Calculus (5): Integration

Integration or anti-differentiation

You have probably wondered if it is possible to find a function given its derivative.

For example, if $\frac{dy}{dx} = x^2 + 2x - 3$, is it possible to find y?

We can think of this as a process of anti-differentiation, the inverse operation to differentiation, though it is actually called integration, and the result an integral.

The principle is easily found for single terms. We remember that if we differentiate x^n we get nx^{n-1}. So if we differentiate x^{n+1} we get $(n + 1)x^n$. and if we differentiate $\frac{1}{n+1}x^{n+1}$ we get x^n.

Hence if $\frac{dy}{dx} = x^n$, $y = \frac{x^{n+1}}{n+1}$ i.e., the integral of x^n (wrt x) is $\frac{x^{n+1}}{n+1}$. $(n \neq -1)$.

For example, if $\frac{dy}{dx} = x^5$, $y = \frac{x^6}{6}$;

If $\frac{dy}{dx} = 3x^2$, $y = \frac{3x^3}{3} = x^3$;

If $\frac{dy}{dx} = 6$, $y = 6x$;

If $\frac{dy}{dx} = \frac{1}{x^2} = x^{-2}$, $y = \frac{x^{-1}}{-1} = -\frac{1}{x}$.

Check each of these results by differentiating y w.r.t x, However, there is one important point to notice before proceeding further. If we differentiate $x^3 - x + 5$, $x^3 - x - 5$, $x^3 - x$ wrt x we obtain $3x^2 - 1$ in each case.

On integrating $3x^2 - 1$ the constant term cannot be recovered, without further information. To show that there is a constant tem' in the integral, we add an arbitrary constant c (which may be zero).

Hence if $\frac{dy}{dx} = ax^n$

$$y = \frac{ax^{n+1}}{n+1} + c. \qquad (n \neq -1),$$

This is known as the indefinite integral of ax^n, and the constant c should always be added. We shall discuss this matter further below.

Our working rule is: increase the index of the term by 1 and divide by the new index, leaving coefficients as they are, and add an arbitrary constant. The result can always be tested by differentiation.

Example 1

Integrate wrt x

(a) 5; (b) $x^{\frac{3}{2}}$; (c) $4\sqrt{x}$ (d) $\frac{1}{\sqrt{x}}$.

(a) If $\frac{dy}{dx} = 5$

Then $y = 5x + c$. (In using the rule, 5 can be thought of as $5x^o$).

(b) If $\frac{dy}{dx} = x^{\frac{3}{2}}$,

Then $y = \frac{x^{\frac{5}{2}}}{\frac{5}{2}}$

$= \frac{2}{5}x^{\frac{5}{2}} + c.$

(c) If $\frac{dy}{dx} = 4\sqrt{x} = 4x^{\frac{1}{2}}$,

then $y = \frac{4x^{\frac{3}{2}}}{\frac{3}{2}}$

$$= \frac{8}{3}x^{\frac{3}{2}} + c.$$

(d) If $\frac{dy}{dx} = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$,

Then $y = \frac{x^{\frac{1}{2}}}{\frac{1}{2}}$

$= 2x^{\frac{1}{2}} + c$ or $2\sqrt{x} + c.$

If $\frac{dy}{dx}$ is given as a polynomial, integrate term by term. In this way it is possible to integrate $\frac{x^4 + x - 3}{x^3}$ as $x + \frac{1}{x^2} - \frac{3}{x^3}$ but not $\frac{x^4 + x - 3}{x + 1}$; $(2x + 3)^2$ can be integrated if expanded first.

Example 2

If $\frac{dy}{dx} = 3x^3 - 4x^2 + 5x - 1 + \frac{1}{x^2}$, find y.

$$y = \frac{3x^4}{4} - \frac{4x^3}{3} + \frac{5x^2}{2} - x + \frac{x^{-1}}{-1} + c$$

$$= \frac{3x^4}{4} - \frac{4x^3}{3} + \frac{5x^2}{2} - x - \frac{1}{x} + c.$$

Note: There is one important exception. If $\frac{dy}{dx} = \frac{1}{x} = x^{-1}$ what is y?

If we use the rule, then $y = \frac{x^0}{0}$ which is not defined. Hence $\frac{1}{x}$ cannot be integrated by this method. There is an integral, a surprising one. The integral is a logarithm function but the work is too ad vanced for this stage.

Summarizing, If $\frac{dy}{dx} = ax^n$,

$y = \frac{ax^{n+1}}{n+1} + c$ Provided $n \neq -1$.

Exercise 18.1

Integrate w.r.t x, simplifying your results where appropriate. Check the first ten by differentiation.

1. x^2
2. x^3
3. $2x^4$
4. $3x$
5. $4x^5$
6. 8
7. x^7
8. $\frac{1}{x^3}$
9. $2\sqrt{x}$

10 $4x^{\frac{1}{2}}$

11.

$x^{\frac{2}{3}}$ 12. $x^{-\frac{1}{2}}$

13. 1 14. $x^{\frac{3}{4}}$ 15. $\frac{x^6}{2}$

16. $\frac{1}{\sqrt[3]{x}}$

17. $x^3 + x^2 + x + 1$ 18. $3x^4 - x + 2$

19. $x - \sqrt{x}$

20. $\frac{x^4 + x + 2}{x^3}$ 21. $\frac{3x^3 + x - 2}{2x^3}$ 22. $(x + 3)^2$

23. $(x - 1)^2$

24. $(x - \frac{1}{x})^2$ 25. $(1 + x^2)^2$ 26. $(x - 1)^3$

27. $\frac{(x + 3)^2}{2x^4}$.

