OUR LADY OF AFRICA S.S NAMILYANGO (OLAN)

A LEVEL PURE MATHEMATICS SEMINAR SOLUTIONS 2023

1(a).	**ALGEBRA** Let $\sqrt{14+6\sqrt{5}} = \pm(\sqrt{a}+\sqrt{b})$ $\left(\sqrt{14+6\sqrt{5}}\right)^2 = \left(\pm(\sqrt{a}+\sqrt{b})\right)^2$ $14+6\sqrt{5} = a+2\sqrt{ab}+b$ $2\sqrt{ab} = 6\sqrt{5}$ $ab = 45,\ \ a = \frac{45}{b}$ $a+b = 14 \quad ; \quad \frac{45}{b}+b = 14$ $45+b^2 = 14b$ $b^2-15b+45 = 0$ $(b-9)(b-5) = 0$ $b = 9\ or\ b = 5$ $when\ b = 9, a = \frac{45}{9} = 5$ $when\ b = 5, a = \frac{45}{5} = 9$ $\therefore \sqrt{14+6\sqrt{5}} = \pm(\sqrt{5}+\sqrt{9}) = \pm(3+\sqrt{5})$
1(b).	$4x^4+17x^3+8x^2+17x+4 = 0$ $dividing\ through\ by\ x^2$ $4x^2+17x+8+\frac{17}{x}+\frac{4}{x^2} = 0$ $4\left(x^2+\frac{1}{x^2}\right)+17\left(x+\frac{1}{2}\right)+8 = 0$ $From\ \ y = x+\frac{1}{x}$ $Squaring\ both\ sides,$ $y^2 = x^2+2+\frac{1}{x^2}$ $\Rightarrow x^2+\frac{1}{x^2} = y^2-2$ $4(\ y^2-2)+17y+8 = 0$ $4y^2-8+17y+8 = 0$ $4y^2+17y = 8$ $y(4y+17) = 0$ $Either\ y = 0\ or\ y = -\frac{17}{4}$ $When\ y = 0;$ $x+\frac{1}{x} = 0$ $x^2+1 = 0$ $x^2 = -1, x\ is\ undefined$ $When\ y = -\frac{17}{4}$ $x+\frac{1}{x} = -\frac{17}{4}$ $4x^2+4 = -17x$

	$4x^2+17x+4=0$ $(4x+1)(x+4)=0$ $Either\ x=-\frac{1}{4}\ \ or\ x=-4$ $\therefore x=-\frac{1}{4}\ \ and\ \ x=-4$
1(c).	$(1+x)^n=1+n(x)+\frac{n(n-1)}{2!}+\cdots$ $\sqrt{(1+x)(1+x^2)}=(1+x)^{\frac{1}{2}}(1+x^2)^{\frac{1}{2}}$ $(1+x)^{\frac{1}{2}}=1+\frac{1}{2}x+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2+\cdots$ $(1+x)^{\frac{1}{2}}=1+\frac{1}{2}x-\frac{1}{8}x^2+\cdots$ $(1+x^2)^{\frac{1}{2}}=1+\frac{1}{2}\ x^2+\cdots$ $(1+x)^{\frac{1}{2}}(1+x^2)^{\frac{1}{2}}=\left(1+\frac{1}{2}x-\frac{1}{8}x^2+\cdots\right)\left(1+\frac{1}{2}\ x^2+\cdots\right)$ $(1+x)^{\frac{1}{2}}(1+x^2)^{\frac{1}{2}}=1+\frac{1}{2}x-\frac{1}{8}x^2+\cdots+\frac{1}{2}\ x^2+\cdots$ $(1+x)^{\frac{1}{2}}(1+x^2)^{\frac{1}{2}}=1+\frac{1}{2}x+\frac{3}{8}x^2+\cdots$
1(d).	$x^2+3x+2=0$ $old\ roots\ are\ k\ and\ l$ $Sum\ of\ old\ roots;\ \ k+l=-3$ $Product\ of\ old\ roots; kl=2$ $Sum\ of\ new\ roots;\ \frac{k}{l^2}+\frac{l}{k^2}=\frac{k^3+l^3}{(kl)^2}=\frac{(k+l)^3-3kl(k+l)m}{(kl)^2}=\frac{(-3)^3-3(2)(-3)}{(2)^2}=-\frac{9}{4}$ $Product\ of\ new\ roots;\ ;\ \frac{k}{l^2}.\frac{l}{k^2}=\frac{kl}{(kl)^2}=\frac{1}{kl}=\frac{1}{2}$ $Required\ equation;\ x^2-\left(\frac{-9}{4}\right)x+\frac{1}{2}=0$ $4x^2+9x+2=0$
2(a).	$\sum_{r=1}^{n}(3r+1)(r+2)=n(n+2)(n+3)$ $12+28+\cdots+(3n+1)(n+2)=n(n+2)(n+3)$ $For\ n=1$ $LHS=12;\ \ RHS=12; since\ LHS=RHS=12\ so\ it\ holds\ true\ for\ n=1$ $For\ n=2$ $L.H.S=12+28=40;\ RHS=40; Since\ LHS=RHS=40, so\ it\ holds\ true\ for\ n=2$ $Assuming\ it\ holds\ for\ n=k;\ 12+28+\cdots+(3k+1)(k+2)=k(k+2)(k+3)\ldots\ldots\ldots(i)$ $For\ n=k+1; RHS=\ k(k+2)(k+3)+(3(k+1)+1)(k+3)$ $=(k+3)[k(k+2)+3k+4]$ $=(k+3)[k^2+5k+4]$ $=(k+3)(k+1)(k+4); for\ n=k+1$ $=(k+1+2)(k+1)(k+1+3)=n(n+1)(n+3)$ $Since\ it\ holds\ true\ for\ n=1, n=2, n=k, and\ n$ $=k+1\ then\ it\ holds\ true\ for\ all\ positive\ values\ of\ n$
2(b).	$f(n)=4^n+5^n+6^n$ $For\ n=1; f(1)=4+5+6=15\ which\ is\ divisible\ by\ 15$ $For\ n=k; f(k)=4^k+5^k+6^k=15a_k;\ \ 4^k=15a_k-\left(5^k+6^k\right)\ldots\ldots\ldots(i)$ $For\ n=k+2; f(k+2)=4^{k+2}+5^{k+2}+6^{k+2}$ $=16.4^k+25.5^k+36.6^k$

	$From\ (i); f(k+2) = 16\left[15a_k - \left(5^k + 6^k\right)\right] + 25.5^k + 36.6^k$ $= 240a_k - 16.5^k - 16.6^k + 25.5^k + 36.6^k$ $= 240a_k + 20.6^k - 9.5^k$ $= 240a_k - 9 \times 5.5^{k-1} + 20 \times 6.6^{k-1}$ $= 15[16a_k - 3.5^{k-1} + 8.6^{k-1}]$ $Since\ the\ statement\ is\ true\ for\ n = 1,$ $n = k\ and\ n = k+1, then\ it\ is\ true\ for\ all\ positive\ odd\ integers, n.$
2(c).	$(cos\theta + isin\theta)^5$ $= cos^5\theta + 5cos^4\theta(isin\theta) + 10cos^3\theta(isin\theta)^2 + 10cos^2\theta(isin\theta)^3 + 5cos\theta(isin\theta)^4 + (isin\theta)^5$ $cos5\theta + isin5\theta$ $= cos^5\theta + 5icos^4\theta sin\theta - 10cos^3\theta sin^2\theta - 10icos^2\theta sin^3\theta + 5cos\theta sin^4\theta + isin^5\theta$ $For\ immaginary\ part;$ $sin5\theta = 5cos^4\theta sin\theta - 10cos^2\theta sin^3\theta + sin^5\theta$ $sin5\theta = 5(1 - sin^2\theta)^2 sin\theta - 10(1 - sin^2\theta)sin^3\theta + sin^5\theta$ $sin5\theta = 5(1 - 2sin^2\theta + sin^4\theta)sin\theta - 10(1 - sin^2\theta)sin^3\theta + sin^5\theta$ $sin5\theta = 5sin\theta - 10sin^3\theta + 5sin^5\theta - 10sin^3\theta + 10sin^5\theta + sin^5\theta$ $sin5\theta = 5sin\theta - 20sin^3\theta + 16sin^5\theta$
2(d).	$sin5\theta = 5sin\theta - 20sin^3\theta + 16sin^5\theta; 2 + 10x - 40x^2 + 32x^5 = 0$ $Comparing\ the\ two\ equations;\ -1 = 5x - 20x^2 + 16x^5$ $\Rightarrow sin5\theta = -1; 5\theta = sin^{-1}(1)$ $5\theta = -90^0, 270^0, 630^0, 990^0, 1350^0, 1710^0$ $\theta = -18^0, 54^0, 126^0, 198^0, 270^0, 342^0$ $x = \sin(-18^0) = -0.3090$ $x = \sin(54^0) = 0.8090$ $x = \sin(126^0) = 0.8090$ $x = \sin(198^0) = -0.3090$ $x = \sin(270^0) = -1$ $x = \sin(342^0) = -0.3090$ $\therefore x = -0.3090, 0.8090, -1$
3(a).	$S_n = 2^{2n} - n$ $For\ n = 1;\ S_1 = 2^2 - 1$ $S_1 = 3;\ U_1 = 3$ $For\ n = 2;\ S_2 = 2^4 - 2 = 14$ $U_1 + U_2 = 14$ $3 + U_2 = 14;\ \ U_2 = 11$ $For\ n = 3;\ S_3 = 2^6 - 3 = 64 - 3 = 61$ $U_1 + U_2 + U_3 = 61$ $14 + U_3 = 61;\ \ U_3 = 47$ $the\ first\ three\ terms\ are\ 3, 11, 47$
3(b).	$Selections\ made$ $^{14}C_5$ and $^{10}C_5$ or $^{14}C_8$ and $^{10}C_2$ $=$ $^{14}C_5$ x $^{10}C_5$ + $^{14}C_8$ x $^{10}C_2$ $= (2002)(252) + (3003)(45)$ $= 504504 + 135135$ $= 639639\ ways$
3(c).	$p^3 + q^3 = 4$ and $pq = \frac{1}{2}(p^3 + q^3) + 1$ $Sum\ of\ roots = P^6 + P^6 = (p^3 + q^3)^2 - 2(pq)^2 = 4^2 - 2\left[\frac{1}{2}(4) + 1\right]^3 = -38$ $Product\ of\ roots = P^6 \times q^6 = (pq)^6 = \left(\frac{1}{2}(p^3 + q^3) + 1\right)^6 = \left(\frac{1}{2}(4) + 1\right)^6 = 729$

