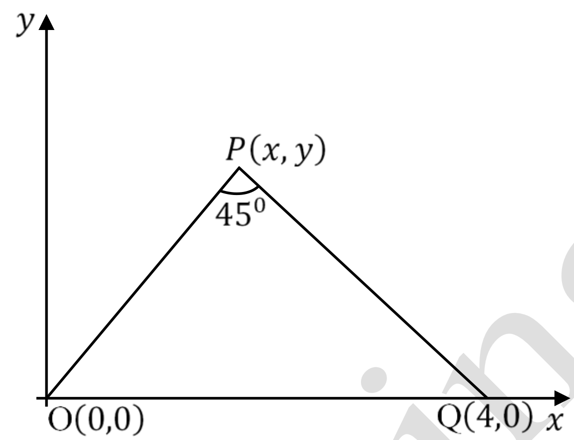
PROPOSED MARKING GUIDE UACE 2024
PURE MATHEMATICS UTEC
P425/1

NO	SOLUTION	MKS	COMMENT
1	$\sin 3\theta = \cos\theta$ $\sin(2\theta+\theta) = \cos\theta$ $\sin 2\theta\cos\theta + \cos 2\theta\sin\theta = \cos\theta$ Dividing through by $\cos\theta$ $\sin 2\theta + \cos 2\theta\tan\theta = 1$ $\left(\frac{2t}{1+t^2}\right) + \left(\frac{1-t^2}{1+t^2}\right)t = 1$ where $t = \tan\theta$ $2t + t - t^3 = 1 + t^2$ $t^3 + t^2 - 3t + 1 = 0$ Put $t = 1$; $1^3 + 1^2 - 3(1) + 1 = 0$ $0 = 0$ $t = 1$ is a root, then $t - 1$ is a factor. $\begin{array}{r} t^2+2t-1 \\ (t-1)\overline{)\,t^3+t^2-3t+1} \\ -\underline{t^3-t^2\quad\quad\quad} \\ 2t^2-3t+1 \\ -\underline{2t^2-2t\quad} \\ -t+1 \\ \underline{-t+1} \\ -\quad - \end{array}$ $t^3 + t^2 - 3t + 1 = 0$ $(t-1)(t^2+2t-1) = 0$ $t = 1$ or $t^2 + 2t - 1 = 0$		

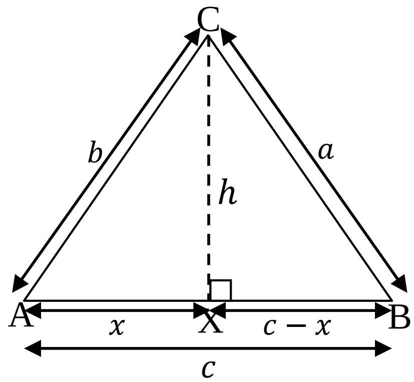
	$t = \frac{-2\pm\sqrt{2^2-4\times1\times-1}}{2\times1}$ $t = 0.4142, t = -2.4142$ When $t = 1, \tan\theta = 1$ $\theta = \tan^{-1}(1)$ $= 45^0$ When $t = 0.4142;\ \tan\theta = 0.4142$ $\theta = \tan^{-1}(0.4142)$ $= 22.50^0$ When $t = -2.4142;\ \tan\theta = -2.4142$ $\theta = 112.50^0$ $\therefore \theta = \{22.50^0, 45^0, 112.50^0\}$		
		05	
2	$x^3 = (y-3x)^2$ $3x^2 = 2\left(\frac{dy}{dx} - 3\right)(y-3x)$ Multiplying through by x; $3x^3 = 2x\left(\frac{dy}{dx} - 3\right)(y-3x)$ $3(y-3x)^2 = 2x\left(\frac{dy}{dx} - 3\right)(y-3x)$ $3(y-3x) = 2x\left(\frac{dy}{dx} - 3\right)$ $3y - 9x = 2x\frac{dy}{dx} - 6x$ $3y - 3x = 2x\frac{dy}{dx}$ $\therefore 2x\frac{dy}{dx} = 3y - 3x$		
		05	

