LOGARITHMS

A logarithm of a number to base x is the power to which x must be raised to give a number e.g.

$$x = a^n \Rightarrow a = \log_x m$$

The following 3 statements are equivalent

(i) $16 = 2^4$ and $\log_2 16 = 4$
(ii) $27 = 3^3$ and $\log_3 27 = 3$
(iii) $64 = 8^2$ and $log_8\ 64 = 2$

Logarithms appear in all sorts of calculations in engineering and science, business and economics. Before days of calculators, they were used to assist in the process of multiplication by replacing the operation of multiplication by addition similarly they enabled the operation of division to be replaced by subtraction.

Laws of Logarithms

(1) $log_a x + log_a y = log_a xy$
(2) $\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$
(3) $log_a x^m = mlog_a x$
(4) $log_a 1 = 0$
(5) $log_a b = \frac{log_c b}{log_c a}$ (Change of base law)

Example I

Prove the following laws of logarithms

(1) $log_a xy = log_a x + log_a y$
(2) $log_a \frac{x}{y} = log_a x - log_a y$
(3) $log_a x^m = mlog_a x$
(4) $log_a b = \frac{log_c b}{log_c a}$
(5) $log_a b = \frac{1}{log_a b}$

Solution:

(1) Suppose $x = a^n$ and $y = a^m$ then equivalent logarithmic forms are

$$log_a x = n \text{ and } log_a y = m$$

Using the first rule of indices,

$$xy = a^n x\, a^m$$

$$xy = a^{n+m}$$

Now the logarithmic form of the statement

$xy = a^{n+m}$ is $log_a xy = n + m$,

but $n = log_a x$ and $m = log_a y$

Putting this result together we have:

$$log_a xy = log_a x + log_a y$$

Therefore, $\mathbf{log_a xy = log_a x + log_a y}$ (as required.)

(2) For $\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$

Suppose $x = a^n$ and $y = a^m$ with equivalent respective logarithmic forms
$log_a x = n$ and $log_a y = m$.

$$\frac{x}{y} = a^n \div a^m$$

$$\frac{x}{y} = a^{n-m}$$

$$log_a \left(\frac{x}{y}\right) = log_a a^{n-m}$$

$$log_a \left(\frac{x}{y}\right) = n - m$$

$$log_a \left(\frac{x}{y}\right) = log_a x - log_a y$$

$$log_a x - log_a y = log_a \frac{x}{y}$$

$\therefore log_a \left(\frac{x}{y}\right) = log_a x - log_a y$ as required

(3) For $\mathbf{log_a x^m = mlog_a x}$

Suppose $x = a^n$ or $log_a x = n$.
Suppose we raise both sides of $x = a^n$ to the power m

$$x^m = (a^n)^m$$

Using the rules of indices this can be written as;

$$x^m = a^{nm}$$

Thinking of the quantity x^m as a single term, the logarithmic form is

$$\log_a x^m = \log_a a^{nm}$$

$$\log_a x^m = nm \log_a a$$

$$\log_a x^m = nm$$

$$m \log_a x^m = m(n)$$

$$\log_a x^m = m \log_x a$$

(4) For $\boldsymbol{log_a b = \frac{log_c b}{log_c a}}$

Let $log_a b = y$

$\therefore\ a^y = b$…………………… (1)

Introducing log_c on both sides of equation (1)

$$\log_c a^y = \log_c b$$
$$y\log_c a = \log_c b$$
$$y = \frac{\log_c b}{\log_c a}$$
$$log_a b = \frac{log_c b}{log_c a}$$

(5) For $log_a b = \frac{1}{log_a b}$

Let $y = log_a b$

$$b = a^y$$
$$log_b b = log_b a^y$$
$$log_b b = ylog_b a$$
$$y = \frac{log_b b}{log_b a}$$
$$y = \frac{1}{log_b a}$$
$$\therefore\ log_a b = \frac{1}{log_b a}$$

Example II

Solve $\log_5 x + 2\log_x 5 = 3$

Solution

$\log_5 x + 2\log_x 5 = 3$

But $log_x 5 = \frac{1}{log_5 x}$

$$\therefore\ log_5 x + \frac{2}{log_5 x} = 3$$

Let $log_5 x = m$

$$m + \frac{2}{m} = 3$$

$m^2 + 2 = 3m$

$m^2 - 3m + 2 = 0$

$(m-1)(m-2) = 0$

$m = 1\ and\ m = 2$

Since $log_5 x = 1,$

$\therefore\ 5^1 = x$

$\log_5 x = 2$

$5^2 = x$

$x = 25$

$\therefore x = 5\ or\ x = 25$

Example III

Solve $log_2 x - log_x 8 = 2$

Solution:

$$log_2 x - log_x 8 = 2$$
$$log_2 x - \frac{log_2 8}{log_2 x} = 2$$
$$log_2 x - \frac{log_2 2^3}{log_2 x} = 2$$
$$log_2 x - \frac{3log_2 2}{log_2 x} = 2$$
$$log_2 x - \frac{3}{log_2 x} = 2$$

Let $m = log_2 x$

$$m - \frac{3}{m} = 2$$

$m^2 - 3 = 2m$

$m^2 - 2m - 3 = 0$

$(m-3)(m+1) = 0$

$m = 3, m = -1$

$log_2 x = 3$

$2^3 = x$

$x = 8$

$\log_2 x = -1$

$2^{-1} = x$

$x = \frac{1}{2}$

Example IV

Solve for x: $3\log_2 x - \log_x 2 = 2$

Solution

$$3\log_2 x - \log_x 2 = 2$$

But $\log_x 2 = \frac{1}{log_2 x}$

$$\Rightarrow 3log_2 x - \frac{1}{log_2 x} = 2$$

Let $log_2 x = y$

$$3y - \frac{1}{y} = 2$$

$3y^2 - 1 = 2y$

$3y^2 - 2y - 1 = 0$

$3y^2 - 3y + y - 1 = 0$

$3y(y-1) + 1(y-1) = 0$

$(3y+1)(y-1) = 0$

$y = -\frac{1}{3}, y = 1$

Since $log_2 x = y$

$$\therefore log_2 x = -\frac{1}{3}$$
$$2^{-\frac{1}{3}} = x$$
$$log_2 x = 1$$

$$2^1 = x$$

Example IV

Solve $\ln(6x-5) = 3$

Solution

$$\ln(6x-5) = 3$$
$$log_e(6x-5) = 3$$
$$\therefore\ e^3 = 6x-5$$
$$6x = e^3 + 5$$
$$x = \frac{e^3+5}{6}$$
$$x = \frac{e^3+5}{6}$$

Example V

Prove that $log_6 x = \frac{log_3 x}{1+log_3 2}$

Solution

$$log_6 x = \frac{log_3 x}{log_3 6}$$
$$= \frac{log_3 x}{log_3 3 + log_3 2}$$
$$= \frac{log_3 x}{1 + log_3 2}$$
$$\therefore\ log_6 x = \frac{log_3 x}{1 + log_3 2}$$

Example VI (UNEB Question)

Given that $log_3 x = p\ and\ log_{18} x = q$. Prove that $log_6 3 = \frac{q}{p-q}$

Solution

$log_{18} x = q$

$\therefore\ 18^q = x$…………………….. (1)

$log_3 x = p$

$\therefore\ 3^p = x$……………………… (2)

Equating equation (1) and (2)

$18^q = 3^p$

$3^p = 18^q$

$log_6 3^p = log_6 18^q$

$plog_6 3 = qlog_6 18$

$plog_6 3 = q(log_6 6 + log_6 3)$

$plog_6 3 = qlog_6 6 + qlog_6 3$

$plog_6 3 - qlog_6 3 = qlog_6 6$

$(p-q)log_6 3 = q$

$log_6 3 = \frac{q}{p-q}$

Example VII

If $log_4 m = a, log_{12} m = b$. Prove that

$$log_3 48 = \frac{a+b}{a-b}$$

Solution

$log_4 m = a$

$4^a = m$

$4 = m^{\frac{1}{a}}$

$log_{12} m = b$

$12^b = m$

$12 = m^{\frac{1}{b}}$

$4 \times 12 = m^{\frac{1}{a}} \times m^{\frac{1}{b}}$

$48 = m^{\frac{1}{a}+\frac{1}{b}}$

$48 = m^{\frac{b+a}{ab}}$

$log_3 48 = log_3 m^{\frac{b+a}{ab}}$

$log_3 48 = \frac{b+a}{ab} log_3 m$ …………. (1)

But $m = 4^a$

Substituting for $m = 4^a$ in Eqn (1);

$$log_3 48 = \frac{b+a}{ab} log_3 4^a$$
$$log_3 48 = \frac{b+a}{ab} \times a log_3 4$$
$$log_3 48 = \frac{b+a}{b} log_3 4 \text{……………….. (2)}$$

Since $m = 4^a,\ 12^b = m$

$4^a = 12^b$

$log_3 4^a = log_3 12^b$

$alog_3 4 = blog_3 12$

$alog_3 4 = b[log_3 3 + log_3 4]$

$alog_3 4 = b + blog_3 4$

$alog_3 4 - blog_3 4 = b$

$log_3 4 = \frac{b}{a-b}$…………………….. (3)

Substituting eqn. (3) in eqn. (2)

$$log_3 48 = \left(\frac{b+a}{b}\right) \times \left(\frac{b}{a-b}\right)$$
$$log_3 48 = \frac{b+a}{a-b}$$

Example VIII (UNEB Question)

Prove that $\log_8 x = \frac{2}{3}\log_4 x$. Hence find $\log_8 6$ if $\log_4 3$ = 0.7925.

Solution

$$\log_8 x = \frac{\log_4 x}{\log_4 8} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

But $\log_4 8 = \frac{\log_2 8}{\log_2 4}$

$$= \frac{\log_2 2^3}{\log_2 2^2}$$

$$= \frac{3\log_2 2}{2\log_2 2}$$

$$\log_4 8 = \frac{3}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

substituting Eqn (2) in Eqn (1);

$$log_8 x = \frac{log_4 x}{3/2}$$

$$log_8 x = {}^2/_3\, log_4 6$$

$$= {}^2/_3\, [log_4 2 + log_4 3]$$

But, $log_4 2 = \frac{log_2 2}{log_2 4}$

$$= \frac{log_2 2}{log_2 2^2} = \frac{1}{2}$$

$$log_8 6 = {}^2/_3\, [0.5 + 0.7925]$$

$$= 0.867$$

Example (UNEB Question)

Solve $\log_x 5 + 4\log_5 x = 4$

Solution

$\log_x 5 + 4\log_5 x = 4$

$$\log_x 5 + 4\log_5 x = 4$$

$$\log_5 x = \frac{1}{\log_x 5}$$

$$\Rightarrow \log_x 5 + \frac{4}{\log_x 5} = 4$$

Let $m = \log_x 5$

$$m + \frac{4}{m} = 4$$

$\Rightarrow m^2 + 4 = 4m$

$m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

$m = 2$

$\log_x 5 = 2$

$x^2 = 5$

$x = \sqrt{5}$, $x = -\sqrt{5}$

Example IX

Solve $6\log_3 x + 6\log_{27} y = 7$

$4\log_9 x + 4\log_3 y = 9$

Solution

$$6log_3 x + 6log_{27} y = 7 \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$4log_9 x + 4log_3 y = 9 \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

From equation (1)

$$6log_3 x + \frac{6log_3 y}{log_3 27} = 7$$

$$6log_3 x + \frac{6log_3 y}{3log_3 3} = 7$$

$$6log_3 x + 2log_3 y = 7 \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

From equation (2)

$$\frac{4log_3 x}{log_3 9} + 4log_3 y = 9$$

$$\frac{4log_3 x}{2log_3 3} + 4log_3 y = 9$$

$$2log_3 x + 4log_3 y = 9 \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Let $A = \log_3 x$, $B = log_3 y$

$$6A + 2B = 7 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

$$2A + 4B = 9 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Solving equation 5 and 6 simultaneously

$$A = \frac{1}{2} \text{ and } B = 2$$

But, $A = log_3 x$

$$\therefore \frac{1}{2} = log_3 x$$

$$x = 3^{\frac{1}{2}}$$

$B = \log_3 y$

$2 = \log_3 y$

$y = 3^2$

$y = 9$

$x = 3^{1/2}$, $y = 9$

Example X

Given that $log_2 x + 2log_4 y = 4$, show that $xy = 16$. Hence solve for x and y in the equations.

$$log_2 x + 2log_4 y = 4$$

$$log_{10} x + y = 1$$

Solution

$$log_2 x + 2log_4 y = 4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

From eqn. (1)

$$log_2 x + \frac{2log_2 y}{log_2 4} = 4$$

But $\log_2 4 = 2\log_2 2$

$$= 2$$

$$\log_2 x + \frac{2\log_2 y}{2} = 4$$

$\log_2 x + \log_2 y = 4$

$\log_2 xy = 4$

$2^4 = xy$

$16 = xy$………………………… (2)

From

$log_{10}x + y = 1$

$\therefore\ 10^1 = x + y$

$y = 10 - x$ ………………………… (3)

Substituting Eqn (3) in Eqn (2);

Substituting Eqn (3) in Eqn (1)

$16 = x(10 - x)$

$16 = 10x - x^2$

$x^2 - 10x + 16 = 0$

$(x - 2)(x - 8) = 0$

$x = 2, x = 8$

$10 = x + y$

If $x = 2$, $10 = 2 + y$

$y = 8$

If $x = 8$, $y = 2$

Example XI

Solve for x and y;

$log_2(x - 3y + 2) = 0$

$(\log_2 x + 1) - 1 = 2\log_2 y$

Solution

$log_2(x - 3y + 2) = 0$

$2^0 = x - 3y + 2$

$1 = x - 3y + 2$

$x - 3y = -1$ …………………… (1)

$log_2(x + 1) - 1 = 2log_2 y$

$log_2(x + 1) - log_2 2 = 2log_2 y$

$\log_2 2(x + 1) = log_2 y^2$

$y^2 = 2(x + 1)$………………………… (2)

From eqn. (1)

$\frac{x+1}{3} = y$ ……………… (3)

Substituting Eqn (3) in Eqn (2)

$$\frac{(x + 1)^2}{9} = 2(x + 1)$$

$$\frac{(x + 1)^2}{9} + -2(x + 1) = 0$$

$$(x + 1)\left[\frac{x + 1}{9} - 2\right] = 0$$

$$x + 1 = 0 \quad \therefore x = -1$$

$$\frac{x + 1}{9} = 2$$

$$x + 1 = 18$$

$$x = 17$$

$$x = -1, x = 17$$

Example

Solve the equation $\log_4 x^2 - 6\log_x 4 - 1 = 0$

Solution

$\log_4 x^2 - 6\log_x 4 - 1 = 0$

But $\log_x 4 = \frac{1}{\log_4 x}$

$$\Rightarrow \log_4 x^2 - \frac{6}{\log_4 x} - 1 = 0$$

$$\Rightarrow 2\log_4 x - \frac{6}{\log_4 x} - 1 = 0$$

$$2y - \frac{6}{y} - 1 = 0$$

$$2y^2 - 6 - y = 0$$

$$(2y + 3)(y - 2) = 0$$

$$y = \frac{-3}{2},\ y = 2$$

If $y = 2 \Rightarrow \log_4 x = 2$

$4^2 = x$

$x = 16$

If $y = \frac{-3}{2}, \Rightarrow \log_4 x = \frac{-3}{2}$

$4^{-3/2} = x$

$x = \frac{1}{8}$

Application of Indices

Example XIII

Solve the equation $2^{2x+8} - 32(2^x) + 1 = 0$

Solution

$$2^{2x+8} - 32(2^x) + 1 = 0$$

$$(2^x)^2 . 2^8 - 32 . 2^x + 1 = 0$$

$$256(2^x)^2 - 32(2^x) + 1 = 0$$

Let $2^x = y$

$$256y^2 - 32y + 1 = 0$$

$$256y^2 - 16y - 16y + 1 = 0$$

$$16y(16y - 1) - 1(16y - 1) = 0$$

$$(16y - 1)(16y - 1) = 0$$

$$y = \frac{1}{16}$$

$$2^x = \frac{1}{16}$$

$$2^x = \frac{1}{2^4}$$

$$2^x = 2^{-4}$$

$$x = -4$$

Example IV

Solve the following equations

(i) $5^{2x} - 5^{x+1} + 4 = 0$

(ii) $9^x - 12(3^x) + 27 = 0$

(iii) $4^x + 2 = 3 \times 2^x$

(iv)