<table>
<tr><td></td><td>$x^2 - (sum\ of\ roots\ )x + product\ of\ roots = 0$
$x^2 + 38x + 729 = 0$</td></tr>
<tr><td>3(d).</td><td>$From\ A = \frac{PR(R^n - 1)}{R - 1}$
$Given\ P = 800{,}000,\ R = 1 + r = 1.05$
$n = 4\ number\ of\ times\ interest\ has\ been\ calculated$
$A = 800{,}000(1.05)\frac{[1.05^4 - 1]}{1.05 - 1}$
$A = shs\ 3{,}620{,}505$</td></tr>
<tr><td>4(a).</td><td>$\frac{x+3}{X-2} = \frac{x+1}{x-2} \geq 0$
$\frac{(x+3)(x-2) - (x+1)(x-2)}{(x-2)^2} \geq 0$
$\frac{x^2_{2x} + 3x - 6 - (x^2 - 2x + x - 2)}{(x-2)^2} \geq 0$
$\frac{x^2 + x - 6 - x^2 + x + 2}{(x-2)^2} \geq 0$
$\frac{2(x-4)}{(x-2)^2} \geq 0;\ \ x = 2$
$\frac{2}{x-2} \geq 0$
<table><tr><td></td><td>$x < 2$</td><td>$x > 2$</td></tr><tr><td>$x - 2$</td><td>$-$</td><td>$+$</td></tr><tr><td>$\frac{2}{x-2}$</td><td>$-$</td><td>$+$</td></tr></table>$x \geq 2$</td></tr>
<tr><td>4(b)
(i)

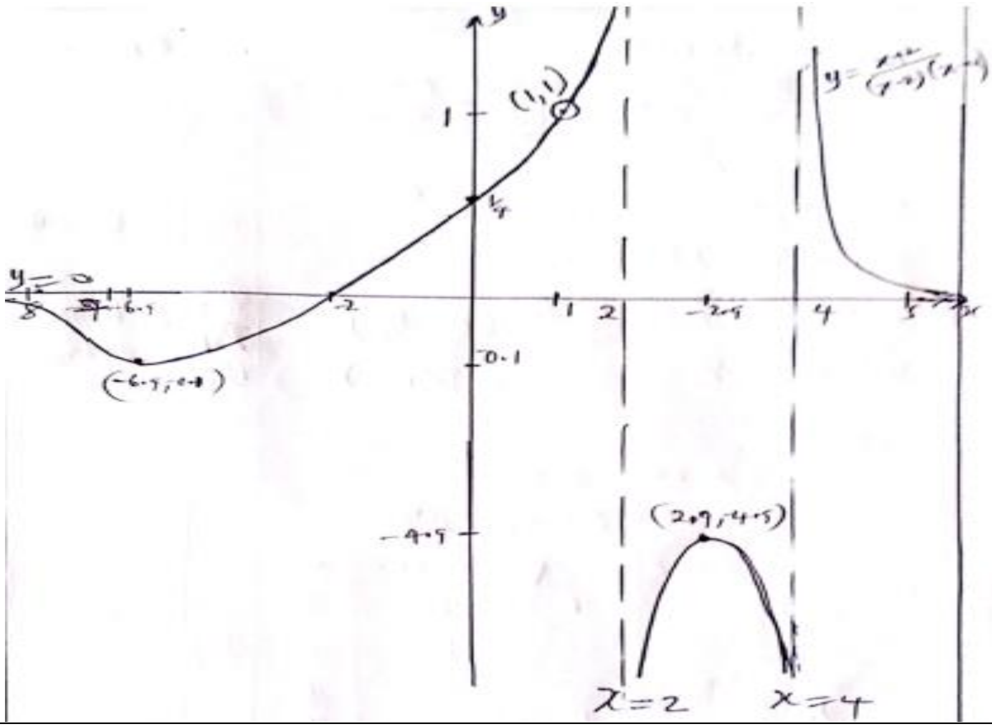
(ii)

(iii)</td><td>$y = \frac{x^2 + x - 2}{x^{3-7x^2} + 14x - 8} = \frac{(x+2)(x-1)}{(x-1)(x-2)(x-4)}$
$y = \frac{x+2}{(x-2)(x-4)}$
$x - 1 = 0; x = 2$
$y = \frac{1+2}{(1-2)(1-4)} = \frac{3}{3} = 1$
$\therefore\ Coordinates\ of\ the\ hole, (1{,}1)$

$Vertical\ asymptotes; As\ y \to \infty, (x-2)(x-4) \to 0$
$\Rightarrow (x-2)(x-4) = 0$
$x = 2\ and\ x = 4$
$Horizontal\ asymptote$
$y = \frac{x+2}{(x-2)(x-4)}$
$yx^2 - (6y+1)y + 8y - 2 = 0$
$x = \frac{6y + 1 \pm \sqrt{(6y+1)^2 - 4y(8y-2)}}{2y}$
$As\ x \to \infty, 2y \to 0$
$\Rightarrow y = 0$
$y = \frac{x+2}{(x-2)(x-4)}$
$\frac{dy}{dx} = \frac{(x^2 - 6x + 8) - (x+2)(2x-6)}{(x^2 - 6x + 8)^2} = \frac{20 - 4x - x^2}{(x^2 - 6x + 8)^2}$
$\Rightarrow 20 - 4x - x^2 = 0$</td></tr>
</table>

$$x^2 + 4x - 20 = 0$$
$$either\ x = 2.9, x = -6.9,$$
$$when\ x = 2.9\ , y = -4.9, (2.9, -4.9) - maximum\ point$$
$$when\ x = -6.9, y = -0.1; (-6.9, -0.1) - Minimum\ point$$
$$Intercepts, x = 0\ \ y = \frac{1}{4}\ ; \left(0, \frac{1}{4}\right)$$
$$y = 0, x = -2\ ;\ (-2,0)$$
$$y = \frac{x+2}{(x-2)(x-4)}$$
$$Critical\ values;\ -2, 2, 4$$

	$x < -2$	$-2 < x < 2$	$2 < x < 4$	$x > 4$
$x + 2$	-	+	+	+
$x - 2$	-	-	+	+
$x - 4$	-	-	-	+
y	-	+	-	+



5(a).

TRIGONOMETRY

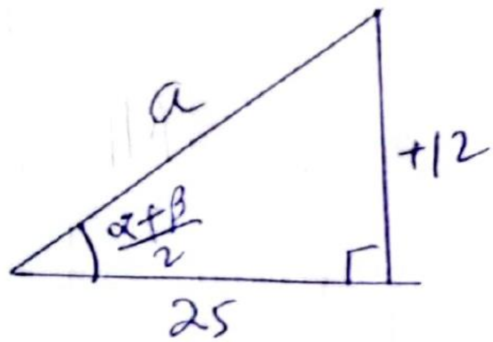
$$For\ \ t = tan\frac{x}{2};\ \cos x = \frac{1-t^2}{1+t^2};\quad sinx = \frac{2t}{1+t^2}$$
$$f(x) = \frac{3}{13 + 6\left(\frac{2t}{1+t^2}\right) - 5\left(\frac{1-t^2}{1+t^2}\right)} = \frac{3(1+t^2)}{13(1+t^2) + 12t - 5 - 5t^2}$$
$$= \frac{3(1+t^2)}{2(9t^2 + 6t + 1) + 6} = \frac{3(1+t^2)}{2(3t+1)^2 + 6}$$

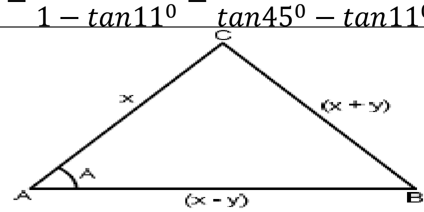
5(b).

$$cot^2\theta + 3cosec^2\theta = 7$$
$$cot^2\theta + 3(1 + cot^2\theta) = 7$$
$$4cot^2\theta + 3 = 7$$
$$4cot^2\theta = 4;\ \ cot^2\theta = 1$$
$$tan^2\theta = 1;\ \ tan = \pm 1$$

5(c)(i).	$y = 3cosx - \sqrt{3}sinx$; $Rcos(\theta+\alpha)$ $R = \sqrt{3^2 + (\sqrt{3})^2} = \sqrt{12} = 2\sqrt{3}$ $\alpha = tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = 30^0$ $\therefore y = 2\sqrt{3}\cos(\theta + 30^0)$
(ii).	$At\ minimum;\ \cos(\theta+30^0) = -1$ $\theta + 30^0 = 180^0, 360^0$ $\theta = 150^0, 330^0$ $\theta = \frac{5}{6}\pi, \quad \frac{11}{6}\pi$ $For\ \ \theta = \frac{5}{6}\pi;\ \ y = 2\sqrt{3}(-1) = -2\sqrt{3}\ ; Point\ \left(\frac{5}{6}\pi, -2\sqrt{3}\right)$ $For\ \ \theta = \frac{11}{6}\pi;\ \ y = 2\sqrt{3}(-1) = -2\sqrt{3}\ ; Point\ \left(\frac{11}{6}\pi, -2\sqrt{3}\right)$
(iii).	$At\ the\ point\ where\ the\ curve\ y = 0$ $2\sqrt{3}\cos(\theta + 30^0) = 0$ $\theta + 30^0 = cos^{-1}(0)$ $\theta + 30^0 = 90^0, 270^0$ $x = 60^0, 240^0$ $x = \frac{\pi}{3}, \frac{4\pi}{3}$
6(a).	$Cos3\theta = \cos(2\theta + \theta)$ $= cos2\theta cos\theta - sin2\theta sin\theta$ $= (2cos^2\theta - 1)cos\theta - 2sin\theta cos\theta sin\theta$ $= 2cos^3\theta - cos\theta - 2sin^2\theta cos\theta$ $= 2cos^3\theta - cos\theta - 2(1 - cos^2\theta)cos\theta$ $= 2cos^3\theta - cos\theta - 2cos\theta - 2cos^3\theta$ $= 4cos^3\theta - 3cos\theta$ $cos\theta = \frac{1}{2}\left(a + \frac{1}{a}\right)$ $cos3\theta = 4\left[\frac{1}{2}\left(\left(a + \frac{1}{a}\right)\right)\right]^3 - 3.\frac{1}{2}\left(a + \frac{1}{a}\right) = \frac{1}{2}\left(a + \frac{1}{a}\right)\left[\left(a + \frac{1}{a}\right)^2 - 3\right]$ $= \frac{1}{2}\left(a + \frac{1}{a}\right)\left(a^2 + 2 + \frac{1}{a^2} - 3\right) = \frac{1}{2}\left(a + \frac{1}{a}\right)\left(a^2 + \frac{1}{a^2} - 1\right)$ $= \frac{1}{2}\left(a^3 + \frac{1}{a} - a + a + \frac{1}{a^3} - \frac{1}{a}\right) = \frac{1}{2}\left(a^3 + \frac{1}{a^3}\right)$
6(b).	Let $K = sin^2\frac{1}{2}A + sin^2\frac{1}{2}B + sin^2\frac{1}{2}C$ $= \frac{1}{2}(1 - cosA) + \frac{1}{2}(1 - cosB) + \frac{1}{2}(1 - cosC) = \frac{3}{2} - \frac{1}{2}(cosA + cosB + cosC)$ $= \frac{3}{2} - \frac{1}{2}\left(2cos\frac{A+B}{2}cos\frac{A-B}{2} + 1 - 2sin^2\frac{C}{2}\right)$ $A + B + C = 180; \frac{A+B}{2} = 90 - \frac{C}{2}$ $Cos\left(\frac{A+B}{2}\right) = cos\left(90 - \frac{C}{2}\right) = Sin\left(\frac{C}{2}\right)$ $K = \frac{3}{2} - \frac{1}{2}\left[2Sin\left(\frac{C}{2}\right)cos\left(\frac{A-B}{2}\right) - 2sin^2\frac{C}{2} + 1\right]$ $= \frac{3}{2} - \frac{1}{2}\left[2Sin\left(\frac{C}{2}\right)\left(cos\left(\frac{A-B}{2}\right) - Cos\left(\frac{A+B}{2}\right)\right) + 1\right]$