3	Using $\frac{\mu\boldsymbol{a}+\lambda\boldsymbol{b}}{\lambda+\mu}$ for $\lambda:\mu$ a) $c = \frac{1\binom{3}{5}+3\binom{-5}{-1}}{1+3} = \frac{1}{4}\binom{-12}{2} = \binom{-3}{0.5}$ $\therefore C(-3, 0.5)$ b) $c = \frac{-1\binom{3}{5}+3\binom{-5}{-1}}{-1+3} = \frac{1}{2}\binom{-18}{-8} = \binom{-9}{-4}$ $\therefore C(-9, -4)$		
		05	
4	$y = 1 + 2\sin x$ When $x = \frac{\pi}{4}, y = 1 + 2\sin\left(\frac{\pi}{4}\right) = 1 + \sqrt{2}$ $\frac{dy}{dx} = 2\cos x$ At $x = \frac{\pi}{4}$, $\frac{dy}{dx} = 2\cos\left(\frac{\pi}{4}\right) = \sqrt{2}$ Equation; $y - (1 + \sqrt{2}) = \sqrt{2}\left(x - \frac{\pi}{4}\right)$ $\therefore y = x\sqrt{2} + 1 + \sqrt{2} - \frac{\pi\sqrt{2}}{4}$		
		05	
5	$x(1+3i) + y(1-3i) = 7 + 3i$ $x + 3xi + y - 3yi = 7 + 3i$ $x + y + (3x - 3y)i = 7 + 3i$ Equating components; $x + y = 7$ …………………….(i)		

	$3x - 3y = 3$ $x - y = 1$ ……………………….(ii) (i)+(ii); $2x = 8$ $x = 4$ From (i); $4 + y = 7$ $y = 3$ $\therefore x = 4, y = 3$		
		05	
6	 Gradient OP, $m_1 = \frac{y-0}{x-0} = \frac{y}{x}$ Gradient PQ, $m_2 = \frac{y-0}{x-4} = \frac{y}{x-4}$ Using $\tan\theta = \left\lvert\frac{m_1 - m_2}{1 + m_1 m_2}\right\rvert$ $\tan 45^0 = \left\lvert\frac{\frac{y}{x} - \frac{y}{x-4}}{1 + \frac{y}{x} \times \frac{y}{x-4}}\right\rvert$ $1 = \frac{\frac{y}{x} - \frac{y}{x-4}}{1 + \frac{y}{x} \times \frac{y}{x-4}}$ $1 + \frac{y}{x} \times \frac{y}{x-4} = \frac{y}{x} - \frac{y}{x-4}$		

	$1 + \frac{y^2}{x^2-4x} = \frac{xy-4y-xy}{x^2-4x}$ $x^2 - 4x + y^2 = -4y$ $\therefore x^2 + y^2 - 4x + 4y = 0$ is the locus		
		05	
7	From $\mathrm{P}(x) = \mathrm{Q}(x)(x-a) + \mathrm{R}(x)$ $\mathrm{P}(x) = \mathrm{Q}(x)(x-2)(x+2) + 3x + 7$ a) When $x = 2, \mathrm{P}(2) = R$ $\mathrm{P}(2) = 3 \times 2 + 7$ $= 13$ b) When $x = -2, \mathrm{P}(-2) = \mathrm{R}$ $\mathrm{P}(-2) = 3(-2) + 7$ $= 1$		
		05	
8	$\int_0^{\sqrt{3}} \frac{dx}{\sqrt{3-x^2}} = \frac{1}{\sqrt{3}} \int_0^{\sqrt{3}} \frac{dx}{\sqrt{1-\left(\frac{x}{\sqrt{3}}\right)^2}}$ Let $\frac{x}{\sqrt{3}} = \sin u$ $x = \sqrt{3}\sin u$ $dx = \sqrt{3}\cos u \ du$ $\begin{array}{\|c\|c\|} \hline x & u \\ \hline 0 & 0 \\ \hline \sqrt{3} & \frac{\pi}{2} \\ \hline \end{array}$ $\Rightarrow \frac{1}{\sqrt{3}} \int_0^{\pi/2} \frac{\sqrt{3}\cos u \, du}{\sqrt{1-sin^2 u}}$ $= \Big[u \Big]_0^{\frac{\pi}{2}}$ $= \frac{\pi}{2} - 0$		

