately positive and negative.

It is easy to give a formula for the i th term. To get from u_1 to u_i you multiply by the common ratio $i - 1$ times, so $u_i = r^{i-1} \times u_1$, which gives $u_i = ar^{i-1}$.

Example 14.1.1

A geometric sequence has first term $u_1 = 1$ and common ratio 1.1. Which is the first term greater than (a) 2, (b) 5, (c) 10, (d) 1000?

On many calculators you can keep multiplying by 1.1 by repeatedly pressing a single key. This makes it easy to display successive terms of a geometric sequence.

- (a) This can easily be done experimentally, counting how many times you press the key until the display exceeds 2. You should find that after 8 presses you get 2.143 588 81, which is 1.1^8 . The i th term of the sequence is $u_i = 1 \times 1.1^{i-1}$, so this is first greater than 2 when $i - 1 = 8$, or $i = 9$.

(b) Go on pressing the key. After another 8 presses you reach $(2.143\ 588\ 81)^2$, which is certainly greater than 4, so you will reach 5 quite soon. In fact it turns out that 1.1^{15} is already greater than 4, and two more presses take you to $1.1^{17} = 5.054\dots$. So $u_{18} = 1 \times 1.1^{17}$ is the first term greater than 5.

(c) Since $1.1^8 > 2$ and $1.1^{17} > 5$, it is certainly true that $1.1^{25} = 1.1^8 \times 1.1^{17}$ is greater than 10. Rather than continuing to multiply, you can just use the power key to find $1.1^{25} = 10.834\dots$. But you must check that 1.1^{24} is not already greater than 10. In fact it isn't, since $1.1^{24} = 9.849\dots$. So the first term greater than 10 is $u_{26} = 1 \times 1.1^{25}$.

(d) Since $1.1^{24} < 10$ and $1.1^{25} > 10$, you can cube both sides to find that $1.1^{72} = (1.1^{24})^3 < 1000$ and $1.1^{75} = (1.1^{25})^3 > 1000$. So you only need to check 1.1^{73} and 1.1^{74} . Using the power key, $1.1^{73} = 1051.1\dots$. So the first term greater than 1000 is $u_{74} = 1.1^{73}$.

The first terms greater than 2, 5, 10 and 1000 are u_9 , u_{18} , u_{26} and u_{74} .

You can see from this example that, even with a common ratio only slightly greater than 1, the terms of a geometric sequence get big quite quickly.

14.2 Summing geometric series

Geometric sequences have many applications in finance, biology, mechanics and probability, and you often need to find the sum of all the terms. In this context it is usual to call the sequence a **geometric series**.

The method used in Chapter 8 to find the sum of an arithmetic series does not work for geometric series. You can see this by taking a simple geometric series like $1 + 2 + 4 + 8 + 16$ and placing an upside-down copy next to it, as in Fig. 14.1. When you did this with arithmetic series the two sets of crosses and noughts made a perfect join (see Fig. 8.7), so they could easily be counted; but for the geometric series there is a gap in the middle.

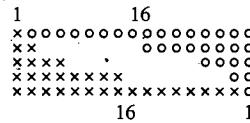


Fig. 14.1

For geometric series a different method is used. If you multiply the equation $S = 1 + 2 + 4 + 8 + 16$ by 2, then you get $2S = 2 + 4 + 8 + 16 + 32$. Notice that the right sides in these two equations have the terms $2 + 4 + 8 + 16$ in common; the sum of these terms is equal to $S - 1$, from the first equation, and $2S - 32$, from the second. So

$$S - 1 = 2S - 32, \text{ giving } S = 31.$$

You can use this method to find the sum of any geometric series. Let S be the sum of the first n terms of the series. Then

$$S = a + ar + ar^2 + \dots + ar^{n-2} + ar^{n-1}.$$

If you multiply this equation by r , you get

$$Sr = ar + ar^2 + ar^3 + \dots + ar^{n-1} + ar^n.$$

The right sides in these two equations have the terms $ar + ar^2 + \dots + ar^{n-2} + ar^{n-1}$ in common; so

$$S - a = ar + ar^2 + \dots + ar^{n-2} + ar^{n-1} = Sr - ar^n,$$

which gives

$$S(1 - r) = a(1 - r^n), \quad \text{or} \quad S = \frac{a(1 - r^n)}{1 - r}.$$

The sum of the geometric series $a + ar + ar^2 + \dots + ar^{n-1}$, with n terms, is

$$\frac{a(1 - r^n)}{1 - r}.$$

You should notice that it has nowhere been assumed that r is positive. The formula is valid whether r is positive or negative. When $r > 1$, some people prefer to avoid fractions with negative numerators and denominators by using the result in the alternative form

$$S = \frac{a(r^n - 1)}{r - 1}.$$

Example 14.2.1

A child lives 200 metres from school. He walks 60 metres in the first minute, and in each subsequent minute he walks 75% of the distance he walked in the previous minute. Show that he takes between 6 and 7 minutes to get to school.

The distances walked in the first, second, third, ..., n th minutes are 60 m, 60×0.75 m, 60×0.75^2 m, ..., $60 \times 0.75^{n-1}$ m. In the first n minutes the child walks S_n metres, where

$$\begin{aligned} S_n &= 60 + 60 \times 0.75^1 + 60 \times 0.75^2 + \dots + 60 \times 0.75^{n-1} \\ &= \frac{60(1 - 0.75^n)}{1 - 0.75} = \frac{60(1 - 0.75^n)}{0.25} = 240(1 - 0.75^n). \end{aligned}$$

From this formula you can calculate that

$$S_6 = 240(1 - 0.75^6) = 240(1 - 0.177\dots) = 197.2\dots, \text{ and}$$

$$S_7 = 240(1 - 0.75^7) = 240(1 - 0.133\dots) = 207.9\dots.$$

So he has not reached school after 6 minutes, but (if he had gone on walking) he would have gone more than 200 m in 7 minutes. That is, he takes between 6 and 7 minutes to walk to school.

Example 14.2.2

Find a simple expression for the sum $p^6 - p^5q + p^4q^2 - p^3q^3 + p^2q^4 - pq^5 + q^6$.

This is a geometric series of 7 terms, with first term p^6 and common ratio $-\frac{q}{p}$.
Its sum is therefore

$$\frac{p^6(1 - (-q/p)^7)}{1 - (-q/p)} = \frac{p^6(1 - (-q^7/p^7))}{1 + q/p} = \frac{p^7(1 + q^7/p^7)}{p(1 + q/p)} = \frac{p^7 + q^7}{p + q}.$$

Another way of writing the result of this example is

$$p^7 + q^7 = (p + q)(p^6 - p^5q + p^4q^2 - p^3q^3 + p^2q^4 - pq^5 + q^6).$$

You can use a similar method for any odd number n to express $p^n + q^n$ as the product of $p + q$ and another factor.

Exercise 14A

- 1 For each of the following geometric sequences find the common ratio and the next two terms.

(a) 3, 6, 12, ...	(b) 2, 8, 32, ...	(c) 32, 16, 8, ...
(d) 2, -6, 18, -54, ...	(e) 1.1, 1.21, 1.331, ...	(f) $x^2, x, 1, \dots$
- 2 Find an expression for the i th term of each of the following geometric sequences.

(a) 2, 6, 18, ...	(b) 10, 5, 2.5, ...	(c) 1, -2, 4, ...
(d) 81, 27, 9, ...	(e) x, x^2, x^3, \dots	(f) $pq^2, q^3, p^{-1}q^4, \dots$
- 3 Find the number of terms in each of these geometric progressions.

(a) 2, 4, 8, ..., 2048	(b) 1, -3, 9, ..., 531 441
(c) 2, 6, 18, ..., 1458	(d) 5, -10, 20, ..., -40 960
(e) 16, 12, 9, ..., 3.796 875	(f) $x^{-6}, x^{-2}, x^2, \dots, x^{42}$
- 4 Find the common ratio and the first term in the geometric progressions where
 - (a) the 2nd term is 4 and the 5th term is 108,
 - (b) the 3rd term is 6 and the 7th term is 96,
 - (c) the 4th term is 19 683 and the 9th term is 81,
 - (d) the 3rd term is 8 and the 9th term is 64,
 - (e) the n th term is 16 807 and the $(n+4)$ th term is 40 353 607.
- 5 Find the sum, for the given number of terms, of each of the following geometric series. Give decimal answers correct to 4 places.

(a) $2 + 6 + 18 + \dots$	10 terms	(b) $2 - 6 + 18 - \dots$	10 terms
(c) $1 + \frac{1}{2} + \frac{1}{4} + \dots$	8 terms	(d) $1 - \frac{1}{2} + \frac{1}{4} - \dots$	8 terms
(e) $3 + 6 + 12 + \dots$	12 terms	(f) $12 - 4 + \frac{4}{3} - \dots$	10 terms
(g) $x + x^2 + x^3 + \dots$	n terms	(h) $x - x^2 + x^3 - \dots$	n terms
(i) $x + \frac{1}{x} + \frac{1}{x^3} + \dots$	n terms	(j) $1 - \frac{1}{x^2} + \frac{1}{x^4} + \dots$	n terms

- 6** Use the method in Section 14.2 to find the sum of each of the following geometric series. Give numerical answers as rational numbers.

$$\begin{array}{ll}
 \text{(a)} & 1+2+4+\dots+1024 \\
 \text{(c)} & 3+12+48+\dots+196\,608 \\
 \text{(e)} & 1-\frac{1}{3}+\frac{1}{9}-\dots-\frac{1}{19\,683} \\
 \text{(g)} & \frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\dots+\frac{1}{1024} \\
 \text{(i)} & 16+4+1+\dots+\frac{1}{2^{2n}} \\
 \text{(b)} & 1-2+4-\dots+1024 \\
 \text{(d)} & 1+\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{512} \\
 \text{(f)} & 10+5+2.5+\dots+0.156\,25 \\
 \text{(h)} & 1+\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2^n} \\
 \text{(j)} & 81-27+9-\dots+\frac{1}{(-3)^n}
 \end{array}$$

- 7** A well-known story concerns the inventor of the game of chess. As a reward for inventing the game it is rumoured that he was asked to choose his own prize. He asked for 1 grain of rice to be placed on the first square of the board, 2 grains on the second square, 4 grains on the third square and so on in geometric progression until all 64 squares had been covered. Calculate the total number of grains of rice he would have received. Give your answer in standard form!

- 8** A problem similar to that of Question 7 is posed by the child who negotiates a pocket money deal of 1 cent on 1 February, 2 cents on 2 February, 4 cents on 3 February and so on for 28 days. How much should the child receive in total during February?

- 9** If x , y and z are the first three terms of a geometric sequence, show that x^2 , y^2 and z^2 form another geometric sequence.

- 10** Different numbers x , y and z are the first three terms of a geometric progression with common ratio r , and also the first, second and fourth terms of an arithmetic progression.

(a) Find the value of r .

(b) Find which term of the arithmetic progression will next be equal to a term of the geometric progression.

- 11** Different numbers x , y and z are the first three terms of a geometric progression with common ratio r and also the first, second and fifth terms of an arithmetic progression.

(a) Find the value of r .

(b) Find which term of the arithmetic progression will next be equal to a term of the geometric progression.

- 12** Consider the geometric progression

$$q^{n-1} + q^{n-2}p + q^{n-3}p^2 + \dots + qp^{n-2} + p^{n-1}.$$

(a) Find the common ratio and the number of terms.

(b) Show that the sum of the series is equal to $\frac{q^n - p^n}{q - p}$.

(c) By considering the limit as $q \rightarrow p$ deduce expressions for $f'(p)$ in the cases

(i) $f(x) = x^n$, (ii) $f(x) = x^{-n}$, for all positive integers n .

14.3 Convergent sequences

Take any sequence, such as the sequence of triangle numbers $t_1 = 1, t_2 = 3, t_3 = 6, \dots$ (see Section 8.2). Form a new sequence whose terms are the sums of successive triangle numbers:

$$S_1 = t_1 = 1, \quad S_2 = t_1 + t_2 = 1 + 3 = 4, \quad S_3 = t_1 + t_2 + t_3 = 1 + 3 + 6 = 10, \text{ and so on.}$$

This is called the sum sequence of the original sequence.

Notice that $S_2 = S_1 + t_2, S_3 = S_2 + t_3, \dots$. This property can be used to give an inductive definition for the sum sequence:

For a given sequence u_i , the **sum sequence** $S_i = u_1 + \dots + u_i$ is defined by $S_1 = u_1$ and $S_{i+1} = S_i + u_{i+1}$.

(If the original sequence begins with u_0 rather than u_1 , the equation $S_1 = u_1$ in the definition is replaced by $S_0 = u_0$.)

Geometric sequences have especially important sum sequences. Here are four examples, each with first term $a = 1$:

(a) $r = 3$

u_i	1	3	9	27	81	243	729	...
S_i	1	4	13	40	121	364	1093	...

Table 14.2

(b) $r = 0.2$

u_i	1	0.2	0.04	0.008	0.0016	0.00032	0.000064	...
S_i	1	1.2	1.24	1.248	1.2496	1.24992	1.249984	...

Table 14.3

(c) $r = -0.2$

u_i	1	-0.2	0.04	-0.008	-0.0016	-0.00032	-0.000064	...
S_i	1	0.8	0.84	0.832	0.8336	0.83328	0.833344	...

Table 14.4

(d) $r = -3$

u_i	1	-3	9	-27	81	-243	729	...
S_i	1	-2	7	-20	61	-182	547	...

Table 14.5

The sum sequences for (b) and (c) are quite different from the others. You would guess that in (b) the values of S_i are getting close to 1.25, but never reach it. This can be proved, since the formula for the sum of the first n values of u_i gives, with $a = 1$ and $r = 0.2$,

$$\frac{1 - 0.2^n}{1 - 0.2} = \frac{1 - 0.2^n}{0.8} = 1.25(1 - 0.2^n).$$

Now you can make 0.2^n as small as you like by taking n large enough, and then the expression in brackets comes very close to 1, though it never equals 1. You can say that the sum tends to the limit 1.25 as n tends to infinity.

It seems that the sum in (c) tends to 0.833 33... (the recurring decimal for $\frac{5}{6}$) as n tends to infinity, but here the values are alternately above and below the limiting value. This is because the formula for the sum is

$$\frac{1 - (-0.2)^n}{1 - (-0.2)} = \frac{1 - (-0.2)^n}{1.2} = \frac{5}{6}(1 - (-0.2)^n).$$

In this formula the expression $(-0.2)^n$ is alternately positive and negative, so $1 - (-0.2)^n$ alternates above and below 1.

The other two sequences, for which the sum formulae are (a) $\frac{1}{2}(3^n - 1)$ and (d) $\frac{1}{4}(1 - (-3)^n)$, do not tend to a limit. The sum (a) can be made as large as you like by taking n large enough; it is said to **diverge to infinity** as n tends to infinity. The sum (d) can also be made as large as you like; the sum sequence is said to **oscillate infinitely**.

It is the expression r^n in the sum formula $\frac{a(1 - r^n)}{1 - r}$ which determines whether or not the sum tends to a limit. If $|r| > 1$, then $|r^n|$ increases indefinitely; but if $|r| < 1$, then $|r^n|$ tends to 0 and the sum tends to the value $\frac{a(1 - 0)}{1 - r} = \frac{a}{1 - r}$. As long as $|r| < 1$, even if r is very close to 1, r^n becomes very small if n is large enough; for example, if $r = 0.9999$ and $n = 1\ 000\ 000$, then $r^n \approx 3.70 \times 10^{-44}$.

If $|r| < 1$, the sum of the geometric series with first term a and common ratio r tends to the limit $S_\infty = \frac{a}{1 - r}$ as the number of terms tends to infinity.

The infinite geometric series is then said to be **convergent**.

S_∞ is called the **sum to infinity** of the series.

Example 14.3.1

Express the recurring decimal 0.296 296 296... as a fraction.

The decimal can be written as

$$\begin{aligned} 0.296 + 0.000\ 296 + 0.000\ 000\ 296 + \dots \\ = 0.296 + 0.296 \times 0.001 + 0.296 \times (0.001)^2 + \dots , \end{aligned}$$

which is a geometric series with $a = 0.296$ and $r = 0.001$. Since $|r| < 1$, the series is convergent with limiting sum $\frac{0.296}{1 - 0.001} = \frac{296}{999}$.

Since $296 = 8 \times 37$ and $999 = 27 \times 37$, this fraction in its simplest form is $\frac{8}{27}$.

Example 14.3.2

A beetle starts at a point O on the floor. It walks 1 m east, then $\frac{1}{2}$ m west, then $\frac{1}{4}$ m east, and so on, halving the distance at each change of direction. How far from O does it end up?

The final distance from O is $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots$, which is a geometric series with common ratio $-\frac{1}{2}$. Since $|- \frac{1}{2}| < 1$, the series converges to a limit

$$\frac{1}{1 - (-\frac{1}{2})} = \frac{1}{\frac{3}{2}} = \frac{2}{3}.$$

The beetle ends up $\frac{2}{3}$ m from O .

Notice that a point of trisection was obtained as the limit of a process of repeated halving.

Exercise 14B

- 1 Find the sum to infinity of the following geometric series. Give your answers to parts (a) to (j) as whole numbers, fractions or exact decimals.

(a) $1 + \frac{1}{2} + \frac{1}{4} + \dots$	(b) $1 + \frac{1}{3} + \frac{1}{9} + \dots$
(c) $\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \dots$	(d) $0.1 + 0.01 + 0.001 + \dots$
(e) $1 - \frac{1}{3} + \frac{1}{9} - \dots$	(f) $0.2 - 0.04 + 0.008 - \dots$
(g) $\frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \dots$	(h) $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \dots$
(i) $10 - 5 + 2.5 - \dots$	(j) $50 + 10 + 2 + \dots$
(k) $x + x^2 + x^3 + \dots$, where $-1 < x < 1$	(l) $1 - x^2 + x^4 - \dots$, where $x^2 < 1$
(m) $1 + x^{-1} + x^{-2} + \dots$, where $ x > 1$	(n) $x^2 - x + 1 - \dots$, where $ x > 1$
- 2 Express each of the following recurring decimals as exact fractions.

(a) 0.363 636...	(b) 0.123 123 123...
(c) 0.555...	(d) 0.471 471 471...
(e) 0.142 857 142 857 142 857...	(f) 0.285 714 285 714 285 714...
(g) 0.714 285 714 285 714 285...	(h) 0.857 142 857 142 857 142...
- 3 Find the common ratio of a geometric series which has a first term of 5 and a sum to infinity of 6.
- 4 Find the common ratio of a geometric series which has a first term of 11 and a sum to infinity of 6.
- 5 Find the first term of a geometric series which has a common ratio of $\frac{3}{4}$ and a sum to infinity of 12.

- 6 Find the first term of a geometric series which has a common ratio of $-\frac{3}{5}$ and a sum to infinity of 12.
- 7 In Example 14.3.2 a beetle starts at a point O on the floor. It walks 1 m east, then $\frac{1}{2}$ m west, then $\frac{1}{4}$ m east and so on. It finished $\frac{2}{3}$ m to the east of O . How far did it actually walk?
- 8 A beetle starts at a point O on the floor and walks 0.6 m east, then 0.36 m west, 0.216 m east and so on. Find its final position and how far it actually walks.
- 9 A ‘supa-ball’ is thrown upwards from ground level. It hits the ground after 2 seconds and continues to bounce. The time it is in the air for a particular bounce is always 0.8 of the time for the previous bounce. How long does it take for the ball to stop bouncing?
- 10 A ‘supa-ball’ is dropped from a height of 1 metre onto a level table. It always rises to a height equal to 0.9 of the height from which it was dropped. How far does it travel in total until it stops bouncing?
- 11 A frog sits at one end of a table which is 2 m long. In its first jump the frog goes a distance of 1 m along the table, with its second jump $\frac{1}{2}$ m, with its third jump $\frac{1}{4}$ m and so on.
- What is the frog’s final position?
 - After how many jumps will the frog be within 1 cm of the far end of the table?

14.4 Exponential growth and decay

Many everyday situations are described by geometric sequences. Of the next two examples, the first has a common ratio greater than 1, and the second has a common ratio between 0 and 1.

Example 14.4.1

A person invests \$1000 in a savings bank account which pays interest of 6% annually. Calculate the amount in the account over the next 8 years.

The interest in any year is 0.06 times the amount in the account at the beginning of the year. This is added on to the sum of money already in the account. The amount at the end of each year, after interest has been added, is 1.06 times the amount at the beginning of the year. So

$$\text{Amount after 1 year} = \$1000 \times 1.06 = \$1060$$

$$\text{Amount after 2 years} = \$1060 \times 1.06 = \$1124$$

$$\text{Amount after 3 years} = \$1124 \times 1.06 = \$1191, \text{ and so on.}$$

Continuing in this way, you get the amounts shown in Table 14.6, to the nearest whole number of dollars.

Number of years	0	1	2	3	4	5	6	7	8
Amount (\$)	1000	1060	1124	1191	1262	1338	1419	1504	1594

Table 14.6

These values are shown in Fig. 14.7.

Notice that in the first year the interest is \$60, but in the eighth year it is \$90. This is because the amount on which the 6% is calculated has gone up from \$1000 to \$1504. This is characteristic of **exponential growth**, in which the increase is proportional to the current amount. As the amount goes up, the increase goes up.

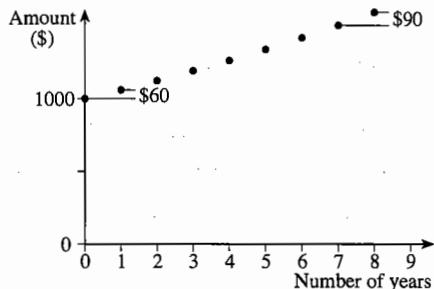


Fig. 14.7

Example 14.4.2

A car cost \$15 000 when new, and each year its value decreases by 20%. Find its value on the first five anniversaries of its purchase.

The value at the end of each year is 0.8 times its value a year earlier. The results of this calculation are given in Table 14.8.

Number of years	0	1	2	3	4	5
Value (\$)	15000	12000	9600	7680	6144	4915

Table 14.8

These values are shown in Fig. 14.9.

The value goes down by \$3000 in the first year, but by only \$1229 in the fifth year, because by then the 20% is calculated on only \$6144 rather than \$15 000. This is characteristic of **exponential decay**, in which the decrease is proportional to the current value. Notice that, if the 20% rule continues, the value never becomes zero however long you keep the car.

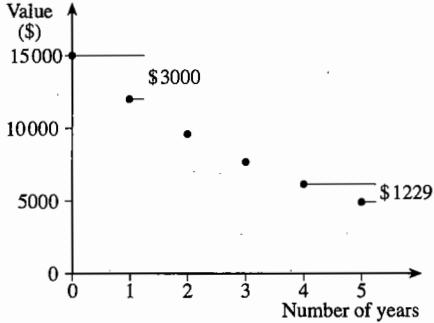


Fig. 14.9

Notice that in both these examples it is more natural to think of the first term of the sequence as u_0 rather than u_1 , so that \$ u_i is the amount in the account, or the value of the car, after i years. The sequence in Example 14.4.1 has

$$u_0 = 1000 \text{ and } u_{i+1} = 1.06u_i \text{ for } 0 \leq i \leq 7.$$

From this you can deduce that $u_1 = 1000 \times 1.06$, $u_2 = 1000 \times 1.06^2$, and more generally $u_i = 1000 \times 1.06^i$. The sequence in Example 14.4.2 has

$$u_0 = 15000 \text{ and } u_{i+1} = 0.8u_i \text{ for } 0 \leq i \leq 4.$$

In this case $u_1 = 15000 \times 0.8$, $u_2 = 15000 \times 0.8^2$ and $u_i = 15000 \times 0.8^i$.

These are both examples of exponential sequences. (The word ‘exponential’ comes from ‘exponent’, which is another word for index. The reason for the name is that the variable i appears in the exponent of the formula for u_i .) An exponential sequence is a special kind of geometric sequence, in which a and r are both positive. If the first term is denoted by u_0 , the sequence can be defined inductively by

$$u_0 = a \quad \text{and} \quad u_{i+1} = ru_i,$$

or by the formula

$$u_i = ar^i.$$

If $r > 1$ the sequence represents exponential growth; if $0 < r < 1$ it represents exponential decay.

It may or may not be useful to find the sum of the terms of an exponential sequence. In Example 14.4.2 there would be no point in adding up the year-end values of the car. But many investment calculations (such as for pensions and mortgages) require the terms of an exponential sequence to be added up. This is illustrated by the next example.

Example 14.4.3

Saria’s grandparents put \$1000 into a savings bank account for her on each birthday from her 10th to her 18th. The account pays interest at 6% for each complete year that the money is invested. How much money is in the account on the day after her 18th birthday?

Start with the most recent deposit. The \$1000 on her 18th birthday has not earned any interest. The \$1000 on her 17th birthday has earned interest for one year, so is now worth $\$1000 \times 1.06 = \1060 . Similarly, the \$1000 on her 16th birthday is worth $\$1000 \times 1.06^2 = \1124 , and so on. So the total amount is now $\$S$, where

$$S = 1000 + 1000 \times 1.06 + 1000 \times 1.06^2 + \dots + 1000 \times 1.06^8.$$

Method 1 The terms of this series are just the amounts calculated in Example 14.4.1. The sum of the nine entries in Table 14.6 is 11 492.

Method 2 The sum is a geometric series with $a = 1000$, $r = 1.06$ and $n = 9$. Using the general formula in the alternative version for $r > 1$,

$$S = \frac{a(r^n - 1)}{r - 1} = \frac{1000(1.06^9 - 1)}{1.06 - 1} = 11\,491.32.$$

There is a small discrepancy between the two answers because the amounts in Table 14.6 were rounded to the nearest dollar. The amount in the account just after Saria’s 18th birthday is \$11 491.32.

Exercise 14C

- 1 Trudy puts \$500 into a savings bank account on the first day of January each year from 2000 to 2010 inclusive. The account pays interest at 5% for each complete year of investment. How much money will there be in the account on 2 January 2010?
- 2 Jayesh invests \$100 in a savings account on the first day of each month for one complete year. The account pays interest at $\frac{1}{2}\%$ for each complete month. How much does Jayesh have invested at the end of the year (but before making a thirteenth payment)?
- 3 Neeta takes out a 25-year mortgage of \$40 000 to buy her house. Compound interest is charged on the loan at a rate of 8% per annum. She has to pay off the mortgage with 25 equal payments, the first of which is to be one year after the loan is taken out. Continue the following argument to calculate the value of each annual payment.
 - After 1 year she owes $(40\ 000 \times 1.08)$ (loan plus interest) less the payment made, P , that is, she owes $(40\ 000 \times 1.08 - P)$.
 - After 2 years she owes $((40\ 000 \times 1.08 - P) \times 1.08 - P)$.
 - After 3 years she owes $(((40\ 000 \times 1.08 - P) \times 1.08 - P) \times 1.08 - P)$.

At the end of the 25 years this (continued) expression must be zero. Form an equation in P and solve it.

- 4 Fatima invests \$100 per month for a complete year, with interest added every month at the rate of $\frac{1}{2}\%$ per month at the end of the month. How much would she have had to invest at the beginning of the year to have the same total amount after the complete year?
- 5 Charles borrows \$6000 for a new car. Compound interest is charged on the loan at a rate of 2% per month. Charles has to pay off the loan with 24 equal monthly payments. Calculate the value of each monthly payment.
- 6 The population of Pascalia is increasing at a rate of 6% each year. On 1 January 1990 it was 35 200. What was its population on
 - (a) 1 January 2000,
 - (b) 1 July 1990,
 - (c) 1 January 1980?
- 7 The population of the United Kingdom in 1971 was 5.5615×10^7 ; by 1992 it was estimated to be 5.7384×10^7 . Assuming a steady exponential growth estimate the population in
 - (a) 2003,
 - (b) 1981.
- 8 The population of Pythagora is decreasing steadily at a rate of 4% each year. The population in 1998 was 21 000. Estimate the population in
 - (a) 2002,
 - (b) 1990.
- 9 A man of mass 90 kg plans to diet and to reduce his mass to 72 kg in four weeks by a constant percentage reduction each day.
 - (a) What should his mass be 1 week after starting his diet?
 - (b) He forgets to stop after 4 weeks. Estimate his mass 1 week later.

- 10** A savings account is opened with a single payment of \$2000. It attracts compound interest at a constant rate of 0.5% per month.
- Find the amount in the account after two complete years.
 - Find, by trial, after how many months the value of the investment will have doubled.
- 11** The Bank of Utopia offers an interest rate of 100% per annum with various options as to how the interest may be added. Gopal invests \$1000 and considers the following options.
- Option A Interest added annually at the end of the year.
- Option B Interest of 50% credited at the end of each half-year.
- Option C, D, E, ... The Bank is willing to add interest as often as required, subject to
 $(\text{interest rate}) \times (\text{number of credits per year}) = 100$.
- Investigate to find the maximum possible amount in Gopal's account after one year.

Miscellaneous exercise 14

- In a geometric progression, the fifth term is 100 and the seventh term is 400. Find the first term.
- A geometric series has first term a and common ratio $\frac{1}{\sqrt{2}}$. Show that the sum to infinity of the series is $a(2 + \sqrt{2})$. (Hint: $(\sqrt{2} - 1)(\sqrt{2} + 1) = 1$.)
- The n th term of a sequence is ar^{n-1} , where a and r are constants. The first term is 3 and the second term is $-\frac{3}{4}$. Find the values of a and r .
 Hence find the sum of the first n terms of the sequence.
- Evaluate, correct to the nearest whole number,

$$0.99 + 0.99^2 + 0.99^3 + \dots + 0.99^{99}.$$
- Find the sum of the infinite series $\frac{1}{10^3} + \frac{1}{10^6} + \frac{1}{10^9} + \dots$, expressing your answer as a fraction in its lowest terms.
 Hence express the infinite recurring decimal 0.108 108 108 ... as a fraction in its lowest terms.
- A geometric series has first term 1 and common ratio r . Given that the sum to infinity of the series is 5, find the value of r .
 Find the least value of n for which the sum of the first n terms of the series exceeds 4.9.
- In a geometric series, the first term is 12 and the fourth term is $-\frac{3}{2}$. Find the sum, S_n , of the first n terms of the series.
 Find the sum to infinity, S_∞ , of the series and the least value of n for which the magnitude of the difference between S_n and S_∞ is less than 0.001.
- A geometric series has non-zero first term a and common ratio r , where $0 < r < 1$. Given that the sum of the first 8 terms of the series is equal to half the sum to infinity, find the value of r , correct to 3 decimal places. Given also that the 17th term of the series is 10, find a .

- 9 An athlete plans a training schedule which involves running 20 km in the first week of training; in each subsequent week the distance is to be increased by 10% over the previous week. Write down an expression for the distance to be covered in the n th week according to this schedule, and find in which week the athlete would first cover more than 100 km.
- 10 At the beginning of 1990, an investor decided to invest \$6000, believing that the value of the investment should increase, on average, by 6% each year. Show that, if this percentage rate of increase was in fact maintained for 10 years, the value of the investment will be about \$10 745.
- The investor added a further \$6000 at the beginning of each year between 1991 and 1995 inclusive. Assuming that the 6% annual rate of increase continues to apply, show that the total value, in dollars, of the investment at the beginning of the year 2000 may be written as $6000(1.06^5 + 1.06^6 + \dots + 1.06^{10})$ and evaluate this, correct to the nearest dollar.
- 11 A post is being driven into the ground by a mechanical hammer. The distance it is driven by the first blow is 8 cm. Subsequently, the distance it is driven by each blow is $\frac{9}{10}$ of the distance it was driven by the previous blow.
- The post is to be driven a total distance of at least 70 cm into the ground. Find the smallest number of blows needed.
 - Explain why the post can never be driven a total distance of more than 80 cm into the ground.
- 12 When a table-tennis ball is dropped vertically on to a table, the time interval between any particular bounce and the next bounce is 90% of the time interval between that particular bounce and the preceding bounce. The interval between the first and second bounces is 2 seconds. Given that the interval between the n th bounce and the $(n+1)$ th bounce is the first such interval less than 0.02 seconds, find n . Also find the total time from the first bounce to the n th bounce, giving 3 significant figures in your answer.
- 13 An investment of \$100 in a savings scheme is worth \$150 after 5 years. Calculate as a percentage the annual rate of interest which would give this figure.
- 14 A geometric series G has positive first term a , common ratio r and sum to infinity S . The sum to infinity of the even-numbered terms of G (the second, fourth, sixth, ... terms) is $-\frac{1}{2}S$. Find the value of r .
- Given that the third term of G is 2, show that the sum to infinity of the odd-numbered terms of G (the first, third, fifth, ... terms) is $\frac{81}{4}$.
 - In another geometric series H , each term is the modulus of the corresponding term of G . Show that the sum to infinity of H is $2S$.
- 15 An infinite geometric series has first term a and sum to infinity b , where $b \neq 0$. Prove that a lies between 0 and $2b$.
- 16 The sum of the infinite geometric series $1+r+r^2+\dots$ is k times the sum of the series $1-r+r^2-\dots$, where $k > 0$. Express r in terms of k .

- 17** A person wants to borrow \$100 000 to buy a house. He intends to pay back a fixed sum of \$ C at the end of each year, so that after 25 years he has completely paid off the debt. Assuming a steady interest rate of 4% per year, explain why

$$100\ 000 = C \left(\frac{1}{1.04} + \frac{1}{1.04^2} + \frac{1}{1.04^3} + \dots + \frac{1}{1.04^{25}} \right).$$

Calculate the value of C .

- 18** A person wants to buy a pension which will provide her with an income of \$10 000 at the end of each of the next n years. Show that, with a steady interest rate of 5% per year, the pension should cost her

$$\$10\ 000 \left(\frac{1}{1.05} + \frac{1}{1.05^2} + \frac{1}{1.05^3} + \dots + \frac{1}{1.05^n} \right).$$

Find a simple formula for calculating this sum, and find its value when $n = 10, 20, 30, 40, 50$.

- 19** Find the sum of the geometric series

$$(1-x) + (x^3 - x^4) + (x^6 - x^7) + \dots + (x^{3n} - x^{3n+1}).$$

Hence show that the sum of the infinite series $1 - x + x^3 - x^4 + x^6 - x^7 + \dots$ is equal to $\frac{1}{1+x+x^2}$, and state the values of x for which this is valid.

Use a similar method to find the sum of the infinite series $1 - x + x^5 - x^6 + x^{10} - x^{11} + \dots$

- 20** Find the sums of the infinite geometric series

(a) $\sin^2 x^\circ + \sin^4 x^\circ + \sin^6 x^\circ + \sin^8 x^\circ + \dots$,

(b) $1 - \tan^2 x^\circ + \tan^4 x^\circ - \tan^6 x^\circ + \tan^8 x^\circ - \dots$,

giving your answers in as simple a form as possible. For what values of x are your results valid?

- 21** Use the formula to sum the geometric series $1 + (1+x) + (1+x)^2 + \dots + (1+x)^6$ when $x \neq 0$. By considering the coefficients of x^2 , deduce that

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} + \binom{6}{2} = \binom{7}{3}.$$

Illustrate this result on a Pascal triangle.

Write down and prove a general result about binomial coefficients, of which this is a special case.

- 22** Make tables of values of $1+x$, $1+x+x^2$, $1+x+x^2+x^3$, $1+x+x^2+x^3+x^4$ and $\frac{1}{1-x}$ and use them to draw graphs of these functions of x for $-1.5 \leq x \leq 1.5$.

What do your graphs suggest about the possibility of using the polynomial

$$1+x+x^2+x^3+\dots+x^n$$
 as an approximation to the function $\frac{1}{1-x}$?

15 Second derivatives

This chapter extends the idea of differentiation further. When you have completed it, you should

- understand the significance of the second derivative for the shape of graphs and in real-world applications
- be able to use second derivatives where appropriate to distinguish minimum and maximum points
- understand that at a point of inflexion the second derivative is zero.

15.1 Interpreting and sketching graphs

The results in Chapter 7, linking features of the graph of a function with values of the derivative, were restricted to functions which are continuous within their domains. These results used the idea that the derivative doesn't just measure the gradient at a particular point of a graph, but could itself be regarded as a function.

In this chapter a further restriction needs to be made, to functions which are 'smooth'; that is, functions whose graphs do not have sudden changes of direction. This means that, with a function such as $x^{\frac{3}{2}}(1-x)$ (from Example 7.2.3), you must exclude the 'awkward' point (the origin in this example) from the domain.

The 'smooth' condition means that the derivative, considered as a function, is continuous and can itself be differentiated. The result is called the **second derivative** of the function, and it is denoted by $f''(x)$. It is sometimes called the 'second order derivative'. If you are using the $\frac{dy}{dx}$ notation, the second derivative is written as $\frac{d^2y}{dx^2}$. (The reason for this rather curious symbol is explained in Section 15.5.)

Example 15.1.1

In the graph of $y = f(x) = x^3 - 3x^2$, identify the intervals in which $f(x)$, $f'(x)$ and $f''(x)$ are positive, and interpret these graphically.

$$\frac{dy}{dx} = f'(x) = 3x^2 - 6x, \quad \text{and} \quad \frac{d^2y}{dx^2} = f''(x) = 6x - 6.$$

Fig. 15.1 shows the graphs of the function and its first and second derivatives.

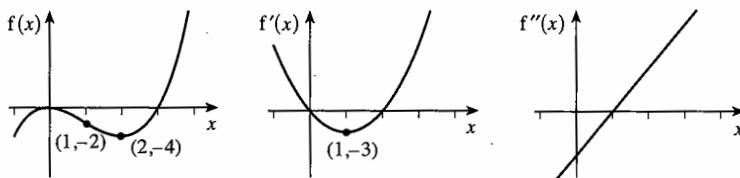


Fig. 15.1

Notice first that $f(x) = x^2(x - 3)$, so that $f(x) > 0$ when $x > 3$. These are the values of x for which the graph of $f(x)$ lies above the x -axis.

Since $f'(x) = 3x(x - 2)$, $f'(x) > 0$ when $x < 0$ or $x > 2$. In the graph of $f(x)$, the gradient is positive in these intervals, so that $f(x)$ is increasing.

Lastly, $f''(x) = 6(x - 1)$, so that $f''(x) > 0$ when $x > 1$. It appears that this is the interval in which the graph of $f(x)$ can be described as bending upwards.

To make this idea of 'bending upwards' more precise, it is helpful to use the letter g to denote the gradient of the graph on the left of Fig. 15.1, so that $g = f'(x)$. Then $f''(x) = \frac{dg}{dx}$, which is the rate of change of the gradient with respect to x . In an interval where $f''(x) > 0$, the gradient increases as x increases.

This can be seen in the middle graph of Fig. 15.1, which is a quadratic graph with its vertex at $(1, -3)$. So the gradient of the graph on the left increases from a value of -3 at the point $(1, -2)$, through zero at the minimum point $(2, -4)$ and then becomes positive and continues to increase when $x > 2$.

Fig. 15.2 shows three curves which would be described as bending upwards, for which $f''(x) > 0$, and three bending downwards for which $f''(x) < 0$. The important thing to notice is that this property does not depend on the sign of the gradient. A curve can bend upwards whether its gradient is positive, negative or zero.

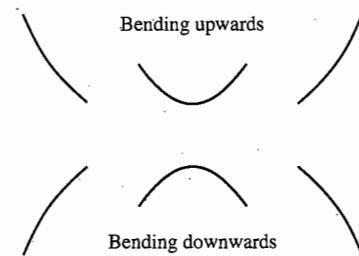


Fig. 15.2

Example 15.1.2

Investigate the graph of $y = f(x)$, where $f(x) = \frac{1}{x} - \frac{1}{x^2}$ with domain $x > 0$.

You can write $f(x)$ either as $\frac{x-1}{x^2}$ or, with negative indices, as $x^{-1} - x^{-2}$. So

$$f'(x) = -x^{-2} + 2x^{-3} = -\frac{1}{x^2} + \frac{2}{x^3} = \frac{-x+2}{x^3},$$

$$\text{and } f''(x) = 2x^{-3} - 6x^{-4} = \frac{2}{x^3} - \frac{6}{x^4} = \frac{2(x-3)}{x^4}.$$

It follows that, in the given domain,

$$f(x) < 0 \text{ for } x < 1 \quad \text{and} \quad f(x) > 0 \text{ for } x > 1;$$

$$f'(x) > 0 \text{ for } x < 2 \quad \text{and} \quad f'(x) < 0 \text{ for } x > 2;$$

$$f''(x) < 0 \text{ for } x < 3 \quad \text{and} \quad f''(x) > 0 \text{ for } x > 3.$$

So the graph lies below the x -axis when $0 < x < 1$ and above it when $x > 1$, crossing the axis at $(1, 0)$. It has positive gradient when $0 < x < 2$ and negative gradient for $x > 2$, with a maximum point at $(2, \frac{1}{4})$. And the graph bends

downwards for $0 < x < 3$ and upwards for $x > 3$.

This is enough information to give a good idea of the shape of the graph for values of x in an interval covering the critical values $x = 1, 2$ and 3 , but to complete the investigation it would be helpful to know more about the graph for very small and very large values of x . This suggests calculating, say,

$$f(0.01) = 100 - 10000 = -9900 \quad \text{and} \quad f(100) = 0.01 - 0.0001 = 0.0099.$$

So when x is small, y is a negative number with large modulus; and when x is large, y is a small positive number.

Try to sketch the graph for yourself using the information found in the example. If you have access to a graphic calculator use it to check your sketch.

The skill in sketching a graph is to work out the coordinates of only those points where something significant occurs. Example 15.1.2 draws attention to the point $(1, 0)$, where the graph crosses the x -axis, and to the maximum point $(2, \frac{1}{4})$. Another interesting point is $(3, \frac{2}{9})$, where the graph changes from bending downwards to bending upwards. Notice that $f''(x)$ changes from $-$ to $+$ at this point, and that $f''(3) = 0$.

A point of a graph which separates a part of the curve which bends one way from a part which bends the other way is called a **point of inflection** of the graph. If $(p, f(p))$ is a point of inflection of the graph of a smooth function, $f''(p) = 0$.

15.2 Second derivatives in practice

There are many real-world situations in which second derivatives are important, because they give advance warning of future trends.

