BAHATI ABDULKARIM

A'LEVEL APPLIED MATHEMATICS

CONTINUOUS RANDOM VARIABLES

SUITABLE FOR S.5 AND S.6

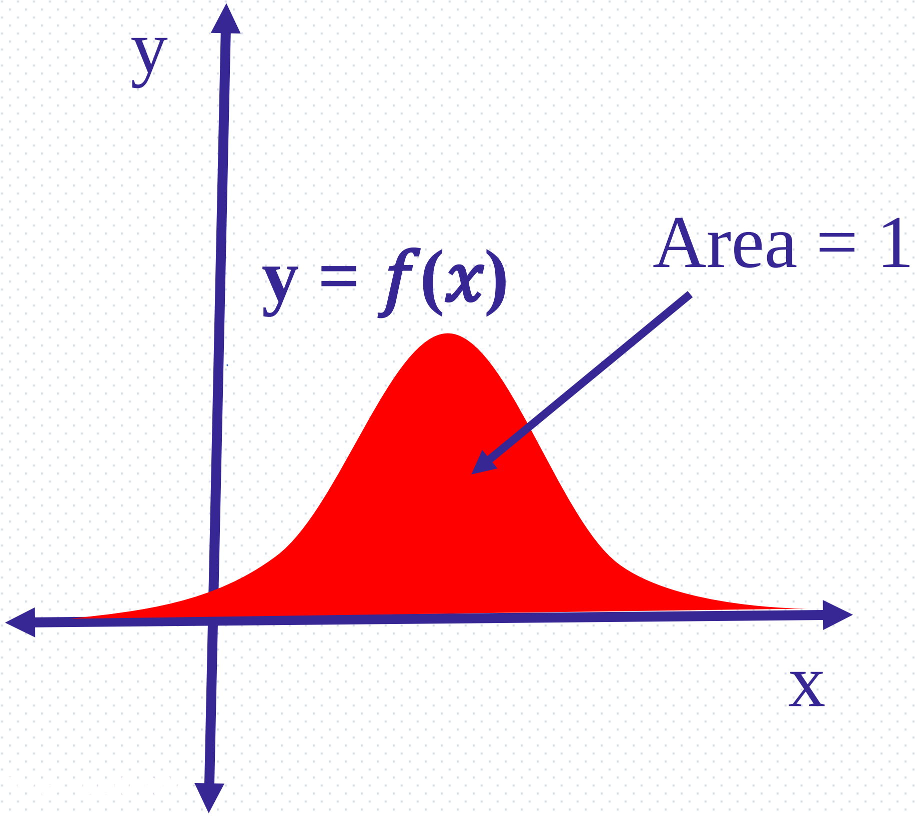
Continuous random variables

Contents

- **Probability density functions**
- Mode
- Cumulative distribution functions
- Median and quartiles
- Expectation
- Variance
- Rectangular/uniformly distributed functions

Probability density functions (p.d.f's)

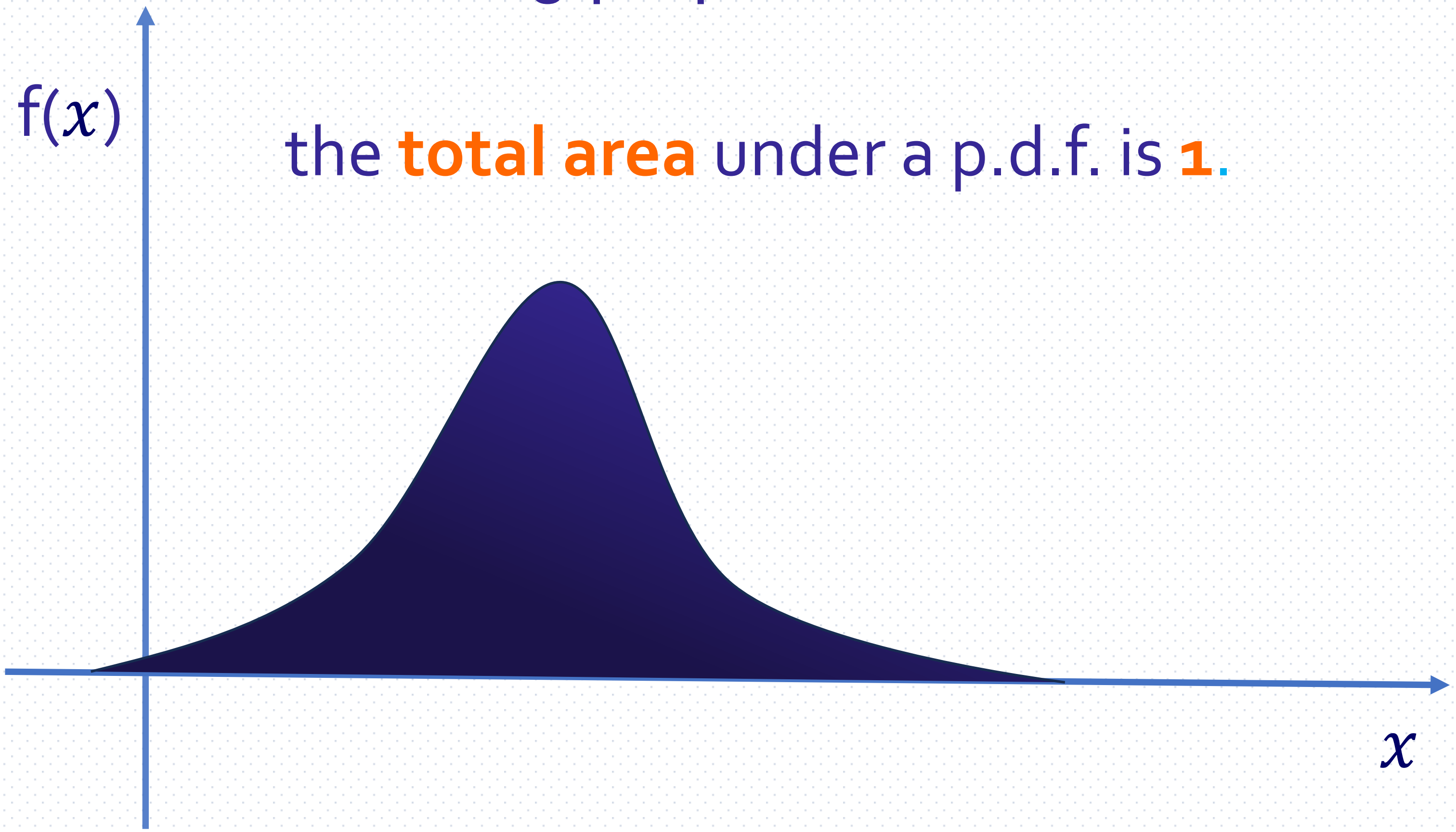
A probability density function (or p.d.f.) is a curve that models the shape of the distribution corresponding to a continuous random variable.



Properties probability density functions (**p.d.f's**)

If $\mathrm{f}(x)$ is the p.d.f corresponding to a continuous random variable X and if $\mathrm{f}(x)$ is defined for $a \le x \le b$ then the following properties must hold:

1. $\int_a^b \mathrm{f}(x)dx = 1$



the **total area** under a p.d.f. is **1**.

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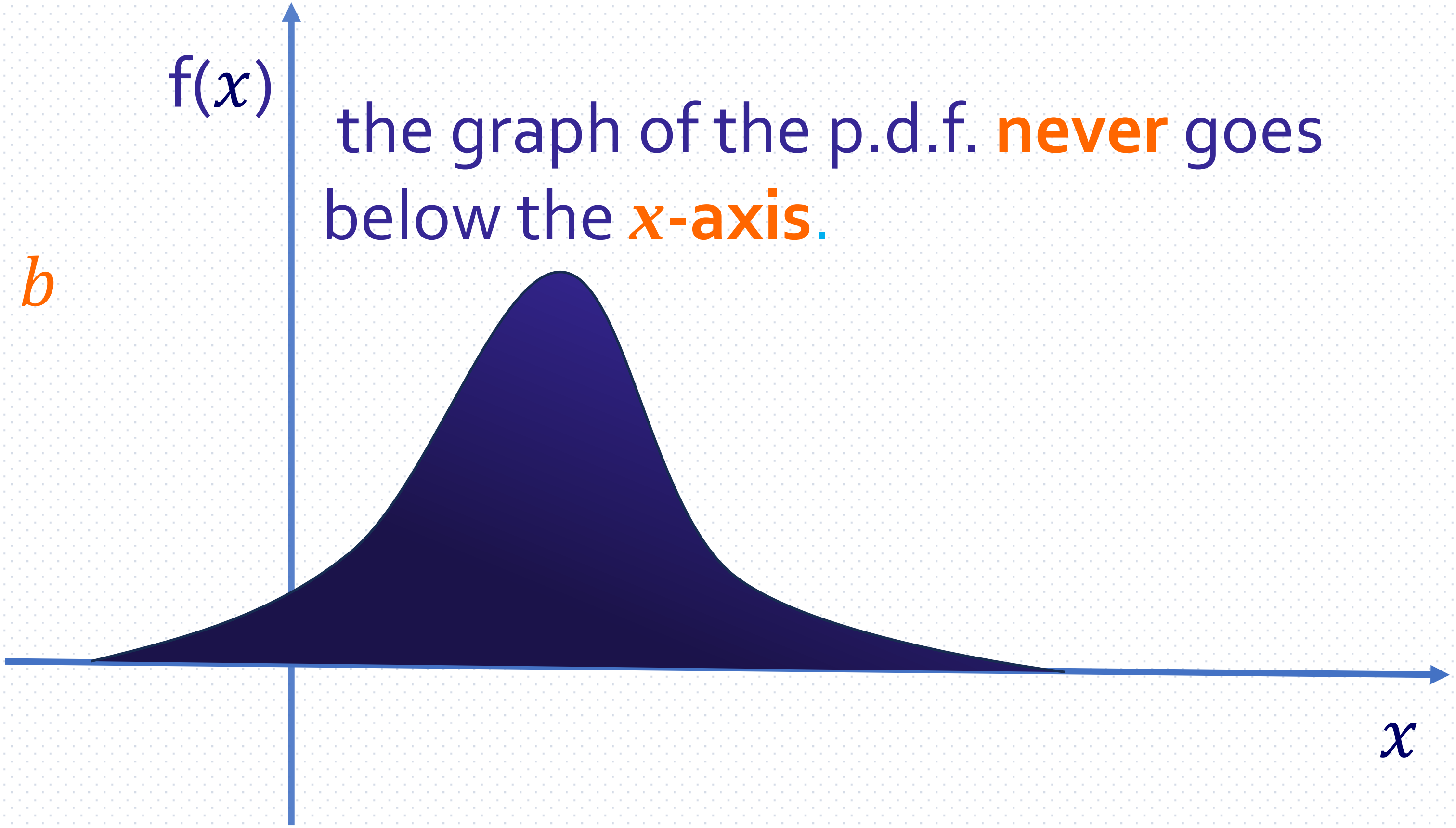
Properties probability density functions (p.d.f's)