	$=\frac{3}{2}-\frac{1}{2}\left[2Sin\left(\frac{C}{2}\right)(-2)sin\left(\frac{A}{2}\right)sin\left(\frac{-B}{2}\right)+1\right]$ $but\ sin\left(\frac{-B}{2}\right)=-sin\left(\frac{B}{2}\right)$ $=\frac{3}{2}-\frac{1}{2}-\frac{1}{2}.4sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}=1-2sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}$
6(c).	$\sin(\theta+\alpha)=a;\quad \cos(\theta+\alpha)=\sqrt{1-a^2}$ $\sin(\theta+\beta)=b;\ \cos(\theta+\beta)=\sqrt{1-b^2}$ $\cos(\alpha-\beta)=cos[(\theta+\alpha)-(\theta+\beta)]$ $\cos(\alpha-\beta)=\cos(\theta+\alpha)\cos(\theta+\beta)+\sin(\theta+\alpha)\sin(\theta+\beta)$ $\cos(\alpha-\beta)=\sqrt{1-a^2}\sqrt{1-b^2}+ab$ $\cos(\alpha-\beta)=\mathrm{ab}+\sqrt{1-a^2-b^2+a^2b^2}$ $Cos2(\alpha-\beta)-4abCos(\alpha-\beta)=2cos^2(\alpha-\beta)-1-4abCos(\alpha-\beta)$ $=2Cos(\alpha-\beta)[Cos(\alpha-\beta)-2ab]-1$ $=2\left(\mathrm{ab}+\sqrt{1-a^2-b^2+a^2b^2}\right)\left(\mathrm{ab}+\sqrt{1-a^2-b^2+a^2b^2}-2ab\right)-1$ $=2\left(\sqrt{1-a^2-b^2+a^2b^2}+ab\right)\left(\sqrt{1-a^2-b^2+a^2b^2}-ab\right)-1$ $2[1-a^2-b^2+a^2b^2-a^2b^2]-1=2-2a^2-2b^2-1$ $\therefore Cos2(\alpha-\beta)-4abCos(\alpha-\beta)=1-2a^2-2b^2$
7(a).	$cos45^0=\frac{1}{\sqrt{2}}$ $\sin\left(292\tfrac{1}{2}^0\right)=-\tfrac{1}{2}\sqrt{2+\sqrt{2}}$ $Sin\left(270+22\frac{1}{2}^0\right)=sin270^0cos22\frac{1}{2}^0+cos270^0sin22\frac{1}{2}^0=-cos22\frac{1}{2}^0$ But $cos^2\theta=\tfrac{1}{2}(cos2\theta+1)$ $sin\left(292\frac{1}{2}^0\right)=-cos22\frac{1}{2}^0=-\sqrt{\frac{1}{2}\ (cos45^0+1)}=-\sqrt{\frac{1}{2}\left(\frac{1}{\sqrt{2}}+1\right)}$ $=-\sqrt{\frac{1}{2\sqrt{2}}\ (1+\sqrt{2})}=-\sqrt{\frac{\sqrt{2}}{4}\ (1+\sqrt{2})}=-\frac{1}{2}\sqrt{(\sqrt{2}+2)}$ $Sin\left(292\frac{1}{2}^0\right)=-\frac{1}{2}\sqrt{2+\sqrt{2}}$
7(b).	$cos\alpha-cos\beta=\frac{2}{5}$ $-2sin\frac{\alpha+\beta}{2}sin\frac{\alpha-\beta}{2}=\frac{2}{5}\dots\dots\dots\dots\dots\dots.(i)$ $sin\alpha-sin\beta=\frac{5}{6}$ $2cos\frac{\alpha+\beta}{2}sin\frac{\alpha-\beta}{2}=\frac{5}{6}\dots\dots\dots\dots\dots\dots(ii)$ $(i)\div(ii);tan\frac{\alpha+\beta}{2}=\frac{-12}{25}$

	 $a^2 = 12^2 + 25^2; a = \sqrt{769}$ $Sin\frac{\alpha+\beta}{2} = \frac{12}{\sqrt{769}} = 0.4327$ $Cos(\alpha+\beta) = 1 - 2sin^2\frac{\alpha+\beta}{2} = 1 - 2\left(\frac{144}{769}\right) = \frac{481}{769} = 0.6255$
(c)(i)	$b^2 = a^2 + c^2 - 2acCosB; \quad 2acCosB = a^2 - b^2 + c^2$ $c^2 = a^2 + b^2 - 2abcCosC; \quad 2abcCosC = a^2 + b^2 - c^2$ $\frac{a^2+b^2-c^2}{a^2-b^2+c^2} = \frac{2abcCosC}{2acCosB} = \frac{bCosC}{cCosB}$ $From, \frac{a}{Sin\,A} = \frac{b}{SinB} = \frac{c}{SinC} = 2R;$ $a = 2RsinA, \quad b = 2RsinB, \quad c = 2Rsin\,C$ $\frac{a^2+b^2-c^2}{a^2-b^2+c^2} = \frac{2abcCosC}{2acCosB} = \frac{bCosC}{cCosB} = \frac{2Rsin\,BcosC}{2RsinCCosB} = tanBcotC$
(ii)	$Given\ \ Area, A = 1008cm^2$ $b + c = 97cm \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(i)$ $A = \sqrt{s(s-a)(s-b)(s-c)}$ $s = \frac{a+b+c}{2} = \frac{97+65}{2} = 81$ $A = \sqrt{81(81-65)(81-b)(81-c)}$ $1008^2 = 1296(81-b)(81-c)$ $784 = 6561 - 81c - 81b + bc$ $-5777 = -81(b+c) + bc = -81(97) + bc$ $bc = 2080 \ldots\ldots\ldots\ldots\ldots\ldots.(ii)$ $b = \frac{2080}{c}$ $\frac{2080}{c} + c = 97$ $2080 + c^2 = 97c$ $c^2 - 97c + 2080 = 0$ $(c-32)(c-65) = 0$ $either\ c = 32\ or\ c = 65$ $for\ c = 32, \quad b = \frac{2080}{c} = \frac{2080}{32} = 65$ $for\ c = 65, \quad b = \frac{2080}{c} = \frac{2080}{65} = 32$
8(a)	$tan\left(\frac{X-Y}{2}\right) = \left(\frac{x-y}{x+y}\right)Cot\left(\frac{Z}{2}\right),$ $from\ LHS;$

	$$\left(\frac{x-y}{x+y}\right)Cot\left(\frac{Z}{2}\right)=\frac{2R(SinX-SinY)Cos\frac{Z}{2}}{2R(SinX+SinY)Sin\frac{Z}{2}}=\frac{2Sin\left(\frac{X-Y}{2}\right)Cos\left(\frac{X+Y}{2}\right)Cos\frac{Z}{2}}{2Sin\left(\frac{X+Y}{2}\right)Cos\left(\frac{X-Y}{2}\right)Sin\frac{Z}{2}}$$ $$But\ Cos\frac{Z}{2}=Cos\left(90-\frac{X+Y}{2}\right)=Sin\left(\frac{X-Y}{2}\right)$$ $$Sin\frac{Z}{2}=Sin\left(90-\frac{X+Y}{2}\right)=Cos\left(\frac{X+Y}{2}\right)$$ $$\left(\frac{x-y}{x+y}\right)Cot\left(\frac{Z}{2}\right)=\frac{2Sin\left(\frac{X-Y}{2}\right)Cos\left(\frac{X+Y}{2}\right)Sin\left(\frac{X-Y}{2}\right)}{2Sin\left(\frac{X+Y}{2}\right)Cos\left(\frac{X-Y}{2}\right)Cos\left(\frac{X+Y}{2}\right)}=tan\left(\frac{X-Y}{2}\right)$$
8(b).	$$\frac{Cos11^0+sin11^0}{Cos11^0-Sin11^0}=tan56^0$$ $$From\ LHS=\frac{Cos11^0+sin11^0}{Cos11^0-Sin11^0}=\frac{1+tan11^0}{1-tan11^0}=\frac{tan45^0+tan11^0}{tan45^0-tan11^0}=tan(45^0+11^0)=tan56^0$$
(c).	 $$By\ the\ cosine\ rule;$$ $$(x+y)^2=x^2+(x-y)^2-2x(x-y)cosA$$ $$x^2+2xy+y^2=x^2+x^2-2xy+y^2-2x(x-y)\cos A$$ $$4xy-x^2=-2x(x-y)\cos A$$ $$x-4y=2(x-y)cosA$$ $$cosA=\frac{x-4y}{2(x-y)}$$
d(i).	$$tan^{-1}\left(\frac{1}{2}\right)+tan^{-1}\left(\frac{1}{5}\right)=tan^{-1}\left(\frac{7}{9}\right)$$ $$Let\ x=tan^{-1}\left(\frac{1}{2}\right);tanx=\frac{1}{2}$$ $$y=tan^{-1}\left(\frac{1}{5}\right);tany=\frac{1}{5}$$ $$let\ \theta=tan^{-1}\left(\frac{1}{2}\right)+tan^{-1}\left(\frac{1}{5}\right)$$ $$\theta=x+y;tan\theta=\tan(x+y);tan\theta=\frac{tanx+tany}{1-tanxtany}$$ $$tan\theta=\frac{\frac{1}{2}+\frac{1}{5}}{1-\frac{1}{10}}=\frac{7}{10}\div\frac{9}{10}=\frac{7\times 10}{10\times 9}=\frac{7}{9}$$ $$\theta=tan^{-1}\left(\frac{7}{9}\right)\quad\therefore tan^{-1}\left(\frac{1}{2}\right)+tan^{-1}\left(\frac{1}{5}\right)=tan^{-1}\left(\frac{7}{9}\right)$$
(ii)	$$Sin(2sin^{-1}x+cos^{-1}x)=\sqrt{1-x^2}$$ $$From\ \ LHS=Sin(2sin^{-1}x+cos^{-1}x)$$ $$let\ A=sin^{-1}x$$ $$sinA=x;\ \ cosA=\sqrt{1-x^2}$$ $$Let\ B=cos^{-1}x;CosB=x,\ Sin\ B=\sqrt{1-x^2}$$ $$\sin(2A+B)=sin2AcosB+cos2AsinB$$ $$=2sinAcosAcosB+(1-2sin^2A)cosB$$ $$=2x^2\sqrt{1-x^2}\ +\sqrt{1-x^2}-2x^2\sqrt{1-x^2}=\sqrt{1-x^2}$$ $$\therefore Sin(2sin^{-1}x+cos^{-1}x)=\sqrt{1-x^2}$$

9(a).