For example, the number of households possessing a computer has been increasing for a long time. Manufacturers will estimate the number of such households, H , in year t , and note that the graph of H against t has a positive gradient $\frac{dH}{dt}$. But to plan ahead they need to know whether this rate of increase is itself increasing (so that they should increase production of models for first-time users) or decreasing (in which case they might target existing customers to upgrade their equipment). So it is the value of $\frac{d^2H}{dt^2}$ which affects such decisions.

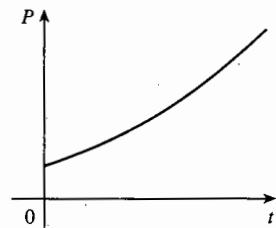
Similarly, a weather forecaster observing the pressure p at time t may not be too concerned if $\frac{dp}{dt}$ is negative; but if she also notices that $\frac{d^2p}{dt^2}$ is negative, it may be time to issue a warning of severe weather.

Exercise 15A

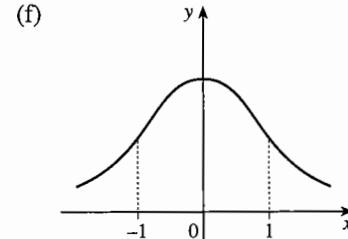
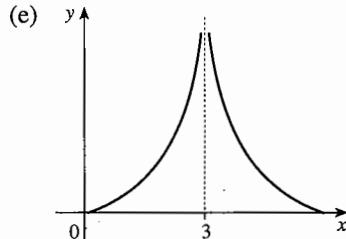
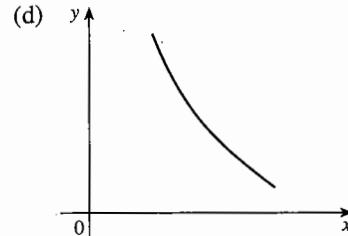
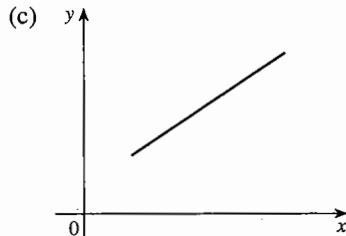
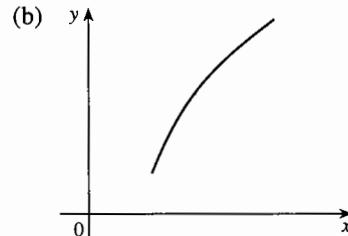
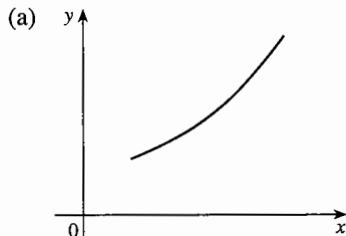
In this exercise try to sketch the graphs using information about the first and second derivatives. When you have drawn your sketch, check it from a graphic calculator or computer display if you have one available.

- 1 Consider the graph of $y = f(x)$ where $f(x) = x^3 - x$.
 - (a) Use the fact that $f(x) = x(x^2 - 1) = x(x-1)(x+1)$ to find where the graph cuts the x -axis and hence sketch the graph.
 - (b) Find $f'(x)$ and sketch the graph of $y = f'(x)$.
 - (c) Find $f''(x)$ and sketch the graph of $y = f''(x)$.
 - (d) Check the consistency of your sketches: for example, check that the graph of $y = f(x)$ is bending upwards where $f''(x) \geq 0$.
- 2 For the graph of $y = x^3 + x$
 - (a) use factors to show that the graph crosses the x -axis once only;
 - (b) find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$;
 - (c) find the interval in which the graph is bending upwards;
 - (d) use the information gained to sketch the graph of $y = x^3 + x$.
- 3 Use information about $f'(x)$ and $f''(x)$ to sketch the graph of $y = f(x)$, where $f(x) = x^3 - 3x^2 + 3x - 9$. (Note that $x^3 - 3x^2 + 3x - 9 = (x-3)(x^2 + 3)$.)
- 4 Sketch the graphs of the following, giving the coordinates of any points at which
 - (i) $\frac{dy}{dx} = 0$,
 - (ii) $\frac{d^2y}{dx^2} = 0$.

(a) $y = x^4 - 4x^2$	(b) $y = x^3 + x^2$	(c) $y = x + \frac{1}{x}$
(d) $y = x - \frac{1}{x}$	(e) $y = x + \frac{4}{x^2}$	(f) $y = x - \frac{4}{x^2}$
- 5 (a) This graph shows prices (P) plotted against time (t).
 The rate of inflation, measured by $\frac{dP}{dt}$, is increasing. What does $\frac{d^2P}{dt^2}$ represent and what can be said about its value?
 (b) Sketch a graph showing that prices are increasing but that the rate of inflation is slowing down with an overall increase tending to 20%.

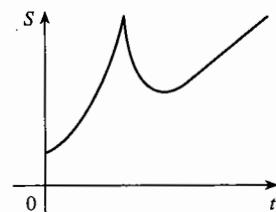


- 6 Write down the signs of $f'(x)$ and $f''(x)$ for the following graphs of $y = f(x)$. In parts (e) and (f) you will need to state the relevant intervals.



- 7 The graph shows the price S of shares in a certain company.

- (a) For each stage of the graph, comment on $\frac{dS}{dt}$ and $\frac{d^2S}{dt^2}$.
- (b) Describe what happened in non-technical language.



- 8 Colin sets off for school, which is 800 m from home. His speed is proportional to the distance he still has to go. Let x metres be the distance he has gone, and y metres be the distance that he still has to go.

- (a) Sketch graphs of x against t and y against t .
- (b) What are the signs of $\frac{dx}{dt}$, $\frac{d^2x}{dt^2}$, $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$?

- 9 The rate of decay of a radioactive substance is proportional to the number, N , of radioactive atoms present at time t .

- (a) Write an equation representing this information.
- (b) Sketch a graph of N against t .
- (c) What is the sign of $\frac{d^2N}{dt^2}$?

10 Sketch segments of graphs of $y = f(x)$ in each of the following cases.

(For example, in (a), you can only sketch the graph near the y -axis because you have no information for other values of x .)

- | | |
|--|--|
| (a) $f(0) = 3, \quad f'(0) = 2, \quad f''(0) = 1$ | (b) $f(5) = -2, \quad f'(5) = -2, \quad f''(5) = -2$ |
| (c) $f(0) = -3, \quad f'(0) = 0, \quad f''(0) = 3$ | |

15.3 Minima and maxima revisited

In the last exercise you will sometimes have found that different pieces of information reinforce each other. This is especially true at points where a graph has a minimum or maximum. If you have identified a minimum from changes in the sign of $f'(x)$, you will also have found from $f''(x)$ that the graph is bending upwards.

The curves in Fig. 15.2 suggest a general result:

If $f'(q) = 0$ and $f''(q) > 0$, then $f(x)$ has a minimum at $x = q$.

If $f'(q) = 0$ and $f''(q) < 0$, then $f(x)$ has a maximum at $x = q$.

It is often simpler to use this instead of considering the change in sign of $f'(x)$ to decide whether a point on a graph is a minimum or a maximum. The procedure described in Section 7.3 can then be amended as follows.

To find the minimum and maximum points on the graph of $y = f(x)$:

Step 1 Decide the domain in which you are interested.

Step 2 Find an expression for $f'(x)$.

Step 3 List the values of x in the domain for which $f'(x)$ is 0.
(If there are values where $f'(x)$ is undefined, use the old procedure, in Section 7.3.)

Step 4 Find an expression for $f''(x)$.

Step 5 For each value of x in Step 3, find the sign of $f''(x)$. If the sign is +, the graph has a minimum point; if -, a maximum.
(If the value of $f''(x)$ is 0, follow the old procedure.)

Step 6 For each value of x which gives a minimum or maximum, calculate $f(x)$.

Notice that there are two ways in which this procedure can break down.

First, the method only works for the graphs of smooth functions, so that it does not apply at points where $f'(x)$ is undefined.

Secondly, if $f'(q) = 0$ and $f''(q) = 0$, it is possible for $f(x)$ to have a minimum, or a maximum, or neither, at $x = q$. This can be shown by comparing $f(x) = x^3$ with $g(x) = x^4$ at $x = 0$. You can easily check that $f'(0) = f''(0) = 0$ and that $g'(0) = g''(0) = 0$. But $g(x)$ has a minimum at $x = 0$, whereas $f(x)$ has neither a minimum nor a maximum. (In fact the graph of $y = f(x)$ has a point of inflection at the origin, since $f''(x) = 6x$, which is negative when $x < 0$ and positive when $x > 0$.)

You will also find later on that for some functions it can be very laborious to find the second derivative. In that case, it is more efficient to use the old procedure.

Example 15.3.1

Find the minimum and maximum points on the graph of $f(x) = x^4 + x^5$.

Step 1 The function is defined for all real numbers.

Step 2 $f'(x) = 4x^3 + 5x^4 = x^3(4 + 5x)$.

Step 3 $f'(x) = 0$ when $x = 0$ or $x = -0.8$.

Step 4 $f''(x) = 12x^2 + 20x^3 = 4x^2(3 + 5x)$.

Step 5 $f''(-0.8) = 4 \times (-0.8)^2 \times (3 - 4) < 0$, so $x = -0.8$ gives a maximum.

$f''(0) = 0$, so follow the old procedure. For $-0.8 < x < 0$, $x^3 < 0$ and $4 + 5x > 0$, so $f'(x) < 0$; for $x > 0$, $f'(x) > 0$. So $x = 0$ gives a minimum.

Step 6 The maximum point is $(-0.8, 0.08192)$; the minimum point is $(0, 0)$.

Example 15.3.2

Find the minimum and maximum points on the graph of $y = \frac{(x+1)^2}{x}$.

The function is defined for all real numbers except 0.

To differentiate, write $\frac{(x+1)^2}{x}$ as $\frac{x^2 + 2x + 1}{x} = x + 2 + x^{-1}$.

Then $\frac{dy}{dx} = 1 - x^{-2} = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}$, so $\frac{dy}{dx} = 0$ gives $x^2 - 1 = 0$, or $x = \pm 1$.

The second derivative is $\frac{d^2y}{dx^2} = 2x^{-3} = \frac{2}{x^3}$. This has values -2 when $x = -1$, and 2 when $x = 1$. So $(-1, 0)$ is a maximum point and $(1, 4)$ is a minimum point.

The minimum value is greater than the maximum value. How can this happen?

Exercise 15B

Use first and second derivatives to locate and describe the stationary points on the graphs of the following functions and equations. If this method fails, then use the change of sign of $\frac{dy}{dx}$ or $f'(x)$ to distinguish maxima, minima and points of inflexion.

- | | | |
|----------|--|-------------------------------------|
| 1 | (a) $f(x) = 3x - x^3$ | (b) $f(x) = x^3 - 3x^2$ |
| | (c) $f(x) = 3x^4 + 1$ | (d) $f(x) = 2x^3 - 3x^2 - 12x + 4$ |
| | (e) $f(x) = \frac{2}{x^4} - \frac{1}{x}$ | (f) $f(x) = x^2 + \frac{1}{x^2}$ |
| | (g) $f(x) = \frac{1}{x} - \frac{1}{x^2}$ | (h) $f(x) = 2x^3 - 12x^2 + 24x + 6$ |
| 2 | (a) $y = 3x^4 - 4x^3 - 12x^2 - 3$ | (b) $y = x^3 - 3x^2 + 3x + 5$ |
| | (c) $y = 16x - 3x^3$ | (d) $y = \frac{4}{x^2} - x$ |
| | (e) $y = \frac{4+x^2}{x}$ | (f) $y = \frac{x-3}{x^2}$ |
| | (g) $y = 2x^5 - 7$ | (h) $y = 3x^4 - 8x^3 + 6x^2 + 1$ |

15.4 Logical distinctions

You have seen that, for the graphs of smooth functions, it is true that

if $(q, f(q))$ is a minimum or maximum point, then $f'(q) = 0$;

but the **converse** statement, that

if $f'(q) = 0$, then $(q, f(q))$ is a minimum or maximum point,

is false.

You can show that it is false by finding a **counterexample**; that is, an example of a function for which the ‘if ...’ part of the statement holds, but the ‘then ...’ part does not.

Such a function is $f(x) = x^3$ with $q = 0$. Since $f'(x) = 3x^2$, $f'(0) = 0$, but $(0,0)$ is not a minimum or maximum point of the graph of $y = x^3$.

A similar situation arises with points of inflexion. For the graphs of smooth functions it is true that

if $(p, f(p))$ is a point of inflexion, then $f''(p) = 0$;

but the converse, that

if $f''(p) = 0$, then $(p, f(p))$ is a point of inflexion,

is false.

A suitable counterexample in this case is $f(x) = x^4$ with $x = 0$. Since $f''(x) = 12x^2$, $f''(0) = 0$, but $(0,0)$ is a minimum point on the graph of $y = x^4$, not a point of inflection.

Much of advanced mathematics involves applying general theorems to particular functions. There are many theorems (such as Pythagoras' theorem) whose converses are also true. But if, as in the examples above, the converse of a theorem is false, it is very important to be sure that you are applying the (true) theorem rather than its (false) converse.

15.5 Extending $\frac{dy}{dx}$ notation

Although $\frac{dy}{dx}$ is a symbol which should not be split into smaller bits, it can usefully be adapted by separating off the y , as $\frac{d}{dx}y$, so that if $y = f(x)$, you can write

$$f'(x) = \frac{d}{dx} f(x).$$

This can be used as a convenient shorthand. For example, instead of having to write

$$\text{if } y = x^4, \text{ then } \frac{dy}{dx} = 4x^3$$

you can abbreviate this to

$$\frac{d}{dx}x^4 = 4x^3.$$

You can think of $\frac{d}{dx}$ as an instruction to differentiate whatever comes after it.

You may have seen calculators which do algebra as well as arithmetic. With these, you can input a function such as x^4 , key in 'differentiate', and the output $4x^3$ appears in the display. The symbol $\frac{d}{dx}$, sometimes called the **differential operator**, is the equivalent of pressing the 'differentiate' key.

This explains the notation used for the second derivative, which is what you get by differentiating $\frac{dy}{dx}$; that is, $\frac{d}{dx}\frac{dy}{dx}$. If you collect the elements of this expression into a single symbol, the top line becomes d^2y , and the bottom line $(dx)^2$. Dropping the brackets, this takes the form $\frac{d^2y}{dx^2}$.

15.6* Higher derivatives

There is no reason to stop at the second derivative. Since $\frac{d^2y}{dx^2}$ is also a function, provided it is smooth it can be differentiated to give a third derivative; and the process can continue indefinitely, giving a whole sequence of higher derivatives

$$\frac{d^3y}{dx^3}, \quad \frac{d^4y}{dx^4}, \quad \frac{d^5y}{dx^5}, \quad \dots$$

In function notation these are written as

$$f'''(x), \quad f^{(4)}(x), \quad f^{(5)}(x), \quad \dots$$

Notice that, from the fourth derivative onwards, the dashes are replaced by a small numeral in brackets.

These further derivatives do not often have useful interpretations in graph sketching or in real-world applications. But they are important in some applications, for example in finding approximations and for expressing functions in series form.

Exercise 15C*

1 Find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$ and $\frac{d^4y}{dx^4}$ for the following.

- | | | |
|------------------------|----------------------------------|---------------------------|
| (a) $y = x^2 + 3x - 7$ | (b) $y = 2x^3 + x + \frac{1}{x}$ | (c) $y = x^4 - 2$ |
| (d) $y = \sqrt{x}$ | (e) $y = \frac{1}{\sqrt{x}}$ | (f) $y = x^{\frac{1}{4}}$ |

2 Find $f'(x)$, $f''(x)$, $f'''(x)$ and $f^{(4)}(x)$ for the following.

- | | | |
|---------------------------|------------------------------|------------------------------|
| (a) $f(x) = x^2 - 5x + 2$ | (b) $f(x) = 2x^5 - 3x^2$ | (c) $f(x) = \frac{1}{x^4}$ |
| (d) $f(x) = x^2(3 - x^4)$ | (e) $f(x) = x^{\frac{3}{4}}$ | (f) $f(x) = x^{\frac{3}{8}}$ |

3 Find $\frac{d^n y}{dx^n}$ for $y = x^n$ in the case where n is a positive integer.

4 Find an expression for $\frac{d^n y}{dx^n}$ for $y = x^{n+2}$ where n is a positive integer.

5 Find $\frac{d^n y}{dx^n}$ where $y = x^m$ in the case where m is a positive integer and $n > m$.

Miscellaneous exercise 15

1 Find the maximum and minimum values of $x^3 - 6x^2 + 9x + 6$, showing carefully how you determine which is which.

2 Find any maximum and minimum values of the function $f(x) = 16x + \frac{1}{x^2}$, indicating how you decide whether they are maxima or minima.

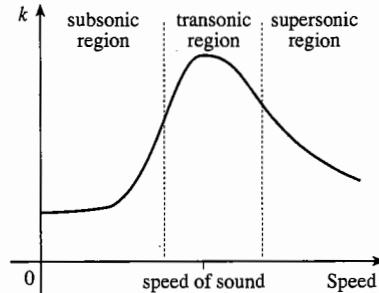
3 Find any maximum and minimum values of the function $f(x) = \sqrt{x} + \sqrt{30 - 5x}$, and give the corresponding values of x .

4 Find the coordinates of the maximum and minimum points on the graph of $y = \frac{1}{x} + \frac{1}{1-4x}$.

- 5 The rate at which Nasreen's coffee cools is proportional to the difference between its temperature, θ° , and room temperature, α° . Sketch a graph of θ against t given that $\alpha = 20$ and that $\theta = 95$ when $t = 0$. State the signs of θ , $\frac{d\theta}{dt}$ and $\frac{d^2\theta}{dt^2}$ for $t > 0$.

- 6 Aeroplanes in flight experience a resistance known as drag. For a particular aeroplane at low speeds the drag is equal to kS^2 , where k is the (constant) drag coefficient and S is the speed of the aeroplane.

At high speeds, however, k increases with speed, and a typical graph of k against S is shown here. (The transonic region is commonly known as the 'sound barrier'.)



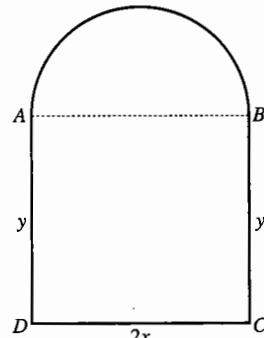
- (a) Give the signs of $\frac{dk}{dS}$ and $\frac{d^2k}{dS^2}$ for each of the three sections of the graph and, in particular, say where each is zero.
 (b) Where is k changing most rapidly?
 (c) What does the graph imply about k at even higher speeds?

- 7 A window consists of a lower rectangular part $ABCD$ of width $2x$ metres and height y metres and an upper part which is a semicircle of radius x metres on AB as diameter, as shown in the diagram.

The perimeter of the window is 10 metres.

Find an expression in terms of x and π for the total area of the window, and find the value of x for which the area is a

maximum. Use the value of $\frac{d^2y}{dx^2}$ to verify that the area is a maximum for this value of x .



- 8 Investigate the maxima and minima of the following functions, where $a > 0$.
 (a) $x^2(x-a)$ (b) $x^3(x-a)$ (c) $x^2(x-a)^2$ (d) $x^3(x-a)^2$
- Make a conjecture about $x^n(x-a)^m$.

- 9 Find an expression for $f^{(n)}(x)$ where

(a) $f(x) = \frac{1}{x^3}$, (b) $f(x) = \sqrt{x}$.

- 10* Find the coordinates of any points of inflection on the curves with equations

(a) $y = x^4 - 8x^3 + 18x^2 + 4$, (b) $y = x^2 - \frac{1}{x} + 2$.

16 Integration

Integration is the reverse process of differentiation. When you have completed this chapter, you should

- understand the term ‘indefinite integral’ and the need to add an arbitrary constant
- be able to integrate functions which can be expressed as sums of powers of x , and be aware of any exceptions
- know how to find the equation of a graph given its derivative and a point on the graph
- know how to evaluate a definite integral
- be able to use definite integrals to find areas.

16.1 Finding a function from its derivative

It was shown in Chapter 7 that some features of the graph of a function can be interpreted in terms of the graph of its derived function.

Suppose now that you know the graph of the derived function. What does this tell you about the graph of the original function?

It is useful to begin by trying to answer this question geometrically. Fig. 16.1 shows the graph of the derived function $f'(x)$ of some function. The problem is to sketch the graph of $f(x)$. Scanning the domain from left to right, you can see that:

For $x < 1$ the gradient is negative, so $f(x)$ is decreasing.

At $x = 1$ the gradient changes from $-$ to $+$, so $f(x)$ has a minimum.

For $1 < x < 3$ the gradient is positive, so $f(x)$ is increasing. Notice that the gradient is greatest when $x = 2$, so that is where the graph climbs most steeply.

At $x = 3$ the gradient changes from $+$ to $-$, so $f(x)$ has a maximum.

For $x > 3$ the gradient is negative, so $f(x)$ is again decreasing.

Using this information you can make a sketch like Fig. 16.2, which gives an idea of the shape of the graph of $f(x)$. But there is no way of deciding precisely where the graph is located. You could translate it in the y -direction by any amount, and it would still have the same gradient $f'(x)$. So there is no unique answer to the problem; there are many functions $f(x)$ with the given derived function.

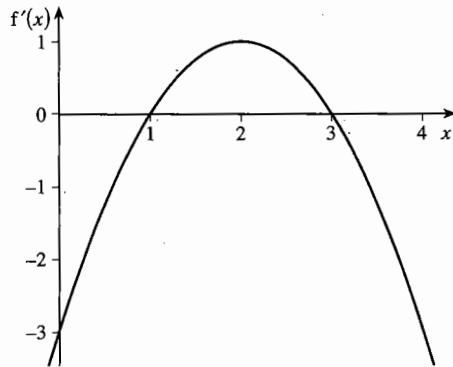


Fig. 16.1

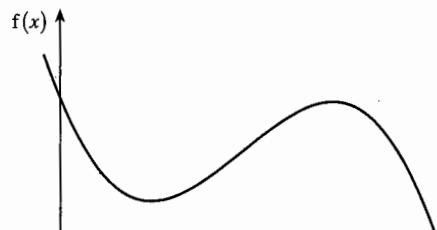


Fig. 16.2

This can be shown algebraically. The graph in Fig. 16.1 comes from the equation

$$f'(x) = (x - 1)(3 - x) = 4x - x^2 - 3.$$

What function has this expression as its derivative? The key is to note that in differentiating x^n the index decreases by 1, from n to $n-1$. So to reverse the process the index must go up by 1. The three terms $4x$, $-x^2$ and -3 must therefore come from multiples of x^2 , x^3 and x . These functions have derivatives $2x$, $3x^2$ and 1, so to get the correct coefficients in $f'(x)$ you have to multiply by 2 , $-\frac{1}{3}$ and -3 . One possible answer is therefore

$$f(x) = 2x^2 - \frac{1}{3}x^3 - 3x.$$

But, as argued above, this is only one of many possible answers. You can translate the graph of $f(x)$ in the y -direction by any amount k without changing its gradient. This is because the derivative of any constant k is zero. So the complete solution to the problem is

$$f(x) = 2x^2 - \frac{1}{3}x^3 - 3x + k \quad \text{for any constant } k.$$

The process of getting from $f'(x)$ to $f(x)$ is called **integration**, and the general expression for $f(x)$ is called the **indefinite integral** of $f'(x)$. Integration is the reverse process of differentiation.

The indefinite integral always includes an added constant k , which is called an **arbitrary constant**. The word ‘arbitrary’ means that, in any application, you can choose its value to fit some extra condition; for example, you can make the graph of $y = f(x)$ go through some given point.

It is easy to find a rule for integrating functions which are powers of x . Because differentiation reduces the index by 1, integration must increase it by 1. So the function x^n must be derived from some multiple of x^{n+1} . But the derivative of x^{n+1} is $(n+1)x^n$; so to reduce the coefficient in the derivative to 1 you have to multiply x^{n+1} by $\frac{1}{n+1}$. The rule is therefore that one integral of x^n is $\frac{1}{n+1}x^{n+1}$.

But notice an important exception to this rule. The formula has no meaning if $n+1$ is 0, so it does not give the integral of x^{-1} , or $\frac{1}{x}$. You will find in P2 that the integral of $\frac{1}{x}$ is not a power of x , but a quite different kind of function.

The extension to functions which are sums of powers of x then follows from the equivalent rules for differentiation:

The indefinite integral of a function made up of the sum of multiples of x^n , where $n \neq -1$, is the corresponding sum of multiples of $\frac{1}{n+1}x^{n+1}$, together with an added arbitrary constant.

Example 16.1.1

The graph of $y = f(x)$ passes through $(2, 3)$, and $f'(x) = 6x^2 - 5x$. Find its equation.

The indefinite integral is $6\left(\frac{1}{3}x^3\right) - 5\left(\frac{1}{2}x^2\right) + k$, so the graph has equation

$$y = 2x^3 - \frac{5}{2}x^2 + k$$

for some constant k . The coordinates $x = 2$, $y = 3$ have to satisfy this equation, so

$$3 = 2 \times 8 - \frac{5}{2} \times 4 + k, \text{ giving } k = 3 - 16 + 10 = -3.$$

The equation of the graph is therefore $y = 2x^3 - \frac{5}{2}x^2 - 3$.

Example 16.1.2

A gardener is digging a plot of land. As he gets tired he works more slowly; after t minutes he is digging at a rate of $\frac{2}{\sqrt{t}}$ square metres per minute. How long will it take him to dig an area of 40 square metres?

Let A square metres be the area he has dug after t minutes. Then his rate of digging is measured by the derivative $\frac{dA}{dt}$. So you know that $\frac{dA}{dt} = 2t^{-\frac{1}{2}}$, in this case $n = -\frac{1}{2}$, so $n+1 = \frac{1}{2}$ and the indefinite integral is

$$A = 2\left(\frac{1}{\frac{1}{2}}t^{\frac{1}{2}}\right) + k = 4\sqrt{t} + k.$$

To find k , you need to know a pair of values of A and t . Since $A = 0$ when he starts to dig, which is when $t = 0$, $0 = 4\sqrt{0} + k$ and so $k = 0$.

The equation connecting A with t is therefore $A = 4\sqrt{t}$.

To find how long it takes to dig 40 square metres, substitute $A = 40$:

$$40 = 4\sqrt{t}, \text{ so that } \sqrt{t} = 10, \text{ and hence } t = 100.$$

It will take him 100 minutes to dig an area of 40 square metres.

Exercise 16A

- 1** Find a general expression for the function $f(x)$ in each of the following cases.
- | | |
|--|---|
| (a) $f'(x) = 4x^3$
(c) $f'(x) = 2x$
(e) $f'(x) = 10x^9 - 8x^7 - 1$ | (b) $f'(x) = 6x^5$
(d) $f'(x) = 3x^2 + 5x^4$
(f) $f'(x) = -7x^6 + 3x^2 + 1$ |
|--|---|

2 Find a general expression for the function $f(x)$ in each of the following cases.

(a) $f'(x) = 9x^2 - 4x - 5$

(b) $f'(x) = 12x^2 + 6x + 4$

(c) $f'(x) = 7$

(d) $f'(x) = 16x^3 - 6x^2 + 10x - 3$

(e) $f'(x) = 2x^3 + 5x$

(f) $f'(x) = x + 2x^2$

(g) $f'(x) = 2x^2 - 3x - 4$

(h) $f'(x) = 1 - 2x - 3x^2$

3 Find y in terms of x in each of the following cases.

(a) $\frac{dy}{dx} = x^4 + x^2 + 1$

(b) $\frac{dy}{dx} = 7x - 3$

(c) $\frac{dy}{dx} = 2x^2 + x - 8$

(d) $\frac{dy}{dx} = 6x^3 - 5x^2 + 3x + 2$

(e) $\frac{dy}{dx} = \frac{2}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{3}x + \frac{1}{6}$

(f) $\frac{dy}{dx} = \frac{1}{2}x^3 - \frac{1}{3}x^2 + x - \frac{1}{3}$

(g) $\frac{dy}{dx} = x - 3x^2 + 1$

(h) $\frac{dy}{dx} = x^3 + x^2 + x + 1$

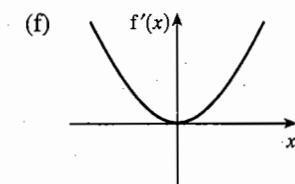
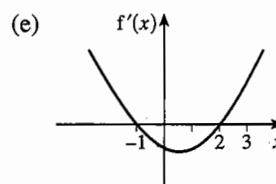
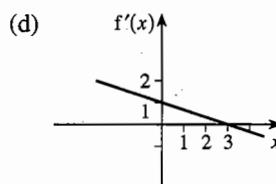
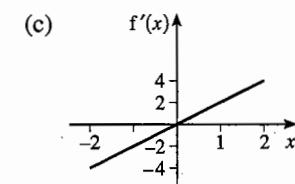
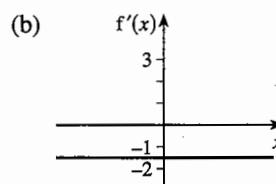
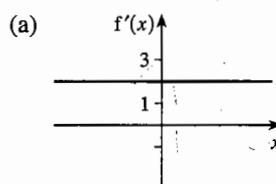
4 The graph of $y = f(x)$ passes through the origin and $f'(x) = 8x - 5$. Find $f(x)$.

5 A curve passes through the point $(2, -5)$ and satisfies $\frac{dy}{dx} = 6x^2 - 1$. Find y in terms of x .

6 A curve passes through $(-4, 9)$ and is such that $\frac{dy}{dx} = \frac{1}{2}x^3 + \frac{1}{4}x + 1$. Find y in terms of x .

7 Given that $f'(x) = 15x^2 - 6x + 4$ and $f(1) = 0$, find $f(x)$.

8 Each of the following diagrams shows the graph of a derived function $f'(x)$. In each case, sketch the graph of a possible function $f(x)$.



9 The graph of $y = f(x)$ passes through $(4, 25)$ and $f'(x) = 6\sqrt{x}$. Find its equation.

10 Find y in terms of x in each of the following cases.

(a) $\frac{dy}{dx} = x^{\frac{1}{2}}$

(b) $\frac{dy}{dx} = 4x^{-\frac{2}{3}}$

(c) $\frac{dy}{dx} = \sqrt[3]{x}$

(d) $\frac{dy}{dx} = 2\sqrt{x} - \frac{2}{\sqrt{x}}$

(e) $\frac{dy}{dx} = \frac{5}{\sqrt[3]{x}}$

(f) $\frac{dy}{dx} = \frac{-2}{\sqrt[3]{x^2}}$

11 Find a general expression for the function $f(x)$ in each of the following cases.

(a) $f'(x) = x^{-2}$

(b) $f'(x) = 3x^{-4}$

(c) $f'(x) = \frac{6}{x^3}$

(d) $f'(x) = 4x - \frac{3}{x^2}$

(e) $f'(x) = \frac{1}{x^3} - \frac{1}{x^4}$

(f) $f'(x) = \frac{2}{x^2} - 2x^2$

12 The graph of $y = f(x)$ passes through $(\frac{1}{2}, 5)$ and $f'(x) = \frac{4}{x^2}$. Find its equation.

13 A curve passes through the point $(25, 3)$ and is such that $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$. Find the equation of the curve.

14 A curve passes through the point $(1, 5)$ and is such that $\frac{dy}{dx} = \sqrt[3]{x} - \frac{6}{x^3}$. Find the equation of the curve.

15 In each of the following cases, find y in terms of x .

(a) $\frac{dy}{dx} = 3x(x+2)$

(b) $\frac{dy}{dx} = (2x-1)(6x+5)$

(c) $\frac{dy}{dx} = \frac{4x^3+1}{x^2}$

(d) $\frac{dy}{dx} = \frac{x+4}{\sqrt{x}}$

(e) $\frac{dy}{dx} = (\sqrt{x}+5)^2$

(f) $\frac{dy}{dx} = \frac{\sqrt{x}+5}{\sqrt{x}}$

16 A tree is growing so that, after t years, its height is increasing at a rate of $\frac{30}{\sqrt[3]{t}}$ cm per year. Assume that, when $t = 0$, the height is 5 cm.

(a) Find the height of the tree after 4 years.

(b) After how many years will the height be 4.1 metres?

17 A pond, with surface area 48 square metres, is being invaded by a weed. At a time t months after the weed first appeared, the area of the weed on the surface is increasing at a rate of $\frac{1}{2}t$ square metres per month. How long will it be before the weed covers the whole surface of the pond?

18 The function $f(x)$ is such that $f'(x) = 9x^2 + 4x + c$, where c is a particular constant. Given that $f(2) = 14$ and $f(3) = 74$, find the value of $f(4)$.

16.2 Calculating areas

An important application of integration is to calculate areas and volumes. Many of the formulae you have learnt, such as those for the volume of a sphere or a cone, can be proved by using integration. This chapter deals only with areas.

The method can be illustrated by finding the area in Fig. 16.1 between the x -axis and the graph of $y = (x - 1)(3 - x)$ from $x = 1$ to $x = 3$. The key is to begin by asking a more general question: what is the area, A , between the x -axis and the graph from $x = 1$ as far as any value of x ? This is illustrated by the region with dark shading in Fig. 16.3.

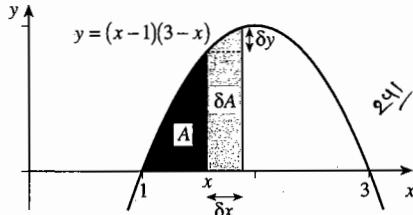


Fig. 16.3

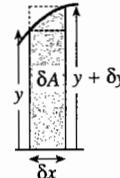


Fig. 16.4

The point of doing this is that x can now be varied. Suppose that x is increased by δx . Since both y and A are functions of x , you can write the corresponding increases in y and A as δy and δA . This is represented in Fig. 16.3 by the region with light shading.

This region is drawn by itself in Fig. 16.4. Dotted lines have been added to show that the area δA of the region is between the areas of two rectangles, each with width δx and having heights of y and $y + \delta y$. So

$$\delta A \text{ is between } y \delta x \text{ and } (y + \delta y) \delta x,$$

from which it follows that

$$\frac{\delta A}{\delta x} \text{ is between } y \text{ and } y + \delta y.$$

Now consider the effect of making δx tend to 0. From the definition, $\frac{\delta A}{\delta x}$ tends to the derivative $\frac{dA}{dx}$. Also δy tends to 0, so that $y + \delta y$ tends to y . It follows that

$$\frac{dA}{dx} = y.$$

So A is a function whose derivative is $y = (x - 1)(3 - x)$; that is, A is an integral of $(x - 1)(3 - x)$. You found in Section 16.1 that this has equation

$$A = 2x^2 - \frac{1}{3}x^3 - 3x + k, \quad \text{for some number } k.$$

To find k , you need to know A for some value of x . In this case it is obvious that $A = 0$ when $x = 1$, so that $0 = 2 - \frac{1}{3} - 3 + k$, which gives $k = \frac{4}{3}$.

The original problem was to find the area when $x = 3$, which is

$$2 \times 3^2 - \frac{1}{3} \times 3^3 - 3 \times 3 + \frac{4}{3} = 18 - 9 - 9 + \frac{4}{3} = \frac{4}{3}.$$

The answer has been given without a unit, because it is not usual to attach a unit to the variables x and y when graphs are drawn. But if in a particular application x and y each denote numbers of units, then when stating the answer a corresponding unit (the x -unit \times the y -unit) should be attached to the value of A .

16.3 The area algorithm

When you want to calculate an area you do not need to go through the argument in Section 16.2 each time. The procedure can be reduced to a set of rules, called an algorithm.

For any function, the problem is to calculate the area bounded by the x -axis, the graph of $y = f(x)$, and the lines $x = a$ and $x = b$. This is illustrated in Fig. 16.5, and is described as ‘the area under the graph from a to b ’. For this section, you should assume that $f(x) > 0$ for $a < x < b$.

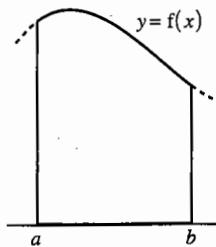


Fig. 16.5

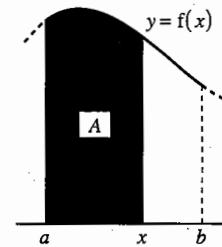


Fig. 16.6

As in Section 16.2, let A denote the area under the graph from a to any value of x (see Fig. 16.6). Then, by the same argument, $\frac{dA}{dx} = y$, so that A is an integral of $f(x)$.

There are many such functions, but if the ‘simplest’ one is denoted by $I(x)$, then you know that

$$A = I(x) + k$$

for some number k . And since $A = 0$ when $x = a$,

$$0 = I(a) + k, \text{ giving } k = -I(a).$$

Therefore

$$A = I(x) - I(a).$$

You find the required area by putting $x = b$ in this expression. That is, the area from $x = a$ to $x = b$ is $I(b) - I(a)$.

To find the area under the graph $y = f(x)$ from $x = a$ to $x = b$:

Step 1 Find the ‘simplest’ integral of $f(x)$; call it $I(x)$.

Step 2 Work out $I(a)$ and $I(b)$.

Step 3 The area is $I(b) - I(a)$.

Example 16.3.1

Find the area under $y = \frac{1}{x^2}$ from $x = 2$ to $x = 5$.

Step 1 Let $f(x) = y$. You can write $f(x)$ as x^{-2} , so $I(x)$ is $\frac{1}{-1} x^{-1}$, or $-\frac{1}{x}$.

Step 2 $I(2) = -\frac{1}{2} = -0.5$, $I(5) = -\frac{1}{5} = -0.2$.

Step 3 The area is $I(5) - I(2) = (-0.2) - (-0.5) = -0.2 + 0.5 = 0.3$.

You can shorten the operation of this algorithm still further by using a special notation.

The ‘area under $y = f(x)$ from $x = a$ to $x = b$ ’ is denoted by

$$\int_a^b f(x) dx.$$

This is called a **definite integral**. Notice that a definite integral has a specific value. Unlike an indefinite integral, it is not a function of x , and it involves no arbitrary constant. The numbers a and b are often called the **limits**, or the bounds, of integration. (But notice that they are not ‘limits’ in the sense in which the word has been used in relation to differentiation.) The function $f(x)$ is called the **integrand**.

The symbol \int was originally a letter S, standing for ‘sum’. Before the link with differentiation was discovered in the 17th century, attempts were made to calculate areas as the sums of areas of rectangles of height $f(x)$ and width denoted by δx , or dx . (Look at the lower rectangle in Fig. 16.4, where the lower rectangle has area $f(x)\delta x$.) The notation is based on this idea.

There is also an abbreviation for $I(b) - I(a)$: it is written

$$\left[I(x) \right]_a^b.$$

Using this notation, you would write the calculation of the area in Example 16.3.1 as

$$\text{Area} = \int_2^5 \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_2^5 = (-0.2) - (-0.5) = -0.2 + 0.5 = 0.3.$$

Example 16.3.2

Find the area under $y = \sqrt{x}$ from $x = 1$ to $x = 4$.

$$\text{Area} = \int_1^4 \sqrt{x} dx = \int_1^4 x^{\frac{1}{2}} dx = \left[\frac{2}{3} x^{\frac{3}{2}} \right]_1^4 = \frac{2}{3} \times 8 - \frac{2}{3} \times 1 = \frac{14}{3}.$$

The symbol $\int f(x) dx$ by itself, without limits a and b , is used to stand for the **indefinite integral**. For example, you can write

$$\int \frac{1}{x^2} dx = -\frac{1}{x} + k.$$

Exercise 16B**1** Find the following indefinite integrals.

(a) $\int 4x \, dx$

(b) $\int 15x^2 \, dx$

(c) $\int 2x^5 \, dx$

(d) $\int 9 \, dx$

(e) $\int \frac{1}{2}x^8 \, dx$

(f) $\int \frac{2}{3}x^4 \, dx$

2 Evaluate the following definite integrals.

(a) $\int_1^2 3x^2 \, dx$

(b) $\int_2^5 8x \, dx$

(c) $\int_0^2 x^3 \, dx$

(d) $\int_{-1}^1 10x^4 \, dx$

(e) $\int_0^{\frac{1}{2}} \frac{1}{2}x \, dx$

(f) $\int_0^1 2 \, dx$

3 Find the following indefinite integrals.

(a) $\int (6x + 7) \, dx$

(b) $\int (6x^2 - 2x - 5) \, dx$

(c) $\int (2x^3 + 7x) \, dx$

(d) $\int (3x^4 - 8x^3 + 9x^2 - x + 4) \, dx$

(e) $\int (2x + 5)(x - 4) \, dx$

(f) $\int x(x + 2)(x - 2) \, dx$

4 Evaluate the following definite integrals.

(a) $\int_0^2 (8x + 3) \, dx$

(b) $\int_2^4 (5x - 4) \, dx$

(c) $\int_{-2}^2 (6x^2 + 1) \, dx$

(d) $\int_0^1 (2x + 1)(x + 3) \, dx$

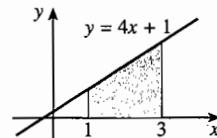
(e) $\int_{-3}^4 (6x^2 + 2x + 3) \, dx$

(f) $\int_{-3}^3 (6x^3 + 2x) \, dx$

5 Find the area under the curve $y = x^2$ from $x = 0$ to $x = 6$.**6** Find the area under the curve $y = 4x^3$ from $x = 1$ to $x = 2$.**7** Find the area under the curve $y = 12x^3$ from $x = 2$ to $x = 3$.**8** Find the area under the curve $y = 3x^2 + 2x$ from $x = 0$ to $x = 4$.**9** Find the area under the curve $y = 3x^2 - 2x$ from $x = -4$ to $x = 0$.**10** Find the area under the curve $y = x^4 + 5$ from $x = -1$ to $x = 1$.**11** The diagram shows the region under $y = 4x + 1$ between $x = 1$ and $x = 3$. Find the area of the shaded region by

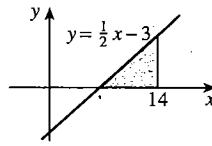
(a) using the formula for the area of a trapezium,

(b) using integration.



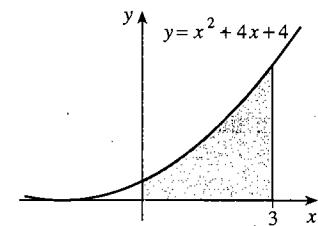
- 12** The diagram shows the region bounded by $y = \frac{1}{2}x - 3$, by $x = 14$ and the x -axis. Find the area of the shaded region by

- using the formula for the area of a triangle,
- using integration.