If $\mathrm{f}(x)$ is the p.d.f corresponding to a continuous random variable X and if $\mathrm{f}(x)$ is defined for $a \leq x \leq b$ then the following properties must hold:

2. $\mathrm{f}(x) \geq 0 \quad \text{for } a \leq x \leq b$

the graph of the p.d.f. **never** goes below the ***x*-axis**.



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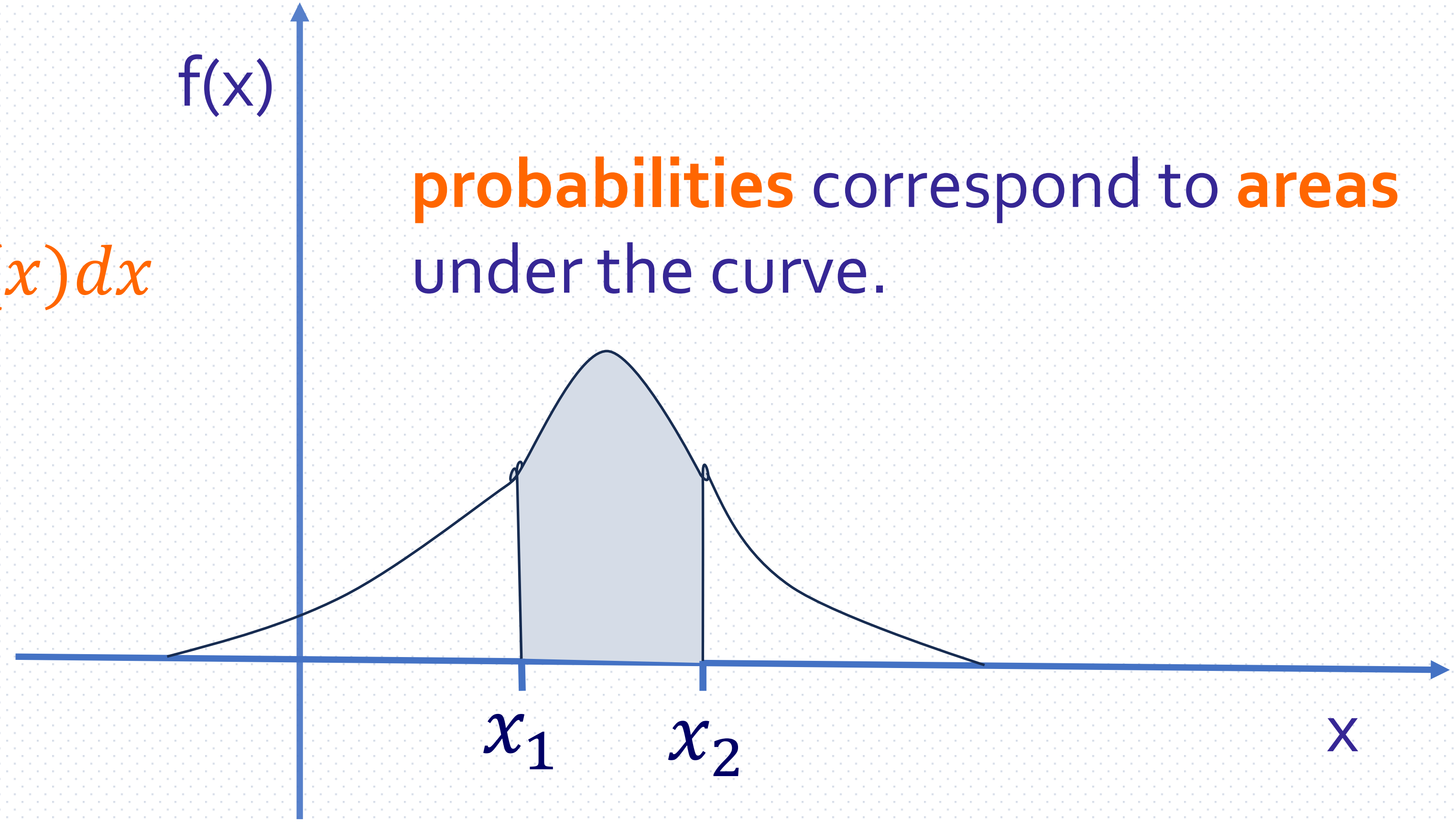
Properties probability density functions (**p.d.f's**)

If $f(x)$ is the p.d.f corresponding to a continuous random variable X and if $f(x)$ is defined for $a \le x \le b$ then the following properties must hold:

3. $P(x_1 \le X \le x_2) = \int_{x_1}^{x_2} f(x)\,dx$

probabilities correspond to **areas** under the curve.



Probability density functions Qn. 1

A random variable X of a continuous p.d.f is given by

$$f(x) = \begin{cases} kx, & 0 < x < 4 \\ 0, & elsewhere \end{cases}$$

Find the value of k.

Answer From $\int f(x)dx = 1$

$$\int_0^4 kx\,dx = k\left[\frac{x^2}{2}\right]_0^4 = 1$$

$$k\left(\frac{4^2}{2} - \frac{0^2}{2}\right) = 1$$

$$k\left(\frac{16}{2}\right) = 1$$

$$8k = 1$$

$$k = \frac{1}{8}$$

Probability density functions Qn. 2

A random variable X of a continuous p.d.f is given by

$$f(x) = \begin{cases} kx, & 0 < x < 2 \\ 2k(x-1), & 2 \le x \le 4 \\ 0, & elsewhere \end{cases}$$

Find the value of k and sketch f(x)

Answer From $\int f(x)dx = 1$

$$\int_0^4 kx\,dx + \int_0^4 2k(x-1)\,dx = 1$$

$$k\left[\frac{x^2}{2}\right]_0^2 + 2k\left[\frac{x^2}{2} - x\right]_0^2 = 1$$

$$k\left(\frac{2^2}{2} - \frac{0^2}{2}\right) + 2k\left(\left(\frac{4^2}{2} - 4\right) - \left(\frac{2^2}{2} - 2\right)\right) = 1$$

$$2k + 8k = 1$$

$$k = \frac{1}{10}$$

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Probability density functions Qn. 3

A random variable X of a continuous p.d.f is given by

$$f(x) = \begin{cases} kx, & 0 < x < 6 \\ 0, & elsewhere \end{cases}$$

Find;

(i) the value of k

(ii) P(X > 4)

(iii) P(X< 3)

(iv) P(1< x < 3)

(v) P$(X > 2/X \leq 4)$

Probability density functions Qn. 3

Answer

From $\int f(x)dx = 1$

$$\int_0^6 kx\,dx = k\left[\frac{x^2}{2}\right]_0^6 = 1$$

$$k\left(\frac{6^2}{2} - \frac{0^2}{2}\right) = 1$$

$$k\left(\frac{36}{2}\right) = 1$$

$$18k = 1$$

$$k = \frac{1}{18}$$

Probability density functions Qn. 3

Answer;

$$P(X>4) = \frac{1}{18}\int_4^6 x\,dx$$

$$= \frac{1}{18}\left[\frac{x^2}{2}\right]_4^6 = 1$$

$$= \frac{1}{18}\left[\frac{6^2}{2} - \frac{4^2}{2}\right] = 1$$

$$= \frac{5}{9}$$

Probability density functions Qn. 3

Answer;

$$P(X< 3) = \frac{1}{18}\int_0^3 x\,dx$$

$$= \frac{1}{18}\left[\frac{x^2}{2}\right]_0^3 = 1$$

$$= \frac{1}{18}\left[\frac{3^2}{2} - \frac{0^2}{2}\right] = 1$$

$$= \frac{1}{4}$$

Probability density functions Qn. 3

Answer;

$$P(1<X<3) = \frac{1}{18}\int_1^3 x\,dx$$

$$= \frac{1}{18}\left[\frac{x^2}{2}\right]_1^3 = 1$$

$$= \frac{1}{18}\left[\frac{3^2}{2} - \frac{1^2}{2}\right] = 1$$

$$= \frac{2}{9}$$

Probability density functions Qn. 3

Answer;

$$P(X > 2 / X \leq 4) = \frac{P(X > 2 \cap X \leq 4)}{P(X \leq 4)}$$

$$= \frac{P(2 < X \leq 4)}{P(X \leq 4)}$$

$$= \frac{\frac{1}{18}\int_2^4 x\,dx}{\frac{1}{18}\int_0^4 x\,dx}$$

$$P(X > 2 / X \leq 4) = \frac{\left[\frac{x^2}{2}\right]_2^4}{\left[\frac{x^2}{2}\right]_0^4} = \frac{\left[\frac{4^2}{2} - \frac{2^2}{2}\right]}{\left[\frac{4^2}{2} - \frac{0^2}{2}\right]}$$

$$= \frac{6}{8}$$

$$= \frac{3}{4}$$

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Sketching f(x)

Procedures:

- Find the initial and final points of f(x)
- Join the initial and final points of f(x) using a line or curve.

