A

Understanding Pure Mathematics

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7. Binomial theorem

Binomial theorem

Pascal's triangle

The Pascal's triangle below and its further extension is used to determine the coefficients of the expansion of $(p + q)^n$

1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Observations

- The coefficients are symmetrical, they are the same irrespective of which side they are read from, for instance the coefficients of $(p + q)^6$ are
  1 6 15 20 15 6 1
- The coefficient of the 2nd term in the expansion is the index of a given expansion. E.g. in the expansion of $(p + q)^6$, the coefficient of the 2nd term is 6.
- The number of terms in the expansion exceeds the index by one, e.g. $(p + q)^4$ with index 4, has 5 terms
- The index of the first term of the expansion decreases by one, from the index given till zero, whereas, the index of the second term increases by one from zero to the given index
  For $(p + q)^3 = 1p^3q^0 + 3p^2q^1 + 3p^1q^2 + 1p^0q^3$
  $=p^3 + 3p^2q + 3pq^2 + q^3$
- The sum of indices in each term s constant and equal to the index of the expansion.

Example 1

Use Pascal's triangle to expand

(a) $(p + q)^4$

The coefficients are 1 4 6 4 1

Terms are $1p^4 + 4p^3q^1 + 6p^2q^2 + 4p^1q^3 + 1q^4$

$= p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4$

(b) $(2x + 3y)^3$

The coefficients are 1 3 3 1

Terms are

$1(2x)^3(3y)^0 + 3(2x)^2(3y)^1 + 3(2x)^1(3y)^2 + 1(2x)^0(3y)^3$

$= 8x^3 + 36x^2y + 54xy^2 + 27y^3$

(c) $(a + b + c)^2$

The terms can be grouped into two ways:

Either a and (b + c) or (a + b) and c

The coefficients are 1 2 1

Either

Terms are: $1a^2(b + c)^0 + 2a^1(b + c)^1 + 1a^0(b + c)^2$

$= a^2 + 2a(b + c) + (b + c)^2$

$= a^2 + 2ab + 2ac + b^2 + 2bc + c^2$

Or

$(a + b + c)^2$

$= 1(a + b)^2c^0 + 2(a + b)_1c^1 + 1c^2$

$= (a + b)^2 + 2c(a + b) + c^2$

$=a^2 + 2ab + b^2 + 2ac + 2bc + c^2$

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Example 2

Expand $(a + b)^4$ using Pascal's triangle. Hence find $(1.996)^4$ correct to 3 decimal places

Solution

From Pascal's triangle the coefficients are
1 4 6 4 1

$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$(1.996)^4 = (2- 0.004)^4$

Substituting a = 2 and b = -0.004

$24 + 4(2)^3(-0.004) + 6(2)^2(-0.004)^2 + 4(2)(-0.004)^3 + (-0.004)^4$

16 – 0.128 + 0.000384 = 15.872 (3D)

Example 3

(a) Expand $(2 – 3x)^4$ using Pascal's triangle. Hence evaluate $(1.97)^4$ correct to 3 decimal places.

Solution

Coefficients: 1 4 6 4 1

$(2 – 3x)^4 = 1\ (2)^4(-3x)^0 + 4(2)^3(-3x)^1 + 6(2)^2(-3x)$

$+ 4(2)^1(-3x)^3 + 1(2)^0(-3x)^4$

$= 16 – 96x + 216x^2 - 216x^3 + 81x^4$

Now $(1.97)^4 = (2 – 0.03)^4 = (2 - 3(0.01))^4$

⇨ x = 0.01

$(1.97)^4 = 16 – 96\ (0.01) + 216(0.01)^2 -$

$216(0.01)^3 + 81(0.01)^4 = 15.0613848$

= 15.061 (3d.p)

(b) Use Pascal's triangle to evaluate $(1.02)^3$ correct 5 significant figures

Solutions

$(1.02)^3 = (1 + 02)^3$

Coefficients are : 1 3 3 1

$(1 + 02)^3 = 1^3 + 3(1)^2(0.02) + 3(1)(0.02)^2 +$

$(0.02)^3 = 1.061208$

= 1.0612 (5 significant figures)

The idea of factorial and combination can also be used to determine the coefficients of the expansions

Example 3

Expand

(a) $(p + 3q)^3$

Solution

The coefficients are ${}^3C_0\ \ {}^3C_1\ \ {}^3C_2\ \ {}^3C_3$

Terms are $p^3\ \ p^2(3q)\ \ p(3q)^2\ \ (3q)^3$

$(p + 3q)^3 = {}^3C_0p^3 + {}^3C_1p^2(3q) + {}^3C_2p(3q)^2 +$

$^3C_3(3q)^3$

$P^3 + 9p^2q + 27pq^2 + 27q^3$

(b) $(1 – x)^4$. Hence evaluate $(0.99)^4$ correct to 4 decimal places

Solution

Coefficients are ${}^4C_0\ \ {}^4C_1\ {}^4C_2\ {}^4C_3\ {}^4C_4$

Or simply $\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4}$

$1 – x)^4 = {}^4C_01^4(-x)^0 + {}^4C_11^3(-x)^1 + {}^4C_21^2(-x)^2 +$

$^4C_31^1(-x)^3 + {}^4C_41^0(-x)4^0$

$= 1 – 4x + 6x^2 – 4x^3 + x^4$

Now 0.99 = 1 – 0.01) = x = 0.01

$(0.99)^4$

$= 1 – 4(0.01) + 6(0.01)^2 – 4(0.01)^3 + (0.01)^4$

= 0.96059601

= 0.9606 (4d.p)

The binomial theorem for positive integral index

Consider the expansion of $(1 + x)^4$, the result is

$(1 + x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$

Observations

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(i) The indices of x increase by 1 from term to term

(ii) The index of the last term being the same as the power to which (1 + x) s raised

(iii) The coefficients of the terms of expansion are ${}^4C_0$ ${}^4C_1$ ${}^4C_2$ ${}^4C_3$ ${}^4C_4$

Hence the expansion of $(1 + x)^n$ is

$$(1 + x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \dots x^n$$

The expansion of $(a + x)^n$ is

$$(a + x)^n = a^n\left(1 + \frac{x}{a}\right)^n$$

$$= a^n\left[1 + \binom{n}{1}\left(\frac{x}{a}\right) + \binom{n}{2}\left(\frac{x}{a}\right)^2 + \cdots x\left(\frac{x}{a}\right)^n\right]$$

$$= a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + \cdots x^n$$

Example 4

Expand $(1 + 4x)^{14}$ in ascending power of x up to and include the 4th term. Hence evaluate $(1.0004)^{14}$ correct to four decimal places

Solution

$(1 + 4x)^{14}$

$= 1 + \binom{14}{1}(4x) + \binom{14}{2}(4x)^2 + \binom{14}{3}(4x)^3$

$= 1 + 56x + 1456x^2 + 23296x^3$

Now $(1.0004)^{14} = (1 + 0.0004)^{14}$ => x = 0.0001

$(1 + 4x)^{14}$

$= 1 + 56(0.0001) + 1456(0.0001)^2 +$

$23296(0.0001)^3$

= 1.005614583

= 1.0056 (4d.p)

Note the next term in the expansion is

$\binom{14}{4}(4x)^4 = 256256x^4$

Its value = $256256(0.0001)^4 = 2.56256 \times 10^{-11}$

Which when added to the above answer there will negligible change in value

Example 5

Expand $(3 - 2x)^{12}$ in ascending powers of x up to and including the term x^3. Hence evaluate $(2.998)^{12}$ correct to the nearest whole number.

Solution

$$(3 - 2x)^{12} = 3^{12} + \binom{12}{1}(3)^{11}(-2x)^1 + \binom{12}{2}(3)^{10}(-2x)^2 + \dots$$

$= 531441 - 4251528x + 1588936x^2$

Now $(2.998)^{12} = (3 - 0.002)^{12}$ => x = 0.001

$(2.998)^{12} = 531441 - 4251528(0.001) +$

$1588936(0.001)^2$

= 527[illegible]06[illegible]

[illegible]272[illegible] (nearest whole number)

Particular terms of binomial expansion

As earlier seen, the expansion of

$(a + x)^5 = a^5x^0 + 5a^4x^1 + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5$

In general if r is the power of x in the expansion $(a + x)^n$, then

the U_{r+1} term of x = ${}^nC_r a^{n-r}x^r$

Example 6

Find the term of x^4 in the expression of

(i) $(1 + x)^9$

By using U_{n+1}; term of x = ${}^nC_r a^{n-r}x^r$

n = 9, r = 4, a = 1

$U_5 = {}^9C_4 1^{9-4}x^4 = {}^9C_4x^4 = 126x^4$

(ii) (3 + x)7

n = 7, a = 3, r = 4

$U_5 = {}^7C_4 3^{7-4}x^4 = {}^7C_4(3)^3x^4 = 945x^4$

(iii) $\left(2 - \frac{x}{2}\right)^{12}$

$n = 12, a = 2$

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$U_5 = {}^{12}C_4 2^{12-4}\left(-\frac{x}{2}\right)^4 = {}^{12}C_4(2)^8\left(-\frac{x}{2}\right)^4 = 7920x^4$

Example 7

Find the term indicated in expansion of the following expression

(i) $\left(3x - \frac{2}{x}\right)^5$ $[x^3]$

$$\left(3x - \frac{2}{x}\right)^5 = 3^5x^5\left(1 - \frac{2}{3x^2}\right)^5$$

The term in $x^3 = 3^5x^5.{}^5C_1\left[1^4\left(-\frac{2}{3x^2}\right)^1\right]$

$= 3^5x^5.{}^5C_1\left(-\frac{2}{3x^2}\right) = -810x^3$

(ii) $\left(2x + \frac{5}{x}\right)^6$ $[x^4]$

$$\left(2x + \frac{5}{x}\right)^5 = 2^6x^6\left(1 + \frac{5}{2x^2}\right)^6$$

The term in $x^3 = 2^6x^6.{}^6C_1\left[1^5\left(\frac{5}{2x^2}\right)^1\right]$

$= 2^6x^6.{}^6C_1\left(\frac{5}{2x^2}\right)\ 960x^4$

Finding terms independent of x

The term is said to be independent of x when the power of x is zero

Example 8

Find the term independent of x in the following expansion

(a) $\left(2x + \frac{1}{x^2}\right)^{12}$

Solution

$$\left(2x + \frac{1}{x^2}\right)^{12} = 2^{12}x^{12}\left(1 + \frac{1}{2x^3}\right)^{12}$$

The term independent of $x = 2^{12}x^{12}$ multiplied by the term in x^{-12} in the expansion $\left(2x + \frac{1}{x^2}\right)^{12}$

⇨ The term independent of x

$= 2^{12}x^{12}.{}^{12}C_4\left(\frac{1}{2x^3}\right)^4 = 2^{12}x^{12} \times {}^{12}C_4 \times \frac{1}{2^4x^{12}}^4$

$=2^8 \times 495 = 126720$

Alternatively

By using $U_{r+1} = {}^nC_r a^{n-r}x^r$

The term independent of x is got by equating the index of x to zero

$U_{r+1} = {}^{12}C_r(2x)^{12-r}\left(\frac{1}{x^2}\right)^r$

$= {}^{12}C_r(2x)^{12-r}x^{-2r}$

$= {}^{12}C_r.\ 2^{12-r}.x^{12-3r}$

Equating the $x^{12-3r} = 1$

⇨ $12 - 3r = 0$
$r = 4$

Term independent of $x = {}^{12}C_4.\ 2^{12-4}$

$= {}^{12}C_4.\ 2^8$

$= 126720$

(b) $\left(2x + \frac{3}{x^2}\right)^{12}$

Solution

$$\left(2x + \frac{3}{x}\right)^{12} = 2^{12}x^{12}\left(1 + \frac{3}{2x^2}\right)^{12}$$

$U_{n+r} = {}^{12}C_r(2x^2)^{12-r}(3x^{-1})^r$

$= {}^{12}C_r(2x^2)^{12-r}(3^r)x^{-r}$

$= {}^{12}C_r.\ 2^{12-r}(3^r)x^{24-3r}$

Equating the $x^{12-3r} = 1$

⇨ $24 - 3r = 0$
$r = 8$

Term independent of $x = {}^{12}C_8.\ 2^{12-4}.3^8$

$= {}^{12}C_8.\ 2^4.3^8$

$= 51,963,120$

Binomial expansion of terms with fractional or negative powers

As noted earlier

$(1 + x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots x^n$

The following is noted

(i) For positive integral value of n, i.e. $n \geq 1$, the series above terminates at the term x^n and its sum is $(1 + x)^n$.

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(ii) For fractional or negative values of n, the series above does not terminate but instead converges to $(1 + x)^n$ as the limit of its sum only $-1 < x < 1$ or $|x| < 1$

Example 9

Expand $\sqrt{(1-2x)}$ up to the term x^3. Hence evaluate $\sqrt{0.98}$ correct to four decimal places.

Solution

$$\sqrt{(1-2x)} = (1-2x)^{\frac{1}{2}}$$

Using

$$(1 + x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots x^n$$

Comparing with $(1-2x)^{\frac{1}{2}}$ => x ≡ -2x and n = ½

$$\sqrt{(1-2x)} = 1 + \frac{1}{2}(-2x) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)(-2x)^2}{2!} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-2x)^3}{3!}$$

$$= 1 - x - \frac{1}{2}x^2 - \frac{1}{2}x^3$$

Now $\sqrt{0.98} = \sqrt{(1-0.02)}$

Comparing with $\sqrt{(1-2x)}$; x = 0.0[illegible]

Substituting

$$\sqrt{0.98} = 1 - (0.01) - \frac{1}{2}(0.01)^2 - \frac{1}{2}(0.01)^3$$

$$= 0.9899495$$

$\therefore \sqrt{0.98} = 0.9899$ (4d.p)

Example 10

Given that x is very small that its cube and higher powers can be neglected, show that

$$\sqrt{\left(\frac{1-x}{1+x}\right)} = 1 - x + \frac{1}{2}x^2$$

By putting x = $\frac{1}{8}$ show that $\sqrt{7} = \frac{339}{128}$

Solution

By rationalizing the denominator

$$\sqrt{\left(\frac{1-x}{1+x}\right)\left(\frac{1-x}{1-x}\right)} = \sqrt{\frac{(1-x)^2}{(1-x^2)}}$$

$$= (1-x)(1-x^2)^{\frac{1}{2}}$$

But $(1-x^2)^{\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)(-x^2) + \ldots..$

$$= 1 + \frac{1}{2}x^2$$

$$\sqrt{\left(\frac{1-x}{1+x}\right)} = (1-x)\left(1 + \frac{1}{2}x^2\right)$$

$$= 1 - x + \frac{1}{2}x^2$$

Putting x = $\frac{1}{8}$

$$\sqrt{\left(\frac{1-\frac{1}{8}}{1+\frac{1}{8}}\right)} = 1 - \frac{1}{8} + \frac{1}{2}\left(\frac{1}{8}\right)$$

$$\sqrt{\frac{7}{9}} = \frac{\sqrt{7}}{3} = 1 - \frac{1}{8} + \frac{1}{128} = \frac{113}{128}$$

$$\sqrt{7} = \frac{339}{128}$$

Example 11

Expand $\sqrt{\left(\frac{1+x}{1-x}\right)}$ up to the term x^3

Use your expansion to evaluate $\sqrt{23}$ correct to 3 decimal places taking x = $\frac{1}{24}$.

Solution

By rationalizing the denominator

$$\sqrt{\left(\frac{1+x}{1-x}\right)\left(\frac{1+x}{1+x}\right)} = (1+x)(1-x^2)^{\frac{1}{2}}$$

But $(1-x^2)^{\frac{1}{2}} = 1 + \frac{1}{2}x^2 + \cdots$

$$\sqrt{\left(\frac{1+x}{1-x}\right)} = (1+x)\left(1 + \frac{1}{2}x^2\right)$$

$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{2}x^3$$

Hence putting x = $\frac{1}{24}$

$$\sqrt{\left(\frac{1+\frac{1}{24}}{1-\frac{1}{24}}\right)} = 1 + \frac{1}{24} + \frac{1}{2}\left(\frac{1}{24}\right)^2 + \frac{1}{2}\left(\frac{1}{24}\right)^3$$

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$\sqrt{\frac{25}{23}} = \frac{5}{\sqrt{23}} = \frac{28825}{27648}$

$\sqrt{23} = \frac{5\ x\ 27648}{28825} = 4.7958$

$\therefore \sqrt{23} = 4.7958$ (4d.p)

Example 12

Use the binomial theorem to expand $\sqrt[3]{(1-x)}$ up to x^3. Use your expansion to evaluate $\sqrt[3]{7}$ correct to four decimal places

Solution

$\sqrt[3]{(1-x)} = (1-x)^{\frac{1}{3}}$

$= 1 + \frac{1}{3}(-x) + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)(-x)^2}{2!} + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)(-x)^3}{3!} + \ldots$

$= 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{1}{81}x^3$

Hence $\sqrt[3]{7} = \sqrt[3]{(8-1)} = \sqrt[3]{8\left(1-\frac{1}{8}\right)}$

$= 2\left(1-\frac{1}{8}\right)^{\frac{1}{3}}$ => x = $\frac{1}{8}$

$\sqrt[3]{7} = 2\left[1 + \frac{1}{3}\left(\frac{1}{8}\right) - \frac{1}{9}\left(\frac{1}{8}\right)^2 + \frac{1}{81}\left(\frac{1}{8}\right)^3\right]$

=1.9130(4d.p)

Example 13

Write down the expansion of $\sqrt{(1-x)}$ in ascending powers of x as far as the term x^4. Use your expansion to find $\sqrt{80}$ correct to four significant figures.