The arbitrary constant

If we differentiate $y = 3x^2 + 2x + 5$ we obtain $\frac{dy}{dx} = 6x + 2$.

On integrating $\frac{dy}{dx}$ we must write $y = 3x^2 + 2x + c$.

Without further information the actual solution could be for example,

$y = 3x^2 + 2x + 5$

or $y = 3x^2 + 2x$

or $y = 3x^2 + 2x - 3$ **(fig 18.1)**.

Each of the curves shown is the graph of a solution of the differential equation $\frac{dy}{dx} = 6x + 2$. They are identical in position, which depends on the shape (all parabolas) and differ only value of the constant term. For any particular value of x, $\frac{dy}{dx}$ is the same for all the curves, so the tangents at 5 corresponding points are parallel, i.e. curves are parallel.

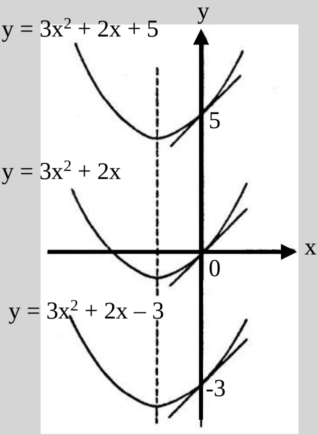


Fig 18.1

Summarizing, if $\frac{dy}{dx} = 6x + 2$, the indefinite integral or general solution is

$$y = 3x^2 + 2x + c$$

which represents a family of parallel curves. c is an arbitrary constant. To find the actual value of c, thus identifying a particular member of the family, a pair of values of x and y (i.e., one point on the curve) must be given. See Example 4 below.

Notation for integration

If $\frac{dy}{dx} = 6x + 2$, then we write $y = \int(6x+2)\,dx$ (read 'integral (6x + 2)dx').

$\int$ is the sign of integration or the integral sign and $\int$ and dx must both be written. The function to be integrated, called the integrand, is placed between them. dx is written to show that the integrand is to be integrated w.r.t x.

So if $\frac{dy}{dx} = f(x)$,

$y = \int f(x)dx + c$ where c is any constant.

Example 3

$$\int (x^2 + 3x + 4)dx = \frac{x^3}{3} + \frac{3x^2}{2} + 4x + c;$$

$$\int (s^2 + 4s)ds = \frac{s^3}{3} + 2s^2 + c;$$

$$\int \frac{t^3 + 3t^2 - 1}{t^2}\, dt = \int \left(t + 3 - \frac{1}{t^2}\right) dt$$

$$= \frac{t^2}{2} + 3t + \frac{1}{t} + c.$$

Example 4

If $\frac{dy}{dx} = 6x + 2$, find y given that y = 3 when x = 1.

$y = \int (6x + 2)dx$

$= 3x^2 + 2x + c.$

Substitute the given information.

Then $3 = 3 + 2c$

Giving $c = -2.$

Hence $y = 3x^2 + 2x - 2.$

Example 5

A particle moves in a straight line such that its acceleration after time t s is a ms^{-2} where $a = 2t^2 + t$. if its initial velocity was 3 ms^{-1} find an expression for s, the distance (in m) traveled from the start in t s.

$a = \frac{dv}{dt}$

$= 2t^2 + t.$

Hence $v = \int (2t^2 + t)\, dt$

$$= \frac{2t^3}{3} + \frac{t^2}{2} + c.$$

When t = 0, v = 3 which gives c = 3.

Hence v $= \frac{2t^3}{3} + \frac{t^2}{2} + 3.$

Now v $= \frac{ds}{dt}$

and hence s $= \int \left(\frac{2t^3}{3} + \frac{t^2}{3} + 3 \right) dt$

$$= \frac{2t^4}{12} + \frac{t^3}{6} + 3t + c.$$

(A different arbitrary constant, through the same letter is used).

When t = 0, s = 0 which gives c = 0.

Hence s $= \frac{t^4}{6} + \frac{t^3}{6} + 3t$ which is the expression required.

Integration of trigonometrical functions

If y = sin x, $\frac{dy}{dx}$ = cos x.

Hence $\int$ cos x dx = sin x + c (x is in radians).

If $y = \cos x$, $\frac{dy}{dx} = -\sin x$.

Hence $\int \sin x\, dx = -\cos x + c$ (x is in radians).

If $y = \tan x$, $\frac{dy}{dx} = \sec^2 x$.

Hence $\int \sec^2 x\, dx = \tan x + c$ (x is in radians).

Further, if $y = \sin ax$, $\frac{dy}{dx} = a \cos ax$, where a is a constant.

Hence $\int \cos ax\, dx = \frac{1}{a} \sin ax + c$.

Similarly $\int \sin ax\, dx = -\frac{1}{a} \cos ax + c$.

And $\int \sec^2 ax\, dx = \frac{1}{a} \tan ax + c$.