Note:

- A line is in the form of $y = mx + c$.
- A curve has a power of x being 2 and above or fractional power e.g. $y = x^2$.
- A curve has a positive coefficient of x^2 has a minimum turning point while a curve with a negative coefficient has a maximum turning point

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Probability density functions Qn. 4

A random variable X of a continuous p.d.f is given by

$$f(x) = \begin{cases} kx, & 0 < x < 2 \\ 2k(x-1), & 2 \leq x \leq 4 \\ 0, & elsewhere \end{cases}$$

Find;

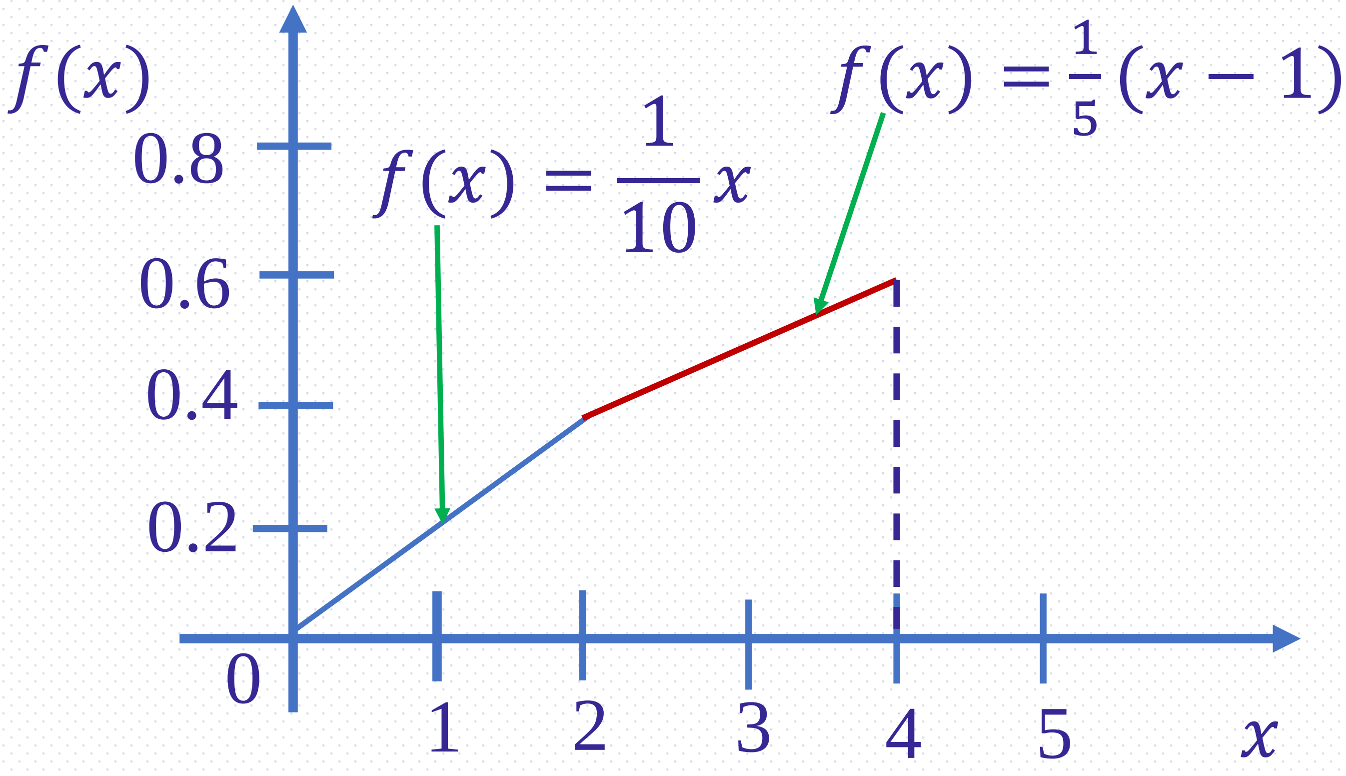
(i) the value of k and sketch f(x).

$$we\ have\ already\ calculated\ k\ in\ one\ of\ the\ previous\ examples\ as = \frac{1}{10}$$

$$f(x) = \begin{cases} \frac{1}{10}x, & 0 < x < 2 \\ \frac{1}{5}(x-1), & 2 \leq x \leq 4 \\ 0, & elsewhere \end{cases}$$

Probability density functions Qn. 4

$$f(x) = \begin{cases} \frac{1}{10}x, & 0 < x < 2 \\ \frac{1}{5}(x-1), & 2 \le x \le 4 \\ 0, & elsewhere \end{cases}$$



When x = 0, f(x) = 0

When x = 2, f(x) = $\frac{1}{5}$

When x = 4, f(x) = $\frac{3}{5}$

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Probability density functions Qn. 5

A random variable X of a continuous p.d.f is given by

$$f(x) = \begin{cases} 2kx, & 0 < x < 1 \\ k(3-x), & 1 \leq x \leq 2 \\ 0, & elsewhere \end{cases}$$

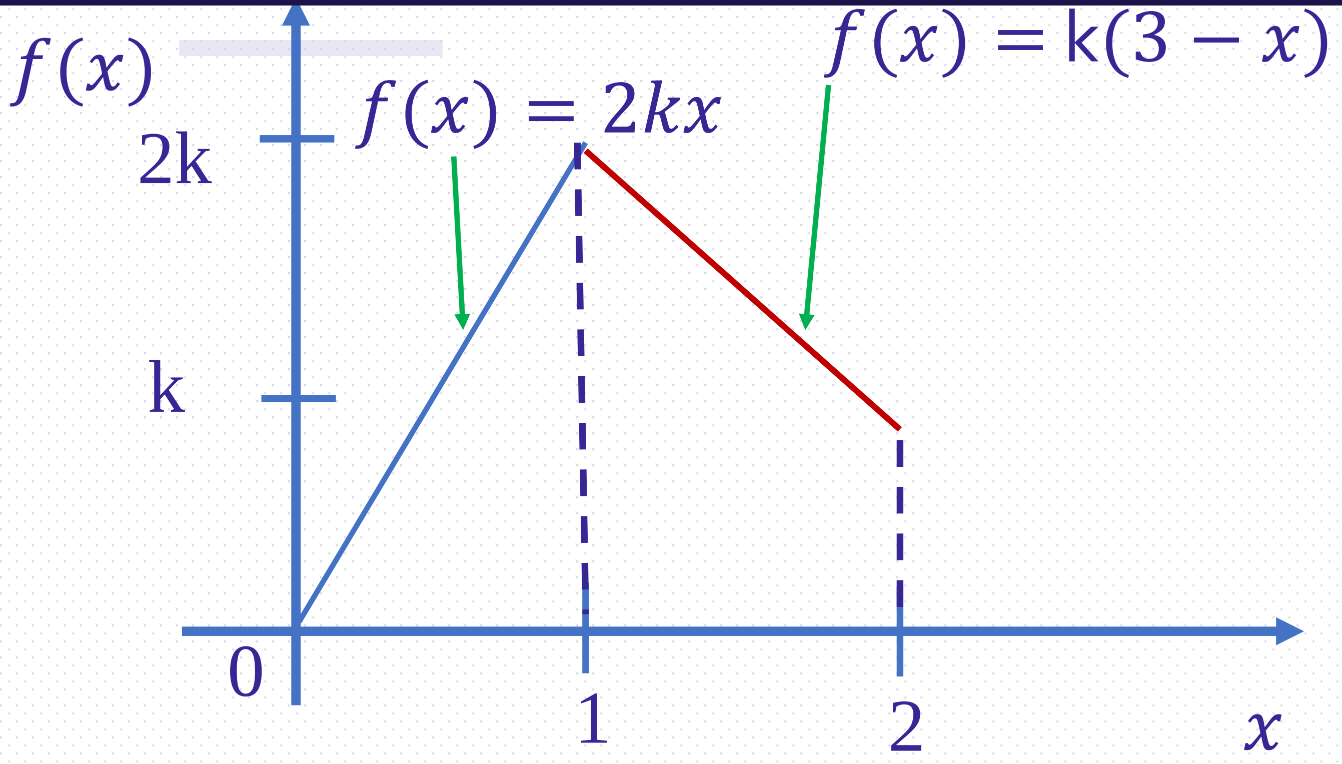
sketch f(x) and hence find the value of the constant k.