ANALYSIS

$$\frac{1}{\sqrt{0.97}} = (0.97)^{\frac{-1}{2}} = (1-0.03)^{\frac{-1}{2}}$$

$$x = 1, \qquad \Delta x = -0.03$$

$$y = (x)^{\frac{-1}{2}} = (1)^{\frac{-1}{2}} = 1$$

$$\frac{dy}{dx} = -\frac{1}{2}(x)^{\frac{-3}{2}}$$

$$\Delta y = \frac{dy}{dx}.\Delta x = -\frac{1}{2}(1)^{\frac{-3}{2}} \times (-0.03) = 0.015$$

$$\frac{1}{\sqrt{0.97}} = y + \Delta y = 1 + 0.015 = 1.015$$

(b).

$$\int_0^1 \frac{8x-8}{(x+1)^2(x-3)^2}dx = 8\int_0^1 \frac{x-1}{(x^2-3x+x-3)^{\,2}}dx = 8\int_0^1 \frac{x-1}{(x^2-2x-3)^2}dx$$

$$Let\ u = x^2-2x-3; \qquad du = (2x-2)dx; \qquad du = 2(x-1)dx; \qquad (x-1)dx = \frac{du}{2}$$

$$\frac{8}{2}\int\frac{du}{u} = 4\int u^{-2}du = 4\left[\frac{u^{-1}}{-1}\right] = 4\left[\frac{-1}{4}\right] = -4\left[\frac{1}{x^2-2x-3}\right]_0^1 = -4\left[\left(\frac{1}{-4}\right)-\left(\frac{1}{-3}\right)\right] = \frac{-1}{3}$$

(c)(i).

$$\frac{x^4+x^3-6x^2-13x-6}{x^3-7x-6} = \frac{x^4+x^3-6x^2-13x-6}{(x-1)(x-3)(x+2)}$$

$$Let\ \frac{x^4+x^3-6x^2-13x-6}{(x-1)(x-3)(x+2)} \equiv Ax+B+\frac{C}{x+1}+\frac{D}{x-3}+\frac{E}{x+2}$$

$$x^4+x^3-6x^2-13x-6$$
$$\equiv (Ax+B)(x-3)(x+2)(x+1)+C(x-3)(x+2)+D(x+1)(x+2)$$
$$+E(x+1)(x-3)$$

$$Put\ x=3;\quad 81+27-54-39-6 = 20D;\ \therefore D = \frac{9}{20}$$

$$Put\ x=-2;\ \ 16-8-24+26-6 = 5C;\ \ \therefore C = \frac{4}{5}$$

$$Put\ x=-1; 1-1-6+13-6 = -4C; \therefore C = \frac{-1}{4}$$

$$Compare\ coefficients\ of\ x^4; \quad 1 = A$$

$$Put\ x=0;\ -6 = -6B-6C+2D-2E$$

$$-6 = -6B-6\left(\frac{-1}{4}\right)+2\left(\frac{9}{20}\right)-3\left(\frac{4}{5}\right)$$

$$-6 = -6B;\ \ B = 1$$

$$\therefore f(x) \equiv (x+1)-\frac{1}{4(x+1)}+\frac{9}{20(x-3)}+\frac{4}{5(x+2)}$$

(c) (ii).

Hence; $\int_4^5 f(x)dx = \int_4^5(x+1)dx - \frac{1}{4}\int_4^5\frac{dx}{(x+1)}+\frac{9}{20}\int_4^5\frac{dx}{(x-3)}+\frac{4}{5}\int_4^5\frac{dx}{(x+2)}$

$$= \left[\frac{x^2}{2}+x-\frac{1}{4}\ln(x+1)+\frac{9}{20}\ln(x-3)+\frac{4}{5}\ln(x+2)\right]_4^5$$

$$= \left(\frac{5^2}{2}+5-\frac{1}{4}\ln(5+1)+\frac{9}{20}\ln(5-3)+\frac{4}{5}\ln(5+2)\right)$$
$$-\left(\frac{4^2}{2}+4-\frac{1}{4}\ln(4+1)+\frac{9}{20}\ln(4-3)+\frac{4}{5}\ln(4+2)\right)$$

$$= 5.8896967 \approx 5.8897(4dps)$$

(d).

$$\int_0^{\frac{\pi}{4}} \frac{4}{1+cos2x}dx = \int_0^{\frac{\pi}{4}} \frac{4}{1+2cos^2x-1}dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{4}{2cos^2x}dx = 2\int_0^{\frac{\pi}{4}} \frac{1}{cos^2x}dx = 2\int_0^{\frac{\pi}{4}} sec^2x dx = 2[tanx]_0^{\frac{\pi}{2}} = 2\left(tan\frac{\pi}{2} - tan0\right)$$

$$= 2$$

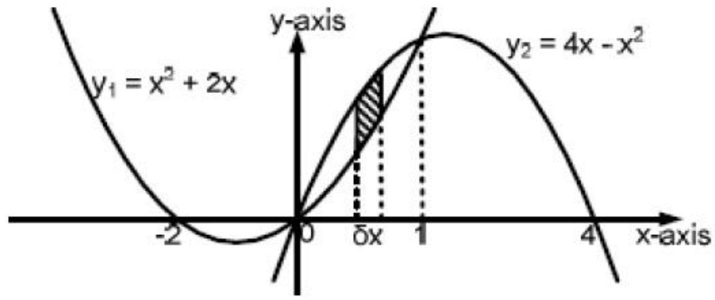
10(a).

$$For\ point\ of\ intersection;\ x(x+2) = x(4-x)$$

$$x^2+2x = 4x - x^2$$

$$2x^2 - 2x = 0$$

$$2x(x-1) = 0 \quad ; \quad Either\ x = 0\ or\ x = 1$$



(ii).

$$\text{Element of area } \Delta A = y\Delta x$$

$$Required\ area, A = \int_0^1 (y_2 - y_1)dx = \int_0^1 [(4x-x^2)-(x^2+2x)]dx$$

$$= \int_0^1 (2x - 2x^2)dx = \left[x^2 - \frac{2}{3}x^3\right]_0^1 = \left(1-\frac{2}{3}\right) - (0) = \frac{1}{3} sq\ units$$

(iii).

$$Element\ volume\ \Delta V = \pi(y_2-y_1)^2\Delta x$$

$$Required\ volume\ V = \int_0^1 \pi(y_2-y_1)^2 dx = \pi\int_0^1 (2x-2x^2)^2 dx$$

$$= \pi\int_0^1 (4x^2 - 8x^3 + 4x^4)\,dx = \pi\left[\frac{4}{3} - 2 + \frac{4}{5}x^5\right]_0^1 = \pi\left(\frac{4}{3} - 2 + \frac{4}{5}\right) - (0) = \frac{2\pi}{15}\ cubic\ units$$

(c).

$$\int_2^6 \frac{\sqrt{x-2}}{x}dx \quad ; \quad let\ u = \sqrt{x-2}$$

$$u^2 = x-2;\ \ 2udu = dx$$

x	u
6	2
2	0

$$\int_2^6 \frac{\sqrt{x-2}}{x}dx = \int_0^2 \frac{u}{u^2+2}.2udu = \int_0^2 \frac{2u^2}{u^2+2}du$$

$$by\ long\ division;\ \frac{2u^2}{u^2+2} = 2 - \frac{4}{2+u^2}$$

$$\int_0^2 \frac{2u^2}{u^2+2}du = \int_2^2 \left[2 - 4\left(\frac{1}{2+u^2}\right)\right]du = \left[2u - 4.\frac{1}{\sqrt{2}}tan^{-1}\left(\frac{1}{\sqrt{2}}u\right)\right]_0^2$$

$$= \left(4 - 2\sqrt{2}tan^{-1}\sqrt{2} - 0\right) = 1.2980$$

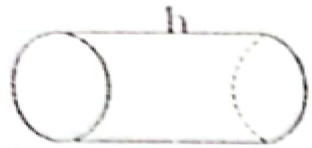
10(d).

$$Inner\ surface\ area; A_1 = 2xh + \frac{3x^2}{2} + 3xh$$

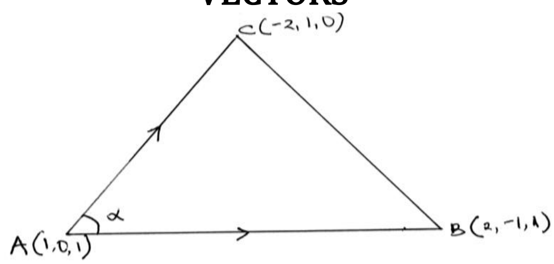
$$Outer\ surface\ area;\ A_2 = 2\left(\frac{3}{2}xh\right) + 2\left(\frac{3x^2}{2}\right) = 3xh + 3x^2$$

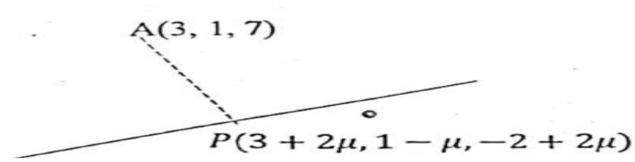
$$S = 2xh + \frac{3x^2}{2} + 3xh + 3xh + 3x^2 = 8xh + \frac{9x^2}{2}$$

$$V = l \times w \times h$$

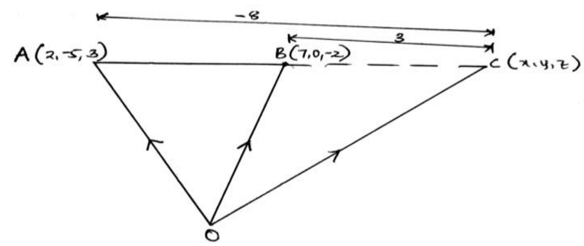
	$$25 = \frac{3x^2}{2}h; \quad h = \frac{50}{3x^2}$$ $$S = \frac{400}{3x} + \frac{9x^2}{2}; \ \frac{ds}{dx} = -\frac{-400}{3x^2} + 9x$$ $$For\ least\ area;\ \frac{ds}{dx} = 0$$ $$27x^3 = 400; \quad x = \sqrt[3]{\left(\frac{400}{27}\right)} = 2.456$$ $$Length = (2.456 \times 1.5) = 3.684 \approx 3.7$$
11(a).	 $$2\pi r + h = 6$$ $$V = \pi r^2 h$$ $$V = \pi r^2(6 - 2\pi r)$$ $$V = 6\pi r^2 - 2\pi^2 r^3$$ $$\frac{dV}{dr} = 12\pi r - 6\pi^2 r^2$$ $$for\ the\ largest\ parcel, \frac{dV}{dr} = 0$$ $$6\pi r(2 - \pi r) = 0; \quad r = 0, \qquad r = \frac{2}{\pi}$$ $$\frac{d^2V}{dr^2} = 12\pi - 12\pi^2 r$$ $$\frac{d^2V}{dr^2}\left(r = \frac{2}{\pi}\right) = 12\pi - 24\pi = -12\pi < 0$$ $$r = \frac{2}{\pi}\ gives\ the\ maximum\ volume$$ $$h = 6 - 2\pi.\left(\frac{2}{\pi}\right) = 2$$ $$\therefore r = \frac{2}{\pi}cm, \quad h = 2cm$$
11(b)(i).	$$(1+t)\frac{dp}{dt} + p = (1+t)\sqrt{t}$$ $$\frac{dp}{dt} + \frac{1}{1+t}p = \sqrt{t}$$ $$IF = e^{\int \frac{1}{1+t}dt} = e^{\ln(1+t)} = 1 + t$$ $$(1+t)\left(\frac{dp}{dt} + \frac{1}{1+t}p\right) = (1+t)\sqrt{t}$$ $$\frac{d}{dt}(1+t)P = (1+t)\sqrt{t}$$ $$\int \frac{d}{dt}(1+t)Pdt = \int (1+t)\sqrt{t}\,dt$$ $$(1+t)P = \int \left(t^{\frac{1}{2}} + t^{\frac{3}{2}}\right)dt = \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + \frac{t^{\frac{5}{2}}}{\frac{5}{2}} + c$$ $$At\ t = 0. \quad p_o = 5000\ \therefore c = 0$$