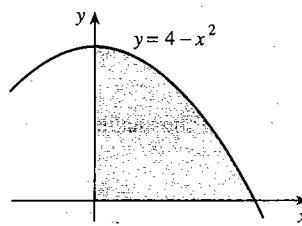


- 13** Find the area of the region shaded in each of the following diagrams.

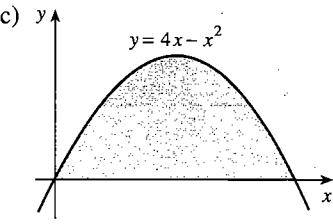
(a)



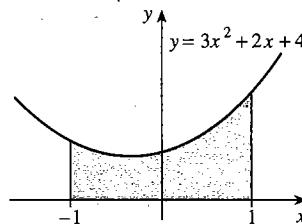
(b)



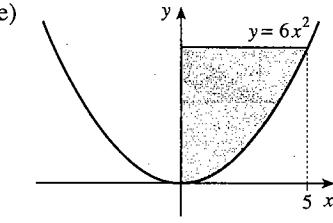
(c)



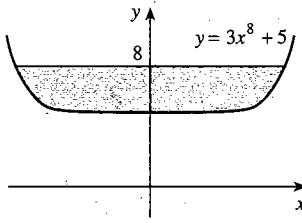
(d)



(e)



(f)



- 14** Find the following indefinite integrals.

(a) $\int \frac{1}{x^3} dx$

(b) $\int \left(x^2 - \frac{1}{x^2}\right) dx$

(c) $\int \sqrt{x} dx$

(d) $\int 6x^{\frac{2}{3}} dx$

(e) $\int \frac{6x^4 + 5}{x^2} dx$

(f) $\int \frac{1}{\sqrt{x}} dx$

- 15** Evaluate the following definite integrals.

(a) $\int_0^8 12\sqrt[3]{x} dx$

(b) $\int_1^2 \frac{3}{x^2} dx$

(c) $\int_1^4 \frac{10}{\sqrt{x}} dx$

(d) $\int_1^2 \left(\frac{8}{x^3} + x^3\right) dx$

(e) $\int_4^9 \frac{2\sqrt{x} + 3}{\sqrt{x}} dx$

(f) $\int_1^8 \frac{1}{\sqrt[3]{x^2}} dx$

- 16** Find the area under the curve $y = \frac{6}{x^4}$ between $x = 1$ and $x = 2$.

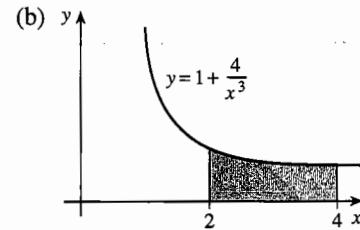
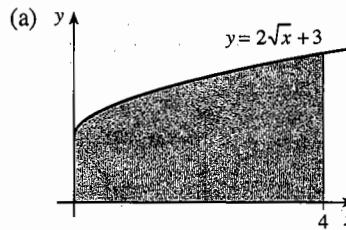
17 Find the area under the curve $y = \sqrt[3]{x}$ between $x = 1$ and $x = 27$.

18 Find the area under the curve $y = \frac{5}{x^2}$ between $x = -3$ and $x = -1$.

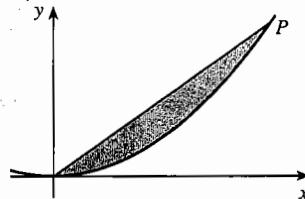
19 Given that $\int_0^a 12x^2 dx = 1372$, find the value of the constant a .

20 Given that $\int_0^9 p\sqrt{x} dx = 90$, find the value of the constant p .

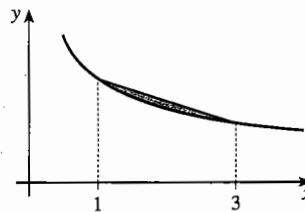
21 Find the area of the shaded region in each of the following diagrams.



22 The diagram shows the graph of $y = 9x^2$. The point P has coordinates $(4, 144)$. Find the area of the shaded region.



23 The diagram shows the graph of $y = \frac{1}{\sqrt{x}}$. Show that the area of the shaded region is $3 - \frac{5\sqrt{3}}{3}$.



24 Find the area of the region between the curve $y = 9 + 15x - 6x^2$ and the x -axis.

16.4 Some properties of definite integrals

In definite integral notation the calculation in Section 16.2 of the area in Fig. 16.1 would be written

$$\int_1^3 (x-1)(3-x) dx = \left[2x^2 - \frac{1}{3}x^3 - 3x \right]_1^3 = (0) - \left(-\frac{4}{3} \right) = \frac{4}{3}.$$

But how should you interpret the calculation

$$\int_0^3 (x-1)(3-x) dx = [2x^2 - \frac{1}{3}x^3 - 3x]_0^3 = (0) - (0) = 0 ?$$

Clearly the area between the graph and the x -axis between $x = 0$ and $x = 3$ is not zero as the value of the definite integral suggests.

You can find the clue by calculating the integral between $x = 0$ and $x = 1$:

$$\int_0^1 (x-1)(3-x) dx = [2x^2 - \frac{1}{3}x^3 - 3x]_0^1 = \left(-\frac{4}{3}\right) - (0) = -\frac{4}{3}.$$

This shows that you need to be careful in identifying the definite integral as an area. In Fig. 16.1 the area of the region contained between the curve and the two axes is $\frac{4}{3}$, and the negative sign attached to the definite integral indicates that between $x = 0$ and $x = 1$ the graph lies below the x -axis.

The zero answer obtained for the integral from $x = 0$ to $x = 3$ is then explained by the fact that definite integrals are added exactly as you would expect:

$$\int_0^3 (x-1)(3-x) dx = \int_0^1 (x-1)(3-x) dx + \int_1^3 (x-1)(3-x) dx = -\frac{4}{3} + \frac{4}{3} = 0.$$

This is a special case of a general rule:

$$\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx.$$

To prove this, let $I(x)$ denote the simplest integral of $f(x)$.

Then the sum of the integrals on the left side is equal to

$$[I(x)]_a^b + [I(x)]_b^c = \{I(b) - I(a)\} - \{I(c) - I(b)\} = I(c) - I(a) = \int_a^c f(x) dx.$$

Negative definite integrals can also arise when you interchange the bounds of integration. Since

$$[I(x)]_b^a = I(a) - I(b) = -\{I(b) - I(a)\} = -[I(x)]_a^b,$$

it follows that

$$\int_b^a f(x) dx = - \int_a^b f(x) dx.$$

You are not likely to use this in numerical examples, but such integrals may turn up if a or b are algebraic expressions.

16.5 Infinite and improper integrals

Example 16.5.1

Find the areas under the graphs of (a) $y = \frac{1}{x^2}$, (b) $y = \frac{1}{\sqrt{x}}$ in the intervals

- (i) $x = 1$ to $x = s$, where $s > 1$, (ii) $x = r$ to $x = 1$, where $0 < r < 1$.

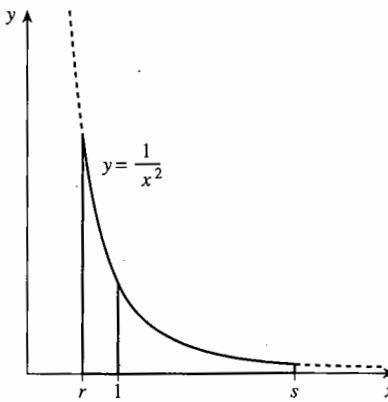


Fig. 16.7

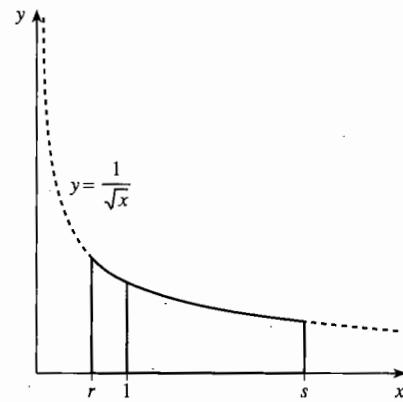


Fig. 16.8

These areas are shown in Figs. 16.7 and 16.8.

The functions are (a) x^{-2} and (b) $x^{-\frac{1}{2}}$, so the simplest integrals are

$$(a) -x^{-1} = -\frac{1}{x} \quad \text{and} \quad (b) 2x^{\frac{1}{2}} = 2\sqrt{x}.$$

$$(i) \int_1^s \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_1^s = 1 - \frac{1}{s}.$$

$$(ii) \int_r^1 \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_r^1 = \frac{1}{r} - 1.$$

$$(b) (i) \int_1^s \frac{1}{\sqrt{x}} dx = \left[2\sqrt{x} \right]_1^s = 2\sqrt{s} - 2.$$

$$(ii) \int_r^1 \frac{1}{\sqrt{x}} dx = \left[2\sqrt{x} \right]_r^1 = 2 - 2\sqrt{r}.$$

The interesting feature of these results appears if you consider what happens in (i) if s becomes indefinitely large, and in (ii) if r comes indefinitely close to 0.

Consider s first. By taking a large enough value for s , you can make $1 - \frac{1}{s}$ as close to 1 as you like, but it always remains less than 1. You can say that the integral (a)(i) ‘tends to 1 as s tends to infinity’ (written ‘ $\rightarrow 1$ as $s \rightarrow \infty$ ’).

A shorthand for this is

$$\int_1^\infty \frac{1}{x^2} dx = 1.$$

This is called an **infinite integral**.

However, $2\sqrt{s} - 2$ can be made as large as you like by taking a large enough value for s , so the integral (b)(i) ‘tends to infinity as s tends to infinity’ (or ‘ $\rightarrow \infty$ as $s \rightarrow \infty$ ’). Since ‘infinity’ is not a number, you cannot give a meaning to the symbol

$$\int_1^\infty \frac{1}{\sqrt{x}} dx.$$

In the case of r , the situation is reversed. The expression $2 - 2\sqrt{r}$ tends to 2 as r tends to 0. So you can write

$$\int_0^1 \frac{1}{\sqrt{x}} dx = 2$$

even though the integrand $\frac{1}{\sqrt{x}}$ is not defined when $x = 0$. This is called an **improper integral**. But in (a)(ii), $\frac{1}{r} - 1$ tends to infinity as r tends to 0, so you cannot give a meaning to the symbol

$$\int_0^1 \frac{1}{x^2} dx.$$

You can see from the graphs that, as you would expect, the cases where the integrals are defined correspond to regions in which the graph is very close to one of the axes. You can then say that the region has a finite area, even though it is unbounded.

16.6 The area between two graphs

You sometimes want to find the area of a region bounded by the graphs of two functions $f(x)$ and $g(x)$, and by two lines $x = a$ and $x = b$, as in Fig. 16.9.

Although you could find this as the difference of the areas of two regions of the kind illustrated in Fig. 16.5, calculated as

$$\int_a^b f(x) dx - \int_a^b g(x) dx,$$

it is often simpler to find this as a single integral

$$\int_a^b (f(x) - g(x)) dx.$$

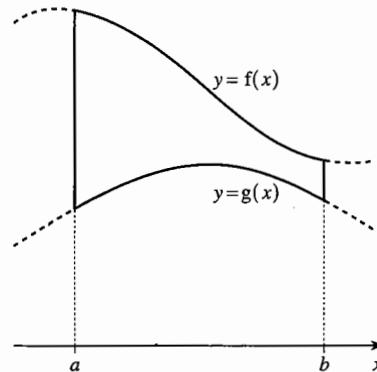


Fig. 16.9

Example 16.6.1

Show that the graphs of $f(x) = x^3 - x^2 - 6x + 8$ and $g(x) = x^3 + 2x^2 - 1$ intersect at two points, and find the area enclosed between them.

The graphs intersect where

$$\begin{aligned}x^3 - x^2 - 6x + 8 &= x^3 + 2x^2 - 1, \\0 &= 3x^2 + 6x - 9, \\3(x+3)(x-1) &= 0.\end{aligned}$$

The points of intersection are therefore $(-3, -10)$ and $(1, 2)$.

If you draw the graphs between $x = -3$ and $x = 1$, you will see that $f(x) > g(x)$ in this interval.

The area between the graphs is

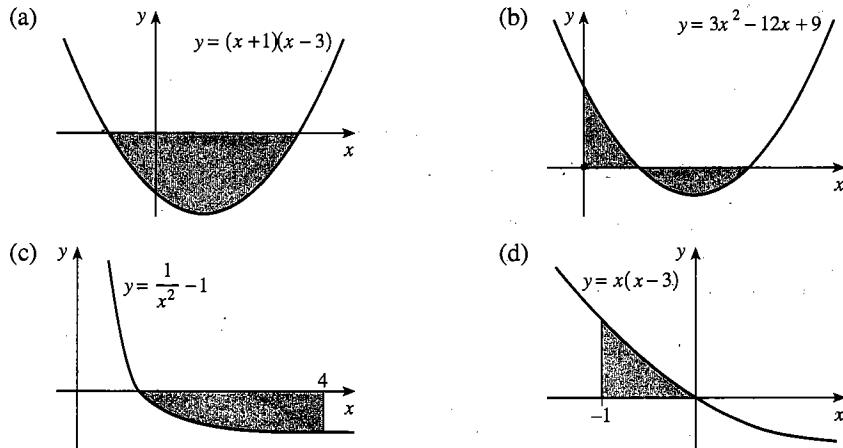
$$\begin{aligned}\int_{-3}^1 (f(x) - g(x)) dx &= \int_{-3}^1 (9 - 6x - 3x^2) dx = [9x - 3x^2 - x^3]_{-3}^1 \\&= (9 - 3 - 1) - (-27 - 27 + 27) = 5 - (-27) = 32.\end{aligned}$$

Notice that in this example, integrating $f(x) - g(x)$, rather than $f(x)$ and $g(x)$ separately, greatly reduces the amount of calculation.

Exercise 16C

1 Evaluate $\int_0^2 3x(x-2) dx$ and comment on your answer.

2 Find the total area of the region shaded in each of the following diagrams.



3 Find the values of the improper integrals

$$(a) \int_0^{16} \frac{1}{\sqrt[4]{x}} dx, \quad (b) \int_0^{16} \frac{1}{\sqrt[4]{x^3}} dx, \quad (c) \int_0^1 x^{-0.99} dx.$$

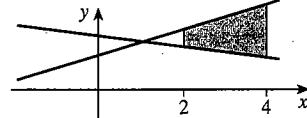
4 Find the values of the infinite integrals

$$(a) \int_2^\infty \frac{6}{x^4} dx, \quad (b) \int_4^\infty \frac{6}{x\sqrt{x}} dx, \quad (c) \int_1^\infty x^{-1.01} dx.$$

5 Find an expression for $\int_1^s \frac{1}{x^m} dx$ in terms of m and s , where m is a positive rational number, $m \neq -1$ and $s > 1$. Show that the infinite integral $\int_1^\infty \frac{1}{x^m} dx$ has a meaning if $m > 1$, and state its value in terms of m .

6 Find an expression for $\int_r^1 \frac{1}{x^m} dx$ in terms of m and r , where m is a positive rational number, $m \neq -1$ and $0 < r < 1$. For what values of m does the improper integral $\int_0^1 \frac{1}{x^m} dx$ have a meaning? State its value in terms of m .

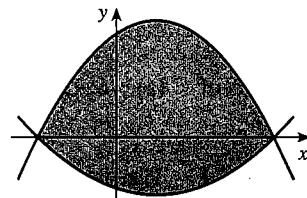
7 The diagram shows the graphs of $y = 2x + 7$ and $y = 10 - x$.
Find the area of the shaded region.



8 Find the area enclosed between the curves $y = x^2 + 7$ and $y = 2x^2 + 3$.

9 Find the area enclosed between the straight line $y = 12x + 14$ and the curve $y = 3x^2 + 6x + 5$.

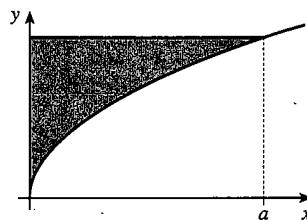
10 The diagram shows the graphs of $y = 16 + 4x - 2x^2$ and $y = x^2 - 2x - 8$.
Find the area of the region, shaded in the diagram, between the curves.



11 Find the area between the curves $y = (x-4)(3x-1)$ and $y = (4-x)(1+x)$.

12 Parts of the graphs of $f(x) = 2x^3 + x^2 - 8x$ and $g(x) = 2x^3 - 3x - 4$ enclose a finite region.
Find its area.

13 The diagram shows the graph of $y = \sqrt{x}$. Given that the area of the shaded region is 72, find the value of the constant a .



16.7 Integrating $(ax+b)^n$

You can also use the differentiation result in Section 12.1 in reverse for integration. For example, to integrate $(3x+1)^3$, you should recognise that it comes from differentiating $(3x+1)^4$.

A first guess at the integral $\int (3x+1)^3 dx$ is $(3x+1)^4$. If you differentiate $(3x+1)^4$, you get $4(3x+1)^3 \times 3 = 12(3x+1)^3$. Therefore

$$\int (3x+1)^3 dx = \frac{1}{12}(3x+1)^4 + k.$$

You can formalise the guessing process by reversing the last result from Section 12.1.

$$\int g(ax+b) dx = \frac{1}{a} f(ax+b) + k \text{ where } f(x) \text{ is the simplest integral of } g(x).$$

Applying this to the previous example, $g(x) = x^3$, $a = 3$ and $b = 1$. Then $f(x) = \frac{1}{4}x^4$, so

$$\begin{aligned} \int (3x+1)^3 dx &= \int g(3x+1) dx = \frac{1}{3}f(3x+1) + k = \frac{1}{3} \times \frac{1}{4}(3x+1)^4 + k \\ &= \frac{1}{12}(3x+1)^4 + k. \end{aligned}$$

Example 16.7.1

Find the integrals of (a) $\sqrt{5-2x}$, (b) $\frac{1}{(3-x)^2}$.

(a) **Method 1** The first guess at $\int \sqrt{5-2x} dx = \int (5-2x)^{\frac{1}{2}} dx$ is $(5-2x)^{\frac{3}{2}}$. Differentiating $(5-2x)^{\frac{3}{2}}$, you obtain $\frac{3}{2}(5-2x)^{\frac{1}{2}} \times (-2) = -3(5-2x)^{\frac{1}{2}}$. Therefore

$$\int (5-2x)^{\frac{1}{2}} dx = -\frac{1}{3}(5-2x)^{\frac{3}{2}} + k.$$

Method 2 Using the result in the shaded box, $g(x) = \sqrt{x} = x^{\frac{1}{2}}$, $a = -2$, $b = 5$.

So $f(x) = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3}x^{\frac{3}{2}}$, and $\int \sqrt{5-2x} dx = \frac{1}{-2} \times \frac{2}{3}(5-2x)^{\frac{3}{2}} + k = -\frac{1}{3}(5-2x)^{\frac{3}{2}} + k$.

(b) First write $\frac{1}{(3-x)^2}$ as $(3-x)^{-2}$. Then

$$\int \frac{1}{(3-x)^2} dx = \int (3-x)^{-2} dx = \frac{1}{-1} \times \frac{1}{-1}(3-x)^{-1} + k = \frac{1}{3-x} + k.$$

With practice you might find that you can write down the correct integral, but check your answer by differentiation, because it is easy to make a numerical mistake.

Example 16.7.2

Find the area between the curve $y = 16 - (2x+1)^4$ and the x -axis. (See Fig. 16.10.)

To find where the graph cuts the x -axis,
solve the equation $16 - (2x+1)^4 = 0$.
Thus $(2x+1)^4 = 16$, so $(2x+1) = 2$ or
 $(2x+1) = -2$, leading to the limits of
integration, $x = \frac{1}{2}$ and $x = -\frac{3}{2}$.

The area is given by

$$\begin{aligned} & \int_{-\frac{3}{2}}^{\frac{1}{2}} (16 - (2x+1)^4) dx \\ &= \left[16x - \frac{1}{10}(2x+1)^5 \right]_{-\frac{3}{2}}^{\frac{1}{2}} \\ &= \left(16 \times \frac{1}{2} - \frac{1}{10}(2 \times \frac{1}{2} + 1)^5 \right) - \left(16 \times \left(-\frac{3}{2} \right) - \frac{1}{10}(2 \times \left(-\frac{3}{2} \right) + 1)^5 \right) \\ &= \left(8 - \frac{1}{10} \times 2^5 \right) - \left(-24 - \frac{1}{10} \times (-2)^5 \right) \\ &= 4.8 - (-20.8) = 25.6. \end{aligned}$$

The required area is then 25.6.

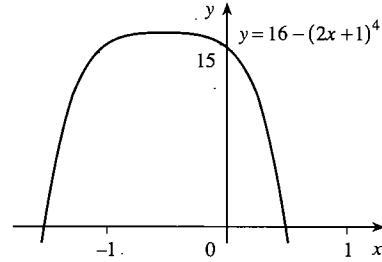


Fig. 16.10

Exercise 16D

1 Integrate the following with respect to x .

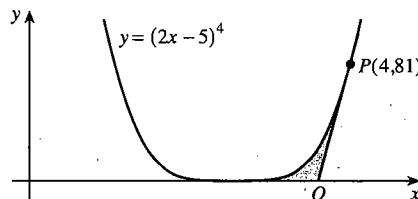
- | | | | |
|--------------------|-----------------------------|--------------------------------------|--------------------------------|
| (a) $(2x+1)^6$ | (b) $(3x-5)^4$ | (c) $(1-7x)^3$ | (d) $(\frac{1}{2}x+1)^{10}$ |
| (e) $(5x+2)^{-3}$ | (f) $2(1-3x)^{-2}$ | (g) $\frac{1}{(x+1)^5}$ | (h) $\frac{3}{2(4x+1)^4}$ |
| (i) $\sqrt{10x+1}$ | (j) $\frac{1}{\sqrt{2x-1}}$ | (k) $(\frac{1}{2}x+2)^{\frac{1}{3}}$ | (l) $\frac{8}{\sqrt[4]{2+6x}}$ |

2 Evaluate the following integrals.

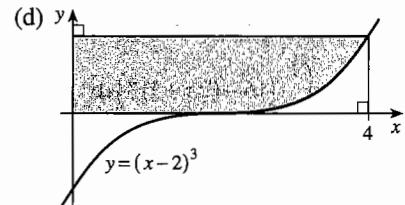
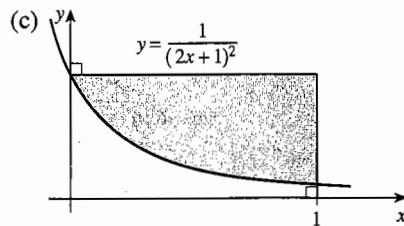
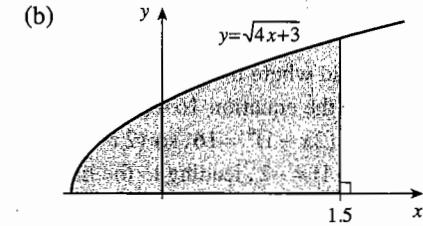
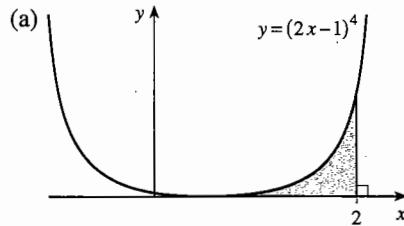
- | | | | |
|----------------------------|-------------------------------|-------------------------------------|-------------------------------------|
| (a) $\int_1^5 (2x-1)^3 dx$ | (b) $\int_1^5 \sqrt{2x-1} dx$ | (c) $\int_1^3 \frac{1}{(x+2)^2} dx$ | (d) $\int_1^3 \frac{2}{(x+2)^3} dx$ |
|----------------------------|-------------------------------|-------------------------------------|-------------------------------------|

- 3** Given that $\int_{1.25}^p (4x-5)^4 dx = 51.2$, find the value of p .

- 4** The diagram shows the curve $y = (2x-5)^4$. The point P has coordinates $(4, 81)$ and the tangent to the curve at P meets the x -axis at Q . Find the area of the region (shaded in the diagram) enclosed between the curve, PQ and the x -axis.



5 Find the area of each of the following shaded regions.

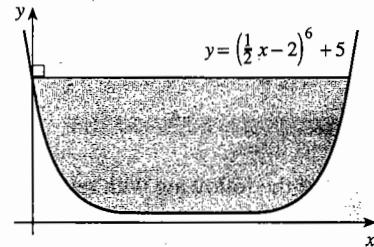


6 Find the area of the region enclosed between the curves $y = (x - 2)^4$ and $y = (x - 2)^3$.

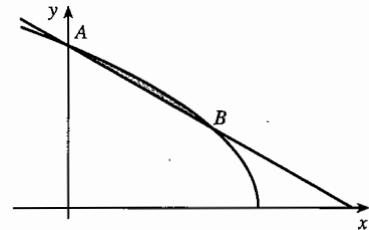
7 The diagram shows the curve

$$y = \left(\frac{1}{2}x - 2\right)^6 + 5.$$

Find the area of the shaded region.



8 The diagram shows a sketch of the curve $y = \sqrt{4 - x}$ and the line $y = 2 - \frac{1}{3}x$. The coordinates of the points A and B where the curve and line intersect are $(0, 2)$ and $(3, 1)$ respectively. Calculate the area of the region between the line and the curve (shaded in the diagram), giving your answer as an exact fraction. (OCR)



Miscellaneous exercise 16

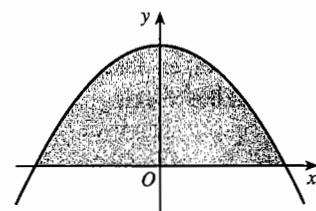
1 Find $\int 6\sqrt{x} dx$, and hence evaluate $\int_1^4 6\sqrt{x} dx$. (OCR)

2 The diagram shows the graph of $y = 12 - 3x^2$.

Determine the x-coordinate of each of the points where the curve crosses the x-axis.

Find by integration the area of the region (shaded in the diagram) between the curve and the x-axis.

(OCR)



3 Evaluate $\int_0^{\frac{2}{3}} (3x - 2)^3 dx$. (OCR)

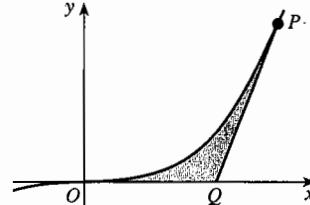
4 Find $\int_0^4 \sqrt{2x+1} dx$. (OCR)

5 (a) Find $\int \left(\frac{1}{x^3} + x^3 \right) dx$. (b) Evaluate $\int_0^8 \frac{1}{\sqrt[3]{x}} dx$. (OCR)

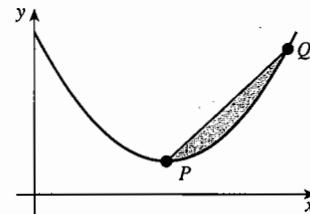
6 Find the area of the region enclosed between the curve $y = 12x^2 + 30x$ and the x -axis.

7 Given that $\int_{-a}^a 15x^2 dx = 3430$, find the value of the constant a .

- 8 The diagram shows the curve $y = x^3$. The point P has coordinates $(3, 27)$ and PQ is the tangent to the curve at P . Find the area of the region enclosed between the curve, PQ and the x -axis.



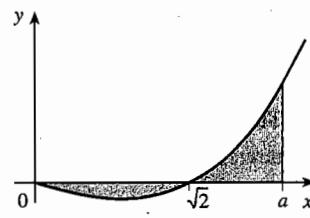
- 9 The diagram shows the curve $y = (x-2)^2 + 1$ with minimum point P . The point Q on the curve is such that the gradient of PQ is 2. Find the area of the region, shaded in the diagram, between PQ and the curve.



- 10 Evaluate $\int_0^2 x(x-1)(x-2) dx$ and explain your answer with reference to the graph of $y = x(x-1)(x-2)$.

11 (a) Find $\int x(x^2 - 2) dx$.

- (b) The diagram shows the graph of $y = x(x^2 - 2)$ for $x \geq 0$. The value of a is such that the two shaded regions have equal areas. Find the value of a . (OCR)

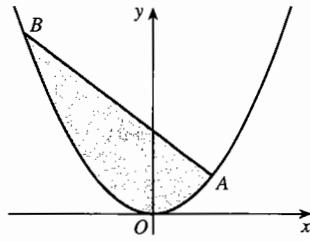


- 12 Given that $\int_1^p (8x^3 + 6x) dx = 39$, find two possible values of p . Use a graph to explain why there are two values.

- 13 Show that the area enclosed between the curves $y = 9 - x^2$ and $y = x^2 - 7$ is $\frac{128\sqrt{2}}{3}$.

- 14** The diagram shows a sketch of the graph of $y = x^2$ and the normal to the curve at the point $A(1,1)$.

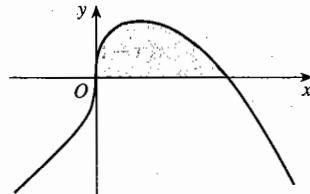
- (a) Use differentiation to find the equation of the normal at A . Verify that the point B where the normal cuts the curve again has coordinates $(-\frac{3}{2}, \frac{9}{4})$.
- (b) The region which is bounded by the curve and the normal is shaded in the diagram. Calculate its area, giving your answer as an exact fraction. (OCR)



- 15** Given that $f(x)$ and $g(x)$ are two functions such that $\int_0^4 f(x) dx = 17$ and $\int_0^4 g(x) dx = 11$, find, where possible, the value of each of the following.

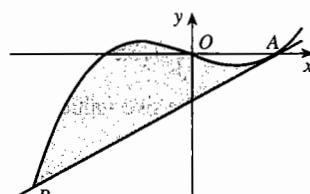
- | | |
|---|-----------------------------------|
| (a) $\int_0^4 (f(x) - g(x)) dx$ | (b) $\int_0^4 (2f(x) + 3g(x)) dx$ |
| (c) $\int_0^2 f(x) dx$ | (d) $\int_0^4 (f(x) + 2x + 3) dx$ |
| (e) $\int_0^1 f(x) dx + \int_1^4 f(x) dx$ | (f) $\int_4^0 g(x) dx$ |
| (g) $\int_1^5 f(x-1) dx$ | (h) $\int_{-4}^0 g(-t) dt$ |

- 16** The diagram shows the graph of $y = \sqrt[3]{x} - x^2$. Show by integration that the area of the region (shaded in the diagram) between the curve and the x -axis is $\frac{5}{12}$. (OCR)



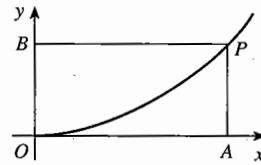
- 17** The diagram shows a sketch of the graph of the curve $y = x^3 - x$ together with the tangent to the curve at the point $A(1,0)$.

- (a) Use differentiation to find the equation of the tangent to the curve at A , and verify that the point B where the tangent cuts the curve again has coordinates $(-2, -6)$.
- (b) Use integration to find the area of the region bounded by the curve and the tangent (shaded in the diagram), giving your answer as a fraction in its lowest terms. (OCR)



- 18 The diagram shows part of the curve $y = x^n$, where $n > 1$.

The point P on the curve has x -coordinate a . Show that the curve divides the rectangle $OAPB$ into two regions whose areas are in the ratio $n:1$.



- 19 Find the stationary points on the graph of $y = x^4 - 8x^2$. Use your answers to make a sketch of the graph. Show that the graphs of $y = x^4 - 8x^2$ and $y = x^2$ enclose two finite regions. Find the area of one of them.
- 20 Using the same axes, make sketches of the graphs of $y = x^3$ and $y = (x+1)^3 - 1$. Then sketch on a larger scale the finite area enclosed between them.

Find the area of the region.

- 21 A function $f(x)$ with domain $x > 0$ is defined by $f(x) = \frac{6}{x^4} - \frac{2}{x^3}$.

(a) Find the values of $\int_2^3 f(x) dx$ and $\int_2^\infty f(x) dx$.

(b) Find the coordinates of

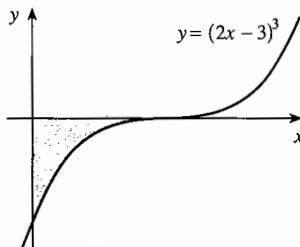
- (i) the point where the graph of $y = f(x)$ crosses the x -axis,
(ii) the minimum point on the graph.

Use your answers to draw a sketch of the graph, and hence explain your answers to part (a).

- 22 The diagram shows the curve $y = (2x - 3)^3$.

(a) Find the x -coordinates of the two points on the curve which have gradient 6.

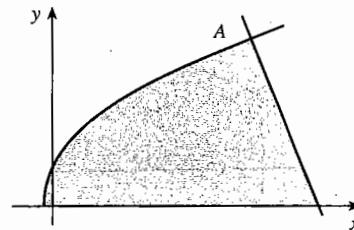
(b) The region shaded in the diagram is bounded by part of the curve and by the two axes. Find, by integration, the area of this region. (OCR)



- 23 The diagram shows the curve with equation $y = \sqrt{4x + 1}$ and the normal to the curve at the point A with coordinates $(6, 5)$.

(a) Show that the equation of the normal to the curve at A is $y = -\frac{5}{2}x + 20$.

(b) Find the area of the region (shaded in the diagram) which is enclosed by the curve, the normal and the x -axis. Give your answer as a fraction in its lowest terms. (OCR)



17 Volume of revolution

This chapter is about using integration to find the volume of a particular kind of solid, called a solid of revolution. When you have completed it, you should

- be able to find a volume of revolution about either the x - or y -axis.

17.1 Volumes of revolution

Let O be the origin, and let OA be a line through the origin, as shown in Fig. 17.1. Consider the region between the line OA and the x -axis, shown shaded. If you rotate this region about the x -axis through 360° , it sweeps out a solid cone, shown in Fig. 17.2. A solid shape constructed in this way is called a **solid of revolution**. The volume of a solid of revolution is sometimes called a **volume of revolution**.

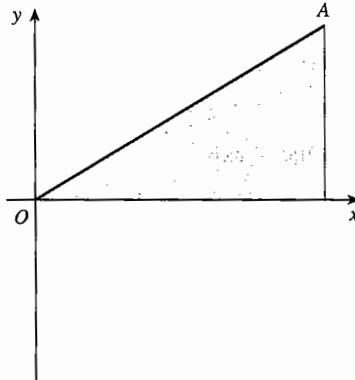


Fig. 17.1

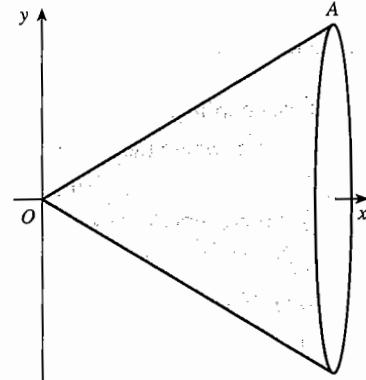


Fig. 17.2

Calculating a volume of revolution is similar in many ways to calculating the area of a region under a curve, and can be illustrated by an example.

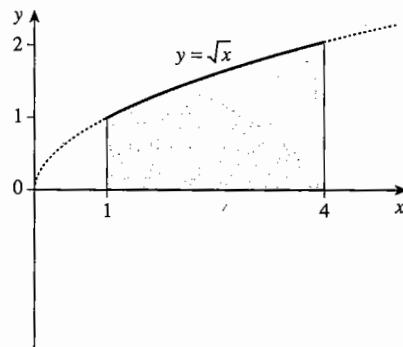


Fig. 17.3

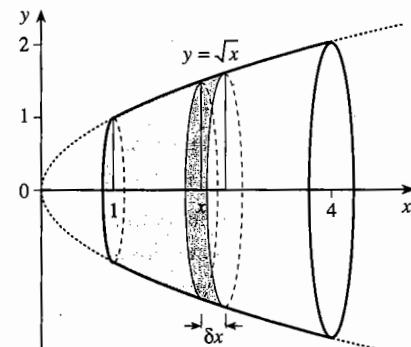


Fig. 17.4

Suppose that the region between the graph of $y = \sqrt{x}$ and the x -axis from $x = 1$ to $x = 4$, shown in Fig. 17.3, is rotated about the x -axis to form the solid of revolution in Fig. 17.4.

The key is to begin by asking a more general question: what is the volume, V , of the solid of revolution from $x = 1$ as far as any value of x ? This solid is shown by the light shading in Fig. 17.4.

Suppose that x is increased by δx . Since y and V are both functions of x , the corresponding increases in y and V can be written as δy and δV . The increase δV is shown by darker shading in Fig. 17.4. Examine this increase δV in the volume more closely. It is shown in more detail in the left diagram in Fig. 17.5.

The increase δV in the volume is between the volumes of two disc-like cylinders, each of width δx and having radii y and $y + \delta y$. (These two cylinders are shown in the centre and right diagrams in Fig. 17.5.) So

$$\delta V \text{ is between } \pi y^2 \delta x \text{ and } \pi(y + \delta y)^2 \delta x$$

from which it follows that

$$\frac{\delta V}{\delta x} \text{ is between } \pi y^2 \text{ and } \pi(y + \delta y)^2.$$

Now let δx tend to 0. From the definition in Section 7.4, $\frac{\delta V}{\delta x}$ tends to the derivative $\frac{dV}{dx}$. Also, δy tends to 0, so that $y + \delta y$ tends to y . It follows that

$$\frac{dV}{dx} = \pi y^2.$$

So V is a function whose derivative is πy^2 , and since $y = \sqrt{x}$, $\frac{dV}{dx} = \pi x$. Therefore

$$V = \frac{1}{2} \pi x^2 + k$$

for some number k .

Since the volume $V = 0$ when $x = 1$, $0 = \frac{1}{2} \pi \times 1^2 + k$, giving $k = -\frac{1}{2} \pi$. Thus

$$V = \frac{1}{2} \pi x^2 - \frac{1}{2} \pi.$$

To find the volume up to $x = 4$, substitute $x = 4$ in this expression for V . The volume is $\frac{1}{2} \pi \times 4^2 - \frac{1}{2} \pi = \frac{1}{2} \pi(16 - 1) = \frac{15}{2} \pi$.

You can shorten the last part of this work by using the integral notation introduced in Section 16.3:

$$V = \int_1^4 \pi y^2 dx = \int_1^4 \pi x dx = \left[\frac{1}{2} \pi x^2 \right]_1^4 = \frac{1}{2} \pi \times 16 - \frac{1}{2} \pi \times 1 = \frac{15}{2} \pi.$$

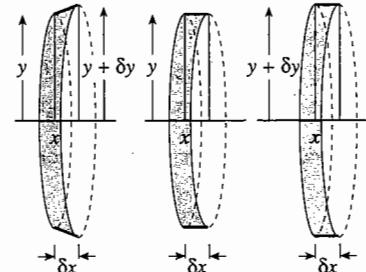


Fig. 17.5

Notice that the argument used at the beginning of the example was completely general, and did not depend in any way on the equation of the original curve.

When the region under the graph of $y = f(x)$ between $x = a$ and $x = b$ (where $a < b$) is rotated about the x -axis, the volume of the solid of revolution formed is

$$\int_a^b \pi(f(x))^2 dx, \text{ or } \int_a^b \pi y^2 dx.$$

Example 17.1.1

Find the volume generated when the region under the graph of $y = 1 + x^2$ between $x = -1$ and $x = 1$ is rotated through four right angles about the x -axis.

The phrase 'four right angles' is sometimes used in place of 360° for describing a full rotation about the x -axis.

The required volume is V , where

$$\begin{aligned} V &= \int_{-1}^1 \pi y^2 dx = \int_{-1}^1 \pi(1+x^2)^2 dx = \int_{-1}^1 \pi(1+2x^2+x^4) dx \\ &= \left[\pi \left(x + \frac{2}{3}x^3 + \frac{1}{5}x^5 \right) \right]_{-1}^1 \\ &= \pi \left\{ \left(1 + \frac{2}{3} + \frac{1}{5} \right) - \left((-1) + \frac{2}{3}(-1)^3 + \frac{1}{5}(-1)^5 \right) \right\} = \frac{56}{15}\pi. \end{aligned}$$

The volume of the solid is $\frac{56}{15}\pi$.

It is usual to give the result as an exact multiple of π , unless you are asked for an answer correct to a given number of significant figures or decimal places.

You can also use the method to obtain the formula for the volume of a cone.

Example 17.1.2

Prove that the volume V of a cone with base radius r and height h is $V = \frac{1}{3}\pi r^2 h$.

The triangle which rotates to give the cone is shown in Fig. 17.6, where the 'height' has been drawn across the page. The gradient of OA is $\frac{r}{h}$, so its equation is $y = \frac{r}{h}x$.

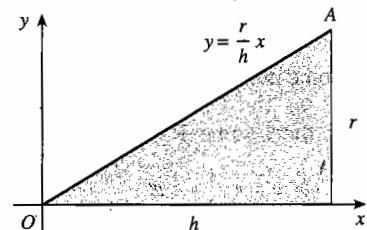


Fig. 17.6

Therefore, remembering that π , r and h are constants and do not depend on x ,

$$\begin{aligned} V &= \int_0^h \pi y^2 dx = \int_0^h \pi \left(\frac{r}{h}x \right)^2 dx = \int_0^h \pi \frac{r^2}{h^2} x^2 dx \\ &= \pi \frac{r^2}{h^2} \int_0^h x^2 dx = \pi \frac{r^2}{h^2} \left[\frac{1}{3}x^3 \right]_0^h = \pi \frac{r^2}{h^2} \times \frac{1}{3}h^3 = \frac{1}{3}\pi r^2 h. \end{aligned}$$

17.2 Volumes of revolution about the y -axis

In Fig. 17.7, the region between the graph of $y = f(x)$ between $y = c$ and $y = d$ is rotated about the y -axis to give the solid shown in Fig. 17.8.

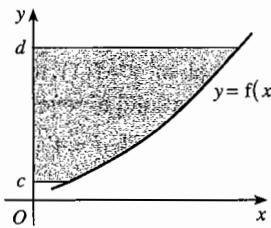


Fig. 17.7

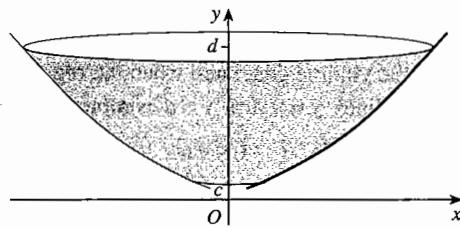


Fig. 17.8

To find the volume of this solid of revolution about the y -axis, you can reverse the roles of x and y in the discussion in Section 17.1.