When $x = 0$, $f(x) = 0$

When $x = 1$, $f(x) = 2k$

When $x = 2$, $f(x) = k$

Probability density functions Qn. 5



Total area = $\frac{1}{2} \times 1 \times 2k + \frac{1}{2} \times 1 \times (k \times 2k) = 1$

$$\left(\frac{5}{2}\right)k = 1$$

$$k = \frac{2}{5}$$

$$f(x) = \begin{cases} \frac{4}{5}x, & 0 < x < 1 \\ \frac{2}{5}(3 - x), & 1 \leq x \leq 2 \\ 0, & elsewhere \end{cases}$$

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Probability density functions Qn. 6

A random variable X of a continuous p.d.f is given by

$$f(x) = \begin{cases} k(x+2)^2, & -2 < x < 0 \\ 4k, & 0 \le x \le \frac{4}{3} \\ 0, & elsewhere \end{cases}$$

Find;

(i) the value of k and sketch f(x).

$$\int_{-2}^{0} k(x+2)^2 dx + \int_{0}^{\frac{4}{3}} 4k\, dx = 1$$

$$k\left[\frac{(x+2)^3}{3}\right]_{-2}^{0} + 4k[x]_0^{\frac{4}{3}} = 1$$

$$k\left(\frac{2^3}{3} - \frac{(0)^3}{3}\right) + 4k\left(\frac{4}{3} - 0\right) = 1$$

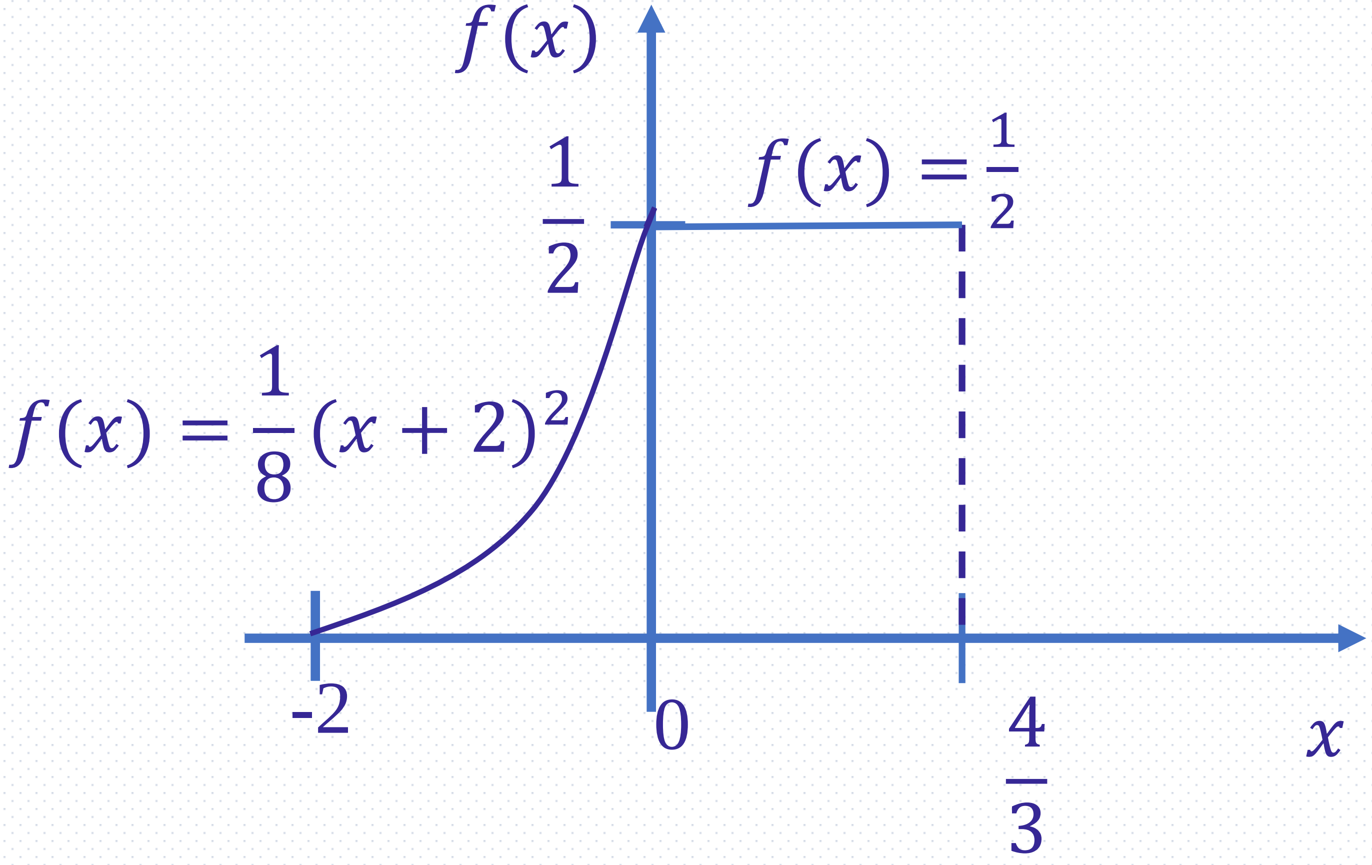
$$\frac{8}{3}k + \frac{16}{3}k = 1$$

$$8k = 1$$

$$k = \frac{1}{8}$$

Probability density functions Qn. 6

$$f(x)=\begin{cases}\frac{1}{8}(x+2)^2, -2<x<0\\ \frac{1}{2}, \quad 0\le x\le \frac{4}{3}\\ 0, \quad elsewhere\end{cases}$$



When x = -2, f(x) = 0

When x = 0, f(x) = $\frac{1}{2}$

When x = $\frac{4}{3}$, f(x) = $\frac{1}{2}$

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Probability density functions Qn. 6

$$P(-1<X<1) = \frac{1}{8}\int_{-1}^{0}(x+2)^2dx + \frac{1}{8}\int_{0}^{1}4dx$$

$$= \frac{1}{8}\left[\frac{(x+2)^3}{3}\right]_{-1}^{0} + \frac{1}{2}[x]_0^1$$

$$= \frac{1}{8}\left(\frac{2^3}{3} - \frac{1^3}{3}\right) + \frac{1}{2}(1-0)$$

$$= \frac{1}{8}\left(\frac{7}{3}\right) + \frac{1}{2} \qquad = \frac{19}{24}$$

Probability density functions Qn. 6

$$P(X > 1) = \frac{1}{8}\int_{1}^{\frac{4}{3}} 4k\,dx$$

$$= \frac{1}{2}[x]_{1}^{\frac{4}{3}}$$

$$= \frac{1}{2}\left[\frac{4}{3} - 1\right]$$

$$= \frac{1}{6}$$

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Probability density functions Qn. 7

Question 1: A continuous random variable X is defined by the probability density function

$$f(x) = \begin{cases} k(5-x) & 0 \leq x \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

a) Sketch the probability density function.

b) Find the value of the constant k.

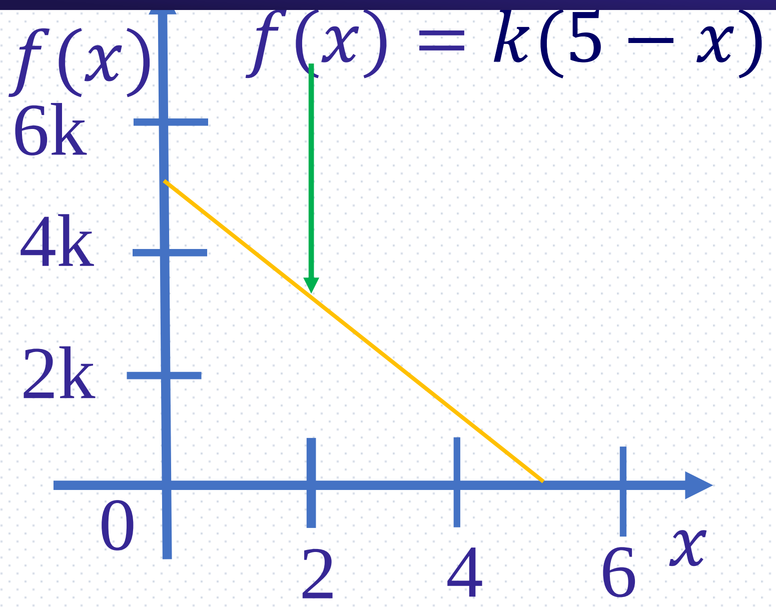
c) Find $P(1 \leq X \leq 3)$.

Probability density functions Qn. 7

a) $f(x) = \begin{cases} k(5-x) & 0 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$

When x = 0, f(x) = 5k

When x = 5, f(x) = 0



b) To find k, we can use the property that

$$\int_0^5 k(5-x)dx = k\left[5x - \frac{1}{2}x^2\right]_0^5 = k((25-12.5)-0) = 12.5k$$

Therefore, $12.5k = 1$ $\Rightarrow k = \frac{2}{25}$

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Probability density functions Qn. 7

c) $P(1 \leq X \leq 3)$ $= \int_1^3 \frac{2}{25}(5-x)dx = \frac{2}{25}\left[5x - \frac{x^2}{2}\right]_1^3$

$$= \frac{2}{25}((15-4.5)-(5-0.5))$$

$= 0.48$

Question. 8

A continuous random variable X is defined by the probability density function

$$f(x) = \begin{cases} k(x-1) & 1 \leq x \leq 3 \\ k(5-x)(x-2) & 3 < x \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

a) Sketch the probability density function.

b) Find the value of the constant k.

c) Find $P(X > 2)$.

