Understanding
Applied Mathematics

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6.NORMAL DISTRIBUTION

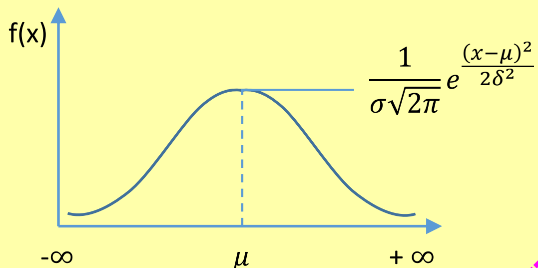
Normal distribution

A continuous random variable X follows a normal distribution with mean, μ and variance, σ^2 if

X~N(μ, σ²) root

Its p.d.f is given by fx = $\frac{1}{\sigma\sqrt{2\pi}} e^{\frac{(x-\mu)^2}{2\delta^2}}$, $-\infty < x < \infty$

A sketch of f(x) gives a normal curve



Properties of the curve

- It is bell shaped
- It is symmetrical about μ
- It extends from $-\infty < x < \infty$

The maximum value of f(x) = $\frac{1}{\sigma\sqrt{2\pi}} e^{\frac{(x-\mu)^2}{2\delta^2}}$

- The total area under the curve = 1

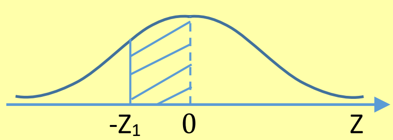
How to read the cumulative normal distribution table

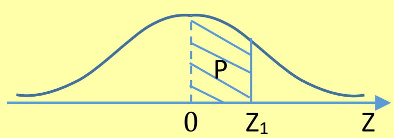
(i) Between 0 and any z value

(a) $P(0 \le Z \le Z_1) = \phi(Z_1)$ = region P

(b) $P(-Z_1 \le Z \le 0) = P(0 \le Z \le Z_1) = \phi(Z_1)$ = region P

By symmetrical



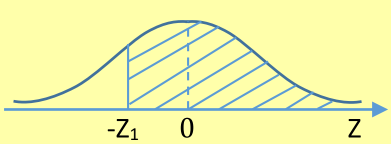


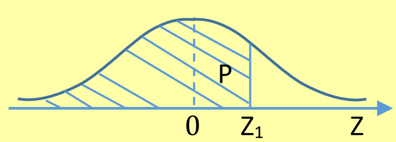
(ii) Less than any positive z value

(a) $P(Z < Z_1) = 0.5 + P(0 \le Z \le Z_1) = \phi(Z_1)$ = region P

(b) $P(Z > -Z_1) = P(Z < Z_1) = 0.5 + P(0 \le Z \le Z_1) = 0.5 + \phi(Z_1)$ = region P

By symmetrical



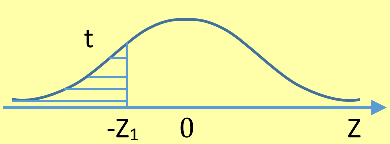


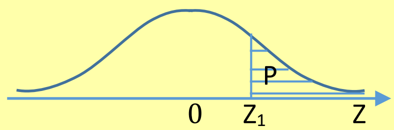
(iii) Greater than any positive z value

$P(Z > Z_1) = 0.5 - P(0 \le Z \le Z_1) = 0.5 - \phi(Z_1)$ = region P

$P(Z < -Z_1) = P(Z > Z_1) = 0.5 - P(0 \le Z \le Z_1) = 0.5 - \phi(Z_1)$ = region P

By symmetrical





Example 1

Find (i) P(Z< 2) (ii) P(Z > 0.85) (iii) P(X< 0.345)

Solution

(i) P(X < 2) = 0.5 + ϕ(2) = 0.5 + 0.4772 = 0.9772

(ii) P(Z > 0.85) = 0.5 - ϕ(0.85) = 0.5- 0.3023 = 0.1977

(iii) P(X< 0.345) = 0.5 + - ϕ(0.345) =0.5 + 0.1331 + 0.0019 = 0.6350

Example 2

Find (i)P(Z < -0.25) (ii) P(Z > -1.377) (iii) P(Z < -1.377)

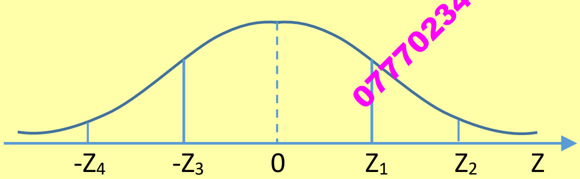
Solution

(i) P(Z < -0.25) = P(Z > 0.25) = 0.5 - ϕ(0.25)= 0.5 – 0.0987 = 0.4013

(ii) P(Z > -1.377) = P(Z < 1.377) = 0.5 + ϕ(1.377)= 0.5 + 0.4147 + 0.0011 = 0.9158

(iii) P(Z < -1.377) = P(Z > 1.377 = 0.5 - ϕ(1.377) = 0.5 – (0.4147 + 0.0011) = 0.0842

Other important results



(i) Between two Z values on the same side of the mean

(a) $P(Z_1 < Z < Z_2) = P(0 < Z < Z_2) - P(0 < Z < Z_1) = \phi(Z_2) - \phi(Z_1)$

(b) $P(-Z_4 < Z < -Z_3) = P(0 < Z < Z_4) - P(0 < Z < Z_3) = \phi(Z_4) - \phi(Z_3)$

(ii) Between two Z values on the opposite side of the mean

(a) $P(-Z_3 < Z < Z_1) = P(0 < Z < Z_3) + P(0 < Z < Z_1) = \phi(Z_3) + \phi(Z_1)$

(b) $P(|Z|< Z_1) = P(-Z_1 < Z < Z_1) = 2 \text{ x } P(0 < Z < Z_1) = 2 \text{ x } \phi(Z_1)$

(c) $P(|Z|> Z_1) = 1 - P(-Z_1 < Z < Z_1) = 1 -2 \text{ x } \phi(Z_1)$

Example 3

Find (i) P(1.5 < Z < 1.88) (iii) P(-2.696 < Z < 1.865) (v) P(|Z|< 1.75)
(ii) P(-2.5 < Z < 1) (iv) P (-1.4 < Z < -0.6) (vi) P(|Z|> 1.433)

Solution

(i) P(1.5 < Z < 1.88) = ϕ(1.88) - ϕ(1.5) = 0.4699 – 0.4332 = 0.0367

(ii) P(-2.5 < Z < 1) = ϕ(1) + ϕ(2.5) = 0.3413 + 0.4938 = 0.8351

(iii) P(-2.696 < Z < 1.865) = ϕ(1.865) + ϕ(2.696) = 0.469 + 0.4964 = 0.9654

(iv) P (-1.4 < Z < -0.6) = ϕ(1.4) + ϕ(0.6) = 0.4192 – o.2257 =0.1935

(v) P(|Z|< 1.75) = P(-1.75 < Z < 1.75) = 2 x ϕ(1.75) = 2 x 0.4625= 0.925

(vi) P(|Z|> 1.433) = 1 - P(|Z|< 1.433) = 1 – 2 x ϕ(1.433) = 1 – 2 x 0.424 = 0.152

Standardizing a random variable X

If a random variable X follows a normal distribution with mean, μ and variance, σ^2, then $X \sim N(\mu, \sigma^2)$ and can be standardized using the equation below and read from a cumulative normal table

$Z = \frac{X-\mu}{\sigma}$

Example 4

Given that the random variable X is $X \sim N(300, 25)$. Find

(i) P(X > 305) (ii) P(X < 291) (iii) P(X < 312) (iv) P(X > 286)

Solution

(i) P(X> 305) $= P\left(Z < \frac{305-300}{5}\right)$ = P(X > 1) = 0.5 – ϕ(1) = 0.5 – 0.3413 = 0.1587

(ii) P(X < 291) $= P\left(Z < \frac{291-300}{5}\right)$ = P(Z < -1.8)
= P(Z >1.8) = 0.5 - ϕ(1.8) = 0.5 – 0.4641 = 0.0359

(iii) P(X < 312) $= P\left(Z < \frac{312-300}{5}\right)$ = P(X < 2.4) = 0.5 + ϕ(2.4)
= 0.5 – 0.4918 = 0.9918

(iv) P(X > 286) $= P\left(Z < \frac{286-300}{5}\right)$ = P(Z < -2.8)
= P(Z < 2.8) = 0.5 + ϕ(2.8) = 0.5 + 0.4974= 0.9974

Example 5

Given that the random variable X is $X \sim N(10, 4)$. Find

Find (i) P(X < 7) (ii) P(X > 12) (iii) P(7 < X < 12) (iv) P(9 < X < 11)

Solution

(i) P(X < 7) $= P\left(Z < \frac{7-10}{2}\right)$ = P(Z < - 0.15) = P(Z > 1.5)
= 0.5 – ϕ(1.5) = 0.5 – 0.4332 0.0668

(ii) P(X > 12) $= P\left(Z > \frac{12-10}{2}\right)$ = P(Z > 1) = 0.5 – ϕ(1) = 0.5 – 0.3413 = 0.1587

(iii) P(7 < X < 12) = $P\left(\frac{7-10}{2} < Z > \frac{12-10}{2}\right)$
= P(-1.5 < Z < 1) = ϕ(1.5) + ϕ(1) = 0.4332 + 0.3413 = 0.7745

(iv) P(9 < X < 11) = $P\left(\frac{9-10}{2} < Z > \frac{11-10}{2}\right)$
= P(-0.5 < Z < 0.5) = ϕ(0.5) + ϕ(0.5) = 2 x 0.1915 = 0.3830

Example 6

Given that the random variable X is $X \sim N(50, 8)$. Find

(i) P(48 < X < 54) (ii) P(52 < X < 55) (iii) P(46 < X < 49) (iv) P(|X- 50|< $\sqrt{8}$)

Solution

(i) P(48 < X < 54) = $P\left(\frac{48-50}{\sqrt{8}} < Z > \frac{54-50}{\sqrt{8}}\right)$ = P(-0.707 < Z < 1.414)

= ϕ(1.414) + ϕ(0.707) = 0.4213 + 0.2601 = 0.6814

(ii) P(52 < X < 55) = $P\left(\frac{52-50}{\sqrt{8}} < Z > \frac{55-50}{\sqrt{8}}\right)$ = P(0.707 < Z < 1.768)

= ϕ(1.768) - ϕ(0.707) = 0.4615 -0.2601 = 0.2014

(iii) P(46 < X < 49) = $P\left(\frac{46-50}{\sqrt{8}} < Z > \frac{49-50}{\sqrt{8}}\right)$ = P(-1.414 < Z < -0.354)

=ϕ(1.414) – ϕ(0.354) = 0.4213 – 0.1383 -0.283

(iv) P(|X- 50|< $\sqrt{8}$) = $P\left(\frac{-\sqrt{8}+50-50}{\sqrt{8}} < Z > \frac{\sqrt{8}+50-50}{\sqrt{8}}\right)$ P(-1 < X < 1) = 2 x ϕ(1) = 2 x 0.3413 = 0.6826

Example 6

A random variable X is normally distributed with mean 65 and variance 100, find the probability that X assumes a value between 50 and 90.

P(50 < X < 90) = $P\left(\frac{50-65}{10} < Z < \frac{90-65}{10}\right)$= P(-1.5 < Z < 2.5) = ϕ(1.5) + ϕ(2.5) = 0.4332 + 0.4938 = 0.927

Example 7

Lengths of metal strips produced by a machine are normally distributed with mean length of 150cm and standard deviation of 10cm. find the probability that the length of a randomly selected strip is

(i)shorter than 165 (ii) within 5cm of the mean

Solution

P(X < 165) = $P\left(Z < \frac{165-150}{10}\right)$ =P(Z< 1.5) = 0.5 + ϕ(1.5) = 0.5 + 0.4332 = 0.9332

P(150 – 5 < X < 150 + 5) = $P\left(\frac{-5}{10} < Z < \frac{5}{10}\right)$= P(-0.5 < Z < 0.5) = 2 x ϕ(0.5) = 2 x 0.1915 = 0.383

Example 8

In end of year exams, the marks are normally distributed with a mean mark of 50 and standard deviation 5. If a mark 45 is required to pass the exam, what percentage of the students failed the exam.

P(X < 45) = $P\left(Z < \frac{45-50}{5}\right)$ = P(Z < -1) = P(Z > 1) = 0.5 – ϕ(1) = 0.5 – 0.3413 = 0.1587

Example 9

A bakery supplies bread to a shop every day. The time to deliver bread to the shop is normally distributed with mean 12 minutes and standard deviation of 2 minutes. Estimate the number of days the year when he takes

(i) longer than 17 minutes (ii) less than 10 minutes (iii) between 9 and 13 minutes

Solution

(i) P(X > 17) = $P\left(Z > \frac{17-12}{2}\right)$= P(Z > 2.5) = 0.5 – ϕ(2.5) = 0.5 – 0.4938 = 0.0062

The number of days = 0.0062 x 365 = 2 days

(ii) P(X < 10) = $P\left(Z > \frac{10-12}{2}\right)$= P(Z < -1) = P(Z> 1) = 0.5 – ϕ(1) + 0.5 – 0.3413 0.1587

The number of days = 0.1587 x 365 = 58 days.

(iii) P(9 < X <13) = $P\left(\frac{9-12}{2} < Z < \frac{13-12}{2}\right)$= P(-1.5 < Z < 0.5) = ϕ(1.5) + ϕ(0.5) 0.4332 + 0.1915 = 0.6247

Number of days = 0.6247 x 365 = 228 days.

Example 10

(a) In a certain athletics competition, points are awarded according to level of performance. The average grade was 82 points with standard deviation of 5 points. All competitors whose grades ranged between 88 to 94 points received certificates. If the grades are normally distributed and 8 competitors received certificates. How many participants took part in the competition?

P(88 < X < 94) = $P\left(\frac{88-82}{5} < Z < \frac{94-82}{5}\right)$= P(1.2 < Z < 2.4) =$\frac{8}{n}$

ϕ(2.4) – ϕ(1.2) = 0.4918 – 0.3849 = 0.1069 = $\frac{8}{n}$; n = 74.84

hence 75 participants took part.

(b) If certificates were to be awarded to only those having between 90 and 94 points. What proportion of the participants would acquire certificates.

P(90 < X < 94 = $P\left(\frac{90-82}{5} < Z < \frac{94-82}{5}\right)$=P(1.6 < Z < 2.4 = ϕ(2.4) – ϕ(1.6)

= 0.4918 – 0.4452 = 0.0466

= 0.0466 x 100% = 4.66%

Revision exercise 1

1. The amount of meat sold by a butcher is normally distributed with mean 43kg and standard deviation 4kg. Determine the probability that the amount of meat sold is between 40kg and 50kg. (0.7333)
2. Given that a random variable X is X∼N(2, 2.89). Find P(X<0)(0.1198)
3. In a school of 800 students their average weight is 54.5kg and standard deviation 6.8kg. given that the weight of students are normally distributed, find
   (i) Probability that the weight of any student randomly selected is 52.8 kg or less = 0.4014
   (ii) Number of students who weigh over 75kg = 1
   (iii) Weight of the middle 56% of the students (49.251< X < 59.750)
4. A sugar factory sells sugar in bags of mean weight 50kg and standard deviation 2.5kg. given that the weight of the bags is normally distributed, find the
   (i) Probability that the weight of any bag of sugar randomly selected lies between 51.5kg and 53kg = 0.1592
   (ii) Percentage of bags whose weight exceeds 54kg = 5.48%
   (iii) Number of bags that will be rejected out of 1000 bags purchased for weighing below 45.0kg = 23
5. A certain maize firm sells maize in bags of mean weight 40kg and standard deviation 2kg,. given that the weight of the bags are normally distributed, find
   (i) Probability that the weight of any bag of maize randomly selected lies between 41.0 and 42.5kg = 0.2029
   (ii) Percentage of bags whose weight exceeds 43kg = 6.68%
   (iii) Number of bags that will be rejected out of 500 bags purchased for weighing below 38.5kg = 113
6. Given that the random variable X is X∼N(300, 25) Find
   (i) P(X>308) = 0.0548   (ii) P(X> 311.5) = 0.0107   (iii) P(X < 294) = 0.8849

(iv) $P(X < 290.5) = 0.9713$ (v) $P(X > 302) = 0.6554$ (vi) $P(X > 312) = 0.9918$

7. If $X \sim N(50, 20)$. Find
   (i) $P(X > 60.3) = 0.0106$ (ii) $P(X < 47.3) = 0.273$ (iii) $P(X > 48.9) = 0.5972$
   (iv) $P(X > 53.5) = 0.2831$ (v) $P(X < 59.8) = 0.9857$ (vi) $P(X < 62.3) = 0.9970$
8. If $X \sim N(-8, 12)$. Find
   (i) $P(X < -9.8) = 0.1587$ (ii) $P(X > 0) = 0.8413$ (iii) $P(X < -3.4) = 0.9079$
   (iv) $P(X > -5.7) = 0.2533$ (v) $P(X < 10.8) = 0.2097$ (vi) $P(X > -1.6) = 0.0323$
9. If $X \sim N(\alpha, \alpha^2)$. Find
   (i) $P(X < 0) = 0.1587$ (ii) $P(X > 0) = 0.8413$ (iii) $P(X < 0.5\alpha) = 0.6915$ $P(X > 0.5\alpha) = 0.3085$
10. If $X \sim N(100, 80)$. Find
    (i) $P(85 < X < 112) = 0.8634$ (ii) $P(105 < X < 115) = 0.2413$
    (iii) $P(85 < X < 92) = 0.1388$ (iv) $P(|X| < \sqrt{80} = 0.6826$
11. If $X \sim N(84, 12)$. Find
    (i) $P(80 < X < 89)= 0.8014$ (ii) $P(X < 79 \text{ or } X > 92) = 0.085$ (iii) $P(76 < X < 82) = 0.2714$
    (iv) $P(|X- 84| > 2.9) = 0.4028$ (v) $P(87 < X < 93) = 0.1886$
12. The masses of packages from a particular machine are normally distributed with a mean of 200g and standard deviation of 2g, find the probability that a randomly selected package from the machine weighs
    (i) less than 197g = 0.0668
    (ii) more than 200.5g = 0.4013
    (iii) between 198.5g and 199.5g = 0.1747
13. The heights of boys at a certain school follow a normal distribution with mean = 150.3cm and variance 25cm, find the probability that a boy picked at random from the group has a height;
    (i) less than 153cm = 0.7054
    (ii) more than 158cm = 0.018
    (iii) between 150 cm and 158 cm= 0.4621
    (iv) more than 10cm difference from the mean height = 0.0046
14. The masses of a certain type of cabbages are normally distributed with mean of 1000g and standard deviation of 0.15kg. In a batch of 800 cabbages, estimate how many have a mass between 750g and 1290g = 740
15. Cartons of milk from quality super market are advertised as containing 1 litre, but in fact the volume of the content is normally distributed with a mean of 1012ml and standard deviation of 15ml.
    (i) Find the probability that a randomly chosen carton contains more than 1010ml =0.6554
    (ii) In a batch of 1000 cartons, estimate the number of cartons containing less than the advertised volume of milk =8
16. A random variable X is such that $X \sim N(-5, 9)$. Find the probability that;
    (i) A randomly chosen item from the population will have positive value = 0.0478
    (ii) Out of 10 items chosen randomly, exactly 4 will have a positive value = 0.00082
17. The life of a laptop is normally distributed with a mean of 2000 hours and standard deviation of 120 hours. Estimate the probability that the life of such a laptop will be
    (i) greater than 2150 hours = 0.1056
    (ii) greater than 1910hours = 0.7734
    (iii) within a range 1850 hours to 2090 hours = 0.6678

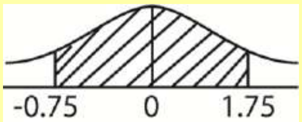
Solutions to revision questions 1

1. The amount of meat sold by a butcher is normally distributed with mean 43kg and standard deviation 4kg. Determine the probability that the amount of meat sold is between 40kg and 50kg.

   X ~N(43,4)

   P(40<x<50) = $P\left(\frac{40-43}{4} < Z < \frac{50-43}{4}\right)$

   = P(- 0.75 < Z < 1.75)



   P(40<x<50) = P(-0.75 < Z < 0) + P(0 < Z < 1.75)

   By property of symmetry

   P(40<x<50) = P(-0.75 < Z < 0) + P(0 < Z < 1.75)

   = 0.2735 + 0.4599

   = 0.733

2. Given that a random variable X is X~N(2, 2.89). Find P(X<0)

   μ = 2, σ = $\sqrt{2.89}$ = 1.7

   P(X < 0) = $P(Z < \frac{0-2}{1.7})$ = P(Z < - 1.176) = P(X> 1.176)

   = 0.5 – P(0 < Z < 1.176)

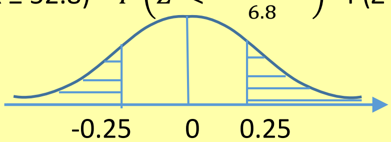
   = 0.5 – 0.3802 = 0.1198

3. In a school of 800 students their average weight is 54.5kg and standard deviation 6.8kg. given that the weight of students are normally distributed, find

   (i) Probability that the weight of any student randomly selected is 52.8 kg or less

   Let x be the weight of the student

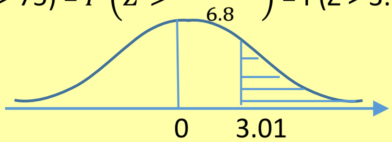
   P(x ≤ 52.8) = $P\left(Z < \frac{52.8-54.5}{6.8}\right)$= P(Z <-0.25)



   = P(Z> 0.25) = 0.5- P(0 < Z < 0.25) = 0.5 -0.0987 = 0.4013

   (ii) Number of students who weigh over 75kg = 1

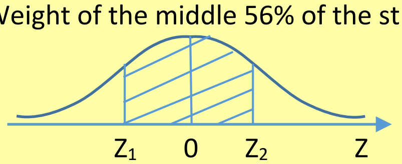
   P(Z > 75) = $P\left(Z > \frac{75-54.5}{6.8}\right)$ = P(Z > 3.01)



   P(Z > 3.01) = 0.5 – P(0 < Z < 3.01) = 0.5 – 0.4990= 0.001

   Number of students who weigh more than 75g = 800 x 0.001 =1

   (iii) Weight of the middle 56% of the students



   P(X_1 < X < X_2) = P(Z_1< Z < Z_2) = 0.56

   But P(0< Z < Z_2) = 2.8; Z_2 = 0.772 and Z_1 = -0.772

   $Z_1 = \frac{x_1-54.5}{6.8}$

   -0.772 =$\frac{x_1-54.5}{6.8}$; x_1 = 49.251

   $Z_2 = \frac{x_1-54.5}{6.8}$

   0.772 =$\frac{x_1-54.5}{6.8}$; x_2 = 59.750

Hence the weight range of the middle 56% of students of the school is 49.251< X < 59.750

4. A sugar factory sells sugar in bags of mean weight 50kg and standard deviation 2.5kg. given that the weight of the bags is normally distributed, find the
   (i) Probability that the weight of any bag of sugar randomly selected lies between 51.5kg and 53kg

   P(51.5 < X < 53) = $\frac{51.5-50}{2.5} < Z < \frac{53-50}{2.5}$ = P(0.6 < Z 1.2)

   = ϕ(1.2) – ϕ(0.6) = 0.3849 – 0.2257== 0.1592
   (ii) Percentage of bags whose weight exceeds 54kg

   P(X > 54) =P($Z > \frac{54-50}{2.5}) = P(Z > 1.6)$= 0.5 –ϕ(1.6) = 0.5 – 0.4452 = 0.0548

   = 0.0548 x 100 = 5.48%
   (iii) Number of bags that will be rejected out of 1000 bags purchased for weighing below 45.0kg

   P(X < 45) = P($Z < \frac{45-50}{2.5}) = P(Z < -2) = P(Z < 2)$ = 0.5 –ϕ(2)= 0.5- 0.4772 = 0.0228

   Number of bags rejected = 0.0228 x 1000 = 22.8 ≈23
5. A certain maize firm sells maize in bags of mean weight 40kg and standard deviation 2kg, given that the weight of the bags are normally distributed, find
   (i) Probability that the weight of any bag of maize randomly selected lies between 41.0 and 42.5kg

   P(41.0 < X < 42.5) = $\frac{41-40}{2} < Z < \frac{42.5-40}{2}$ = P(0.5 < Z 1.25)

   = ϕ(1.25) – ϕ(0.5) = 0.3944 - 0.1915 = 0.2029
   (ii) Percentage of bags whose weight exceeds 43kg

   P(X > 43) =P($Z > \frac{43-40}{2}) = P(Z > 1.5)$= 0.5 –ϕ(1.5) = 0.5 – 0.4332 = 0.0668

   = 0.0668 x 100 = 6.68%

   (iii) Number of bags that will be rejected out of 500 bags purchased for weighing below 38.5kg

   P(X < 38.5) = P($Z < \frac{38.5-40}{2}) = P(Z < -0.77) = P(Z < 0.75)$ = 0.5 –ϕ(0.75)

   = 0.5- 0.2734 = 0.2266

   Number of bags rejected = 0.2266 x 500 = 113
7. If X~N(50, 20). Find
   (i) P(X > 60.3)

   P(X > 60.3) = P(Z > $\frac{60.3-50}{\sqrt{20}}$) = P(Z > 2.303) = 05 –ϕ(2.303)

   = 0.5 –(0.4893 + 0.0001) = 0.0106
   (ii) P(X < 47.3)

   P(X < 47.3) P(Z > $\frac{47.3-50}{\sqrt{20}}$) = P(Z <-0.6037) =P(Z> 0.6037) = 05 –ϕ(0.6037)

   = 0.5 –(0.2257 + 0.0013) = 0.273
   (iii) P(X > 48.9)

   P(X < 48.9) = P(Z > $\frac{48.9-50}{\sqrt{20}}$) = P(Z <-0.246) = P(Z < 0.246) = 0.5 +ϕ(0.246)

   = 0.5 + 0.0948 + 0.0022 = 0.597
   (iv) P(X > 53.5)

   P(X > 53.5) = P(Z > $\frac{53.5-50}{\sqrt{20}}$) = P(Z > 0.783) =0.2823 + 0.0008 = 0.2831

(v) P(X < 59.8)

$P(X < 59.8) = P(Z < \frac{59.8-50}{\sqrt{20}}) = P(Z < 2.191) = 0.5 + \phi(2.191)$

$= 0.5 + 0.4826 + 0.0001 = 0.9857$

(vi) P(X < 62.3)

$P(X < 62.3) = P(Z < \frac{62.3-50}{\sqrt{20}}) = P(Z < 2.750) = 0.5 + \phi(2.730$

$= 0.5 + 0.4970 = 0.9970$

How to obtain Z-values from a given probabilities

If you are interested in finding the Z-values whose probabilities are given, it is important to note that the Z-value may be positive or negative.

Sign	Probability	Z-value
<	< 0.5	-
>	>0.5	-
<	>0.5	+
>	<0.5	+

Note: for the above table the probability given in the question always correspond to Q in the critical table

Example 11

$P(Z<Z_1) = 0.5$, find Z_1

$P(Z<Z_1) = 0.5$, find Z_1

Solution

$P(Z<Z_1) = 0.5(Q)$

$P(Z<Z_1) = 0.5$, find Z_1

$Z_1 = -0.674$ (negative since 0.25 < 0.5 read directly from a critical table)

Example 12

$P(Z < Z_1) = 0.0968$, find Z_1

Since 0.0968(Q) is not on critical table

P(Z < Z1) = 0.5 – 0.0968 = 0.403(P)

$Z_1 = -1.3$ (negative since 0.0968 < 0.5 read directly from a critical table)

Example 13

$P(Z < Z_1) = 0.5$, find Z_1

Solution

$P(Z < Z_1) = 0.05(Q)$

$Z_1 = -1.645$ (negative since 0.05 < 0.5 read directly from a critical table)

Example 14

P(Z< a) = 0.787, find a

Since 0.787 is not on critical table

P(Z < a) = 0.787 – 0.5 = 0.287

From the table0.287 lies between 0.2852 and 0.2881. Since the extra information to the right hand side is **add**, we consider the smallest value i.e. 0.2852 but 2852 corresponds to 0.79

to get the next

0.2870 – 0.285 = 0.0018

So we look for 0.0018 on the add column which gives 6

$\therefore$ a = 0.79 + 0.006 = 0.796

Example 15

P(Z > b) = 0.01, find b

P(Z > b) = 0.01(Q)

b = 2.326 read directly from critical table

Inverse process (De-standardizing Z)

It involves converting the Z-value to raw data (X) form

Example 16

If X~(100, 36) and P(X > α) = 0.8907, find the value of α

Since 0.8907 is not critical on a critical table

P(Z < $\frac{\alpha-100}{6}$) = 0.8907 – 0.5 = 0.390(P)

From the table Z =1.23

1.23 = $\frac{\alpha-100}{6}$

α = 100 + 1.23 x 6 =107.38

Example 17

If X~(24, 9) and P(X > b) = 0.974, find the value of b

P(Z < $\frac{b-24}{3}$) = 0.974 – 0.5 = 0.474(P)

From the table Z =-1.943

1.943 = $\frac{b-24}{3}$

α = 24 -1.943 x 3 =18.171

Example 18

The height of flowers in a farm is normally distributed with the mean 169 cm and standard deviation 9cm. if X stands for the height of flowers in cm, find X values for

(a) P(X < a) = 0.8

Solution

P(X < a) = 0.8(Q)

$P(Z < \frac{a-169}{9})$ = 0.8 – 0.5 = 0.3(P)

From the table Z = 0.842

a = 0.842 x 9 + 169 = 176.38

(b) P(X > b) = 0.6

Solution

P(X > b) = 0.6 (Q)

$P(Z < \frac{a-169}{9})$ = 0.6 – 0.5 = 0.1(P)

From the table Z = -0.253

b = -0.253 x 9 + 169 = 166.72

Example 19

The period of a certain machine approximately follows a normal distribution with mean of five years and standard deviation of 1 year. Given that the manufacturer of this machine replaces the machine that fails under guarantee, determine the

(i) Length of the guarantee required so that not more than 2% of the machine that fail are replaced.

$P(X < X_0) = 0.02(Q)$

$P\left(Z < \frac{X_0-5}{1}\right) = 0.02(Q)$

From the table Z = -2.054

$X_0 = -2.054 \times 1 + 5 = 2.946$

∴ the guarantee period is 2.946 years

(ii) The proportion of the machines that would be replaced if the guarantee period was four years

$P(X < 4) = P\left(Z < \frac{4-5}{1}\right) = P(Z < -1) = P(Z > 1) = 5 - \phi(1) = 5 - 0.3413 = 0.1587$

$P(Z < 4) = 0.1587 \times 100 = 15 .87\%$

Example 20

The marks of 500 students in a mock examination for which the ass mar was 50%. Their marks are normally distributed with mean 45 marks and standard deviation 20 marks.

(a) Given that the pass mar is 41, estimate the number of candidates who passed the examination.

$P(X \geq 41) = P\left(Z < \frac{41-45}{20}\right) = P(Z \geq -0.2) = P(Z \leq 0.2) = 0.5 + \phi(0.2) = 0.5 + 0.0793 = 0.5793$

Number of candidates who assed =0.5793 x 500 = 290

(b) If 5% of the candidates obtain a distinction by scoring X marks or more, estimate the value of X.

$P(X > X_0) = 0.05(Q)$

$P\left(Z < \frac{X_0-45}{20}\right) = 0.5 - 0.05 = 0.45(P)$; from the table Z = 1.645

$X_0 = 1.645 \times 20 + 45 = 78$

∴the distinction starts at 78%

(c) Estimate the interquartile range of the distribution

Interquartile range = q3 – q1

$P\left(0 < Z < \frac{q_3-55}{20}\right) = 0.25(P)$; from the table Z = 0.674

$q_3 = 0.674\, x\, 20 + 45 = 58.48$

$P\left(\frac{q_1-45}{20} < Z < 0\right) = 0.25(P)$; from the table Z = -0.674

$q_1 = -0.674\, x\, 20 + 45 = 31.52$

∴ interquartile range = 58.48 – 31.52 = 26.96

Example 21

If X~N(70, 25) and P(|X - 70|< a) = 0.8, find the value of a and hence the limits within which the central 80% of the distribution lies.

$P(|X - 70| < a) = P(-a < X - 70 < a) = P(-a + 70 < X < a + 70) = 0.8$

$$P\left(\frac{-a+70-70}{5} < X < \frac{a+70-70}{5}\right) = P\left(\frac{-a}{5} < X < \frac{a}{5}\right) = 0.8$$

$2 \; x \; P\left(0 < X < \frac{a}{5}\right) = 0.8$; $P\left(0 < X < \frac{a}{5}\right) = 0.4$(P)

From table Z = 1.282

$1.282 = \frac{a}{5}$; a = 6.41

But P(-a + 70 < X < a + 70) = 0.8

P(63.59 < X < 76.41) = 0.8

∴ Central 80% of the distribution lies between 63.59 and 76.41

Revision Exercise 2

1. Find the value of the following
   (i) P(Z < a) = 0.506 [a = 0.015]   (ii) P(Z < a) = 0.787 [a= 0.796]   (iii)P(Z< a) =0.0296[a= -0.1887]
   (iv) P(Z > a) = 0.713 [a = -0.562]   (iii)P(Z< a) = 0.325 [a=-0.454] (vi) P(|Z| > a) = 0.5 [a =0.674]
   (vii) P(|Z|> a) = 0.6 [a = 0.842]   (viii) P(Z< a) = 0.9738 [a =1.94] (ix) P(Z< a)= 0.2435[a = -0.695]
   (x) P(Z > a) = 0.82 [a = - 0.915] (xi) P(Z >a) = 0.2351 [a= 0.628] (xii) P(|Z|<a)= 0.6372[a=0.91]
   (xiii) P(Z>a) = 0.097 a=1.66]   (xv) P(|Z|> a) = 0.0404[a= 2.05]

2. Find the value of a if
   (i) P(Z< a) =0.9693 [a =1.87] (ii) P(Z > a) = 0.3802 [a = 0.305] (iii) P(Z > a)= 0.7367 [a = -0.633]
   (iv) P(Z< a) = 0.0793[a = -1.41] (v) P(|Z|< a) = 0.9 [a = 1.645]
3. If X~N(60, 25) find a if
   (i) P(X > a) = 0.2324 [a = 63.66]   (ii) P(X > a) = 0.0702 [a =67.37]
   (iii) P(X > a) = 0.837 [a = 55.09]   (iv) P(X > a) =0.7461 [a = 56.69]
4. If X~N(45, 16) find a if
   (i) P(X < a) = 0.0317 [a = 37.57]   (ii) P(X < a) = 0.895 [a = 50.01]
   (iii) P(X < a) = 0.0456 [a = 38.24]   (iv) P(X < a) = 0.996 [a = 55.6]
5. If X~N(400, 64) find a if
   (i) P(|X-400|< a) = 0.75 [9.2]   (ii) P(|X-400|< a) = 0.98 [18.61]
   (iii) P(|X-400|< a) = 0.95 [15.68]   (iv) P(|X-400|< a) = 0.975 [17.92]
   (v) The limits within which the central 95% of distribution lies. [384.32 < X < 415.68]
   (vi) Interquartile range of distribution [394.61, 405.39]
6. Bags of flour paced by a particular machine have masses which are normally distributed with mean 500g and standard deviation 20g. 2% of the bags are rejected for being overweight. Between what ranges of values should the mass of a bag of flour lie if it is to be accepted. [0458.92, 546.52]
7. The masses of mangoes sold at a market are normally distributed with mean mass 600g and standard deviation 20g.
   (i) If a mango is chosen at random, find the probability that its mass lies between 570g and 610g [0.6247]
   (ii) Find the mass exceeded by 7% of mangoes [629.52]
   (iii) In one day 1000 mangoes are sold. Estimate how many weigh less than 545g [3]
8. The length of metal strips are normally distributed with mean of 120cm and standard deviation of 10cm.
   (a) Find the probability that a strip selected at random has a length
      (i) greater than 105cm [0.9332] (ii) within 5cm of the mean =[0.383]

(b) Strips that are shorter than L cm are rejected. Estimate the value of L, if 9% or all the strips are rejected. [106.6cm]

(c) In a sample of 500 strips, estimate the number having a length over 126cm. [137]

9. The number of shirts sold in a week by a boutique are normally distributed with a mean 2080 and standard deviation of 50. Estimate
   (i) The probability that in a given week fewer than 2000 shirts are sold [0.0548]
   (ii) The number of weeks in a year that between 2060 and 2130 shirts are sold [26]
   (iii) The least number n of shirts such that the probability that more than n are sold in a given week is less than 0.02 [2183]
10. Batteries for a transistor radio have a mean life under normal usage of 160 hours, with standard deviation of 30 hours. Assuming the battery life follow normal distribution
    (i) Find the percentage of batteries which have a life between 150 hours and 180 hours. [37.8%]
    (ii) Calculate the range, symmetrical about the mean, within which 75% of the batteries lives lie. [125.5, 194.5]
    (iii) If the radio takes four of these batteries and require all of them to be working, find the probability that the radio will run for at least 135 hours. [0.405]
11. The length of type A rod is normally distributed with mean of 15cm and a standard deviation of 0.1cm. the length of another type B is also normally distributed with mean of 20cm and standard deviation 0.16cm. For type A rod to be acceptable, its length must be between 14.8cm and 15.2 cm and type B rod, the length must be between 19.8cm and 20.2cm.
    (i) What is the proportion of type a rod is of acceptable length? [95.44%]
    (ii) What is the probability that one of them is of acceptable length [0.7528, 0.2375]
12. The marks of 1000 students in an examination were normally distributed with mean 55 marks and standard deviation 8 marks.
    (i) If a mark of 71 or more is required for A-pass, estimate the number of a-passes awarded. [23]
    (ii) If 15% of the candidates failed, estimate the minimum mark required to pass. [47]
    (iii) Calculate the probability that two candidates chosen at random both passes examination [0.7225]
13. The burning life of a bulb approximately follows a normal distribution with mean of 1300hours and standard deviation of 125 hours
    (i) What is the probability that the bulb selected at random will burn for more than 1500 hours. [0.0548]
    (ii) Given that the manufacturer guarantees to replace any bulb that burns for less than 1050hours, what percentage of the bulbs will have to be replaced. [2.28%]
    (iii) If two bulbs are installed at the same time, what is the probability that bot will burn less than 1400 hours but more than 1200 hours [0.3320]
14. The marks in an examination were found to be normally distributed with mean 53.9 and standard deviation 16.5. 20% od the candidates who sat this examination failed. Find the pass mark [40.007]

Finding the value of mean, μ or standard deviation, σ or both

Hint: X = Zσ + μ

Example 22

If X~N(100, σ^2) and P(X < 106) = 0.8849, find the value of standard deviation, σ.

Solution

P(X < 106) = 0.8849 (Q –value)

$P\left(Z < \frac{106-100}{\sigma}\right)$ = 0.8849 – 0.5 = 0.3849 (P – value)

From table Z = 1.2

$\frac{106-100}{\sigma}$ = 1.2

σ = 5

Example 23

The length of a certain item follows a normal distribution with mean, μ cm and standard deviation of 6cm. it is known that 4.78% of the items have length greater than 82cm, find the mean, μ.

Solution

P(X > 82) = 0.0478 (Q – value)

$P\left(Z > \frac{82-\mu}{6}\right)$ = 0.5 – 0.0478 = 0.4522(P – value)

From table Z = 1.667

$\frac{82-\mu}{6}$ = 1.667

μ = 72 cm

Example 24

The masses of boxes of oranges are normally distributed such that 30% of them are greater than 4.00kg and 20% are greater than 4.53kg. Estimate the mean and standard deviation of the masses

Solution

P(X > 4) = 0.3 (Q – value)

$P\left(Z > \frac{4-\mu}{\sigma}\right)$ = 0.3 (Q – value)

$\frac{4-\mu}{\sigma} = 0.524$

4 = $\mu + 0.524\sigma$ ……………….(i)

$P\left(Z > \frac{4.53-\mu}{\sigma}\right)$ = 0.2 (Q – value)

$\frac{4.53-\mu}{\sigma} = 0.842$

4.53 = $\mu + 0.842\sigma$ ……………….(ii)

Eqn. (ii) – eqn. (i): 0.53 = 0.318σ; σ = 1.67kg

From eqn. (i)

μ = 4 – 0.524 x 1.67 = 3.13kg

Example 25

The speed of cars passing certain Entebbe high way can be taken to be normally distributed. 95% of the cars are travelling at less than 85m/s and 10% are travelling at less than 55m/s.

(i) Find the average speed of the cars passing through the high way
(ii) Find the proportion of the cars that travel at more than 70m/s

Solution

P(X < 85) = 0.95 (Q – value)

$P\left(Z < \frac{85-\mu}{\sigma}\right)$ = 0.45 (Q – value)

$\frac{85-\mu}{\sigma} = 1.645$

85 = $\mu + 1.645\sigma$ ………………(i)

P(X < 55) = 0.1 (Q – value)

$P\left(Z < \frac{55-\mu}{\sigma}\right)$ = 0.1 (Q – value)

$\frac{55-\mu}{\sigma} = -1.282$

55 = μ – 1.282σ …………………………… (ii)

Eqn. (i) – eqn. (ii): 30 = 2.927σ; σ = 10.25m/s

From eqn. (i)

μ = 85 – 1.645x 10.25 = 68.14m/s

(ii) P(X > 70) = $P\left(Z < \frac{70-68.14}{10.25}\right)$= P(Z > 0.182)

= 0.5 – 0.0722

= 0.4278

Example 26

The masses of articles produced in a particular shop are normally distributed with mean μ and standard deviation σ. 5% of the articles have greater than 85g and 10% have masses less than 25g.

(i) Find the values of μ and σ
(ii) Find the symmetrical limits, about the mean, within which75% of the masses lie.

Solution

P(X > 85) = 0.05 (Q – value)

$P\left(Z > \frac{85-\mu}{\sigma}\right)$ = 0.05 (Q – value)

$\frac{85-\mu}{\sigma} = 1.645$

85 = $\mu + 1.645\sigma$ ………………(i)

P(X < 25) = 0.1 (Q – value)

$P\left(Z < \frac{25-\mu}{\sigma}\right)$ = 0.1 (Q – value)

$\frac{25-\mu}{\sigma} = -1.282$

55 = μ – 1.282σ …………………………… (ii)

Eqn. (i) – eqn. (ii): 60 = 2.927σ; σ = 20.5m/s

From eqn. (i)

μ = 85 – 1.645x 20.5 = 513g

(ii) P(|X-51.3| < a) = 0.75

P(-a + 51.3 < X < a + 51.3) = 0.75

$P\left(\frac{-a+51.3-51.3}{20.5} < Z < \frac{a+51.3-51.3}{20.5}\right) = 0.75$

$P\left(\frac{-a}{20.5} < Z < \frac{a}{20.5}\right)$ = 0.75

2 x $P\left(0 < Z < \frac{a}{20.5}\right)$= 0.375 (P – value)

$\frac{a}{20.5} = 1.15$; a = 23.575

Lower limit = -23.575 + 51.3 = 27.73

Upper limit = 23.575 + 51.3 = 74.88

Example 27

A total population of 700 students sat a mock examination for which the pass mark was 50%. Their marks were normally distributed. 28 students scored below 40% while 35 students scored above 60%.

(a) Find the mean mark and standard deviation of the students' marks.
(b) What is the probability that a student chosen at random passed the exam?

(c) Suppose the pass mark is lowered by 2%, how many more students will pass.

Solution

P(X < 40) = $\frac{28}{700}$ =0.04 (Q – value)

$P\left(Z > \frac{40-\mu}{\sigma}\right)$ = 0.04 (Q – value)

$\frac{40-\mu}{\sigma} = -1.751$

40 = $\mu - 1.751\sigma$ ……………….(i)

P(X > 60) = $\frac{35}{700}$ =0.05 (Q – value)

$P\left(Z > \frac{60-\mu}{\sigma}\right)$ = 0.05 (Q – value)

$\frac{60-\mu}{\sigma} = 1.645$

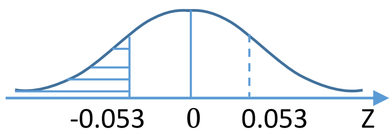
60 = μ + 1.645σ ……………………………. (ii)

Eqn. (ii) – eqn. (ii): 20 = 3.396σ; σ = 5.889

From eqn. (i)

μ = 40 + 1.751 x 5.889 = 50.312

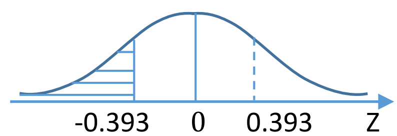
(ii) P(X ≥ 50) = $P\left(Z \geq \frac{50-50.312}{5.889}\right)$= P(Z ≥ - 0.053)



P(Z ≥ -0.053) = 0.5 + P(0 < Z < 0.053)

= 0.5 + 0.0211 = 0.5211

(iii) P(X ≥ 48) = $P\left(Z \geq \frac{48-50.312}{5.889}\right)$= P(Z ≥ - 0.053)



P(Z ≥ -0.392) = 0.5 + P(0 < Z < 0.392)

= 0.5 + 0.1528 = 0.6528

More proportion = 0.6528 – 0.5211 = 0.1317

More students = 0.1317 x 700 = 92

Example 28

A random variable X has a normal distribution with P(X > 55) = 0.2 and P(35 < X < 55) = 0.5. Find

(a) The value of the mean, μ and standard deviation, σ.
(b) The percentage of those with P(X > 45)

Solution

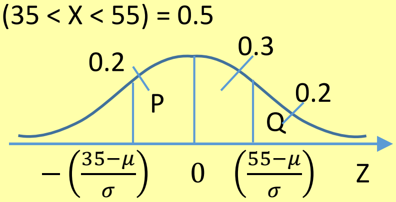
P(X > 55) = $\frac{28}{700}$ =0.2 (Q – value)

$P\left(Z > \frac{55-\mu}{\sigma}\right)$ = 0.2 (Q – value)

$\frac{40-\mu}{\sigma} = 0.842$

55 = $\mu + 0.842\sigma$ ……………….(i)

P(35 < X < 55) = 0.5



$P\left(\frac{35-\mu}{\sigma} < Z < 0\right)$ = 0.2(P – value)

$\frac{35-\mu}{\sigma} = -0.842$

35 = $\mu - 0.842\sigma$ ……………….(ii)

μ = 70.8797, σ = 42.5872

(b) P(X > 45) = $P\left(Z > \frac{45-70.8797}{42.5872}\right)$= P(Z > -0.608)

= 0.5 + P(0 < Z < 0.608)

= 0.5 + 0.2283 = 0.7283

Percentage = 0.7283 x 100 = 72.83

Revision exercise 3

1. $X \sim N(45, \sigma^2)$ and $P(X > 51) = 0.288$. find σ. [$\sigma = 10.7$]
2. $X \sim N(21, \sigma^2)$ and $P(X < 27) = 0.9332$. find σ. [$\sigma = 4$]
3. $X \sim N(\mu, 25)$ and $P(X < 27.5) = 0.3085$. find μ. [$\mu = 30$]
4. $X \sim N(\mu, 12)$ and $P(X > 32) = 0.8438$. find μ. [$\mu = 35.5$]
5. $X \sim N(\mu, \sigma^2)$ and $P(X > 80) = 0.0113$ and $P(X > 30) = 0.9713$. find σ and μ. [$\mu = 52.73$ and $\sigma = 11.96$]
6. $X \sim N(\mu, \sigma^2)$ and $P(X > 102) = 0.42$ and $P(X < 97) = 0.25$. find σ and μ. [$\mu = 100.8$ and $\sigma = 5.71$]
7. $X \sim N(\mu, \sigma^2)$ and $P(X < 57.84) = 0.90$ and $P(X < 50) = 0.5$. find σ and μ. [$\mu = 50$ and $\sigma = 6.12$]
8. $X \sim N(\mu, \sigma^2)$ and $P(X < 35) = 0.20$ and $P(35 < X < 45) = 0.65$. find σ and μ. [$\mu = 39.5$ and $\sigma = 5.32$]
9. The length of rods produced in a workshop follow a normal distribution with mean μ and variance 4. 10%of the rods are less than 17,4cm long. Find the probability that a rod chosen at random will be between 18cm and 23 cm. [0.7725]
10. The length of a stick follow a normal distribution. 10% are of length 250cm or more while 55% have a length over 240cm. Find the probability that a stick chosen at random is less than 235cm long. [0.203]
11. A certain make of car tyres can be safely used for 25000km on average before replaced. The makers guarantee to pay compensation to anyone whose tyre does not last for 22000km. they expect 7.5% of all the tyres sold to qualify for compensation. If the distance X travelled before a tyre is replaced has normal distribution.
    (i) Find the standard deviation [2080]
    (ii) Estimate the number of tyres per 1000 which will not have been replaced when they have covered 26500km. [236]
12. The continuous random variable X is normally distributed with mean μ and standard deviation σ. If $P(X < 53) = 0.04$ and $P(X < 65) = 0.97$, find the interquartile range [4.46]
13. Tea sold in packages marked 750g. The masses are normally distributed with mean 760g and standard deviation σ. What is the maximum value of σ, if less than 1% of the packages are underweight? [4.299]
14. In an examination 30% of the candidates fail and 10% achieve distinction. Last year the pass mark (out of 200) was 84 and the minimum mark required for a distinction was 154. Assuming that the marks of candidates are normally distributed, estimate the mean mark and standard deviation. [$\mu = 104.31$, $\sigma = 38.76$]
15. AT St Noa junior, the heights of students are normally distributed. 10% are over 1.8m and 20% are below 1.6m.
    (i) Find the mean height μ and standard deviation σ. [$\mu = 1.68$, $\mu = 0.09$]
    (ii) Find the interquartile range [0.13]
16. Observation of a very rage number of cars are certain point on a motor way established that the speeds are normally distributed. 90% of the cars have speed less than 77.7km/h and only 5% of cars have speed less than 63.1km/h. find the mean speed μ and standard deviation σ. [$\mu = 71.305$, $\sigma = 4.988$]
17. A sample of 100 apples is taken from a load. The apple have the following distribution of size.

Diameter (cm)	6	7	8	9	10
Frequency	11	21	38	17	13

Assuming that the distribution is approximately normal with mean μ and standard deviation σ.
    (i) Determine μ and σ [$\mu = 8$, $\sigma = 1.16$]
    (ii) Find the range of sizes of apples for packing, if 5% are to be rejected as too small and 5% are to be rejected as too large [6.10, 9.90]

18. The volumes of soda in bottles are normally distributed with mean of 333ml. Given that 20% of the bottles contain more than 340ml, find
    (i) Standard deviation of the volume of bottle. [8.31]
    (ii) Percentage of bottles that contain less than 330ml. [35.9%]
19. The heights of 500 students are normally distributed with a standard deviation of 0.080cm. If the heights of 129 of the students are greater than the mean height but less than 1.806m find the mean height. [1.75]
20. The masses of boxes of apples are normally distributed such that 20% of them are greater than 5.08kg and 15% are greater than 5.62kg; find the mean and standard deviation. [μ= 2.74, σ = 2.78]
21. The masses of sugar are normally distributed. If 5% of the packets have mass greater than 510g and 2% have masses greater than 515g. Find the mean and standard deviation. [μ = 490, σ = 12.2g]
22. Sugar packed in 500g packets is observed to be approximately normally distributed with standard deviation of 4. If only 2% of the packets contained less than 500g of sugar. Find the mean weight of sugar in the packets. [508.216g]
23. Sixty students sat for a mathematics contest whose pass mark was 40marks. Their scores in the contest were approximately normally distributed. 9 students scored less than 20 marks while 3 scored more than 70 marks. Find the
    (i) Mean scored and the standard deviation of the contest. [μ = 39.32, σ = 18.65]
    (ii) Find the probability that a student chosen at random passed the contest. [0.4856]
24. The number of cows owned by residents n a village is assumed to be normally distributed. 15% of the residents have less than 60 cows while 5% of residents have over 90 cows.
    (a) Determine the values of the mean and standard deviations of cows [μ =71.5926, σ= 11.1899]
    (b) If there are 200 residents, find how many have more than 80 cows. [45]
25. A random variable X has a normal distribution when P(X > 9) = 0.9192 and P (X < 11) = 0.7580. find
    (a) the value of the mean and standard deviation. [μ= 10.3333, σ = 0.9524]
    (b) P(X > 10) [0.6386

26. The marks in an examination were normally distributed with mean μ and standard deviation σ. 20% of the candidates scored less than 40marks and 10% more than 75 marks. Find the
    (a) values of μ and σ. (08marks)
    (b) percentage of the candidate who scored more than 50 marks.(04marks)

Solutions to revision exercise 3

20. The masses of boxes of apples are normally distributed such that 20% of them are greater than 5.08kg and 15% are greater than 5.62kg; find the mean and standard deviation.

Solution

P(X > 5.08) = 0.2 (Q – value)

$P\left(Z > \frac{5.08 - \mu}{\sigma}\right)$ = 0.2 (Q – value)

$\frac{5.08-\mu}{\sigma} = 0.842$

5.08 = $\mu + 0.842\sigma$ ………………(i)

P(X > 5.62) = 0.15 (Q – value)

$\frac{5.62-\mu}{\sigma} = 1.036$

5.62 = $\mu + 1.036\sigma$ ………………(ii)

Eqn. (ii) – eqn. (i)

0.54 = 0.194σ; σ = 2.7835kg

From eqn. (i)

μ = 5.08 – 0.842 x 2.7835 = 2.7363kg

$P\left(Z > \frac{5.62-\mu}{\sigma}\right)$ = 0.15 (Q – value)

21. The masses of sugar are normally distributed. If 5% of the packets have mass greater than 510g and 2% have masses greater than 515g. Find the mean and standard deviation.

Solution

P(X > 510) = 0.05 (Q – value)

$P\left(Z > \frac{510-\mu}{\sigma}\right)$ = 0.05 (Q – value)

$\frac{510-\mu}{\sigma} = 1.645$

510 = $\mu + 1.645\sigma$ ……………….(i)

P(X > 515) = 0.02 (Q – value)

$P\left(Z > \frac{515-\mu}{\sigma}\right)$ = 0.02 (Q – value)

$\frac{515-\mu}{\sigma} = 2.054$

515 = $\mu + 2.054\sigma$ ……………….(ii)

Eqn. (ii) – eqn. (i)

5 = 0.409σ; σ = 12.225kg

From eqn. (i)

μ = 510 – 1.645 x 12.225 = 489.89kg

22. Sugar packed in 500g packets is observed to be approximately normally distributed with standard deviation of 4. If only 2% of the packets contained less than 500g of sugar. Find the mean weight of sugar in the packets.

P(X < 500) = 0.02 (Q – value)

$P\left(Z < \frac{500-\mu}{4}\right)$ = 0.02 (Q – value)

$\frac{500-\mu}{\sigma} = -2.054$

Mean weight, μ = 500 + 2.054 x 4 = 508.216g

23. Sixty students sat for a mathematics contest whose pass mark was 40marks. Their scores in the contest were approximately normally distributed. 9 students scored less than 20 marks while 3 scored more than 70 marks. Find the
    (i) Mean scored and the standard deviation of the contest.

Solution

P(X < 20) $= \frac{9}{60}$ =0.15 (Q – value)

$P\left(Z > \frac{20-\mu}{\sigma}\right)$ = 0.15 (Q – value)

$\frac{20-\mu}{\sigma} = -1.036$

20 = $\mu - 1.036\sigma$ ……………….(i)

P(X > 70) = $\frac{3}{60}$ = 0.05 (Q – value)

$P\left(Z > \frac{70-\mu}{\sigma}\right)$ = 0.05 (Q – value)

$\frac{70-\mu}{\sigma} = 1.645$

70 = $\mu + 1.645\sigma$ ……………….(ii)

Eqn. (ii) – eqn. (i)

50 = 2.681; σ = 18.65

From eqn. (i)

μ = 20 + 1.036 x 18.65 = 39.3214

(ii) Find the probability that a student chosen at random passed the contest.

P(X > 40) =$P\left(Z > \frac{40-39.3214}{18.65}\right)$= P(Z > 0.0364) = 0.5 – ϕ(0.036) = 0.5 – 0.0144 = 0.4856

24. The number of cows owned by residents n a village is assumed to be normally distributed. 15% of the residents have less than 60 cows while 5% of residents have over 90 cows.
    (a) Determine the values of the mean and standard deviations of cows
    (b) If there are 200 residents, find how many have more than 80 cows. [45]

Solution

(i) P(X < 60) =0.15 (Q – value)

$P\left(Z < \frac{60-\mu}{\sigma}\right)$ = 0.15 (Q – value)

$\frac{20-\mu}{\sigma} = -1.036$

60 = $\mu - 1.036\sigma$ ……………….(i)

P(X > 90) = 0.05 (Q – value)

$P\left(Z > \frac{70-\mu}{\sigma}\right)$ = 0.05 (Q – value)

$\frac{70-\mu}{\sigma} = 1.645$

90 = $\mu + 1.645\sigma$ ……………….(ii)

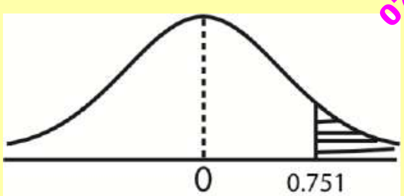
Eqn. (ii) – eqn. (i)

30 = 2.681; σ = 11.1899

From eqn. (i)

μ = 60 + 1.036 x 11.1899= 71.5927

(ii) P(X > 80) = $P\left(Z > \frac{80-71.5927}{11.1899}\right)$= P(Z > 0.751)



P(Z > 0.751) = 0.5 – (0 < Z < 0.751)

= 0.5 – 0.2737

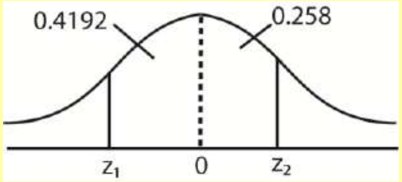
=0.2263

Number of residents =200 x 0.2263 = 45

25. A random variable X has a normal distribution when P(X > 9) = 0.9192 and P (X < 11) = 0.7580. find
    (a) The values of the mean and standard deviation (08marks)

P(X, >9) = P$\left(z_1 > \frac{9-\mu}{\delta}\right)$ =0.9192

P(X, < 11) = P$\left(z_2 > \frac{11-\mu}{\delta}\right)$ =0.0.7580



$z_1 = -\phi(0.4192) = -1.4$

$z_2 = \phi(0.258) = 0.7$

⇨ $\frac{9-\mu}{\delta} = -1.4$

$9 - \mu = -1.4\delta$ .... (i)

⇨ $\frac{11-\mu}{\delta} = 0.7$

$11 - \mu = 0.7\delta$ .... (ii)

$Eqn\ (i) - Eqn\ (ii)$

$-2 = -2.1\delta$

$\delta = \frac{-2}{-2.1} = 0.9524$

From (i)

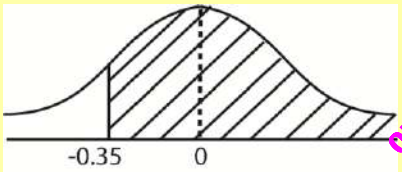
$9 - \mu = -1.4 \times 0.9524$

$\mu = 10.333$

(b) P(X>10) (04marks)

P(X>10) = $P\left(z > \frac{10-10.333}{0.9524}\right)$

= P(z > -0.35)



P(X > 10) = P(0.5 + P(0 < z < 0.35)

= 0.5 + 0.1368

= 0.6368

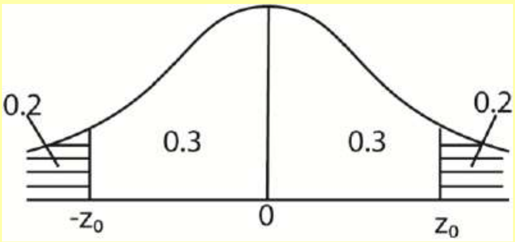
26. The marks in an examination were normally distributed with mean μ and standard deviation σ. 20% of the candidates scored less than 40marks and 10% more than 75 marks. Find the

(c) values of μ and σ. (08marks)

Let x = marks scored

$P(x < z < z_0) = 20\% = 0.2$

$P(x < 40) = P(z < z_0) = 0.2$



$P(0 < x < z_0) = 0.3$

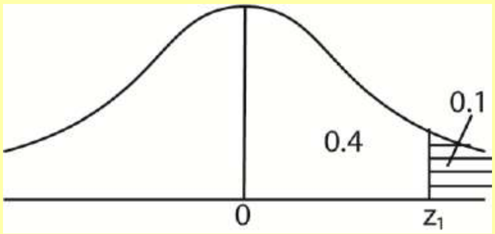
$z_0 = -0.842$

but $z = \frac{40-\mu}{\sigma}$

$-0.842 = \frac{40-\mu}{\sigma}$

$-0.842\sigma = 40 - \mu$ .... (i)

P(x > 75) = 10% = 0.1



$P(0 < z < z_1) = 0.4$

$z_1 = 1.282$

$1.282 = \frac{75-\mu}{\sigma}$

$1.282\sigma = 75 - \mu$ …………………… (ii)

Eqn. (ii) – eqn. (i)

$2.124\sigma = 35$

$\sigma = 16.478\ (3D)$

$substituting\ \sigma\ into\ eqn. (ii)$

$1.282 \times 16.478 = 75 - \mu$

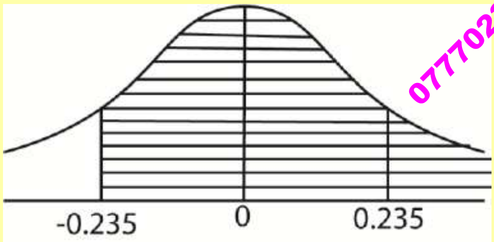
$\mu = 53.875$

Hence μ = 53.875 and σ = 16.478

(d) percentage of the candidate who scored more than 50 marks.(04marks)

$P(x > 50) = P\left(z - \frac{50-53.875}{16.478}\right)$

$= P(z > -0.235)$



= 0.5 + (0< z < 0.235)

= 0.5 + 0.0929

= 0.5929

= 59.29%

Binomial approximation to a normal distribution

Under the following conditions, the normal distribution is used to approximate binomial distribution

Conditions

(i) the number of trials of the binomial experiment should be large, n > 20.

(ii) The probability of success not to small or too large i.e. p constant and very close to 0.5

X~N(np, npq)

The z-value is obtained from $Z = \frac{X \pm 0.5 - np}{\sqrt{npq}}$

Where ± 0.5 is used to make the binomial distribution continuous.

Note; 0.5 must be subtracted from the minimum value and added to the maximum value

(i) P(X≥ x_1) = $P\left(Z \geq \frac{(X-0.5)-np}{\sqrt{npq}}\right)$

(ii) P(X≤ x_1) = $P\left(Z \leq \frac{(X+0.5)-np}{\sqrt{npq}}\right)$

(iii) P(x_1 ≤X≤ x_2) = $P\left(\frac{(X-0.5)-np}{\sqrt{npq}} \leq Z \leq \frac{(X=0.5)-np}{\sqrt{npq}}\right)$

Example 29

In a box containing different pens, the probability that a pen is red is 0.35. Find the probability that in a random sample of 400 pens from the box

(i) Less than 120 are red pens
(ii) More than 160 are red pens
(iii) Between 120 and 150 inclusive are red pens.

Solution

N = 400, p = 0.35, q = 0.65

Mean, μ = np = 400 x 0.35 = 140

σ = $\sqrt{npq} = \sqrt{400\ x\ 0.35\ x\ 0.65} = \sqrt{91}$

(i) P(X < 120) = $P\left(Z \leq \frac{119.5-140}{\sqrt{91}}\right) = P(Z \leq\ 2.149)$

=$P(Z\ \geq 2.149)$ = 0.5 – ϕ(2.149)

=0.5 – 0.4842 = 0.0158

(ii) P(X > 160) = P(X ≥ 161)

= $P\left(Z \leq \frac{116.5-140}{\sqrt{91}}\right) = P(Z \geq\ 2.149)$

=0.5 – ϕ(2.149) = 0.5 – 0.4821 = 0.0158

(iii) P(120 ≤ X ≤ 150)

=$P\left(\frac{119.5-140}{\sqrt{91}}\ \leq Z\ \frac{150.5-140}{\sqrt{91}}\right)$

=P(-2.149 ≤ Z ≤ 1.101)

=0.4842 + 0.3645 = 0.8487

Example 30

In unbiased coin is tossed 100 times, what is the probability that

(i) There will be more than 60 heads (ii) there will be less than 43 head

(iii) there will be between 45 heads and 55 head

Solution

N = 100, p = 0.5, q = 0.5

Mean, μ = np = 100 x 0.5 = 50

σ = $\sqrt{npq} = \sqrt{100\ x\ 0.5\ x\ 0.5} = 5$

(i) P(X > 60) = P(X ≥ 61)

= $P\left(Z \geq \frac{60.5-50}{5}\right) = P(Z \geq\ 2.1)$

=0.5 – ϕ(2.149) =0.5 – 0.4821 = 0.0179

(ii) P(X < 43) = P(X ≤ 42)

= $P\left(Z \leq \frac{42.5-50}{5}\right) = P(Z \leq\ -1.5$

= $P(Z \geq\ 1.5)$ =0.5 – ϕ(1.5)

= 0.5 – 0.4332 = 0.0668

(iii) P(45 ≤ X ≤ 55)

=$P\left(\frac{44.5-50}{5}\ \leq Z\ \frac{55.5-50}{5}\right)$

=P(-1.1 ≤ Z ≤ 1.1) = 2 x ϕ(1.1)

=2 x 0.3643 = 0.7286

Example 31

It is known that 72% f NTV viewer watch news at 9 pm. What is the probability that a sample of 500 viewer chosen at random

(i) More than 350 watch news (ii) fewer than 340 watch news (ii) exactly 350 watch news

Solution

(i) P(X > 350) = P(X ≥ 351)

$= P\left(Z \geq \frac{350.5 - 500\ x\ 0.72}{\sqrt{500\ x\ 0.72\ x\ 0.28}}\right)$

$= P(Z \geq -0.946) = P(Z \leq 0.946$

$= 0.5 + \phi(0.946) = 0.5 + 0.328 = 0.8280$

(ii) P(X < 340) = P(X ≤ 339)

$= P\left(Z \leq \frac{338.5 - 500\ x\ 0.72}{\sqrt{500\ x\ 0.72\ x\ 0.28}}\right)$

$= P(Z \leq -2.042) = P(Z \geq 2.042$

$= 0.5 - \phi(2.042) = 0.5 - 0.4794 = 0.0206$

(iii) P(X = 350) = P(349.5 ≤ X ≤ 350.5)

$= P\left(\frac{349.5 - 500\ x\ 0.72}{\sqrt{500\ x\ 0.72\ x\ 0.28}} \leq Z \leq \frac{350.5 - 500\ x\ 0.72}{\sqrt{500\ x\ 0.72\ x\ 0.28}}\right)$

= ϕ(-1.046 – ϕ0.946)

= 0.3522 – 0.3280 = 0.0242

∴ P(X = 350) = 0.0242

Example 32

A pair of balanced dice, each numbered 1 to 6 is tossed 150 times. Determine the probability that a sum of seven appear at least 26 times

P(sum of 7) = p= $\frac{6}{36} = \frac{1}{6}$

$P(X \geq 26) = P\left(Z \geq \frac{(26-0.5) - 150\ x \frac{1}{6}}{\sqrt{150\ x \frac{1}{6}\ x \frac{5}{6}}}\right)$

$= P\left(Z \geq \frac{25.5 - 25}{4.56}\right)$ = P(Z ≥ 0.11)

= 0.5 – ϕ(0.11) = 0.5 – 0.0438

= 0.4562

Example 33

Two players play a game in which each of them tosses a balanced coin. The game ends in a draw if both get the same result. Determine the probability that in 100 trials, the game ends in a draw.

(i) At least 53 times (ii) at most 53 times

P(sum of 7) = p= $\frac{2}{4} = \frac{1}{2}$

$P(X \geq 53) = P\left(Z \geq \frac{(53-0.5) - 100\ x \frac{1}{2}}{\sqrt{100\ x \frac{1}{2}\ x \frac{1}{2}}}\right)$

$= P\left(Z \geq \frac{52.5 - 50}{5}\right)$ = P(Z ≥ 0.5)

= 0.5 – ϕ(0.5) = 0.5 – 0.1915 = 0.3085

$P(X \leq 53) = P\left(Z \leq \frac{(53+0.5) - 100\ x \frac{1}{5}}{\sqrt{100\ x \frac{1}{2}\ x \frac{1}{2}}}\right)$

$= P\left(Z \leq \frac{53.5 - 50}{5}\right)$ = P(Z ≤ 0.7)

= 0.5 + ϕ(0.5) = 0.5 + 0.2580 = 0.7580

∴ P(X ≤ 53) = 0.7580

Example 34

In a certain book of words per page follow normal distribution with mean 800 words and standard deviation 40 words. Three pages are chosen at random, what is the probability that

(i) None of them has between 830 and 845 words.
(ii) At least tw pages have between 830 and 845 words

Solution

(i) $P(830 \le X \le 845) = P\left(\frac{830-800}{40} \le Z \le \frac{845-800}{40}\right) = P(0.75 \le Z \le 1.25)$

$= \phi(1.25) - \phi(0.75) = 0.3522 - 0.3280 = 0.0962$

$P(X = 0) = {}^3C_0(0.0962)^0(0.9038)^3 = 0.7383$

(ii) $P(X \ge 2) = P(X = 2) + P(X = 3)$

$= {}^3C_2(0.0962)^2(0.9038)^1 + {}^3C_3(0.0962)^3(0.9038)^0 = 0.02509 + 0.00089 = 0.02598$

Revision exercise 4

1. A random variable X~B(200, 0.7). Find
 (i) P(X ≥ 130) [0.9474] (ii) P(136 ≤ X < 148) [0.6325] (iii) P(X < 142) [0.5914] (iv) P(X= 152) [0.0111]
2. An ordinary unbiased die is thrown 120 times. Find the probability of obtaining at least 24 sixes. [0.1958]
3. A pair f dice is tossed 144 times and the sum of the outcomes recorded. Find the probability that a sum of 7 occurs at least 26 times. [0.3688]
4. In a school 45% f the boys are circumcised. Find the probability that in a group of 200 boys 97 are circumcised. [0.1432]
5. 10% of phones imported to Uganda are I-phones, a random sample of 1000 phones is taken. Find the probability that
   (i) Less than 80 are I-phones [0.0154]
   (ii) Between 90 and 115 inclusive are I-phones [0.8145]
   (iii) 120 or more are I-phones [0.02]
6. During Christmas, the probability that message is sent on phone successfully is 0.85.
   (i) When 8 messages are sent, find the probability that at least 7 are successfully sent [0.657]
   (ii) When 50 messages are sent, find the probability that at least 45 are successfully sent [0.2142]
7. One –fifth f tourist have COVID 19. Find the probability that the number of tourist with COVID 19 is
   (i) More than 20 in a sample f 100 people [0.4502]
   (ii) Exactly 20 in a random sample of 100 people [0.0996]
   (iii) More than 200 in a random sample of 1000 people [0.484]
8. If a fair die is thrown 300 times, what is the probability that
   (i) There will be more than 60 sixes [0.0519]
   (ii) There will be fewer than 45 sixes [0.1971]
9. A coin is biased such that head is twice as likely to occur as a tail. The coin is tossed 120 times. find the probability that there will be
   (i) Between 42 and 51 tails inclusive [0.3729]
   (ii) 48 tails or less [0.9501]
   (iii) Less than 34 tails [0.1039]
   (iv) At least 72 and at most 90 heads [0.9290]
10. A lorry of potatoes has average one rotten potato in six. A green grocer tests a random sample of 100 potatoes and decides to turn away the lorry if he finds more than 18 rotten potatoes in the sample. Find the probability that he accepts the consignment. [0.6886]
11. On a certain farm, 20% of all the cows are infected by a tick disease. Find the probability that in a sample of 50 cows selected at random not more than 10% of the cows are infected. [0.0558]

12. A pair of balanced dice, each numbered from 1 to 6 is tossed 180 times. determine the probability that a sum of seven appears;
    (i) Exactly 40 times [0.0108]
    (ii) Between 25 and 35 inclusive times [0.7286]
13. On average 20% of all the eggs supplied by a farm have cracks. Find the probability that in a sample of 900 eggs supplied by a far will have more than 200 cracked eggs. [0.0439]
14. On average 15% of all boiled eggs sold in a restaurant have cracks. Find the probability that in a sample of 300 boiled eggs will have more than 50 cracked eggs [0.215]
15. Among spectators watching a football watch, 80% were the home supporters while 20% were the visiting team supporters. If 2500 of the spectators are selected randomly, what is the probability that there are at least 541 visitors in the sample? [0.0215]
16. A die is tossed 40 times and the probability of getting at any one toss is 0.122, estimate the probability of getting between 6 to 10 sixes. [0.2048]
17. In an examination which consists of 100 questions, a student has a probability of 0.6 of getting each answer correct. A student fails the examination if he obtains a mark less than 55, and obtains a distinction for a mark of 68 or more. Calculate
    (i) The probability that he fails the examination [0.1308]
    (ii) The probability that he obtains a distinction [0.0629]
18. A research station supplies three varieties of seeds s_1, S_2 and S_3 in the ratio 4:2:1. The probabilities of germination of S_1, S_2 and S_3 are 50%, 60% and 80% respectively
    (i) Find the probability that a selected seed will germinate [
    (ii) Given that 150 seeds are selected at random, find the probability that less than 90 seed will germinate.
19. A biased die with faces labelled 1, 2, 3, 4, 5, and 6 is tossed 45 times. calculate the probability that 2 appears
    (i) More than 18 times [0.1342]
    (ii) Exactly 11 times[0.0568]

Solutions to revision exercise 4

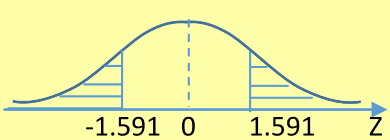
11. On a certain farm, 20% of all the cows are infected by a tick disease. Find the probability that in a sample of 50 cows selected at random not more than 10% of the cows are infected. [0.0558]

    Since n is large, we use the normal distribution to approximate binomial distribution

    Given: n = 50, p = 0.2, q = 1 – 0.2 = 0.8, μ = np = 50 x 0.2 = 10,
    $=\sqrt{npq} = \sqrt{50 \; x \; 0.2 \; x \; 0.8} = \sqrt{8}$

    10% of 50 cows = 0.1 x 50 = 5

    P(x < 5) = $P\left(Z < \frac{5.5}{\sqrt{8}}\right) = P(Z < -1.591)$



$P(Z < -1.591) = P(Z > 1.591)$
= 0.5 – P(0 < Z < 1.591)

= 0.5 – 0.4442 = 0.0558

12. A pair of balanced dice, each numbered from 1 to 6 is tossed 180 times. determine the probability that a sum of seven appears;
    (i) Exactly 40 times [0.0108]
    (ii) Between 25 and 35 inclusive times [0.7286]

Solution

Dice	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Let E = event that the sum 7 is picked when a pair of dice is tossed

n(E) = 7 and P(E) = $\frac{6}{36} = \frac{1}{6}$

Since n is large, we use the normal distribution to approximate binomial distribution

Given: n = 180, p = $\frac{1}{6}$, q = $1 - \frac{1}{6} = \frac{5}{6}$, μ = np = 180 x $\frac{1}{6}$= 30,

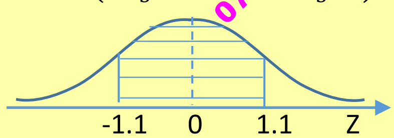
$=\sqrt{npq} = \sqrt{180 \; x \; \frac{1}{6} \; x \; \frac{5}{6}}$ = 5

Let x = number of times a sum of 7 appears

(i) P(x = 40) = P(39.5 < X < 40.5) = $P\left(\frac{39.5-30}{5} < Z < \frac{39.5-30}{5}\right) = P(1.9 < Z < 2.1)$

=P(0 < Z < 2.1) - (0 < Z < 1.9) = 0.4821 – 0.4713 = 0.0108

(ii) P(25 ≤ x ≤ 35) = $P\left(\frac{24.5-30}{5} < Z < \frac{35.5-30}{5}\right)$= P(-.1.1 < Z < 1.1)



P(-.1.1 < Z < 1.1) = 2 x P() < Z < 1.1) = 2 x 0.3643 = 0.7286

Hence P(25 ≤ x ≤ 35) = 0.7286

13. On average 20% of all the eggs supplied by a farm have cracks. Find the probability that in a sample of 900 eggs supplied by a far will have more than 200 cracked eggs.

Since n is large, we use the normal distribution to approximate binomial distribution

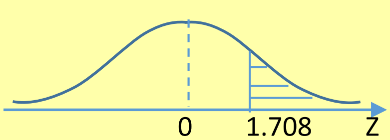
Given: n = 900, p = 0.2, q = 1 – 0.2 = 0.8, μ = np = 900 x 0.2 = 180,

$=\sqrt{npq} = \sqrt{900 \; x \; 0.2 \; x \; 0.8}$ = 12

let x = number of eggs with cracks

P(x > 200) = P(Z > Z_1)

Where $Z_1 = \frac{200.5-180}{12} = 1.708$



P(Z > 1.708) = 0.5 – P(0 < Z < 1.708)

= 0.5 – 0.4561 = 0.0439

14. On average 15% of all boiled eggs sold in a restaurant have cracks. Find the probability that in a sample of 300 boiled eggs will have more than 50 cracked eggs

Since n is large, we use the normal distribution to approximate binomial distribution

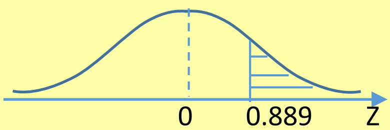
Given: n = 300, p = 0.15, q = 1 – 0.15 = 0.85, μ = np = 300 x 0.15 = 45,

$=\sqrt{npq} = \sqrt{300\ x\ 0.15\ x\ 0.85}$ = 6.1847

let x = number of eggs with cracks

P(x > 50) = P(Z > Z_1)

Where $Z_1 = \frac{50.5-45}{6.1847} = 0.889$



P(Z > 0.889) = 0.5 – P(0 < Z < 0.889)

= 0.5 – 0.2850 = 0.215

15. Among spectators watching a football watch, 80% were the home supporters while 20% were the visiting team supporters. If 2500 of the spectators are selected randomly, what is the probability that there are at least 541 visitors in the sample?

Solution

Since n is large, we use the normal distribution to approximate binomial distribution

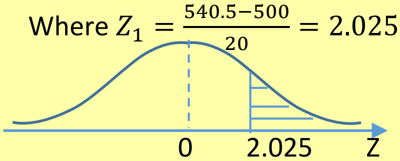
Given: n = 2500, p = 0.2, q = 1 – 0.2 = 0.8, μ = np = 2500 x 0.2 = 500,

σ $=\sqrt{npq} = \sqrt{2500\ x\ 0.2\ x\ 0.8}$ = 20

let x = number of visitors to support their team

P(x > 540) = P(Z > Z_1)

Where $Z_1 = \frac{540.5-500}{20} = 2.025$



P(Z > 2.025) = 0.5 – P(0 < Z < 2.025)

= 0.5 – 0.4785 = 0.0215

16. A die is tossed 40 times and the probability of getting at any one toss is 0.122, estimate the probability of getting between 6 to 10 sixes.