Solution

$\sqrt{(1-x)} = (1-x)^{\frac{1}{2}}$

For binomial expansion

$(1 + x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots. x^n$

Comparing $(1-x)^{\frac{1}{2}}$ with$(1 + x)^n$

n = $\frac{1}{2}$ and x = -x

$(1-x)^{\frac{1}{2}} = 1 + \frac{1}{2}(-x) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)(-x)^2}{2!} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-x)^3}{3!} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(-x)^4}{4!}$

$(1-x)^{\frac{1}{2}} = 1 - \frac{1}{2}x - \frac{x^2}{8} - \frac{x^3}{6} - \frac{x^4}{24}$

Now $\sqrt{80} = \sqrt{81-1} = \sqrt{81\left(1-\frac{1}{81}\right)}$

$= 9\left(1-\frac{1}{81}\right)^{\frac{1}{2}}$

Comparing $\left(1-\frac{1}{81}\right)^{\frac{1}{2}}$ with $(1-x)^{\frac{1}{2}}$

⇨ x = $\frac{1}{81}$

Substituting for

$\left(1 - \frac{1}{81}\right)^{\frac{1}{2}}$

$= 1 - \frac{1}{2}\left(\frac{1}{81}\right) - \frac{\left(\frac{1}{81}\right)^2}{8} - \frac{\left(\frac{1}{81}\right)^3}{6} - \frac{\left(\frac{1}{81}\right)^4}{24}$

$= \frac{52163}{52488}$

$\sqrt{80} = 8.944$ (4 S.F)

Not the termx3 has been neglected as it does not affect the answer to 4 significant figured

Example 14

John operates an account with a certain bank which pays a compound interest rate of 13.5% per annum. He opened the account at the beginning of the year with sh. 500,000 and deposits the same amount of money at the beginning of every year. Calculate how much he will accumulate at the end of 9 years. After how long will the money have accumulated to sh. 3.32 millions?

Solution

The 1st deposit will grow to

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$500000\left(1+\frac{13.5}{100}\right)$= 500000 x 1.135

2nd deposit will grow to 500000 x 1.135^2

nth deposit will grow to 500000 x 1.135^n

9th deposit will grow to 500000 x 1.135^9

The total = 500,000[1.135 +1.135^2 + … + 1.135^9]

$=500000\left[a\left(\frac{r^n-1}{r-1}\right)\right]$

$=500000\ \left[1.135\left(\frac{1.135^9-1}{1.135-1}\right)\right]$

=8,936,381

Finding how long it will take the money to accumulate to sh. 3,320,000

$500000\ \left[1.135\left(\frac{1.135^n-1}{1.135-1}\right)\right]=3320000$

n = 4.6years

Example 15

Expand $(1 + x)^{-2}$ in descending powers of x including the term x^{-4}. If x = 9 find the percentage error in using the first two terms of the expression.

Solution

From

$(1+x)^n = 1+nx+\frac{n(n-1)x^2}{2!}+\cdots+x^n$

Now $(1 + x)^{-2} = x^{-2}\left(1+\frac{1}{x}\right)^{-2}$

$x^{-2}\left(1+\frac{1}{x}\right)^{-2}$

$=x^{-2}\left[1+(-2)\frac{1}{x}+\frac{(-2)(-3)}{2!}\left(\frac{1}{x}\right)^2\right]$

$=x^{-2}\left[1-\frac{2}{x}+\frac{3}{x^2}\right]$

$=x^{-2}[1-2x^{-1}+3x^{-2}]$

$=x^{-2}-2x^{-3}+3x^{-4}$

If x = 9

$(1 + x)^{-2}=9^{-2}-2(9)^{-3}+3(9)^{-4}$

$=\frac{1}{81}-\frac{2}{729}$ (using the first 2 terms)

$=\frac{7}{729}$

The exact value is $(1+9)^{-2}=\frac{1}{100}$

Error = $\frac{1}{100}-\frac{7}{729}=\frac{29}{72900}$

% error = $\frac{29}{72900}\ x100\ x100$= 3.978%(3d.p)

Example 16

(a) Find the three terms of the expansion $(2 - x)^6$ and use it to find $(1.998)^6$ correct to two decimal places (06 marks)

$(2-x)^6 = 2^6+\binom{6}{1}2^5(-x)^1+\binom{6}{2}4(-x)^2$

$= 64 -192x + 240x^2$

$(1.998)^2=(2-0.002)^2$

$=64-192(0.002)+240(0.002)^2$
$=64-0.384+0.00096$
$=63.61696$
$=63.62\ (2D)$

Example 17

(a) Prove by induction

$1.3 + 2.4 + \ldots + n(n + 2) = \frac{1}{6}n(n+1)(2n+7)$ for all values of n.

Suppose n = 1

L.H.S = 1 x 3 = 3

R.H.S = $\frac{1}{6}x1(1+1)(2+7)=3$

L.H.S = R.H.S, hence the series holds for n =1

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Suppose n = 2

L.H.S= 1 x 3 + 2 x 4 = 11

R.H.S = $\frac{1}{6}\ x\ 2(2+1)(4+7) = 11$

L.H.S = R.H.S, hence the series holds for n =2

Suppose n = k

1.3 + 2.4 + … + k(k + 2) = $\frac{1}{6}k(k+1)(2k+7)$

For n = k + 1

1.3 + 2.4 + … + k(k + 2), (k + 1)(k + 3) = $\frac{1}{6}k(k+1)(2k+7)$+(k +1)(k + 3)

=(k + 1)$\left[\frac{1}{6}k(2k+7)+(k+3\right]$

= $\frac{1}{6}(k+1)(2k^2+13k+18)$

= $\frac{1}{6}(k+1)(2k^2+4k+9k+18)$

= $\frac{1}{6}(k+1(k+2)(2k+9)$

= $\frac{1}{6}(k+1)(k+2)[2(k+1)+7]$

Which is equal to R.H.S when n = k + 1

It holds for n = 1, 2, 3 …, hence it holds for all integral values of n

(b) A man deposits Shs. 150,000 at the beginning of every year in a micro finance bank with the understanding that at the end of the seventh years he is paid back his money with 5% per annum compound interest. How much does he receive?

Using amount, A = P$\left(1+\frac{r}{100}\right)^n$

=150000$\left(1+\frac{5}{100}\right)^7$=2[illegible]005.06

Alternatively

1st year

P = 150,000

He is paid back principal plus interest; P$\left(1+\frac{5}{100}\right)$ = 150,000$\left(1+\frac{5}{100}\right)$=157,500

2nd year

P = 157, 500

He is paid back principal plus interest; P$\left(1+\frac{5}{100}\right)$ = 157500$\left(1+\frac{5}{100}\right)$=165375

3rd year

P = 165375

Interest = $\frac{5}{10}\ x\ 165375 = 8268.75$

He is paid back principal plus interest; P$\left(1+\frac{5}{100}\right)$ = 165375 $\left(1+\frac{5}{100}\right)$=173643.75

4th year

P=173643.75

He is paid back principal plus interest; P$\left(1+\frac{5}{100}\right)$ = 173643.75$\left(1+\frac{5}{100}\right)$=182325.94

5th year

P =182325.94

His paid back principal plus interest; P$\left(1+\frac{5}{100}\right)$ = 182325.94$\left(1+\frac{5}{100}\right)$=191442.23

6th year

P = 191442.23

He is paid back principal plus interest; P$\left(1+\frac{5}{100}\right)$ = 191442.23$\left(1+\frac{5}{100}\right)$=201014.35

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7th year

P = 201014.35

He is paid back principal plus interest; $P\left(1 + \frac{5}{100}\right) = 201014.35\left(1 + \frac{5}{100}\right) = 211{,}065.06$

∴ by the 7th year he has accumulated shs. **211,065.06**

Example 18

Expand $\sqrt{\left(\frac{1+2x}{1-x}\right)}$ up to the term x^2. Hence find the value of $\sqrt{\left(\frac{1.04}{0.98}\right)}$ to four significant figures. (12marks)

$$\sqrt{\left(\frac{1+2x}{1-x}\right)} = (1+2x)^{\frac{1}{2}}(1-x)^{\frac{-1}{2}}$$

$$using\ (1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \cdots$$

$$\sqrt{\left(\frac{1+2x}{1-x}\right)} = \left(1 + x - \frac{1}{2}x^2\right)\left(1 + \frac{1}{2}x + \frac{3}{8}x^2\right)$$

$$= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + x + \frac{1}{2}x^2 - \frac{1}{2}x^2$$

$$= 1 + \frac{3}{2}x + \frac{3}{8}x^2$$

$$\therefore \sqrt{\left(\frac{1+2x}{1-x}\right)} \approx 1 + \frac{3}{2}x + \frac{3}{8}x^2$$

$$substituting\ for\ x = 0.02$$

$$\sqrt{\left(\frac{1.04}{0.98}\right)} = \sqrt{\frac{1+2(0.02)}{1-0.02}}$$

$$= 1 + \frac{3}{2}(0.02) + \frac{3}{8}(0.02)^2 = [illegible]$$

Revision exercise

1. Given that the ratio of the 3rd [illegible] the 4[illegible] term of the expansion (2+[illegible])n [illegible] 5:14, find the value of n when x [illegible] 5 [n= 16]
2. Expand $(3 - 4x)^5$ in ascending order of x up to and including the term x^3. Hence evaluate $(4.96)^5$ correct to 2 d.p. [3001.98]
3. (a) Find the coefficient of x2 in the expansion of $(1 - 2x)^n$ is 24 [4]
   (b) Find the term independent of x in the expansion of $\left(2x^3 - \frac{1}{x}\right)^{20}$ [-496128]
   (c) Use the binomial expansion to expand $\sqrt[4]{(1+2x)}$ up to the term x^3. Hence evaluate $\sqrt[4]{83}$ correct to three decimal places [3.018]
4. Expand $\sqrt{\left(\frac{1+2x}{1-2x}\right)}$ up to and including the term x3. Hence find the value of $\sqrt{\frac{1.02}{0.98}}$ to four significant figures. Deduce the value of $\sqrt{51}$ to 3 significant figures [1.0202, 7.14]
5. Five millions shillings are invested each year at a rate of 15%. In how many years will it accumulate to more than 50millions? [6years]
6. Expand $\left(1 - \frac{x}{3}\right)^{\frac{1}{2}}$ as far as x3. Hence evaluate $\sqrt{8}$ correct to 3 decimal places [2.829]
7. A man deposits sh. 800,000 into his saving account on which interest is 15% per annum. If he makes no withdraws, after how many years will his balance exceed sh. 8millions? [16.5 years]
8. Determine the binomial expansion of $\left(1 + \frac{x}{2}\right)^4$. Hence evaluate $(2.1)^4$ correct to 2 decimal places.[19.45]
9. Determine the binomial expansion of $\left(1 - \frac{x}{2}\right)^5$. Hence evaluate $(0.875)^5$ correct to four decimal places [0.5129]

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10. A financial credit society give a compound interest of 2% per annum to its members. If Bbosa deposits sh. 10000 at the beginning of every year. How much would he accumulate at the end of the fifth year if no withdraws within this period [sh. 530812]

11. Expand $\sqrt{\left(\frac{1+x}{1-x}\right)}$ in ascending powers of x up to a term x^2. $\left[1 + x + \frac{x^2}{x} + \cdots\right]$

12. (a) Using the expansion $(1 + x)^{\frac{1}{2}}$ up to the term x^3, find the value of $\sqrt{1.08}$ to four decimal places [1.0392]
(b) Express $\sqrt{1.08}$ in the form $\frac{a}{b}\sqrt{c}$. Hence evaluate $\sqrt{3}$ correct to 3 significant figure $\left[\frac{3}{5}\sqrt{3}, 1.73\right]$

13. (a) obtain the first four non – zero terms of the binomial expansion in ascending powers of x of $(1 - x^2)^{-\frac{1}{2}}$ give that$|x| < 1$ $\left[1 + \frac{x^2}{2} + \frac{3x^4}{8} + \frac{5x^6}{16}\right]$
(b) show that, when x = $\frac{1}{3}$;
$(1 - x^2)^{-\frac{1}{2}} = \frac{3\sqrt{2}}{4}$
(c) substitute x = $\frac{1}{3}$ into your expansion and hence obtain an approximation of $\sqrt{2}$, give your answer to five decimal places [1.41415]

14. (a) show that $\frac{1}{\sqrt{4-x}} = \frac{1}{2}\left(1 - \frac{x}{4}\right)^{-\frac{1}{2}}$
(c) Write the first three terms in the binomial expansion of $\left(1 - \frac{x}{4}\right)^{-\frac{1}{2}}$ in ascending power of x stating the range for which this expansion is valid. $\left[1 + \frac{3}{8}x + \frac{3x^2}{[illegible]}; |x| < 4\right]$
(d) Find the first three terms in the expansion of $\frac{2(1+x)}{\sqrt{4-x}}$ in ascending powers of x for small values of x $\left[1 + \frac{9x}{8} + \frac{19x^2}{128}; \right]$

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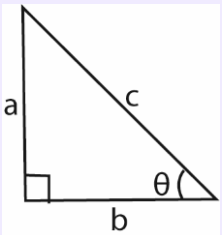
8. Trigonometry

Trigonometry

The word 'trigonometry' suggests 'tri'-three, 'gono'-angle, 'metry'-measurement. As such, trigonometry is basically about triangles, most especially right-angled triangles.

Important to note

(a) For a right angled triangle below



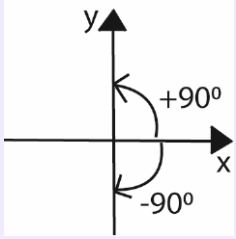
- $\sin\theta = \frac{a}{c}$
- $\cos\theta = \frac{b}{c}$
- $\tan\theta = \frac{\sin\theta}{\cos\theta}\ \frac{a}{b}$

$\text{cosec}\theta = \frac{1}{\sin\theta} = \frac{c}{a}$

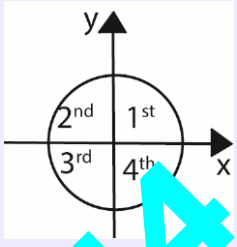
$\sec\theta = \frac{1}{\cos\theta} = \frac{c}{b}$

$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{b}{a}$

(b) All positive angles are measured anticlockwise from positive x-axis

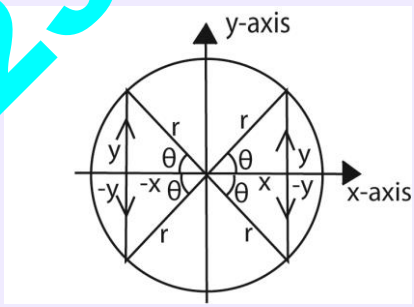


(c) A circle drawn with the centre O, divides the co-ordinate axis into four equal parts called quadrants



The quadrants are also labelled anti-clockwise from the positive x-axis.

The sign of the trigonometric ratios in the quadrants are given below



Ratio	Quadrant			
	1st	2nd	3rd	4th
cosθ	$\frac{+x}{r}$	$\frac{-x}{r}$	$\frac{-x}{r}$	$\frac{+x}{r}$
sinθ	$\frac{+y}{r}$	$\frac{+y}{r}$	$\frac{-y}{r}$	$\frac{-y}{r}$
tanθ	$\frac{+y}{x}$	$\frac{y}{-x}$	$\frac{y}{x}$	$\frac{-y}{x}$
secθ	$\frac{+r}{x}$	$\frac{-r}{x}$	$\frac{-r}{x}$	$\frac{+r}{x}$
cosecθ	$\frac{+r}{y}$	$\frac{+r}{y}$	$\frac{-r}{y}$	$\frac{-r}{y}$
cotθ	$\frac{+x}{y}$	$\frac{-x}{y}$	$\frac{+x}{y}$	$\frac{-x}{y}$

Note

- If θ is the angle in the 1st quadrat
- In the 2nd quadrat the angle is (180 – θ)
- In the 3rd quadrat the angle is (180 + θ)
- In the 4th quadrat the angle is (360 –θ)

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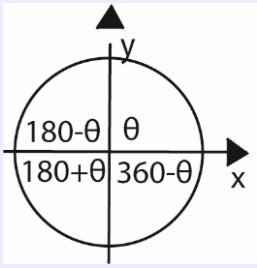

Solving equations

We make use of the quadrants to find the ranges of values within which the angle follows

Example 1

Solve the following equations for $0^0 \le \theta \le 360^0$

(i) $3\cos\theta + 2 = 0$

Solution

$\cos\theta = -\frac{2}{3}$

But cos is negative in the 2nd and 3rd quadrants.