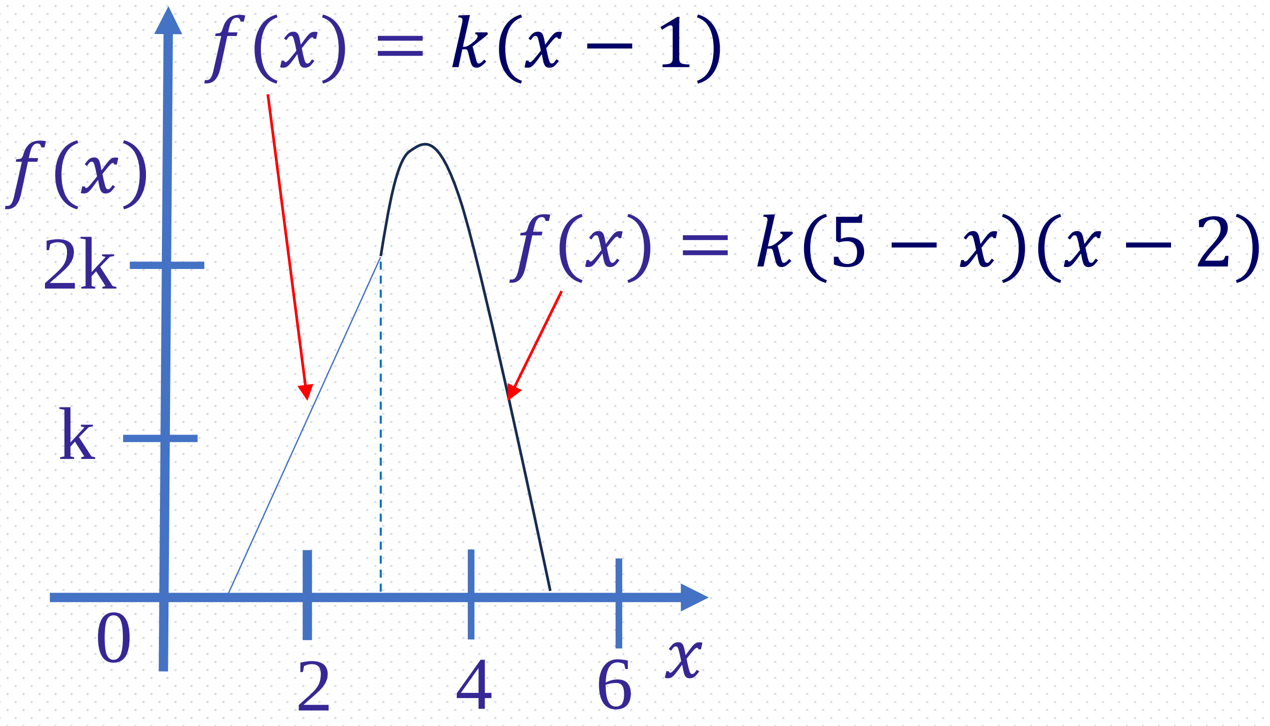
Probability density functions Qn. 8

a) $$f(x) = \begin{cases} k(x-1) & 1 \leq x \leq 3 \\ k(5-x)(x-2) & 3 < x \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

When x = 1, f(x) = 0

When x = 3, f(x) = 2k

When x = 5, f(x) = 0



Probability density functions Qn. 8

b) To find k, we can use the property that $$\int_{\text{all}\,x} f(x)dx = 1$$

Note that $(5 - x)(x - 2) = 7x - 10 - x^2$

So, $$\int_1^3 k(x-1)dx + \int_3^5 k(7x - 10 - x^2)dx = 1$$

$$\Rightarrow k\left[\frac{1}{2}x^2 - x\right]_1^3 + k\left[\frac{7}{2}x^2 - 10x - \frac{1}{3}x^3\right]_3^5 = 1$$

$$\Rightarrow k\left(1\frac{1}{2} + \frac{1}{2}\right) + k\left(-4\frac{1}{6} + 7\frac{1}{2}\right) = 1$$

Therefore $5\frac{1}{3}k = 1$ i.e. $k = \frac{3}{16}$

Probability density functions Qn. 8

c) $P(X > 2) = \int_2^5 f(x)dx$

$$= \int_2^3 \frac{3}{16}(x-1)dx + \int_3^5 \frac{3}{16}(7x - 10 - x^2)dx$$

$$= \frac{3}{16}\left[\frac{1}{2}x^2 - x\right]_2^3 + \frac{3}{16}\left[\frac{7}{2}x^2 - 10x - \frac{1}{3}x^3\right]_3^5$$

$$= \frac{3}{16}\left(1\frac{1}{2} - 0\right) + \frac{3}{16}\left(-4\frac{1}{6} + 7\frac{1}{2}\right)$$

$$= \frac{29}{32}$$

An alternative method would be to utilise $P(X > 2) = 1 - P(X \leq 2)$

Probability density functions Qn. 9

The life, T hours, of an electrical component is modelled by the probability density function

$$f(t) = \begin{cases} ke^{-0.001t} & t \geq 1000 \\ 0 & \text{otherwise} \end{cases}$$

a) Sketch the probability density function.

b) Find the value of the constant k.

c) Find $P(1500 \leq T \leq 2000)$.

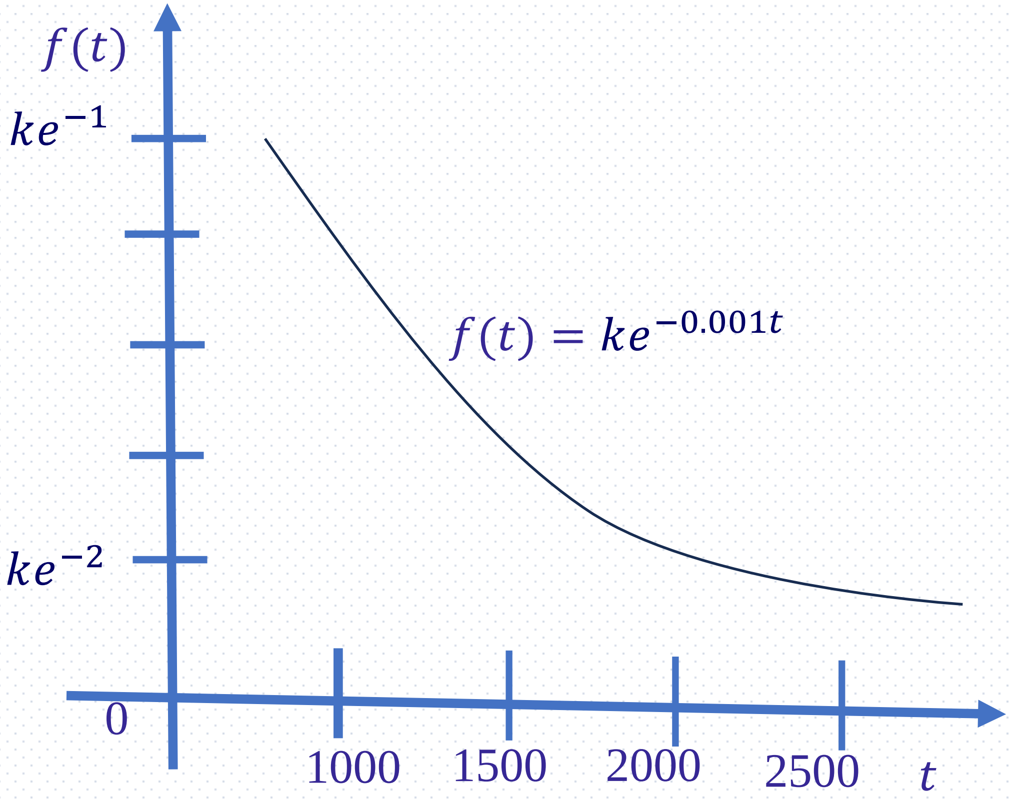
Probability density functions Qn. 9

Solution:

a) $f(t) = \begin{cases} ke^{-0.001t} & t \geq 1000 \\ 0 & \text{otherwise} \end{cases}$

When t = 1000, f(t) =ke^{-1}

When t = 2000, f(t) = ke^{-2}



Probability density functions Qn. 9

$$f(t) = \begin{cases} ke^{-0.001t} & t \geq 1000 \\ 0 & \text{otherwise} \end{cases}$$

b) To find k, we use the fact that

$$\int_{1000}^{\infty} ke^{-0.001t} dt = 1$$

$$\Rightarrow k[-1000e^{-0.001t}]_{1000}^{\infty} = 1$$

$$\Rightarrow k(0 - (-1000e^{-1})) = 1$$

Therefore $k = \dfrac{1}{1000e^{-1}} = \dfrac{e}{1000} = 0.00272$

Probability density functions Qn. 9

$$f(t) = \begin{cases} ke^{-0.001t} & t \geq 1000 \\ 0 & \text{otherwise} \end{cases}$$

c) $P(1500 \leq T \leq 2000) = \displaystyle\int_{1500}^{2000} f(t)\,dt \quad = k\int_{1500}^{2000} e^{-0.001t}\,dt$

$$= k[-1000e^{-0.001t}]_{1500}^{2000}$$

$$= \frac{e}{1000}(-1000e^{-2} + 1000e^{-1.5})$$

= 0.239 (3 s.f.)

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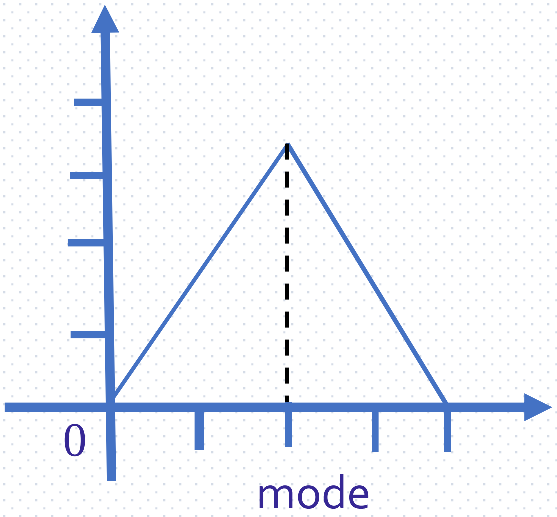
Mode

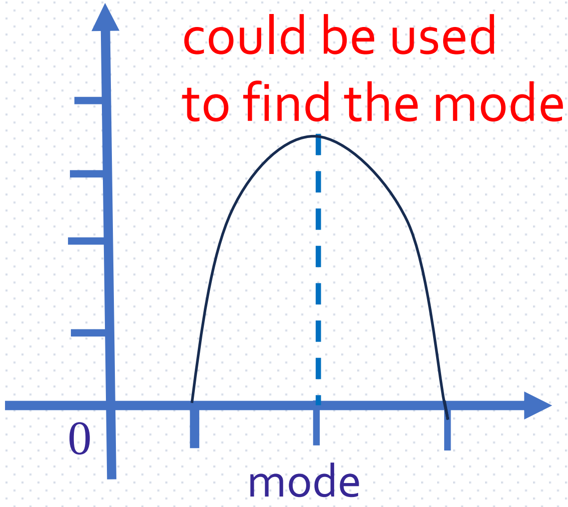
Suppose that a random variable X is defined by the probability density function $\mathrm{f}(x)$ for $a \leq x \leq b$.