Solution

Given: n = 40, p = 0.122, q = 1 – 0.122 = 0.878, μ = np = 40 x 0.122 = 4.88,

σ $=\sqrt{npq} = \sqrt{40\ x\ 0.122\ x\ 0.878}$ = 2.07

Let x be the number of sixes

P(6 < x < 10) = P(7≤ x ≤ 9)

Using normal approximation to binomial

Z = $\frac{X \pm 0.5-\mu}{\sigma}$

P(7≤ x ≤ 9) = $P\left(\frac{6.5-4.88}{2.07} \leq Z \leq \frac{9.5-4.88}{2.07}\right)$

=P($0.78 < Z < 2.23$

= P(0 < Z < 2.23) – P(0 < Z < 0.78)

= 0.4871 – 0.2823

= 0.2048

17. In an examination which consists of 100 questions, a student has a probability of 0.6 of getting each answer correct. A student fails the examination if he obtains a mark less than 55, and obtains a distinction for a mark of 68 or more. Calculate
    (i) The probability that he fails the examination [0.1308]
    (ii) The probability that he obtains a distinction [0.0629]

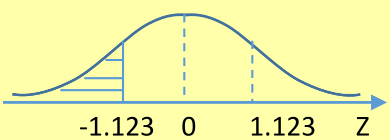
Solution

Given: n = 100, p = 0.6, q = 0.4, μ = np = 100 x 0.6 = 60 and σ = $\sqrt{npq} = \sqrt{100 \ x \ 0.6 \ x \ 0.4} = 4.899$

Let x = mark scored

(i) P(x < 55) = P(x ≤ 54) = $P\left(Z \leq \frac{54.5-60}{4.899}\right)$

= (Z ≤ -1.123)



P(Z < -1.123) = P(Z > 1.123)

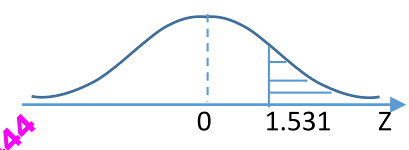
= 0.5 – P(0 ≤ Z ≤1.123)

=0.5 -0.3692

= 0.1308

(ii) P(x ≥ 68) = $P\left(Z \leq \frac{67.5-60}{4.899}\right)$

=P(Z ≥ 1.531)



P(Z ≥ 1.531) = 0.5 – P(0 ≤ Z ≤ 1.531)

= 0.5 -0.4371

= 0.0629

18. A research station supplies three varieties of seeds s_1, S_2 and S_3 in the ratio 4:2:1. The probabilities of germination of S_1, S_2 and S_3 are 50%, 60% and 80% respectively
    (i) Find the probability that a selected seed will germinate [
    (ii) Given that 150 seeds are selected at random, find the probability that less than 90 seed will germinate.

Solution

Given

4:2:1

4 + 2+ 1 = 7

P(S_1) = $\frac{4}{7}$; P(S_2) = $\frac{2}{7}$; P(S_3) = $\frac{1}{7}$

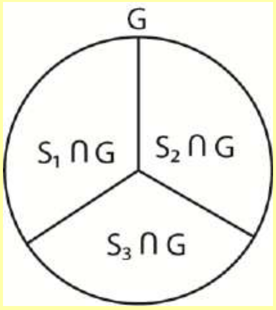
Let G = germination of seeds

P$\left({}^{G}/_{S_1}\right)$ = 50% = 0.5

P$\left({}^{G}/_{S_2}\right)$ = 60% = 0.6

P$\left({}^{G}/_{S_3}\right)$ = 80% = 0.8

(a) Find the probability that a seed selected at random will germinate.



$P(G) = P(s_1 \cap G) + P(s_2 \cap G) + P(s_3 \cap G)$

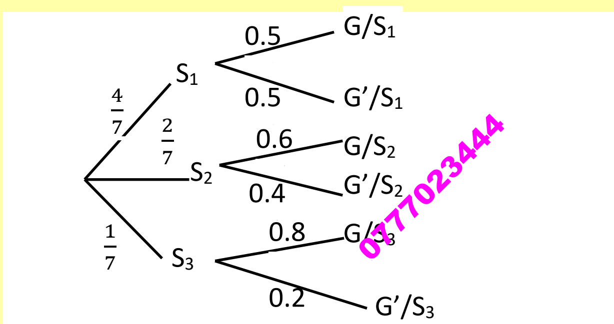
$= P(S_1).P\left(^G/_{S_1}\right) + P(S_2).P\left(^G/_{S_2}\right) + P(S_3).P\left(^G/_{S_3}\right)$

$= \frac{4}{7}\ x\ 0.5 + \frac{2}{7}\ x\ 0.6 + \frac{1}{7}\ x\ 0.8$

$= \frac{2}{7} + \frac{1.2}{7} + \frac{0.8}{7}$

$= \frac{4}{7}$

Or Using factor tree diagram



$P(G) = \frac{4}{7}\ x\ 0.5 + \frac{2}{7}\ x\ 0.6 + \frac{1}{7}\ x\ 0.8$

$= \frac{2}{7} + \frac{1.2}{7} + \frac{0.8}{7}$

$= \frac{4}{7}$

(b) Given that 150 seeds are selected at random, find the probability that less than 90 of the seeds will germinate. Give your answer to two decimal places.

$n = 150;\ P = \frac{4}{7};\ q = \frac{3}{7}$

since n = large (= 150), we use the normal approximate this binomial

$\mu = np = \frac{4}{7}\ x150 = \frac{600}{7}$

$\sigma = \sqrt{npq} = \sqrt{\frac{600}{7}\ x\ \frac{3}{7}} = \frac{30\sqrt{2}}{7}$

Let X = number of seeds that will germinate

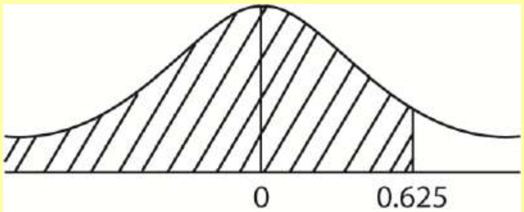
$P(x < 90) = P(x \leq 89)$

$= P\left(z \leq \frac{89.5 - \frac{600}{7}}{\frac{30\sqrt{2}}{7}}\right)$

$= P\left(z \leq \frac{7\left(89.5 - \frac{600}{7}\right)}{30\sqrt{2}}\right)$

$= P\left(z \le \frac{628.5-600}{30\sqrt{2}}\right)$

$= P(z \le 0.6250)$



$= 0.5 + (0 \le z \le 0.625)$

$= 0.5 + 0.2340$

$= 0.7340$

$= 0.73$ (2D)

19. A biased die with faces labelled 1, 2, 3, 4, 5, and 6 is tossed 45 times. calculate the probability that 2 appears

(i) More than 18 times (07marks)

n =45, p $= \frac{2}{6} = \frac{1}{3}$, q $= \frac{2}{3}$

μ = np = 45 x $\frac{1}{3}$ = 15

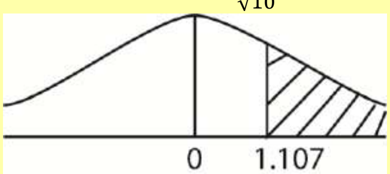
σ $= \sqrt{npq} = \sqrt{45\, x\, \frac{1}{3} x\, \frac{2}{3}} = \sqrt{10}$

Changing binomial to normal distribution.

P(X> x) = P(X> 18 + 0.5) = P(X> 18.5)

Standardizing using z= $\frac{\overline{x-\mu}}{\sigma}$

P(X> 18.5) = P(z > $\frac{18.5-15}{\sqrt{10}}$ = P(z > 1.107)



P(z > 1.107) =0.5 – P(0 < z < 1.107)

= 0.5 – 0.3658

= 0.1342

∴ P(X > 18) = 0.1342

(ii) Exactly 11 times (05marks)

P(X = 11) = P(11 – 0.5 < X < 11 + 0.5)

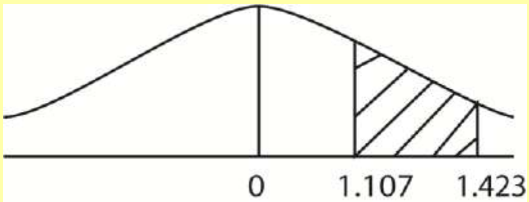
= P(10.5 < X < 11.5)

$= P\left(\frac{10.5-15}{\sqrt{10}} < z < \frac{11.5-15}{\sqrt{10}}\right)$

= P(-1.423 < z < 1.107)

By symmetry

P(-1.423 < z < 1.107) = P(1.107 < z < 1423)



P(1.107 < z < 1423) = P(0 < z < 1.423) – P(0 < z < 1.107)

= 0.4226 – 0.3658

= 0.0568

Distribution of sample mean of a normal distribution population

If a random variable X of a sample of size n from a normal distribution with mean μ and variance σ^2, then distribution of the sample mean $\overline{x}$ is also said to be normally distributed with mean μ and variance $\frac{\sigma^2}{n}$, such that $\overline{x} \approx \left(\mu,\ \frac{\sigma^2}{n}\right)$

Then $Z = \frac{\overline{x}-\mu}{\frac{\sigma^2}{n}}$

Example 35

At a certain school, the masses of students are normally distributed with mean 70kg and standard deviation 5kg. If 4 students are randomly selected, find the probability that their mean is less than 65.

P$(\overline{X} < 65) = P\left(Z < \frac{65-70}{\frac{5}{\sqrt{4}}}\right) = P(Z < \ -2)$

$P(Z < \ -2) = P(Z > 2) = 0.5 - P(0 < Z < 2)$ =0.5 – 0.4772 = 0.0228

Example 36

A random sample of size 15 is taken from a normal population with mean 60 and standard deviation 4. Find the probability that the mean of the sample is less than 58

P$(\overline{X} < 58) = P\left(Z < \frac{58-60}{\frac{4}{\sqrt{15}}}\right) = P(Z < \ -1.936)$

$P(Z < \ -1.936) = P(Z > 1.936) = 0.5 - P(0 < Z < 1.936)$ =0.5 – 0.4736 = 0.0264

Example 37

The height of students are normally distributed with mean 164cm and standard deviation 7.2cm. Calculate the probability that the mean height of a sample of 36 students will be between 162cm and 166cm.

P$(162 < \ \overline{X} < 166) = P\left(\frac{162-164}{\frac{7.2}{\sqrt{36}}} < Z < \ \frac{166-164}{\frac{7.2}{\sqrt{36}}}\right) = P(-1.667 < Z < \ 1.667)$

$P(-1.667 < Z < \ 1.667) = 2\ x\ P(0 < Z < 1.667$ = 2 x 0.4522 = 0.9044

Example 38

The height of a certain plant follows a normal distribution with mean 21cm and standard deviation $\sqrt{90}cm$. A random sample of 10 plants is taken and the mean height calculated. Find the probability that this sample mean lies between 18cm and 27 cm

$$P(18 < \overline{X} < 27) = P\left(\frac{18-21}{\frac{\sqrt{90}}{\sqrt{10}}} < Z < \frac{27-21}{\frac{\sqrt{90}}{\sqrt{10}}}\right) = P(-1 < Z < 2)$$

$$P(-1 < Z < 2) = P(0 < Z < 1) + P(0 < Z < 2) = 0.3413 + 0.4772 = 0.8185$$

Example 39

A large number of random sample of size n is taken from a distribution X where X ~N(74, 36) and the sample mean $\overline{x}$ for each sample is noted. If $P(\overline{x} > 72) = 0.854$, *find the value of n.*

$$P(\overline{X} > 72) = P\left(Z > \frac{72-74}{\frac{6}{\sqrt{n}}}\right) = 0.854$$

$$P\left(Z > \frac{-\sqrt{n}}{3}\right) = 0.854$$

From table Z = - 1.054

$$\frac{-\sqrt{n}}{3} = -1.054$$

n = 10

Example 40

The distribution of a random variable x is X ~N(25, 340) and the sample mean $\overline{x}$ for each sample is calculated. If $P(\overline{x} > 28) = 0.005$, *find the value of n.*

$$P(\overline{X} > 28) = P\left(Z > \frac{28-25}{\frac{\sqrt{340}}{\sqrt{n}}}\right) = 0.005$$

$$P\left(Z > \frac{3\sqrt{n}}{\sqrt{340}}\right) = 0.005$$

From table Z = 2.576

$$\frac{3\sqrt{n}}{\sqrt{340}} = 2.576$$

n = 250

Revision exercise 5

1. If X~N(200, 80) and a random sample of size 5 is taken from the distribution, find the probability that the sample mean
   (i) is greater than 207 [0.0401]
   (ii) lies between 201 and 209 [0.3891]
2. If X~N(200, 10) and a random sample of size 10 is taken from the distribution, find the probability that the sample mean lies outside the range 198 and 205 [0.3206]
3. If X~N(50, 12) and a random sample of size 12 is taken from the distribution, find the probability that the sample mean

(i) Is less than48.5 [0.0668]
(ii) Is less than 52.3[0.9893]
(iii) Lie between 50.7 and 51.7 [0.1974]

4. Biscuits are produced with weight (W g) where W is N(10, 4) and are packed at random into boxes consisting of 25 biscuits. Find the probability that
   (i) a biscuit chosen at random weigh between 9.25g and 10.7g [0.2924]
   (ii) the content of a box weighs between 245g and 255g [0.0796]
   (iii) the average weight of the biscuit in the box lies between 9.7g and 10.3g [0.5468]
5. A normal distribution has a mean of 40 and standard deviation of 4. If 25 items are drawn at random, find the probability that their mean
   (i) 41.4 or more [0.0401]
   (ii) Between 38.7 and 40 .7 [0.7571]
   (iii) Less than 39.5 [0.2660]
6. A random sample of size 25 is taken from a normal population with mean 60 and standard deviation 4. Find the probability that the mean of the sample
   (i) Less than 58 [0.0062]
   (ii) Greater than 58 [0.9918]
   (iii) Between 58 and 62[0.9876]
7. At St. Noa Junior, the marks of the pupils can be modelled by a normal distribution with mean 70% and standard deviation 5%. If four pupils are chosen at random, find the probability that the mean mark is
   (i) Less than 65% [0.9772]
   (ii) Greater than 65% [0.0228]
   (iii) Greater than 75%[0.0228]
   (iv) Between 72% and 75% [0.1891]
8. The volume of soda in bottle are normally distributed with mean 758ml and standard deviation of 12ml. a random sample of 10 bottles is taken and mean volume is found. Find the probability that the sample mean is less than 750ml. [0.0176]
9. The height of cassava plants are normally distributed with mean of 2m and standard deviation of 40cm. a random sample of 50 cassava plants is taken and the mean height found. Find the probability that the sample mean lies between 195cm and 205cm. [0.6234]
10. In an examination, marks are normally distributed with mean 64.5 and variance 64. The mean ark in a random sample of 100 scripts is denoted by X. find
    (i) P(X> 65.5) [0.1056]
    (ii) P(63.8 < $\overline{X}$ < 64.5[0.3092]
11. The marks of an examination were normally distributed. 20% of the students scored below 40 marks while 10%of the students scored above 75 marks
    (i) Find the mean mark and standard deviation of the students [μ = 53.87, σ = 16.473]
    (ii) If 25 students were chosen at random from those who sat for the examination, what is the probability that their average mark exceeds 60. [0.0313]
    (iii) If a sample of 8 students were chosen, find the probability that not more than 3 scored between 45 and 65 marks. [0.5419]
12. The life time of batteries produced by a certain factory is normally distributed. Out of 10,000 batteries selected at random, 668have life tie less than 130 hours and 228 have life time more than 200 hours.
    (i) Find the mean mark and standard deviation of the battery life time[μ =160, σ = 20]
    (ii) Find the percentage of the batteries with life time between 150 and 180 hours.

(iii) If the sample of 25 batteries is selected at random, find the probability that the mean of the life time exceeds 165 hours [0.1056]

13. A normal distribution has a mean of 30 and a variance of 5. Find the probability that
    (i) The average of 10 observation exceeds 30.5 [0.2399]
    (ii) The average of 40 observation exceeds 30.5[0.0787]
    (iii) The average of 100 observation exceeds 30.5 [0.0127]
    (iv) Find n such that the probability that the average of observations exceed 30.5 is less than 1%, [n > 108]
14. The random variable is such that X~N(μ, 4). A random sample size n is taken from the population. Find the least n such that P($|\overline{X} - \mu| < 0.5) = 0.95$ [62]
15. Boxes made in a factory have weight which are normally distributed with a mean of 4.5kg and a standard deviation of 2.0kg. if a sample of 16 boxes is drawn at random, find the probability that their mean is
    (i) between 4.6 and 4.7 kg [0.0761]
    (ii) between 4.3 and 4.7g [0.3108]
16. the masses of soap powder in a certain packet are normally distributed with mean 842g and variance 225g. find the probability that a random sample of 120 packets has sample mean mass
    (i) between 844g and 846g [0.0702]
    (ii) less than 843g[0.7673]

Estimation of population parameters

Statistical estimation is used to describe the unknown characteristics of the population (population parameters) by using sample characteristics.

A sample is a representation of the population parameter such as population mean, μ and population variance, σ^2.

Types of parameter estimation

- point estimation
- interval estimation

(a) point estimates

(i) the unbiased estimate of the population mean, μ is

$\overline{x} = \frac{\sum x}{n}$ or $\overline{x} = \frac{\sum fx}{\sum f}$ where $\overline{x}$ is sample mean

(ii) the unbiased estimate of the population variance, σ^2 is $\hat{\sigma}^2$ where $\hat{\sigma}^2 = \frac{n}{n-1}s^2$ where s^2 is sample variance

OR $\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2\right]$ or $\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2\right]$

Example 41

Find the best unbiased estimate of mean μ and variance σ2 of the population from each of the following sample is drawn

(i) 46, 48, 50, 45, 53, 50, 48, 51

Solution

x	f	fx	fx^2
45	1	45	2025
46	1	46	2116
48	2	96	4608
50	2	100	5000
51	1	51	2601
53	1	53	2809
	$\sum f = 8$	$\sum fx = 391$	$\sum fx^2 = 19159$

Unbiased estimate for the mean $\overline{x} = \frac{\sum fx}{\sum f} = \frac{391}{8} = 48.875$

The unbiased estimate of the population variance, $\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2\right]$

$= \frac{8}{8-1}\left[\frac{19159}{8} - \left(\frac{391}{8}\right)^2\right] = 6.982$

(ii) $\sum x = 100, \sum x^2 = 1028, n = 10$

Unbiased estimate for the mean $\overline{x} = \frac{\sum fx}{\sum f} = \frac{100}{10} = 10$

The unbiased estimate of the population variance, $\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2\right]$

$= \frac{10}{10-1}\left[\frac{1028}{10} - \left(\frac{100}{10}\right)^2\right] = 3.11$

(iii) $\sum x = 120, \sum x^2 = 2102, n = 8$

Unbiased estimate for the mean $\overline{x} = \frac{\sum fx}{\sum f} = \frac{120}{8} = 15$

The unbiased estimate of the population variance, $\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2\right]$

$= \frac{8}{8-1}\left[\frac{2102}{8} - \left(\frac{120}{8}\right)^2\right] = 43.14$

(iv) $\sum x = 330, \sum x^2 = 23700, n = 34$

Unbiased estimate for the mean $\overline{x} = \frac{\sum fx}{\sum f} = \frac{330}{34} = 9.71$

The unbiased estimate of the population variance, $\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2\right]$

$= \frac{34}{34-1}\left[\frac{23700}{34} - \left(\frac{330}{34}\right)^2\right] = 621.12$

(v) $\sum x = 738, \sum x^2 = 16526, n = 50$

Unbiased estimate for the mean $\overline{x} = \frac{\sum fx}{\sum f} = \frac{738}{50} = 14.76$

The unbiased estimate of the population variance, $\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2\right]$

$$= \frac{50}{50-1}\left[\frac{16526}{50} - \left(\frac{738}{50}\right)^2\right] = 114.96$$

Example 42

The fuel consumption of a new car model was being tested. In one trials 8 cars chosen at random were driven under identical conditions and distance x km covered on one litre of petro was recorded. The following results were obtained. $\sum x = 152.98$, $\sum x^2 = 2927.1$. Calculate the unbiased estimate of the mean and variance of the distance covered by the car.

Solution

Unbiased estimate for the mean $\overline{x} = \frac{\sum fx}{\sum f} = \frac{152.98}{8} = 19.1225$

The unbiased estimate of the population variance, $\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2\right]$

$$= \frac{8}{8-1}\left[\frac{2927.1}{8} - \left(\frac{152.98}{8}\right)^2\right] = 0.25$$

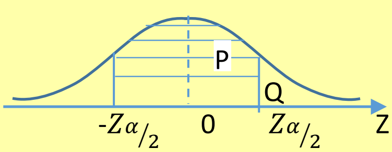
(b) Interval estimate

Here we are interested in obtaining the interval over which the true population mean lies (confidence interval)

The unbiased estimate of the population mean, μ is $\overline{x}$

Z = $\frac{\overline{x}-\mu}{\frac{\sigma}{\sqrt{n}}}$ where n is the sample size

Z is the area under the normal curve leaving an area of $\frac{\alpha}{2}$ on either side of the curve



$P\left(-Z\alpha/_2 < Z < Z\alpha/_2\right) = \frac{\alpha}{2}$

$P\left(-Z\alpha/_2 < \frac{\overline{x}-\mu}{\frac{\sigma}{\sqrt{n}}} < Z\alpha/_2\right) = \frac{\alpha}{2}$

$P\left(\overline{x} - Z\alpha/_2\frac{\sigma}{\sqrt{n}} < \mu < \overline{x} + Z\alpha/_2\frac{\sigma}{\sqrt{n}}\right) = \frac{\alpha}{2}$

Confidence interval $\left[\overline{x} - Z\alpha/_2\frac{\sigma}{\sqrt{n}} < \mu < \overline{x} + Z\alpha/_2\frac{\sigma}{\sqrt{n}}\right]$

Confidence Limits $\left[\overline{x} - Z\alpha/_2\frac{\sigma}{\sqrt{n}} < \mu < \overline{x} + Z\alpha/_2\frac{\sigma}{\sqrt{n}}\right]$

Or $\mu < \overline{x} \pm Z\alpha/_2\frac{\sigma}{\sqrt{n}}$

(i) Confidence interval for population mean μ

- of a normal or non-normal population
- with known population variance σ2 or standard deviation σ
- using any sample size

The confidence interval is obtained from $\mu < \overline{x} \pm Z\alpha/_2\frac{\sigma}{\sqrt{n}}$ where $\overline{x}$ is sample mean

Example 43

The length of a bar of a metal is normally distributed with mean of 115cm and standard deviation of 3cm. find the 95% confidence limits for the length of the bar

$\frac{\alpha}{2} = \frac{0.95}{2} = 0.475$

$\frac{\alpha}{2} = \frac{0.95}{2} = 0.475$ $Z\alpha/_2 = 1.96$	$\mu < \overline{x} \pm Z\alpha/_2 \frac{\sigma}{\sqrt{n}}$ $\mu < 115 \pm 1.96\frac{3}{\sqrt{1}}$	Lower limit =109.12 Upper limit = 120.88

Example 44

The mass of vitamin in a capsule is normally distributed with standard deviation 0.042mg. a random sample of 5 capsules was taken and the mean mass of vitamin e found to be 5.12. calculate a symmetric confidence interval for the population mean mass.

$\frac{\alpha}{2} = \frac{0.95}{2} = 0.475$ $Z\alpha/_2 = 1.96$	$\mu < \overline{x} \pm Z\alpha/_2 \frac{\sigma}{\sqrt{n}}$ $\mu < 5.12 \pm 1.96\frac{0.042}{\sqrt{5}}$	Lower limit =5.08 Upper limit = 5.16

Example 44

It is known that an examination paper is marked in such a way that the standard deviation of the marks is 15.1. in a certain school, 80 candidates took the examination and they had an average mark of 57.4. find

(i) 95% confidence limits for the mean mark in the exam.

$\frac{\alpha}{2} = \frac{0.95}{2} = 0.475$ $Z\alpha/_2 = 1.96$	$\mu < \overline{x} \pm Z\alpha/_2 \frac{\sigma}{\sqrt{n}}$ $\mu < 5.12 \pm 1.96\frac{15.1}{\sqrt{80}}$	Lower limit =54.091 Upper limit = 60.709

(ii) 99% confidence limits for the mean mark in the exam.

$\frac{\alpha}{2} = \frac{0.99}{2} = 0.495$ $Z\alpha/_2 = 2.575$	$\mu < \overline{x} \pm Z\alpha/_2 \frac{\sigma}{\sqrt{n}}$ $\mu < 5.12 \pm 2.595\frac{15.1}{\sqrt{80}}$	Lower limit =53.053 Upper limit = 61.746

Example 45

After a particular rainy night, 12 worms were picked and their length in cm measured;
9.5, 9.5, 11.2, 10.6, 9.9, 11.1, 10.9, 9.8, 10.1, 10.2, 10.9, 11.0

Assuming that this sample came from a normal population with standard deviation 2, find the 95 confidence interval for the mean length of all the worms

$$\overline{x} = \frac{\sum x}{n} = \frac{9.5 + 9.5 + 11.2 + 10.6 + 9.9 + 11.1 + 10.9 + 9.8 + 10.1 + 10.2 + 10.9 + 11.0}{12}$$

= 10.39

$\frac{\alpha}{2} = \frac{0.95}{2} = 0.475$ $Z\alpha/_2 = 1.96$	$\mu < \overline{x} \pm Z\alpha/_2 \frac{\sigma}{\sqrt{n}}$ $\mu < 10.39 \pm 1.96\frac{2}{\sqrt{12}}$	Lower limit =9.81 Upper limit = 10.97

The height of students are normally distributed with mean μ and standard deviation σ. On the bsis of results obtained from a random sample of 100 students from school, the 95% confidence interval of the mean was calculated and found to be (177.22cm, 179.18cm). Calculate

(i) the value of the sample mean

$\frac{\alpha}{2} = \frac{0.95}{2} = 0.475$

$Z\alpha/_2 = 1.96$

$\mu < \overline{x} \pm Z\alpha/_2 \frac{\sigma}{\sqrt{n}}$

$177.22 < \overline{x} - 1.96\frac{\sigma}{\sqrt{100}}$ .... (i)

$179.18 < \overline{x} + 1.96\frac{\sigma}{\sqrt{100}}$ .... (ii)

Eqn. (i) + eqn. (ii)

$2\overline{x}$ = 356.4; $\overline{x}$ = 178.2

(ii) the value of standard deviation

$177.22 < 178.2 - 1.96\frac{\sigma}{\sqrt{100}}$;

σ = 5

(iii) 90% confidence interval of the mean, μ

$\frac{\alpha}{2} = \frac{0.90}{2} = 0.45; Z\alpha/_2 = 1.645$

Example 46

A plant produces steel sheets whose weight are normally distributed with standard deviation of 2.4kg. A random sample of 36 sheets had a mean weight of 31.4kg.

(i) Find the 99% confidence limit for the population

$\frac{\alpha}{2} = \frac{0.99}{2} = 0.495$

$Z\alpha/_2 = 2.575$

$\mu < \overline{x} \pm Z\alpha/_2 \frac{\sigma}{\sqrt{n}}$

$\mu < 31.4 \pm 2.595\frac{2.4}{\sqrt{36}}$

Lower limit =30.37kg

Upper limit = 32.43kg

(ii) Find the width of the 99% confidence limit

=32.43kg - 30.37kg = 2.06kg

Example 47

The marks scored by students are normally distributed with mean μ and standard deviation1.3. it is required to have 95% confidence interval for μ with width less than 2. Find the least number of students that be sampled to achieve this.

$\frac{\alpha}{2} = \frac{0.95}{2} = 0.475$

$Z\alpha/_2 = 1.96$

width = $2\ x Z\alpha/_2 \frac{\sigma}{\sqrt{n}} < 2$

$2\ x\ 1.96\frac{\sigma}{\sqrt{n}} < 2$

$2\ x\ 1.96\frac{\sigma}{2} < \sqrt{n}$; n < 6.49

n > 6.49

the least number = 7

(ii) Confidence interval for population mean μ

- of a normal or non-normal population
- with unknown population variance σ2 or standard deviation σ
- using a large sample size (n ≥ 30)

If the population variance σ2 s not given or unknown, the confidence interval is obtained from $\mu < \overline{x} \pm Z\alpha/_2 \frac{\hat{\sigma}}{\sqrt{n}}$ where $\overline{x}$ is sample mean, $\hat{\sigma} = \frac{n}{n-1} s^2$ and s = sample variance

$$\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2\right]$$

Or

$$\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2\right]$$

Example 48

the fuel consumption of a new car was being tested. In one trials 50 cars chosen at random were driven under identical conditions and the distance x km covered on one litre of petrol was recorded. the following results were obtained. $\sum x = 525$, $\sum x^2 = 5625$. Calculate the 95%confidence interval for the mean petrol consumption, in km per litre of cars of this type..

Unbiased estimate for the mean $\overline{x} = \frac{\sum fx}{\sum f} = \frac{525}{50} = 10.5$

The unbiased estimate of the population variance, $\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2\right]$

$= \frac{50}{50-1}\left[\frac{5625}{50} - \left(\frac{525}{50}\right)^2\right]$ = 2.2952

$\frac{\alpha}{2} = \frac{0.95}{2} = 0.475$ $Z\alpha/_2 = 1.96$	$\mu < \overline{x} \pm Z\alpha/_2 \frac{\sigma}{\sqrt{n}}$ $\mu < 10.5 \pm 1.96 \frac{\sqrt{2.2952}}{\sqrt{50}}$	Lower limit =10.08km/litre Upper limit = 10.92km/litre

Example 49

The height x cm of each man in a random sample of 200 men living in Nairobi was measured. The following results were obtained$\sum x = 35050$, $\sum x^2 = 6163109$.

(a) calculate the unbiased estimate of the mean and variance of the heights of men living Nairobi

Unbiased estimate for the mean $\overline{x} = \frac{\sum fx}{\sum f} = \frac{35050}{200} = 175.25$

The unbiased estimate of the population variance, $\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2\right]$

$= \frac{200}{200-1}\left[\frac{6163109}{200} - \left(\frac{35050}{200}\right)^2\right]$ = 103.5

(b) Determine the 90% confidence interval for the mean height of mean living in Nairobi.

$\frac{\alpha}{2} = \frac{0.90}{2} = 0.45$ $Z\alpha/_2 = 1.645$	$\mu < \overline{x} \pm Z\alpha/_2 \frac{\sigma}{\sqrt{n}}$ $\mu < 175.25 \pm 1.645 \frac{\sqrt{103.5}}{\sqrt{200}}$	Lower limit =174.07cm Upper limit = 176.43cm

Example 50

A random sample of 100 observations from a normal population with mean µ gave the following results $\sum x = 8200$, $\sum x^2 = 686000$.

(a) calculate the unbiased estimate of the mean and variance of the heights of men living Nairobi

Unbiased estimate for the mean $\overline{x} = \frac{\sum fx}{\sum f} = \frac{8200}{100} = 82$

The unbiased estimate of the population variance, $\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2\right]$

$= \frac{200}{200-1}\left[\frac{68600}{100} - \left(\frac{8200}{100}\right)^2\right]$ = 11.72

(b) Determine the 98% confidence interval for the mean

$\frac{\alpha}{2} = \frac{0.98}{2} = 0.49$ $Z\alpha_{/2} = 2.326$	$\mu < \overline{x} \pm Z\alpha_{/2}\frac{\sigma}{\sqrt{n}}$ $\mu < 82 \pm 2.326\frac{11.72}{\sqrt{100}}$	Lower limit =79.274 Upper limit = 84.726

(c) determine the 99% confidence interval for the mean

$\frac{\alpha}{2} = \frac{0.99}{2} = 0.495$ $Z\alpha_{/2} = 2.575$	$\mu < \overline{x} \pm Z\alpha_{/2}\frac{\sigma}{\sqrt{n}}$ $\mu < 82 \pm 2.575\frac{11.72}{\sqrt{100}}$	Lower limit =78.981 Upper limit = 85.726

Example 50

The mean and standard deviation of a random sample of size 100 is 900 and 60 respectively. Given that the population is normally distributed, find 96% confidence interval of the population mean.

$\hat{\sigma} = \sqrt{\frac{n}{n-1}s^2} = \sqrt{\frac{100}{100-1}x\ 60^2} = 60.302$

$\frac{\alpha}{2} = \frac{0.96}{2} = 0.48$ $Z\alpha_{/2} = 2.054$	$\mu < \overline{x} \pm Z\alpha_{/2}\frac{\sigma}{\sqrt{n}}$ $\mu < 900 \pm 2.054\frac{60.302}{\sqrt{100}}$	Lower limit =887.614 Upper limit = 912.386

Revision exercise 6

1. the concentration in mg per litre of a trace element in 7 randomly chosen samples of water from a spring were 240.8, 237.3, 236.7, 236.6, 234.2, 233.9, 232.5.
   Determine the unbiased mean and variance of the concentration of the trace element per litre from spring [236, 7.58]
2. Cartons of oranges are filled by a machine. A sample of 10 cartons selected at random from the population contained the following quantities in ml) 201.2, 205.0, 209.1, 202.3,204.6, 206.4,

210.1,201.9, 203.7, 207.3. Determine the unbiased mean and variance of the population from which the sample was taken. [203.16, 9.223]

3. A factory produces cans of meat whose masses are normally distributed with standard deviation 18g. A random sample of 25 cans is found to have the mean of 458g. find the 99% confidence interval for the population mean of a can of meat produced at the factory. [448.7, 467.3g]
4. The height of bounce of a tennis ball is normally distributed with standard deviation 2cm. A sample of 60 tennis balls is tested and the mean height of bounce is 140cm. Find
   (i) 95% [139.5, 140.51] (ii) 98% [139.4, 140.6] confidence interval for the mean height of bounce of the tennis ball
5. A random sample of 100 is taken from a population. The sample is found to have a mean of 76.0 and standard deviation of 120. Find
   (i) 90% [747.51, 748.49] (ii) 95% [747.42, 748.58] (iii) 98% [747.31, 748.69] confidence interval for the mean of the population
6. 150 bags of flour of a particular brand are weighed and the mean mass is found to be 748g with standard deviation 3.6g. Find
   (i) 90% [74.02, 77.98] (ii) 97% [73.38, 78.62] (iii) 99% [72.89, 79.11] confidence interval for the mean mass of bags of flour of this brand.
7. A random sample of 100 readings taken from a normal population gave the following data: $\overline{x}$ = 82, $\sum x^2 = 686800$. Find
   (i) 98% [79.19, 84.81] (ii) 99% [78.89, 85.11] confidence interval of the population mean
8. 80 people were asked to measure their pulse rates when they woke up in the morning. The mean was 69 beats and standard deviation 4 beats. find
   (i) 95% [68.12, 69.88] (ii) 99% [67.84, 70.16] (iii) 97% [68.0, 70.0] confidence interval of the population mean
9. The 95% confidence interval for the mean length of a particular brand of light bulbs is [1023.3h, 1101.7h]. This is interval is based on results from a sample of 36 light bulbs. Find the 99% confidence interval for the mean length of life of this brand of light bulbs assuming that the length of life is normally distributed. [1011, 1114]
10. A random sample of 6 items taken from a normal population with variance 4.5cm^2 gave the following data: 12.9cm, 13.2cm, 14.6cm, 12.6cm, 11.3cm, and 10.1 cm.
    (i) Find the 95% confidence interval for the population mean. [10.75, 14.15].
    (ii) What is the width of this confidence interval [3.4]
11. A random sample of 60 loaves is taken from a population whose mean masses are normally distributed with mean μ and standard deviation 10g.
    (i) calculate the width of 95% confidence interval for μ bases on the sample [5.06]
    (ii) Find the confidence level having the same width as in (i) but based on a random sample of 40 loaves. [89%]
12. The distribution of measurements of masses of a random sample of bags packed in a factory is shown below

Mass (kg)	72.5	77.5	82.5	87.5	92.5	97.5	102.5	107.5
frequency	6	18	32	57	102	51	25	9

(i) Find the mean and standard deviation of the masses [μ =91.317, σ=7.41]
(ii) find the 95% confidence limits[90.5, 92.2]

7.CORRELATION

Correlations and scatter diagrams

Correlation refers to **the degree of correspondence or relationship between two variables**. Correlated variables tend to change together. If one variable gets larger, the other one systematically becomes either larger or smaller.

The degree of correlation/association is determined by rank correlation coefficients

There are two types

1. Spearman rank correlation coefficient (ρ)
2. Kenddall's rank correlation coefficient (τ

Interpretation of the rank correlation coefficients

A rank correlation coefficient measures the degree of similarity between two rankings

The table below is used

Correlation coefficient	Interpretation
0-0-19	Very low correlation
0.2-0.39	Low correlation
0.4-0.59	Moderate correlation
0.6 – 0.79	High correlation
0.8 – 1.0	Very high correlation

Note: the positive or negative signs indicate positive or negative relationships respectively. Or they the relationships are directly or inversely related.

The closer to zero the lower the relationship

Spearman rank correlation (ρ)

It is given by $\rho = 1 - \left[\frac{6\sum d^2}{n(n^2-1)}\right]$

Where d = difference between ranks

n = total number of pairs

Example 1

Two examiners marked the scripts of 8 candidates. The table shows the marks awarded by two examiners x and y.

x	72	60	56	76	68	52	80	64
y	56	44	60	74	66	38	68	52

Calculate the rank correlation coefficient and comment on your reults

Solution

Rx	Ry	d	d^2
3	5	2	4
6	7	1	1
7	4	3	9
2	1	1	1
4	3	1	1
8	8	0	0
1	2	-1	1
5	5	1	1
			$\sum d^2 = 18$

$\rho = 1 - \left[\frac{6\sum d^2}{n(n^2-1)}\right]$

$= 1 - \left[\frac{6 \, x \, 18}{8(8^2-1)}\right] = 0.786$

There is a high positive correlation between x and y

Example 2

The following shows the marks obtained by 10 students in mathematics and physics exams.

Mathematics	80	80	70	60	65	80	68	90	95	50
Physics	50	45	70	80	70	90	70	80	70	95

Calculate the ranks correlation coefficient and comment on your results

Solution

R_M	R_P	d	d^2
4	9	-5	25
6	6.5	0.5	0.25
9	3.5	5.5	30.25
8	6.5	1.5	2.25
4	2	2	4
7	6.5	0.5	0.25
2	3.5	1.5	2.25
1	6.5	-5.5	30.25
10	1	9	81
			$\sum d^2 = 211.5$

$\rho = 1 - \left[\frac{6\sum d^2}{n(n^2-1)}\right]$

$= 1 - \left[\frac{6 \, x \, 211.5}{10(10^2-1)}\right] = -0.282$

There is a low negative correlation between mathematics and physics

Example 3

The following table gives the order in which six candidates were ranked in two tests x and y

x	E	C	B	F	D	A
y	F	A	D	E	C	C

Calculate the rank correlation coefficient and comment on your results

Solution

Rx	Ry	d	d^2
5	6	1	1
3	1	2	4
2	4	2	4
6	5	1	1
4	2.5	1.5	2.25
1	2.5	-1.5	225
			$\sum d^2 = 14.5$

ρ = 1- $\left[\frac{6\sum d^2}{n(n^2-1)}\right]$

= 1- $\left[\frac{6\ x\ 14.5}{6(6^2-1)}\right]$ = -0.586

There is a moderate negative correlation between x and y

Kendall's Rank correlation coefficient

It is a coefficient that represents the degree of concordance/agreement between toe columns of ranked data. The greater the 'inversions' the smaller the coefficient will

Kendall's Rank correlation coefficient, τ (tau) = $\frac{C-D}{C+D}$

Where C = number of concordant pairs or pairs in agreement

D = number of disconcordant pairs or pairs in disagreement

The Tau correlation coefficient returns a value of 0 to 1, where: 0 is no relationship, 1 is a perfect relationship

- Concordant pairs are the number of observed ranks below a particular rank which are larger than that particular rank.
- Disconcordant pairs are the number of observed rank below a particular rank which are smaller than that particular rank

Example 4

Two examiners marked the scripts of 8 candidates. The table shows the marks awarded by two examiners x and y.

x	72	60	56	76	68	52	80	64
y	56	44	60	74	66	38	68	52

Calculate the rank correlation coefficient and comment on your results

Solution

Ranking values

The subscripts are the ranks

x	72_3	60_6	56_7	76_2	68_4	52_8	80_1	64_5
y	56_5	44_7	60_4	74_1	66_3	38_8	68_2	52_6

- The ranks of x and y are filled in the table as below those of x in ascending order and those of y correspondingly as shown in the table below.
- The values of C are bigger values in the column Ry bigger than and below a particular value in that column. While the values of D are sallerr values in the column Ry bigger than and below a particular value in that column

- For instance, the first value of C in the table below is the number of values bigger than and below 2 in column Ry i.e. (5, 3, 6, 7, 4, 8) =6; the first value D = the number of values smaller than and below 2 in column Ry; i.e. (1) = 1

Rx	Ry	C	D
1	2	6	1
2	1	6	0
3	5	3	2
4	3	3	0
5	6	2	1
6	7	1	1
7	4	1	0
8	8		
		$\sum C = 22$	$\sum D = 5$

τ (tau) = $\frac{C-D}{C+D}$

τ (tau) = $\frac{22-5}{22+5}$ = 0.63

moderate positive correlation

Example 5

The height (cm) an ages (years) of random sample of ten farmers are given in the table below

Height (cm)	156	151	152	160	146	157	149	142	158	140
Ages (years)	47	38	44	55	46	49	54	52	45	30

(a)(i) Calculate the Kendall rank correlation coefficient

(ii) comment on your result (06marks)

Let the farmers be A, B, C, D, E, F, G, H, I, J (the subscripts are the rank)

Farmers	A	B	C	D	E	F	G	H	I	J
Height	156_4	151_6	152_5	160_1	146_8	157_3	149_7	142_9	158_2	140_{10}
Age	47_5	38_9	44_8	55_1	46_6	49_4	54_2	52_3	45_7	30_{10}

By re-arranging the findings we have

Farmers	D	I	F	A	C	B	G	E	H	J
Height	1	2	3	4	5	6	7	8	9	10
Age	1	7	4	5	8	9	2	6	3	10
Agreements (C)	9	3	5	4	2	1	3	1	1	=29
Disagreements (D)	0	4	2	2	3	3	0	1	0	=15

τ (tau) = $\frac{C-D}{C+D}$

τ (tau) = $\frac{29-15}{29+15}$ = 0.32

low positive correlation

Significance of ranks correlation coefficients

The calculated ranks correlation coefficients to be statistically significant; the calculated value should be greater than that from the table critical values associated with the various sample sizes and significance levels (α).

That is

- ⇨ If the $|\rho_c| > |\rho_T|$ or $|\tau_c| > |\tau_T|$, a significant relationship exists
- ⇨ If the $|P_c| < |P_T|$ or $|\tau_c| < |\tau_T|$, no significant relationship exists

Where ρ_c = calculated Spearman's correlation coefficient

ρ_T = Table Spearman's correlation coefficient at either 1% (α=0.01) or 5% (α =0.05)

τ_c = calculated Kendall's coefficient

τ_T = Table Kendall's correlation coefficient at either 1% (α=0.01) or 5% (α =0.05)

Example 6

The following shows the marks obtained by 8students in mathematics and physics exams

Mathematics	65	65	70	75	75	80	85	85
Physics	50	55	58	55	65	58	61	65

Calculate the ranks correlation coefficient and comment of the significance of your results at 5% level (Spearman's $\rho = 0.71$), $Kendall's\ \tau = 0.64$

(a) Using Spearman's correlation coefficient

R_M	R_P	d	d^2
7.5	8	0.5	0.25
7.5	6.5	1	1
6	4.5	1.5	2.25
4.5	6.5	-2	4
4.5	1.5	3	9
3	4.5	-1.5	2.25
1.5	3	-1.5	2.25
1.5	1.5	0	0
			$\sum d^2 = 21$

ρ = 1- $\left[\frac{6\sum d^2}{n(n^2-1)}\right]$

= 1- $\left[\frac{6\ x\ 21}{8(8^2-1)}\right]$ = 0.75

Since ρ_C(0.75)> ρ_T (0.71), a significant relationship exist

Using Kendall's correlation coefficient

R_M	R_P	C	D
1.5	1.5	6	0
1.5	3	5	1
3	4.5	3	1
4.5	1.5	4	0
4.5	6.5	1	1
6	4.5	2	0
7.5	6.5	1	0
7.5	8	=22	3

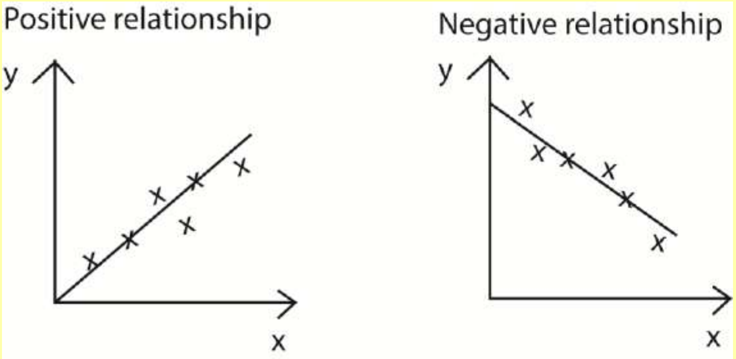
τ (tau) = $\frac{C-D}{C+D}$

τ (tau) = $\frac{22-3}{22+3}$ = 0.76

Since τ_C(0.75)> τ_T (0.64), a significant relationship exist

Scatter graphs

They are graphs showing the relationship between two variables



Example 7

The heights and masses of ten students are given in the table below

Height (cm)	156	152	152	146	160	157	149	142	158	68
Mass (kg)	62	58	63	58	70	60	55	57	68	56

(a)(i) Plot the data on a scatter diagram

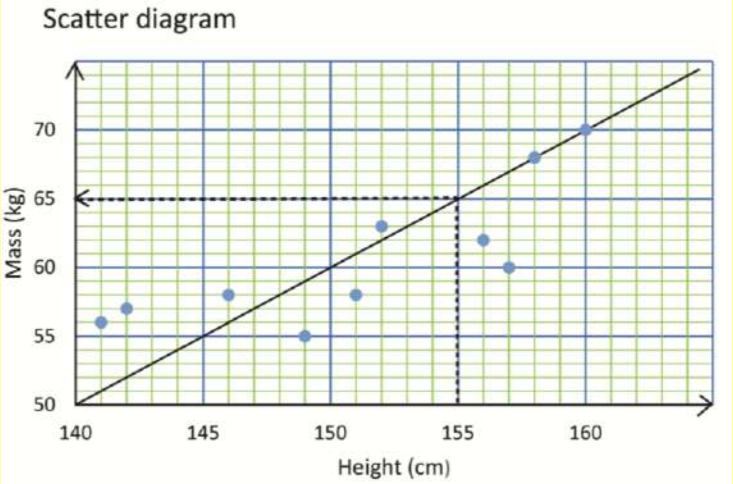
(ii) Draw the line of the best fit. Hence estimate the mass corresponding to height of 155cm

(b) (i) Calculate the rank correlation coefficient for the data.

(ii) Comment on the significance of the heights on the masses of the students.[Spearman's ρ = 0.79 and Kendall's τ = 0.64at 1% level of significance based on 10 observations]

Solution

(a)(i)



The weight corresponding to height 155cm is 65kg

Note: this value may vary from 63 to 67kg depending on how one has drawn the line of the best fit.

(b)(i) Using Spearman' ran correlation coefficient

Heights (x)	Mass (y)	Rx	Ry	d	d^2
156	62	4	4	0	0
151	58	6	6.5	0.5	0.25
152	63	5	3	2	4
146	58	8	6.5	1.5	2.25
160	70	1	1	0	0
157	60	3	5	-2	4
149	55	7	10	-3	9
142	57	9	8	1	1
158	68	2	2	0	0
141	56	10	9	1	1
					$\sum d^2 = 21.5$

ρ = 1- $\left[\frac{6\sum d^2}{n(n^2-1)}\right]$

= 1- $\left[\frac{6 \, x \, 21.5}{10(10^2-1)}\right]$ = 0.8697

Since $\rho_C(0.87) > \rho_T\ (0.79)$, a significant relationship exist between height and weight of students at 1% level

Using Kendall's method

By naming the pairs we have

A(156, 62), B(151, 58), C(152, 63), D(146, 58), E(160, 70), F(157, 60), G149, 55), H(142, 57), I(158, 68), J(141, 56)

	E	I	F	A	C	B	G	D	H	J
x	1	2	3	4	5	6	7	8	9	10
y	1	2	5	4	3	6.5	10	6.5	8	9
C	9	8	5	5	5	3	0	2	1	=38
D	0	0	2	1	0	0	3	0	0	=6

τ (tau) = $\frac{C-D}{C+D}$

τ (tau) = $\frac{38-6}{38+6}$ = 0.73

Since $\tau_C(0.73) > \tau_T\ (0.64)$, a significant relationship exist between the heights and masses of student.

Revision questions

1. UNEB 1990/212

   Eight candidate seeing admission to a University sat for written and oral test. The scores were shown below

Written (x)	55	54	35	62	87	53	71	50
Oral (y)	57	60	47	65	83	56	74	63

   (a) Plot the result on a scatter diagram. Comment on the relationship between the written test and oral test

   (b) Draw the line of the best fit on your graph and use it to estimate y when x = 70.

   (c) Calculate the rank correlation coefficient. Comment on your results

2. UNEB 1990/2/12

   The pairs of observation have been made on two random variables x and y. the ten (x, y) are (0, 20), (-7, 12), (-10, 15), (-12, 22), (-17, 5), (-30, -5), (-32, 13), (10, 30), (15, 40), and (-12, 8).

   (a) Draw the results on a scatter diagram

   (b) Draw the line of the best fit

(c) Estimate the expected value of y corresponding to x = -7

(d) Calculate the rank correlation coefficient and comment on the significance of the results at 1% significance level. (ρ = 0.894, τ= 0.778)

3. UNEB 1991/2/12

Three examiners X, Y and Z each marked scripts of ten candidates who sat for mathematics examination. The table below shows the examiner's ranking of candidates.

	A	B	C	D	E	F	G	H	I	J
X	8	5	9	2	10	1	7	6	3	4
Y	5	3	6	1	4	7	2	10	8	9
Z	6	3	7	2	5	4	1	10	9	8

Calculate the coefficient of rank correlation of the rankings

(i) X and Y

(ii) Y and Z

(iii) Comment on the significance of each at 5% significant level

4. UNEB 1992/2/13

Three weighing scales from three different shops W, X and Y in a market were used to weigh 10 bags of beans (A, B, C……) and the results in (kg) were given in the table below

	A	B	C	D	E	F	G	H	I	J
W	65	68	70	63	64	62	73	75	72	78
X	63	68	68	60	65	60	72	73	70	66
Y	63	74	78	75	64	73	79	70	67	79

Determine the rank correlation coefficient for the performance of the scales

(i) W and X

(ii) X and Y

(iii) Which of the three scales W, X and Y were in good working conditions

5. UNEB 1994/2/14

(a) In many government institution, officers complain about typing errors. A test was designed to investigate the relationship between typing speed and errors made. Twelve typist A, B, C …L were picked at random to type a text. The table below shows the rankings of the typist according to speed and errors made. (N.B lowest ranking in error implies the least errors

Typist	A	B	C	D	E	F	G	H	I	J	K	L
Speed	3	4	2	1	8	11	10	6	7	12	5	9
Errors	2	6	5	1	10	9	8	3	4	12	7	11

(i) Calculate the coefficient of rank correlation

(ii) Comment on the significance at 1% significance level

(b) The cost of travelling at a certain distance away from the city centre is found to depend on the route and distance a given place is away from the centre. The table below gives average rates of travel charged for distances to be travelled away from the city centre

Distance (s km)	9	12	14	21	24	30	33	45	46	50
Rate charged (r shs)	750	1000	1150	1200	1350	1250	1400	1750	1600	2000

(i) Plot the above data on a scatter diagram and draw a line of best fit through the points of the scatter diagram

(ii) Estimate the expected value r corresponding to s = 40km

6. UNEB/1995/2/13

In a certain commercial institution, a speed and error typing examination was administered to 12 randomly selected candidates A, B, C, …, L of the institution. The table below shows their speed (y) in seconds and the number of errors in their typed scripts (x)

	A	B	C	D	E	F	G	H	I	J	K	L
No. of errors (x)	12	24	20	10	32	30	28	15	18	40	27	35
Speed (y) in seconds	130	136	124	120	153	160	155	142	145	172	140	157

(i) Calculate the coefficient of rank correlation of the ranking

(ii) Comment on your results

(iii) Plot the above data on a scatter diagram and draw a line of the best fit through the points of the scatter diagram

7. UNEB 1996/2/16

The following table gives the marks obtained in calculus, physics and statistics by seven students

Calculus	72	50	60	55	35	48	82
Physics	61	55	70	50	30	50	73
Statistics	50	40	62	70	40	40	60

Determine the rank correlation coefficient for the performance of students in

(i) Calculus and physics

(ii) Calculus and statistics

8. UNEB/1999/2/8

Given the table below

x	80	75	86	60	75	92	86	50	64	75
y	62	58	60	45	68	68	81	48	50	70

Determine the rank correlation coefficient between the variable x and y, comment on your results

9. UNEB2003/2/15

The table below shows the percentage of sand y in the soils at different depth x (in cm)

Soil depth (x)(cm)	35	65	55	25	45	75	20	90	51	60
% of sand, y	86	70	84	92	79	68	96	58	86	77

(a) Plot the results on a scatter diagram. Comment on the relationship between the depth of the soil and the percentage of sand in the soil

(b) Draw the line of the best fit on you graph and use it to estimate

(i) The percentage of sand in the soil at a depth of 31cm

(ii) Depth of the soil with 54% sand

(iii) Calculate the rank correlation coefficient

10. UNEB 2004/2/7

Eight applicants for a certain job obtained the following marks in aptitude and written test

Applicants	A	B	C	D	E	F	G	H
Aptitude test	33	45	16	42	45	35	40	48
Written test	57	60	40	75	68	48	54	68

(i) Calculate the coefficient of rank correlation of applicant's performance in the two tests

(ii) Comment on your results

11. UNEB 2005/2/7

The table below shows the marks scored by students in mathematics and fine art tests

students	A	B	C	D	E	F	G	H	I	J
Mathematics,	40	48	79	26	55	35	37	70	60	40
Fine art	59	62	68	47	46	39	63	29	55	67

Calculate the coefficient of rank correlation for the students' performance in the two subjects and comment on your results.

12. UNEB 2007/2/12

Below are marks scored by 8 students A, B, C … H in Mathematics, Economics and geography in the end of term examination.

	A	B	C	D	E	F	G	H
Math	52	75	41	60	81	31	65	52
Economics	50	60	35	65	66	45	60	48
Geography	59	62	68	54	63	40	55	72

Determine the rank correlation coefficient for the performance of students in

(i) Math and economics

(ii) Geography and math

(iii) Comment on the significance of the math in performance of economics and geography. (ρ = 0.86, τ = 0.79 based on 8 observations at a 1% level of significance)

13. UNEB 2011/2/12

The heights and ages of ten students are given in the table below

Height, cm	156	151	152	146	160	157	149	142	158	140
Mass, kg	62	58	63	58	70	60	55	57	68	56

(a) Plot the data on a scatter diagram

(b) Draw the line of best fit on you graph and use it to estimate the mass corresponding to a height of 155cm

(c) Calculate the rank correlation coefficient for the data. Comment on the significance of the height on masses of students (ρ = 0.79, τ 0.64 based on 10 observations at 1% level of significance.

14. UNEB 2013/2/9

The heights and ages of ten farmers are given in the table below

Height, cm	156	151	152	160	146	157	149	142	158	140
Age, years	47	38	44	55	46	49	45	30	45	20

(a) Plot the data on a scatter diagram

(b) Draw the line of best fit on your diagram and use it to estimate

(i) Age when height = 147

(ii) Height when the age is 43

(c) Calculate the rank correlation coefficient for the data. Comment on your results (ρ=0.752, τ= 0.6)

15. UNEB 2015/2/12

The table gives the points awarded to eight schools by three judges, J_1, J_2 and J_3 during a music competition. J_1 was the chief judge.

J_1	72	50	50	55	35	38	82	72
J_2	60	55	70	50	50	50	73	70
J_3	50	40	62	70	40	48	67	67

(a) Determine the rank correlation coefficients between the judges of

(i) J_1 and J_2

(ii) J_1 and J_3

(b) Who of the two judges had a better correlation with the chief judge? Give a reason.

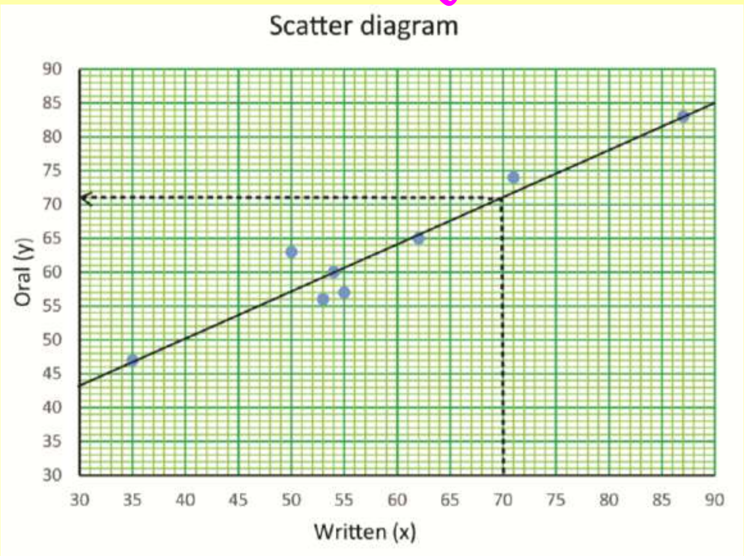
Answers to the revision exercise

1. UNEB 1990/212

Eight candidate seeing admission to a University sat for written and oral test. The scores were shown below

Written (x)	55	54	35	62	87	53	71	50
Oral (y)	57	60	47	65	83	56	74	63

(a) Plot the result on a scatter diagram. Comment on the relationship between the written test and oral test



(b) Draw the line of the best fit on your graph and use it to estimate y when x = 70. The value of y when x = 70 is 71

(c) Calculate the rank correlation coefficient. Comment on your results

Using Spearman's rank correlation method

x	y	Rx	Ry	d	d^2
55	57	4	6	-2	4
54	60	5	5	0	0
35	47	8	8	0	0
62	65	3	3	0	0
87	83	1	1	0	0
53	56	6	7	-1	1
71	74	2	2	0	0
50	63	7	4	3	9
					$\sum d^2 = 14$

$\rho = 1 - \left[\frac{6\sum d^2}{n(n^2-1)}\right]$

$= 1 - \left[\frac{6 \times 14}{8(8^2-1)}\right] = 0.833$

There is very high correlation between x and y

Using Kendall' Rank correlation coefficient method

Rx	Ry	C	D
1	1	7	0
2	2	6	0
3	3	5	0
4	6	3	1
5	5	2	1
6	7	1	1
7	4	1	0
8	8	=25	=3

$\tau = \frac{C-D}{C+D} = \frac{25-3}{25+3} = 0.786$

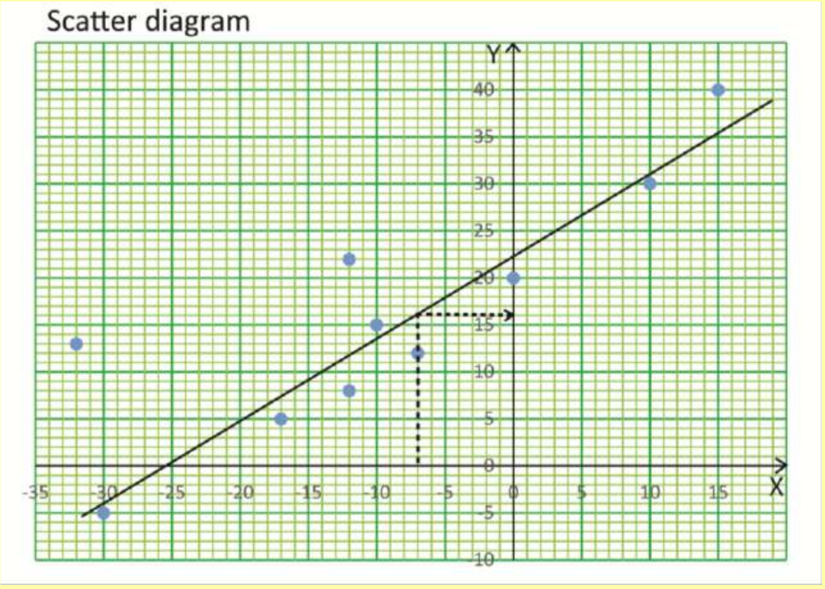
there is high correlation between x and y

2. UNEB 1990/2/12

The pairs of observation have been made on two random variables x and y. the ten (x, y) are (0, 20), (-7, 12), (-10, 15), (-12, 22), (-17, 5), (-30, -5), (-32, 13), (10, 30), (15, 40), and (-12, 8).

(a) Draw the results on a scatter diagram

(b) Draw the line of the best fit



(c) Estimate the expected value of y corresponding to x = -7

The value of y corresponding to x = -7 is 15.7

(d) Calculate the rank correlation coefficient and comment on the significance of the results at 1% significance level. (ρ = 0.894, τ= 0.778)

Using Spearman's rank correlation coefficient

X	Y	Rx	Ry	d	d^2
0	20	3	4	-1	1
-7	12	4	7	-3	9
-10	15	5	5	0	0
-12	22	6.5	3	3.5	12.25
-17	5	8	9	-1	1
-30	-5	9	10	-1	1
-32	13	10	6	4	16
10	30	2	2	0	0
15	40	1	1	0	0
-12	8	6.5	8	-1.5	2.25
					$\sum d^2$ =42.5

$\rho = 1- \left[\frac{6\sum d^2}{n(n^2-1)}\right]$

$= 1- \left[\frac{6 \: x \: 42.5}{10(10^2-1)}\right] = 0.742$

Since ρ_C (0.742)<ρ_T (0.894)

There is no significant relationship between X and Y at 1% significance level

Using Kendall's rank correlation coefficient

Rx	Ry	C	D
1	1	9	0
2	2	8	0
3	4	6	1
4	7	3	3
5	5	4	1
6.5	3	4	0
6.5	8	2	1
8	9	1	1
9	10	0	1
10	6	=37	=8

$\tau = \frac{C-D}{C+D} = \frac{37-8}{37+8} = 0.644$

Since τ_C (0.644)<τ_T (0.778)

There is no significant relationship between X and Y at 1% significance level.

3. UNEB 1991/2/12

Three examiners X, Y and Z each marked scripts of ten candidates who sat for mathematics examination. The table below shows the examiner's ranking of candidates.

	A	B	C	D	E	F	G	H	I	J
X	8	5	9	2	10	1	7	6	3	4
Y	5	3	6	1	4	7	2	10	8	9
Z	6	3	7	2	5	4	1	10	9	8

Calculate the coefficient of rank correlation of the rankings

(i) X and Y

Spearman’s rank correlation

Rx	Ry	d	d^2
8	5	3	9
5	3	2	4
9	6	3	9
2	1	1	1
10	4	6	36
1	7	-6	36
7	2	5	25
6	10	-4	16
3	8	-5	25
4	9	-5	25
			$\sum d^2 = 186$

$\rho = 1 - \left[\frac{6\sum d^2}{n(n^2-1)}\right]$

$= 1 - \left[\frac{6 \times 186}{10(10^2-1)}\right] = -0.127$

There is low negative relationship between X and Y

Kendall’s rank correlation coefficient

Rx	Ry	C	D
1	7	3	6
2	1	8	0
3	8	2	5
4	9	1	5
5	3	4	1
6	10	0	4
7	2	3	0
8	5	1	1
9	6	0	1
10	4	=22	=23

$\tau = \frac{C-D}{C+D} = \frac{22-23}{22+23} = -0.022$

There is very low negative relationship between X and Y at 1% significance level.

(ii) Y and Z

Spearman's rank correlation coefficient

Ry	Rz	d	d^2
5	6	-1	1
3	3	0	0
6	7	-1	1
1	2	-1	1
4	5	-1	1
7	4	3	9
2	1	1	1
10	10	0	0
8	9	-1	1
9	8	1	1
			$\sum d^2 = 16$

$\rho = 1 - \left[\frac{6\sum d^2}{n(n^2-1)}\right]$

$= 1 - \left[\frac{6 \times 16}{10(10^2-1)}\right] = 0.903$

There is high positive relationship between Y and Z

Kendall's rank correlation coefficient

Rz	Ry	C	D
1	2	9	1
2	1	8	0
3	3	7	0
4	7	3	3
5	4	5	0
6	5	4	0
7	6	3	0
8	9	1	1
9	8	1	0
10	10	41	5

τ = $\frac{C-D}{C+D} = \frac{41-5}{41+5}$ = 0.7826

There is very high relationship between X and Y

(iii) Comment on the significance of each at 5% significant level

(From the tables of critical values at 5% level of significance based on 10 observations, ρ = 0.648 , τ = 0.467)

(i) For X and Y since$|\rho_C(-0.127)| < |\rho_T(0.648)|$ and $|\tau_C(-0.022)| < |\tau\rho_T(0.467)|$; there is no significant relationship between X and Y at 5% significance level.

(ii) For Y and Z since$|\rho_C(0.903)| > |\rho_T(0.648)|$ and $|\tau_C(0.7826)| > |\tau\rho_T(0.467)|$; there is significant relationship between Y and Z at 5% significance level.

4. UNEB 1992/2/13

Three weighing scales from three different shops W, X and Y in a market were used to weigh 10 bags of beans (A, B, C……) and the results in (kg) were given in the table below

	A	B	C	D	E	F	G	H	I	J
W	65	68	70	63	64	62	73	75	72	78
X	63	68	68	60	65	60	72	73	70	66
Y	63	74	78	75	64	73	79	70	67	79

Determine the rank correlation coefficient for the performance of the scales

(i) W and X (ρ = 0.8)

Spearman's rank correlation

W	X	RW	RX	d	d^2
65	63	7	8	-1	1
68	68	6	4.5	1.5	2.25
70	68	5	4.5	0.5	0.25
63	60	9	9.5	-0.5	0.25
64	65	8	7	1	1
62	60	10	9.5	0.5	0.25
73	72	3	2	1	1
75	73	2	1	1	1
72	70	4	3	1	1
78	66	1	6	-5	25
					$\sum d^2$ =33

ρ = 1- $\left[\frac{6\sum d^2}{n(n^2-1)}\right]$

= 1- $\left[\frac{6 \times 33}{10(10^2-1)}\right]$ = 0.8

There is high positive relationship between Y and Z

Kendall's correlation coefficient

Rw	Rx	C	D
1	6	4	5
2	1	8	0
3	2	7	0
4	3	6	0
5	4.5	4	0
6	4.5	4	0
7	8	2	1
8	7	2	0
9	9.5	0	0
10	9.5	=37	=6

τ = $\frac{C-D}{C+D}$ =$\frac{37-6}{37+6}$ = 0.721

There is high relationship positive between X and Y

(ii) X and Y (ρ = 0.185)

Spearman's correlation coefficient

X	Y	Rx	Ry	d	d²
63	63	8	10	-2	4
68	74	4.5	4	0.5	0.25
68	78	4.5	3	1.5	2.25
60	75	9.5	4	5.5	30.25
65	64	7	9	-2	4
60	73	9.5	6	3.5	12.25
72	79	2	1.5	0.5	0.25
73	70	1	7	-6	36
70	67	3	8	-5	25
66	79	6	1.5	4.5	20.25
					$\sum d^2$ =134.5

ρ = 1- $\left[\frac{6\sum d^2}{n(n^2-1)}\right]$

= 1- $\left[\frac{6\ x\ 134.5}{10(10^2-1)}\right]$ = 0.185

There is very low positive relationship between Y and Z

Kendall's rank coefficient

Rx	Ry	C	D
1	7	3	6
2	1.5	7	0
3	8	2	5
4.5	4	3	2
4.5	3	4	1
6	1.5	4	0
7	9	1	2
8	10	0	2
9.5	4	1	0
9.5	6	=25	=18

τ = $\frac{C-D}{C+D}$ =$\frac{25-18}{25+18}$ = 0.163

There is very low relationship positive between X and Y

(iii) Which of the three scales W, X and Y were in good working conditions

W and X are in good working conditions because the show high positive correlation

5. UNEB 1994/2/14

(a) In many government institution, officers complain about typing errors. A test was designed to investigate the relationship between typing speed and errors made. Twelve typist A, B, C …L were picked at random to type a text. The table below shows the rankings of the typist according to speed and errors made. (N.B lowest ranking in error implies the least errors

Typist	A	B	C	D	E	F	G	H	I	J	K	L
Speed	3	4	2	1	8	11	10	6	7	12	5	9
Errors	2	6	5	1	10	9	8	3	4	12	7	11

(i) Calculate the coefficient of rank correlation

Spearman's rank coefficient

Rs	Re	d	d^2
3	2	1	1
4	6	-2	4
2	5	-3	9
1	1	0	0
8	10	-2	4
11	9	2	4
10	8	2	4
6	3	3	9
7	4	3	9
12	12	0	0
5	7	-2	4
9	11	-2	4
			$\sum d^2$ =52

$\rho = 1- \left[\frac{6\sum d^2}{n(n^2-1)}\right]$

$= 1- \left[\frac{6 \ x \ 52}{12(12^2-1)}\right] = 0.8182$

There is very low positive relationship between Y and Z

Kendall's rank correlation coefficient

Rs	Re	C	D
1	1	11	0
2	5	7	3
3	2	9	0
4	6	6	2
5	7	5	2
6	3	6	0
7	4	5	0
8	10	2	2
9	11	1	2
10	8	2	0
11	9	1	0
12	12	55	11

$\tau = \frac{C-D}{C+D} = \frac{55-11}{55+11} = 0.667$

There is very low relationship positive between X and Y

(ii) Comment on the significance at 1% significance level

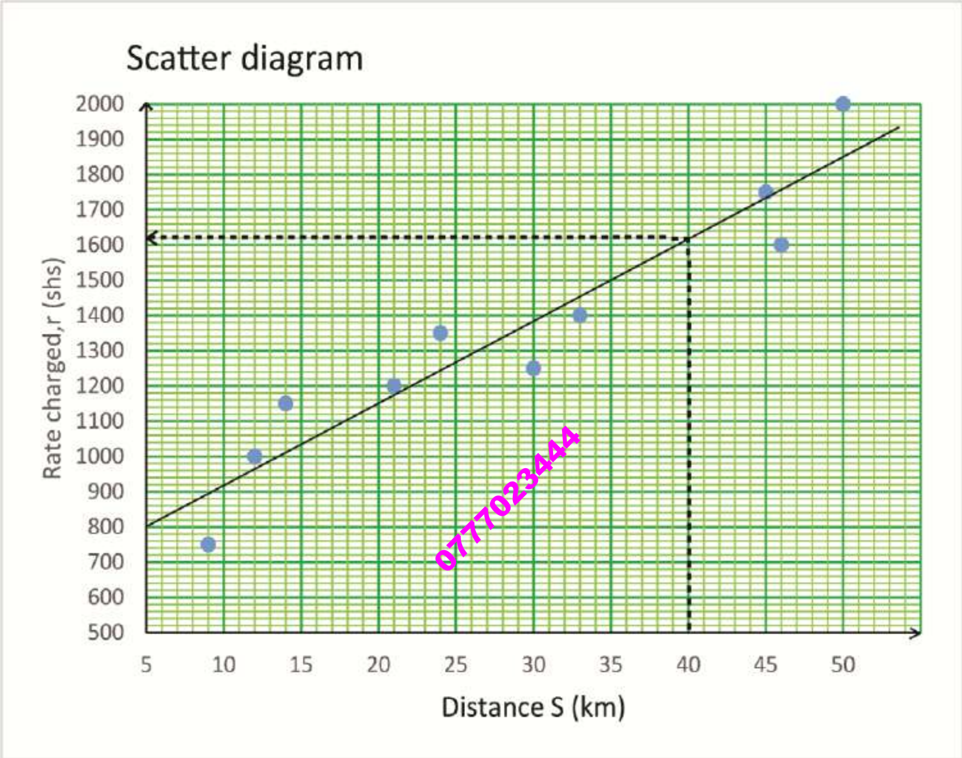
(From the tables of critical values at 1% level of significance based on 12 observations, ρ = 0.727 , τ = 0.545)

Since$|\rho_C(0.8182)| > |\rho_T(0.727)|$ and $|\tau_C(0.667)| > |\tau\rho_T(0.545)|$; there is significant relationship between speed and errors at 1% significance level.

(b) The cost of travelling at a certain distance away from the city centre is found to depend on the route and distance a given place is away from the centre. The table below gives average rates of travel charged for distances to be travelled away from the city centre

Distance (s km)	9	12	14	21	24	30	33	45	46	50
Rate charged (r shs)	750	1000	1150	1200	1350	1250	1400	1750	1600	2000

(i) Plot the above data on a scatter diagram and draw a line of best fit through the points of the scatter diagram



(ii) Estimate the expected value r corresponding to s = 40km

Value of r corresponding to s = 40 is shs 1610

6. UNEB/1995/2/13

In a certain commercial institution, a speed and error typing examination was administered to 12 randomly selected candidates A, B, C, …, L of the institution. The table below shows their speed (y) in seconds and the number of errors in their typed scripts (x)

	A	B	C	D	E	F	G	H	I	J	K	L
No. of errors (x)	12	24	20	10	32	30	28	15	18	40	27	35
Speed (y) in seconds	130	136	124	120	153	160	155	142	145	172	140	157

(i) Calculate the coefficient of rank correlation of the ranking

Spearman's rank correlation

Errors	speed	Re	Rs	d	d^2
12	130	11	10	1	1
24	136	7	9	-2	4
20	124	8	11	-3	9
10	120	12	12	0	0
32	153	3	5	-2	4
30	160	4	2	2	4
28	155	5	4	1	1
15	142	10	7	3	9
18	145	9	6	3	9
40	172	1	1	0	0
27	140	6	8	-2	4
35	157	2	3	-1	1
					46

ρ = 1- $\left[\frac{6\sum d^2}{n(n^2-1)}\right]$

= 1- $\left[\frac{6 \ x\ 46}{12(12^2-1)}\right]$ = 0.839

There is high positive relationship between speed and errors made

Kendall's rank correlation

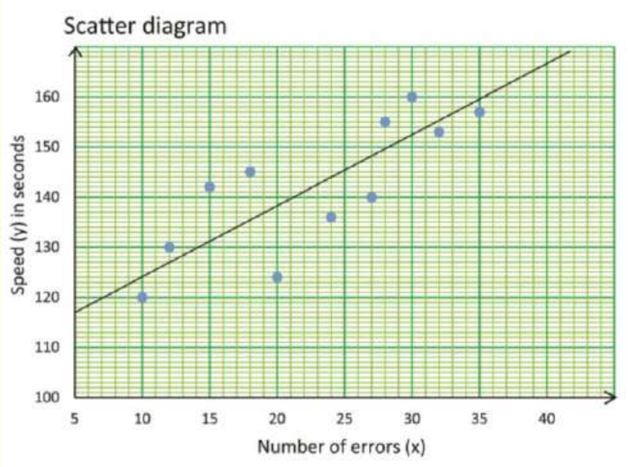
Re	Rs	C	D
1	1	11	0
2	3	9	1
3	5	7	2
4	2	8	0
5	4	7	0
6	8	4	2
7	9	3	2
8	11	2	2
9	6	3	0
10	7	2	0
11	10	1	0
12	12	57	9

$\tau = \frac{C-D}{C+D} = \frac{57-9}{57+9}$ = 0.72

There is high positive relationship between speed and errors made

(ii) Comment on your results

(iii) Plot the above data on a scatter diagram and draw a line of the best fit through the points of the scatter diagram



7. UNEB 1996/2/16

The following table gives the marks obtained in calculus, physics and statistics by seven students

Calculus	72	50	60	55	35	48	82
Physics	61	55	70	50	30	50	73
Statistics	50	40	62	70	40	40	60

Determine the rank correlation coefficient for the performance of students in

(i) Calculus and physics

Spearman's correlation coefficient

C	P	Rc	Rp	d	d^2
72	61	2	3	-1	1
50	55	5	4	1	1
60	70	3	2	1	1
55	50	4	5.5	-1.5	2.25
35	30	7	7	0	0
48	50	6	5.5	0.5	0.25
82	73	1	1	0	0
					$\sum d^2 = 5.5$

$\rho = 1 - \left[\frac{6\sum d^2}{n(n^2-1)}\right]$

$= 1 - \left[\frac{6 \times 5.5}{7(7^2-1)}\right] = 0.902$

There is high positive relationship between calculus and physics

Kendall' correlation coefficient

Rc	Rp	C	D
1	1	6	0
2	3	4	1
3	2	4	0
4	5.5	1	1
5	4	2	0
6	5.5	1	0
7	7	=18	=2

$\tau = \frac{C-D}{C+D} = \frac{18-2}{18+2} = 0.8$

There is high positive relationship between calculus and physics

(ii) Calculus and statistics (ρ = 0.64)

Spearman's correlation

C	S	Rc	Rs	d	d^2
72	50	2	4	-2	4
50	40	5	6	-1	1
60	62	3	2	1	1
55	70	4	1	3	9
35	40	7	6	1	1
48	40	6	6	0	0
82	60	1	3	-2	4
					$\sum d^2 = 20$

$\rho = 1 - \left[\frac{6\sum d^2}{n(n^2-1)}\right]$

$= 1 - \left[\frac{6 \times 20}{7(7^2-1)}\right] = 0.643$

Kendall's rank correlation coefficient

Rc	Rs	C	D
1	3	4	2
2	4	3	2
3	2	3	1
4	1	3	0
5	6	0	0
6	6	0	0
7	6	=13	=5

$\tau = \frac{C-D}{C+D} = \frac{13-5}{13+5} = 0.444$

8. UNEB/1999/2/8

Given the table below

x	80	75	86	60	75	92	86	50	64	75
y	62	58	60	45	68	68	81	48	50	70

Determine the rank correlation coefficient between the variable x and y, comment on your results (ρ = 0.715)

Spearman's correlation

x	y	Rx	Ry	d	d^2
80	62	4	5	-1	1
75	58	6	7	-1	1
86	60	2.5	6	-3.5	12.25
60	45	9	10	-1	1
75	68	6	3.5	2.5	6.25
92	68	1	3.5	-2.5	6.25
86	81	2.5	1	1.5	2.25
50	48	10	9	1	1
64	50	8	8	0	0
75	70	6	2	4	16
					47

$\rho = 1 - \left[\frac{6\sum d^2}{n(n^2-1)}\right]$

$= 1 - \left[\frac{6 \times 47}{10(10^2-1)}\right] = 0.715$

Kendall's correlation coefficient

Rx	Ry	C	D
1	3.5	6	2
2.5	6	4	4
2.5	1	7	0
4	5	4	2
6	7	3	2
6	3.5	3	2
6	2	3	0
8	8	2	0
9	10	0	1
10	9	=32	=13

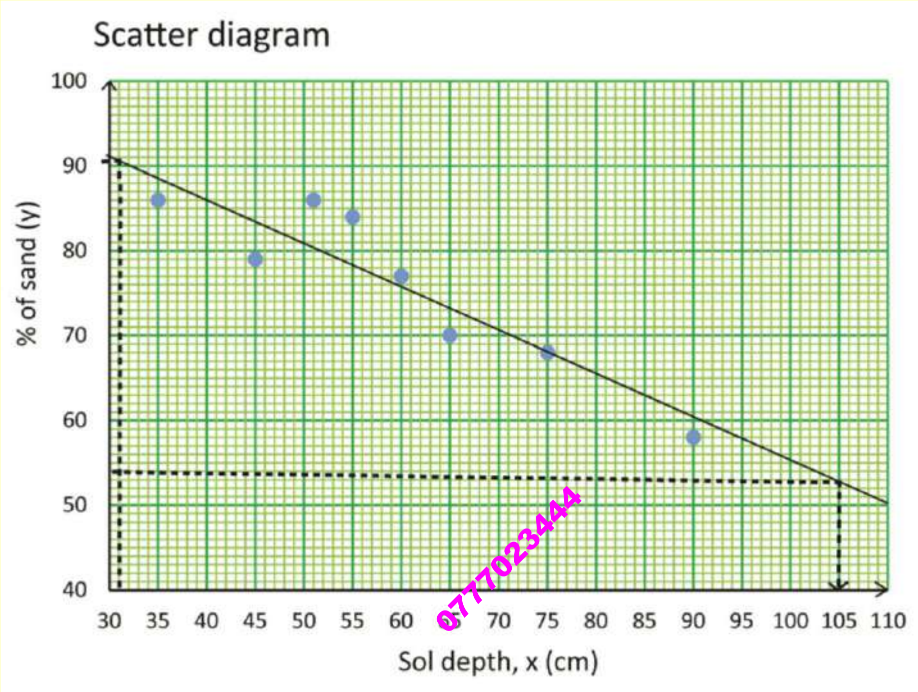
$\tau = \frac{C-D}{C+D} = \frac{32-13}{32+13} = 0.422$

9. UNEB2003/2/15

The table below shows the percentage of sand y in the soils at different depth x (in cm)

Soil depth (x)(cm)	35	65	55	25	45	75	20	90	51	60
% of sand, y	86	70	84	92	79	68	96	58	86	77

(a) Plot the results on a scatter diagram. Comment on the relationship between the depth of the soil and the percentage of sand in the soil



(b) Draw the line of the best fit on you graph and use it to estimate
 (i) The percentage of sand in the soil at a depth of 31cm (91%)
 (ii) Depth of the soil with 54% sand (105cm)
 (iii) Calculate the rank correlation coefficient
 Spearman's rank correlation coefficient

soil depth (x)	%of sand (y)	Rx	Ry	d	d^2
35	86	8	3.5	4.5	20.25
65	70	3	8	-5	25
55	84	5	5	0	0
25	92	9	2	7	49
45	79	7	6	1	1
75	68	2	9	-7	49
20	96	10	1	9	81
90	58	1	10	-9	81
51	86	6	3.5	2.5	6.25
60	77	4	7	-3	9
					$\sum d^2 = 321.5$

$\rho = 1 - \left[\frac{6\sum d^2}{n(n^2-1)}\right]$

$= 1 - \left[\frac{6 \times 321.5}{10(10^2-1)}\right] = -0.948$

Kendall's correlation coefficient

Rx	Ry	C	D
1	10	0	9
2	9	0	8
3	8	0	7
4	7	0	6
5	5	1	4
6	3.5	1	2
7	6	0	3
8	3.5	0	2
9	2	0	1
10	1	2	42

τ = $\frac{C-D}{C+D} = \frac{2-42}{2+42} =$ -0.91

10. UNEB 2004/2/7

Eight applicants for a certain job obtained the following marks in aptitude and written test

Applicants	A	B	C	D	E	F	G	H
Aptitude test	33	45	16	42	45	35	40	48
Written test	57	60	40	75	68	48	54	68

(i) Calculate the coefficient of rank correlation of applicant's performance in the two tests

Aptitude (A)	Written (W)	R_A	R_W	d	d^2
33	57	7	5	2	4
45	60	2	4	-2	4
16	40	8	8	0	0
42	75	4	1	3	9
45	68	3	2.5	0.5	0.25
35	48	6	7	-1	1
40	54	5	6	-1	1
48	68	1	2.5	-1.5	2.25
					21.5

ρ = 1- $\left[\frac{6\sum d^2}{n(n^2-1)}\right]$

= 1- $\left[\frac{6\ x\ 21.5}{8(8^2-1)}\right]$ = 0.744

(ii) There is high positive relationship between Aptitude and written

Kendall's correlation coefficient

Rx	Ry	C	D
1	2.5	5	1
2	4	4	2
3	2.5	4	1
4	1	4	0
5	6	2	1
6	7	1	1
7	5	1	0
8	8	=21	=6

τ = $\frac{C-D}{C+D} = \frac{21-6}{21+6}$ = 0.555

(ii) There is moderate positive relationship between Aptitude and written

(ii) Comment on your results

11. UNEB 2005/2/7

The table below shows the marks scored by students in mathematics and fine art tests

students	A	B	C	D	E	F	G	H	I	J
Mathematics,	40	48	79	26	55	35	37	70	60	40
Fine art	59	62	68	47	46	39	63	29	55	67

Calculate the coefficient of rank correlation for the students' performance in the two subjects and comment on your results.