Ignoring the negative sign, the angle obtained is referred to as the key or principle angle, i.e.

key angle = $\cos^{-1}\frac{2}{3} = 48.2^0$ (1d.p)

In the 2nd quadrant, $\theta = 180 - 48.2 = 131.8^0$

In the 3rd quadrant, $\theta = 180 + 48.2 = 228.2$

$\therefore\{\theta: \theta = 131.8^0, 228.2^0\}$

Note that: the key angle is not part of the solution but only a guide.

(ii) $4\cos^2\theta - 1 = 0$

Solution

$\cos\theta = \sqrt{\frac{1}{4}} = \pm\frac{1}{2}$

Key angle, $\theta = \cos^{-1}\frac{1}{2} = 60^0$

When $\cos\theta = ½$ (positive is 1st and 4th quadrants)

1st quadrant $\theta = 60^0$

4th quadrant $\theta = 360 - 60 = 300^0$

When $\cos\theta = -½$ (positive is 2nd and 3rd quadrants)

3rd quadrant $\theta = 180 - 60 = 120^0$

4th quadrant $\theta = 180 + 60 = 240^0$

$\therefore\{\theta: \theta = 60^0, 120^0, 240^0, 300^0\}$

(iii) $\operatorname{cosec}\theta + 2 = 0$

Solution

$\operatorname{cosec}\theta = -2 \Rightarrow \sin\theta = -½$ (taking reciprocal)

Key angle = $\sin^{-1}\frac{1}{2} = 30^0$

In the 3rd quadrant $\theta = 180 + 30 = 210^0$

In the 4th quadrant $\theta = 360 - 30 = 330^0$

$\therefore\{\theta: \theta = 210^0, 330^0\}$

(iv) $3\sec^2\theta - 4 = 0$

Solution

$\sec\theta = \frac{2}{\sqrt{3}} \Rightarrow \cos\theta = \pm\frac{\sqrt{3}}{2}$

Key angle = $\cos^{-1}\frac{\sqrt{3}}{2} = 30^0$

For $\cos\theta = \frac{\sqrt{3}}{2}$; $\theta = 30^0, 330^0$

For $\cos\theta = -\frac{\sqrt{3}}{2}$; $\theta = 120^0, 210^0$

$\therefore\{\theta: \theta = 30^0, 120^0, 210^0, 330^0\}$

(d) Definitions of angle

(i) **Acute angle** is an angle between 0^0 and 90^0. It lies in the 1st quadrant

(ii) **Right angle** is an angle = 90^0

(iii) **Obtuse angle** is an angle between 90^0 and 180^0. It lies in the 2nd quadrant

(iv) **Reflex angle** is an angle between 180^0 and 360^0. It lies in the 3rd and 4th quadrant

Example 2

(a) If $\sin\theta = \frac{3}{5}$ and $0^0 \le \theta \le 360^0$. Find the possible values of $3\tan\theta - \cot\theta$

Solution

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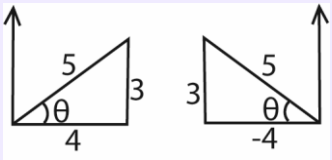
If $\sin\theta = \frac{3}{5}$; θ lies in 1st or 2nd quadrants



In 1st quadrant

$3\tan\theta - \cot\theta = 3\left(\frac{3}{4}\right) - \left(\frac{4}{3}\right) = \frac{11}{12}$

In 2nd quadrant

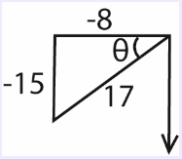
$3\tan\theta - \cot\theta = 3\left(-\frac{3}{4}\right) - \left(-\frac{4}{3}\right) = -\frac{11}{12}$

∴ the possible values are $\pm\frac{11}{12}$

(b) If $\cos\theta = -\frac{8}{17}$ and θ is reflex, find the value of $4\sec^2\theta + \tan\theta$

Solution

If $\cos\theta = -\frac{8}{17}$ and θ is reflex, θ lies in the 3rd quadrant



$4\sec^2\theta + \tan\theta = 4\left(-\frac{17}{8}\right)^2 + \frac{15}{8} = \frac{19}{}$

Example 3

Solve for θ, where $0^0 \le \theta \le 360^0$

(i) $3\tan^2 3\theta = 1$

Solution

$\tan 3\theta = \pm\frac{1}{\sqrt{3}}$

taking $\tan 3\theta = \frac{1}{\sqrt{3}}$

⇨ $3\theta = 30^0, 210^0, 390^0, 570^0, 750^0, 930^0$
$\theta = 10^0, 70^0, 130^0, 190^0, 250^0, 310^0$

taking $\tan 3\theta = -\frac{1}{\sqrt{3}}$

⇨ $3\theta = 150^0, 330^0, 510^0, 690^0, 870^0, 1050^0$
$\theta = 50^0, 110^0, 170^0, 230^0, 290^0, 350^0$

∴{θ: $\theta = 10^0, 50^0, 70^0, 110^0, 130^0, 170^0, 190^0, 230^0, 250^0, 290^0, 310^0, 350^0$}

Note

- If $0^0 \le \theta \le 360^0$ then $0^0 \le 3\theta \le 1080^0$ [multiply the interval through by 3]

(ii) $2\cos 2\theta + \sqrt{3} = 0$

Solution

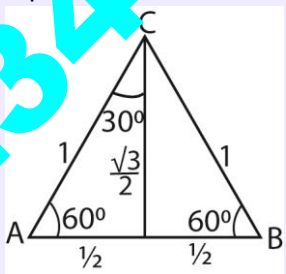
$\cos 2\theta = -\frac{\sqrt{3}}{2}$ and $0^0 \le 2\theta \le 720^0$

$2\theta = 150^0, 210^0, 510^0, 570^0$

∴{θ: $\theta = 75^0, 105^0, 2[illegible]^0, 285^0$}

Set square angles: $3[illegible]^0$, [illegible], and 60^0

(i) From equi[illegible] triangle ABC with side equ[illegible] 1 unit



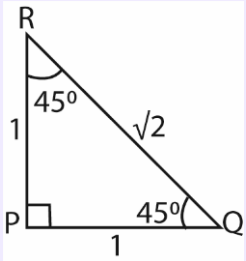
$\cos 60^0 = \sin 30^0 = \frac{1}{2}$

$\cos 30^0 = \sin 60^0 = \frac{\sqrt{3}}{2}$

$\tan 30^0 = \cot 60^0 = \frac{1}{\sqrt{3}}$

$\tan 60^0 = \cot 30^0 = \sqrt{3}$

(ii) From the right angled triangle PQR below



$\cos 45^0 = \sin 45^0 = \frac{1}{\sqrt{2}}$

$\tan 45^0 = 1$

Example 4

Without using tables or calculators find the value of

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(i) $\cos 240^0$

Solution

$\cos 240^0 = -\cos(240 - 180)^0 = -\cos 60^0 = -\frac{1}{2}$

(ii) $\tan 3990^0$

Solution

$\tan 3990^0 = \tan[(360 \times 11) + 30]^0 = \tan 30^0 = \frac{1}{\sqrt{3}}$

(iii) $\sin 570^0$

Solution

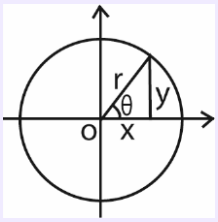
$\sin 570^0 = \sin\{(360 \times 1) + 210]^0 = -\sin 30 = -\frac{1}{2}$

(iv) $\sec 225^0$

Solution

$\sec 225^0 = \sec(225 - 180)0 = \sec 45^0 = -\sqrt{2}$

The Pythagoras theorem



For any acute angle θ

$x = r\cos\theta$ and $y = r\sin\theta$

By Pythagoras theorem

$x^2 + y^2 = r^2$

Substituting for x and y

$(r\cos\theta)^2 + (r\sin\theta)^2 = r^2$

$r^2(\cos^2\theta + \sin^2\theta) = r^2$

$\therefore \cos^2\theta + \sin^2\theta = 1$

Now $\tan\theta = \frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \frac{\sin\theta}{\cos\theta}$

$\therefore \frac{\sin\theta}{\cos\theta} = \tan\theta$

Identities

$\cos^2\theta + \sin^2\theta = 1$ .....(i)

Identity (i) $\div \cos^2\theta$

$1 + \tan^2\theta = \sec^2\theta$ ..... (ii)

Identiy (i) $\div \sin^2\theta$

$1 + \cot^2\theta = \text{cosec}^2\theta$ .....(iii)

Example 5

Show that

(i) $\sin^2\theta + (1 + \cos\theta)^2 = 2(1 + \cos\theta)$

Solution

$\sin^2\theta + (1 + \cos\theta)^2$

$= \sin^2\theta + 1 + 2\cos\theta + \cos^2\theta$

$= \sin^2\theta + \cos^2\theta + 1 + 2\cos\theta$

$= 1 + 1 + 2\cos\theta$ (Recall that $\sin^2\theta + \cos^2\theta = 1$)

$= 2 + 2\cos\theta = 2(1 + \cos\theta)$

$\therefore \sin^2\theta + (1 + \cos\theta)^2 = 2(1 + \cos\theta)$

(ii) $\frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{1+\sec\theta}{1+\csc\theta} = \tan\theta$

Solution

$$\frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{1+\sec\theta}{1+\text{cosec}\theta} = \frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{1+\frac{1}{\cos\theta}}{1+\frac{1}{\sin\theta}}$$

$$= \frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{\frac{\cos\theta+1}{\cos\theta}}{\frac{\sin\theta+1}{\sin\theta}}$$

$$= \frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{\cos\theta+1}{\cos\theta} \div \frac{\sin\theta+1}{\sin\theta}$$

$$= \frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{\cos\theta+1}{\cos\theta} x \frac{\sin\theta}{\sin\theta+1}$$

$$= \frac{\sin\theta}{\cos\theta} = \tan\theta$$

$\therefore \frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{1+\sec\theta}{1+\csc\theta} = \tan\theta$

(iii) $(\tan\theta + \sec\theta)^2 = \frac{1+\sin\theta}{1-\sin\theta}$

Solution

$$(\tan\theta + \sec\theta)^2 = \left(\frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta}\right)^2 = \left(\frac{\sin\theta+1}{\cos\theta}\right)^2$$

$$= \frac{(1+\sin\theta)^2}{\cos^2\theta} = \frac{(1+\sin\theta)^2}{1-\sin^2\theta}$$

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$= \frac{(1+sin\theta)(1+sin\theta)}{(1+sin\theta)(1-sin\theta)} = \frac{1+sin\theta}{1-sin\theta}$

$\therefore(\tan\theta + \sec\theta)^2 = \frac{1+sin\theta}{1-sin\theta}$

Example 6

Solve the following equations for $-180^0 \le x \le 180^0$

(i) $2\cos^2\theta + \sin\theta - 1 = 0$

Solution

$2(1 - \sin^2\theta) + \sin\theta - 1 = 0$

$2\sin2\theta - \sin\theta - 1 = 0$

$(\sin\theta - 1)(2\sin\theta + 1) = 0$

Either $\sin\theta = 1$ or $\sin\theta = -\frac{1}{2}$

When $\sin\theta = 1$; $\theta = 90^0$

When $\sin\theta = -\frac{1}{2}$; $\theta = -150^0, -30^0, 210^0, 330^0$

[θ: $\theta = -150^0, -30^0, 90^0$ for given range]

(ii) $\cos\theta + \sqrt{3}\sin\theta = 1$

Solution

1st approach

$\sqrt{3}\sin\theta = 1 - \cos\theta$

Squaring both sides

$3\sin^2\theta = 1 - 2\cos\theta + \cos$

$3(1 - \cos^2\theta) = 1 - 2\cos \quad \cos^2\theta$

$4\cos^2\theta - 2\cos\theta - 1 = 0$

$(2\cos\theta + 1)(\cos\theta - 1) = 0$

$\cos\theta = -\frac{1}{2}$	$\cos\theta = 1$
$\theta = \pm 120^0$	$\theta = 0^0$

$\therefore[\theta: \theta = 0^0, \pm 120^0]$

2nd approach

$\sqrt{3}\sin\theta = 1 - \cos\theta$

Dividing through by $\cos\theta$

$\sqrt{3}\tan\theta = \sec\theta - 1$

Squaring both sides

$3\tan^2\theta = \sec^2\theta - 2\sec\theta + 1$

$3\tan^2\theta = \sec^2\theta - 2\sec\theta + 1$

$3[\sec^2\theta - 1] = \sec^2\theta - 2\sec\theta + 1$

$2\sec^2\theta + 2\sec\theta - 4 = 0$

$\sec^2\theta + \sec\theta - 2 = 0$

$(\sec\theta + 2)(\sec\theta - 1) = 0$

$\sec\theta = -2$ or $\sec\theta = 1$

$\cos\theta = -\frac{1}{2}$ or $\cos\theta = 1$

$\therefore[\theta: \theta = 0^0, \pm 120^0]$

3rd approach

$\sqrt{3}\sin\theta = \quad \cos$

Dividing [illegible]h by $\sin\theta$

$\sqrt{3} = \cos \quad \operatorname{cosec}\theta - \cot\theta$

Rearranging

$\sqrt{3} + \cot\theta = \operatorname{cosec}\theta$

Squaring both sides

$3 + 2\sqrt{3}\cot\theta + \cot^2\theta = \operatorname{cosec}^2\theta$

$3 + 2\sqrt{3}\cot\theta + \cot^2\theta = 1 + \cot^2\theta$

$\cot\theta = -\frac{1}{\sqrt{3}}$; => $\tan\theta = -\sqrt{3}$

$\therefore[\theta: \theta = -60^0, 120^0]$

Example 7

(a) Given that $7\tan\theta + \cot\theta = 5\sec\theta$, derive a quadratic equation for $\sin\theta$. Hence or otherwise, find all values of θ in the interval $0^0 \le \theta \le 180^0$ which satisfy the equation, giving your answer to the nearest 0.10 where necessary

Solution

$7\tan\theta + \cot\theta = 5\sec\theta$

$7\frac{sin\theta}{cos\theta} + \frac{cos\theta}{sin\theta} = \frac{5}{cos\theta}$

$7\frac{sin^2\theta + cos^2\theta}{cos\theta sin\theta} = \frac{5}{cos\theta}$

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$7\sin^2\theta + \cos^2\theta = 5\sin\theta$

$7\sin^2\theta + (1 - \sin^2\theta) = 5\sin\theta$

$6\sin^2\theta - 5\sin\theta + 1 = 0$

$(3\sin\theta - 1)(2\sin\theta - 1) = 0$

$\sin\theta = \frac{1}{3}$	$\sin\theta = \frac{1}{2}$
$\theta = 19.5^0, 160.5^0$	$\theta = 30^0, 150^0$

$\therefore[\theta: \theta = 19.5^0, 30^0, 150^0, 160.5^0]$

Example 8

Find the solution of $3\cot\theta + \operatorname{cosec}\theta = 2$ for $0^0 \le \theta \le 180^0$.