When the region bounded by the graph of $y = f(x)$, the lines $y = c$ and $y = d$ and the y -axis is rotated about the y -axis, the volume of the solid of revolution formed is

$$\int_c^d \pi x^2 dy.$$

You can only use this result if the inverse function $x = f^{-1}(y)$ is defined for $c \leq y \leq d$ (see Chapter 11). Remember that the limits in the integral are limits for y , not for x .

Example 17.2.1

Find the volume generated when the region bounded by $y = x^3$ and the y -axis between $y = 1$ and $y = 8$ is rotated through 360° about the y -axis.

Since the volume is given by $\int_1^8 \pi x^2 dy$, you need to express x^2 in terms of y .

The equation $y = x^3$ can be inverted to give $x = y^{\frac{1}{3}}$, so that $x^2 = y^{\frac{2}{3}}$. Then

$$\begin{aligned} V &= \int_1^8 \pi y^{\frac{2}{3}} dy = \pi \left[\frac{3}{5} y^{\frac{5}{3}} \right]_1^8 = \pi \left(\frac{3}{5} \times 8^{\frac{5}{3}} \right) - \pi \left(\frac{3}{5} \times 1^{\frac{5}{3}} \right) \\ &= \pi \left(\frac{3}{5} \times 32 \right) - \pi \left(\frac{3}{5} \times 1 \right) = \frac{93}{5} \pi. \end{aligned}$$

The required volume is $\frac{93}{5} \pi$.

Exercise 17

In all the questions in this exercise, leave your answers as multiples of π .

- Find the volume generated when the region under the graph of $y = f(x)$ between $x = a$ and $x = b$ is rotated through 360° about the x -axis.

(a) $f(x) = x$; $a = 3, b = 5$	(b) $f(x) = x^2$; $a = 2, b = 5$
(c) $f(x) = x^3$; $a = 2, b = 6$	(d) $f(x) = \frac{1}{x}$; $a = 1, b = 4$

- 2 Find the volume formed when the region under the graph of $y = f(x)$ between $x = a$ and $x = b$ is rotated through 360° about the x -axis.
- (a) $f(x) = x + 3; a = 3, b = 9$ (b) $f(x) = x^2 + 1; a = 2, b = 5$
 (c) $f(x) = \sqrt{x+1}; a = 0, b = 3$ (d) $f(x) = x(x-2); a = 0, b = 2$
- 3 Find the volume generated when the region bounded by the graph of $y = f(x)$, the y -axis and the lines $y = c$ and $y = d$ is rotated about the y -axis to form a solid of revolution.
- (a) $f(x) = x^2; c = 1, d = 3$ (b) $f(x) = x + 1; c = 1, d = 4$
 (c) $f(x) = \sqrt{x}; c = 2, d = 7$ (d) $f(x) = \frac{1}{x}; c = 2, d = 5$
 (e) $f(x) = \sqrt{9-x}; c = 0, d = 3$ (f) $f(x) = x^2 + 1; c = 1, d = 4$
 (g) $f(x) = x^{\frac{2}{3}}; c = 1, d = 5$ (h) $f(x) = \frac{1}{x} + 2; c = 3, d = 5$
- 4 In each case the region enclosed between the following curves and the x -axis is rotated through 360° about the x -axis. Find the volume of the solid generated.
- (a) $y = (x+1)(x-3)$ (b) $y = 1 - x^2$
 (c) $y = x^2 - 5x + 6$ (d) $y = x^2 - 3x$
- 5 The region enclosed between the graphs of $y = x$ and $y = x^2$ is denoted by R . Find the volume generated when R is rotated through 360° about
- (a) the x -axis, (b) the y -axis.
- 6 The region enclosed between the graphs of $y = 4x$ and $y = x^2$ is denoted by R . Find the volume generated when R is rotated through 360° about
- (a) the x -axis, (b) the y -axis.
- 7 The region enclosed between the graphs of $y = \sqrt{x}$ and $y = x^2$ is denoted by R . Find the volume generated when R is rotated through 360° about
- (a) the x -axis, (b) the y -axis.
- 8 A glass bowl is formed by rotating about the y -axis the region between the graphs of $y = x^2$ and $y = x^3$. Find the volume of glass in the bowl.
- 9 The region enclosed by both axes, the line $x = 2$ and the curve $y = \frac{1}{8}x^2 + 2$ is rotated about the y -axis to form a solid. Find the volume of this solid.

Miscellaneous exercise 17

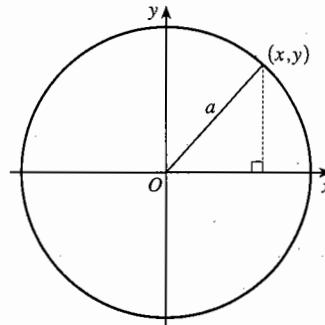
- 1 The region bounded by the curve $y = x^2 + 1$, the x -axis, the y -axis and the line $x = 2$ is rotated completely about the x -axis. Find, in terms of π , the volume of the solid formed.
 (OCR)

- 2 Explain why the coordinates (x, y) of any point on a circle, centre O , radius a satisfy the equation $x^2 + y^2 = a^2$.

The semicircle above the x -axis is rotated about the x -axis through 360° to form a sphere of radius a . Explain why the volume V of this sphere is given by

$$V = 2\pi \int_0^a (a^2 - x^2) dx.$$

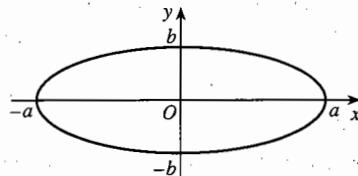
Hence show that $V = \frac{4}{3}\pi a^3$.



- 3 The ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, shown in the diagram, has semi-axes a and b .

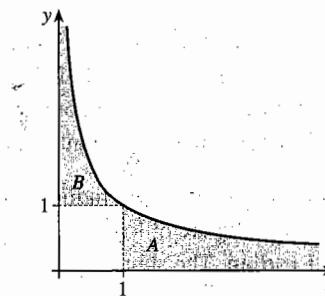
The ellipse is rotated about the x -axis to form an *ellipsoid*. Find the volume of this ellipsoid.

Deduce the volume of the ellipsoid formed if, instead, the ellipse had been rotated about the y -axis.



- 4 The diagram shows the curve $y = x^{-\frac{2}{3}}$.

- Show that the shaded area A is infinite.
- Find the shaded area B .
- Area A is rotated through 360° about the x -axis. Find the volume generated.
- Area B is rotated through 360° about the y -axis. Find the volume generated.



- 5 Investigate the equivalent areas and volumes to those in Question 4 for the equations

$$(i) \quad y = x^{-\frac{1}{3}}, \quad (ii) \quad y = x^{-\frac{1}{2}}.$$

- 6 Sketch the curve $y = 9 - x^2$, stating the coordinates of the turning point and of the intersections with the axes.

The finite region bounded by the curve and the x -axis is denoted by R .

- Find the area of R and hence or otherwise find $\int_0^9 \sqrt{9-y} dy$.
- Find the volume of the solid of revolution obtained when R is rotated through 360° about the x -axis.
- Find the volume of the solid of revolution obtained when R is rotated through 360° about the y -axis.

- 7 The region R is bounded by the part of the curve $y = (x-2)^{\frac{3}{2}}$ for which $2 \leq x \leq 4$, the x -axis, and the line $x = 4$. Find, in terms of π , the volume of the solid obtained when R is rotated through four right angles about the x -axis. (OCR)

18 Radians

This chapter introduces radians, an alternative to degrees for measuring angles. When you have completed it, you should

- know how to convert from degrees to radians and vice versa
- be able to use the formula $r\theta$ for the length of a circular arc, and $\frac{1}{2}r^2\theta$ for the area of a circular sector
- know the graphs and symmetry properties of $\cos\theta$, $\sin\theta$ and $\tan\theta$ when θ is in radians
- know the meaning of $\cos^{-1}x$, $\sin^{-1}x$ and $\tan^{-1}x$, their domains and ranges
- be able to solve trigonometric equations with roots expressed in radians.

18.1 Radians

Suppose that you were meeting angles for the first time, and that you were asked to suggest a unit for measuring them. It seems highly unlikely that you would suggest the degree, which was invented by the Babylonians in ancient times. The full circle, or the right angle, both seem more natural units.

However, the unit used in modern mathematics is the radian, illustrated in Fig. 18.1. This is particularly useful in differentiating trigonometric functions, as you will see if you go on to unit P2 or unit P3.

In a circle of radius 1 unit, radii joining the centre O to the ends of an arc of length 1 unit form an angle called **1 radian**. The abbreviation for radian is **rad**.

You can see immediately from this definition that there are 2π radians in 360° . This leads to the following conversion rule for radians to degrees and vice versa:

$$\pi \text{ rad} = 180^\circ$$

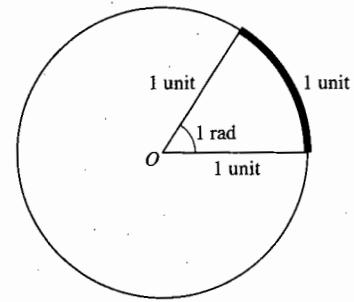


Fig. 18.1

You could calculate that 1 radian is equal to $57.295\dots^\circ$, but no one uses this conversion. It is simplest to remember that $\pi \text{ rad} = 180^\circ$, and to use this to convert between radians and degrees.

You can set your calculator to radian mode, and then work entirely in radians.

You might find on your calculator another unit for angle called the 'grad'; there are 100 grads to the right angle. Grads will not be used in this course.

Example 18.1.1

Convert 40° to radians, leaving your answer as a multiple of π .

Since 40° is $\frac{2}{9}$ of 180° , $40^\circ = \frac{2}{9}\pi$ rad.

It is worthwhile learning a few common conversions, so that you can think in both radians and degrees. For example, you should know and recognise the following conversions:

$$180^\circ = \pi \text{ rad}, \quad 90^\circ = \frac{1}{2}\pi \text{ rad}, \quad 45^\circ = \frac{1}{4}\pi \text{ rad}, \quad 30^\circ = \frac{1}{6}\pi \text{ rad}, \quad 60^\circ = \frac{1}{3}\pi \text{ rad}.$$

18.2 Length of arc and area of sector

Fig. 18.2 shows a circular arc, centre O and radius r , which subtends an angle θ rad at its centre. You can calculate the length of the circular arc by noticing that the length of the arc is the fraction $\frac{\theta}{2\pi}$ of the length $2\pi r$ of the circumference of the circle.

Let s be the arc length. Then

$$s = \frac{\theta}{2\pi} \times 2\pi r = r\theta.$$

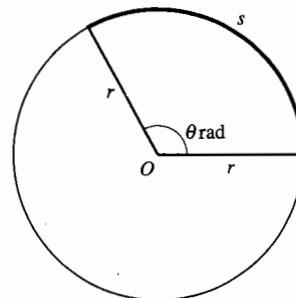


Fig. 18.2

You can use a similar argument to calculate the area of a sector.

The circular sector, centre O and radius r , shown shaded in Fig. 18.3, has an angle θ rad at the centre.

The area of the circular sector is the fraction $\frac{\theta}{2\pi}$ of the area πr^2 of the full circle.

Let A be the required area. Then

$$A = \frac{\theta}{2\pi} \times \pi r^2 = \frac{1}{2} r^2 \theta.$$

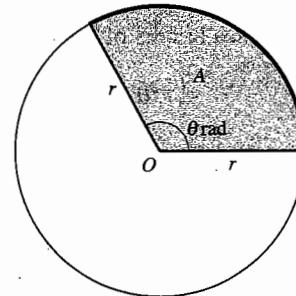


Fig. 18.3

The length of a circular arc with radius r and angle θ rad is $s = r\theta$.

The area of a circular sector with radius r and angle θ rad is $A = \frac{1}{2}r^2\theta$.

No units are given in the formulae above. The units are the appropriate units associated with the length; for instance, length in m and area in m².

Example 18.2.1

Find the perimeter and the area of the segment cut off by a chord PQ of length 8 cm from a circle centre O and radius 6 cm. Give your answers correct to 3 significant figures.

In problems of this type, it is helpful to start by thinking about the complete sector OPQ , rather than just the shaded segment of Fig. 18.4.

The perimeter of the segment consists of two parts, the straight part of length 8 cm, and the curved part; to calculate the length of the curved part you need to know the angle $\angle POQ$.

Call this angle θ . As triangle POQ is isosceles, a perpendicular drawn from O to PQ bisects both PQ and angle $\angle POQ$.

$$\sin \frac{1}{2}\theta = \frac{4}{6} = 0.666\dots, \text{ so } \frac{1}{2}\theta = 0.7297\dots \text{ and } \theta = 1.459\dots$$

Make sure that your calculator is in radian mode.

Then the perimeter d cm is given by $d = 8 + 6\theta = 16.756\dots$; the perimeter is 16.8 cm, correct to 3 significant figures.

To find the area of the segment, you need to find the area of the sector OPQ , and then subtract the area of the triangle OPQ . Using the formula $\frac{1}{2}bc \sin A$ for the area of a triangle, the area of the triangle POQ is given by $\frac{1}{2}r^2 \sin \theta$. Thus the area in cm^2 of the shaded region is

$$\begin{aligned} \frac{1}{2}r^2\theta - \frac{1}{2}r^2 \sin \theta &= \frac{1}{2} \times 6^2 \times 1.459\dots - \frac{1}{2} \times 6^2 \times \sin 1.459\dots \\ &= 8.381\dots \end{aligned}$$

The area is 8.38 cm^2 , correct to 3 significant figures.

It is worthwhile using your calculator to store the value of θ to use in the calculations. If you round θ to 3 significant figures and use the rounded value, you are liable to introduce errors.

In the course of Example 18.2.1, the notations $\sin \frac{1}{2}\theta = \frac{4}{6} = 0.666\dots$, $\sin \theta$ and $\sin 1.459\dots$ were used, without any indication that the angles were in radians. The convention is that when you see, for example, ‘ $\sin 12$ ’, you should read it as the sine of 12 radians. If it were the sine of 12° it would be written ‘ $\sin 12^\circ$ ’.

Example 18.2.2

A chord of a circle which subtends an angle of θ at the centre of the circle cuts off a segment equal in area to $\frac{1}{3}$ of the area of the whole circle.

- (a) Show that $\theta - \sin \theta = \frac{2}{3}\pi$.
- (b) Verify that $\theta = 2.61$ correct to 2 decimal places.

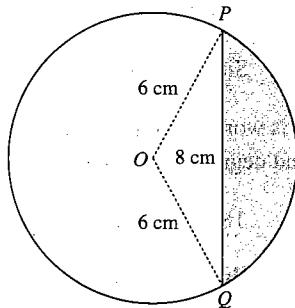


Fig. 18.4

(a) Let r cm be the radius of the circle. Using a method similar to the one in Example 18.2.1, the area of the segment is

$$\frac{1}{2}r^2\theta - \frac{1}{2}r^2 \sin \theta.$$

This is $\frac{1}{3}$ of the area of the whole circle if

$$\frac{1}{2}r^2\theta - \frac{1}{2}r^2 \sin \theta = \frac{1}{3}\pi r^2.$$

Multiplying by 2 and dividing by r^2 you find

$$\theta - \sin \theta = \frac{2}{3}\pi.$$

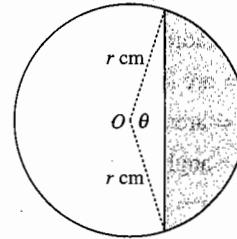


Fig. 18.5

(b) If you substitute $\theta = 2.61$ in the equation $f(\theta) \equiv \theta - \sin \theta$, you get $f(2.61) = 2.103\dots$, which is very close to $\frac{2}{3}\pi = 2.094\dots$.

This suggests that θ is close to 2.61, but it is not enough to show that it is '2.61 correct to 2 decimal places'. To do that you need to show that θ lies between 2.605 and 2.615.

It is obvious from Fig. 18.5 that θ lies between 0 and π , and that the shaded area gets larger as θ increases. So you have to show that the area is too small when $\theta = 2.605$ and too large when $\theta = 2.615$.

$$f(2.605) = 2.605 - \sin 2.605 = 2.093\dots \text{, and}$$

$$f(2.615) = 2.615 - \sin 2.615 = 2.112\dots.$$

The first of these is smaller, and the second larger, than $\frac{2}{3}\pi = 2.094\dots$.

It follows that the root of the equation is between 2.605 and 2.615; that is, the root is 2.61, correct to 2 decimal places.

Exercise 18A

1 Write each of the following angles in radians, leaving your answer as a multiple of π .

- | | | | |
|-----------------|-----------------|-----------------|---------------------------|
| (a) 90° | (b) 135° | (c) 45° | (d) 30° |
| (e) 72° | (f) 18° | (g) 120° | (h) $22\frac{1}{2}^\circ$ |
| (i) 720° | (j) 600° | (k) 270° | (l) 1° |

2 Each of the following is an angle in radians. Without using a calculator change these to degrees.

- | | | | |
|-----------------------|-----------------------|-----------------------|-----------------------|
| (a) $\frac{1}{3}\pi$ | (b) $\frac{1}{20}\pi$ | (c) $\frac{1}{5}\pi$ | (d) $\frac{1}{8}\pi$ |
| (e) $\frac{1}{9}\pi$ | (f) $\frac{2}{3}\pi$ | (g) $\frac{5}{8}\pi$ | (h) $\frac{3}{5}\pi$ |
| (i) $\frac{1}{45}\pi$ | (j) 6π | (k) $-\frac{1}{2}\pi$ | (l) $\frac{5}{18}\pi$ |

3 Without the use of a calculator write down the exact values of the following.

- | | | | |
|---------------------------|---------------------------|---------------------------|-----------------------------|
| (a) $\sin \frac{1}{3}\pi$ | (b) $\cos \frac{1}{4}\pi$ | (c) $\tan \frac{1}{6}\pi$ | (d) $\cos \frac{3}{2}\pi$ |
| (e) $\sin \frac{7}{4}\pi$ | (f) $\cos \frac{7}{6}\pi$ | (g) $\tan \frac{5}{3}\pi$ | (h) $\sin^2 \frac{2}{3}\pi$ |

- 4** The following questions refer to the diagram, where

r = radius of circle (in cm),

s = arc length (in cm),

A = area of sector (in cm^2),

θ = angle subtended at centre (in radians).

(a) $r = 7, \theta = 1.2$. Find s and A .

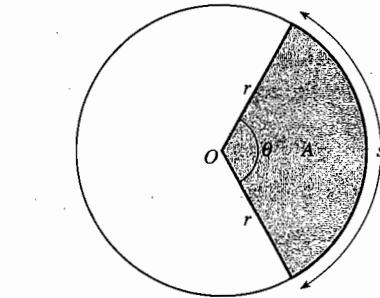
(b) $r = 3.5, \theta = 2.1$. Find s and A .

(c) $s = 12, r = 8$. Find θ and A .

(d) $s = 14, \theta = 0.7$. Find r and A .

(e) $A = 30, r = 5$. Find θ and s .

(g) $A = 64, s = 16$. Find r and θ .



(f) $A = 24, r = 6$. Find s .

(h) $A = 30, s = 10$. Find θ .

- 5** Find the area of the shaded segment in each of the following cases.

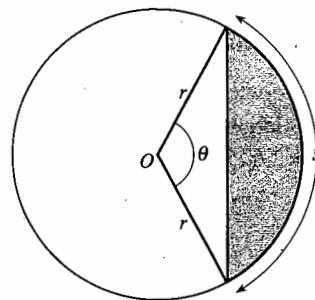
(a) $r = 5 \text{ cm}, \theta = \frac{1}{3}\pi$

(b) $r = 3.1 \text{ cm}, \theta = \frac{2}{5}\pi$

(c) $r = 28 \text{ cm}, \theta = \frac{5}{6}\pi$

(d) $r = 6 \text{ cm}, s = 9 \text{ cm}$

(e) $r = 9.5 \text{ cm}, s = 4 \text{ cm}$



- 6** Find the area of the segment cut off by a chord of length 10 cm from a circle radius 13 cm.

- 7** Find the perimeter of the segment cut off by a chord of length 14 cm from a circle radius 25 cm.

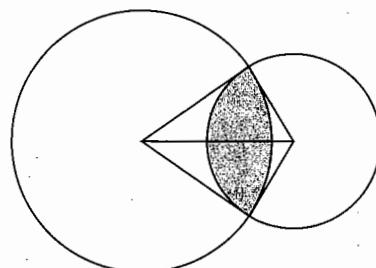
- 8** A chord of a circle which subtends an angle of θ at the centre cuts off a segment equal in area to $\frac{1}{4}$ of the area of the whole circle.

(a) Show that $\theta - \sin \theta = \frac{1}{2}\pi$.

(b) Verify that $\theta = 2.31$, correct to 2 decimal places.

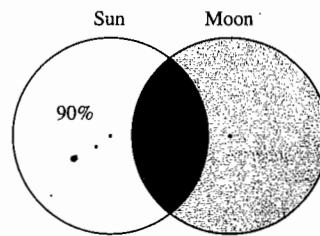
- 9** Two circles of radii 5 cm and 12 cm are drawn, partly overlapping. Their centres are 13 cm apart. Find the area common to the two circles.

- 10** The diagram shows two intersecting circles of radius 6 cm and 4 cm with centres 7 cm apart. Find the perimeter and area of the shaded region common to both circles.



- 11.** An eclipse of the sun is said to be 10% total when 10% of the area of the sun's disc is hidden behind the disc of the moon. A child models this with two discs, each of radius r cm, as shown.

- Calculate, in terms of r , the distance between the centres of the two discs.
- Calculate also the distance between the centres when the eclipse is 80% total.



18.3 Graphs of the trigonometric functions

The graphs of $y = \cos \theta$, $y = \sin \theta$ and $y = \tan \theta$ when the angle is measured in radians have a similar shape to those for $y = \cos \theta^\circ$, $y = \sin \theta^\circ$ and $y = \tan \theta^\circ$ which are drawn in Figs. 10.3, 10.4 and 10.5 on pages 139 and 140. The only change is the scale along the θ -axis.

The graphs of $y = \cos \theta$, $y = \sin \theta$ and $y = \tan \theta$, with θ in radians, are shown in Fig. 18.6, Fig. 18.7 and Fig. 18.8 respectively. These three graphs are drawn with the same scales on each axis.

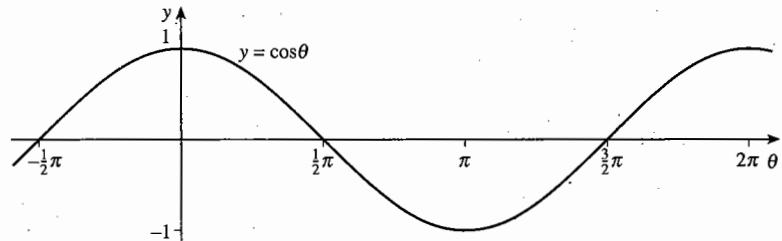


Fig. 18.6

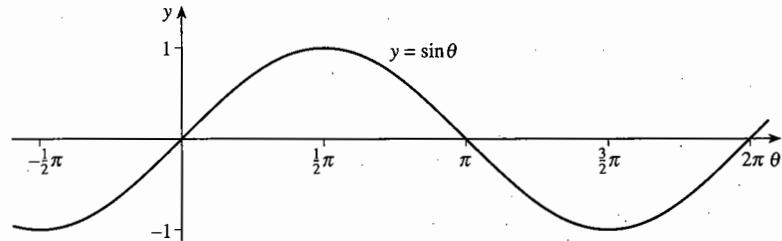


Fig. 18.7

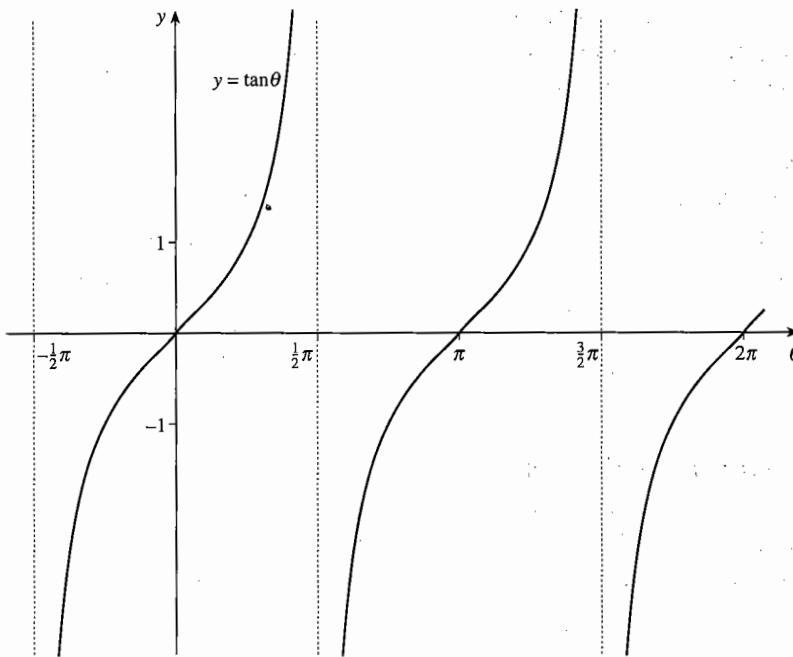


Fig. 18.8

If you were to draw the graphs in Section 10.1 and Section 10.2 with the same scales in each direction, they would be very wide and flat compared with the graphs shown here.

In fact radians are almost always used when you need to find the gradients of the graphs of $y = \cos \theta$, $y = \sin \theta$ and $y = \tan \theta$.

These graphs also have symmetry properties similar to those of the graphs of $y = \cos \theta^\circ$, $y = \sin \theta^\circ$ and $y = \tan \theta^\circ$.

Periodic properties: $\cos(\theta \pm 2\pi) = \cos \theta$ $\sin(\theta \pm 2\pi) = \sin \theta$ $\tan(\theta \pm \pi) = \tan \theta$

Odd/even properties: $\cos(-\theta) = \cos \theta$ $\sin(-\theta) = -\sin \theta$ $\tan(-\theta) = -\tan \theta$

Translation properties: $\cos(\theta - \pi) = -\cos \theta$ $\sin(\theta - \pi) = -\sin \theta$

$\cos(\pi - \theta) = -\cos \theta$ $\sin(\pi - \theta) = \sin \theta$ $\tan(\pi - \theta) = -\tan \theta$

Exercise 18B

- 1 Use the graphs of $y = \cos \theta$ and $y = \sin \theta$ to show that $\sin\left(\frac{1}{2}\pi - \theta\right) = \cos \theta$.

Use this property, and the symmetry properties of the sine, cosine and tangent functions in the box above to establish the following results.

- | | |
|---|--|
| (a) $\sin\left(\frac{3}{2}\pi + \theta\right) = -\cos \theta$ | (b) $\sin\left(\frac{1}{2}\pi + \theta\right) = \cos \theta$ |
| (c) $\cos\left(\frac{1}{2}\pi + \theta\right) = -\sin \theta$ | (d) $\sin\left(-\theta - \frac{1}{2}\pi\right) = -\cos \theta$ |

2 With the same axes, sketch $y = \tan \theta$ and $y = \frac{1}{\tan \theta}$. Show that $\tan(\frac{1}{2}\pi - \theta) = \frac{1}{\tan \theta}$.

3 Find the least positive value of α for which

- | | |
|--|---|
| (a) $\cos(\alpha - \theta) = \sin \theta$, | (b) $\sin(\alpha - \theta) = \cos(\alpha + \theta)$, |
| (c) $\tan \theta = \tan(\theta + \alpha)$, | (d) $\sin(\theta + 2\alpha) = \cos(\alpha - \theta)$, |
| (e) $\cos(2\alpha - \theta) = \cos(\theta - \alpha)$, | (f) $\sin(5\alpha + \theta) = \cos(\theta - 3\alpha)$. |

18.4 Inverse trigonometric functions

You have already met the notation \sin^{-1} , \cos^{-1} and \tan^{-1} a number of times. It is now time to give a more precise definition of the inverse trigonometric functions.

The functions $\cos x$, $\sin x$ and $\tan x$ are not one-one, as you can see from Section 18.3. It follows from Section 11.6 that they do not have inverses unless you restrict their domains of definition. The definitions given here assume that you are working in radians.

Fig. 18.9 shows how the domain of the cosine function is restricted to $0 \leq x \leq \pi$ to define the function \cos^{-1} .

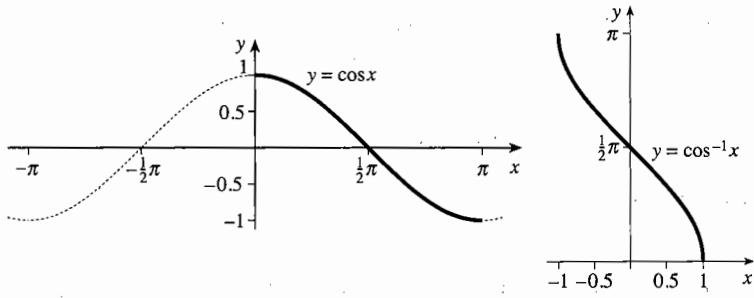


Fig. 18.9

Recall from Section 11.8 that if the graph of a function and its inverse are plotted on the same axes, then each is the reflection of the other in $y = x$. You can see that if the two graphs in Fig. 18.9 were superimposed, then the thicker part of the graph of $y = \cos x$ would be the reflection of $y = \cos^{-1} x$ in $y = x$, and vice versa.

Similarly Fig. 18.10 shows how the domain of the sine function is restricted to $-\frac{1}{2}\pi \leq x \leq \frac{1}{2}\pi$ to define the function \sin^{-1} .

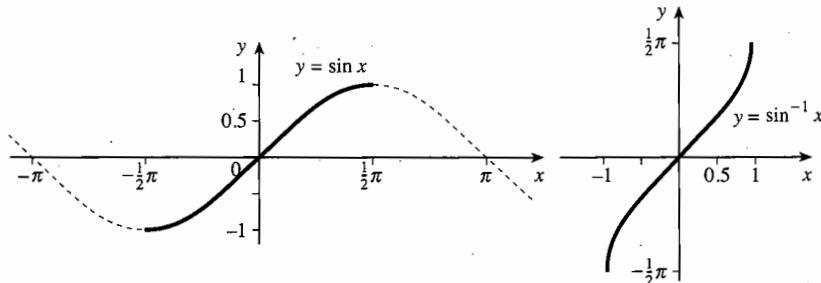


Fig. 18.10

Once again, the thicker part of the graph of $y = \sin x$ is the reflection of $y = \sin^{-1} x$ in the line $y = x$, and vice versa.

Fig. 18.11 shows the graph of the function \tan^{-1} , obtained by restricting the domain of the tangent function to $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$.

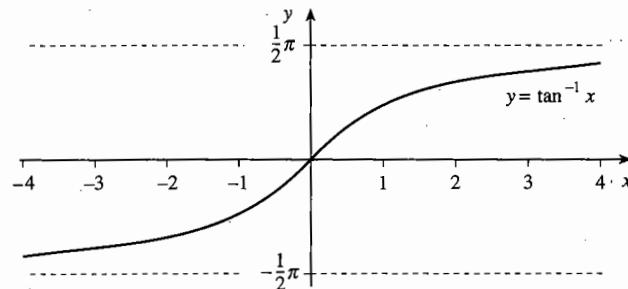


Fig. 18.11

\cos^{-1} has domain $-1 \leq x \leq 1$ and range $0 \leq \cos^{-1} x \leq \pi$.

\sin^{-1} has domain $-1 \leq x \leq 1$ and range $-\frac{1}{2}\pi \leq \sin^{-1} x \leq \frac{1}{2}\pi$.

\tan^{-1} has domain \mathbb{R} and range $-\frac{1}{2}\pi < \tan^{-1} x < \frac{1}{2}\pi$.

Exercise 18C

Do not use a calculator in Questions 1 to 5.

1 Find

- | | | | |
|---------------------------------------|-----------------------|-----------------------|---------------------------------------|
| (a) $\cos^{-1} \frac{1}{2}\sqrt{3}$, | (b) $\tan^{-1} 1$, | (c) $\cos^{-1} 0$, | (d) $\sin^{-1} \frac{1}{2}\sqrt{3}$, |
| (e) $\tan^{-1}(-\sqrt{3})$, | (f) $\sin^{-1}(-1)$, | (g) $\tan^{-1}(-1)$, | (h) $\cos^{-1}(-1)$. |

2 Find

- | | | | |
|--------------------------------------|-------------------------|-------------------------|--|
| (a) $\cos^{-1} \frac{1}{\sqrt{2}}$, | (b) $\sin^{-1}(-0.5)$, | (c) $\cos^{-1}(-0.5)$, | (d) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$. |
|--------------------------------------|-------------------------|-------------------------|--|

3 Find

- | | | | |
|-----------------------------|-----------------------------|----------------------------------|---------------------------|
| (a) $\sin(\sin^{-1} 0.5)$, | (b) $\cos(\cos^{-1}(-1))$, | (c) $\tan(\tan^{-1} \sqrt{3})$, | (d) $\cos(\cos^{-1} 0)$. |
|-----------------------------|-----------------------------|----------------------------------|---------------------------|

4 Find

- | | | | |
|--|---|--|------------------------------|
| (a) $\cos^{-1}(\cos \frac{3}{2}\pi)$, | (b) $\sin^{-1}(\sin \frac{13}{6}\pi)$, | (c) $\tan^{-1}(\tan \frac{1}{6}\pi)$, | (d) $\cos^{-1}(\cos 2\pi)$. |
|--|---|--|------------------------------|

5 Find

- | | | | |
|---|-------------------------------------|-------------------------------|---|
| (a) $\sin(\cos^{-1} \frac{1}{2}\sqrt{3})$, | (b) $\frac{1}{\tan(\tan^{-1} 2)}$, | (c) $\cos(\sin^{-1}(-0.5))$, | (d) $\tan(\cos^{-1} \frac{1}{2}\sqrt{2})$. |
|---|-------------------------------------|-------------------------------|---|

6 Use a graphical method to solve, correct to 3 decimal places, the equation $\cos x = \cos^{-1} x$. What simpler equation has this as its only root?

18.5 Solving trigonometric equations using radians

When you have a trigonometric equation to solve, you will sometimes want to find an angle in radians. The principles are similar to those that you used for working in degrees in Section 10.5, but the functions \cos^{-1} , \sin^{-1} and \tan^{-1} will now have the meanings that were assigned to them in Section 18.4.

Example 18.5.1

Solve the equation $\cos \theta = -0.7$, giving all the roots in the interval $0 \leq \theta \leq 2\pi$ correct to 2 decimal places.

Step 1 $\cos^{-1}(-0.7) = 2.346\dots$. This is one root in the interval $0 \leq \theta \leq 2\pi$.

Step 2 Use the symmetry property $\cos(-\theta) = \cos \theta$ to show that $-2.346\dots$ is another root. Note that $-2.346\dots$ is not in the required interval.

Step 3 Use the periodic property, $\cos(\theta \pm 2\pi) = \cos \theta$, to say that $-2.346\dots + 2\pi = 3.936\dots$ is a root in the required interval.

The roots of the equation $\cos \theta = -0.7$ in $0 \leq \theta \leq 2\pi$ are 2.35 and 3.94, correct to 2 decimal places.

Example 18.5.2

Solve the equation $\sin \theta = -0.2$, giving all the roots in the interval $-\pi \leq \theta \leq \pi$ correct to 2 decimal places.

Step 1 $\sin^{-1}(-0.2) = -0.201\dots$. This is one root in the interval $-\pi \leq \theta \leq \pi$.

Step 2 Another root of the equation is $\pi - (-0.201\dots) = 3.342\dots$, but this is not in the required interval.

Step 3 Subtracting 2π gives $-2.940\dots$, the other root in the interval $-\pi \leq \theta \leq \pi$.

Therefore the roots of $\sin \theta = -0.2$ in $-\pi \leq \theta \leq \pi$ are -2.94 and -0.20 , correct to 2 decimal places.

Example 18.5.3

Solve the equation $\cos(3\theta - 0.1) = 0.3$, giving all the roots in the interval $-\pi \leq \theta \leq \pi$ correct to 2 decimal places.

Let $3\theta - 0.1 = \phi$, so that the equation becomes $\cos \phi = 0.3$. As θ lies in the interval $-\pi \leq \theta \leq \pi$, $\phi = 3\theta - 0.1$ lies in the interval $-3\pi - 0.1 \leq \phi \leq 3\pi - 0.1$ which is $-9.524\dots \leq \phi \leq 9.324\dots$

The first part of the problem is to solve $\cos \phi = 0.3$ for $-9.524\dots \leq \phi \leq 9.324\dots$

Step 1 $\cos^{-1} 0.3 = 1.266\dots$. This is one root in the interval $-9.524\dots \leq \phi \leq 9.324\dots$

Step 2 Using the fact that the cosine function is even, another root is $-1.266\dots$.

Step 3 Adding 2π to and subtracting 2π from $\pm 1.266\dots$ gives $\pm 5.017\dots$ and $\pm 7.549\dots$ as the other roots.

Since $\theta = \frac{1}{3}(\phi + 0.1)$, the roots of the original equation are $-2.48, -1.64, -0.39, 0.46, 1.71, 2.55$, correct to 2 decimal places.

Example 18.5.4

Solve the equation $\tan \theta = \cos \theta$, giving all the roots in radians in the interval $0 \leq \theta \leq 2\pi$, correct to 2 decimal places.

If you have an equation like this it is usually a good idea to use the identity

$\tan \theta \equiv \frac{\sin \theta}{\cos \theta}$ to replace $\tan \theta$. The equation then becomes $\frac{\sin \theta}{\cos \theta} = \cos \theta$, which, on multiplying both sides by $\cos \theta$, gives

$$\sin \theta = \cos^2 \theta.$$

As it stands you cannot solve this equation, but if you use the identity $\cos^2 \theta + \sin^2 \theta \equiv 1$ to replace $\cos^2 \theta$, you get the equation $\sin \theta = 1 - \sin^2 \theta$, which you can rewrite as

$$\sin^2 \theta + \sin \theta - 1 = 0.$$

This is a quadratic equation in $\sin \theta$, which you can solve using the quadratic formula in Section 4.4:

$$\sin \theta = \frac{-1 \pm \sqrt{1^2 - 4 \times 1 \times (-1)}}{2}, \text{ giving } \sin \theta = 0.618\dots \text{ or } \sin \theta = -1.618\dots.$$

One root is $\sin^{-1} 0.618\dots = 0.666\dots$. The other root of $\sin \theta = 0.618\dots$ in the interval, obtained from the symmetry of $\sin \theta$, is $\pi - 0.666\dots = 2.475\dots$.

The equation $\sin \theta = -1.618\dots$ has no roots, as $\sin \theta$ has the property that $-1 \leq \sin \theta \leq 1$.

So the required roots are 0.67 and 2.48 , correct to 2 decimal places.

Exercise 18D

- 1 Find in radians correct to 2 decimal places, the two smallest positive values of θ for which
- | | | |
|--|---|---|
| (a) $\sin \theta = 0.12$, | (b) $\sin \theta = -0.86$, | (c) $\sin \theta = 0.925$, |
| (d) $\cos \theta = 0.81$, | (e) $\cos \theta = -0.81$, | (f) $\cos \theta = \frac{1}{\sqrt{3}}$, |
| (g) $\tan \theta = 4.1$, | (h) $\tan \theta = -0.35$, | (i) $\tan \theta = 0.17$, |
| (j) $\sin(\pi + \theta) = 0.3$, | (k) $\cos\left(\frac{1}{2}\pi - \theta\right) = -0.523$, | (l) $\tan\left(\frac{1}{2}\pi - \theta\right) = -4$, |
| (m) $\sin(2\theta + \frac{1}{3}\pi) = 0.123$, | (n) $\sin\left(\frac{1}{6}\pi - \theta\right) = 0.5$, | (o) $\cos\left(3\theta - \frac{2}{3}\pi\right) = 0$. |

2 Find all values of θ in the interval $-\pi \leq \theta \leq \pi$ which satisfy each of the following equations, giving your answers correct to 2 decimal places where appropriate.

(a) $\sin \theta = 0.84$

(d) $\sin \theta = -0.73$

(g) $4 \sin \theta = 3 \cos \theta$

(b) $\cos \theta = 0.27$

(e) $\cos \theta = -0.15$

(h) $3 \sin \theta = \frac{1}{\sin \theta}$

(c) $\tan \theta = 1.9$

(f) $4 \tan \theta + 5 = 0$

(i) $3 \sin \theta = \tan \theta$

3 Find all the solutions in the interval $0 < x \leq 2\pi$ of each of the following equations.

(a) $\cos 2x = \frac{1}{4}$

(d) $\cos 4x = -\frac{1}{5}$

(b) $\tan 3x = 3$

(e) $\tan 2x = 0.5$

(c) $\sin 2x = -0.62$

(f) $\sin 3x = -0.45$

4 Find the roots in the interval $-\pi < t \leq \pi$ of each of the following equations.

(a) $\cos 3t = \frac{3}{4}$

(d) $\cos 2t = 0.264$

(b) $\tan 2t = -2$

(e) $\tan 5t = 0.7$

(c) $\sin 3t = -0.32$

(f) $\sin 2t = -0.42$

5 Find the roots (if there are any) in the interval $-\pi < \theta \leq \pi$ of the following equations.

(a) $\cos \frac{1}{2}\theta = \frac{1}{3}$

(d) $\cos \frac{1}{3}\theta = \frac{1}{2}$

(b) $\tan \frac{2}{3}\theta = -5$

(e) $\tan \frac{2}{3}\theta = 0.5$

(c) $\sin \frac{1}{5}\theta = -\frac{1}{5}$

(f) $\sin \frac{2}{5}\theta = -0.4$

6 Without using a calculator, find the exact roots of the following equations, if there are any, giving your answers as multiples of π in the interval $0 < \theta \leq 2\pi$.