Example 6

$\int \sin 3x\, dx = -\frac{1}{3} \cos 3x + c$;

$\int (\cos 2x - \sin \frac{x}{2})dx = \frac{1}{2} \sin 2x + 2 \cos \frac{x}{2} + c$;

$\int \sec^2 4\,\theta\, d\,\theta \quad = \quad \frac{1}{4} \tan 4\,\theta + c.$

Exercise 18.2

Find

1. $\int x\, dx$ **2.** $\int 3\, dx$

3. $\int (x^2 + 1)dx$

4. $\int \sin 2x\, dx$ **5.** $\int (x + 3)^2\, dx$

6. $\int \frac{dx}{x^2}$ (abbreviated form of $\int \frac{1}{x^2} dx$) **7.** $\int \cos 5x\, dx$

8. $\int \sqrt{x}\, dx$ **9.** $\int (x - \frac{1}{x})^2\, dx$

10. $\int \sec^2 \frac{x}{2}\, dx$

11. $\int (\sqrt{x} - \frac{1}{\sqrt{x}})dx$ **12.** $\int \frac{x^2 + 1}{x^2}\, dx$

13. $\int (\cos 2x - \sin 4x)\, dx$ **14.** $\int \frac{x^3 - x^2 + 1}{x^2}\, dx$

15. $\int x(x - 3)\, dx$

16. $\int (x + 1)\,(x - 2)dx$ **17.** $\int (3t + 4t^2)\, dt$

18. $\int (x + \cos \frac{x}{3})dx$

19. $\int (t^3 - t)\, dt$ **20.** $\int \sec^2 (\frac{2\theta}{3})\, d\theta$

21. $\int (\sqrt{x} - \frac{1}{x})^2\, dx$

22. $\int (\cos x + \cos 2x + \cos 3x)\, dx$

23. $\int \frac{2(x+3)}{x^3}\, dx.$

24. A curve is given by the differential equation $\frac{dy}{dx} = x + 2$, and it passes through the point (2, 0). Find its equation and sketch the curve.

25. If a curve is given by $\frac{dy}{dx} = 2x + 1$ and passes through the point (1, 2), find its equation and sketch the curve.

26. The rate of change of a quantity A is given by $\frac{dA}{dt} = t^2 - 1$. If $A = \frac{4}{3}$ when $t = 1$ find A in terms of t.

27. The velocity of a particle moving in a straight line at time t s is given by $v = 2t^2 - 3t$. Find an expression for the distance (s m) traveled, if s = 0 when t = 0.

28. A particle starts from rest at a point O and moves in a straight line in such a way that its velocity, v ms^{-1}, after time t s, is given by $v = 12t - 3t^2$, until it comes to rest again at A after 4 s. Calculate

(a) the distance OA;
(b) the greatest velocity of the particle.

29. If $\frac{dy}{d\theta} = \frac{1}{\theta^2} + \frac{1}{2}\cos 2\theta$, find y if y = 0 when $\theta = \frac{\pi}{2}$
.
30. A particle is moving in a straight line and at time t s its acceleration is $(6 - kt)\ ms^{-2}$ where k is a constant. When t = 9, the acceleration of the particle is zero and its velocity is $30\ ms^{-1}$. Find the numerical value of the velocity when t = 0 and the distance between its' positions when t = 0 and t = 9.
31. **(i)** A curve passes through the point (0, 1) and is such that at every point of the curve $\frac{dy}{dx} = x^2$. Sketch the curve.

(ii) A particle is given an initial velocity of $12\ ms^{-1}$ and travels in a straight line so that its retardation after t s is equal to $6t\ ms^{-2}$ until it comes to rest. If the particle then remains stationary, calcu1ate the distance traveled.

32. If $\frac{d^2y}{dx^2} = 6x - 4$ find $\frac{dy}{dx}$ given that $\frac{dy}{dx} = 3$ when x = 0. If also y = 0 when x = 0 find y.
33. Integrate w.r.t x:
(a) $\sin^2 x$ (Use the double angle formula cos 2x = 1 – 2 $\sin^2 x$ in the from $\sin^2 x = \frac{1 - 2\cos 2x}{2}$ and now

integrate).

(b) $\cos^2 2x$ (Use the double angle formula for cos 4x).

Application of integration (1): Areas

Suppose y = f(x) is the equation of a curve. We assume for the moment that the portion of the curve between the ordinates x = a and x = b (b > a) lies entirely above the x-axis, i.e. y > 0 **(fig 18.2)**. We also assume that the curve is 'continuous', i.e. that there are no breaks or gaps in it.

We now find a method of calculating the area enclosed by the curve, the x-axis and **Fig 18.2** the ordinates at A and B, i.e. the area ABCD. You will appreciate that up to now only areas which could be dissected into triangles or trapezium could be found by calculation, other shapes being found by approximate methods. Our new method is therefore very important, as it will apply to areas such as ABCD, bounded partly by a curve.

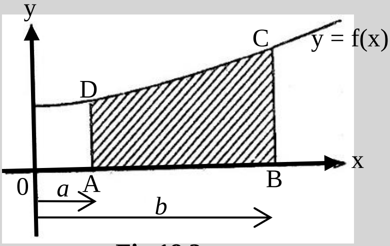


Fig 18.2

Let P be a variable point on the x-axis A and B where OP = x **(fig 18.3)**. Draw the ordinate PQ (length y) and let the shaded area APQD = A. A is thus a function of x and when x = a, A = 0. Now take an increment δx in x and the area A is increased by an amount δA, i.e. the portion PRSQ. RS is $y + \delta y$.