Solution

$3\cot\theta + \operatorname{cosec}\theta = 2$

$3\frac{\cos\theta}{\sin\theta} + \frac{1}{\sin\theta} = 2$

$(3\cos\theta + 1)^2 = (2\sin\theta)^2$

$9\cos^2\theta + 6\cos\theta + 1 = 4\sin^2\theta$

$9\cos^2\theta + 6\cos\theta + 1 = 4(1 - \cos^2\theta)$

$13\cos^2\theta + 6\cos\theta - 3 = 0$

$\cos\theta = \frac{-6 \pm \sqrt{6^2 + 4x3x13}}{2x13}$

$\cos\theta = 0.3021$	$\cos\theta = 0.763$
$\theta = 72.40$	$\theta = 40$

$\therefore[\theta: \theta = 72.4^0, 40.2^0]$

Elimination of trigonometric parameter

This involves the use of identities to eliminate the trigonometric values in equation

Example 9

(a) If $x = \tan\theta + \sec\theta$ and $y = \tan\theta - \sec\theta$; show that $xy + 1 = 0$

Solution

$x + y = \tan\theta$

$x - y = 2\sec\theta$

$\sec\theta = \frac{1}{2}(x - y)$

Using identity: $1 + \tan^2\theta = \sec^2\theta$

$1 + (x + y)\,2 = \left[\frac{1}{2}(x - y)\right]^2$

$4 + x^2 + 2xy + y^2 = x^2 - 2xy + y^2$

$4xy + 4 = 0$

$xy + 1 = 0$ as required

(b) $x = 2 + 3\sin\theta$ and $y = 3 + 2\cos\theta$ show that $4(x - 2)^2 + (y - 3)^2 = 36$

Solution

$x = 2 + 3\sin\theta \Rightarrow \sin\theta = \frac{x-2}{3}$

$y = 3 + 2\cos\theta \Rightarrow \cos\theta = \frac{y-3}{2}$

Using identity $\sin^2\theta + \cos^2\theta = 1$

$\left(\frac{x-2}{3}\right)^2 + \left(\frac{y-3}{2}\right)^2 = 1$

$4(x - 2)^2 + (y - 3)^2 = 36$ as reqyured

(c) $x = 2\sin\theta$ and $y = \tan\theta$, prove that

$x = \pm\frac{2y}{\sqrt{(1+y^2)}}$

Solution

$x = 2\sin\theta; \Rightarrow \operatorname{cosec}\theta = \frac{2}{x}$

$y = \tan\theta, \Rightarrow \cot\theta = \frac{1}{y}$

Using identity: $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

$1 + \left(\frac{1}{y}\right)^2 = \left(\frac{2}{x}\right)^2$

$x = \pm\frac{2y}{\sqrt{(1+y^2)}}$

Revision exercise 1

1. Solve for θ, where $\theta^0 \le \theta \le 360^0$

(a) $\sec\theta\operatorname{cosec}\theta + 2\sec\theta - 2\operatorname{cosec}\theta - 4 = 0$
[$\theta: \theta = 60^0, 210^0, 300^0, 330^0$]

(b) $\tan^2\theta - (\sqrt{3} + 1)\tan\theta + \sqrt{3} = 0$
[$\theta: \theta = 45^0, 60^0, 225^0, 240^0$]

2. Show that

(a) $\frac{1-\cos\theta+\sin\theta}{1-\cos\theta} = \frac{1+\cos\theta+\sin\theta}{\sin\theta}$

(b) $\tan\theta + \cot\theta = \sec\theta\operatorname{cosec}\theta$

(c) $\cos^4\theta - \sin^4\theta + 1 = 2\cos^2\theta$

(d) $\frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} = 2\operatorname{cosec}\theta$

(e) $\sqrt{\frac{1+\sin\theta}{1-\sin\theta}} = \sec\theta + \tan\theta$

3. Solve the following equations for $-180^0 \le x \le 180^0$

(i) $2\cos^2\theta + \sin\theta - 1 = 0$
[$\theta: \theta = -150^0, -30^0, 90^0$]

(ii) $\sin 2\theta + 5\cos 2\theta = 3$
[$\theta: \theta = \pm 45^0, \pm 135^0$]

(iii) $4\cot^2\theta + 24\operatorname{cosec}\theta + 39 = 0$

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[θ: θ = 16.6^0, 23.6^0, 156.4^0, 163.4^0]

4. Solve each of the following equations in the stated range

(a) $4\cos^2\theta + 2\sin\theta = 4$ $\quad 0^0 \le \theta \le 360^0$
[θ: θ = 0^0, 48.6^0, 131.4^0, 180^0, 360^0]

(b) $2\sec^2\theta - 4\tan\theta - 2 =$ $\quad -180^0 \le \theta \le 360^0$
[θ: θ = -135^0, -161.6^0, 18.4^0, 45^0]

(c) $5\cos^2 3\theta = 3(1 + \sin 3\theta)$, $\quad 0^0 \le \theta \le 360^0$
[θ: θ = 7.9^0, 52.1^0, 90^0, 127.9^0, 172.1^0]

5. Solve for θ; $00 \le \theta \le 3600$

(a) $\tan\theta + 3\cot\theta = 4$
[θ: θ = 45^0, 71.6^0, 225^0,251.6^0]

(b) $4\cos\theta - 3\sin\theta = 2$
[θ: θ = 29.50, 256.70]

6. Solve

(a) $\cos\theta + \sqrt{3}\sin\theta = 2$ $\quad 0 \le \theta \le \pi$
$\left[\theta = \frac{\pi}{3}\right]$

(b) $2\cos\theta - \operatorname{cosec}\theta = 0$ $\quad 0^0 \le \theta \le 270^0$
[θ: θ = 45^0, 225^0]

(c) $2\sin^2\theta + 3\cos\theta = 0$ $\quad 0^0 \le \theta \le 360^0$
[θ: θ = 240^0, 120^0]

(d) $3\sin\theta + 4\cos\theta = 2$ $\quad -180^0 \le \theta \le 180$
[θ: θ = - 29.55^0, 103.29^0]

(e) $3\tan^2\theta + 2\sec^2\theta = 2(5 - 3\tan$, fo
$0^0 < \theta < 180^0$
[θ: θ = 38.66^0, 116.57^0]

7. Without using a table or calculator , show that $\tan 15^0 = 2 - \sqrt{3}$

8. Solve equat
$8\cos^4\theta - 10\cos^2\theta + 3$ for $0^0 \le \theta \le 180^0$
[θ: θ = 30^0, 45^0, 135^0, 150^0]

9. Eliminate θ from the following equation

(a) $x = a\sec\theta$ and $y = b + c\cos\theta$
[$ac = x(y - b)$]

(b) $x = \sec\theta + \tan\theta$ and $y = \sec\theta - \tan\theta$
[$xy = 1$]

10. Solve the simultaneous equation
$\cos x + 4\sin y = 1$
$4\sec x - 3\operatorname{cosec} y = 5$ for values of x and y between 0^0 and 360^0
[x = 78.8^0, 281.5^0; y = 11.5^0, 168.5^0]

11. Prove each of the following identities

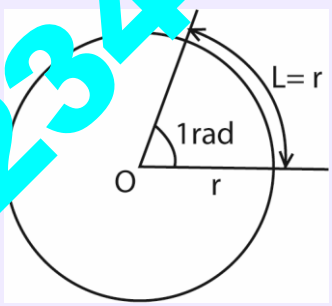
(a) $\sin x\tan x + \cos x = \sec x$
(b) $\operatorname{cosec} x + \tan x\sec x = \operatorname{cosec} x\sec^2 x$
(c) $\operatorname{cosec} x - \sin x = \cot x\cos x$
(d) $(\sin x + \cos x)^2 - 1 = 2\sin x\cos x$

12. Eliminate θ from each of the following pairs of relationships

(a) $x = 3\sin\theta$, $y = \operatorname{cosec}\theta$ [$xy = 3$]
(b) $5x = \sin\theta$, $y = 2\cos\theta$ [$100x^2 + y^2 - 4 = 0$]
(c) $x = 3 + \sin\theta$, $y = \cos\theta$ [$(x-3)^2 + y^2 = 1$]
(d) $x = 2 + \sin\theta$, $\cos\theta = 1 + y$
[$(x-2)^2 + (y+1)^2 = 1$]

Measuring angles in radians

A radian is defined a n angle subtended at the centre of a circle by arc that is equal to the radius of the circle. One radian is represented by where $\pi = \frac{22}{7}$



How to convert between degrees and radians

1 revolution = circumference of a circle

But circumference of a circle subtends an angle 2π at the centre.

⇨ 1 revolution = $2\pi = 360^0$
$\pi = 180^0$
$1^0 = \frac{\pi}{180}$ radians
$x^0 = \frac{\pi}{180}x$ radians

Example 10

Convert the following angles to radians

(a) 330^0
(b) 90^0
(c) 30^0

Solution

(a) $330^0 = \frac{\pi}{180}x\ 330 = \frac{11\pi}{6}$ radians

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(b) $90^0 = \frac{\pi}{180} x\ 90 = \frac{\pi}{2}$ radians

(c) $30^0 = \frac{\pi}{180} x\ 30 = \frac{\pi}{6}$ radians

Converting radians to degrees

2π radians = 360^0

1 radian = $\frac{180^0}{\pi}$

x radians = $\frac{180^0}{\pi} x$

Example 11

Convert each of the following radians to degrees

(i) $\frac{\pi}{3}$ radians

(ii) $\frac{2\pi}{5}$ radians

(iii) π radians

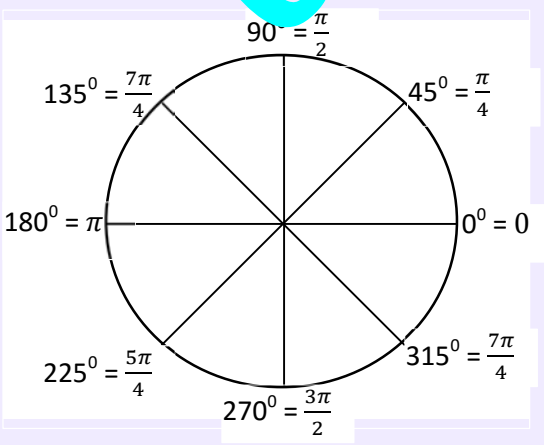
Solution

(i) $\frac{\pi}{3}$ radians = $\frac{180^0}{\pi}\ x \frac{\pi}{3} = 60^0$

(ii) $\frac{2\pi}{5}$ radians= $\frac{180^0}{\pi}\ x \frac{2\pi}{5}$ = 7

(iii) π radians = $\frac{180^0}{\pi}\ x\ \pi$ 80^0

Some equivalent angles in degrees and radians



Example 12

Find each of the following values

(a) $\sin\left(\frac{2\pi}{3}\right)$

(b) $\cos\left(\frac{4\pi}{3}\right)$

(c) $\tan\left(\frac{7\pi}{4}\right)$

Solution

Convert the angles from radian to degrees

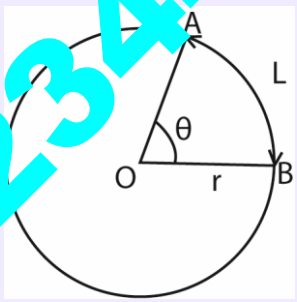
(a) $\sin\left(\frac{2\pi}{3}\right) = \sin\left(\frac{2\ x\ 180}{3}\right) = \sin 120^0 = \frac{\sqrt{3}}{2}$

(b) $\cos\left(\frac{4\pi}{3}\right) = \cos\left(\frac{4\ x\ 180}{3}\right) = \cos 240^0 = -\frac{1}{2}$

(d) $tan\left(\frac{7\pi}{4}\right) = \tan\left(\frac{7x\ 180}{4}\right) = \tan 60^0 = \sqrt{3}$

Length of an arc

Suppose that the angle subtended by the length L of an arc of a circle is θ as shown.



$\frac{L}{\theta} = \frac{2\pi r}{2\pi}$

L = rθ where θ must be in radians

Example 13

Find the length of an arc of a circle of radius 14 if it subtends an angle

(i) $\frac{\pi}{4}$

(ii) 150^0

Solution

(i) L = rθ = $14\ x\ \frac{\pi}{4}$ = 11cm

(ii) Convert degrees to radians

$150^0 = \frac{\pi}{180} x 150 = \frac{5\pi}{6}$ radians

L = $14\ x\ \frac{5\pi}{6}$ = 36.67cm

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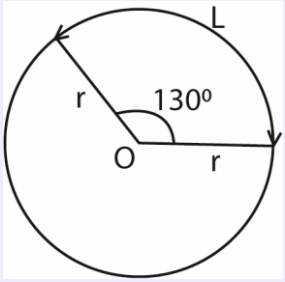
Example 14

A sector was drawn which had a perimeter of 80cm, and centre angle of 130^0. Calculate the radius

Solution

The sides of a sector are composed of an arc, and two more sides which are radii of a circle.



2r + L = 80

L = 80 – 2r

Converting 130^0 to radians

$130^0 = \frac{\pi}{180} x 130 = \frac{13\pi}{18}$

But L = rθ

$80 - 2r = \frac{13\pi r}{18}$

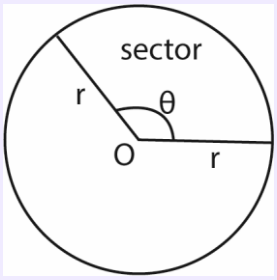
$2r + \frac{13\pi r}{18} = 80$

$\frac{(36+13\pi)r}{18} = 80$

r = 18.74cm

Area of a sector of a circle

A sector of a circle is a portion of the interior of a circle intercepted by a central angle.



The area of a sector of a circle of radius r and central angle θ is given by

$A = \left(\frac{\theta}{2\pi}\right)\pi r^2 = \left(\frac{\theta}{2}\right) r^2$

Where θ must be in radians

Example 15

Find the area of a sector with radius 14cm and angle (i) $\frac{\pi}{4}$ (ii) 1200

Solution

(i) $A = \left(\frac{\theta}{2}\right) r^2 = \left(\frac{\pi}{8}\right).14^2$=77cm^3

(ii) Converting 120^0 to radians

$120^0 = \frac{\pi}{180} x\ 120 = \frac{2\pi}{3}$

$A = \left(\frac{\theta}{2}\right) r^2 = \left(\frac{\pi}{3}\right).$ [illegible] 05.25cm^3

Solving trigonometric functions whose range is in radians

When the range of the trigonometric function is in radians, the answer should be given in radians

Example 16

Solve the following equations for the ranges indicated

(i) cosθ + √3sinθ = 1 $0 \le \theta \le \pi$

Solution

√3sinθ = 1- cosθ

Squaring both sides

$3\sin^2\theta = 1 - 2\cos\theta + \cos^2\theta$

$3(1 - \cos^2\theta) = 1 - 2\cos\theta + \cos^2\theta$

$4\cos^2\theta - 2\cos\theta - 1 = 0$

$(2\cos\theta + 1)(\cos\theta - 1) = 0$

$\text{Cos}\theta = -\frac{1}{2}$	$\cos\theta = 1$
$\theta = \pm 120^0$	$\theta = 0^0$

$\pm 1200 = \pm\frac{\pi}{180} x\ 120 = \pm\frac{2\pi}{3}$ Radians

0^0 = 0 radians

$\therefore \left[\theta: \theta = 0, \pm\frac{2\pi}{3}\right]$

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(ii) $2\cos^2\theta + \sin\theta - 1 = 0 \quad 0 \le \theta \le \pi$

Solution

$2(1 - \sin^2\theta) + \sin\theta - 1 = 0$

$2\sin2\theta - \sin\theta - 1 = 0$

$(\sin\theta - 1)(2\sin\theta + 1) = 0$

Either $\sin\theta = 1$ or $\sin\theta = -\frac{1}{2}$

When $\sin\theta = 1$; $\theta = 90^0$

When $\sin\theta = -\frac{1}{2}$; $\theta = -150^0, -30^0, 210^0, 330^0$

[θ: $\theta = \frac{\pi}{180} x\ 90 = \frac{\pi}{2}$ for given range]

Revision exercise 2

1. Express each of the following in radians
   (a) $30^0 \left[\frac{\pi}{6}\right]$
   (b) $45^0 \left[\frac{\pi}{4}\right]$
   (c) $120^0 \left[\frac{2\pi}{3}\right]$
   (d) $300^0 \left[\frac{5\pi}{3}\right]$

2. Express the following angle in degrees
   (a) $\frac{\pi}{3}$ rad [60^0]
   (b) $\frac{\pi}{8}$ rad [22.5^0]
   (c) 3π rad [540^0]
   (d) 5.2π rad[936^0]

3. A sector of the circle [illegible] radius 7 cm subtends an angle $\frac{\pi}{[illegible]}$ radians at the centre. Calculate the
   (a) Length of the a[illegible] $\left[6\frac{2}{3}cm\right]$
   (b) Perimeter of the sector $\left[20\frac{2}{3}cm\right]$
   (c) Area of the sector $\left[\frac{77}{3}cm^2\right]$

4. AOB is a sector of a circle, centre O, and is such that OA = OB = 7cm and angle AOB is 300. Calculate the
   (a) Perimeter of sector AOB $\left[17\frac{2}{3}cm\right]$
   (b) The area of AOB $\left[\frac{77}{6}cm^2\right]$

5. Find the value each of the following
   (a) $\sin\pi$ [0]
   (b) $\cos 3\pi$ [-1]
   (c) $\tan\frac{\pi}{3}$ [$\sqrt{3}$]

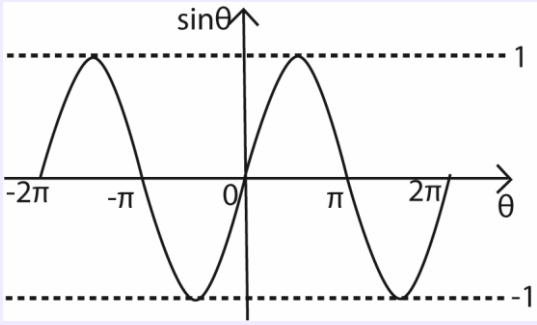
6. Solve the following equations for the ranges indicated
   (a) $2\sec^2\theta = 3 + \tan\theta$ for $0 \le \theta \le 2\pi$
   [θ: $\theta = 0.25\pi, 0.85\pi, 1.25\pi, 1.85\pi$]
   (b) $2\sin^2 x\cos x + \cos x - 1$ for $0 \le \theta \le 2\pi$
   [θ: $\theta = 0.38\pi, 1.62\pi, 2\pi$]
   (c) $2\tan\theta + 4\cot\theta = \operatorname{cosec}\theta$ for $-\pi \le \theta \le \pi$
   $\left[\theta: \theta = \pm\frac{1}{3}\pi, \pm 0.73\pi\right]$

Graphs of trigonometric functions

The following are the characteristic of the three major trigonometric functions

The sine function

- It is continuous [illegible]o breaks)
- The range [illegible] $\le \sin$ [illegible] ≤ 1
- The shape of [illegible] graph from $\theta = 0$ to $\theta = 2\pi$ [illegible]ed every 2π radians
- This [illegible] a periodic or cyclic function [illegible] the width of the repeating pattern th[illegible] is measured on horizontal axis is called a **period**. The sine wave has a period of 2π, a maximum value of +1 and a minimum value of -1.
- The greatest value of sine wave is called the **amplitude**.