Solutions

(i) $5^{2x} - 5^{x+1} + 4 = 0$

$(5^x)^2 - 5^x.5^1 + 4 = 0$

$y^2 - 5y + 4 = 0$

$y^2 - y - 4y + 4 = 0$

$y(y-1) - 4(y-1) = 0$

$(y-4)(y-1) = 0$

$y = 4 \; and \; y = 1$

$5^x = 4$

$5^x = 1$

For $5^x = 4$,

$\Rightarrow \log_{10} 5^x = \log_{10} 4$

$x = \dfrac{\log_{10} 4}{\log_{10} 5} = 0.86135$

For $5^x = 1$, $\Rightarrow 5^x = 5^0$

$x = 0$

(ii) $9^x - 12(3^x) + 27 = 0$

$\Rightarrow (3^x)^2 - 12(3^x) + 27 = 0$

[Since $9^x = (3^2)^x = (3^x)^2$]

Let $y = 3^x$

$y^2 - 12y + 27 = 0$

$(y-3)(y-9) = 0$

$y = 3, y = 9$

But $y = 3^x$

$3^x = 3$

$\therefore x = 1$

$3^x = 9$

$\therefore x = 2$

(iii) $4^x + 2 = 3 \times 2^x$

$(2^2)^x + 2 = 3 \times 2^x$

$(2^x)^2 + 2 = 3 \; X \; 2^x$

Let $y = 2^x$

$y^2 + 2 = 3y$

$y^2 - 3y + 2 = 0$

$(y-1)(y-2) = 0$

$y = 1, y = 2$

$2^x = 1$

$x = 0$

$2^x = 2$

$x = 1$

$\therefore x = 0 \; and \; x = 1$

(iv) $9^x - 3^{x+1} = 10$

$(3^2)^x - 3^x.3^1 = 10$

$(3^x)^2 - 3^x.3^1 = 10$

$y^2 - 3y - 10 = 0$

$(y-5)(y+2) = 0$

$y = 5, y = -2$

$3^x = 5$

$x log_{10} 3 = log_{10} 5$

$x = \dfrac{\log_{10} 5}{\log_{10} 3}$

$x = 1.465$

Example (UNEB Question)

Solve the equations:

$9^x - 3^{x+1} = 10$

Solution

$9^x - 3^{x+1} = 10$

$(3^2)^x - 3^x \times 3 = 10$

$(3^x)^2 - 3 \cdot 3^x - 10 = 0$

Let $3^x = y$

$\Rightarrow y^2 - 3y - 10 = 0$

$(y-5)(y+2) = 0$

$y = 5$, $y = -2$

$3^x = 5$

$\Rightarrow \log_{10} 3^x = \log_{10} 5$

$x \log_{10} 3 = \log_{10} 5$

$x = \dfrac{\log_{10} 5}{\log_{10} 3}$

$x = 1.465$

ROOTS OF QUADRATIC EQUATIONS

Any equation of the form $ax^2 + bx + c = 0$ where $a \neq 0$ is called a quadratic equation.

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is the formula used to solve quadratic equations.

$b^2 - 4ac$ is a discriminant and determines the nature of the roots.

Nature of roots of quadratic Equations

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is the formula used to solve the equation $ax^2 + bx + c = 0$, we see that

Either

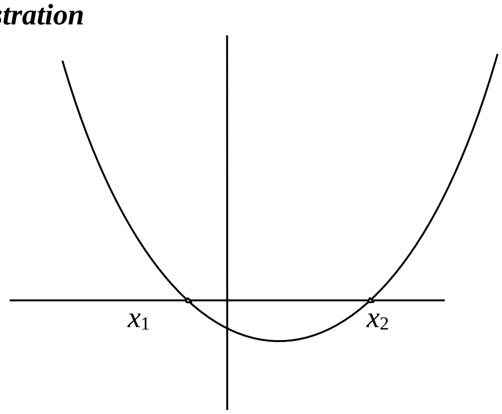
Or $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

Therefore in general, a quadratic equation has two solutions (called roots).

1. If $b^2 - 4ac$ is positive

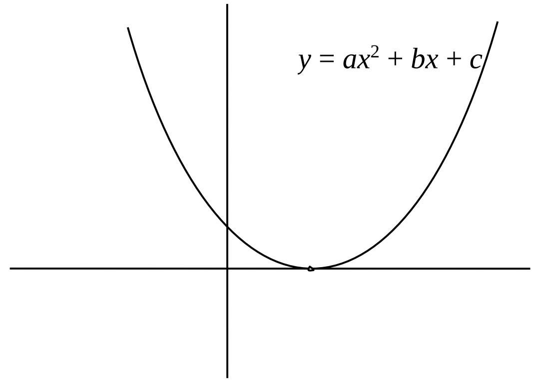
$\sqrt{b^2 - 4ac}$ can be evaluated and the equation has two real and distinct (different) roots.

Illustration



2. If $b^2 - 4ac$ is zero

$\sqrt{b^2 - 4ac}$ is zero; the equation is satisfied by only one value of x; $x = \dfrac{-b}{2a}$ and we say that it has repeated roots or equal roots.



3. If $b^2 - 4ac$ is negative

$\sqrt{b^2 - 4ac}$ has no real values; so the equation has no real roots.

To summarise the equation $ax^2 + bx + c = 0$;

Has two distinct roots if $b^2 - 4ac > 0$ **Has equal roots if $b^2 - 4ac = 0$** **Has no real roots if $b^2 - 4ac < 0$; and $b^2 - 4ac$ is called a discriminant.**

Example I

Determine the nature of roots of the following equations:

(a) $4x^2 - 7x + 3 = 0$
(b) $x^2 + ax + a^2 = 0$
(c) $x^2 - px - q = 0$

Solutions

(a) $4x^2 - 7x + 3 = 0$

Compare $4x^2 - 7x + 3 = 0$ with $ax^2 + bx + c = 0$, it follows that $a = 4$, $b = -7$ and $c = 3$

The discriminant $= b^2 - 4ac$

$b^2 - 4ac$

$\Rightarrow (-7)^2 - 4 \times 4 \times 3$

$= 49 - 48$

$= 1 > 0$

Since $b^2 - 4ac > 0$,

$\Rightarrow$ The equation $4x^2 - 7x + 3 = 0$ has two real distinct roots

(b) $\mathbf{x^2 + ax + a^2 = 0.}$

The discriminant is $b^2 - 4ac$

$b^2 - 4ac = (a)^2 - 4 \times 1(a^2)$

$= a^2 - 4a^2$

$= -3a^2$

Since a^2 is positive irrespective of the value of a, $b^2 - 4ac < 0$. So the equation $x^2 + ax + a^2 = 0$ has no real roots.

(c) $x^2 - px - q^2 = 0$

Comparing $x^2 - px - q^2 = 0$ with $ax^2 + bx + c = 0$, it follows that $a = 1$, $b = -p$, and $c = -q^2$.

The discriminant is $b^2 - 4ac$

$$b^2 - 4ac = (-p)^2 - 4(1)(-q^2)$$
$$= p^2 + 4q^2$$

$p^2 + 4q^2 > 0$ irrespective of the values of p and q.

Since $b^2 - 4ac > 0$,

$\Rightarrow x^2 - px - q^2 = 0$ has two real distinct roots.

Example II

Determine the nature of the roots of the following equations:

(a) $x^2 - 6x + 9 = 0$

(b) $x^2 - 2x + 1 = 0$

(c) $x^2 - 6x + 10 = 0$

(d) $4x^2 - 12x - 9 = 0$

Solution

(a) $x^2 - 6x + 9 = 0$

Comparing $x^2 - 6x + 9 = 0$ with $ax^2 + bx + c = 0$ gives $a = 1$, $b = -6$, $c = 9$

The discriminant is $b^2 - 4ac$

$$= (-6)^2 - 4 \times 1 \times 9$$
$$= 36 - 36$$
$$= 0$$

Since $b^2 - 4ac = 0$,

$\Rightarrow$ The equation $x^2 - 6x + 9 = 0$ has repeated roots.

(b) $x^2 - 2x + 1 = 0$

Comparing $x^2 - 2x + 1 = 0$ with $ax^2 + bx + c = 0$, it follows that $a = 1$, $b = -2$, $c = 1$

The discriminant is $b^2 - 4ac$

$$= (-2)^2 - 4 \times 1 \times 1$$
$$= 4 - 4$$
$$= 0$$

Since $b^2 - 4ac = 0$,

$\Rightarrow$ The equation $x^2 - 2x + 1 = 0$ has repeated roots.

(c) $x^2 - 6x + 10 = 0$

Comparing $x^2 - 6x + 10 = 0$ with $ax^2 + bx + c = 0$ gives $a = 1$, $b = -6$, $c = 10$

The discriminant is $b^2 - 4ac = 0$

$$= (-6)^2 - 4 \times 1 \times 10$$
$$= 36 - 40$$
$$= -4$$

Since $b^2 - 4ac < 0$,

$\Rightarrow$ The equation $x^2 - 6x + 10 = 0$ has no real roots.

(d) $4x^2 - 12x - 9 = 0$

Comparing $4x^2 - 12x - 9 = 0$ with $ax^2 + bx + c = 0$ gives a = 4, $b = -12$, $c = -9$.

The discriminant is $b^2 - 4ac$

$$= (4)^2 - 4 \times 4 \times (-9)$$
$$= 16 - 16 \times -9$$
$$= 16 + 144$$
$$= 160$$
$$b^2 - 4ac > 0$$

$\Rightarrow$ The equation $4x^2 - 12x - 9 = 0$ has two real distinct roots.

Example III

Find the values of k for which the following equations have equal roots.

(i) $3x^2 + kx + 12 = 0$

(ii) $x^2 - 5x + k = 0$

Solution

(i) $3x^2 + kx + 12 = 0$

For a quadratic equation to have equal roots,

$$b^2 = 4ac$$

Comparing $3x^2 + kx + 12 = 0$ with $ax^2 + bx + c = 0$ gives $a = 3$, $b = k$ and $c = 12$

$$b^2 = 4ac$$
$$\Rightarrow (k)^2 = 4 \times 3 \times 12$$
$$k^2 = 144$$
$$k = \pm 12$$
$$\Rightarrow k = 12, k = -12$$

(ii) $x^2 - 5x + k = 0$

For the equation $x^2 - 5x + k = 0$ to have real roots,

$b^2 = 4ac$

$$\Rightarrow (-5)^2 = 4 \times 1 \times k$$
$$25 = 4k$$
$$k = \frac{25}{4}$$

Example IV

Prove that $kx^2 + 2x - (k - 2) = 0$ has real roots for any values of k.

Solution

$$k^2 + 2x - (k - 2) = 0$$

Comparing $kx^2 + 2x - (k - 2) = 0$ with $ax^2 + bx + c = 0$ gives $a = k$, $b = 2$, $c = -(k - 2)$

The discriminant is $b^2 - 4ac$

$$= (2)^2 - 4 \times k[-(k - 2)]$$
$$= 4 + 4k(k - 2)$$
$$= 4 + 4k^2 - 8k$$
$$= 4k^2 - 8k + 4$$
$$= 4(k^2 - 2k + 1)$$
$$= 4(k - 1)^2$$

Since $4(k - 1)^2 > 0$, $\Rightarrow kx^2 + 2x - (k - 2)$ has two real distinct roots for any values of k.

Example V

Find the range of values k can take for $9x^2 + kx + 4 = 0$ to have two real distinct roots.

Solution

Comparing $9x^2 + kx + 4 = 0$ with $ax^2 + bx + c = 0$ gives $a = 9$, $b = k$, $c = 4$.

The discriminant is $b^2 - 4ac$

$$= (k)^2 - 4 \times 9 \times 4$$
$$= k^2 - 144$$

For two distinct real roots, $b^2 - 4ac > 0$

$$k^2 - 144 > 0$$
$$(k + 12)(k - 12) > 0$$

For the boundary conditions, $k = -12$, $k = 12$

	$k < -12$	$-12 < k < 12$	$k > 12$
$k + 12$	–ve	+ve	+ve
$k - 12$	–ve	–ve	+ve
$(k + 12)(k - 12)$	+ve	–ve	+ve

$\Rightarrow$ For $9x^2 + kx + 4 = 0$ to have real roots, the product $(k + 12)(k - 12)$ must be positive.

$\Rightarrow$ $k < -12$ and $k > 12$ are ranges of values for which $9x^2 + kx + 4 = 0$ has real distinct roots.

Example VI (UNEB Question)

Find the value of k for which the equation $\frac{x^2 - x + 1}{x - 1} = k$ has repeated roots. What are the repeated roots?

Solution

$$\frac{x^2 - x + 1}{x - 1} = k$$
$$x^2 - x + 1 = kx - k$$
$$x^2 - x - kx + 1 + k = 0$$
$$x^2 - (k + 1)x + k + 1 = 0$$

For repeated roots, $b^2 = 4ac$

Comparing $x^2 - (k + 1)x + k + 1 = 0$ with $ax^2 + bx + c = 0$ gives $a = 1$, $b = -(k + 1)$, $c = k + 1)$

$$b^2 = 4ac$$
$$[-(k + 1)]^2 = 4 \times 1(k + 1)$$
$$(k + 1)^2 = 4k + 4$$
$$k^2 + 2k + 1 = 4k + 4$$
$$k^2 - 2k - 3 = 0$$
$$(k - 3)(k + 1) = 0$$

$\Rightarrow k = 3$ **OR** $k = -1$

If $k = 3$,

$$x^2 - (k + 1)x + k + 1 = 0$$
$$x^2 - 4x + 4 = 0$$
$$(x - 2)^2 = 0$$
$$x = 2, x = 2$$

When $x = 3$, the repeated roots are $x=2$ and $x = 2$

If $x = -1$;

$$x^2 - (k + 1)x + k + 1 = 0$$
$$x^2 = 0$$
$$x = 0, x = 0$$

When $k= -1$, the repeated roots are $x=0$ and $x =0$.

Maximum and Minimum values of a quadratic function

Consider $y = ax^2 + bx + c$

Using the method of completing squares, the quadratic equation can be reduced to:

(i) $a(x - p)^2 + q$

(ii) $q - a(x - p)^2$

(i) Let $y = a(x - p)^2 + q$

Since $(x - p)^2$ is never negative, the least value of y occurs when $(x - p)^2 = 0$

(ii) For $y = q - a(x - p)^2$;

Since $(x - p)^2$ is never negative

$\Rightarrow$ The maximum value of y is q.

Examples

Find the greatest or least values of the following functions:

(a) $x^2 - 2x + 5$

(b) $5 - 4x - x^2$

(c) $x^2 - 3x + 5$

(d) $2x^2 - 4x + 5$

(e) $7 + x - x^2$

(f) $x^2 - 2$

(g) $2x - x^2$

Solution

(a) $\mathbf{x^2 - 2x + 5}$

By completing squares,

$x^2 - 2x + 5 = x^2 - 2x + \left[\tfrac{1}{2}(-2)\right]^2 - \left[\tfrac{1}{2}(-2)\right]^2 + 5$

$= x^2 - 2x + 1 - 1 + 5$

$= (x-1)^2 + 4$

$y = x^2 - 2x + 5$

$y = 4 + (x-1)^2$

The least value of y is 4 and it occurs when $(x-1)^2 = 0$

(b) $\mathbf{5 - 4x - x^2}$

By completing squares;

$5 - 5x - x^2 = 5 - (x^2 + 4x)$

$= 5 - (x^2 + 4x + 4) - {}^-4$

$= 5 - (x+2)^2 + 4$

$= 9 - (x+2)^2$

$y = 9 - (x+2)^2$

The greatest value is $y = 9$ and it occurs when $(x+2)^2 = 0$

(c) $\mathbf{x^2 - 3x + 5}$

By completing squares;

$x^2 - 3x + 5 = x^2 - 3x + \frac{9}{4} - \frac{9}{4} + 5$

$= \left(x - \frac{3}{2}\right)^2 + \frac{11}{4}$

$y = \left(x - \frac{3}{2}\right)^2 + \frac{11}{4}$

The least value of y is $\frac{11}{4}$ and it occurs when

$\left(x - \frac{3}{2}\right)^2 = 0$

(d) $\mathbf{2x^2 - 4x + 5}$

$2(x^2 - 2x) + 5$

$2(x^2 - 2x + 1) - 2 + 5$

$3 + 2(x-1)^2$

$y = 3 + 2(x-1)^2$

The least value of y is 3 and it occurs when $2(x-1)^2 = 0$

(e) $\mathbf{7 + x - x^2}$

$7 - (x^2 - x)$

$7 - (x^2 - x + \frac{1}{4}) - \frac{-1}{4}$

$7 - \left(x - \frac{1}{2}\right)^2 + \frac{1}{4}$

$y = \frac{29}{4} - \left(x - \frac{1}{2}\right)^2$

The greatest value of y is $\frac{29}{4}$ and it occurs when

$\left(x - \frac{1}{2}\right)^2 = 0$

(f) $\mathbf{x^2 - 2}$

The least value of y is -2 and it occurs when $x^2 = 0$

(g) $\mathbf{2x - x^2}$

$y = 2x - x^2$

$y = -(x^2 - 2x)$

By completing squares;

$y = -(x^2 - 2x + 1) - {}^-1$

$y = -(x-1)^2 + 1$

$y = 1 - (x-1)^2$

The greatest value of $y = 1$ and it occurs when $x = 1$

Sum & Product of the roots of Quadratic Equations

Consider the equation $ax^2 + bx + c = 0$

$\Rightarrow x^2 + \frac{bx}{a} + \frac{c}{a} = 0$ …………………….. (i)

Suppose α and β are the roots of the equation

$$x^2 + \frac{bx}{a} + \frac{c}{a} = 0$$

We can use α and β to form an algebraic equation in which the unknown quantity x is satisfied by putting $x = \alpha$ or $x = \beta$.