(a) $\sin(2\theta - \frac{1}{3}\pi) = \frac{1}{2}$

(d) $\tan(\frac{3}{2}\theta - \frac{1}{6}\pi) = -\sqrt{3}$

(g) $\cos(\frac{1}{5}\theta - \frac{5}{18}\pi) = 0$

(b) $\tan(2\theta - \frac{1}{6}\pi) = 0$

(e) $\cos(2\theta - \frac{5}{18}\pi) = -\frac{1}{2}$

(h) $\tan(3\theta - \pi) = -1$

(c) $\cos(3\theta + \frac{1}{4}\pi) = \frac{1}{2}\sqrt{3}$

(f) $\sin(\frac{1}{2}\theta + \frac{5}{18}\pi) = 1$

(i) $\sin(\frac{1}{4}\theta - \frac{1}{9}\pi) = 0$

7 Find the roots (if there are any) in the interval $-\pi < \theta \leq \pi$ of the following equations.

(a) $\tan \theta = 2 \cos \theta$

(d) $\sin^2 \theta = 2 \cos^2 \theta - 1$

(b) $\sin^2 \theta = 2 \cos \theta$

(e) $2 \sin \theta = \tan \theta$

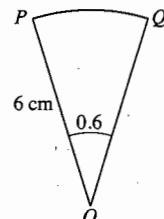
(c) $\sin^2 \theta = 2 \cos^2 \theta$

(f) $\tan^2 \theta = 2 \cos^2 \theta$

Miscellaneous exercise 18

1 The diagram shows a sector of a circle with centre O and radius 6 cm.

Angle $\angle POQ = 0.6$ radians. Calculate the length of arc PQ and the area of sector POQ . (OCR)

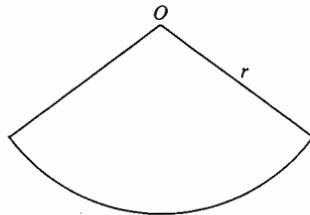


2 A sector OAB of a circle, of radius a and centre O , has $\angle AOB = \theta$ radians. Given that the area of the sector OAB is twice the square of the length of the arc AB , find θ . (OCR)

- 3 The diagram shows a sector of a circle, with centre O and radius r . The length of the arc is equal to half the perimeter of the sector.

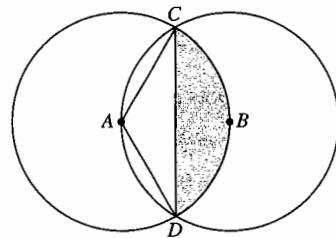
Find the area of the sector in terms of r .

(OCR)



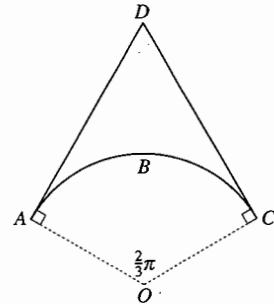
- 4 The diagram shows two circles, with centres A and B , intersecting at C and D in such a way that the centre of each lies on the circumference of the other. The radius of each circle is 1 unit. Write down the size of angle CAD and calculate the area of the shaded region (bounded by the arc CBD and the straight line CD). Hence show that the area of the region common to the interiors of the two circles is approximately 39% of the area of one circle.

(OCR)



- 5 In the diagram, ABC is an arc of a circle with centre O and radius 5 cm. The lines AD and CD are tangents to the circle at A and C respectively. Angle $AOC = \frac{2}{3}\pi$ radians. Calculate the area of the region enclosed by AD , DC and the arc ABC , giving your answer correct to 2 significant figures.

(OCR)



- 6 Find, either to 2 decimal places or an exact multiple of π , all values of x in the interval $-\pi < x \leq \pi$ satisfying the following equations.

(a) $\sin x = -0.16$	(b) $\cos x(1 + \sin x) = 0$	(c) $(1 - \tan x)\sin x = 0$
(d) $\sin 2x = 0.23$	(e) $\cos\left(\frac{3}{4}\pi - x\right) = 0.832$	(f) $\tan(3x - 17) = 3$

- 7 The electric current, c amperes, in a wire is given by the equation

$$c = 5\sin\left(100\pi t + \frac{1}{6}\pi\right),$$

where t denotes the time in seconds.

- (a) Calculate the period of the oscillation, and find the number of oscillations per second.
 (b) Find the first three positive values of t for which $c = 2$, giving your answers correct to 3 decimal places.

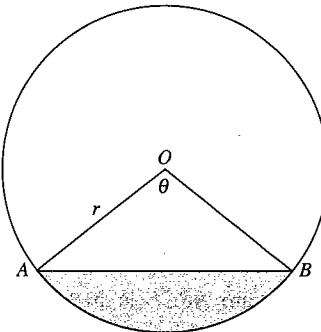
- 8 An oscillating particle has displacement y metres, where y is given by $y = a \sin(kt + \alpha)$, where a is measured in metres, t is measured in seconds and k and α are constants. The time for a complete oscillation is T seconds. Find

- (a) k in terms of T , (b) the number, in terms of k , of complete oscillations per second.

- 9 The diagram shows a circle with centre O and radius r , and a chord AB which subtends an angle θ radians at O . Express the area of the shaded segment bounded by the chord AB in terms of r and θ .

Given that the area of this segment is one-third of the area of triangle OAB , show that $3\theta - 4 \sin \theta = 0$.

Find the positive value of θ satisfying $3\theta - 4 \sin \theta = 0$ to within 0.1 radians, by tabulating values of $3\theta - 4 \sin \theta$ and looking for a sign change, or otherwise. (OCR)

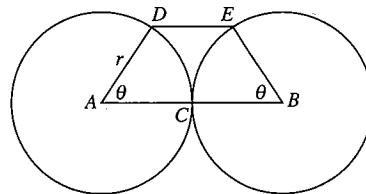


- 10 The diagram shows two circles, with centres A and B , which touch at C . The radius of each circle is r . The points D and E , one on each circle, are such that DE is parallel to the line ACB . Each of the angles DAC and EBC is θ radians, where $0 < \theta < \pi$. Express the length of DE in terms of r and θ .

The length of DE is equal to the length of each of the minor arcs CD and CE .

- Show that $\theta + 2 \cos \theta - 2 = 0$.
- Sketch the graph of $y = \cos \theta$ for $0 < \theta < \frac{1}{2}\pi$. By drawing on your graph a suitable straight line, the equation of which must be stated, show that the equation $\theta + 2 \cos \theta - 2 = 0$ has exactly one root in the interval $0 < \theta < \frac{1}{2}\pi$.

Verify by calculation that θ lies between 1.10 and 1.11. (OCR)



- 11 The diagram shows an arc ABC of a circle with centre O and radius r , and the chord AC . The length of the arc ABC is s , and angle $AOC = \theta$ rad. Express θ in terms of r and s , and deduce that the area of triangle AOC may be expressed as

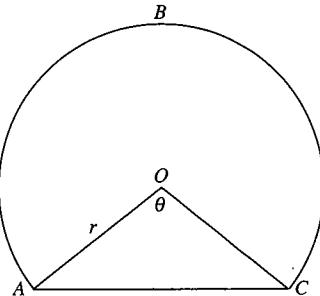
$$\frac{1}{2}r^2 \sin\left(2\pi - \frac{s}{r}\right).$$

Show, by a graphical argument based on a sketch of $y = \sin x$, or otherwise, that

$$\sin(2\pi - \alpha) = -\sin \alpha,$$

where α is any angle measured in radians.

Given that the area of triangle AOC is equal to one-fifth of the area of the major sector $OABC$, show that $\frac{s}{r} + 5 \sin\left(\frac{s}{r}\right) = 0$.



12 By using a graphical method, or otherwise, establish the identities

$$(a) \sin^{-1} x + \cos^{-1} x \equiv \frac{1}{2}\pi,$$

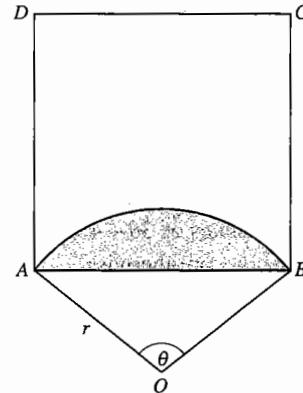
$$(b) \tan^{-1} x + \tan^{-1} \left(\frac{1}{x} \right) \equiv \frac{1}{2}\pi \text{ or } -\frac{1}{2}\pi.$$

13 The diagram shows a sector of a circle with centre O and radius r , and a chord AB which subtends an angle θ radians at O , where $0 < \theta < \pi$.

A square $ABCD$ is drawn, as shown in the diagram. It is given that the area of the shaded segment is exactly one-eighth of the area of the square. Show that

$$2\theta - 2\sin\theta + \cos\theta - 1 = 0.$$

Hence show that θ lies between 1 and 2, and use a tabulation method to find θ correct to 1 decimal place.



14 Give the domains and the ranges of the following functions.

$$(a) 2\sin^{-1} x - 4$$

$$(b) 2\sin^{-1}(x - 4)$$

15 Solve the equation $3\sin^2\theta + 4\cos\theta = 4$, giving all the roots, correct to 2 decimal places, in the interval $0 \leq \theta \leq 2\pi$.

16 Solve the following equations giving any roots in terms of π in the interval $-2\pi \leq 0 \leq 2\pi$.

$$(a) 2\cos^2\theta + \sin^2\theta = 0$$

$$(b) 2\cos^2\theta + \sin^2\theta = 1$$

$$(c) 2\cos^2\theta + \sin^2\theta = 2$$

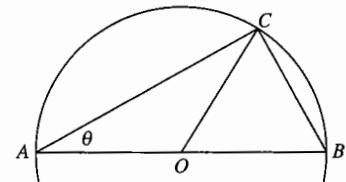
Revision exercise 3

- 1 Sketch the graphs of $y = 4 - x$ and $y = x^2 + 2x$, and calculate the coordinates of their points of intersection. Calculate the area of the finite region between the two graphs.
- 2 (a) Find the coordinates of the stationary points on the graph of $y = x^3 - 3x + 3$.
(b) Calculate the coordinates of the point for which $\frac{d^2y}{dx^2} = 0$.
(c) Find the equation of the normal to the curve at the point where $x = -2$.
(d) Calculate the area enclosed between the curve, the x -axis and the lines $x = 0$ and $x = 2$.
- 3 Without using a calculator, draw a sketch of $y = x^4 - x^5$, indicating those points for which $\frac{d^2y}{dx^2}$ is positive and those for which $\frac{d^2y}{dx^2}$ is negative.
- 4 Let n be a positive integer. Sketch the graphs of $y = x^n$ and $y = x^{\frac{1}{n}}$ for $x \geq 0$ and find the area of the region which they enclose.
- 5 A curve has an equation which satisfies $\frac{d^2y}{dx^2} = 5$. The curve passes through the point $(0, 4)$ and the gradient of the tangent at this point is 3. Find y in terms of x .
- 6 The part of the curve $y = kx^2$, where k is a constant, between $y = 1$ and $y = 3$ is rotated through 360° about the y -axis. Given that the volume generated is 12π , calculate the value of k .
- 7 Find the value of $\int_1^3 (x^3 - 6x^2 + 11x - 6) dx$. Interpret your result geometrically.
- 8 The region R is bounded by the x -axis, the line $x = 16$ and the curve with equation $y = 6 - \sqrt{x}$, where $0 \leq x \leq 36$. Find, in terms of π , the volume of the solid generated when R is rotated through one revolution about the x -axis. (OCR, adapted)
- 9 Calculate the area of the region in the first quadrant bounded by the curve with equation $y = \sqrt{9 - x}$ and the axes.
- 10 (a) Draw a sketch of the part of the curve $y = \frac{1}{\sqrt{x}}$ from $x = 1$ to $x = 4$.
(b) Calculate the area of the region R bounded by the curve, the x -axis and the lines $x = 1$ and $x = 4$.
- 11 The angle made by a wasp's wings with the horizontal is given by the equation $\theta = 0.4 \sin 600t$ radians, where t is the time in seconds. How many times a second do its wings oscillate?

12 Determine whether the point $(1, 2, -1)$ lies on the line passing through $(3, 1, 2)$ and $(5, 0, 5)$.

13 The figure shows part of a circle with centre O and radius r . Points A , B and C lie on the circle such that AB is a diameter. Angle $BAC = \theta$ radians.

- (a) Find angle AOC in terms of θ , and use the cosine rule in triangle AOC to express AC^2 in terms of r and θ .
- (b) By considering triangle ABC , write down the length of AC in terms of r and θ , and deduce that $\cos 2\theta = 2\cos^2 \theta - 1$. (OCR)



14 Sketch the graph of $y = \frac{9}{2x+3}$ for positive values of x .

The part of the curve between $x = 0$ and $x = 3$ is rotated through 2π radians about the x -axis. Calculate the volume of the solid of revolution formed.

15 Differentiate each of the following functions with respect to x .

$$(a) (x^3 + 2x - 1)^3 \quad (b) \sqrt{\frac{1}{x^2 + 1}}$$

16 A teacher received a salary of £12 800 in his first full year of teaching. He models his future salary by assuming it to increase by a constant amount of £950 each year up to a maximum of £20 400.

- (a) How much will he earn in his fifth year of teaching?
- (b) In which year does he first receive the maximum salary?
- (c) Determine expressions for the total amount he will have received by the end of his n th year of teaching, stating clearly for which values of n each is valid.

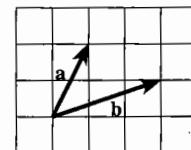
His twin sister chose accountancy as her profession. She started her career in the same year as he did. Her first year's salary was £13 500, and she can expect her salary to increase at a constant rate of 5% each year.

- (d) Select an appropriate mathematical model and use this to determine her annual salary in her n th year as an accountant.
- (e) Show that she earns less than he in their 4th year of working.
- (f) Which is the first year after that in which he earns less than she? (OCR)

17 A geometric progression has first term 6 and common ratio 0.75. Find the sum of the first ten terms of this geometric progression, giving your answer correct to 2 decimal places. Write down the sum to infinity of the geometric progression.

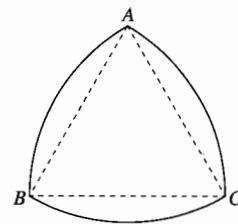
18 The vectors \mathbf{a} and \mathbf{b} are shown on the grid of unit squares.

- (a) Calculate $|\mathbf{a} + \mathbf{b}|$.
- (b) Calculate $\mathbf{a} \cdot \mathbf{b}$.
- (c) Calculate the angle between \mathbf{a} and \mathbf{b} . (OCR)



- 19** A coin is made by starting with an equilateral triangle ABC of side 2 cm. With centre A an arc of a circle is drawn joining B to C . Similar arcs join C to A and A to B .

Find, exactly, the perimeter of the coin and the area of one of its faces.



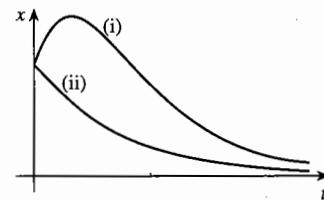
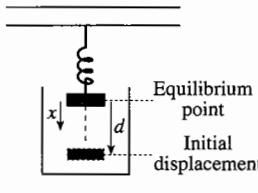
- 20** Find the value of t such that the variable vector $\begin{pmatrix} 4 \\ 6 \\ 10 \end{pmatrix} + t \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$ is perpendicular to the vector $\begin{pmatrix} 4 \\ 2 \\ -7 \end{pmatrix}$. Find also the angle between the vectors $\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 2 \\ -7 \end{pmatrix}$. Give your answer in degrees correct to 1 decimal place.

- 21** Draw sketches of possible graphs for which the following data hold. Consider only the domain $0 \leq x \leq 5$, and assume that the graph of $y = f''(x)$ is smooth.

- $f(0) = 0$, $f(2) = 5$, $f'(2) = 0$, $f''(2) < 0$, $f(4) = 3$, $f'(4) = 0$, $f''(4) > 0$
- $f(0) = 0$, $f''(1) < 0$, $f(2) = 5$, $f'(2) = 0$, $f''(3) > 0$, $f(4) = 7$
- $f(0) = 0$, $f'(0) = -1$, $f'(1) = 0$, $f(3) = 0$, $f'(3) = 2$, $f''(3) = 0$, $f''(4) < 0$
- $f(0) = 1$, $f'(0) = 1$, $f''(0) = 1$ and $f''(x)$ increases as x increases
- $f(0) = 1$, $f'(0) = 0$, $f'(x) < 0$ for $0 < x < 5$, $f(5) = f'(5) = 0$
- $f(0) = 3$, $f'(0) = -2$, $f''(x) > 0$ for $0 < x < 5$, $f(5) = f'(5) = 0$

- 22** The diagram shows a mass M suspended in a viscous liquid by an elastic spring with one end fixed to a beam. The mass has a natural position of equilibrium, and its displacement downwards from this position is given by x .

The mass is given a displacement d from its equilibrium position and is then given an initial velocity. Graph (i) shows x plotted against time t when the initial velocity is away from the equilibrium position. Graph (ii) shows x plotted against time when the initial velocity is small but towards the equilibrium position.



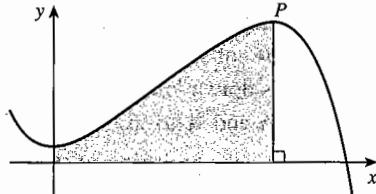
Make a sketch copy of the graph and add to it a sketch of x against t

- when the initial velocity is large and towards the equilibrium position, and
- when M is simply released at displacement d .

Describe the four graphs, (i), (ii) and your answers to (a) and (b), in terms of $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$.

- 23 The diagram shows the curve $y = (1 - 4x)^5 + 20x$. The curve has a maximum point at P as shown.

Show that the curve has a minimum point which lies on the y -axis and calculate the area of the region shaded in the diagram.



- 24 (a) Sketch the graphs of $y = x^2$ and $y = (x - 4)^2$, and find the coordinates of the point P where they intersect.
 (b) The region bounded by the x -axis between $x = 0$ and $x = 4$, the graph of $y = x^2$ between the origin and P , and the graph of $y = (x - 4)^2$ between P and the x -axis is rotated through 2π radians about the x -axis. Calculate in terms of π the volume of the solid of revolution formed.

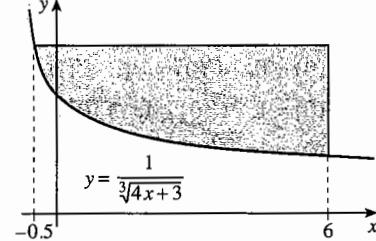
- 25 A pyramid has a square base $ABCD$ of side 8 cm. The diagonals AC and BD of the base meet at O . The point E is midway between O and A . The vertex V is at a height of 6 cm vertically above E .

Calculate to the nearest tenth of a degree the angle between DV and BV .

- 26 The diagram shows the curve

$$y = \frac{1}{\sqrt[3]{4x+3}}.$$

Find the area of the shaded region.



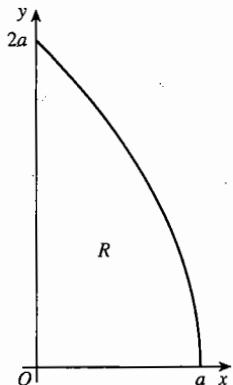
- 27 Use the methods of Exercise 6A to calculate an approximation to the gradient of the graph $y = \sin x$ for $x = 0$ and $x = \frac{1}{4}\pi$. Use the symmetry properties of $y = \sin x$ to predict the gradients at other values of x in the interval $0 \leq x \leq 2\pi$, and use your results to sketch a graph of $\frac{dy}{dx}$ against x . Make a conjecture about the equation of this graph.

- 28 The diagram shows the region R , which is bounded by the axes and the part of the curve $y^2 = 4a(a - x)$ lying in the first quadrant. Find, in terms of a ,

- (a) the area of R ,
 (b) the volume, V_x , of the solid formed when R is rotated completely about the x -axis.

The volume of the solid formed when R is rotated completely about the y -axis is V_y . Show that $V_y = \frac{8}{15} V_x$.

The region S , lying in the first quadrant, is bounded by the curve $y^2 = 4a(a - x)$ and the lines $y = a$ and $y = 2a$. Find, in terms of a , the volume of the solid formed when S is rotated completely about the y -axis. (OCR, adapted)



Practice examination 1

Time 1 hour 45 minutes

Answer all the questions.

The use of an electronic calculator is expected, where appropriate.

- 1** The 3rd and 4th terms of a geometric progression are 12 and 8 respectively. Find the sum to infinity of the progression. [5]

2 (i) Find $\int \left(\frac{1}{x^2} - \frac{1}{\sqrt{x}} \right) dx$. [4]

- (ii) A curve passes through the point $(1,0)$ and is such that $\frac{dy}{dx} = \frac{1}{x^2} - \frac{1}{\sqrt{x}}$. Find the equation of the curve. [2]

- 3** Points A and B have coordinates $(3,2)$ and $(-1,4)$ respectively.

- (i) Find the equation of the straight line which is perpendicular to AB and which passes through its mid-point. [4]

- (ii) Verify that the point C with coordinates $(0,1)$ lies on this line, and calculate the area of the triangle ABC . [3]

- 4** (i) Show that the equation $3\tan\theta = 2\cos\theta$ may be written in the form

$$2\sin^2\theta + 3\sin\theta - 2 = 0. \quad [3]$$

- (ii) Hence solve the equation $3\tan\theta = 2\cos\theta$, for $0 \leq \theta \leq 2\pi$. [4]

- 5** (i) Express the quadratic polynomial $x^2 + 6x + 3$ in completed square form. [2]

- (ii) Hence, or otherwise,

- (a) find the coordinates of the vertex of the graph of $y = x^2 + 6x + 3$, [2]

- (b) solve the inequality $x^2 + 6x + 3 < 0$, leaving your answer in an exact form. [3]

- 6** The functions f and g are defined by

$$f:x \mapsto \frac{1}{x}, \quad 0 < x \leq 3,$$

$$g:x \mapsto 2x - 1, \quad x \in \mathbb{R}.$$

- (i) Using a graphical method, or otherwise, find the range of f . [1]

- (ii) Calculate $gf(2)$. [1]

- (iii) Find an expression in terms of x for $g^{-1}(x)$. [2]

- (iv) Sketch, in a single diagram, the graphs of $y = g(x)$ and $y = g^{-1}(x)$, and state a geometrical relationship between these graphs. [3]

- 7 (i) Solve the simultaneous equations

$$\begin{aligned}2x^2 + xy &= 10, \\x + y &= 3.\end{aligned}$$

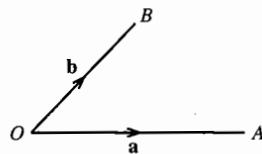
[5]

- (ii) Show that the simultaneous equations

$$\begin{aligned}2x^2 + xy &= 10, \\x + y &= a.\end{aligned}$$

always have two distinct solutions, for all possible values of the constant a . [3]

8



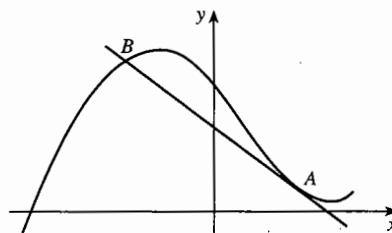
The diagram shows the origin O , and points A and B whose position vectors are denoted by \mathbf{a} and \mathbf{b} respectively.

- (i) Copy the diagram, and show the positions of the points P and Q such that $\vec{OP} = 3\mathbf{a}$ and $\vec{OQ} = \mathbf{a} + \mathbf{b}$. [2]
- (ii) Given that $\mathbf{a} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, evaluate the scalar product $\vec{OQ} \cdot \vec{BP}$. [3]
- (iii) Calculate the acute angle between the lines OQ and BP , giving your answer correct to the nearest degree. [3]

- 9 A circle with centre O has radius r cm. A sector of the circle, which has an angle of θ radians at O , has perimeter 6 cm.

- (i) Show that $\theta = \frac{6}{r} - 2$, and express the area A cm^2 of the sector in terms of r . [4]
- (ii) Show that A is a maximum, and not a minimum, when $r = \frac{3}{2}$, and calculate the corresponding value of θ . [5]

10



The diagram shows the curve $y = 4x^3 - 4x^2 - 10x + 12$ and the tangent at the point A where $x = 1$.

- (i) Find the equation of this tangent. [4]
- (ii) Verify that this tangent meets the curve again at the point B with x -coordinate -1 . [2]
- (iii) Calculate the area of the region which lies between the curve and the tangent AB . [5]

Practice examination 2

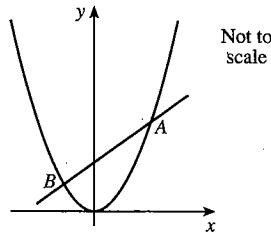
Time 1 hour 45 minutes

Answer all the questions.

The use of an electronic calculator is expected, where appropriate.

- 1 Use the binomial theorem to expand $(3x + 2)^4$ in powers of x . [3]
- 2 A solid is made by rotating the part of the curve with equation $y = (3x - 1)^{\frac{3}{2}}$ from $x = \frac{1}{3}$ to $x = \frac{2}{3}$ through four right angles about the x -axis. Find the volume of the solid, giving your answer in terms of π . [5]
- 3 The point A has coordinates $(7, 7)$ and the line l has equation $x + 3y = 8$.
 - (i) Find the equation of the line that passes through A and is perpendicular to l . [3]
 - (ii) Hence find the coordinates of the foot of the perpendicular from A to l . [3]
- 4 (i) Sketch the graph of $y = \cos 2x^\circ$ for $0 \leq x \leq 360$. [2]
(ii) Solve the equation $\cos 2x^\circ = -0.7$, for x in the interval $0 \leq x \leq 360$. [4]

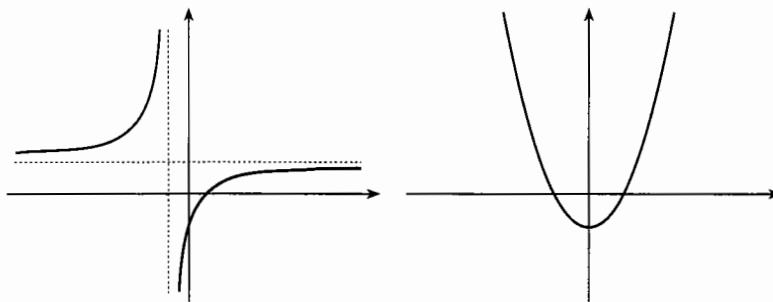
5



The diagram shows the curve $y = x^2$ and the line $y = 3 + 2x$.

- (i) Prove algebraically that A is the point $(3, 9)$ and calculate the coordinates of the other point of intersection, B , of the line with the curve. [5]
- (ii) Calculate the area of the region enclosed between the curve and the line segment AB . [3]
- 6 An arithmetic progression has first term a and common difference d .
 - (i) Write down expressions, in terms of a and d , for the second and sixth terms of the progression. [1]
 - (ii) The first, second and sixth terms of this arithmetic progression are also the first three terms of a geometric progression. Prove that $d = 3a$. [3]
 - (iii) Given that $a = 2$, find the sum of the first 15 terms of each progression. [4]

7



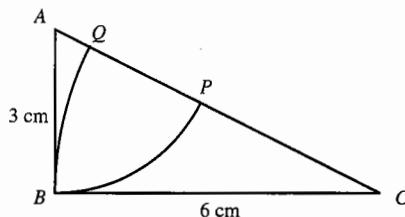
The diagram shows the graphs of two functions f and g defined by

$$f: x \mapsto \frac{3x-1}{3x+1}, \quad x \in \mathbb{R}, x \neq -\frac{1}{3},$$

$$g: x \mapsto 3x^2 - 1, \quad x \in \mathbb{R}.$$

- (i) Find the range of each of these functions. [2]
- (ii) One of these functions does not have an inverse. Identify this function, and give a clear reason why it does not have an inverse. [3]
- (iii) Find the inverse of the other function, expressing your answer in a form similar to that given in the definition of the function. [4]

8



The diagram shows triangle ABC , in which angle B is a right angle, $AB = 3$ cm and $BC = 6$ cm. The circular arc BP has centre A and radius 3 cm, and the circular arc BQ has centre C and radius 6 cm. Calculate

- (i) the size of angle A , giving your answer in radians correct to 4 significant figures; [2]
- (ii) the area of the region BPC , bounded by the arc BP and the lines PC and CB ; [2]
- (iii) the area of the region BPQ , bounded by the line PQ and the arcs BP and BQ ; [3]
- (iv) the perimeter of the region BPQ , bounded by the line PQ and the arcs BP and BQ . [3]

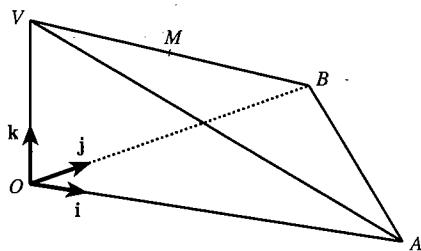
9 (a) Differentiate $\sqrt{x} + \frac{1}{2x+1}$ with respect to x . [4]

(b) A curve has equation $y = x^3 - 3x^2 + 12x$.

(i) Find $\frac{dy}{dx}$. [2]

(ii) Hence, by completing the square or otherwise, prove that the gradient of the curve is never less than 9. [4]

10



The diagram shows a triangular pyramid $OABV$, whose base is the right-angled triangle OAB and whose vertical height is OV . The perpendicular unit vectors \mathbf{i} , \mathbf{j} and \mathbf{k} are directed along OA , OB and OV as shown, and the position vectors of A , B and V are given by

$$\vec{OA} = 10\mathbf{i}, \quad \vec{OB} = 8\mathbf{j}, \quad \vec{OV} = 6\mathbf{k}.$$

- (i) The point M is the mid-point of VB . Find the position vector of M and the length of OM . [3]
- (ii) The point P lies on OA , and has position vector $p\mathbf{i}$. Show that the value of the scalar product $\overrightarrow{VB} \cdot \overrightarrow{MP}$ is -14 . [3]
- (iii) Explain briefly how you can deduce from part (ii) that MP is never perpendicular to VB for any value of p . [1]
- (iv) For the case where P is at the mid-point of OA , find angle PMB , giving your answer correct to the nearest degree. [3]

Answers

1 Coordinates, points and lines

Exercise 1A (page 6)

- 1 (a) 13 (b) 5
 (c) $\sqrt{50}$ (d) $\sqrt{52}$
 (e) $17a$ (f) $\sqrt{20}$
 (g) 23 (h) $9a$
 (i) $\sqrt{2(p-q)^2}$ (j) $\sqrt{50q^2}$
- 5 (a) (4,13) (b) (1,8)
 (c) $(-\frac{1}{2}, -4\frac{1}{2})$ (d) $(-5\frac{1}{2}, 4\frac{1}{2})$
 (e) $(2p+3, 2p-3)$ (f) $(p+4, -2)$
 (g) $(3p, 3q)$ (h) $(a+3, b+1)$
- 6 (2,3)
 7 (7,10)
 9 A
- 10 (a) 2 (b) -3
 (c) $\frac{1}{2}$ (d) $-\frac{3}{4}$
 (e) -2 (f) -1
 (g) $(p-3)/(q-1)$ (h) 0
- 11 $\frac{1}{2}, \frac{1}{2}$; points are collinear
- 12 $\frac{y}{x-3}, \frac{6-y}{5-x}$
- 13 5
- 14 (a) M is $(\frac{1}{2}, -1\frac{1}{2})$, N is $(1\frac{1}{2}, 4)$
- 15 (a) $MN = 5$, $BC = 10$
- 16 (a) Gradients PQ and SR both -1 ,
 QR and PS both $\frac{3}{5}$
 (b) Parallelogram
- 17 (d) Rhombus
- 18 (d) Rectangle
- 19 (a) Gradients PQ and SR both $-\frac{1}{3}$,
 $QR 4$, $PS \frac{3}{4}$
 (b) Trapezium
- 21 (a) $DE = DG = \sqrt{10}$, $EF = FG = \sqrt{40}$
 (b) Kite
- 22 (a) M is $(6,1)$ (c) N is $(8, 5\frac{1}{2})$
- Exercise 1B (page 11)
- 1 (a) Yes (b) No (c) Yes
 (d) No (e) Yes (f) No
 (g) Yes (h) Yes
- 2 (a) $y = 5x - 7$ (b) $y = -3x + 1$
 (c) $2y = x + 8$ (d) $8y = -3x + 2$
 (e) $y = -3x$ (f) $y = 8$
- (g) $4y = -3x - 19$ (h) $2y = x + 3$
 (i) $8y = 3x + 1$ (j) $2y = -x + 11$
 (k) $y = -2x + 3$ (l) $y = 3x + 1$
 (m) $y = 7x - 4$ (n) $y = -x + 2$
 (o) $8y = -5x - 1$ (p) $5y = -3x + 9$
 (q) $y = 7x - 7d$ (r) $y = mx + 4$
 (s) $y = 3x + c$ (t) $y = mx - mc$
- 3 (a) $y = 3x + 1$ (b) $y = 2x - 3$
 (c) $2x + 3y - 12 = 0$ (d) $x - 3 = 0$
 (e) $3x - 5y - 45 = 0$ (f) $3x + y - 8 = 0$
 (g) $y = -3$ (h) $x + 3y - 2 = 0$
 (i) $5x + 3y + 14 = 0$ (j) $2x + 7y + 11 = 0$
 (k) $x + 3 = 0$ (l) $x + y + 1 = 0$
 (m) $y = 3x + 1$ (n) $5x + 3y + 13 = 0$
 (o) $3x + 5y = 0$ (p) $qx - py = 0$
 (q) $x + 3y - p - 3q = 0$ (r) $x - p = 0$
 (s) $x - y - p + q = 0$ (t) $qx + py - pq = 0$
- 4 (a) -2 (b) $\frac{3}{4}$ (c) $-2\frac{1}{2}$
 (d) 0 (e) $1\frac{1}{2}$ (f) Undefined
 (g) -1 (h) 3 (i) $-\frac{1}{2}$
 (j) $2\frac{1}{3}$ (k) m (l) $-p/q$
- 5 $y = \frac{1}{2}x + 2$
- 6 $y + 2x = 5$
- 7 $3x + 8y = 19$
- 8 $x + y = 12$
- 9 $y = 7$
- 10 $y = mx - md$
- 11 (a) (7,3) (b) (2,7)
 (c) $(-\frac{1}{4}, -\frac{7}{8})$ (d) (-3,-1)
 (e) (2,4) (f) (6,5)
 (g) No intersection (h) (2,-1)
 (i) Same lines
 (j) $(c/a(1+2b), 2c/(1+2b))$
 (k) $((d-c)/2m, (c+d)/2)$
 (l) $(1/(a-b), 1/(a-b))$
- Exercise 1C (page 14)
- 1 (a) $-\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $-1\frac{1}{3}$
 (d) $1\frac{1}{5}$ (e) 1 (f) $-\frac{4}{7}$
 (g) m (h) $-1/m$ (i) $-q/p$
 (j) Undefined (k) $1/m$ (l) $(c-b)/a$
- 2 (a) $x + 4y = 14$ (b) $y = 2x + 7$
 (c) $x - 5y = 27$ (d) $x = 7$
 (e) $3x - 2y = -11$ (f) $5x + 3y = 29$
 (g) $y + 3 = 0$ (h) $x + 2y = 6$
 (i) $x + my = 0$ (j) $x + my = a + mb$
 (k) $nx + y = d + nc$ (l) $bx - ay = 2a - b$

3 $x + 3y = 13$, (1,4)

4 $3x + 2y = 5$, (3,-2)

5 $x + 2y = 8$

6 (a) (i) $x = 8$ (ii) $x - 2y = 2$ (b) (8,3)

Miscellaneous exercise 1 (page 15)

2 (2,-1)

3 (b) (3.6,0)

4 (a) $3x - y = 9$ (b) (3,0) and (5,6)

6 (a) $4x - 3y = 13$ (b) (4,1) (c) 5

7 78

8 (0,0), (12,12) and (14,4)

9 $2x + 7y = 23$

10 $3x - 4y = 1$

11 $(3\frac{1}{2}, 4)$, (4,3)

12 (a) $x + 3y = 9$ (b) (0.9,2.7)
(c) $\sqrt{0.9}$

14 $3x + 5y - 4 = 0$, $(1\frac{1}{3}, 0)$

15 $(3\frac{1}{2}, -1\frac{1}{2})$, -7; $x - 7y - 14 = 0$

16 (a) (-3,0), (0,1.5) (b) $x - 2y + 3 = 0$
(c) (1.8,2.4)

17 $x - 2y - 8 = 0$; (2,-3), $\sqrt{80}$

19 25

20 (a) $x + y = 1$ (b) (2,-1), (0,1)

21 $y = 2$, $3x + 4y = 18$, $y = 3x - 8$

22 Draw gradient triangles for both lines. Because $1/(-m_2) = m_1/1$, one triangle is similar to the other, but rotated through 90° .