The coosine function

- It is continuous (with no breaks)
- The range $-1 \le \sin\theta \le 1$
- Has a period of 2π
- The shape is the same as the sine wave but displaced a distance $\frac{\pi}{2}$ to the left on the horizontal axis. This is called a **phase shift**

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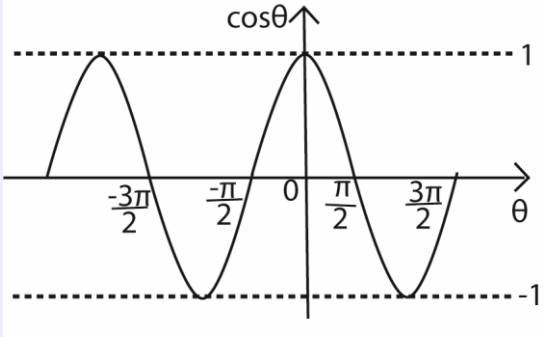
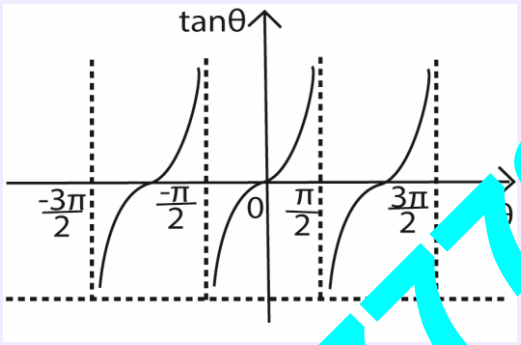


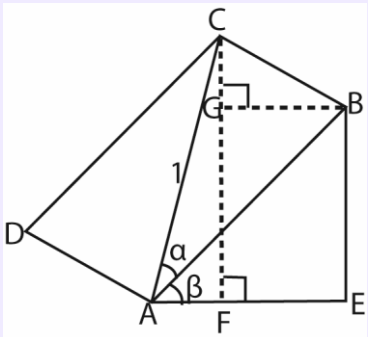
The tan function

- The tan function is found using; $tan\theta = \frac{sin\theta}{cos\theta}$. It follows that tanθ = 0 when sin θ = 0; and tanθ is undefined when cosθ = 0
- The graph is continuous, but undefined when θ= $-\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$
- The range of values for tanθ is unlimited
- It has a period π



Compound angles

Consider a cardboard ABCD of unit diagonal that stands on the edge making an angle β with the horizontal ground. Let the unit diagonal AC be inclined at an angle α to the side AB (see diagram)



Angles EAB= ABG(Alternative angles)

∴ Angle ABG = β

Angle [ABG + GBC] = 90^0

∴ Angle GBC = 90 – β

From triangle GBC,

Angle BCG = 180 – (90 + 90 – β)

∴ Angle BCG = β

From

(1) Triangle ABC:
cosα = $\frac{AB}{AC} = \frac{AB}{1}$; => AB = cosα

(2) Triangle ABE:
cosβ = $\frac{AE}{AB} = \frac{AE}{cos\alpha}$; => AE = cosβcosα
sinβ = $\frac{BE}{AB} = \frac{BE}{cos\alpha}$; => BE = cosαsinβ

(3) Triangle BCG:
cosβ = $\frac{CG}{BC} = \frac{CG}{sin\alpha}$; => CG = sinαcosβ
sinβ = $\frac{BG}{BC} = \frac{BG}{sin\alpha}$; => BG = sinαsinβ

(4) Triangle ACF:
cos(α + β) = $\frac{AF}{AC} = \frac{AF-BG}{1}$ = AE –BG
cos(α + β) = cosαcosβ – sinαsinβ
sin(α + β) = $\frac{CF}{AC} = \frac{CG-GF}{1}$ = CG + GF
∴ sin(α + β) = sinαcosβ + cosαsinβ

It follows that

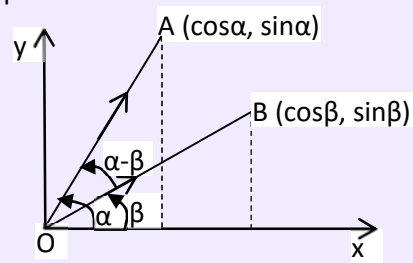
(i) cos(α + β) = cosαcosβ – sinαsinβ
(ii) cos(α - β) = cosαcosβ + sinαsinβ

[substituting –β for β)

(iii) **sin(α + β) = sinαcosβ + cosαsinβ**
(iv) **sin(α - β) = sinαcosβ - cosαsinβ**

[substituting –β for β)

These can also be derived using vector approach.
Consider two unit vectors $\underline{OA}$ and $\underline{OB}$ each inclined at angles α and β, respectively to the positive x-axis



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Using the definition of a vector product:

$\underline{OA}.\underline{OB} = |\underline{OA}|.|\underline{OB}|\cos(\alpha - \beta)$

Since $\underline{OA}$ and $\underline{OB}$ are unit vectors,

$|\underline{OA}| = |\underline{OB}| = 1$

$\therefore \underline{OA}.\underline{OB} = \cos(\alpha - \beta)$

⇨ $(\cos\alpha\underline{i} + \sin\alpha\underline{j}).(\cos\beta\underline{i} + \sin\beta\underline{j}) = \cos(\alpha - \beta)$

$\therefore \cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha - \beta)$

Substituting 90 –α for α

$\cos(90 - \alpha)\cos\beta + \sin(90 - \alpha)\sin\beta$

$= \cos(90 - \alpha - \beta)$

$\therefore \sin\alpha\cos\beta + \cos\alpha\sin\beta = \sin(\alpha + \beta)$

Other expansions can be similar substitutions

i.e. $\tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)}$

$= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$

Dividing through by cosαcosβ

$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

Similarly

$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

The following is a summary of compound angles

1. $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
2. $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
3. $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$
4. $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$
5. $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$
6. $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

Example 17

Calculate the value of $\sin 15^0$ given that $\sin 45^0 = \cos 45 = \frac{1}{\sqrt{2}}$, $\sin 30^0 = \frac{1}{2}$ and $\cos 30^0 = \frac{\sqrt{3}}{2}$

$\sin 15^0 = \sin(45^0 - 30^0)$

$= \sin 45^0\cos 30^0 - \sin 30^0\cos 45^0$

$= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} - \frac{1}{2}.\frac{1}{\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}} = 0.2588$

Example 18

Prove that $\tan(45^0 + A) = \frac{1 + \tan A}{1 - \tan A}$

From $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$\tan(45^0 + A) = \frac{\tan 45^0 + \tan\beta}{1 - \tan 45^0\tan\beta}$

$= \frac{1 + \tan A}{1 - \tan A}$

Example 19

Acute angles A and B are such that: $\cos A = \frac{1}{2}$, $\sin B \frac{1}{3}$. Show without using tables or calculator that $\tan(A + B) = \frac{9\sqrt{3} + 8\sqrt{2}}{5}$

Solution

$\sin^2\theta + \cos^2\theta = 1$

$\left(\frac{1}{2}\right)^2 + \sin^2 A = 1$

$\sin^2 A = \frac{3}{4} \Rightarrow \sin A = \frac{\sqrt{3}}{2}$

$\tan A = \frac{\sqrt{3}}{2} \div \frac{1}{2} = \sqrt{3}$

Similarly;

$\cos^2 B + \left(\frac{1}{3}\right)^2 = 1$

$\cos B = \frac{2\sqrt{2}}{3}$

$\tan B = \frac{2\sqrt{2}}{3} \div \frac{1}{3} = \frac{1}{2\sqrt{2}}$

But

From $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$= \frac{\sqrt{3} + \frac{1}{2\sqrt{2}}}{1 - \sqrt{3}.\frac{1}{2\sqrt{2}}}$

$= \frac{(2\sqrt{2}\sqrt{3} + 1)(2\sqrt{2} + \sqrt{3})}{(2\sqrt{2} - \sqrt{3})(2\sqrt{2} + \sqrt{3})}$

$\tan(A + B) = \frac{9\sqrt{3} + 8\sqrt{2}}{5}$

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Example 20

Solve $\cos(\theta + 35^0) = \sin(\theta + 25^0)$
for $0^0 \le \theta \le 360^0$

$\cos\theta\cos35^0 - \sin\theta\sin35^0 = \sin\theta\cos25^0 + \cos\theta\sin25^0$

Dividing through by $\cos\theta$

$\cos35^0 - \tan\theta\sin35^0 = \tan\theta\cos25^0 + \sin25^0$

$\tan\theta = \frac{\cos35^0 - \sin25^0}{\cos35^0 + \sin25^0} = \frac{0.3965337825}{1.479884223}$

$\theta = 15^0, 195^0$ for $0^0 \le \theta \le 360^0$

Example 21

(a) Prove than $\frac{2\tan\theta}{1+\tan^2\theta} = \sin2\theta$

Solution

$$\frac{2\tan\theta}{1+\tan^2\theta} = \frac{2\sin\theta}{\cos\theta} \div \left(1 + \frac{\sin^2\theta}{\cos^2\theta}\right)$$
$$= \frac{2\sin\theta}{\cos\theta} \div \left(\frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}\right)$$
$$= \frac{2\sin\theta}{\cos\theta} \div \left(\frac{1}{\cos^2\theta}\right)$$
$$= 2\sin\theta\cos\theta = \sin2\theta$$

(b) Solve $\sin2\theta = \cos\theta$ for $0^0 \le \theta \le 90^0$

Solution

$\sin2\theta = \cos\theta$

$2\sin\theta\cos\theta = \cos\theta$

$\sin\theta = \frac{1}{2}$

$\theta = 30^0$ for $0^0 \le \theta \le 9$

Example 22

Given that α, β and γ a[illegible]ngles of a triangle, show that $\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$

Hence find $\tan\gamma$ if $\tan\alpha = 1$ and $\tan\gamma = 2$.

Solution

$\alpha + \beta + \gamma = 180^0$ (angle sum of a triangle)

$\tan(\alpha + \beta + \gamma) = \tan180^0 = 0$

$\tan[(\alpha + \beta) + \gamma] = 0$

$$\frac{\tan(\alpha+\beta) + \tan\gamma}{1-\tan(\alpha+\beta)\tan\gamma} = 0$$

⇨ $\tan(\alpha + \beta) + \tan\gamma = 0$

$$\frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta} = -\tan\gamma$$

$\tan\alpha + \tan\beta = -\tan\gamma + \tan\alpha\tan\beta\tan\gamma$

$\therefore \tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$

Example 23

In a triangle ABC, prove that

$\cot\frac{1}{2}A + \cot\frac{1}{2}B + \cot\frac{1}{2}C$

$= \cot\frac{1}{2}A\cot\frac{1}{2}B\cot\frac{1}{2}C$

Solution

$\frac{1}{2}(A + B + C) = \frac{1}{2}(180^0) = 90^0$

$\cot\left[\frac{1}{2}(A + B + C)\right] = \cot90^0 = 0$

⇨ $\frac{1-\tan\left(\frac{1}{2}A+\frac{1}{2}B\right)\tan[illegible]}{\tan\left(\frac{1}{2}A+\frac{1}{2}B\right)[illegible]}$

$1 = \tan\left([illegible] + [illegible]\right)\tan\frac{1}{2}C$

$\left([illegible]\frac{\tan\frac{1}{2}B}{1-t[illegible]\frac{1}{2}A\tan\frac{1}{2}B}\right)\tan\frac{1}{2}C$

$1 [illegible] \tan\frac{1}{2}A\tan\frac{1}{2}B$

$= \tan\frac{1}{2}A\tan\frac{1}{2}C + \tan\frac{1}{2}B\tan\frac{1}{2}C$

$1 = \tan\frac{1}{2}A\tan\frac{1}{2}B + \tan\frac{1}{2}A\tan\frac{1}{2}C + \tan\frac{1}{2}B\tan\frac{1}{2}C$

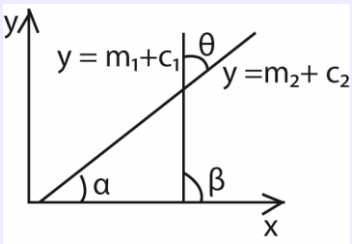
Dividing each side by $\tan\frac{1}{2}A\tan\frac{1}{2}B\tan\frac{1}{2}C$

$\cot\frac{1}{2}A\cot\frac{1}{2}B\cot\frac{1}{2}C = \cot\frac{1}{2}A + \cot\frac{1}{2}B + \cot\frac{1}{2}C$

Example 24

Prove that the angle θ, between the straight line $y = m_1x + c_1$ and the straight line $y = m_2x + c_2$ is given by $\tan\theta = \frac{m_2-m_1}{1+m_2m_1}$

Let the lines be inclines at angles α and β with the x-axis respectively



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From the diagram above

$\theta = \beta - \alpha$

⇨ $\tan\theta = \tan(\beta - \alpha)$

$= \frac{tan\beta - tan\alpha}{1 + tan\beta tan\alpha}$

$\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1}$

Revision exercise 3

1. (a) show that $\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2\cos\alpha\sin\beta$
   (b) If $\sin(\alpha + \beta) = 5\cos(\alpha - \beta)$ show that $\tan\alpha = \frac{5 - tan\beta}{1 + tan\beta}$
   (c) Without using tables or calculator, show that $\cos 15^0 = \sin 75^0$
   (d) If $\alpha + \beta = 45^0$, show that $\tan\alpha = \frac{1 - tan\beta}{1 + tan\beta}$
2. Prove that:
   (i) $\frac{\sin(\alpha+\beta)}{\cos(\alpha - \beta)} + 1 = \frac{(1+tan\beta)(1+cot\alpha)}{cos\alpha + tan\beta}$
   (ii) $tan\alpha - tan\beta = \frac{\sin(\alpha-\beta)}{cos\alpha cos\beta}$
   (iii) $cot\alpha + cot\beta = \frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}$
   (iv) $\frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)} = \frac{tan\alpha - tan\beta}{tan\alpha + tan\beta}$
   (v) $\frac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)} = \frac{cot\alpha cot\beta + 1}{cot\alpha cot\beta - 1}$
   (vi) $\cot(A + B) = \frac{cotAcotB - 1}{cotA + cotB}$
3. (a) Determine solution of $\cos 2x + 2\sin x = 0$ for $0^0 \le x \le 180^0$ [x: $x = 0^0, 150^0, 120^0, 180^0$]
   (vii) Show that in triangle ABC, $\tan A + \tan B + \tan C = \tan A\tan B\tan C$
4. Find the values of tan for each of the following
   (a) $\sin(\alpha - 30^0) = \cos\alpha$ [$\sqrt{3}$]
   (b) $\sin(\alpha + 45^0) = \cos\alpha$ [$\sqrt{2} - 1$]
   (c) $\cos(\alpha + 60^0) = \sin\alpha$ [$2 - \sqrt{3}$]
   (d) $\sin(\alpha + 60^0) = \cos(\alpha - 60^0)$ [1]
   (e) $\cos(\alpha + 60^0) = 2\cos(\alpha + 30^0)$ [$4 + 3\sqrt{3}$]
   (f) $\sin(\alpha + 60^0) = \cos(45^0 - \alpha)\left[\frac{\sqrt{3} - \sqrt{2}}{\sqrt{2} - 1}\right]$
5. Given that
   (a) $\tan(\alpha - \beta) = ½$ and $\tan\alpha = 3$ find the value of $\tan\beta$ [1]
   (b) $\tan(\alpha + \beta) = 5$ and $\tan\beta = 2$ find the value of $\tan\alpha\left[\frac{3}{11}\right]$
6. Given that
   (a) $\tan(\theta - 45^0) = 4$, find the value of θ $\left[-\frac{5}{3}\right]$
   (b) $\tan(\theta + 60^0)$ find the value of $\cot\theta$ $[8 + 5\sqrt{3}]$

Double angles and half angles

(b) From $\cos(\theta + \theta) = \cos\theta\cos\theta - \sin\theta\sin\theta$

⇨ $\cos 2\theta = \cos^2\theta - \sin^2\theta$..... (i)