Math	Fine Art	R_M	R_F	d	d^2
40	59	6.5	5	1.5	2.25
48	62	5	4	1	1
79	68	1	1	0	0
26	47	10	7	3	9
55	46	4	8	-4	16
35	39	9	9	0	0
37	63	8	3	5	25
70	29	2	10	-8	64
60	55	3	6	-3	9
40	67	6.5	2	4.5	20.25
					$\sum d^2$ =146.5

ρ = 1- $\left[\frac{6\sum d^2}{n(n^2-1)}\right]$

= 1- $\left[\frac{6\ x\ 146.5}{10(10^2-1)}\right]$

= 0.112

Kendall's correlation coefficient

RM	RF	C	D
1	1	9	0
2	10	0	8
3	6	3	4
4	8	1	5
5	4	3	2
6.5	5	2	2
6.5	2	3	0
8	3	2	0
9	9	0	1
10	7	23	22

τ = $\frac{C-D}{C+D}$ = $\frac{23-22}{23+22}$ = 0.022

12. UNEB 2007/2/12

Below are marks scored by 8 students A, B, C … H in Mathematics, Economics and geography in the end of term examination.

	A	B	C	D	E	F	G	H
Math	52	75	41	60	81	31	65	52
Economics	50	60	35	65	66	45	69	48
Geography	35	40	60	54	63	40	65	72

Determine the rank correlation coefficient for the performance of students in

(i) Math and economics

Spearman's correlation coefficient

Math, M	Economic C	R_M	R_E	d	d^2
52	50	5.5	5	0.5	0.25
75	60	2	4	-2	4
41	35	7	8	-1	1
60	65	4	3	1	1
81	66	1	2	-1	1
31	45	8	7	1	1
65	69	3	1	2	4
52	48	5.5	6	-0.5	0.25
					$\sum d^2 = 12.5$

$\rho = 1- \left[\frac{6\sum d^2}{n(n^2-1)}\right]$

$= 1- \left[\frac{6 \ x \ 12.5}{10(10^2-1)}\right]$

$= 0.851$

Kendall's correlation coefficient

RM	RE	C	D
1	2	6	1
2	4	4	2
3	1	5	0
4	3	4	0
5.5	5	3	0
5.5	6	2	0
7	8	1	1
8	7	=25	=4

$\tau = \frac{C-D}{C+D} = \frac{25-4}{25+4} = 0.724$

(ii) Geography and math

Spearman's rank correlation coefficient

Geog, G	Math, M	R_G	R_M	d	d^2
35	52	8	5.5	2.5	6.25
40	75	6.5	2	4.5	20.25
60	41	4	7	-3	9
54	60	5	4	1	1
63	81	3	1	2	4
40	31	6.5	8	-1.5	2.25
65	65	2	3	-1	1
72	52	1	5.5	-4.5	20.25
					64

$\rho = 1- \left[\frac{6\sum d^2}{n(n^2-1)}\right]$

$= 1- \left[\frac{6 \ x \ 64}{8(8^2-1)}\right]$

$= 0.238$

Kendall's rank coefficient

RG	RM	C	D
1	5.5	2	4
2	3	4	2
3	1	5	0
4	7	1	3
5	4	2	1
6.5	2	2	0
6.5	8	0	1
8	5.5	16	11

$\tau = \frac{C-D}{C+D} = \frac{16-11}{16+11} = 0.185$

(iii) Comment on the significance of the math in performance of economics and geography. (ρ = 0.86, τ = 0.79 based on 8 observations at a 1% level of significance)

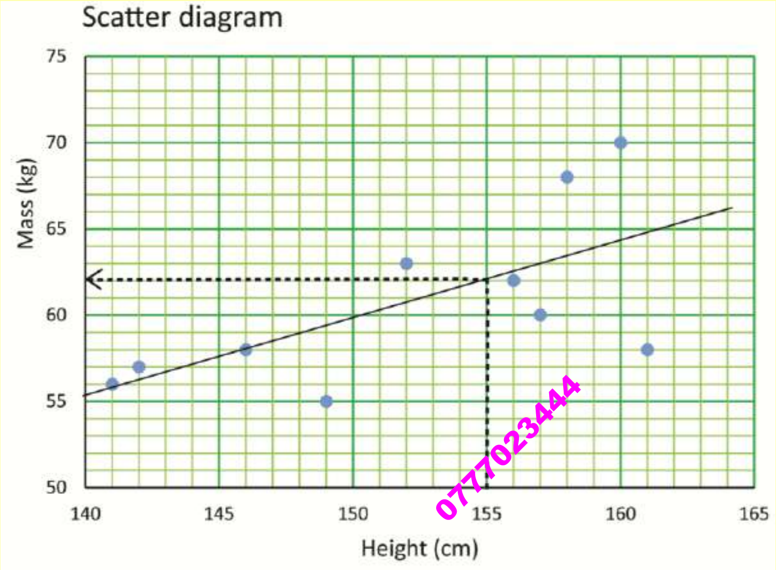
Since$|\rho_C(0.851)| < |\rho_T(0.86)|$ and $|\tau_C(0.724)| < |\tau\rho_T(0.79)|$; there is no significant relationship between math and economics at 1% significance level.

13. UNEB 2011/2/12

The heights and ages of ten students are given in the table below

Height, cm	156	151	152	146	160	157	149	142	158	140
Mass, kg	62	58	63	58	70	60	55	57	68	56

(a) Plot the data on a scatter diagram



(b) Draw the line of best fit on you graph and use it to estimate the mass corresponding to a height of 155cm(63kg)

(c) Calculate the rank correlation coefficient for the data. Comment on the significance of the height on masses of students (ρ = 0.79, τ =0.64 based on 10 observations at 1% level of significance.)

Method I: Using Spearman's rank correlation coefficient

Height (x)	Mass (y)	Rx	Ry	Rx – Ry = d	d^2
156	62	4	4	0	0
151	58	6	6.5	-0.5	0.25
152	63	5	3	2	4
146	58	8	6.5	1.5	2.25
160	70	1	1	0	0
157	60	3	5	-2	4
149	55	7	10	-3	9
142	57	9	8	1	1
158	68	2	2	0	0
140	56	10	9	1	1
					$\sum d^2 = 21.5$

$\rho = 1 - \left[\frac{6\sum d^2}{n(n^2-1)}\right] = 1 - \left[\frac{6 \, x \, 21.5}{10(10^2-1)}\right] = 0.87$

Since $|\rho_C(0.0.87)| > |\rho_T(0.79)|$ there is no significant relationship between height and age at 1% significance level.

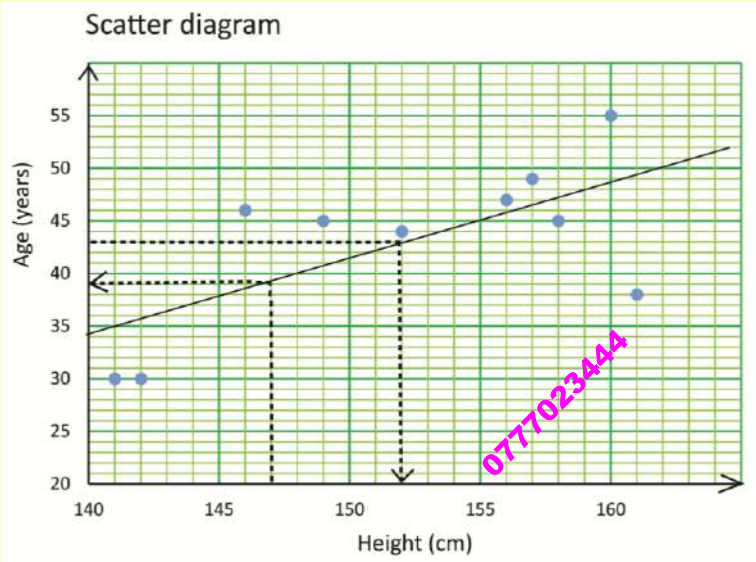
Since $|\tau_C(0.6\,)| < |\tau\rho_T(0.0.64\,)|$;

14. UNEB 2013/2/9

The heights and ages of ten farmers are given in the table below

Height, cm	156	151	152	160	146	157	149	142	158	140
Age, years	47	38	44	55	46	49	45	30	45	20

(a) Plot the data on a scatter diagram



(b) Draw the line of best fit on your diagram and use it to estimate

(i) Age when height = 147 (39)

(ii) Height when the age is 43 (152)

(c) Calculate the rank correlation coefficient for the data. Comment on your results

Method I: Using Spearman's rank correlation coefficient

Height (x)	Age (y)	Rx	Ry	Rx – Ry = d	d^2
156	47	4	3	1	1
151	38	6	8	-2	4
152	44	5	7	-2	4
160	55	1	1	0	0
146	46	8	4	4	16
157	49	3	2	1	1
149	45	7	5.5	1.5	2.25
142	30	9	9.5	-0.5	0.25
158	45	2	5.5	-3.5	12.25
140	30	10	9.5	0.5	0.25
					$\sum d^2 = 41$

p =1 - $\frac{6\sum d^2}{n(n^2-1)} = 1 - \frac{6\,x\,41}{10(10^2-1)} = 0.7515(4D)$

Method II: using Kendall's rank correlation coefficient

Let the farmers be A, B, C, D, E, F, G, H, I, J

Farmers	A	B	C	D	E	F	G	H	I	J
Height	156	151	152	160	146	157	149	142	158	140
Age	47	38	44	55	46	49	45	30	45	30

By re-arranging the findings we have

Farmers	D	I	F	A	C	B	G	E	H	J
Height	1	2	3	4	5	6	7	8	9	10
Age	1	5.5	3	3	7	8	5.5	4	9.5	9.5
agreements	9	4	7	6	3	2	2	2	0	=35
Disagreements	0	3	0	0	2	2	1	0	0	=8

s = total agreements – total disagreements

= 35 – 8 = 27

$\tau = \frac{2s}{n(n-1)} = \frac{2\,x\,27}{10(10-1)} = \frac{54}{90} = 0.6$ Or τ = $\frac{35-8}{35+8} = 0.63$

Comment

Since$|\rho_C(0.7515)| < |\rho_T(0.79)|$ and $|\tau_C(0.6\ )| < |\tau\rho_T(0.0.64\ )|$; there is no significant relationship between height and age at 1% significance level.

Kendall's rank correlation coefficient

By naming the pairs we have

A(156, 62), B(151, 58), C(152, 63), D(146, 58), E(160, 70), F(157, 60), G149, 55), H(142, 57), I(158, 68), J(141, 56)

	E	I	F	A	C	B	G	D	H	J
x	1	2	3	4	5	6	7	8	9	10
y	1	2	5	4	3	6.5	10	6.5	8	9
C	9	8	5	5	5	3	0	2	1	=38
D	0	0	2	1	0	0	3	0	0	=6

τ (tau) = $\frac{C-D}{C+D}$

τ (tau) = $\frac{38-6}{38+6}$ = 0.73

Since $\tau_C(0.73) > \tau_T\ (0.64)$, a significant relationship exist between the heights and masses of student.

15. UNEB 2015/2/12

The table gives the points awarded to eight schools by three judges, J_1, J_2 and J_3 during a music competition. J_1 was the chief judge.

J_1	72	50	50	55	35	38	82	72
J_2	60	55	70	50	50	50	73	70
J_3	50	40	62	70	40	48	67	67

(a) Determine the rank correlation coefficients between the judges of

(i) J_1 and J_2
(ii) J_1 and J_3

(b) Who of the two judges had a better correlation with the chief judge? Give a reason.

Solution

(i) J_1 and J_2

J_1	J_2	R_{J_1}	R_{J_2}	D_1	D_1^2
72	60	2.5	4	-1.5	2.25
50	55	5.5	5	0.5	0.25
50	70	5.5	2.5	3	9
55	50	4	7	-3	9
35	50	8	7	1	1
38	50	7	7	0	0
82	73	1	1	0	0
72	70	2.5	2.5	0	0
					$\sum D_1^2$ = 21.5

$$\rho_1 = 1 - \frac{6\sum D_1^2}{n(n^2-1)}$$

$$= 1 - \frac{6 \ x \ 21.5}{8(8^2-1)}$$

$$= \frac{125}{168} = 0.7440$$

(ii) J_1 and J_3 (10marks)

J_1	J_3	R_{J_1}	R_{J_3}	D_2	D_2^2
72	50	2.5	5	-2.5	6.25
50	40	5.5	7.5	-4	4
50	62	5.5	4	0.5	2.25
55	70	4	1	3	9
35	40	8	7.5	0.5	0.25
38	48	7	6	1	1
82	67	1	2.5	-1.5	2.25
72	67	2.5	2.5	0	0
					$\sum D_1^2$ = 25

$$\rho_2 = 1 - \frac{6\sum D_2^2}{n(n^2-1)}$$

$$= 1 - \frac{6 \ x \ 25}{8(8^2-1)}$$

$$= 0.7023$$

(a) Who of the two other judges had a better correlation with the chief judge? Give a reason. (02marks)

J_2 has a better correlation with the Chief Judge because the coefficient of correlation is smaller showing a stronger mutual relationship

Upper Critical Values for Kendall's Rank Correlation Coefficient $\hat{\tau}$

Note: In the table below, the critical values give significance levels as close as possible to but not exceeding the nominal α.

	Nominal α					
n	0.10	0.05	0.025	0.01	0.005	0.001
4	1.000	1.000	-	-	-	-
5	0.800	0.800	1.000	1.000	-	-
6	0.600	0.733	0.867	0.867	1.000	-
7	0.524	0.619	0.714	0.810	0.905	1.000
8	0.429	0.571	0.643	0.714	0.786	0.857
9	0.389	0.500	0.556	0.667	0.722	0.833
10	0.378	0.467	0.511	0.600	0.644	0.778
11	0.345	0.418	0.491	0.564	0.600	0.709
12	0.303	0.394	0.455	0.545	0.576	0.667
13	0.308	0.359	0.436	0.513	0.564	0.641
14	0.275	0.363	0.407	0.473	0.516	0.604
15	0.276	0.333	0.390	0.467	0.505	0.581
16	0.250	0.317	0.383	0.433	0.483	0.567
17	0.250	0.309	0.368	0.426	0.471	0.544
18	0.242	0.294	0.346	0.412	0.451	0.529
19	0.228	0.287	0.333	0.392	0.439	0.509
20	0.221	0.274	0.326	0.379	0.421	0.495
21	0.210	0.267	0.314	0.371	0.410	0.486
22	0.203	0.264	0.307	0.359	0.394	0.472
23	0.202	0.257	0.296	0.352	0.391	0.455
24	0.196	0.246	0.290	0.341	0.377	0.449
25	0.193	0.240	0.287	0.333	0.367	0.440
26	0.188	0.237	0.280	0.329	0.360	0.428
27	0.179	0.231	0.271	0.322	0.356	0.419
28	0.180	0.228	0.265	0.312	0.344	0.413
29	0.172	0.222	0.261	0.310	0.340	0.404

8.INDEX NUMBERS

Index numbers

This the percentage ratio of one quantity to the other, e.g. price index

Index number is a technique of measuring changes in a variable or group of variables with respect to time, geographical location or other characteristics. It is a statistical measure of change in a representative group of individual data points.

For example, if the price of a certain commodity rises from shs. 100 in the year 2007 to shs. 150 in the year 2017, the price index number will be 150 showing that there is a 50% increase in the prices over this period

Simple index numbers

The simple index numbers include;

(i) Price index or price relative

Price relative = $\frac{P_1}{P_o}\ x\ 100$ where P_1= price the current year and P_0 – price in the base year

(ii) Wage index

Wage index = $\frac{W_1}{W_o}\ x\ 100$

(iii) Quantity index (quantum) index

Quantum index = $\frac{Q_1}{Q_o}\ x\ 100$

Example 1

A loaf of bread costed shs. 1200/= in 2008 and shs. 1800/= in 2014. Taking 2008 as the base year, find the price relative in 2014

Solution

Price relative = $\frac{P_1}{P_o}\ x\ 100 = \frac{1800}{1200}\ x\ 100 = 150$

Example 2

In 2020, the price index of a commodity using 2019 as the base was 180. In 2021, the price index using 2020 as the base year was 150. What is the price index in 2021 using 2019 as the base year?

Solution

$\frac{P_{2020}}{P_{2019}}\ x\ 100 = 180$

$\frac{P_{2020}}{P_{2019}} = 1.80$ .... (i)

$\frac{P_{2021}}{P_{2020}}\ x\ 100 = 150$

$\frac{P_{2021}}{P_{2020}} = 1.50$ .... (ii)

Eqn. (i) x Eqn. (ii)

$\frac{P_{2020}}{P_{2019}} \cdot \frac{P_{2021}}{P_{2020}}$ = 1.8 x 1.5

$\frac{P_{2021}}{P_{2019}} = 2.7$

$\frac{P_{2021}}{P_{2019}} x\ 100 = 2.7x\ 100 = 270$

∴ the price index in 2021 using 2019 as the base year = 270

Example 3

The wage of a nurse in Uganda in 2010 was 350,000/=, the wage of the nurse in 2015 was increase by shs. 150,000/=. Using 2010 as the a base year calculate the nurses wage index in 2015.

Wage index = $\frac{W_1}{W_o}\ x\ 100$ = $\frac{500,000}{350,000}\ x\ 100 = 142.9$

Price indices

Price indices are divided into

(a) Simple price index

(b) Simple Aggregate price index

(c) Weighed price index

(a) Simple price index

This is the average of the price relative

It is given by simple price index = $\frac{\sum\frac{P_1}{P_0}}{n}$x100

Where n = number of items.

(b) Simple aggregate price index

It is given by simple aggregate price index = $\left(\frac{\sum P_1}{\sum P_0}\ x\ 100\right)$

Example 4

The table shows the prices 0f bread and meat per kg in 2000 and 2008

	Year	
item	2000	2008
Beans	700	1200
Meat	2500	4500

Using 2000 as the base year, find

(a) Price relatives of each commodity

(b) Simple price index

(c) Simple aggregate price index

Solution

(a) Price index = $\frac{P_1}{P_o}\ x\ 100$

For beans: P.R = $\frac{1200}{700}\ x\ 100 = 171.43$

For meat: P.R = $\frac{4500}{2500}\ x\ 100 = 180$

(b) Simple price index (S.P.I) = $\frac{171.43+180}{2} = 175.72$

(c) Simple aggregate price index (S.A.P.I) = $\frac{1200+450}{700+2500}\ x\ 100 = 178.13$

Example 5

In 2014 the price of a shirt, a dress and a pair of shoes were shs. 20,000; shs. 35,000 and shs. 45, 000 respectively. Given that in 2017 the prices were shs. 25,000, shs. 50,000 and shs. y respectively. Find the value of y if the aggregate price index was 130 while taking 2014 as the base year.

Solution

Simple aggregate price index = $\left(\frac{\sum P_1}{\sum P_0}\ x\ 100\right)$

$\left(\frac{25{,}000+50{,}000+y}{20{,}000+35{,}000+45{,}000}\right)\ x\ 100 = 130$

$\left(\frac{75.000+y}{1000}\right) = 130$

$y = shs. 55{,}000$

Weighted price index (composite index)

If the weight or quantity in the base year and current year are the same, we use

(i) Weighted aggregate price index

Weighted aggregate price index = $\left(\frac{\sum P_1 w}{\sum P_0 w}\right) x\ 100$

Example 6

The table below shows the prices (shs.) and amounts of items bought for assembling a phone in 2012 and 2015

Items	Prices (shs		Quantity
	2012	2015	
Transistor	12,000	18,000	8
Resistor	16,500	21,000	22
Capacitor	15,000	17,000	9
Diode	16,000	18,000	2
Circuit	20,000	25,000	1

Calculate the composite price index for a phone taking 2012 as the base year

W.A.P.I = $\left(\frac{18{,}000\ x\ 8 + 21{,}000\ x\ 22 + 17{,}000\ x\ 9 + 18{,}000\ x\ 2 + 25{,}000\ x\ 1}{12{,}000\ x\ 8 + 16{,}500\ x\ 22 + 15{,}000\ x\ 9 + 16{,}000\ x\ 2 + 20{,}000\ x\ 1}\right)\ x100 = 126.94$

Example 7

The table below shows the prices (shs) and amount of items bought for making a cake in 2008 and 2009.

Items	Prices (shs		Quantity
	2008	2009	
Flour per kg	6,000	7,800	3
Sugar per kg	5,000	4,000	1
Milk per litre	1,000	1,500	2
Eggs per egg	200	300	8

(a) Calculate the weighted aggregate price index taking 2008 as the base year

(b) In 2009, the cost of making a cake was shs. 80,000/=. Using the weighted aggregate price index above, find the cost of the coke in 2008.

Solution

(a) W.A.P.I = $\left(\frac{7800\ x\ 3+4000\ x\ 1+1500\ x\ 2+3\ x\ 8}{6000\ x\ 3+5000\ x\ 1+1000\ x\ 2+200\ x\ 8}\right) x\ 100 = 123.3083$

(b) $\frac{P_1}{P_o}\ x\ 100 = 123.3083$

$\frac{80{,}000}{P_o}\ x\ 100 = 123.3083$

$P_0 = shs.\ 64{,}878.0333$

(ii) Average weighted price index

Average weighted price index = $\frac{\sum\frac{P_1}{P_0}w}{\sum w}\ x\ 100$

When the price relative (P.R) is given then

Average weighted price index =$\frac{\sum(P.R\ x\ w)}{\sum w}$

Example 8

The table shows the expenditure (Ug. Shs.) of a student during the first and second terms

Items	Expenditure (shs.)		Amount
	1st term	2nd term	
clothing	46,500	49,350	5
Pocket money	55,200	37,500	3
Books	80,000	97,500	8

Using the first term expenditure as the base, find the average weighted price index

Average weighted price index = $\frac{\sum\frac{P_1}{P_0}w}{\sum w}\ x\ 100$

$= \left(\frac{\frac{49{,}350}{46{,}500}x\ 5+\frac{37{,}500}{55{,}200}x\ 3+\frac{97{,}500}{80{,}000}x\ 8}{5+3+8}\right) x\ 100$

= 106.841

Example 9

The table below shows the price relatives together with their weights for a certain family

Item	Weight	Price relative
Food	172	120
Water	160	124
Housing	170	125
Electricity	210	135
Clothing	140	104

Find the:

(i) simple price index

(ii) cost of living

Solution

(i) S.P.I = $\frac{\sum\left(\frac{P_1}{P_0}\right)}{n} = \frac{120+124+125+135+104}{5} = 121.6$

(ii) Cost of living = $\frac{\sum(P.R\ x\ w)}{\sum w} = \frac{120\ x\ 172\ +124x\ 160\ +125\ x\ 170\ +135\ x\ 210+104\ x\ 140}{172+160+170+210+140}$ = 122.82

Weighted aggregate price indices/ Paache's theory/value index

If the weight or quantity in the base year and current year are different, we use

Weighted aggregate price index = $\frac{\sum P_1W_1}{\sum P_0W_0}\ x\ 100$

Example 10

The table below shows the prices of items per kg in the year 2001 and 2002

Item	2001 = 100		2002	
	Price (shs.)	Quantity (kg)	Price (shs.)	Quantity (kg)
Rice	2800	20	3200	30
Millet	1500	10	1900	10
Beans	2000	5	2500	70

Calculate for 2002

(i) Price index

(ii) Simple aggregate price index

(iii) Simple aggregate quantity index

(iv) Weighted aggregate price index

Solution

(i) Price index = $\frac{P_{2002}}{P_{2001}} x\ 100$

For rice, price index = $\frac{3200}{2800} x\ 100 = 114.29$

For millet, price index = $\frac{1900}{1500} x\ 100 = 126.67$

For beans, price index = $\frac{2500}{2000} x\ 100 = 125$

(ii) S.A.P.I = $\left[\frac{3200+1900+250}{2800+1500+2000}\right]\ x\ 100 = 120.63$

(iii) S.A.Q.I = $\frac{\sum Q_{2002}}{\sum Q_{2001}}\ x\ 100$

= $\left[\frac{30+10+70}{20+20+5}\right] x\ 100 = 314.29$

(iv) W.A.P.I = $\frac{\sum P_1W_1}{\sum P_0W_0}\ x\ 100$

= $\left[\frac{3200\ x\ 30\ +1900\ x\ 10+2500\ x\ 70}{2800\ x\ 20+1500\ x\ 10+2000\ x\ 5}\right] x\ 100$

= 358.02

Revision exercise

1. UNEB 2020/2/5

The table below shows the price indices of beans, maize, rice and meat with corresponding weights

Item	Price index 2008 (2007 = 100%)	Weight
Beans	105	4
Maize	x	7
Rice	104	2
Meat	113	5

Calculate the;

(a) Value of x given that the price indices of maize in 2007 and 2008 using 2006 as the base year are 112 and 130 respectively. (**116.0714**)

(b) Weighted price index for 2008 using 2007 as the base year (**111.4167**)

2. UNEB 2018/2/5

The price index of an article in 2000 based on 1998 was 130. The price index for the article in 2005 based on 2000 was 80. Calculate

(a) Price index of the article in 2005 based on 1998. (104)

(b) Price of the article in 1998 if the price of the article was 45,000 in 2005.

UNEB 2017/2/7

The table below shows the price (shs.) and amount of items bought weekly by a restaurant in 2002 and 2003.

Items	Price (shs.)		Amount
	2002	2003	
Mil k per litre	400	500	200
Eggs per tray	2500	3000	18
Cooking oil per litre	2400	2100	2
Flour per packet	2000	2200	15

Calculate

(a) the weighted aggregate price index taking 2002 as the base year. (**119.63**)

(b) In 2003, the restaurant spent shs. 450,000/=. Using the weighted aggregate price index, find how the restaurant could have spent in 2002. (**376,096.95**)

3. The table below shows the prices (shs.) an amount of items bought in 2006 and 2007

Items	Price (shs.)		Amount
	2006	2007	
Mil k per litre	300	400	1
Eggs per tray	2500	3000	3
Cooking oil per litre	3000	8000	4
Flour per packet	1500	1800	15

Taking 2006 as the base year

(a) Calculate the simple aggregate price index (**180.92**)

(b) Calculate the weighted aggregate price index(**161.702**)

4. The table below shows the prices in US dollars and weights of five components of an engine, in 1998 and 2005

components	weight	Prices (US D)	
		1998	2005
A	6	35	60
B	5	70	135
C	3	43	105
D	2	180	290
E	1	480	800

(a) Taking 1998 as the base year, calculate the

(i) Simple aggregate price index (**172.03**)

(ii) Price relative of each component (**171.4, 192.9, 244.2, 161.1, 166.7**)

(iii) Weighted price index (**178.55**)

(b) Using the price index in (a)(i) estimate the cost of the engine in 1998 if the cost of the engine in 2005 was 1600 USD (**896.11**)

5. The table below shows the prices and amounts of items bought in 2004 and 2005

Item	Prices (shs)		Amount
	2004	2005	
A	635	887.5	6
B	720	815	4
C	730	1045	3
D	362	503	7

(a) Calculate the simple aggregate price index (**132.836**)
(b) Calculate the weighted aggregate price index (**133.52**)
(c) Calculate the price of an item costing 500 in 2004 using weighted aggregate price index above. (**667.64**)

6. The table below shows the prices of items per kg in the year 2005 and 2007

Item	Posho	Beans	Rice	Beef	Chicken
Price in 2005	1200	2000	1200	4000	8000
Price in 2007	1600	2500	1600	6000	9500

Calculate for 2007 using 2005a as the base year
(a) Simple price index (**132.0833**)
(b) Simple aggregate price index (**129.2683**)

7. The table below shows the prices in the year 2010 and 2018

Item	Price in 2010	Price in 2018
Flour per kg	3000	5400
Eggs per dozen	5,000	7800

Calculate for 2018 using 2010 as the base year
(i) Simple price index (**168**)
(ii) Simple aggregate price index (**165**)

8. The table below shows the prices and quantities of four items in the year 2020 and 2021

Item	Price per unit		Quantities	
	2020	2021	2020	2021
A	100	120	36	42
B	110	100	96	88
C	50	65	10	12
D	80	85	11	10

(a) Calculate the price index (120, 90.91, 130, 106.25)
(b) Simple aggregate price index (108.82)
(c) Weighted aggregate price index (99.55)
(d) Cost in 2021, A, B, C and Dare ingredients to make chapatti and in 2020a price of chapatti costed shilling 600 using index in (iii) above. (**shs. 597.3**)

9. MOTION IN A STRAIHT LINE

Motion in straight line

Distance and displacement

Distance is a length between 2 fixe points

Displacement is the distance covered in a specific direction

Speed and velocity

Speed is the rate of change of distance with time

Velocity is the rate of change of displacement with time

Average speed = $\frac{total\ distance}{total\ time\ taken}$

Average velocity = $\frac{total\ displacement}{total\ time\ taken}$

Example 1

Find the distance travelled in 5s by a body moving with a constant speed of 3.2ms^{-1}

Solution

Average speed = $\frac{total\ distance}{total\ time\ taken}$	3.2 = $\frac{total\ distance}{5}$	distance = 16m

Example 2

John ran 1500m in 3minutes and 33s, find his average speed.

Average speed = $\frac{total\ distance}{total\ time\ taken}$	speed = $\frac{1500}{(3\ x\ 60+33)}$ = 7.04ms^{-1}

Acceleration

It is the rate of change of velocity

Acceleration =$\frac{change\ in\ velocity}{total\ time\ taken}$	a = $\frac{v-u}{t}$ where v = final velocity, u = initial velocity, t = time

Uniform acceleration

This is the constant rate of change of velocity with time

Equations of uniform acceleration

1st equation

Suppose a body moving in a straight line with uniform acceleration a, increases its velocity from u to v in time t, then from the definition of acceleration

a = $\frac{v-u}{t}$	at = v -u	**v = u + at** 1

2nd equation

Suppose an object with velocity u moves with uniform acceleration a time t and attains a velocity v, the distance s travelled by the object is given by: s = average velocity x time

s = $\left(\frac{v+u}{2}\right)t$ but v = u + at

s = $\left(\frac{u+u+at}{2}\right)t$

s = $\left(\frac{2ut+at^2}{2}\right)$

s = ut + $\boldsymbol{\frac{1}{2}at^2}$ 2

3rd equation

s = average velocity x time

s = $\left(\frac{v+u}{2}\right)t$ but t = $\frac{v-u}{a}$

s = $\left(\frac{v+u}{2}\right)\left(\frac{v-u}{a}\right)$ =

s = $\left(\frac{v^2-u^2}{2a}\right)$

$\boldsymbol{v^2 = u^2 - 2as}$ 3

Example 3

A car is initially at rest at a point O. The car moves from O in a straight line with an acceleration of $4ms^{-2}$. find how far the car

(i) is from O after 2s

From **s = ut +** $\boldsymbol{\frac{1}{2}at^2}$**;** **s =** 0 x 2 + $\frac{1}{2}x\ 4\ x\ 2^2$ = 8m

(ii) is from O after 3s

s = 0 x 2 + $\frac{1}{2}x\ 4\ x\ 2^2$ = **1**8m

(iii) distance travelled in the third second = 18 – 8 = 10m

Example 4

A body at O moving with a velocity $10ms^{-2}$ decelerates at $2ms^{-2}$.

(a) find the displacement of the body from O after 7s

From s = ut + $\frac{1}{2}at^2$

s = 10 x 7 + $\frac{1}{2}x - 2\ x\ 7^2$ = 21m

(b) how far from O does the body come to rest and how long does it take

s = $\left(\frac{v^2-u^2}{2a}\right) = \frac{0^2-10^2}{2\ x-2}$ = 25m

t = $\frac{v-u}{a} = \frac{0-10}{-2}$ = 5s

Example 5

A taxi approaching a stage runs two successive half kilometres in 16s and 20s respectively. Assuming the retardation is uniform, find

(i) Initial speed of the taxi

s = ut + $\frac{1}{2}at^2$

For the first half kilometre or 500m

500 =16u + $\frac{1}{2}a\ (16)^2$ ……… (i)

for the kilometre or 1000m

1000 = 36u +$\frac{1}{2}a\ (36)^2$…….. (ii)

from eqn. (i) and eqn. (ii)

a =$\frac{25}{72}$ and u = 34.028ms^{-1}

(ii) the further distance, the taxi runs before stopping

s = $\left(\frac{v^2 - u^2}{2a}\right) = \ s\ = \left(\frac{0^2 - (34.028)^2}{2\left(\frac{25}{72}\right)}\right) = 1667.3\text{m}$

Extra distance = 1667.3 – 1000 = 667.3m

Example 6

An overloaded taxi travelling at constant velocity of 90km/h overtakes a stationary traffic police car. 2s later, the police car sets in pursuit, accelerating at a uniform rate of 6ms^{-2}. How far does the traffic car travel before catching up with the taxi?

Solution

t_1 = time taken by the taxi

t_2 = time taken by the police car

t_1 = 2 + t_2

speed of the taxi in m/s

90km/h = $\frac{90\ x\ 1000}{3600}$ = 25ms^{-1}

s = ut + $\frac{1}{2}at^2$

s_T = 25t_1

s_C = 0 x t_2 + $\frac{1}{2}x\ 6\ x\ {t_2}^2$ = 3 ${t_2}^2$

For the car to catch taxi; s_T = s_C

25t_1 =3 ${t_2}^2$

25(2 + t_2) =3 ${t_2}^2$

t = 10s or t = $\frac{4}{3}$s

the car leaves 2s later then 10s is the correct time since it gives positive distance

s_C = 3 ${t_2}^2$ =3 x 10^2 = 300m

Example 7

A lorry starts from a point A and moves along a straight horizontal road with a constant acceleration of 2ms^{-2}. At the same time a car moving with a speed of 20ms^{-1} and a constant acceleration of 3ms^{-1} is 400m behind the point A and moving in the same direction as the lorry. find:

(a) how far from A the car overtakes the lorry.
a car over takes the lorry; both move in the same time, t

s = ut + $\frac{1}{2}at^2$

distance moved by the car = 400 + distance moved by the lorry

20t + $\frac{1}{2}\ x\ 3x\ t^2$ = 400 + $\frac{1}{2}\ x\ 2\ x\ t^2$

t^2 + 40t – 800 = 0; t = 14.64s or t = -54.64s

Hence t = 14.64s

sL = $\frac{1}{2}\ x\ 2\ x\ (14.64)^2$ = 214.33m

(b) the speed of the lorry when it is being overtaken

v = u + at

= 0 + 2 x 14.64 = 29.28ms^{-1}

Example 8

The seed of a taxi decreases from 90kmh^{-1} to 18kmh^{-1} in a distance of 120 metres. Find the speed of the taxi when it had covered a distance of 50metres. (05marks)

Given u = 90kmh^{-1}, v = 18kmh^{-1}, s = 120m = 0.12km
Using $v^2 = u^2 + 2as$
$18^2 = 90^2 + 2a(0.12)$
a = -32400kmh^{-2}
When s = 50m = 0.05km, u = 90kmh^{-1}, a = -32400kmh^{-2}
Using $v^2 = u^2 + 2as$
$v^2 = 90^2$ – 2 x 32400 x 0.05 = 4860
v = $\sqrt{4860} = 69.71kmh^{-1}$

Example 9

(a) Show that the final velocity v of a body which starts with an initial velocity u and moves with uniform acceleration a consequently covering a distance x, is given by v = $[u^2 + 2ax]^{\frac{1}{2}}$

x = average velocity x time

x = $\left(\frac{v+u}{2}\right)t$ but t = $\frac{v-u}{a}$

x = $\left(\frac{v+u}{2}\right)\left(\frac{v-u}{a}\right)$ = $\left(\frac{v^2-u^2}{2a}\right)$

$$\boldsymbol{v^2 = u^2 + 2ax}$$

v = $[u^2 + 2ax]^{\frac{1}{2}}$

(b) Find the value of x in (a) if v = 300m/s, u = 10m/s and a = 5m/s

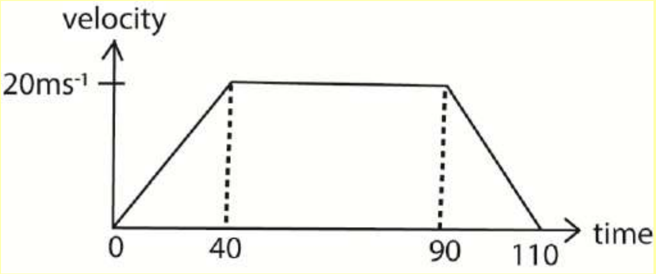
30 = $[10^2 + 2x\ 5x]^{\frac{1}{2}}$
900 = 100 + 10x
x =80m

Velocity-time graphs

Example 10

A car started from rest and attained a velocity of 20m/s in 40s. It then maintained the velocity attained for 50s. After that it was brought to rest by a constant breaking force in 20s.

(i) Draw a velocity-time graph for the motion



(ii) using the graph, find the total distance travelled by the car

Total distance = total area under the graph

$= \frac{1}{2}bh + lw + \frac{1}{2}bh$

$= \frac{1}{2}x\ 40\ x\ 20 + 50x\ 20 + \frac{1}{2}x\ 20\ x\ 20$ = 1600m

Method II (area of a trapezium)

A = $\frac{1}{2}h(a+b) = \frac{1}{2}x\ 20(50+110)$ = 1600m

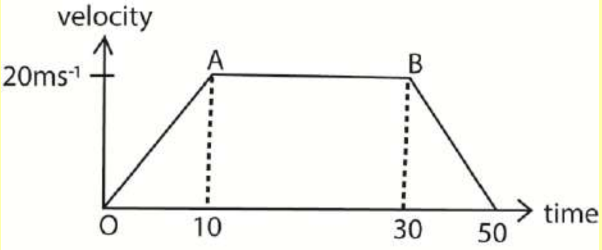
(iii) what is the acceleration of the car?

a= $\frac{v-u}{t} = \frac{20-0}{40} = 0.5ms^{-2}$

Example 11

A car from rest accelerates steadily to 10s up to a velocity f 20ms. It continues with uniform velocity for further 20s and then decelerates so that it stops in 20s.

(a) Draw a velocity-time graph to represent the motion



(b) Calculate

(i) acceleration

a= $\frac{v-u}{t} = \frac{20-0}{10} = 2ms^{-2}$

(ii) deceleration

a= $\frac{v-u}{t} = \frac{0-20}{20} = -1ms^{-2}$

(i) Distance = area under the graph

A = $\frac{1}{2}x\ 10\ x\ 20 + 20x\ 20 + \frac{1}{2}x\ 20\ x\ 20$

= 700m

Method II (area of a trapezium)

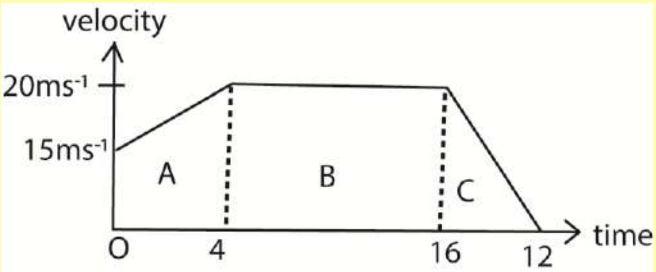
A = $\frac{1}{2}\ x\ 20(50+20)$= 700m

Average speed = $\frac{distance}{time}$

= $\frac{700}{50} = 14m/s$

Example 12

The graph below shows the motion in the body.



(a) Describe the motion of the body
A body with initial velocity of 15m/s accelerates steadily to a velocity of 20m/s in 4s, it then continues with a uniform velocity for 6s and brought to rest in 2s.

(b) Calculate the total distance travelled
Distance = 4 x 15 + $\frac{1}{2}x\ 4\ \ x\ 5 + 20\ x\ 6 + \ \frac{1}{2}x\ 20\ \ x\ 2$ = 210m

Revision exercise

1. P, Q and R are points on a straight road such that PQ = 20m and QR = 55m. A cyclist moving with uniform acceleration passes O and then notices that it takes him 10s and 15s to travel between P and Q and Q and R respectively. find the acceleration [a = $\frac{2}{15}ms^{-2}$]
2. A car travels from Kampala to Jinja and back. It takes average speed on the return journey is 4km/h greater than that on the outward journey and it takes 12 minutes less. Given that Kampala and Jinja are 80km apart, find the average speed on the outward journey.[30.05kmh]
3. Car A traveling at 35ms^{-1} along a straight horizontal road, accelerates uniformly at 0, 4ms^{-2}. At the same time, another car B moving at 44ms^{-1} and accelerating uniformly at 0.5ms^{-2} is 200m behind A
   (i) Find the time taken before car B over takes car A. [20s]
   (ii) speed with which B over takes A. [55m/s]
4. A car is being driven along a road at 72kmh^{-1} notices a fallen tree on the road 800m ahead and suddenly reduces the speed to 36kh^{-1} by applying brakes. For how long were the brakes applied [53.33s]
5. A train starts from station a with a uniform acceleration of 0.2ms^{-2} for 2 minutes and attains a maximum speed and moves uniformly for 15 minutes. it is then brought to rest at constant retardation of 5/3ms^{-2} at station B. find the distance between A and B. [23212.8m]
6. A motorcycle decelerated uniformly from 20kmh^{-1} to 8kmh^{-1} in travelling 896m. find the rate of deceleration in ms2 [0.0145ms^{-2}]
7. A body moves with a uniform acceleration and covers a distance of 27m in 3s; it then moves with a uniform velocity and covers a distance of 60m in 5s. Find the initial velocity and acceleration of the body. [6ms^{-1}, 2ms^{-2}]
8. A particle is projected away from an origin O with initial velocity of 0.25ms^{-1}. The particle travels in a straight line and accelerates at 1.5ms^{-2}. find
   (i) how far the particle is from O after 4s [7.5m]
   (ii) the distance travelled by the particle during the fourth second after projection. [5.5m]
9. A taxi which is moving with a uniform acceleration is observed to take 20s and 30s to travel successive 400m. find
   (i) initial speed of the taxi. [$\frac{68}{3}ms^{-1}$]

(ii) the further distance it covers before stopping [163.3m]

10. Two cyclist A and B are 36m apart on a straight road. Cyclist B starts from rest with an acceleration of $6ms^{-2}$ while A is in pursuit of B with velocity of $20ms^{-1}$ and acceleration of $4ms^{-1}$. Find the time taken when A overtakes B [13466s]

10.VERTICAL MOTION

Vertical motion under gravity

When a body is projected **vertically downwards**, it is subjected to an acceleration of $9.8ms^{-2}$. i.e. $a = g = 9.8ms^{-2}$

Equations of motion become

$v = u + gt$; $h = ut + \frac{1}{2}gt^2$; $v^2 = u^2 + 2gh$

When a body is projected **vertically upwards**, it is subjected to a retardation of $9.8ms^{-2}$. i.e. $a = g = 9.8ms^{-2}$

Equations of motion become

$v = u - gt$; $h = ut - \frac{1}{2}gt^2$; $v^2 = u^2 - 2gh$

Maximum /greatest height

When a particle is projected vertically upwards, the final velocity is 0ms-1 at its maximum height

$v^2 = u^2 - 2gh$

$0 = u^2 - 2gh_{max}$

$h_{max} = \frac{u^2}{2g}$

Time to reach maximum height

$v = u – gt$

$0 = u – gt$

$t = \frac{u}{g}$

Time of flight

$T = \frac{2u}{g}$

Example 1

A stone is dropped from a point which is 40m above the ground. Find the time taen for the stone to reach the ground

$h = ut + \frac{1}{2}gt^2$

$40 = 0 \text{ x } t + \frac{1}{2} \text{ x } 9.8 \text{ x } t^2$

$t = \sqrt{\frac{40}{9.8}} = 2.857s$

Example 2

A ball is thrown vertically upwards with an initial speed of $30ms^{-1}$. Calculate

(i) Time taken to reach thrower

$T = \frac{2u}{g} = \frac{2 \times 30}{9.8} = 6.12s$

(ii) maximum height reached

$h_{max} = \frac{u^2}{2g} = \frac{30^2}{2 \times 9.8} = 45.92m$

Example 3

A particle is projected from the ground level vertically upwards with velocity of $19.6ms^{-1}$. Find

(i) greatest height reached

$h_{max} = \frac{u^2}{2g} = \frac{19.6^2}{2 \times 9.8} = 19.6m$

(ii) time taken by the particle to reach maximum height

$t = \frac{u}{g} = \frac{19.6}{9.8}\ 2s$

(iii) Time of flight

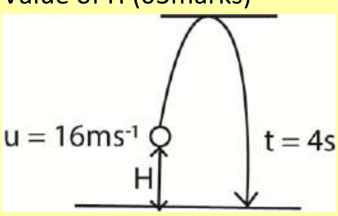
$T = 2t = 2 \times 2 = 4s$

Example 4

1. A stone is thrown vertically upwards with velocity $16ms^{-1}$ from a point H meters above the ground level. The stone hits the ground 4 seconds later.

Calculate the

(a) Value of H (03marks)



Using $s = ut + \frac{1}{2}at^2$; s = -H(below point of projection), $u = 16ms^{-1}$, a = -g, t = 4s

$-H = 16 \times 4 - \frac{1}{2} \times 9.8 \times 4^2$

H = 14.4m

(b) Velocity of the stone as it hits the ground (02marks)

Using v = u + at; v = -v(below point of projection), a = -g, t = 4s

$-v = 16 - 9.8 \times 4$

$v = 23.2ms^{-1}$

$\therefore$ the velocity of the stone as it hits the ground is $23.2ms^{-1}$

Example 6

A stone is thrown vertically upwards with a velocity of $21ms^{-1}$. Calculate the

(a) Maximum height attained by the stone (03marks)

$H = \frac{u^2}{2g} = \frac{21^2}{2 \times 9.8} = 225m$

(b) Time the stone takes to reach the maximum height. (02marks)

$t = \frac{u}{g} = \frac{21}{9.8} = 2.143s$

Example 7

A particle is projected vertically upwards with velocity ums-1. After t seconds another particle is projected vertically upwards from the same point of projection and with the same initial velocity. Prove that the particles collide after$\left(\frac{2}{2}+\frac{u}{g}\right)s$. Hence show that they meet at a height of $\frac{u^2-(gt)^2}{8g}$.

Solution

t_1 = time taken by 1st particle

t_2 = time taken by 2nd particle

$t_1 - t_2$ = t .... (i)

t_1 and t_2 are roots of the equation

h = ut - $\frac{1}{2}gt^2$ or gt^2 – 2ut + 2h = 0

$t_1 = \frac{2u+\sqrt{4u^2-8gh}}{2g}$

$t_2 = \frac{2u-\sqrt{4u^2-8gh}}{2g}$

$\frac{2u+\sqrt{4u^2-8gh}}{2g} - \frac{2u-\sqrt{4u^2-8gh}}{2g}$ = t

$\sqrt{4u^2-8gh} = gt$ ...... (ii)

From eqn (ii)

$h = \frac{4u^2-(gt)^2}{8g}$

$t_1 = \frac{2u+\sqrt{4u^2-8gh}}{2g}$ putting eqn. (ii)

$$t_1 = \frac{2u+gt}{2g} = \left(\frac{2}{2}+\frac{u}{g}\right)s$$

Example 8

A particle is projected upwards with velocity of 10ms**-1**. After 2s another particle is projected vertically upwards from the same point of projection with the same initial velocity. Find the height above the level of projection where the particle meet and time taken by the first particle before they meet.

t_1 = time taken by 1st particle

t_2 = time taken by 2nd particle

$t_1 - t_2$ = t .... (i)

t_1 and t_2 are roots of the equation

h = ut - $\frac{1}{2}gt^2$ or gt^2 – 2ut + 2h = 0

$t_1 = \frac{20+\sqrt{400-8gh}}{2g}$

$t_2 = \frac{20-\sqrt{400-8gh}}{2g}$

$\frac{20+\sqrt{400-8gh}}{2g} - \frac{20-\sqrt{400-8gh}}{2g}$ = t

$\sqrt{400-8gh} = gt$ ...... (ii)

From eqn (ii)

$h = \frac{400-(2\,x\,9.8)^2}{8\,x\,9.8} = 0.202m$

$t_1 = \frac{2u+\sqrt{4u^2-8gh}}{2g}$ putting eqn. (ii)

$t_1 = \frac{2u+gt}{2g} = \left(\frac{2}{2}+\frac{u}{g}\right) = \frac{2\,x10+9.8\,x\,2}{2\,x\,9.8} = 2.02\,s$

Revision exercise

1. A particle is projected vertically upwards with a velocity of 21ms^{-1}. How long it takes to reach a point 280m below the point of projection. [10s]
2. A particle is projected vertically upwards with a velocity of 17.5ms^{-1}. Find
   (i) how high the particle goes. [15.6m]

(ii) what time elapse before it's at a height of 10m [$\frac{5}{7}s$; $\frac{22}{7}s$

3. A particle is projected vertically upwards with velocity of 24.5ms^{-1}. Find
   (a) when its velocity is 4.9ms^{-1} [2s]
   (b) how long it takes to return to the point of projection.[5s]
   (c) at what time it will be 19.6m above the point of projection. [1s and 4s]
4. A particle is projected vertically upwards with a velocity of 35ms^{-1}. find
   (a) how long it takes to reach the greatest height.[3.57s]
   (b) distance it ascends during the 3rd second of motion. [10.5m]
5. Two objects are dropped from a cliff of height H. the second is dropped when the first has travelled a distanc e d. Prove that the instant when the first object reaches the bottom, the second is a distance $2\sqrt{DH} - D$ from the top of the cliff.
6. A particle is projected vertically upwards from point O with a speed of $\frac{4}{3}v\ ms^{-1}$. After it has travelled a distance of $\frac{2}{5}X\ m$ above O on its upward motion, another particle is projected vertically upwards from the same point with the same initial speed. Given that the particles collide at a height $\frac{2}{5}X\ m$ above O, prove that
   (i) the maximum height, H is given by $8v^2 = 9gH$
   (ii) when the particle collide 9X = 20H.
7. A particle is projected vertically upwards with velocity um/s. After t seconds another particle is projected vertically upwards from the same point of projection and with the same initial velocity. Prove that the particles collide each other having a velocity of $\frac{1}{2}gt$.
8. A particle is projected vertically upwards with velocity 28m/s. After 2s another particle is projected vertically upwards from the same point of projection and with an initial velocity of 21m/s. Find when the two particles are at the same height and the velocity of each body at that instant. [4.9s after the first particle is projected, 20m/s, 7.4m/s]
9. A particle is projected vertically upwards with velocity 25m/s. After 4s another particle is projected vertically upwards from the same point of projection and with the same initial velocity. Find the time and height when the two particles meet.[4.55s after the first particle is projected, 12.288m]
10. A stone is dropped from the top of a tower. In the last second of its motion, it falls through a distance which is a fifth of the height of the tower. Find the height of the tower. [439.6m]

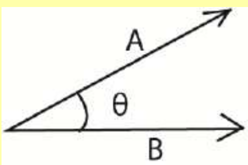
11.RESULTANT FORCES

Resultant of forces

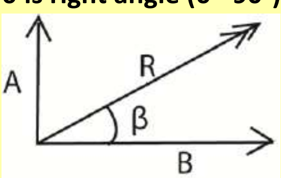
A force is anything which change a body's state of rest or uniform motion in a straight line. Examples are weight, tension, reaction, friction, resistance force.

Resultant of two forces

Consider two forces A and B inclined to each other at an angle θ



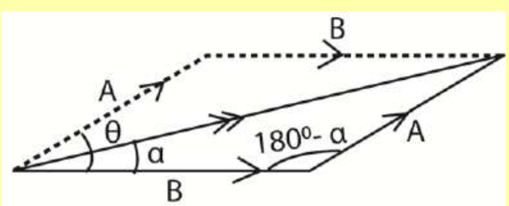
(i) θ is right angle (θ =90⁰)



Resultant, R = $\sqrt{A^2 + B^2}$

Direction of resultant, β =$\tan^{-1}\left(\frac{A}{B}\right)$

(ii) **θ is acute ($0^0 \le \theta \le 90^0$)**

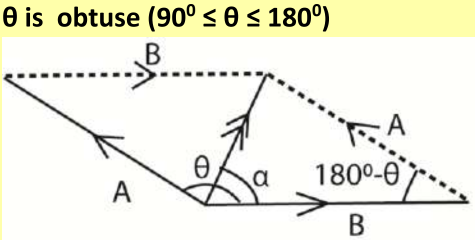


Direction of resultant, $\frac{\sin\alpha}{A} = \frac{\sin(180-\theta)}{R}$

$\alpha = \sin^{-1}\left(\frac{A\sin(180-\theta)}{R}\right)$

Resultant, R =$\sqrt{[A^2 + B^2 - 2ABcos\ (180-\theta)]}$

(iii) **θ is obtuse ($90^0 \le \theta \le 180^0$)**



Direction of resultant, $\frac{\sin\alpha}{A} = \frac{\sin(180-\theta)}{R}$

$\alpha = \sin^{-1}\left(\frac{A\sin(180-\theta)}{R}\right)$

Resultant, R =$\sqrt{[A^2 + B^2 - 2ABcos\ (180-\theta)]}$

Example 1

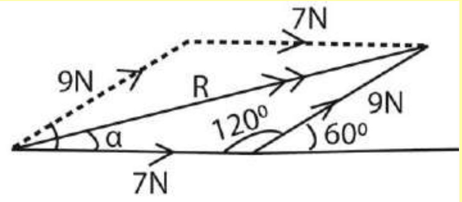
Two forces of magnitude 5N and 12N act on a particle with their direction inclined at 90^0. Find the magnitude and direction of the resultant

$R = \sqrt{5^2 + 12^2}$ = 13N $\alpha = \tan^{-1}\left(\frac{5}{12}\right) = 22.6^0$

The resultant = 13N at 22.6^0 to 12N force

Example 2

Forces of magnitude 7N and 9N act on a particle at an angle of 60^0 between them. Find the magnitude and direction of the resultant.



Direction of resultant, $\frac{\sin\alpha}{9} = \frac{\sin(180-\theta)}{13.89}$

$\alpha = \sin^{-1}\left(\frac{9\sin(180-60)}{13.89}\right) = 34.13^0$

Resultant, R $= \sqrt{[A^2 + B^2 - 2ABcos\ (180-\theta)]}$

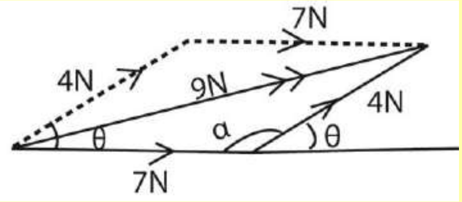
$= \sqrt{[7^2 + 9^2 - 2x7\ x\ 9cos\ (180-60)]}$

= 13.89N

Example 3

Find the angle between a force of 7N and 4N their resultant has a magnitude of 9N

Solution



$9^2 = 7^2 + 4^2 - 2 \times 7 \times 4 \times \cos\alpha$

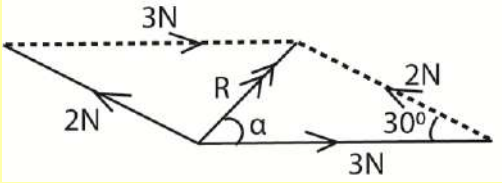
$\alpha = \cos^{-1}\left(-\frac{2}{7}\right) = 106.6^0$

the angle θ between the forces = 180 – 106.6

$= 73.4^0$

Example 4

Forces of 3N and 2N act on a particle at an angle of 150^0 between them. Find the magnitude and direction of the resultant.



Direction of resultant, $\frac{\sin\alpha}{2} = \frac{\sin(30)}{1.61}$

$\alpha = \sin^{-1}\left(\frac{2\sin(180-60)}{1.61}\right) = 38.3^0$

$R^2 = 2^2 + 3^2 - 2 \times 2 \times 2 \times \cos(30)$

R = 1.61N

Revision exercise

1. Two forces of magnitude 7N and 24N act on a particle with their direction at 90^0. Find the magnitude and direction of the resultant. [25N, 16.26^0 with 24N force]
2. Forces of 5N and 8N act on a particle at an angle of 50^0 between them. Find the magnitude and direction of the resultant. [11.9N at 19^0 with 8N force]
3. Forces of 4N and 6N act on a particle at angle 60^0 between them. Find the magnitude and the direction of the resultant. [5.29N, at 40.9^0 with 6N force]
4. Forces of 9N and 10N act on a particle at angle 40^0 between them. Find the magnitude and the direction of the resultant. [17.9N, at 18.9^0 with 10N force]
5. Forces of 12N and 10N act on a particle at angle 105^0 between them. Find the magnitude and the direction of the resultant. [13.5N, at 45.7^0 with 12N force]
6. Forces of 8N and 3N act on a particle at angle 160^0 between them. Find the magnitude and the direction of the resultant. [5.28N, at 11.2^0 with 8N force]
7. Find the angle between a force of 10N and 4N their resultant has a magnitude of 8N. [130.5^0]
8. The angle between a force αN and a force of 3N is 120^0. If the resultant of the two forces has magnitude 7N, find the value of α. [8N]
9. The angle between a force βN and a force of 8N is 45^0. If the resultant of the two forces has a magnitude 15N, find the value of β. [8.24N]

12. FORCE AND NEWTON'S LAWS

Force and Newton's laws of motion

Law I: A body continues in its state of rest or uniform motion in a straight line unless acted upon by an external force

Law II: The rate of change of momentum of a body is directly proportional to the applied force and takes place in the direction of the force.

F = m$\left(\frac{v-u}{t}\right)$ but $\left(\frac{v-u}{t}\right)$ =a

= ma

NB: F must be the resultant force

Example 1

Find the acceleration produced when a body of mass 5kg experiences a resultant force of 10N

F = ma | 10 = 5a | a = 2ms^{-2}

Example 2

A car of mass 600kg travels a distance of 24m while uniformly accelerated from rest to 12ms^{-1}

(i) Find the acceleration of the car
$v^2 = u^2 + 2as$
$12^2 = 0^2 + 2a \times 24$
a = 3ms^{-2}

(ii) determine the accelerating force
F = ma = 600 x 3 = 1800N

Example 3

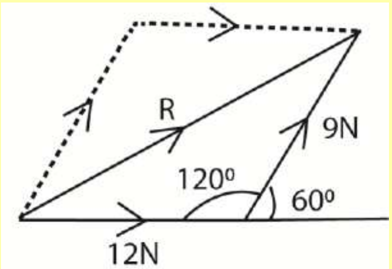
A body of mass 500g experiences a resultant force 3N. Find

(i) Acceleration produced
F = ma | 3 = 0.5 x a | a = 6ms^{-2}

(ii) Distance travelled by the body while increasing speed from 1ms^{-1} to 7ms^{-1}
$v^2 = u^2 + 2as$
$7^2 = 1^2 + 2 \times 6 \times s$
s = 4m

Example 4

Two forces of magnitude 12N and 9N act on a particle producing an acceleration of 3.65ms^{-2}. The two forces act at an angle of 600 to each other. Find the mass of the particle.



$R^2 = 12^2 + 9^2 - 2 \times 12 \times 9 \cos 120^0$

R = 18.25N

F = ma

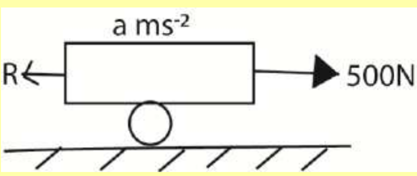
18.25N = 3.65m

m = 5kg

When resistance or friction is involved

Example 5

A car moves along a level road at constant velocity of 22ms^{-2}. If its engine is exerting a forward force of 500N, what resistance is the car experiencing.



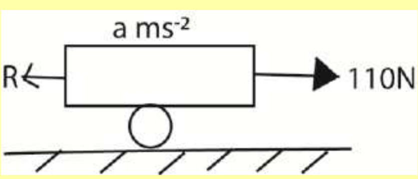
F = ma

500 – R = m x 0

R = 500N

Example 6

A car of mass 500kg moves along a level road with acceleration of 2ms^{-2}. Its Engine is exerting a forward force of 110N. What is the resistance a car is experiencing?



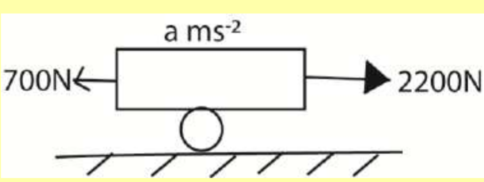
F = ma

500 – R = 500 x 2

R = 100N

Example 7

A van of mass 2tonnes moves along a level road against resistance of 700N. If its engine is exerting a forward force of 2200N. Find the acceleration of the van



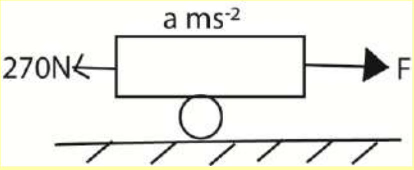
F = ma

2200 – 700 = 2000 x a

a = 0.75ms^{-2}

Example 8

Find the constant force necessary to accelerate a car of mass 1000kg from $15ms^{-1}$ to $20ms^{-1}$ in 10s against a resistance of 270N



v = u + at

20 = 15 + 10a

a = $0.5ms^{-2}$

F = ma

F – 270 = 1000 x 0.5

F = 770N

Calculations involving vector form

Find the resultant force required to make a body of mass 2kg at $(5i + 2j)ms^{-2}$.

F = ma | F = $2\binom{5}{2} = \binom{10}{4}N$

Example 9

F = ma $\quad \binom{4}{2} + \binom{1}{1} = 0.5a$ | a = $\binom{6}{6}ms^2$

Find the acceleration produced in a body of mass 500N is subjected to forces of (4i + 2j)N and (-i + j)N

F = ma | $\binom{4}{2} + \binom{1}{1} = 0.5a$ | a = $\binom{6}{6}ms^2$

Example 10

Find the magnitude of the acceleration produced in a body of mass 2kg subjected to forces of (2i – 3j +4k)N and (i + 5j + 2k)N

F = ma | $\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} = 2a$ | a = $\begin{pmatrix} 1.5 \\ 1 \\ 3 \end{pmatrix}ms^2$ | $|a| = \sqrt{1.5^2 + 1^2 + 3^2} = 2.3ms^{-2}$

Example 10

A particle of mass 2.5kg is acted on by a resultant force of 15N acting in the direction (2i – j – 2k). find the magnitude of the acceleration

F = 15 x $\frac{2i-j-2k}{\sqrt{2^2+(-1)^2+(-2)^2}}$

= 15 x $\frac{2i-j-2k}{3}$

F = 10i – 5j -10k

F = ma

$\begin{pmatrix} 10 \\ -3 \\ -10 \end{pmatrix} = 2a$

a = $\begin{pmatrix} 4 \\ -2 \\ -4 \end{pmatrix}$

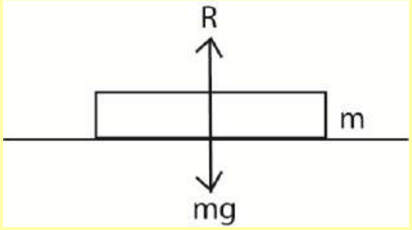
$|a| = \sqrt{10^2 + (-3)^2 + (-10)^2}$

=6ms-2

Law III: To every action there is an equal but opposite reaction

Consider

1. a body of mass m place on a smooth horizontal surface

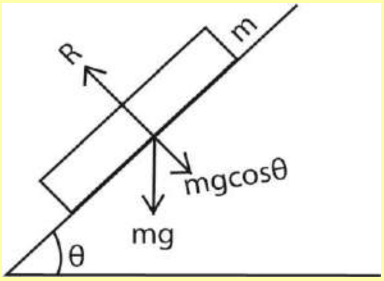


R = mg

R = normal reaction

mg gravitational pull (weight)

2. Mass m laced on a smooth inclined lane of angle of inclination θ



R = mgcosθ

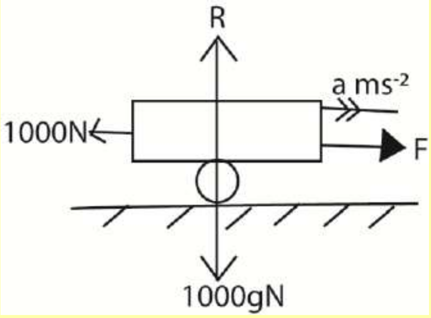
- All objects placed on or moving on an inclined plane experience a force mgsinθ **down** the plane no matter the direction of movement.
- If the plane is rough, the body experience a frictional force whose direction is opposite of the direction of motion

Motion on horizontal plane

Example 11

A car of 1000kg is accelerating at 2ms^{-2}. If the resistance to motion is 100N

(i) Find the normal reaction of the car on the road surface



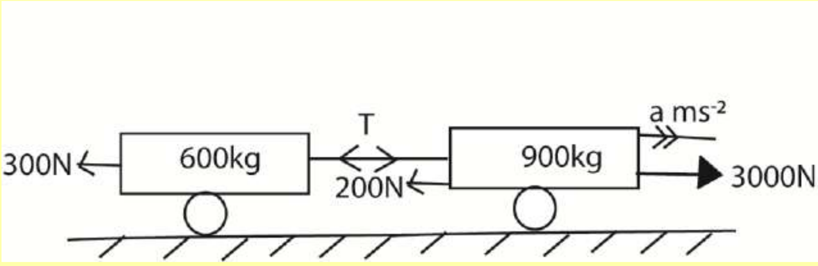
R = 1000gN

= 1000 x 9.8 = 9800N

(ii) What accelerating force acts on the car?
F = ma
F – 1000 = 1000 x 2; F = 3000N

Example 12

A car of mass 900kg tows a trailer of mass 600kg along a level road by means of a rigid bar. The car experiences a resistance of 200N and the trailer a resistance of 300N, if the car engine exerts a force of 3kN, find the acceleration produced and the tension in the tow bar



For 900kg: 3000 – (T + 200) = 900a … (i)

For 600kg: T – 300 = 600a ……………. (ii)

(i) and (ii) a= 1.6667ms^{-2}

Alternatively

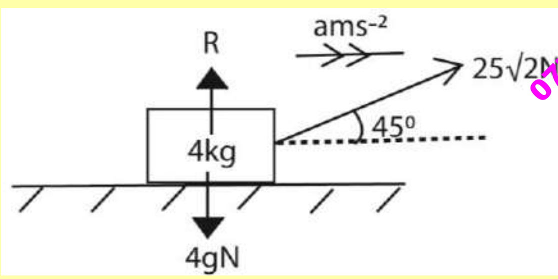
3000 – (200 + 300) = (900 + 600)a

a = 1.6667ms^{-2}

Force inclined at an angle to the horizontal

Example 13

A body of mass 4kg is acted on by force of $25\sqrt{2}$N which is inclined at 450 to a smooth horizontal surface. Find the acceleration of the body and the normal reaction between the body and the surface.



(→) $25\sqrt{2}$Ncos45 = 4a

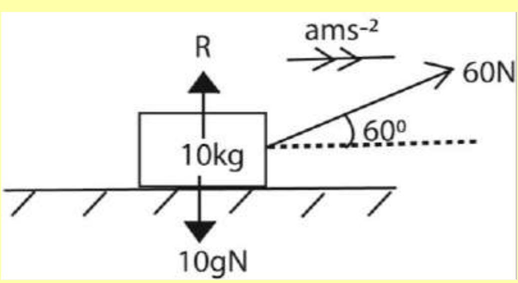
a = 6.25ms^{-2}

(↑) R + $25\sqrt{2}$Nsin 45 - = 0

R = 14.2N

Example 14

A body of mass 10kg is initially at rest on a rough horizontal surface. It is pulled along the surface by constant force of 60N inclined at 600 above the horizontal. If the resistance to motion totals 10n, find the acceleration of the body and the distance travelled in the first 3s.



(→) 60sos60 - 10 = 10a

a = 2ms^{-2}

s = ut +$\frac{1}{2}at^2$

s = 0 x 3 + $\frac{1}{2}\ x\ 2\ x\ 3^2$ =9m

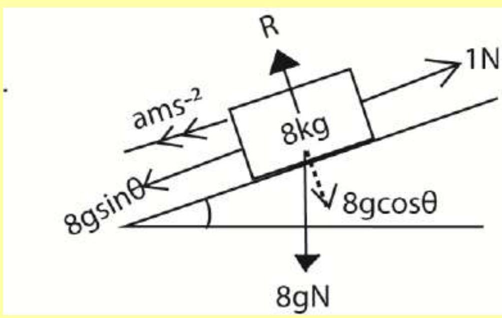
Revision exercise 1

1. A railway engine of mass 100tonnes is attached to a line of trucks of total mass 80 tonnes. Assuming there is no resistance to motion, find the tension in the coupling between the engine and the leading truck when the train has acceleration of 0.020ms^{-2} [25.6kN]
2. A body of mass 5kg, initially at rest on a smooth horizontal surface is pulled along the surface by a constant force P inclined at 450 above the horizontal. In the first 5 seconds of motion, the body moves a distance of 10m along the surface. Find the
   (i) acceleration of the body [0.8ms^{-2}]
   (ii) magnitude of P [$4\sqrt{2}$N]
   (iii) normal reaction between the body and the surface. [45N]
3. A body of mass m kg, initially at rest on a smooth horizontal surface is pulled along the surface by a constant force P inclined at θ above the horizontal. Show that the body moves a distance in time t along the surface given by $\frac{Pt^2cos\theta}{2m}$.
4. A body of mass m kg, initially at rest on a rough horizontal surface is pulled along the surface by a constant force P inclined at θ above the horizontal. If the mass acquire velocity v in a distance d. Show that the resistance to motion is given by Pcosθ = $\frac{mv^2}{2d}$

Motion on an inclined plane

Example 15

A body of mass 8kg is released from on the surface of a plane at 1 in 40. If the resistance to motion is 1N, find the acceleration of the body and the speed it acquired after 6s.



sinθ= $\frac{1}{40}$

F = ma

8gsinθ – 1 = 8a

a = 0.12ms^{-2}

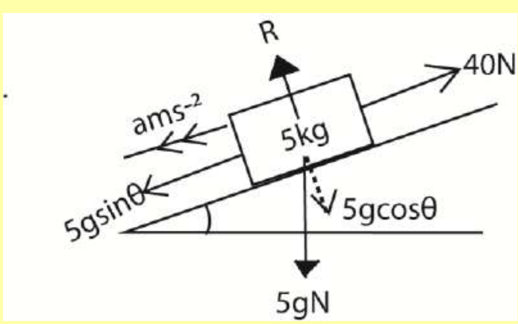
v = u + at

= 0 + 0.12 x 6 =0.72ms^{-2}

Example 16

A body of mass 5kg is pulled u a smooth plane inclined at 30^0 to the horizontal by a force of 40N acting parallel to the plane. Find

(i) acceleration of the body
(ii) force exerted on the body by the plane R



F = ma

40 -5gsin30^0 = 5a

a = 3.095ms^{-2}

(ii) R = 5gcos30

= 5 x 9.8cos30 = 42.4N

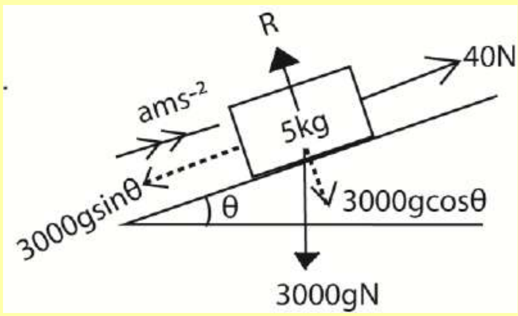
Example 17

A lorry of mass 3tonnes travelling at 90k/h starts to climb an incline of 1 in 5. Assuming the attractive pull between its tyres and the road remains constant and that its velocity reduces to 54kmh in a distance of 500m. Find the attractive pull.

u = 90kmh = $\frac{90 \times 1000}{3600}$ = 25ms^{-1}

V = 54kmh = $\frac{54 \times 1000}{3600}$ = 15ms^{-1}

a = $\frac{v^2 - u^2}{2s} = \frac{15^2 - 25^2}{2 \times 500}$ = -0.4ms^{-2}



sinθ = $\frac{1}{40}$

F = ma

F - 3000gsinθ = 3000a

F – 3000 x 9.8 x $\frac{1}{5}$ = 3000 x 0.4

F = 4686N

Revision exercise 2

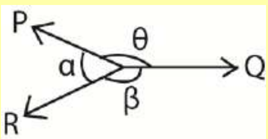
1. A particle of mass 5kg resting on smooth plane inclined at $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$to the horizontal. Find the magnitude of the horizontal force required to keep the particle in equilibrium and the normal reaction. [28.29N, 56.58N]
2. The engine of a train exerts a force of 35,000N on a train of mass 240 tonnes and draws up a slope of 1 in 120 against resistance totalling to 60N per tonne. Find the acceleration of the train. [0.004167ms^{-2}]
3. A car of mass 2.5 metric tonnes is drawn up a slope of 1 in 10 from rest with acceleration of 1.2ms^{-2} against a constant frictional force of $\frac{1}{100}$ of the weight of the vehicle using a cable. Find the tension in the cable. [5695N]
4. A mass 5kg is initially at the bottom of a smooth slope which is inclined at $\sin^{-1}\left(\frac{3}{5}\right)$ to the horizontal. The mass is pushed up the slope by horizontal force 50N, find
   (i) the normal reaction between the mass and the plane [69.2N]
   (ii) calculate the acceleration up the slope [2.12ms^{-2}]
   (iii) how far up the slope the mass travels in the first 4s [16.96m]
5. A body of mass 100kg is released from rest at the top of a smooth slope which is inclined at 30^0 to the horizontal. Find
   (i) velocity of the body when it has travelled 20m down the slope. [14ms^{-1}]
   (ii) velocity, if the mass of the body was 50kg. [14ms^{-1}]
6. A body of mass 20kg is released from rest at the top of a smooth slope which is inclined at 30^0 to the horizontal. If the body accelerates down the slope at 3ms^{-2}, find the constant resistance to motion experienced by the body. [38N]
7. A body of mass 20kg is released from rest at the top of a rough slope which is inclined at 30^0 to the horizontal. 6s later the body has a velocity of 21ms^{-1} down the slope, find the constant resistance to motion experienced by the body. [28N]

8. A car of 1 tonne accelerated from $36kmh^{-1}$ to $72kmh^{-1}$ while moving 0.5km up a road inclined at an angle of α to the horizontal where $\sin\alpha = \frac{1}{20}$. If the total resistive force to its motion is 0,3kN, find the driving force of the car engine. [1090N]
9. A railway truck of mass 6.0 tonnes moves with an acceleration of $0.050ms^{-2}$ down a track which is inclined to the horizontal at an angle α where $\sin\alpha = \frac{1}{120}$. find the resistance to motion. [190N]
10. A body of mass 5.0kg is pulled along a smooth horizontal ground by means of 40N acting at 60^0 above the horizontal. find
    (i) Accelerating force [$4ms^{-2}$]
    (ii) Force the body exerts on the ground [14.4N]
11. A body of mass 3.0kgslides down a plane which is inclined at 300 to the horizontal. Find the acceleration of the body if
    (i) the plane is smooth [$4.9ms^{-2}$]
    (ii) there is frictional resistance of 9.0N [$1.9ms^{-2}$]
12. A car of mass 1000kg tows a caravan of mass 600kg u a road which rises 1m vertically for every 20m of its length. There is a constant frictional resistance of 200N and 100N to the motion of the car and caravan respectively. The combination has an acceleration of $1.2ms^{-2}$ with the engine exerting a constant driving force. (Take $g = 10ms^{-2}$). Find
    (a) driving force[3020N]
    (b) Tension in the tow-bar [1120N]

13. EQUILIBRIUM OF THREE FORCES.

Equilibrium of three forces L ami's theorem

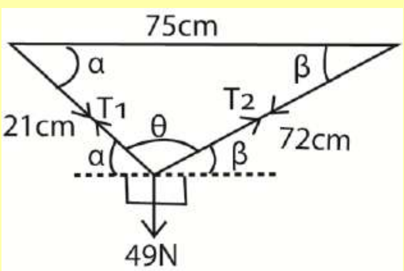
For any three forces acting on a particle in equilibrium where none of them is parallel to each other, Lami's theorem is applicable



$$\frac{P}{sin\beta} = \frac{Q}{sin\alpha} = \frac{R}{sin\theta}$$

Example 1

A weight of 49N is suspended by two strings of length 21 cm and 72cm attached to 2 points in a horizontal line a distance of 75cm apart. Find the tension in the strings so that the particle remain in equilibrium



By cosine rule:

$75^2 = 21^2 + 72^2 - 2 \times 21 \times 72\cos\theta$

$\theta = 90^0$

Similarly, $\beta = 16.26^0$ and $\alpha = 73.74^0$

$\frac{T_1}{\sin(16.26+90)} = \frac{49}{\sin 90}$;

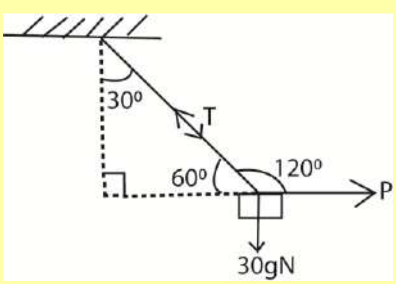
$\therefore T_1 = 47N$

$\frac{T_2}{\sin(73.74+90)} = \frac{49}{\sin 90}$;

$\therefore T_2 = 13.72N$

Example 2

Mass of 30kg hangs vertically at the end of a light string. If the mass is pulled by a horizontal force P so that the string makes 30^0 with the vertical. Find the magnitude of the force and the tension in the string so that the particle remain in equilibrium.