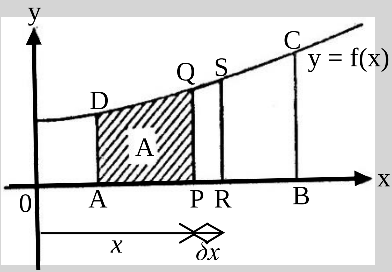


Fig 18.3

Now from **fig 18.4** it is seen that area PRTQ < ΔA < area PRSU where QT, US are parallel to the x-axis.

fig 18.4

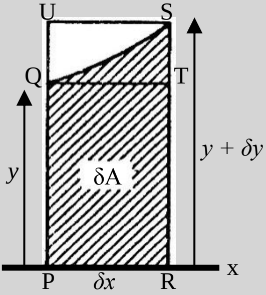


Fig 18.4

$\therefore \quad y \times \delta x < \Delta A < (y + \delta y) \times \delta x$

or $\quad y < \dfrac{\delta A}{\delta x} < (y + \delta y).$

If now $\delta x \rightarrow 0$, $\delta y \rightarrow 0$

and $\dfrac{\delta A}{\delta x} \rightarrow \dfrac{dA}{dx}$ and hence $\dfrac{dA}{dx} = y$, as the right hand term of the above inequality tends to y.

Therefore $A = \int y\, dx + c$

$= \int f(x)\, dx + c.$

The value of c can be found from the fact that when x = 0, A = 0. We then have A expressed as a function of x and obtain the area ABCD of **fig 18.2**.

Note: If the curve has a negative gradient in the range considered **(fig 18.5)** the above must be modified as follows.

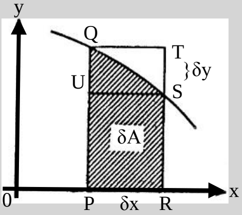


Fig 18.5

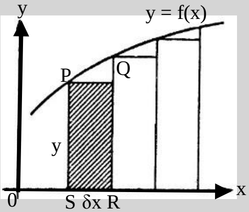
The inequality will now be area PRTQ > ΔA > area PRSU

Or $y\,\delta x > \Delta A > (y + \delta y)\,\delta x$

i.e. $y > \dfrac{\delta A}{\delta x} > y + \delta y$ and in the limit the same result is obtained.

If the curve contains a turning point in the range, further modification could easily be devised.

The origin of the sign $\int$ may be of interest and will give some idea of an alternative approach to the question of area calculation. P is a point (x, y) on the curve y = f(x) and PQRS is a rectangle whose side SR is δx **(fig 18.6)**. The area under the curve will contain a series of such rectangles, of area y.δx.



(fig 18.6)

Then the area under the curve will be approximately the sum of the areas of these rectangles, i.e. sum ($y\delta x$) for the range considered.

As $\delta x \to 0$, the hunt of this sum (assuming it exists and can be found) will be the actua1 area under the curve. The initials S of sum is then written in the form $\int$ and δx is written as dx, to show that the limit has been taken.

Example 7

Find the area bounded by the curve $y = x^2 + 3$, the x-axis and the coordinates $x = 1$, $x = 3$. **(fig 18.7)**.

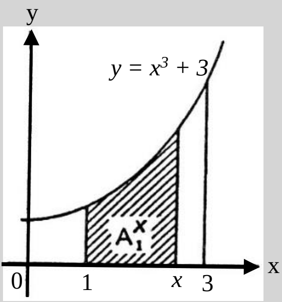


Fig 18.7

By the above, A $= \int y\,dx$

$= \int (x^2 + 3)\,dx$

$= \dfrac{x^3}{3} + 3x + c.$

When x = 1, A = 0.

Hence 0 $= \dfrac{1}{3} + 3 + c,$

giving c $= -3\dfrac{1}{3}.$

Therefore $A_1^x = \dfrac{x^3}{3} + 3x - \dfrac{10}{3}$, A_1^x meaning the area from 1 to x.

Now put x = 3.

Then A_1^3 = the required area

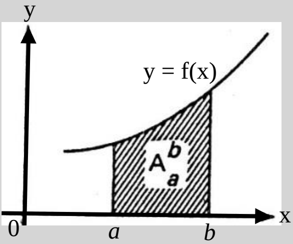
$= \dfrac{27}{3} + 9 - \dfrac{10}{3}$

$= \dfrac{44}{3}$ square units.

The definite integral

We can now generalize the above process, introducing a very important technique.

Consider the curve y = f(x), y > 0 in the range of x = a to x = b **(fig 18.8)**. Then the area A between the curve and the x-axis is given by



Fig

$A = \int f(x)\,dx + c$
$= g(x) + c$ say.
When x = a, the area = 0.
Thus $0 = g(a) + c$
or $c = -g(a)$
Then $A_a^x = g(x) - g(a)$
= (value of integral when x = b) – (value of integral when x = a)
Which is written $\int_a^b f(x)\,dx$. This is called the definite integral of f(x) w.r.t x between the limits a (the lower limit) and b (the upper limit). It is a function of a and b. the arbitrary constant c disappears in the subtraction.
Hence if y = f(x) is the equation of a curve, the area between the curve, the x-axis and the coordinates x = a, x = b (b > a) is given by $\int_a^b y\,dx = \int_a^b f(x)\,dx$. In the next section we examine some complications which may arise in finding areas.

The actual technique in evaluating definite integrals is shown in the following examples.