Either

$\cos 2\theta = \cos^2\theta - 1 + \cos^2\theta$ $(\cos^2\theta + \sin^2\theta = 1]$

$\cos 2\theta = 2\cos^2\theta - 1$ ..... (ii)

Or

$\cos 2\theta = 1 - \sin^2\theta - \sin^2\theta$

⇨ $\cos 2\theta = 1 - 2\sin^2\theta$ .....(iii)

It follows that

$\cos^2\theta = \frac{1}{2}(1 + cos2\theta)$ .....(iv)

$\sin^2\theta = \frac{1}{2}(1 - cos2\theta)$ .....(iv)

The identities imply

$\text{Cos}6\theta = \cos^2 3\theta - \sin^2 3\theta$

$= 2\cos^2 3\theta - 1 = 1 - 2\sin^2 3\theta$

$\text{Cos}\,\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$

$= 2\cos^2\frac{\theta}{2} - 1 = 1 - 2\sin^2\frac{\theta}{2}$

(c) $\sin(\theta + \theta) = \sin\theta\cos\theta + \cos\theta\sin\theta$

⇨ $\sin 2\theta = 2\sin\theta\cos\theta$

It follows that

$\sin 6\theta = 2\sin 3\theta\cos 3\theta$

$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

(d) $\tan(\theta + \theta) = \frac{tan\theta + tan\theta}{1 - tan\theta tan\theta}$

⇨ $tan2\theta = \frac{2tan\theta}{1 - tan^2\theta}$

It follows that

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$$tan\theta = \frac{2tan\frac{\theta}{2}}{1-tan^2\frac{\theta}{2}}$$

$$tan6\beta = \frac{2tan3\beta}{1-tan^2 3\beta}$$

Note that in all cases, the angles on the right hand side are half the angles on the left hand side [***half angle formulae]***

Example 25

Show that

(a) cosec2θ + cot2θ = cotθ

Solution

$$cosec2\theta + cot2\theta = \frac{1}{sin2\theta} + \frac{cos2\theta}{sin2\theta}$$

$$= \frac{1+cos2\theta}{sin2\theta}$$

$$= \frac{1+2cos^2\theta-1}{2sin\theta cos\theta}$$

$$= \frac{2cos^2\theta}{2sin\theta cos\theta} = cot\theta$$

(b) $tan3\theta = \frac{3tan\theta-tan^3\theta}{1-3tan^2\theta}$

Hence deduce that if 3θ + α = 45^0, then

tanα = $\frac{1-3tan\theta-3tan^2\theta+tan^3\theta}{1+2tan\theta-3tan^2\theta-tan^3\theta}$

Solution

tan3θ=tan(2θ + θ) = $\frac{tan2\theta+tan\theta}{1-tan2\theta tan\theta}$

$$=\left\{\left(\frac{2tan\theta}{1-tan^2\theta}\right)+tan\theta\right\} \div \left\{1-\left(\frac{2tan\theta}{1-tan^2\theta}\right)tan\theta\right\}$$

$$=\frac{2tan\theta+tan\theta-tan^3\theta}{1-tan^2\theta-2tan^2\theta} = \frac{3tan\theta-tan^3\theta}{1-3tan^2\theta}$$

$$\therefore tan3\theta = \frac{3tan\theta-tan^3\theta}{1-3tan^2\theta}$$

Hence 3θ + α = 45^0 => α = 45^0 - 3θ

Tanα = tan(45^0 - 3θ)

$$= \frac{tan\,45^0-tan3\theta}{1+tan45^0 tan3\theta} = \frac{1-tan3\theta}{1+tan3\theta}$$

$$= \frac{1-\left(\frac{3tan\theta-tan^3\theta}{1-3tan^2\theta}\right)}{1+\left(\frac{3tan\theta-tan^3\theta}{1-3tan^2\theta}\right)}$$

$$=\frac{1-3tan\theta-3tan^2\theta+tan^3\theta}{1+2tan\theta-3tan^2\theta-tan^3\theta}$$

$$\therefore tan\alpha = \frac{1-3tan\theta-3tan^2\theta+tan^3\theta}{1+2tan\theta-3tan^2\theta-tan^3\theta}$$

Example 26

If tanα = $\frac{3}{4}$ and α is acute, without using tables or calculator work out the value of

(a) tan2α

$$tan2\alpha = \frac{2tan\alpha}{1-tan^2\alpha} = \frac{2x\frac{3}{4}}{1-\left(\frac{3}{4}\right)^2} = \frac{\frac{2}{2}}{\frac{3}{4}} = \frac{24}{7}$$

(b) tan$\frac{\alpha}{2}$

similarly $tan\alpha = \frac{2tan\frac{\alpha}{2}}{1-tan^2\frac{\alpha}{2}} = \frac{3}{4}$

⇨ $3tan^2\frac{\alpha}{2} + 8tan\frac{\alpha}{2} - 3 = 0$

$(3tan\frac{\alpha}{2} - 1)(tan\frac{\alpha}{2} + 3) = 0$

$tan\frac{\alpha}{2} = \frac{1}{3}$ or $tan\frac{\alpha}{2} = -3$

Since α is acute, tanα cannot be negative

$\therefore tan\frac{\alpha}{2} = \frac{1}{3}$

Example 27

(a) Show that cos3α = $4cos^2$ α -3cos α. Hence solve the equation $4x^3 - 3x - \frac{\sqrt{3}}{3} = 0$ for $0^0 \le \alpha \le 180^0$

Solution

Cos3α = cos(2α +α)

= cos2αcosα – sin2αsinα

=$(2cos^2\alpha - 1)cos\alpha - 2sin^2\alpha cos\alpha$

=$(2cos^2\alpha - 1)cos\alpha - 2(1-cos^2\alpha)cos\alpha$

= $2cos^3\alpha - cos\alpha - 2cos\alpha + 2cos^3\alpha$

=$4cos^2$ α -3cos α

Hence $4x^3 - 3x = \frac{\sqrt{3}}{3}$

i.e. $4cos^2\alpha - 3cos\alpha = \frac{\sqrt{3}}{3}$

$0^0 \le \alpha \le 180^0$; cos3α = $\frac{\sqrt{3}}{3}$

For the range $0^0 \le \alpha \le 180^0$

⇨ $0^0 \le 3\alpha \le 540^0$

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$3\alpha = 54.7^0, 414.7^0$

$\alpha = 18.23^0, 138.23^0$ (2d.p)

[α: α= $18.23^0, 138.23^0$]

(b) Given that $t = \tan 22\frac{1}{2}$, show that $t^2 + 2t - 1 = 0$,

Hence show that $\tan 22\frac{1}{2}^0 = -1 + \sqrt{2}$

Solution

$\tan 45^0 = \frac{2\tan 22\frac{1}{2}^0}{1-\tan^2 22\frac{1}{2}^0}$

$1 = \frac{2t}{1-t^2}$

$1 - t^2 = 2t$

$t^2 + 2t - 1 = 0$ (as required)

solving

$t = \frac{-2\pm\sqrt{2^2-(4x1\ x-1)}}{2x1}$

$t = \frac{-2\pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}$

Since $22\frac{1}{2}^0$ is an acute angle,

$\tan 22\frac{1}{2}^0 = -1 + \sqrt{2}$ is positive

$\therefore \tan 22\frac{1}{2}^0 = -1 + \sqrt{2}$

Example 28

(a) Show that $3\sin\theta = 3\sin\theta - 4$[illegible]. Hence solve the equation $\sin 3\theta$[illegible] $\sin\theta$[illegible]0 for $0^0 \le \theta \le 360^0$

Solution

$\sin 3\theta = \sin(2\theta$[illegible]$\theta)$

$= \sin 2\theta\cos$[illegible]$s2\theta\sin\theta$

$= 2\sin\theta\cos^2\theta + (1 - 2\sin^2\theta)\sin\theta$

$= 2\sin\theta(1 - \sin^2\theta) + (1 - 2\sin^2\theta)\sin\theta$

$= 3\sin\theta - 4\sin^3\theta$

Hence $\sin 3\theta + \sin\theta = 0$

$3\sin\theta - 4\sin^3\theta + \sin\theta = 0$

$4\sin\theta - 4\sin^3\theta = 0$

$4\sin\theta(1 - \sin^2\theta) = 0$

$4\sin\theta(1 - \sin\theta)(1 + \sin\theta) = 0$

$\sin\theta = 0$; $\theta = 0^0, 180^0, 360^0$

$\sin\theta = 1$; $\theta = 90^0$

sino = -1; $\theta = 270^0$

$\therefore\theta$: $\theta = 0^0, 90^0, 180^0, 270^0, 360^0$

(b) Prove that $\cot 2\theta = \frac{\cot^2\theta - 1}{2\cot\theta}$. Hence solve the equation $\cot 2\theta + 2\cot\theta = 2$ for $0^0 \le \theta \le 360^0$

Solution

$\cot 2\theta = \frac{\cos 2\theta}{\sin 2\theta} = \frac{\cos^2\theta - \sin^2\theta}{2\sin\theta\cos\theta}$

dividing through by $\sin^2\theta$

$\cot 2\theta = \frac{\cot^2\theta - 1}{2\cot\theta}$

Hence, $\cot 2\theta + 2\cot\theta = 0$

$\frac{\cot^2\theta - 1}{2\cot\theta} + 2\cot\theta = 0$

$5\cot^2\theta - 4\cot\theta - 1 =$ [illegible]

$(5\cot\theta + 1)(\cot$ [illegible] $= 0$

$\cot\theta = -$ [illegible] $\cot\theta = 0$

[illegible] $\tan\theta = -5$ ot $\tan\theta = 1$

Whe[illegible]anθ= -5; θ= $101.3^0, 281.3^0$

[illegible]hen $\tan\theta = 1$, $\theta = 45^0, 225^0$

$\therefore\{\theta: \theta = 45^0, 101.3^0, 225^0, 281.3^0\}$

Revison exercise 4

1. Prove that

(a) $\sin\alpha\operatorname{cosec}\beta + \cos\alpha\sec\beta = 2\sin(\alpha + \beta)\operatorname{cosec}2\beta$

(b) $\cos^6\theta + \sin^6\theta = 1 - \frac{3}{4}\sin^2 2\theta$

(c) $\frac{\sin 3\alpha}{1+2\cos 2\alpha} = \sin\alpha$ and hence deduce that $\sin 15^0 = \frac{\sqrt{3}-1}{2\sqrt{2}}$

2. (a) Solve the equation for θ, $0^0 \le \theta \le 360^0$

$\sin^2\theta - 2\sin\theta\cos\theta - 3\cos^2\theta = 0$

[θ: $\theta = 71.6^0, 135^0, 251.6^0, 315^0$]

(b) show that $\frac{\cos\theta}{1+\sin\theta} = \cot(\frac{\theta}{2} + 45^0)$.

Hence or otherwise solve the equation $\frac{\cos\theta}{1+\sin\theta} = \frac{1}{2}$ for $0^0 \le \theta \le 360^0$ [$\theta = 36.8^0$]

3. (a) solve the equation $4\cos 2\theta - 2\cos\theta + 3 = 0$ for $0^0 \le \theta \le 360^0$

[θ: $\theta = 60^0, 104.5^0, 255.5^0, 300^0$]

(c) Solve the equation $\sin\theta + \sin\frac{\theta}{2} = 0$ for $0^0 \le \theta \le 360^0$

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[θ: θ= -360^0, -240^0, 0^0, 2405^0, 360^0]

4. (a) Prove that $\tan\left(\frac{\pi}{4}+\theta\right)$ - $\tan\left(\frac{\pi}{4}-\theta\right)$ = 2tan2θ
   (b) By expressing 2sinθsin(θ + α) as difference of cosines of two angles or otherwise, where α is constant, find its least value $\left[\frac{-a}{2}\right]$
   (c) Solve for θ in the equation Cosθ – cos(θ + 60^0) = 0.4 for 0^0 ≤ θ ≤ 360^0 [θ = 126.4^0, 353.6^0]
5. (a) Show that cos3θ = 4cos^{3}θ – 3cosθ. Hence solve the equation 4x^3 – 3x $\frac{\sqrt{3}}{3}$ =0 [x: x = -0.746, -0.204, 0.959]
   (b) Find all solutions of the equation 5cosx – 4sinx = 6 in the range -180^0 ≤ x ≤ 180^0 [x: x = -59.1^0, -18.3^0]
6. (a) Express $\sqrt{\left(\frac{sin2\theta-cos2\theta-1}{2-2sin2\theta}\right)}$ in terms of tanθ$\left[\frac{1}{\sqrt{(tan\theta-1)}}\right]$
   (b) Find the solution of the equation $\sqrt{3sin\theta}-cos\theta+1=0$ for 0≤θ≤2π
   $$\left[\theta: \theta=\frac{4}{3}\pi, 2\pi\right]$$
   (c) Factorize cosθ – cos3θ – cos7θ + cos5θ in form Acospθsinqθsinrθ where A, p, q and r are constants [A= -4, p =5, q = 1, r = 2]
7. (a) Given that sinα + sinβ = p and cosα + cosβ = q show that
   (i) $\tan\left(\frac{\alpha+\beta}{2}\right) = \frac{p}{q}$
   (ii) cos(α + β)= $\frac{q^2-p^2}{q^2+p^2}$
   (b) Solve the simultaneous equation: cosα + 4sinβ = 1
   4secα – 3cosecβ = 5 [θ= 78.5^0, 281.5^0]
8. (a) Express sinθ + sin3θ in form pcosθsinqθ where p and q are constant [p =2, q = 2]
   (b) Find the solution of cos7θ + cos5θ = 2cosθ for 0^0 ≤ θ ≤ 360^0 [0^0, 60^0, 270^0, 360^0]
   (c) Prove that $\frac{sinA+sin4A+sin7A}{cosA+cos4A+cos7A}=tan4A$
9. Eliminate θ from each of the following pairs of expression
   (a) x+1 = cos2θ, y = sinθ [x + 2y^2 = 0]
   (b) x = cos2θ, y = cosθ -1 [x = 2y^2 + 4y + 1]
   (c) y – 3 = cos2θ, x = 2 – sinθ [2x^2 – 8x + y + 4 = 0]
10. Solve the following equations for -180^0 ≤ θ ≤ 180^0
   (a) Sin2θ + sinθ = 0 [±120^0, ±180^0]
   (b) Sin2θ – 2cos^{2}θ = 0 [-135^0, 45^0, ±90^0]
   (c) 3cos2θ + 2+ cosθ = 0 [±70.5^0, ±120^0]
   (d) sin2θ = tanθ [0^0, ±45^0, ±135^0, ±180^0]
11. Solve the following equations for -360^0 ≤ θ ≤ 360^0, giving your answer correct to 1 decimal place
   (a) sinθ = $\sin\left(\frac{\theta}{2}\right)$[0^0, ±120^0, ±360^0]
   (b) 3cos$\left(\frac{\theta}{2}\right)$ = 2sinθ [±180^0, 97.2^0, 262.8^0]
   (c) 2sinθ = tan[illegible] [0^0, ±120^0, ±[illegible], ±360^0]
   (d) 2cosθ [illegible] $\cos\left(\frac{\theta}{2}\right)+2$ [±209^0]
12. Prove the following identities
   (a) [illegible] – cos2θ = 1
   (b) 2cosec2θ = cosecθsecθ
   (c) cos^{3}θ + sin2θsinθ =2cosθ
   (d) tanθ + cotθ = 2cosec2θ
   (e) cos^{4}θ – sin^{4}θ = cos2θ
   (f) $\frac{1-cos2\theta}{1+cos2\theta}=tan^2\theta$
   (g) Cotθ – tanθ = 2cot2θ
   (h) cot2θ + cosecθ = cotθ
   (i) $\frac{cos2\theta}{cos\theta+sin\theta}=cos\theta-sin\theta$
   (j) $\frac{sin2\theta}{1-cos2\theta}=cot\theta$
   (k) $cos2\theta=\frac{1-tan^2\theta}{1+tan^2\theta}$
   (l) $tan3\theta=\frac{3tan\theta-tan^3\theta}{1-3tan^2\theta}$
   (m) $tan\left(\frac{\theta}{2}\right)=\frac{sin\theta}{1+cos\theta}$
13. Express tan22$\frac{1}{2}^0$ in the form a + b√2 where a and b are integers [a = -1, b= ±1]
14. Solve the equation
   (i) 4cosθ – 2cos2θ = 3 for 0 ≤ θ ≤ π $\left[\frac{\pi}{3}\right]$
   (ii) Cos2θ + cos3θ + cosθ = 0 for 0^0 ≤ θ ≤ 360^0 [θ = 45^0, 120^0, 135^0]
   (iii) cosθ + sin2θ = 0 for 0^0 ≤ θ ≤ 360^0 [θ = 90^0, 210^0, 270^0, 330^0]
   (iv) 2sin2θ = 3cosθ for -180^0 ≤ θ ≤ 180^0 [θ = -90^0, 48.6^0, 90^0, 132.4^0]
   (v) Sinθ – 4sin4θ = sin2θ – sin3θ for –π ≤ θ ≤ π $\left[\frac{-\pi}{5},\frac{-\pi}{2},\frac{-3\pi}{5},0,\frac{\pi}{2},\frac{\pi}{5},\frac{3\pi}{5}\right]$

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Harmonic form

These are trigonometric functions expressed in the form of **Rcos(x ± α) and Rsin(x ±α).** They are in two ways

(i) solving equations in the form
$a\cos\theta + b\sin\theta + c = 0$

(ii) determining the maximum and minimum values of the function
$a\cos\theta + b\sin\theta + c = 0$
where a, b and c are constants

A: Solving equations

Example 29

(a) Express $3\cos\theta - 4\sin\theta$ in the form $R\cos(\theta + \alpha)$, where R and α are constants

Solution

Let $3\cos\theta - 4\sin\theta = R\cos(\theta + \alpha)$
$= R(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$
$= R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

Comparing coefficient of cosθ and sinθ

$R\cos\alpha = 3$ ..... (i)

$R\sin\alpha = 4$ ..... (ii)

Eqn (ii) ÷ eqn (i)

$\tan\alpha = \frac{4}{3}$; $\alpha = 53.1^0$

Eqn. $(i)^2$ + eqn. $(ii)^2$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 3^2 + 4^2 = 25$

$R^2[\cos^2\alpha + \sin^2\alpha] = 25$

$R^2 = 25$

$R = 5$

$\therefore 3\cos\theta - 4\sin\theta = 5\cos(\theta + 53.1^0)$

(b) Solve the equation $3\cos\theta - 4\sin\theta = 5$ for $0^0 \le \theta \le 360^0$.