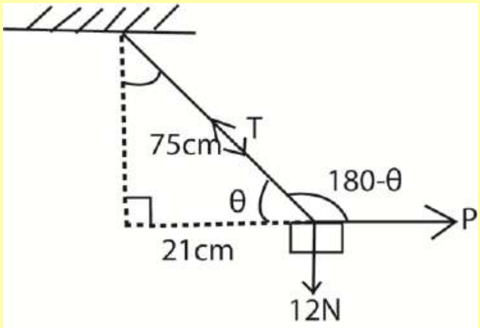


$\frac{T}{\sin 90} = \frac{30 \, x \, 9.8}{\sin 120}$; T = 339.48N

$\frac{P}{\sin(60+90)} = \frac{30 \, x \, 9.8}{\sin 120}$; P = 16974N

Example 3

One end of a light inextensible string of length 75cm is fixed to a point on a rigid pole. The particle of weight 12N is attached to the other end of the string. The particle is held 21cm away from the pole by a horizontal force, P. Find the magnitude of the force, P and the tension of the string so that the particle remain in equilibrium



$\theta = \cos^{-1}\left(\frac{21}{75}\right) = 73.74^0$

$$\frac{T}{\sin 90} = \frac{12}{\sin(180-73.74)}$$

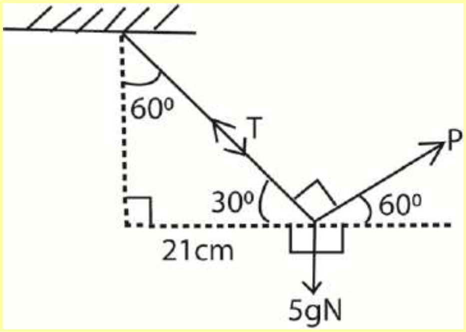
T = 12.5N

$$\frac{P}{\sin(90+73.4)} = \frac{12}{\sin(180-73.74)}$$

P = 3.5N

Example 4

A light inextensible string AB whose end A is fixed has end B attached to a particle of mass 5kg. A force P acting perpendicular to the string is applied on the particle keeping it in equilibrium with the string inclined at 60^0 to the vertical. Find the value of P and the tension in the string



$$\frac{T}{\sin(90+60)} = \frac{5\ x\ 98}{\sin 90}$$

T = 24.5N

$$\frac{5\ x\ 98}{\sin 90} = \frac{P}{\sin\ (90+30)}$$

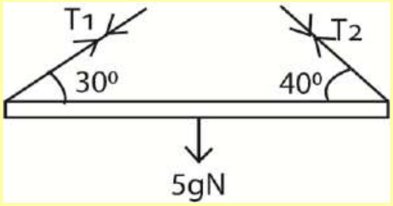
P = 42.44N

Example 5

A non-uniform beam of mass 5kg rests horizontally in equilibrium supported by two strings attached to the ends of the beam.

The strings makes 300 and 400 with the horizontal beam as shown above. Find the tension in the strings.

Solution



(→) $T_1\cos30 = T_2\cos40$; $T_1 = 0.8846T_2$

(↑)$T_1\sin30 + T_2\sin40 = 5g$

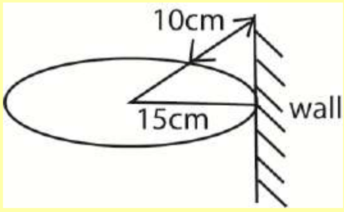
$0.8846T_2\sin30 + T_2\sin40 = 5 \text{ x } 9.8$

$T_2 = 45.159N$

$T_1 = 0.8846 \text{ x } 45.159 = 39.94N$

Example 6

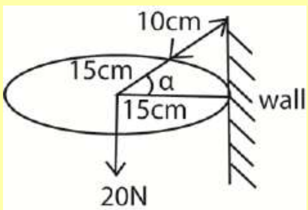
A sphere of weight 20N and radius 15cm rests against a smooth vertical wall. A sphere is supported in its position by a string of length 10cm attached to a point on the sphere and to a point on the wall as shown.



(i) calculate the reaction on the sphere due to the wall

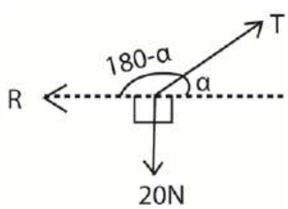
(ii) Find the tension in the string

Solution



α = $\cos^{-1}\left(\frac{15}{25}\right)$ = 53.13^0

Using Lami's theory



$\frac{T}{sin90} = \frac{20}{\sin(180-53.13)}$; T =25N

$\frac{R}{\sin(90+53.13)} = \frac{20}{\sin(180-53.13)}$; R = 15N

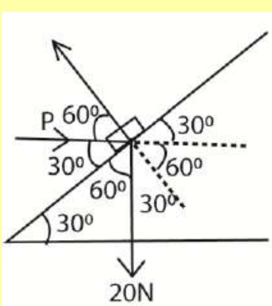
Example 7

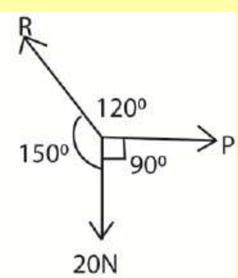
A particle of weight 20N is held at equilibrium on a smooth plane inclined at 30^0 to the horizontal by a horizontal force P.

(i) Find the value of P and the reaction between the particle and the plane.

(ii) If the force P is removed and a string parallel to the plane is used to hold the particle, find the tension in the string and the new value of the reaction.

Solution

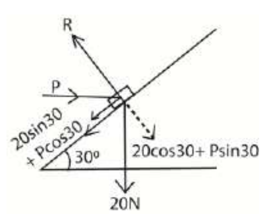




$\frac{P}{sin150} = \frac{R}{sin90} = \frac{20}{sin120}$

R = 23.09N and P = 11.55N

Alternatively: by resolving forces



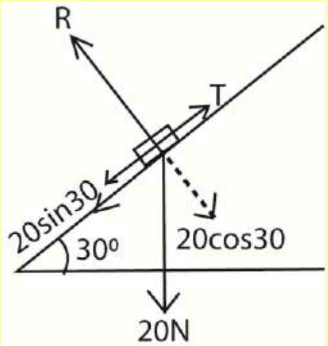
At equilibrium parallel to plane forces = 0

Pcos30 + 20sin30 = 0; P = 11.55N

R = 20cos30 + Psin30

R = 20cos30 + 11.55sin30 = 23.09N

(ii)



Parallel to the plane T = 20sin30 = 10N

Perpendicular to the plane R = 20cos30 = 13N

Alternatively by Lami's theory

$\frac{T}{sin150} = \frac{R}{sin120} = \frac{20}{sin90}$

T = 10N

R = 1.3N

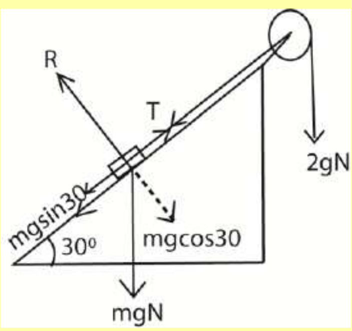
Example 8

A light inextensible string passes over a smooth fixed pulley at the top of a smooth plane inclined at 300 to the horizontal. A particle of mass 2kg is attached to one end of the string and rests vertically in equilibrium when the particle of ma resting on the surface of the plane is attached to the other end of the string. Find

(i) the normal reaction between m and the plane
(ii) tension in the string and the value of m.

Solution

By resolving forces



For 2kg mass: T – 2 x 9.8 = 0; T = 19.62N

Parallel to the plane

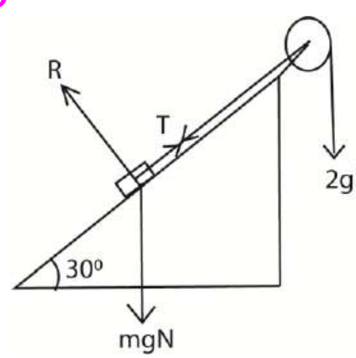
T – mgsin30 = 0; m = 4kg

Perpendicular to the plane

R = mgco30

R = 4 x 98cos30 = 33.98

Alternatively by using Lami's theorem



For 2kg mass: T – 2 x 9.8 = 0; T = 19.62N

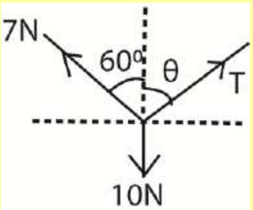
$\frac{T}{sin150} = \frac{mg}{sin90} = \frac{R}{sin120}$

$\frac{19.62}{sin150} = \frac{mg}{sin90} = \frac{R}{sin120}$

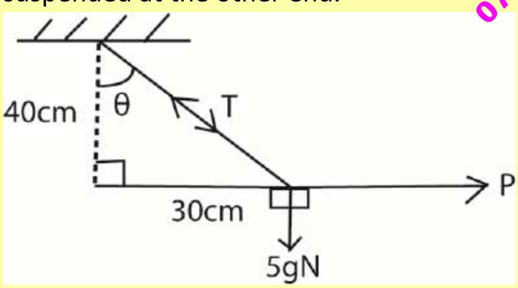
m = 4kg and R = 33.98N

Revision exercise

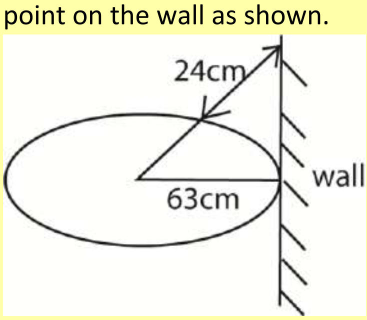
1. A particle P of mass 2kg is suspended from a fixed point O by means of a light inextensible string. The string is taut and makes an angle of 300 with the downward vertical through O and a particle is held in equilibrium by means of a horizontal force of magnitude F acting on the particle. Find the value of F and the tension in the string [F = 11.3161, T = 22.6321N]
2. A particle of mass 3kg lies on a smooth plane inclined at angle θ to the horizontal, where $\tan\theta = \frac{3}{4}$. The particle is held in equilibrium by horizontal force of magnitude FN. The line of action of this force is the same vertical plane as a line of greatest slope of inclined plane. Find the value of F. [22.05N]
3. The diagram below shows a body of weight 10N supported in equilibrium by two light inextensible strings. The tension in the strings are 7N and T and the angle the string makes with the upward vertical are 60^0 and θ respectively.



Find T and θ. [T = 8.9N, θ = 43^0]
4. A particle of weight 8N is attached to a point B by a light inextensible string AB. It hangs in equilibrium with point A fixed and AB at an angle of 30^0 to the downward vertical. A force F at B acting at right angles to AB, keeps the particle in equilibrium. Find the magnitude of force F and the tension in the string. [4N, $4\sqrt{3}N$]
5. The diagram shows a light inextensible string with one end fixed at A and a mass of 5kg suspended at the other end.

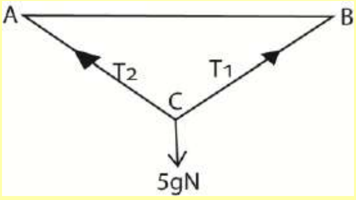


The mass is held in equilibrium at an angle θ to the downward vertical by a horizontal force P. Find the value of θ, P and the tension in the string [θ = 36.9^0, P = 36.75N, T = 61.25N)
6. A sphere of mass 5kg and radius 63cm rests against a smooth vertical wall. A sphere is supported in its position by a string of length 24cm attached to a point on the sphere an to a point on the wall as shown.



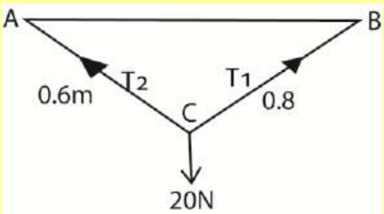
Find the tension in the string. [71.05N]
7. A particle whose weight is 50N is suspended by a light string which is 35^0 to the vertical under the action of a horizontal force F. Find the force F and the tension in the string. [35.0N, 61.0N]

8. A particle of weight w rests on a smooth plane which inclined at 40^0 to horizontal. The particle is prevented from slipping by a force of 50.0N acting parallel to the plane and up a line of greatest slope. Calculate w and reaction due to the plane. [77.8N, 59.6N]
9. A mass of 2kg is suspended by two light inextensible strings. One making an angle of 600 with the upward vertical and the other 300with the upward vertical. Find the tension in each string. [9.8N, 17.0N]
10. A heavy uniform rod of weight W is hung from a point by two equal strings, one attached to each end of the rod. A body of weight w is hang half-way between A and the center of the rod. Prove that the ratio of tension in the string is$\frac{2W+3w}{2W+w}$.
11. A non-uniform beam AB of length 8m and its weight 10N acts from a point G between A and B such that AG = 6m. The beam is supported horizontally by strings attached to A and B. The string attached to A makes an angle of 30^0 with AB. Find the angle that the string attached to B makes with AB and find the tension in the strings. [60^0, 5N, 8.66N]
12. A light inextensible string of length 40cm has its upper end fixed to a point A and carries a mass of 2kg at its lower end. A horizontal force applied to the mass keeps it in equilibrium, 20cm from the vertical through A. Find the magnitude of this horizontal force and the tension in the string. [11.3N, 22.6N]
13. The diagram shows a body of mass 5kg supported by two light inextensible strings, the other ends of which are attached to two points A and B on same level as each other end 7m apart.



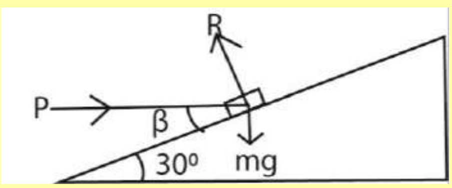
The body rests in equilibrium at 3m vertically below AB. If angle CBA = 45^0, find T_1 and T_2 the tensions in the strings. [35N, $28\sqrt{2}N$]

14. The diagram shows a body of weight 20N supported by two light inextensible strings of length 0.6m and 0.8m from two points 1m apart on a horizontal beam.



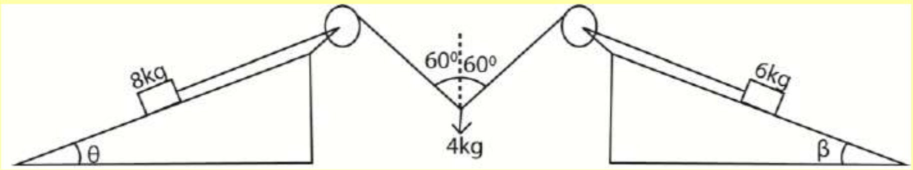
The body rests in equilibrium, find T_1 and T_2 the tensions in the strings. [16N, 12N]

15. A light inextensible string of length 50cm has its upper end fixed at point A and carries a particle of 8kg at its lower end. A horizontal force P applied to the particle in equilibrium 30cm from the vertical through A, find the magnitude of P and the tension in the string. [58.8N, 98N}
16. A article is in equilibrium under the action of forces 4N due north, 8N due west, $5\sqrt{2}$N south east and P, find the magnitude and direction of P. [3.16N, N71.6^0E]
17. A force P holds a particle of mass mkg in equilibrium on a smooth plane which is inclined at 300 to the horizontal.



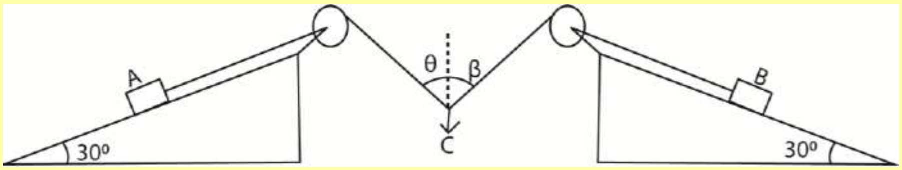
If P makes an angle β with the plane, find β when R the normal reaction between the particle and the plane is 15mg [51.7^0]

18. The diagram below shows masses of 8kg and 6kg lying on smooth planes of inclination θ and β respectively



Light inextensible strings attached to these masses pass along the line of greatest slopes over smooth pulleys and are connected to 4kg mass hanging freely. The strings both make an angle of 600 with the upward vertical as shown above. If the system rest in equilibrium find θ and β. [θ = 30^0 and β 41.8^0]

19. The diagram below shows masses A and B each lying on smooth planes of inclination30^0.



Light inextensible strings attached to A and B pass along the lines of greatest slopes, over smooth pulleys and are connected to a third mass C hanging freely. The strings make angles of θ and β with the upward vertical as shown above. If A, B and C have masses 2m, m, and m respectively and the system rests in equilibrium show that sinθ = 2sinβ and cosβ +2cosθ =2. Hence find θ and β. [29.0^0, 75.5^0]

14.RESOLUTION OF FORCES ACTING ON A POLYGON

Resolutions of forces acting on a polygon

For any regular polygon

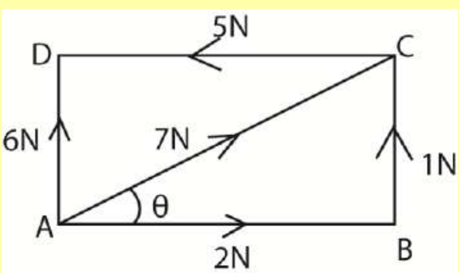
- all sides are equal
- all angles are equal
- an exterior angle $=\frac{360}{n}$ where n is the number of sides

Example 1

ABCD is a rectangle with AB= 4cm and BC = 3cm. Forces of magnitude 2N, 1N, 5N,6N and 7N act along AB, BC, CD, AD and AC respectively. In each case the direction of the force being given by the order of the letters. Given that AB is horizontal determine

(i) the magnitude of the resultant force

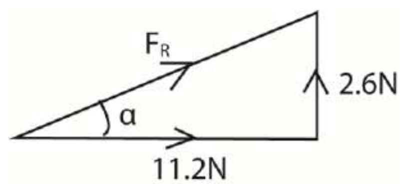
(ii) direction of the resultant with AB



$\theta = \tan^{-1}\left(\frac{3}{4}\right) = 36.87$

R = $\begin{pmatrix}2\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}-5\\0\end{pmatrix} + \begin{pmatrix}0\\6\end{pmatrix} + \begin{pmatrix}7cos36.87\\7sin36.87\end{pmatrix} = \begin{pmatrix}2.6\\11.2\end{pmatrix}$

$R = \sqrt{2.6^2 + 11.2^2}$ = 11.498N



Direction,$\alpha = \tan^{-1}\left(\frac{2.6}{11.2}\right)$

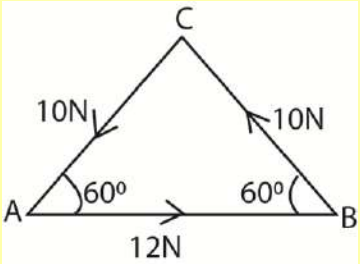
=13.069^0

Direction is 13.069^0 above AB

Example 2

ABC is an equilateral triangle. Forces of magnitude 12N, 10N and 10N act along AB, BC and CA respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal determine

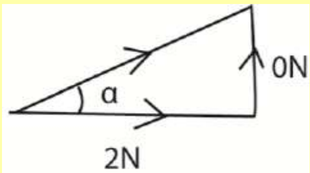
(i) the magnitude of the resultant force



$R = \begin{pmatrix}12\\0\end{pmatrix} + \begin{pmatrix}-10cos60\\10sin60\end{pmatrix} + \begin{pmatrix}-10cos60\\-10sin60\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix}$

$R = \sqrt{2^2 + 0^2}$ = 2N

(ii) Direction of the resultant with AB

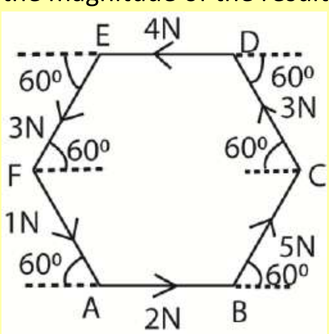


Direction $\alpha = \tan^{-1}\left(\frac{0}{2}\right) = 0^2$

Example 3

ABCDEF is a regular hexagon. Force of magnitude 2N, 5N, 3N, 4N, 3N and 1N act along the line AB, BC, CD, DE, EF and FA respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, determine

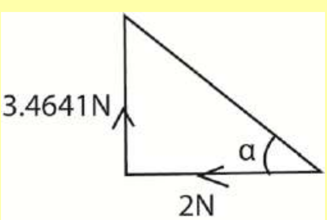
(i) the magnitude of the resultant force and



$$R = \begin{pmatrix}2\\0\end{pmatrix} + \begin{pmatrix}5cos60\\5sin60\end{pmatrix} + \begin{pmatrix}-3cos60\\3sin60\end{pmatrix} + \begin{pmatrix}-4\\0\end{pmatrix} + \begin{pmatrix}-3cos60\\-3sin60\end{pmatrix} + \begin{pmatrix}1\cos 60\\-1sin60\end{pmatrix} = \begin{pmatrix}-2\\3.4641\end{pmatrix}$$

$$R = \sqrt{(-2)^2 + 3.4641^2} = 4N$$

(ii) direction of the resultant with AB.

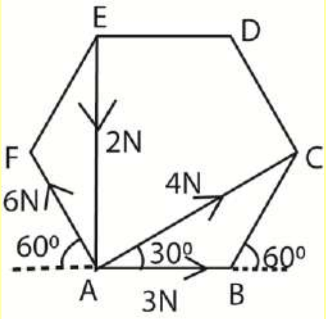


$\alpha = \tan^{-1}\left(\frac{3.461}{2}\right)$ = 60^0 to AB

Example 4

ABCDEF is a regular hexagon. Forces of magnitude 3N, 4N, 2N and 6N act along the line AB, AC, EA and AF respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, determine

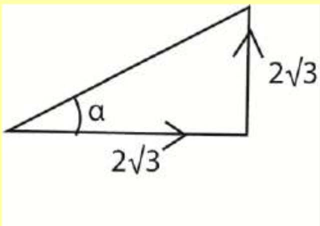
(i) the magnitude of the resultant force



$$R = \binom{3}{0} + \binom{4cos30}{4sin30} + \binom{-6cos60}{6sin60} + \binom{0}{-2} = \binom{2\sqrt{3}}{3\sqrt{3}}$$

$$R = \sqrt{(2\sqrt{3})^2 + (3\sqrt{3})^2} = 6.245N$$

(ii) direction of the resultant force



$$\theta = \tan^{-1}\left(\frac{3\sqrt{3}}{2\sqrt{3}}\right) = 56.3^0$$

1. ABCD is a square. Forces of magnitude 6N, 4N and $2\sqrt{2}N$ act along AD, AB and AC respectively in each case the direction of force being the order of the letters. Given that AB is horizontal, determine the magnitude and direction of the resultant force.
   [10N at 53.1^0 with AB]
2. ABCD is a square. Forces of magnitude 2N, 1N, $\sqrt{2}N$ and 4N act along AB, BC and AC and DA respectively in each case the direction of force being the order of the letters. Given that AB is horizontal, determine the magnitude and direction of the resultant force.
   [5.13N at 33.7^0 with AB]
3. ABCD is a square. Three forces of magnitude 4N, 10N and $7N$ act along AB, AD and CA respectively in each case the direction of force being the order of the letters. Given that AB is horizontal, determine the magnitude [5.1388N]
4. In equilateral triangle PQR, three forces of magnitude 5N, 10N and 8N act along the side PQ, QR and PR respectively. Their direction are the order the letters. Find the magnitude of the resultant force. [16.1N]
5. ABCD is a square. Forces of magnitude $6\sqrt{3}N$ N, 2N and $4\sqrt{3}N$ act along AB, CB and CD respectively in each case the direction of force being the order of the letters. Given that AB is horizontal, determine the magnitude and direction of the resultant force.
   [4N at 30^0 to AB]
6. ABCD is a rectangle with AB= 4cm and BC = 3cm. Forces of magnitude 3N, 1N, and 10N act along AB, DC and AC respectively. In each case the direction of the force being given by the order of the letters. Given that AB is horizontal determine the magnitude and direction of the resultant force. [13.4N at 26.6^0 with AB]
7. ABCD is a rectangle. Forces of magnitude 8N, 4N, 10N and 2N act along AB, CB, CD and AD respectively. In each case the direction of the force being given by the order of the letters. Given that AB is horizontal determine the magnitude and direction of the resultant force.
   [283N at 45^0 at AB]
8. In equilateral triangle ABC, forces of magnitude 10N each act along the side AB, BC and AC respectively. Their direction are the order the letters. Find the magnitude of the resultant force and the angle it makes with AB. [20N at $60^0$to AB]
9. In equilateral triangle ABC, forces of magnitude 5N, 9N and 7N act along the side AB, BC and CA respectively. Their direction are the order the letters. Find the magnitude of the resultant force and the angle it makes with AB. [$2\sqrt{3}$N at $30^0$to AB]

10. In equilateral triangle ABC, forces of magnitude 4N, 4N and 6N act along the side AB, BC and AC respectively. Their direction are the order the letters. Find the magnitude of the resultant force and the angle it makes with AB. [10N at 60^0 to AB]
11. ABCDEF is a regular hexagon. Forces of magnitude 2N, 5N, 3N, 4N, 3N and 1N act along the line AB, BC, CD,DE, EF and AF respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, determine the magnitude and direction of the resultant force. [6N at 60^0 to AB]
12. ABCDEF is a regular hexagon. Forces of magnitude 8N, 7N, 6N, 4N, 7N, and 6N act along the line AB, BC, CD, DE, EF and FA respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, determine the magnitude and direction of the resultant force. [12.5N at 76^0 to AB]
13. PQRSTU is a regular hexagon. Forces of magnitude 4N, 5N, 2N, and 6N act along the line PQ, PR, PT and PU respectively, in each case the direction of the force being given by the order of the letters. Given that PQ is horizontal, determine the magnitude and direction of the resultant force. [11.065N at 61.2^0 to PQ]
14. ABCD is a square. Forces of magnitude $10N$ N, 9N, 8N and $5N$ act along AB, BC, CD and AD respectively in each case the direction of force being the order of the letters. Given that AB is horizontal, determine the magnitude and direction of the resultant force.
[$2\sqrt{5}$N at 63.43^0 to AB]
15. ABCD is a rectangle with AB= 4cm and BC = 3cm. Forces of magnitude 3N, 10N, 4N, 6N and 5N act along AB, BC, CD, DA, and AC respectively. In each case the direction of the force being given by the order of the letters. Given that AB is horizontal determine the magnitude and direction of the resultant force. [7.62N at 66.8^0 with AB]

15. CONNECTED PARTICLES

Connected particles

Simple connections

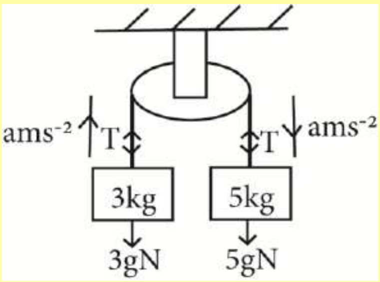
When two particles are connected by a light inextensible string passing over a smooth pulley and allowed to move freely, then as long as the string is tout, the following must be observed.

- acceleration of the particles is the same
- tension in the uninterrupted string is constant
- tensions in interrupted strings are different.

Example 1

Two particles of masses 5kg and 3kg are connected by a light inextensible string passing over a smooth fixed pulley. Find

(a) acceleration of the particle
(b) the tension in the string



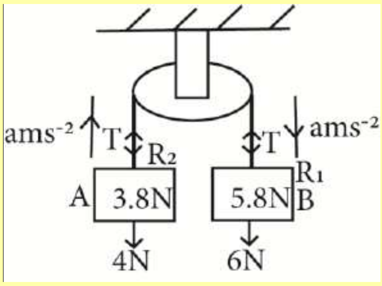
For 5kg mass: 5g – T = 5a …. (i)
For 3kg mass: T – 3g = 3a …. (ii)

(i) and (ii)
2g = 8a
a = $\frac{2 \times 9.8}{8}$ = 2.45ms^{-2}
(ii) tension in the string
T -3g = 3a
T = 3 x 2.45 + 3 x 9.8 = 36.78N
(iii) Force on the pulley
R = 2T = 3 x 36.78 = 73.56N

Example 2

An inextensible string attached to two scale A and B each of weight 20g passes over a smooth fixed pulley. Particles of weight 3.8N and 5.8N are placed in pans A and B respectively. If the system s released from rest (take g = 10ms^{-2}). Find the

(a) Tension in the string
(b) Reaction of the scale pan holding the 3.8N weight



Weight of the scale pan = $\frac{20}{1000} x\ 10 = 0.2N$

Total weight of A = 3.8 + 0.2 = 4N

Total weight of B = 5.8 + 0.2 = 6N

For 6N: 6 – T = 0.6a …….. (i)

For 4N: T – 4 = 0.4a …….. (i)

Adding (i) and (ii)

a = 2ms^{-2}

T = 4 + 0.4 x 2 = 4.8N

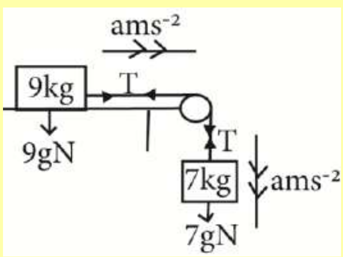
For scale pan A R_2 – 3.8 = 0.38a

R_2 = 3.8 + 2 x 0.38 x 2 = 4.56N

Example 3

A mass of 9kg resting on a smooth horizontal table is connected by a light inextensible string passing over a smooth pulley at the edge of the table to the pulley is a 7kg mass hanging freely 1.5m above the ground. Find

(a) common acceleration
(b) tension in the string
(c) force on the pulley when the system is allowed to move freely
(d) time taken for the 7kg mass to hit the ground



F = ma

For 7kg mass: 7g – T = 7a

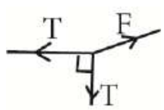
For 9kg mass: T = 9a

(i) + (ii): 7g = 16a

a = $\frac{7\ x\ 9.8}{16} = 4.29ms^{-2}$

(b) Tension: T = 9a = 9 x 4.29 = 38.61N

(c) The force on the pulley, F:



F =$\sqrt{T^2 + T^2} = T\sqrt{2}$ = 38.61$\sqrt{2}$ = 54.603N

(d) s = ut + $\frac{1}{2}at^2$

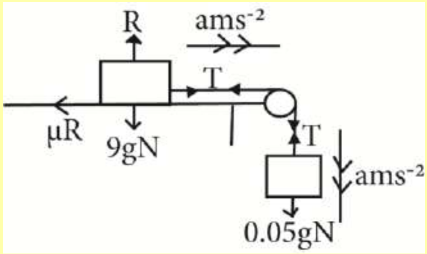
1.5 = 0 x t + $\frac{1}{2} x\ 4.29\ x\ t^2$

t = 0.84s

Example 4

A mass of 90g resting on a rough horizontal table is connected by a light inextensible string passing over a smooth pulley at the edge of the table attached to a 50g mass hanging freely. The coefficient of friction between the 90g mass and the table is $\frac{1}{3}$ and the system is released from rest, find

(a) common acceleration (b) the tension in the string



For 50g mass: $0.05g - T = 0.05a$ ....(i)

For 90g mass: $T - \mu R = 0.09a$

$T - \frac{1}{3}x\ 0.09\ x\ 9.8 = 0.09a$ .... (ii)

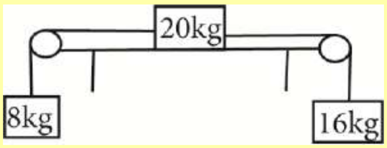
(i) + (ii): $0.05g - \frac{1}{3}x\ 0.09\ x\ 9.8 = 0.14a$

$a = \frac{0.02g}{0.14} = 1.4ms^{-2}$

(b) $0.05g - T = 0.05a$

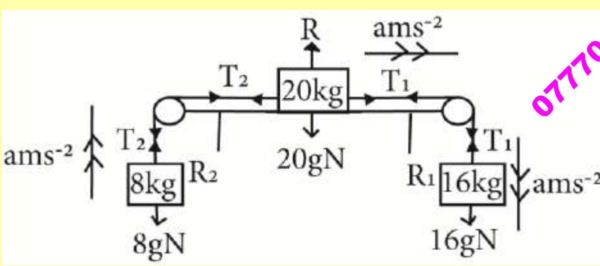
$T = 0.05 \times 9.8 - 0.05 \times 1.4 = 0.42N$

Example 5



The figure shows a block of mass 20kg resting on a smooth horizontal table. It is connected by light inextensible string which pass over fixed pulleys at the edges of the table to two loads of masses 8kg and 16kg which hang freely vertically. When the system is released freely calculate:

(a) Acceleration of 16kg mass
(b) tension in the string
(c) reaction on each pulley



For 16kg mass: $16g - T_1 = 16a$ …… (i)

For 20kg mass: $T_1 - T_2 = 20a$ ……. (ii)

For 8kg mass: $T_2 - 8g = 8a$ ……… (iii)

(i) + (ii) + (iii): $8g = 44a$

$a = \frac{8\ x\ 9.8}{44} = 1.782ms^{-2}$

(b) Tension in the string

$16g - T_1 = 16a$

$T_1 = 16 \times 9.8 - 16 \times 1.782 = 128.288N$

$T_2 - 8g = 8a$

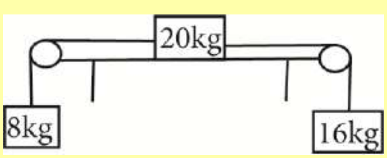
$T_2 = 8 \times 9.8 + 8 \times 1.782 = 92.656N$

(c) Reaction on each pulley

$R_1 = \sqrt{T_1^{\ 2} + T_1^{\ 2}} = T_1\sqrt{2} = \sqrt{2} \times 128.288 = 181.427N$

$R_2 = \sqrt{T_2^{\ 2} + T_2^{\ 2}} = T_2\sqrt{2} = \sqrt{2} \times 92.626 = 131\ N$

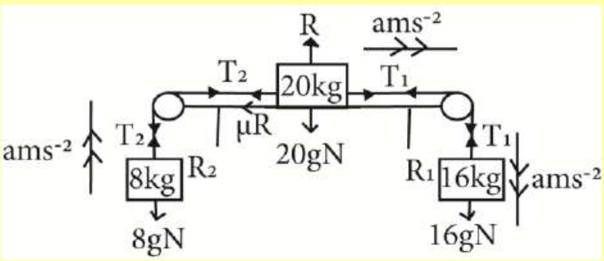
Example 6



The figure shows a block of mass 20kg resting on a rough horizontal table of coefficient of friction 0.21. It is connected by light inextensible string which pass over fixed pulleys at the edges of the table to two loads of masses 8kg and 16kg which hang freely vertically. When the system is released freely calculate:

(a) acceleration of the 16kg mass
(b) Tension in each string
(c) reaction on each pulley

Solution



For 16kg mass: $16g - T_1 = 16a$ …… (i)

For 20kg mass: $T_1 - T_2 - 20g\mu = 20a$ ……. (ii)

For 8kg mass: $T_2 - 8g = 8a$ ……… (iii)

(i) + (ii) + (iii): $8g - 20g\mu = 44a$

$a = \frac{8 \times 9.8 - 20 \times 9.8 \times 0.21}{44} = 0.846ms^{-2}$

(b) Tension in the string

$16g - T_1 = 16a$

$T_1 = 16 \times 9.8 - 16 \times 0.846 = 143.264N$

$T_2 - 8g = 8a$

$T_2 = 8 \times 9.8 + 8 \times 0.846 = 85.168N$

(c) Reaction on each pulley

$R_1 = \sqrt{T_1^{\ 2} + T_1^{\ 2}} = T_1\sqrt{2} = 2 \times 128.291 = 202.606N$

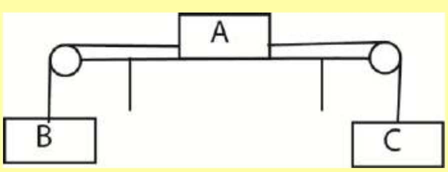
$R_2 = \sqrt{T_2^{\ 2} + T_2^{\ 2}} = T_2\sqrt{2} = \sqrt{2} \times 85.168 = 120.446N$

Revision exercise 1

1. Two particles of masses 7kg and 3kg are connected by a light inelastic string passing over a smooth fixed pulley. Find
   (a) acceleration of the particles [3.92ms^{-2}]
   (b) the tension in the string [41.16N]
   (c) the force on the pulley [82.32N]
2. Two particles of masses 6kg and 2kg are connected by a light inextensible string passing over a smooth fixed pulley. With the masses hanging vertically, the system is released from rest. Find
   (a) acceleration of the particles [4.9ms^{-2}]
   (b) the tension in the string [29.4N]
   (c) distance moved by the 6kg mass in the first 2s of motion [9.8m]
3. A man of mass 70kg and a bucket of bricks of mass 100kg are tied to the opposite ends of a rope which passes over a frictionless pulley so that they hang vertically downwards.
   (a) what is the tension in the section of the rope supporting the man [807.06N]
   (b) what is the acceleration of the bucket [1.73ms^{-2}]
4. Two particles of masses 200g and 300g are connected to a light inelastic string passing over a smooth pulley; when released freely find
   (i) common acceleration [1.96ms-2]
   (ii) the tension in the string [2.352N]
   (iii) the force on the pulley [4.704N]

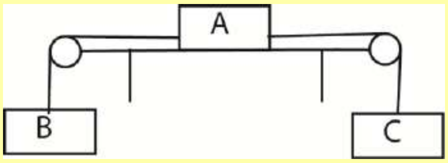
5. The diagram below shows a particles of mass 8kg connected to a light scale pan by a light inextensible string which passes over a smooth fixed pulley. The scale pan holds two blocks A and B of mass 3kg and 4kg, with B resting on top of A. If the system is released from rest find
   (a) acceleration of the system [$0.653ms^{-2}$]
   (b) the reaction between A and B[41.813N]
6. A mass of 5kg is placed on a smooth horizontal table and connected by a light string to a 3kg mass passing over a smooth pulley at the edge of the table and hanging freely. If the system is allowed to move, calculate
   (a) the common acceleration of the masses [$3.675ms^{-2}$]
   (b) the tension in the string [18.375N]
   (c) the force acting on the pulley[26N]
7. A mass of 3kg on a smooth horizontal table is attached by a light inextensible sting passing over a smooth pulley at the edge of the table, to another mass of 2kg hanging freely 2.1m above the ground; find
   (a) common acceleration[$3.92ms^{-2}$]
   (b) the tension in the string [11.76N]
   (c) The force on the pulley in the system if it's allowed to move freely. [16.63N]
   (d) the velocity with which the 2kg mass hits the ground [$4.06ms^{-1}$]
8. A mass of 5kg rests on a rough horizontal table and is connected by a light inextensible string passing over a smooth pulley at the edge of the table to a 3kg mass hanging freely. the coefficient of friction between the 5kg mass and the table is 0.25 and the system is released from rest find
   (a) common acceleration[$2.144ms^{-2}$]
   (b) tension in the string [22.97N]

9. A mass of 11kg rests on a rough horizontal table and is connected by a light inextensible string passing over a smooth pulley at the edge of the table to 500g mass hanging freely. The coefficient of friction between the 1kg mass and the table is 0.1 and the system is released from rest find
   (a) common acceleration [$2.61ms^{-2}$]
   (b) the tension in the string [3.593N]
10. The objects of mass 3kg and 5kg are attached to ends of a cord which passes over a fixed frictionless pulley placed at 4.5m above the floor. The objects are held at rest with 3kg mass touching the floor and the 5kg mass at 4m above the floor and then release, what is
    (a) the acceleration of the system[$2.45ms^{-2}$]
    (b) tension in the chord[36.75N]
    (c) the time that will elapse before the 5kg object hits the floor [1.81s]
11.



The diagram shows a particle A of mass 2kg resting on a rough horizontal table of coefficient of friction 0.5. It is attached to the particle B of mass 5kg and C of mass 3kgby light inextensible strings hanging over smooth pulleys. If the system is released from rest find the
   (a) common acceleration [$0.98ms^{-2}$]
   (b) the tension of each string [12.37N, 44.15N]

12.

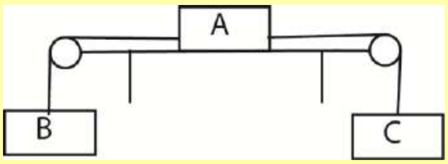


The diagram shows a particle A of mass 3kg resting on a rough horizontal table of coefficient of friction 0.5. It is attached to the particle B of mass 4kg and C of mass 6kgby light inextensible strings hanging over smooth pulleys. If the system is released from rest find the

(a) common acceleration [0.75ms^{-2}]

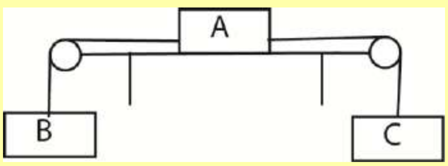
(b) the tension of each string [54.277N, 31.662N]

13.



The diagram shows a particle A of mass 5kg resting on a rough horizontal table of coefficient of friction 0.5. It is attached to the particle B of mass 3kg and C of mass 2kgby light inextensible strings hanging over smooth pulleys. If the system is released from rest body B descend with an acceleration of 0.28ms-2, find the coefficient of friction between the body A and the surface of the table. [0.143]

14.



The diagram shows a particle A of mass 10kg resting on a smooth horizontal table. It is attached to the particle B of mass 4kg and C of mass 7kgby light inextensible strings hanging over smooth pulleys. If the system is released from rest find the

(c) common acceleration [1.4ms^{-2}]

(d) the tension of each string [44.8N, 58.8N]

Connected particles on inclined planes

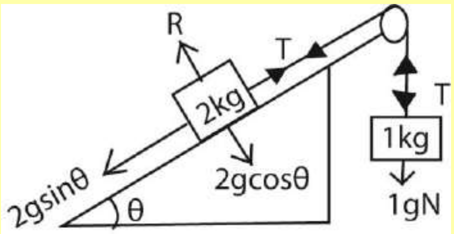
Example 7

A mass of 2kg lies on a smooth plane of inclination 1 in3. One end of a light inextensible sting is attached to this mass and the string passes up the line of greatest slope over a smooth pulley fixed at the top of the plane is freely suspended mass of 1kg at its end. If the system is released from rest , find the

(i) acceleration of the masses

(ii) tension in the string

(iii) distance each particle travels in the first 2s.



$\sin\theta = \frac{1}{3}$ F =ma

For 2kg mass: T – 2gsinθ = 2a….. (i)

For 1kg mass: 1g – T = 1a ……….. (ii)

(ii) + (i): = 1g – 2gsinθ = 3a

$a = \frac{9.8 - 2 \times 9.8 \times \frac{1}{3}}{3} = 1.089\text{ms}^{-2}$

(ii) Tension: 1g – T = 1a

T = 9.8 – 1.089 = 8.71N

(iii) $s = ut + \frac{1}{2}at^2$

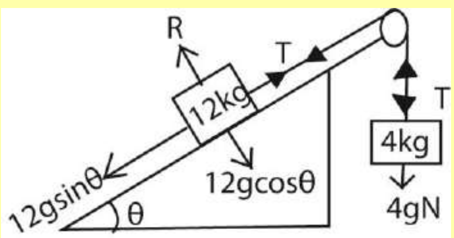
$s = 0 \times 2 + \frac{1}{2} x\ 1.089\ x\ 2^2 = 2.178m$

Example 8

A mass of 12kg lies over a smooth incline plane 6m long and 1m high. One end of a light inextensible string is attached to this mass and the string passes up the line of greatest slope over a smooth pulley fixed at the top of the plane to freely suspended mass of 4kg at its other end. If the system is released from rest, find the

(a) acceleration of the system
(b) velocity with which the 4kg mass hits the ground
(c) time the 4kg mass takes to hit the ground.

Solution



$\sin\theta = \frac{1}{6}$ F =ma

For 12kg mass: T – 12gsinθ = 2a….. (i)

For 4kg mass: 4g – T = 4a ……….. (ii)

(ii) + (i): = 4g – 12gsinθ = 16a

$a = \frac{4 \times 9.8 - 12 \times 9.8 \times \frac{1}{6}}{16} = 1.225\text{ms}^{-2}$

(ii) Tension: 4g – T = 4a

T = 4 x 9.8 – 4 x 1.225 = 34.3N

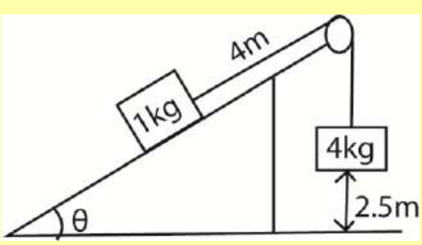
(iii) $s = ut + \frac{1}{2}at^2$

$1 = 0 \times t + \frac{1}{2} x\ 1.225\ x\ t^2$

t = 1.28s

Example 9

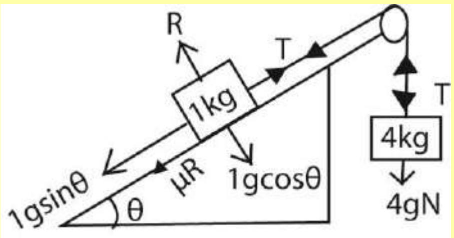
A mass of 1kg lies on a rough plane with coefficient of friction 0.25. One end of a light inextensible string is attached to 1kgmass and passes up the line of greatest slope over a smooth fixed pulley at the top of the plane and the other end of a string is tied to a mass of 4kg hanging freely.



The plane makes an angle θ with the horizontal where $\sin\theta = \frac{3}{5}$. When the system is released from rest, find:

(i) the acceleration of the system
(ii) tension in the string
(iii) velocity with which the 4kg mass hits the floor
(iv) velocity with which the 1kg mass hits the pulley

Solution



$\sin\theta = \frac{3}{5}$; $\cos\theta = \frac{4}{5}$ F =ma

For 12kg mass: T – 1gsinθ- 0.25R = 1a….. (i)

For 4kg mass: 4g – T = 4a ……….. (ii)

(ii) + (i): = 4g – 1gsinθ - 0.25R = 5a

$a = \frac{4 x 9.8 - 1 x 9.8 x \frac{3}{5} - 0.25 x 1 x 9.8 x \frac{4}{5}}{5} = 6.272ms^{-2}$

(ii) Tension: 4g – T = 4a

T = 4 x 9.8 – 4 x 6.272 = 14 .112N

(iii) $v^2 = u^2 + 2as$ but u = 0

$v = \sqrt{2 x 6.272 x 2.5} = 5.6ms^{-1}$

(iv) When a 4kg mass hits the floor, the 1kg mass has still to move 4 – 2.5 = 1.5m before hitting the pulley It will experience a retarding force due to gravity and friction

F = 1a = 1gsinθ + 0.25R

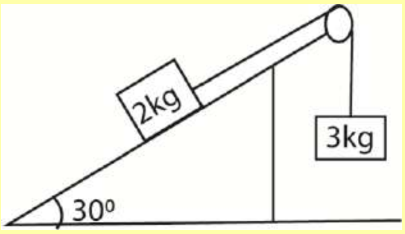
$=(1 x 9.8 x \frac{3}{5} + 0.25 x 1 x 9.8 x \frac{4}{5})$

$a = - 7.84ms^{-2}$

$v^2 = u^2 + 2as$

$v = \sqrt{5.6^2 - 2 x 7.84 x 1.5} = 2.8ms^{-1}$

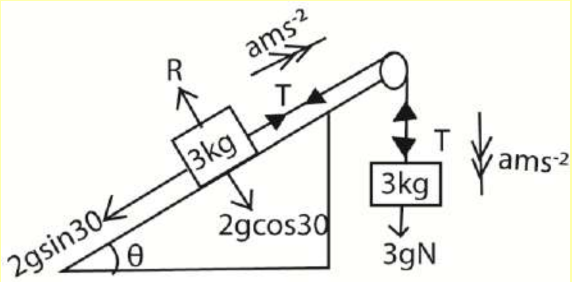
Example 10



A particle of mass 2kg on a rough plane inclined at 300 to the horizontal is attached by means of light inextensible string passing over a smooth pulley at the top edge of the plane to a particle of mass 3kg which hangs freely. If the system is released from rest with above parts of the strings taut, the 3kg mass travels a distance of 0.75m before attains a speed of $2.25ms^{-1}$. Calculate

(a) acceleration
(b) coefficient of friction between the plane and 2kgmass
(c) reaction of the pulley on the string

Solution



(i) $v^2 = u^2 + 2as$

$a = \frac{2.25^2 - 0^2}{2 \times 0.75} = 3.375ms^{-2}$

(ii) F = ma

For 2kg mass: $T - 2g\sin\theta - \mu R = 2x\ a$ ….. (i)

For 4kg mass: $3g - T = 3\ a$ ……….. (ii)

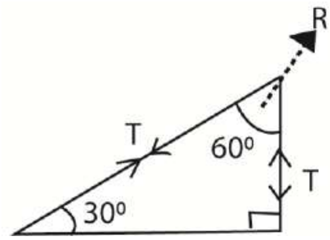
(ii) + (i): $3g - 2g\sin\theta - \mu(2g\cos\theta) = 5a$

$$\mu = \frac{(3 \times 9.8) - (2 \times 9.8\sin 30 + 5 \times 3.375)}{2 \times 9.8 \times \cos 30} = 0.161$$

(iii)

Tension: $3g - T = 3a$

$T = 3 \times 9.8 - 3 \times 3.375 = 19.275N$



Using parallelogram law of force

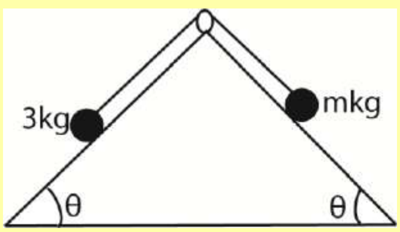
$R^2 = T^2 + T^2 + 2 \times T\cos 60 = 3T^2$

$R = 19.275\sqrt{3} = 33.4N$

Double inclined plane

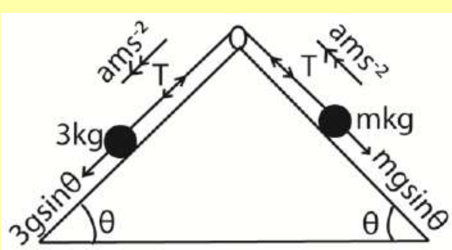
Example 11

The diagram below shows two smooth fixed slopes each inclined at an angle θ to the horizontal where sinθ = 0.6. Two particles of mass 3kg and mkg, where m< 3kg are connected by a light inextensible string passing over a smooth fixed pulley.



The particles are released from rest with a string taut. After travelling a distance of 1.08m, the speed of the particle is 1.8ms-1. Calculate

(i) acceleration
(ii) tension in the string
(iii) value of m



(i) $v^2 = u^2 + 2as$

$1.8^2 = 0^2 + 2 \times a \times 1.08$

$a = 1.5ms^{-2}$

(ii) F = ma

For 3kg mass: $3g\sin\theta - T = 3a$

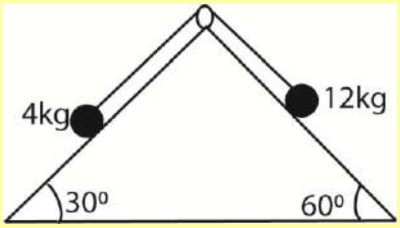
$T = 3 \times 9.8 \times 0.6 - 3 \times 1.5 = 13.14N$

(iii) For mkg mass; $T - mg\sin\theta = ma$

$13.14 = m(9.8 + 1.5)$; $m = 1.78kg$

Example 12

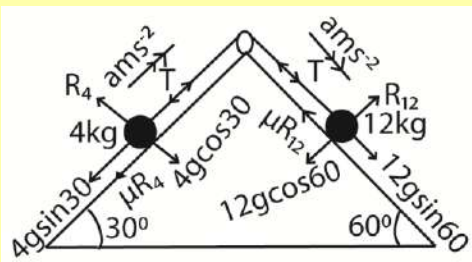
Two rough planes inclined at 300 and 600 to the horizontal and of the same height are placed back to back. Masses of 4kg and 12 kg are placed on the faces and connected by a light string passing over smooth pulley on the top of the planes.



If the coefficient of friction is 0.5 on both faces, find

(a) acceleration
(b) Tension in the strings

Solution



For 4kg mass: T -4gsin30 – 0.5 x 4gcos30 = 4a .... (i)

For 12kg mass: 12gsin60 –T – 0.5 x 12gcos 60 = 12a ..... (ii)

(i) + (ii)

12gsin60 – 4gsin30 - 0.5(4gcos30 + 12gcos 60) = 16a

a = 2.25ms^{-2}

(b) For 4kgmass

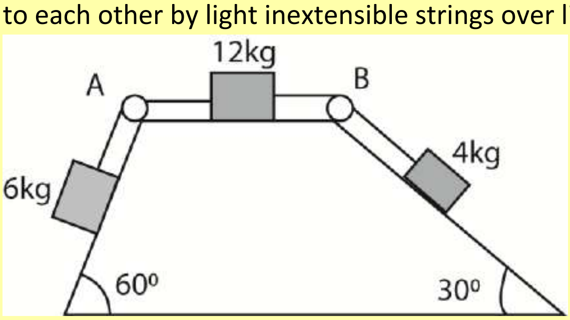
T -4gsin30 – 0.5 x 4gcos30 = 4a

T = 4gsin30 + 0.5 x 4gcos30 + 4x2.25

T = 45.54

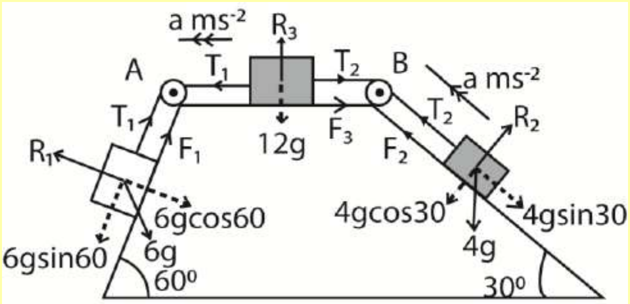
Example 13

1. The diagram below shows a 12kg mass on a horizontal rough plane. The 6kg and 4kg masses are on rough planes inclined at angles 600 and 300 respectively. The masses are connected to each other by light inextensible strings over light smooth pulleys A and B.



The planes are equally rough with coefficient of friction $\frac{1}{12}$. If the system is released from rest find the;

(a) Acceleration of the system (08marks)



For 6kg mass

6gsin60-(T$_1$ + $\frac{1}{12}\ x\ 6gcos60) = 6a$

6gsin60 – T$_1$ - $\frac{1}{2}cos60 = 6a$ ………………………… (i)

For 4kg mass

T$_2$ – ($\frac{1}{12}\ x\ 4gcos30 + 4gsin30) = 4a$

T$_2$ - $\frac{1}{3}gcos30 - 4gsin30$ = 4a …………………. (ii)

For 12kg mass

T$_1$ – (T$_2$ +$\frac{1}{12}\ R_3$) = 12a

T$_1$ – (T$_2$ +$\frac{1}{12}\ x\ 12g$) = 12a

T$_1$ –T$_2$ - g = 12a……………………………………. (iii)

Eqn. (i) + Eqn. (ii) + Eqn. (iii)

6gsin60 -$\frac{1}{2}gcos60 - \frac{1}{3}gcos30 - 4gsin\ 30 - g = 22a$

16.24327742 = 22a

a = $\frac{16.24327742}{22} = 0.73833ms^{-2}$

(b) Tensions in the strings. (04marks)

From equation (i)

$T_1 = 6gsin60 - \frac{1}{2}cos60 - 6a$

= $6gsin60 - \frac{1}{2}cos60 - 6\ x\ 0.73833$

= 44.0423N

From eqn. (ii)

T$_2$ = $\frac{1}{3}gcos30 + 4gsin30 + 4a$

= $\frac{1}{3}gcos30 + 4gsin30 + 4\ x\ 0.73833$

= 25.3823N

Revision exercise 2

1. A mass of 2kg lies on a smooth inclined plane 9m long and 3m high. One end of a light inextensible string is attached to this mass and the string passes up the line of greatest slope

over a smooth pulley fixed at the top of the plane is freely suspended mass of 1kg at its other end. If the system is released from rest, find

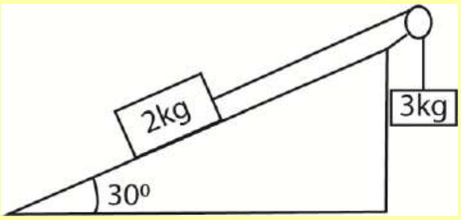
(i) acceleration of the system[1.089ms^{-2}]

(ii) tension in the string. [8.711N]

(iii) velocity with which the 1kg mass will hit the ground[2.556ms^{-1}]

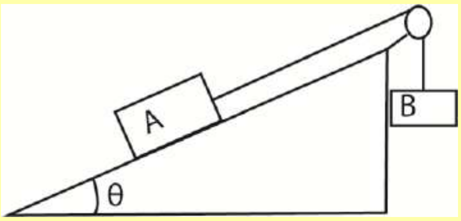
(iv) time the 1kg mass will hit the ground[2.347s]

2. A mass of 15kg lies on a smooth plane of inclination in 49. One end of a light inextensible string is attached to this mass and the string passes up a line of greatest slope, over a smooth pulley fixed at the top of the plane is freely suspended mass of 10kg at its other end. If the system is released from rest, find the acceleration of the masses and the distance each travel in the first 2s. [3.8ms^{-2}, 7.6m]
3. A mass of 2kg lies on a rough plane which is inclined at 300 to the horizontal. One end of a light inextensible string is attached to this mass and the string passes up a line of greatest slope, over a smooth pulley fixed at the top of the plane is freely suspended mass of 5kg at its other end. The system is released from rest as the 2kg mass accelerates up the slope, it experiences a constant resistance to motion of 14N down the slope due to friction. Find the tension of the string. [31N]
4. A mass of 10kg lies on a smooth plane which is inclined at θ to the horizontal. The mass is 5m from the top, measured along the plane. One end of a light inextensible string is attached to this mass and the string passes up a line of greatest slope, over a smooth pulley fixed at the top of the plane is freely suspended mass of 15kg at its other end. The 15kg mass is 4m above the floor. The system is released from rest and the string first goes slack $1\frac{3}{7}s$ latter. Find the value of θ. [30^0]
5. One of two identical masses lies on a smooth plane, which is inclines at $\sin^{-1}\left(\frac{1}{4}\right)$ to the horizontal and is 2m from the top. A light inextensible string attached to this mass passes along the line of greatest slope over a smooth pulley fixed at the top of the incline, the other end carries the other mass hanging freely 1m above the floor. If the system is released from rest, find the time taken for the hanging mass to reach the floor. [0.663s]
6. A particle of mass 2kg on a smooth plane inclined at 300 to the horizontal is attached by means of a light inextensible string passing over a smooth pulley at the edge of the plane to a particle of mass 4kg which hangs freely.



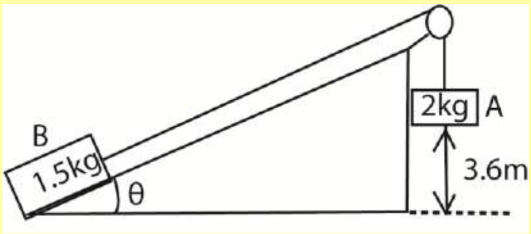
If the system is released from rest with above parts of the string taut, find the speed acquired by the particles when both have moved a distance of 1m [2.8ms^{-1}}

7. A body A of mass 13kg lying on a rough inclined plane, coefficient of friction, μ. From A, a light inextensible string passes up the line of greatest slope and over a smooth fixed pulley to a body B of mass mkg hanging freely, the plane makes an angle θ with the horizontal where sinθ = $\frac{5}{13}$.



When m = 1kgand the system is released from rest, B has upward acceleration of a ms^{-2}. When m = 11kg and the system released from rest, B has downward acceleration of ams^{-2}. Find a and μ. [1.96ms^{-2}, 0.1]

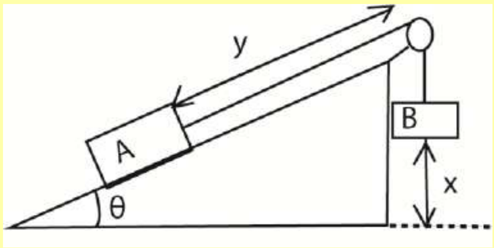
8. A particle A of mass 2kg and B of mass 1.5kg are connected by light inextensible string passing over a smooth pulley. The system is released from rest with A at height of 3.6m above the horizontal ground and B at the foot of a smooth slope inclined at an angle θ to the horizontal where $\sin\theta = \frac{1}{6}$. Take g = 10ms^{-2}.



Calculate

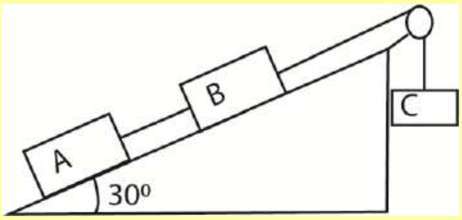
(i) the magnitude of the acceleration of particles [6ms^{-1}]
(ii) the speed with which A reaches the ground[5ms^{-2}]
(iii) the distance B moves up the slope before coming to instantaneous rest. [14.4m]

9. A mass A of 4kg and a mass B of 3kg are connected by a light inextensible string passing over a smooth pulley. The system is released from rest and mass accelerates up along a smooth slope inclined at an angle θ to horizontal where θ = 30^0.



If y = 3m and x = 2.8m, calculate the velocity with which A hits the pulley [2.42ms^{-1}]

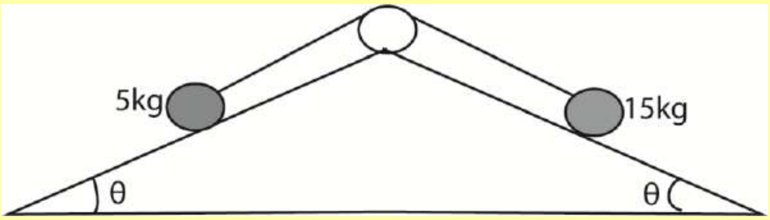
10. The diagram below shows particles A, B and C of masses 10kg, 8kg and 2kg respectively connected by a light inextensible strings. The string connecting B and C passes over a smooth light pulley fixed at the top of the plane.



If the coefficient of friction between the plane and particles A and B are 0.22 and 0.25 respectively, calculate

(i) acceleration of the system [1.6477ms^{-2}]
(ii) tension in the strings[22.89N, 13.851N]

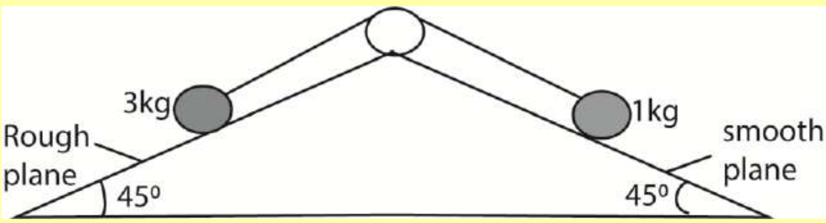
11. The diagram below shows two smooth fixed slopes each inclined at an angle θ to the horizontal where $\sin\theta = \frac{3}{5}$. Two particles of mass 5kg and 15kg are connected by a light inextensible string over a smooth fixed pulley.



The particles are releases from rest with a string taut calculate

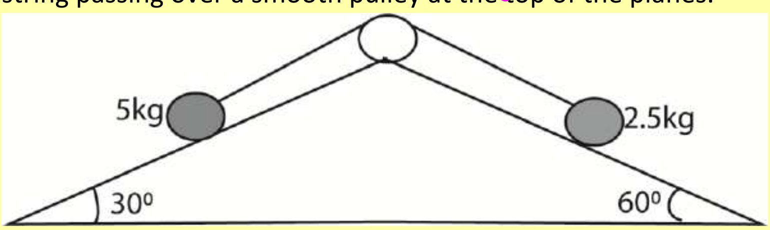
(i) Acceleration of the particles
(ii) Tension in the string

12. The diagram below shows two smooth plane and a rough plane both inclined at 450 to the horizontal. Two particles of mass of mass 1kg and 3kg are connected by light inextensible string passing over a smooth fixed pulley.



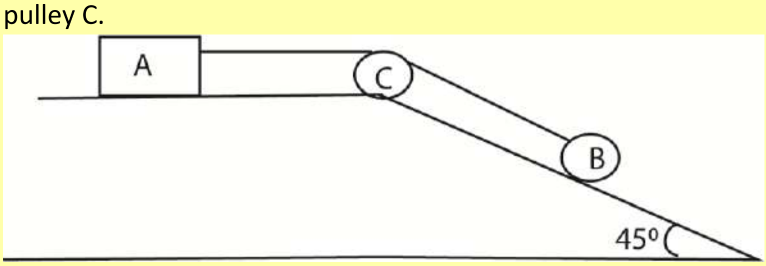
The particle are released from rest with a string taut. Calculate
(i) acceleration of the particle [$1.4ms^{-2}$]
(ii) tension in the string [.48N
(iii) coefficient of friction [0.4]

13. Two equally rough planes inclined at 30^0 and 60^0 to the horizontal and of the same height are placed back to back. Masses of 5kg and 2.5kg are placed on the faces and connected by a light string passing over a smooth pulley at the top of the planes.



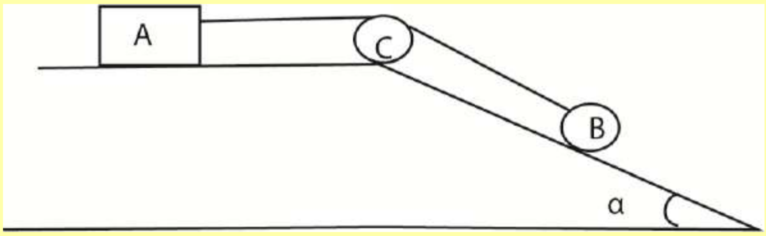
If the string is taut and 5kg is just about to slip downwards find the
(i) coefficient of friction[0.06]
(ii) tension in the string [21.9538N]

14. In the diagram, particle A and particle B are masses of 10kg and 8kg respectively and rest on planes as shown below. They are connected by a light inextensible string passing over a smooth pulley C.



Find the acceleration in the system and the tension in the string if
(i) the particles are in contact with smooth planes [$3.08ms^{-2}$, 30.N]
(ii) the particles are in contact with rough planes with coefficient of friction 0.25. [$0.95ms^{-2}$, 33.98N]

15. In the diagram particles A and B are of masses mkg and 5mkg respectively and rest on the planes as shown below. They are connected by a light inextensible string passing over a smooth fixed pulley at C

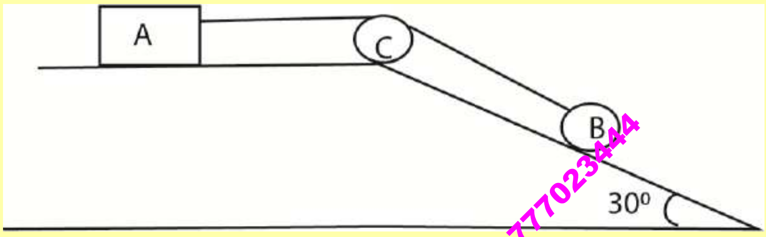


Find the acceleration of the system and the tension in the string if sinα = $\frac{4}{5}$ when;

(i) the particles are in contact with smooth plane[6.533ms-2, 6.533N]

(ii) the particles are in contact with rough plane of coefficient of friction $\frac{1}{3}$. [4.356ms^{-2}, 7.622N]

16. In the diagram particles A and B of masses 2.4kg and 3.6kg respectively. A rests on a rough horizontal plane (coefficient of friction 0.5), it is connected by a light inextensible string passing over a smooth pulley C to particle B resting on smooth plane inclined at 30^0 to the horizontal.



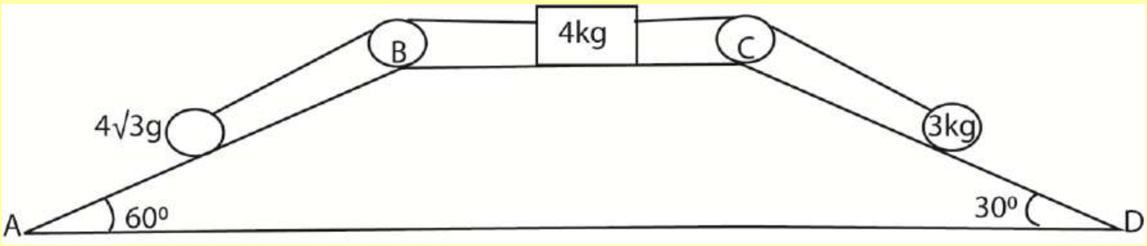
When the system is released from rest find

(i) acceleration of the system and tension in the string [0.98ms^{-2}, 14.112N]

(ii) the force on the pulley C [7.3049N]

(iii) the velocity of A mass after 2 seconds[1.96ms^{-2}]

17. The diagram below shows a 4kg mass on a horizontal rough plane with coefficient of friction 0.25. The $4\sqrt{3}kg$ mass rests on a smooth plane inclined at angle 600 to the horizontal while the 3kg mass rests on a rough plane inclined at an angle 300 to the horizontal and coefficient of friction $\frac{1}{\sqrt{3}}$. the masses are connected to each other by a light inextensible strings over light smooth fixed pulleys B and C.



Find the

(i) acceleration of the system[1.407ms-2]

(ii) tension in the string [49.051N, 33.622N]

(iii) work done against frictional force when the particles each moved 0.5m [12.25J]

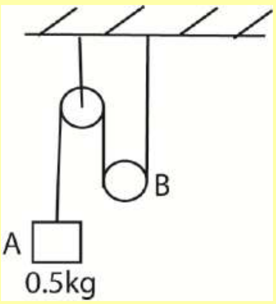
Multiple connections

- Acceleration of a particle moving between to portions of the string is equal to half the net acceleration of the particle (s) attached to the end of the string
- The tension in uninterrupted string is constant
- The tensions in interrupted strings are different.

Case I: A pulley moving between two portions of a string

Example 15

The diagram below shows particle A of mass 0.5kg attached to one end of alight inextensible string passing over a fixed pulley and under a movable light pulley B. The other end of the string is fixed as shown

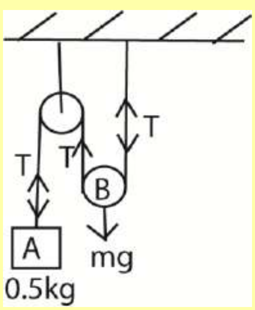


(i) What mass should be attached at B for the system to be in equilibrium
(ii) If B is 0.8kg what are the accelerations of particles A and pulley B?
(iii) find the tension in the string in (ii)

Solution

(i) Let T = tension in string

m = mass at B



For the system to be in equilibrium upward forces are equal to downward force. by resolving vertically

For mass A: T= 0.5g ……… (i)

For pulley B: 2T = mg

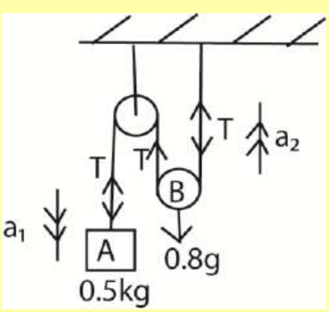
$T = \frac{mg}{2}$ ……… (i)

equating (i) to (ii)

$\frac{mg}{2} = 0.5g$; m = 1kg

(ii) Let a_1 = acceleration of A

a_2 = acceleration of B



For mass A: $0.5g - T = 0.5a_1$……… (i)

For pulley B: $2T - 0.8g = 0.8a_2$ but $a_2 = \frac{1}{2} a_1$

⇨ $2T - 0.8g = 0.4a_1$
$T - 0.4g = 0.2a_1$ …….. (i)

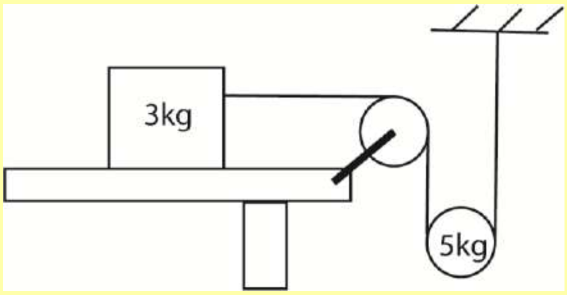
(i) + (ii) $a1 = \frac{9.9}{7} = 1.4ms^{-2}$ and $a2 = 0.7\ ms^{-2}$

From eqn. (i)

T = 0.5 x 9.8 - 0.5 x 1.4 = 4.2N

Example 16

A particle of mass 3kg on a smooth horizontal table is tied to one end of the string which passes over a fixed pulley at the edge and then under a movable pulley of mass 5kg, its other end being fixed so that the string beyond the table are vertical.

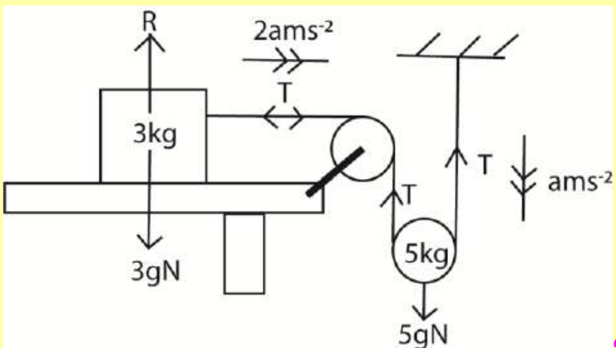


Find

(i) acceleration of 3kg and 5g

(ii) Tension in the string

Solution



F = ma

For 3kg: T = 3 x 2a ……… (i)

For 5kg: 5g – 2T = 5a …… (ii)

(ii) + 2 x (i)

5x 9.8 = 17a

a = $\frac{49}{17}$ = 2.8824ms^{-2}

Acceleration of 5kg: = 2.8824ms^{-2}

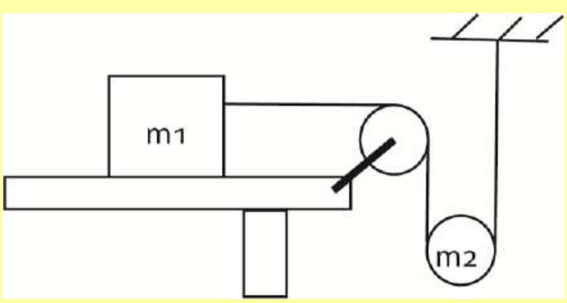
Acceleration of 3kg: = 2.8824 x 2ms^{-2}

= 5.7648ms^{-2}

T = 6a = 2.8824 x 6 = 17.2944N

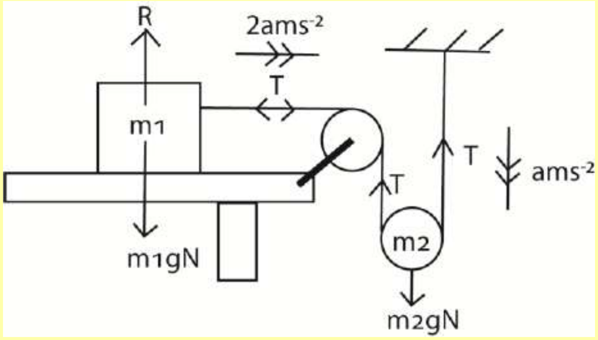
Example 17

A particle of mass m_1 on a smooth horizontal table is tied to one end of the string which passes over a fixed pulley at the edge and then under a movable pulley of mass m_2, its other end being fixed so that the parts of the string beyond the table is vertical.



Show that m_2 descends with acceleration $\frac{m_2 g}{4m_1+m_2}$

Solution



F = ma

For m_1 kg mass: T = m_1 x 2a …… (i)

For m_2 kg mass: m_2g – 2T = m_2a … (ii)

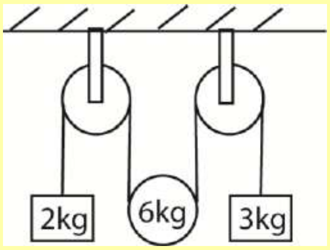
(ii) + 2 x (i)

$m_2g = 4m_1a + m_2a$

a = $\frac{m_2 g}{4m_1+m_2}$

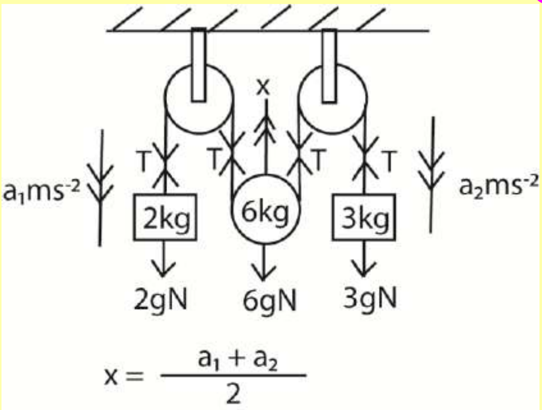
Example 18

A string has a load of mass 2kg attached at one end. The string passes over a smooth fixed pulley then under a movable pulley of mass 6kg and over another fixed pulley and has a load of mass 3kg attached to its end.