Example 8

Evaluate $\int_1^3 (x^2 - 1)\, dx$.

$$\int_1^3 (x^2 - 1)\, dx = \left[\frac{x^3}{3} - x\right]_1^3$$

Square brackets round the indefinite integrals but omit the arbitrary constant.

$$= \left(\frac{27}{3} - 3\right) - \left(\frac{1}{3} - 1\right)$$

$$= 6 + \frac{2}{3}$$

$$= 6\frac{2}{3}$$

Example 9

Find the value of $\int_{-2}^{1} (3t - 2)\, dt$.

$$\int_{-2}^{1} (3t - 2)\, dt = \left[\frac{3x^2}{2} - 2t\right]_{-2}^{1}$$

$$= \left(\frac{3\times(1)^2}{2} - 2\right) -$$

$$\left(\frac{3\times(-2)^2}{2}-2(-2)\right)$$

$$= \quad -\frac{1}{2} - 10$$

$$= \quad -10\frac{1}{2}.$$

Example 10

Find $\int_0^{\pi/4}$ (cos 4x + sin 2x) dx.

$$\text{Integral} \quad = \quad \left[\frac{\sin 4x}{4}-\frac{\cos 2x}{2}\right]_0^{\pi/4}$$

$$= \quad \left(\frac{\sin\frac{4\pi}{4}}{4}-\frac{\cos\frac{2\pi}{4}}{2}\right) - \left(\frac{\sin 0}{4}-\frac{\cos 0}{2}\right)$$

$$= \quad (0-0)-(0-\frac{1}{2})$$

$$= \quad \frac{1}{2}.$$

Exercise 18.3

Evaluate

1. $\int_0^1 dx$

2. $\int_0^3 x\,dx$

3. $\int_0^2 x^3 dx$

4. $\int_1^2 (2x-1)dx$

5. $\int_0^\pi \cos x\, dx$

6. $\int_0^{\pi/2} \sin 2x\ dx$

7. $\int_{-1}^1 2x^3 dx$

8. $\int_2^3 \frac{1}{x^2} dx$

9. $\int_0^2 (x+1)^2 dx$

10. $\int_{-2}^{-1} (x^2+x-1)dx$

11. $\int_0^{\pi/4} \sec^2 x\ dx$

12. $\int_{-1}^0 x(x-1)dx$

13. $\int_0^t (x^2-3)dx$

14. $\int_0^1 (s^2+3x-2)ds$

15. $\int_0^\pi (\sin 2\theta - \cos\theta)d\theta$.

Further notes on areas

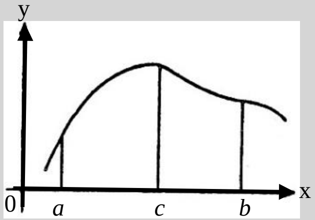


Fig 18.9

1. From fig 18.9 it is clear that $c\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx.$

2. If y is negative in the range a to b, then the value obtained from the integral $\int_a^b f(x)dx$ will also be negative **(fig 18.10)** as dx is essentially positive. Thus the numerical value of the shown shaded will be $-\int_a^b f(x)dx$.

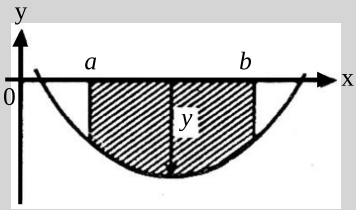


Fig 18.10

3. If the range includes both positive and negative values of y **(fig 18.11)** the total area must be found in two parts and will be.

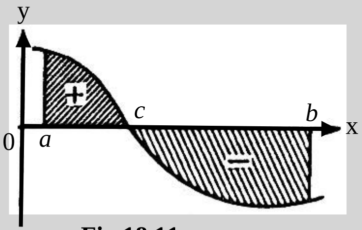


Fig 18.11

Important note: $\int_a^b f(x)dx$ is the value of the definite integral between the limits of a and b but it is NOT necessary the correct value for the area under the curve y = f(x) from a to b. it is the algebraic sum of the areas above and below the x-axis. It is wise to sketch a graph before integrating when finding an area.

4. The area between a curve and the y-axis and the lines y = a, y = b **(fig 18.12)** will be $\int_a^b x dy$. This can be proved in a manner as before.

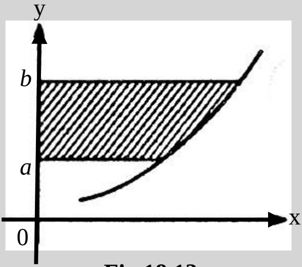


Fig 18.12

5. The area between any two curves y = f(x) and y = g(x) is easily found if the points of intersection or the limits are known **(fig 18.13)**.

The area below y = g(x) is $\int_a^b g(x)dx$.

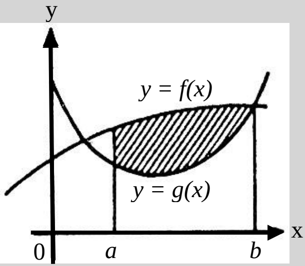


Fig 18.13

Hence the enclosed area (shown shaded) is the difference between two areas above, i.e.

$\int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b [f(x) - g(x)]dx$ assuming f(x) > g(x).

A sketch should always be made to show the relative positions o f the curves.

Example 11

Find the area between the curve y = x^2, the x-axis and the coordinates x = 0 and x = 2.