	$$(1+t)P=\frac{t^{\frac{3}{2}}}{\frac{3}{2}}+\frac{t^{\frac{5}{2}}}{\frac{5}{2}}+5000$$ $$At\ t=8\ hours, p=?$$ $$(1+8)P=\frac{(8)^{\frac{3}{2}}}{\frac{3}{2}}+\frac{(8)^{\frac{5}{2}}}{\frac{5}{2}}+5000;\quad p=565.2770\approx 565\ bacteria$$
11(b)(ii).	$$At\ t=4hours; p=?$$ $$(1+t)P=\frac{t^{\frac{3}{2}}}{\frac{3}{2}}+\frac{t^{\frac{5}{2}}}{\frac{5}{2}}+5000$$ $$(1+4)P=\frac{(4)^{\frac{3}{2}}}{\frac{3}{2}}+\frac{(4)^{\frac{5}{2}}}{\frac{5}{2}}+5000;\quad p=1003.6267\approx 1004\ bacteria$$ $$(1+t)\frac{dp}{dt}+p=(1+t)\sqrt{t}$$ $$(1+4)\frac{dp}{dt}=(1+4)\sqrt{4}-1004;\quad \frac{dp}{dt}=-198.8\ bacteria\ per\ hour$$
12(a).	$$y=\log_e\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)=\log_e\left(\frac{1-tan\frac{x}{2}}{1+tan\frac{x}{2}}\right)$$ $$y=\log_e\left(1-tan\frac{x}{2}\right)-\log_e\left(1+tan\frac{x}{2}\right)$$ $$\frac{dy}{dx}=\frac{-\frac{1}{2}sec^2\frac{x}{2}}{1-tan\frac{x}{2}}-\frac{\frac{1}{2}sec^2\frac{x}{2}}{1+tan\frac{x}{2}}$$ $$=\frac{-\frac{1}{2}sec^2\frac{x}{2}\left(1+tan\frac{x}{2}+1-tan\frac{x}{2}\right)}{1-tan^2\frac{x}{2}}=\frac{-sec^2\frac{x}{2}}{2-sec^2\frac{x}{2}}$$ $$=\frac{-1}{cos^2\frac{x}{2}}\div\left(2-\frac{1}{cos^2\frac{x}{2}}\right)=\frac{-1}{cos^2\frac{x}{2}}\div\left(\frac{2cos^2\frac{x}{2}-1}{cos^2\frac{x}{2}}\right)=\frac{-1}{2cos^2\frac{x}{2}-1}=\frac{-1}{cosx}=-secx$$
12(b).	$$x\frac{dy}{dx}=2x-y$$ $$x\frac{dy}{dx}+y=2x$$ $$\frac{d}{dx}(xy)=2x$$ $$\int\frac{d}{dx}(xy)dx=\int 2xdx\quad;\quad xy=x^2+c$$
12(c).	$$\frac{dx}{dt}\ \alpha\ x(1-x)$$ $$\frac{dx}{dt}=k\ x(1-x)$$ $$\int\frac{dx}{x(1-x)}=\int kdt$$ $$\frac{1}{x(1-x)}=\frac{A}{x}+\frac{B}{1-x}$$ $$A(1-x)+Bx=1$$ $$For\ ;\ x=0, A=1$$

	$$For\ ;\ x = 1, B = 1$$ $$\int \left(\frac{1}{x} + \frac{1}{1-x}\right) dx = \int kdt$$ $$lnx - \ln(1-x) = kt + c$$ $$ln\left(\frac{x}{1-x}\right) = kt + c$$ $$For;\ t = 0, \quad x = \frac{1}{2}$$ $$For;\ t = 6, \quad x = \frac{3}{4}$$ $$ln\left(\frac{\frac{1}{2}}{1-\frac{1}{2}}\right) = c\ ;\ \ c = 0$$ $$ln\left(\frac{x}{1-x}\right) = kt$$ $$ln\left(\frac{\frac{3}{4}}{1-\frac{3}{4}}\right) = 6k; \quad k = \frac{1}{6}ln3$$ $$ln\left(\frac{x}{1-x}\right) = \frac{1}{6}tln3\ ; t = 12$$ $$ln\left(\frac{x}{1-x}\right) = ln9$$ $$\frac{x}{1-x} = 9; \quad x = \frac{9}{10}$$ $$Population\ destroyed = 90\%$$
13(a).	**VECTORS**  $$\overline{AB} = \overline{OB} - \overline{OA} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$ $$\overline{AC} = \overline{OC} - \overline{OA} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix}$$ $$\overline{AB}.\overline{AC} = \lvert\overline{AB}\rvert.\lvert\overline{AC}\rvert cos\alpha$$ $$cos\alpha = \frac{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.\begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix}}{\sqrt{1^2 + (-1)^2 + 0^2}\sqrt{(-3)^2 + 1^2 + (-1)^2}} = \frac{-3 +\ -1 + 0}{\sqrt{2}\sqrt{11}} = \frac{-4}{\sqrt{22}}$$ $$\alpha = cos^{-1}\left(\frac{-4}{\sqrt{22}}\right) = 148.52^0$$
b(i).	$$r = 2i + 4j - k + \lambda(i + 2j - 3k) + \mu(-i + 2j + k)$$ $$Cartesian\ equation\ is\ given\ by \quad r.n = n.a$$ $$a = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$$ $$n = \begin{vmatrix} 1 & 2 & -3 \\ -1 & 2 & 1 \end{vmatrix} = i[(2 \times 1) - (2 \times -3)] - j[(1 \times 1) - (-1 \times -3)] + k[(1 \times 2) - (-1 \times 2)]$$

	$$n = 8i - 2j + 4k; \quad n = \begin{pmatrix} 8 \\ -2 \\ 4 \end{pmatrix}$$ $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} . \begin{pmatrix} 8 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 8 \\ -2 \\ 4 \end{pmatrix} . \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$$ $$8x - 2y + 4z = 16 - 8 - 4$$ $$8x - 2y + 4z = 4$$
b(iii).	$$\frac{x-1}{5} = \frac{y-3}{-3} = \frac{z+3}{4} = \lambda$$ $$x = 1 + 5\lambda; \quad y = 3 + 3\lambda; \quad z = -3 + 4\lambda$$ $$8x - 2y + 4z = 4$$ $$8(1 + 5\lambda) - 2(3 + 3\lambda) + 4(-3 + 4\lambda) = 4$$ $$8 + 40\lambda - 6 - 6\lambda - 12 + 16\lambda = 4$$ $$50\lambda = 14; \quad \lambda = \frac{14}{50}$$ $$x = 1 + 5\left(\frac{14}{50}\right) = 2.4$$ $$y = 3 + 3\left(\frac{14}{50}\right) = 3.84$$ $$z = -3 + 4\left(\frac{14}{50}\right) = -1.88$$ $$point\ (2.4, 3.84, -1.88)$$
14(a)(i).	$$r = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$ $$r. \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} = d$$ $$for\ \mu = 0, \quad r = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$$ $$\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} . \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} = d$$ $$d = 3 - 2 + 4; \quad d = 5$$
(ii).	 $$\overline{AP} = \begin{pmatrix} 3 + 2\mu \\ 1 - \mu \\ -2 + 2\mu \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix}$$ $$\overline{AP} = \begin{pmatrix} 2\mu \\ -\mu \\ -9 + 2\mu \end{pmatrix}$$ $$\begin{pmatrix} 2\mu \\ -\mu \\ -9 + 2\mu \end{pmatrix} . \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = 0$$ $$4\mu + \mu - 18 + 4\mu = 0$$ $$9\mu = 18; \ \mu = 2$$ $$\overline{AP} = \begin{pmatrix} 4 \\ -2 \\ -5 \end{pmatrix}$$ $$\lvert\overline{AP}\rvert = \sqrt{4^2 + (-2)^2 + (-5)^2} = \sqrt{45} = 6.7082\ units$$

(b).



$$\overline{OC} = \frac{\lambda}{\lambda-\mu}\overline{OB} - \frac{\mu}{\lambda-\mu}\overline{OA}$$

$$\overline{OC} = \frac{3}{3-5}\begin{pmatrix}7\\0\\-2\end{pmatrix} - \frac{8}{3-8}\begin{pmatrix}2\\-5\\3\end{pmatrix}$$

$$\overline{OC} = \frac{-1}{5}\left[\begin{pmatrix}21\\0\\-6\end{pmatrix} - \begin{pmatrix}18\\-40\\24\end{pmatrix}\right] = \frac{-1}{5}\begin{pmatrix}5\\40\\-30\end{pmatrix} = \begin{pmatrix}-1\\-8\\6\end{pmatrix}$$

$$The\ cordinates\ of\ point\ C(-1,-8,6)$$

ALT

$$Given\ points\ \ A(2,-5,3)\ and\ B(7,0,-2)$$

$$\overline{AC} = \overline{CB} = 3:-8$$

$$\frac{\overline{AC}}{\overline{CB}} = \frac{3}{-8}$$

$$-8(\overline{OC} - \overline{OA}) = 3(\overline{OB} - \overline{OC})$$

$$-8\overline{OC} + 8\overline{OA} = 3(\overline{OB} - 3\overline{OC})$$

$$50\overline{OC} = 8\overline{OA} - 3\overline{OC}$$

$$50\overline{OC} = 8\begin{pmatrix}2\\-5\\3\end{pmatrix} - 3\begin{pmatrix}7\\0\\-2\end{pmatrix}$$

$$50\overline{OC} = \begin{pmatrix}-5\\-40\\30\end{pmatrix}$$

$$\overline{OC} = \begin{pmatrix}-1\\-8\\6\end{pmatrix}$$

$$The\ cordinates\ of\ point\ C(-1,-8,6)$$

(c).

$$\overline{PR} = \overline{OR} - \overline{OP} = \begin{pmatrix}3\\8\\1\end{pmatrix} - \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}2\\6\\-2\end{pmatrix}$$

$$\overline{PT} = \overline{OT} - \overline{OP} = \begin{pmatrix}7\\20\\-3\end{pmatrix} - \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}6\\18\\-6\end{pmatrix}$$

$$For\ parallel\ vectors\ \overline{PR} = k\overline{PT}$$

$$\begin{pmatrix}2\\6\\-2\end{pmatrix} = k\begin{pmatrix}6\\18\\-6\end{pmatrix}$$

$$2 = 6k; \quad k = \frac{1}{3}$$

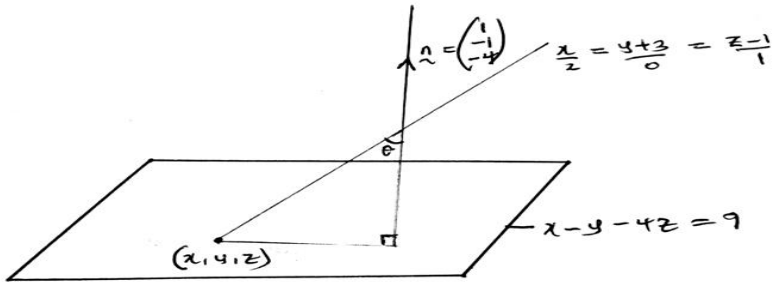
$$6 = 18k\ ; k = \frac{1}{3}$$

$$-2 = -6k; \ \ k = \frac{1}{3}$$

$$since\ the\ value\ of\ k = \frac{1}{3}\ is\ consistent, then\ the\ points\ P, R\ and\ T\ \ are\ collinear$$