Find the acceleration of the movable pulley and the tension in the string

Solution



For 2kg mass: 2g – T = $2a_1$ …………………….. (i)

For 3kg mass: 3g – T = $3a_2$ …………………….. (ii)

For 6kg mass: 2T – 6g = $6\,x\,\frac{1}{2}(a_1 + a_2)$…. (iii)

(ii) – (i): g = $(3a_2 - 2a_1)$ ……………………………. (iv)

2 x (ii) + (iii): 0 = $9a_2 + 3a_1$ …………………….. (v)

3 x (iv)- (v): 3g = $-9a_1$

a1 = $\frac{-g}{3}$ = -3.267ms^{-2}

From (v): 0 = $9a_2 + 3a_1$

0 = $9a_2$ + 3(-3.267)

a_2 = 1.089ms^{-2}

Acceleration of pulley = $\frac{1}{2}(a_1 + a_2)$

= $\frac{1}{2}(-3.267 + 1.089)$

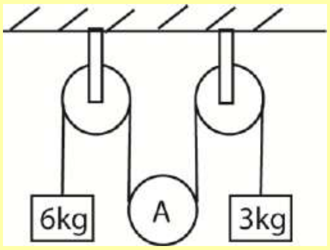
= -1.089ms^{-2}

Tension: T = 2g – $2a_1$

T = 2 x 9.8 – 2 x -3.267 = 26.134N

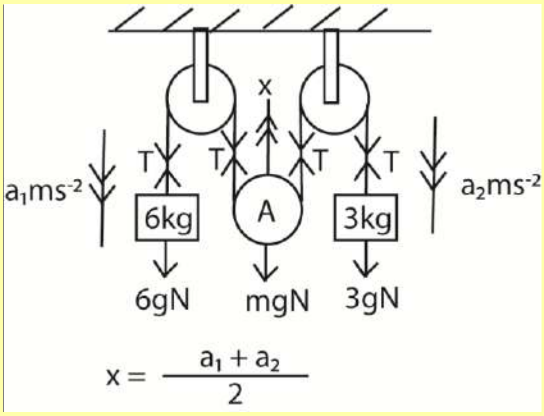
Example 19

In the pulley system below, A is a heavy pulley which is free to move



Find the mass of pulley A if it does not move upwards of downwards when the system is released from rest.

Solution



For 2kg mass: $6g - T = 6a_1$ …………………….. (i)

For 3kg mass: $3g - T = 3a_2$ …………………….. (ii)

For mkg mass: $2T - mg = 0$…………………….. (iii)

$a_1 = -a_2$

$6g - T = -6a_2$ ……….. (iv)

$3g - T = 3a_2$ …………. (v)

(iv) – (v)

$3g = 9a_2$

$a_2 = \frac{-g}{3} = -3.267ms^{-2}$

$3g - T = 3a2$

$T = 3 \times 9.8 - 3(-3.267) = 39.201$

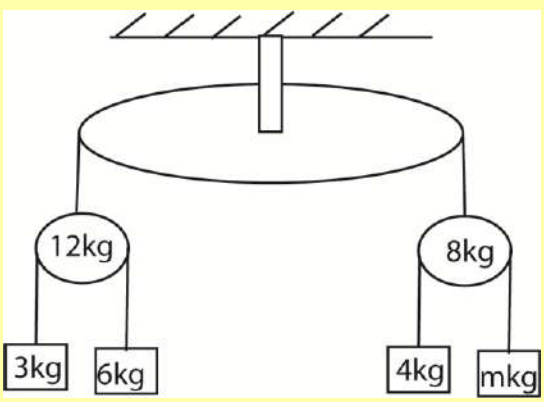
$2T - mg = 0$

$m = \frac{2 \times 39.201}{9.8} = 8kg$

Case 2: A pulley moving on one portion of a string

Example 20

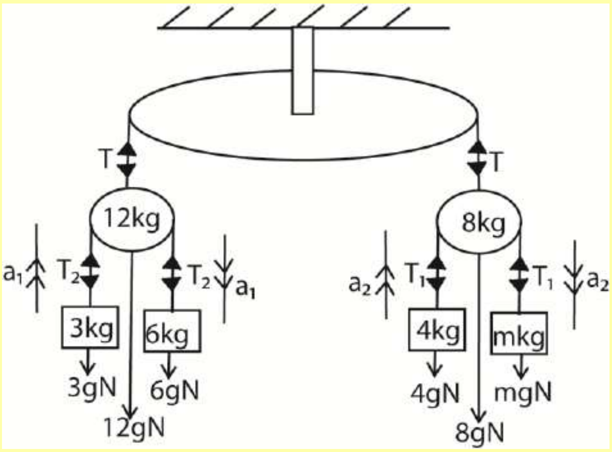
The diagram below shows two pulley to pulleys of masses 8kg and 12kg connected by a light inextensible string hanging over a fixed pulley.



The hanging portions of the strings are vertical. Given that the string of the fixed pulley remains stationary, find the

(i) tensions in the string
(ii) value of m

Solution.



For 3kg mass: $T_2 - 3g = 3a_2$ .... (i)

For 6kg mass: $6g - T_2 = 6a_2$ .... (ii)

For 4kg mass: $T_1 - 4g = 4a_1$ .... (iii)

For mkg mass: $mg - T_1 = ma_1$ ....(iv)

For 8kg mass: $2T_1 + 8g = T$ .... (v)

For 12kg mass: $2T_2 + 12g = T$ .... (vi)

eqn. (i) + eqn. (ii): $3g = 3a_2$

$a_2 = \frac{3 \times 9.8}{9} = 3.2667\text{ms}^{-2}$

eqn. (i): $T_2 - 3g = 3a_2$

$T_2 = 3 \times 9.8 + 3 \times 3.2667 = 39.2001\text{N}$

eqn. (vi): $2T_2 + 12g = T$

$T = 2 \times 39.2001 + 12 \times 9.8 = 196.0002\text{N}$

eqn. (v): $2T_1 + 8g = T$

$T_1 = \frac{196.0002 - 8 \times 9.8}{2} = 58.8001\text{N}$

eqn. (iii): $T_1 - 4g = 4a_1$

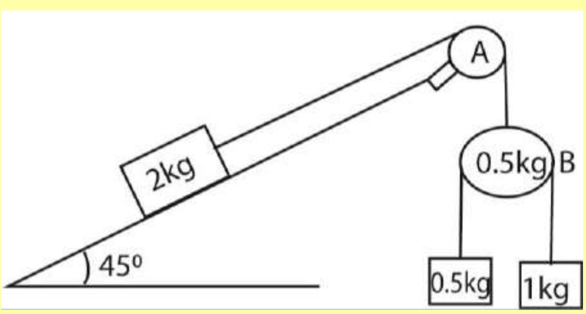
$a_1 = \frac{58.8001 - 4 \times 9.8}{4} = 4.9\text{ms}^{-2}$

eqn. $mg - T_1 = ma_1$

$m = \frac{58.8001}{9.8 - 4.9} = 12\text{kg}$

Example 21

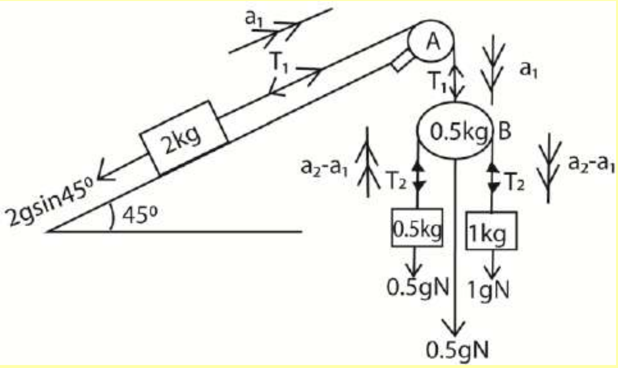
The diagram shows a particle of mass 2kg on a smooth plane inclined at 450 to the horizontal and attached by means of a light inextensible string over a smooth pulley, A at the top of the plane to pulley B of mass 0.5kg which hangs freely. Pulley B carries to particles of mass 0.5kg and 1kg on either side



Find

(a) acceleration of 2kg, 0.5kg and 1kg mass
(b) the tension in the strings

Solution



For 2kg mass: T_1 - 2gsin45 = $2a_1$ .... (i)

For 0.5kg mass: $T_2 - 0.5g = (a_2 - a_1)$ ...... (ii)

For 1kg mass: $gN - T_2 = 1(a_1 + a_2)$ .... (iii)

For pulley B: $2T_2 + 0.5g - T_1 = 0.5a_1$ .... (iv)

eqn. (ii) + eqn (iii): 0.5g = 1.5a2 + 0.5a1

$9.8 = 3a_2 + a_1$ .... (v)

eqn. (i) + eqn. (iv): $2T_2 - 2g\sin 45 + 0.5g = 2.5a_1$

$2T_2 - 8.9593 = 2.5a_1$ .... (vi)

2 x eqn. (iii) + eqn. (vi): $10.6407 = 4.5a_1 + 2a_2$

5.3204 = $2.5a_1$ + a2................. (vii)

eqn. (vii) – eqn. (v): $5.75a_1 = 6.1612$

$a_1 = \frac{6.1612}{5.75} = 1.0715\text{ms}^{-2}$

from eqn. (v): $9.8 = 3a_2 + a_1$

$a_2 = \frac{9.8-1.0715}{3} = 2.9095\text{ms}^{-2}$

Acceleration of 2kg mass = 1.0715ms^{-2}

Acceleration of 0.5kg mass = 2.9095ms^{-2}

Acceleration of 1kg = 2.9095 + 1.0715

= 3.981ms^{-2}

From eqn. (i): T_1 - 2gsin45 = $2a_1$

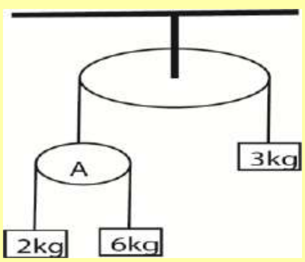
T1 = 2 x 1.0715 + 2 x 9.8sin45 =16.0023N

from eqn. (iv): $2T_2 + 0.5g - T_1 = 0.5a_1$

$T_2 = \frac{0.5 \times 1.0715 + 16.0023 - 4.9}{2}$ =5.8190N

Example 22

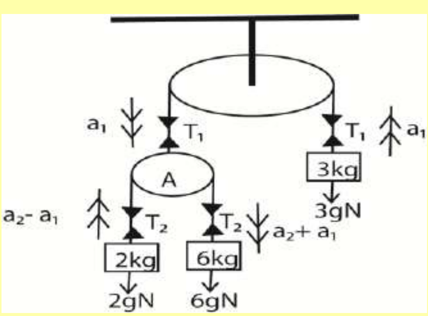
The diagram shows a fixed pulley carrying a string which has a mass of 3kg attached at one end and a light pulley A attached at the other end. Another string passes over pulley A and carries a mass of 6kg at one and a mass of 2kg at the other end.



Find

(a) acceleration of pulley A
(b) acceleration of 2kg, 6kg and 3kg masses
(c) tension in the string

solution



For 3kg mass: $T_1 - 3g = 3a_1$ .... (i)

For 6kg mass: $6g - T_2 = 6(a_2 + a_1)$ .... (ii)

For 2kg mass: $T_2 - 2g = 2(a_2 - a_1)$ .... (iii)

For pulley A: $2T_2 - T_1 = 0 \times a_1$ .... (iv)

eqn. (ii) and eqn. (iii): $4g = 8a_2 + 4a_1$ ..... (v) eqn. (i) + eqn. (iv): $2T_2 - 3g = 3a_1$...... (vi)

2 x eqn. (iii) - eqn. (vi)

$-g = 4a_2 - 7a_1$ .... (vii)

2eqn. (vii) – eqn. (v)

$-18a_1 = -6g$

$a_1 = \frac{6 \times 9.8}{18} = 3.27ms^{-2}$

$4g = 8a_2 + 4a_1$

$a_2 = \frac{9.8 - 3.27}{2} = 3.27ms^{-2}$

Acceleration of pulley A $= 3.27ms^{-2}$

Acceleration of 2kg $= 3.27ms^{-2} - 3.27ms^{-2} = 0$

Acceleration of 6kg $= 3.27ms^{-2} + 3.27ms^{-2}$

$= 6.54ms^{-2}$

Acceleration of 3kg $= 3.27ms^{-2}$

$T_1 - 3g = 3a_1$

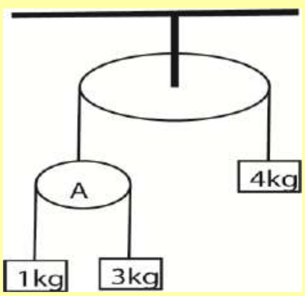
$T_1 = 3 \times 3.27 + 3 \times 9.8 = 39.21N$

$2T_2 - T_1 = 0 \times a_1$

$T_2 = \frac{39.21}{2} = 19.61N$

Example 23

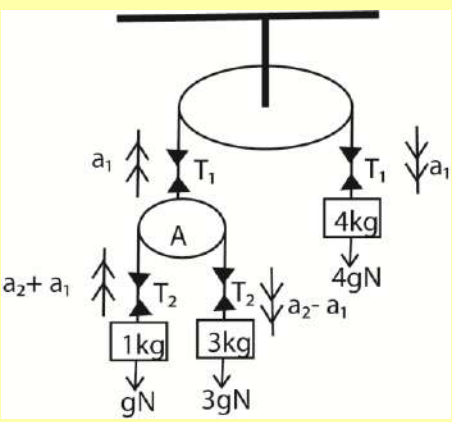
The diagram shows a fixed pulley carrying a string which has mass of 4kg attached at one end and a light pulley A at the other end. Another string passes over pulley A and carries a mass of 3kg at one end and a mass of 1kg at the other end.



Find

(a) acceleration of pulley A
(b) acceleration of 1kg, 3kg and 4kg masses
(c) tension in the string

Solution



For 4kg mass: $4g - T_1 = 4a_1$ .... (i)

For 3kg mass: $3g - T_2 = 3(a_2 - a_1)$ .... (ii)

For 1kg mass: $T_2 - g = (a_2 + a_1)$ .... (iii)

For pulley A: $T_1 - 2T_2 = 0 \times a_1$ .... (iv)

eqn. (ii) and eqn. (iii): $g = 2a_2 - a_1$ ..... (v)

eqn. (i) + eqn. (iv): $4g - 2T_2 = 4a1$ .... (vi)

2 x eqn.(iii) + eqn. (v): $2g = 2a_2 + 6a_1$ ..... (vii)

eqn. (vii) – eqn. (i): $7a_1 = g$

$a_1 = \frac{9.8}{7} = 1.4ms^{-2}$

$g = 2a_2 - a_1$

$a_2 = \frac{9.8 + 1.4}{2} = 5.6ms^{-2}$

Acceleration of pulley A $= 1.4ms^{-2}$

Acceleration of1kg mass $= 5.6 + 1.4 = 7ms^{-2}$

Acceleration of 3kg mass $= 5.6 - 1.4 = 4.2ms^{-2}$

Acceleration of 4kg mass $= 1.4ms^{-2}$

$4g - T_1 = 4a_1$

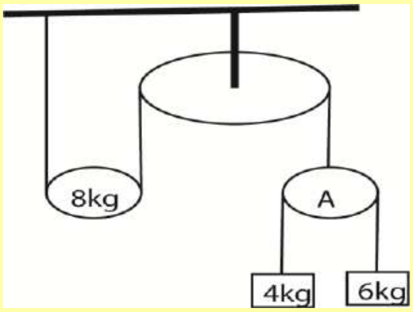
$T_1 = 4 \times 9.8 - 4 \times 1.4 = 33.6N$

$T_1 - 2T_2 = 0 \times a_1$

$T_2 = \frac{33.6}{2} = 16.8N$

Example 24

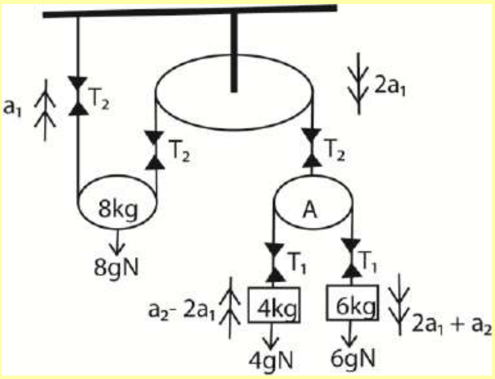
The diagram below shows a fixed pulley carrying a movable pulley of mass 8kg at one end and a light pulley A attached at the other end. A string passes over pulley A and carries a mass of 4kg at one end and a mass of 6kg at the other end.



Find

(a) acceleration of pulley A
(b) acceleration of 8kg, 6kg and 4kg masses
(c) tension in the string

Solution



For 8kg mass: $2T_2 - 8g = 8a_1$ .... (i)

For 4kg mass: $T_1 - 4g = 4(a_2 - 2a_1)$ .... (ii)

For 6kg mass: $6g - T_1 = 6(2a_1 + a_2)$ .... (iii)

For pulley A: $2T_1 - T_2 = 0 \times a_1$ .... (iv)

eqn. (ii) and eqn. (iii): $2g = 10a_2 + 4a_1$

$4.9 = 2.5a_2 + a_1$ ..... (v)

eqn. (i) + 2 x eqn. (iv): $4T_1 - 8g = 8a_1$.... (vi)

4 x eqn. (iii) + eqn. (vi): $16g = 56a_1 + 24a_2$

$2g = 7a_1 + 3a2$ .... (vii)

7 x eqn. (v) – eqn. (vii): $14.5a_2 = 14.7$

$a_2 = \frac{14.7}{14.5} = 1.0138ms^{-2}$

eqn. (v); $4.9 = 2.5a_2 + a_1$

$a_1 = 4.9 - 2.5 \times 1.0138 = 2.3655ms^{-2}$

Acceleration of pulley = $2a_1 = 2 \times 2.3655$

$= 4.731ms^{-2}$

Acceleration of 6kg = 4.731 + 1.0138

$= 5.7448ms^{-2}$

Acceleration of 3kg = $a_2 - 2a_1$

$1.0138 - 4.731 = -3.7172ms^{-2}$

From eqn. (i): $2T_2 - 8g = 8a_1$

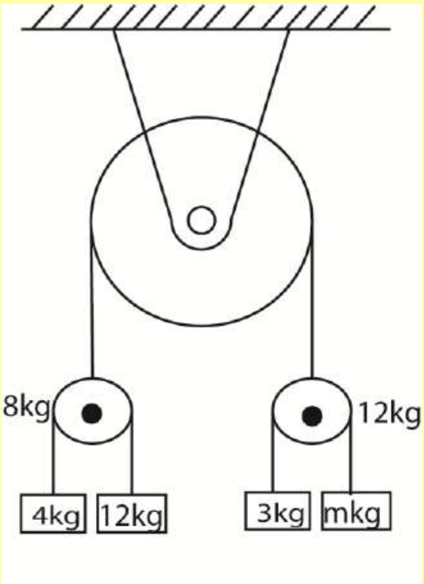
$T_2 = \frac{8 \times 4.731 + 8 \times 9.8}{2} = 58.124N$

From eqn. (iv)

$T_1 = \frac{T_2}{2} = \frac{58.124}{2} = 29.062N$

Example 25

The diagram below shows two pulleys of mass 8kg and 12kg connected by a light inextensible string hanging over a fixed pulley.



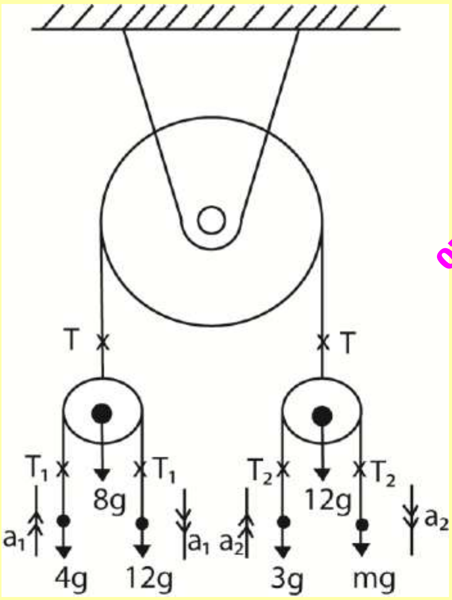
The acceleration of 4kg and 12kg masses are $\frac{g}{2}$ upward and $\frac{g}{2}$ downward respectively. The acceleration of the 3kg and m masses are $\frac{g}{3}$ upwards and $\frac{g}{3}$ downwards respectively. The hanging portions of the strings are vertical. Given that the string of the fixed pulley remains stationary, find the

(a) tensions in the strings (09marks)

Let T = tension n the string joining masses 8kg and 12kg

T_1 = tension in the string joining masses 4kg and 12kg

T_2 = tension in string joining masses 3kg and mkg



Resolving vertically

2T = 8g + 12g

T = 10g = 10 x 9.8 = 98N

For 4kg mass; resultant force = T_1- 4g

$4a_1 = T_1 - 4g$

$T_1 = 4g + 4a_1 = 4g + 4 \times \frac{g}{2} = 6g$

= 6 x 9.8 = 58.8N

For 3kg mass; resultant force = T_2- 3g

⇨ $3a_2 = T_2 - 3g$

$T_2 = 3g + 3a_2 = 3g + 3 \times \frac{g}{3} = 4g$

= 4 x 9.8 = 39.2N

Hence the tensions in the strings are 98N, 58.8N and 39.2N

Since the string of the fixed pulley remains stationary, this means the pulleys of the 8kg and 12kg are stationary or fixed

(b) value of m. (03marks)

For the m kg mass

Resultant force = mg –T_2

$ma_2 = mg - T_2$

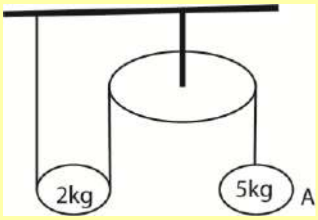
$m(\frac{g}{3}) = mg - 4g$

$\frac{2}{3}mg = 4g$

$m = \frac{12}{2} = 6kg$

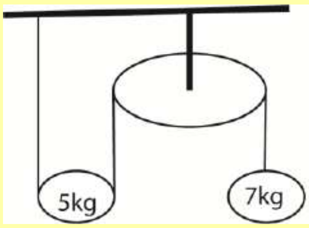
Revision exercise 3

1. A string with one end fixed passes under a movable pulley of mass 2kg, over a fixed pulley and carries a 5kg mass at its end



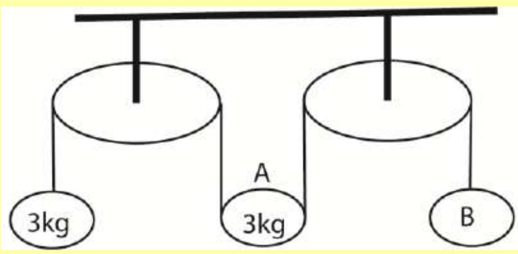
Find the acceleration of the movable pulley and the tension in the string. [$3.56ms^{-2}$, 13.36N]

2. a string with one end fixed passes under a movable pulley of mass 5kg, over a fixed pulley and carries a mass of7kg at its other end.



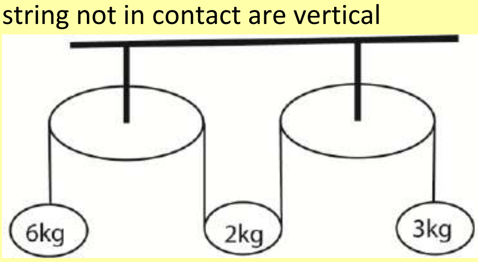
Find the acceleration of the movable pulley and the particle [$2.673ms^{-2}$, $5.146ms^{-2}$]

3. In the pulley system below, A is a heavy pulley which is free to move.



Find the mass B, if it does not move upwards or downwards when the system is released from rest. [1kg]

4. Two particles of mass 3kg and 6kg are connected by a light inextensible string passing over two fixed smooth pulleys and under a heavy smooth movable pulley of mass 2kg, the portions of the string not in contact are vertical

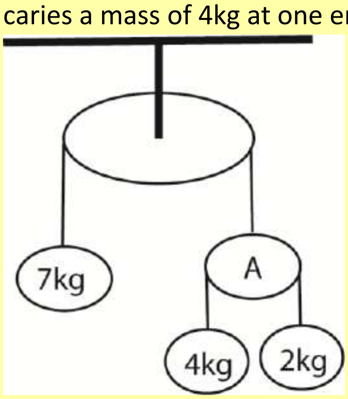


If the system is released from rest, find

(a) acceleration of movable pulley [$5.88ms^{-2}$]

(b) tension in the string [15.6N]

5. The diagram shows a fixed pulley carrying a string which has a mass of 7kg attached at one end and a light pulley A attached at the other end. Another string passes over the pulley A and caries a mass of 4kg at one end and a mass of 2kg at the other end.

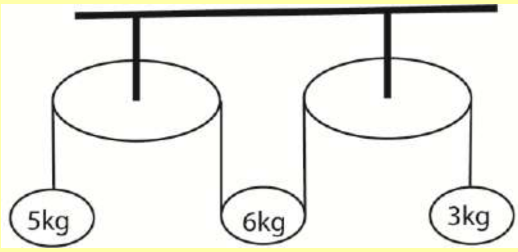


If the system is released from rest, find

(a) acceleration of 4kg mass [$2.38ms^{-2}$]

(b) tension in the strings [59.33N, 29.66N]

6. Two particles of mass 3kg and 5kg are connected by a light inextensible string passing over two fixed smooth pulleys and under a heavy smooth movable pulley of mass 6kg, the portions of the string not in contact are vertical

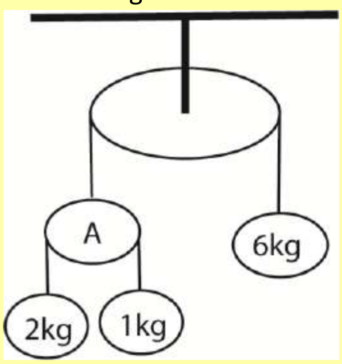


If the system is released from rest, find

(a) acceleration of movable pulley [$1.089ms^{-2}$]

(b) tension in the string [32.667N]

7. The diagram shows a fixed pulley carrying a string which has a mass of 7kg attached at one end and a light pulley A attached at the other end. Another string passes over pulley A and carries a mass of 4kg at one end and a mass of 2kg at the other end.

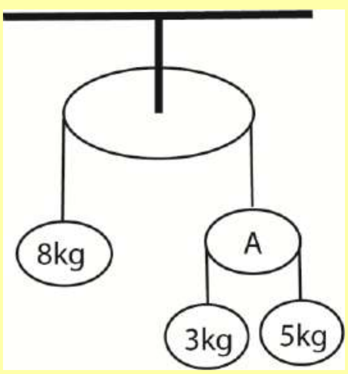


If the system is released from rest, find

(a) acceleration of 1kg mass [$8.2923ms^{-2}$]

(b) tension in the strings
[18.0923N, 36.1846N]

8. The diagram shows a system of masses and pulleys.

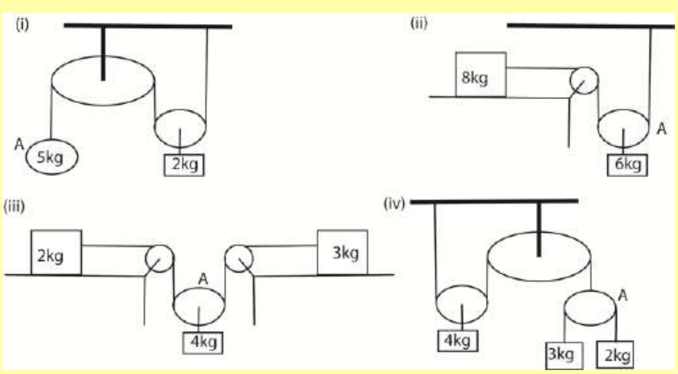


If the system is released from rest, find

(a) acceleration of 5kg mass [$2.8451ms^{-2}$]

(b) tension in the strings
[75.8712N, 37.9356N]

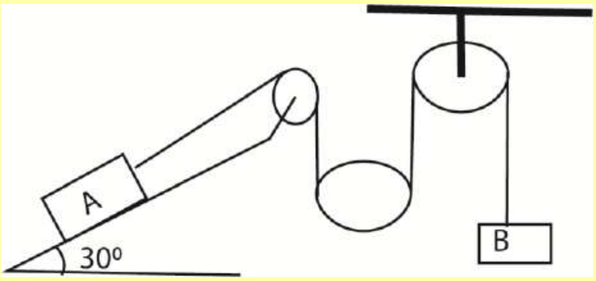
9. For each of the systems below: all the strings are light and inextensible, all pulleys are light and smooth and all surface are smooth. In each case find the acceleration of A and the tension in the string.



(i) [$7.127ms^{-2}$, 13.364N] (ii) [$1.547ms^{-2}$, 24.758N]

(iii) [$3.564ms^{-2}$, 10.691N] (iv) [$4.731ms^{-2}$, 12.166N, 24.331N]

10. Two particles A and B of mass 4kg and 2kg respectively and a movable pulley c of mass 6kg are connected by a light inextensible string as shown below



Given that the coefficient of friction between A and the plane is 0.2 and the system is released from rest, find the acceleration of A, B, C and the tension in the string.
[A = $-0.25ms^{-2}$, B = $2.9ms^{-2}$, C = $1.325ms^{-2}$]

16. FRICTION

Friction

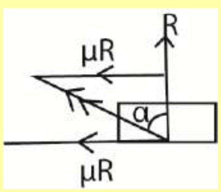
Friction is a force that opposes relative motion or attempted motion between two bodies in contact.

Friction force F = μR where R = normal reaction and μ = coefficient of friction

At limiting equilibrium, the body is at the point of moving (slip or slide) and friction force is maximum.

Angle of friction

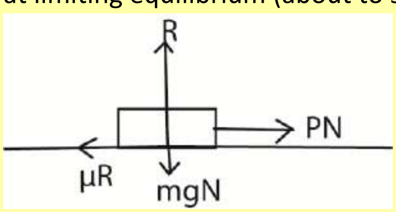
This is the angle between the resultant force and the normal reaction



tanα = $\frac{\mu R}{R} = \mu$

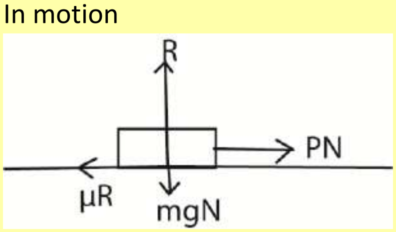
A horizontal plane

(i) at limiting equilibrium (about to slip or slid)



P = μR

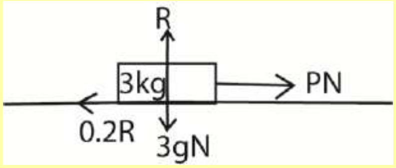
(ii) In motion



P – μR = ma

Example 1

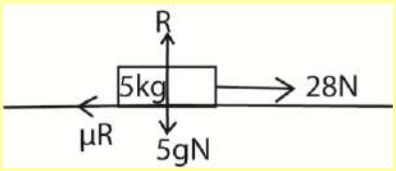
Calculate the maximum frictional force which can act when a block of mass 3kg rests on a rough horizontal surface, the coefficient of friction between the surface being 0.2



F = μR = 0.2 x 3 x 9.8 = 5.88N

Example 2

When a horizontal force of 28N is applied to a body of mass 5kg which is resting on a rough horizontal surface, the body is found to in limiting equilibrium. Find the coefficient of friction between the body and the plane



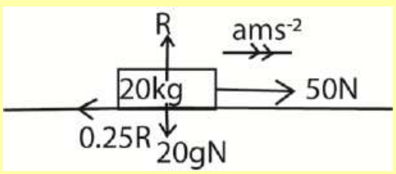
28 = μR

28 = μ x 5 x 9.8

$\mu = 0.57$

Example 3

A block of mass 20kg rests on a rough horizontal plane. The coefficient of friction between the block and the plane is 0.25. If a horizontal force of 50N acts no the body, find the acceleration of the body.



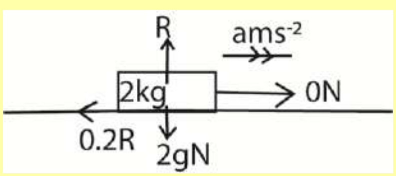
50 - μR = 20a

50 - $(0.25 \; x \; 20 \; x \; 9.8)$ = 20a

a = 0.05ms^{-2}

Example 4

A block of mass 2kg sliding along a smooth surface at a constant speed of 2ms-1. When the mass encounter a rough surface of coefficient of friction 0.2, it comes to rest. Find the distance the body will move across the rough surface before it comes to rest.



F = ma

0 – μR = 20a

-0.2 x 2 x 9.8 = 20a

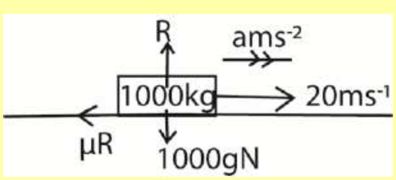
a = -1.96ms^{-2}

s = $\frac{v^2 - u^2}{2a} = \frac{0^2 - 20^2}{2x\,(-1.96)}$ = 1.02m

Example 5

A car of mass 1000kg moving along a straight road with speed of 72kmh^{-1} is brought to rest by a speedy application of brakes in a distance of 5m. Find the coefficient of kinetic friction between the tyres and the road.

u = $\frac{72 \; x \; 1000}{2600}$ = 20ms^{-1}



a = $\frac{v^2 - u^2}{2s} = \frac{0^2 - 20^2}{2x\;5}$ = -4ms^{-2}

ma = μR

4 x 1000 = 1000 x 9.8 x μ

μ = 0.41

26. CIRCULAR MOTION

Circular motion

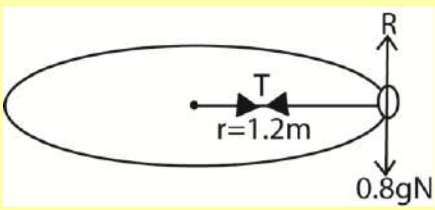
Circular motion on a smooth horizontal surface

Example 1

A particle of mass 0.8kg is attached to one end of a light inextensible string of length 1.2m. The other end is fixed to a point P on a smooth horizontal table. The particle is set moving in a circular path. If the speed of the particle is $16ms^{-1}$;

(i) Determine the tension in the string and the reaction on the table

(ii) If the string snaps when the tension in the string exceed 100N, find the greatest angular velocity at which the particle can travel.



(i) T = F= $\frac{mv^2}{r} = \frac{0.8\,x\,16^2}{1.2}$ =170.667N

R = 0.8gN = 0.8 x 9.8 = 7.4N

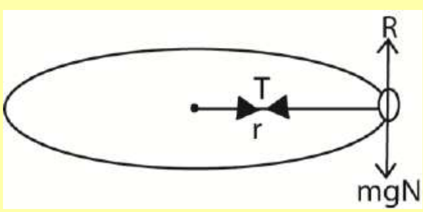
(ii) T = $m\omega^2 r$

100 = 0.8 x ω^2 x 1.2

ω = 10.21$rads^{-1}$

Example 2

A ball is tied on an elastic sting of natural length 30m to a fixed point on a smooth horizontal table upon which a ball is describing a circle around a point at a constant speed. If the modulus of elasticity of the string is 100times the weight of the ball and the number of revolution per minute is 20. Show that extension in the string is approximately 4.7m



T= F = $m\omega^2 r$

$\omega = 2\pi f = 2\pi\left(\frac{20}{60}\right) = \frac{2}{3}\pi$ and r = 30 + e

$T = m\left(\frac{2}{3}\pi\right)^2(30+e)$ ………………(i)

T= $\frac{\lambda}{L}e = \frac{100mg}{30}e$ ………. (ii)

Eqn. (i) and (ii)

$\frac{100mg}{30}e = m\left(\frac{2}{3}\pi\right)^2(30+e)$

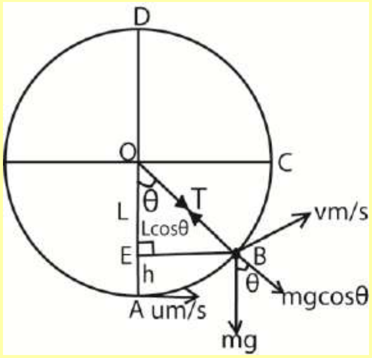
98 x 3e = $4\pi^2(30+e)$

$e = \frac{4\pi^2 x\, 30}{294-4\pi^2}$ = 4.6532m ≈ 4.7m

Motion in a vertical plane

Particle in fourth quadrant

Consider a body of mass m attached to a string (light rod) of length L and whirled in a vertical circle with a constant speed v. if there is no air resistance to the motion, then the net force towards the centre is the centripetal force. Tension in the string acts in the same way as the reaction R



At equilibrium at B: T- mgcosθ = $\frac{v^2}{L}$

$T = \frac{v^2}{L} + \text{ mgcos}\theta$ …………. (i)

But $v^2 = u^2 + 2as$

a = -g, s = h = L –Lcosθ

$v^2 = u^2 - 2g(L - L\cos\theta)$ ……… (ii)

when the particle comes momentarily to rest at some point A, then v = 0

$o = u^2 - 2g(L - L\cos\theta)$

$cos\theta = L - \frac{u^2}{2gL}$ ……………… (iii)

If the particle is attached to <u>a rod</u>, it can it can complete circle when v > 0 and θ= 180^0

$v^2 = u^2 - 2g(L - L\cos\theta)$

$u^2 - 2g(L - L\cos 180) > 0$

$u^2 > 2g(L + L)$

$u^2 > 4gL$ ……………(iv)

Put (ii) into (i)

$T = \frac{m[u^2 - 2g(L - Lcos\theta)]}{L} + mgcos\theta$

$T = \frac{m[u^2 - 2g(L - Lcos\theta)] + mgLcos\theta}{L}$

$T = \frac{m}{L}(u^2 - 2gL\ + 3gLcos\theta)$ ….. (v)

If the particle attached to <u>a string</u>, it can complete circle when T > 0 and θ = 180^0

$T = \frac{m}{L}(u^2 - 2gL\ + 3gLcos\theta)$

$\frac{m}{L}(u^2 - 2gL\ + 3gLcos\theta) > 0$

$\frac{m}{L}(u^2 - 2gL\ + 3gLcos180) > 0$

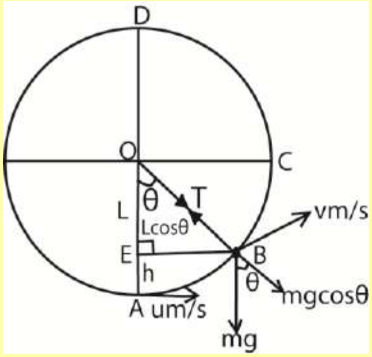
$(u^2 - 2gL - 3gL) > 0$

$u^2 > 5gL$ ……………… (vi)

Example 3

A particle P of mass 5kg is suspended from a fixed point O by light inextensible string of length 1m. The particle is projected from its lowest position at A, with horizontal speed of $4ms^{-1}$. When the angle AOB = 60^0, find

(a) Speed at P (b) the tension in the string at P



T.M.E$_A$ = T.M.E$_B$

$$5\left(\frac{1}{2}x\ 4^2 + 9.8\ x\ 0\right) = 5\left(\frac{1}{2}x\ 4^2 + 9.8\ x\ h\right)$$

$$5\left(\frac{1}{2}x\ 4^2 + 9.8\ x\ 0\right) = 5\left(\frac{1}{2}x\ 4^2 + 9.8\ (1 - cos60\right)$$

v = 2.49ms^{-1}

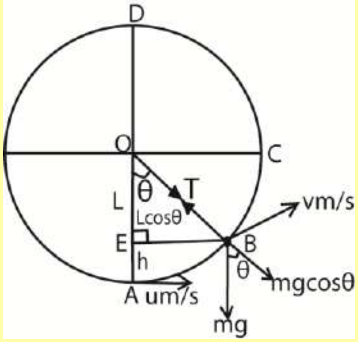
$$T = \frac{mv^2}{L} + mgcos\theta$$

$$T = \frac{5(2.49)^2}{1} + 5\ x\ 9.8cos60$$ = 55.5N

Example 4

A light rod of length 2m is pivoted at one end O and has a particle of mass 8kg attached at the other end. The rod is held vertically with the particle at A, directly below O and the particle is given an initial speed ums^{-1}; find

(a) Find an expression in terms of u and θ for the speed of the particle when at B where angle AOB = θ
(b) Restriction on u2 if the particle to perform complete oscillation



At B: $v^2 = u^2 + 2as$

a = -g, s = h = L –Lcosθ

$v^2 = u^2 – 2g(L –Lcos\theta)$

v = $\sqrt{u^2 - 2gL(1 - cos\theta)}$

If the particle to perform complete oscillation

v > 0 and θ = 180^0

$0^2 = u^2 – 2g(L –Lcos\theta)$

$u^2 – 2g(L –Lcos180)> 0$

$u^2 > 2g(L + L)$

$u^2 > 4gL$

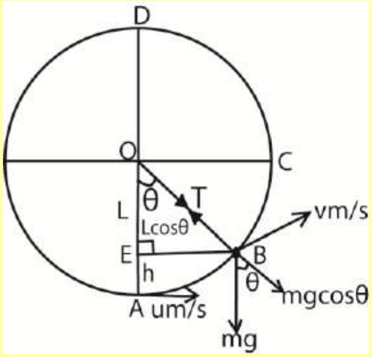
$u^2 > 4 \times 9.8 \times 2$

$u^2 > 78.4$

Example 5

A particle P of mass 8kg is suspended from a fixed point O by an inextensible string of length 2m. the particle is projected from its lowest position A with an initial speed ums^{-1}. Find

(a) An expression in terms of u and θ for the tension in the string when the particle is at B where angle AOB = θ
(b) Restriction on u2 if the particle is to perform complete oscillation



At equilibrium at B:

T – mgcosθ = $\frac{v^2}{L}$

$T = \frac{v^2}{L} + mgcos\theta$ ………… (i)

$v^2 = u^2 + 2ah$

a = -g, s = h = L –Lcosθ

$v^2 = u^2 - 2g(L - Lcos\theta)$

v = $\sqrt{u^2 - 2gL(1 - cos\theta)}$………(ii)

Put (ii) into (i)

$T = \frac{m[u^2 - 2g(L - Lcos\theta)]}{L} + mgcos\theta$

$T = \frac{m[u^2 - 2g(L - Lcos\theta)] + mgLcos\theta}{L}$

$T = \frac{m}{L}(u^2 - 2gL + 3gLcos\theta)$

for the particle attached to circle when T > 0 and θ = 180^0

$T = \frac{m}{L}(u^2 - 2gL + 3gLcos\theta)$

$\frac{m}{L}(u^2 - 2gL + 3gLcos\theta) > 0$

$\frac{m}{L}(u^2 - 2gL + 3gLcos180) > 0$

$(u^2 - 2gL - 3gL) > 0$

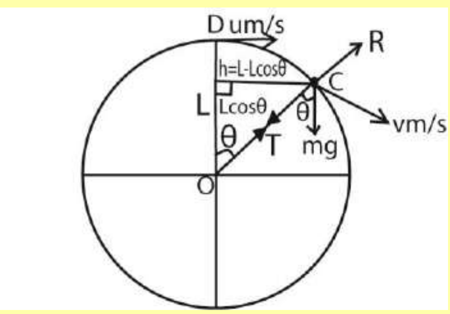
$u^2 > 5gL$

$u^2 > 5 x 9.8 x 2$

$u^2 > 98$

Particle in first quadrant

Consider a body of mass m rolled from the top of a sphere of radius L. The normal reaction R acts outwards



At equilibrium at C

mgcosθ – R = $\frac{v^2}{L}$

R = mgcosθ - $\frac{v^2}{L}$ ……….(i)

$v^2 = u^2 + 2ah$

a = g, s = h = L –Lcosθ

$v^2 = u^2 + 2g(L - Lcos\theta)$ ……. (ii)

Putting (ii) into (i)

R = mgcosθ - $\frac{m[u^2 + 2g(L - Lcos\theta)]}{L}$

R = $\frac{mgLcos\theta - mu^2 - 2mgL + 2mgLcos\theta}{L}$

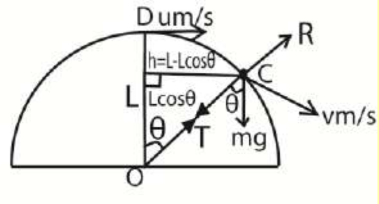
$R = \frac{m}{L}(3gLcos\theta - 2gL - u^2)$……..(iii)

Example 6.

A particle of mass 5kg is slightly disturbed from rest on the top of a smooth hemisphere, radius 4m and centre O, resting with its plane face on horizontal ground.

(a) Show that the particle leaves the surface of hemisphere at point C, where the angle between the radius OC and the upward vertical is $\cos^{-1}\frac{2}{3}$

(b) Find the speed at C



At equilibrium at C

mgcosθ – R = $\frac{v^2}{L}$

R = mgcosθ - $\frac{v^2}{L}$ ……….(i)

$v^2 = u^2 + 2ah$

u = 0, a = g, s = h = L –Lcosθ

$v^2 = 2g(L - L\cos\theta)$ ……. (ii)

Putting (ii) into (i)

R = mgcosθ - $\frac{[2g(L - L\cos\theta)]}{L}$

$R = \frac{3mgL\cos\theta - 2mgL}{L}$

When the particle leaves the surface of the sphere R = 0

$0 = \frac{3mgL\cos\theta - 2mgL}{L}$

3cosθ =2

θ = $\cos^{-1}\frac{2}{3}$

From eqn. (ii)

$v^2 = 2g(L - L\cos\theta) = 2g(L - L \times \frac{2}{3})$

$v^2 = \frac{2}{3}gL = \frac{2}{3} \times 9.8 \times 4 = 26.1333$

v = 5.1121ms^{-1}

27. HOOKE'S LAW

Hooke’s law, elastic strings and Simple harmonic motion

Hooke’s law states that the tension in a stretched string is proportional to the extension,e from its natural (unstretched) length, L.

$T = \lambda -$ where λ is modulus of elasticity.

Example 1

An elastic string is of natural length 4m and modulus 25N. Find

(i) The extension in the string when the extension is 20cm

$T = \lambda -$ = 25 $x\frac{0.2}{4}$= 1.25N

(ii) The extension of the string when the tension is 6N

$e = T\frac{-}{\lambda} = 6\,x\,\frac{4}{25} = 0.96m$

Example 2

A spring is of natural length 1.6 and modulus 20N. Find the thrust in the spring when it is compresses to a length of 1m

$T = \lambda -$ = 20 $x\frac{0.6}{1.6}$= 7.5N

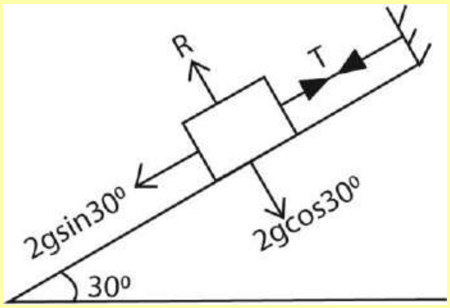
Example 3

A body of mass mkg lies on a smooth horizontal surface and is connected to a point O on the surface by a light elastic string of natural length 60cm and modulus 90N. When the body moves in a horizontal circular path about O with constant speed of 3ms^{-1}, the extension in the string is 30cm. find the mass of the body.

$T = \lambda -$ = 90 $x\frac{0.3}{0.6}$= 45N	T = $m\frac{v^2}{r}$	45 = $m\frac{3^2}{(0.6+0.3)}$	m = 4.5kg

Example 4

A smooth surface inclined at 30^0 to the horizontal has a body A of mass 2kg that is held at rest on the surface by a light elastic string which has one end attached to A and the other to a fixed point on the surface 1.5m away from A up to the line of greatest slope. If the modulus of elasticity of the string is 2gN, find the natural length



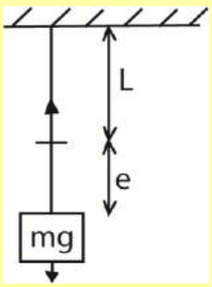
$T = \lambda\frac{e}{L}$

2gsin30 = 2g x $\frac{1.5-L}{L}$

L = 1m

Equilibrium of a suspended body

When an elastic string one end fixed and a mass attached to its other end so that the mass is suspended in equilibrium, the string is stretched due to the mass.



T = mg

mg = $\lambda\frac{e}{L}$ at equilibrium

Example 5

A light elastic string of natural length 65 has one end fixed and a mass of 500g freely suspended from the other end. Find the modulus of elasticity of the string if the total length of the string in equilibrium position is 85cm

mg = $\lambda -$	0.5 x 9.8 = $\lambda\, x\, \frac{0.85-0.65}{0.65}$	λ = 15.93N

Example 6

A light spring of natural length 1.5m has one end fixed and a mass of 400g freely suspended from the other. The modulus of the spring is 44.1N

(a) Find the extension of the spring when the body hangs in equilibrium

mg = $\lambda -$	0.4 x 9.8 = $44.1\, x\, \frac{}{1.5}$	e= 0.13m

(b) The mass is pulled vertically downwards a distance of 10cm and released, find the acceleration of the body when released

$T_1 = \lambda\frac{+x}{}$

$T_1 = 44.1\, x\frac{0.13+0.1}{1.5} = 6.762N$

F =ma

$T_1 - T = ma$

6.762- 3.92 = 0.4a

A = 7.11ms^{-2}

Potential energy stored in an elastic string

Work done = average force x extension = $\lambda\frac{{}^{2}}{2L}$

Example 7

An elastic string is of natural length 6.4m and modulus 55N. Find the work done in stretching it from 6.4m to 6.8m

Work done = $\lambda\frac{\ ^2}{2L} = 55\,x\,\frac{0.4^2}{2\,x\,6.4}$ = 0.688J

Example 8

An elastic string of natural length 4m is fixed at one end and stretched to 5.6m by a force of 8N. Find the modulus of elasticity and the work done

$T = \lambda -$

8 = $\lambda\frac{5.6-4}{4}$

λ = 20N

Work done = $\lambda\frac{e^2}{2L} = 20\,x\,\frac{1.6^2}{2\,x\,4}$ = 6.4J

Example 9

An elastic string of natural length 1.2m and modulus of elasticity 8N is stretched until the extending force is 6N. Find the extension and work done

$T = \lambda -$

6 = $8\,x\,\frac{\ }{1.2}$

e = 0.9m

Work done = $\lambda\frac{e^2}{2L} = 8\,x\,\frac{0.9^2}{2\,x 1.2}$ = 2.7J

Example 10

A light elastic string of natural length 1.2m has one end fixed at A and a mass of 5kg freely suspended from the other. The modulus of the string is such that a 5kg mass hanging vertically would stretch the string by 15cm. the mass is held at A and allowed to fall vertically, how far from A it comes to rest.

$mg = \lambda -$

$5\,x\,9.8 = \lambda\,x\,\frac{0.15}{1.2}$

λ = 392N

At A; u = $0ms^{-1}$, s = 1.2m, g = $9.8ms^{-2}$

$v^2 = u^2 + 2gs = 0^2 + 2 \times 9.8 \times 1.2 = 23.52m^2s^{-2}$

w = k.e + p.e

$\lambda\frac{e^2}{2L} = \frac{mv^2}{2} + mge$

$392\,x\,\frac{e^2}{2\,x\,1.2} = \frac{5\,x\,23.52}{2} + 5\,x\,9.8\,x\,e$

e = 0.769m

Depth = 1.2 + 0.769 = 1.969m

Revision exercise

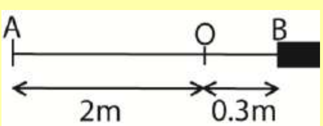
1. An elastic string of natural length 1m and modulus 20N. find the tension in the string when the extension is 20cm [4N]
2. A spring of natural length 50cm and modulus 10N. Find the thrust in the spring when it is compressed to a length 40cm [2N]
3. When the length of a spring is 60% of its original length, the thrust in the spring is 10N. find the modulus of the spring [25N]

4. An elastic string of natural length 60cm and modulus 18N. Find the extension of the string when the tension in the string is 6N [20cm]
5. The tension in an elastic string is 8N when the extension in the string is 25cm. If the modulus of the string is 8N. Find the un-stretched length. [25cm]
6. A light elastic string of natural length 20cm and modulus 2gN has one end fixed and a mass of 500g freely suspended from the other. Find the total length of the string [25cm]
7. When a mass of 5kg is freely suspended from one end of a light elastic string, the other end of it fixed, the string extends to twice its natural length. Find the modulus of the string [49N]
8. A body of mass 4kg lies on a smooth horizontal surface and is connected to point O of the surface by a light elastic string of natural length 64cm and modulus 25N. when the body moves in a horizontal circular path about O with constant speed of vms^{-1}, the extension in the string is 36cm. Find v [$1.875ms^{-1}$]
9. A body of mass 5kg lies on a horizontal surface and is connected to a point O on the surface by a light elastic string of natural length 2m and modulus 30N. When the body moves in a horizontal circular path about O with a constant speed $3ms^{-1}$, find the extension in the string[1m]
10. An elastic string of natural length 2m and modulus 10N. Find the energy stored when it is extended to a length of 3m [2.5J]
11. An elastic string of natural length 1m and modulus 20N. Find the energy stored when it is extended by a length of 30cm [0.9J]

Simple harmonic motion in strings

Example 11

One end of a light elastic string of natural length 2m and modulus 10N is fixed to point A on a smooth horizontal surface. A body of mass 200g is attached to the other end of the string and is held at rest at point B on the surface causing the string to extend by 30cm. Show that when released, the body will move with S.H.M. State its amplitude and find the maximum speed.



$T = \lambda\frac{x}{\ } = 10\,x\frac{x}{2} = 5x$

F = ma

5x = -0.2a

a = -25x

Since $a = -\omega^2 x$

$\omega^2 = 25$

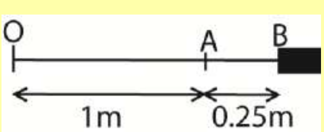
$\omega = 5rads^{-1}$

$T = \frac{2\pi}{\omega} = \frac{2\pi}{2}s$ and r = 0.3m

$v_{max} = \omega r = 5 \times 0.3 = 1.5ms^{-1}$

Example 12

One end of a light elastic string of natural length 1m and modulus 5N is fixed to a point O on a smooth horizontal surface. A body of mass 1kg is attached to the other end, A of the string and is held at rest at point B where OB = 1.25m. Show that when released, the body will move with S.H.M. Find the maximum speed and the total time taken from B to O



$T = \lambda\frac{x}{L} = 5\,x\frac{x}{1} = 5x$

F = ma

5x = -1a

a = 5x

Since a = $-\omega^2 x$

$\omega^2 = 5$

$\omega = 2.236 \text{rads}^{-1}$

$T = \frac{2\pi}{\omega} = \frac{2\pi}{2.236} s = 2.81s$ and r = 0.25m

$v_{max} = \omega r = 2.236 \times 0.25 = 0.559 \text{ms}^{-1}$

$t_{OA} = \frac{distance}{speed} = \frac{1}{0.559} = 1.788s$

$\frac{distance}{time} = \frac{1}{T} = \frac{0.25}{t_{AB}}$

$t_{AB} = \frac{0.25 \ x\ 2.81}{1} = 0.7025s$

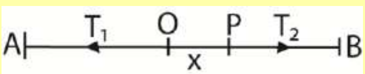
$t_{OB} = t_{OA} + t_{AB}$

$t_{OB} = 1.788 + 0.7025 = 2.491s$

Example 13

A light elastic string of natural length 2.4m and modulus 15 is stretched between two points A and B, 3m apart on a smooth horizontal surface. A body of mass 4kgis attached to the midpoint of the string is pulled 10 cm towards B and released

(i) Show that the subsequent motion is simple harmonic
(ii) Find the speed of the body when it is 158cmfrom A



At equilibrium $T_1 = T_2$

$\lambda \frac{1}{\ } = \lambda \frac{2}{\ }$

$e_1 = e_2$ ………. (i)

$e_1 + e_2 + 2.4 = 3$

$e_1 = 0.3m$

F = ma

$T_2 - T_1$ = ma

$\lambda\left(\frac{0.3-x}{1.2}\right) - \lambda\left(\frac{0.3+x}{1.2}\right)$ = 4a

4.8a = $-2\lambda x$

a = $-\frac{2 \ x\ 15x}{4.8}$ = -6.25x

It is in form of a =ω^2x hence S.H.M

$\omega^2 = 6.25$

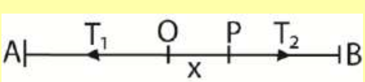
$\omega = 2.5\text{rads}^{-1}$

$v^2 = \omega^2(r^2 - x^2)$ when 158 from A, x = 8cm

$v = \sqrt{0.1^2 - 0.08^2} = 0.15ms^{-1}$

Example 14

A particle of mass m is attached by means of a light string AP and BP of the same natural length a m and modulus of elasticity mgN and 2mgN respectively, to point A and B on a smooth horizontal surface. The particle is released from the midpoint AB where AB = 3a. Show that the subsequent motion is simple harmonic with period $T = \left(\frac{4\pi^2 a}{3g}\right)^{\frac{1}{2}}$.



At equilibrium $T_1 = T_2$

$mg \frac{1}{a} = 2mg \frac{2}{a}$

$e_1 = 2e_2$ ………. (i)

$e_1 + e_2 + 2a = 3a$

$2e_2 + e_2 + 2a = 3a$

$e_2 = \frac{a}{3}$ and $e_1 = \frac{2a}{3}$

F = ma

$T_2 - T_1 = ma$

$2mg\left(\frac{a/_3 - x}{a}\right) - mg\left(\frac{2a/_3 + x}{a}\right) = ma$

$a = -\frac{3g}{a}x$

It is in form of a $= \omega^2 x$ hence S.H.M

$\omega^2 = \frac{3g}{a}$

$\omega = \left(\frac{3g}{a}\right)^{1/2}$ rads^{-1}

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\left(\frac{3g}{a}\right)^{1/2}} = \left(\frac{4\pi^2 a}{3g}\right)^{\frac{1}{2}}$$

Example 15

A particle of mass 1.5kglies on a smooth horizontal table and attached to two light elastic string fixed at points P and Q 12m apart. The strings are of natural length 4m and 5m and their modulus are λ and 2.5λ respectively.

(a) Show that the particles stays in equilibrium at a point R midway between P and Q

At equilibrium $T_1 = T_2$

$\lambda \frac{1}{4} = 2.5\lambda \frac{2}{5}$

$e_1 = 2e_2$ .... (i)

$e_1 + e_2 + 4 + 5 = 12$

$2e_2 + e_2 + 9 = 12$

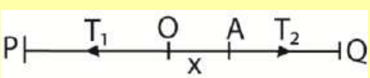
$e_2 = 1$

$e_1 = 2$

At R (midpoint) $4 + e_1 = 5 + e_2$

4 + 2 = 5 + 1 = 6

(b) If the particle is held at some point S in the line PQ with PS = 4.8m and then released. Show that the particle performs S.H.M and find the
   (i) Period of oscillation
   (ii) Velocity when the particle is 5.5m from P



F= ma

$T_2 - T_1 = ma$

$2.5\lambda\left(\frac{1-x}{5}\right) - \lambda\left(\frac{2+x}{4}\right) = 15a$

$a = -\frac{\lambda}{2}x$

It is in form of a $= \omega^2 x$ hence S.H.M

$\omega^2 = \frac{\lambda}{2}$

$\omega = \left(\frac{\lambda}{2}\right)^{1/2}$ rads^{-1}

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\left(\frac{\lambda}{2}\right)^{1/2}} = \left(\frac{8\pi^2}{\lambda}\right)^{\frac{1}{2}}$$

$v^2 = \omega^2(r^2 - x^2)$ when 5.5m from P, x = 0.5m

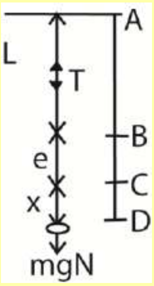
$$v = \sqrt{\frac{\lambda}{2}((6 - 4.8)^2 - 0.5^2)}$$

$$v = \sqrt{\frac{\lambda}{2}((1.2)^2 - 0.5^2)} = \sqrt{0.595\lambda}\ ms^{-1}$$

Elastic strings or springs hanging vertically

Example 16

A particle of mass m is suspended by a string from a fixed point A and has a natural length L. If the spring is extended from B to C where BC = e and this extension is due to weight of the body (mg), CD = x is the length a particle is pulled vertically downwards.



At equilibrium T = mg

T = $\lambda -$

When pulled a distance x:

$T - T_1 = ma$

$$\lambda\frac{e}{L} - \lambda\frac{e+x}{L} = ma$$

$$a = -\frac{\lambda}{mL}x$$

It is in form of a $=\omega^2 x$ hence S.H.M

$$\omega^2 = \frac{\lambda}{mL}$$

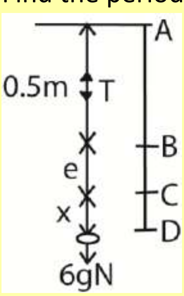
$$\omega = \left(\frac{\lambda}{mL}\right)^{1/2} \text{rads}^{-1}$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\left(\frac{\lambda}{mL}\right)^{1/2}} = 2\pi\sqrt{\frac{mL}{\lambda}}$$

Example 17

A light elastic spring of natural length 50cm and modulus 20gN, hangs vertically with its upper and fixed and the body of mass 6kg attached to its lower end. The body initially rests in equilibrium and then pulled down a distance of 25cm and released.

(a) Show that the ensuing motion will simple harmonic and
(b) Find the period of motion and maximum speed of the body



At equilibrium T = mg

6g = $20g\frac{}{0.5}$

e = 0.15m

When pulled a distance x:

$T - T_1 = ma$

$$6g - \lambda\frac{e+x}{0.5} = 6a$$

$$a = -\frac{196}{3}x$$

It is in form of a $=\omega^2 x$ hence S.H.M

$$\omega^2 = \frac{196}{3}$$

$$\omega = \left(\frac{196}{3}\right)^{1/2} = 8.083 \text{ rads}^{-1}$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\left(\frac{196}{3}\right)^{1/2}} = 0.773s$$

$$v_{max} = \omega r = 8.083\ x\ 0.25 = 2.021ms^{-1}$$

28.SIMPLE HARMONIC MOTION

Simple harmonic motion

This is a periodic motion of a body whose acceleration is directly proportional to the displacement from a fixed point and is directed towards the fixed point.

$a \propto -x$ or;

$a = -\omega\ \ x$ where ω is angular velocity, x, is displacement from fixed point

The negative sign means that the acceleration and displacement are always in opposite direction

Maximum acceleration

$a_{max} = -\omega\ \ r$ where r is the maximum displacement or amplitude.

Force F

F = ma = $m\omega\ \ x$

Maximum force, $F_{max} = m\omega\ \ r$

Velocity in terms of displacement

Velocity of a body executing S.H.M can be expressed as a function of displacement x.

$a = -\omega\ \ x$

$a = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt}$

But $\frac{dx}{dt} = v$

$a = v\frac{dv}{dx}$

$v\frac{dv}{dx} = -\omega\ \ x$

$vdv = -\omega\ \ xdx$

Integrating both sides

$\int v\, dv = -\omega^2 \int xdx$

$\frac{v^2}{} = -\frac{\omega^2 x^2}{} + c$ ……………..(i)

Where c is a constant of integration

At momentary rest v = 0

x = r (amplitude

$$\frac{0}{2} = -\frac{\omega^2 x}{2} + c$$

$c = \frac{\omega^2 r^2}{}$

Substituting c in eqn. (i)

$\frac{v^2}{} = -\frac{\omega^2 x^2}{} + \frac{\omega^2 r^2}{}$

$v^2 = (\omega^2 r^2 - \omega^2 x\ \ )$

$\boldsymbol{v^2 = \omega^2(r^2 - x^2)}$

Velocity is maximum when x = 0

$v\ \ = \omega\ \ r$

$\boldsymbol{v_{max} = \omega r}$

Displacement at any time t

$\frac{dx}{dt} = v$

$\frac{dx}{dt} = \omega\sqrt{(r \quad - x \quad)}$

$\int \frac{dx}{\sqrt{(r^2 - x^2)}} = \int \omega dt$

$\sin^{-1}\frac{x}{r} = \omega t + \varepsilon$

x = rsin$(\omega t + \varepsilon)$

When timing at the centre, t = 0, x = 0

x = rsin$\boldsymbol{\omega t}$ particle moves away from the centre

When timing at the amplitude, t = 0, x = r

x = rsin$\boldsymbol{\omega t}$ particle moves towards from the centre

Period

This is the time taken for one complete oscillation

T = $\frac{distance}{speed} = \frac{2\pi r}{v}$

But v = rω

T = $\frac{2\pi r}{r\omega}$

T = $\frac{2\pi}{\omega}$

Example 1

A particle moves in a straight line with simple harmonic motion about mean position O with a periodic time of $\frac{\pi}{\ }$s. Find the magnitude of acceleration of the particle when 1m from O.

Solution

From $a = -\omega \ \ x$ and $\omega = \frac{\pi}{T}$

Negative ignored

$a = \left(\frac{\pi}{\pi/2}\right) \ \ x1 = 16ms^{-2}$

Example 2

A particle moves with S.H.M about a mean position O. When the particle is 25cm from O, its acceleration is 1ms^{-1} towards O. Find the

(i) Periodic time of motion

(ii) Magnitude of acceleration of the particle when 20cm from O

Solution

$a = -\omega \ \ x$

$1 = \omega \ \ (0.25)$

$\omega \ \ = 4$

$\omega = 2rads^{-1}$

T = $\frac{2\pi}{\omega} = \frac{2\pi}{\ } = \pi$s

(ii) $a = -\omega^2 x$

$a = 2^2(0.2)$ = 0.8ms^{-2}

Example 3

A particle move with S.H.M of the periodic time $\frac{\pi}{\ }$s and has a maximum speed of 3ms^{-1}. Find the maximum acceleration experienced by the particle

$\boldsymbol{v_{max} = \omega r}$

3 = $\left(\frac{\pi}{\pi/2}\right)r$; r = 0.75m

$a_{max} = -\omega^2 r$

$a_{max} = \left(\frac{2\pi}{\pi/}\right)$ x 0.75 = 12ms^{-2}

Example 4

A particle moves with S.H.M about a mean position O. the amplitude of the motion is 5m and the period is 8πs. Find the

(i) maximum speed of the particle (ii) speed of the particle when 3m from O

Solution

(i) $\boldsymbol{v_{max} = \omega r} = \frac{\pi}{8\pi}x5$ = 1.25ms^{-1}

(ii) $v \ = \omega\ (r \ - x\ )$

$v = \sqrt{\left(\frac{2\pi}{8\pi}\right)\ (5 \ - 3^2)}$ =1ms^{-1}

Example 5

A body of mass 200g executes S.H.M with amplitude of 20mm. The maximum force which acts on it is 0.064N, calculate (a) its maximum velocity (ii) it period of oscillation

Solution

$F_{max} = m\omega\ r$

0.064 = 0.2 $x\ \omega\ \ x$ 0.02

$\omega = 4rads^1$

$v_{max} = \omega r$ = 4 x 0.02 = 0.08ms^{-1}

(ii) T = $\frac{2\pi}{\omega} = \frac{2\pi}{4} = \frac{\pi}{}$ s

Example 6

A particle moves with S.H.M about O with a period of 2π seconds. It passes a point A with a velocity of 4ms^{-1} away from O. Given that OA = 4m, find

(i) The amplitude
(ii) The speed at B where OB = 3m

Solution

$\omega = \frac{\pi}{T} = \frac{\pi}{\pi} = 1rads^{-1}$

From $v \ = \omega\ (r \ - x\ )$

$4 \ = 1\ (r \ - 4\ )$

r = 5.66m

Amplitude = 5.66m

(ii) $v = \ \omega\sqrt{r^2 - x}$

$v = \ \omega\sqrt{5.66 \ - 3^2}$ = 4.8ms^{-1}

Example 7

A particle moving with S.H.M has velocities of 4ms^{-1} and 3ms^{-1} at distance of 3m and 4m respectively from equilibrium position. Find

(a) amplitude (b) period

$v^2 = \omega^2(r^2 - x^2)$

$4^2 = \omega^2(r^2 - 3^2)$

$16 = \omega^2(r^2 - 9)$ …….. (i)

Also,

$3^2 = \omega^2(r^2 - 4^2)$

$9 = \omega^2(r^2 - 16)$ ……..(ii)

(i) ÷ (ii) $\frac{16}{9} = \frac{\omega^2(r^2-9)}{\omega^2(r^2-16)}$

r = 5m

Amplitude = 5m

(b) eqn. (i)

$16 = \omega^2(5^2 - 9)$

ω=1rads^{-1}

Period, $T = \frac{2\pi}{\omega} = \frac{2\pi}{1} = 2\pi$s

Example 8

A particle moves with S.H.M about a mean position O. the particle is initially projected from O with speed $\frac{\pi}{6}ms^{-1}$ and just reaches a point A, 2m from O.

(a) Find how far the particle from O, 3 seconds after projection

(b) How many second after projection is the particle a distance of 1m from O

(i) For the first time, (ii) second time (iii) third time

Solution

(a) At equilibrium position, $v_{max} = \omega r$

$\frac{\pi}{6} = \omega\, x2;\ \omega = \frac{\pi}{12} rads^{-1}$

x = rsinωt since particle moves away from O

x =2sin$\left(\frac{\pi}{12}x3\right)$= 1.414m

x = rsinωt

1 = rsin$\left(\frac{\pi}{12}\right)$t

$\left(\frac{\pi}{12}\right)$t =$\sin^{-1}0.2$= 30^0, 150^0, 210^0

T = 2s, 10s, 14s

Example 9

A particle is released from rest at point A. 1m from a second point O. the particle accelerates towards O and moves with S.H.M of period 12s and O is the centre of oscillation

(a) Find how far the particle is from O, 1s after release

(b) How many seconds after release is the particle at the midpoint of OA

(i) For the first time (ii) second time

Solution

(a) x = rcosωt since particles moves towards centre

$\omega = \frac{\pi}{12} = \frac{\pi}{6} rads^{-1}$

$x = 1cos\frac{\pi}{6}\ x\ 1 = \frac{\sqrt{3}}{2}m$

(b) x = rcosωt

$0.5 = 1cos\frac{\pi}{6}\ x\ t$

$\frac{\pi}{6}t = \cos^{-1}0.5$ =60^0, 300^0

t = 2s, 10s

Example 10

A particle of mass 2kgmoving with S.H.M along the x-axis, is attracted towards the origin O by a force of 32x newton. Initially the particle is at x = 20. Find

(a) amplitude and period of oscillation
(b) velocity of the particle at any time t > 0
(c) speed when t = $\frac{\pi}{4}$s

Solution

(a) F = $m\omega^2x$

$32x = 2\omega^2x$

$\omega = 4rads^{-1}$

$T = \frac{\pi}{\omega} = \frac{\pi}{4} = 1.571s$

$v \ = \omega \ (r \ \ - x \ )$

$0 \ = 4 \ (r \ \ - 20 \ )$

r =20m

(b) x = rcosωt

$v = \frac{d}{dt}(rcos\omega t)$

v = -rωsinωt = -20 x 4sin4t

v = -80sin4t

(c) speed = -80sin$\left(4\ x\frac{\pi}{4}\right)$ = $0ms^{-1}$

Example 11

A particle is initially released from rest at point A and performs S.H.M about mean position B. The particle just returns to A during each oscillation and AB = $2\sqrt{2}m$. If the particle passes through B with speed $\pi\sqrt{2}ms^{-1}$, find

(i) The time when the particle is first travelling with speed of πms^{-1}
(ii) How far from B the particle during this time

Solution

(i) At equilibrium position (B)

$v_{max} = \omega r$

$\pi\sqrt{2} = \ \omega\ x\ 2\sqrt{2}$

$\omega = \frac{\pi}{}rads^{-1}$

$v \ = \omega \ (r \ \ - x \ )$

$\pi \ \ = \left(\frac{\pi}{}\right) \ \left(\left(2\sqrt{2}\right) \ - x \ \right)$

x = 2m

(ii) x = rcosωt

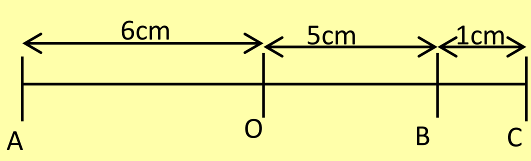
2 = $2\sqrt{2}cos\frac{\pi}{}t$

$\frac{\pi}{}t = \cos^{-1}\frac{1}{\sqrt{2}}$ = 45^0

t = 0.5s

Example 12

The points A, O, B, C lie in the that order on a straight line AO= OOC = 6cm and OB = 5c. A particle perform S.H.M of period 3s and amplitude 6cm between A and C. find the time taken for the particle from A to B



Time for AO is half the period = 1.5s

B is 5cm from O

x = rcosωt

5 = 6cos$\frac{\pi}{3}$t

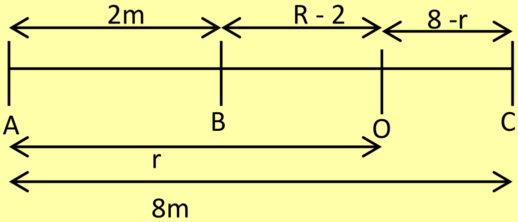
$\frac{2\pi}{3}$t = $\cos^{-1}\frac{5}{6}$ = 33.6^0

t = 0.28

Time for AB = 1.5 + 0.28 = 1.78s

Example 13

A particle passes through 3 points A, B and C in that order with velocity 0ms^{-1}, 2ms^{-1} and -1ms^{-1} respectively. The particle is moving with S.H.M in a straight line. What is the amplitude and period of the motion if AB = 2m and AC = 8m.



At B: v = 2ms^{-1}, x = r-2

Using $v \ = \omega \ (r \ - x \ )$

$2 \ = \omega \ (r \ - (r-2) \ )$

$1 = \omega \ (r-1)$ .... (i)

At C: v =-1ms^{-1}, x = 8-r

$(-1) \ = \omega \ (r \ - (8-r) \ )$

$1 = \omega^2(16r-64)$ .... (ii)

(i)÷ (ii) $\frac{1}{1} = \frac{\omega^2(r-1)}{\omega^2(16r-64)}$

r = 4.2

Amplitude = 4.2m

Using equation (i)

$1 = \omega^2(4.2-1)$

ω = 0.56rads^{-1}

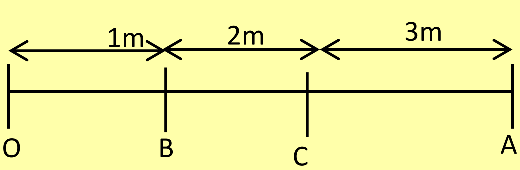
T = $\frac{2\pi}{\omega} = \frac{2\pi}{0.56} = 11.22$s

Example 14

A particle is performing S.H.M with centre O, amplitude 6m and period 2π. Points B and C lis between O and A with OB = 1m, OC = 3m and OA = 6m. find the least time taken while travelling from

(a) A to B (ii) A to C

Solution



$\omega = \frac{\pi}{T} = \frac{\pi}{\pi} = 1$

x = rcosωt

1 = 6cos (1x t)

t= 1.403s

(ii)

x = rcosωt

3 = 6cos (1x t)

t = 1.047s

Example 15

The velocity of a particle at any time is given by v(t) = -aωsinωt + bωcosωt.

(a) Find the expression for displacement x at any time that x = 0 when time t = 0

v(t) = -aωsinωt + bωcosωt.

But v(t) = $\frac{dx}{dt}$

⇨ $\frac{dx}{dt}$ =-aωsinωt + bωcosωt.

$\int dx = \int(-a\omega\sin\omega t + b\omega\cos\omega t)dt$

x = $\frac{a\omega}{\omega}\cos\omega t + \frac{b\omega}{\omega}\sin\omega t + c$

Since a and b are expressed in terms of amplitude and phase angle ε

If at t =0, x = 0, this means that A = 0, hence a = b = 0

By substituting, 0= 0 + 0 + c

c = 0

hence : x = acosωt+ bsinωt

(b) Show that the motion of the particle is simple harmonic [x

$\frac{dx}{dt}$ =-aωsinωt + bωcosωt.

$\frac{d^2x}{dt^2}$ =-aω^2cosωt - bω^2sinωt

- $\omega^2(a\cos\omega t + b\sin\omega t)$

But x = acosωt+ bsinωt

⇨ $\frac{d^2x}{dt^2}$ =- $\omega^2 x$, hence S.H.M

Example 16

A particle is moving with simple harmonic motion (SHM). When the particle is 15m from the equilibrium, its speed is 6ms^{-1}. When the particle is 13 m from equilibrium, its speed is 9ms^{-1}. Find the amplitude of the motion (05marks)

$v^2=\omega^2(A^2 - x^2)$

$6^2= \omega^2(A^2 - 15^2)$ ……. (i)

$9^2=\omega^2(A^2 - 13^2)$ …….. (ii)

Dividing (i) by (ii)

$\frac{36}{81} = \frac{A^2-225}{A^2-169}$

Amplitude A = 16.4256m

Revision exercise 1

1. A particle moving with S.H.M about a mean position O has velocities of 5ms^{-1} and 8ms^{-1} at distances of 16m and 12m respectively from O.
   (a) Amplitude [18.1m]
   (b) Period [10.63s]
2. A particle is describing S.H.M in a straight line directly towards a fixed point O. when it is a distance from O is 3m, its velocity is 25ms^{-1} and acceleration is 75ms^{-2}. Determine the
   (a) Period [$\left[\frac{2\pi}{5}\right]$ and amplitude [5.83m]
   (b) Time taken by the particle to reach O$\left[\frac{\pi}{10}\right]$
   (c) Velocity of the particle as it passes through O [29.15ms^{-1}]
3. A particle moving with simple harmonic motion about a mean position O has velocities of $3\sqrt{3}ms^{-1}$ and 3ms^{-1} at distances of 1m and 0.268m respectively. Find the amplitude of motion [2m]
4. A mass oscillates with S.H.M of period 1second and amplitude of oscillation is 5cm. Given that the particle begins from the centre of the motion, state the relationship between displacement x of the mass at any time t. Hence find the first two times when the is 3cm from its end position [x = rsinωt, 0.066s, 0.434s]
5. A particle moves in a straight line with S.H.M of period5s. the greatest speed is 4ms^{-1}, find the

(a) amplitude $\left[\frac{10}{2\pi}m\right]$.