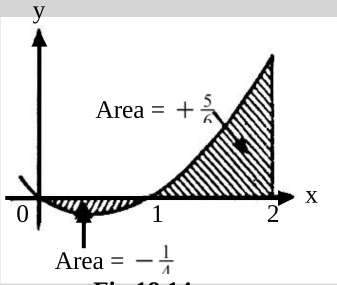


Fig 18.14

The curve crosses the x-axis where x = 0 and x = 1 **(fig 18.14)**. Hence the total area numerically

$$= \quad -\int_0^1 y\,dx + \int_1^2 y\,dx$$

$$= \quad -\int_0^1 (x^2 - x)dx + \int_1^2 (x^2 - x)dx$$

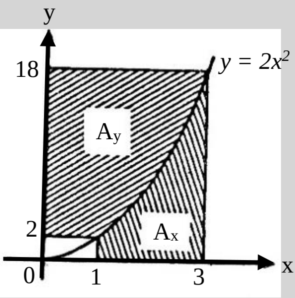
$$= \quad -\left[\frac{x^3}{3} - \frac{x^2}{2}\right]_0^1 + \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^2$$

$$= \quad -\left[\left(\tfrac{1}{3}-\tfrac{1}{2}\right)-0\right] + -\left[\left(\tfrac{8}{3}-\tfrac{4}{2}\right)-\left(\tfrac{1}{3}-\tfrac{1}{2}\right)\right]$$

$$= \quad +\tfrac{1}{6}+\tfrac{2}{3}+\tfrac{1}{6} = 1\text{square unit.}$$

Example 12

Find the areas between the curve $y = 2x^2$ and
(i) the x-axis,
(ii) the y-axis cut off but lines parallel to the axes through the points on the curve where x = 1 and x = 3 **(fig 81.15)**.



(i) A_x is required area $= \displaystyle\int_1^3 y\,dx$

$$= \int_1^3 2x^2\,dx$$

$$= \left[\frac{2x^3}{3}\right]_1^3$$

$$= \quad 17\tfrac{1}{3}.$$

(ii) $A_y = \int_2^{18} x\,dy$

$= \int_2^{18} \sqrt{\frac{y}{2}}\,dy$

$= \frac{1}{\sqrt{2}} \int_2^{18} \sqrt{y}\,dy$

$= \frac{1}{\sqrt{2}} \left[\frac{2}{3} y^{\frac{3}{2}}\right]_2^{18}$

$= \frac{1}{\sqrt{2}} \left[\left(\frac{2}{3} \times 18^{\frac{2}{3}}\right) - \left(\frac{2}{3} \times 2^{\frac{3}{2}}\right)\right]$

$= \frac{1}{\sqrt{2}} \left[\frac{2}{3} \times 27 \times 2\sqrt{2} - \frac{2}{3} \times 2\sqrt{2}\right]$

As $18^{\frac{3}{2}} = (\sqrt{18})^3$

$= (3\sqrt{2})^3$

$= 27 \times 2\sqrt{2}$

$A_y = 36 - \frac{4}{3}$

$= 34\frac{2}{3}$.

Note: In this part the limits must be the limit s which y takes to cover the required range, and the equation of the curve must be written in the form $x = +\sqrt{\frac{y}{2}}$.

Example 13

Find the area enclosed between the curve $y = x^2 + 2$ and the line $y = 4x - 1$.

The intersections are given by

$$x^2 + 2 = 4x - 1$$

or $$x^2 - 4x + 3 = 0$$

Which gives $x = 1$ or 3.

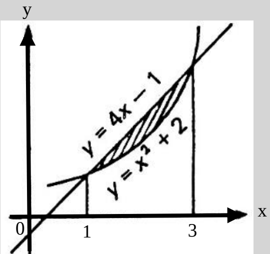


Fig 18.16

The graphs are sketched in **fig 18.16**. Between $x = 1$ and $x = 3$ the line is above the curve. Area enclosed =

$$\int_1^3 (4x-1)dx - \int_1^3 (x^2+2)dx$$

$$= \int_1^3 (4x-1-x^2-2)dx$$

$$= \int_1^3 (4x-x^2-3)dx$$

$$= \left[2x^2 - \frac{x^3}{3} - 3x\right]_1^3$$

$$= (18 - 9 - 9) - (2 - \tfrac{1}{3} - 3)$$

$$= 1\tfrac{1}{3}.$$

Exercise 18.4

Find the areas between the following curves and the x-axis between the ordinates at the values given. In each case sketch the curve.

1. $y = x^2$; $x = 0, x = 3$
2. $y = 3x - 2$; $x = 3, x = 4$
3. $y = x^2 - 3x$; $x = 0, x = 3$
4. $y = x^2 - 5x + 6$; $x = 2, x = 4$
5. $y = x^3$; $x = 1, x = 3$
6. $y = \dfrac{1}{x^2}$; $x = 1, x = 3$
7. $y = (x - 1)(x - 3)$; $x = 1, x = 3$
8. $y = (1 - x)(x - 2)$: $x = 0, x = 3$
9. Sketch the curve $y = 3x - x^2$ and calculate the area between the curve and the x-axis.

10. Find the area between the curve $y = \sin x$ and the x-

axis from $x = 0$ to $x = \pi$.

11. Find the area between the curve $y = \dfrac{1}{x^2}$, the y-axis and the lines $y = 4$, $y = 9$.

12. Find the area enclosed between the curves $y = 2x^2$ and $y^2 = 4x$.

13. Find the area between the x-axis and the part of the curve $y = (x - 3)(2 - x)$ which is above the x-axis.

14. Make a rough sketch of the curve $y = x(x + l)(x + 3)$ from $x = -4$ to $x = +1$. What are the slopes of the graph at the points where it closes the x-axis?