	$= \frac{\pi}{2}$ or 1.5708		
		05	
9	a) Let $\sqrt{3}\sin\theta + \cos\theta \equiv R\sin(\theta + \alpha)$ $\sqrt{3}\sin\theta + \cos\theta \equiv (R\cos\alpha)\sin\theta + (R\sin\alpha)\cos\theta$ Comparing coefficients of; $\sin\theta\,;\, R\cos\alpha = \sqrt{3}$ ………………(i) $\cos\theta\,;\, R\sin\alpha = 1$ ………………..(ii) $(R\cos\alpha)^2 + (R\sin\alpha)^2 = 3 + 1^2$ $R^2 = 4$ $R = 2$ (ii)÷(i); $\tan\alpha = \frac{1}{\sqrt{3}}$ $\alpha = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ $= 30^0$ $\therefore \sqrt{3}\sin\theta + \cos\theta = 2\sin(\theta + 30^0)$ Hence; $2\sin(\theta + 30^0) = \sqrt{2}$ $\sin(\theta + 30^0) = \frac{\sqrt{2}}{2}$ $\theta + 30^0 = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$ $\theta + 30^0 = 45^0, 135^0$ $\therefore \theta = \{15^0, 105^0\}$ b) Consider triangle ABC,		



Form ΔAXC, $x^2 + h^2 = b^2$……..(i)

ΔBXC, $(c - x)^2 + h^2 = a^2$

$c^2 - 2cx + x^2 + h^2 = a^2$ ………(ii)

Putting (i) in (ii)

$a^2 = b^2 + c^2 - 2cx$

But $x = b\cos A$

$\therefore a^2 = b^2 + c^2 - 2bc\cos A$

Solving,

$a^2 = 5^2 + 8^2 - 2 \times 5 \times 8\cos 60^0$

$a^2 = 25 + 64 - 40$

$a^2 = 49$

$a = 7\ cm$

$\frac{a}{\sin A} = \frac{b}{\sin B}$

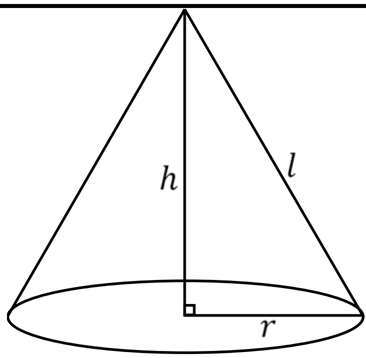
$\frac{7}{\sin 60^0} = \frac{5}{\sin B}$

$B = \sin^{-1}\left(\frac{5\sin 60^0}{7}\right) = 38.21^0$

$A + B + C = 180^0$

$60^0 + 38.21^0 + C = 180^0$

	$C = 81.79^0$		
		12	
10	a) Let $y = \sin x$ $\frac{dy}{dx} = \cos x$ $\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$ $\delta y \approx \frac{dy}{dx} \cdot \delta x$ $\approx \cos x \cdot \delta x$ Set $x = 30^0, \delta x = -0.5^0 = -\frac{\pi}{360}$ $\delta y \approx \cos 30^0 \times -\frac{\pi}{360}$ $\delta y \approx \frac{\sqrt{3}}{2} \times -\frac{\pi}{360}$ $\delta y \approx -\frac{\sqrt{3}\,\pi}{720}$ ⇨ $\sin 29.5^0 \approx y - \delta y$ $\approx \sin 30^0 - \frac{\sqrt{3}\,\pi}{720}$ $\approx 0.5 - \frac{\sqrt{3}\,\pi}{720}$ ≈ 0.492442502 $= 0.4924$ (4dps) b) Let l =slant height, h =height, r = radius.		