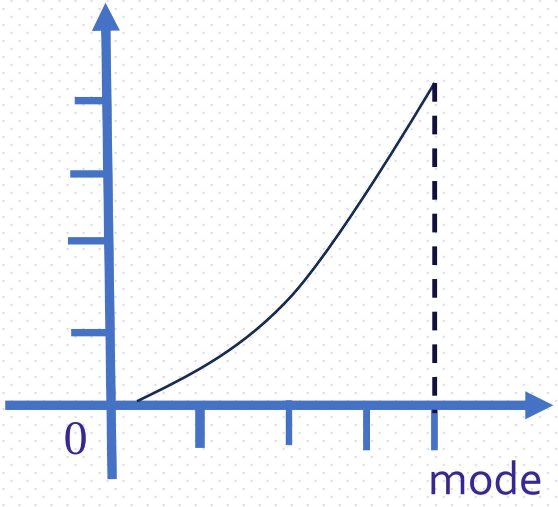
The **mode** of X is the value of x that produces the largest value for $\mathrm{f}(x)$ in the interval $a \leq x \leq b$.

A sketch of the probability density function can be very helpful when determining the mode.



Differentiation could be used to find the mode here.





Mode: Question. 1

Example: A random variable X has p.d.f. $\mathrm{f}(x)$, where

$$\mathrm{f}(x) = \begin{cases} x^2(2-x) & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

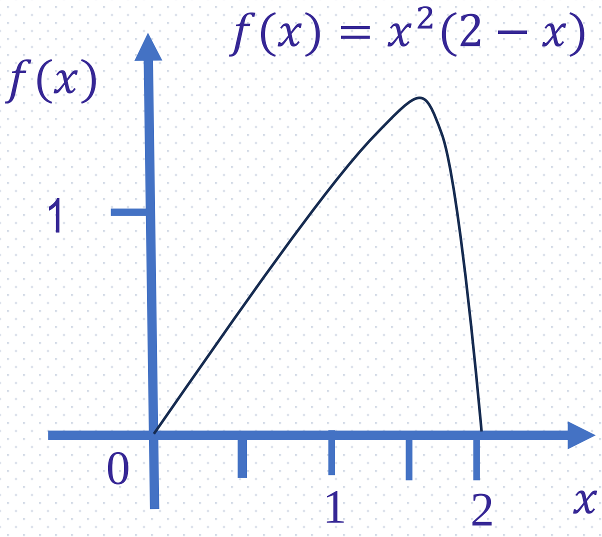
Find the mode.

Sketch of $\mathrm{f}(x)$:

When x = 0, f(x) = 0

When x = 1, f(x) = 1

When x = 2, f(x) = 0



Mode: Question. 1

The mode can be found using differentiation:

$$f(x) = 2x^2 - x^3 \Rightarrow f'(x) = 4x - 3x^2$$

To find a turning point, we solve

Factorize: $x(4 - 3x) = 0 \Rightarrow$ $x = 0 \text{ or } x = \frac{4}{3}$

Check that $x = \frac{4}{3}$ gives the maximum value:

$$f''(x) = 4 - 6x \Rightarrow f''(\frac{4}{3}) = 4 - 8 = -4 < 0$$

So the mode is. $x = \frac{4}{3}$

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Cumulative distribution functions Qn. 1

The **cumulative distribution function (c.d.f.)** F(x) for a continuous random variable X is defined as F(x) = P($X \leq x$).

Therefore, the c.d.f. is found by **integrating** the p.d.f..

Example 1: A random variable X has p.d.f. f(x), where

$$f(x) = \begin{cases} \frac{1}{6}(x^3 + 1) & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Find the c.d.f. and find P(X < 1).

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Cumulative distribution functions Qn. 1

Solution: The c.d.f., F(x) is given by:

For $0 \leq x \leq 2$

$$F(x) = \int_0^x f(t)dt$$

$$= \int_0^x \left(\frac{1}{6}t^3 + \frac{1}{6}\right) dt$$

$$F(x) = \left[\frac{1}{24}t^4 - \frac{1}{6}t\right]_0^x$$

$$= \left(\frac{1}{24}x^4 + \frac{1}{6}x\right) - \left(\frac{1}{24}0^4 + \frac{1}{6}0\right)$$

$$= \frac{1}{24}x^4 + \frac{1}{6}x$$

$F(0) = 0$

$F(2) = 1$

So the c.d.f. is

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{24}x^4 + \frac{1}{6}x & 0 \leq x \leq 2 \\ 1 & x > 2 \end{cases}$$

$$P(X < 1) = F(1) = \frac{1}{24} + \frac{1}{6} = \frac{5}{24}$$

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Median and quartiles

The **median**, m, of a random variable X is defined to be the value such that

$$\mathrm{F}(m) = \mathrm{P}(X \le m) = 0.5$$

where F is the cumulative distribution function of X.

Likewise the **lower quartile** is the solution to the equation

$$\mathrm{F}(x) = 0.25$$

and the **upper quartile** is the solution to

$$\mathrm{F}(x) = 0.75.$$

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Median and quartiles Examples

Example 1: A random variable X is defined by the cumulative distribution function:

$$F(x) = \begin{cases} 0 & x < 2 \\ \frac{1}{24}(x^2 + x - 6) & 2 \le x \le 5 \\ 1 & x > 5 \end{cases}$$

a) Calculate and sketch the probability density function.

b) Find the median value.

c) Work out $P(3 \le X \le 4)$.

Median and quartiles

a) We can get the p.d.f. by differentiating the c.d.f.

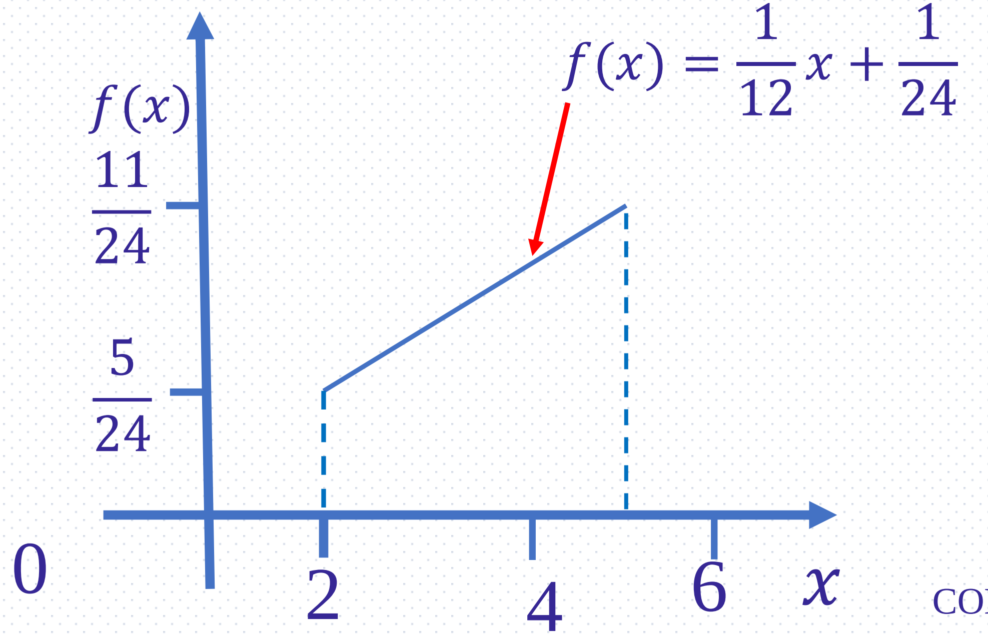
$$f(x) = F'(x) = \frac{1}{12}x + \frac{1}{24}$$

So the p.d.f. is $f(x) = \begin{cases} \frac{1}{12}x + \frac{1}{24} & 2 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$

Sketch of f(x):

When x = 2, f(x) = $\frac{5}{24}$

When x = 5, f(x) = $\frac{11}{24}$



Median and quartiles

b) The median, m, satisfies $\mathrm{F}(m) = 0.5$.

Therefore $$\frac{1}{24}(m^2 + m - 6) = 0.5$$

$$\Rightarrow m^2 + m - 6 = 12$$

$$\Rightarrow m^2 + m - 18 = 0$$

$$\Rightarrow m = \frac{-1 \pm \sqrt{1 - 4 \times 1 \times (-18)}}{2}$$

$$\Rightarrow m = -4.77 \mathrm{or} m = 3.77$$

The median must be **3.77** (as the p.d.f. is only non-zero for values in the interval [2, 5]).

Median and quartiles

c) $P(3 \leq X \leq 4) = F(4) - F(3)$

$$= \frac{1}{24}(4^2 + 4 - 6) - \frac{1}{24}(3^2 + 3 - 6)$$

$$= \frac{7}{12} - \frac{1}{4}$$

$$= \frac{1}{3}$$

Median and quartiles

Example 2: A random variable X has p.d.f. $\mathrm{f}(x)$, where

$$\mathrm{f}(x) = \begin{cases} \frac{3}{4}x^2 - \frac{3}{2}x + \frac{3}{4} & 1 \leq x \leq 2 \\ & \end{cases}$$

a) Calculate the cumulative distribution function and verify that the lower quartile is at $x = 2$.

b) Work out the median value of X.

Median and quartiles

a) For $f(x) = \frac{3}{4}x^2 - \frac{3}{2}x + \frac{3}{4}$

(worked out using method 2 for finding F(X))

$$F(x) = \frac{1}{4}x^3 - \frac{3}{4}x^2 + \frac{3}{4}x + c$$

We know that $P(X \leq 1) = 0$, i.e., that $F(1) = 0$.

