# Calculus (4): Differentiation of Trigonometrical Functions

In this Chapter we extend our technique of differentiation to include the trigonometrical functions. We begin by finding a very important limit which is required later:

$$\lim_{x \to 0} \frac{\sin x}{x}.$$

Look at the sine table in your book of tables, where the angle is small and compare the value of sin x with x (expressed in radians).

| Angle $x$ | | $\sin x$ |
|---|---|---|
| in degrees | in radians | |
| 10 | 0.1745 | 1.1736 |
| 5 | 0.0873 | 0.0872 |
| 3 | 0.0524 | 0.0523 |
| 1 | 0.0175 | 0.0175 |
| 30' | 0.00873 | 0.0087 |
| 6' | 0.00175 | 0.0017 |
| 3' | 0.0009 | 0.0009 |

Clearly when x is small, say less than $5^o$ , $\sin x \approx x$ or $\frac{\sin x}{x} \approx 1$ provided x is in radians. It is impossible to make any more precise comparison using 4 figure tables, but it would seem reasonable to predict that $\lim \frac{\sin x}{x} = 1$, $x \to 0$ provided we work in radian measure.

Here is a simple proof of this result.

In **fig 16.1** OAB is a sector of a circle radius r, AOB (acute) = θ radians, AB is a chord and AC is the tangent at A.

$AC = r \tan \theta$.

Then $\Delta AOB <$ area of sector $OAB < \Delta OAC$.

i.e., $\frac{1}{2} r^2 \sin \theta < \frac{1}{2} r^2 \theta < \frac{1}{2} r^2 \tan \theta$

hence $1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta}$.

Now as $\theta \to 0$, $\cos \theta \to 1$ and $\frac{1}{\cos \theta}$ also $\to 1$.

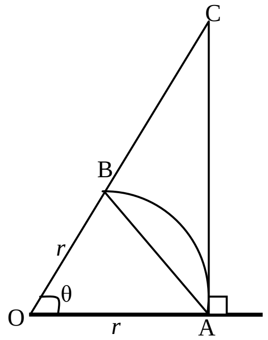


**Fig 16.1**

The left hand term of the inequality is fixed at I and the right hand term decreases and tends to 1, as $\theta \to 0$, and the middle term must also tend to 1.

Hence $\lim_{\theta \to 0} \frac{\theta}{\sin \theta} = 1$ and reverting to x as the variable

$$\lim_{\theta \to 0} \frac{\sin x}{x} = 1 \qquad \text{where x is in rad.}$$

This result plays a crucial part in finding the derivative of sin x. To take advantage of it, we must work in radians throughout.

**Derivative of sin x**

Working as before from first principles (Chapter 9) let y = sin x, x is in radians. Take an increment $\delta x$ in x to produce a corresponding increment $\delta y$ in y.

Then $\quad y + \delta y = \sin(x + \delta x)$.

Hence $\quad \delta y = \sin(x + \delta x) - \sin x$

$\quad = 2\cos(x + \tfrac{1}{2}\delta x)\sin(\tfrac{1}{2}\delta x)$

using one of the factor formulae.

Therefore $$\frac{\delta y}{\delta x} = 2\cos\left(x + \frac{\delta x}{2}\right)\frac{\sin\left(\frac{\delta x}{2}\right)}{\delta x}$$

$$= \cos\left(x + \frac{\delta x}{2}\right)\frac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)}.$$

Now as $\delta x \to 0$, $\quad \frac{\delta y}{\delta x} \to \frac{\delta y}{\delta x}$,

$$\cos\left(x + \frac{\delta x}{2}\right) \to \cos x$$

and using the above limit,

$$\frac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)} \to 1.$$

Hence $\quad \frac{\delta y}{\delta x} = \cos x.$

$$\frac{d(\sin x)}{\delta x} = \cos x \qquad \text{whre x is in rad.}$$

The gradient at any point on the sine curve (y = sin x) is the value of cos x for that value of x. A comparison of the two curves **(fig 16.2)** illustrates this.