From Pythagoras theorem,

$h^2 + r^2 = l$

$r^2 = 81 \times 3 - h^2$

$r^2 = 243 - h^2$

$v = \frac{1}{3}\pi r^2 h$

$v = \frac{1}{3}\pi(243 - h^2)h$

$v = \frac{1}{3}\pi(243h - h^3)$

$\frac{dv}{dh} = \frac{1}{3}\pi(243 - 3h^2)$

For maximum volume, $\frac{dv}{dh} = 0$

$\frac{1}{3}\pi(243 - 3h^2) = 0$

$3h^2 = 243$

$h^2 = 81$

$h = 9\ cm$

From $r^2 = 243 - h^2$

$r^2 = 243 - 81$

$r^2 = 162$

$r = \sqrt{81 \times 2} = 9\sqrt{2}\ cm$

	$v_{max} = \frac{1}{3} \times \pi \times 162 \times 6$ $v_{max} = 324\pi\ cm^3$ or $v_{max} = 1017.87602\ cm^3$		
		12	
11	a) Using $u_{r+1} = {}^{n}C_r \cdot a^{n-r}x^r$ $u_{r+1} = {}^{9}C_r\,(x^2)^{9-r}\left(\frac{1}{2}x^{-1}\right)^r$ $u_{r+1} = {}^{9}C_r\,x^{18-2r} \cdot \left(\frac{1}{2}\right)^r x^{-r}$ For the coefficient of x^{-3}; $18 - 2r - r = -3$ $3r = 21$ $r = 7$ ⇨ $u_8 = {}^{9}C_7\,x^4 \cdot \left(\frac{1}{2}\right)^7 x^{-7}$ $= \frac{9}{32}x^{-3}$ ∴ The coefficient of x^{-3} is $\frac{9}{32}$. b) $\left(1 - \frac{1}{4}x\right)^{1/2} = 1 + \frac{1}{2}\left(-\frac{1}{4}x\right) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{1}{4}x\right)^2}{2!} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{1}{4}x\right)^3}{3!} + \cdots$ $= 1 - \frac{1}{8}x - \frac{1}{128}x^2 - \frac{1}{1024}x^3 + \cdots$ (i) $\sqrt{15} = (16 - 1)^{1/2}$ $= 4\left(1 - \frac{1}{16}\right)^{1/2}$ ⇨ $\frac{1}{4}x = \frac{1}{16}$ $x = \frac{1}{4}$		

	$\sqrt{15} \approx 4\left[1-\frac{1}{8}\left(\frac{1}{4}\right)-\frac{1}{128}\left(\frac{1}{4}\right)^2-\frac{1}{1024}\left(\frac{1}{4}\right)^3\right]$ $\approx 4(0.96824646)$ ≈ 3.87298584 $= 3.873$ (3dps) (ii) $\sqrt{7} = (9-2)^{1/2}$ $= 3\left(1-\frac{2}{9}\right)^{1/2}$ $\Rightarrow \frac{1}{4}x = \frac{2}{9}$ $x = \frac{8}{9}$ $\sqrt{7} \approx 3\left[1-\frac{1}{8}\left(\frac{8}{9}\right)-\frac{1}{128}\left(\frac{8}{9}\right)^2-\frac{1}{1024}\left(\frac{8}{9}\right)^3\right]$ $\approx 3(0.882030178)$ ≈ 2.6460905335 $= 2.6461$ (4dps)		
		12	
12	x^2 $(x^4-16)\overline{)\,x^6+64}$ $-\;x^6-16x^2$ $16x^2+64$ $\frac{x^6+64}{x^4-16} = x^2 + \frac{16x^2+64}{x^4-16}$ For $\frac{16x^2+64}{x^4-16} = \frac{16x^2+64}{(x+2)(x-2)(x^2+4)}$ Let $\frac{16x^2+64}{(x+2)(x-2)(x^2+4)} \equiv \frac{A}{x+2}+\frac{B}{x-2}+\frac{Cx+D}{x^2+4}$ $16x^2+64 \equiv A(x-2)(x^2+4)+B(x+2)(x^2+4)+(Cx+D)(x+2)(x-2)$		

	Put $x = 2; 128 = 32B$ $B = 4$ Put $x = -2; 128 = -32A$ $A = -4$ Put $x = 0; 64 = -8A + 8B - 4D$ $64 = -8(-4) + 8(4) - 4D$ $D = 0$ Comparing coefficients of x^3; $0 = A + B + C$ $0 = -4 + 4 + C$ $C = 0$ $\therefore \frac{x^6+64}{x^4-16} = x^2 - \frac{4}{x+2} + \frac{4}{x-2}$ Hence; $\int_3^4 \frac{x^6+64}{x^4-16} dx = \int_3^4 x^2\, dx - 4\int_3^4 \frac{1}{x+2} dx + 4\int_3^4 \frac{1}{x-2} dx$ $= \left[\frac{x^3}{3}\right]_3^4 - 4[\ln(x+2)]_3^4 + 4[\ln(x-2)]_3^4$ $= \frac{1}{3}(64 - 27) - 4(\ln 6 - \ln 5) + 4(\ln 2 - \ln 1)$ $= \frac{37}{3} - 0.729286227 + 2.772588722$ $= 14.37663583$ $= 14.3766$		
		12	
13	a) At point of intersection,		

$$\begin{pmatrix}2\\-4\\4\end{pmatrix}+\lambda\begin{pmatrix}2\\3\\-1\end{pmatrix}=\begin{pmatrix}10\\1\\7\end{pmatrix}+\mu\begin{pmatrix}3\\1\\2\end{pmatrix}$$