$x = \alpha$ or $x = \beta$

$x - \alpha = 0$ or $x - \beta = 0$

$(x - \alpha)(x - \beta) = 0$

$x^2 - \beta x - \alpha x + \alpha\beta = 0$

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$ ……………………… (ii)

Eqn (i) and Eqn (ii) have the same roots, must be precisely the same equation written in two different ways.

Equating coefficients of the same monomials in Eqn (i) and Eqn (ii);

$\Rightarrow$ $-(\alpha + \beta) = \frac{b}{a}$

$(\alpha + \beta) = \frac{-b}{a}$

Similarly, $\alpha\beta = \frac{c}{a}$

$\Rightarrow$ For a quadratic function with roots α and β,

Sum of roots = $\alpha + \beta = \frac{-b}{a}$

Product of roots $\alpha\beta = \frac{c}{a}$

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$\alpha^3 + \beta^3 = (\alpha\beta)^3$

$(\alpha - \beta) = \sqrt{(\alpha - \beta)^2}$

$$\Rightarrow \alpha - \beta = \sqrt{\alpha^2 - 2\alpha\beta + \beta^2}$$

$$\alpha - \beta = \sqrt{\alpha^2 + \beta^2 - 2\alpha\beta}$$

$$\alpha - \beta = \sqrt{(\alpha + \beta)^2 - 2\alpha\beta - 2\alpha\beta}$$

$$(\alpha - \beta) = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}.$$

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta}$$

$$\frac{1}{\alpha} \times \frac{1}{\beta} = \frac{1}{\alpha\beta}$$

$\Rightarrow$ **The following are important formulae used under roots of quadratic equations.**

$\boldsymbol{(\alpha^2 + \beta^2) = (\alpha + \beta)^2 - 2\alpha\beta}$

$\boldsymbol{\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)}$

$\boldsymbol{\alpha - \beta = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}}$

Example I

If α and β are roots of the equation $x^2 + 8x + 1 = 0$, find the values of

(a) $\alpha\beta$
(b) $\alpha + \beta$
(c) $\alpha^2\beta + \alpha\beta^2$
(d) $\alpha^2 + \beta^2$

Solution

(a) $x^2 + 8x + 1 = 0$

Comparing $x^2 + 8x + 1 = 0$ with $ax^2 + bx + c = 0$ gives

$a = 1$, $b = 8$, $c = 1$

$\alpha\beta = \frac{c}{a}$

$\alpha\beta = \frac{1}{1} = 1$

(b) $\boldsymbol{\alpha + \beta = \frac{-b}{a}}$.

$\alpha + \beta = \frac{-8}{1}$

$\alpha + \beta = -8$

(c) $\boldsymbol{\alpha^2\beta + \alpha\beta^2}$

$= \alpha\beta(\alpha + \beta)$
$= 1(-8)$
$= -8$

$\therefore \alpha^2\beta + \alpha\beta^2 = -8$

(d) $\boldsymbol{\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta}$

$\alpha + \beta = -8$
$\alpha\beta = 1$
$\alpha^2 + \beta^2 = (-8)^2 - 2 \times 1$
$= 64 - 2$
$= 62$

Example II

If α and β are roots of the equation $x^2 - x - 3 = 0$, state the values of $\alpha + \beta$ and $\alpha\beta$ and find the values of:

(a) $\alpha^2 + \beta^2$
(b) $(\alpha - \beta)^2$
(c) $\alpha^3 + \beta^3$

Solution

Comparing $x^2 - x - 3 = 0$ with $ax^2 + bx + c = 0$ gives $a = 1$, $b = -1$, $c = -3$

$\alpha + \beta = \frac{-b}{a}$

$\Rightarrow \alpha + \beta = \frac{-^{-}1}{1} = 1$

$\alpha\beta = \frac{c}{a}$

$\alpha\beta = \frac{-3}{1} = -3$

(a) $\alpha^2 + \beta^2$

$(\alpha + \beta)^2 - 2\alpha\beta$
$= (1)^2 - 2(-3)$
$= 1 - {}^{-}6$
$= 7$

(b) $(\alpha - \beta)^2 = \alpha^2 - 2\alpha\beta + \beta^2$
$= \alpha^2 + \beta^2 - 2\alpha\beta$

But $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= (\alpha + \beta)^2 - 4\alpha\beta$

$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

$= 1^2 - 4 \times (-3)$

$= 13$

(c) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\,\alpha\beta(\alpha + \beta)$

$\alpha^3 + \beta^3 = (1)^3 - 3(-3)(1)$

$= 1 + 9$

$= 10$

Example III

If α and β are roots of the equation $2x^2 - 5x + 1 = 0$, find the values of:

(a) $\boldsymbol{\alpha + \beta}$

(b) $\alpha\,\beta$

(c) $\alpha^2 + 3\alpha\beta + \beta^2$

(d) $\alpha^2 - 3\alpha\beta + \beta^2$.

(e) $\alpha^3\beta + \alpha\beta^3$

(f) $\frac{1}{\beta} + \frac{1}{\alpha}$

Solution

Comparing $2x^2 - 5x + 1 = 0$ with $ax^2 + bx + c = 0$ gives $a = 2$, $b = -5$ and $c = 1$

$$\alpha + \beta = \frac{-b}{a}$$

$$\alpha + \beta = -\frac{-5}{2}$$

$$\Rightarrow \alpha + \beta = \frac{5}{2}$$

$$\alpha\,\beta = \frac{c}{a}$$

$$\Rightarrow \alpha\,\beta = \frac{1}{2}$$

(c) $\boldsymbol{\alpha^2 + 3\alpha\beta + \beta^2}$

$= (\alpha^2 + \beta^2) + 3\alpha\beta$

$= (\alpha + \beta)^2 - 2\alpha\beta + 3\alpha\beta$

$= (\alpha + \beta)^2 + \alpha\beta$

$$= \left(\frac{5}{2}\right)^2 + \frac{1}{2}$$

$$= \frac{25}{4} + \frac{1}{2}$$

$$= \frac{27}{4}$$

(d) $\boldsymbol{\alpha^2} - 3\alpha\beta + \beta^2$

$\alpha^2 + \beta^2 - 3\alpha\beta$

$(\alpha + \beta)^2 - 2\,\alpha\beta - 3\alpha\beta$

$(\alpha + \beta)^2 - 5\alpha\beta$

$$= \left(\frac{5}{2}\right)^2 - 5\left(\frac{1}{2}\right)$$

$$= \frac{25}{4} - \frac{5}{2}$$

$$= \frac{15}{4}$$

(d) $\boldsymbol{\alpha^3}\,\beta + \alpha\beta^3$

$= \alpha\,\beta\,(\alpha^2 + \beta^2)$

$= \alpha\,\beta[(\alpha + \beta)^2 - 2\,\alpha\beta]$

$$= \frac{1}{2}\left[\left(\frac{5}{2}\right)^2 - 2 \times \frac{1}{2}\right]$$

$$= \frac{1}{2}\left(\frac{5}{4} - 1\right) = \frac{1}{8}$$

(f) $\frac{1}{\beta} + \frac{1}{\alpha}$

$$= \frac{\alpha + \beta}{\alpha\beta}$$

$$= \frac{5/2}{1/2}$$

$= 5$

Example IV

If α and β are roots of the equation $6x^2 + 2x - 3 = 0$, find the values of:

(a) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ (b) $\frac{1}{\beta^2} + \frac{1}{\alpha^2}$

(c) $\frac{2\alpha\beta}{1 + \frac{\alpha}{\beta}}$ (d) $\frac{1}{\alpha\beta} - \frac{1}{\alpha} - \frac{1}{\alpha}$

(e) $\alpha^3 + \beta^3$ (f) $\frac{1}{\alpha^3} + \frac{1}{\beta^3}$

Solution

Comparing $6x^2 + 2x - 3 = 0$ with $ax^2 + bx + c = 0$, gives $a = 6$, $b = 2$, $c = -3$

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta}$$

$$\frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}$$

$$\alpha + \beta = \frac{-b}{a}$$

$$\alpha + \beta = \frac{-2}{6} = \frac{-1}{3}$$

$$\alpha\beta = \frac{c}{a}$$

$$\alpha\beta = \frac{-3}{6} = \frac{-1}{2}$$

$$\frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{(\frac{-1}{3})^2 - 2(\frac{1}{2})}{-1/2}$$

$$= \frac{1/9 + 1}{-1/2}$$

$$= \frac{-20}{9}$$

$$\Rightarrow \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{-20}{9}$$

(b) $$\frac{1}{\beta^2} + \frac{1}{\alpha^2} = \frac{\alpha^2 + \beta^2}{\alpha^2\beta^2}$$

$$= \frac{(\alpha+\beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$$

$$= \frac{(-1/3)^2 - 2(-1/2)}{(-1/2)^2}$$

$$= \frac{\frac{1}{9} + 1}{1/4}$$

$$= \frac{10/9}{1/4} = \frac{40}{9}$$

$$\frac{1}{\beta^2} + \frac{1}{\alpha^2} = \frac{40}{9}$$

(c) $$\frac{2\beta}{1 + \beta/\alpha} = \frac{2\beta}{\frac{\alpha+\beta}{\alpha}}$$

$$= \frac{2\alpha\beta}{\alpha+\beta}$$

$$= \frac{2(\frac{-1}{2})}{\frac{-1}{3}}$$

$$= \frac{-1}{-1/3} = 3$$

$$\Rightarrow \frac{2\beta}{1+\frac{\beta}{\alpha}} = 3$$

(c) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\,\alpha\,\beta(\alpha + \beta)$

$$= \left(\frac{-1}{3}\right)^3 - 3\left(\frac{-1}{2}\right)\left(\frac{-1}{3}\right)$$

$$= \frac{-1}{27} - \frac{1}{2}$$

$$= \frac{-29}{54}$$

(f) $$\frac{1}{\alpha^3} + \frac{1}{\beta^3} = \frac{\beta^3 + \alpha^3}{\alpha^3\beta^3}$$

$$= \frac{(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)}{(\alpha\beta)^3}$$

$$= \frac{(\frac{-1}{3})^3 - 3(\frac{-1}{2})(\frac{-1}{3})}{(\frac{-1}{2})^3}$$

$$= \frac{\frac{-1}{27} - \frac{1}{2}}{-1/8} = \frac{\frac{-29}{54}}{-1/8}$$

$$= \frac{116}{27}$$

Example V

If α^2 and β^2 are roots of the equation $x^2 - 21x + 4 = 0$, and α and β are both positive, find $\alpha\,\beta$ and $\alpha + \beta$.

Solution

Comparing $x^2 - 21x + 4 = 0$ with $ax^2 + bx + c = 0$ gives $a = 1$, $b = -21$, $c = 4$

$$\alpha^2 + \beta^2 = \frac{-b}{a}$$

$$= \frac{21}{1}$$

$$= 21$$

$\alpha^2\,\beta^2 = 4$

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\,\alpha\,\beta$

$(\alpha + \beta)^2 - 2\alpha\beta = 21$

$(\alpha + \beta)^2 - 2 \times 2 = 21$

$(\alpha + \beta)^2 = 25$

$(\alpha + \beta) = 5$

Example VI

Write down the equation whose roots are:

(a) 3, 4 (b) -2, ½

(c) $\frac{1}{3}, \frac{-2}{5}$ (d) $\frac{-1}{4}, 0$

(e) a^2, a^2 (f) $-(k + 1), k^2 - 3$

(g) $\frac{b}{a}, \frac{c^2}{b}$

Solution

Any quadratic equation is given by
x^2 – (sum of roots)x + product of roots = 0

(a) $x = 3, 4$

Sum of roots = 3 + 4
= 7

Product of roots = 3 × 4
= 12

x^2 – (sum of roots)x + product = 0
$x^2 - (7)x + 12 = 0$
$x^2 - 7x + 12 = 0$

(b) $x = -2, x = \frac{1}{2}$

Sum of roots = $-2 + \frac{1}{2}$

Sum of roots = $\frac{-3}{2}$

Product of roots = -1

x^2 – (sum of roots)x + product = 0

$x^2 - \left(\frac{-3}{2}\right)x + -1 = 0$

$x^2 + \frac{3}{2}x - 1 = 0$

$2x^2 + 3x - 2 = 0$

(c) $x = \frac{1}{3}, x = \quad x = \frac{-2}{5}$

Sum of the roots = $\frac{1}{3} + \frac{-2}{5}$

$= \frac{-1}{15}$

Product of the roots = $\frac{1}{3} \times \frac{-2}{5}$

$= \frac{-2}{15}$

x^2 – (sum of roots)x + product of roots = 0

$x^2 - \left(\frac{-1}{5}\right)x + \frac{-2}{15} = 0$

$15x + x - 2 = 0$

(d) $x = \frac{-1}{4}, \ x = 0$

Sum of roots = $\frac{-1}{4} + 0 = \frac{-1}{4}$

Product of roots = 0

x^2 – (sum of roots)x + product of roots = 0

$x^2 - \frac{-1}{4}x = 0$

$4x^2 + x = 0$

(e) $x = a^2, x = a^2$

Sum of the roots = $a^2 + a^2$
$= 2a^2$

Product of the roots = $a^2 \times a^2$
$= a^4$

x^2 – (sum of roots)x + product of roots = 0
$x^2 - (2a^2)x + a^4 = 0$
$x^2 - 2a^2x + a^4 = 0$

(f) $-(k + 1), k^2 - 3$

Sum of roots = $-k + 1 + k^2 - 3$
$= k^2 - k - 2$

Product of roots = $-(k + 1)(k^2 - 3)$
$= -(k^3 - 3k + k^2 - 3)$

Product of roots = $-k^3 + 3k - k^2 + 3$

x^2 – (sum of roots)x + product of roots = 0
$x^2 - (k^2 - k - 2)x + 3k + 3 - k^2 - k^3 = 0$

(g) $x = \frac{b}{a}, x = \frac{c^2}{b}$

Sum of roots = $\frac{b}{a} + \frac{c^2}{b}$

$= \frac{b^2 + ac^2}{ab}$

Product of the roots = $\frac{b}{a} \times \frac{c^2}{b} = \frac{c^2}{a^2}$

$x^2 - \left(\frac{b^2 + ac^2}{ab}\right)x + \frac{c^2}{a^2} = 0$

$a^2bx^2 - a(b^2 + a^2c)x + c^2 = 0$

Example VII

The roots of the equation $x^2 - 2x + 3 = 0$ are α and β. Find the equation whose roots are:

(a) $\alpha + 2, \ \beta + 2$

(b) $\frac{1}{\beta}, \ \frac{1}{\alpha}$

(c) α^2, β^2

(d) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$

Solution

$x^2 - 2x + 3 = 0$

Comparing $x^2 - 2x + 3 = 0$ with $ax^2 + bx + c = 0$ gives $a = 1$, $b = -2$, and $c = 3$.