(b) Speed when it is $\frac{6}{\pi}m$ from the centre. [$3.2ms^{-1}$]

6. A particle moves with S.H.M about mean position O with a periodic time $\frac{2\pi}{3}s$. When the particle is 0.8m from one extreme end, its speed is $3.6ms^{-1}$. Find the amplitude of motion [1.3m]
7. A body of mass 0.30kg executes S.H.M with a period 2.5s and amplitude 0.04m. Determine
   (i) Maximum velocity of the body [$0.101ms^{-1}$]
   (ii) The maximum acceleration of the body [$0.25ms^{-2}$]
8. A particle moving with S.H.M about a mean position O has velocities of $1.6ms^{-1}$ and $1.2ms^{-1}$ at distances of 60cm and 80cm respectively from O. find
   (i) Amplitude [1m]
   (ii) Period [πs]
9. A particle moves with S.H.M in a straight line with amplitude 0.05m and period 12s. find
   (a) speed as it passes equilibrium position [$0.026ms^{-1}$]
   (b) maximum acceleration of the particle [$0.014ms^{-2}$]
10. A particle moves in straight line with S.H.M about mean position with periodic time $\frac{\pi}{2}s$ and amplitude 2m. Find the maximum speed of the particle [$8ms^{-1}$]
11. A body of mass 500g moves horizontally with S.H.M of periodic time $\frac{\pi}{2}s$ and amplitude 1m. Determine the magnitude of the greatest horizontal force experienced by the body during the motion. [8N]
12. A body of mass 100g moves horizontally with S.H.M about a mean position O. When the body is 50cm from O, the horizontal force on the body is of magnitude 5N, find the period of motion $\left[\frac{\pi}{2}s\right]$
13. A particle moves in a straight line with S.H.M about a mean position O with a periodic time $\frac{\pi}{4}s$ and amplitude 65cm. find how far the particle from O when its speed is $2ms^{-1}$ [60cm]
14. A particle moves in a straight line with S.H.M about a mean position O. The particle has zero velocity at a point which is 50cm from O and speed of $3ms^{-1}$ at O. Find
    (a) The maximum speed of the particle [$3ms^{-1}$]
    (b) The amplitude of motion[50cm]
    (c) The periodic time of motion[$\frac{\pi}{2}s$]
15. A particle moving with S.H.M about a mean position O has velocities $3ms^{-1}$ and $1.4ms^{-1}$ at distances of 2m and 2,4m respectively from point O. Find the
    (a) Amplitude of motion [2.5m]
    (b) Greatest speed attained by the particle [$5ms^{-1}$]
16. A particle is initially projected from a point A performs S.H.M about mean position A with periodic time of 3s and amplitude 50cm. find the
    (a) Maximum speed of projection [$1.047ms^{-1}$]
    (b) Speed of the particle 2s after projection [$0.524ms^{-1}$]
    (c) Distance of the particle from A 2s after projection[0.433m]
17. The head of piston moves with S.H.M of amplitude $\frac{\sqrt{3}}{10}$m about mean position O. How far from O the head of the piston when travelling with a speed equal to half of its maximum speed [15cm]
18. A particle is fastened to the midpoint of a stretched spring lying on a smooth horizontal surface. The particle is set in motion so that it moves with S.H.M about a mean position O. If one metre is the greatest distance the particle is from O during the motion. Find how far from O the particle is when it is travelling with speed equal to four fifth of its greatest speed. [60cm]

19. A particle performs S.H.M about mean position O with a periodic time of 3s and amplitude 6cm. Find time it takes the particle to travel from O to a point P, a distance of 3cm from O [0.25s]
20. A particle performs S.H.M about mean position O with a periodic time of 4s and amplitude 2cm. Find the time it takes the particle to travel from O to a point P, a distance of $\sqrt{2}$cm from O[0.5s]
21. A particle performs S.H.M about mean position O with a periodic time of 10s and amplitude 8cm. After passing through O, the particle moves through a point A which is 1cm from to a point B which is 2cm from O. find the time it takes the particle to move from A to B[0.186s]
22. A particle performs S.H.M about mean position O with a periodic time of 4.5s and amplitude 6cm. After passing through O, the particle moves through point P which is 3cm from O. Find the time that elapse before the particle passes through P [1.5s]
23. The points A, O, B, C lie in that order on a straight line with AO = OC = 4cm and OB = 2cm. A particle perform S.H.M of period 6s and amplitude 4cm between A and C. Find the time taken for the particle to travel from A to B [2s]

29. VARIABLE ACCELERATION

Variable acceleration

This occurs when the rate of change of velocity is not constant.

Differential calculus

Let r = displacement, v = velocity and a = acceleration and are functions of time, t.

velocity, v = $\frac{dr}{dt}$ while acceleration, a = $\frac{dv}{dt} = \frac{d^2r}{dt^2}$

Differentiation; displacement => velocity => acceleration

Example 1

A particle moves along a straight line such that after t seconds, its displacement from a fixed point is r metres where r = $8t^2$i – t^4j. Find

(a) velocity after t seconds (b) velocity after 1s (c) speed after 1 s

Solution

$v_{(t=t)} = \frac{dr}{dt} = \frac{d(8t^2i-t^4j)}{dt}$

= (16ti – $4t^3$j)ms^{-1}

(b) when t = 1

$v_{(t=1)}$= (16i -4j)ms^{-1}

(c) speed = $|v_{(t=1)}| = \sqrt{16^2-(-4)^2}$

=16.49ms^{-2}

Example 2

A particle move along a straight line such that after t seconds its displacement from a fixed point is s in metres where s = 2sinti + 3costj. Find

(a) acceleration after t seconds (b) acceleration after $\frac{\pi}{2}s$ (c) magnitude of acceleration after $\frac{\pi}{2}s$

(a) $v_{(t=t)} = \frac{dr}{dt} = \frac{d(\text{2sinti + 3costj})}{dt}$

= (2costi – 3sintj)ms^{-1}

$a_{(t=t)} = \frac{dv}{dt} = \frac{d(\text{2costi – 3sintj})}{dt}$

=(-2sinti – 3costj)ms^{-2}

(b) when t = $\frac{\pi}{2}$s

$a_{(t=\frac{\pi}{2})}$ = (-2sin$\left(\frac{\pi}{2}\right)$i – 3cos$\left(\frac{\pi}{2}\right)$j) = -2i ms^{-2}

(c) $\left|a_{(t=\frac{\pi}{2})}\right| = \sqrt{(-2)^2-0^2}$ = 2ms^{-2}

Example 3

A particle moves along a straight line such that after t seconds its displacement from a fixed point is s where $s = \begin{pmatrix} \sin 2t \\ t+1 \\ cost + \sin t \end{pmatrix}$. Find

(a) velocity when t = $\frac{\pi}{2}s$ (b) speed when t= $\frac{\pi}{2}s$ (c) acceleration when t = $\frac{\pi}{2}s$

Solution

$v_{(t=t)} = \frac{dr}{dt} = \frac{d[\text{sin2ti} + (\text{t+1})\text{j} + (\text{cost+sint})\text{k}]}{dt}$

=2cos2ti + j + (sint –cost)k

When t = $\frac{\pi}{2}s$

$v_{(t=\frac{\pi}{2})} = 2\cos\left(\frac{2\pi}{2}\right)i + j + (\sin\frac{\pi}{2} - \cos\frac{\pi}{2})k$

= (-2i + j –k)ms^{-1}

(b) $\left|v_{(t=\frac{\pi}{2})}\right| = \sqrt{(-2)^2 + 1^2 + (-1)^2}$

= 2.45ms^{-1}

(c) $a_{(t=t)} = \frac{dv}{dt} = \frac{d[2cos2\text{ti} + \text{j} + (\text{sint}-\text{cost})\text{k}]}{dt}$

= -4sin2ti + (cost + sint)k

$a_{(t=\frac{\pi}{2})} = -4sin2\left(\frac{\pi}{2}\right)i + [\cos\frac{\pi}{2} + \sin\frac{\pi}{2}]k$

a = kms^{-2}

Example 4

A particle moves in x-y plane such that its position at any time t is given by r= $(3t^2-1)i + (4t^3 + t - 1)j$.
Find (a) speed after time t = 2 (b) magnitude of acceleration after t = 2s

$v_{(t=t)} = \frac{dr}{dt} = \frac{d[(3t^2-1)i + (4t^3+t-1)]}{dt}$

= $6ti + (12t^2 + 1)j\ ms^{-1}$

when t = 2s

$v_{(t=2)} = [6\ x\ 2i + (12\ x\ 2^2 + 1)j]ms^{-1}$

= [12i + 49j] ms^{-1}

speed = $\left|v_{(t=2)}\right| = \sqrt{12^2 + 49^2}$ = 50.45ms^{-1}

(b) $a_{(t=t)} = \frac{dv}{dt} = \frac{d[6ti + (12t^2+1)j]}{dt}$

= (6i + 24tj)ms^{-2}

when t = 2s

$a_{(t=2)}$= 6i =+ (24 x 2)j = (6i + 48i)ms^{-2}

$\left|a_{(t=2)}\right| = \sqrt{6^2 + 48^2}$ = 48ms^{-2}

Revision exercise 1

1. The position vector of a particle at any time (t) is given by r(t) = $[(t^2 + 4t)I + (3t - t^3)j]$m. Find the speed of the particle at t =3seconds. [26ms^{-1}]
2. The displacement of a particles after t seconds is given by r = $t^3i + 9tj$. Find the speed when t = 2s. [15ms^{-1}]
3. The displacement of a particle after t seconds is given by s = $2\sqrt{3}$ sinti + 8costj. find the speed when t = $\frac{\pi}{6}s$. [5ms^{-1}]
4. The displacement of a particle after t seconds is given by r = $8t^3i + 2t^2j$. Find
   (i) acceleration when t = 1s [(6i+ 4j)ms^{-2}]
   (ii) magnitude of the acceleration when t = 1s [7.21ms^{-2}]

5. The velocity of a particle after t seconds is given by v = sin2ti – costj. Find
   (i) acceleration when t = $\frac{\pi}{6}s$ [(-2i + j)ms^{-2}]
   (ii) magnitude of acceleration when t = $\frac{\pi}{6}s$ [2.24ms^{-2}]
6. The velocity of a particle after t seconds is given by v = 2t^{2}i + 6j. find the magnitude of acceleration when t = 3. [12ms^{-2}]
7. A particle of mass 6kg moves such that its displacement s = $\begin{pmatrix} t^2-5 \\ t^2-3t+2 \end{pmatrix}$m. Find the
   (a) velocity after time t [2ti + (2t – 3)j ms^{-1}]
   (b) speed of the particle at t = 2s [15ms^{-1}]
   (c) acceleration and hence determine the force acting on the particle [2i + 2j)ms^{-2}, (12i + 12j)N]
8. A particle of mass 2kg moves such that its displacement s = $\begin{pmatrix} t^2-4t-5 \\ t^2-4t+3 \end{pmatrix}$m. Find
   (a) speed of the particle at t= 2s [0ms^{-1}]
   (b) force acting on the particle [(4i + 4j)N]
9. A particle of mass 4kg moves such that its displacement s = (t^3 –t^2 – 4t + 3)I + (t^3 -2t^2 + 3t -7)j. Find the
   (i) the speed of the particle at t= 4s [50.21ms^{-1}]
   (ii) magnitude of the force acting on the particle when t = $\frac{2}{3}s$ [8N]
10. A particle of mass 0.5kgmoves such that its displacement r =$\begin{pmatrix} 4sin2t \\ 2cost-1 \end{pmatrix}$m. Find the
    (a) velocity of the particle after t = $\frac{\pi}{6}s$ [(4i – j)ms^{-1}]
    (b) force acting on the particle at any time t [(-8sin2ti –costj)N]
11. A particle of mass 2kg moves such that its displacement r = (2 – cos3t)I + (6sin2t)j. Find the
    (a) velocity of the particle at t = $\frac{\pi}{6}s$[(3i + 6j)ms^{-1}]
    (b) force acting on the particle at t = πs[-18N]
12. A particle moves such that its displacement s = (2sint +sin2t)I + (4cost + cos2t)j. Find the
    (a) velocity of the particle at t = $\frac{\pi}{3}s$ [-3$\sqrt{3}jms^{-1}$
    (b) acceleration of the particle at t= $\frac{\pi}{2}s$ [(-2i + 4j)ms^{-2}]

Integral calculus

If r, v or a are function of time t;

velocity, v = $\int adt + c$ and displacement, r = $\int vdt + c$

integration; acceleration => velocity => displacement

Example 5

The velocity of the particle v = 3t^{2}i + 10tj. Given that the displacement is 4i- 4j at t = 0. Find the distance of the body from the origin when t = 2s.

r = $\int vdt + c$

$r_{(t=t)} = \int(3t^2i + 10tj)dt + C$

$r_{(t=t)} = t^3i + 5t^2j + C$

$At\ t = 0, r = 4i - 4j$

4i -4j = 0^{3}i + 5x 0^{2}j +C

c = 4i -4j

$r_{(t=t)} = [(t^3 + 4)i + (5t^2 - 4)j]$

when t = 2s

$r_{(t=2)} = [(2^3 + 4)i + (5x2^2 - 4)j]$ = (12i + 16j)m

$|r_{(t=2)}|$ = $\sqrt{12^2 + 16^2}$ = 20m

Example 6

A particle is accelerated from rest at the origin with an acceleration of (2t + 4)ms^{-2}. Find

(a) velocity attained after t = 2s (b) distance travelled at t = 1s

Solution

v = $\int adt + c$

v = $\int(2t+4)dt + c$

$v_{(t=t)} = 2t^2 + 4t + c$

At t = 0, v = 0

0 = 2(0)^2 + 4(0) + c

c =0

$v_{(t=t)} = 2t^2 + 4t$

$v_{(t=2)} = 2x2^2 + 4x2$ = 12ms^{-1}

r = $\int vdt + c$

$r_{(t=t)} = \int(2t^2 + 4t\,)dt + c$

$r_{(t=t)} = \frac{t^3}{3} + 2t^2 + c$

At t = 0, r = 0

$0 = \frac{0^3}{3} + 2x\,0^2 + c$

c = 0

$r_{(t=t)} = \frac{t^3}{3} + 2t^2$

$r_{(t=1)} = \frac{1^3}{3} + 2x1^2$ = 2.33m

Example 7

A particle is accelerated from rest at the origin with acceleration of (4t + 2)I – 3j. Find

(i) velocity attained after t = 3s (ii) speed after 3s

Solution

v = $\int adt + c$

v = $\int\{(4t+2)i - 3j\}dt + c$

$v_{(t=t)} = (2t^2 + 2t)i - 3tj + c$

t = 0, v = 0

0 =$(2\,x\,0^2 + 2\,x\,0)i - 3\,x\,0j + c$

c = 0

$v_{(t=t)} = (2t^2 + 2t)i - 3tj$

$v_{(t=3)} = (2\,x\,3^2 + 2\,x\,3)i - 3x\,3j$

=24i -9j

Speed = $|v_{(t=3)}| = \sqrt{24^2 + (-9)^2}$ = 25.63ms^{-1}

Example 8

A particle starts from origin (0, 0). Its acceleration in ms^{-2} at time t seconds is given by a = 6ti -4j.

Find its speed after t =2s

v = $\int adt + c$

v = $\int(6ti - 4j)dt + c$

$v_{(t=t)}$ = 3t^{2}i -4tj + c

At t = 0, v= 0

0 = 3 x 0^{2}i -4 x 0j + c

c = 0

$v_{(t=t)}$ =3t^{2}i -4tj

$v_{(t=2)}$ =3 x 2^{2}i -4 x 2tj

=12i- 8j

$v_{(t=2)} = \sqrt{12^2 + \ (-8)^2} = 14.42ms^{-2}$

Example 9

An object of mass 5kg is initially at rest at a point whose position vector is -2i+ j. If it is acted on by a force F = 2i + 3j -4k. Find

(i) acceleration

F = ma

$a_{(t=t)} = \frac{1}{5}(2i + 3j - 4k)ms^{-2}$

(ii) speed after t= 3s

v = $\int adt + c$

$v_{(t=t)} = \frac{1}{5}\int(2i + 3j - 4k) + c$

$= \frac{1}{5}(2ti + 3tj - 4tk) + c$

At t = 0, v = 0

$0 = \frac{1}{5}(2 \text{ x } 0i + 3 \text{ x } 0j - 4 \text{ x } 0k) + c$

c =0

$v_{(t=t)} = \frac{1}{5}(2ti + 3tj - 4tk)$

$v_{(t=3)} = \frac{1}{5}(2 \text{ x } 3i + 3 \text{ x } 3j - 4 \text{ x } 3k)$

$= \frac{1}{5}(6i + 9j - 12tk)$

$|v_{(t=3)}| = \frac{1}{5}\sqrt{6^2 + 9^2 + (-12)^2}$ = 3.23ms^{-1}

(iii) its distance from the origin after 3 seconds

r = $\int vdt + c$

$r_{(t=t)} = \frac{1}{5}\int(2ti + 3tj - 4tk) + c$

$r_{(t=t)} = \frac{1}{5}(t^2i + 1.5t^2j - 2t^2k) + c$

At t = 0, r = -2i+ j

-2i+ j = $\frac{1}{5}(0^2i + 1.5 \text{ x } 0^2j - 2 \text{ x } 0^2k) + c$

c = -2i+ j

$r_{(t=t)} = \frac{1}{5}\{(t^2 - 10)i + (1.5t^2 + 5)j - 2t^2k\}$

$r_{(t=3)} = \frac{1}{5}\left((3^2 - 10)i + (1.5 \text{ x } 3^2 + 5)j - 18k\right)$

$= \frac{1}{5}(-i + 18.5j - 2 \text{ x } 3^2k)$

$|r_{(t=3)}| = \sqrt{(-1)^2 + (18.5)^2 + 18^2}$=5.166m

Example 10

A particle starts from rest at point (2, 0, 0) and moves such that its acceleration in ms-2 at time t seconds is given by a = [16cos4ti + 8sin2tj + (sint -2sin2t)k]ms^{-2}. Find the

(a) speed when t = $\frac{\pi}{4}$.

a = [16cos4ti + 8sin2tj + (sint – 2sin2t)k]ms^{-2}

v = $\int adt$

= $\int[16\cos4ti + 8\sin2tj + (\sin t - 2\sin2t)k]dt$

= [4sin4ti – 4cos2tj + (-cos t + cos2t)k]+ c

At t = 0

0 = [4sin0i – 4cos0j + (-cos 0 + cos0)k]+ c

0 = -4j + c

c = 4j

⇨ v = [4sin4ti + (-4cos2t + 4)j + (-cos t + cos2t)k

At t = $\frac{\pi}{4}$

⇨ v = [4sinπi + (-4cos$\frac{\pi}{2}$ + 4)j + (-cos $\frac{\pi}{4}$ + cos$\frac{\pi}{2}$)k

= 4j – cos $\frac{\pi}{4}$ k

$|v| = \sqrt{4^2 + \left(-\cos\frac{\pi}{4}\right)^2} = \sqrt{16 + \frac{2}{4}} = \sqrt{16.5}$ = 4.062ms^{-1}

(b) distance from the origin when t = $\frac{\pi}{4}$

s = $\int vdt$

$= \int[4\sin4ti + (-4\cos2t + 4)j + (-\cos t + \cos2t)k$

=-cos4ti +(-2sin2t + 4t)j + (-sint + $\frac{1}{2}sin2t)k$ + c

At t = 0, s = 2i

By substitution

2i = -cos0 i +(-2sin0 + 4(0))j + (-sin0 + $\frac{1}{2}sin2\ (0))k$ + c

2i = -I + c

c = 3i

⇨ s = (-cos4t + 3)i +(-2sin2t + 4t)j + (-sint + $\frac{1}{2}sin2t)k$

At t = $\frac{\pi}{4}$

⇨ s = (-cosπ + 3)i +(-2sin$\frac{\pi}{2}$t + π)j + (-sin$\frac{\pi}{4}$ + $\frac{1}{2}sin\frac{\pi}{2})k$

= 4i + (π -2)j + $\left(-\frac{\sqrt{2}}{2} + \frac{1}{2}\right)k$

= 4i + 1.416j – 0.207k

$|s|=\sqrt{4^2 + (1.416)^2 + (-0.207)^2}$

= 4.24828

= 4.248 (3D)

Vector approach of finding work and power

Work done by a variable force is a dot product

W= F.d or $W_{(t=t)} = \int Fvdt$

Since power, P = F.v

Example 11

A particle of mass 4kg starts frm rest at the origin. It is acted on by a force F = (2ti + 3t^{2}j + 5k)N

Find the work done by the force after 3 second

Solution

F = ma

$a_{(t=t)} = \frac{1}{4}(2\mathrm{ti} + 3\mathrm{t}^2\mathrm{j} + 5\mathrm{k})$

v = $\int adt$ + c

$v_{(t=t)} = \frac{1}{4}\int(2\mathrm{ti} + 3\mathrm{t}^2\mathrm{j} + 5\mathrm{k})\,dt + c$

$v_{(t=t)} = \frac{1}{4}(\mathrm{t}^2\mathrm{i} + \mathrm{t}^3\mathrm{j} + 5\mathrm{tk}) + c$

At t = 0, v = 0

0 = $\frac{1}{4}(0^2\mathrm{i} + 0^3\mathrm{j} + 5\,\mathrm{x}\,0\mathrm{k}) + c$

c = 0

$v_{(t=t)} = \frac{1}{4}(\mathrm{t}^2\mathrm{i} + \mathrm{t}^3\mathrm{j} + 5\mathrm{tk})$

$v_{(t=3)} = \frac{1}{4}(3^2\mathrm{i} + 3^3\mathrm{j} + 5\mathrm{x}\,3\mathrm{k})$

$v_{(t=t)} = \frac{1}{4}(9\mathrm{i} + 27\mathrm{j} + 15\mathrm{k})$

$|v_{(t=t)}| = \frac{1}{4}\sqrt{9^2 + 27^2 + 15^2}$

= $\frac{1}{4}\sqrt{1{,}025}ms^{-1}$

$W_{(t=0\ and\ t=3)}$ =$\frac{1}{2}m({v_{t=3}}^2 - {v_{t=0}}^2)$

= $\frac{1}{2}$ x 4 $\left(\frac{1}{16}\ x\ 1{,}025\ - 0\right)$ = 129.375J

Alternatively

$P_{(t=t)} = F.v$

$P_{(t=t)} = \begin{pmatrix} 2t \\ 3t^2 \\ t \end{pmatrix} . \frac{1}{4}\begin{pmatrix} t^2 \\ t^3 \\ 5t \end{pmatrix} = \frac{1}{4}(2t^3 + 3t^5 + 5t^2)$

$W_{(t=0\ and\ t=3)} = \int_0^3 F.vdt$

$= \frac{1}{4}\int_0^3 (2t^3 + 3t^5 + 5t^2)\, dt$

$= \frac{1}{4}\left[\frac{1}{2}t^4 + \frac{1}{2}t^6 + \frac{25}{2}t^2\right]_0^3$

$= \frac{1}{4}\left\{\left(\frac{1}{2}x\ 3^4 + \frac{1}{2}\ x\ 3^6 + \frac{25}{2}x\ 3^2\right) - 0\right\}$

= 129.375J

Example 12

A particle of mass 3kg moves along a straight line such that after t seconds its position vector is s metres where s = $\begin{pmatrix} 25t + 4t^2 \\ 50 + 2t^2 \\ 15 + 3t^2 \end{pmatrix}$. Find

(a) Magnitude of force

$v_{(t=t)} = \frac{ds}{dt} = \frac{d}{dt}\begin{pmatrix} 25t + 4t^2 \\ 50 + 2t^2 \\ 15 + 3t^2 \end{pmatrix} = \begin{pmatrix} 25 + 8t \\ 4t \\ 6t \end{pmatrix} ms^{-1}$

$a_{(t=t)} = \frac{dv}{dt} = \frac{d}{dt}\begin{pmatrix} 25 + 8t \\ 4t \\ 6t \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 6 \end{pmatrix} ms^{-2}$

$F = 3\begin{pmatrix} 8 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 24 \\ 12 \\ 18 \end{pmatrix}$

$|F| = \sqrt{12^2 + 12^2 + 18^2}$

= 32.31N

(b) Power when t = 2s

$P_{(t=t)} = F.v$

$v_{t=2} = \begin{pmatrix} 25 + 8\ x\ 2 \\ 4\ x\ 2 \\ 6\ x\ 2 \end{pmatrix} = \begin{pmatrix} 41 \\ 8 \\ 12 \end{pmatrix} ms^{-1}$

$P_{(t=2)} = \begin{pmatrix} 24 \\ 12 \\ 18 \end{pmatrix} . \begin{pmatrix} 41 \\ 8 \\ 12 \end{pmatrix} = 1{,}296ms^2$

(c) Work done on the particle between t = 1s and t=2s

$W_{(t=1\ and\ t=2)} = \frac{1}{2}m({v_{t=2}}^2 - {v_{t=1}}^2)$

$v_{(t=1)} = \begin{pmatrix} 25 + 8\ x1 \\ 4\ x\ 1 \\ 6\ x\ 1 \end{pmatrix} = \begin{pmatrix} 33 \\ 4 \\ 6 \end{pmatrix} ms^{-1}$

$|v_{(t=1)}| = \sqrt{33^2 + 4^2 + 6^2} = \sqrt{1141}$ms^{-1}

$|v_{(t=1)}| = \sqrt{41^2 + 8^2 + 12^2} = \sqrt{1889}$ms^{-1}

$W_{(t=1\ and\ t=2)} = \frac{1}{2}m({v_{t=2}}^2 - {v_{t=1}}^2)$

$= \frac{1}{2}$ x 3 $(1889 - 1141)$ = 1122J

Alternatively

$P_{(t=1)} = F.v$

$= \begin{pmatrix} 24 \\ 12 \\ 18 \end{pmatrix} . \begin{pmatrix} 25 + 8t \\ 4t \\ 6t \end{pmatrix} = (600 + 348t)$

$W_{(t=1\ and\ t=2)} = \int_1^2 F.vdt$

$= \int_1^2 (600 + 348t)dt$

$[600t + 174t^2]_1^2$

$= (12000 + 696) - (6000 + 174)$

= 1122J

Example 13

A particle of mass 10kg starts from rest at a point A with position vector (4i + 3j + 2k)m. It is acted on by a constant force, F = (8i + 4j + 6k)N causing it to accelerate to B after 4s. Find the

(a) Magnitude of acceleration

F= ma

$a_{(t=t)} = \frac{1}{10}((8\text{i} + 4\text{j} + 6\text{k}))ms^{-2}$

$|a_{(t=t)}| = \frac{1}{10}\sqrt{8^2 + 4^2 + 6^2}$

= 1.077ms^{-2}

(b) velocity at any time t

$v_{(t=t)} = \int adt + c$

$= \frac{1}{10}\int(8\text{i} + 4\text{j} + 6\text{k})\, dt$

$v_{(t=t)} = \frac{1}{10}(8ti + 4tj + 6tk) + c$

At t = 0; v=0

$0 = \frac{1}{10}(8\ x\ 0i + 4\ x\ 0j + 6\ x\ 0k) + c$

c = 0

$v_{(t=t)} = \frac{1}{10}(8ti + 4tj + 6tk)$

(c) position vector point B

$r_{(t=t)} = \int vdt + c$

$r_{(t=t)} = \frac{1}{10}\int(8ti + 4tj + 6tk)\, dt + c$

$r_{(t=t)} = \frac{1}{10}(4t^2i + 2t^2j + 3t^2k) + c$

At t = 0, OA = (4i + 3j + 2k)

(2i + 3j + 2k) $= \frac{1}{10}(4\ x\ 0^2i + 2x\ 0^2j + 3\ x\ 0^2k) + c$

c = (4i + 3j + 2k)

$r_{(t=t)} = \frac{1}{10}(4t^2i + 2t^2j + 3t^2k)$ +(4i + 3j + 2k)

$r_{(t=t)} = \frac{1}{10}\{(4t^2 + 40)i + (2t^2 + 30)j + (3t^2 + 2)k\}$

(d) the displacement vector AB

$$OB_{(t=4)} = \frac{1}{10}\begin{pmatrix} 4\ x\ 4^2 + 40 \\ 2\ x\ 4^2 + 30 \\ 3\ x\ 4^2 + 20 \end{pmatrix} = \begin{pmatrix} 10.4 \\ 3.2 \\ 6.8 \end{pmatrix} m$$

$\overline{AB}$ = OB – OA = $\begin{pmatrix} 10.4 \\ 3.2 \\ 6.8 \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 6.4 \\ 3.2 \\ 4.8 \end{pmatrix} m$

(e) work done by the force F after 4s.

$$W_{(t=4)} = F.r = \begin{pmatrix} 8 \\ 4 \\ 6 \end{pmatrix}.\begin{pmatrix} 6.4 \\ 3.2 \\ 4.8 \end{pmatrix} = 92.8J$$

Alternative 1

$$v_{(t=4)} = \frac{1}{10}\begin{pmatrix} 8\ x 4 \\ 4\ x\ 4 \\ 6\ x\ 4 \end{pmatrix} = \begin{pmatrix} 3.2 \\ 1.6 \\ 2.4 \end{pmatrix} ms^{-1}$$

$v_{(t=4)} = \sqrt{3.2^2 + 1.6^2 + 2.4^2} = \sqrt{18.56} ms^{-1}$

$W_{(t=0\ and\ t=4)} = \frac{1}{2}m\left({v_{(t=4)}}^2 - {v_{(t=0)}}^2\right)$

$= \frac{1}{2}x\ 10(18.56 - 0)$ =92.8J

Alternative 2

$P_{(t=t)} = F.v$

$$P_{(t=t)} = \begin{pmatrix} 8 \\ 4 \\ 6 \end{pmatrix}.\frac{1}{10}\begin{pmatrix} 8t \\ 4t \\ 6t \end{pmatrix} = 11.6t$$

$W_{(t=0\ and\ t=4)} = \int F.vdt = \int_0^4 11.6tdt$

$=[5.8t^2]_0^4$

= [5.8 x 4^2 – 0]

= 92.8J

Example 14

A particle move along a curve such that after t seconds its position vector is r where r = $\begin{pmatrix} t+1 \\ \frac{10}{3}t^3 - 6 \\ 4 - \frac{3}{2}t^2 \end{pmatrix}$.

The particle is acted on by a force F = (2ti + t^2j -2tk)N. Find

(a) Power at any time t.

$$v_{(t=t)} = \frac{dr}{dt} = \frac{d}{dt}\begin{pmatrix} t+1 \\ \frac{10}{3}t^3 - 6 \\ 4 - \frac{3}{2}t^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 10t^2 \\ -3t \end{pmatrix} ms^{-1}$$

$$P_{(t=t)} = F.v = \begin{pmatrix} 2t \\ t^2 \\ -2t \end{pmatrix}.\begin{pmatrix} 1 \\ 10t^2 \\ -3t \end{pmatrix} = 2t + 10t^4 + 6t^2$$

(b) work done by the force in the interval t = 1s and t = 3s

$$W_{(t=1\ and\ t=3)} = \int_1^3 F.v dt = \int_1^3 2t + 10t^4 + 6t^2 dt$$

$$= [t^2 + 2t^3 + 2t^5]_1^3$$

$W = (9 + 54 + 486) - (1 + 2 + 2)$ = 544J

Example 15

A particle of mass 2kg has a displacement vector s = $\begin{pmatrix} 2t^2 \\ 4t \\ 8t^3 \end{pmatrix}$m and a force F= $\begin{pmatrix} 3t \\ 4t^3 \\ 5t \end{pmatrix}N$. Find

(a) work done at t = 2s

W = F.v

$$W_{(t=t)} = \begin{pmatrix} 3t \\ 4t^3 \\ 5t \end{pmatrix}.\begin{pmatrix} 2t^2 \\ 4t \\ 8t^3 \end{pmatrix} = (6t^3 + 56t^4)J$$

$W_{(t=2)} = (6\ x\ 2^3 + 56\ x\ 2^4)$ = 944J

(b) Power when t = 4s

$$v_{(t=t)} = \frac{ds}{dt} = \frac{d}{dt}\begin{pmatrix} 2t^2 \\ 4t \\ 8t^3 \end{pmatrix} = \begin{pmatrix} 4t \\ 4 \\ 24t^2 \end{pmatrix} ms^{-1}$$

$$P_{(t=t)} = F.v = \begin{pmatrix} 3t \\ 4t^3 \\ 5t \end{pmatrix}.\begin{pmatrix} 4t \\ 4 \\ 24t^2 \end{pmatrix} = (12t^2 + 136t^3)W$$

$P_{(t=4)} = (12\ x\ 4^2 + 136\ x\ 4^3)W$ = 8,896W

Example 16

1. A force F = (2t i + j – 3t k)N acts on a particle of mass 2kg. The particle is initially at a point (0,0,0) and moving with a velocity (i + 2j-k)ms^{-1}. Determine the:

   (a) Magnitude of the acceleration of the particle after 2 seconds (04marks)

   F = (2t i + j – 3t k) = $\begin{pmatrix} 2t \\ 1 \\ -3t \end{pmatrix}$N

$$a = \frac{F}{m} = \frac{1}{2}\begin{pmatrix} 2t \\ 1 \\ -3t \end{pmatrix} = \begin{pmatrix} t \\ 0.5 \\ -1.5t \end{pmatrix} ms^{-1}$$

At t = 2s

$\underline{a}$=2i + 0.5j – 3k

$|\underline{a}| = \sqrt{2^2 + 0.5^2 + (-3)^2}$ = 3.64ms^{-2}

(b) Velocity of the particle after 2seconds (04marks)

$$\underline{v} = \int \underline{a}dt = \int \begin{pmatrix} t \\ 0.5 \\ -1.5t \end{pmatrix} dt = \begin{pmatrix} 0.5t^2 \\ 0.5t \\ -7.5t^2 \end{pmatrix} + C$$

At t = 0 initial velocity = (i + 2j –k)

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + C \Rightarrow C = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$\therefore \underline{v} = \begin{pmatrix} 0.5t^2 + 1 \\ 0.5t + 2 \\ -7.5t^2 - 1 \end{pmatrix}$$

At t = 2s

$$\underline{v} = \begin{pmatrix} 0.5(2)^2 + 1 \\ 0.5(2) + 2 \\ -7.5(2)^2 - 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -4 \end{pmatrix} ms^{-1}$$

(c) Displacement of the particle after 2 seconds (04marks)

$$\underline{r} = \int \underline{v}dt \int \begin{pmatrix} 0.5t^2 + 1 \\ 0.5t + 2 \\ -7.5t^2 - 1 \end{pmatrix} dt = \begin{pmatrix} \frac{t^3}{6} + t \\ \frac{t^2}{4} + 2t \\ -t - \frac{t^3}{4} \end{pmatrix} + C$$

At t = 0; $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + C$

C = $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\underline{r} = \begin{pmatrix} t^3 + t \\ \frac{t^2}{4} + 2t \\ -t - \frac{t^3}{4} \end{pmatrix}$$

At t = 2s

$$\underline{r} = \begin{pmatrix} \frac{2^3}{6} + 2 \\ \frac{2^2}{4} + 2x2 \\ -2 - \frac{2^3}{4} \end{pmatrix} = \begin{pmatrix} \frac{10}{3} \\ 5 \\ -4 \end{pmatrix} m$$

Example 17

1. A particle of mass 4kg starts from rest at point (2i- 3j + k)m. it moves with acceleration a= (4i + 2j -3k)ms^{-2} when a constant force F acts on it.
   Find the:
   (a) Force F (02marks)
   F = ma
   $= 4\begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 16 \\ 8 \\ -12 \end{pmatrix}$N or F = (16i + 8j -12k)N
   (b) Velocity at any time t. (04marks)
   $v = \int adt$
   v = at + c
   $= \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} t + c$
   At time t = 0, v = u = c = $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

   Substituting for c

   v = $\begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} t = \begin{pmatrix} 4t \\ 2t \\ -3t \end{pmatrix} ms^{-1}$
   or v = (4t i + 2t j -3t k)ms^{-1}
   (c) Work done by the force F after 6 seconds (06marks)
   Work done = force (F) x distance ($\underline{r}$)

$$\underline{r} = \int v dt = \int \begin{pmatrix} 4t \\ 2t \\ -3t \end{pmatrix} dt = \begin{pmatrix} 2t^2 \\ t^2 \\ -1.5t^2 \end{pmatrix} + c$$

$$At\ t = 0, \underline{r} = c = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

$$Hence\ \underline{r} = \begin{pmatrix} 2t^2 \\ t^2 \\ -1.5t^2 \end{pmatrix} + \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

After time t= 6seconds

$$\underline{r} = \begin{pmatrix} 2(6)^2 \\ (6)^2 \\ -1.5(6)^2 \end{pmatrix} + \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 74 \\ 33 \\ -53 \end{pmatrix}$$

Work done = force (F) x distance ($\underline{r}$)

$$= \begin{pmatrix} 16 \\ 8 \\ -12 \end{pmatrix} x \begin{pmatrix} 74 \\ 33 \\ -53 \end{pmatrix}$$

= 1184 + 264 + 636 = 2,084J

Revision exercise

1. A particle of mass 3kg is acted on by a force F = (24t3i + (36t -16)j + 12k)N at time t. at time t = 0, the particle is at the pint with position vector (3, -1, 4) and moving with velocity (16i + 15j -8k)ms^{-1}. Determine the
   (i) acceleration of the particle at time t= 2s. [28.4253ms^{-2}]
   (ii) speed of the particle at t= 2s [42.9534ms^{-1}]
   (iii) distance of the particle from the origin at t = 2s [56.5155m]
2. A particle of mass 4kg is acted on by a force F = (6i -36t^{2}j + 54tk)N at time t. At time t = 0, the particle is the point with position vector (i -5j –k) and it velocity is (3i + 3j)ms^{-1}. Determine the
   (a) position vector of the particle a time t = 1s. $\begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix}$m.
   (b) Distance of the particle from the origin a time t = 1s [6.1644m]

3. The acceleration f a particle is 6ti + 2j. Given that the velocity is (4i –j)ms^{-1} and displacement is (2i + 3j)m when t = 1s. Find the displacement when t =3s [(30i +5j)m]
4. If the velocity of a particle is 4cos2ti + 2sin2tj, given the displacement is 6i – 2j when t = $\frac{\pi}{4}s$. Find the distance of the body from the origin when t = πs. [5m]
5. If the acceleration of a particle is 9sin3ti + 2costj and the body id initially at rest. Find its velocity when t = $\frac{\pi}{6}s$. [(3i +j)ms^{-1}]
6. If the acceleration of a particle is 6sin6ti + 9cos3tj, given that the velocity is (i + 3j)ms^{-1} and displacement is (5i +2j)m when t = $\frac{\pi}{6}s$. Find the displacement when t = $\frac{\pi}{6}s$. [(5i +3j)m]
7. If the acceleration of the particle is 6ti + 6j -2k, given that the velocity is (3i + 6j -3k)ms^{-1} and displacement is (2i + 5j -2k)m when t =1s. Find
   (a) velocity when t= 2s [(12i + 12tj -5k)ms^{-1}]
   (b) displacement when t = 3s [(28i + 29j – 12k)m
8. If the acceleration f a particle is 2i +6j + 12t^{2}k, given that the velocity is (3i + k)ms^{-1} and displacement is (-i + k)m when t = 1s. Find the
   (i) velocity when t = 2s [(-i + 3j + 5k)
   (ii) displacement when t = 2s [(-3i + 8j + 19k)m
9. If the acceleration of a particle is (6ti – 2k)ms^{-2}, given that the velocity is (i + 12j -4k)ms^{-1} and the displacement is (3i + 6j)m when t = 2s, Find the
   (i) velocity when t = 4s [37i + 12j -8k)ms^{-1}
   (ii) displacement when t = 3s [11i + 18j -5k)m

10. The velocity of a particle v = $4t^3i + 6tj - 3t^2k$. Given that the displacement is 14i + 6j-3k at t = 1s. Find the
    (i) acceleration when t = 5s [(108i + 6j -18k)ms^{-2}
    (ii) displacement when t = 0 [(13i + 3j – 2k)m]
11. The velocity of a particle v = $(3t^2i - 10t)i + 2j - 6tk$. Given that the displacement is -9i + 3j – 13k at t = 2s. Find the
    (i) acceleration when t = 5s [(20i – 6k)ms^{-2}]
    (ii) displacement when t = 3s [15i + 5j -28k)m]
12. The particle starts from rest at the origin moving with velocity of $v = \begin{pmatrix} 2cos2t + 11 \\ 3sin3t \\ 4 \end{pmatrix}$. Find the
    (i) speed when t = $\frac{\pi}{6}s$. [13ms^{-1}]
    (ii) displacement when t = $\frac{\pi}{2}s$. [(1.5πi + j -2πk)m]
    (iii) acceleration when t = πs [9jms^{-2}]

30. LOCATION OF ROOTS

Location of real roots

The range where the root of an equation lie can be located using the following methods

(i) sign change
(ii) Graphical method

(a) Sign change method

Example 1

Show that equation $x^3 + 6x^2 + 9x + 2 = 0$ has a root between -1 and 0

Solution

$f(x) = x^3 + 6x^2 + 9x + 2$

$f(-1) = (1)^3 + 6(1)^2 + 9(-1) + 2 = -14$

$f(0) = (0)^3 + 6(0)^2 + 9(0) + 2 = 2$

Since there is a sign change the root lies between 0 and -1.

Example 2

Show that the equation $e^{2x}\sin x - 1 = 0$ has a root between 0 and 1

Solution

Note that in trigonometric function the calculator must be in radian mode

f(x) $= e^{2x}\sin x - 1$

f(0) $= e^{2(0)}\sin 0 - 1$ = -1

f(1) $= e^{2}\sin 1 - 1$ = 5.2177

Since there is a sign change the root lies between 0 and 1.

(b) Using graphical method

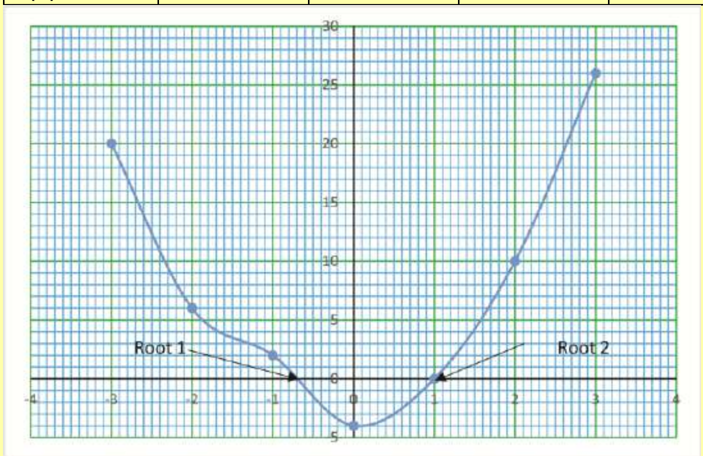
One or two graph(s) can be drawn to locate the root.

(i) Single graph method
When one graph is drawn then the root lies between the two points where the curve crosses the xaxis.

Example 3

Using a suitable graph locate the interval over which the root of the equation $3x^2 + x - 4 = 0$ lie.

x	-3	-2	-1	0	1	2	3
f(x)	20	6	-2	-4	0	10	26

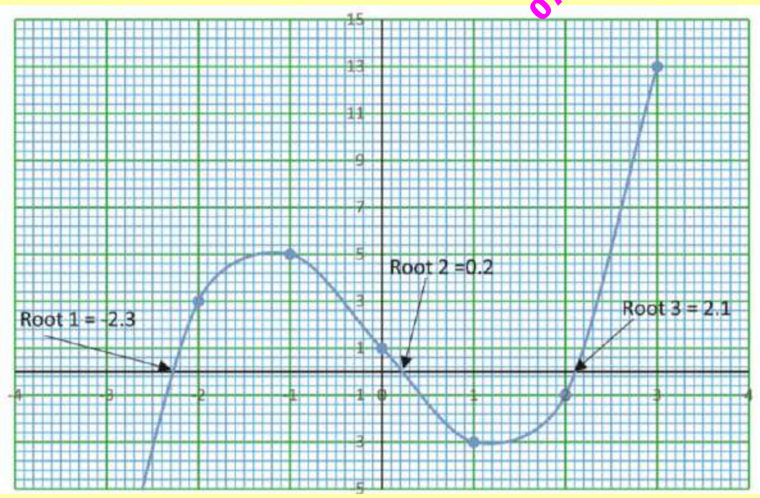


The root lies between -1 and 1

Example 4

Show graphically that there is positive real root of equation $x^3 - 5x + 1 = 0$

x	-3	-2	-1	0	1	2	3
f(x)	-11	3	5	1	-3	-1	13



(ii) Double graph method

When two graphs are drawn, the root lies between the points where the two curves meet.

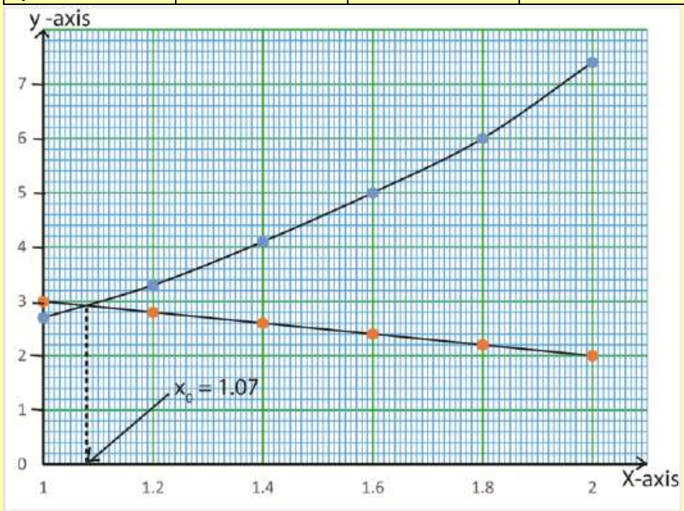
Note

(i) Both curves must have a consistent scale and should be labelled.
(ii) A line must be drawn using a ruler while a curve must be drawn using a freehand
(iii) Both graphs must be labelled
(iv) The initial approximation of the root must be located and indicated in the graph

Example 5

By plotting graph of y = e^x and y = 4 - x on the same axes, show the root of the equation e^x +x - 4 = 0 lie between 1 and 2

x	1	1.2	14	1.6	1.8	2.0
y = e^x	2.7	3.3	4.1	5.0	6.0	7.4
y = 4 - x	3.0	2.8	2.6	2.4	2.2	2

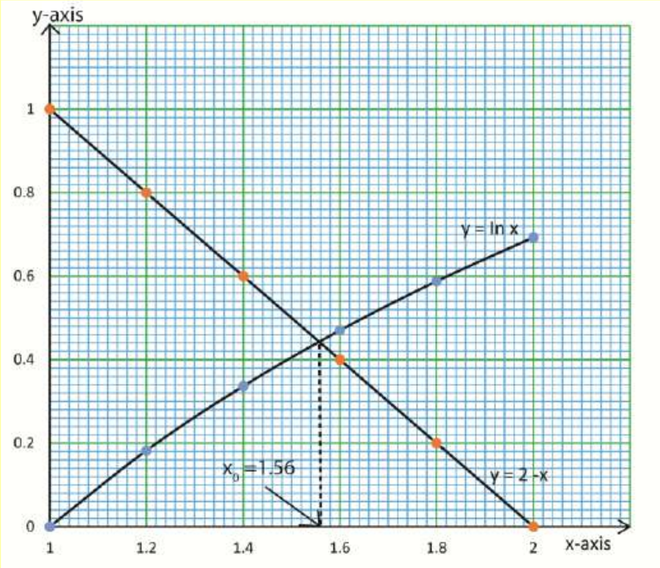


Therefore the root(1.07) lies between 1 and 2.

Example 6

Show that the equation lnx + x – 2 = 0 has a real root between x = 1 and x = 2

x	1	1.2	1.4	1.6	1.8	2.0
y = lnx	0	0.1823	0.3365	0.4700	0.5878	0.6731
y = 2-x	1	0.8	0.6	0.4	0.2	0

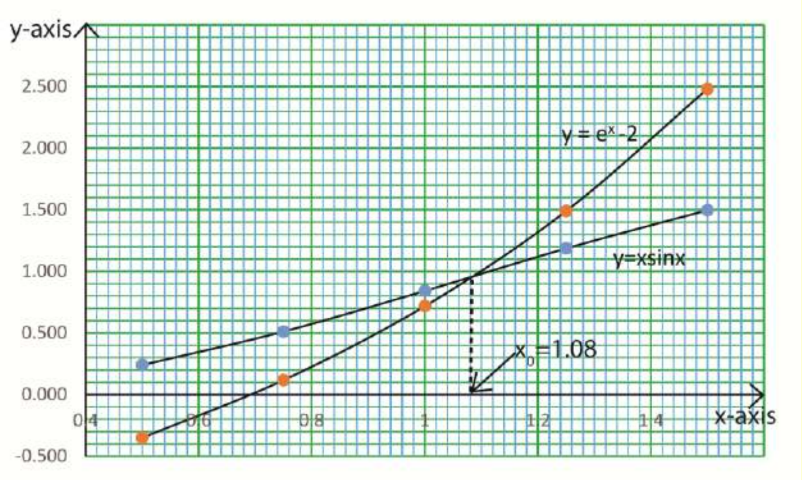


Therefore the root lies between x =1 and x =2

Example 7

By plotting graphs $y = e^x - 2$ an $y = x \sin x$ on the same axis show that the root of the equation $e^x - 2 - x\sin x = 0$ lies between x = 0.5 and x = 1.5

x	0.5	0.75	1.00	1.25	1.5
y = xsinx	0.240	0.511	0.841	1.186	1.496
$y = e^x - 2$	-0.351	0.117	0.718	1.490	2.481



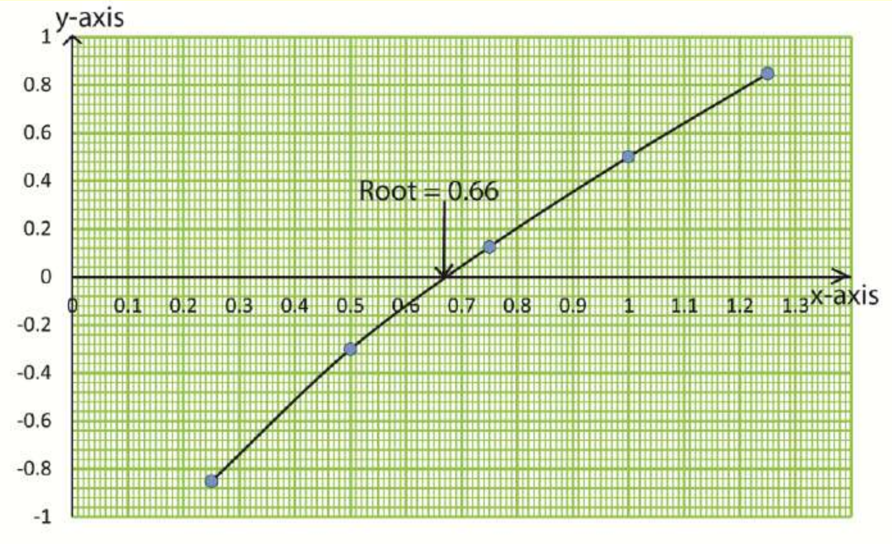
Example 8

Show graphically the equation x + logx = 0.5 has only one real root that lie between 0.5 and 1.

Solution

let y = x + logx - 0.5

x	0.25	0.5	0.75	1.00	1.25
y	-.852	-0.301	0.125	0.5	0.847



Therefore the root (0.66) lies between 0.5 and 1

Revision exercise 1

1. By sketching graphs of y = 2x and y = tanx show that the equation 2x = tanx has only one root between x = 1.1 and 1.2. Use linear interpolation to find the value of the root correct to 2dp.
2. Given the equation y = sinx - $\frac{x}{3}$, show by plotting two suitable graphs on the same axis that positive root lies between $\frac{2\pi}{3}$ and $\frac{5\pi}{6}$.
3. Show graphically that the positive real root of the equation $2x^2 + 3x - 3 = 0$ lies between 0 and 1 [0.7]
4. Use a graphical method to show that the equation $e^x - x - 2 = 0$ has only one real root between 2 and -1 by drawing two graphs $y = e^x$ and y = x + 2 [-1.8]
5. On the same axes, draw graphs of y = 3 – 3x and $y = 2x^2$ to show that the root of the equation $2x^2 + 3x\ \ 3 = 0$ lies between -3 and -2 [-2.2]
6. Show graphically that the positive real root of the equation $x^3 - 3x - 1 = 0$, lies between 1 and 2 [1.6]
7. on the same axes, draw graph y = 3x -1 and $y = x^3$ to show that the root of the equation $X^3 - 3x - 1 = 0$lies between 0 and 1.[0.35]
8. Using suitable graphs and plotting them on the same axes. Find the root of the equation $e^{2x}\sin x - 1 = 0$, in the interval x = 0.1 and x = 0.8. [0.44]
9. Show graphically that equation $e^{-x} = x$ has only one real root between 0.5 and 1. [0.56]
10. Show graphically that equation $e^x = -2x + 2$ has only one real root between 0 and 1.0.
11. on the same axes, draw graphs of y = 9x -4 and $y = x^3$ show that the root of equation $x^3 - 9x + 4 = 0$ lie between 2.5 and 3
12. Show that the positive real root of equation $4 + 5x^2 - x^3 = 0$ lies between 5 and 6.
13. On the same axes, draw graphs of y = x +1 and y = tanx to show that the equation tanx - x -1 = 0 lie between 1 and 1.5.
14. Using suitable graphs and plotting them on the same axes, find the roots of the equation $5e^x = 4x + 6$ in the interval x = 2 and x = -1.
15. On the same axes, draw graphs of y = 2x + 1 and y = $\log_e(x+2)$ to show that the root of equation $\log_e(x+2) - 2x - 1 = 0$ lies between 1 and 0.
16. Using suitable graphs and plotting them on the same axes, find the real root of the equation $9\log_{10} x$ = 2(X -1) in the interval x = 3 and x =4.

Method of solving for roots

The following methods can be used

(a) Interpolation

Example 9

Show that the equation $x^4 - 12x^2 + 12 = 0$ has root between 1 and 2. Hence use linear interpolation to get the first approximation of the root.

Solution

$f(x) = x^4 - 12x^2 + 12$

$f(1) = 1^4 - 12(1)^2 + 12 = 1$

$f(2) = 2^4 - 12(2)^2 + 12 = -20$

Since there is a sign change,

then the root lies between

1 and 2.

x	1	x_0	2
f(x)	1	0	-20

$$\frac{x_0-1}{0-1} = \frac{2-1}{-20-1}$$

$x_0 = 1.05$

Example 10

Show that the equation $2x - 3\cos\left(\frac{x}{2}\right) = 0$ has a root between 1 and 2.Hence use linear interpolation twice to get the approximation of the root.

solution

Note: for trigonometric functions the

calculator must be strictly in radian mode

$f(x) = 2x - 3\cos\left(\frac{x}{2}\right)$

$f(1) = 2 \times 1 - 3\cos\left(\frac{1}{2}\right) = -0.633$

$f(2) = 2 \times 1 - 3\cos\left(\frac{2}{2}\right) = 2.379$

Since there is a sign change,

then the root lies between

1 and 2

x	1	x_0	2
f(x)	-0.633	0	2.379

$$\frac{x_0-1}{0- -0.633} = \frac{2-1}{2.379--0.633}$$

$x_0 = 1.2102$

x	1.2102	x_0	2
f(x)	-0.047	0	2.379

$$\frac{x_0-1}{0- -0.047} = \frac{2-1}{2.379--0.047}$$

$x_0 = 1.226$

Example 11

Show that the equation $3x^2 + x - 5 = 0$ has a real root between x = 1 and x = 2. Hence use linear interpolation twice to calculate the root to 2 dp.

Solution

$f(x) = 3x^2 + x - 5$

$f(1) = 3(1)^2 + 1 - 5 = 1$

$f(2) = 3(2)^2 + 2 - 5 = 9$

Since there is a sign change,

then the root lies between

x	1	x_0	2
f(x)	-1	0	9

$$\frac{x_0-1}{0- -1} = \frac{2-1}{9--1}$$

$x_0 = 1.1$

x	1.1	x_0	2
f(x)	-0.27	0	9

$$\frac{x_0-1.1}{0- -0.27} = \frac{2-1.1}{9--0.27}$$

$x_0 = 1.13$

(b) General iterative method

This involves generating equation by splitting the original equation into several equations by making x the subject.

Example 12

Given $x^2 + 4x - 2 = 0$. Find the possible equations for estimating the roots

Solution

Let x_{n+1} be a better approximation

x_n be the next approximation

$x_{n+1} = \frac{2}{x_n} - 4$ | $x_{n+1} = \sqrt{(2 - 4x_n)}$ | $x_{n+1} = \frac{2 - x^2}{4}$

Example 13

Given $f(x) = x^3 - 3x - 12 = 0$. Generate equations in form of $x_{n+1} = g(x_n)$ that can be used to solve the equation $f(x) = 0$

Solution

Let x_{n+1} be a better approximation

x_n be the next approximation

$x_{n+1} = \frac{x_n^3 - 12}{3}$ | $x_{n+1} = \sqrt[2]{(3x_n + 12)} = \frac{12}{x_n^2 - 3}$ | $x_{n+1} = \sqrt{\left(3 + \frac{12}{x_n}\right)} = \frac{3x_n + 12}{x_n^2}$

Testing for convergence

From the several iterative equations obtained, the equation whose $|f^1(x_n| < 1$ is the one which converges the correct root.

Example 14

Given the two iterative formulas

(i) $x_{n+1} = \frac{x_n^3 - 1}{5}$ (ii) $x_{n+1} = \sqrt{\left(5 + \frac{1}{x_n}\right)}$

Using $x_0 = 2$ deduce a more suitable formula for solving the equation. Hence find the root correct to 2dp

$x_{n+1} = \frac{x_n^3 - 1}{5}$

$f(x_n) = x_{n+1} = \frac{x_n^3 - 1}{5}$; $f^1(x_n) = \frac{3x_n^2}{5}$

$f^1(2) = \frac{3(2)^2}{5} = 2.4$

since $|f^1(2)| > 1$ it will not converge

$x_{n+1} = \sqrt{\left(5 + \frac{1}{x_n}\right)}$

f(x_n) = $\sqrt{\left(5 + \frac{1}{x_n}\right)}$; $f^1(x_n) = -\frac{1}{2}x_n^{-2}\left(5 + \frac{1}{x_n}\right)$

$f^1(2) = -\frac{1}{2}(2)^{-2}\left(5 + \frac{1}{2}\right)$ = -0.0533

since $|f^1(2)|$<1 it will converge so this equation gives the root

$x_{n+1} = \sqrt{\left(5 + \frac{1}{x_n}\right)}$, $|e|$= 0.005, x_0 = 2

x_1 = $\sqrt{\left(5 + \frac{1}{2}\right)}$ = 2.3452

$|x_1 - x_0|$ = 2.3452 – 2 = 0.3452 > 0.005

x_2 = $\sqrt{\left(5 + \frac{1}{2.3452}\right)}$ = 2.3295

$|x_2 - x_1|$ = 2.3452 – 2.3295 = 0.0157 > 0.005

x_3 = $\sqrt{\left(5 + \frac{1}{2.3295}\right)}$ = 2.3301

$|e|$=$|2.3301 - 2.3295|$= 0.0006 < 0.005

Hence root is 2.33

Example 15

Show that the iterative formula for solving the equation x^3 = x + 1 is $x_{n+1} = \sqrt{\left(1 + \frac{1}{x_n}\right)}$ starting with x_0 = 1 find the solution of the equations to 3sf.

Solution

$x_{n+1} = \sqrt{\left(1 + \frac{1}{x_n}\right)}$ $|e|$= 0.005, x_0 = 1

$x_1 = \sqrt{\left(1 + \frac{1}{1}\right)}$ =1.41421

$|x_1 - x_0|$ = $|1.41421 - 1|$ = 0.41421 > 0.005

$x_2 = \sqrt{\left(1 + \frac{1}{1.41421}\right)}$ =1.30656;

$|x_2 - x_1|$ = $|1.30656 - 1.41421|$ = 0.10765 > 0.005

$x_3 = \sqrt{\left(1 + \frac{1}{1.30656}\right)}$ = 1.32869

$|x_3 - x_2|$ = $|1.32869 - 1.30656|$ =0.03691 > 0.005

$x_4 = \sqrt{\left(1 + \frac{1}{1.32869}\right)}$ =1.32389

$|e|=|1.32389 - 1.32869|$= 0.0048 < 0.005

Hence the root is 1.32

Example 16

Given two iterative formulae I and II (shown below) for calculating the positive root of the quadratic equation f(x) = 0

$x_{n+1} = \frac{1}{2}(x_n^2 - 1)$ and $x_{n+1} = \frac{1}{2}\left(\frac{x_n^2+1}{x_n-1}\right)$ for n = 1, 2, 3 …………

Taking x_0 = 2.5, use each formula thrice to two decimal places to decide which is the more suitable formula. Give a reason for your answer.

Solution

Iterative formula $x_{n+1} = \frac{1}{2}(x_n^2 - 1)$

x_0 = 2.5

$x_1 = \frac{1}{2}(2.5^2 - 1)$ = 2.625

$|x_1 - x_0|$ = 0.125

$x_2 = \frac{1}{2}(2.625^2 - 1)$ 2.99453125

$|x_2 - x_1|$ = 0.3200125

$x_3 = \frac{1}{2}(2.99453125^2 - 1)$ = 3.837432861

$|x_3 - x_2|$ = 0.89212036

Iterative formula $x_{n+1} = \frac{1}{2}\left(\frac{x_n^2+1}{x_n-1}\right)$

x_0 = 2.5

$x_1 = \frac{1}{2}\left(\frac{2.5^2+1}{2.5-1}\right)$ = 2.416666667

$|x_1 - x_0|$ = 0.083333

$x_2 = \frac{1}{2}\left(\frac{2.416666667^2+1}{2.416666667-1}\right)$ = 2.414215686

$|x_2 - x_1|$ = 0.002450781

$x_2 = \frac{1}{2}\left(\frac{2.414215686^2+1}{2.414215686-1}\right)$ = 2.414215686

$|x_3 - x_2|$ = 0.000002124

The more suitable formula is $x_{n+1} = \frac{1}{2}\left(\frac{x_n^2+1}{x_n-1}\right)$.

Because the absolute difference between $x_3 - x_2$ is less than absolute error, where as in the first formula the absolute difference between$x_3 - x_2$ is greater than absolute error. In all the 2[nd] formula converge whereas the first formula diverges.

Example 17

(a) (i) Show that the equation $e^x - 2x - 1 = 0$ has a root between x = 1 and x = 1.5.
(ii) Use linear interpolation to obtain an approximation for the root

(b) (i) Solve the equation in (a)(i), using each formula below twice
Take the approximation in (a)(i) as the initial value
Formula I: $x_{n+1} = \frac{1}{2}(e^{x_n} + 1)$.
Formula II: $x_{n+1} = \frac{e^{x_n}(x_n-1) + 1}{e^{x_n} - 2}$
(ii) Deduce with a reason which of the two formulae is appropriate for solving the given equation in (a)(i). Hence write down a better approximate root, correct to two decimal places

Solution

(a) (i) using sign change method
let f(x) = $e^x - 2x - 1$
f(1) = $e^1 - 2(1) - 1 = -2.817$
f(1.5) = $e^{1.5} - 2(1.5) - 1 = 0.4817$

Since f(1).f(1.5)< 0, the root lies between x = 1 and x = 1.5

(a)(ii) Extract

1	x_0	1.5
-0.2817	0	0.4817

$\frac{x_0-1}{0--0.2817} = \frac{1.5-1}{0.4817--0.2817}$; $x_0 = 1.1845$

Hence the approximation to the root is 1.18 (2 dp)

(b)(i)

Solution

formula 1: $x_{n+1} = \frac{1}{2}(e^{x_n} + 1)$

$x_0 = 1.18$

$x_1 = \frac{1}{2}(e^{1.18} + 1)$= 2.1272

$|x_1 - x_0|$ = 0.9472

$x_2 = \frac{1}{2}(e^{2.127187} + 1)$= 4.6956

$|x_2 - x_1|$ = 2.5684

formula 2: $x_{n+1} = \frac{e^{x_n}(x_n-1) + 1}{e^{x_n}-2}$

$x_0 = 1.18$

$x_1 = \frac{e^{1.18}(1.18-1) + 1}{e^{1.18}-2}$= 1.2642

$|x_1 - x_0|$ = 0.0842

$x_2 = \frac{e^{1.2642}(1.2642-1) + 1}{e^{1.2642}-2}$= = 1.2565

$|x_2 - x_1|$ = 0.0077

Formula 1, the sequence 1.18, 2.1272, 4.6956 diverge, hence the formula is not suitable

Formula 2, the sequence 1.18, 1.2642, 1.2565 converge, hence the formula is suitable solving the equation

A better approximation = 1.26 (2 dp)

Revision exercise 2

1. Given the following iterative formula
   (i) $x_{n+1} = 5 - \frac{3}{x_n}$ (ii) $x_{n+1} = \frac{1}{5}(x_n^2 + 3)$

   Taking $x_0 = 5$ deduce a more suitable iterative formula for solving the equation

2. Show that the iterative formula for solving the equation $x^2 - 5x + 2 = 0$ can be written in two ways as $x_{n+1} = 5 - \frac{2}{x_n}$ or $x_{n+1} = \frac{x_n^2+2}{5}$.
   Starting with $x_0 = 4$, deduce the more suitable formula for the equation and hence find the root correct to 2 dp [4.56]

3. Show that the iterative formula for solving the equation $x^3 - x - 1 = 0$ is $x_{n+1} = \sqrt{\left(1 + \frac{1}{x_n}\right)}$.
   Starting with xo = 1 find the root of the equation correct to 3 s.f. [1.33]
4. (a) Show that the iterative formula for solving the equation $2x^2 - 6x - 3 = 0$ is $x_{n+1} = \frac{2x_n^2+3}{4x_n+6}$
   (b) Show that the positive root for $2x^2 - 6x - 3 = 0$ lies between 3 and 4. find the root correct to 2 decimal places [3.44]
5. (a) If b is the first approximation to the root of equation x2 = a, show that the second approximation to the root is given by $\frac{b+\frac{a}{b}}{2}$. Hence taking b = 4, estimate $\sqrt{17}$ correct to 3 dp [4.123]
   (b) Show that the positive real root of the equation $x^2 - 17 = 0$ lies between 1.5 and 1.8. Hence use the formula in (a) above to determine the root to 3 dp

(c) Newton Raphson's Method

It is given by $x_{n+1} = x_n - \left[\frac{f(x_n)}{f^1(x_n)}\right]$ n = 1, 2, 3 …

Example 18

Use Newton Raphson's method to find the root of equation $x^3 + x - 1 = 0$ using x0 = 0.5 as the initial approximation, correct your answer to 2 decimal places

Solution

$f(x) = x^3 + x - 1$, $f^1(x) = 2x^2 + 1$

$x_{n+1} = x_n - \left[\frac{(x^3+x-1)}{3x^2+1}\right]$

$x_1 = 0.5 - \left[\frac{((0.5)^3 + 0.5-1)}{3(0.5)^2+1}\right]$ =0.7142

$|x_1 - x_0| = 0.7142 - 0.5 = 0.2142 > 0.005$

$x_2 = 0.7142 - \left[\frac{((0.7142)^3 + 0.7142-1)}{3(0.7142)^2+1}\right]$ =0.6831

$|x_2 - x_1| = |0.6831 - 0.7142|$

$= 0.0311 > 0.005$

$x_2 = 0.6831 - \left[\frac{((0.6831)^3 + 0.6831-1)}{3(0.6831)^2+1}\right]$ =0.6824

$|3 - x_2| = |0.6824 - 0.6831|$

$= 0.0007 < 0.005$

∴ Root = 0.68

Example 19

Show that the equation 5x - 3cos2x = 0 has a root between 0 and 1. Hence use Newton Raphson's method to find the root of equation correct to 2 decimal places using $x_0 = 0.5$.

Solution

Using sign change method to locate

The roots. Note for trigonometric

functions the calculator is used

in radians mode

f(x) =5x - 3cos2x

f(0) = 5(0) – 3cos2(0) = -3

f(1) = 5(1) – 3cos2(1) = 2.455

Since there is change sign the root lies between x = 0 and x = 1 f(x) = 5x - 3cos2x, $f^1(x)$ = 5 + 6sin2x

$x_{n+1} = x_n - \left[\frac{(5x_n - 3\cos 2x_n)}{5 + 6\sin 2x_n}\right]$

$x_0 = 0.5$, $|e| = 0.005$

$x_1 = 0.5 - \left[\frac{(5(0.5) - 3\cos 2(0,5))}{5 + 6\sin 2(0.5)}\right] = 0.4125$

$|x_1 - x_0| = |0.4125 - 0.5| = 0.0875 > 0.005$

$x_1 = 0.4125 - \left[\frac{(5(0.4125) - 3\cos 2(0.4125))}{5 + 6\sin 2(0.4125)}\right] = 0.4096$

$|x_2 - x_1| = |0.4096 - 0.4125| = 0.0029 < 0.005$

∴ Root = 0.41

Example 20

Use Newton Raphson's iterative formula to show that the cube root of a number N is given by $\frac{1}{3}\left(2x_n + \frac{N}{x_n^2}\right)$. Hence taking $x_0 = 2.5$ determine $\sqrt[3]{10}$ correct to 3 dp.

Solution

$x = N^{\frac{1}{3}}$

$x^3 - N = 0$

$f(x) = x^3 - N$; $f^1(x) = 3x^2$

$x_{n+1} = x_n - \left[\frac{(x_n{}^3 - N)}{3x_n{}^2}\right] = \frac{x_n(3x_n{}^2) - (x_n{}^3 - N)}{3x_n{}^2}$

$= \frac{2x_n{}^3 + N}{3x_n{}^2} = \frac{1}{3}\left(2x_n + \frac{N}{x_n^2}\right).$

$x_0 = 2.5$, N = 10, $|e| = 0.005$

$x_{n+1} = \frac{1}{3}\left(2x_n + \frac{N}{x_n^2}\right)$

$x_1 = \frac{1}{3}\left(2(2.5) + \frac{N}{2.5^2}\right) = 2.2$

$|x_1 - x_0| = |2.2 - 2.5| = 0.3 > 0.005$

$x_2 = \frac{1}{3}\left(2(2.2) + \frac{N}{2.2^2}\right) = 2.1554$

$|x_2 - x_1| = |2.1554 - 2.2| = 0.0446 > 0.005$

$x_3 = \frac{1}{3}\left(2(2.1554) + \frac{N}{2.1554^2}\right) = 2.1544$

$|x_3 - x_2| = |2.1544 - 2.1554| = 0.001 < 0.005$

∴ Root = 2.154

Example 21

(a) Show that the equation x -3sinx = 0 has a root between 2 and 3. (03marks)

f(x) = x – 3sinx
f(2) = 2 – 3sin 2 = -0.7279
f(3) = 3 – 3sin3 = 2.5766
since f(2).f(3) = -1.8755<0
there exist a root of x-3sinx =0 between 2 and 3

(b) Show that Newton- Raphson iterative formula for estimating the root of the equation in (a) is given by

$$X_{n+1} = \frac{3(\sin x_n - x_n \cos x_n)}{1 - 3\cos x_n}, n = 0, 1, 2 \ldots.$$

Hence find the root of the equation corrected to 2 decimal places (09 marks)

f'(x) = 1 – 3cos x

$x_{n+1} = x_n - \frac{f(x)}{f'(x)}$

$= x_n - \frac{x_n - 3\sin x_n}{1-3\cos x_n}$

$= \frac{x_n - 3x_n\cos x_n - x_n + 3\sin x_n}{1-3\cos x_n}$

$x_{n+1} = \frac{3(\sin x_n - x_n\cos x_n}{1-3\cos x_n}$

Taking $x_0 = \frac{2+3}{2} = 2.5$

$x_1 = \frac{3(\sin 2.5 - 2.5\cos 2.5}{1-3\cos 2.5} = 2.293$

Error = $|2.293 - 2.5| = 0.207 > 0.005$

$x_2 = \frac{3(\sin 2.293 - 2.5\cos 2.293}{1-3\cos 2.293} = 2.279$

Error = $|2.279 - 2.293| = 0.014 > 0.005$

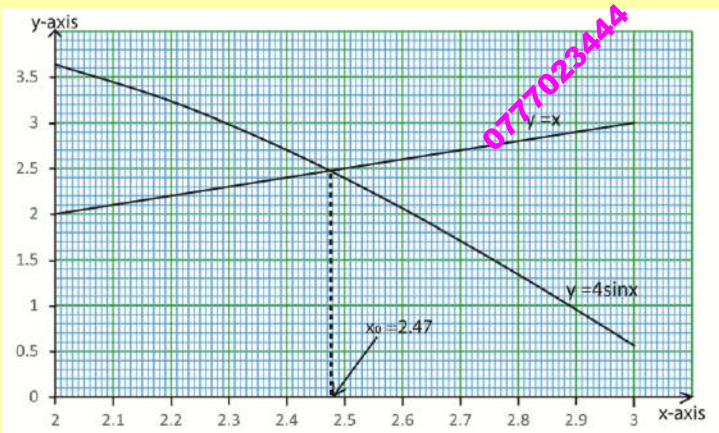
$x_3 = \frac{3(\sin 2.279 - 2.5\cos 2.279}{1-3\cos 2.279} = 2.279$

Error = $|2.279 - 2.279| = 0.000 < 0.005$

∴ root = 2.279 = 2.28(2D)

Example 22

(a) On the same axis, draw graphs of y = x and y = 4sinx to show that the root of the equation x-4sinx = 0 lies between x= 2 and x = 3



Therefore the root (2.47) lies between x = 2 and x = 3

(b) Use Newton Raphson's method to calculate the root of the equation x – 4sinx = 0, taking approximate root in (a) as the initial approximation to the root. correct your answer to 3 decimal places.

f(x) = x – 4sinx

f'(x) = 1 – 4cos x

$x_{n+1} = x_n - \frac{x_n - 4\sin x_n}{1-4\cos x_n}$

$Taking\ x_0 = 2.47$

$x_1 = 2.47 - \frac{2.47 - 4\sin 2.45}{1-4\cos 2.47} = 2.4746$

$Error = |2.4746 - 2.47| = 0.0046 > 0.0005$

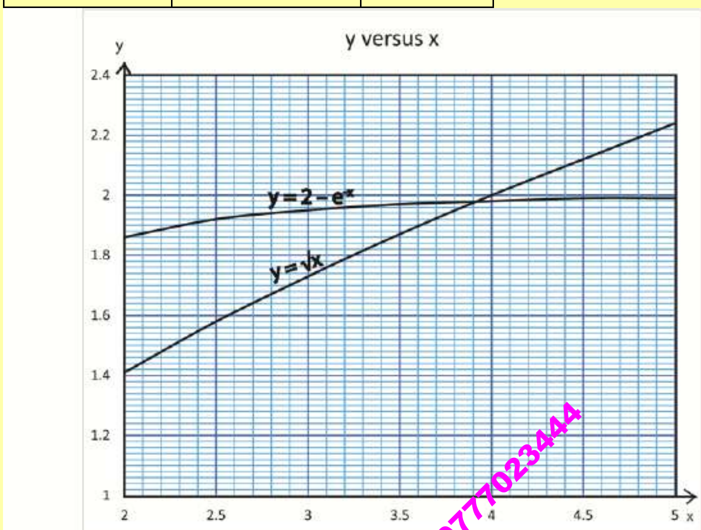
$x_2 = 2.4746 - \frac{2.4746 - 4\sin 2.4546}{1-4\cos 2.4746} = 2.4746$

$Error = |2.4746 - 2.4746| = 0.000 < 0.0005$ ∴ the root = 2.475 (3D)

Example 23

(a) Draw on the same axes the graphs of the curves y = 2 – e^{-x} and y = $\sqrt{x}$values 2≤ x ≤ 5. (04marks)

x	y = 2 – e^{-x}	y = $\sqrt{x}$
2.0	1.86	1.41
2.5	1.92	1.58
3.0	1.95	1.73
3.5	1.97	1.87
4.0	1.98	2.00
4.5	1.99	2.12
5.0	1.99	2.24



(b) Determine from your graph the interval within which the roots of the equation
$e^{-x} + \sqrt{x - 2 = 0}$ lies

Hence, use Newton-Raphson's method to find the root of the equation correct to 3 decimal places (07marks)

Root lies between 3.9 and 4

f(x) = 2 – e^{-x} - $\sqrt{x}$

f'(x) = $e^{-x} - \frac{1}{2\sqrt{x}}$

f(x_n) = $e^{-x_n} - \frac{1}{2\sqrt{x_n}}$

x_{n+1} =x_n - $\frac{2- e^{x_n} - \sqrt{x_n}}{2e^{-x_n}\sqrt{x_n} - 1}$

x_0 = $\frac{3.9+4}{2}$ = 3.95

$x_1 = 3.95 - \frac{2\sqrt{3.95}(2 - e^{-3.95} - \sqrt{3.95})}{2e^{-3.95}\sqrt{3.95} - 1}$ = 3.9211

$Error = |3.9211 - 3.95| = 0.0289$

$x_2 = 3.9211 - \frac{2\sqrt{3.9211}(2 - e^{-3.9211} - \sqrt{3.9211})}{2e^{-3.9211}\sqrt{3.9211} - 1}$ = 3.9211

∴ Root = 3.921 (3dp)

Example 24

Given the equation $X^3 - 6x^2 + 9x + 2 = 0$

(a) Show that the equation has a root between -1 and 0.

Let $f(x) = X^3 - 6x^2 + 9x + 2$

$f(-1) = (-1)^3 - 6(-1)^2 + 9(-1) + 2$

$= -1 - 6 - 9 + 2 = -14$

$f(0) = 0 + 0 + 0 + 2$

$= 2$

$f(-1).f(0) = -14 \times 2 = -28$

since $f(-1).f(0) < 0$; the root exist between -1 and 0.

(b) (i) Show that the Newton Raphson formula approximating the root of the equation is given by $X_{n+1} = \frac{2}{3}\left[\frac{x_n^3 - 3x_n^2 - 1}{x_n^2 - 4x_n + 3}\right]$

$f(x) = X^3 - 6x^2 + 9x + 2$

$f(x_n) = x_n^3 - 6x_n^2 + 9x_n + 2$

$f'(x_n) = 3x_n^2 - 12x_n + 9$

$$x_{n+1} = x_n - \left(\frac{x_n^3 - 6x_n^2 + 9x_n + 2}{3x_n^2 - 12x_n + 9}\right)$$

$$= \frac{x_n(3x_n^2 - 12x_n + 9) - (x_n^3 - 6x_n^2 + 9x_n + 2)}{3x_n^2 - 12x_n + 9}$$

$$= \frac{(3x_n^3 - 12x_n^2 + 9x_n) - (x_n^3 - 6x_n^2 + 9x_n + 2)}{3x_n^2 - 12x_n + 9}$$

$$= \frac{2x_n^3 - 6x_n^2 - 2}{3x_n^2 - 12x_n + 9}$$

$$= \frac{2}{3}\left[\frac{x_n^3 - 3x_n^2 - 1}{x_n^2 - 4x_n + 3}\right]$$

(ii) Use the formula in (b)(i) above with initial approximation of $x_0 = -0.5$, to find the root of the given equation correct to two decimal places

Taking x = -0.5

$x_1 = \frac{2}{3}\left[\frac{(-0.5)^3 - 3(-0.5)^2 - 1}{(-0.5)^2 - 4(-0.5) + 3}\right] = -0.2381$

$|e| = |0.2381 - (-0.5)| = 0.2619$

$x_2 = \frac{2}{3}\left[\frac{(-0.2381)^3 - 3(-0.2381)^2 - 1}{(-0.2381)^2 - 4(-0.2381) + 3}\right] = -0.1968$

$|e| = |0.1968 - (-0.2381)| = 0.0.0413$

$x_3 = \frac{2}{3}\left[\frac{-0.1968^3 - 3(-0.1968)^2 - 1}{(-0.1968)^2 - 4(-0.1968) + 3}\right] = -0.1958$

$|e| = |-0.1958 - (-0.1968)| = 0.001 < 0.005$

Hence the root = -0.20 (2D)

Revision Exercise 3

1. Using the Newton Raphson's formula, show that the reciprocal of a number N is $x_n(2 - Nx_{n)}$
2. Use Newton Raphson's iterative formula to show that the cube root of a number N is given by $\frac{1}{3}\left(2x_n + \frac{N}{{x_n}^2}\right)$. Hence use the iterative formula to find $\sqrt[3]{96}$ correct to3 decimal places. use $x_0 = 5$. [4.579]
3. (a) Show that the equation 3x3 + x -5 = 0 has real root between x = 1 and x = 2.