$2+2\lambda=10+3\mu$

$2\lambda-3\mu=8$ ……………..(i)

$-4+3\lambda=1+\mu$

$3\lambda-\mu=5$ ……………..(ii)

$4-\lambda=7+2\mu$

$-\lambda-2\mu=3$ ……………(iii)

(i)+2(iii); $-7\mu=14$

$\mu=-2$

From (i); $2\lambda-3(-2)=8$

$2\lambda=2$

$\lambda=1$

$$\begin{pmatrix}2\\-4\\4\end{pmatrix}+\lambda\begin{pmatrix}2\\3\\-1\end{pmatrix}=\begin{pmatrix}10\\1\\7\end{pmatrix}+\mu\begin{pmatrix}3\\1\\2\end{pmatrix}$$

$$\begin{pmatrix}2\\-4\\4\end{pmatrix}+\begin{pmatrix}2\\3\\-1\end{pmatrix}=\begin{pmatrix}10\\1\\7\end{pmatrix}-2\begin{pmatrix}3\\1\\2\end{pmatrix}$$

$$\begin{pmatrix}4\\-1\\3\end{pmatrix}=\begin{pmatrix}4\\-1\\3\end{pmatrix}$$

$\therefore (4,-1,3)$ is the point of intersection

Let θ be the acute angle,

$$\boldsymbol{d}_1=\begin{pmatrix}2\\3\\-1\end{pmatrix},\boldsymbol{d}_2=\begin{pmatrix}3\\1\\2\end{pmatrix}$$

	$\boldsymbol{d}_1 \cdot \boldsymbol{d}_2 = \lvert\boldsymbol{d}_1\rvert\,\lvert\boldsymbol{d}_2\rvert\ \cos\theta$ $\begin{pmatrix}2\\3\\-1\end{pmatrix}\cdot\begin{pmatrix}3\\1\\2\end{pmatrix} = \sqrt{2^2+3^2+(-1)^2}\,\sqrt{3^2+1^2+2^2}\ \cos\theta$ $6+3-2=\sqrt{14}\,\sqrt{14}\ \cos\theta$ $7 = 14\cos\theta$ $\theta = \cos^{-1}\left(\frac{7}{14}\right)$ $\theta = 60^0$ b) $\boldsymbol{r} = \begin{pmatrix}4\\-1\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\3\\-1\end{pmatrix} + \mu\begin{pmatrix}3\\1\\2\end{pmatrix}$		
		12	
14	a) $y = \frac{2x-5}{x^2-4}$ $\frac{dy}{dx} = \frac{(x^2-4)\cdot 2-(2x-5)\cdot 2x}{(x^2-4)^2} = 0$ $2x^2-8-4x^2+10x=0$ $-2x^2+10x-8=0$ $x^2-5x+4=0$ $(x-4)(x-1)$ $x=1, x=4$ When $x=1, y=\frac{2(1)-5}{1^2-4} = 1;\ (1,1)$ When $x=4, y=\frac{2(4)-5}{4^2-4} = 0.25;\ (4,0.25)$ Nature; For (1,1)		

x	0	1	2
Sign of $\frac{dy}{dx}$	$-ve$	0	$+ve$
	\	—	/

$(1,1)$ is a minimum point.

For $(4,0.25)$

x	3	4	5
Sign of $\frac{dy}{dx}$	$+ve$	0	$-ve$
	/	—	\

$(4,0.25)$ is a maximum point.

Hence ;