$\alpha + \beta = \dfrac{-b}{a} = \dfrac{-(-2)}{1} = 2$

$\alpha\beta = \dfrac{c}{a} = \dfrac{3}{1} = 3$

New sum of roots $= \alpha + 2 + \beta + 2$
$= \alpha + \beta + 4$
$= 2 + 4$
$= 6$

New product of the roots $= (\alpha + 2)(\beta + 2)$
$= \alpha\beta + 2\alpha + 2\beta + 4$
$= \alpha\beta + 2(\alpha + \beta) + 4$
$= 3 + 2(2) + 4$
$= 3 + 4 + 4$
$= 11$

Any quadratic equation is given by:

$x^2 - (\text{sum of roots})x + \text{product of roots} = 0$

$x^2 - (2)x + 11 = 0$

$x^2 - 2x + 11 = 0$

(b) $\dfrac{1}{\alpha}, \dfrac{1}{\beta}$

$\alpha + \beta = 2$

$\alpha\beta = 3$

New sum of roots $= \dfrac{1}{\alpha} + \dfrac{1}{\beta}$

$= \dfrac{\alpha + \beta}{\alpha\beta} = \dfrac{2}{3}$

New product of roots $= \dfrac{1}{\alpha} \times \dfrac{1}{\beta} = \dfrac{1}{\alpha\beta}$

$= \dfrac{1}{3}$

$x^2 - (\text{sum of the roots})x + \text{product of roots} = 0.$

$x^2 - \left(\dfrac{2}{3}\right)x + \dfrac{1}{3} = 0$

$3x^2 - 2x + 1 = 0$

(c) α^2, β^2

$\alpha + \beta = 2$

$\alpha\beta = 3$

New sum of the roots $= \alpha^2 + \beta^2$
$= (\alpha + \beta)^2 - 2\alpha\beta$
$= 2^2 - 2(3)$
$= 4 - 6$
$= -2$

New product of roots $= \alpha^2\beta^2$
$= (\alpha\beta)^2$
$= 3^2$
$= 9$

$x^2 - (\text{sum of the roots})x + \text{product of the roots} = 0$

$x^2 - (-2)x + 9 = 0$

$x^2 + 2x + 9 = 0$

(d) $\dfrac{\alpha}{\beta}, \dfrac{\beta}{\alpha}$

$\alpha + \beta = 2; \qquad \alpha\beta = 3$

New sum of roots $= \dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$

$= \dfrac{\alpha^2 + \beta^2}{\alpha\beta}$

$= \dfrac{(\alpha+\beta)^2 - \alpha\beta}{\alpha\beta}$

$= \dfrac{2^2 - 2\times 3}{3} = \dfrac{4-6}{3}$

$= \dfrac{-2}{3}$

New product of roots $= \dfrac{\alpha}{\beta} \times \dfrac{\beta}{\alpha}$

$= 1$

$x^2 - (\text{sum of roots})x + \text{product of roots} = 0$

$x^2 - \left(\dfrac{-2}{3}\right)x + 1 = 0$

$3x^2 + 2x + 3 = 0$

(e) $\alpha - \beta, \ \beta - \alpha$

$\alpha + \beta = 2, \ \alpha\beta = 3$

New sum of the roots $= \alpha - \beta + \beta - \alpha$
$= \alpha + \beta - (\alpha + \beta)$
$= 0$

New product of the roots $= (\alpha - \beta)(\beta - \alpha)$
$= \alpha\beta - \alpha^2 - \beta^2 + \alpha\beta$
$= 2\alpha\beta - [(\alpha^2 + \beta^2)]$
$= 2\alpha\beta - [(\alpha + \beta)^2 - 2\alpha\beta]$
$= 4\alpha\beta - (\alpha + \beta)^2$
$= 4 \times 3 - (2^2)$
$= 8$

$x^2 - (\text{sum of the roots})x + \text{product of roots} = 0$

$x^2 - (0)x + 8 = 0$

$x^2 + 8 = 0$

Example VIII

The roots of the equation $2x^2 + 7x - 3 = 0$ are α and β. Find the equation whose roots are $\left(\alpha + \dfrac{5}{\beta}\right)$ and $\left(\beta + \dfrac{5}{\alpha}\right)$.

Solution

Sum of the roots = $\alpha + \beta = \frac{-b}{a}$

$= \frac{-7}{2}$

Product of the roots = $\alpha\beta = \frac{c}{a} = \frac{-3}{2}$

New sum of the roots = $\alpha + \frac{5}{\beta} + \beta + \frac{5}{\alpha}$

$= \alpha + \beta + 5\left(\frac{1}{\beta} + \frac{1}{\alpha}\right)$

$= \alpha + \beta + 5\left(\frac{\alpha+\beta}{\alpha\beta}\right)$

$= \frac{-7}{2} + 5\left(\frac{-7/2}{-3/2}\right)$

$= \frac{-7}{2} + \frac{-35}{3}$

$= \frac{-91}{6}$

New product of roots = $\left(\alpha + \frac{5}{\beta}\right)\left(\beta + \frac{5}{\alpha}\right)$

$= \alpha\beta + 5 + 5 + \frac{25}{\alpha\beta}$

$= \alpha\beta + \frac{25}{\alpha\beta} + 10$

$= \frac{-3}{2} + \frac{25}{-3/2} + 10$

$= \frac{-49}{10}$

x^2 – (sum of roots)x + product of roots = 0

$x^2 - \left(\frac{-91}{6}\right)x + \frac{-49}{6} = 0$

$6x^2 + 91x - 49 = 0$

Example IX

Given that α and β are roots of the equation $4x^2 + 7x - 5 = 0$. Find the equation whose roots are $2\alpha - 1$ and $2\beta - 1$

Solution

Comparing $4x^2 + 7x - 5 = 0$ with $ax^2 + bx + c = 0$ gives $a = 4$, $b = 7$, and $c = -5$

Sum of the roots = $\alpha + \beta = \frac{-b}{a}$

$\Rightarrow \alpha + \beta = \frac{-7}{4}$

Product of the roots = $\alpha\beta = \frac{c}{a}$

$= \frac{-5}{4}$

New sum of roots $= 2\alpha - 1 + 2\beta - 1$

$= 2(\alpha + \beta) - 2$

$= 2\left(\frac{-7}{4}\right) - 2$

$= \frac{-14}{4} - 2$

$= \frac{-22}{4} = \frac{-11}{2}$

New product of roots = $(2\alpha - 1)(2\beta - 1)$

$= 4\alpha\beta - 2\alpha - 2\beta + 1$

$= 4\alpha\beta - 2(\alpha + \beta) + 1$

$= 4\left(\frac{-5}{4}\right) - 2\left(\frac{-7}{4}\right) + 1$

$= -5 + \frac{14}{4} + 1$

$= \frac{-20 + 14 + 4}{4}$

$= \frac{-1}{2}$

x^2 – (sum of roots)x + product of roots = 0

$x^2 - \left(\frac{-11}{2}\right)x + \frac{1}{2} = 0$

$x^2 + 11x - 1 = 0$

Example (UNEB Question)

If the roots of the equation $x^2 + 2x + 3 = 0$ are α and β, form an equation whose roots are $\alpha^2 - \beta$ and $\beta^2 - \alpha$.

Solution

Comparing $x^2 + 2x + 3 = 0$ with $ax^2 + bx + c = 0$ gives $a = 1$, $b = 2$ and $c = 3$

Sum of roots = $\alpha + \beta = \frac{-b}{a}$

Product of the roots = $\frac{c}{a}$

$\Rightarrow \alpha + \beta = \frac{-2}{1} = -2$

$\Rightarrow \alpha\beta = \frac{c}{a} = \frac{3}{1} = 3$

New sum of the roots = $\alpha^2 - \beta + \beta^2 - \alpha$

$= \alpha^2 + \beta^2 - (\alpha + \beta)$

$= (\alpha + \beta)^2 - 2\alpha\beta - (\alpha + \beta)$

$= (-2)^2 - 2(3) - (-2)$

$= 4 - 6 + 2$

$= 0$

New product of roots $= (\alpha^2 - \beta)(\beta^2 - \alpha)$

$= \alpha^2\beta^2 - \alpha^3 - \beta^3 + \alpha\beta$

$= (\alpha\beta)^2 - (\alpha^3 + \beta^3) + \alpha\beta$

$= (\alpha\beta)^2 - [(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)] + \alpha\beta$

$= 3^2 - [(-2)^3 - 3(3)(-2)] + 3$

$= 9 - [-8 + 18] + 3$

$= 9 - [10] + 3$

$= 2$

x^2 – (sum of the roots)x + product of roots = 0

$x^2 - (0)x + 2 = 0$

$x^2 + 2 = 0$

Example (UNEB Question)

If α and β are roots of the equation $x^2 - px + q = 0$, find the equation whose roots are $\frac{\alpha^3 - 1}{\alpha}$ and $\frac{\beta^3 - 1}{\beta}$

Solution

Comparing $x^2 - px + q = 0$ with $ax^2 + bx + c = 0$ gives $a = 1$, $b = -p$, $c = q$

Sum of the roots $= \alpha + \beta = \frac{-b}{a}$

$= \frac{-(-p)}{1} = p$

Product of the roots $= \alpha\beta = \frac{c}{a} = \frac{q}{1} = 1$

New sum of the roots $= \frac{\alpha^3 - 1}{\alpha} + \frac{\beta^3 - 1}{\beta}$

$$= \frac{\alpha^3\beta - \beta + \alpha\beta^3 - \alpha}{\alpha\beta}$$

$$= \frac{-(\alpha + \beta) + \alpha\beta(\alpha^2 + \beta^2)}{\alpha\beta}$$

$$= \frac{-(\alpha + \beta) + \alpha\beta[(\alpha + \beta)^2 - 2\alpha\beta]}{\alpha\beta}$$

$$= \frac{-(p) + q[p^2 - 2q]}{q}$$

$$= \frac{-p + p^2q - 2q^2}{q}$$

New product of the roots $= \left(\frac{\alpha^3 - 1}{\alpha}\right)\left(\frac{\beta^3 - 1}{\beta}\right)$

$$= \frac{\alpha^3\beta^3 - \alpha^3 - \beta^3 + 1}{\alpha\beta}$$

$$= \frac{(\alpha\beta)^3 - (\alpha^3 + \beta^3) + 1}{\alpha\beta}$$

$$= \frac{(\alpha\beta)^3 - [(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)] + 1}{\alpha\beta}$$

$$= \frac{q^3 - [p^3 - 3q(p)] + 1}{q}$$

$$= \frac{q^3 - p^3 + 3pq + 1}{q}$$

x^2 – (sum of the roots)x + product of the roots = 0

$$x^2 - \frac{(p^2q - p - 2q^2)}{q}x + \frac{q^3 - p^3 + 3pq + 1}{q} = 0$$

$$qx^2 - (p^2q - p - 2q^2)x + q^3 - p^3 + 3pq + 1 = 0$$

Example (UNEB Question)

If α and β are roots of the equation $ax^2 + bx + c = 0$, express $(\alpha - \beta)(\beta - 2\alpha)$ in terms of a, b and c. Hence deduce the condition for the root to be twice the other.

Solution

$\alpha + \beta = \frac{-b}{a}$

$\alpha\beta = \frac{c}{a}$

$(\alpha - 2\beta)(\beta - 2\alpha)$

$\alpha\beta - 2\alpha^2 - 2\beta^2 + 4\alpha\beta$

$\alpha\beta - 2(\alpha^2 + \beta^2) + 4\alpha\beta$

$= \alpha\beta - \alpha[(\alpha + \beta)^2 - 2\alpha\beta] + 4\alpha\beta$

$= 5\alpha\beta - 2[(\alpha + \beta)^2 - 2\alpha\beta]$

$$= \frac{5c}{a} - 2\left[\left(\frac{-b}{a}\right)^2 - 2\left(\frac{c}{a}\right)\right]$$

$$= \frac{5c}{a} - 2\left[\frac{b^2}{a^2} - \frac{2c}{a}\right]$$

$$= \frac{5c}{a} - 2\left(\frac{b^2 - 2ac}{a^2}\right)$$

$$= \frac{5ac - 2b^2 + 4ac}{a^2}$$

$$= \frac{9ac - 2b^2}{a^2}$$

$\Rightarrow (\alpha - 2\beta)(\beta - 2\alpha)$

For one root to be twice the other, $(\alpha - 2\beta)(\beta - 2\alpha) = 0$

$$\Rightarrow = \frac{9ac - 2b^2}{a^2} = 0$$

$9ac = 2b^2$

Example (UNEB Question)

Given that α and β are roots of the equation $x^2 + px + q = 0$, express $(\alpha - \beta^2)(\beta - \alpha^2)$ in terms of p and q.

Deduce that for one root to be a square of another root, $p^3 - 3pq + q^2 + q = 0$

Solution

Sum of the roots = $\alpha + \beta = \frac{-b}{a}$

$\alpha + \beta = -p$

$\alpha\beta = q$

$(\alpha + \beta^2)(\beta - \alpha^2)$

$= \alpha\beta - \alpha^3 - \beta^3 + (\alpha\beta)^2$

$= \alpha\beta - \left[(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)\right] + (\alpha\beta)^2$

$= q - [(-p^3 - 3q(-p)] + q^2$

$= q + p^3 - 3pq + q^2$

For one root to be a square of the other,

$(\alpha - \beta^2)(\beta - \alpha^2) = 0$

$\alpha = \beta^2, \quad \beta = \alpha^2$

$(\alpha - \beta^2)(\beta - \alpha^2) = q + p^3 - 3pq + q^2$

$\Rightarrow p^3 - 3pq + q + q^2 = 0$

Example (UNEB Question)

Given that α and β are roots of the equation $ax^2 + bx + c = 0$, determine the equation whose roots are $\alpha + \beta$ and $\alpha^3 + \beta^3$.

Solution

Sum of roots = $\alpha + \beta = \frac{-b}{a}$

Product of roots = $\alpha\beta = \frac{c}{a}$

New sum of roots = $(\alpha + \beta) + \alpha^3 + \beta^3$

$$= (\alpha+\beta) + \left[(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)\right]$$

$$= \frac{-b}{a} + \left[\frac{-b^3}{a^3} - \frac{3c}{a}\left(\frac{-b}{a}\right)\right]$$

$$= \frac{-b}{a} + \left[\frac{-b^3}{a^3} + \frac{3bc}{a^2}\right]$$

$$= \frac{-b}{a} + \left[\frac{3abc - b^3}{a^3}\right]$$

$$= \frac{-a^2b + 3abc - b^3}{a^3}$$

$$= -\left(\frac{a^2b + b^3 - 3abc}{a^3}\right)$$

New product of roots = $(\alpha + \beta)(\alpha^3 + \beta^3$

$= (\alpha + \beta)[(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)]$

$$= \frac{-b}{a}\left[\frac{-b^3}{a^3} - \frac{3c}{a}\left(\frac{-b}{a}\right)\right]$$

$$= \frac{-b}{a}\left[\frac{-b^3}{a^3} + \frac{3cb}{a^2}\right]$$

$$= \frac{-b}{a}\left[\frac{-b^3 + 3abc}{a^3}\right]$$

$$= \frac{-3ab^2c + b^4}{a^4}$$

$$= \frac{b^4 - 3ab^2c}{a^4}$$

x^2 – (sum of roots)x + product of roots = 0

$$x^2 + \left(\frac{a^2b + b^3 - 3abc}{a^3}\right)x + \frac{b^4 - 3ab^2c}{a^4} = 0$$

$a^4x^2 + (a^2b + b^3 - 3abc)x + b^4 + 3ab^2c = 0$

Example (UNEB Question)

Given that equations $y^2 + py + q = 0$ and $y^2 + my + k = 0$ have a common root, show that $(q - k)^2 = (m - p)(pk - mq)$

Solution

Let the common root be α.