Calculate the area enclosed by the x-axis and the curve between $x = -3$ and $x = -l$.

15. Draw a rough sketch of the curve $y = x(x - 1)(x - 2)$. If this curve crosses the axis of x at O, A and B (in that order), show that the area included between the arc OA and the x-axis is equal to the area, included between the arc AB and the x-axis.

16. The curve $y = ax^2 + bx + c$ passes through the points (1, 0) and (2, 0) and its gradient at the point (2, 0) is 2. Find the numerical value of the area included between the curve and the axis of x.

17. Draw a rough sketch of the curve $y^2 = 16x$. Calculate

the area enclosed by the curve and the line $x = 4$.

18. The curve $y = ax^2 + b$ passes through the points $(0, k)$ and $(h, 2k)$. Express a and b in terms of h and k. Show that the area bounded by the curve, the x-axis, the y-axis and the line $x = h$ is $\frac{4}{3}hk$.

19. Calculate the coordinates of the points of intersection of the line $x - y - 1 = 0$ and the curve $y = 5x - x^2 - 4$. In the same diagram sketch the line and the curve for values of x from 0 to +5. Calculate the area contained between the line and. the curve.

20. Sketch the curve whose equation is $y = (x - 1)^3$. Find the equation of the tangent to this curve at the point where $x = 3$. Calculate the area enclosed by the curve, the tangent at the point where $x = 3$ and the x-axis.

Applications of integration (2): Volumes of revolution

A solid which has a central axis of symmetry is a solid of revolution for example, a cone, a cylinder, a flower, etc. Imagine the area under a portion AB of the curve $y = f(x)$ revolved about the x-axis through four right angles or 360°, the x-axis acting as a kind of hinge **(fig 18.17)**. Each point of the curve describes a circle centered on the x-axis. A solid of revolution can be thought of as created this way, with two circular plane ends, cutting the x-axis at $x = a$ and $x = b$.

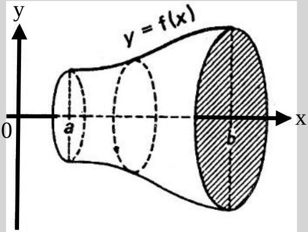


Fig 18.17

Let V be the volume of the solid from $x = a$ up to an arbitrary value of x between a and b **(fig 18.18)**. Given an increment δx in x, y takes an increment δy and V an increment δV.

Fig 18.19 shows a section through x-axis and from this it is seen that the slice δV of thickness δx is enclosed between two cylinders of outer $y + \delta y$, and inner radius y.

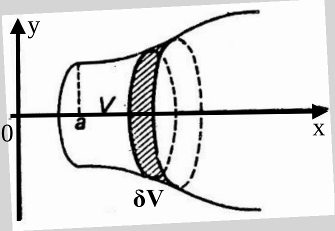


Fig 18.18

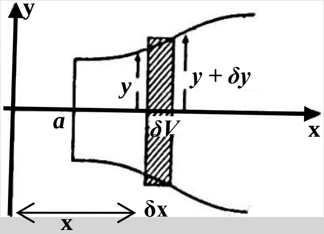


Fig 18.19

Then $\pi y^2\ \delta x < \Delta v < \pi\ (y + \delta y)^2\ \delta x$, with appropriate modification if the curve is falling at this point.

Then $\quad \pi y^2 < \dfrac{\delta V}{\delta x} < \pi\ (y + \delta y)^2$.

Now let $\delta x \to 0$ and $\delta y \to 0$ and $\dfrac{\delta V}{\delta x} \to \dfrac{dV}{dx}$.

Hence from the above inequality, $\dfrac{dV}{dx} = \pi y^2$

or $V = \int_a^b \pi y^2 dx$

where y = f(x) and V is the volume e of solid generated when the curve y = f(x) between limits x = a and x = b is rotated completely around the x-axis.

Example 14

The portion of the curve $y = x^2$ between x = 0 and x = 2 is rotated completely round the x-axis. Find the volume of the solid created.

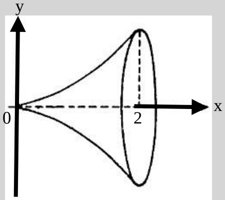


Fig 18.20

$$V = \int_0^2 \pi y^2 dx$$

$$= \int_0^2 \pi x^4 dx$$

$$= \pi \left[\frac{x^5}{5} \right]_0^2$$

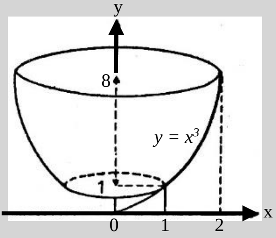
$$= \frac{32\pi}{5} \ (\approx 20.1) \text{ units of volume.}$$

Similarly if a portion of the curve y = f(x) between the limits y = a and y = b is rotated completely round the y-axis, the volume of the solid generated will be given by

$$V = \int_a^b \pi x^2 dy$$ which can be proved in the same way.

Example 15

The part of the curve $y = x^3$ from x = 1 to x = 2 is rotated completely round the y-axis. Find the volume of the solid generated (**fig 18.21**).



Fig

$$V = \int_1^8 \pi x^2 dy\ .$$

Note the limits: these are the limits of y corresponding to x = 1, x = 2. We must also express the integrand in terms of y, as we are integrating w.r.t y.