$y(x^2 - 4) = 2x - 5$

$yx^2 - 4y = 2x - 5$

$yx^2 - 2x + 5 - 4y = 0$

For real values of $x, b^2 - 4ac \geq 0$

$(-2)^2 - 4 \times y \times (5 - 4y) \geq 0$

$4 - 20y + 16y^2 \geq 0$

$4y^2 - 5y + 1 \geq 0$

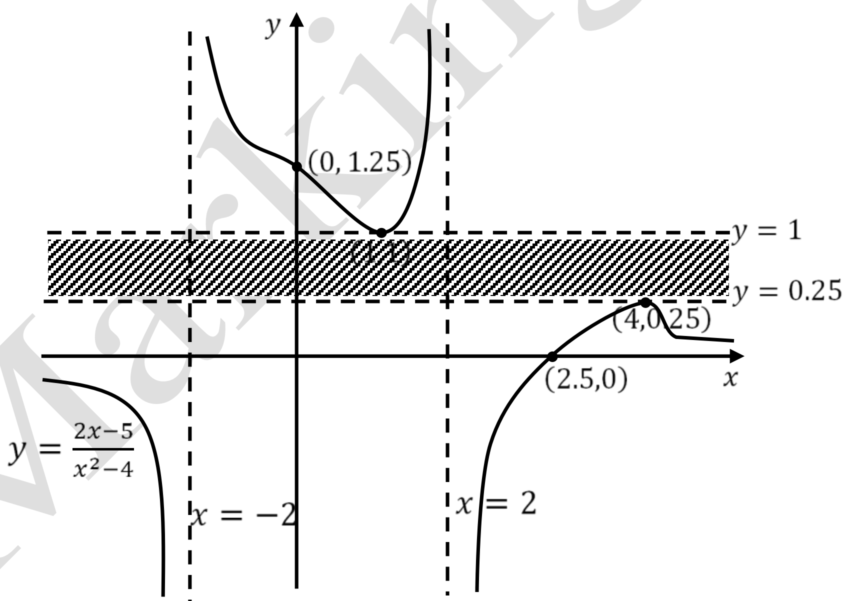
$(y - 1)(4y - 1) \geq 0$

Critical values,

$y = 1, y = 0.25$

y	$y < 0.25$	$0.25 < y < 1$	$y > 1$
$(y-1)(4y-1)$	$+$	$-$	$+$

Hence the curve does not lie in the range $0.25 < y < 1$

	b) ***Intercepts***; $x; y = 0$ $0 = 2x - 5$ $x = 2.5, (2.5, 0)$ $y;\ x = 0$ $y = \frac{2(0)-5}{0^2-4} = 1.25, (0,1.25)$ ***Asymptotes***, Vertical $x^2 - 4 = 0$ $x = -2, x = 2$ Horizontal $y = \frac{\frac{2}{x}-\frac{5}{x^2}}{1-\frac{4}{x^2}}$ As $x \to \pm, y \to 0$ i.e $y = 0$  		
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15	a) $y^2 = 4ax$		

$2y\frac{dy}{dx} = 4a$

$\frac{dy}{dx} = \frac{2a}{y}$

At $(at^2, 2at)$; $\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$

Equation;

$\frac{y-2at}{x-at^2} = \frac{1}{t}$

$ty - 2at^2 = x - at^2$

$\therefore x - ty + at^2 = 0$

b) From $x - ty + at^2 = 0$

At $A(-6a, a)$;

$-6a - at + at^2 = 0$

$t^2 - t - 6 = 0$

$(t-3)(t+2) = 0$

$t = 3, t = -2$

When $t = 3, x - 3y + 9a = 0$

When $t = -2, x + 2y + 4a = 0$

Hence

(i) For $x - 3y + 9a = 0$

$x = 3y - 9a$

$y^2 = 4a(3y - 9a)$

$y^2 = 12ay - 36a^2$

$y^2 - 12ay + 36a^2 = 0$

$y = \frac{-b}{2a}$

	$y = \frac{12a}{2} = 6a$ When $y = 6a, x = 18a - 9a = 9a \quad \therefore (9a, 6a)$ For $x + 2y + 4a = 0$ $x = -2y - 4a$ $y^2 = 4a(-2y - 4a\ )$ $y^2 = -8ay - 16a^2$ $y^2 + 8ay + 16a^2 = 0$ $y = \frac{-b}{2a}$ $y = \frac{-8a}{2} = -4a$ When $y = -4a, x = -12a - 9a = -21a$ $\therefore (-21a, -4a)$ (ii) Let $m_1 = \frac{1}{3}$ and $m_2 = -\frac{1}{2}$ From $\tan\theta = \left\lvert\frac{m_1 - m_2}{1 + m_1 m_2}\right\rvert$ $\tan\theta = \left\lvert\frac{\frac{1}{3}+\frac{1}{2}}{1+\frac{1}{3}\times -\frac{1}{2}}\right\rvert$ $\tan\theta = \frac{5/6}{5/6}$ $\theta = \tan^{-1}(1)$ $\theta = 45^0$		
		12	
16	a) b)		
		12	