$\Rightarrow \alpha^2 + p\alpha + q = 0$ ……………………. (i)

$\alpha^2 + m\alpha + k = 0$ …………………… (ii)

Eqn (i) – Eqn (ii);

$\Rightarrow (p - m)\alpha + q - k = 0$

$-(m - p)\alpha + q - k = 0$

$\alpha = \frac{q-k}{m-p}$ ……………………….. (iii)

Substituting Eqn (iii) in Eqn (i);

$$\frac{(q-k)^2}{(m-p)^2} + p\left(\frac{q-k}{m-p}\right) + q = 0$$

$(q - k)^2 + p(m - p)(q - k) + q(m - p)^2 = 0$

$(q - k)^2 + (m - p)(pq - pk + qm - pq) = 0$

$(q - k)^2 + (m - p)(qm - pk) = 0$

$(q - k)^2 - (m - p)(pk - qm) = 0$

$(q - k)^2 = (m - p)(pk - qm)$

Example

If α and β are roots of $px^2 + qx + r = 0$, form an equation with algebraic integral coefficients whose roots are $\frac{1-\alpha}{1+\alpha}$ and $\frac{1-\beta}{1+\beta}$

Solution

$\alpha + \beta = \frac{-b}{a}$

$$\Rightarrow \alpha + \beta = \frac{-q}{p}$$

$$\alpha\beta = \frac{c}{a}$$

$$\Rightarrow \alpha\beta = \frac{r}{p}$$

New sum of the roots = $\dfrac{1-\alpha}{1+\alpha} + \dfrac{1-\beta}{1+\beta}$

$$= \frac{(1-\alpha)(1+\beta)+(1-\beta)(1+\alpha)}{(1+\alpha)(1+\beta)}$$

$$= \frac{1+\beta-\alpha-\alpha\beta+1+\alpha-\beta-\alpha\beta}{1+\beta+\alpha+\alpha\beta}$$

$$= \frac{2-2\alpha\beta}{1+(\alpha+\beta)+\alpha\beta}$$

$$= \frac{2-\frac{2r}{p}}{1+\frac{-q}{p}+\frac{r}{p}}$$

$$= \frac{2p-2r}{p-q+r}$$

$$= \frac{2(p-r)}{p-q+r}$$

New product of the roots = $\dfrac{(1-\alpha)(1-\beta)}{(1+\alpha)(1+\beta)}$

$$= \frac{1-\beta-\alpha+\alpha\beta}{1+\beta+\alpha+\alpha\beta}$$

$$= \frac{1-(\alpha+\beta)+\alpha\beta}{1+(\alpha+\beta)+\alpha\beta}$$

$$= \frac{1-(\frac{-q}{p})+\frac{r}{p}}{\frac{p-q+r}{p}}$$

$$= \frac{p+q+r}{p-q+r}$$

x^2 – (sum of the roots)x + product of the roots = 0

$$x^2 - \frac{2(p-r)x}{p-q+r} + \frac{p+q+r}{p-q+r} = 0$$

$$(p-q+r)x^2 - 2(p-r)x + p + q + r = 0$$

Revision Exercise

1. The roots of the equation $4x^2 + 4x - 1 = 0$ are α and β. Find the values of: (a) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$
   (b) $\alpha^2 + \beta^2$
2. If α and β are roots of the equation $3x^2 - 6x + 2 = 0$. Find
   (a) $\alpha^2 - 3\alpha\beta + \beta^2$
   (b) $\alpha^3\beta + \alpha\beta^3$
   (c) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$
3. If α and β are roots of the quadratic equation $x^2 - 2x - 5 = 0$. Find the quadratic equation whose roots are:
   (a) $\alpha - 4,\ \beta - 4$
   (b) $\dfrac{1}{\alpha+2}, \dfrac{1}{\beta+2}$
   (c) $\dfrac{1}{\alpha^2}, \dfrac{1}{\beta^2}$
4. The roots of the equation $3x^2 - 8x + 2 = 0$ are α and β. Find an equation whose roots are $\dfrac{\alpha}{\beta}$ and $\dfrac{\beta}{\alpha}$.
5. If the roots of the equation $ax^2 + bx + c = 0$ differ by 4, show that $\dfrac{b^2}{4a} = 4a + c$.
6. Prove that if the roots of the equation $ax^2 + bx + c = 0$ is three times the other, then $3b^2 = 16ac$.
7. The roots of the equation $x^2 + 2px + q = 0$ differ by 8. Show that $p^2 - 16 = q$.
8. The roots of the equation $x^2 + 2x + k$ are β and $\beta - 1$. Find the value of k.
9. The roots of the equation $ax^2 + bx + c = 0$ is a square of the other. Prove that $c(a - b)^3 = a(c - b)^3$.
10. If α and β are roots of the equation $px^2 + qx + r = 0$, form an equation with integral coefficients whose roots are $\dfrac{1-\alpha}{1+\alpha}$ and $\dfrac{1-\beta}{1+\beta}$.
11. Given that α and β are roots of the equation $2x^2 - 8x + 2 = 0$, show that $\alpha^3 + \beta^3 = 52$. Hence that $\alpha^6 + \beta^3 = 27$.

12. Find the relationship between p, q and r if the roots of the equation $px^2 + qx + r = 0$ double each other. Show that

$$\frac{\log_3 243 + \log_3(\tfrac{1}{3})^8 + \log_3(27^{\frac{1}{2}})^{8/3} + \log_3 a^3}{\log_3 a^2 + 2} = \frac{3}{2}$$

13. If the roots of the equation $2x^2 - 3x - 1 = 0$ are α and β, find the value of $\alpha^2 + \beta^2$ and hence form the equation whose roots are α^2 and β^2.
14. Given that α is a common root of the equations $x^2 - 2x - k = 0$ and $x^2 - 5x + 2k = 0$, where $k \neq 0$. Find the numerical values of k and α.
15. In the equation $m^2x^2 + 2mnx + n^2 + 1 = 0$, m and n are constants which are real numbers. Show that the equation has no real roots for any values of m and n.
16. The roots of the quadratic equation $ax^2 + bx + c = 0$ are α and β. Write down the expression for $(\alpha + \beta)$ and $\alpha\beta$. Express in terms of α and β

    (i) $\frac{-c}{b}$ (ii) $\frac{a-b+c}{a}$

    (b) The roots of the equation $2x^2 - 3x + 4 = 0$ are α and β. Prove that $\frac{2}{\alpha}$ and $\frac{2}{\beta}$ are also roots of the equation.
17. Solve the equation $2^{2(x+1)} - 5 \times 2^x + 1 = 0$
18. Solve the simultaneous equations $2x + y + 3 = x + y + 2 = 2x^2 - 11y^2 + 3$
19. The roots of the equation $x^2 + ax + b = 0$ is the square of the other. Find the roots in terms of a and b.
20. The roots of the equations $2x^2 - 3x + 5 = 0$ are α and β. And the roots of the equation $px^2 + x = q = 0$ are $\alpha - 1$ and $\beta - 1$. Find the value of p and q.
21. If α and β are the roots of the equation $2x^2 - x = 5$, find the equation whose roots are $\alpha + 2\beta$ and $\beta + 2\alpha$.
22. If α and β are roots of the equation $2x^2 - 3x - 4 = 0$, find the equation whose roots are $\frac{1}{\alpha} + \beta$ and $\frac{1}{\beta} + \alpha$.
23. If the roots of the equation $x^2 - 5x + 1 = 0$ are α and β, form an equation with roots $\alpha + 3\beta$, $3\alpha + \beta$.
24. If α and β are roots of the equation $3x^2 - 3x - 1 = 0$, form an equation whose roots are $\alpha - \frac{1}{\alpha}$ and $\beta - \frac{1}{\beta}$.
25. If α and β are roots of the equation $3x^2 + x + 2 = 0$,

    (a) Evaluate $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$

    (b) Find the equation whose roots are $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$

    (c) Show that $27\alpha^4 = 11\alpha + 10$
26. The roots of the equation $x^2 + 6x + c = 0$ differ by $2n$, where n is real and non-zero. Show that $n^2 = 9 - c$. Given that the roots have opposite signs, find the set of all possible values of n.
27. Prove that the equation $x(x - 2p) = q(x - p)$ has real roots for all values of p and q. If $p = 3$, find the non-zero value for q.
28. If the roots of the equation $x^2 + bx + c$ are α and β and the roots of the equation $x^2 + \lambda bx + \lambda^2c = 0$ are γ and δ. Show that the equation whose roots are $\alpha\gamma + \beta\delta$ and $\alpha\delta + \beta\gamma$ is $x^2 - \lambda b^2x + 2\lambda^2c(b^2 - 2c) = 0$
29. The roots of the quadratic equation $x^2 - px + q = 0$ are α and β. Determine the equation having the roots $\alpha^2 + \beta^{-2}$ and $\beta^2 + \alpha^{-2}$.
30. Prove that the roots of the equation $(\gamma + 3)x^2 + (6 - 2\gamma)x + \gamma - 1 = 0$ are real if and only if γ is not greater than $\frac{3}{2}$. Find the values of γ if one root is six times the other.
31. Form the equation whose roots are the cubes of the roots of the equation $x^2 - 3x + 4 = 0$.
32. Show that if the equations $x^2 + bx + c = 0$ and $x^2 + px + q = 0$ have a common root, then
$$(c - q)^2 = (b - p)(cp - bq)$$
33. (i) Write $x^2 + 6x + 16$ in the form $(x + a)^2 + b$, where a and b are integrals to be found.

    (ii) Find the minimum values of $x^2 + 6x + 16$ and state the value of x for which this minimum value occurs.

34. The roots of the equation $2x^2 + 3x - 4 = 0$ are $\alpha\beta$. Find the values of: (a) $\alpha^2 + \beta^2$

(b) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$

(c) $(\alpha + 1)(\beta + 1)$

(d) $\dfrac{\beta}{\alpha} + \dfrac{\alpha}{\beta}$

35. If the roots of the equation $3x^2 - 5x + 1 = 0$ are α and β, find the values of:

(a) $\alpha\beta^2 + \alpha^2\beta$ (b) $\alpha^2 - \alpha\beta + \beta^2$

(c) $\alpha^3 + \beta^3$ (d) $\dfrac{\alpha^2}{\beta} + \dfrac{\beta^2}{\alpha}$

36. The equation $4x^2 + 8x - 1 = 0$ has roots α and β. Find the values of:

(a) $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2}$ (b) $(\alpha - \beta)^2$

(c) $\alpha^3\beta + \alpha\beta^3$ (d) $\dfrac{1}{\alpha^2\beta} + \dfrac{1}{\alpha\beta^2}$

37. If the roots of the equation $x^2 - 5x - 7 = 0$ are α and β, find the equations whose roots are:

(a) α^2, β^2 (b) $\alpha + 1, \beta + 1$

(c) $\alpha^2\beta,\ \alpha\beta^2$

38. The roots of the equation $2x^2 - 4x + 1$ are α and β. Find the equations with integral coefficients whose roots are:

(a) $\alpha - 2, \beta - 2$ (b) $\dfrac{1}{\alpha}, \dfrac{1}{\beta}$ (c) $\dfrac{\alpha}{\beta}, \dfrac{\beta}{\alpha}$

39. Find the equation with integral coefficients whose roots are the squares of the roots of the equation $2x^2 + 5x - 6 = 0$

40. The roots of the equation $x^2 + 6x + q = 0$ are α and $\alpha - 1$. Find the values of q.

41. The roots of the equation $x^2 - px + 8 = 0$ are α and $\alpha + 2$. Find the two possible values of p.

42. The roots of the equation $x^2 + 2px + q = 0$ differ by 2. Show that $p^2 = 1 + q$

43. If the roots of the equation $ax^2 + bx + c = 0$ are α and β, find expressions in terms of a, b, and c for:

(a) $\alpha^2\beta + \alpha\beta^2$ (b) $\alpha^2 + \beta^2$

(c) $\alpha^3 + \beta^3$ (d) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$

(e) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$ (f) $\alpha^4 + \beta^4$

44. The equation $ax^2 + bx + c = 0$ has roots α and β. Find equations whose roots are:

(a) $-\alpha, -\beta$ (b) $\alpha + 1, \beta + 1$ (c) $\alpha^2,\ \beta^2$

(d) $\dfrac{-1}{\alpha}, \dfrac{-1}{\beta}$ (e) $\alpha - \beta, \beta - \alpha$ (f) $2\alpha + \beta, \alpha + 2\beta$

45. Prove that, if the difference between the roots of the equation $ax^2 + bx + c = 0$ is 1, then $a^2 = b^2 - 4ac$

46. Prove that if one root of the equation $ax^2 + bx + c = 0$ is twice the other, then $2b^2 = 9ac$

47. Prove that if the sum of the squares of the roots of the equation $ax^2 + bx + c = 0$ is 1, then $b^2 = 2ac + a^2$.

48. Prove that if the sum of the reciprocals of the roots of the equation $ax^2 + bx + c = 0$ is 1, then $b + c = 0$.

Answers

1. (a) 4 (b) $\dfrac{3}{4}$
2. (a) $\dfrac{2}{3}$ (b) $\dfrac{16}{9}$ (c) 3
3. (a) $x^2 + 6x + 3$ (b) $3x^2 - 6x + 1 = 0$ (c) $25x^2 - 14x + 1 = 0$
4. $3x^2 - 26x + 3 = 0$
8. $k = \dfrac{3}{4}$
10. $(p - q + r)x^2 + 2(r - p)x + p + q + r = 0$
13. $\dfrac{13}{4}$, $4x^2 - 13x + 1 = 0$
14. $k = 3, \alpha = 3$ 17. $x = 0,\ x = -2$
18. $x = -1,\ y = \dfrac{-1 \pm \sqrt{177}}{22}$
20. $p = -2, q = -4$ 21. $2x^2 - 3x - 4 = 0$
22. $4x^2 - 3x - 1$. 23. $x^2 - 20x + 79$.
24. $x^2 - 4x - 1 = 0$
25. (i) $\dfrac{-11}{4}$ (ii) $4x^2 + 11x + 9 = 0$
29. $q^2x^2 - (p^2 - 2q)(q^2 + 1)x + (q^2 + 1)^2 = 0$
30. -11, $\dfrac{33}{25}$
31. $x^2 + 9x + 64 = 0$
33. (i) $(x + 3)^2 + 7$ (ii) 7 at $x = -3$

34. (a) $\frac{25}{4}$, (b) $\frac{3}{4}$ (c) $\frac{-5}{2}$ (d) $\frac{-25}{8}$

35. (a) $\frac{5}{9}$, (b) $\frac{16}{9}$ (c) $\frac{80}{27}$ (d) $\frac{80}{9}$

36. (a) 72, (b) $5\frac{16}{9}$ (c) $\frac{-9}{8}$ (d) -32

37. (a) $x^2 - 39x + 49 = 0$
(b) $x^2 - 7x - 1 = 0$
(c) $x^2 + 35x - 343 = 0$

38. (a) $2x^2 + 4x + 1 = 0$
(b) $x^2 - 4x + 2 = 0$
(c) $x^2 - 6x + 1 = 0$

39. $4x^2 - 49x + 36 = 0$

40. $\frac{35}{4}$ 41. ± 6

43. (a) $\frac{-bc}{a^2}$ (b) $\frac{b^2 - 2ac}{a^2}$ (c) $\frac{b(3ac - b^2)}{a^3}$
(d) $\frac{-b}{c}$ (e) $\frac{b^2 - 2ac}{ac}$ (f) $\frac{b^4 - 4ab^2c + 2a^2c^2}{a^4}$

44 (a) $ax^2 - bx + c = 0$
(b) $ax^2 + (b - 2a)x + a - b + c = 0$
(c) $a^2x^2 + (2ac - b^2)x + c^2 = 0$
(d) $cx^2 - bx + a = 0$
(e) $a^2x^2 - (b^2 - 4ac) = 0$
(f) $a^2x^2 + 3abx + (2b^2 + ac) = 0$

POLYNOMIALS

A polynomial is an expression consisting of variables and co-efficient which only employs the operations of addition, multiplication and non-negative integer exponent.

Consider the expression

$P(x) = C_0 x^n + C_1 x^{n-1} + C_2 x^{n-2} + \cdots k$

Where $C_0 \neq 0$ *then* $P(x)$ is said to be a polynomial of degree n. When we solve $P(x) = 0$ we get n unequal roots. When x is equal to each of the unequal values

$x = a_1,\ x = a_2,\ x = a_3, \ldots x = a_n$ then

$x - a_1, x - a_2, x - a_3, \ldots x - a_n$ are factors of $P(x)$

Polynomials must have whole numbers as exponents for example $x^2 - 4x + 7$ is a polynomial but $9x^{-1} + 12x^{\frac{1}{2}}$ is not a polynomial.

Polynomials appear in a wide variety of areas of mathematics and science. For example they are used to form polynomial equations which encode a wide range of problems from elementary word problems to complicated problems in science. They are used in calculus to approximate other functions.