2 Surds and indices

Exercise 2A (page 20)

1 (a) 3 (b) 10

(c) 16 (d) 4

(e) 8 (f) 6

(g) 15 (h) 30

(i) 72 (j) 60

(k) 28 (l) 27

(m) 5 (n) 48

(o) 256 (p) 5

2 (a) $3\sqrt{2}$ (b) $2\sqrt{5}$

(c) $2\sqrt{6}$ (d) $4\sqrt{2}$

(e) $2\sqrt{10}$ (f) $3\sqrt{5}$

(g) $4\sqrt{3}$ (h) $5\sqrt{2}$

(i) $3\sqrt{6}$ (j) $6\sqrt{2}$

(k) $3\sqrt{15}$ (l) $15\sqrt{3}$

3 (a) $5\sqrt{2}$ (b) $3\sqrt{3}$

(c) $\sqrt{5}$ (d) $2\sqrt{2}$

(e) 0 (f) $6\sqrt{3}$
(g) $6\sqrt{11}$ (h) $12\sqrt{2}$
(i) $13\sqrt{5}$ (j) $\sqrt{13}$
(k) $30\sqrt{5}$ (l) $6\sqrt{6} - \sqrt{3}$

4 (a) 2 (b) 3
(c) 2 (d) 5
(e) 5 (f) 3
(g) $\frac{1}{4}$ (h) $\frac{1}{2}$

5 (a) $\frac{1}{3}\sqrt{3}$ (b) $\frac{1}{5}\sqrt{5}$
(c) $2\sqrt{2}$ (d) $\sqrt{6}$
(e) $\sqrt{11}$ (f) $\frac{1}{2}\sqrt{2}$

(g) $4\sqrt{3}$ (h) $2\sqrt{7}$
(i) $\sqrt{3}$ (j) $\frac{1}{3}\sqrt{3}$
(k) $\sqrt{15}$ (l) $\frac{4}{5}\sqrt{30}$

(m) $\frac{7}{6}\sqrt{6}$ (n) $\frac{2}{3}\sqrt{6}$
(o) $\frac{3}{2}\sqrt{6}$ (p) $\frac{1}{9}\sqrt{6}$

6 (a) $7\sqrt{3}$ (b) $4\sqrt{3}$
(c) $\sqrt{3}$ (d) $\sqrt{3}$

(e) $12\sqrt{3}$ (f) $-5\sqrt{3}$

7 (a) $20\sqrt{2} \text{ cm}^2$ (b) $3\sqrt{10} \text{ cm}$

8 (a) $x = 5\sqrt{2}$ (b) $y = -4\sqrt{2}$
(c) $z = 3\sqrt{2}$

9 (a) $2\sqrt[3]{3}$ (b) $4\sqrt[3]{3}$
(c) $3\sqrt[3]{3}$ (d) $5\sqrt[3]{3}$

10 (a) $4\sqrt{13} \text{ cm}$ (b) $5\sqrt{11} \text{ cm}$
(c) $2\sqrt{15} \text{ cm}$ (d) $3\sqrt{5} \text{ cm}$

11 (a) 10.198 039 027 2

(b) 25.495 097 568 0

(c) 2.549 509 756 8

12 (a) $x = 3\sqrt{5}$, $y = 4$

13 (a) 1 (b) 1

(c) 4 (d) 7

(e) 46 (f) 5

(g) 107 (h) -3

14 (a) $\sqrt{3} + 1$ (b) $\sqrt{5} - 1$
(c) $\sqrt{6} + \sqrt{2}$ (d) $2\sqrt{7} - \sqrt{3}$
(e) $\sqrt{11} - \sqrt{10}$ (f) $3\sqrt{5} + 2\sqrt{6}$

16 (a) $2 + \sqrt{3}$ (b) $\frac{1}{20}(3\sqrt{5} + 5)$

(c) $4\sqrt{2} - 2\sqrt{6}$

Exercise 2B (page 25)

1 (a) a^{12} (b) b^8 (c) c^4
(d) d^9 (e) e^{20} (f) x^6y^4
(g) $15g^8$ (h) $3h^8$ (i) $72a^8$
(j) p^7q^{15} (k) $128x^7y^{11}$ (l) $4c$
(m) $108m^{14}n^{10}$ (n) $7r^3s$ (o) $2xy^2z^3$

- 2 (a) 2^{26} (b) 2^{12} (c) 2^6 (d) 2^6
 (e) 2^2 (f) 2^1 (g) 2^0 (h) 2^0
- 3 (a) $\frac{1}{8}$ (b) $\frac{1}{16}$ (c) $\frac{1}{5}$
 (d) $\frac{1}{9}$ (e) $\frac{1}{10\,000}$ (f) 1
 (g) 2 (h) 27 (i) $\frac{2}{5}$
 (j) $\frac{1}{128}$ (k) $\frac{1}{216}$ (l) $\frac{27}{64}$
- 4 (a) $\frac{1}{2}$ (b) $\frac{1}{512}$ (c) $\frac{1}{32}$
 (d) 8 (e) $\frac{1}{8}$ (f) 8
- 5 (a) $\frac{1}{10}$ (b) $\frac{2}{5}$ (c) $\frac{2}{5}$
 (d) $\frac{1}{10}$ (e) 10 (f) $\frac{2}{5}$
- 6 (a) a (b) b (c) c^{-6}
 (d) 2 (e) e^{-9} (f) f^{-5}
 (g) $3g^{-1}$ (h) $\frac{1}{9}h^{-4}$ (i) $\frac{1}{9}t^4$
 (j) $8j^6$ (k) $8x^9y^{-3}$ (l) $p^{-8}q^{-16}r^{-12}$
 (m) $2m$ (n) $9n^{-9}$ (o) $8x$
 (p) $\frac{25}{2}a^7c^{-4}$ (q) $\frac{1}{64}q^6$ (r) $144x^{-2}y^4$
- 7 (a) $x = -2$ (b) $y = 0$ (c) $z = 4$
 (d) $x = -2$ (e) $y = 120$ (f) $t = 0$
- 8 (a) $2.7 \times 10^{-5} \text{ m}^3$ (b) $5.4 \times 10^{-3} \text{ m}^2$
- 9 26.7 km h^{-1} (to 1 d.p.)
- 10 (a) $1.0 \times 10^{-3} \text{ m}^3$ (to 2 s.f.)
 (b) 101.9 m (to 1 d.p.)
 (c) $5.6 \times 10^{-3} \text{ m}$ (to 2 s.f.)
- 11 (a) 4.375×10^{-4} (b) 4.5×10^{-7}
- 12 $x = -3$

Exercise 2C (page 28)

- 1 (a) 5 (b) 2 (c) 6 (d) 2
 (e) 3 (f) $\frac{1}{3}$ (g) $\frac{1}{2}$ (h) $\frac{1}{7}$
 (i) $\frac{1}{10}$ (j) -3 (k) 16 (l) $\frac{1}{625}$
- 2 (a) 2 (b) $\frac{1}{16}$ (c) 16 (d) $\frac{1}{2}$
 (e) 2 (f) $\frac{1}{2}$ (g) 16 (h) $\frac{1}{2}$
- 3 (a) 4 (b) 8 (c) $\frac{1}{27}$ (d) 81
 (e) 4 (f) 8 (g) $\frac{1}{32}$ (h) 32
 (i) $\frac{1}{1000}$ (j) 625 (k) $2\frac{1}{4}$ (l) $\frac{2}{3}$
- 4 (a) a^2 (b) $12b^{-1}$ (c) $12c^{3/4}$
 (d) 1 (e) $4x^4y^5$ (f) 2
 (g) $5pq^2$ (h) $4m^{5/4}n^{7/4}$ (i) $2^{5/4}x^{-1}y^{5/4}$
- 5 (a) 64 (b) 27 (c) 8 (d) 9
 (e) $\frac{1}{4}$ (f) $\frac{1}{27}$ (g) 2 (h) 2
- 6 (a) 1.9 (to 1 d.p.)
 (b) 2.2 m (to 1 d.p.)
- 7 6.5 cm (to 1 d.p.)
- 8 (a) $x = \frac{5}{2}$ (b) $y = -\frac{3}{2}$ (c) $z = \frac{1}{4}$ (d) $x = \frac{3}{2}$
 (e) $y = \frac{4}{3}$ (f) $z = -\frac{7}{3}$ (g) $t = \frac{6}{5}$ (h) $y = -\frac{7}{4}$

Miscellaneous exercise 2 (page 29)

- 1 (a) $11 + \sqrt{2}$ (b) 29
 (c) $10 - 4\sqrt{5}$ (d) $128\sqrt{2}$
- 2 (a) $4\sqrt{3}$ (b) $\sqrt{7}$
 (c) $111\sqrt{10}$ (d) $3\sqrt[3]{2}$
- 3 (a) $\frac{3}{2}\sqrt{3}$ (b) $\frac{1}{25}\sqrt{5}$
 (c) $\frac{1}{3}\sqrt{2}$ (d) $\frac{2}{15}\sqrt{30}$
- 4 (a) $\sqrt{2}$ (b) $4\sqrt{5}$
 (c) $-2 + \frac{1}{2}\sqrt{2}$ (d) $4\sqrt{3}$
- 5 $\frac{5}{7}\sqrt{7}$
- 8 (a) 14 cm^2
- 9 18
- 10 $3\sqrt{7} \text{ cm}$
- 11 $x = 3\sqrt{2} + 4$, $y = 5\sqrt{2} - 7$
- 12 (a) 0.001 442 1 (b) 1.2991
- 13 $2\sqrt{10}$, (3,0)
- 14 (a) $x + 2y = 8$ (c) $\sqrt{5}$ or $-\sqrt{5}$
 (d) $t = -\frac{1}{5}, \frac{8}{5}\sqrt{5}$
- 15 $a = 2\sqrt{10}$, $b = 6\sqrt{10}$
- 16 Either $3\sqrt{2}$ and $\frac{9}{2}\sqrt{2}$ or $3\sqrt{5}$ and $\frac{9}{5}\sqrt{5}$; 27
- 17 (a) 6 (b) $\frac{1}{16}$ (c) $\frac{1}{2}$ (d) $2\frac{10}{27}$
- 18 $\frac{1}{3a^2}$
- 19 -1 and 8
- 20 $x = \frac{8}{7}$
- 21 $a^{-2/3}$
- 22 (a) $2p^{1/8}q^{-3/2}$ (b) $\frac{1}{10}b^{-3}$
 (c) $2xy^2$ (d) $m^{-2/3}n^{1/3}$
- 23 (a) 2.5×10^{179} (b) 1×10^{292}
 (c) 2×10^{56} (d) 3×10^{-449}
- 24 (b) $1.5 \times 10^{11} \text{ m}$ (to 2 s.f.)
- 25 (a) $\frac{15}{4}\sqrt{2}$ (b) $\frac{13}{3} + \frac{40}{9}\sqrt{3}$
- 26 (a) 2^{71} (b) 2^{-399} (c) $2^{7/3}$
 (d) 2^{99} (e) $2^{3.3}$
- 27 $x = -\frac{5}{6}$
- 28 (a) $S = 2^{2/3}3^{2/3}\pi^{1/3}V^{2/3}$
 (b) $V = 2^{-1}3^{-1}\pi^{-1/2}S^{3/2}$
- 29 $8 \times 10^3 \text{ joules, or } 8000 \text{ joules}$

3 Functions and graphs

Exercise 3A (page 35)

- 1 (a) 11 (b) 5 (c) -3 (d) 0

- 2** (a) 50 (b) 5 (c) 29 (d) 29
3 (a) 15 (b) $5\frac{1}{4}$ (c) 0 (d) 0

- 4** (a) 17 (b) 9 (d) 33

5 4

6 $a = 5, b = -3$

- 7** (a) $x \geq 0$ (b) $x \leq 0$
(c) $x \geq 4$ (d) $x \leq 4$
(e) $x \leq 0$ and $x \geq 4$ (f) $x \leq 0$ and $x \geq 4$
(g) $x \leq 3$ and $x \geq 4$ (h) $x \geq 2$
(i) All real numbers except 2
(j) $x > 2$ (k) $x \geq 0$
(l) All real numbers except 1 and 2

- 8** (a) $f(x) > 7$ (b) $f(x) < 0$
(c) $f(x) > -1$ (d) $f(x) > -1$
(e) $f(x) > 3$ (f) $f(x) \geq 2$

- 9** (a) $f(x) \geq 4$ (b) $f(x) \geq 10$
(c) $f(x) \geq 6$ (d) $f(x) \leq 7$
(e) $f(x) \geq 2$ (f) $f(x) \geq -1$

- 10** (a) $0 \leq f(x) \leq 16$ (b) $-1 \leq f(x) \leq 7$
(c) $0 \leq f(x) \leq 16$ (d) $4 \leq f(x) \leq 25$

- 11** (a) $f(x) \geq 0$ (b) All real numbers
(c) All real numbers except 0
(d) $f(x) > 0$ (e) $f(x) \geq 5$
(f) All real numbers (g) $0 \leq f(x) \leq 2$
(h) $f(x) \geq 0$

12 $0 < w < 12, 0 < A \leq 36$

13 $0 < x < 4$

Exercise 3B (page 40)

- 2** (a) RQP (b) PQR (c) QPR (d) QRP

- 3** (a) $x > 10$
(b) $x < -50$ or $x > 50$ (i.e., $|x| > 50$)
(c) $-\sqrt{10}/10 \leq x \leq \sqrt{10}/10, x \neq 0$
(i.e., $0 < |x| < \frac{1}{10}\sqrt{10}$)
(d) $x < -20$ or $x > 20$ (i.e., $|x| > 20$)

7 (a) and (c) are odd; (b) is even.

- 8** (a) 7 (b) $\frac{1}{200}$ (c) 5
(d) 5 (e) $\pi - 3$ (f) $4 - \pi$

- 9** (a) 2 (b) $\frac{1}{4}$ (c) 0
(d) 2 (e) 1

- 10** (a) $0 < y < 0.0001$ (b) $y > 10000$

- 11** (a) $|y| > 10^{-9}$ (b) $|x| < 0.1$

12 $|N - 37000| < 500$

13 $|m - n| \leq 5$

14 The length is 5.23 cm correct to 2 d.p.

Exercise 3C (page 41)

- 11** (c)

- 12** (a)

Exercise 3D (page 44)

- 1** (a) $(3,14)$ (b) $(1,3), (4,3)$
(c) $(-4,8), (2,8)$ (d) $(-3,-3), (\frac{1}{2}, -3)$

- 2** (a) $(1,2), (3,4),$
(c) $(-1\frac{1}{2}, 6\frac{1}{2}), (2,17)$ (d) $(-2,-7), (2,9)$
(e) $(-\frac{1}{3}, 2), (2\frac{1}{2}, -6\frac{1}{2})$

- 3** (a) $(2,6)$ (b) $(-3,-1)$

- 4** (a) $(0,0), (2,2)$ (b) $(1,0)$

- 5** (a) $(-4,14)$ (b) $(-6,24), (-2,12)$

- 6** (a) $(-1,4), (3,8)$ (b) $(1,6)$

- 7** (a) $(5,51)$ (b) $(-2,3)$ (c) $(0,1)$

- 8** (a) $(-1, \frac{1}{2}), (1, \frac{1}{2})$ (b) $(1,9), (2,18)$
(c) $(-3,1), (-2,3)$ (d) $(-1,-5), (2,19)$
(e) $(-3,65), (2,4, 7.76)$
(f) $(0,0), (8,80)$

- 9** (a) $(1,8)$
(b) $(3, \frac{1}{3})$
(c) $(\sqrt{2}, \sqrt{2}), (-\sqrt{2}, -\sqrt{2})$
(d) $(\frac{1}{2}, 32), (-\frac{1}{2}, 32)$
(e) $(\frac{1}{3}, 243), (-\frac{1}{3}, -243)$
(f) $(2,4), (-2,4)$

Exercise 3E (page 47)

- 4** (a) $y = x^2 - 7x + 10$ (b) $y = x^2 + 17x + 70$
(c) $y = x^2 + 2x - 15$ (d) $y = x^2 - 2x - 15$

- 6** (a) $y = 3x^2 - 18x + 15$
(b) $y = 4x^2 - 20x - 56$
(c) $y = -\frac{1}{2}x^2 - 4x - 6$
(d) $y = -4x^2 - 4x + 24$
(e) $y = 5x^2 + 15x - 350$

- 8** (a) A, B, G, H (b) B, D, F (c) F, G, H
(d) D (e) G (f) I
(g) B, E (h) A, C, E

Miscellaneous exercise 3 (page 48)

- 1** (a) $45, -\frac{1}{2}, -39$ (b) 2
(c) $\frac{2}{3}$ (d) 37

- 2** $-1, 4$

- 3** \square

- 4** $(\frac{1}{2}, 1\frac{3}{4}), (4, -7)$

- 5** $(-2\frac{1}{2}, -2), (2, 7)$

- 6** $(1, -1)$

- 7** $(2, 0)$

- 9** $a = 2, b = -7, c = 6$

- 10** $(-\frac{3}{4}, 6)$

12 6

13 (0,108)

14 $p = 35, q = -\frac{1}{2}$ or 4

15 (3,33)

16 $c = 2, k = 14, (3,2)$

17 2

18 $p = q = 13$

19 (1,1), (9,81)

21 $(\sqrt{3}, -5\sqrt{3}), (-\sqrt{3}, 5\sqrt{3})$ 22 $(2 + \sqrt{2}, 13 + 6\sqrt{2})$

23 (a) 2 (b) 0 (c) 1

24 (a) $f(x) = 1$ (b) $f(x) = -1$

4 Quadratics

Exercise 4A (page 54)

1 (a) (i) (2,3) (ii) $x = 2$ (b) (i) (5,-4) (ii) $x = 5$ (c) (i) (-3,-7) (ii) $x = -3$ (d) (i) $\left(\frac{3}{2}, 1\right)$ (ii) $x = \frac{3}{2}$ (e) (i) $\left(-\frac{3}{5}, 2\right)$ (ii) $x = -\frac{3}{5}$ (f) (i) $\left(-\frac{7}{3}, -4\right)$ (ii) $x = -\frac{7}{3}$ (g) (i) (3,c) (ii) $x = 3$ (h) (i) (p, q) (ii) $x = p$ (i) (i) $(-b/a, c)$ (ii) $x = -b/a$

2 (a) (i) -1 (ii) -2 (b) (i) 2 (ii) 1

(c) (i) 5 (ii) -3 (d) (i) -7 (ii) $-\frac{1}{2}$ (e) (i) 3 (ii) 4 (f) (i) q (ii) $-p$ (g) (i) $-q$ (ii) p (h) (i) r (ii) t (i) (i) c (ii) $-b/a$ 3 (a) $3 \pm \sqrt{3}$ (b) 0, -4(c) $-3 \pm \frac{1}{2}\sqrt{10}$ (d) $\frac{1}{3}(7 \pm 2\sqrt{2})$ (e) $-p \pm \sqrt{q}$ (f) $-b \pm \sqrt{c/a}$ 4 (a) $(x+1)^2 + 1$ (b) $(x-4)^2 - 19$ (c) $(x+1\frac{1}{2})^2 - 9\frac{1}{4}$ (d) $(x-3)^2 - 4$ (e) $(x+7)^2$ (f) $2(x+3)^2 - 23$ (g) $3(x-2)^2 - 9$ (h) $11 - 4(x+1)^2$ (i) $2(x+1\frac{1}{4})^2 - 6\frac{1}{8}$ 5 (a) $(x-7)(x+5)$ (b) $(x-22)(x+8)$ (c) $(x+24)(x-18)$ (d) $(3x+2)(2x-3)$ (e) $(2+7x)(7-2x)$ (f) $(4x+3)(3x-2)$ 6 (a) 3 when $x = 2$ (b) $2\frac{3}{4}$ when $x = 1\frac{1}{2}$ (c) 13 when $x = 3$ (d) $-1\frac{1}{8}$ when $x = 1\frac{1}{4}$ (e) $-4\frac{1}{3}$ when $x = -\frac{1}{3}$ (f) $7\frac{1}{12}$ when $x = -1\frac{1}{6}$ 7 (a) $f(x) \geq 1$ (b) $f(x) \geq -\frac{45}{4}$ (c) $f(x) \geq \frac{7}{4}$

- 8 (a) (i) (2,2) (ii) $x = 2$
 (b) (i) (-3,-11) (ii) $x = -3$
 (c) (i) (-5,32) (ii) $x = -5$
 (d) (i) $\left(-1\frac{1}{2}, -1\frac{1}{4}\right)$ (ii) $x = -1\frac{1}{2}$
 (e) (i) $\left(1\frac{3}{4}, -4\frac{1}{8}\right)$ (ii) $x = 1\frac{3}{4}$
 (f) (i) (2,-7) (ii) $x = 2$

- 9 (a) $f(x) \geq -\frac{1}{4}$ (b) $f(x) \geq -\frac{1}{4}$
 (c) $f(x) \geq -\frac{9}{8}$

Exercise 4B (page 58)

- 1 (a) $\frac{1}{2}(-3 \pm \sqrt{29})$ (b) $2 \pm \sqrt{11}$
 (c) -3 (repeated) (d) $\frac{1}{2}(-5 \pm \sqrt{17})$
 (e) No solution (f) $\frac{1}{6}(5 \pm \sqrt{97})$
 (g) -3 and $-\frac{1}{2}$ (h) $\frac{1}{2}(-3 \pm \sqrt{41})$
 (i) $\frac{1}{6}(2 \pm \sqrt{34})$

- 2 (a) 2 (b) 1 (c) 0 (d) 0 (e) 2

- (f) 2 (g) 1 (h) 2 (i) 2 (j) 2

- 3 (a) $-\frac{9}{4}$ (b) $-\frac{25}{32}$ (c) 81
 (d) $\pm 2\sqrt{6}$ (e) $\pm 4\sqrt{6}$ (f) $p^2/4q$

- 4 (a) $k < \frac{9}{4}$ (b) $k = \frac{49}{4}$

- (c) $k > \frac{9}{20}$ (d) $k > -\frac{25}{12}$

- (e) $k = \frac{4}{3}$ (f) $k > \frac{25}{28}$

- (g) $k > 4$ or $k < -4$ (h) $-6 < k < 6$

- 5 (a) 2 (b) 0 (c) 1 (d) 0 (e) 2
 (f) 2 (g) 0 (h) 0 (i) 1

6 Intersects x -axis twice, faces up.

7 Intersects x -axis twice, faces down.

Exercise 4C (page 61)

- 1 (a) $x = 3, y = 4$ or $x = -4, y = -3$

- (b) $x = 3, y = 4$ or $x = 4, y = 3$

- (c) $x = 5, y = 2$ or $x = -1, y = -4$

- (d) $x = 3, y = -1$

- (e) $x = 0, y = 5$ or $x = 4, y = -3$

- (f) $x = 1, y = 0$ or $x = \frac{1}{2}, y = \frac{1}{2}$

- (g) $x = 0, y = 7$

- (h) $x = 3, y = -2$ or $x = \frac{1}{7}, y = -10\frac{4}{7}$

- 2 (a) (2,5) and (1,3)

- (b) (1,5) and (-2.2, -4.6)

- (c) (3,4) and (-1,-4)

- (d) (1,1) and $(-4, 3\frac{1}{2})$

- (e) (4,3)

- (f) (2,-1) and $(-\frac{7}{8}, 4\frac{3}{4})$

- (g) (5,-2) and (27,42)

- (h) $(\frac{1}{2}, -1)$ and $(-\frac{1}{8}, -1\frac{17}{20})$

- 3 (a) 2 (b) 0 (c) 2

- (d) 1 (e) 0 (f) 1

- 4** (a) $\pm 1, \pm 2$ (b) $\pm 1, \pm 3$ (c) ± 2
 (d) $\pm \sqrt{6}$ (e) $-1, 2$ (f) $\sqrt[3]{3}, -\sqrt[3]{4}$
5 (a) $-2, 5$ (b) $-6, 1$ (c) $-3, \frac{1}{2}$
 (d) $-4, 3$ (e) 36 (f) 9
 (g) $1, -\frac{8}{3}$ (h) $-7, 2$ (i) $5, -\frac{11}{3}$
 (j) $\frac{1}{2}, -\frac{4}{11}$ (k) ± 2 (l) ± 1
6 (a) 16 (b) $25, 9$ (c) 49
 (d) 25 (e) $-8, 27$ (f) $-1, 64$

Miscellaneous exercise 4 (page 62)

- 1** $x = 3, y = -1$ or $x = -\frac{1}{3}, y = \frac{7}{3}$
2 $a = 5, b = -8$. Least value is -8 when $x = 5$.
3 $x = 2, y = -1$ or $x = -\frac{5}{4}, y = \frac{11}{2}$
4 8 and -8
5 $f(x) = 2\left(x - \frac{5}{4}\right)^2 - \frac{121}{8}, f(x) \geq -15\frac{1}{8}$
6 (a) $2\sqrt{3}, 4\sqrt{3}$ (b) $\pm 1.86, \pm 2.63$
8 $(3x-6)^2 + 16$. Takes values ≥ 16 .
9 $(-1.64, 6.63), (0.24, -1.67)$
10 (a) $(3x+2)^2 + 3$ (b) $0 < f(x) \leq \frac{1}{3}$
11 ± 0.991 and ± 0.131
12 $a = 3, b = -\frac{5}{6}, c = -\frac{13}{12}; \left(\frac{5}{6}, -\frac{13}{12}\right)$
13 $(1, 6), (2, 3)$
14 (a) $\left(b/a, c - b^2/a\right)$ (b) $c = b(b+1)/a$
15 (b) (i) Line is tangent to curve.
 (ii) Line and curve do not intersect.
17 (a) $2\sqrt{5}/5$ (b) $4\sqrt{5}/5$ (c) 1
19 (a) $2\sqrt{5}$ m (b) $8\sqrt{5}$ m
 (c) They collide!
20 (a) $6 - (2+x)^2, 8 + (4+x)^2$
 (c) For example, $y = 2 - (x-3)^2$ and $y = x^2$.
21 \$4800, 100

5 Inequalities**Exercise 5A (page 68)**

- 1** (a) $x > 14$ (b) $x < 4$ (c) $x \leq 2\frac{1}{2}$
 (d) $x \geq 7$ (e) $x > -4$ (f) $x \leq -3\frac{1}{5}$
 (g) $x < -3\frac{1}{2}$ (h) $x \leq -4$
2 (a) $x > 7$ (b) $x \leq 22$ (c) $x < -11\frac{1}{2}$
 (d) $x \leq 6$ (e) $x \geq -2\frac{3}{4}$ (f) $x > -2$
 (g) $x < 3\frac{1}{3}$ (h) $x \geq -4$
3 (a) $x \geq -4$ (b) $x \leq 4$ (c) $x > 9$
 (d) $x \geq -2$ (e) $x \geq \frac{1}{3}$ (f) $x < 1$
 (g) $x < 1\frac{2}{5}$ (h) $x > 2$
4 (a) $x > -7$ (b) $x \leq 2$ (c) $x < -6$
 (d) $x > 3$ (e) $x \geq 2\frac{3}{4}$ (f) $x < 19$
 (g) $x < 8\frac{1}{2}$ (h) $x \geq 9$

- 5** (a) $x \geq -9$ (b) $x \geq 4$ (c) $x > 6$
 (d) $x > 2\frac{1}{4}$ (e) $x \leq \frac{6}{7}$ (f) $x \geq -1$
 (g) $x < -\frac{3}{4}$ (h) $x > -3$ (i) $x \leq -\frac{1}{4}$
6 (a) $x > 5\frac{1}{2}$ (b) $x < -3$ (c) $x \geq -1\frac{5}{8}$
 (d) $x < -2$ (e) $x \leq 5$ (f) $x \leq 1\frac{7}{9}$
 (g) $x < -1\frac{2}{5}$ (h) $x \geq 4\frac{8}{13}$

Exercise 5B (page 71)

- 1** (a) $2 < x < 3$
 (b) $x < 4$ or $x > 7$
 (c) $1 < x < 3$
 (d) $x \leq -1$ or $x \geq 4$
 (e) $x < -3$ or $x > \frac{1}{2}$
 (f) $-2\frac{1}{2} \leq x \leq \frac{2}{3}$
 (g) $x \leq -2$ or $x \geq -1\frac{1}{4}$
 (h) $x < -3$ or $x > 1$
 (i) $x < 1\frac{1}{2}$ or $x > 5$
 (j) $-5 < x < 5$
 (k) $-1\frac{1}{3} < x < \frac{3}{4}$
 (l) $x \leq -\frac{2}{3}$ or $x \geq \frac{2}{3}$
2 (a) $3 < x < 6$
 (b) $x < 2$ or $x > 8$
 (c) $-5 \leq x \leq 2$
 (d) $x \leq -1$ or $x \geq 3$
 (e) $x < -1\frac{1}{2}$ or $x > 2$
 (f) $-5 \leq x \leq \frac{2}{3}$
 (g) $x \leq -3$ or $x \geq -\frac{4}{5}$
 (h) $x < -5$ or $x > 2$
 (i) $x < 2\frac{1}{2}$ or $x > 3$
 (j) $x \leq -\frac{1}{3}$ or $x \geq \frac{1}{3}$
 (k) $x < -1\frac{1}{3}$ or $x > \frac{2}{7}$
 (l) $x \leq -1\frac{2}{3}$ or $x \geq \frac{1}{3}$
3 (a) $x < \frac{1}{2}(-3 - \sqrt{29})$ or $x > \frac{1}{2}(-3 + \sqrt{29})$
 (b) True for no x
 (c) $\frac{1}{2}(5 - \sqrt{17}) < x < \frac{1}{2}(5 + \sqrt{17})$
 (d) True for all x
 (e) $-3 < x < 3$
 (f) $x = -1$ only
 (g) $\frac{1}{4}(3 - \sqrt{17}) < x < \frac{1}{4}(3 + \sqrt{17})$
 (h) $\frac{1}{2}(-3 - \sqrt{41}) < x < \frac{1}{2}(-3 + \sqrt{41})$
 (i) $x \leq \frac{1}{4}(-7 - \sqrt{41})$ or $x \geq \frac{1}{4}(-7 + \sqrt{41})$
4 (a) $x < -3$ or $x > -2$
 (b) $3 < x < 4$
 (c) $-3 \leq x \leq 5$
 (d) $x \leq -3$ or $x \geq 3$
 (e) $x \leq 1$ or $x \geq 1\frac{1}{2}$
 (f) $-\frac{2}{3} < x < 1\frac{1}{2}$
 (g) $x < \frac{1}{2}(-5 - \sqrt{17})$ or $x > \frac{1}{2}(-5 + \sqrt{17})$

- (h) $x < -\frac{1}{3}\sqrt{21}$ or $x > \frac{1}{3}\sqrt{21}$
 (i) True for no x
 (j) True for all x
 (k) $x < -\frac{3}{4}$ or $x > \frac{1}{3}$
 (l) $\frac{1}{6}(7-\sqrt{37}) \leq x \leq \frac{1}{6}(7+\sqrt{37})$

Miscellaneous exercise 5 (page 72)

- 1 $-6 \leq x \leq 7$
 2 $-4 < x < 2$
 3 $-4 < x < 3$
 4 $-1 < x < 0$ or $x > 1$
 5 $-3 \leq x \leq 0$ or $x \geq 2$
 6 (a) $k < 0$ or $k > 8$
 (b) $-1\frac{1}{2} < k < 1\frac{1}{2}$ provided $k \neq 0$ (if $k = 0$ the equation is linear, and has just one root)
 (c) $k < -2$ or $k > 2$
 7 (a) $0 \leq k < 5$ (b) $k = 0$
 (c) $-\frac{8}{25} < k < 0$
 8 $k \leq 0$ or $k \geq \frac{4}{9}$
 9 $x < -2$ or $x > \frac{2}{3}$
 10 $-\frac{1}{2} < x < 0$ or $x > 2$
 11 (a) $x < 2$ or $x > 2\frac{1}{2}$ (b) $1-\sqrt{6} < x < 1+\sqrt{6}$

Revision exercise 1
(page 73)

- 1 $(-4, -16)$
 3 $(x+5)^2 + 13$;
 (a) $13, -5$ (b) $x \leq -8$ or $x \geq -2$
 4 $8x^2$
 5 (a) $\frac{1}{2} \leq x \leq 2$ (b) $-\frac{1}{2} < x < \frac{7}{2}$ (c) $x > 6.3$
 6 6
 7 $\pm\sqrt{65}$
 8 $(0,0), (-1,-1), (1,1)$
 9 $\$(12(x+y) + 15xy); 5z^2 + 8z - 4 = 0, 0.4$
 10 $x+y=5; 7$
 11 (a) $x > \frac{4}{3}$ (b) $x > \frac{3}{2}$ (c) $x \leq 0$ or $x \geq 5$
 12 $0, 5$
 13 $x = 1, y = 1$ or $x = 4, y = -1$
 14 25
 15 $(2,1)$ The line is a tangent to the curve.
 16 $2x+3y=7, 3x-2y=4; (4,4)$
 17 (a) $(3,-1)$ (b) 5 (c) $2\sqrt{6}-1$
 18 (a) $x < -1$ or $x > 2$ (b) $-1 < x < 2$ or $x > 3$
 19 $2\sqrt{19}$ cm
 20 $x = 7, 3y+x = 10; (7,1)$; all $5\sqrt{2}, 30, 2\sqrt{5}$

- 21 $5, y=1; 2\sqrt{2}, y=x+1$
 22 $3, -18, 35; 3(x-3)^2+8; 8$
 23 $-2, 0$
 24 $1.52, -8.52$
 25 (a) $\left(\frac{c}{a}\right)^{\frac{1}{2}}$ (b) $\frac{c^2}{a^2}\sqrt{\frac{c}{a}}$

6 Differentiation**Exercise 6A (page 77)**

- 1 $y = 3x - 2$
 2 (a) 2.001 (b) 1.9999
 (c) 4.002 (d) 3.999
 (e) 6.000 001 (f) 5.999 99

Exercise 6B (page 79)

- 1 (a) 2 (b) 8 (c) 0 (d) -4
 (e) -0.4 (f) -7 (g) $2p$ (h) $4p$
 2 (a) 2 (b) 8 (c) 0 (d) -4
 (e) -0.4 (f) -7 (g) $2p$ (h) $4p$
 3 4 and -4
 4 (a) $y = 4x - 4$ (b) $y = -2x + 1$
 (c) $y = 2x - 3$ and $y = -2x - 3$
 (d) $y = -2$
 5 (a) $2y = -x + 3$ (b) $4y = x + 22$
 (c) $x = 0$
 (d) $2y\sqrt{c} = -x + \sqrt{c}(4c + 1)$
 6 $12y = -4x + 33$
 7 $4y = 4x + 3$
 8 $(-2\frac{1}{4}, 5\frac{1}{16})$

Exercise 6C (page 82)

- 1 (a) $2x$ (b) $2x-1$ (c) $8x$
 (d) $6x-2$ (e) -3 (f) $1-4x$
 (g) $4-6x$ (h) $\sqrt{2}-2\sqrt{3}x$
 2 (a) 3 (b) $-6x$ (c) 0
 (d) $2+6x$ (e) $-2x$ (f) $6-6x$
 (g) $2-4x$ (h) $4x+1$
 3 (a) 6 (b) 3 (c) -3 (d) 8
 (e) -8 (f) -6 (g) 4 (h) -17
 4 (a) $\frac{3}{4}$ (b) $\frac{1}{2}$ (c) $-\frac{3}{2}$
 (d) -1 (e) $\frac{3}{2}$ (f) $\frac{1}{2}$
 5 (a) $y = -2x - 1$ (b) $y = -x$
 (c) $y = 2x - 1$ (d) $y = 6x + 10$
 (e) $y = 1$ (f) $y = 0$
 6 (a) $2y = x - 3$ (b) $4y = -x + 1$
 (c) $8y = -x - 58$ (d) $x = 0$
 (e) $2y = x + 9$ (f) $x = \frac{1}{2}$
 7 $4y = 4x - 1$

8 $y = 0$

9 $y = -2x$

10 $12y = 12x - 17$

11 $x = 1$

12 $7y = -x + 64$

- 14 (a) 0.499 875... (b) 0.500 012...
 (c) 0.249 968... (d) 0.250 015...
 (e) 0.999 999... (f) 1.000 001...

Exercise 6D (page 86)

- | | | |
|------------------------------|------------------------------|-------------------------------|
| 1 (a) $3x^2 + 4x$ | (b) $-6x^2 + 6x$ | |
| (c) $3x^2 - 12x + 11$ | (d) $6x^2 - 6x + 1$ | |
| (e) $4x - 24x^3$ | (f) $-3x^2$ | |
| 2 (a) -10 | (b) 6 | (c) 58 |
| (d) -1 | (e) 8 | (f) 12 |
| 3 (a) -2, 2 | (b) $-\frac{4}{3}, 2$ | (c) -5, 7 |
| (d) 1 | (e) $-1, -\frac{1}{3}$ | (f) No values |
| 4 (a) $\frac{1}{\sqrt{x}}$ | (b) $\frac{1}{\sqrt{x}} + 1$ | (c) $1 - \frac{1}{4\sqrt{x}}$ |
| (d) $1 - \frac{1}{\sqrt{x}}$ | (e) $1 + \frac{1}{x^2}$ | (f) $2x + 1 - \frac{1}{x^2}$ |
| (g) $1 - \frac{2}{x^2}$ | (h) $1 + \frac{1}{\sqrt{x}}$ | |

5 $y = 4x + 2$

6 $y = x + 2$

7 $4y = x + 4$

8 $4y = -x + 4$

9 $x = 1$

11 $y = -2x - 6$

12 $y = -a^2x, y = 2a^2x + 2a^3, y = 2a^2x - 2a^3$

13 $(\frac{1}{2}, -2)$

- | | | |
|--------------------------------------|-------------------------------|-----------------------------------|
| 14 (a) $-\frac{1}{4x^2}$ | (b) $-\frac{6}{x^3}$ | (c) 0 |
| (d) $\frac{3}{4\sqrt[4]{x}}$ | (e) $\frac{2}{\sqrt[3]{x^2}}$ | (f) $-\frac{2}{\sqrt{x^3}}$ |
| (g) $-\frac{3}{x^2} - \frac{1}{x^4}$ | (h) $10\sqrt{x^3}$ | (i) $\frac{3}{2}\sqrt{x}$ |
| (j) $-\frac{1}{6\sqrt[3]{x^4}}$ | (k) $\frac{4-x}{x^3}$ | (l) $\frac{3x-1}{4\sqrt[4]{x^5}}$ |

15 $3y - x = 4, y + 3x = 28$

16 $(0, 12), (\frac{3}{4}, 0)$

Exercise 6E (page 93)

1 $f'(p) = 3p^2$

2 $f'(p) = 8p^7$

3 $f'(p) = -\frac{2}{p^3}$

Miscellaneous exercise 6 (page 94)

1 $y = 13x - 16$

2 (a) -9 (b) $-\frac{19}{3}, 3$

3 $80y = 32x - 51$

4 $(-3, -\frac{1}{3})$

5 2

6 $(-\frac{1}{3}, -4\frac{17}{27}), (2, 13)$

7 $x + 19y - 153 = 0$

8 13

9 (2, 12)

10 2

11 Both curves have gradient 12.

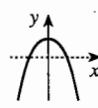
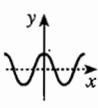
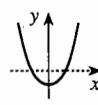
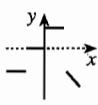
12 -183

13 $mn = -1$

14 $(\frac{11}{20}, \frac{4}{5})$

15 $(\frac{67}{32}, \frac{5}{8})$

7 Applications of differentiation**Exercise 7A (page 96)**

- 3 (a)  (b) 
 (c)  (d) 
 4 (a)  (b) 

Exercise 7B (page 103)

1 (a) $2x - 5, x \geq \frac{5}{2}$ (b) $2x + 6, x \geq -3$

(c) $-3 - 2x, x \leq -\frac{3}{2}$ (d) $6x - 5, x \geq \frac{5}{6}$

(e) $10x + 3, x \geq -\frac{3}{10}$ (f) $-4 - 6x, x \leq -\frac{2}{3}$

2 (a) $2x + 4, x \leq -2$ (b) $2x - 3, x \leq \frac{3}{2}$

(c) $-3 + 2x, x \leq \frac{3}{2}$ (d) $4x - 8, x \leq \frac{5}{2}$

(e) $7 - 4x, x \geq \frac{7}{4}$ (f) $-5 - 14x, x \geq -\frac{5}{14}$

3 (a) $3x^2 - 12, x \leq -2$ and $x \geq 2$

(b) $6x^2 - 18, x \leq -\sqrt{3}$ and $x \geq \sqrt{3}$

(c) $6x^2 - 18x - 24, x \leq -1$ and $x \geq 4$

(d) $3x^2 - 6x + 3, \text{ all } x$

(e) $4x^3 - 4x, -1 \leq x \leq 0$ and $x \geq 1$

(f) $4x^3 + 12x^2, x \geq -3$

(g) $3 - 3x^2, -1 \leq x \leq 1$

(h) $10x^4 - 20x^3, x \leq 0$ and $x \geq 2$

(i) $3(1 + x^2), \text{ all } x$

- 4** (a) $3x^2 - 27, -3 \leq x \leq 3$
 (b) $4x^3 + 8x, x \leq 0$
 (c) $3x^2 - 6x + 3, \text{none}$
 (d) $12 - 6x^2, x \leq -\sqrt{2} \text{ and } x \geq \sqrt{2}$
 (e) $6x^2 + 6x - 36, -3 \leq x \leq 2$
 (f) $12x^3 - 60x^2, x \leq 5$
 (g) $72x - 8x^3, -3 \leq x \leq 0 \text{ and } x \geq 3$
 (h) $5x^4 - 5, -1 \leq x \leq 1$
 (i) $nx^{n-1} - n; x \leq 1 \text{ if } n \text{ is even,}$
 $-1 \leq x \leq 1 \text{ if } n \text{ is odd}$
- 5** (a) $\frac{1}{2}x^{1/2}(5x - 3); 0 < x < \frac{3}{5}; x \geq \frac{3}{5}$
 (b) $\frac{1}{4}x^{-1/4}(3 - 14x); x \geq \frac{3}{14}; 0 \leq x \leq \frac{3}{14}$
 (c) $\frac{1}{3}x^{-1/3}(5x + 4); -\frac{4}{5} \leq x \leq 0;$
 $x \leq -\frac{4}{5} \text{ and } x \geq 0$
 (d) $\frac{13}{5}x^{-2/5}(x^2 - 3); -\sqrt{3} \leq x \leq \sqrt{3};$
 $x \leq -\sqrt{3} \text{ and } x \geq \sqrt{3}$
 (e) $1 - \frac{3}{x^2}; -\sqrt{3} \leq x < 0 \text{ and } 0 < x \leq \sqrt{3};$
 $x \leq -\sqrt{3} \text{ and } x \geq \sqrt{3}$
 (f) $\frac{x-1}{2x\sqrt{x}}; 0 < x \leq 1; x \geq 1$
- 6** (a) (i) $(4, -12)$ (ii) minimum (iv) $f(x) \geq -12$
 (b) (i) $(-2, -7)$ (ii) minimum (iv) $f(x) \geq -7$
 (c) (i) $(-\frac{3}{5}, \frac{1}{5})$ (ii) minimum (iv) $f(x) \geq \frac{1}{5}$
 (d) (i) $(-3, 13)$ (ii) maximum (iv) $f(x) \leq 13$
 (e) (i) $(-3, 0)$ (ii) minimum (iv) $f(x) \geq 0$
 (f) (i) $(-\frac{1}{2}, 2)$ (ii) maximum (iv) $f(x) \leq 2$
- 7** (a) $(-4, 213)$, maximum; $(3, -130)$, minimum
 (b) $(-3, 88)$, maximum; $(5, -168)$, minimum
 (c) $(0, 0)$, minimum; $(1, 1)$, neither
 (d) $(-2, 65)$, maximum; $(0, 1)$, neither;
 $(2, -63)$, minimum
 (e) $(-\frac{1}{3}, -\frac{11}{27})$, minimum; $(\frac{1}{2}, \frac{3}{4})$, maximum
 (f) $(-1, 0)$, neither
 (g) $(-1, -2)$, maximum; $(1, 2)$, minimum
 (h) $(3, 27)$, minimum
 (i) none
 (j) $(\frac{1}{4}, -\frac{1}{4})$, minimum
 (k) $(6, \frac{1}{12})$, maximum
 (l) $(-2, 17)$, minimum
 (m) $(1, 3)$, maximum
 (n) $(-1, -5)$, minimum
 (o) $(0, 0)$, minimum; $(\frac{4}{5}, \frac{256}{3125})$, maximum
- 8** (a) $f(x) \geq \frac{3}{4}$
 (b) $f(x) \geq -16$
 (c) $f(x) \leq -2, f(x) \geq 2$

Exercise 7C (page 109)

- 1** (a) Gradient of road
 (b) Rate of increase of crowd
 (c) Rate of increase (or decrease if negative) of magnetic force with respect to distance
 (d) Acceleration of particle
 (e) Rate of increase of petrol consumption with respect to speed
- 2** (a) $\frac{dp}{dh}$, p in millibars, h in metres
 (b) $\frac{d\theta}{dt}$, θ in degrees C, t in hours
 (c) $\frac{dh}{dt}$, h in metres, t in hours
 (d) $\frac{dW}{dt}$, W in kilograms, t in weeks
- 3** (a) $6t + 7$ (b) $1 - \frac{1}{2\sqrt{x}}$
 (c) $1 - \frac{6}{y^3}$ (d) $2t - \frac{1}{2t\sqrt{t}}$
 (e) $2t + 6$ (f) $12s^5 - 6s$
 (g) 5 (h) $-\frac{2}{r^3} - 1$
- 4** (a) Velocity
 (b) (i) Increasing (ii) Decreasing
 (c) 9, occurs when velocity is zero and direction of motion changes
- 5** (a) $\frac{dx}{dt} = c$
 (b) $\frac{dA}{dt} = kA$; A stands for the amount deposited
 (c) $\frac{dx}{dt} = f(\theta)$; x stands for diameter, θ for air temperature
- 6** 80 km h^{-1}
7 20 m
8 36
9 $4\sqrt{5}$
10 Greatest $V = 32\pi$ when $r = 4$, least $V = 0$ when $r = 0$ or $h = 0$
11 25
12 (b) 1800 m^2
13 $0 < x < 20$, 7.36 cm
14 20 cm
15 (b) 38400 cm^3 (to 3 s.f.)
16 2420 cm^3 (to 3 s.f.)

Miscellaneous exercise 7 (page 110)

- 1** Maximum at $(-2, -4)$; minimum at $(2, 4)$; y increases with x for $x \leq -2$ and $x \geq 2$
- 2** (a) $\frac{dn}{dt} = kn$ (b) $\frac{d\theta}{dt} = -k\theta$
 (c) $\frac{d\theta}{dt} = -k(\theta - \beta)$
- 4** $(20 - 4t) \text{ m s}^{-1}$, -4 m s^{-2} ; for $0 \leq t \leq 5$
- 5** (a) 20 m (b) 6 s (c) 40 m s^{-1}
- 6** 50
- 7** (a) $9\sqrt{2} \text{ cm}$ (b) $40\frac{1}{2} \text{ cm}^2$
- 8** (a) $(-1, -7), (2, 20)$
 (b) Graph crosses the x -axis three times.
 (c) $y = -5$ has three intersections with graph.
 (d) (i) $-20 < k < 7$ (ii) $k < -20$ and $k > 7$
- 9** $(-2, 4), (2, -28); -28 \leq k \leq 4$
- 10** $(-\frac{2}{3}, \frac{4}{27}), (0, 0); 0 < k < \frac{4}{27}$
- 11** $(-1, 5), (2, -22), (0, 10);$
 (a) $5 < k < 10$ (b) $-22 < k < 5$ and $k > 10$
- 12** $(\frac{1}{3}, \frac{4}{27}), (1, 0); k < -\frac{2}{3\sqrt{3}}$ and $k > \frac{2}{3\sqrt{3}}$
- 13** (a) $P = 2x + 2r + \frac{1}{2}\pi r, A = \frac{1}{4}\pi r^2 + rx$
 (b) $x = \frac{1}{4}r(4 - \pi)$
- 14** Maximum at $(2, \frac{1}{4})$
 (a) $(2, 5\frac{1}{4})$
 (b) $(3, \frac{1}{2})$; that is, when $x - 1 = 2$
- 16** (a) $1100 - 20x$ (b) $\text{£}x(1100 - 20x)$
 (c) $\text{£}(24000 - 400x)$; $\text{£}37.50$
- 17** (a) The gradient at P' is the negative of the gradient at P . So $f'(-p) = -f'(p)$. The derivative of an even function is odd.