So, $\frac{1}{4} - \frac{3}{4} + \frac{3}{4} + c = 0 \Rightarrow \quad c = -\frac{1}{4}$

Therefore $F(x) = \frac{1}{4}x^3 - \frac{3}{4}x^2 + \frac{3}{4}x - \frac{1}{4}$

For $f(x) = \frac{3}{2} - \frac{3}{8}x$

$$F(x) = \frac{3}{2}x - \frac{3}{16}x^2 + c$$

We know that $P(X \leq 4) = 1$, i.e. that $F(4) = 1$.

So $\frac{3}{2} \times 4 - \frac{3}{16} \times 4^2 + c = 1 \Rightarrow \quad c = -2$

Therefore $F(x) = \frac{3}{2}x - \frac{3}{16}x^2 - 2$

Median and quartiles

So $$F(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{4}x^3 - \frac{3}{4}x^2 + \frac{3}{4}x - \frac{1}{4} & 1 \le x \le 2 \\ \frac{3}{2}x - \frac{3}{16}x^2 - 2 & 2 \le x \le 4 \\ 1 & x > 4 \end{cases}$$

To verify that the lower quartile is 2, we simply need to check that F(2) = 0.25:

$$F(2) = \frac{1}{4} \times 2^3 - \frac{3}{4} \times 2^2 + \frac{3}{4} \times 2 - \frac{1}{4} = 0.25$$

Therefore the lower quartile is **2**.

Median and quartiles

b) The median, m, must lie in the interval [2, 4] because $\mathrm{F}(2) = 0.25$.

To find the median we must solve $\mathrm{F}(m) = 0.5$:

$$\text{i.e.}\mathrm{F}(m) = \frac{3}{2}m - \frac{3}{16}m^2 - 2 = \frac{1}{2}$$

This can be rearranged to give the quadratic equation:

$$3m^2 - 24m + 40 = 0$$

Using the quadratic formula, $$m = \frac{24 \pm \sqrt{576 - 4 \times 3 \times 40}}{6}$$

$m = 5.63$ or $m = 2.37$

As 5.63 does not lie in the interval [2, 4], the median must be **2.37**.

Continuous random variables

Contents

- Probability density functions
- Mode
- Cumulative distribution functions
- Median and quartiles
- **Expectation**
- Variance
- Rectangular/uniformly distributed functions

Expectation

If X is a continuous random variable defined by the probability density function f(x) over the domain $a \leq x \leq b$, then the **mean** or **expectation** of X is given by

$$\mathrm{E}[X] = \int_a^b x\mathrm{f}(x)dx$$

E[X] is the value you would expect to get, on average.

This mean value of X is also sometimes denoted μ.

[**Note**: if the p.d.f. is symmetrical, then the expected value of X will be the value corresponding to the line of symmetry].

We can also find the expected value of g(X), i.e. any function of X:

$$\mathrm{E}[\mathrm{g}(X)] = \int_a^b \mathrm{g}(x)\mathrm{f}(x)dx$$

Expectation Examples

Example 1: A random variable X is defined by the probability density function

$$\mathrm{f}(x) = \begin{cases} \dfrac{2}{x^3} & x \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

Calculate the value of E[X] and E[1/X].

$$\mathrm{E}[X] = \int_{\text{all}\,x} x\mathrm{f}(x)dx = \int_1^\infty x.\frac{2}{x^3}dx \quad = \int_1^\infty \frac{2}{x^2}dx$$

$$\int_1^\infty 2x^{-2}\,dx = [-2x^{-1}]_1^\infty \quad = (0) - (-2) = 2$$

Expectation

$$E\left[\frac{1}{X}\right] = \int_{\text{all}\,x} \frac{1}{x} f(x)dx = \int_1^{\infty} \frac{1}{x} \cdot \frac{2}{x^3}\, dx$$

$$= \int_1^{\infty} 2x^{-4}\, dx$$

$$= \left[-\frac{2}{3}x^{-3}\right]_1^{\infty}$$

$$E\left[\frac{1}{X}\right] = (0) - \left(-\frac{2}{3}\right)$$

So, $E[\frac{1}{X}] = \frac{2}{3}$

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Continuous random variables

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Variance

If X is a continuous random variable defined by the probability density function f(x) over the domain $a \leq x \leq b$, then the **variance** of X is given by

$$\mathrm{Var}[X] = \mathrm{E}[X^2] - \{\mathrm{E}[X]\}^2$$

or

$$\mathrm{Var}[X] = \int_a^b x^2 \mathrm{f}(x)dx - \mu^2$$

The standard deviation of X is the square root of the variance.

The standard deviation is sometimes denoted by the symbol σ.

Variance Examples

Example 1: A continuous random variable Y has a probability density function f(y) where

$$f(y) = \begin{cases} \frac{3}{32}y(4-y) & 0 \le y \le 4 \\ 0 & \text{otherwise} \end{cases}$$

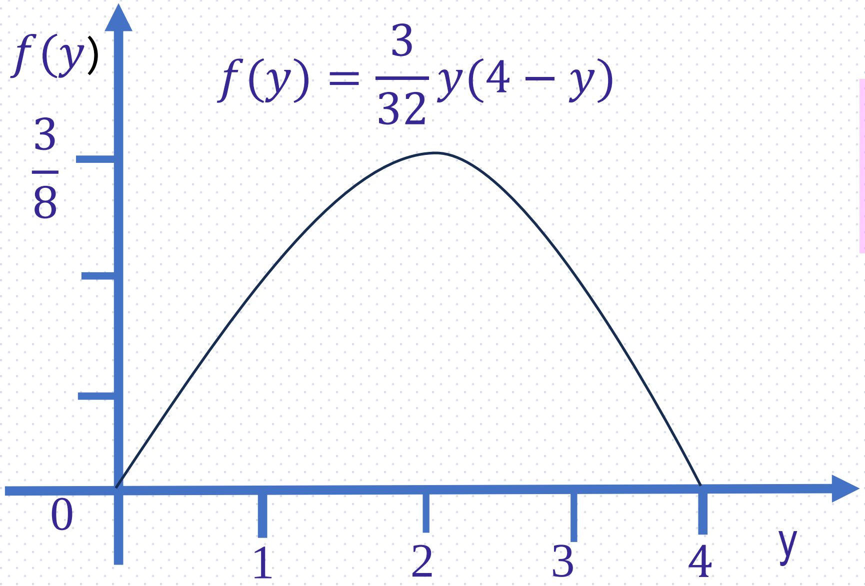
Calculate the value of Var[Y].

Sketch of f(y):

When y = 0, f(y) = 0

When y = 2, f(y) = $\frac{3}{8}$

When y = 4, f(y) = 0



The p.d.f. is symmetrical at y = 2. Therefore E[Y] = 2.

Variance Examples

$$E[Y^2] = \int_0^4 y^2\, f(y)dy = \int_0^4 y^2 . \frac{3}{32} y(4-y)dy$$

$$= \frac{3}{32}\int_0^4 (4y^3 - y^4)dy$$

$$= \frac{3}{32}\left[y^4 - \frac{1}{5}y^5\right]_0^4$$

$$E[Y^2] = \frac{3}{32}\left(\left(4^4 - \frac{1}{5}\times 4^5\right) - 0\right)$$

$$= 4\frac{4}{5}$$

Therefore $Var[Y] = 4\frac{4}{5} - 2^2 = \frac{4}{5}$

Variance Examples

Example 2: A continuous random variable x has a probability density function $f(x)$ where

$$f(x) = \begin{cases} \frac{1}{4} & 0 \leq x \leq 2 \\ \frac{3}{8} - \frac{x}{16} & 2 \leq x \leq 6 \\ 0 & \text{otherwise} \end{cases}$$

Calculate

a) the mean value, μ.

b) the standard deviation, σ.

Variance

$$f(x) = \begin{cases} \frac{1}{4} & 0 \le x \le 2 \\ \frac{3}{8} - \frac{x}{16} & 2 \le x \le 6 \\ 0 & \text{otherwise} \end{cases}$$

a) $$E[X] = \int_0^2 x.\frac{1}{4}dx + \int_2^6 x.\left(\frac{3}{8} - \frac{x}{16}\right)dx$$

$$= \int_0^2 \frac{x}{4}dx + \int_2^6 \left(\frac{3x}{8} - \frac{x^2}{16}\right)dx$$

$$E[X] = \left[\frac{x^2}{8}\right]_0^2 + \left[\frac{3x^2}{16} - \frac{x^3}{48}\right]_2^6$$

$$= \left(\frac{4}{8} - 0\right) + \left(2\frac{1}{4} - \frac{7}{12}\right)$$

$$= 2\frac{1}{6}$$

Variance

$$f(x) = \begin{cases} \frac{1}{4} & 0 \le x \le 2 \\ \frac{3}{8} - \frac{x}{16} & 2 \le x \le 6 \\ 0 & \text{otherwise} \end{cases}$$

b) $$E[X^2] = \int_0^2 x^2 . \frac{1}{4} dx + \int_2^6 x^2 . \left(\frac{3}{8} - \frac{x}{16}\right) dx$$

$$= \int_0^2 \frac{x^2}{4} dx + \int_2^6 \left(\frac{3x^2}{8} - \frac{x^3}{16}\right) dx$$

$$E[X^2] = \left[\frac{x^3}{12}\right]_0^2 + \left[\frac{x^3}{8} - \frac{x^4}{64}\right]_2^6$$

$$= \left(\frac{2}{3} - 0\right) + \left(6\frac{3}{4} - \frac{3}{4}\right)$$

$$= 6\frac{2}{3}$$

So, $\text{Var}[X] = 6\frac{2}{3} - \left(2\frac{1}{6}\right)^2 = 1\frac{35}{36}$

Therefore $\sigma = \sqrt{\frac{71}{36}}$ $= \mathbf{1.40}$ (3 s.f.)