Remainder Theorem

If when a polynomial $P(x)$ is divided by $x - a$, the quotient is $Q(x)$ and remainder is R then;

$$P(x) = (x - a)Q(x) + R$$

If $P(a) = 0$ then $x - a$ is a factor of $P(x)$
This approach can be extended to the division of the polynomial $f(x)$ by polynomial $g(x)$ of the degree less or equal to the degree of $f(x)$.
If the division gives the quotient $Q(x)$ and remainder $R(x)$ then $f(x) = g(x)Q(x) + R(x)$
Where $R(x)$ is of lower degree than $g(x)$

The Remainder Theorem

When $P(x)$ is divided by $x - a$, the remainder is $P(a)$
Proof $P(x) = (x - a)Q(x) + R$
Equating the divisor to zero

$$x - a = 0$$

$$x = a$$
$$P(a) = (a - a)Q(x) + R$$
$$P(a) = R$$

Example I

Find the remainders when

(i) $3x^2 - 4x^2 + 5x - 8$ is divided by $x - 2$

(ii) $2x^3 - 3x^2 - 5x + 6$ is divided by $x + 2$

(iii) $2x^3 - 7x + 6$ is divided by $x - 3$

(iv) $x^5 + x - 9$ is divided by $x + 1$

Solutions

$$P(x) = Q(x)D(x) + R$$

Where $Q(x)$ is the quotient, $D(x)$ is the divisor and R is the remainder

(i) $P(x) = 3x^3 - 4x^2 + 5x - 8$

$$D(x) = x - 2$$
$$x - 2 = 0$$
$$x = 2$$
$$P(2) = 3(2^3) - 4(2^2) + 5(2) - 8$$
$$= 24 - 16 + 10 - 8$$
$$= 34 - 24$$
$$= 10$$
$$R = 10$$

The remainder when $3x^3 - 4x^2 + 5x - 8$ is divided by $x - 2\ is\ 10$

(ii) $P(x) = Q(x)D(x) + R$

$P(x) = 2x^3 - 3x^2 - 5x + 6$

$D(x) = x + 2$

Equating the divisor to 0

$\Rightarrow x + 2 = 0$

$x = -2$

$P(-2) = 2(-2)^3 - 3(-2)^2 - 5(-2) + 6$

$$P(-2) = -12$$

The remainder when $2x^3 - 3x^2 - 5x + 6$ is divided by $x + 2$ is -12

(iii) $P(x) = Q(x)D(x) + R$

$$D(x) = x - 3$$

Equating the divisor to 0

$x - 3 = 0$

$x = 3$

$P(x) = 2x^3 - 7x + 6$

$P(3) = 2(3)^3 - 7(3) + 6$

$P(3) = 54 - 21 + 6$

$P(3) = 39$

The remainder when $2x^3 - 2x + 6$ is divided by $x - 3$ is 39

(iv) $x^5 + x - 9 = P(x)$

$$D(x) = x + 1$$

Equating the divisor to 0

$$x + 1 = 0$$
$$x = -1$$
$$P(-1) = (-1)^5 + (-1) - 9$$
$$= -1 - 10$$
$$= -11$$

The remainder when $x^5 + x - 9$ is divided by $x + 1$ is -11

> **Suppose a polynomial $f(x)$ has a repeated factor $x - a$. So that $f(x) = (x - a)^2 . g(x)$**
> **So by differentiating**
> $$f^1(x) = (x - a)^2\, g^1(x) + 2(x - a)g(x)$$
> ***(differentiation by product rule)***
>
> **Hence if $f(x)$ has a repeated factor of $x - a$ then $(x - a)$is also a factor of $f^1(x)$**
>
> $\Rightarrow$ **If $(x - a)^2$ is a factor of a polynomial $f(x)$ if and only if $f(a) = f^1(a) = 0$**

Example II

Given that the polynomial $f(x) = x^3 + 3x^2 - 9x + k$ has a repeated linear factor, find the possible values of k.

Solution

$f(x) = x^3 + 3x^2 - 9x + k$

$f^1(x) = 3x^2 + 6x - 9$

$f^1(x) = 3(x^2 + 2x - 3)$

$f^1(x) = 3(x - 1)(x + 3)$

The repeated factor of $f(x)$ is either $(x - 1)$ or $(x + 3)$

If $x - 1$ is a factor of $f(x)$, then $f(1) = 0$

$\Rightarrow 1 + 3 - 9 + k = 0$

$$k = 5$$

If $x + 3$ is a factor $f(x)$, then $f(-3) = 0$

$$-27 + 27 + 27 + k = 0$$
$$k = -27$$

The possible values of k are $k = 5\ and\ k = -27$

Obtaining the remainder by long division

Here are the steps required for dividing by a polynomial containing more than one term

Step I: Make sure the polynomial is written in descending order. If any terms are missing, use a zero to fill in the missing term (this will help with the spacing)

Step II: Divide the term with the highest power inside the division symbol by the term with the highest power outside the division symbol

Step III: Multiply (or distribute) the answer obtained in the previous step by the polynomial in front of the division symbol

Step IV: Subtract and bring down the next term

Step V: Repeat step (II), (III) and (IV) until there are no more terms to bring down.

Step VI: Write the final answer. The term remaining after the last subtract step is the remainder and must be written as a fraction in the final answer.

Example

Find the remainder when $x^3 - 4x^2 + 2x - 3$ is divided by $x + 2$

Solution

STEP I Make sure the polynomial is written in descending order. If any term is missing, use zero to fill in the missing terms (this will help with the spacing). In this case, the problem is ready as it is.	$x + 2 \overline{)\, x^3 - 4x^2 + 2x - 3}$
STEP II Divide the term with the highest power inside the division symbol by the term with the highest power outside the division symbol. In this case, we have x^3 divided by x which is x^2	x^2 $x + 2 \overline{)\, x^3 - 4x^2 + 2x - 3}$
STEP III Multiply (or distribute) the answer obtained in the previous step by polynomial in front of division symbol. In this case, we need x^2 and $x + 2$	x^2 $x + 2 \overline{)\, x^3 - 4x^2 + 2x - 3}$ $x^3 + 2x^2$

STEP IV Subtract and bring down the next term	x^2 $x + 2 \overline{)\, x^3 - 4x^2 + 2x - 3}$ $\underline{x^3 + 2x^2}$ $-6x^2 + 2x - 3$
STEP V Divide the term with the highest power inside the division symbol by the term with the highest power outside the division symbol. In this case, we have $-6x^2$ divided by x which is $-6x$.	$x^2 - 6x$ $x + 2 \overline{)\, x^3 - 4x^2 + 2x - 3}$ $\underline{x^3 - 2x^2}$ $-6x^2 + 2x - 3$
STEP VI Multiply (or distribute) the answer obtained in the previous step by the polynomial in front of division symbol. In this case, we need to multiply $(-6x)$ by $x+2$	$x^2 - 6x$ $x + 2 \overline{)\, x^3 - 4x^2 + 2x - 3}$ $\underline{x^3 - 2x^2}$ $-6x^2 + 2x$ $-6x^2 - 12x$
STEP VII Subtract and bring down the next term	$x^2 - 6x$ $x + 2 \overline{)\, x^3 - 4x^2 + 2x - 3}$ $\underline{x^3 - 2x^2}$ $-6x^2 + 2x$ $\underline{-6x^2 - 12x}$ $14x - 3$
STEP VIII Divide the term with the highest power inside the division symbol by the term with the highest power outside the division symbol. in this case, we have $14x$ divided by x which is $+14$	$x^2 - 6x + 14$ $x + 2 \overline{)\, x^3 - 4x^2 + 2x - 3}$ $\underline{x^3 - 2x^2}$ $-6x^2 + 2x$ $\underline{-6x^2 - 12x}$ $14x - 3$
STEP IX Multiply (or distribute) the answer obtained in the previous step by the polynomial in front of the division symbol. In this case, we need to multiply 14 by $x + 2$	$x^2 - 6x + 14$ $x + 2 \overline{)\, x^3 - 4x^2 + 2x - 3}$ $\underline{x^3 - 2x^2}$ $-6x^2 + 2x$ $\underline{-6x^2 - 12x}$ $14x - 3$ $14x + 28$

STEP X Subtract and notice there are no more terms to bring down	$\begin{array}{r l} & x^2-6x+14 \\ \cline{2-2} x+2 & \big) \; x^3-4x^2+2x-3 \\ & \; x^3-2x^2 \\ \cline{2-2} & \quad -6x^2+2x \\ & \quad -6x^2-12x \\ \cline{2-2} & \qquad 14x-3 \\ & \qquad 14x+28 \\ \cline{2-2} & \qquad\qquad -31 \end{array}$
STEP XI Write the final answer. The term remaining after the last subtract step is the remainder and must be written as a fraction.	$x^2-6x+14+\dfrac{-31}{x+2}$

$\Rightarrow$ The remainder when x^3-4x^2+2x-3 is divided by $x+2$ is -31.

Example II

By using long division, obtain remainders and quotients when

(i) $x^3+3x^2-4x-12$ is divided by x^2+x-6

(ii) $2x^4-8x^3+5x^2+4$ is divided by $x-3$

(iii) $5x^3-6x^2+3x+14$ is divided by $x+1$

(iv) $2x^4+6x^3-7x^2+9x+11$ is divided by $x+4$

(v) x^4-16 is divided by $x-2$

Solution

(i) $x^3+3x^2-4x-12$ is divided by x^2+x-6

$$\begin{array}{r l} & x+2 \\ \cline{2-2} x^2+x-6 & \big) \; x^3+3x^2-4x-12 \\ & \; x^3+x^2-6x \\ \cline{2-2} & \; 2x^2+2x-12 \\ & \; 2x^2+2x-12 \\ \cline{2-2} & \qquad 0 \end{array}$$

$$\frac{x^3+3x^2-4x-12}{x^2+x-6}=(x+2)-\frac{0}{x^2+x-6}$$

$$R=0$$

$Q(x)=x+2$ (the quotient)

(ii) $\mathbf{2x^4-8x^3+5x^2+4}$ is divided by $x-3$

$2x^4-8x^3+5x^2+0x+4$

$$\begin{array}{r l} & 2x^3-2x^2-x-3 \\ \cline{2-2} x-3 & \big) \; 2x^4-8x^3+5x^2+0x+4 \\ & \; 2x^4-6x^3 \\ \cline{2-2} & \quad -2x^3+5x^2+0x+4 \\ & \quad -2x^3+6x^2 \\ \cline{2-2} & \qquad -x^2+0x+4 \\ & \qquad -x^2+3x \\ \cline{2-2} & \qquad\quad -3x+4 \\ & \qquad\quad -3x+9 \\ \cline{2-2} & \qquad\qquad -5 \end{array}$$

$$2x^3-2x^2-x-3+\frac{-5}{x-3}$$

$$Q(x)=2x^3-2x^2-x-3$$

$$R=-5$$

Where $Q(x)=$ Quotient and remainder $=R$

(iii) $5x^3-6x^2+3x+14$

$$\begin{array}{r l} & 5x^2-11x+14 \\ \cline{2-2} x+1 & \big) \; 5x^3-6x^2+3x+14 \\ & \; 5x^3+5x^2 \\ \cline{2-2} & \quad -11x^2+3x+14 \\ & \quad -11x^2-11x \\ \cline{2-2} & \qquad 14x+14 \\ & \qquad 14x+14 \\ \cline{2-2} & \qquad\quad 0 \end{array}$$

$$\frac{5x^3-6x^2+3x+14}{x+1}=5x^2-11x+14$$

$$R(x)=0, Q(x)=5x^2-11x+14$$

(iv) $2x^4+6x^3-7x^2+9x+11$

$$\begin{array}{r l} & 2x^3-2x^2+x+5 \\ \cline{2-2} x+4 & \big) \; 2x^4+6x^3-7x^2+9x+11 \\ & \; 2x^4+8x^3 \\ \cline{2-2} & \quad -2x^3-7x^2+9x+11 \\ & \quad -2x^3-8x^2 \\ \cline{2-2} & \qquad x^2+9x+11 \\ & \qquad x^2+4x \\ \cline{2-2} & \qquad\quad 5x+11 \\ & \qquad\quad 5x+20 \\ \cline{2-2} & \qquad\qquad -9 \end{array}$$

$$\frac{2x^4+6x^3-7x^2+9x+11}{x+4}=2x^3-2x^2+x+5-\frac{9}{x+4}$$

$$Q(x) = 2x^3 - 2x^2 + x + 5$$

$$R = -9$$

(v) $x^4 - 16$ is divided by $x - 2$

$$
\begin{array}{r|l}
 & x^3 + 2x^2 + 4x + 8 \\ \hline
x - 2 & x^4 + 0x^3 + 0x^2 + 0x - 16 \\
 & \underline{x^4 - 2x^3} \\
 & \quad 2x^3 + 0x^2 + 0x - 16 \\
 & \quad \underline{2x^3 - 4x^2} \\
 & \qquad 4x^2 + 0x - 16 \\
 & \qquad \underline{4x^2 - 8x} \\
 & \qquad\quad 8x - 16 \\
 & \qquad\quad \underline{8x - 16} \\
 & \qquad\qquad 0
\end{array}
$$

$$r = 0$$

$$Q(x) = x^3 + 2x^2 + 4x + 8$$

Were $r = remainder$

And $Q(x) = x^3 + 2x^2 + 14x + 8$

Obtaining the remainder by synthetic approach

Definitions :

Dividend: The number or expression you are dividing into

Divisor: The number or expression you are dividing by

Synthetic division: is a quick method of dividing a polynomial when the divisor is of the form $ax + b$ or $x - c$

Steps involved when obtaining the remainder by synthetic approach

(1) Write the value obtained after equating the divisor to 0 and the coefficients of the dividend in descending order in the first row. If any x terms are missing, place a zero in its place

(2) Bring the leading coefficient in the top row down to bottom (third) row

(3) Next multiply the number in the bottom row by c and place this product in the second row under the next coefficient and add these two terms together

(4) Continue with this process until you reach the last column

(5) The numbers in the bottom are coefficients of the quotient and the remainder. The quotient will have one degree less than the dividend

Example I

Use synthetic approach to obtain the remainder when $2x^4 - 8x^3 + 5x^2 + 4$ is divided by $x - 3$

Solution

First note that the x term is missing so we must record zero in its place

	x^4	x^3	x^2	x	x^0
$x = 3$	2	-8	5	0	4
		6	-6	-3	-9
	2	-2	-1	-3	-5

Therefore, the quotient is $2x^3 - 2x^2 - x - 3$ and the remainder is -5.

Example II

Use synthetic approach to obtain the remainders when

(i) $x^4 - 16$ is divided by $x + 1$

(ii) $5x^3 - 6x^2 + 3x + 14$ is divided by $x + 1$

(iii) $2x^4 + 6x^3 - 7x^2 + 9x + 11$ is divided by $x + 4$

Solution

(i) Equating the divisor to zero

$$x + 1 = 0$$

$$x = -1$$

	x^4	x^3	x^2	x	x^0
$x = -1$	1	0	0	0	16
		-1	1	-1	1
	1	-1	1	-1	-15

$$R = -15$$

$$Q(x) = x^3 - x^2 + x - 1$$

Where $R =$ remainder

And $Q(x) =$ Quotient

Note: Synthetic method only work where

the divisor is of the form
$x - c$ or $ax + b$

(ii) $5x^3 - 6x^2 + 3x + 14$

Equating the divisor to zero

$x + 1 = 0$, $x = -1$

	x^3	x^2	x	x^0
$x = -1$	5	-6	3	14
		-5	11	-14
	5	-11	14	0

$$Q(x) = 5x^2 - 11x + 14$$
$$R = 0$$

Where $Q(x) = Quotient$

R is the remainder

(iii) $2x^4 + 6x^3 - 7x^2 + 9x + 11 = P(x)$

$$D(x) = x + 4$$

Equating the divisor to zero

$$x + 4 = 0$$
$$x = -4$$

	x^4	x^3	x^2	x	x^0
$x = -4$	2	6	-7	9	11
		-8	8	-4	-20
	2	-2	1	5	-9

$Q(x) = 2x^3 - 2x^2 + x + 5$

$R = -9$

Where $Q(x)$ = quotient, R = remainder

More examples on polynomials

Find the values a in the expression below if the following conditions are satisfied

(i) $x^3 + ax^2 + 3x - 5$ has a remainder -3 when divided by $x - 2$

(ii) $x^3 + x^2 - 2ax + a^2$ has a remainder 8 when divided by $x - 2$

Solution

(i) $P(x) = Q(x)D(x) + R$

$x^3 + ax^2 + 3x - 5 = Q(x)(x-2) + -3$ …… (1)

Equating the divisor to zero

$$x - 2 = 0$$
$$x = 2$$

Substituting $x = 2$ in Eqn (1);

$$2^3 + a(2^2) + 3 \times 2 - 5 = 0 + -3$$
$$8 + 4a + 6 - 5 = -3$$
$$9 + 4a = -3$$
$$4a = -12$$
$$a = -3$$