15(a).

$$Parallel\ planes\ have\ the\ same\ vector\ normal\ to\ them; n = \begin{pmatrix}1\\-1\\-4\end{pmatrix}$$

	$Known\ point\ on\ the\ plane\ a = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ $Equation\ of\ the\ plane\ is\ r.n = n.a$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix}.\begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix}.\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ $x - y - 4z = 2 - 1 + 8$ $x - y - 4z = 9$
15(b)(i).	 $\frac{x}{2} = \frac{y+3}{0} = \frac{z-1}{1} = \lambda$ $x = 2\lambda$ $y = -3$ $z = 1 + \lambda$ $x - y - 4z = 9$ $2\lambda + 3 - 4(1 + \lambda) = 9$ $2\lambda + 3 - 4 - 4\lambda = 9$ $-2\lambda = 10;\ \ \lambda = -5$ $x = 2\lambda = 2(-5) = -10$ $y = -3$ $z = 1 + \lambda = 1 - 5 = -4$ $the\ point\ of\ intersection\ is\ (-10, -3, -4)$
15(b)(ii).	$\frac{x}{2} = \frac{y+3}{0} = \frac{z-1}{1}; \qquad d = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ $x - y - 4z = 9; \quad n = \begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix}$ $d.n = \|n\|sin\theta$ $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.\begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix} = \sqrt{2^2 + 0^2 + 1^2}\sqrt{1^2 + (-1)^2 + (-4)^2}\sin\theta$ $\sin\theta = \frac{-2}{\sqrt{5}\sqrt{18}}$ $\theta = sin^{-1}\left(\frac{-2}{\sqrt{5}\sqrt{18}}\right)$ $\theta = -12.17^0$
16(a).	$r.\begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} = 6 \text{ and } r.\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 4.$ $let\ r = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $x + y - 3z = 6$……………………(i) $2x - y + z = 4 \ldots\ldots\ldots\ldots\ldots\ldots..(ii)$ $(i) + (ii); 3x - 2z = 10$

$$let\ z = \mu;$$

$$3x - 2\mu = 10; \quad 3x = 10 + 2\mu; \quad x = \frac{10}{3} + \frac{2}{3}\mu$$

$$from\ (i);$$

$$\frac{10}{3} + \frac{2}{3}\mu + y - 3\mu = 6$$

$$y = \frac{8}{3} + \frac{7}{3}\mu$$

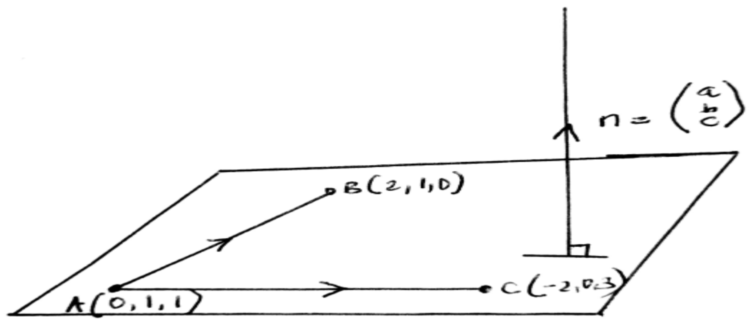
$$x = \frac{10}{3} + \frac{2}{3}\mu$$

$$y = \frac{8}{3} + \frac{7}{3}\mu$$

$$z = \mu$$

$$r = \begin{pmatrix} \frac{10}{3} \\ \frac{8}{3} \\ 0 \end{pmatrix} + \mu\begin{pmatrix} \frac{2}{3} \\ \frac{7}{3} \\ 1 \end{pmatrix} \quad or \quad r = \begin{pmatrix} 10 \\ 8 \\ 0 \end{pmatrix} + \mu\begin{pmatrix} 2 \\ 7 \\ 3 \end{pmatrix}$$

16(b).



$$\overline{AB} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

$$\overline{AC} = \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}$$

$$\overline{AB}.n = 0$$

$$\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}.\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0; \quad 2a - c = 0$$

$$\overline{AC}.n = 0$$

$$\begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}.\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0; \quad -2a - b + 2c = 0$$

$$let\ c = \lambda; \quad 2a = \lambda; \quad a = \frac{\lambda}{2}$$

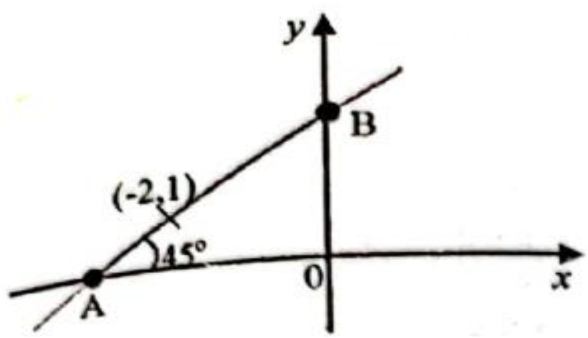
$$-2\left(\frac{\lambda}{2}\right) - b + 2\lambda = 0; \quad b = \lambda$$

$$n = \begin{pmatrix} \frac{\lambda}{2} \\ \lambda \\ \lambda \end{pmatrix} = \frac{\lambda}{2}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}; \quad n = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

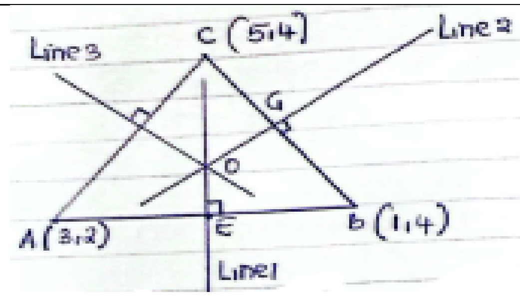
$$r.n = n.a$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}.\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$x + 2y + 2z = 0 + 2 + 2$$

	$\therefore x + 2y + 2z = 4$
$16(c)$.	*The vector equation of the line through points* $(2, 1, 4)$ and $(a-1, 4, -1)$is $r = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} a-3 \\ 3 \\ -5 \end{pmatrix}$ *The vector equation of the line through points* $(0, 2, b-1)$ *and* $(5, 3, -2)$. is $r = \begin{pmatrix} 0 \\ 2 \\ b-1 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ 1 \\ -1-b \end{pmatrix}$ *but for parallel lines;* $d_1 = kd_2$ $\begin{pmatrix} a-3 \\ 3 \\ -5 \end{pmatrix} = k \begin{pmatrix} 5 \\ 1 \\ -1-b \end{pmatrix}$ $k = 3$ $a - 3 = (5 \times 3);\quad a = 15 + 3;\quad a = 18$ $-5 = 3(-1-b);\quad -5 = -3 - 3b;\quad -2 = -3b;\quad b = \frac{2}{3}$
17(a).	**COORDINATE GEOMETRY** $\overline{BC} = \sqrt{(0 - -1)^2 + (n - -2)^2} = \sqrt{1^2 + (n+2)^2} = \sqrt{n^2 + 4n + 5}$ $\overline{AC} = \sqrt{(0 - -3)^2 + (n-2)^2} = \sqrt{3^2 + (n-2)^2} = \sqrt{n^2 - 4n + 13}$ $But\ \overline{BC} = \frac{1}{5}\overline{AC}$ $\sqrt{n^2 + 4n + 5} = \frac{1}{5}\sqrt{n^2 - 4n + 13}$ $5\sqrt{n^2 + 4n + 5} = \sqrt{n^2 - 4n + 13}$ *squaring both sides;* $25(n^2 + 4n + 5) = n^2 - 4n + 13$ $24n^2 + 104n + 112 = 0$ $3n^2 + 13n + 14 = 0$ $(3n + 7)(n + 2) = 0$ $Either\ n = \frac{-7}{3}\ \ or\ \ n = -2$ $Hence\ the\ values\ of\ n\ are\ \frac{-7}{2}\ \ and\ \ n = -2$
17(b)(i).	 $Gradient\ of\ line\ L;\ m = tan45^0 = 1$ $so\ y = 1(x) + c;\ \ y = x + c$ The line passes through (-2, 1) $by\ substitution; 1 = -2 + c; c = 3$ $\therefore the\ line\ is\ y = x + 3$
17(b)(ii).	$At\ point\ A; y = 0, x = -3; hence\ the\ coordinates\ of\ A\ are\ (-3,0)$ $At\ point\ B, x = 0, y = 3\ ; hence\ the\ coordinates\ of\ B\ are\ (0,3)$ $\overline{AB} = \sqrt{(0 - -3)^2 + (3-0)^2} = \sqrt{9+9} = \sqrt{18} = 4.2426\ units$

17(c).



$$Point\ E = \left(\frac{3+1}{2}, \frac{2+4}{2}\right) = (2,3)$$

$$Gradient\ of\ \overline{AB} = \frac{4-2}{1-3} = -1$$

$$Gradient\ of\ Line\ 1 = 1$$

$$Equation\ of\ line\ 1; y - 3 = x - 2; \quad y = x + 1 \ldots\ldots\ldots.(i)$$

$$Point\ G = \left(\frac{1+5}{2}, \frac{4+4}{2}\right) = (3,4)$$

$$Gradient\ of\ \overline{BC} = \frac{4-4}{1-5} = 0$$

$$Gradient\ of\ Line\ 2 = \frac{1}{0}$$

$$Equation\ of\ line\ 2; \ \ y - 4 = \frac{1}{0}(x - 3)$$

$$x - 3 = 0(y - 4)$$

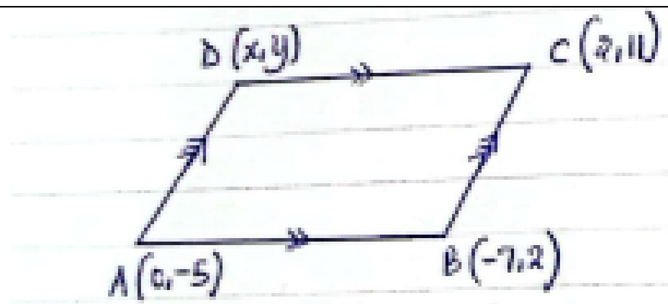
$$x = 3 \ldots\ldots\ldots\ldots\ldots\ldots.(ii)$$

$$solving\ (ii) and\ (i) simultaneously;$$

$$y = 3 + 1 = 4;$$

$$\therefore The\ coordinates\ of\ the\ circumcenter\ are\ (3,4)$$

17(d).