Trial question 1

The mass, X kg, of luggage taken on board an aircraft by a passenger can be modelled by the probability density function

$$f(x) = \begin{cases} kx^3(30-x) & 0 \leq x \leq 30 \\ 0 & \text{otherwise} \end{cases}$$

a) Sketch the probability density function and find the value of k.

b) Verify that the median weight of luggage is about 20.586 kg.

c) Find the mean and the variance of X.

Answer to trial question 1.

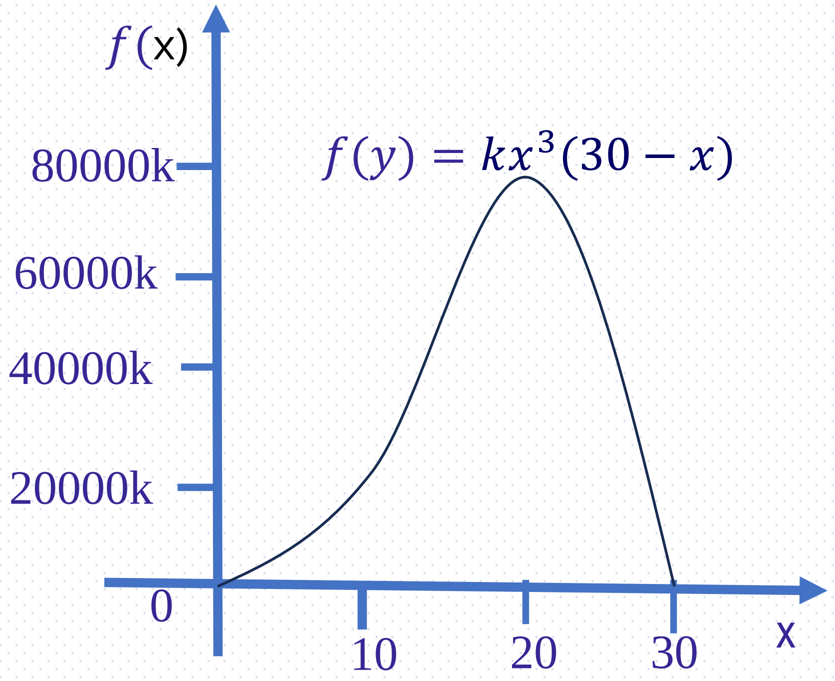
a) $f(x) = \begin{cases} kx^3(30-x) & 0 \le x \le 30 \\ 0 & \text{otherwise} \end{cases}$

When x = 0, f(x) = 0

When x = 10, f(x) = 20000k

When 20 = 4, f(x) = 80000k

When 30 = 4, f(x) = 0



To find k we use $\int_0^{30} kx^3(30-x)dx = 1$

$$\Rightarrow k\int_0^{30}(30x^3 - x^4)dx = 1$$

$$\Rightarrow k\left[\frac{30}{4}x^4 - \frac{1}{5}x^5\right]_0^{30} = 1$$

$$\Rightarrow k(1215000 - 0) = 1$$

$$\Rightarrow k = \frac{1}{1215000}$$

Answer to trial question 1.

b) $$f(x) = \begin{cases} kx^3(30-x) & 0 \le x \le 30 \\ 0 & \text{otherwise} \end{cases}$$

To verify that the median is about 20.586, we need to check that P(X ≤ 20.586) = 0.5

$$P(X \le 20.586) = \int_0^{20.586} kx^3(30-x)dx$$

$$= \frac{1}{1215000}\left[\frac{30}{4}x^4 - \frac{1}{5}x^5\right]_0^{20.586}$$

$$= \frac{1}{1215000}(607525 - 0)$$

$$= 0.500$$

Answer to trial question 1.

c) $$f(x) = \begin{cases} kx^3(30 - x) & 0 \le x \le 30 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \int_0^{30} x.kx^3(30 - x)dx = k\int_0^{30} (30x^4 - x^5)dx$$

$$= \frac{1}{1215000}\left[6x^5 - \frac{1}{6}x^6\right]_0^{30}$$

$$= \frac{1}{1215000}(24300000 - 0)$$

$$= 20$$

Answer to trial question 1.

c) $$f(x) = \begin{cases} kx^3(30-x) & 0 \le x \le 30 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X^2] = \int_0^{30} x^2 . kx^3(30-x)dx = k\int_0^{30} (30x^5 - x^6)dx$$

$$= \frac{1}{1215000}\left[5x^6 - \frac{1}{7}x^7\right]_0^{30}$$

$$= 428.5714$$

Therefore, $\text{Var}[X] = 428.5714 - 20^2 = 28.57$ (to4 s.f.)

Continuous random variables

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- Probability density functions
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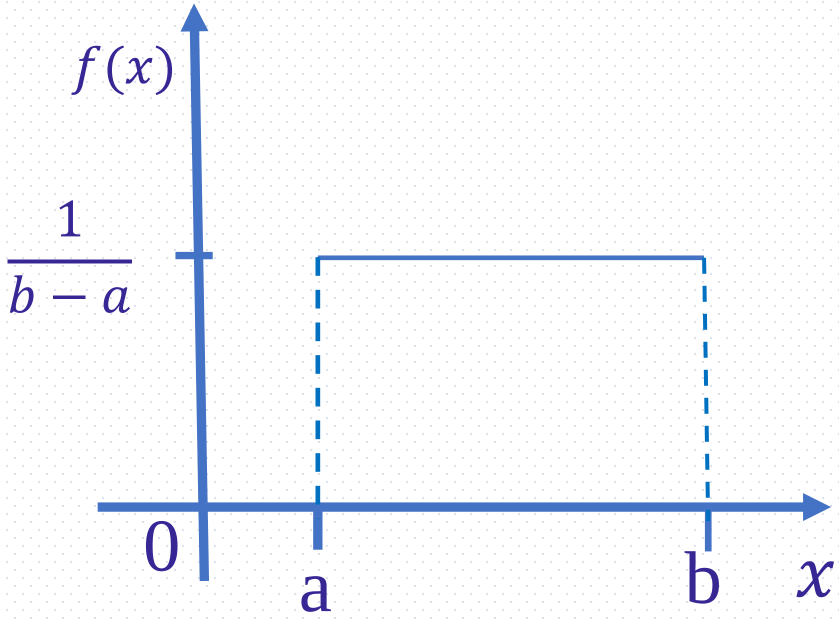
Uniform distribution

A continuous random variable X is said to be uniformly distributed over the interval a and b, if the p.d.f is given by

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b \\ 0, & otherwise \end{cases}$$



Uniform distribution Example 1.

A continuous random variable X is uniformly distributed between 6 and 9.

(i). Write the probability density function

(ii). Find P(7.2 < x < 8.4)

Answer

(i) $$f(x) = \begin{cases} \frac{1}{9-6}, & 6 \leq x \leq 9 \\ 0, & otherwise \end{cases}$$

(ii) P(7.2 < x < 8.4) $$= \int_{7.2}^{8.4} \frac{1}{3} dx$$

$$= \frac{1}{3}[x]_{7.2}^{8.4}$$

$$= 4$$

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Uniform distribution Example 2.

A continuous random variable X is uniformly distributed between o and $\frac{\pi}{2}$.

(i). Write the probability density function

(ii). Find $P(\frac{\pi}{3} < x < \frac{\pi}{2})$

Answer

(i) $$f(x) = \begin{cases} \frac{1}{\frac{\pi}{2} - 0}, & 0 \leq x \leq \frac{\pi}{2} \\ 0, & otherwise \end{cases}$$

(ii) $$P(\frac{\pi}{3} < x < \frac{\pi}{2}) = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\pi}{2} dx = \frac{\pi}{2} [x]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \frac{1}{3}$$

Expectation

$$E[X] = \int_{\text{all}x} x\mathrm{f}(x)dx$$

$$E[X] = \int_a^b \frac{1}{b-a} . xdx$$

$$= \frac{1}{2(b-a)}\left[\boldsymbol{x^2}\right]_{\boldsymbol{a}}^{\boldsymbol{b}}$$

$$= \frac{1}{2(b-a)}(\boldsymbol{b^2} - \boldsymbol{a^2})$$

$$E[X] = \frac{(\boldsymbol{b} - \mathbf{a})(\boldsymbol{b} + \mathbf{a})}{2(b-a)}$$

$$= \frac{(\boldsymbol{b} + \mathbf{a})}{2}$$

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Variance

$$\mathrm{Var}[X] = \int_a^b x^2 f(x)dx - \mu^2$$

$$E[X] = \int_a^b \frac{1}{b-a} . x^2 dx - \left[\frac{(b+a)}{2}\right]^2$$

$$= \frac{1}{3(b-a)}\left[x^3\right]_a^b - \left[\frac{(b+a)}{2}\right]^2$$

$$= \frac{(b-a)(b^2+ab+a^2)}{3(b-a)} - \left[\frac{(b+a)}{2}\right]^2$$

$$\mathrm{Var}[X] = \frac{(b-a)(b^2+ab+a^2)}{3(b-a)} - \left[\frac{b^2+2ab+a^2}{4}\right]$$

$$= \frac{(4b^2+4ab+4a^2-3b^2-6ab-3a^2)}{12}$$

$$= \frac{(b^2-2ab+a^2)}{12}$$

$$= \frac{(b-a)^2}{12}$$

Exercise qn.1.

A continuous random variable X uniformly distributed between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ is given by.

$$f(x) = \begin{cases} \frac{1}{\pi}, -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ 0, \quad otherwise \end{cases}$$

(i). Find $P(-\frac{\pi}{3} < x < \frac{\pi}{3})$

(ii). E(X)

(ii). Var(X)

(iv). Standard deviation of X.

(v). Median of X.