(ii) $x^3 + x^2 - 2ax + a^2 = Q(x)(x-2) + 8$ …. (1)

$$x - 2 = 0$$
$$x = 2$$

Substituting $x = 2$ in Eqn (1);

$$8 + 4 - 4a + a^2 = 0 + 8$$
$$a^2 - 4a + 4 = 0$$
$$(a-2)^2 = 0$$
$$a = 2$$

Example II

Show that $12x^3 + 16x^2 - 5x - 3$ is divisible by $2x - 1$ and find other factors

Solution

$$\begin{array}{r|l} & 6x^2 + 11x + 3 \\ \hline 2x - 1 & 12x^3 + 16x^2 - 5x - 3 \\ & 12x^3 - 6x^2 \\ \hline & \quad 22x^2 - 5x - 3 \\ & \quad 22x^2 - 11x \\ \hline & \qquad 6x - 3 \\ & \qquad 6x - 3 \\ \hline \end{array}$$

Since the remainder is zero

$12x^3 + 16x^2 - 5x - 3$ is divisible by $2x - 1$

$12x^3 + 16x^2 - 5x - 3 = (2x-1)(6x^2 + 11x + 3)$

For $6x^2 + 11x + 3$

The other Factors are 2, 9 and product of factors is 18

$6x^2 + 2x + 9x + 3$

$2x(3x + 1) + 3(3x + 1)$

$\Rightarrow (2x + 3)(3x + 1)$

$6x^2 + 11x + 3 = (2x + 3)(3x + 1)$

$12x^3 + 16x^2 - 5x - 3 = (2x - 1)(2x + 3)(3x + 1)$

The other factors of $12x^3 + 16x^2 - 5x - 3$ are $(2x + 3)$and $(3x + 1)$

Example III

$x + 2$ is a factor of $2x^3 + 6x^2 + bx - 5$. Find the remainder when the expression is divided by $2x - 1$

Solution

$$P(x) = 2x^3 + 6x^2 + bx - 5$$

Since $x + 2$ is a factor,

$x + 2 = 0$ $\quad\Rightarrow$ $x = -2$

$P(-2) = 0$ (Since $x + 2$ is a factor of $P(x)$

(Remainder = 0)

$P(-2) = 2(-2^3) + 6(-2^2) + b(-2) - 5$

$P(-2) = -16 + 24 - 2b - 5$

$P(-2) = -21 + 24 - 2b$

$P(-2) = 3 - 2b$

$3 - 2b = 0$

$b = 1.5$

Example IV

The remainder obtained when $2x^3 + ax^2 - 6x + 1$ is divided by $x + 2$ is twice the remainder obtained when the same expression is divided by $x - 3$. Find a

Solution

$$P(x) = 2x^3 + ax^2 - 6x + 1$$

$$P(-2) = 2(-2)^3 + a(-2^2) - 6(-2) + 1$$

$$= -16 + 4a + 12 + 1$$

$$P(2) = -3 + 4a$$

$$P(3) = 2(3^3) + a(3^2) - 6(3) + 1$$

$$= 54 + 9a - 18 + 1$$

$$\mathrm{P}(3) = 37 + 9a$$

$$P(-2) = 2P(3)$$

$$3 + 4a = (37 + 9a) \times 2$$

$$-3 + 4a = 74 + 18a$$

$$-77 = 14a$$

$$a = -5.5$$

Example V

A cubic polynomial $6x^3 + 7x^2 + ax + b$ has a remainder 72 when divided by $x - 2$ and exactly divisible by $x + 1$. Calculate the values of a and b. Show that $2x - 1$ is also a factor. Obtain the other factor factor

Solution:

$P(x) = 6x^3 + 7x^2 + ax + b$

$P(2) = 72$

$P(-1) = 0$

$P(2) = 6(2^3) + 7(2^2) + a(2) + b$

$= 48 + 28 + 2a + b$

$P(2) = 76 + 2a + b$

$72 = 76 + 2a + b$

$-4 = 2a + b$………………………….. (1)

$P(-1) = 6(-1)^3 + 7(-1)^2 + a(-1) + b$

$P(-1) = -6 + 7 - a + b$

$0 = 1 - a + b$

$-1 = -a + b$………………………… (2)

Eqn. (1) – eqn. (2)

-3 = 3a

a = -1

Substituting a = -1 in Eqn (2);

$\Rightarrow$ -1 = 1 + b

b = -2

P(x) = $6x^3 + 7x^2 - x - 2$

We can apply long division to obtain other factors

$$\begin{array}{r|l}
 & 6x^2 + x - 2 \\
\hline
x + 1 & 6x^3 + 7x^2 - x - 2 \\
 & \underline{6x^3 + 6x^2} \\
 & \quad x^2 - x - 2 \\
 & \quad \underline{x^2 + x} \\
 & \qquad -2x - 2 \\
 & \qquad \underline{-2x - 2}
\end{array}$$

$\Rightarrow (6x^2 + x - 2)(x + 1)$

But $6x^2 + x - 2 = 6x^2 + 4x - 3x - 2$

$= 2x(3x + 2) - 1(3x + 2)$

$\Rightarrow 6x^3 + 7x^2 - x - 2 = (2x - 1)(3x + 2)(x + 1)$

$\Rightarrow$ The other factors are $(3x + 2)$ and $(x + 1)$

Example VI

$x - 1$and $x + 1$ are factors of $x^3 + ax^2 + bx + c$ and it leaves a remainder of 12 when divided by $x - 2$.find the values of a, b, and c.

Solution

$P(x) = x^3 + ax^2 + bx + c$

$P(1) = 0$

$P(-1) = 0$

$P(2) = 12$

$P(1) = 1^3 + a(1^2) + b(1) + c$

$P(1) = 1 + a + b + c$

$0 = 1 + a + b + c$

$\Rightarrow a + b + c = -1$…………………… (1)

$P(-1) = (-1)^3 + a(-1)^2 + b(-1) + c$

$P(-1) = -1 + a - b + c$

$0 = -1 + a - b + c$

$a - b + c = 1$………………………… (2)

$P(2) = 2^3 + a(2^2) + b(2) + c$

$12 = 8 + 4a + 2b + c$

$4a + 2b + c = 4$………………….. (3)

Eqn. (1) – eqn.(2)

$2b = -2$

$b = -1$

Eqn.(1) – eqn.(3)

$-3a - b = -5$

But b = -1

$-3a + 1 = -5$

$-3a = -6$

$a = 2$

Substituting $a = 2, b = -1$ in eqn. (1)

$2 - 1 + c = -1$

$c = -2$

$\therefore a = 2, b = -1 \text{ and } c = -2$

Example IX

When a polynomial $P(x)$ is divided by $x - 2$, the remainder is 4 and when $P(x)$ is divided by $x - 3$, the remainder is 7. Find the remainder when $P(x)$ is divided by $(x - 2)(x - 3)$

Solution

P(2) = 4 and P(3) = 7

The remainder is of the form

$R(x) = ax + b$

$P(x) = Q(x)D(x) + R(x)$

$P(x) = Q(x)(x - 2)(x - 3) + ax + b$

Equating the divisor to zero

$(x - 2)(x - 3) = 0$

$x = 2, x = 3$

$P(2) = 2a + b$

$P(3) = 3a + b$

But $P(2) = 4 \text{ and } P(3) = 7$

$2a + b = 4$…………………………… (1)

$3a + b = 7$…………………………… (2)

Eqn (2) – Eqn (1)

$a = 3$

Substituting $a = 3$ in eqn. (1)

$2 \times 3 + b = 4$

$6 + b = 4$

$b = 4 - 6$

$b = -2$

$R(x) = ax + b$

$R(x) = 3x - 2$

Example VIII

When a polynomial $p(x)$is divided by $x - 1$, the remainder is 5 and when $p(x)$ is divided by $x - 2$, the remainder is 7. Find the remainder when the same expression is divided by $(x - 1)(x - 2)$.

Solution

The remainder takes the form $R(x) = ax + b$.

$$p(x) = Q(x)(x - 1)(x - 2) + ax + b$$

Equating the divisor to zero

$(x - 1)(x - 2) = 0$

$x = 1 \text{ and } x = 2$

$p(1) = a + b$

$p(2) = 2a + b$

$5 = a + b$…………………………… (1)

$7 = 2a + b$………………………….. (2)

Eqn. (2) – eqn. (1)

$2 = a$

$a = 2$

Substituting $a = 2$ in eqn. (1)

$5 = 2 + b$

$b = 3$

The remainder is $2x + 3$

Example IX

Given that the polynomial $f(x) = Q(x)g(x) + R(x)$ where $Q(x)$ is a quotient, $g(x) = (x - \alpha)(x - \beta)$and $R(x)$ is a remainder. Show that

$$R(x) = \frac{(x - \beta)f(\alpha) + (\alpha - x)f(\beta)}{\alpha - \beta}$$

Solution

$f(x) = Q(x)D(x) + R(x)$

$f(x) = Q(x)(x - \alpha)(x - \beta) + ax + b$

Where $R(x) = ax + b$

Equating the divisor to zero

$(x - \alpha)(x - \beta) = 0$

$x = \alpha, x = \beta$

$f(\alpha) = a\alpha + b$……………………… (1)

$f(\beta) = a\beta + b$……………………… (2)

Eqn. (1) – eqn. (2)

$a(\alpha - \beta) = f(\alpha) - f(\beta)$

$$a = \frac{f(\alpha) - f(\beta)}{\alpha - \beta}$$

Substituting $a = \frac{f(\alpha) - f(\beta)}{\alpha - \beta}$ in equation (1)

$$f(\alpha) = \alpha\left(\frac{f(\alpha) - f(\beta)}{\alpha - \beta}\right) + b$$

$$b = f(\alpha) - \left(\frac{\alpha f(\alpha) - \alpha f(\beta)}{\alpha - \beta}\right)$$

$$b = \frac{\alpha f(\alpha) - \beta f(\alpha) - \alpha f(\alpha) + \alpha f(\beta)}{\alpha - \beta}$$

$$b = \frac{\alpha f(\beta) - \beta f(\alpha)}{\alpha - \beta}$$

But since $R(x) = ax + b$

$$R(x) = \left(\frac{f(\alpha) - f(\beta)}{\alpha - \beta}\right)x + b$$

$$R(x) = \frac{xf(\alpha) - f(\beta)x}{\alpha - \beta} + \frac{\alpha f(\beta) - \beta f(\alpha)}{\alpha - \beta}$$

$$R(x) = \frac{xf(\alpha) - \beta f(\alpha) + \alpha f(\beta) - xf(\beta)}{\alpha - \beta}$$

$$R(x) = \frac{(x - \beta)f(\alpha) + (\alpha - x)f(\beta)}{\alpha - \beta}$$

Example

If $x^2 + 1$ a factor of $3x^4 + x^3 - 4x^2 + px + q$. Find the values of p and q

Solution

$$3x^4 + x^3 - 4x^2 + px + q = (x^2+1)\left(ax^2 + bx + c\right)$$

But

$$(x^2+1)\left(ax^2 + bx + c\right) = ax^4 + bx^3 + cx^2 + ax^2 + bx + c$$

$$= ax^4 + bx^3 + \left(a + c\right)x^2 + bx + c$$

$$3x^4 + x^3 - 4x^2 + px + q = ax^4 + bx^3 + (a+c)x^2 + bx + c$$

Equating co-efficients of the same monomial;

$\Rightarrow a = 3, b = 1$

$$a + c = -4$$
$$3 + c = -4$$
$$c = -7$$
$$b = p$$
$$1 = p$$
$$c = q$$
$$q = -7$$

$\Rightarrow p = 1$ and $q = -7$

Example XII

If $f(x)$ and $g(x)$ are polynomials. $f(x) = (x - a)^2 g(x) + Ax + B$. Find $f^1(x)$ and hence find A and B in terms of $f(a)$ and $f^1(a)$ and deduce that $x - a$ is a repeated factor of $f(x)$ if and only if $f(a) = f^1(a)$

Solution

$$f(x) = (x - a)^2 g(x) + Ax + B$$
$$f^1(x) = (x-a)^2 g^1(x) + g(x)2(x-a) + A$$
$$f^1(a) = 0 + A$$
$$A = f^1(a)$$
$$f(a) = 0 + Aa + B$$
$$f(a) = af^1(a) + B$$
$$B = f(a) - af^1(a)$$

If $(x - a)^2$ is a repeated factor of $f(x)$

$$f(x) = (x - a)^2 Q_1(x)$$
$$f(a) = (a - a)^2 Q_1(x) = 0$$
$$f(a) = 0$$
$$f^1(x) = (x - a)^2 Q_1^1(x) + Q_1(x)2(x - a)$$
$$f^1(a) = 0$$

$\Rightarrow$ If $(x - a)^2$ is a repeated factor of $f(x)$ if and only if

$$f(a) = f^1(a) = 0$$

Example (UNEB 2015)

(a) Given that $f(x) = (x - \alpha)^2 g(x)$, show that $f'(x)$ is divisible by $(x - \alpha)$

(b) A polynomial $P(x) = x^3 + 4ax^2 + bx + 3$ is divisible by $(x - 1)^2$. Use your results above to find the values of a and b. Hence solve the equation $p(x) = 0$

Solution

$f(x) = (x - \alpha)^2 g(x)$

$f'(x) = (x - \alpha)^2 g'(x) + g(x)^2(x - \alpha)$

$\Rightarrow$ $f'(x)$ is divisible by $(x - \alpha)$

$p(x) = x^3 + 4ax^2 + bx + 3$

$p'(x) = 3x^2 + 8ax + b$

Since $x - 1$ is a factor of $p(x)$ and $p'(x)$,

$\Rightarrow p(1) = 0$ and $p'(1) = 0$

$1 + 4a + b + 3 = 0$

$4a + b = -4$ …………………. (i)

$p'(1) = 0$

$3 + 8a + b = 0$

$8a + b = -3$ …………………. (ii)

Eqn (ii) – Eqn (i);

$4a = 1$

$$a = \frac{1}{4}$$

Substituting $a = \frac{1}{4}$ in Eqn (ii)

$$\Rightarrow 8\left(\frac{1}{4}\right) + b = -3$$

$2 + b = -3$

$b = -5$

$p(x) = x^3 + x^2 + -5x + 3$

$p(1) = 1 + 1 + 3 - 5 = 0$

$p'(x) = 3x^2 + 2x - 5$

$p'(1) = 3 + 2 - 5$

$p'(1) = 0$

$P(x) = x^3 + x^2 - 5x + 3$

$(x^3 + x^2 - 5x + 3) = (x - 1)^2 g(x)$

$(x^3 + x^2 - 5x + 3) = (x^2 - 2x + 1)g(x)$

$$\begin{array}{r|l} & x + 3 \\ \hline x^2 - 2x + 1 & x^3 + x^2 - 5x + 3 \\ & x^3 - 2x^2 + x \\ \hline & \quad 3x^2 - 6x + 3 \\ & \quad 3x^2 - 6x + 3 \end{array}$$

$\Rightarrow (x^3 + x^2 - 5x + 3) = (x^2 - 2x + 1)(x + 3)$

$(x^2 - 2x + 1)(x + 3) = 0$

$(x - 1)^2(x + 3) = 0$

$x - 1 = 0$ OR $x + 3 = 0$

$x = 1$ $\quad x = -3$

Example **(UNEB Question)**

When the quadratic expression $ap^2 + bp + c$ is divided by $p - 1$, $p - 2$ and $p + 1$, the remainders are 1, 1 and 25 respectively. Determine the factors of the expression.

b) Express $2x^3 + 5x^2 - 4x - 3$ in the form $(x^2 + x - 2)\,Q(x) + Ax + B$; where $Q(x)$ is a polynomial in x and A and B are constants. Determine the values of A and B and the expression $Q(x)$.