   (b) Using linear interpolation, find the first approximation for this root to 2dp. [1.04]

(c) Using Newton Raphson's method twice find the value of this root correct to 2 dp. [1.09]

4. (a) Show graphically that there is a positive real root of equation $xe^{-x} - 2x + 5 = 0$ between $x = 2$ and $x = 3$

   (b) Using Newton Raphson's method, find this root correct to 1 dp. [2.6]

5. Using the iterative formula for NRM, show that the fourth root of a number N is $\frac{3}{4}\left(x_n + \frac{N}{3{x_n}^3}\right)$. Starting with $x_0 = 2.5$ show that $(45.7)^{\frac{1}{4}} = 2.600$ (3dp)

6. On the same axes, draw graphs of $y = x^3$ and $y = 2x + 5$. Using NRM twice find the positive root of the equation $x^3 - 2x - 5 = 0$ correct to 2 decimal places. [2.09]

7. (a) Show that the Newton Raphson's formula for finding the smallest positive root of the equation $3\tan x + x = 0$ is $\frac{6x_n - 3\sin 2x_n}{6 + 2\cos 2x_n}$

   (b) By sketching the graphs of y = tans, $y = \frac{-x}{3}$ 0r otherwise, find the first approximation to the required root and use it to find the actual root correct to 3 dp . [2.456]

8. (a) Show that the root of the equation $f(x) = e^x + x^3 - 4x = 0$ has a root between $x = 1$ and $x = 2$

   (b) Use the Newton Raphson's method to find the root of equation in (a) correct to 2 decimal places. [$x_0 = 1$, root =1.12]

9. (a) Show that the iterative formula for approximation of the root of f(x) = 0 by NRM process for the equation $xe^x + 5x - 10 = 0$ is $x_{n+1} = \frac{{x_n}^2 e^{x_n} + 10}{x_n e^{x_n} + e^{x_n} + 5}$.

   (b) Show that the root of the equation in (i) above lies between x = 1 and x = 2. Hence find the root of the equation correct to 2 dp. [1.20]

10. (a) Use a graphical method to find a first approximation to the real root of $x^3 + 2x - 2 = 0$.

    (b) Use the Newton Raphson's method to find the root of the equation in (a) correct to 2 dp. [0.77]

11. (a) Show that equation x = In (8-x) has a root between x= 1 and x = 2.

    (b) Use the Newton Raphson's method to find the root of the equation in (a) correct to 2 decimal places [1.82]

12. (a) Use graphical method to find the first approximation to the root of $x^3 - 3x + 4 = 0$. [-2]

    (b) Use NRM to find the root of the equation in (a) correct to 2 d.p. [-2.20]

13. Show graphically that equation $e^x + x - 4 = 0$ has only one root between x = 1 and x = 2. Use NRM to find the approximation of the equation correct to 3dp. [1.07]

14. Show that the NRM for approximating the K[th] root of a number N is given by $\frac{1}{K}\left((K-1)x_n + \frac{N}{{x_n}^{K-1}}\right)$. Hence use your formula to find the positive square root of 67 correct to 4 s.f. [8.185].

15. (a) Show that equation $x^3 + 3x - 9 = 0$ has a root between x= 1 and x = 2.

    (b) Use the Newton Raphson's method to find the root of the equation in (a) correct to 2 One places [1.6]

16. (a) Show graphically that there is a positive real root of equation $xe^{-x} - 2x - 1 = 0$ between x = 1 and x = 2

    (b) Using Newton Raphson's method, find this root for the equation in (a) correct to 2 dp. [1.26]

17. (a) Show that equation $2x - 3\cos\left(\frac{x}{2}\right) = 0$ has a root between x= 1 and x = 2.

    (b) Use the Newton Raphson's method to find the root of the equation in (a) correct to one places [1.23]

18. (a) If a is the first approximation to the root of the equation $x^5 - b = 0$, show that the second approximation is given by $\frac{4a+\frac{b}{a^4}}{5}$.

    (b) Show that the positive real root of the equation $x^5 - 17 = 0$ lies between 1.5 and 1.8. Hence use the formula in (a) above to determine the root to 3 decimal places. [1.762]

19. (a)(i) On the same axes, draw graphs of $y = x^2$ and y = cox x for $0 \le x \le \frac{\pi}{2}$ at intervals of $\frac{\pi}{8}$.

    (ii) Use your graphs, to find to 1 decimal place an approximate root of the equation $x^2 - \cos x = 0$ [0.8]

    (b) Use the NRM to calculate the root of the equation $x^2 - \cos x = 0$ taking the approximate root in (a) as the initial approximation. Correct your answer to 3 dp. [0.824]

20. (a) (i) Draw on same axes the graphs of equation y = xsinx and $y = e^x - 2$ for $0 \le x \le 1.5$.

    (ii) Use your graphs to find an approximate root of the equation $2 - e^x + x\sin x =$ [1.1]

(c) Use the Newton Raphson's method to find the root of the equation in (a)(ii) correct to three decimal places [1.085]

21. Show graphically that equation $e^x + x - 8 = 0$ has only one real root between x = 1 and x = 2. Use NRM to find approximation of x = In(x 8) correct to 3 dp [1.821]

22. Draw using the same axes, graphs of $y = x^2$ and y = sin2x for $0 \le x \le \frac{\pi}{2}$. From the graphs obtain to one decimal place an approximation of the non-zero root of the equation $x^2 - \sin 2x = 0$. Using NRM, calculate to 2 dp a more suitable approximation. [0.97]

23. Given the equation In(1 + 2x) – x = 0.

    (i) show the root of the equation above lies between x = 1 and x = 1.5

    (ii) Use NRM twice to estimate the root of the equation, correct to 2 dp. [1.26]

31. ERRORS

Errors

An error, commonly known as absolute error is the absolute difference between exact value and approximate value.

Source of errors

(a) Rounding off

These errors that arise as a result of simply approximating the exact value of different numbers.

Example 1

Round off the following numbers to the given number of decimal places or significant figures.

(i) 3.896234 to 4 dp [3.8962]
(ii) $\frac{2}{3}$ to 3dp [0.667]
(iii) 5.002570 to 3s.f [5.00]

(iv) 0.006526736 to4 s.f [0.006527]

(v) 7.00214 to 4 s.f. [7.002]

(vi) 5415678 to 3 s.f. [5420000]

(b) Truncation

These occur when an infinite number is terminated/cutoff (without rounding off] at some point.

Example 2

Truncate the following number to the given number of decimal places (d.p) or significant figures. s.f.

(i) 4.56172 to 2dp [4.56] (ii) $\frac{2}{3}$ to 3dp [0.666] (iii) 1.345618 to 4 s.f. [1.345]

Common terms used

(a) Error or absolute error
If x represent an approximate value of X and Δx is the error approximation
$|Error| = |exact\ value - approximate\ value|$
$|\Delta x| = |X - x|$

Example 3

Round off 32.5263 to 2 dp and determine the absolute error.

Solution

X = 32.5263, x = 32.53

$|\Delta x| = |X - x|$ = $|32.5263 - 32.53|$ = 0.0037

(b) Relative error

Relative error = $\frac{absolute\ error}{exact\ value} = \frac{|\Delta x|}{X} = \frac{|X-x|}{X}$

(c) Percentage error or percentage relative error

Percentage relative error = $\frac{absolute\ error}{exact\ value} x\ 100\% = \frac{|\Delta x|}{X} x\ 100\% = \frac{|X-x|}{X}\ x\ 100\%$

Example 4

Find the percentage error in rounding off $\sqrt{3}$ 2 dp

Solution

X = $\sqrt{x}$ = 1.732050808, x = 1.73

Percentage error = $\frac{|X-x|}{X}\ x\ 100\%$ = $\frac{|1.732050808 - 1.73|}{1.732050808}\ x\ 100\%$ = 0.118%

(d) Error bound or minimum possible error in an approximated number

This depends on the number of decimal places the number is rounded to. If the number is rounded to n dp, then the maximum possible error in that number is = 0.5 x 10^{-n}.

Example 4

If a student weighs 50kg. Find the range where his weight lies

Solution

n = 0 dp, e = 0.5 x 10^{-0} = 0.5

Range = 50 $\pm$ 0.5 =(49.5, 50.5)

Example 5

If x is given to stated level of accuracy stat the lower and upper bounds of x

(a) 6.45

n = 2 dp, e = 0.5 x 10^{-2} = 0.005

Lower bound = 6.45 – 0.005 = 6.445

upper bound = 6.45 + 0.005 = 6.455

(b) 0.278

n = 3 dp, e = 0.5 x 10^{-3} = 0.0005

(c) Lower bound = 0.278 – 0.0005 = 0.2775

upper bound = 0.278 + 0.0005 = 0.2785

Example 6

A value of w = 150.58m was obtained when measuring the width of the football pitch. Given that the relative error in this value as 0.07%, find the limit within which the value w lies.

% relative error = $\frac{|\Delta w|}{w}\ x100\%$

0.07 =$\frac{|\Delta w|}{150.58}\ x\ 100$

$|\Delta w|$ = 0.105

Lower limit = 150.58 – 0.105 = 150 .475

Upper limit = 150.58 + 0.105 = 150 .685

Absolute error in an operation

When the minimum and maximum value is known then.

absolute error = $\frac{1}{2}[maximum\ value - minimum\ value]$

Absolute error in addition

Given two numbers a and b with errors Δa + Δb

$(a + b)_{max} = a_{max} + b_{max} = (a + \Delta a) + (b + \Delta b)$

$(a + b)_{min} = a_{min} + b_{min} = (a - \Delta a) + (b - \Delta b)$

Absolute error in subtraction

Given two numbers a and b with errors Δa + Δb

$(a - b)_{max} = a_{max} - b_{min} = (a + \Delta a) - (b - \Delta b)$

$(a + b)_{min} = a_{min} - b_{max} = (a - \Delta a) + (b + \Delta b)$

Example 7

Given that a = 2.453, b = 6.79, find the limits and hence absolute error of

(i) a + b

solution

a = 2.453, Δa = 0.0005 and b = 6.79, Δb = 0.005

$(a + b)_{max} = a_{max} + b_{max} = (a + \Delta a) + (b + \Delta b)$

$= (2.453 + 0.0005) + (6.79 + 0.005) = 9.2485$

$(a + b)_{min} = a_{min} + b_{min} = (a - \Delta a) + (b - \Delta b)$

$= (2.453 - 0.0005) + (6.79 - 0.005) = 9.2375$

lower limit = 9.2375; upper limit = 9.2485

absolute error $= \frac{1}{2}[9.2485 - 9.2375] = 0.0055$

(ii) a - b

solution

$(a + b)_{max} = a_{max} - b_{min} = (a + \Delta a) - (b - \Delta b)$

$= (2.453 + 0.0005) - (6.79 - 0.005) = -4.3315$

$(a + b)_{min} = a_{min} - b_{max} = (a - \Delta a) - (b + \Delta b)$

$= (2.453 - 0.0005) - (6.79 + 0.005) = -4.3425$

lower limit = -4.3425; upper limit = -4.3315

absolute error $= \frac{1}{2}[-4.3315 - -4.3425] = 0.0055$

Absolute error in multiplication

Given two numbers a and b with errors Δa + Δb
$(ab)_{max} = a_{max}b_{max}$ = (a + Δa)(b+Δb)
$(ab)_{min} = a_{min}b_{min}$ = (a - Δa)(b - Δb)

Example 8

Given that a = 4.617, and b = 3.65 find the absolute error in ab

solution

a = 4.617, Δa= 0.0005, b = 3.65, Δb = 0.005

$(ab)_{max} = a_{max}b_{max}$ = (a + Δa)(b+Δb) = (4.617 + 0.0005)(3.65b+ 0.005) =16.87696

$(ab)_{min} = a_{min}b_{min}$ = (a - Δa)(b - Δb) = (4.617 - 0.0005)(3.65b - 0.005)=16.82853

absolute error = $\frac{1}{2}[maximum\ value - minimum\ value]$ = $\frac{1}{2}(16.87696 - 16.82853)$ =0.02422

Example 9

Given that a = 4.617, and b = -3.65 find the

(i) Limits of values where ab lies
Solution

a = 4.617, Δa= 0.0005, b = -3.65, Δb = 0.005

$(ab)_{max} = a_{max}b_{max}$ = (4.617 + 0.0005)(-3.65b+ 0.005) = -16.83079
$(ab)_{min} = a_{min}b_{min}$ = (4.617 - 0.0005)(-3.65b - 0.005 = -16.87331
Lower limit = -16.87331; upper limit = -16.83079

(ii) the interval of values where ab lies
(-16.87331, -16.83079)

(iii) the absolute error
Absolute error =$\frac{1}{2}[-16.83079 - \ -16.87331]$= 0.02126

Absolute error n division

Given two numbers a and b with errors Δa + Δb

$$\left(\frac{a}{b}\right)_{max} = \frac{a_{max}}{b_{min}} = \frac{(a+\Delta a)}{(b-\Delta b)}$$

$$\left(\frac{a}{b}\right)_{min} = \frac{a_{min}}{b_{max}} = \frac{(a-\Delta a)}{(b+\Delta b)}$$

Example 10

Given a = 1.26, b = 0.435. Find the absolute error of

(i) Range of value where $\frac{a}{b}$ lies

$$\left(\frac{a}{b}\right)_{max} = \frac{a_{max}}{b_{min}} = \frac{(1.25+0.005)}{(0.435-0.0005)} = 2.91139$$

$\left(\frac{a}{b}\right)_{min} = \frac{a_{min}}{b_{max}} = \frac{(1.25-0.005)}{(0.435+0.0005)} = 2.88175$

Range of values is (2.88175, 2.91139)

(ii) Absolute error

$=\frac{1}{2}(2.91139 - 2.88175) = 0.01482$

Example 11

(a) Given that y = e^x and x = 0.62correct to two decimal places, find the interval within which the exact value of y lies. (05marks)

$e_x = 0.005$
$y_{max} = e^{0.625} = 1.8682$
$y_{min} = e^{0.615} = 1.8497$
The interval = (1.8497, 1.8682)

(b) Show that the maximum possible relative error in ysin^2x is

$\left|\frac{\Delta y}{y}\right| + 2\cot x\,|\Delta x|$, where$\Delta x$ and Δy are errors in x and y respectively

Hence find the percentage error in calculating ysin^2x if y = 5.2 $\pm$ 0.05 and x = $\frac{\pi}{6} \pm \frac{\pi}{360}$

(07 marks)

z = y$\sin^2$x

$e_z = \Delta y sin^2x + 2y\Delta x cos\, x sinx$

$\frac{e_z}{z} = \frac{\Delta y sin^2x}{y\sin^2 x} + \frac{2y\Delta x cos\, x sinx}{y\sin^2 x}$

$\left|\frac{e_z}{z}\right| = \left|\frac{\Delta y}{y} + 2cotx.\Delta x\right|$

$\leq \left|\frac{\Delta y}{y}\right| + 2cotx.|\Delta x|$

∴ Maximum possible error is $\left|\frac{\Delta y}{y}\right| + 2cotx.|\Delta x|$

$percentage\ error = \left[\frac{0.05}{5.2} + 2\cot\frac{\pi}{6}.\left|\frac{\pi}{360}\right|\right] x\ 100\%$ = 3.9845%

Example 12

Two numbers A and B have maximum possible error e_a and e_b respectively.

(a) Write an expression for the maximum possible error in their sum
Maximum possible error = $|e_a||e_b|$

(b) If A = 2.03 and B = 1.547, find the maximum possible error in A + B (05marks)
$e_a = 0.005, e_b = 0.0005$
$|e_{(A+B)}| = |0.005| + |0.0005|$
= 0.0055

Example 13

Given that y = $\frac{1}{x} + x$ and x = 2.4 correct to one decimal place, find the limits within which y lies. (05marks)

Error in 2.4 = $\frac{1}{2}\ x\ \frac{1}{10} = 0.05$

y_{max} = 2.45 + $\frac{1}{2.35} = 2.8755$

y_{min} = 2.35 + $\frac{1}{2.45} = 2.7582$

∴ the limits are [2.7582, 2.8755]

Example 14

The numbers X = 1.2, Y = 1.33 and Z = 2.245 have been rounded off to the given decimal places. find the maximum possible value of $\frac{Y}{Z-X}$ correct to 3 decimal places

Maximum value = $\frac{(Y+\Delta Y)}{(Z-\Delta Z)-(X+\Delta X)} = \frac{(1.33+0.005)}{(2.245-0.0005)-(\ 1.2+0.05)}$ = 1.342

Revision exercise 1

1. Given the numbers x = 2.678 and y = 0.8765 measured the nearest possible decimal places indicated.
   (i) state the maximum possible error in x ans y [Δx = 0.0005, Δy = 0.00005]
   (ii) find the limits within which the product xy lie [2.3467, 2.3478]
   (iii) determine the maximum possible error in xy [0.000572]

2. The length, width and height of water all rounded off to 3.65m, 2.14m and 2.5m respectively. Determine the least and greatest amount of water the tank can contain [19.066, 19.992]
3. Given that the values x = 4, y = 6 and z = 8 each has been approximate to the nearest integer. find the maximum and minimum values of
   (i) $\frac{y}{x}$ [1.85714, 1.22222]
   (ii) $\frac{z-x}{y}$ [0.90909, 0.46154]
   (iii) $(x+y)z$ [93.5, 67.5]

Error propagation

Triangular inequality state that $|a \pm b| \leq \ |\Delta a| + \ |\Delta b|$

Addition

Consider two numbers X and Y are approximated by x and y with errors Δx and Δy.

$|e_{x+y}|$ = $|(x+\Delta x)+(y+\ \Delta y)-(x+y)|$

$=|\Delta x+\ \Delta y| = |\Delta x| + |\Delta y|$

R.E$_{max}$ = $\left[\left|\frac{\Delta x}{X+Y}\right| + \left|\frac{\Delta y}{X+Y}\right|\right]$

Alternatively

absolute error = $\frac{1}{2}[\max - min]$

$= \frac{1}{2}[(x+\Delta x)+(y+\ \Delta y)] - [(x-\Delta x)+(y-\ \Delta y)]$

$|e_{x+y}|=|\Delta x + \Delta y|= |\Delta x| + |\Delta y|$

R.E$_{max}$ = $\left[\left|\frac{\Delta x}{X+Y}\right| + \left|\frac{\Delta y}{X+Y}\right|\right]$

Example 15

Given numbers x = 7.824 and y = 3.36 rounded to the given number of decimal places. Find the limits within which (x + y) lies

Solution

Δx = 0.0005, Δy = 0.005

$|e_{x+y}|=|\Delta x| + |\Delta y|$= 0.0005 + 0.005 = 0.0055

working value (x + y) = 7.824 + 3.36 = 10.184

Upper limit = 10.184 + 0.0055 = 10.1895

Lower limit = 10.184 - 0.0055 = 10.1785

Alternatively

$(x + y)_{max}$ = 7.8245 + 3.365 =10.1895

$(x + y)_{min}$ = 7.8235 + 3.355 = 10.1785

Example 16

If x = 4.95 and y = 2.2 are each rounded off to the given number of decimal places. Calculate

(i) The percentage error in x + y

Solution

Δx = 0.005, Δy = 0.05

%error = $\left[\left|\frac{\Delta x}{X+Y}\right| + \left|\frac{\Delta y}{X+Y}\right|\right] x\ 100\%$ = $\left[\left|\frac{0.005}{4.95+2.2}\right| + \left|\frac{0.05}{4.95+2.2}\right|\right] x\ 100\%$ = 0.769

Alternatively

Working value x + y = $4.95 + 2.2$ = 7.15

$|e_{x+y}|=|\Delta x| + |\Delta y|$= 0.005 + 0.05 = 0.055

% error = $\frac{0.055}{7.15} x\ 100\%$ = 0.769

(ii) Find the limit within which (x + y) is expected to lie. Give your answer to two decimal places.
Upper limit = 7.15 + 0.055 = 7.21; lower limit = 7.15 – 0.055 = 7.10
Alternatively
Upper limit = 4.955 + 2.25 = 7.21; lower limit = 4.945 + 2.15 = 7.10

Subtraction

Consider two numbers X and Y are approximated by x and y with errors Δx and Δy.

$|e_{x-y}| = |(x + \Delta x) - (y + \Delta y) - (x - y)|$

$=|\Delta x - \Delta y|= |\Delta x| + |\Delta y|$

R.E$_{max}$ = $\left[\left|\frac{\Delta x}{X-Y}\right| + \left|\frac{\Delta y}{X-Y}\right|\right]$

Alternatively

absolute error = $\frac{1}{2}[\max - min]$

$=\frac{1}{2}\{[(x+\Delta x)-(y-\Delta y)]-[(x+\Delta x)-(y+\Delta y)]\}$

$|e_{x-y}| = \quad |\Delta x + \Delta y| = |\Delta x| + |\Delta y|$

R.E$_{max}$ = $\left[\left|\frac{\Delta x}{X-Y}\right| + \left|\frac{\Delta y}{X-Y}\right|\right]$

Example 17

Given number x = 6.375 and y 4.46 rounded off to the given number of decimal places. Find the limit within which (x – y) lies

Solution

Δx = 0.0005, Δy = 0.005

$|e_{x-y}| = |\Delta x| + |\Delta y| = |0.0005| + |0.005|$ = 0.0055

working value = 6.375 – 4.46 = 1.915

Upper limit = 1.915 + 0.0055 = 1.9205; Lower limit = 1.915 – 0.0055 = 1.9095

Alternatively

$(x – y)_{max}$ = 6.3755 – 4.455 = 1.9205

$(x – y)_{min}$ = 6.3745 – 4.465 = 1.9095

Example 18

If x = 1.563 and y = 9.87 are each rounded off to the given number of decimal places. Calculate

(i) the percentage error in (x – y)

% error = $\left[\left|\frac{\Delta x}{X-Y}\right| + \left|\frac{\Delta y}{X-Y}\right|\right] x\ 100\% = \left[\left|\frac{0.0005}{1.563-9.87}\right| + \left|\frac{0.005}{1.563-9.87}\right|\right] x100\%$ = 0.0662

Alternatively

Working value = x – y = 1.563 9.87 = -8.307

$|e_{x-y}| = |\Delta x| + |\Delta y| = |0.0005| + |0.005|$ =0.0055

$\%\ error = \frac{0.0055}{-8.307}\ x\ 100\% = 0.0662$

(ii) the limit within which (x – y) is expected to lie. Give your answer to three decimal places

Upper limit = -8.307 + 0.0055 = -8.302

Lower limit = -8.307 - 0.0055 = -8.313

Alternatively

$(x – y)_{max}$ = 1.5635 – 9.865 =-8.302

$(x – y)_{min}$ =1.5625 – 9.875 =-8.313

Multiplication

Consider two numbers X and Y are approximated by x and y with errors Δx and Δy.

$|e_{xy}| = |(x+\Delta x)(y+\ \Delta y)-(xy)|$

$= |xy + y\Delta x + x\Delta y + \Delta x\Delta y - xy|$

Since Δx and Δy are very small, $\Delta x\Delta y \approx 0$

$|e_{xy}| = |y\Delta x + x\Delta y| = |y\Delta x| + \ |x\Delta y|$

$R.E_{max} = \left|\frac{y\Delta x}{xy}\right| + \left|\frac{x\Delta y}{xy}\right|$

$= \left|\frac{\Delta x}{x}\right| + \left|\frac{\Delta y}{y}\right|$

Alternatively

absolute error = $\frac{1}{2}|\max - min|$

$=\frac{1}{2}\left[(x+\Delta x)(y+\ \Delta y) - [(x-\Delta x)(y-\ \Delta y)]\right]$

$|e_{xy}| = |y\Delta x + x\Delta y| = |y\Delta x| + \ |x\Delta y|$

$R.E_{max} = \left|\frac{y\Delta x}{xy}\right| + \left|\frac{x\Delta y}{xy}\right|$

Example 19

Given numbers x = 6.375 and y = 4.46 rounded off to eh given number of decimal places. Find the limit within which (xy) lies

Solution

Δx =0.0005 Δy = 0.005

$|e_{xy}| = |y\Delta x| + \ |x\Delta y| = |6.375\ x\ 0.005| + \ |4.46\ x\ 0.0005|$ = 0.0341

working value = xy = 6.375 x 4.46 = 28.4325

Upper limit = 28.4325 + 0.0341 =28.4666

Lower limit = 28.4325 - 0.0341 = 28.3984

Alternatively

$(xy)_{max}$ = 6.3755 x 4.465 =28.4666

$(xy)_{min}$ = 6.3745 x 4.455 = 28.3984

Example 20

If x = 1.563 and y = 9.87 are each rounded off to the given number of decimal places. Calculate

(i) Percentage error in (xy)

$\Delta x = 0.0005$, $\Delta y = 0.005$

% error $=\left\{\left|\frac{\Delta x}{x}\right| + \left|\frac{\Delta y}{y}\right|\right\} x\ 100\% = \left\{\left|\frac{0.0006}{1.563}\right| + \left|\frac{0.005}{9.87}\right|\right\} x\ 100\% = 0.0826$

Alternatively

Working value = 1.563 x 9.87 = 15.4268

$|e_{xy}|$ = $|y\Delta x| + |x\Delta y|$ = 9.87 x 0.0005 + 1.563 x 0.005 =0.0128

$$\%\ error = \frac{0.0128}{15.4268}\ x\ 100\% = 0.0826$$

(ii) the limit within which (xy) is expected to lie. Give your answer to three decimal places.

Upper limit = 15.4268 + 0.0128 = 15.440

Lower limit =15.4268 - 0.0128 =15.414

Alternatively

Upper limit = 1.5635 x 9.875 = 15.440

Lower limit = 1.5625 x 9.865 =15.414

Division

Consider two numbers X and Y are approximated by x and y with errors Δx and Δy.

$\left|e_{x/y}\right| = \left|\frac{x+\Delta x}{y+\Delta y} - \frac{x}{y}\right| = \left|\frac{xy+y\Delta x-x\Delta y-xy}{y^2+y\Delta y}\right|$

$= \left|\frac{y\Delta x-x\,\Delta y}{y^2\left(1+\frac{\Delta y}{y}\right)}\right|$

Since Δx and Δy are very small, then $\frac{\Delta y}{y} \approx 0$

$\left|e_{x/y}\right| = \left|\frac{y\Delta x-x\,\Delta y}{y^2}\right|$

$\left|e_{x/y}\right| \leq \frac{|y\Delta x|+|x\,\Delta y|}{|y^2|}$

$e_{max} = \frac{|y\Delta x|+|x\,\Delta y|}{|y^2|}$

$R.E_{max} = \frac{|y\Delta x|-|x\,\Delta y|}{|y^2|} \div \frac{x}{y}$

$R.E_{max} = \left|\frac{\Delta x}{x}\right| + \left|\frac{\Delta y}{y}\right|$

Alternatively

absolute error = $\frac{1}{2}|\max - min|$

$= \frac{1}{2}\left|\frac{(x+\Delta x)}{(y-\Delta y)} - \frac{(x-\Delta x)}{(y+\Delta y)}\right|$

$e_{x/y} = \left|\frac{x\Delta y+y\Delta x}{y^2-\Delta y^2}\right|$

Since Δx and Δy are very small, then $\Delta y^2 \approx 0$

$\left|e_{x/y}\right| = \left|\frac{y\Delta x-x\,\Delta y}{y^2}\right|$

$\left|e_{x/y}\right| \leq \frac{|y\Delta x|+|x\,\Delta y|}{|y^2|}$

$e_{max} = \frac{|y\Delta x|+|x\,\Delta y|}{|y^2|}$

$R.E_{max} = \frac{|y\Delta x|-|x\,\Delta y|}{|y^2|} \div \frac{x}{y}$

$R.E_{max} = \left|\frac{\Delta x}{x}\right| + \left|\frac{\Delta y}{y}\right|$

Example 21

Given numbers x = 5.794 and y = 0.28 rounded off to the given number of decimal places. Find limit within which $\frac{x}{y}$ lies

Solution

Δx = 0.0005, Δy = 0.005

$\left|e_{x/y}\right| = \frac{|y\Delta x|+|x\,\Delta y|}{|y^2|}$

= $\frac{|0.28\,x\,0.0005|+|5.794\,x\,0.005|}{|0.28^2|}$

= 0.3713

Working value =$\frac{x}{y}$ = $\frac{5.794}{0.28}$ = 20.6929

Upper limit = 20.6929 + 0.3713 =21.0642

Lower limit =20.6929 - 0.3713 = 20 .3198

Alternatively

Upper limit = $\frac{5.7945}{0.275}$ =21.079

Lower limit = $\frac{5.7935}{0.285}$ = 20.3281

Example 22

If x = 7.37 and y = 2.00 are each rounded off to the given number of decimal places. Calculate

(i) Percentage error

% error = =$\left\{\left|\frac{\Delta x}{x}\right| + \left|\frac{\Delta y}{y}\right|\right\} x100\% = \left\{\left|\frac{0.005}{7.37}\right| + \left|\frac{0.005}{2.00}\right|\right\} x100\% = 0.318$

Alternatively

$\left|e_{x/y}\right| = \frac{|y\Delta x|+|x\,\Delta y|}{|y^2|}$ = $\frac{|2.00\,x\,0.005|+|7.37\,x\,0.005|}{|2.00^2|}$ = 0.0117

Working value = $\frac{x}{y} = \frac{7.37}{2.00} = 3.685$

% error = $\frac{0.0117}{3.685}\ x\ 100 = \ 0.318$

(ii) the limit within which $\left(\frac{x}{y}\right)$ is expected to lie. Give your answer to three decimal places.

Upper limit = $3.685 + \ 0.318$ = 3.697

Lower limit = $3.685 - \ 0.318$ = 3.673

Alternatively

Upper limit = $\frac{7.375}{1.995} = 3.697$

Lower limit = $\frac{7.365}{2.005} = 3.673$

Error in functions

Given a function f(x) with a maximum possible error Δx.

Absolute error, $|e|$ = $|\Delta x| f^1(x)$

Maximum possible relative error, R.E = $\frac{|\Delta x|f^1(x)}{f(x)}$

Example 23

Find the absolute error and maximum relative error in each of the following functions

(i) y = x^4

$|e| = |\Delta x|f^1(x) = 4x^3|\Delta x|$

R.E = $\frac{|\Delta x|f^1(x)}{f(x)} = \frac{4x^3|\Delta x|}{x^4} = \frac{4|\Delta x|}{x}$

(ii) y = $x^{\frac{3}{2}}$

$|e| = |\Delta x|f^1(x) = \frac{3}{2}x^{\frac{1}{2}}|\Delta x|$

R.E = $\frac{|\Delta x|f^1(x)}{f(x)} = \frac{\frac{3}{2}x^{\frac{1}{2}}|\Delta x|}{x^{\frac{3}{2}}} = \frac{3}{2}\frac{|\Delta x|}{x}$

(iii) y = sin x

$|e| = |\Delta x|f^1(x) = \cos x|\Delta x|$

R.E = $\frac{|\Delta x|f^1(x)}{f(x)} = \frac{\cos x|\Delta x|}{sin\ x} = |\Delta x||\cot x|$

Example 24

Given that the error in measuring an angle is 0.4^0. find the maximum possible error and relative error in tanx if x = 60^0.

Solution

$|e| = |\Delta x|f^1(x) = (1 + \tan^2 x)|\Delta x|$

$|e| = (1 + \tan^2 60)\left|\frac{0.4}{180}\pi\right| = 0.0280$

R.E = $\frac{0.0280}{\tan 60} = 0.0162$

Error in a function that has more variables

Given a function f(x, y) with a maximum possible error Δx and Δy respectively

Absolute error, $|e| = |\Delta x|f^1(x) + |\Delta y|f^1(y)$

Maximum possible relative error = $\frac{|\Delta x|f^1(x)+|\Delta y|f^1(y)}{f(x,y)}$

Example 25

Given that X and Y are rounded off to give x and y with error Δx and Δy respectively. Show that the maximum relative error recorded in x^4y is given by $4\left|\frac{\Delta x}{x}\right| + \left|\frac{\Delta y}{y}\right|$

Solution

$|e| = |\Delta x|f^1(x) + |\Delta y|f^1(y) = |\Delta x 4x^3y| + \Delta y x^4$

$|e| \leq 4|x^3y||\Delta x| + |x^4||\Delta y|$

$|e_{max}| = 4|x^3y||\Delta x| + |x^4||\Delta y|$

R.E = $\frac{4|x^3y||\Delta x| + |x^4||\Delta y|}{x^4y}$

$= 4\left|\frac{\Delta x}{x}\right| + \left|\frac{\Delta y}{y}\right|$

Example 26

Show that the maximum possible relative error in ysin^2x is

$\left|\frac{\Delta y}{y}\right| + 2\cot x\,|\Delta x|$, where$\Delta x$ and Δy are errors in x and y respectively

Hence find the percentage error in calculating ysin^2x if y = 5.2 $\pm$ 0.05 and x = $\frac{\pi}{6} \pm \frac{\pi}{360}$ (07 marks)

z = y$\sin^2$x

$e_z = \Delta y sin^2 x$ + 2y$\Delta x cos\ x sinx$

$\frac{e_z}{z} = \frac{\Delta y sin^2 x}{\mathrm{ysin^2x}} + \frac{2\mathrm{y}\Delta x cos\ x sinx}{\mathrm{ysin^2x}}$

$\left|\frac{e_z}{z}\right| = \left|\frac{\Delta y}{y} + 2cotx.\Delta x\right|$

$\leq \left|\frac{\Delta y}{y}\right| + 2cotx.|\Delta x|$

$\therefore$ Maximum possible error is $\left|\frac{\Delta y}{y}\right| + 2cotx.|\Delta x|$

$percentage\ error = \left[\frac{0.05}{5.2} + 2\cot\frac{\pi}{6}.\ \left|\frac{\pi}{360}\right|\right] x\ 100\%$ = 3.9845%

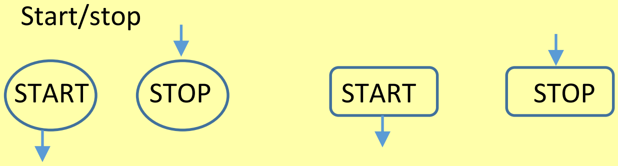
32. FLOW CHARTS

Flowcharts in mathematics

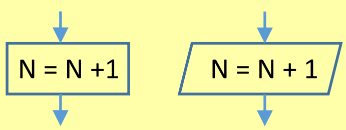
A flow chart is a diagram comprising of systematic steps followed in order to solve a problem.

Shapes used

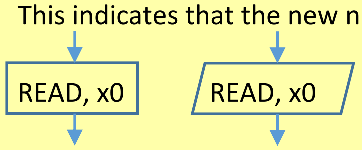
1. Start/stop



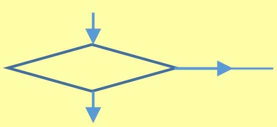
2. OPERATIONASSIGNMENT



This indicates that the new number is obtained by adding one to the previous N



3. Decision box



Note: all other shapes can be interchanged except for the decision box

Dry run or trace

This is the method of predicting the outcome of a given flow chart using a table

UACE MATHEMATICS PAPER 2 2016 guide

SECTION A (40 marks)

Answer all questions in this section

1. A ball s projected vertically upwards and returns to its point of projection 3seconds later. Fin the
   (a) speed with which the ball is projected
   (b) the greatest height reached.
2. The table shows the values of two variables P and Q

P	14	15	25	20	15	7
Q	30	25	20	18	15	22

   Calculate the rank correlation coefficient between the two variables.
3. Use the trapezium rule with 4 sub-divisions to estimate $\int_0^{\frac{\pi}{2}} \cos x \, dx$.
   Correct to **three** decimal places
4. A body of mass 4kg is moving with initial velocity $5ms^{-1}$ on a plane. The kinetic energy of the body is reduced by 16 joules in a distance of 40m. Find the deceleration of the body.
5. A continuous random variable X has a cumulative distribution function:

$$F(x) = \begin{cases} 0, x \leq 0 \\ \lambda x2, 0 \leq x \leq 4 \\ 1, x \geq 4 \end{cases}$$

   Find the
   (a) Value of λ
   (b) Probability density function (x)
6. The table below shows values of f(x) foe the given values of x.

X	0.4	0.6	0.8
f(x)	-0.9613	-0.5108	-0.2231

   Use linear interpolation to determine $f^{-1}(-0.4308)$ correct to 2 decimal places.
7. A particle of mass 2kg rests in limiting equilibrium on a rough plane inclined at 30^0 to the horizontal. Find the value of coefficient of friction.
8. A bag contains 5 Pepsi cola and 4 Mirinda bottle tops. Three bottle tops are picked at random from the bag one after the other without replacement. Find the probability that the bottle tops picked are of the same type.

SECTION B (60 MARKS)

Answer any five questions from this section. All questions carry equal marks

9. The data below shows the length in centimetres of different calendars produced by the printing press. Accumulative frequency distribution was formed.

Length (cm)	<10	<30	<35	<40	<50	<60
Cumulative frequency	4	20	32	42	48	50

(a) Construct frequency distribution table

(b) Draw a histogram and use it to estimate the modal length

(c) Find the mean length of the calendars

10. Five forces of magnitude 3N, 4N, 4N, 3N and 5N act along the lines AB, BC, CD, DA and AC respectively, of a square of side 1m. The direction of the forces is given by the order of the letters. Taking AB and AD as reference axes, find the

(a) magnitude and direction of resultant forces.

(b) point where the line of action of the resultant force cuts the side AB.

11. Given the equation $X^3 - 6x^2 + 9x + 2 = 0$

(a) Show that the equation has a root between -1 and 0.

(b) (i) Show that the Newton Raphson formula approximating the root of the equation is given by $X_{n+1} = \frac{2}{3}\left[\frac{x_n^3 - 3x_n^2 - 1}{x_n^2 - 4x_n + 3}\right]$

(ii) Use the formula in (b)(i) above, with initial approximation of $x_0 = -0.5$, to find the root of the given equation correct to two decimal places

12. A newspaper vender by 12 copies of a sport magazine every week. The probability distribution for the number of copies sold in a week is given in the table below

Number of copies sold	9	10	11	12
probability	0.2	0.35	0.3	0.15

(a) Estimate the

(i) Expected number of copies that she sells in a week

(ii) Variance of the number of copies sold in a week.

(b) The vendor buys the magazine at shs. 1,200 and sells it at shs. 1,500. Any copies not sold are destroyed. Construct a probability distribution table for the vendor's weekly profit from the sales. Hence calculate her mean profit.

13. A particle starts from rest at a point (2, 0, 0) and moves such that its acceleration at any time t>0 is given by a = [16cos4ti + 8sin2tj + (sint – 2sin2t)k]ms^{-2}. Fins the

(a) speed when $t = \frac{\pi}{4}$.

(b) distance from the origin when $t = \frac{\pi}{4}$

14. The numbers x and y are approximate by X and Y with errors Δx and Δy respectively

(a) Show that the maximum relative error in xy is given by $\left|\frac{\Delta x}{x}\right| + \left|\frac{\Delta y}{y}\right|$.

(b) If x = 4.95 and y = 2.013 are rounded off to the given number of decimal places, calculate

(i) percentage error in xy.

(ii) Limits within which xy is expected to lie, give your answer to three decimal places.

15. The drying time of any newly manufactured paint is normally distributed with mean 110.5 minutes and standard deviation 12 minutes.
    (a) Find the probability that paint dries between 104 and 109 minutes
    (b) If a random sample of 20 tins of paint was taken, find the probability that the mean drying time of the samples is between 108 and 112 minutes.
16. A particle of mass 2kg moving with Simple Harmonic Motion (SHM) along the x-axis, is attracted towards the origin 0 by a force of 32x Newton. Initially the particle is at rest x = 20. Find the
    (a) Amplitude and period of oscillation
    (b) Velocity of the particle at any time, t>0
    (c) Speed when t = $\frac{\pi}{4}$ seconds

Solutions

SECTION A (40 marks)

Answer all questions in this section

1. A ball s projected vertically upwards and returns to its point of projection 3seconds later. Fin the
   (a) speed with which the ball is projected
   time taken by the ball to reach maximum point = $\frac{3}{2} = 1.5s$
   v = u – gt at maximum height, v= 0
   0 = u – 9.8 x 1.5
   u = 9.8 x 1.5 = 14.7ms^{-1}
   (b) the greatest height reached.
   H = ut – ½t^2
   = 14.7 x 1.5 – ½ (1.5)^2 = 11.025m

2. The table shows the values of two variables P and Q

P	14	15	25	20	15	7
Q	30	25	20	18	15	22

Calculate the rank correlation coefficient between the two variables.
Using Spearman's approach

P	Q	R_P	R_Q	d	d^2
14	30	5	1	4	16
15	25	3.5	2	1.5	2.25
25	20	1	4	-3	9
20	18	2	5	-3	9
15	15	3.5	6	-2.5	6.25
7	22	6	3	3	9
					$\sum d^2 = 51.5$

p = 1 - $\frac{6\sum d^2}{n(n^2-1)}$ = $1 - \frac{6 \times 51.5}{6(35-1)}$ = $1 - \frac{309}{210}$ = -0.4714

Using Kendall's approach

Using agreements and disagreements

R_P	R_Q	A	D	A-D
1	5	1	4	-3
2	3.5	1	2	-1
3	6	0	3	-3
4	1	2	0	2
5	2	1	0	1
6	3.5	-	-	S = -4

$\tau = \frac{2S}{n(n-1)} = \frac{2\,x{-4}}{6(6-1)} = \frac{-8}{30} = 0.2667$

Using crossing

Let the variables be A, B, C, D, E, F

P	RP		Q	RQ	
25	1	C	30	1	A
20	2	D	25	2	B
15	3.5	B	22	3	F
15	3.5	E	20	4	C
14	5	A	18	5	D
7	6	F	15	6	E

C = 9

$\tau = 1 - \frac{4C}{n(n-1)} = 1 - \frac{4\,x9}{6(6-1)} = 1 - \frac{36}{30} = -0.2$

3. Use the trapezium rule with 4 sub-divisions to estimate $\int_0^{\frac{\pi}{2}} \cos x\, dx$.

Correct to **three** decimal places

h = $\frac{\frac{\pi}{2}-0}{4} = \frac{\pi}{8}$

let y = $\int_0^{\frac{\pi}{2}} \cos x\, dx$.

x	y	
0	1	
$\frac{\pi}{8}$		0.92388
$\frac{\pi}{4}$		0.70711
$\frac{3\pi}{8}$		0.38268
$\frac{\pi}{2}$	0	
Sum	2	2.01367

$\int_0^{\frac{\pi}{2}} \cos x\, dx. = \frac{h}{2}[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$

$= \frac{\pi}{16}[1 + 2(2.01367)]$

= 0.9871159 = 0.987 (3D)

4. A body of mass 4kg is moving with initial velocity 5ms^{-1} on a plane. The kinetic energy of the body is reduced by 16 joules in a distance of 40m. Find the deceleration of the body.

Work done = change in K.E

F x d =change in K.E

ma x d = -16

160a = -16

a = $\frac{-16}{160} = -0.1ms^{-2}$

∴ the deceleration is 0.1ms^{-2}

Alternatively

Change in K.E = $\frac{1}{2}mv^2 - \frac{1}{2}mu^2$

16 = $\frac{1}{2}(4)(v^2 - u^2)$

16 = 2

Change

5. A continuous random variable X has a cumulative distribution function:

$$F(x) = \begin{cases} 0, x \leq 0 \\ \lambda x^3, 0 \leq x \leq 4 \\ 1, x \geq 4 \end{cases}$$

Find the

(a) Value of λ

F(4) = 1

$\lambda(4^3) = 1$

64λ= 1

λ = $\frac{1}{64}$

(b) Probability density function (x)

For x ≤ 0, f(x) = 0

For x ≥ 4, f(x) = $\frac{d}{dx}(4) = 0$

For 0≤ x ≤ 4.

f(x) = $\frac{d}{dx}(\lambda x^3)$ = $3\lambda x^2 = \frac{3}{16}x^2$

$$\therefore f(x) = \begin{cases} \frac{3}{16}x^2 & 0 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

6. The table below shows values of f(x) foe the given values of x.

X	0.4	0.6	0.8
f(x)	-0.9613	-0.5108	-0.2231

Use linear interpolation to determine $f^{-1}(-0.4308)$ correct to 2 decimal places.

Extract

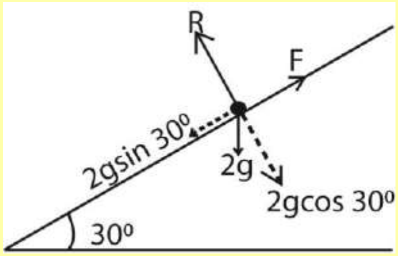
x	0.6	x_1	0.8
f(x)	-0.5108	-0.4308	-0.2231

$$\frac{-0.4308-(-0.5108)}{x_1-0.6} = \frac{-0.2231-(-0.5108)}{0.8-0.6}$$

$$\frac{0.08}{x_1-0.6} = \frac{0.2877}{0.2}$$

x_1 = 0.66

7. A particle of mass 2kg rests in limiting equilibrium on a rough plane inclined at 30^0 to the horizontal. Find the value of coefficient of friction.



R = 2gcos 30^0
F = 2gsin 3^0
μR = 2gsin30^0
$\mu[2gcos\ 30^0]$ = 2gsin30^0
$\mu = \frac{2gsin\ 30^0}{2gcos\ 30^0}$ = tan 30^0 = 0.57735

8. A bag contains 5 Pepsi cola and 4 Mirinda bottle tops. Three bottle tops are picked at random from the bag one after the other without replacement. Find the probability that the bottle tops picked are of the same type.

P(all the same type) = P($P_1 \cap P_2 \cap P_3$) + P($M_1 \cap M_2 \cap M_3$)
$= \frac{5}{9}\ x\ \frac{4}{8}\ x\ \frac{3}{7} + \frac{4}{9}\ x\ \frac{3}{8}\ x\ \frac{2}{7}$
$= \frac{60+24}{504}$
$= \frac{84}{504}$
= 0.1667

SECTION B (60 MARKS)

Answer any five questions from this section. All questions carry equal marks

9. The data below shows the length in centimetres of different calendars produced by the printing press. Accumulative frequency distribution was formed.

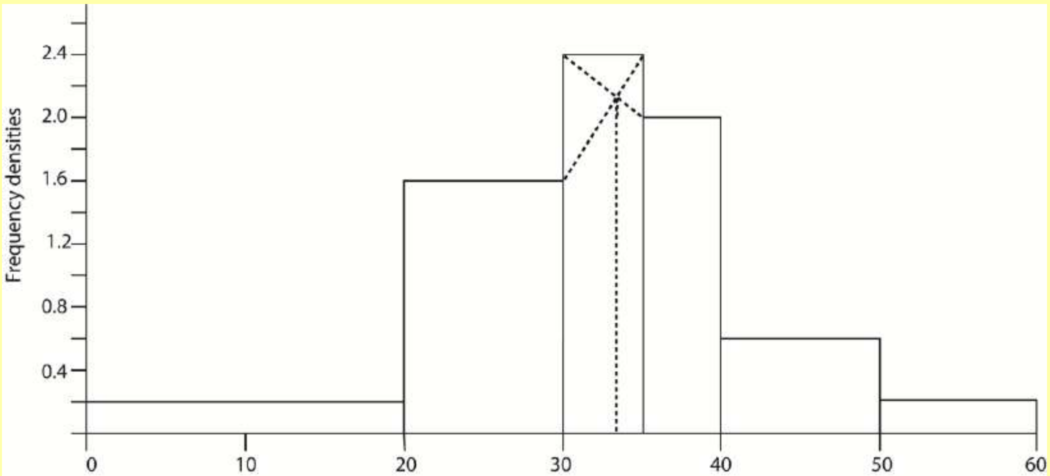
Length (cm)	<10	<30	<35	<40	<50	<60
Cumulative frequency	4	20	32	42	48	50

(a) Construct frequency distribution table

Length (cm)	F	cf
0 - < 20	4	4
20 - < 30	16	20
30 - < 35	12	32
35 - < 40	10	42
40 - < 50	6	40
50 - < 60	2	50
	$\sum f = 50$	

(b) Draw a histogram and use it to estimate the modal length

Length (cm)	x	c	f	cf	fx	fd
0 - < 20	10	20	4	4	40	0.2
20 - < 30	25	10	16	20	400	1.6
30 - < 35	32.5	5	12	32	390	2.4
35 - < 40	37.5	5	10	42	375	2.0
40 - < 50	45	10	6	40	270	0.6
50 - < 60	55	10	2	50	110	0.2
			$\sum f = 50$		$\sum fx = 1585$	



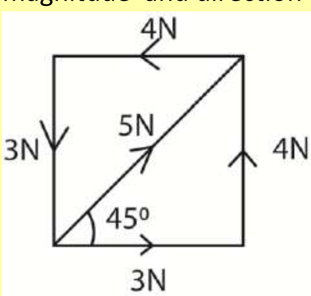
Mode = 33.5

(c) Find the mean length of the calendars

Mean = $\frac{\sum fx}{\sum f} = \frac{1585}{50} = 31.7$

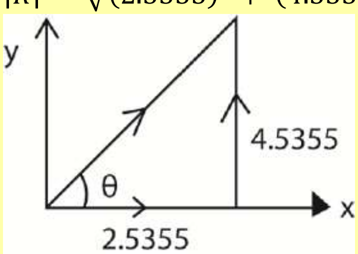
10. Five forces of magnitude 3N, 4N, 4N, 3N and 5N act along the lines AB, BC, CD, DA and AC respectively, of a square of side 1m. The direction of the forces is given by the order of the letters. Taking AB and AD as reference axes, find the

(a) magnitude and direction of resultant forces.



$\binom{x}{y} = \binom{3}{0} + \binom{0}{4} + \binom{-4}{0} + \binom{0}{-3} + \binom{5\cos 45^0}{5\sin 45^0} = \binom{2.5355}{4.5355}$

$|R| = \sqrt{(2.5355)^2 + (4.5355)^2}$ =5.1961N



$\theta = tan^{-1}\left(\frac{4.5355}{2.5355}\right) = 60.79^0$

(b) point where the line of action of the resultant force cuts the side AB.

Equation for line of action is given by

G – Xy + yx = 0

A): G = 4 x 1+ 4 x 1 = 0

G = 4 + 4 = 8Nm

By substitution

8- x(4.5355) + y(2.5355) = 0

8 = 4.5355x + 2.5355y = 0

The line cuts AB when y = 0

x = $\frac{8}{4.5355} = 1.764m$

∴ the line of action of the resultant cuts AB 1.764m from A

11. Given the equation $X^3 - 6x^2 + 9x + 2 = 0$
    (a) Show that the equation has a root between -1 and 0.
    Let $f(x) = X^3 - 6x^2 + 9x + 2$
    $f(-1) = (-1)^3 - 6(-1)^2 + 9(-1) + 2$
    $= -1 - 6 - 9 + 2 = -14$
    $f(0) = 0 + 0 + 0 + 2$
    $= 2$
    $f(-1).f(0) = -14 \times 2 = -28$
    since $f(-1).f(0) < 0$; the root exist between -1 and 0.
    (b) (i) Show that the Newton Raphson formula approximating the root of the equation is given by $X_{n+1} = \frac{2}{3}\left[\frac{x_n^3 - 3x_n^2 - 1}{x_n^2 - 4x_n + 3}\right]$
    $f(x) = X^3 - 6x^2 + 9x + 2$
    $f(x_n) = x_n^3 - 6x_n^2 + 9x_n + 2$
    $f'(x_n) = 3x_n^2 - 12x_n + 9$
    $x_{n+1} = x_n - \left(\frac{x_n^3 - 6x_n^2 + 9x_n + 2}{3x_n^2 - 12x_n + 9}\right)$
    $= \frac{x_n(3x_n^2 - 12x_n + 9) - (x_n^3 - 6x_n^2 + 9x_n + 2)}{3x_n^2 - 12x_n + 9}$
    $= \frac{(3x_n^3 - 12x_n^2 + 9x_n) - (x_n^3 - 6x_n^2 + 9x_n + 2)}{3x_n^2 - 12x_n + 9}$
    $= \frac{2x_n^3 - 6x_n^2 - 2}{3x_n^2 - 12x_n + 9}$
    $= \frac{2}{3}\left[\frac{x_n^3 - 3x_n^2 - 1}{x_n^2 - 4x_n + 3}\right]$

    (ii) Use the formula in (b)(i) above, with initial approximation of $x_0 = -0.5$, to find the root of the given equation correct to two decimal places
    Taking x = -0.5
    $x_1 = \frac{2}{3}\left[\frac{(-0.5)^3 - 3(-0.5)^2 - 1}{(-0.5)^2 - 4(-0.5) + 3}\right]$ = -0.2381
    $|e| = |0.2381 - (-0.5)|$ = 0.2619
    $x_2 = \frac{2}{3}\left[\frac{(-0.2381)^3 - 3(-0.2381)^2 - 1}{(-0.2381)^2 - 4(-0.2381) + 3}\right]$ = -0.1968
    $|e| = |0.1968 - (-0.2381)|$ = 0.0.0413
    $x_3 = \frac{2}{3}\left[\frac{-0.1968^3 - 3(-0.1968)^2 - 1}{(-0.1968)^2 - 4(-0.1968) + 3}\right]$ = -0.1958
    $|e| = |-0.1958 - (-0.1968)|$ = 0.001< 0.005

    Hence the root = -0.20 (2D)

12. A newspaper vender by 12 copies of a sport magazine every week. The probability distribution for the number of copies sold in a week is given in the table below

Number of copies sold	9	10	11	12
probability	0.2	0.35	0.3	0.15

(a) Estimate the

x	9	10	11	12
P(X = x)	0.2	0.35	0.30	0.15
xP(X = x)	1.8	3.5	3.3	1.8
x^2P(X = x)	16.2	35	36.3	21.6

(i) Expected number of copies that she sells in a week
Let x be the number of copies
E(X) = $\sum xP(X = x)$= 1.8 + 3.5 + 3.3 + 1.8 = 10.4

(ii) Variance of the number of copies sold in a week.
Var (X) = $\sum x^2P(X = x)$ - $(E(X))^2$
E(X^2) = $\sum x^2P(X = x)$ = 16.2 + 35 + 36.3 + 21.6 = 109.1
Var (X) = 109.1 – $(10.4)^2$
= 109.1 – 108.16
= 0.94

(b) The vendor buys the magazine at shs. 1,200 and sells it at shs. 1,500. Any copies not sold are destroyed. Construct a probability distribution table for the vendor's weekly profit from the sales. Hence calculate her mean profit.

Profit for 9 copies = 9 x 1500 – 12 x 1200
= 13500 – 14400
= -900/=
Profit for 10 copies = 10 x 1500 – 12 x 1200
= 15000 – 14400
= 600/=
Profit for 11 copies = 11 x 1500 – 12 x 1200
= 16500 – 14400
= 2100/=
Profit for 12 copies = 12 x 1500 – 12 x 1200
= 18000 – 14400
= 3600/=

Note; the vendor buys 12 copies every week
Let y = weekly profit
Probability distribution is given by

y	-900	600	2100	3600
P(Y = y)	0.2	0.35	0.30	0.15
yP(Y = y)	-180	210	630	540

E(y) = $\sum yP(Y = y)$ = -180 + 210 + 630 + 540 = 1200/=
Hence the expected weekly profit is shs. 1200/=

13. A particle starts from rest at a point (2, 0, 0) and moves such that its acceleration at any time t>0 is given by a = [16cos4ti + 8sin2tj + (sint – 2sin2t)k]ms^{-2}. Fins the

(a) speed when t = $\frac{\pi}{4}$.
a = [16cos4ti + 8sin2tj + (sint – 2sin2t)k]ms^{-2}
v = $\int adt$
= $\int[16\cos 4ti + 8\sin 2tj + (\sin t - 2\sin 2t)k]dt$
= [4sin4ti – 4cos2tj + (-cos t + cos2t)k]+ c
At t = 0
0 = [4sin0i – 4cos0j + (-cos 0 + cos0)k]+ c
0 = -4j + c

c = 4j

⇨ v = [4sin4ti + (-4cos2t + 4)j + (-cos t + cos2t)k

At t = $\frac{\pi}{4}$

⇨ v = [4sinπi + (-4cos$\frac{\pi}{2}$ + 4)j + (-cos $\frac{\pi}{4}$ + cos$\frac{\pi}{2}$)k

= 4j – cos $\frac{\pi}{4}$ k

$|v| = \sqrt{4^2 + \left(-\cos\frac{\pi}{4}\right)^2} = \sqrt{16 + \frac{2}{4}} = \sqrt{16.5}$ = 4.062ms^{-1}

(b) distance from the origin when t = $\frac{\pi}{4}$

s = $\int vdt$

=$\int[4\sin4t\mathrm{i} + (-4\cos2t + 4)\mathrm{j} + (-\cos t + \cos2t)\mathrm{k}$

=-cos4ti +(-2sin2t + 4t)j + (-sint + $\frac{1}{2}sin2t)k$ + c

At t = 0, s = 2i

By substitution

2i = -cos0 i +(-2sin0 + 4(0))j + (-sin0 + $\frac{1}{2}sin2\ (0))k$ + c

2i = -I + c

c = 3i

⇨ s = (-cos4t + 3)i +(-2sin2t + 4t)j + (-sint + $\frac{1}{2}sin2t)k$

At t = $\frac{\pi}{4}$

⇨ s = (-cosπ + 3)i +(-2sin$\frac{\pi}{2}$t + π)j + (-sin$\frac{\pi}{4}$ + $\frac{1}{2}sin\frac{\pi}{2})k$

= 4i + (π -2)j + $\left(-\frac{\sqrt{2}}{2} + \frac{1}{2}\right)k$

= 4i + 1.416j – 0.207k

$|s|=\sqrt{4^2 + (1.416)^2 + (-0.207)^2}$

= 4.24828

= 4.248 (3D)

14. The numbers x and y are approximate by X and Y with errors Δx and Δy respectively

(a) Show that the maximum relative error in xy is given by $\left|\frac{\Delta x}{x}\right| + \left|\frac{\Delta y}{y}\right|$.

Let z = xy and Z = XY

x =X + Δx and y = Y + Δy

z + Δz = (X + Δx)(Y + Δy)

= XY + XΔy + YΔx + ΔxΔY

Δz = XY + XΔy + YΔx + ΔxΔY – XY

= XΔy + YΔx + ΔxΔY

But Δx<< X, Δy << Y, => ΔxΔY = 0

⇨ Δz = XΔy + YΔx

R.E = $\left|\frac{\Delta z}{Z}\right|$

$=\left|\frac{X\Delta y}{XY} + \frac{Y\Delta x}{XY}\right|$

$=\left|\frac{\Delta y}{Y}+\frac{\Delta x}{X}\right|$

R.E$\leq \left|\frac{\Delta x}{X}\right|+\left|\frac{\Delta y}{Y}\right|$

R.Emax = $\left|\frac{\Delta x}{X}\right|+\left|\frac{\Delta y}{Y}\right|$

(b) If x = 4.95 and y = 2.013 are rounded off to the given number of decimal places, calculate

(i) percentage error in xy.

Percentage error = $\left[\left|\frac{\Delta x}{X}\right|+\left|\frac{\Delta y}{Y}\right|\right]x100\%$

X = 4.95, Δx= 0.005

Y = 2.013, Δy = 0.0005

Percentage error = $\left[\left|\frac{0.005}{4.95}\right|+\left|\frac{0.0005}{2.013}\right|\right]x100\%$= 0.126%

(ii) Limits within which xy is expected to lie, give your answer to three decimal places.

Minimum value = 4.945 x 2.0125 = 9.9518125 = 9.952 (3D)

Maximum value = 4.955 x 2.013 = 9.9768925 = 9.977 (3D)

Limits ar [9.952, 9.977]

15. The drying time of any newly manufactured paint is normally distributed with mean 110.5 minutes and standard deviation 12 minutes.

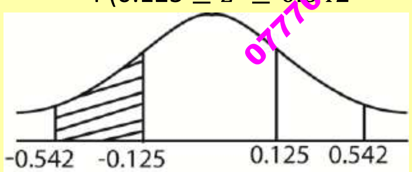
(a) Find the probability that paint dries between 104 and 109 minutes

Let x = drying time of the paint

P(104 $\leq x \leq 109$ = $P\left(\frac{104-110.5}{12}\leq z \leq \frac{109-110.5}{12}\right)$

= P(-0.542 $\leq z \leq -0.125$

= P(0.125 $\leq z \leq 0.542$



= P(0$\leq z \leq 0.542) - P(0 \leq z \leq 0.125$

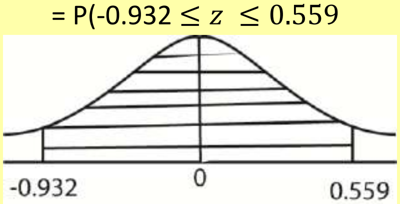
= 0.2061 – 0.0498

= 0.1563

(b) If a random sample of 20 tins of paint was taken, find the probability that the mean drying time of the samples is between 108 and 112 minutes.

P(108 $\leq \bar{X} \leq 112$) = $P\left(\frac{108-110.5}{\frac{12}{\sqrt{20}}}\leq z \leq \frac{112-110.5}{\frac{12}{\sqrt{20}}}\right)$

= P(-0.932 $\leq z \leq 0.559$



= P(-0.932 $\leq z \leq 0) + P(0 \leq z \leq 0.559)$

= 0.3243 + 0.2119

= 0.5362

16. A particle of mass 2kg moving with Simple Harmonic Motion (SHM) along the x-axis, is attracted towards the origin 0 by a force of 32x Newton. Initially the particle is at rest x = 20. Find the
    (a) Amplitude and period of oscillation
    ma = F
    ⇨ 2a = -32x
    a = -16x
    where $\omega^2 = 16$
    $\omega = 4$
    since the particle s momentarily at rest, when x = 20, the amplitude = A = 20m
    Period $=\frac{2\pi}{\omega} = \frac{2\pi}{4} = \frac{\pi}{2} seconds$
    (b) Velocity of the particle at any time, t>0
    x = rcosωt
    v $= \frac{dx}{dt} = \frac{d\,(20\cos 4t)}{dt}$ = -80sin4t
    Hence velocity of the particle at any time, t is -80sin4t
    (c) Speed when t = $\frac{\pi}{4}$ seconds
    Speed = $|v| = \left|-80\sin\left(4\ x\ \frac{\pi}{4}\right)\right|$ = -80sinπ =0ms^{-1}

UACE MATHEMATICS PAPER 2 2017 guide

SECTION A (40 marks)

Answer all questions in this section

1. A particle is projected from a point O with speed20m/s at an angle 60^0 to the horizontal. Express in vector form its velocity v and its displacement, r, from O at any time t seconds. (05marks)
2. The probability that a patient suffering from a certain disease recovers is 0.4. If 15 people contract the disease, find the probability that:
   (a) More than 9 will recover. (02marks)
   (b) Between five and eight will recover. (03marks)
3. The table below gives values of x and corresponding values of f(x).

X	0.1	0.2	0.3	0.4	0.5	0.7
f(x)	4.21	3.83	3.25	2.85	2.25	1.43

   Use linear interpolation/extrapolation to find
   (a) f(x) when x = 0.6 (03marks)
   (b) the value of x when f(x) = 0.75 (02mars)
4. In a square ABCD, three forces of magnitudes 4N, 10N and 7N and CA respectively. Their directions are in the order of the letters. Find magnitude of the resultant force. (05marks)
5. A box A contains 1 white ball and I blue ball. Box B contains only 2 white balls. If a ball is pice at random, find the probability that it is
   (a) White (02mark)
   (b) From box A given that it is white(03marks)
6. Given that y = $\frac{1}{x} + x$ and x = 2.4 correct to one decimal place, find the limits within which y lies. (05marks)
7. The table below shows the retail prices (shs) and amount of each item bought weekly by a restaurant in 2002 and 2003.

Item	Prices(shs)		Amount bought
	2002	2003	
Milk (per litre)	400	500	200
Eggs (per tray)	2,500	3,000	18
Cooking oil (per litre)	2,400	2,100	2
Baking flour (per packet)	2,000	2,200	15

   (a) Taking 2002 as the base year, calculate the weighed aggregate price index. (03 marks)

(b) In 2003, the restaurant spent shs. 450,000 on buying these items. Using the weighed aggregate price index obtained in (a), calculate what the restaurant could have spent in 2002. (02marks)

8. The engine of a lorry of mass 5,000kg is working at a steady rate of 350 kW against a constant resistance force of 1,000N. The lorry ascends a slope of inclination θ to the horizontal. If the maximum speed of the lorry is $20ms^{-1}$, find the value of θ (05marks)

SECTON B (60 MARKS)

Answer any five questions from this section. All questions carry equal marks

9. A discrete random variable X has a probability distribution given by

$$P(X = x) = \begin{cases} kx\ , x = 1, 2, 3, 4. 5 \\ 0. \quad \text{Otherwise,} \end{cases}$$

Where k is a constant

Determine;

(a) the value of k (03 marks)
(b) $P(2 < X < 5)$ (02marks)
(c) Expectation, E (X) (03marks)
(d) Variance, Var(X) (04mars)

10. A particle of mass 3g is acted on by a force $F = 6i - 36t^2j + 54k$ Newton at time t. At time t = 0 the particle is at the point with position vector $i - 5j - k$ and its velocity is $3i + 3j\ ms^{-1}$.
Determine the
(a) position vector of the particle at time t= 1second (09marks)
(b) distance of the particle from the origin at time t = 1 second

11. A student used the trapezium rule with five sub-intervals to estimate $\int_2^3 \frac{x}{(x^2-3)}dx$ correct to **three** decimal places

Determine;

(a) The value the student obtained (06marks)
(b) The actual value of the integral (03marks)
(c) (i) the error the student made in the estimate
(ii) how the student can reduce the error(03marks)

12. The times taken for 55 students to have their lunch to the nearest minute are given in the table below

Time (minutes)	3 -4	5-9	10-19	20 – 29	30 – 44
Number of students	2	7	16	21	9

(a) Calculate the mean time for the student to have lunch (04marks)
(b) (i) Draw a histogram for the given data
(ii) Use your histogram to estimate the modal time for the students to have lunch. (08marks)

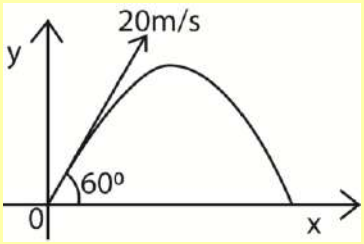
13. A non-uniform rod AB of mass 10k has its centre of gravity a distance ¼ AB. The rod is smoothly hinged at A. it is maintained in equilibrium at 60^0 above the horizontal by a light inextensible string tied at B and at a right angle to AB. Calculate the magnitude and direction of the reaction at A. (12marks)
14. By plotting graphs of y = x and y = 4siin x on the same axes. Show that the root of the equation x – 4sinx = 0 lies between 2 and 3.
    Hence use Newton Raphson's method to find the root of the equation correct to 3 decimal places. (12marks)
15. The number of cows owned by residents in a village is assumed to be normally distributed. 15% of the residents have less than 60 cows, 5% of the residents have over90 cows.
    (a) Determine the values of the mean and standard deviation of the cows. (08marks)
    (b) If there are 200 residents, find how many have more than 80 cows. (04marks)
16. At 12 noon a ship A is moving with constant velocity of $20.4kmh^{-1}$ in the direction $N\theta^0E$ where tan θ = $\frac{1}{5}$. A second ship B is 15km due to north of A. Ship B is moving with constant velocity of 5kmh-1 in the direction Sα0W, where tan α= ¼. If the shortest distance between the ships is 4.2km, find the time to the nearest minute when the distance between the ships is shortest. (12mars)

Solutions

SECTION A (40 marks)

Answer all questions in this section

1. A particle is projected from a point O with speed20m/s at an angle 60^0 to the horizontal. Express in vector form its velocity v and its displacement, r, from O at any time t seconds . (05marks)



At time t = 0

V = 20cos60^0i + 20sin60^0j = 10i + 10$\sqrt{3}$j

But A = -gj

At any time t,

V = $\int adt$ = -gj$\int dt$

= -gtj + c

At t = 0

10i + 10$\sqrt{3}$j = 0 + c

⇨ At time t

V= 10i + $(10\sqrt{3} - gt)$j = 10i + $(10\sqrt{3} - 9.8t)$j

Or

V = $\begin{pmatrix} 10 \\ 10\sqrt{3} - 9.8t \end{pmatrix}$

r = $\int vdt = \int \begin{pmatrix} 10 \\ 10\sqrt{3} - 9.8t \end{pmatrix} dt$ = $\begin{pmatrix} 10t \\ 10\sqrt{3}t - 4.9t^2 \end{pmatrix}$+ c

at t = 0, r = $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

⇨ $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + c$, c = $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Hence at time t, r = $\begin{pmatrix} 10t \\ 10\sqrt{3}t - 4.9t^2 \end{pmatrix}$

2. The probability that a patient suffering from a certain disease recovers is 0.4. If 15 people contract the disease, find the probability that:
   (a) More than 9 will recover. (02marks)
   Given n = 15, p = 0.4
   Let x = number of people who will recover
   P$(X \geq 10)$= B$(X \geq 10, 15, 0.4)$
   (cumulative probabilities)
   P$(X \geq 10) = 0.0338$
   (b) Between five and eight will recover. (03marks)
   P(5 < X < 8) = P(X = 6,7)
   = P(X = 6) + P(X =7)

= B(6, 15, 0.4) + B(7, 15, 0.4)
= 0.2066 + 0.1771
= 0.3837
Or
P(5 < X < 8) = P(X = 6,7)
= P(X $\geq$ 6) – P(X $\geq$ 8)
= 0.5968 – 0.2131
= 0.3837

3. The table below gives values of x and corresponding values of f(x).

X	0.1	0.2	0.3	0.4	0.5	0.7
f(x)	4.21	3.83	3.25	2.85	2.25	1.43

Use linear interpolation/extrapolation to find
(a) f(x) when x = 0.6 (03marks)
Using linear interpolation
Extract

0.5	0.6	0.7
2.25	y	1.43

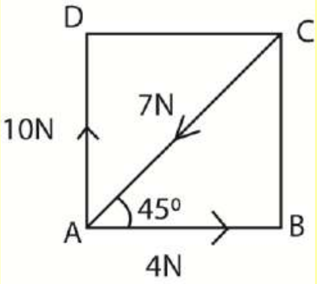
$\frac{1.43-2.25}{0.7-0.5} = \frac{y-2.25}{0.6-0.5}$
y = 1.84

(b) the value of x when f(x) = 0.75 (02mars)
Using linear extrapolation
Extract

0.5	0.7	X
2.25	1.43	0.75

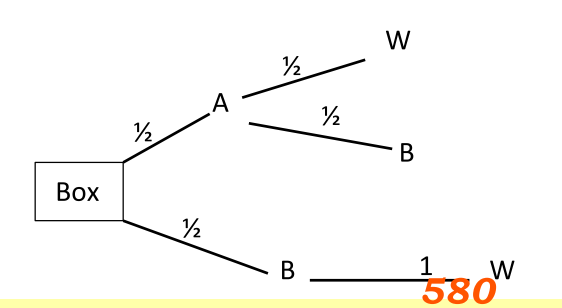
$\frac{x-0.5}{0.75-2.25} = \frac{0.7-0.5}{1.43-2.25}$; x = 0.9

4. In a square ABCD, three forces of magnitudes 4N, 10N and 7N and CA respectively. Their directions are in the order of the letters. Find magnitude of the resultant force. (05marks)



$\binom{X}{Y} = \binom{4}{0} + \binom{-7\cos 45^0}{-7\sin 45^0} + \binom{0}{10} = \binom{-0.9497}{5.0503}$
$|R| = \sqrt{(-0.9497)^2 + (5.0503)^2}$ = 5.1388N

5. A box A contains 1 white ball and I blue ball. Box B contains only 2 white balls. If a ball is pice at random, find the probability that it is
(a) White (02mark)



$P(W) = \frac{1}{2}\ x\ \frac{1}{2} + \frac{1}{2}\ x\ 1 = \frac{3}{4}$

(b) From box A given that it is white(03marks)

$P(A/W) = \frac{P(A \cap W)}{P(W)} = \frac{P(W/_A).P(A)}{P(W)} = \frac{\frac{1}{2}x\frac{1}{2}}{\frac{3}{4}} = \frac{1}{3}$

6. Given that $y = \frac{1}{x} + x$ and x = 2.4 correct to one decimal place, find the limits within which y lies. (05marks)

Error in 2.4 = $\frac{1}{2}\ x\ \frac{1}{10} = 0.05$

$y_{max} = 2.45 + \frac{1}{2.35} = 2.8755$

$y_{min} = 2.35 + \frac{1}{2.45} = 2.7582$

∴ the limits are [2.7582, 2.8755]

7. The table below shows the retail prices (shs) and amount of each item bought weekly by a restaurant in 2002 and 2003.

Item	Prices(shs)		Amount bought
	2002	2003	
Milk (per litre)	400	500	200
Eggs (per tray)	2,500	3,000	18
Cooking oil (per litre)	2,400	2,100	2
Baking flour (per packet)	2,000	2,200	15

(a) Taking 2002 as the base year, calculate the weighed aggregate price index. (03 marks)

P_0	W	P_0W	P_1	P_1W
400	200	80,000	500	100,000
2500	18	45,000	3,000	54,000
2400	2	4800	2100	4200
200	15	30,000	2200	33,000
		$\sum P_0W = 159800$		$\sum P_1W = 191200$

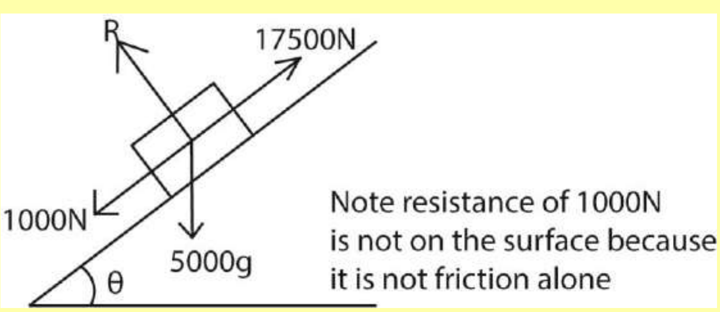
W.A.P.I = $\frac{191200}{159800}\ x\ 100$ = 119.65

(b) In 2003, the restaurant spent shs. 450,000 on buying these items. Using the weighed aggregate price index obtained in (a), calculate what the restaurant could have spent in 2002. (02marks)

$\frac{450,000}{P_0}$ x 100 = 119.65

$P_0 = 376,097$

8. The engine of a lorry of mass 5,000kg is working at a steady rate of 350 kW against a constant resistance force of 1,000N. The lorry ascends a slope of inclination θ to the horizontal. If the maximum speed of the lorry is 20ms^{-1}, find the value of θ (05marks)



Resultant force = 17500 – (1000 + 5000sinθ)

5000a = 17500 – (1000 + 5000sinθ)

At maximum speed a = 0

⇨ 0 = 17500 – (1000 + 5000sinθ)
16500 = 5000sinθ
$\theta = 19.7^0$

SECTON B (60 MARKS)

Answer any five questions from this section. All questions carry equal marks

9. A discrete random variable X has a probability distribution given by

$$P(X = x) = \begin{cases} kx\text{ , } x = 1, 2, 3, 4. 5 \\ 1.\quad \text{Otherwise,} \end{cases}$$

Where k is a constant

Determine;

(a) the value of k (03 marks)

x	1	2	3	4	5	sum
P(X =x)	k	2k	3k	4k	5k	15k

15k = 1

$k = \frac{1}{15}$

x	1	2	3	4	5	sum
P(X =x)	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{1}{15}$	1
xP(X= x)	$\frac{1}{15}$	$\frac{4}{15}$	$\frac{9}{15}$	$\frac{16}{15}$	$\frac{25}{15}$	$\frac{55}{15}$
X^2P(X =x)	$\frac{1}{15}$	$\frac{8}{15}$	$\frac{27}{15}$	$\frac{64}{15}$	$\frac{125}{15}$	$\frac{225}{15}$

(b) P(2 < X < 5) (02marks)
P(2<X<5) = P(X = 3, 4)

$= \frac{3}{15} + \frac{4}{15}$

$= \frac{7}{15}$

(c) Expectation, E (X) (03marks)

E(X) = $\sum xP(X = x) = \frac{55}{15} = \frac{11}{3}$

(d) Variance, Var(X) (04mars)

Var(X) = $E(X^2) – E(X)^2$

$= \frac{225}{15} - \left(\frac{55}{15}\right)^2$

$= \frac{14}{9}$

10. A particle of mass 3kg is acted on by a force F = 6i – $36t^2$j + 54k Newton at time t. At time t = 0 the particle is at the point with position vector i – 5j –k and its velocity is 3i + 3j ms^{-1}.