8 Sequences**Exercise 8A (page 115)**

- 1** (a) 7, 14, 21, 28, 35 (b) 13, 8, 3, -2, -7
 (c) 4, 12, 36, 108, 324 (d) 6, 3, 1.5, 0.75, 0.375
 (e) 2, 7, 22, 67, 202 (f) 1, 4, 19, 364, 132499
- 2** (a) $u_1 = 2, u_{r+1} = u_r + 2$
 (b) $u_1 = 11, u_{r+1} = u_r - 2$
 (c) $u_1 = 2, u_{r+1} = u + 4$
 (d) $u_1 = 2, u_{r+1} = 3u_r$
 (e) $u_1 = \frac{1}{3}, u_{r+1} = \frac{1}{3}u_r$
 (f) $u_1 = \frac{1}{2}a, u_{r+1} = \frac{1}{2}u_r$
 (g) $u_1 = b - 2c, u_{r+1} = u_r + c$
 (h) $u_1 = 1, u_{r+1} = -u_r$

- (i) $u_1 = \frac{p}{q^3}, u_{r+1} = qu_r$
 (j) $u_1 = \frac{a^3}{b^2}, u_{r+1} = \frac{bu_r}{a}$
 (k) $u_1 = x^3, u_{r+1} = \frac{5u_r}{x}$
 (l) $u_1 = 1, u_{r+1} = (1+x)u_r$
- 3** (a) 5, 7, 9, 11, 13; $u_1 = 5, u_{r+1} = u_r + 2$
 (b) 1, 4, 9, 16, 25; $u_1 = 1, u_{r+1} = u_r + 2r + 1$
 (c) 1, 3, 6, 10, 15; $u_1 = 1, u_{r+1} = u_r + r + 1$
 (d) 1, 5, 14, 30, 55; $u_1 = 1, u_{r+1} = u_r + (r+1)^2$
 (e) 6, 18, 54, 162, 486; $u_1 = 6, u_{r+1} = 3u_r$
 (f) 3, 15, 75, 375, 1875; $u_1 = 3, u_{r+1} = 5u_r$
- 4** (a) $u_r = 10 - r$ (b) $u_r = 2 \times 3^r$
 (c) $u_r = r^2 + 3$ (d) $u_r = 2r(r+1)$
 (e) $u_r = \frac{2r-1}{r+3}$ (f) $u_n = \frac{r^2+1}{2^r}$

Exercise 8B (page 119)

- 2** (a) r (c) $1^3 = t_1^2 - t_0^2, 2^3 = t_2^2 - t_1^2,$
 $3^3 = t_3^2 - t_2^2, \dots, n^3 = t_n^2 - t_{n-1}^2$
- 3** (a) 5040 (b) 6720 (c) 35
- 4** (a) $\frac{8!}{4!}$ (b) $\frac{12!}{8!}$ (c) $\frac{n!}{(n-3)!}$
 (d) $\frac{(n+1)!}{(n-2)!}$ (e) $\frac{(n+3)!}{(n-1)!}$ (f) $\frac{(n+6)!}{(n+3)!}$
 (g) 8! (h) $n!$
- 5** (a) 12 (b) $22 \times 22!$
 (c) $n+1$ (d) $n \times n!$
- 7** (a) 1, 5, 10, 10, 5, 1, 0, 0, ...
 (b) 1, 6, 15, 20, 15, 6, 1, 0, 0, ...
 (c) 1, 8, 28, 56, 70, 56, 28, 8, 1, 0, 0, ...
- 8** (a) $\frac{11!}{4! \times 7!}$ (b) $\frac{11!}{7! \times 4!}$ (c) $\frac{10!}{5! \times 5!}$
 (d) $\frac{12!}{3! \times 9!}$ (e) $\frac{12!}{9! \times 3!}$

10 $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$

11 The sum of the terms in the sequence is 2^n .

Exercise 8C (page 124)

- 1** (a), (d), (f), (h); 3, -2, q , x respectively
- 2** (a) 12, 2r (b) 32, 14 + 3r
 (c) -10, 8 - 3r (d) 3.3, 0.9 + 0.4r
 (e) $3\frac{1}{2}, \frac{1}{2} + \frac{1}{2}r$ (f) 43, 79 - 6r
 (g) $x + 10, x - 2 + 2r$ (h) $1 + 4x, 1 - 2x + xr$
- 3** (a) 14 (b) 88 (c) 36
 (d) 11 (e) 11 (f) 11
 (g) 16 (h) 28

Miscellaneous exercise 8 (page 125)

- 1** (a) (i) 1, 2, 5, 14, 41 (ii) 2, 5, 14, 41, 122
 (iii) 0, -1, -4, -13, -40
 (iv) $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$

(b) (i) $b = \frac{1}{2}$ (ii) $b = \frac{3}{2}$ (iii) $b = -\frac{1}{2}$
 (iv) $b = 0$

2 1444

3 (a) 3, 1, 3; alternately 1 and 3
 (b) All terms after the first are 2 (c) 2

4 (a) Alternately 0 and -1; 1, then alternately 0 and -1; gets increasingly large
 (b) $\frac{1}{2}(1 \pm \sqrt{5})$

5 $n(2n+3)$

6 168

7 -750

8 $2n - \frac{1}{2}$

9 167 167; 111 445

10 40

11 (a) 991 (b) 50 045.5

12 (a) 18 (b) $2(2n+1)$
 (c) $a = 6, d = 4$, sum = 2 004 000

13 71 240

14 (a) 47, 12 years left over (b) £345 450

15 $a = \frac{10\ 000}{n} - 5(n-1); 73$

16 (a) 0, 1, 4, 9, 16; r^2 (b) 0, 1, 2, 3, 4; r
 (c) 1, 2, 4, 8, 16; 2^r

9 The binomial theorem

Exercise 9A (page 130)

- 1** (a) $4x^2 + 4xy + y^2$
 (b) $25x^2 + 30xy + 9y^2$
 (c) $16 + 56p + 49p^2$
 (d) $1 - 16t + 64t^2$
 (e) $1 - 10x^2 + 25x^4$
 (f) $4 + 4x^3 + x^6$
 (g) $x^6 + 3x^4y^3 + 3x^2y^6 + y^9$
 (h) $27x^6 + 54x^4y^3 + 36x^2y^6 + 8y^9$

2 (a) $x^3 + 6x^2 + 12x + 8$
 (b) $8p^3 + 36p^2q + 54pq^2 + 27q^3$
 (c) $1 - 12x + 48x^2 - 64x^3$
 (d) $1 - 3x^3 + 3x^6 - x^9$

3 (a) 42 (b) 150

4 (a) 240 (b) 54

5 (a) $1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5$
 (b) $p^6 + 12p^5q + 60p^4q^2 + 160p^3q^3$
 $+ 240p^2q^4 + 192pq^5 + 64q^6$
 (c) $16m^4 - 96m^3n + 216m^2n^2 - 216mn^3 + 81n^4$
 (d) $1 + 2x + \frac{3}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{16}x^4$

6 (a) 270 (b) -1000

7 $1 + 2x + 5x^2 + 4x^3 + 4x^4$

8 $x^3 + 12x^2 + 48x + 64;$
 $x^4 + 13x^3 + 60x^2 + 112x + 64$

9 $72x^5 + 420x^4 + 950x^3 + 1035x^2 + 540x + 108$

10 7

11 $x^{11} + 11x^{10}y + 55x^9y^2 + 165x^8y^3 + 330x^7y^4$
 $+ 462x^6y^5 + 462x^5y^6 + 330x^4y^7$
 $+ 165x^3y^8 + 55x^2y^9 + 11xy^{10} + y^{11}$

12 59 136

Exercise 9B (page 134)

1 (a) 35 (b) 28 (c) 126 (d) 715
 (e) 15 (f) 45 (g) 11 (h) 1225

2 (a) 10 (b) -56 (c) 165 (d) -560

3 (a) 84 (b) -1512 (c) 4032 (d) $-\frac{99}{4}$

4 (a) 3003 (b) 192 192
 (c) 560 431 872 (d) 48 048

5 (a) $1 + 13x + 78x^2 + 286x^3$
 (b) $1 - 15x + 105x^2 - 455x^3$
 (c) $1 + 30x + 405x^2 + 3240x^3$
 (d) $128 - 2240x + 16800x^2 - 70\ 000x^3$

6 (a) $1 + 22x + 231x^2$
 (b) $1 - 30x + 435x^2$
 (c) $1 - 72x + 2448x^2$
 (d) $1 + 114x + 6156x^2$

- 7 $1+16x+112x^2$; 1.17
 8 $4096+122880x+1689600x^2$; 4220.57
 9 $1+32x+480x^2+4480x^3$; 5920
 10 $1-30x+405x^2$; 234
 11 7
 12 $2+56x^2+140x^4+56x^6+2x^8$;
 $2.005\ 601\ 400\ 056\ 000\ 2$
 13 $a=4, n=9$

Miscellaneous exercise 9 (page 135)

- 1 $27+108x+144x^2+64x^3$
 2 (a) $1+40x+720x^2$ (b) $1-32x+480x^2$
 3 (a) -48.384 (b) $\frac{875}{4}$
 4 $2187+25515x+127575x^2$; 2455
 5 $256+256x+112x^2+28x^3$; 258.571
 6 $256-3072x+16128x^2$; 253
 7 $x^6+3x^3+3+\frac{1}{x^3}$
 8 $16x^4-96x+\frac{216}{x^2}-\frac{216}{x^5}+\frac{81}{x^8}$
 9 $2x^6+\frac{15x^2}{2}+\frac{15}{8x^2}+\frac{1}{32x^6}$
 10 48
 11 20 000
 12 495; (a) 40 095 (b) 7920 (c) $\frac{495}{16}$
 13 30
 14 $1+40x+760x^2$;
 (a) 1.0408 (b) 0.9230
 15 $1024-\frac{2560}{x^2}+\frac{2880}{x^4}$; 999
 16 B, D
 17 $\frac{7}{16}$
 18 5376
 19 $\frac{6435}{128}$
 20 -2024
 21 $1+12x+70x^2$; 1.127
 22 $270x^2+250$; $\pm\frac{4}{3}$
 23 $\pm\frac{5}{6}$
 24 (a) $1+5\alpha t+10\alpha^2t^2$ (b) $1-8\beta t+28\beta^2t^2$;
 $10\alpha^2-40\alpha\beta+28\beta^2$
 25 (b) (i) 4 or 7 (ii) 3 or 13
 (iii) 7 or 13 (iv) 17 or 28
 27 (b) (i) $a=3, b=6, c=8$
 (ii) $a=6, b=4, c=9$
 29 1.005 413 792 056 807
 30 (a) $217+88\sqrt{6}$ (b) $698\sqrt{2}+569\sqrt{3}$

- 31 (a) 568; 567 and 568 (b) 969 and 970
 32 $2n(n+1)$
 33 $a=-\frac{1}{6}, b=\frac{1}{2}, c=-\frac{1}{2}, d=\frac{1}{6}$

10 Trigonometry**Exercise 10A (page 141)**

- 1 (a) (i) 0.9063 (ii) 0.4226 (iii) 0.4663
 (b) (i) -0.5736 (ii) 0.8192 (iii) -1.4281
 (c) (i) -0.7071 (ii) -0.7071 (iii) 1
 (d) (i) 0.8192 (ii) -0.5736 (iii) -0.7002
 (e) (i) -0.3420 (ii) 0.9397 (iii) -2.7475
 (f) (i) 0.3843 (ii) 0.9232 (iii) 2.4023
 (g) (i) -0.5721 (ii) 0.8202 (iii) -1.4335
 (h) (i) -0.9703 (ii) -0.2419 (iii) 0.2493
 2 (a) max 3 at $x = 90$, min 1 at $x = 270$
 (b) max 11 at $x = 180$, min 3 at $x = 360$
 (c) max 13 at $x = 180$, min -3 at $x = 90$
 (d) max 4 at $x = 90$, min 2 at $x = 270$
 (e) max 10 at $x = 27\frac{1}{2}$, min 8 at $x = 72\frac{1}{2}$
 (f) max 5 at $x = 90$, min 1.875 at $x = 450$
 3 (a) 160 (b) 320 (c) 240
 (d) 50 (e) 220 (f) 340
 (g) 40, 140 (h) 30, 330 (i) 70, 250
 (j) 80, 100 (k) 160, 200 (l) 100, 280
 4 (a) 160 (b) -40 (c) -120
 (d) 50 (e) -140 (f) -20
 (g) 40, 140 (h) 30 (i) -110, 70
 (j) 80, 100 (k) ± 160 (l) -80, 100
 5 (a) $\frac{1}{2}\sqrt{2}$ (b) $-\frac{1}{2}$ (c) $-\frac{1}{2}$ (d) $\sqrt{3}$
 (e) $-\frac{1}{2}\sqrt{2}$ (f) $\frac{1}{2}\sqrt{3}$ (g) -1 (h) $-\frac{1}{2}\sqrt{3}$
 (i) $-\frac{1}{2}\sqrt{2}$ (j) 0 (k) 1 (l) $\frac{1}{2}\sqrt{2}$
 (m) $-\frac{1}{2}$ (n) -1 (o) $-\frac{1}{2}$ (p) 0
 6 (a) 60 (b) 240 (c) 120 (d) 30
 (e) 30 (f) 135 (g) 210 (h) 90
 7 (a) 120 (b) 60 (c) -90 (d) 180
 (e) 60 (f) -30 (g) -45 (h) 0
 8 $A=5, B=2.8$; 7.42 m
- Exercise 10B (page 144)**
- 3 (a) 90 (b) 45 (c) 180 (d) 30
 (e) 360 (f) $11\frac{1}{4}$
- Exercise 10C (page 148)**
- 1 (a) 5.7, 174.3 (b) 237.1, 302.9
 (c) 72.0, 108.0 (d) 36.9, 323.1
 (e) 147.1, 212.9 (f) 35.3, 324.7
 (g) 76.0, 256.0 (h) 162.3, 342.3
 (i) 6.3, 186.3 (j) 203.6, 336.4
 (k) 214.8, 325.2 (l) 161.6, 341.6

- (m) 49.5, 160.5 (n) 240, 360
 (o) 10, 70
- 2** (a) 53.1, 126.9 (b) ± 75.5
 (c) -116.6, 63.4 (d) -137.9, -42.1
 (e) ± 96.9 (f) -36.9, 143.1
 (g) -128.7, 51.3 (h) $\pm 45, \pm 135$
 (i) 0, $\pm 60, \pm 180$
- 3** (a) 35.3, 144.7, 215.3, 324.7
 (b) 21.1, 81.1, 141.1, 201.1, 261.1, 321.1
 (c) 108.4, 161.6, 288.4, 341.6
 (d) 26.1, 63.9, 116.1, 153.9, 206.1, 243.9,
 296.1, 333.9
 (e) 10.9, 100.9, 190.9, 280.9
 (f) 68.3, 111.7, 188.3, 231.7, 308.3, 351.7
- 4** (a) $\pm 16.1, \pm 103.9, \pm 136.1$
 (b) -125.8, -35.8, 54.2, 144.2
 (c) -176.2, -123.8, -56.2, -3.8, 63.8, 116.2,
 (d) $\pm 37.9, \pm 142.1$
 (e) -172.3, -136.3, -100.3, -64.3, -28.3, 7.7
 43.7, 79.7, 115.7, 151.7
 (f) -78.5, -11.5, 101.5, 168.5
- 5** (a) ± 96.4 (b) -107.3, 162.7
 (c) -56.0 (d) No roots in the interval
 (e) 35.4 (f) -43.6
- 6** (a) 30, 90, 210, 270
 (b) 22.5, 112.5, 202.5, 292.5
 (c) 65, 85, 185, 205, 305, 325
 (d) 110, 230, 350
 (e) 85, 145, 265, 325
 (f) 80
 (g) No roots in the interval
 (h) 45, 105, 165, 225, 285, 345
 (i) 80
- 7** (a) -170.8, -9.2
 (b) -90, 90
 (c) -180, -135, 0, 45, 180
 (d) -173.4, -96.6, 6.6, 83.4
 (e) 11.3, 78.7
 (f) -150.5, -90.5, -30.5, 29.5, 89.5, 149.5
- 8** (a) 27, 63, 207, 243
 (b) 4, 68, 76, 140, 148, 212, 220, 284, 292, 356
 (c) 20, 80, 140, 200, 260, 320
- 9** 0, 60, 120, 180
- 10** For example,
 (a) (i) $\sin 4\theta^\circ$ (ii) $\cos 4\theta^\circ$ (iii) $\tan 2\theta^\circ$
 (b) (i) $\sin 18\theta^\circ$ (ii) $\cos 18\theta^\circ$ (iii) $\tan 9\theta^\circ$
 (c) (i) $\sin \frac{15}{2}\theta^\circ$ (ii) $\cos \frac{15}{2}\theta^\circ$ (iii) $\tan \frac{15}{4}\theta^\circ$
 (d) (i) $\sin 3\theta^\circ$ (ii) $\cos 3\theta^\circ$ (iii) $\tan \frac{3}{2}\theta^\circ$
 (e) (i) $\sin \frac{1}{2}\theta^\circ$ (ii) $\cos \frac{1}{2}\theta^\circ$ (iii) $\tan \frac{1}{4}\theta^\circ$
 (f) (i) $\sin \frac{3}{5}\theta^\circ$ (ii) $\cos \frac{3}{5}\theta^\circ$ (iii) $\tan \frac{3}{10}\theta^\circ$
- 11** (a) 120 (b) 180 (c) 90
 (d) 540 (e) 720 (f) 720
 (g) 120 (h) 90 (i) 360
- 12** (a) 0.986
 (b) $A = 12, B = 6, 6$ hours 7 minutes
 (c) Days 202, 345
- Exercise 10D (page 152)**
- 1** (a) (i) 11 (ii) $\frac{4}{5}, \frac{3}{5}, \frac{4}{3}$
 (b) (i) 37.5 (ii) $\frac{15}{17}, \frac{8}{17}, \frac{15}{8}$
 (c) (i) $\sqrt{13}$ (ii) $\frac{3}{13}\sqrt{13}, \frac{2}{13}\sqrt{13}, \frac{3}{2}$
 (d) (i) 11 (ii) $\frac{5}{14}\sqrt{3}, \frac{11}{14}, \frac{5}{11}\sqrt{3}$
 (e) (i) $17\sqrt{5}$ (ii) $\frac{22}{85}\sqrt{5}, \frac{31}{85}\sqrt{5}, \frac{22}{31}$
 (f) (i) $12\sqrt{2}$ (ii) $\frac{1}{3}, \frac{2}{3}\sqrt{2}, \frac{1}{4}\sqrt{2}$
- 2** (a) $-\frac{11}{14}$ (b) $\frac{20}{29}$ (c) $\pm \frac{1}{2}\sqrt{3}$
 (d) 0, ± 78.5 (to 1 d.p.)
- 4** (a) 30, 150, 210, 330 (b) 0, 180, 360
 (c) 36.9, 143.1, 199.5, 340.5
 (d) 0, 51.0, 180, 309.0, 360
- 5** -116.6, -26.6, 63.4, 153.4
- Miscellaneous exercise 10 (page 152)**
- 1** (a) 360 (b) 90
- 2** (a) $\cos x^\circ$ (b) $-\cos x^\circ$
- 3** (0, 1), ($\pm 180, 0$)
- 4** (a) 21.8, 201.8 (b) 11.8, 78.2, 191.8, 258.2
- 5** 24.1, 155.9
- 6** (a) Examples are $\tan x^\circ$, $\sin 2x^\circ$, $\cos 2x^\circ$
 (b) 10, 50, 130, 170
- 7** 30, 270
- 8** (a) $2\sin 2x^\circ$ (b) 105, 165, 285, 345
- 9** (a) 168.5 (b) 333.4 (c) 225 (d) 553.1
- 11** (a) 2, 0; 180, 90 (b) 9, 1; 240, 60
 (c) 49, 9; 105, 45 (d) 8, 5; 90, 180
 (e) 6, 3; 180, 360 (f) 60, 30; $7\frac{1}{2}, 52\frac{1}{2}$
- 12** (a) 0, 180, 360 (b) 0, 30, 150, 180, 360
 (c) $67\frac{1}{2}, 157\frac{1}{2}, 247\frac{1}{2}, 337\frac{1}{2}$
 (d) 30, 120, 210, 300
- 13** (a) 60 (b) 8.9, 68.9, 128.9
 (c) (i) 51.1 (ii) 21.1
- 14** (a) $4.8 \pm 1.2 \sin 15t^\circ$ or $4.8 \pm 1.2 \cos 15t^\circ$
 (b) $21500 \pm 6500 \sin 36t^\circ$ or
 $21500 \pm 6500 \cos 36t^\circ$
 (c) $12 \pm 10 \sin t^\circ$ or $12 \pm 10 \cos t^\circ$

- 15** (a) 0.1 cm, 0.0009 seconds
 (b) 0.0036 seconds
 (c) 278 (d) 0.002 13 seconds

- 16** (a) 110 cm and 90 cm (b) 0.36 seconds
 (c) 0.72 seconds (d) 0.468

17 (a) $k = \frac{360}{T}$ (b) $\frac{k}{360}$

- 18** (a) 7.2
 (b) (i) $N - C$ (ii) $N + C$, after 37.5 weeks

- 19** (a) 30 (b) 3.25 m (c) 27

20 $A = 0.5, B = 4.5, \alpha = 1, \beta = 12$

11 Combining and inverting functions

Exercise 11A (page 162)

1 (a) 121 (b) 4 (c) 676

2 (a) 17 (b) 8 (c) 152

3 (a) 1 (b) 4 (c) $\frac{1}{2}$

4 (a) 1 (b) 4 (c) $6\frac{1}{3}$

5 (a) 0 (b) 21
 (c) $-3\frac{3}{4}$ (d) $x^2 - 4$

6 (a) 27 (b) 8
 (c) $3\frac{3}{8}$ (d) $(\cos x^\circ + 2)^3$

7 (a) 16 (b) 4
 (c) 81 (d) $(2\sqrt{x} - 10)^2$

8 (a) $\times, 4, +, 9$ (b) $+, 9, =, \times, 4$
 (c) square, $\times, 2, -, 5$ (d) $-5, =, \text{square}, \times, 2$
 (e) $\sqrt{}, -, 3, =, \text{cube}$ (f) $\pm, 2, =, \text{square}, +, 10, =, \sqrt{}$

9 (a) $\mathbb{R}, f(x) \geq 0$ (b) $\mathbb{R}, -1 \leq f(x) \leq 1$
 (c) $x \geq 3, f(x) \geq 0$ (d) $\mathbb{R}, f(x) \geq 5$
 (e) $x > 0, f(x) > 0$ (f) $\mathbb{R}, f(x) \leq 4$
 (g) $0 \leq x \leq 4, 0 \leq f(x) \leq 2$
 (h) $\mathbb{R}, f(x) \geq 6$ (i) $x \geq 3, f(x) \geq 0$

10 (a) 4 (b) -14 (c) -9 (d) 33
 (e) 23 (f) -17 (g) 16 (h) 0

11 (a) 49 (b) 59 (c) 35
 (d) $\frac{1}{16}$ (e) 9 (f) 899

12 (a) 7 (b) -19 (c) 1 (d) $\frac{1}{2}$
 (e) $\frac{1}{2}$ (f) -1 (g) $3\frac{2}{3}$ (h) 2

13 (a) $x \mapsto 2x^2 + 5$ (b) $x \mapsto (2x+5)^2$
 (c) $x \mapsto \frac{2}{x} + 5$ (d) $x \mapsto \frac{1}{2x+5}$
 (e) $x \mapsto 4x+15$ (f) $x \mapsto x$
 (g) $x \mapsto \left(\frac{2}{x} + 5\right)^2$ (h) $x \mapsto \frac{1}{(2x+5)^2}$

- 14** (a) $x \mapsto \sin x^\circ - 3$ (b) $x \mapsto \sin(x-3)^\circ$
 (c) $x \mapsto \sin(x^3 - 3)^\circ$ (d) $x \mapsto \sin(x^3)^\circ$
 (e) $x \mapsto x - 9$ (f) $x \mapsto (\sin x^\circ)^3$

- 15** (a) fh (b) fg
 (c) hh (d) hg or ggh
 (e) gf or fffg (f) gffh
 (g) fgfg or fffffgg (h) hf
 (i) hffg

- 16** (a) $x \geq 0, gf(x) \geq -5$
 (b) $x \geq -3, gf(x) \geq 0$
 (c) $x \neq 2, gf(x) \neq 0$
 (d) $\mathbb{R}, 0 \leq gf(x) \leq 1$
 (e) $\mathbb{R}, gf(x) \geq 0$
 (f) $-4 \leq x \leq 4, 0 \leq gf(x) \leq 2$
 (g) $x \leq -2 \text{ or } x \geq 3, gf(x) \geq 0$
 (h) $x < -2, gf(x) > 0$

17 (a) $-2\frac{2}{3}$ or 4 (b) 7 (c) 1

18 $a = 3, b = -7$ or $a = -3, b = 14$

19 $a = 4, b = 11$ or $a = 4\frac{1}{2}, b = 10$

21 (a) $t : r \mapsto \frac{1}{2}r(r+1)$ (b) $f : r \mapsto a + (r-1)d$

- 22** For p : (a) \mathbb{N}
 (b) 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31
 (c) The set of prime numbers
 For π : (a) \mathbb{N}
 (b) 0, 1, 2, 2, 3, 3, 4, 4, 4, 4
 (c) $\mathbb{N} \cup \{0\}$

Exercise 11B (page 169)

1 (a) $x \mapsto x - 4$ (b) $x \mapsto x + 5$
 (c) $x \mapsto \frac{1}{2}x$ (d) $x \mapsto 4x$
 (e) $x \mapsto \sqrt[3]{x}$ (f) $x \mapsto x^5$

2 (a) 10 (b) 7 (c) 3
 (d) 5 (e) -4

3 (a) 4 (b) 20 (c) $\frac{7}{5}$
 (d) 15 (e) -6

4 (a) 8 (b) $\frac{1}{8}$ (c) 512
 (d) -27 (e) 5

5 a, d, f, g, k

6 a, c, e, f, g, h, j

7 (a) 0 (b) -1 (c) $\frac{2}{3}$
 (d) 4 (e) -5 (f) -1
 (g) $1\frac{1}{2}$ (h) 1 (i) 4

8 (a) $x \mapsto \frac{1}{3}(x+1)$ (b) $x \mapsto 2(x-4)$
 (c) $x \mapsto \sqrt[3]{x-5}$ (d) $x \mapsto (x+3)^2, x > -3$
 (e) $x \mapsto \frac{1}{5}(2x+3)$ (f) $x \mapsto 1 + \sqrt{x-6}, x \geq 6$

9 (a) $y \mapsto \frac{1}{6}(y-5)$ (b) $y \mapsto 5y-4$
 (c) $y \mapsto \frac{1}{2}(4-y)$ (d) $y \mapsto \frac{1}{2}(3y-7)$
 (e) $y \mapsto \sqrt[3]{\frac{1}{2}(y-5)}$ (f) $y \mapsto \frac{1}{y-4}, y \neq 4$

- (g) $y \mapsto \frac{5}{y} + 1, y \neq 0$
 (h) $y \mapsto \sqrt{y-7} - 2, y \geq 7$
 (i) $y \mapsto \frac{1}{2}(3 + \sqrt{y+5}), y \geq -5$
 (j) $y \mapsto 3 + \sqrt{y+9}, y \geq -9$
- 10 (a) $x \mapsto \frac{1}{4}x$ (b) $x \mapsto x-3$
 (c) $x \mapsto x^2, x \geq 0$ (d) $x \mapsto \frac{1}{2}(x-1)$
 (e) $x \mapsto \sqrt{x+2}, x \geq 0$ (f) $x \mapsto \frac{1}{3}(1-x)$
 (g) $x \mapsto \frac{3}{x}, x \neq 0$ (h) $x \mapsto 7-x$
- 12 (a) $y \mapsto \frac{2y}{y-1}, y \neq 1$ (b) $y \mapsto \frac{4y+1}{y-2}, y \neq 2$
 (c) $y \mapsto \frac{5y+2}{y-1}, y \neq 1$ (d) $y \mapsto \frac{3y-11}{4y-3}, y \neq \frac{3}{4}$
- 13 (a) $f(x) > -1$
 (b) $x \mapsto 2 + \sqrt{x+1}, x > -1; f^{-1}(x) > 2$
- 14 (a) $f(x) > 3$
 (b) $x \mapsto 2 + (x-3)^2, x > 3; f^{-1}(x) > 2$
- 15 $k = -1$
 (a) $f(x) \geq 5$
 (b) $x \mapsto -1 - \sqrt{x-5}, x \geq 5; f^{-1}(x) \leq -1$
- 16 $a = \frac{1}{8}, b = \frac{3}{8}$
 17 6
 18 $x \mapsto \sqrt{x-5}^{\frac{3}{4}} - \frac{1}{2}, x > 6; f^{-1}(x) > 0$
 19 $x \mapsto 1 - \sqrt{-\frac{1}{2}(x+5)}, x < -5; f^{-1}(x) < 1$
- Miscellaneous exercise 11 (page 172)**
- 1 (a) 29 (b) 61 (c) 2
 (d) -3 (e) 290 (f) 497
- 2 (a) $\mathbb{R}, f(x) \leq 4$ (b) $\mathbb{R}, f(x) \geq -7$
 (c) $x \geq -2, f(x) \geq 0$ (d) \mathbb{R}, \mathbb{R}
 (e) $\mathbb{R}, f(x) \geq 0$ (f) $x \geq 0, f(x) \leq 2$
- 3 (a) $x \mapsto (1-2x)^3$ (b) $x \mapsto 1-2x^3$
 (c) $x \mapsto 1-2x^9$ (d) $x \mapsto 4x-1$
 (e) $x \mapsto \frac{1}{2}(1-x)$
- 4 (a) 48 (b) 3 (c) -1
 (d) 4 (e) 4
- 5 (a) 12 (b) 27
- 6 $x \mapsto -3 + \sqrt{x-1}, x > 10$
- 7 $x \mapsto \sqrt[3]{\frac{1}{4}(x-3)}, x \in \mathbb{R};$ reflections in $y = x$
- 8 (a) $\sqrt{\frac{1}{3}(x+4)}$ (b) $3x^2 + 24x + 44$
- 9 (a) gf (b) ff
 (c) g^{-1} (d) fgh or fng
 (e) hfg (f) f^{-1}
 (g) $g^{-1}fgh$ or $g^{-1}fhg$ (h) hf^{-1}
- 10 $f^{-1}(x) = \sqrt{x-1}, x \geq 1;$
 $gf(x) = x^2 - 2, gf(x) \geq -2$
- 11 $1 \pm \sqrt{3}$
 12 $k = 1; x \mapsto 1 - \sqrt{x-6}, x \geq 6$
- 13 $a = -2, b = 11$ or $a = 2, b = -13$
- 14 (b) $-\sqrt{1-x}, x \leq 1$ (c) $-\frac{1}{2}$
- 15 (a) $x \mapsto \frac{1}{4}(x-5)$ (b) $x \mapsto \frac{1}{2}(3-x)$
 (c) $x \mapsto -\frac{1}{8}(7+x)$ (d) $x \mapsto -8x-7$
 (e) $x \mapsto -\frac{1}{8}(7+x)$
- 16 (a) $x \mapsto \frac{1}{2}(x-7)$ (b) $x \mapsto \sqrt[3]{x+1}$
 (c) $x \mapsto \sqrt[3]{\frac{1}{2}(x-5)}$ (d) $x \mapsto \frac{1}{2}(\sqrt[3]{x+1}-7)$
 (e) $x \mapsto 2x^3 + 5$ (f) $x \mapsto (2x+7)^3 - 1$
 (g) $x \mapsto \sqrt[3]{\frac{1}{2}(x-5)}$ (h) $x \mapsto \frac{1}{2}(\sqrt[3]{x+1}-7)$
- 17 (a) 3 (b) 7 (c) 3 (d) 7
- 18 (a) x (b) $\frac{x+5}{2x-1}$ (c) x
 (d) x (e) $\frac{x+5}{2x-1}$
- 19 (a) $\frac{4}{2-x}$ (b) $\frac{4}{2-x}$ (c) x
 (d) $\frac{2x-4}{x}$ (e) x (f) $\frac{2x-4}{x}$
- 12 Extending differentiation**
- Exercise 12A (page 174)**
- 1 (a) $2(x+3)$ (b) $4(2x-3)$
 (c) $-9(1-3x)^2$ (d) $3a(ax+b)^2$
 (e) $-3a(b-ax)^2$ (f) $-5(1-x)^4$
 (g) $8(2x-3)^3$ (h) $-8(3-2x)^3$
- 2 $na(ax+b)^{n-1}$
- 3 (a) $10(x+3)^9$ (b) $10(2x-1)^4$
 (c) $-28(1-4x)^6$ (d) $15(3x-2)^4$
 (e) $-12(4-2x)^5$ (f) $72(2+3x)^5$
 (g) $10(2x+5)^4$ (h) $18(2x-3)^8$
- Exercise 12B (page 176)**
- 1 (a) $20(4x+5)^4$ (b) $16(2x-7)^7$
 (c) $-6(2-x)^5$ (d) $2\left(\frac{1}{2}x+4\right)^3$
- 2 (a) $\frac{-3}{(3x+5)^2}$ (b) $\frac{2}{(4-x)^3}$
 (c) $\frac{-6}{(2x+1)^4}$ (d) $\frac{-64}{(4x-1)^5}$
- 3 (a) $\frac{1}{\sqrt{2x+3}}$ (b) $\frac{2}{\sqrt[3]{(6x-1)^2}}$
 (c) $\frac{-2}{\sqrt{(4x+7)^3}}$ (d) $\frac{-10}{\sqrt[3]{(3x-2)^5}}$

4 60

5 $(-6,125)$

6 $y = -\frac{3}{4}x - \frac{5}{4}$

7 $y = -3x + 48$

Exercise 12C (page 179)

1 (a) $30(5x+3)^5$ (b) $\frac{5}{2}(5x+3)^{-\frac{1}{2}}$
(c) $\frac{-5}{(5x+3)^2}$

2 (a) $-20(1-4x)^4$ (b) $12(1-4x)^{-4}$
(c) $\frac{-2}{\sqrt{1-4x}}$

3 (a) $15x^2(1+x^3)^4$ (b) $-12x^2(1+x^3)^{-5}$
(c) $\frac{x^2}{(1+x^3)^{\frac{2}{3}}}$

4 (a) $24x(2x^2+3)^5$ (b) $\frac{-4x}{(2x^2+3)^2}$
(c) $\frac{2}{\sqrt[3]{(2x^2+3)^3}}$

5 $24x^3(3x^4+2)$

6 (a) $72x^8 + 72x^5 + 18x^2$, (b) $18x^2(2x^3+1)^2$

7 (a) $20x^4(x^5+1)^3$ (b) $48x^2(2x^3-1)^7$
(c) $\frac{5}{2\sqrt{x}}(\sqrt{x}-1)^4$

8 (a) $8x(x^2+6)^3$ (b) $45x^2(5x^3+4)^2$
(c) $28x^3(x^4-8)^6$ (d) $-45x^8(2-x^9)^4$

9 (a) $\frac{2}{\sqrt{(4x+3)}}$ (b) $12x(x^2+4)^5$
(c) $-36x^2(6x^3-5)^{-3}$ (d) $3x^2(5-x^3)^{-2}$

10 (a) $-\frac{4}{25}$ (b) 0

11 $\frac{3}{8}$

12 (a) $6(x^2+3x+1)^5(2x+3)$ (b) $\frac{-3(2x+5)}{(x^2+5x)^4}$

13 $y = 12x - 25$

14 $x+4y=8$

15 $x+6y=23$

16 $6x(x^2-1)^{-\frac{1}{2}}(\sqrt{x^2-1}+1)^5$

17 $\frac{1}{\sqrt{(4x+3)(1+\sqrt{4x+3})}}$

18 $(0,3)$; minimum

Exercise 12D (page 182)

1 4500 per hour

2 0.622°C per minute3 (a) 4.8 cm s^{-1} (b) $24\text{ cm}^2\text{ s}^{-1}$ 4 (a) $240\text{ mm}^2\text{ s}^{-1}$ (b) $2400\text{ mm}^3\text{ s}^{-1}$ 5 $942\text{ mm}^2\text{ s}^{-1}$ 6 0.25 m min^{-1} 7 0.0076 m s^{-1} 8 0.011 m s^{-1} 9 0.0040 cm s^{-1} **Miscellaneous exercise 12 (page 184)**

1 $80(4x-1)^{19}$

2 $\frac{8}{(3-4x)^3}$

3 $40x^3(x^4+3)^4$

4 $24x+y=49$

6 $-\frac{10}{27}$

7 $y=20x+11$

8 $(0,\frac{1}{4})$

9 $3x+4y=18$

10 $0.377\text{ cm}^2\text{ s}^{-1}$

11 (a) $\frac{10}{\sqrt{\pi}}\text{ cm}$ (b) $\frac{1}{4\sqrt{\pi}}\text{ cm s}^{-1}$

12 $8x+5y-34=0$

13 $(2,-4); (0,0), (4,0)$

(0,16), (4,-16); (2,0), $(2 \pm 2\sqrt{3}, 0)$, (0,16)

14 $\frac{(1-1/x^2)}{2\sqrt{(x+1/x)}}$

15 $4\text{ m}^2\text{ s}^{-1}$

16 $y=2x-3$

17 $\frac{-12t}{(3t^2+5)^3}$

18 (a) $\frac{2-x}{\sqrt{4x-x^2}}$, (2,2)

19 (a) Minimum (b) 20

20 $\frac{3}{20\pi}\text{ cm s}^{-1}$

21 0.052 m s^{-1}

22 $(-\sqrt{3}, -4)$, minimum; $(-1, 0)$, maximum;

$(0, -4)$, minimum; $(1, 0)$, maximum;

$(\sqrt{3}, -4)$, minimum

23 $\left(\frac{1}{2}, \frac{1}{4}\right)$, maximum

24 $\left(2\frac{1}{2}, 6\right)$, minimum

Revision exercise 2
(page 187)

1 $9x - y = 16$

2 $(-1, 7)$

3 $(1, 0)$

4 $\left(-\frac{9}{4}, \frac{81}{16}\right)$

5 $\left(-\frac{1}{4}, \frac{1}{16}\right), \left(2\frac{1}{4}, 5\frac{1}{16}\right)$

6 $y + 2x = 3; \left(\frac{9}{4}, -\frac{3}{2}\right)$

7 (a) 22.5 cm (b) 45 cm, 15 cm
(c) 0.33, 3.67 (d) 15

8 (a) $4x^{-\frac{3}{2}}$; $y = x + 16$ (b) $23y - 22x = 360$
(c) $(-8, 8)$

9 $1 - \frac{1}{x^2}; \frac{1}{\sqrt{x}}; \frac{-3}{2x\sqrt{x}}; 1 - \frac{1}{x\sqrt{x}} - \frac{2}{x^3}$

10 -4, maximum; -8, minimum

11 Maximum

12 (a) $\pm 50.8, \pm 129.2$ (b) -150, -30

(c) -166.7, -76.7, 13.3, 103.3

(d) $\pm 165, \pm 15$

13 (a) 15, 75, 105, 165, 195, 255, 285, 345
(c) 180; e.g., (45, 1.5)

14 $\frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}}; 1 - \frac{1}{x^2}$

15 3

16 $x \mapsto \frac{1-x}{1+2x}, x \in \mathbb{R}, x \neq -\frac{1}{2}$

17 $\frac{1}{2}(2n+1)$ (b) 20

18 (a) $x^8 - 8x^6 + 28x^4$ (b) 216

19 $-(2x+3)^{-\frac{3}{2}}$

20 (a) $(x-1)^2 - 2; -2 \leq f(x) \leq 14$; not one-one
(b) $x \geq 1$

21 (b) 22

22 (a) 5 (b) $\frac{1}{2}$

23 (a) $f(x) \leq 4$ (b) Reflect in $y=2$
(c) $\sqrt{4-x}, x \leq 4$ (d) 1.56

24 1215

25 21

26 (a) $3(3x^2 + 2)(x^3 + 2x - 1)^2$

(b) $-x(x^2 + 1)^{-\frac{1}{2}}$

27 (a) $2\pi \times 10^{15} \text{ km}^3 \text{ s}^{-1}$ (b) $4\pi \times 10^9 \text{ km}^2 \text{ s}^{-1}$

28 (a) 360 (b) 720 (c) 240

29 (a) 89 (b) \$1 341 600

31 $\left(\frac{1}{2}(p-q), \frac{1}{2}(p+q)\right), \left(\frac{1}{2}(p+q), \frac{1}{2}(q-p)\right);$
 $\frac{p+q}{p-q}$ and $\frac{q-p}{p+q}$

13 Vectors

Exercise 13A (page 194)

2 (a) $(4\mathbf{i} + \mathbf{j}) + (-3\mathbf{i} + 2\mathbf{j}) = \mathbf{i} + 3\mathbf{j}$

(b) $3(\mathbf{i} - 2\mathbf{j}) = 3\mathbf{i} - 6\mathbf{j}$

(c) $4\mathbf{j} + 2(\mathbf{i} - 2\mathbf{j}) = 2\mathbf{i}$

(d) $(3\mathbf{i} + \mathbf{j}) - (5\mathbf{i} + \mathbf{j}) = -2\mathbf{i}$

(e) $3(-\mathbf{i} + 2\mathbf{j}) - (-4\mathbf{i} + 3\mathbf{j}) = \mathbf{i} + 3\mathbf{j}$

(f) $4(2\mathbf{i} + 3\mathbf{j}) - 3(3\mathbf{i} + 2\mathbf{j}) = -\mathbf{i} + 6\mathbf{j}$

(g) $(2\mathbf{i} - 3\mathbf{j}) + (4\mathbf{i} + 5\mathbf{j}) + (-6\mathbf{i} - 2\mathbf{j}) = \mathbf{0}$

(h) $2(3\mathbf{i} - \mathbf{j}) + 3(-2\mathbf{i} + 3\mathbf{j}) + (-7\mathbf{j}) = \mathbf{0}$

3 (a) $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ (b) $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ (c) $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ (d) $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$

4 $s = 2$

5 $s = 4; \mathbf{q} = \frac{1}{4}(\mathbf{r} - \mathbf{p})$

6 2, 3

7 $1\frac{1}{2}, -\frac{1}{2}$

8 $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} -6 \\ 3 \end{pmatrix}$ are parallel, $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ is in a different direction; $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ is not parallel to $\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

9 Any multiple of 1, -1, 2

10 (a) No (b) -2, 0
 \mathbf{p} is parallel to \mathbf{r} , but \mathbf{q} is in a different direction

Exercise 13B (page 197)

1 (a) (9, 3) (b) (-1, -2) (c) (2, -1)

(d) (-8, -1) (e) (10, 5) (f) $\left(5, 2\frac{1}{2}\right)$

2 (a) (-13, -23) (b) (-1, 1)

3 $\mathbf{c} = 2\mathbf{b} - \mathbf{a}$

4 $\frac{3}{7}\mathbf{a} + \frac{4}{7}\mathbf{b}$

6 $\mathbf{b} - \mathbf{a} = \mathbf{c} - \mathbf{d}$

7 $\frac{1}{2}(\mathbf{b} + \mathbf{c} - 2\mathbf{a}), \frac{1}{4}(\mathbf{b} + \mathbf{c} - 2\mathbf{a})$; G is the mid-point of AD

8 $\mathbf{b} = \mathbf{a} + \mathbf{c}$, $\mathbf{m} = \frac{1}{2}(\mathbf{a} + 2\mathbf{c})$, $\mathbf{p} = \frac{1}{3}(\mathbf{a} + 2\mathbf{c})$;
 O , P and M are collinear, and $OP = \frac{2}{3}OM$

9 $\mathbf{d} = \frac{1}{2}(\mathbf{b} + \mathbf{c})$, $\mathbf{e} = \frac{1}{4}(2\mathbf{a} + \mathbf{b} + \mathbf{c})$, $\mathbf{f} = \frac{1}{3}(2\mathbf{a} + \mathbf{c})$,
 $\mathbf{g} = \frac{1}{4}(\mathbf{b} + 2\mathbf{a} + \mathbf{c})$

10 $\frac{4}{15}\mathbf{b} - \frac{13}{15}\mathbf{a}$, $k = \frac{15}{13}$, $\mathbf{r} = \frac{4}{13}\mathbf{b}$;
 R is on OB , with $OR:RB = 4:9$
 S is on OA , with $OS:SA = 2:9$

Exercise 13C (page 200)

1 $\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$

2 (a) $\begin{pmatrix} 3 \\ -6 \\ -6 \end{pmatrix}$ (b) $3\mathbf{i} - 6\mathbf{j} - 6\mathbf{k}$

- 3 (a) ABC is not a straight line.
(b) ABC is a straight line.