Composite functions involving the sine can now be differentiated.

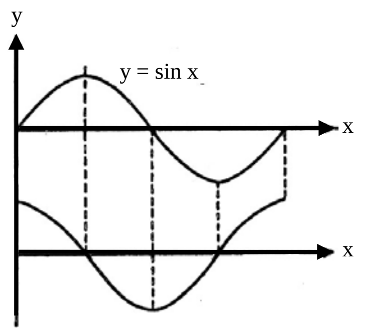


**Fig 16.2**

**Example 1**

$$\frac{d(\sin 3x)}{\delta x} = \cos 3x \times 3.$$

First differentiate sin … wrt (3x) and then differentiate 3x wrt x.

So, if $\quad y = \sin(2x + 5)$,

$$\frac{\delta y}{\delta x} = [\cos(2x + 5)] \times 2$$
$$= 2\cos(2x + 5).$$

Again, if $y = \sin^2 3x$, $\frac{\delta y}{\delta x} = 2 \sin3x \times \cos3x \times 3$

| | | |
|---|---|---|
| Differentiate $\sin^2 3x$ as a power wrt sin3x | differentiate sin3x wrt 3x | differentiate 3x wrt x |

$$= 6 \sin3x \cos3x$$
$$= 3 \sin6x.$$

**Example 2**

$y = \sin x^o$. Find $\frac{\delta y}{\delta x}$.

We must first convert the angle to radians.

$$x^o = \frac{\pi}{180} \text{ x radians.}$$

Hence $y = \sin x^o = \sin \frac{\pi}{180} x.$

Then $\frac{\delta y}{\delta x} = \frac{\pi}{180} \cos \frac{\pi}{180} x = \frac{\pi}{180} \cos x^o.$

## Derivative of cos x

Since $\cos x = \sin(\frac{\pi}{2} - x)$,

$$\frac{d(\cos x)}{\delta x} = \cos(\frac{\pi}{2} - x)\times(-1)$$
$$= -\cos(\frac{\pi}{2} - x)$$
$$= -\sin x.$$

Alternatively we can work again from first principles.

Let $y = \cos x$, x is in radians, then $y + \delta y = \cos(x + \delta x)$ taking as before increments $\delta x$ and $\delta y$.

Then $\delta y = \cos(x + \delta x) - \cos x$
$$= -2\sin(x + \tfrac{1}{2}\delta x)\sin(\tfrac{1}{2}\delta x).$$

Therefore $\frac{\delta y}{\delta x} = -2\sin(x + \tfrac{1}{2}\delta x)\frac{\sin(\frac{\delta x}{2})}{\delta x}$
$$= -\sin(x + \tfrac{1}{2}\delta x)\frac{\sin(\frac{\delta x}{2})}{(\frac{\delta x}{2})}$$

And hence in the limit, when $\delta x \to 0$, $\frac{\delta y}{\delta x} = -\sin x$.

$$\frac{d(\cos x)}{\delta x} = \sin x \qquad \text{where x is in rad.}$$

**Example 3**

If $y = \cos 5x$, $\frac{\delta y}{\delta x} = -\sin 5x \times 5 = -5\sin 5x.$

If $y = \cos^2(2x - 3)$, $\frac{\delta y}{\delta x} = 2\cos(2x - 3) \times [-\sin(2x - 3)] \times 2$

$$= \quad - 4 \cos (2x - 3) \sin (2x - 3)$$
$$= \quad - 2 \sin 2 (2x - 3).$$

The methods previously used for differentiating products, quotients, and for finding maximum and minimum values can also be applied where necessary.

## Derivative of tan x

**Example 4**

Find the derivative of tan x.

If $y = \tan x \quad = \quad \frac{\sin x}{\cos x}$, then by the quotient rule,

$$\frac{dy}{dx} \quad = \quad \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x}$$

$$= \quad \frac{\cos^2 x \sin^2 x}{\cos^2 x}$$

$$= \quad \frac{1}{\cos^2 x}.$$

Hence $\frac{d(\tan x)}{\delta x} \quad = \quad \sec^2 x$ where x is in rad.