Solution

$3\cos\theta - 4\sin\theta = 5\cos(\theta + 53.1^0)$

⇨ $5\cos(\theta + 53.1^0) = 5$

$\cos(\theta + 53.1^0) = 1$

$x + 53.1^0 = 0^0, 360^0$

$x = -53.1^0, 306.9^0$

Hence $x = 306.9^0$

Example 30

(a) Express $\sin\theta - \sqrt{3}\cos\theta$ in the form $R\sin(\theta - \alpha)$

Solution

Let $\sin\theta - \sqrt{3}\cos\theta = R\sin(\theta - \alpha)$
$= R(\sin\theta\cos\alpha - \cos\theta\sin\alpha)$

Equating coefficients

$R\cos\alpha = 1$ ..... (i)

$R\sin\alpha = \sqrt{3}$..... (ii)

Eqn. (ii) ÷ eqn. (i)

$\tan\alpha = \sqrt{3}$; => $\alpha = 60^0$

$R^2[\cos^2\alpha + \sin^2\alpha] = 4$

$R^2 = 4$; $R = 2$

$\therefore \sin\theta - \sqrt{3}\cos\theta = 2\sin(\theta - 60^0)$

(b) Solve the equation
$\sin\theta - \sqrt{3}\cos\theta + 1 = 0$ for $0^0 \le \theta \le 360^0$

$\sin\theta - \sqrt{3}\cos\theta = 2\sin(\theta - 60^0)$

⇨ $2\sin(\theta - 60^0) + 1 = 0$

$\sin(\theta - 60^0) = -\frac{1}{2}$

$\theta - 60^0 = 210^0, 330^0$

$\theta = 270^0, 390^0$

Hence $\theta = 270^0$ for the given range

Example 31

(a) Express $4\cos\theta - 5\sin\theta$ in the form $A\cos(\theta + \beta)$, where A is constant and β is an acute angle

Let $4\cos\theta - 5\sin\theta = A\cos(\theta + \beta)$
$= A(\cos\theta\cos\beta - \sin\theta\sin\beta)$
$= A\cos\theta\cos\beta - R\sin\theta\sin\beta$

Comparing coefficient of cosθ and sinθ

$A\cos\beta = 4$ ..... (i)

$A\sin\beta = 5$ ..... (ii)

Eqn (ii) ÷ eqn (i)

$\tan\alpha = \frac{5}{4}$; $\alpha = 51.3^0$

Eqn. $(i)^2$ + eqn. $(ii)^2$

$A^2\cos^2\beta + A^2\sin^2\beta = 4^2 + 5^2 = 41$

$A^2[\cos^2\alpha + \sin^2\alpha] = 41$

$A^2 = 41$

$A = \sqrt{41}$

$\therefore 3\cos\theta - 4\sin\theta = \sqrt{41}\cos(\theta + 51.3^0)$

(b) Solve the equation $3\cos\theta - 4\sin\theta = 2.2$ for $0^0 \le \theta \le 360^0$

Solution

$3\cos\theta - 4\sin\theta = \sqrt{41}\cos(\theta + 51.3^0)$

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⇨ $\sqrt{41}\cos(\theta + 51.3^0) = 2.2$

$\cos(\theta + 51.3^0) = \frac{2.2}{\sqrt{41}} = 0.3436$

$(\theta + 51.3^0) = 69.9^0,\ 290.1^0$

$\therefore \theta = 18.6^0, 238.3^0$

B: Maximum and minimum values

The maximum and minimum values of a circular function may be obtained using three methods

(i) Express the given function either in for $R\cos(\theta \pm \alpha)$ or $R\sin(\theta \pm \alpha)$ if possible, where R and α are constants.

(ii) Differentiating the given function with respect to the given function say θ

(iii) Sketching the graphs of the function given and noting their maximum and minimum points.

In this chapter approach I will be considered.

Example 32

Determine the maximum and minimum values of the following, stating the value of θ for which they occur

(a) $\sqrt{3}\sin\theta + \cos\theta + 7$

Let $\sqrt{3}\sin\theta + \cos\theta = R\sin(\theta +$

$= R(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$

Equating coefficients

$R\sin\alpha = 1$ ..... (i)

$R\cos\alpha = \sqrt{3}$..... (ii)

Eqn. (i) ÷ eqn. (

$\tan\alpha = \frac{1}{\sqrt{3}}$; => $\alpha = 30^0$

$R^2[\cos^2\alpha + \sin^2\alpha] = \left[1^2 + \left(\sqrt{3}\right)^2\right] = 2$

$R^2 = 4$; $R = 2$

$\therefore \sqrt{3}\sin\theta + \cos\theta = 2\sin(\theta + 30^0)$

⇨ $\sqrt{3}\sin\theta + \cos\theta + 7 = 2\sin(\theta + 30^0) + 7$

The minimum value occurs when

$\sin(\theta + 30^0) = -1$

⇨ Minimum value = 2(-1) + 7 = 5

Now for $\sin(\theta + 30^0) = -1$

$\theta + 30^0 = 270^0$

$\theta = 240^0 = \frac{4\pi}{3}$

The minimum value is $\left(\frac{4\pi}{3}, 5\right)$

And maximum value occurs when

$\sin(\theta + 30^0) = 1$

⇨ Minimum value = 2(1) + 7 = 9

Now for $\sin(\theta + 30^0) = 1$

$\theta + 30^0 = 90^0$

$\theta = 60^0 = \frac{\pi}{3}$

The maximum value is $\left(\frac{\pi}{3}, 9\right)$

(b) $5\cos\theta - 12\sin\theta - 1$

Solution

Let $5\cos\theta - 1$ $\theta = R\cos(\theta - \alpha)$

$= R(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$

$= R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

Comparing coefficient of cosθ and sinθ

$R\cos\alpha = 5$ ..... (i)

$R\sin\alpha = 12$ ..... (ii)

Eqn (ii) ÷ eqn (i)

$\tan\alpha = \frac{12}{5}$; $\alpha = 67.4^0$

Eqn. $(i)^2$ + eqn. $(ii)^2$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 5^2 + 12^2 = 169$

$R^2[\cos^2\alpha + \sin^2\alpha] = 169$

$R^2 = 169$

$R = 13$

$\therefore 2\cos\theta - 12\sin\theta = 13\cos(\theta - 67.4^0)$

⇨ $5\cos\theta - 12\sin\theta - 13 = 13\cos(\theta - 67.4^0) - 13$

The minimum value occurs when

$\cos(\theta - 67.4^0) = -1$

⇨ Minimum value = 13(-1) -13 = -26

Now for $\cos(\theta - 67.4^0) = -1$

$\theta - 67.4^0 = 180^0$

$\theta = 247.4^0$

The minimum value is $(247.4^0, -26)$

And maximum value occurs when

$\cos(\theta - 67.4^0) = 1$

⇨ Minimum value = 13(1) - 13 = 0

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Now for $\cos(\theta - 67.4^0) = 1$

$\theta - 67.4^0 = 0^0$

$\theta = 67.4^0$

The maximum value is $(67.4^0, 0)$

Example 33

(a) Given that $p = 2\cos\theta + 3\cos2\theta$ and $q = 2\sin\theta + 3\sin2\theta$, show that $1 \le p^2 + q^2 \le 25$
If $p^2 + q^2 = 19$ and θ is acute, find θ and show that $pq = \frac{-5\sqrt{3}}{4}$

Solution

$p^2 = 4\cos^2\theta + 12\cos\theta\cos2\theta + 9\cos^2 2\theta$ ....(i)

$q^2 = 4\sin^2\theta + 12\sin\theta\sin2\theta + 9\sin^2 2\theta$ .....(ii)

Eqn. (i) + eqn. (ii)

$p^2 + q^2 = 4 + 12(\cos\theta\cos2\theta + \sin\theta\sin2\theta) + 9$

$p^2 + q^2 = 13 + 12\cos\theta$ [$\cos(-\theta) = \cos\theta$]

But $-1 \le \cos\theta \le 1$

Multiplying through by 12

$-12 \le 12\cos\theta \le 12$

Adding 13 throughout

$1 \le 12\cos\theta + 12$

$\therefore 1 \le p^2 + q^2 \le 25$ as required

If $p^2 + q^2 = 19$, $\Rightarrow 13 + 12\cos\theta = 19$

$\cos\theta = \frac{1}{2}$; $\theta = 60^0$ [θ is acute]

⇨ $p = 2\cos60^0 + 3\cos120^0 = 1 - \frac{3}{2} = -\frac{1}{2}$

$q = 2\sin60^0 + 3\sin120^0 = \sqrt{3} + 3\frac{\sqrt{3}}{2} = \frac{5\sqrt{3}}{2}$

$\therefore pq = \left(-\frac{1}{2}\right)\left(\frac{5\sqrt{3}}{2}\right) = \frac{5\sqrt{3}}{4}$

(b) Express $f(x) = 5\sin^2\theta - 3\sin\theta\cos\theta + \cos^2\theta$ in the form $p + q\cos(2\theta - \alpha)$
Hence show that $\frac{1}{2} \le f(x) \le 5\frac{1}{2}$
Solution

Using $\sin^2\theta = \frac{1}{2}(1 - \cos2\theta)$ and

$\cos^2\theta = \frac{1}{2}(1 + \cos2\theta)$

$f(x) = \frac{5}{2}(1 - \cos2\theta) - 3\sin\theta\cos\theta + \frac{1}{2}(1 + \cos2\theta)$

$= \frac{5}{2} - \frac{5}{2}\cos2\theta - 3.\frac{2}{2}\sin\theta\cos\theta + \frac{1}{2} + \frac{1}{2}\cos2\theta$

$= 3 - 2\cos2\theta - \frac{3}{2}\sin2\theta$

$= 3 - [2\cos\theta + \frac{3}{2}\sin2\theta]$

Now:

$3 - [2\cos\theta + \frac{3}{2}\sin2\theta] \equiv p + q\cos(2\theta - \alpha)$

$= 3 + [q\cos2\theta\cos\alpha + q\sin2\theta\sin\alpha]$

By comparing: $p = 3$, $q\sin\alpha = \frac{3}{2}$ and $q\cos\alpha = 2$

⇨ $\tan\alpha = \frac{3}{4}$, $\alpha = 36.9^0$

And $q = \sqrt{\left\{\left(\frac{3}{2}\right)^2 + (2)^2\right\}} = \frac{5}{2}$

$[2\cos\theta + \frac{3}{2}\sin2\theta] = 3 - \frac{5}{2}\cos(2\theta - 36.9^0)$

But $-1 \le \cos(2\theta - 36.9^0) \le 1$

Multiplying through by $-\frac{5}{2}$

$\frac{5}{2} \ge -\frac{5}{2}\cos(2\theta - 36.9^0) \ge -\frac{5}{2}$

Adding 3 throughout

$3 - \frac{5}{2} \le 3 - \frac{5}{2}\cos(2\theta - 36.9^0) \le 3 + \frac{5}{2}$

$\therefore \frac{1}{2} \le f(x) \le 5\frac{1}{2}$

(c) Find the maximum and minimum points of the function; $f(x) = 3\cos\theta - 4\sin\theta + 20$ for $0^0 \le \theta \le 360^0$
Solution
Let $3\cos\theta - 4\sin\theta = R\cos(\theta + \alpha)$

$= R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

Comparing coefficient of $\cos\theta$ and $\sin\theta$

$R\cos\alpha = 3$ ..... (i)

$R\sin\alpha = 4$ ..... (ii)

Eqn (ii) ÷ eqn (i)

$\tan\alpha = \frac{4}{3}$; $\alpha = 53.1^0$

Eqn. $(i)^2$ + eqn. $(ii)^2$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 3^2 + 4^2 = 25$

$R^2[\cos^2\alpha + \sin^2\alpha] = 25$

$R^2 = 25$

$R = 5$

$\therefore 3\cos\theta - 4\sin\theta = 5\cos(\theta - 53.1^0)$

⇨ $3\cos\theta - 4\sin\theta + 20 = 5\cos(\theta - 53.1^0) + 20$

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The minimum value occurs when

$\cos(\theta - 53.1^0) = -1$

⇨ Minimum value = 5(-1) +20 = 15

Now for $\cos(\theta - 53.1) = -1$

$\theta - 53.1^0 = 180^0$

$\theta = 126.8^0$

The minimum value is $(126.8^0\ , 15)$

And maximum value occurs when

$\cos(\theta - 53.1^0) = 1$

⇨ Minimum value = 5(1) 20 = 25

Now for $\cos(\theta - 53.1^0) = 1$

$\theta + 53.1^0 = 0^0, 360^0$

$\theta = -53.1^0, 306.8^0$

The maximum value is $(306.8^0, 25)$

Example 34

Find the maximum and minimum points of the following

(a) $f(\theta) = \frac{1}{3+sin\theta-2cos\theta}$

Solution

Let $\sin\theta - 2\cos\theta = R\sin(\theta - \alpha)$

$= R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$

Comparing coefficient of $\cos\theta$ and $\sin\theta$

$R\cos\alpha = 1$ ..... (i)

$R\sin\alpha = 2$ ..... (ii)

Eqn (ii) ÷ eqn (i)

$\tan\alpha = 2$; $\alpha = 63.4^0$

Eqn. $(i)^2$ + eqn. $(ii)^2$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 1^2 + 2^2 = 5$

$R^2[\cos^2\alpha + \sin^2\alpha] = 5$

$R^2 = 5$

$R = \sqrt{5}$

$\therefore \sin\theta - 2\cos\theta = \sqrt{5}\sin(\theta - 63.4^0)$

⇨ $3 + sin\theta - 2cos\theta = \sqrt{5}\sin(\theta - 63.4^0) + 3$

⇨ $f(\theta) = \frac{1}{3+\sqrt{5}\sin(\theta-63.4^0)}$

Note: for a fractional function, a maximum point is obtained when the denominator is minimum and the vice versa for the maximum point

The minimum value occurs when

$\sin(\theta - 63.4^0) = 1$

⇨ Minimum value = $\frac{1}{3+\sqrt{5}} = 0.31$

Now for $\sin(\theta - 63.4) = 1$

$\theta - 63.4^0 = 90^0$

$\theta = 153.4^0$

The minimum value is $(153.4^0. 0.31)$

And maximum value occurs when

$\sin(\theta - 63.4^0) = -1$

⇨ Maximum value = $\frac{1}{3+\sqrt{5(-1)}} = 1.31$

Now $\sin(\theta - 63.4^0) = -1$

$\theta - 63.4^0 = 270^0$

$\theta = 333.4^0$

The maximum value is $(333.4^0, 1.31)$

(b) $f(\theta) = \frac{1}{4sin\theta-3cos\theta+6}$

Solution

Let $4\sin\theta - 3\cos\theta = R\sin(\theta - \alpha)$

$= R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$

Comparing coefficient of $\cos\theta$ and $\sin\theta$

$R\cos\alpha = 4$ ..... (i)

$R\sin\alpha = 3$ ..... (ii)

Eqn (ii) ÷ eqn (i)

$\tan\alpha = 0.75$; $\alpha = 36.9^0$

Eqn. $(i)^2$ + eqn. $(ii)^2$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 3^2 + 4^2 = 25$