4 (a) $\begin{pmatrix} 1 \\ 2 \\ -7 \end{pmatrix}, \begin{pmatrix} -6 \\ 4 \\ -8 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ \frac{3}{2} \end{pmatrix}$ (b) $(-4, 7, -17\frac{1}{2})$

5 $\mathbf{i} + \mathbf{j} - 6\mathbf{k}$, $\mathbf{i} + \mathbf{j} - 6\mathbf{k}$; it is a parallelogram

6 (a) $\begin{pmatrix} -3 \\ 6 \\ 1 \end{pmatrix}$ (b) $\begin{pmatrix} -2 \\ 4 \\ \frac{2}{3} \end{pmatrix}$ (c) $\begin{pmatrix} 2 \\ 3 \\ 2\frac{2}{3} \end{pmatrix}$

7 (a) $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ (b) $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$

Exercise 13D (page 206)

1 $-8, 11, 3$

2 $11, -3, 8$

3 $18, 0, 0$; \mathbf{r} is perpendicular to both \mathbf{p} and \mathbf{q} .

4 (a) and (d) are perpendicular; so are (b) and (c).

5 $-4, -8, -12$

6 (a) 5 (b) $\sqrt{5}$ (c) $\sqrt{5}$ (d) 1
(e) 3 (f) 13 (g) 5 (h) $\sqrt{6}$
(i) $\sqrt{5}$ (j) $\sqrt{13}$ (k) $\sqrt{30}$ (l) 2

7 5, $\begin{pmatrix} \frac{4}{5} \\ -\frac{3}{5} \end{pmatrix}$

8 $\begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{pmatrix}, \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$

9 (a) 45° (b) 167.3° (c) 180°
(d) 136.7° (e) 7.0° (f) 90°

10 $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$; the distance between the points with position vectors \mathbf{r}_1 and \mathbf{r}_2 .

11 172.2° (or 7.8°)

12 99.6° (or 80.4°)

13 70.5°

14 76.4°

15 48.2°

16 48.2°

Miscellaneous exercise 13 (page 207)

1 \mathbf{a} and \mathbf{b} , \mathbf{a} and \mathbf{c} , \mathbf{b} and \mathbf{c} , \mathbf{b} and \mathbf{d}

2 58.5°

3 (a) 7, 7 (b) -32 (c) 130.8°

4 (a) $\mathbf{d} = \begin{pmatrix} 9 \\ 4 \\ -5 \end{pmatrix}$ (b) $\begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}$ (c) 120.5°

5 (a) $x\mathbf{i}, -\frac{1}{2}x\mathbf{i} + \frac{1}{2}\sqrt{3}x\mathbf{j}, -\frac{1}{2}x\mathbf{i} - \frac{1}{2}\sqrt{3}x\mathbf{j}$

(b) (i) $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$
(ii) $-\mathbf{x}\mathbf{i} + 30\mathbf{k}, \frac{1}{2}x\mathbf{i} - \frac{1}{2}\sqrt{3}x\mathbf{j} + 30\mathbf{k}$

(c) $30\sqrt{2}$

6 $\frac{\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}}{\sqrt{14}}, \frac{3\mathbf{i} + \mathbf{j} - 2\mathbf{k}}{\sqrt{14}}, 73.4^\circ$

7 (a) 8.5°

8 (a) $\sqrt{29}$ (b) 119.9° (or 60.1°)

9 (a) $\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix}$ (b) 45°

10 (a) 60° (b) $\frac{3}{2}\sqrt{3}$

12 $\frac{4}{3}, \frac{9}{5}$; break up 4 Individual bags and 9 Jumbo bags, and use the fruit to make 5 King-size bags.

13 (a) $\begin{pmatrix} 14 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} -5 \\ 10 \\ 10 \end{pmatrix}; (13, 14, 18)$

(b) $\frac{1}{3}, \frac{2}{15}$; the origin lies in the plane of the parallelogram.

14 (11.4, 3, 0) at 8.04 a.m.

14 Geometric sequences**Exercise 14A (page 213)**

1 (a) 2; 24, 48 (b) 4; 128, 512

(c) $\frac{1}{2}; 4, 2$ (d) -3; 162, -486

(e) 1.1; 1.4641, 1.61051 (f) $\frac{1}{x}; \frac{1}{x}, \frac{1}{x^2}$

2 (a) $2 \times 3^{i-1}$ (b) $10 \times \left(\frac{1}{2}\right)^{i-1}$ (c) $1 \times (-2)^{i-1}$
(d) $81 \times \left(\frac{1}{3}\right)^{i-1}$ (e) x^i (f) $p^{2-i}q^{i+1}$

3 (a) 11 (b) 13 (c) 7
(d) 14 (e) 6 (f) 13

4 (a) $3; 1\frac{1}{3}$ (b) $2; 1\frac{1}{2}$ or $-2; 1\frac{1}{2}$

(c) $\frac{1}{3}; 531\,441$ (d) $\pm\sqrt{2}; 4$

(e) $\pm 7; \frac{16807}{(\pm 7)^{n-1}}$

5 (a) 59 048 (b) -29 524

(c) 1.9922 (d) 0.6641

(e) 12 285 (f) 8.9998

(g) $\frac{x(1-x^n)}{1-x}$ (h) $\frac{x(1-(-x)^n)}{1+x}$

(i) $\frac{x^{2n}-1}{x^{2n-3}(x^2-1)}$ (j) $\frac{x^{2n}-(-1)^n}{x^{2n-2}(x^2+1)}$

6 (a) 2047 (b) 683

(c) 262 143 (d) $\frac{1023}{512}$

(e) $\frac{29\,525}{39\,366}$ (f) 19.843 75

(g) $\frac{341}{1024}$ (h) $2 - \left(\frac{1}{2}\right)^n$

(i) $\frac{1}{3} \left(64 - \left(\frac{1}{4}\right)^n\right)$ (j) $\frac{1}{4} \left(243 + \left(-\frac{1}{3}\right)^n\right)$

7 $2^{64} - 1 \approx 1.84 \times 10^{19}$

8 \$2 684 354.55

10 (a) 2 (b) 8th

11 (a) 3 (b) 14th

12 (a) $\frac{p}{q}; n$ (c) (i) np^{n-1} (ii) $-np^{-(n+1)}$

Exercise 14B (page 217)

1 (a) 2 (b) $\frac{3}{2}$ (c) $\frac{1}{4}$ (d) $\frac{1}{9}$

(e) $\frac{3}{4}$ (f) $\frac{1}{6}$ (g) 3 (h) $\frac{1}{3}$

(i) $\frac{20}{3}$ (j) 62.5 (k) $\frac{x}{1-x}$ (l) $\frac{1}{1+x^2}$

(m) $\frac{x}{x-1}$ (n) $\frac{x^3}{x+1}$

2 (a) $\frac{4}{11}$ (b) $\frac{41}{333}$ (c) $\frac{5}{9}$ (d) $\frac{157}{333}$

(e) $\frac{1}{7}$ (f) $\frac{2}{7}$ (g) $\frac{5}{7}$ (h) $\frac{6}{7}$

3 $\frac{1}{6}$

4 $-\frac{5}{6}$

5 3

6 19.2

7 2 m

8 0.375 m east of O, 1.5 m

9 10 seconds

10 19 m

11 (a) Edge of table (b) 8

Exercise 14C (page 221)

1 \$7103.39

2 \$1239.72

3 $40\,000 \times 1.08^{25} - P(1+1.08+\dots+1.08^{24}) = 0$,
\$3747.15

4 \$1167.70

5 \$317.23

6 (a) 63 000 (b) 36 200 (c) 19 700

7 (a) 5.83×10^7 (b) 5.65×10^7

8 (a) 17 800 (b) 29 100

9 (a) 85.1 kg (b) 68.1 kg

10 (a) \$2254.32 (b) 139

11 \$2718.28

Miscellaneous exercise 14 (page 222)

1 $6\frac{1}{4}$

3 $a=3, r=-\frac{1}{4}; 2.4 \left(1 - \left(-\frac{1}{4}\right)^n\right)$

4 62

5 $\frac{1}{999}; \frac{4}{37}$

6 $\frac{4}{5}; 18$

7 $8 \left(1 - \left(-\frac{1}{2}\right)^n\right); 8, 13$

8 $r=0.917; a=40$

9 $20 \times 1.1^{n-1}; 17$

10 \$56 007

11 (a) 20

(b) The sum of the infinite series is only 80 cm.

12 $n=45, 19.8$ seconds

13 8.45%

14 $r=-\frac{1}{3}$

16 $r=\frac{k-1}{k+1}$

17 \$6401

18 $\$200\,000 \left(1 - \frac{1}{1.05^n}\right); \$77\,217, \$124\,622$,
\$153 725, \$171 591, \$182 559

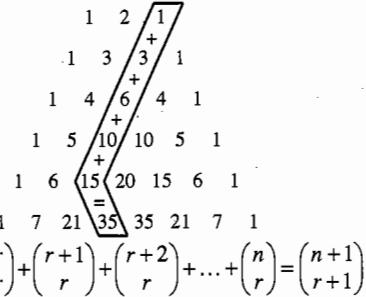
19 $\frac{(1-x)(1-x^{3n+3})}{1-x^3}; |x|<1$

$\frac{1}{1+x+x^2+x^3+x^4}, |x|<1$

20 (a) $\tan^2 x^\circ, x \neq 90(2n+1), n \in \mathbb{Z}$

(b) $\cos^2 x^\circ, 180n-45 < x < 180n+45, n \in \mathbb{Z}$

21 $\frac{(1+x)^7 - 1}{x}$



22 Possible if $|x| < 1$; within these bounds, the larger the value of n the better the approximation.

15 Second derivatives

Exercise 15A (page 228)

1 (a) At $x = -1, 0, 1$ (b) $3x^2 - 1$
(c) $6x$

2 (b) $3x^2 + 1, 6x$ (c) $x \geq 0$

4 (a) (i) $(0, 0), (\pm\sqrt{2}, 4)$ (ii) $(\pm\frac{1}{3}\sqrt{6}, -\frac{8}{9})$

(b) (i) $(0, 0), (-\frac{2}{3}, \frac{4}{27})$ (ii) $(-\frac{1}{3}, \frac{2}{27})$

(c) (i) $(\pm 1, \pm 2)$ (ii) None

(d) (i) None (ii) None

(e) (i) $(2, 3)$ (ii) None

(f) (i) $(-2, -3)$ (ii) None

5 (a) Rate of increase of inflation, positive

6 (a) $f'(x) +, f''(x) +$ (b) $f'(x) +, f''(x) -$

(c) $f'(x) +, f''(x) 0$ (d) $f'(x) -, f''(x) +$

(e) For $0 < x < 3$, $f'(x) +, f''(x) +$;

for $x > 3$, $f'(x) -, f''(x) +$

(f) For $x < -1$, $f'(x) +, f''(x) +$;

for $-1 < x < 0$, $f'(x) +, f''(x) -$;

for $0 < x < 1$, $f'(x) -, f''(x) -$;

for $x > 1$, $f'(x) -, f''(x) +$

7 (a) Both positive, sudden change (drop in S),

then $\frac{dS}{dt}$ is negative changing to positive

with $\frac{d^2S}{dt^2}$ positive.

(b) Price rising sharply, sudden 'crash', price continues to drop but less quickly and then recovers to give steadier growth.

8 (b) $+, -, -, +$

9 (a) $\frac{dN}{dt} = -kN, k > 0$ (c) $+$

Exercise 15B (page 232)

- 1 (a) $(-1, -2)$, minimum; $(1, 2)$, maximum
(b) $(0, 0)$, maximum; $(2, -4)$, minimum
(c) $(0, 1)$, minimum
(d) $(-1, 1)$, maximum; $(2, -16)$, minimum
(e) $(2, \frac{3}{8})$, minimum
(f) $(-1, 2)$, minimum; $(1, 2)$, minimum
(g) $(2, \frac{1}{4})$, maximum
(h) $(2, 22)$, inflection

- 2 (a) $(-1, -8)$, minimum; $(0, -3)$, maximum;
 $(2, -35)$, minimum
(b) $(1, 6)$, inflection
(c) $(-\frac{4}{3}, -14\frac{2}{9})$, minimum; $(\frac{4}{3}, 14\frac{2}{9})$, maximum
(d) $(-2, 3)$, minimum
(e) $(-2, -4)$, maximum; $(2, 4)$, minimum
(f) $(6, \frac{1}{2})$, maximum
(g) $(0, -7)$, inflection
(h) $(0, 1)$, minimum; $(1, 2)$, inflection

Exercise 15C (page 234)

- 1 (a) $2x+3, 2, 0, 0$
(b) $6x^2 + 1 - \frac{1}{x^2}, 12x + \frac{2}{x^3}, 12 - \frac{6}{x^4}, \frac{24}{x^5}$
(c) $4x^3, 12x^2, 24x, 24$
(d) $\frac{1}{2}x^{-1/2}, -\frac{1}{4}x^{-3/2}, \frac{3}{8}x^{-5/2}, -\frac{15}{16}x^{-7/2}$
(e) $-\frac{1}{2}x^{-3/2}, \frac{3}{4}x^{-5/2}, -\frac{15}{8}x^{-7/2}, \frac{105}{16}x^{-9/2}$
(f) $\frac{1}{4}x^{-3/4}, -\frac{3}{16}x^{-7/4}, \frac{21}{64}x^{-11/4}, -\frac{231}{256}x^{-15/4}$
2 (a) $2x-5, 2, 0, 0$
(b) $10x^4 - 6x, 40x^3 - 6, 120x^2, 240x$
(c) $-4x^5, 20x^6, -120x^7, 840x^8$
(d) $6x-6x^5, 6-30x^4, -120x^3, -360x^2$
(e) $\frac{3}{4}x^{-1/4}, -\frac{3}{16}x^{-5/4}, \frac{15}{64}x^{-9/4}, -\frac{135}{256}x^{-13/4}$
(f) $\frac{3}{8}x^{-5/8}, -\frac{15}{64}x^{-13/8}, \frac{195}{512}x^{-21/8}, -\frac{4095}{4096}x^{-29/8}$

3 $n(n-1)(n-2) \times \dots \times 3 \times 2 \times 1$ (i.e., $n!$)

4 $(n+2)(n+1)n(n-1) \times \dots \times 3x^2$

5 0

Miscellaneous exercise 15 (page 234)

- 1 Minimum 6, maximum 10
2 Minimum $32\frac{1}{4}$
3 Maximum 6 when $x = 1$
4 Maximum $(\frac{1}{2}, -\frac{1}{2})$, minimum $(\frac{1}{6}, 9)$
5 $+, -, +$

- 6** (a) Subsonic: $\frac{dk}{dS} +$, initially small then increasing; $\frac{d^2k}{dS^2} +$, decreasing to zero.
 Transonic: $\frac{dk}{dS} +$ at first, zero at speed of sound, then $-$; $\frac{d^2k}{dS^2}$ zero, $-$, zero again.
 Supersonic: $\frac{dk}{dS} -$; $\frac{d^2k}{dS^2}$ zero, then $+$.
 (b) At the boundaries between the regions.
 (c) Possibly levelling out, becoming constant again.

7 $10x - 2x^2 - \frac{1}{2}\pi x^2, \frac{10}{4+\pi} \approx 1.40$

- 8** (a) Maximum 0 when $x = 0$, minimum $-\frac{4}{27}a^2$ when $x = \frac{2}{3}a$
 (b) Minimum $-\frac{27}{256}a^4$ when $x = \frac{3}{4}a$
 (c) Minimum 0 when $x = 0$, maximum $\frac{1}{16}a^4$ when $x = \frac{1}{2}a$, minimum 0 when $x = a$
 (d) Maximum $\frac{108}{3125}a^5$ when $x = \frac{3}{5}a$, minimum 0 when $x = a$

- 9** (a) $(-1)^n(n+2)(n+1)n(n-1)\times\ldots\times 3x^{-(n+3)}$
 (b) $(-1)^{n-1} \times \frac{3 \times 5 \times 7 \times \ldots \times (2n-3)}{2^n} x^{-\frac{1}{2}(2n-1)}$ if $n \geq 3$

- 10** (a) (1,15), (3,31) (b) (1,2)

16 Integration

Exercise 16A (page 238)

- 1** (a) $x^4 + k$ (b) $x^6 + k$
 (c) $x^2 + k$ (d) $x^3 + x^5 + k$
 (e) $x^{10} - x^8 - x + k$ (f) $-x^7 + x^3 + x + k$
- 2** (a) $3x^3 - 2x^2 - 5x + k$
 (b) $4x^3 + 3x^2 + 4x + k$
 (c) $7x + k$
 (d) $4x^4 - 2x^3 + 5x^2 - 3x + k$
 (e) $\frac{1}{2}x^4 + \frac{5}{2}x^2 + k$
 (f) $\frac{1}{2}x^2 + \frac{2}{3}x^3 + k$
 (g) $\frac{2}{3}x^3 - \frac{3}{2}x^2 - 4x + k$
 (h) $x - x^2 - x^3 + k$
- 3** (a) $\frac{1}{5}x^5 + \frac{1}{3}x^3 + x + k$
 (b) $\frac{7}{2}x^2 - 3x + k$
 (c) $\frac{2}{3}x^3 + \frac{1}{2}x^2 - 8x + k$

- (d) $\frac{3}{2}x^4 - \frac{5}{3}x^3 + \frac{3}{2}x^2 + 2x + k$
 (e) $\frac{1}{6}x^4 + \frac{1}{6}x^3 + \frac{1}{6}x^2 + \frac{1}{6}x + k$
 (f) $\frac{1}{8}x^4 - \frac{1}{9}x^3 + \frac{1}{2}x^2 - \frac{1}{3}x + k$
 (g) $\frac{1}{2}x^2 - x^3 + x + k$
 (h) $\frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + k$
- 4** $4x^2 - 5x$
- 5** $y = 2x^3 - x - 19$
- 6** $y = \frac{1}{8}x^4 + \frac{1}{8}x^2 + x - 21$
- 7** $5x^3 - 3x^2 + 4x - 6$
- 9** $y = 4x\sqrt{x} - 7$
- 10** (a) $y = \frac{2}{3}x^{3/2} + k$ (b) $y = 12x^{1/3} + k$
 (c) $y = \frac{3}{4}x^{4/3} + k$
 (d) $y = \frac{4}{3}x\sqrt{x} - 4\sqrt{x} + k$
 (e) $y = \frac{15}{2}\sqrt[3]{x^2} + k$ (f) $y = -6\sqrt[3]{x} + k$
- 11** (a) $-\frac{1}{x} + k$ (b) $-\frac{1}{x^3} + k$
 (c) $-\frac{3}{x^2} + k$ (d) $2x^2 + \frac{3}{x} + k$
 (e) $-\frac{1}{2x^2} + \frac{1}{3x^3} + k$ (f) $-\frac{2}{x} - \frac{2}{3}x^3 + k$
- 12** $y = -4x^{-1} + 13$
- 13** $y = \sqrt{x} - 2$
- 14** $y = \frac{3}{4}x^{4/3} + 3x^{-2} + \frac{5}{4}$
- 15** (a) $y = x^3 + 3x^2 + k$
 (b) $y = 4x^3 + 2x^2 - 5x + k$
 (c) $y = 2x^2 - \frac{1}{x} + k$
 (d) $y = \frac{2}{3}x\sqrt{x} + 8\sqrt{x} + k$
 (e) $y = \frac{1}{2}x^2 + \frac{20}{3}x\sqrt{x} + 25x + k$
 (f) $y = x + 10\sqrt{x} + k$

- 16** (a) 118 cm (to 3 s.f.) (b) 27

- 17** 17 months (to the nearest month)

- 18** 192

Exercise 16B (page 244)

- 1** (a) $2x^2 + k$ (b) $5x^3 + k$ (c) $\frac{1}{3}x^6 + k$
 (d) $9x + k$ (e) $\frac{1}{18}x^9 + k$ (f) $\frac{2}{15}x^5 + k$
- 2** (a) 7 (b) 84 (c) 4
 (d) 4 (e) $\frac{1}{16}$ (f) 2
- 3** (a) $3x^2 + 7x + k$
 (b) $2x^3 - x^2 - 5x + k$
 (c) $\frac{1}{2}x^4 + \frac{7}{2}x^2 + k$
 (d) $\frac{3}{5}x^5 - 2x^4 + 3x^3 - \frac{1}{2}x^2 + 4x + k$
 (e) $\frac{2}{3}x^3 - \frac{3}{2}x^2 - 20x + k$
 (f) $\frac{1}{4}x^4 - 2x^2 + k$

4 (a) 22 (b) 22 (c) 36
 (d) $7\frac{1}{6}$ (e) 210 (f) 0

5 72

6 15

7 195

8 80

9 80

10 $10\frac{2}{5}$

11 18

12 16

13 (a) 39 (b) $5\frac{1}{3}$ (c) $10\frac{2}{3}$
 (d) 10 (e) 500 (f) $5\frac{1}{3}$

14 (a) $-\frac{1}{2x^2} + k$ (b) $\frac{1}{3}x^3 + \frac{1}{x} + k$
 (c) $\frac{2}{3}x\sqrt{x} + k$ (d) $\frac{18}{5}x^{5/3} + k$
 (e) $2x^3 - \frac{5}{x} + k$ (f) $2\sqrt{x} + k$

15 (a) 144 (b) $1\frac{1}{2}$ (c) 20
 (d) $6\frac{3}{4}$ (e) 16 (f) 3

16 $1\frac{3}{4}$

17 60

18 $3\frac{1}{3}$

19 7

20 5

21 (a) $22\frac{2}{3}$ (b) $2\frac{3}{8}$

22 96

24 $42\frac{7}{8}$ **Exercise 16C (page 250)**1 -4; the graph lies below the x-axis for $0 < x < 2$.

2 (a) $10\frac{2}{3}$ (b) 8 (c) $2\frac{1}{4}$ (d) $1\frac{5}{6}$

3 (a) $10\frac{2}{3}$ (b) 8 (c) 100

4 (a) $\frac{1}{4}$ (b) 6 (c) 100

5 $\frac{s^{1-m}-1}{1-m}; \frac{1}{m-1}$

6 $\frac{1-r^{1-m}}{1-m}; m < 1, \frac{1}{1-m}$

7 12

8 $10\frac{2}{3}$

9 32

10 108

11 $42\frac{2}{3}$ 12 $4\frac{1}{2}$

13 36

Exercise 16D (page 253)

1 (a) $\frac{1}{14}(2x+1)^7 + k$ (b) $\frac{1}{15}(3x-5)^5 + k$
 (c) $-\frac{1}{28}(1-7x)^4 + k$ (d) $\frac{2}{11}\left(\frac{1}{2}x+1\right)^{11} + k$
 (e) $-\frac{1}{10}(5x+2)^{-2} + k$ (f) $\frac{2}{3}(1-3x)^{-1} + k$
 (g) $-\frac{1}{4}(x+1)^{-4} + k$ (h) $-\frac{1}{8}(4x+1)^{-3} + k$
 (i) $\frac{1}{15}(10x+1)^{\frac{3}{2}} + k$ (j) $\sqrt{2x-1} + k$
 (k) $\frac{6}{5}\left(\frac{1}{2}x+2\right)^{\frac{5}{3}} + k$ (l) $\frac{16}{9}(2+6x)^{\frac{3}{4}} + k$
 2 (a) 820 (b) $\frac{26}{3}$ (c) $\frac{2}{15}$ (d) $\frac{16}{225}$
 3 2.25
 4 9.1125
 5 (a) $\frac{243}{10}$ (b) $4\frac{1}{2}$ (c) $\frac{2}{3}$ (d) 28
 6 $\frac{1}{20}$
 7 $438\frac{6}{7}$
 8 $\frac{1}{6}$

Miscellaneous exercise 16 (page 254)

1 $4x\sqrt{x} + k$; 28
 2 ± 2 ; 32
 3 $-\frac{4}{3}$
 4 $8\frac{2}{3}$
 5 (a) $-\frac{1}{2x^2} + \frac{1}{4}x^4 + k$ (b) 6
 6 $31\frac{1}{4}$
 7 7
 8 $6\frac{3}{4}$
 9 $1\frac{1}{3}$
 10 0; integrand is > 0 for $0 < x < 1$, < 0 for $1 < x < 2$
 11 (a) $\frac{1}{4}x^4 - x^2 + k$ (b) 2
 12 ± 2
 14 (a) $y = -\frac{1}{2}x + \frac{3}{2}$ (b) $2\frac{29}{48}$
 15 (a) 6 (b) 67 (c) — (d) 45
 (e) 17 (f) -11 (g) 17 (h) 11
 17 (a) $y = 2x - 2$ (b) $6\frac{3}{4}$
 19 (0,0), ($\pm 2, -16$); 32.4
 20 $\frac{1}{2}$
 21 (a) $\frac{1}{27}, 0$ (b) (i) (3,0) (ii) $(4, -\frac{1}{128})$
 22 (a) 1, 2 (b) $10\frac{1}{8}$
 23 (b) $25\frac{5}{6}$

17 Volume of revolution

Exercise 17 (page 261)

1 (a) $\frac{98}{3}\pi$ (b) $\frac{3093}{5}\pi$
(c) $\frac{279808}{7}\pi$ (d) $\frac{3}{4}\pi$

2 (a) 504π (b) $\frac{3498}{5}\pi$
(c) $\frac{15}{2}\pi$ (d) $\frac{16}{15}\pi$

3 (a) 4π (b) 9π (c) 3355π
(d) $\frac{3}{10}\pi$ (e) $\frac{648}{5}\pi$ (f) $\frac{9}{2}\pi$
(g) 156π (h) $\frac{2}{3}\pi$

4 (a) $\frac{512}{15}\pi$ (b) $\frac{16}{15}\pi$
(c) $\frac{1}{30}\pi$ (d) $\frac{81}{10}\pi$

5 (a) $\frac{2}{15}\pi$ (b) $\frac{1}{6}\pi$
6 (a) $\frac{2048}{15}\pi$ (b) $\frac{128}{3}\pi$

7 (a) $\frac{3}{10}\pi$ (b) $\frac{3}{10}\pi$
8 $\frac{1}{10}\pi$
9 9π

Miscellaneous exercise 17 (page 262)

1 $\frac{206}{15}\pi$
3 $\frac{4}{3}\pi ab^2; \frac{4}{3}\pi a^2b$
4 (b) 3 (c) 3π (d) $\frac{1}{2}\pi$

5 (i) (a) Infinite (b) $1\frac{1}{2}$
(c) 5π (d) $\frac{3}{7}\pi$
(ii) (a) Infinite (b) $\frac{1}{3}$
(c) Infinite (d) $\frac{1}{7}\pi$

6 (a) 36, 18 (b) $\frac{1296}{5}\pi$ (c) $\frac{81}{2}\pi$
7 4π

18 Radians

Exercise 18A (page 267)

1 (a) $\frac{1}{2}\pi$ (b) $\frac{3}{4}\pi$ (c) $\frac{1}{4}\pi$ (d) $\frac{1}{6}\pi$
(e) $\frac{2}{5}\pi$ (f) $\frac{1}{10}\pi$ (g) $\frac{2}{3}\pi$ (h) $\frac{1}{8}\pi$
(i) 4π (j) $\frac{10}{3}\pi$ (k) $\frac{3}{2}\pi$ (l) $\frac{1}{180}\pi$

2 (a) 60° (b) 9° (c) 36° (d) $22\frac{1}{2}^\circ$
(e) 20° (f) 120° (g) $112\frac{1}{2}^\circ$ (h) 108°
(i) 4° (j) 1080° (k) -90° (l) 50°

3 (a) $\frac{1}{2}\sqrt{3}$ (b) $\frac{1}{2}\sqrt{2}$ (c) $\frac{1}{3}\sqrt{3}$ (d) 0
(e) $-\frac{1}{2}\sqrt{2}$ (f) $-\frac{1}{2}\sqrt{3}$ (g) $-\sqrt{3}$ (h) $\frac{3}{4}$

4 (a) $s = 8.4, A = 29.4$
(b) $s = 7.35, A = 12.8625$
(c) $\theta = 1.5, A = 48$ (d) $r = 20, A = 140$
(e) $\theta = 2.4, s = 12$ (f) $s = 8$
(g) $r = 8, \theta = 2$ (h) $\theta = \frac{5}{3}$
5 (a) 2.26 cm^2 (b) 1.47 cm^2 (c) 830 cm^2
(d) 9.05 cm^2 (e) 0.556 cm^2

6 6.72 cm^2

7 28.2 cm

9 26.3 cm^2

10 $15.5 \text{ cm}, 14.3 \text{ cm}^2$

11 (a) $1.61r$ (b) $0.32r$

Exercise 18B (page 270)

3 (a) $\frac{1}{2}\pi$ (b) $\frac{1}{4}\pi$ (c) π
(d) $\frac{1}{6}\pi$ (e) 2π (f) $\frac{1}{16}\pi$

Exercise 18C (page 272)

1 (a) $\frac{1}{6}\pi$ (b) $\frac{1}{4}\pi$ (c) $\frac{1}{2}\pi$ (d) $\frac{1}{3}\pi$
(e) $-\frac{1}{3}\pi$ (f) $-\frac{1}{2}\pi$ (g) $-\frac{1}{4}\pi$ (h) π
2 (a) $\frac{1}{4}\pi$ (b) $-\frac{1}{6}\pi$ (c) $\frac{2}{3}\pi$ (d) $\frac{1}{6}\pi$
3 (a) 0.5 (b) -1 (c) $\sqrt{3}$ (d) 0
4 (a) $\frac{1}{2}\pi$ (b) $\frac{1}{6}\pi$ (c) $\frac{1}{6}\pi$ (d) 0
5 (a) $\frac{1}{2}$ (b) $\frac{1}{2}$ (c) $\frac{1}{2}\sqrt{3}$ (d) 1
6 0.739; $\cos x = x$

Exercise 18D (page 274)

1 (a) 0.12, 3.02 (b) 4.18, 5.25
(c) 1.18, 1.96 (d) 0.63, 5.66
(e) 2.51, 3.77 (f) 0.96, 5.33
(g) 1.33, 4.47 (h) 2.80, 5.95
(i) 0.17, 3.31 (j) 3.45, 5.98
(k) 3.69, 5.73 (l) 2.90, 6.04
(m) 0.99, 2.68 (n) 4.19, 6.28
(o) 0.17, 1.22

2 (a) 1.00, 2.14 (b) -1.30, 1.30
(c) -2.06, 1.09 (d) -2.32, -0.82
(e) -1.72, 1.72 (f) -0.90, 2.25
(g) -2.50, 0.64
(h) -2.53, -0.62, 0.62, 2.53
(i) $-\pi, -1.23, 0, 1.23, \pi$

3 (a) 0.66, 2.48, 3.80, 5.63
(b) 0.42, 1.46, 2.51, 3.56, 4.61, 5.65
(c) 1.91, 2.81, 5.05, 5.95
(d) 0.44, 1.13, 2.01, 2.70, 3.58, 4.27, 5.16, 5.84
(e) 0.23, 1.80, 3.37, 4.94
(f) 1.20, 1.94, 3.30, 4.03, 5.39, 6.13

- 4** (a) -2.33, -1.85, -0.24, 0.24, 1.85, 2.33
 (b) -2.12, -0.55, 1.02, 2.59
 (c) -3.03, -2.20, -0.94, -0.11, 1.16, 1.99
 (d) -2.49, -0.65, 0.65, 2.49
 (e) -3.02, -2.39, -1.76, -1.13, ±0.51, 0.12, 0.75, 1.38, 2.01, 2.64
 (f) -1.35, -0.22, 1.79, 2.92
- 5** (a) -2.46, 2.46 (b) -2.06, 2.65
 (c) -1.01 (d) π
 (e) 0.70 (f) -1.03
- 6** (a) $\frac{1}{4}\pi, \frac{7}{12}\pi, \frac{5}{4}\pi, \frac{19}{12}\pi$
 (b) $\frac{1}{12}\pi, \frac{7}{12}\pi, \frac{13}{12}\pi, \frac{19}{12}\pi$
 (c) $\frac{19}{36}\pi, \frac{23}{36}\pi, \frac{43}{36}\pi, \frac{47}{36}\pi, \frac{67}{36}\pi, \frac{71}{36}\pi$
 (d) $\frac{5}{9}\pi, \frac{11}{9}\pi, \frac{17}{9}\pi$
 (e) $\frac{17}{36}\pi, \frac{29}{36}\pi, \frac{53}{36}\pi, \frac{65}{36}\pi$
 (f) $\frac{4}{9}\pi$
 (g) No roots
 (h) $\frac{1}{4}\pi, \frac{7}{12}\pi, \frac{11}{12}\pi, \frac{5}{4}\pi, \frac{19}{12}\pi, \frac{23}{12}\pi$
 (i) $\frac{4}{9}\pi$
- 7** (a) 0.90, 2.25 (b) -1.14, 1.14
 (c) -2.19, -0.96, 0.96, 2.19
 (d) -2.53, -0.62, 0.62, 2.53
 (e) -1.04, 0, 1.04
 (f) -2.36, -0.79, 0.79, 2.36

Miscellaneous exercise 18 (page 275)

- 1** 3.6 cm, 10.8 cm²
- 2** $\frac{1}{4}$
- 3** r^2
- 4** $\frac{2}{3}\pi, \frac{1}{3}\pi - \frac{1}{4}\sqrt{3}$
- 5** 17 cm²
- 6** (a) -2.98, -0.16 (b) $-\frac{1}{2}\pi, \frac{1}{2}\pi$
 (c) $-\frac{3}{4}\pi, 0, \frac{1}{4}\pi$
 (d) -3.02, -1.69, 0.12, 1.45
 (e) 1.77, 2.94
 (f) -2.29, -1.25, -0.20, 0.85, 1.89, 2.94
- 7** (a) 0.02 seconds, 50
 (b) 0.007, 0.020, 0.027
- 8** (a) $k = \frac{2\pi}{T}$ (b) $\frac{k}{2\pi}$
- 9** $\frac{1}{2}r^2(\theta - \sin\theta)$, $1.2 < \theta < 1.3$
- 10** $DE = 2r - 2r\cos\theta$

- 11** $\theta = 2\pi - \frac{s}{r}$
- 13** 1.4
- 14** (a) $-1 \leq x \leq 1, -\pi - 4 \leq y \leq \pi - 4$
 (b) $3 \leq x \leq 5, -\pi \leq y \leq \pi$
- 15** 0, 1.23, 5.05, 2π
- 16** (a) No roots (b) $-\frac{3}{2}\pi, -\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{3}{2}\pi$
 (c) $-2\pi, -\pi, 0, \pi, 2\pi$

Revision exercise 3

(page 279)

- 1** (-4,8), (1,3); $20\frac{5}{6}$
- 2** (a) (-1,5), (1,1) (b) (0,3)
 (c) $9y + x = 7$ (d) 4
- 4** $\frac{n-1}{n+1}$
- 5** $y = \frac{5}{2}x^2 + 3x + 4$
- 6** $\frac{1}{3}$
- 7** 0; between $x=1$ and $x=3$, there is the same area above and below the x -axis.
- 8** 24π
- 9** 18
- 10** (b) 2
- 11** $\frac{300}{\pi}$
- 12** The point does lie on the line.
- 13** (a) $\pi - 2\theta, 2r^2 + 2r^2 \cos 2\theta$ (b) $2r\cos\theta$
- 14** 9π
- 15** (a) $3(3x^2 + 2)(x^3 + 2x - 1)^2$
 (b) $-x(x^2 + 1)^{-\frac{3}{2}}$
- 16** (a) £16 600 (b) Year 9
 (c) £(475n² + 12 325n) for $0 \leq n \leq 9$;
 £(20 400n - 34 200) for $n > 9$
 (d) £13 500 $\times 1.05^{n-1}$
 (f) Year 10
- 17** 22.65, 24
- 18** (a) 5 (b) 5 (c) $\frac{1}{4}\pi$
- 19** 2π cm, $2(\pi - \sqrt{3})$ cm²
- 20** -2, 132.5°

22 (i) $\frac{dx}{dt}$ is $+, 0, -$ with $\frac{d^2x}{dt^2} - 0, +$

(ii) $\frac{dx}{dt}$ is $-$, $\frac{d^2x}{dt^2}$ is $+$

(a) Graph dips below $x = 0$ with a minimum, then tends to $x = 0$. Hence

$$\frac{dx}{dt}$$
 is $-$, $0, +$ with $\frac{d^2x}{dt^2} +, 0, -$.

(b) $\frac{dx}{dt}$ is $0, -$ with $\frac{d^2x}{dt^2} - 0, +$.

23 $2\frac{1}{2}$

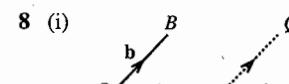
24 (a) $(2,4)$ (b) $\frac{64}{5}\pi$

25 80.9°

26 $3\frac{1}{2}$

27 1, 0.71

28 (a) $\frac{4}{3}a^2$ (b) $2\pi a^3$; $\frac{53}{240}\pi a^3$



(ii) 14

(iii) 30°

9 (i) $A = 3r - r^2$ (ii) $\theta = 2$

10 (i) $y + 6x = 8$ (ii) $\frac{16}{3}$

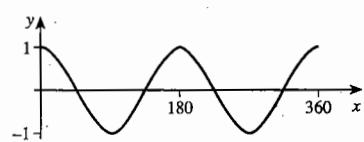
Practice examination 2 (page 285)

1 $81x^4 + 216x^3 + 216x^2 + 96x + 16$

2 $\frac{1}{12}\pi$

3 (i) $y = 3x - 14$ (ii) $(5,1)$

4 (i)



(ii) 67.2, 112.8, 247.2, 292.8

5 (i) $(-1,1)$ (ii) $\frac{32}{3}$

6 (i) $a+d, a+5d$ (iii) 660, 715 827 882

7 (i) $f(x) \in \mathbb{R}, f(x) \neq 1; g(x) \in \mathbb{R}, g(x) \geq -1$
(ii) g is not one-one

(iii) $f^{-1}: x \mapsto \frac{1+x}{3(1-x)}, x \in \mathbb{R}, x \neq 1$

8 (i) 1.107 (ii) 4.02 cm^2

(iii) 4.33 cm^2 (iv) 8.40 cm

9 (a) $\frac{1}{2\sqrt{x}} - \frac{2}{(2x+1)^2}$

(b) (i) $3x^2 - 6x + 12$

10 (i) $4\mathbf{j} + 3\mathbf{k}, 5$
(iii) The scalar product is non-zero.
(iv) 101°

Practice examinations

Practice examination 1 (page 283)

1 81

2 (i) $-\frac{1}{x} - 2\sqrt{x} + k$ (ii) $y = -\frac{1}{x} - 2\sqrt{x} + 3$

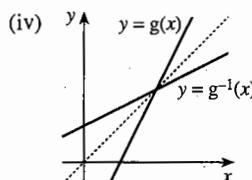
3 (i) $y = 2x + 1$ (ii) 5

4 (ii) $\frac{1}{6}\pi, \frac{5}{6}\pi$

5 (i) $(x+3)^2 - 6$

(ii) (a) $(-3, -6)$ (b) $-3 - \sqrt{6} < x < -3 + \sqrt{6}$

6 (i) $f(x) \geq \frac{1}{3}$ (ii) 0 (iii) $\frac{1}{2}(x+1)$



Reflections in $y = x$

7 (i) $x = -5, y = 8$ or $x = 2, y = 1$

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