**Example 5**

If $y = (1 + x^2) \sin 2x$, find $\frac{dy}{dx}$ and its value when $x = \frac{\pi}{2}$.

By the product rule,

$$\frac{dy}{dx} \quad = \quad (1 + x^2)\, 2 \cos 2x + (2x) \sin 2x.$$

When $x = \frac{\pi}{2}$, $\frac{dy}{dx} \quad = \quad (1 + \frac{\pi^2}{2})2 \cos (2 \times \frac{\pi}{2}) + (2 \times \frac{\pi}{2}) \sin (2 \times \frac{\pi}{2})$

$$= \quad -2(1 + \frac{\pi^2}{2}) \text{ as } \cos \pi = -1 \text{ and } \sin \pi = 0.$$

**Example 6**

If $xy = \sin 2x$, prove that $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4xy = 0$.

Differentiate wrt x, treating xy as a product:

$$x\frac{dy}{dx} + y \quad = \quad 2\cos 2x.$$

Now differentiate again wrt x, treating $x\frac{dy}{dx}$ as a product:

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} \quad = \quad - 2 \sin 2x \times 2$$
$$= \quad - 4\sin 2x$$
$$= \quad - 4xy$$

Hence $\quad \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4xy = 0.$

**Example 7**

Differentiate $\tan^3 2x$.

If $\quad y \quad = \quad \tan^3 2x$,

$$\frac{dy}{dx} = 3\tan^2 2x \times \sec^2 2x \times 2$$
$$= 6\tan^2 2x \sec^2 2x.$$

($\tan^3$ … is first differentiated as a power, then tan … is differentiated, and finally 2x is differentiated).

**Example 8**

Find the maximum and minimum value of 3 sin x + 4 cos x and the values of x at which they occur.

Take $y = 3\sin x + 4\cos x.$

$$\frac{dy}{dx} = 3\cos x - 4\sin x$$

and $\frac{dy}{dx} = 0$ at the turning points.

If $\frac{dy}{dx} = 0$, then $3\cos x - 4\sin x = 0$ or $\tan x = \frac{3}{4}$.

In the range $0^o - 360^o$ this gives $x = 36^o\ 52'$ or $216^o\ 52'$.

Our usual test to distinguish between the maximum and minimum values is a little awkward in this case, but we can ease the working if we modify the expression for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = 3\cos x - 4\sin x$$
$$= \cos x\,(3 - 4\tan x)$$
$$= 4\cos x\,(\tfrac{3}{4} - \tan x). \quad \text{(see table below)}$$

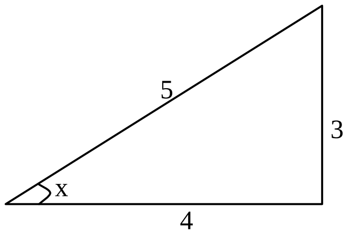


**Fig 16.3**

When $\tan x = \frac{3}{4}$, $\sin x = \frac{3}{5}$ and $\cos x = \frac{4}{5}$ (x in the first quadrant) (fig 16.3) or $\sin x = -\frac{3}{5}$, $\cos x = -\frac{4}{5}$ (x in the third quadrant). Therefore the maximum value of y when $x = 36^o\ 52'$ is $3(\frac{3}{5}) + 4(\frac{4}{5})$ = and the minimum value of y when $x = 216^o\ 52'$ is -5. (These values will recur every $360^o$).

| $x$ | $<36^o\ 52'$ | $36^o\ 52'$ | $>36^o\ 52'$ | $<216^o\ 52'$ | $216^o\ 52'$ | $>216^o\ 52'$ |
|---|---|---|---|---|---|---|
| | $\cos x > 0$<br>$\tan x < 0{\cdot}75$ | | $\cos x > 0$<br>$\tan x > 0{\cdot}75$ | $\cos x < 0$<br>$\tan x < 0{\cdot}75$ | | $\cos x < 0$<br>$\tan x > 0{\cdot}75$ |
| $\frac{dy}{dx}$ | +<br>/ | 0<br>— | –<br>\ | –<br>\ | 0<br>— | +<br>/ |
| | | maximum | | | minimum | |