Then V $= \int_1^8 \pi x^{2\frac{2}{3}} dy$

$= \pi\left[\frac{3}{5}y^{\frac{5}{3}}\right]_1^8$

$= \pi\left(\frac{3}{5}\times 32\right) - \pi\left(\frac{3}{5}\times 1\right)$

$= \frac{93\pi}{5}.$

Exercise 18.5

Leave your answers in terms of π, as in Example 15.

1. Find the volume generated by rotating the curve $y = x + 1$ from $x = 1$ to $x = 2$ completely round the x-axis.

2. The portion of the curve $y = \frac{1}{2}x^2$ from $x = 0$ to $x = 2$ is rotated about the x-axis through four right angles. Find the volume generated.

3. Sketch the curve $y = x^2 - x$. The part below the x-axis is rotated about the x-axis to form a solid of revolution. Find its volume.

4. If the part of the curve $y = x^2$ from $x = 1$ to $x = 2$ is rotated completely about the y-axis, find the volume of the solid so formed.

5. The part of the line $y = mx$ from $x = 0$ to $x = h$ is rotated about the x-axis through four right angles. Find the volume generated and hence show that the volume of a right circular cone of base radius r and height h is $\frac{1}{3}\pi r^2 h$.

6. If the area enclosed between the curves $y = x^2$ and the line $y = 2x$ is rotated around the x-axis through four right angles, find the volume of the solid generated.

7. Calculate

(a) the area bounded by the x-axis and the curve $y = x - 3\sqrt{x}$;

(b) the volume generated by revolving this area through four right angles about the x-axis. Leave this result in terms of π.

8. Find the area included between the curves $y^2 = x^3$ and $y^3 = x^2$. Find also the volume obtained by rotating this area through four right angles about the axis of x.

9. An area is bounded by the curve $y = x + \dfrac{3}{x}$, the x-axis and the ordinates at $x = 1$ and $x = 3$. Calculate the volume of the solid obtained by rotating this area through four right angles about the x-axis.

10. The equation $x^2 + y^2 = r^2$ represents a circle radius r, centre the origin. The quarter circle in the first quadrant is rotated completely round the x-axis to form a hemisphere. Find its volume and deduce a formula for the volume of a sphere of radius r.

11. The area contained between the curve $y^2 = x - 2$, the x-axis, the x-axis, the y-axis and the line $y = 1$ is rotated about the y-axis through four right angles. Find the volume of the solid generated.

12. Sketch the curve $y^2 = x - 1$. The area contained by this curve, the y-axis and the lines $y = \pm 2$ is completely rotated about the y-axis. Find the volume of the solid so formed.

SUMMARY

If $\frac{dy}{dx} = ax^n$, then $y = \frac{ax^{n+1}}{n+1} + c (n \neq 1)$.

If $\frac{dy}{dx} = f(x)$, then $y = f(x)\,dx + c$ and conversely.

$\int a\, f(x)\,dx = a \int f(x)\,dx$ where a is a constant.

$\int \cos ax\,dx = \frac{\sin ax}{a} + c$ (x is in radians).

$\int \sin ax\,dx = -\frac{\cos ax}{a} + c$ (x is in radians).

$\int \sec^2 ax\,dx = \frac{\tan ax}{a} + c$ (x is in radians).

Definite integral $\int_a^b f(x)dx = g(b) - g(a)$

where $g(x) = \int f(x)\,dx$.

The area between the curve $y = f(x)$ and the x-axis between $x – a$ and $x – b$ is $\int_a^b f(x)dx$.

This area is positive if above the axis, negative if below. To find the total area, parts above and below the x-axis must be evaluated separately if necessary.

Similarly, the area between $y = f(x)$ and the y-axis between $y = a$ and $y = b$ is $\int_a^b x dy$, where x is to be expressed in terms of y.

Volume of a solid of revolution where the x-axis is the axis of rotation = $\int_a^b \pi y^2 dx$.

If the y-axis is the axis of revolution, volume = $\int_c^d \pi x^2 dy$, where c and d are the limits for y.

Exercise 18.6 Miscellaneous

1. Find the area enclosed between the curve $y = x^2 – x – 2$ and the x-axis.

2. Find the area enclosed by the curve $y = x^2 + 1$ and the line $y = x + 7$.

3. Find the volume generated when the area (above the x-axis) between the curve $y = \sqrt{3(x-2)}$, the x-axis and the line $x = 3$ is rotated about the x-axis through 360^o.

4. Find the area between the curves $y = x^2 - 2x + 3$ and $y = 6 - x - x^2$.

5. Find the area (in the first quadrant) enclosed by the curves $y = \sin x$, $y = \cos x$ and the y-axis.

6. Sketch the curve $y = x(x - l)(x - 2)$. Find the equation of the tangent to the curve at the point where $x = 1$. Calculate the numerical area between the curve and the part of the x-axis from $x = 0$ to $x = 2$.

7. Find the area enclosed between the curves $y^2 = x$ and $x^2 = 8y$.

8. A particle moves along a straight line so that its distance (s) from a fixed point O on the line after t s is given by $s = t^3 - 12t^2 + 45t$. Find the distances from O when the particle is momentarily at rest and the accelerations at these times.

9. Find the area between the curve $y = 2\cos x + \cos 2x$, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{4}$.

10. Sketch the curve $y = 6x - x^2$. Calculate the area contained by this curve, the tangent to the curve at the point where $x = 2$ and the y-axis.