$$Gradient\ of\ \overline{BC} = \frac{11-2}{2-7} = \frac{9}{9} = 1$$

$$Gradient\ of\ \overline{AD} = \frac{y+5}{x}$$

$$Gradient\ of\ \overline{AD} = Gradient\ of\ \overline{BC}$$

$$\frac{y+5}{x} = 1; \quad x = y + 5 \ldots\ldots\ldots\ldots\ldots.(i)$$

$$Gradient\ of\ \overline{AB}; = \frac{2 - -5}{-7 - 0} = \frac{7}{-7} = -1$$

$$Gradient\ of\ \overline{DC} = \frac{y-11}{x-2}$$

$$Gradient\ of\ \overline{AB} = Gradient\ of\ \overline{DC}$$

$$\frac{y-11}{x-2} = -1$$

$$y - 11 = -x + 2; \quad y = -x + 13 \ldots\ldots\ldots\ldots\ldots(ii)$$

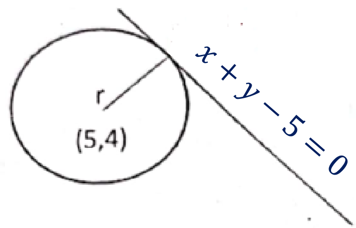
$$By\ substitution; \ \ y = -(y + 5) + 13,$$

$$y = -y - 5 + 13$$

$$2y = 8; \quad y = 4$$
$$x = -4 + 13 = 9$$
$$Point\ D(9,4)$$

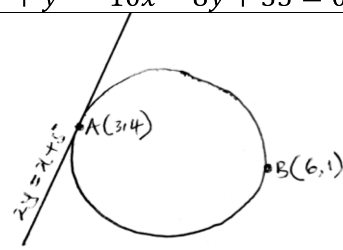
18(a).

$$Gradient\ of\ the\ tangent = \frac{5-1}{0-4} = -1$$
$$Equation\ of\ the\ tangent;\ \frac{y-5}{x} = -1$$
$$y - 5 = -x; \qquad x + y - 5 = 0$$



$$r = \left|\frac{5+4-5}{\sqrt{1^2+1^2}}\right| = \frac{4}{\sqrt{2}}\ units$$
$$Equation\ of\ the\ circle$$
$$(x-a)^2 + (y-b)^2 = r^2$$
$$(x-5)^2 + (y-4)^2 = 8$$
$$x^2 + y^2 - 10x - 8y + 33 = 0$$

18(b)(i).



$$2y = x + 5; \ \ m = \frac{1}{2}$$
$$from\ x^2 + y^2 + 2gx + 2fy + c = 0$$
$$2x + 2y\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0 \ \left(m = \frac{1}{2}, x = 3, y = 4\right)$$
$$6 + 4 + 2g + f = 0; \qquad 2g + f = -10 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$
$$At\ point\ (3,4);\ 9 + 16 + 6g + 8f + c = 0; \ \ 6g + 8f + c = -25 \ldots\ldots\ldots\ldots\ldots\ldots (ii)$$
$$At\ point\ (6,1);\ 36 + 1 + 12g + 2f + c = 0; \ \ 12g + 2f + c = -37 \ldots\ldots\ldots\ldots (iii)$$

On solving (ii) and (iii)

$$6g + 8f + c = -25$$
$$\text{-}12g + 2f + c = -37$$

$$-6g + 6f = 12; \ \ -g + f = 2 \ldots\ldots\ldots\ldots\ldots\ldots (iv)$$

On solving (i) and (iv)

$$2g + f = -10$$
$$(\text{-}) -g + f = 2$$

$$3g = -12; \quad g = -4$$
$$f = 2 + g = 2 - 4 = -2$$
$$\therefore Centre\ (4,2\ )$$

18(b)(ii).

$$Using\ the\ points;\ A(3,4)\ and\ Centre\ (4,2)$$
$$r = \sqrt{(4-3)^2 + (2-4)^2} = \sqrt{5}\,units$$

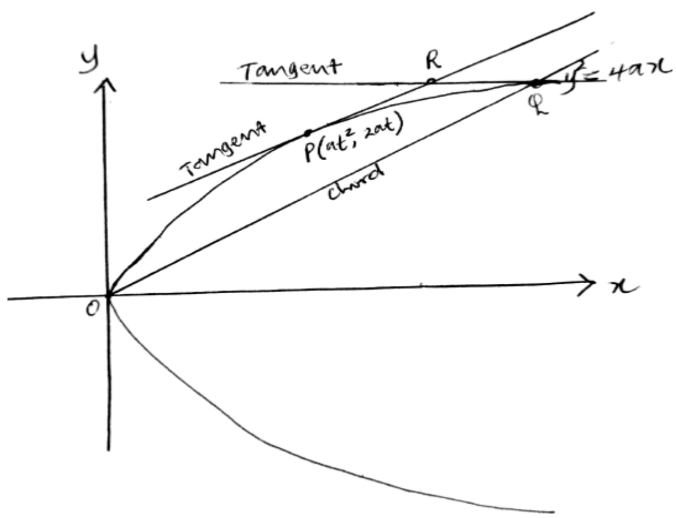
18(b)(iii).

$$Using\ the\ Centre\ (4,2)\ and\ r = \sqrt{5}\,units$$
$$(x-4)^2 + (y-2)^2 = 5$$
$$x^2 - 8x + 16 + y^2 - 2y + 4 - 5 = 0$$
$$x^2 + y^2 - 8x - 2y + 15 = 0$$

18(c).	$y = mx$ $x^2 + y^2 + 2fy + c = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$ $x^2 + (mx)^2 + 2fmx + c = 0$ $(1 + m^2)x^2 + 2fmx + c = 0$ $For\ tangency;\ b^2 = 4ac$ $(2fm)^2 = 4(1 + m^2)c$ $4f^2m^2 = 4c(1 + m^2)$ $c = \frac{f^2m^2}{1 + m^2}$ $Compare;\ \ x^2 + y^2 - 10y + 20 = 0 \ldots\ldots\ldots\ldots\ldots. (ii) with\ (i)$ $2f = -10, \quad f = -5; \quad c = 20$ $therefore;\ 20 = \frac{(-5)^2m^2}{1 + m^2}$ $20 = 20m^2 = 25m^2$ $5m^2 = 20; \quad m^2 = 4; \quad m = \pm 2$ $The\ tangents\ are\ y = 2x\ and\ y = -2x$
18(d).	$Comparing\ x^2 + y^2 - 4y - 5 = 0\ \ with\ x^2 + y^2 + 2gx + 2fy + c = 0$ $2g = 0, \quad g = 0$ $2f = -4;\ \ f = -2$ $C_1 = -5$ $The\ centre\ (0,2)$ $Radius\ r_1 = \sqrt{0^2 + (-2)^2 + 5} = 3\ units$ $Also\ comparing\ x^2 + y^2 - 8x + 12y + 1 = 0\ with\ x^2 + y^2 + 2gx + 2fy + c = 0$ $2g = -8;\ \ g = -4$ $2f = 12; \quad f = 6$ $C_2 = 1$ $The\ centre\ (4, -1)$ $Radius\ r_2 = \sqrt{4^2 + (-1)^2 - 1} = \sqrt{16} = 4\ units$ $For\ orthogonality, r_1^2 + r_2^2 = \lvert C_1C_2 \rvert^2$ $r_1^2 + r_2^2 = 4^2 + 3^2 = 25 units$ $\lvert C_1C_2 \rvert = \sqrt{(0 - 4)^2 + (2 - -1)^3} = \sqrt{16 + 9} = 5\ units$ $\lvert C_1C_2 \rvert^2 = 25$ $Since\ r_1^2 + r_2^2 = \lvert C_1C_2 \rvert, then\ the\ two\ circles\ are\ orthogonal$
19(a).	$P(at^2, 2at), \quad Q\left(\frac{a}{t^2}, \frac{-2a}{t}\right), \quad S(a,0)$ $SP = \sqrt{(at^2 - a)^2 + (2at - 0)^2} = a(t^2 + 1)$ $SQ = \sqrt{\left(\frac{a}{t^2} - a\right)^2 + \left(\frac{-2a}{t} - 0\right)^2} = \frac{a(t^2 + 1)}{t^2}$ $Now;\ \frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a(t^2 + 1)} + \frac{t^2}{a(t^2 + 1)} = \frac{t^2 + 1}{a(t^2 + 1)} = \frac{1}{a}$ $\frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a}$
19(b).	$x = 3t^2 + 1$ and $2y = 3t + 1$ $From;\ 2y = 3t + 1;\ \ t = \frac{2y - 1}{3}$ $x = 3\left(\frac{2y - 1}{3}\right)^2 + 1$ $x = \frac{(2y - 1)^2}{3} + 1$ $(2y - 1)^2 = 3x - 3$

$$4y^2 - 4y + 1 = 3x - 3$$
$$y^2 - y = \frac{3}{4}x - 1$$
$$\left(y - \frac{1}{2}\right)^2 = \frac{3}{4}x - 1 + \left(\frac{1}{2}\right)^2$$
$$\left(y - \frac{1}{2}\right)^2 = \frac{3}{4}x - \frac{3}{4}$$
$$\left(y - \frac{1}{2}\right)^2 = \frac{3}{4}(x - 1) \text{ hence a parabola}$$
$$\text{Comparing with } (y-k)^2 = 4a(x-h)$$
$$4a = \frac{3}{4} \quad \therefore a = \frac{3}{16}, \qquad k = \frac{1}{2}, \qquad h = 1$$
$$Vertex, (h,k) = \left(1, \frac{1}{2}\right)$$
$$Focus, s(h+a, k) = s\left(\frac{19}{16}, \frac{1}{2}\right)$$
$$\text{Length of the latus rectum} = 2(2a+k) = 2\left(\frac{7}{8}\right) = \frac{7}{4} units$$

19(c).