**Example**

If y = $x^3$ + cos x –In + 4

$$\frac{dy}{dx} = 3x^2 - \sin x - \frac{1}{x}$$

If $y = 7x^4$

$$\frac{dy}{dx} = 7(4x^3)$$

$$= 28x^3$$

**Example**

$y = xe^x$

let $u = x$ and $v = e^x$

Then $\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = e^x$

Using $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{du}{dx}$

$$\frac{dy}{dx} = e^x.1 + x.e^x$$

$$= e^x(1+x)$$

**Example**

(a) $(2x^3 - 1)\sin x$

(b) $\frac{\text{In}(5x)}{x^2}$.

(a) Let $y = (2x^3 - 1)\sin x$, a product,

So let $u = 2x^3 - 1$ and $v = \sin x$.

Now $\frac{du}{dx} = 6x^2$ and $\frac{dv}{dx} = \cos x$,

Using $\frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$,

$$\frac{dy}{dx} = \sin x.\ 6x^2 + (2x^3 - 1).\cos x$$

$$= 2x^3 \cos x + 6x^2 \sin x - \cos x.$$

(b) Let $y = \frac{\text{In}(5x)}{x^2}$, a quotient,

So let $u = \text{In}(5x)$ and $v = x^2$

Now $\frac{du}{dx} = 5\frac{1}{5x}$ and $\frac{dv}{dx} = 2x$

Using $\frac{du}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{x^2\frac{1}{x} - \text{In}(5x).2x}{x^4}$$

$$= \frac{1 - 2\text{In}(5x)}{x^3}$$

**Example**

A curve has equation $y = \sin^{-1} x$. Find $\frac{dy}{dx}$.

$y = \sin -1\ x \Rightarrow x = \sin y$.

So, $\dfrac{dx}{dy} = \cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$ .

Hence , $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}} = \dfrac{1}{\sqrt{1-x^2}}$ .

**Example**

Given y = secx, find $\dfrac{dy}{dx}$

Y = secx = $\dfrac{1}{\cos x}$

Using the quotient rule (see P20),

$$\frac{dy}{dx} = \frac{\cos x.0 - 1.(-\sin x}{co^2 x})$$

$$= \frac{\sin x}{co^2 x}$$

$$= \frac{1}{\cos x} \tan x$$

= sec x tan x.

## Example

If y = $x^6 + 4x^2 - \dfrac{3}{x}$

$\dfrac{dy}{dx} = 6x^5 + 8x + \dfrac{3}{x^2}$ first derivative

$\dfrac{d^2y}{dx^2} = 30x^4 + 8 - \dfrac{6}{x^3}$ second derivative

$\dfrac{d^3y}{dx^3} = 120x^3 + \dfrac{18}{x^4}$ third derivative

## Standard results

Algebraic

| function | derivative |
|---|---|
| constant | 0<br>$nx^{n-1}$ |
| sin x<br>cos x<br>tan x | cos x<br>-sin x<br>$\sec^2 x$ |

Trigonometrically
(x in radians)

| function | derivative |
|---|---|
| In x<br>$e^x$ | $\dfrac{1}{x}$<br>$e^x$ |

| function | derivative |
|---|---|
| cosec x<br>sec x<br>cot x | -cosec x cot x<br>sec x tan x<br>$-\ \text{cosec}^2 x$ |
| $\sin^{-1} x$<br>$\cos^{-1} x$ | $\dfrac{1}{\sqrt{1-x^2}}$<br>$-1$ |

**Exercise 16**

Differentiate wrt x (x in radians):