Determine the

(a) position vector of the particle at time t= 1second (09marks)

F = ma

$6i - 36t^2j + 54k = 3a$

a = 2i + 12t2j + 18k

V = $\int adt$ = $\int 2i - 12t^2j + 18k$ dt

= $2ti - 4t^3j + 9t^2k + c$

At t = 0, V = 3i + 3j

⇨ 3i + 3j = 0 + c

c = 3i + 3j

By substitution

V = $(2ti - 4t^3j + 9t^2k) + (3i + 3j) = (2t + 3)I + (4t^3 + 3)j + 9t^2k$

r = $\int vdt$ = $\int (2t + 3)i + (-4t^3 + 3)j + 9t^2k$ dt

= $(t^2 + 3t)i + (-t^4 + 3t)j + 3t^3k + c$

At t = 0, r = i -5j – k

⇨ i -5j – k = 0 +c

c = i – 5j –k

substituting for c

r = $[(t^2 + 3t)i + (-t^4 + 3t)j + 3t^3k] + [i – 5j –k]$

= $[(t^2 + 3t + 1)i + (-t^4 + 3t - 5)j + (3t^3 -1) k] + [i – 5j –k]$

At t = 1

r = (1 + 3+ 1)i + (-1 + 3 -5)j + (3 – 1)k

= 5i -3j + 2k

(b) distance of the particle from the origin at time t = 1 second

$|r| = \sqrt{5^2 + (-3)^2 + 2^2}$ = $\sqrt{38}$ = 6.164

11. A student used the trapezium rule with five sub-intervals to estimate $\int_2^3 \frac{x}{(x^2-3)}dx$ correct to **three** decimal places

Determine;

(a) The value the student obtained (06marks)

h = $\frac{3-2}{5}$ =0.2

X	y_1, y_6	y_2 ……y_5
2.0	2.0	
2.2		1.1956
2.4		0.8696
2.6		0.6915
2.8		0.5785
3	0.5	
Sum	2.5	3.3352

$\int_2^3 \frac{x}{(x^2-3)}dx = \frac{1}{2}\ x\ 0.2[2.5 + 2(3.3352)]$

= 0.91704 = 0.917 (3D)

(b) The actual value of the integral (03marks)

$\int_2^3 \frac{x}{(x^2-3)}dx$ = $\left[\frac{1}{2}Inx^2 - 3\right]_2^3$

$=\frac{1}{2}(In\ 6 - In\ 1)$

= 0.896

(c) (i) the error the student made in the estimate

Error = $|0.896 - 0.917|$ = 0.021

(ii) how the student can reduce the error(03marks)

Increasing on the number of sub-intervals or ordinates or reducing the width of h

12. The times taken for 55 students to have their lunch to the nearest minute are given in the table below

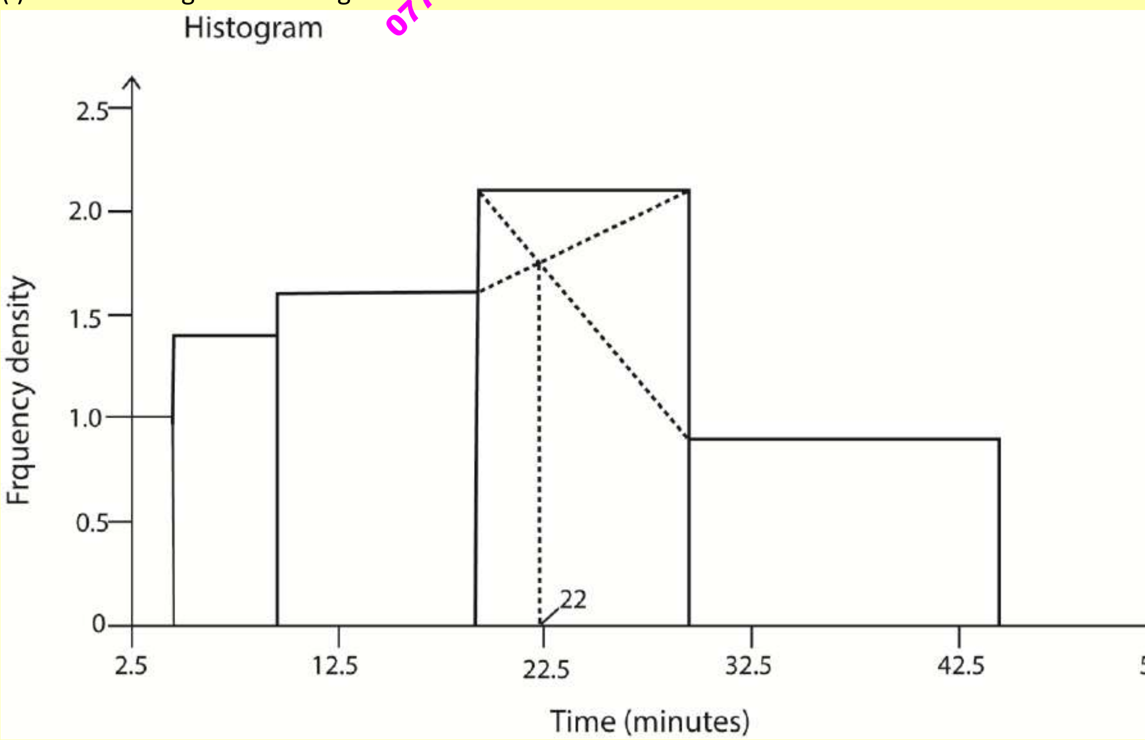
Time (minutes)	3 -4	5-9	10-19	20 – 29	30 – 44
Number of students	2	7	16	21	9

Class boundaries	f	x	fx	c	fd
2.5 – 4.5	2	3.5	7	2	1.0
4.5 – 9.4	7	7	49	5	1.4
9.5 – 19.5	16	14.5	232	10	1.6
19.5 – 29.5	21	24.5	514.5	10	2.1
29.5 – 44.5	9	37	333	15	0.6
	$\sum f = 55$		$\sum fx = 1135.5$		

(a) Calculate the mean time for the student to have lunch (04marks)

Mean = $\frac{\sum fx}{\sum f} = \frac{1135.5}{55} = 20.65$

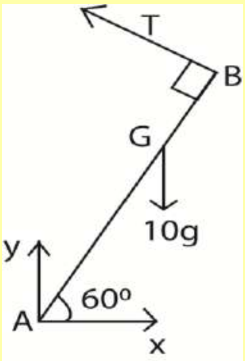
(b) (i) Draw a histogram for the given data



(ii) Use your histogram to estimate the modal time for the students to have lunch. (08marks)

Mode = 22 minutes

13. A non-uniform rod AB of mass 10k has its centre of gravity a distance ¼ AB. The rod is smoothly hinged at A. it is maintained in equilibrium at 60^0 above the horizontal by a light inextensible string tied at B and at a right angle to AB. Calculate the magnitude and direction of the reaction at A. (12marks)



Taking moments about point A

T x (AB) = $10g\left(\frac{3}{4}AB\cos 60^0\right)$

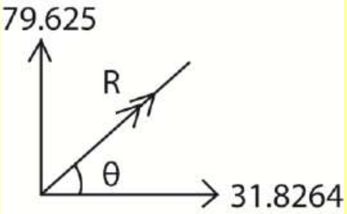
T = $\frac{15g}{4} = \frac{15 \, x \, 9.8}{4}$ = 36.75N

Resolving forces horizontally

X = $T\cos30^0 = 36.75\cos30^0$ = 31.8264N

Y = $10g - 36.75\sin 30^0$ = 79.625N

$|R| = \sqrt{(31.8264)^2 + (79.625)^2}$ = 85.75N



θ = $\tan^{-1}\left(\frac{79.625}{31.8264}\right) = 68.2^0$

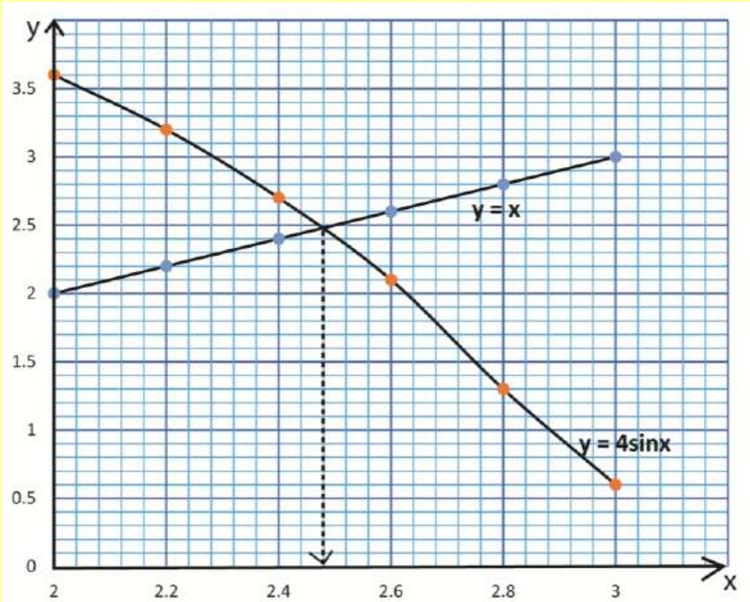
The direction of resultant force is 68.2^0 or E68.2^0N or N21.8^0E

14. By plotting graphs of y = x and y = 4siin x on the same axes. Show that the root of the equation x – 4sinx = 0 lies between 2 and 3.
Hence use Newton Raphson's method to find the root of the equation correct to 3 decimal places. (12marks)

Table of results

x	2.0	2.2	2.4	2.6	2.8	3.0
y = x	2.0	2.2	2.4	2.6	2.8	3.0
Y = 4sinx	3.6	3.2	2.7	2.1	1.3	0.6

Graph



From the graph the root lies between 2.4 and 2.6

Using N.R.M

f(x) = x – 4sinx

f'(x) = 1 – 4cos x

$x_{n+1} = x_n - \frac{x_n - 4\sin x_n}{1 - 4\cos x_n}$

$Taking\ x_0 = 2.47$

$x_1 = 2.47 - \frac{2.47 - 4\sin 2.45}{1 - 4\cos 2.47}$ = 2.4746

$Error = |2.4746 - 2.47| = 0.0046 > 0.0005$

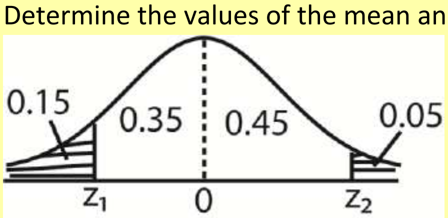
$x_2 = 2.4746 - \frac{2.4746 - 4\sin 2.4546}{1 - 4\cos 2.4746}$ = 2.4746

$Error = |2.4746 - 2.4746| = 0.000 < 0.0005$

∴ 2.475 (3D)

15. The number of cows owned by residents in a village is assumed to be normally distributed. 15% of the residents have less than 60 cows, 5% of the residents have over90 cows.

(a) Determine the values of the mean and standard deviation of the cows. (08marks)



P(0 < Z < Z_1) = 0.35; Z_1 = -1.036

$\frac{60 - \mu}{\sigma}$ = -1.036

$60 - \mu$ = -1.036σ …………… (i)

P(0 < Z < Z_2) = 0.35; Z_2 = 1.645

$\frac{90 - \mu}{\sigma}$ = 1.645

$90 - \mu$ = 1.645σ ………………(ii)

Eqn. (ii) – Eqn. (i)

30 = 2.681σ; σ= 11.1899

From eqn. (i)

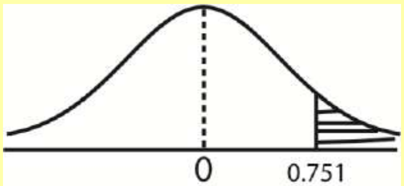
$60 - \mu$ = -1.036 x 11.1899

$\mu = 71.5926$

If there are 200 residents, find how many have more than 80 cows. (04marks)

P(X > 80) = P$\left(Z > \frac{80-71.5926}{11.1899}\right)$

= P(Z > 0.751)



P(Z > 0.751) = 0.5 – (0 < Z < 0.751)

= 0.5 – 0.2737

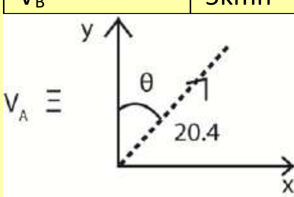
=0.2263

Number of residents =200 x 0.2263 = 45

16. At 12 noon a ship A is moving with constant velocity of 20.4kmh^{-1} in the direction Nθ^0E where tan θ = $\frac{1}{5}$. A second ship B is 15km due to north of A. Ship B is moving with constant velocity of 5kmh-1 in the direction Sα0W, where tan α= ¼. If the shortest distance between the ships is 4.2km, find the time to the nearest minute when the distance between the ships is shortest. (12mars)

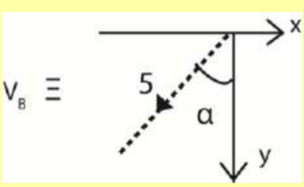
Table of results

Vector	Magnitude	Direction
V_A	20.4kmh^{-1}	NθE
V_B	5kmh^{-1}	SαW



tan θ = $\frac{1}{5}$; θ = 11.3^0

$$V_A = \begin{pmatrix} 20.4 \sin 11.3^0 \\ 20.4 \cos\ 11.3^0 \end{pmatrix} = \begin{pmatrix} 4 \\ 20 \end{pmatrix}$$



tan α = $\frac{1}{5}$; α = 36.87^0

$$V_{-B} = \begin{pmatrix} -5 \sin 36.87^0 \\ -5 \cos\ 36.87^0 \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$$

$r_A = \int V_A dt = \int \begin{pmatrix}4\\20\end{pmatrix} dt = \begin{pmatrix}4t\\20t\end{pmatrix} + c$

At t = 0, $r_A = \begin{pmatrix}0\\0\end{pmatrix}$ => c = $\begin{pmatrix}0\\0\end{pmatrix}$

Hence $r_A = \begin{pmatrix}4t\\20t\end{pmatrix}$

$R_B = \int V_B dt = \int \begin{pmatrix}-3\\-4\end{pmatrix} dt = \begin{pmatrix}-3t\\-4t\end{pmatrix} + c$

At t = 0, $r_B = \begin{pmatrix}0\\15\end{pmatrix}$ => c = $\begin{pmatrix}0\\15\end{pmatrix}$

Hence $r_B = \begin{pmatrix}-3t\\15-4t\end{pmatrix}$

${}_Ar_B = r_A - r_B = \begin{pmatrix}4t\\20t\end{pmatrix} - \begin{pmatrix}-3t\\15-4t\end{pmatrix} = \begin{pmatrix}7t\\24t-15\end{pmatrix}$

$d_s = |{}_Ar_B| = \sqrt{(7t)^2 + (24t-15)^5}$

but ds = 4.2

⇨ $\sqrt{(7t)^2 + (24t-15)^5} = 4.2$

$\left(\sqrt{(7t)^2 + (24t-15)^5}\right)^2 = 4.2^2$

$(7t)^2 + (24t-15)^5 = 4.2^2$

$47t^2 + 576t^2 - 720t + 225 = 17.64$

$625t^2 - 720t + 207.36 = 0$

t = $\frac{720 \pm\sqrt{(-720)^2-4(625)(207.36)}}{2(625)}$ = 0.576hours

= 0.576 x 60 = 35minutes

Hence the time at which the distance is shortest is 12:35pm

UACE MATHEMATICS PAPER 2 2018 guide

SECTION A (40 marks)

Answer all questions in this section

1. A stone is thrown vertically upwards with a velocity of $21ms^{-1}$
   Calculate the
   (a) Maximum height attained by the stone (03marks)
   (b) Time the stone takes to reach the maximum height. (02marks)
2. Two events A and B are such that (A/B)= $\frac{2}{5}$,P(B) = $\frac{1}{4}$ and P(A) = $\frac{1}{5}$.
   Find
   (a) P(A∩B) (02marks)
   (b) P(A∪B) (03marks)
3. The table below shows how T varies with S.

T	-2.9	-0.1	2.9	3.1
S	30	20	12	9

   Use linear interpolation/extrapolation to estimate the value of
   (a) T when S = 26 (03marks)
   (b) S when T = 3.4 (02marks)
4. A particle of mass 15kg is pulled up a smooth slope by a light inextensible string parallel to the slope. The slope is 10.5m long and inclined at $\sin^{-1}\left(\frac{4}{7}\right)$ to the horizontal. The acceleration of the particle is $0.98ms^{-2}$. Determine the
   (a) Tension in the string (03marks)
   (b) Work done against gravity when the particle reached the end of the slope. (02marks)
5. The price index of an article in 2000 based on 1998 was 130. The price index for the article in 2005 based on 2000 was 80. Calculate the:
   (a) Price index of the article in 2005 based on 1998. (03marks)
   (b) Price of the article in 1998 if the price of the article was 45,000 in 2005. (02marks)
6. Two numbers A and B have maximum possible error e_a and e_b respectively.
   (a) Write an expression for the maximum possible error in their sum
   (b) If A = 2.03 and B = 1.547, find the maximum possible error in A + B (05marks)
7. In an equilateral triangle PQR, three forces of magnitude 5N, 10N and 8N act along the sides PQ, QR and PR respectively. Their directions are in the order of the letters. Find the magnitude of the resultant force. (05marks)

8. A biased coin is that a head is three times as likely to occur as a tail. The coin is tossed 5 times, Find the probability that at most two tails occur. (05marks)

SECTION B (60 MARKS)

Answer any five questions from this section. All questions carry equal marks

9. The frequency distribution below shows the age of 240 students admitted to a certain University.

Age (years)	Number of student
18 - < 19	24
19 -< 20	70
20 -< 24	76
24 -< 26	48
26 -< 30	16
30 -< 32	6

(a) Calculate the mean age of the students. (04mark)

(b) (i) Draw a histogram for the given data

(ii) Use the histogram to estimate the modal age (08mark)

10. A particle of mass 4kg starts from rest at point (2i- 3j + k)m. it moves with acceleration a= (4i + 2j -3k)ms^{-2} when a constant force F acts on it.

Find the:

(a) Force F (02marks)

(b) Velocity at any time t. (04marks)

(c) Work done by the force F after 6 seconds (06marks)

11. (a) Use the trapezium rule with 6-ordinated to estimate the value of $\int_0^{\frac{1}{2}}(x + sinx)dx$, correct to three decimal places,

(b)(i) Evaluate $\int_0^{\frac{1}{2}}(x + sinx)dx$, correct to three decimal places

(ii) Calculate the error in your estimation in (a) above

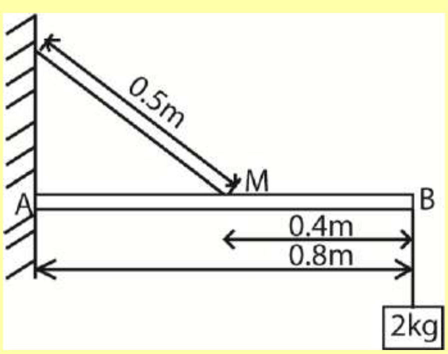
(iii) suggest how the error may be reduced (06marks)

12. A random variable X has a normal distribution where P(X>x9) = 0.9192 and P(X<11) = 0.7580. Find

(a) The values of the mean and standard deviation (08marks)

(b) P(X>10) (04marks)

13. The figure below shows a uniform beam of length 0.8metres and mass 1kg. the beam is hinged at A and has a load of mass 2kg attached at B



The beam is held in a horizontal position by a light inextensible string of length 0.5metres. the string joins the mid-point M of the beam to a point C vertically above A.

Find the

(a) Tension in the string(08marks)
(b) Magnitude and direction of the force exerted by the hinge. (04marks)

14. (a) Draw on the same axes the graphs of the curves y = 2 – e^{-x} and y = $\sqrt{x}$values 2≤ x ≤ 5. (04marks)
(b) Determine from your graph the interval within which the roots of the equation
$e^{-x} + \sqrt{x - 2 = 0}$ lies
Hence, use Newton-Raphson's method to find the root of the equation correct to 3 decimal places (07marks)

15. The table below shows the number of red and green balls put in three identical boxes A, B and C.

Boxes	A	B	C
Red balls	4	6	3
Green balls	2	7	5

A box is chosen at random and two balls are then drawn from it successively without replacement. If the random variable X is "the number of green balls drawn".
(a) Draw a probability distribution table for X (06marks)
(b) Calculate the mean and variance of X (06marks)

16. At 10:00 am, ship A and ship B are 16 km apart. Ship A on a bearingN35^{0}E from ship B. Ship A is travelling at 14kmh^{-1} on a bearing S29^{0}E. Ship B is travelling at 17kmh^{-1} on a bearing N50^{0}E. Determine the
(a) Velocity of ship B relative to ship A (05marks)
(b) Closest distance between the two ship and the time when it occurs (07marks)

Solutions

1. A stone is thrown vertically upwards with a velocity of 21ms^{-1}
Calculate the
(a) Maximum height attained by the stone (03marks)
H =$\frac{u^2}{2g} = \frac{21^2}{2\ x\ 9.8} = 225m$
(b) Time the stone takes to reach the maximum height. (02marks)
t = $\frac{u}{g} = \frac{21}{9.8} = 2.143s$

2. Two events A and B are such that (A/B)= $\frac{2}{5}$,P(B) = $\frac{1}{4}$ and P(A) = $\frac{1}{5}$.
Find
(a) P(A∩B) (02marks)
$\frac{2}{5} = \frac{P(A\cap B)}{\frac{1}{4}}$
$P(A \cap B) = \frac{2}{5}\ x\ \frac{1}{4} = \frac{1}{10} = 0.1$
(b) P(A∪B) (03marks)
$P(A\ \cup B) = P(A) + P(B) - P(A \cap B) = \ \frac{1}{5} + \frac{1}{4} - \frac{1}{10} = \frac{7}{20} = 0.35$

3. The table below shows how T varies with S.

T	-2.9	-0.1	2.9	3.1
S	30	20	12	9

Use linear interpolation/extrapolation to estimate the value of

(a) T when S = 26 (03marks)

-2.9	T	-0.1
30	26	20

$\frac{-0.1-T}{20-26} = \frac{-0.1-(-2.9)}{20-30}$

$T = -1.78$

(b) S when T = 3.4 (02marks)

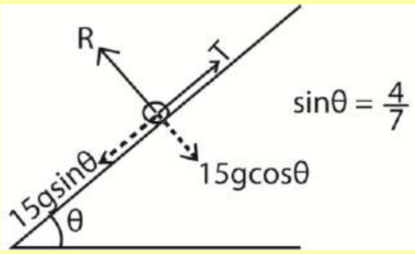
2.9	3.1	3.4
12	9	S

$\frac{s-9}{3.4-3.1} = \frac{9-12}{3.1-2.9}$

S = 4.5

4. A particle of mass 15kg is pulled up a smooth slope by a light inextensible string parallel to the slope. The slope is 10.5m long and inclined at $\sin^{-1}\left(\frac{4}{7}\right)$ to the horizontal. The acceleration of the particle is $0.98ms^{-2}$. Determine the

(a) Tension in the string (03marks)



T – 15gsinθ = 15a

T – 15 x 9.8 x $\frac{4}{7}$ = 15 x 0.98

T- 84 = 14.7

T = 98.7N

(b) Work done against gravity when the particle reached the end of the slope. (02marks)

Work = force x distance

= 15gdsinθ

= 15 x 9.8 x 10.5 x $\frac{4}{7}$

= 882J

5. The price index of an article in 2000 based on 1998 was 130. The price index for the article in 2005 based on 2000 was 80. Calculate the:

(a) Price index of the article in 2005 based on 1998. (03marks)

$\frac{P_{2000}}{P_{1998}}\ x\ 100 = 130, \frac{P_{2005}}{P_{2000}} = 80$

$\frac{P_{2005}}{P_{1998}}\ x100 = \frac{P_{2000}}{P_{1998}}\ x\ \frac{P_{2005}}{P_{2000}}\ x\ 100$

$= \frac{80}{100}\ x\ \frac{130}{100}\ x\ 100$

$= 104$

(b) Price of the article in 1998 if the price of the article was 45,000 in 2005. (02marks)

$\frac{P_{2005}}{P_{1998}}\ x100 = 104$

$P_{1998} = \frac{45,00}{104}\ x\ 100$ = 43269.23077

6. Two numbers A and B have maximum possible error e_a and e_b respectively.

(a) Write an expression for the maximum possible error in their sum

Maximum possible error = $|e_a||e_b|$

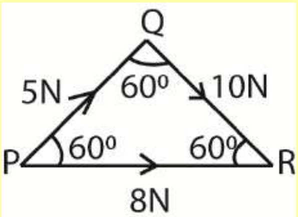
(b) If A = 2.03 and B = 1.547, find the maximum possible error in A + B (05marks)

$e_a = 0.005, e_b = 0.0005$

$|e_{(A+B)}| = |0.005| + |0.0005|$

= 0.0055

7. In an equilateral triangle PQR, three forces of magnitude 5N, 10N and 8N act along the sides PQ, QR and PR respectively. Their directions are in the order of the letters. Find the magnitude of the resultant force. (05marks)



(→);x = 8 + 5cos60^0 + 10sincos 60^0 = 15.5N

(↑); y = 5sin60^0 – 10cos60^0 = -4.3301N

R = $\sqrt{15.5^2 + (-4.3301)^2}$ = 16.094N

8. A biased coin is that a head is three times as likely to occur as a tail. The coin is tossed 5 times, Find the probability that at most two tails occur. (05marks)

Head (H)	Tail (T)	Total
3x	x	1

3x + x = 1

4x = 1

x = $\frac{1}{4}$

hence P(T) = 0.25 and P(H) = 0.75

P(x≤ 2) = 1 – P(X ≥ 3)

= 1 - $\sum_3^5 B(3, 5, 0, 2.5)$

= 1 – 0.1035

= 0.8965

SECTION B (60 MARKS)

Answer any five questions from this section. All questions carry equal marks

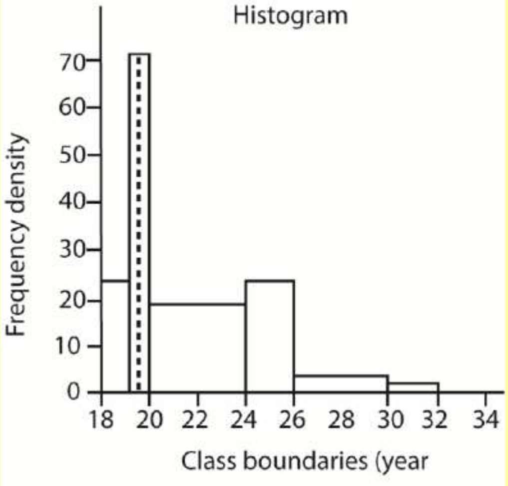
9. The frequency distribution below shows the age of 240 students admitted to a certain University.

Age (years)	Number of student, f	x	fx	c	fd
18 - < 19	24	18.5	444	1	24
19 -< 20	70	19.5	1365	1	70
20 -< 24	76	22	1672	4	19
24 -< 26	48	25	1200	2	24
26 -< 30	16	28	448	4	4
30 -< 32	6	31	186	2	3
	$\sum f = 240$		$\sum fx = 5315$		

(a) Calculate the mean age of the students. (04mark)

Mean = $\frac{\sum fx}{\sum f} = \frac{5315}{240} = 22.1458$

(b) (i) Draw a histogram for the given data



NB. On a histogram the area of the bar = frequency density

(ii) Use the histogram to estimate the modal age (08mark)

Modal age = 19.5 years

10. A particle of mass 4kg starts from rest at point (2i- 3j + k)m. it moves with acceleration a= (4i + 2j -3k)ms^{-2} when a constant force F acts on it.

Find the:

(a) Force F (02marks)

F = ma

$= 4\begin{pmatrix}4\\2\\-3\end{pmatrix} = \begin{pmatrix}16\\8\\-12\end{pmatrix}$N or F = (16i + 8j -12k)N

(b) Velocity at any time t. (04marks)

$v = \int adt$

v = at + c

$=\begin{pmatrix}4\\2\\-3\end{pmatrix}t + c$

At time t = 0, v = u = c = $\begin{pmatrix}0\\0\\0\end{pmatrix}$

Substituting for c

v = $\begin{pmatrix}4\\2\\-3\end{pmatrix}t = \begin{pmatrix}4t\\2t\\-3t\end{pmatrix}ms^{-1}$ or v = (4t i + 2t j -3t k)ms^{-1}

(c) Work done by the force F after 6 seconds (06marks)

Work done = force (F) x distance ($\underline{r}$)

$\underline{r} = \int vdt = \int\begin{pmatrix}4t\\2t\\-3t\end{pmatrix}dt = \begin{pmatrix}2t^2\\t^2\\-1.5t^2\end{pmatrix} + c$

$At\ t = 0, \underline{r} = c = \begin{pmatrix}2\\-3\\1\end{pmatrix}$

$Hence\ \underline{r} = \begin{pmatrix}2t^2\\t^2\\-1.5t^2\end{pmatrix} + \begin{pmatrix}2\\-3\\1\end{pmatrix}$

After time t= 6seconds

$$\underline{r} = \begin{pmatrix} 2(6)^2 \\ (6)^2 \\ -1.5(6)^2 \end{pmatrix} + \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 74 \\ 33 \\ -53 \end{pmatrix}$$

Work done = force (F) x distance($\underline{r}$)

$$= \begin{pmatrix} 16 \\ 8 \\ -12 \end{pmatrix} x \begin{pmatrix} 74 \\ 33 \\ -53 \end{pmatrix}$$

= 1184 + 264 + 636

= 2,084J

11. (a) Use the trapezium rule with 6-ordinated to estimate the value of $\int_0^{\frac{1}{2}}(x + sinx)dx$, correct to three decimal places.

$h = \frac{\frac{1}{2}-0}{5} = \frac{\pi}{10}$

x	y	
0	0	
$\frac{\pi}{10}$		0.6232
$\frac{2\pi}{10}$		1.2161
$\frac{3\pi}{10}$		1.7515
$\frac{4\pi}{10}$		2.2077
$\frac{\pi}{2}$	2.5708	
Sum	2.5708	5.7985

$\int_0^{\frac{1}{2}}(x + sinx)dx = \frac{1}{2}\ x\ \frac{\pi}{10}(2.5708 + 2\ x\ 5.7985)$

= 2.225

(b)(i) Evaluate $\int_0^{\frac{1}{2}}(x + sinx)dx$, correct to three decimal places

$\int_0^{\frac{1}{2}}(x + sinx)dx = \left|\frac{x^2}{2} - \cos x\right|_0^{\frac{1}{2}}$

$= \frac{1}{2}\left(\frac{\pi^2}{4} - 0\right) - \left(\cos\frac{\pi}{2} - \cos 0\right)$

$= \frac{\pi^2}{8} + 1$

= 2.234

(ii) Calculate the error in your estimation in (a) above

Error = |2.234 − 1.225| = 0.009

(iii) Suggest how the error may be reduced (06marks)

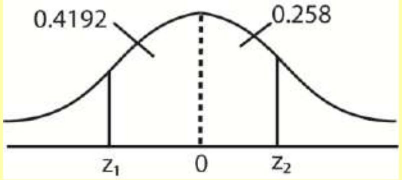
Increasing on number of strips or subintervals

12. A random variable X has a normal distribution where P(X>x9) = 0.9192 and P(X<11) = 0.7580. Find

(a) The values of the mean and standard deviation (08marks)

P(X, >9) = P$\left(z_1 > \frac{9-\mu}{\delta}\right)$ =0.9192

P(X, < 11) = P$\left(z_2 > \frac{11-\mu}{\delta}\right)$ =0.0.7580



$z_1 = -\phi(0.4192) = -1.4$

$z_2 = \phi(0.258) = 0.7$

⇨ $\frac{9-\mu}{\delta} = -1.4$

$9 - \mu = -1.4\delta$ …………………… (i)

⇨ $\frac{11-\mu}{\delta} = 0.7$

$11 - \mu = 0.7\delta$ …………………… (ii)

$Eqn\ (i) - Eqn\ (ii)$

$-2 = -2.1\delta$

$\delta = \frac{-2}{-2.1} = 0.9524$

From (i)

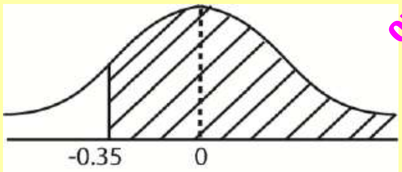
$9 - \mu = -1.4 \times 0.9524$

$\mu = 10.333$

(b) P(X>10) (04marks)

P(X>10) = P$\left(z > \frac{10-10.333}{0.9524}\right)$

= P(z > -0.35)

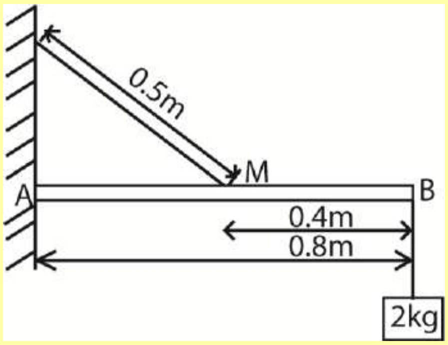


P(X > 10) = P(0.5 + P(0 < z < 0.35)

= 0.5 + 0.1368

= 0.6368

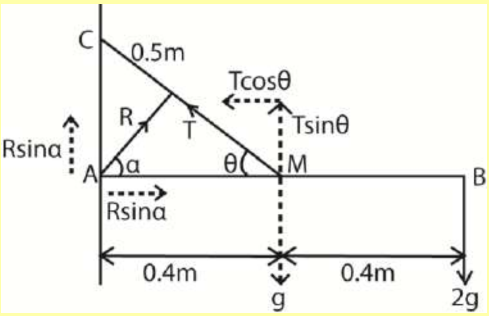
13. The figure below shows a uniform beam of length 0.8metres and mass 1kg. the beam is hinged at A and has a load of mass 2kg attached at B



The beam is held in a horizontal position by a light inextensible string of length 0.5metres. the string joins the mid-point M of the beam to a point C vertically above A.

Find the

(a) Tension in the string(08marks)



$AC^2 = (05)^2 – (0.4)^2$

AC = 0.3

$\cos\theta = \frac{0.4}{0.5} = 0.8$

$\sin\theta = \frac{0.3}{0.5} = 0.6$

Taking moments at A

(9.8 x 0.4) + (2 x 9.8 x 0.8) = T x 0.4sinθ

(9.8 x 0.4) + (2 x 9.8 x 0.8) = T x 0.4 x 0.6

T = 81.667N

(b) Magnitude and direction of the force exerted by the hinge. (04marks)

Resolving forces

(→); Rcosα = Tcosθ = $\frac{245}{3}\ x\ 0.8$

Rcosα = $\frac{196}{3}$ ....(i)

(↑); Rsinα + Tsinθ = g + 2g

Rsinα = 3 x 9.8 - $\frac{245}{3}\ x\ 0.6$

Rsinα = -19.6 .... (ii)

Equn (ii) ÷ Eqn (i)

$\frac{R\sin\alpha}{R\cos\alpha} = \frac{-19.6}{\frac{196}{3}}$ = -0.3

tan α = -0.3

α = -16.7^0

Hence the direction of force at the hinge is 16.7^0 with the beam

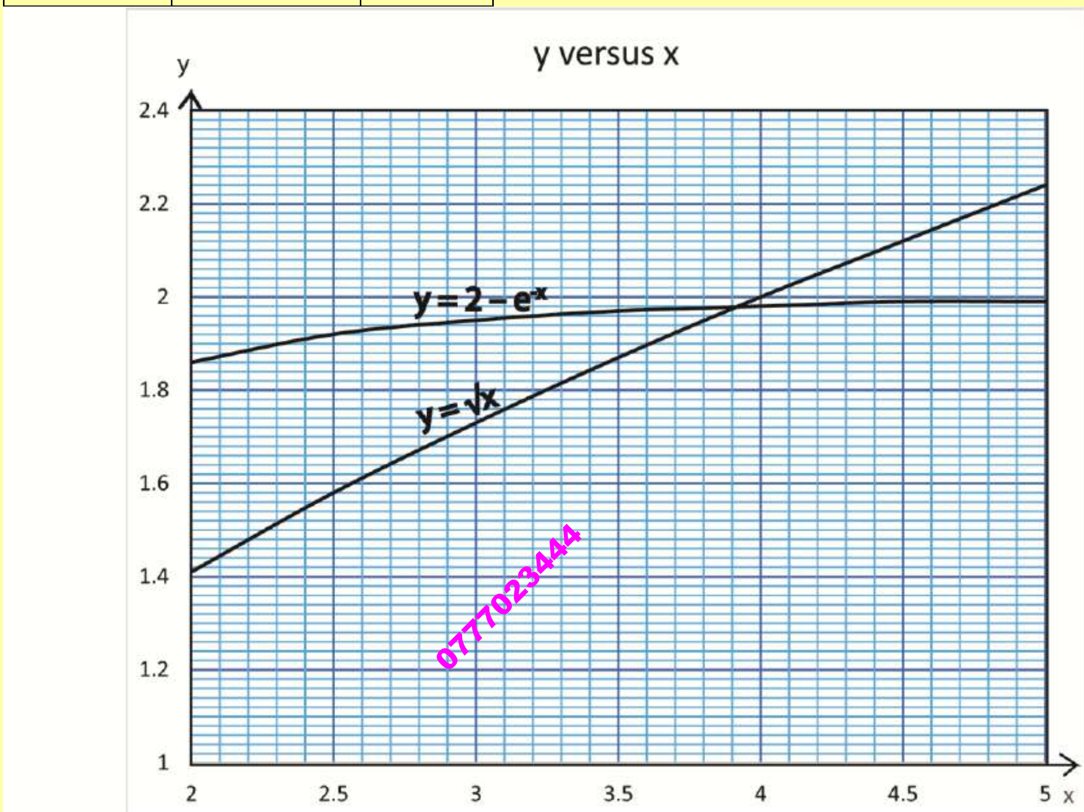
From eqn (i)

R cos $16.7^0 = \frac{196}{3}$

R = 68.21N

14. (a) Draw on the same axes the graphs of the curves $y = 2 – e^{-x}$ and $y = \sqrt{x}$values $2 \leq x \leq 5$. (04marks)

x	y = 2 – e^{-x}	y = $\sqrt{x}$
2.0	1.86	1.41
2.5	1.92	1.58
3.0	1.95	1.73
3.5	1.97	1.87
4.0	1.98	2.00
4.5	1.99	2.12
5.0	1.99	2.24



(b) Determine from your graph the interval within which the roots of the equation

$e^{-x} + \sqrt{x - 2 = 0}$ lies

Hence, use Newton-Raphson's method to find the root of the equation correct to 3 decimal places (07marks)

Root lies between 3.9 and 4

f(x) = 2 – e^{-x} - $\sqrt{x}$

f'(x) = $e^{-x} - \frac{1}{2\sqrt{x}}$

f(x_n) = $e^{-x_n} - \frac{1}{2\sqrt{x_n}}$

x_{n+1} = x_n - $\frac{2- e^{x_n} - \sqrt{x_n}}{2e^{-x_n}\sqrt{x_n} - 1}$

x_0 = $\frac{3.9+4}{2}$ = 3.95

$x_1 = 3.95 - \frac{2\sqrt{3.95}(2- e^{-3.95} - \sqrt{3.95})}{2e^{-3.95}\sqrt{3.95} - 1}$ = 3.9211

$Error = |3.9211 - 3.95| = 0.0289$

$x_2 = 3.9211 - \frac{2\sqrt{3.9211}(2-e^{-3.9211}-\sqrt{3.9211})}{2e^{-3.9211}\sqrt{3.9211}-1} = 3.9211$

∴ Root = 3.921 (3dp)

15. The table below shows the number of red and green balls put in three identical boxes A, B and C.

Boxes	A	B	C
Red balls	4	6	3
Green balls	2	7	5

A box is chosen at random and two balls are then drawn from it successively without replacement. If the random variable X is "the number of green balls drawn".

(a) Draw a probability distribution table for X (06marks)

Using combination

$P(X = 0) = \frac{1}{3}\left[\frac{{}^4C_2}{{}^6C_2} + \frac{{}^6C_2}{{}^{13}C_2} + \frac{{}^3C_2}{{}^8C_2}\right]$

$= \frac{1}{3}\left[\frac{2}{5} + \frac{5}{26} + \frac{2}{28}\right] = \frac{1273}{5460}$

$P(X = 1) = \frac{1}{3}\left[\frac{{}^2C_1\, x\, {}^4C_1}{{}^6C_2} + \frac{{}^7C_1\, x\, {}^6C_1}{{}^{13}C_2} + \frac{{}^3C_1\, x\, {}^3C_1}{{}^8C_2}\right]$

$= \frac{1}{3}\left[\frac{8}{15} + \frac{7}{13} + \frac{15}{28}\right] = \frac{8777}{16380}$

$P(X = 2) = \frac{1}{3}\left[\frac{{}^2C_2}{{}^6C_2} + \frac{{}^7C_2}{{}^{13}C_2} + \frac{{}^5C_2}{{}^8C_2}\right]$

$= \frac{1}{3}\left[\frac{1}{15} + \frac{7}{26} + \frac{5}{14}\right] = \frac{946}{4095}$

x	0	1	2
P(X =x)	$\frac{1273}{5460}$	$\frac{8777}{16380}$	$\frac{946}{4095}$

(b) Calculate the mean and variance of X (06marks)

1	0	1	2
P(X =x)	$\frac{1273}{5460}$	$\frac{8777}{16380}$	$\frac{946}{4095}$
xP(X =x)	0	$\frac{8777}{16380}$	$\frac{1892}{4095}$
x^2P(X =x)	0	$\frac{8777}{16380}$	$\frac{3784}{4095}$

$E(X) = \frac{8777}{16380} + \frac{1892}{4095} = 0.9979$

$E(X^2) = \frac{8777}{16380} + \frac{3784}{4095} = 1.4599$

Var(X) = 1.4599 – 0.9979

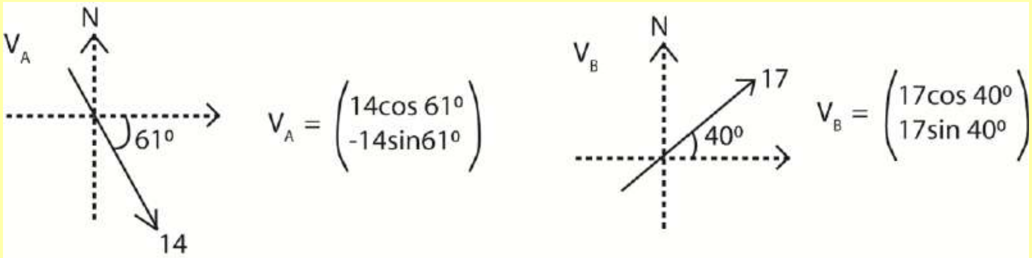
= 0.462

16. At 10:00 am, ship A and ship B are 16 km apart. Ship A on a bearingN35^{0}E from ship B. Ship A is travelling at 14kmh^{-1} on a bearing S29^{0}E. Ship B is travelling at 17kmh^{-1} on a bearing N50^{0}E. Determine the

(a) Velocity of ship B relative to ship A (05marks)

Using vector approach

Vector	Direction	magnitude
V_A	S29^{0}E	14kmh^{-1}
V_B	N50^{0}E	17kmh^{-1}
${}_AV_B$	?	?



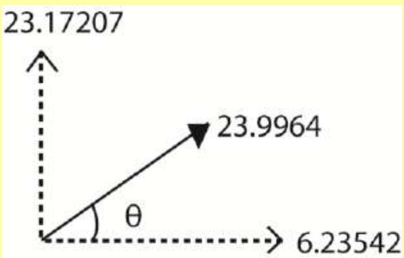
${}_BV_A = V_B - V_A$

$= \begin{pmatrix} 17\cos 40^0 \\ 17\sin 40^0 \end{pmatrix} - \begin{pmatrix} 14\cos 61^0 \\ -17\sin 61^0 \end{pmatrix}$

$= \begin{pmatrix} 6.23542 \\ 23.17207 \end{pmatrix}$

$|{}_BV_A| = \sqrt{(6.23542)^2 + \sqrt{(23.17207)^2}}$

$= 23.9964\text{kmh}^{-1}$



$\theta = \tan^{-1}\left(\frac{23.17207}{6.23542}\right) = 74.94^0$

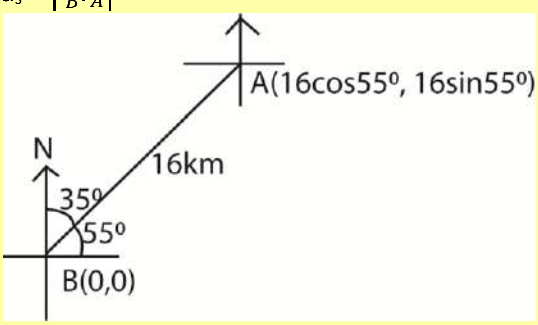
∴ the direction of the velocity of ship B relative to ship A is N15.06^{0}E

(b) Closest distance between the two ships and the time when it occurs (07marks)

Method I: vector method

${}_BV_A . {}_Br_A = 0$

$d_s = |{}_Br_A|$



${}_Br_A = r_B - r_A$

$= (OB + VBt) - (OA + Vat)$

$= (OB - OA) + {}_BV_At$

$= \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 16\cos 55^0 \\ 16\sin 55^0 \end{pmatrix}\right] + \begin{pmatrix} 6.23542 \\ 23.17207 \end{pmatrix} t$

$= \begin{pmatrix} -9.17722 + 6.23542t \\ -13.10643 + 23.17207t \end{pmatrix}$

⇨ $\begin{pmatrix} 6.23542 \\ 23.17207 \end{pmatrix} . \begin{pmatrix} -9.17722 + 6.23542t \\ -13.10643 + 23.17207t \end{pmatrix} = 0$

$6.23541\, x(-9.17722 + 6.23542t) + 23.17207(-13.10643 + 23.17207t) = 0$

$t = 0.6268h$

$d_s = \left|\begin{pmatrix} -9.17722 + 6.23542\, x\, 0.6268 \\ -13.10643 + 23.17207\, x\, 0.6268 \end{pmatrix}\right| = \begin{vmatrix} -5.26886 \\ 1.41782 \end{vmatrix}$

$= \sqrt{(-5.26886)^2 + \sqrt{(1.41782)^2}}$

= 5.4563km

Method II differential approach

Closest approach is given by $d_s = |{}_Br_A|$ and time, t, taken $=\frac{d}{dt}|{}_Br_A| = 0$

${}_Br_A = r_B - r_A$

$= (OB + VBt) - (OA + Vat)$

$= (OB - OA) + {}_BV_At$

$= \left[\begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}16\cos 55^0\\16\sin 55^0\end{pmatrix}\right] + \begin{pmatrix}6.23542\\23.17207\end{pmatrix}t$

$= \begin{pmatrix}-9.17722 + 6.23542t\\-13.10643 + 23.17207t\end{pmatrix}$

But $\frac{d}{dt}|{}_Br_A| = \frac{d}{dt}|{}_Br_A|^2 = 0$

$|{}_Br_A|^2 = (-9.17722 + 6.23542t)^2 + (-13.10643 + 23.17207t)^2 = 0$

$$\frac{d}{dt}[(-9.17722 + 6.23542t)^2 + (-13.10643 + 23.17207t)^2] = 0$$

$2[6.23541(-9.17722 + 6.23542t) + 23.17207(-13.10643 + 23.17207t)] = 0$

$360.9269345 = 575.8252907t$

$t = 0.6268h$

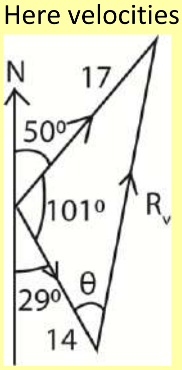
$d_s = \left|\begin{pmatrix}-9.17722 + 6.23542\ x\ 0.6268\\-13.10643 + 23.17207\ x\ 0.6268\end{pmatrix}\right| = \left|\begin{matrix}-5.26886\\1.41782\end{matrix}\right|$

$= \sqrt{(-5.26886)^2 + \sqrt{(1.41782)^2}}$

= 5.4563km

Method III: Geometrical approach

(a) Here velocities originate from a common point



Using cosine rule

R_v^2 =172 + 142 -2 x 17 x 14cos 101^0

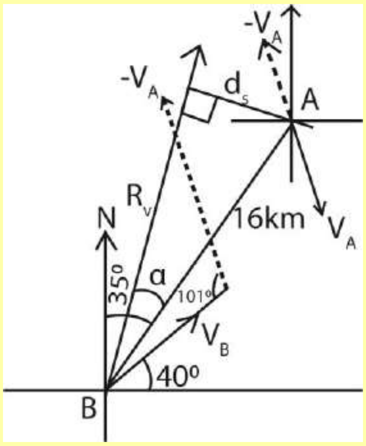
$R_v = 23.9964kmh^{-1}$

Using sine rule

$$\frac{R_v}{\sin 101^0} = \frac{17}{sin\theta}$$

$\frac{23.9964}{\sin 101^0} = \frac{17}{sin\theta}$; θ = 44.06^0

(b)



$\alpha = 35^0 - 15.06^0 = 19.94^0$

ds= $16\sin 19.94^0$

$= 5.457$km

UACE MATHEMATICS PAPER 2 2019 guide

SECTION A

1. The table shows the masses of bolts bought by a carpenter.

Mass (grams)	98	99	100	101	102	103	104
Number of bolts	8	11	14	20	17	6	4

Calculate the:

(a)median mass

(b) mean mass of the bolt (05mark

2. A uniform rod AB of length 3m and mass 8kg is freely hinged to a vertical wall at A. A string BC of length 4m attached at b and to point C on the wall, keeps the rod in equilibrium. If C is 5m vertically above A, find the

(a) Tension in the string (03marks)

(b) Magnitude of the normal reaction at A. (02marks)

3. Use the trapezium rule with seven coordinates to estimate

$\int_0^3 [(1.2)^x - 1]^{\frac{1}{2}} dx$ correct to 2 decimal places (05marks)

4. A discrete random variable X has the following probability distribution

X	0	1	2	3	4	5
P(X = x)	0.11	0.17	0.2	0.13	p	0.09

Find the

(a) Value of p (02marks)

(b) Expected value of X (03marks)

5. A stone is thrown vertically upwards with velocity 16ms^{-1} from a point H meters above the ground level. The stone hits the ground 4 seconds later.

Calculate the

(a) Value of H (03marks)

(b) Velocity of the stone as it hits the ground (02marks)

6. The table below shows the commuter bus fare from stages A to B, C, D and E

Stage	A	B	C	D	E
Distance (km)	0	12	16	19	23
Fare (shs)	0	1300	1700	2200	2500

(a) Jane boarded from A and stopped at a place 2km after E. How much did she pay? (03marks)

(b) Okello paid shs 2000. How far from A did the bus leave him? (02marks)

7. The amount of meat sold by a butcher is normally distributed with a mean of 43kg and standard deviation 4kg. Determine the probability that the amount of meat sold is between 40kg and 50kg. (05marks)
8. A particle is moving with simple harmonic motion (SHM). When the particle is 15m from the equilibrium, its speed is 6ms^{-1}. When the particle is 13 m from equilibrium, its speed is 9ms^{-1}. Find the amplitude of the motion (05marks)

SECTION B

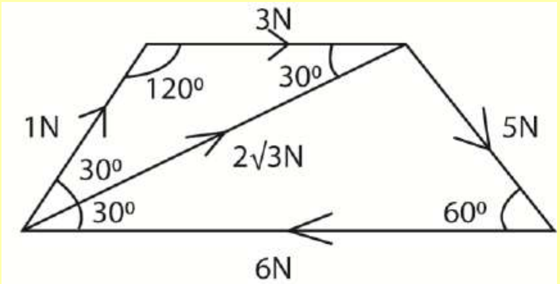
9. Car A is 80m North West of point O. Car B is 50m N 30^{0}E of O. Car A s moving at 20ms^{-1} while car B is moving at 10ms^{-1} each on a straight road towards O. Determine the
   (a) Initial distance between the two cars (03mark)
   (b) Velocity of A relative to B (05marks)
   (c) The shortest distance between the two cars as they approach O (04marks)
10. The table below shows the marks obtained in a mathematic test by a group of student

marks	5 -<15	15-<25	25-<35	35-<45	45-<55	55-<65	65-<75	75-<100
Number of students	5	7	19	17	7	4	2	3

   (a) Construct a cumulative frequency (O give) for the data (05 marks)
   (b) Use your Ogive to find the
      (i) Range between the 10th and 70th percentiles
      (ii) Probability that a student selected at random scored below 50 marks. (07 marks)
11. (a) Show that the equation x -3sinx = 0 has a root between 2 and 3. (03marks)
   (b) Show that Newton- Raphson iterative formula for estimating the root of the equation in (a) is given by

$$X_{n+1} = \frac{3(\sin x_n - x_n \cos x_n)}{1 - 3\cos x_n}, n = 0, 1, 2 \ldots.$$

   Hence find the root of the equation corrected to 2 decimal places (09 marks)
12. A force F = (2t i + j – 3t k)N acts on a particle of mass 2kg. The particle is initially at a point (0,0,0) and moving with a velocity (i + 2j-k)ms^{-1}. Determine the:
   (a) Magnitude of the acceleration of the particle after 2 seconds (04marks)
   (b) Velocity of the particle after 2seconds (04marks)
   (c) Displacement of the particle after 2 seconds (04marks)
13. Two events A and B are such that P(B) = $\frac{1}{8}$, P(A∩B) = $\frac{1}{10}$ and P(B/A) = $\frac{1}{3}$
   Determine the
   (a) P(A) (03marks)
   (b) P(A∪ B) (03marks)
   (c) P(A/$\bar{B}$) (06marks)
14. (a) Given that y = e^x and x = 0.62correct to two decimal places, find the interval within which the exact value of y lies. (05marks)
   (b) Show that the maximum possible relative error in ysin^2x is $\left|\frac{\Delta y}{y}\right| + 2\cot x\,|\Delta x|$, where$\Delta x$ and Δy are errors in x and y respectively
   Hence find the percentage error in calculating ysin^2x if y = 5.2 $\pm$ 0.05 and x = $\frac{\pi}{6} \pm \frac{\pi}{360}$ (07 marks)
15. The diagram below shows a trapezium rule ABCD, AD = DC = CB = 1and AB = 2meters. Forces of magnitude 1N, 3N, 5N, 6N and 2$\sqrt{3}$ N respectively.



(a) Calculate the magnitude of the resultant force and the angle it makes with side AB (09marks)

(b) Given that the line of action of the resultant force meets AB at X, find AX. (03marks)

16. A biased die with faces labelled 1, 2, 3, 4, 5 and 6 is tossed 45 times
Calculate the probability that 2 will appear;
(a) More than 18 times (07marks)
(b) Exactly 11 times (05marks)

Solutions

SECTION A

17. The table shows the masses of bolts bought by a carpenter.

Mass (grams)	98	99	100	101	102	103	104
Number of bolts	8	11	14	20	17	6	4
fx	784	1084	1400	2020	1734	618	416
c.f	8	19	33	53	70	76	80

$$\sum f = 80, \sum fx = 8061$$

Calculate the:

(a)median mass

Medan position = $\left(\frac{N}{2}\right)^{th}$ $value$ = $\left(\frac{80}{2}\right)^{th}$ value = 40^{th} value

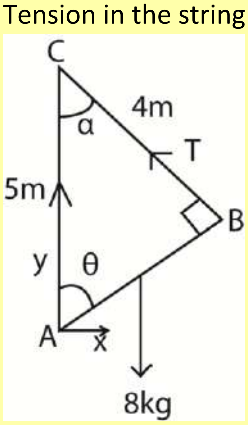
∴ median = 101

(b) mean mass of the bolt (05mark

Mean = $\frac{\sum fx}{\sum f} = \frac{8061}{80} = 100.76g$

18. A uniform rod AB of length 3m and mass 8kg is freely hinged to a vertical wall at A. A string BC of length 4m attached at b and to point C on the wall, keeps the rod in equilibrium. If C is 5m vertically above A, find the

(c) Tension in the string (03marks)



$AB^2 + 4^2 = 5^2$

AB = $\sqrt{(25-16)}$ = 3

Let T be tension in the string, from the diagram

cos θ = $\frac{3}{5}$, cos α = $\frac{4}{5}$

Equation of moment about A

T x 3 = 8g x 1.5cosα

3T = 8 x 9.8 x $\frac{4}{5}$; T = 31.36N

∴ tension in the string is 31.36N

(d) Magnitude of the normal reaction at A. (02marks)

x = Tcosθ = 31.36 x $\frac{3}{5}$ = 18.816N

∴ the magnitude of normal reaction at A is 18.816N

19. Use the trapezium rule with seven coordinates to estimate $\int_0^3 [(1.2)^x - 1]^{\frac{1}{2}}$dx correct to 2 decimal places (05marks)

Solution

For 7 ordinates, there are 6 subintervals

Width, h = $\frac{b-a}{subinterval} = \frac{3-0}{6} = 0.5$

Let y = $\sqrt{(1.2)^x - 1}$

x	y	
0	0	
0.5		0.309
1		0.447
1.5		0.561
2		0.663
2.5		0.760
3	0.853	
Sum	0.853	2.74

Using the trapezium rule

$\int_0^3 [(1.2)^x - 1]^{\frac{1}{2}}$dx = $\frac{0.5}{2}[0.853 + 2(2.74)]$ = 1.58

20. A discrete random variable X has the following probability distribution

X	0	1	2	3	4	5
P(X = x)	0.11	0.17	0.2	0.13	p	0.09

Find the

(c) Value of p (02marks)

Using $\sum P(X = x)$ = 1

0.11 + 0.17 + 0.2 + 0.13 + p + 0.09 = 1

p = 0.3

(d) Expected value of X (03marks)

E(X) = $\sum x.P(X = x)$

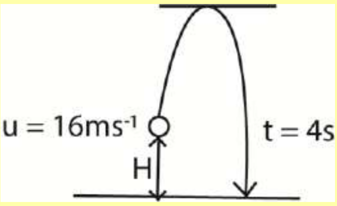
= 0 x 0.11 + 1 x 0.17 + 2 x 0.2 + 3 x 0.13 + 4 x 0.3 + 5 x 0.09

= 2.61

21. A stone is thrown vertically upwards with velocity $16ms^{-1}$ from a point H meters above the ground level. The stone hits the ground 4 seconds later.

Calculate the

(c) Value of H (03marks)



Using $s = ut + \frac{1}{2}at^2$; s = -H(below point of projection), u = 16ms^{-1}, a = -g, t = 4s

-H = 16 x 4 - $\frac{1}{2}\, x\, 9.8\, x\, 4^2$

H = 14.4m

(d) Velocity of the stone as it hits the ground (02marks)

Using v = u + at; v = -v(below point of projection), a = -g, t = 4s

-v = 16 -9.8 x 4

v = 23.2ms^{-1}

∴ the velocity of the stone as it hits the ground is 23.2ms^{-1}

22. The table below shows the commuter bus fare from stages A to B, C, D and E

Stage	A	B	C	D	E
Distance (km)	0	12	16	19	23
Fare (shs)	0	1300	1700	2200	2500

(c) Jane boarded from A and stopped at a place 2km after E. How much did she pay? (03marks)

2kn after E = 25km from A, let x be the fare

Extract

D	E	
19	23	25
2200	2500	x

Using linear extrapolation

$$\frac{x-2500}{25-23} = \frac{2500-2200}{23-19}$$

x = sh 2650

(d) Okello paid shs 2000. How far from A did the bus leave him? (02marks)

Let y be the distance

Extract

C		D
16	y	19
1700	200	2200

Using linear extrapolation

$$\frac{y-16}{2000-1700} = \frac{19-16}{2200-1700}$$

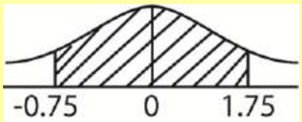
y = 17.8km

23. The amount of meat sold by a butcher is normally distributed with a mean of 43kg and standard deviation 4kg. Determine the probability that the amount of meat sold is between 40kg and 50kg. (05marks)

X ~N(43,4)

P(40<x<50) = $P\left(\frac{40-43}{4} < Z < \frac{50-43}{4}\right)$

= P(- 0.75 < Z < 1.75)



P(40<x<50) = P(-0.75 < Z < 0) + P(0 < Z < 1.75)

By property of symmetry

P(40<x<50) = P(-0.75 < Z < 0) + P(0 < Z < 1.75)

= 0.2735 + 0.4599

= 0.733

24. A particle is moving with simple harmonic motion (SHM). When the particle is 15m from the equilibrium, its speed is $6ms^{-1}$. When the particle is 13 m from equilibrium, its speed is $9ms^{-1}$. Find the amplitude of the motion (05marks)

$v^2=\omega^2(A^2 - x^2)$

$6^2= \omega^2(A^2 - 15^2)$ ……. (i)

$9^2=\omega^2(A^2 - 13^2)$ …….. (ii)

Dividing (i) by (ii)

$\frac{36}{81} = \frac{A^2-225}{A^2-169}$

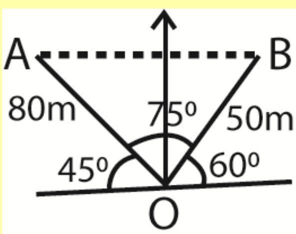
Amplitude A = 16.4256m

SECTION B

25. Car A is 80m North West of point O. Car B is 50m N 30^0E of O. Car A s moving at $20ms^{-1}$ while car B is moving at $10ms^{-1}$ each on a straight road towards O. Determine the

(d) Initial distance between the two cars (03mark)

Method I: using geometrical approach



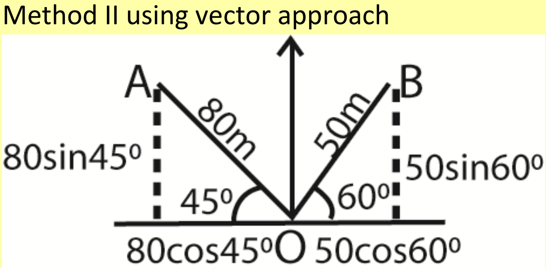
Initial distance = $\overline{AB}$

Using cosine rule

$\overline{AB}^2 = 80^2 + 50^2 - 2\ x\ 80\ x 50 \cos 75^0$

$\overline{AB} = 82.64m$

Method II using vector approach



AB = OB – OA

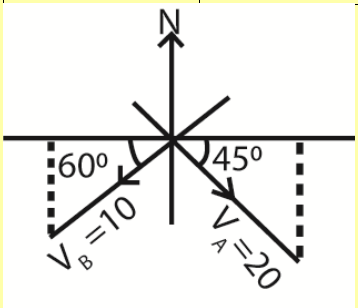
$= \begin{pmatrix} 50\cos 60^0 \\ 50\sin 60^0 \end{pmatrix} - \begin{pmatrix} -80\cos 45^0 \\ 80\sin 45^0 \end{pmatrix} = \begin{pmatrix} 81.569 \\ -13.267 \end{pmatrix}$

$\overline{AB} = \sqrt{81.569^2 + (-13.267)^2} = 82.64m$

(e) Velocity of A relative to B (05marks)

Method I: using vector approach

Vector	Direction	Magnitude
V_A	South east	$20ms^{-1}$
V_B	$S30^0W$	$10ms^{-1}$
$_AV_B$	?	?



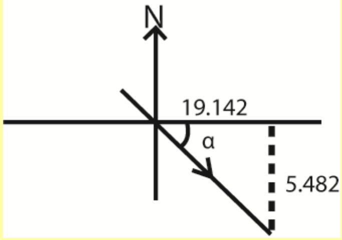
$_AV_B = V_A - V_B$

$$= \begin{pmatrix} 20\cos 45^0 \\ -50\sin 45^0 \end{pmatrix} - \begin{pmatrix} -10\cos 60^0 \\ -10\sin 60^0 \end{pmatrix} = \begin{pmatrix} 19.142 \\ -5.482 \end{pmatrix}$$

$\therefore {_AV_B} = 19.142i - 5.482j$

Or : expressing it in terms of magnitude and direction

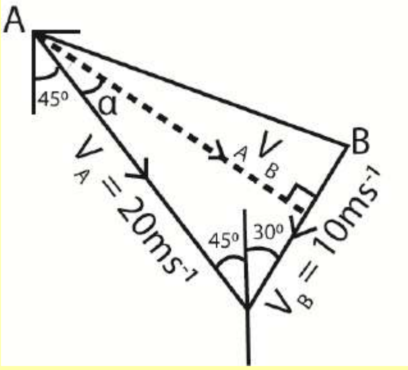
$\left|{_AV_B}\right| = \sqrt{(19.142)^2 + (-5.482)^2} = 19.912m$



$\alpha = \tan^{-1}\left(\frac{5.482}{19.142}\right) = 15.98^0$

Hence the relative velocity of A relative B is $19.912ms^{-1}$ in the direction $E15.98^0S$

Method II: Using geometric approach



Using cosine rule

$\left|{_AV_B}\right|^2 = 20^2 + 10^2 - 2 \, x \, 20 \, x 10 \cos 75^0$

$\left|{}_AV_B\right| = 19.912ms^{-1}$

$\dfrac{\left|{}_AV_B\right|}{\sin 75^0} = \dfrac{10}{\sin\alpha}$

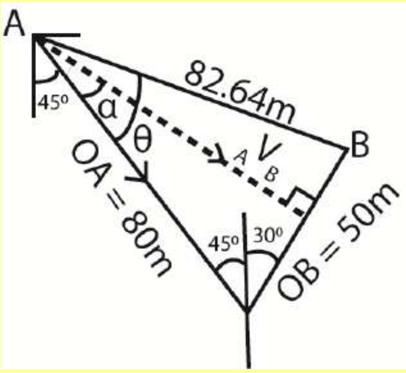
$\alpha = 29.02^0$

$45^0 + 29.02^0 = 74.02^0$

∴ The velocity of A relative to B is $19.912ms^{-1}$ due $S74.02^0E$

(f) The shortest distance between the two cars as they approach O (04marks)

Using geometrical approach



$\dfrac{50}{\sin\theta} = \dfrac{82.64}{\sin 75^0}$; θ = 35.76^0

$< BAD$ = 35.76 – 29.02 = 6.74^0

$\sin 6.74^0 = \dfrac{\left|{}_Ar_B\right|}{AB}$

${}_Ar_B$ = 82.64sin 6.74^0 = 9.699m

∴ The shortest distance between the two cars they approach O is 9.699m

26. The table below shows the marks obtained in a mathematic test by a group of student

marks	5 -<15	15-<25	25-<35	35-<45	45-<55	55-<65	65-<75	75-<100
Number of students	5	7	19	17	7	4	2	3

(c) Construct a cumulative frequency (O give) for the data (05 marks)

Class boundaries	F	Cf
5 – 15	5	5
15 - 25	7	12
25 – 35	19	31
35 – 45	17	48
45 – 55	7	55
55 – 65	4	59
65 – 75	2	61
75 – 85	3	64

(d) Use your Ogive to find the

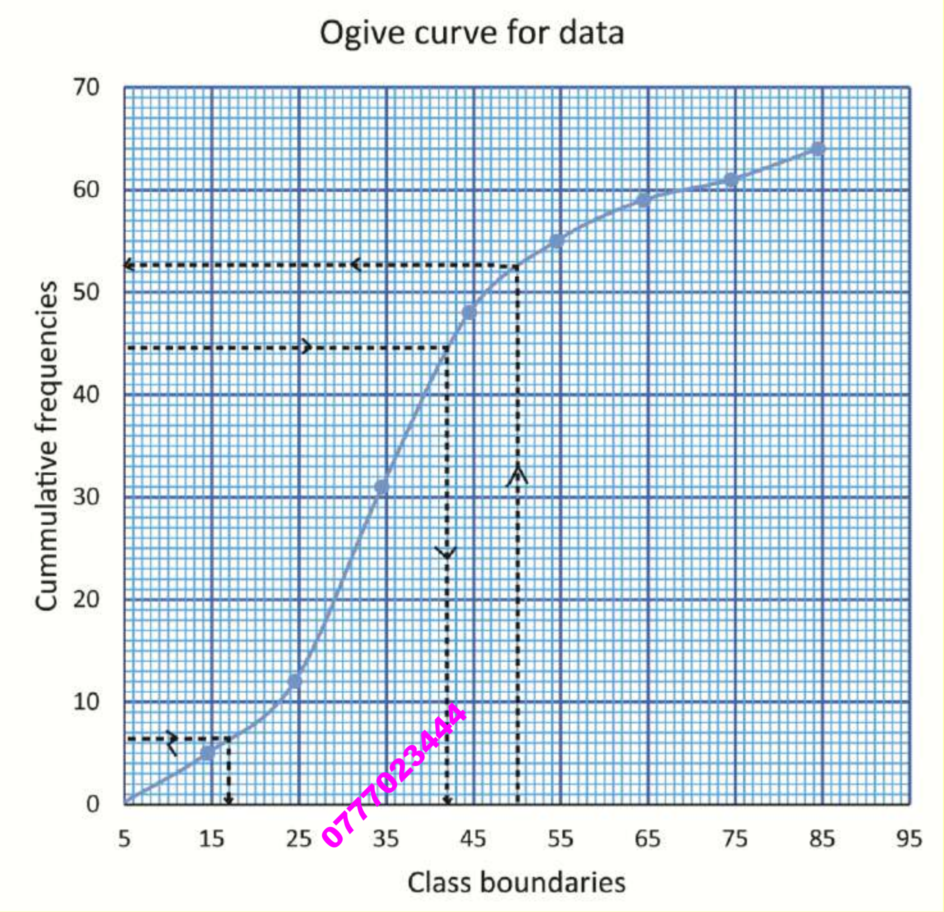
(iii) Range between the 10th and 70th percentiles

10th percentile = $\left(\frac{10}{100}\ x\ 64\right)^{th}$ value = 6.4th value

From the graph below P_{10} = 17

10th percentile = $\left(\frac{70}{100}\ x\ 64\right)^{th}$ value = 44.8th value

From the graph below $P_{70} = 43$



Percentile range = 43 – 17 =26

(iv) Probability that a student selected at random scored below 50 marks. (07 marks)

From the graph number of students who scored below 50 marks = 52

Probability = $\frac{52}{64}$ = 0.8125

27. (a) Show that the equation x -3sinx = 0 has a root between 2 and 3. (03marks)

f(x) = x – 3sinx

f(2) = 2 – 3sin 2 = -0.7279

f(3) = 3 – 3sin3 = 2.5766

since f(2).f(3) = -1.8755<0

there exist a root of x-3sinx =0 between 2 and 3

(b) Show that Newton- Raphson iterative formula for estimating the root of the equation in (a) is given by

$$X_{n+1} = \frac{3(\sin x_n - x_n \cos x_n)}{1 - 3\cos x_n}, n = 0, 1, 2 ....$$

Hence find the root of the equation corrected to 2 decimal places (09 marks)

f'(x) = 1 – 3cos x

$x_{n+1} = x_n - \frac{f(x)}{f'(x)}$

$= x_n - \frac{x_n - 3\sin x_n}{1 - 3\cos x_n}$

$= \frac{x_n - 3x_n \cos x_n - x_n + 3\sin x_n}{1 - 3\cos x_n}$

$x_{n+1} = \frac{3(\sin x_n - x_n \cos x_n}{1 - 3\cos x_n}$

Taking $x_0 = \frac{2+3}{2} = 2.5$

$x_1 = \frac{3(\sin 2.5 - 2.5\cos 2.5}{1 - 3\cos 2.5} = 2.293$

Error = $|2.293 - 2.5| = 0.207 > 0.005$

$x_2 = \frac{3(\sin 2.293 - 2.5\cos 2.293}{1 - 3\cos 2.293} = 2.279$

Error = $|2.279 - 2.293| = 0.014 > 0.005$

$x_3 = \frac{3(\sin 2.279 - 2.5\cos 2.279}{1 - 3\cos 2.279} = 2.279$

Error = $|2.279 - 2.279| = 0.000 < 0.005$

∴ root = 2.279 = 2.28(2D)

28. A force F = (2t i + j – 3t k)N acts on a particle of mass 2kg. The particle is initially at a point (0,0,0) and moving with a velocity (i + 2j-k)ms^{-1}. Determine the:

(d) Magnitude of the acceleration of the particle after 2 seconds (04marks)

F = (2t i + j – 3t k) = $\begin{pmatrix} 2t \\ 1 \\ -3t \end{pmatrix}$N

a = $\frac{F}{m} = \frac{1}{2}\begin{pmatrix} 2t \\ 1 \\ -3t \end{pmatrix} = \begin{pmatrix} t \\ 0.5 \\ -1.5t \end{pmatrix} ms^{-1}$

At t = 2s

$\underline{a}$=2i + 0.5j – 3k

$|\underline{a}| = \sqrt{2^2 + 0.5^2 + (-3)^2}$ = 3.64ms^{-2}

(e) Velocity of the particle after 2seconds (04marks)

$\underline{v} = \int \underline{a}dt = \int \begin{pmatrix} t \\ 0.5 \\ -1.5t \end{pmatrix} dt = \begin{pmatrix} 0.5t^2 \\ 0.5t \\ -7.5t^2 \end{pmatrix} + C$

At t = 0 initial velocity = (i + 2j –k)

$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + C$ => C = $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

∴ $\underline{v} = \begin{pmatrix} 0.5t^2 + 1 \\ 0.5t + 2 \\ -7.5t^2 - 1 \end{pmatrix}$

At t = 2s

$\underline{v} = \begin{pmatrix} 0.5(2)^2 + 1 \\ 0.5(2) + 2 \\ -7.5(2)^2 - 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -4 \end{pmatrix} ms^{-1}$

(f) Displacement of the particle after 2 seconds (04marks)

$\underline{r} = \int \underline{v}dt \int \begin{pmatrix} 0.5t^2 + 1 \\ 0.5t + 2 \\ -7.5t^2 - 1 \end{pmatrix} dt = \begin{pmatrix} \frac{t^3}{6} + t \\ \frac{t^2}{4} + 2t \\ -t - \frac{t^3}{4} \end{pmatrix} + C$

At t = 0; $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + C$

C = $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\underline{r} = \begin{pmatrix} t^3 + t \\ \frac{t^2}{4} + 2t \\ -t - \frac{t^3}{4} \end{pmatrix}$$

At t = 2s

$$\underline{r} = \begin{pmatrix} \frac{2^3}{6} + 2 \\ \frac{2^2}{4} + 2x2 \\ -2 - \frac{2^3}{4} \end{pmatrix} = \begin{pmatrix} \frac{10}{3} \\ 5 \\ -4 \end{pmatrix} m$$

29. Two events A and B are such that P(B) = $\frac{1}{8}$, P(A∩B) = $\frac{1}{10}$ and P(B/A) = $\frac{1}{3}$

Determine the

(d) P(A) (03marks)

P(B/A) = $\frac{P(B\cap A)}{P(A)}$

$\frac{1}{3} = \frac{1}{10} \div P(A)$

$P(A) = 3\,x\,\frac{1}{10} = \frac{3}{10}$

(e) P(A∪ B) (03marks)

P(A∪ B) = P(A) + P(B) - P(A∩B)

$= \frac{3}{10} + \frac{1}{8} - \frac{1}{10} = \frac{13}{40}$

(f) P(A/$\bar{B}$) (06marks)

P(A/$\bar{B}$) = $\frac{P(A\cap\bar{B})}{P(\bar{B})}$

$= \frac{P(A) - P(A\cap B)}{1 - P(B)}$

$= \frac{\frac{3}{10} - \frac{1}{10}}{1 - \frac{1}{8}}$

$= \frac{8}{35}$

30. (a) Given that y = e^x and x = 0.62correct to two decimal places, find the interval within which the exact value of y lies. (05marks)

$e_x = 0.005$

$y_{max} = e^{0.625} = 1.8682$

$y_{min} = e^{0.615} = 1.8497$

The interval = (1.8497, 1.8682)

(c) Show that the maximum possible relative error in ysin^2x is

$\left|\frac{\Delta y}{y}\right| + 2\cot x\,|\Delta x|$, where$\Delta x$ and Δy are errors in x and y respectively

Hence find the percentage error in calculating ysin^2x if y = 5.2 $\pm$ 0.05 and x = $\frac{\pi}{6} \pm \frac{\pi}{360}$

(07 marks)

z = y$\sin^2$x

$e_z = \Delta y sin^2x$ + 2y$\Delta xcos\ xsinx$

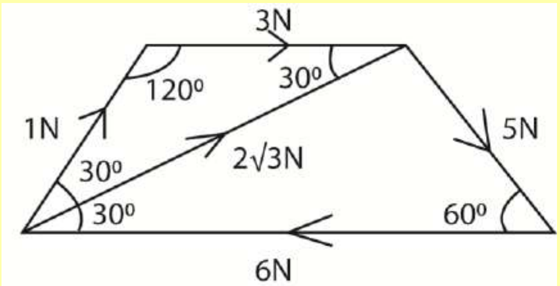
$\frac{e_z}{z} = \frac{\Delta y sin^2 x}{\mathrm{ysin^2x}} + \frac{2\mathrm{y}\Delta x cos\ xsinx}{\mathrm{ysin^2x}}$

$$\left|\frac{e_z}{z}\right| = \left|\frac{\Delta y}{y} + 2cotx.\Delta x\right|$$
$$\leq \left|\frac{\Delta y}{y}\right| + 2cotx.|\Delta x|$$

∴ Maximum possible error is $\left|\frac{\Delta y}{y}\right| + 2cotx.|\Delta x|$

$percentage\ error = \left[\frac{0.05}{5.2} + 2\cot\frac{\pi}{6}.\left|\frac{\pi}{360}\right|\right] x\ 100\%$ = 3.9845%

31. The diagram below shows a trapezium rule ABCD, AD = DC = CB = 1 and AB = 2meters. Forces of magnitude 1N, 3N, 5N, 6N and $2\sqrt{3}$ N respectively.



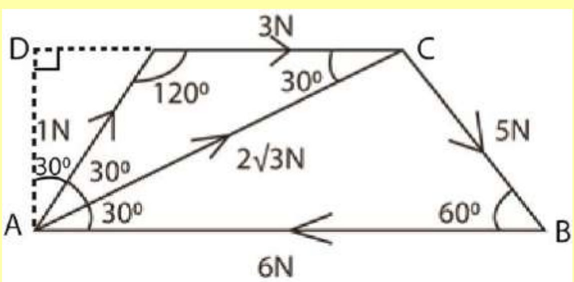
(c) Calculate the magnitude of the resultant force and the angle it makes with side AB (09marks)

$\binom{x}{y} = \binom{-6}{0} + \binom{3}{0} + \binom{2\sqrt{3}\cos 30^0}{2\sqrt{3}\sin 30^0} + \binom{\cos 60^0}{\sin 60^0} + \binom{5\cos 60^0}{-5\sin 60^0} = \binom{3}{-\sqrt{3}}$

$Resultant\ force, R = \sqrt{(3)^2 + (-\sqrt{3})^2}$ = 3.464N

$Direction, \alpha = tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$ = 30^0

(d) Given that the line of action of the resultant force meets AB at X, find AX. (03marks)



Equation of the line action of the resultant is given by G-xY + yX = 0

Taking moments about A

G= -3 x 1cos 30^0 – 5 x 2 cos 30^0

= -3 x $\frac{\sqrt{3}}{2}$ -10 x $\frac{\sqrt{3}}{2}$ = $\frac{-13\sqrt{3}}{2}$

By substitution

$\frac{-13\sqrt{3}}{2} + \sqrt{3}x + 3y = 0$

The line of action of the resultant cuts AB when y = 0

$\frac{-13\sqrt{3}}{2} + \sqrt{3}x + 3\ x\ 0 = 0$

$x = 6.5m$

Hence $\overline{AX}$= 6.5m

32. A biased die with faces labelled 1, 2, 3, 4, 5 and 6 is tossed 45 times
Calculate the probability that 2 will appear;

(c) More than 18 times (07marks)

n =45, p = $\frac{2}{6} = \frac{1}{3}$, q = $\frac{2}{3}$

μ = np = 45 x $\frac{1}{3}$ = 15

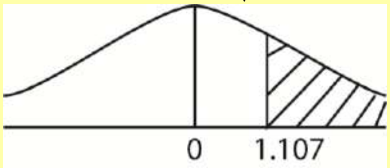
σ = $\sqrt{npq} = \sqrt{45\ x\ \frac{1}{3} x\ \frac{2}{3}} = \sqrt{10}$

Changing binomial to normal distribution.

P(X> x) = P(X> 18 + 0.5) = P(X> 18.5)

Standardizing using z= $\frac{\overline{x-\mu}}{\sigma}$

P(X> 18.5) = P(z > $\frac{18.5-15}{\sqrt{10}}$ = P(z > 1.107)



P(z > 1.107) =0.5 – P(0 < z < 1.107)
= 0.5 – 0.3658
= 0.1342

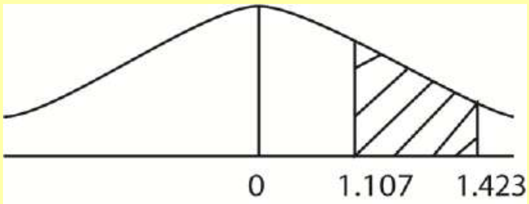
∴ P(X > 18) = 0.1342

(d) Exactly 11 times (05marks)

P(X = 11) = P(11 – 0.5 < X < 11 + 0.5)
= P(10.5 < X < 11.5)
= P$\left(\frac{10.5-15}{\sqrt{10}} < z < \frac{11.5-15}{\sqrt{10}}\right)$
= P(-1.423 < z < 1.107)

By symmetry

P(-1.423 < z < 1.107) = P(1.107 < z < 1423)



P(1.107 < z < 1423) = P(0 < z < 1.423) – P(0 < z < 1.107)
= 0.4226 – 0.3658
= 0.0568