$$x = at^2; \ \frac{dx}{dt} = 2at \quad ; \quad y = 2at; \ \frac{dy}{dt} = 2a$$
$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dx}{dt} = \frac{2a}{2at} = \frac{1}{t} \text{ this is the gradient of the tangent at point } P(at^2, 2at)$$
$$\text{Equation of the tangent through } P(at^2, 2at);$$
$$\frac{y - 2at}{x - at^2} = \frac{1}{t}; \qquad y = \frac{1}{t}x + at \ldots\ldots\ldots\ldots\ldots\ldots (i)$$
$$\text{Equation of the chord } OQ \text{ ; let } Q(x,y)$$
$$\frac{y-0}{x-0} = \frac{1}{t}; x = yt$$
$$But \ y^2 = 4ax$$
$$y^2 = 4ayt$$
$$y^2 - 4ayt = 0$$
$$y(y - 4at) = 0$$
$$Either \ y = 0, y = 4at$$
$$\therefore y = 4at$$
$$\therefore x = yt = 4att = 4at^2$$
$$\therefore Q(4at^2, 4at)$$
$$\text{Equation of the tangent at } Q(4at^2, 4at)$$

	$$x = 4at^2;\ \frac{dx}{dt} = 8at$$ $$y = 4at;\quad \frac{dy}{dt} = 4a$$ $$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dx}{dt} = \frac{4a}{8at} = \frac{1}{2t}\ \textit{this is the gradient of the tangent at point } Q(4at^2, 4at)$$ $$\frac{y - 4at}{x - 4at^2} = \frac{1}{2t}\quad ;\quad y = \frac{x}{2t} + 2at \ldots\ldots\ldots\ldots\ldots\ldots (ii)$$ $$\textit{Solving Equations } (i) \textit{and } (ii) \textit{ to get their point of intersection } R$$ $$\frac{x}{t} + at = \frac{x}{2t} + 2at$$ $$\frac{x}{2t} = at;\quad x = 2at^2$$ $$y = \frac{2at^2}{2t} + 2at = 3at$$ $$\therefore R(2at^2, 3at)$$
20(a).	$$x^2 + 2y^2 + 6x - 8y = 7$$ $$(x^2 + 2x) + 2(y^2 - 4y) = 7$$ $$(x^2 + 2x + 1^2 - 1^2) + 2(y^2 - 4y + 2^2 - 2^2) = 7$$ $$(x + 1)^2 - 1 + 2(y - 2)^2 - 8 = 7$$ $$(x + 1)^2 + 2(y - 2)^2 = 16$$ $$\frac{(x + 1)^2}{16} + \frac{(y - 2)^2}{8} = 1$$ $$\textit{Centre } (-1, 2)$$ $$b^2 = a^2(1 - e^2)$$ $$a^2 = 16,\qquad b^2 = 8$$ $$8 = 16(1 - e^2);\quad e^2 = \frac{1}{2}$$ $$e = \frac{1}{\sqrt{2}}$$
20(b).	$$\frac{x^2}{36} + \frac{y^2}{16} = 1$$ $$\textit{The focus } S \textit{ and } S' \textit{are given by } S\ (ae, 0) \textit{and } S'(-ae, 0)$$ $$a^2 = 36,\qquad a = 6$$ $$b^2 = 16;\ b = 4$$ $$b^2 = a^2(1 - e^2);\ 16 = 36(1 - e^2);\quad 36e^2 = 36 - 16$$ $$e^2 = \frac{20}{36};\quad e = \frac{2\sqrt{5}}{6}$$ $$S\left(6 \times \frac{2\sqrt{5}}{6}, 0\right) = S(2\sqrt{5}\ \ , 0)$$ $$S'\left(-6 \times \frac{2\sqrt{5}}{6}, 0\right) = S(-2\sqrt{5}\ \ , 0)$$

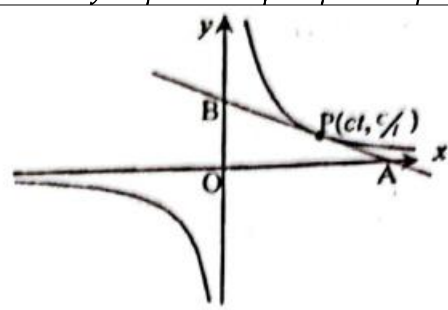
20(c).	$$\frac{x^2}{9}+\frac{y^2}{1}=1$$ $$a^2=9;\ \ a=3$$ $$b^2=1;\ \ b=1$$ $$For\ b^2=a^2(1-e^2)$$ $$1^2=9(1-e^2)$$ $$\frac{1}{9}=1-e^2$$ $$e^2=1-\frac{1}{9};\ \ e^2=\frac{8}{9}=\frac{2\sqrt{2}}{3}$$
20(d)(i).	$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1;\quad \text{line } y=mx+c$$ $$\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1$$ $$b^2x^2+a^2m^2x^2+2a^2mcx+(a^2c^2-a^2b^2)=0$$ $$(b^2+a^2m^2)x^2+(2a^2mc)x+(a^2c^2-a^2b^2)=0$$ $$For\ tangency;\ b^2-4ac=0$$ $$(2a^2mc)^2-4(b^2+a^2m^2)(a^2c^2-a^2b^2)=0$$ $$4a^4m^2c^2-4b^2a^2c^2+4b^4a^2-4a^4m^2c+4a^4m^2b^2=0$$ $$\frac{4b^2a^2c^2}{4b^2a^2}=\frac{4b^4a^2}{4b^2a^2}+\frac{4a^4m^2b^2}{4b^2a^2}$$ $$c^2=b^2+a^2m^2$$ $$\frac{x^2}{23}+\frac{y^2}{3}=1\ \text{ and } \frac{x^2}{14}+\frac{y^2}{4}=1$$ $$Given\ that\ a^2=23\ and\ b^2=3$$ $$c^2=3+23m^2\ \ldots\ldots\ldots\ldots\ldots..(i)$$ $$Given\ that\ a^2=14\ and\ b^2=4$$ $$c^2=4+14m^2\ \ldots\ldots\ldots\ldots\ldots..(ii)$$ $$(i)-(ii)$$ $$c^2=3+23m^2$$ $$-c^2=4+14m^2$$ $$0=-1+9m^2$$ $$m=\pm\frac{1}{3}$$ $$c^2=4+14\left(\frac{1}{9}\right)$$ $$9c^2=36+14;\ \ 9c^2=50;\quad c=\pm\frac{\sqrt{50}}{3}$$ $$\therefore y=\pm\frac{1}{3}x\pm\frac{\sqrt{50}}{3}\ or\ 3y=\pm x\pm\sqrt{50}\quad or\ 3y=\pm x\pm 5\sqrt{2}$$
(ii)	$$\frac{x^2}{16}+\frac{y^2}{9}=1$$ $$Given\ a^2=16\ and\ b^2=9$$ $$but\ c^2=b^2+a^2m^2$$ $$c^2=9+16m^2\ \ldots\ldots\ldots\ldots..(i)$$ $$From\ y=mx+c$$ $$3=-3m+c;\ \ c=3(1+m)\ \ldots\ldots\ldots\ldots\ldots..(ii)$$ $$[3(1+m)]^2=9+16m^2$$ $$9(1+2m+m^2)=9+16m^2$$ $$9+18m+9m^2=9+16m^2$$ $$18m=7m^2$$

$$(7m-18)m=0; \quad m=0; m=\frac{18}{7}$$
$$When\ m=0; c=3(1+0)=3$$
$$y=3\ is\ the\ equation\ of\ the\ tangent\ at\ (-3,3)$$
$$When\ m=\frac{18}{7}; c=3\left(1+\frac{18}{7}\right)=\frac{75}{7}$$
$$y=\frac{18}{7}x+\frac{75}{7}\ \ or\ 7y=18x+75\ is\ the\ other\ equation\ of\ the\ tangent\ at\ (-3,3)$$

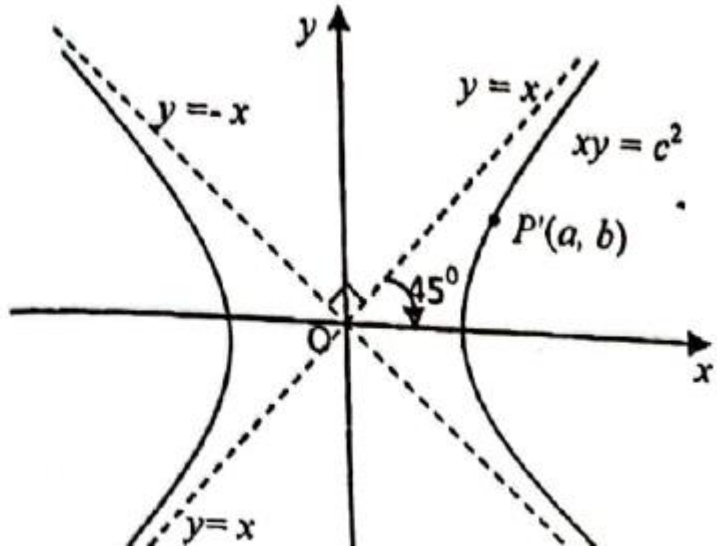
21(a).

$$x^2-9y^2=1$$
$$2x-18y\frac{dy}{dx}=0$$
$$18y\frac{dy}{dx}=2x$$
$$\frac{dy}{dx}=\frac{x}{9y}=\frac{sec\beta}{9\left(\frac{1}{3}tan\beta\right)}=\frac{1}{3sin\beta}$$
$$Let\ a\ point\ (x,y) be\ on\ the\ hyperbola;$$
$$\frac{y-\frac{1}{3}tan\beta}{x-sec\beta}=\frac{1}{3sin\beta}$$
$$3ysin\beta-tan\beta=x-sec\beta$$
$$x=3ysin\beta=xtan\beta sin\beta+sec\beta$$

21(b)(i).



$$From\ xy=c^2$$
$$y=\frac{c^2}{x}; \quad \frac{dy}{dx}=\frac{-c^2}{x^2}$$
$$Now\ at\ P\left(ct,\frac{c}{t}\right); \frac{dy}{dx}=\frac{-c^2}{(ct)^2}=\frac{-c^2}{c^2t^2}=\frac{-1}{t^2}$$
$$Let\ point\ Q(x,y) lie\ on\ the\ tangent. Then:$$
$$\frac{y-\frac{c}{t}}{x-ct}=\frac{-1}{t^2}$$
$$t^2y-tc=-x+ct$$
$$t^2y+x-2ct=0$$
$$The\ tangent\ meets\ the\ y-axis\ when\ x=0$$
$$t^2y=2ct$$
$$y=\frac{2ct}{t^2}=\frac{2c}{t}$$
$$Hence\ B\left(0,\frac{2c}{t}\right)$$
$$The\ tangent\ meets\ the\ x-axis\ when\ y=0$$
$$x=2ct$$
$$Hence\ A(2ct,0)$$
$$AP=\sqrt{(2ct-ct)^2+\left(0-\frac{c}{t}\right)^2}=\sqrt{c^2t^2+\frac{c^2}{t^2}}$$

	$$PB = \sqrt{(0-ct)^2 + \left(\frac{2c}{t} - \frac{c}{t}\right)^2} = \sqrt{c^2t^2 + \frac{c^2}{t^2}}$$ $Hence\ AP = PB$
(ii).	$$Area\ of\ \Delta AOB = \frac{1}{2} \times b \times h = \frac{1}{2} \times \frac{2c}{t} \times 2ct = 2c^2\ sq\ units$$ *Since the area is independent of t, it implies that it is constant*
(iii).	 *The matrix of transformation about the origin through angle θ is given by* $$M = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$$ *Since Point P is mapped onto point P' by M,* $$\begin{array}{ccc} P' & M & P \\ \begin{pmatrix} a \\ b \end{pmatrix} = & \begin{pmatrix} cos45 & -sin45 \\ sin45 & cos45 \end{pmatrix} & \begin{pmatrix} x \\ y \end{pmatrix} \end{array}$$ $$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$ $$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$ $$a = \frac{\sqrt{2}}{2}(x-y)$$ $$b = \frac{\sqrt{2}}{2}(x+y)$$ *Since point $P'(a, b)$ lies on $xy = c^2$* $$ab = c^2$$ $$\frac{\sqrt{2}}{2} . \frac{\sqrt{2}}{2}(x+y)(x-y) = c^2$$ $$\frac{2}{4}(x^2 - y^2) = c^2$$ $$(x^2 - y^2) = 2c^2$$ *Hence the new equation of the curve is $(x^2 - y^2) = 2c^2$*

END