1. sec x [take as $(\cos x)^{-1}$ ] 2. cosecx 3. cotx
4. sin 4x 5. cos 6x 6. tan2x 7. cot3x
8. cos(2x – 8) 9. $\sin(3x + \frac{\pi}{3})$ 10. xtan x
11. xcos2x 12. xsin3x 13. cos2x + sin x
14. sin x – cos x 15. $\cos^2 x$ 16. $\sin^2 5x$
17. $\tan^2 x$ 18. $\sin^2 (x - \frac{\pi}{4})$ 19. $4x^2 + \sin 4x$
20. $\frac{\sin x}{1+\cos x}$ 21. $\frac{\cos x + \sin x}{\cos x - \sin x}$ 22. $\frac{2}{1+\cos 2x}$
23. sinx tan x 24. $\sec^2 x$ 25. $\tan\frac{x}{2}$ 26. $x^2 \tan x$
27. $(1 + x^2)\tan x$ 28. $\frac{1+x}{\sin x}$ 29. $2\cos^2 x - \sin^2 x$
30. (1 + sin x)(1 – cosx)
31. If y = sin 2x, find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$.
32. Find the maximum and minimum values of 2 cosx + sin x and the values of x at which they occur (in the range $0^o$ to $360^o$).
33. If x cosy = sin x, prove that $\frac{dy}{dx} = \frac{\cos y(\cos y - \cos x)}{\sin x \sin y}$.
34. Given that y = x sin 2x, prove that $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y + 4x^2y = 0$.
35. Given that y = $\frac{x}{2+\cos x}$, find the values of $\frac{dy}{dx}$ when x = 0, $\frac{\pi}{4}$ and π.
36. If y = 3x sin 3x + cos3x, show that $x\frac{d^2y}{dx^2} + 9xy = 2\frac{dy}{dx}$.
37. A particle is moving In a straight line and its distance s from a fixed point of the line .after t s is given by s = sin2t.
    Find its velocity and acceleration at this time and prove that its acceleration is always numerically 4 times its distance from the fixed point and directed towards the point. What is Its velocity and acceleration at times 0, $\frac{\pi}{4}$, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, 2π s? Make a diagrammatic sketch of the motion. (Such a motion is called Simple Harmonic Motion).

38. If y = P cos2x + Q sin 2x, where P and Q are constants, prove that $\frac{d^2y}{dx^2} + 4y = 0$. if y = 1 when x = 0 and $\frac{dy}{dx} = 2$ when $x = \frac{\pi}{2}$ find the values of P and Q.

39. If y = 2 cosx + 3 sin x – cos2x, prove that $\frac{d^2y}{dx^2} + y = 3\cos 2x$.

40. If y = sin x + 3 cos2x, solve the equation $\frac{dy}{dx} = 0$ in the range 0 to $2\pi$ and hence find the maximum and minimum values of y in that range.

41. If y = tan 2x, prove that $\frac{dy}{dx} = 2(1 + y^2)$.

42. If $y = \frac{3\sin 2x}{x}$, prove that $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 3xy = 0$.

43. Prove that the maximum value of cosx – sin x $\sqrt{2}$.

44. If y = sin x, prove that $\dfrac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}} = -\frac{2}{3\sqrt{3}}$ when $x = \frac{\pi}{4}$.

45. If r = 1 – cos θ, use the result of No 33 in Ex. 10.5 to prove that $r\frac{d\theta}{dr} = \cot\frac{\theta}{2}$.

**SUMMARY**

$\lim_{x \to 0} \frac{\sin x}{x} = 1$ (x in radians).

**Note:** The following are only true if x is in radians.

$\frac{d(\sin x)}{dx} = \cos x$; $\frac{d(\sin x)}{dx} = -\sin x$;

$\frac{d[\sin(ax+b)]}{dx} = a\cos(ax+b)$; $\frac{d[\cos(ax+b)]}{dx} = -a\sin(ax+b)$

$\frac{d(\tan x)}{dx} = \sec^2 x$; $\frac{d[\tan(ax+b)]}{dx} = a\sec^2(ax+b)$;