Solution

Let $f(p) = ap^2 + bp + c$

Now $f(1) = a + b + c$

But $f(1) = 1$

$\Rightarrow \quad a + b + c = 1$ ……………… (i)

$f(2) = 4a + 2b + c$

But $f(2) = 1$

$\Rightarrow 4a + 2b + c = 1$…………….. (ii)

$f(-1) = a - b + c$

But $f(-1) = 25$

=> $a - b + c = 25$ …………….. (iii)

Eqn (i) – Eqn (ii)

$-3a - 3b = 0$

$-3a = b$…………………….(iv)

Eqn (ii) – Eqn (iii)

$3a + 3b = -24$

$a + b = -8$ ………………….(v)

$a - 3a = -8$

$-2a = -8$

$a = 4$

Substituting for a into Eqn (iv)

$b = -12$

Substituting for and b into Eqn (i)

$4 - 12 + c = 1$

$-8 + c = 1$

$c = 9$

Hence $f(p) = 4p^2 - 12p + 9$

By factorization,

$4p^2 - 12p + 9 = 4p^2 - 6p - 6p + 9$

$= 2p(2p - 3) - 3(2p - 3)$

$= (2p - 3)(2p - 3)$

Hence the factors of $4p^2 - 12p + 9$ are $(2p - 3)$ and $(2p - 3)$

b) Let $2x^3 + 5x^2 - 4x - 3$

$\equiv (x^2 + x - 2)(2x + D) + Ax + B$

By opening brackets on the L.H.S

$2x^3 + 5x^2 - 4x - 3 \equiv 2x^3 + Dx^2 + 2x^2 + Dx - 4x - 2D + Ax + B$

$2x^3 + 5x^2 - 4x - 3 \equiv 2x^3 + (D + 2)x^2 + (D - 4)x - 2D + Ax + B$

$2x^3 + 5x^2 - 4x - 3 \equiv 2x^3 + (D + 2)^2 + (D + A - 4)x - 2D + B$

Equating corresponding coefficients,

For x^2,

$D + 2 = 5$

$D = 3$

For x

$-4 = D + A - 4$

$D = -A$

$3 = -A$

$A = -3$

For constant

$-3 = -6 + B$

$B = 6 - 3$

$B = 3$

Hence $2x^3 + 5x^2 - 4x - 3 \equiv (x^2 + x - 2)\,(2x + 3) - 3x + 3$

Alternatively

By using long division,

$$\begin{array}{r|l} & \qquad\qquad 2x + 3 \\ \hline (x^2 + x - 2) & 2x^3 + 5x^2 - 4x - 3 \\ & \underline{2x^3 + 2x^2 - 4x} \\ & \qquad 3x^2 - 3 \\ & \qquad \underline{3x^2 + 3x - 6} \\ & \qquad\qquad -3x + 3 \end{array}$$

Hence $2x^3 + 5x^2 - 4x - 3 \equiv (x^2 + x - 2)\,(2x + 3) - 3x + 3$

Revision Exercise

1. Find the constants p, q and r such that
$2y^2 - 9y + 14 = p(y - 1)(y - 2) + q(y - 1) + r$
2. Find the relationship between p and r so that $A^2 + 3qA^2 + PA + R$ shall be a perfect cube for all values of A.
3. When the expression $p^6 + 4p^2 + ap + b$ is divided by $p^2 - 1$, the remainder is $2p + 3$. Find the values of a and b.
4. Find the remainder when:
   (a) $4x^3 - 5x + 4$ is divided by $-(1 - 2x)$
   (b) $y^5 + y - 9$ is divided by $y + 1$
5. Find the values of β in the expressions below when the following conditions are satisfied:
   (a) $y^3 + \beta y^2 + 3y - 5$ has remainder -3 when divided by $y - 2$.
   (b) $x^5 + 4x^4 - 6x^2 - \beta x + 2$ has a remainder 6 when divided by $\dfrac{1}{(x+2)^{-1}}$.
6. $(p - 1)$ and $(p + 1)$ are factors of the expression $p^3 + ap^2 + bp + c$ and it leaves a remainder of 12 when divided by $p - 2$. Find the values of a, b, c.
7. The expression $ax^4 + bx^3 + 3x^2 - 2x + 3$ has a remainder $x + 1$ when divided by $x^2 - 3x + 2$. Find the values of a and b.
8. What is the value of a if $2x^2 - x - 6$, $3x^2 - 8x + 4$ and $ax^3 - 10x - 4$ have a common factor?
9. Factorise the expression $3k^3 - 11k^2 - 19k - 5$.
10. Find the values of a and b which make $y^4 + 6y^3 + 13y^2 + ay + b$ a perfect square.
11. If $x^2 + nx + \mathrm{q}$ and $x^2 + dx + m$ have a common factor $(x - p)$. show that $p = \dfrac{m-q}{n-d}$.
12. The remainder obtained when $2x^3 + ax^2 - 6x + 1$ is divided by $(x + 2)$ is twice the remainder obtained when the same expression is divided by $(x - 1)$. Find the values of a and b.
13. Given that $(x + 2)$ is a factor of $2x^3 + 6x^2 + bx - 5$, find the remainder when the expression is divided by $(2x - 1)$
14. Find the values of p and q if the expression $2y^3 - 15y^2 + py + q$ is divisible both by $y - 4$ and $2y - 1$.
15. Use the remainder theorem to find the factors of $x^4 + 3x^2 - 4$.
16. Find p and q so that
$y^4 - 7y^3 + 17y^2 - 17y + 6 = (y - 1)2(y^2 + py + q)$
Hence find all the factors of the quadratic equation.
17. Factorise (a) $2y^3 - y^2 + 2y - 1$
    (b) $2y^3 + 5y^2 + y - 2$
18. Use the synthetic approach to find the remainder when:
    (a) $8y^3 - 10y^2 + 7y + 3$ is divided by $2y - 1$
    (b) $5 + 6x + 7x^2 - x^3$ is divided by $x + 2$
19. Find the range of values of q for which $(2 - 3q)x^2 + (4 - q)x + 2 = 0$ has no real roots.
20. Find the value of k for which the line $y = mx + c$ is a tangent to the curve $x^2 + xy + 2 = 0$.
21. Express the polynomial $f(x) = 2y^4 + y^3 - y^2 + 8y - 4$ as a product of two linear factors and a quadratic factor $q(y)$. Prove that there are no real values of y for which $q(y) = 0$.
22. The polynomial $ax^3 + bx^2 - 5x + 1$ has $2x - 1$ and $x - 1$ as its two factors. Find a and b.
23. $f(x) = 2x^3 + px^2 + qx + 6$ where p and q are constants. When f(x) is divided by $x - 1$, the remainder is -6, when divided by $(x + 1)$ the remainder is 12. Show that $f(½) = 0$ hence write $f(x)$ as a product of linear factors.
24. Find the remainder when
    (a) $3x^5 - x^2 + 1$ is divided by $x + 2$
    (b) $x^4 - 2x^2 + 3x - 6$ is divided by $x^2 + 4x + 3$
25. Use long division to find the missing factors:
    (a) $x^5 + x^4 + 3x^3 + 5x^2 + 2x + 8 = (x^2 - x + 2)(\ldots)$
    (b) $6x^5 + x^4 - x^3 - 15x + 5 = (3x - \ldots)(\ldots.)$
26. The expression $2x^3 + ax^2 + bx + 6$ is exactly divisible by $(x - 2)$ and on division by $(x + 2)$ gives a remainder of -12. Calculate the values of a and b and factorise the expression completely.
27. $f(x) = x^2 + ax + b$ when $f(x)$ is divided by $x - 2$ the remainder is 8 and when $f(x)$ is divided by $x + 3$ the remainder is 18. Find the values of constants a and b.
28. If $f(x)$ denotes the polynomial $2x^3 - 3x^2 - 8x - 3$, find the remainders when $f(x)$ is divided by:
    (i) $x - 1$ (ii) $x + 3$ (iii) $2x + 1$
29. State the remainder when the cubic polynomial $x^3 + ax^2 - 3x + 4$ is divided by $(x - 3)$ the remainder obtained is twice the remainder obtained when the polynomial is divided by $(x - 2)$. Calculate a.
30. When $f(x) = x^4 - 2x^3 + ax^2 - bx + c$ is divided by $x - 2$, the remainder is -24 and when divided by $x + 4$, the remainder is 240. Given that $x + 1$ is a factor of $f(x)$, show that $x - 1$ is also a factor.
31. Given that $f(x) = x^3 + kx^2 - 2x + 1$, When $f(x)$ is divided by $(x - k)$, the remainder is k. Find the possible values of k.
32. When the polynomial $p(x)$ is divided by $(x - 1)$, the remainder is 5 and when $p(x)$ is divided by $(x - 2)$, the remainder is 7. Find the remainder when $p(x)$ is divided by $(x - 1)(x - 2)$.

33. When the polynomial $p(x)$ is divided by $(x - 2)$, the remainder is 4 and when $p(x)$ is divided by $(x - 3)$ the remainder is 7. Find by writing $p(x) = (x - 2)(x - 3)q(x) + ax$, the remainder when $p(x)$ is divided by $(x - 2)(x - 3)$. If $p(x)$ is cubic when the coefficient of $x3$ is unity and $p(1) = 1$ determine $q(x)$.
34. Find the quotient and remainder when:
    (a) $6x^2 - x + 2$ is divided by $2x + 1$
    (b) $6x^2 - 7x + 5$ is divided by $2x - 3$
    (c) $x^3 + 3x^2 - 2x + 1$ is divided by $x - 2$
    (d) $2x^3 - 3x^2 - 4x + 1$ is divided by $x - 4$
    (e) $4x^2 - 3x^2 + x + 2$ is divided by $2x + 3$
35. Use the remainder theorem to find the remainder when
    (a) $3x^2 + 2x - 4$ is divided by $x - 2$
    (b) $2x^3 + 4x^2 - 6x + 5$ is divided by $x - 1$
    (c) $8x^3 + 4x + 3$ is divided by $2x - 1$
    (d) $6x^3 - 2x^2 + 5x - 4$ is divided by x
    (e) $3x^3 + 6x - 8$ is divided by $x + 3$
36. The expression $2x^3 - 3x^2 + ax - 5$ gives a remainder of 7 when divided by $x - 2$. Find the value of the constant a.
37. The remainder when $x^3 - 2x^2 + ax + 5$ is divided by $(x - 3)$ is twice the remainder when the same expression is divided by $x + 1$. Find the value of the constant a.
38. The remainder when $cx^3 + 2x^2 - 5x + 7$ is divided by $x - 2$ is equal to the remainder when the same expression is divided by $x + 1$. Find the value of the constant c.
39. Given that $x - 4$ is a factor of $2x^3 - 3x^2 - 7x + b$, where b is a constant. Find the remainder when the same expression is divided by $2x - 1$.
40. The expression $cx^3 + dx^2 + 3x + 8$ leaves a remainder of -6 when divided by $x - 2$ and a remainder of -34 when divided by $x + 2$. Find the value of the constants c and d.
41. The expression $x^3 - x^2 + ax + \mathrm{b}$ has a factor of $x + 3$, and leaves a remainder of 6 when divided by $x - 3$. Find the values of the constants a and b and hence factorise the expression.
42. The remainder when the expression $x^3 - 2x^2 + ax + b$ is divided by $x - 2$ is five times the remainder when the same expression is divided by $x - 1$, and 12 less than the remainder when the same expression is divided by $x - 3$. Find the values of constants a and b.
43. Show that $(x - 2)$ is a factor of $x^3 - 9x^2 + 26x - 24$. Find the set of values of x for which $x^3 - 9x^2 + 26x - 24 < 0$
44. The expression $6x^2 + x + 7$ leaves the same remainder when divided by $x - a$ and by $x + 2a$, where $a \neq 0$. Calculate the value of a.
45. Given that $x^2 + px + q$ and $3x^2 + q$ have a common factor $x - \mathrm{b}$, where p, q and b are non-zero. Show that $3p^2 + 4q = 0$.
46. Express the polynomial $f(x) = 2x^4 + x^3 - x^2 + 8x - 4$ as a product of two linear factors and a quadratic factor $q(x)$. Prove that there are no real values of x for which $q(x) = 0$.
47. Find the remainder when:
    (a) $x^3 + 3x^2 - 4x + 2$ is divided by $x - 1$
    (b) $x^3 - 2x^2 + 5x + 8$ is divided by $x - 2$
    (c) $x^5 + x - 9$ is divided by $x + 1$
    (d) $x^3 + 3x^2 + 3x + 1$ is divided by $x + 2$
48. Find the values of a in the expressions below when the following conditions are satisfied.
    (a) $x^3 + ax^2 + 3x - 5$ has remainder -3 when divided by $x - 2$.
    (b) $x^3 + x^2 + \mathrm{a}x + 8$ is divisible by $x - 1$
    (c) $x^3 + x^2 - 2ax + a^2$ has remainder 8 when divided by $x - 2$
    (d) $x^4 - 3x^2 + 2x + \mathrm{a}$ is divisible by $x + 1$
49. Show that $2x^3 + x^2 - 13x + 6$ is divisible by $x - 2$, and hence find the other factors of the expression.
50. Show that $12x^3 + 16x^2 - 5x - 3$ is divisible by $2x - 1$ and find the factors of the expression.
51. Factorise:
    $x^3 - 2x^2 - 5x + 6$
    $x^3 - 4x^2 + x + 6$
    $2x^3 + x^2 - 8x - 4$
    $2x^3 + 5x^2 + x - 2$
52. Find the values of a and b if $ax^4 + bx^3 - 8x^2 + 6$ has remainder $2x + 1$ when divided by $x^2 - 1$
53. The expression $px^4 + qx^3 + 3x^2 - 2x + 3$ has remainder $x + 1$ when divided by $x^2 - 3x + 2$. Find the values of p and q.
54. The expression $ax^2 + bx + c$ is divisible by $x - 1$, has remainder 2 when divided by $x + 1$ and has remainder 8 when divided by $x - 2$. Find the values of a, b and c.
55. $(x - 1)$ and $(x + 1)$ are factors of the expression $x^3 + ax^2 + bx + c$ and leaves a remainder of 12 when divided by $x - 2$. Find the values of a, b, and c.
56. What are the values of a and b if $x - 3$ and $x + 7$ are factors of the quadratic equation $ax^2 + 12x + b$?
57. Show that $3x^3 + x^2 - 8x + 4$ is zero when $x = 2/3$, hence factorise the expression.

Answers

1. $p = 2, q = -3, r = 7$ 2. $P^3 = 27R^2$
3. $a = 1, b = -1$ 4. (a) 2 (b) -11
5. (a) -3 (b) -2 6. $a = 2, b = -1, c = -2$
7. $a = 1, b = -3$ 8. 3
9. $(k + 1)(x – 3)(3k + 1)$ 10. $a = 12, b = 4$
12. 1½ 13. -2.5
14. $p = 31, q = -12$ 15. $(x + 1)(x – 1)(x – 2)$
16. $p = -5, q = 6$; $(y – 1)(y – 2)(y – 3)$
17. (a) $(y^2 + 1)(2y – 1)$ (b) $(y + 1)(y + 2)(2y – 1)$
18. (a) 5 (b) 29 19. $-16 < q < 0$
20. ± 4 21. $(2y – 1)(y – 2)(y2 – y – 2)$
22. $a = 2, b =1$
23.(a) $p = 3, q = -11$ b) $(2x – 1)(x + 2)(x – 1)$
24. (a) -99 (b) $35x – 39$
25. (a) $x^3 + 2x^2 + 3x + 4 = 0$ (b) $2x^4 + x^3 – 5$
26. $a = -9, b = 7$, $[(x – 1)(x – 3)(2x + 1)]$
27. $a = -1, b = 6$ 28. (i) -12 (ii) -60 (iii) 0
29. $a = -10$ 30. $a = -9, b = 2, c = 8$
31. $k = 1, ½ \pm \sqrt{\frac{3}{4}}$ 32. $(2x + 3)$
33. $[3x – 2, (x – 1)]$
34.(a) $3x – 2$, 4 (b) $3x + 1$, 8
(c) $x^2 + 5x + 8$, 17 (d) $2x^2 + 5x + 16$, 65
(e) $2x^2 – 3x^2 + 3x – 4$, 14
35. (a) 24 (b) 5 (c) 6 (d) -4 (e) 1
36. $a = 4$ 37. $a = -2$
38. $c = 1$ 39. $b = 52$, remainder = -56
40. $c = 1, d = -7$ 41. -8, 12, $(x – 1)2(x + 3)$
42. 3, -1 43. $x < 2$ or $3 < x < 4$
44. $\frac{1}{6}$ 46. $(2x – 1)(x + 2)(x^2 – x + 2)$
47. (a) 2 (b) 18 (c) -11 (d) -1
48. (a) -3 (b) -10 (c) 2 (d) 4
49. $(x + 3)(2x – 1)$ 50. $(2x – 1)(2x + 3)(3x + 1)$
51. (a) $(x – 1)(x + 2)(x – 3)$ (b) $(x + 1)(x – 2)(x – 3)$
(c) $(2x + 1)(x – 2)(x + 2)$ (d) $x + 1)(x + 2)(2x – 1)$
52. $a = 3, b = 2$ 53. $p = 1, q = -3$
54. $a = 3, b = -1, c = -2$ 55. $a = 2, b = -1, c = -2$
56. $a = 3, b = -63$ 57. $(x – 1)(x + 2)(3x – 2)$