$R^2[\cos^2\alpha + \sin^2\alpha] = 25$

$R^2 = 25$

$R = 5$

$\therefore 4\sin\theta - 3\cos\theta = 5\sin(\theta - 36.9^0)$

⇨ $4\sin\theta - 3\cos\theta + 6 = 5\sin(\theta - 36.9^0) + 6$

⇨ $f(\theta) = \frac{1}{5\sin(\theta-36.9^0)+6}$

The minimum value occurs when

$\sin(\theta - 36.9^0) = 1$

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⇨ Minimum value = $\frac{1}{5(1)+6} = \frac{1}{11}$

Now for sin(θ – 63.4) = 1

θ – 36.9^0= 90^0

θ = 126.9^0

The minimum value is (126.9^0. $\frac{1}{11}$)

And maximum value occurs when

sin(θ – 36.9^0) = -1

⇨ Maximum value = $\frac{1}{5(-1)+6} = 1$

Now for sin(θ – 36.9^0) = -1

θ – 36.9^0 = 270^0

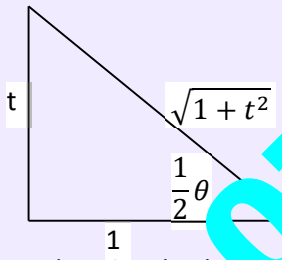
θ = 306.9^0

The maximum value is $(306.9^0, 1)$

The t-formula

Although this form has been tackled indirectly, it is formally stated here

Suppose that t = tan$\frac{\theta}{2}$, we have



From the triangle above

$cos\frac{1}{2}\theta = \frac{1}{\sqrt{1+t^2}}$ and $sin\frac{1}{2}\theta = \frac{t}{\sqrt{1+t^2}}$

But $cos\theta = cos^2\frac{1}{2}\theta - sin^2\frac{1}{2}\theta$

$= \left(\frac{1}{\sqrt{1+t^2}}\right)^2 - \left(\frac{t}{\sqrt{1+t^2}}\right)^2$

$\therefore cos\theta = \frac{1-t^2}{1+t^2}$

And $sin\theta = 2sin\frac{1}{2}\theta cos\frac{1}{2}\theta$

$= 2\left(\frac{t}{\sqrt{1+t^2}}\right)\left(\frac{1}{\sqrt{1+t^2}}\right)$

$\therefore sin\theta = \frac{2t}{1+t^2}$

The t- formula is used widely in solving equations and proving trigonometric identities. These can be extended as follows

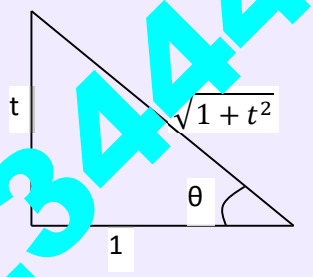
(i) For t= tanθ, sin2θ =$\frac{2t}{1+t^2}$ and $cos2\theta = \frac{1-t^2}{1+t^2}$

(ii) For t= tan$\left(\frac{5x}{4}\right)$, sin$\left(\frac{5x}{2}\right)$ =$\frac{2t}{1+t^2}$ and $cos\left(\frac{5x}{2}\right) = \frac{1-t^2}{1+t^2}$

Example 35

Show that if t = tanθ, then sin2θ =$\frac{2t}{1+t^2}$ and $2\theta = \frac{1-t^2}{1+t^2}$. Hence solve the equation $\sqrt{3}$cos2θ + sin 2θ = 1 for 0^0 ≤ θ ≤ 360^0.

Solution



From the triangle above

$cos\theta = \frac{1}{\sqrt{1+t^2}}$ and $sin\theta = \frac{t}{\sqrt{1+t^2}}$

But $cos2\theta = cos^2\theta - sin^2\theta$

$= \left(\frac{1}{\sqrt{1+t^2}}\right)^2 - \left(\frac{t}{\sqrt{1+t^2}}\right)^2$

$\therefore cos2\theta = \frac{1-t^2}{1+t^2}$

And $sin2\theta = 2sin\theta cos\theta$

$= 2\left(\frac{t}{\sqrt{1+t^2}}\right)\left(\frac{1}{\sqrt{1+t^2}}\right)$

$\therefore sin2\theta = \frac{2t}{1+t^2}$

Hence $\sqrt{3}$cos2θ + sin 2θ = 1

⇨ $\sqrt{3\left(\frac{1-t^2}{1+t^2}\right)} + \left(\frac{2t}{1+t^2}\right) = 1$

$\sqrt{3} - \sqrt{3t^2} + 2t = 1 + t^2$

$(1+\sqrt{3})t^1 - 2t + 1 - \sqrt{3} = 0$

$t = \frac{2\pm\sqrt{2^2-4(1+\sqrt{3})(1-\sqrt{3})}}{2(1+\sqrt{3})} = \frac{2\pm\sqrt{12}}{2(1+\sqrt{3})} = \frac{1\pm\sqrt{3}}{1+\sqrt{3}}$

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$t = \frac{1+\sqrt{3}}{1+\sqrt{3}} = 1$ or

$t = \left(\frac{1-\sqrt{3}}{1+\sqrt{3}}\right)\left(\frac{1-\sqrt{3}}{1-\sqrt{3}}\right) = -2 + \sqrt{3}$

If $\tan\theta = 1$; $\theta = 450, 2250$

If $\tan\theta = -2 + \sqrt{3}$; $\theta = 165^0, 345^0$

$\therefore \theta$: $\theta = 45^0, 165^0, 225^0, 345^0$

Example 36

Find all the solutions of the equation
$5\cos\theta - 4\sin\theta = 6$ for $-180^0 \le \theta \le 180^0$

Solution

Let $t = \tan\frac{\theta}{2}$ then

$cos\theta = \frac{1-t^2}{1+t^2}$

$sin\theta = \frac{2t}{1+t^2}$

⇨ $5\left(\frac{1-t^2}{1+t^2}\right) - 4\left(\frac{2t}{1+t^2}\right) = 6$

$5(1-t^2) - 8t = 6(1+t^2)$

$5 - 5t^2 - 8t = 6 + 6t^2$

$11t^2 + 8t + 1 = 0$

$t = \frac{-8\pm\sqrt{8^2-4\ x\ 11\ x\ 1}}{2\ x\ 11} = \frac{-8\pm 4.47}{2}$

$t = \frac{-8+4.4721}{22} = -0.1604$ or

$t = \frac{-8-4.4721}{22} = -0.56$

Taking t = -0.1604

$tan\frac{\theta}{2} = -0.1604;\ \theta = -18.2^0$

Taking t = -0.5669

$tan\frac{\theta}{2} = -0.5669;\ \theta = -59.1^0$

$\therefore \theta = -59.1^0, -18.2^0$

Example 37

Solve the equation

$3\tan^2\theta + 2\sec^2\theta = 2(5 - 3\tan\theta)$ for
$0^0 < \theta < 180^0$

Let $t = \tan\theta$

$3t^2 - 2(1 + t^2) = 2(5 - 3t)$

$5t^2 + 6t - 8 = 0$

$t = \frac{-6\pm\sqrt{6^2-4(5)(-8)}}{2(5)} = \frac{-6\pm 14}{10} = -2$ or $\frac{4}{5}$

Taking $t = -2$; $\theta = \tan^{-1}(-2) = 116.57^0$

Taking $t = \frac{4}{5}$; $\theta = \tan^{-1}(\frac{4}{5}) = 38.66^0$

Hence $\theta = 38.66^0, 116.57^0$

Example 38

Show that $\tan4\theta = \frac{4t(1-t^2)}{t^4-6t^2+1}$, where $t = \tan\theta$.

Solution

$\tan4\theta = \frac{2\tan 2}{1-tan^2}$ and $\tan2\theta = \frac{2\tan\theta}{1-tan^2\theta}$

$= \frac{2\left(\frac{2t}{1-t}\right)}{1-\left(\frac{2t}{1-t}\right)}$

$= \frac{4t(1-t^2)}{t^4-6t^2+1}$

Example 39

Solve the equation $\cos\theta + \sin\theta + 1 = 0$ for
$0^0 < \theta \le 360^0$

Solution

$\cos\theta + \sin\theta + 1$

Let $t = \tan\frac{\theta}{2}$

$\frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2} = 1$

$1 - t^2 + 2t = 1(1 + t^2)$

$2t + 2 = 0$; $t = -1$

$\therefore \tan\frac{\theta}{2} = -1$

$\frac{\theta}{2} = 135^0, 315^0$

$\theta = 270^0, 630^0$

Hence $\theta = 270^0$

Revision exercise 5

1. Solve equation $3\cos\theta + 4\sin\theta = 2$ for $0^0 \le \le 360^0$ [119.6^0, 346.7^0]
2. (a) Show that $\cos4x = \frac{tan^4x - 6tan^2x+1}{tan^4x+2tan^2x+1}$
   (b) Show that if $q = \cos2x + \sin2x$, then $(1+q)\tan^2x - 2\tan x + q - 1 = 0$.

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Deduce that if the roots of the above equation are $\tan x_1$ and $\tan x_2$, the $\tan(x_1 + x_2) = 1$

3. Find the values of R and tanα in each of the following equations
   (a) $2\cos\theta + 5\sin\theta = R\sin(\theta + \alpha)$ $\left[\sqrt{29}, \frac{2}{5}\right]$
   (b) $2\cos\theta + 5\sin\theta = R\cos(\theta - \alpha)$ $\left[\sqrt{29}, \frac{5}{2}\right]$
   (c) $\sqrt{3}\cos\theta + \sin\theta = R\cos(\theta - \alpha)$ $\left[2, \frac{1}{\sqrt{3}}\right]$
   (d) $5\sin\theta - 12\cos\theta = R\sin(\theta - \alpha)$ $\left[13, \frac{12}{5}\right]$
   (e) $\text{Cos}\theta - \sin\theta = R\cos(\theta + \alpha)$ $\left[\sqrt{2}, 1\right]$

4. Find the greatest and least values and state the smallest non-negative value of x for which each occurs
   (i) $12\sin x + 5\cos x$ [13, 67.4^0; -13, 247.4^0]
   (ii) $2\cos x + \sin x$
   $[\sqrt{5}, 26.6^0;\ -\sqrt{5}, 206.6^0]$
   (iii) $7 + 3\sin x - 4\cos x$
   [12, 143.1^0; 2, 323.1^0]
   (iv) $10 - 2\sin x + \cos x$
   $[10 + \sqrt{5}, 296.6^0; 10 - \sqrt{5}, 116.6^0]$
   (v) $\frac{1}{2+\sin x+\cos x}\left[\frac{2+\sqrt{2}}{2}, 225^0; \frac{2-\sqrt{2}}{2}, 45^0\right]$
   (vi) $\frac{1}{7-2\cos x+\sqrt{5}\sin x}\left[\frac{1}{4}, 311.8^0; \frac{1}{10}, [illegible].8^0\right]$
   (vii) $\frac{3}{5\cos x-12\sin x+16}$ [1, 112.6^0, $\frac{3}{9}$, 292.[illegible]^0]

5. Solve each of the following equations for $0^0 \le x \le 360^0$
   (a) $\sin x + \sqrt{3}\cos x = [illegible]$ [90^0, [illegible]30^0]
   (b) $4\sin x - 3\cos x = 2$ [[illegible]0.4^0, 193.3^0]
   (c) $\sin x + \cos x = \frac{1}{\sqrt{2}}$ [105^0, 345^0]
   (d) $5\sin x + 12\cos x = 7$ [80.0^0, 325.2^0]
   (e) $7\sin x - 4\cos x = 3$ [51.6^0, 187.9^0]
   (f) $\cos x - 3\sin x = 2$ [237.7^0, 339.2^0]
   (g) $5\cos x + 2\sin x = 4$[63.8^0, 339.8^0]
   (h) $9\cos 2x - 4\sin 2x = 6$ [14.2^0, 141.8^0, 194.2^0, 321.8^0]
   (i) $7\cos x + 6\sin x = 2$ [118.1^0, 323.1^0]
   (j) $9\cos x - 8\sin x = 12$ [313.6^0, 323.1^0]

The factor formulae

The following identities were developed from compound angles

$\cos(A + B) = \cos A\cos B - \sin A\sin B$ .....(i)

$\cos(A - B) = \text{coa}A\cos B + \sin A\sin B$ ..... (ii)

$\sin(A + B) = \sin A\cos B + \sin B\cos A$ .....(iii)

$\sin(A - B) = \sin A\cos B - \sin B\cos A$ .....(iv)

eqn. (i) + eqn (ii)

$\cos(A + B) + \cos(A - B) = 2\cos A\cos B$

eqn. (i) – eqn (ii)

$\cos(A + B) - \cos(A - B) = -2\cos A\cos B$

eqn. (iii) + eqn (iv)

$\sin(A + B) + \sin(A - B) = 2\sin A\cos B$

eqn. (iii) - eqn (iv)

$\sin(A + B) - \sin(A - B) = 2\sin B\cos A$

For simplification [illegible] B =α and A – B = β

Add: 2A [illegible] i.e. A = $\left(\frac{\alpha+\beta}{2}\right)$

Subtract 2B = α - β i.e. A = $\left(\frac{\alpha-\beta}{2}\right)$

Substituting for A and B in the above equation

$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$\cos\alpha - \cos\beta = -2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

Example 40

Show that if X, Y and Z are angles of a triangle, then

(a) $\cos X + \cos Y + \cos Z - 1 = 4\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2}$

solution

LHS $\cos X + \cos Y + \cos Z - 1$

$= 2\cos\frac{X+Y}{2}\cos\frac{X-Y}{2} + 1 - 2\sin^2\frac{Z}{2} - 1$

(to eliminate -1)

$= 2\cos\frac{180^0 - Z}{2}\cos\frac{X-Y}{2} - 2\sin^2\frac{Z}{2}$

(since X + Y = 180^0 – Z)

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$= 2\sin\frac{Z}{2}\cos\frac{X-Y}{2} - 2\sin^2\frac{Z}{2}$

(Since $\cos(90^0 - A) = \sin A$)

$= 2\sin\frac{Z}{2}\left[\cos\frac{X-Y}{2} - 2\sin^2\left\{\frac{180^0-(X+Y)}{2}\right\}\right]$

$= 2\sin\frac{Z}{2}\left[\cos\frac{X-Y}{2} - \cos\left\{\frac{X+Y)}{2}\right\}\right]$

(Since $\sin(90^0 - A) = \cos A$)

$= 2\sin\frac{Z}{2}\left[-2\sin\frac{X}{2}\sin\frac{-y}{2}\right]$

$=2\sin\frac{Z}{2}\left[2\sin\frac{X}{2}\sin\frac{Y}{2}\right]$

(Since $\sin(-A) = -\sin A$)

$4\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2}$ as required

(b) $\sin 3X + \sin 3Y + \sin 3Z = -4\cos\frac{3X}{2}\cos\frac{3Y}{2}\cos\frac{3Z}{2}$

Solution

LHS: $\sin 3X + \sin 3Y + \sin 3Z$

$= 2\sin\frac{3(X+Y)}{2}\cos\frac{3(X-Y)}{2} + 2\sin\frac{3Z}{2}\cos\frac{3Z}{2}$

$=2\sin\frac{3(180^0-Z)}{2}\cos\frac{3(X-Y)}{2} + 2\sin\frac{3Z}{2}\cos\frac{3Z}{2}$

$= -2\cos\frac{3Z}{2}\cos\frac{3(X-Y)}{2} + 2\sin\frac{3Z}{2}\cos\frac{3Z}{2}$

Since $\sin(270^0 - A) = -\cos A$

$=-2\cos\frac{3Z}{2}\left[\cos\frac{3(X-Y)}{2} - \sin\frac{3Z}{2}\right]$

$=-2\cos\frac{3Z}{2}\left[\cos\frac{3(X-Y)}{2} - \cos\frac{\{180^0-(X+Y)\}}{2}\right]$

$=-2\cos\frac{3Z}{2}\left[\cos\frac{3(X-Y)}{2} - \cos\frac{3(X+Y)}{2}\right]$

$=-2\cos\frac{3Z}{2}\left[2\cos\frac{3X}{2} + \cos\frac{-3Y}{2}\right]$

Since $\cos(-A) = \cos A$

$=- 4\cos\frac{3X}{2}\cos\frac{3Y}{2}\cos\frac{3Z}{2}$

(c) $\cos 4X + \cos 4Y + \cos 4Z + 1 = 4\cos 2X\cos 2Y\cos 2Z$

Solution

LHS: $\cos 4X + \cos 4Y + \cos 4Z + 1$

$= 2\cos 2(X + Y)\cos 2(X - Y) + 2\cos^2 2Z - 1 + 1$

$= 2\cos 2(180^0 - Z)\cos 2(X - Y) + 2\cos^2 2Z$

$= 2\cos 2Z[\cos 2(X - Y) + \cos 2\{180^0 - (X + Y)\}]$

$= 2\cos 2Z[\cos 2(X - Y) + \cos 2 (X + Y)]$

$= 2\cos 2Z[2\cos 2X \cos -2Y]$

Since $\cos(-A) = \cos A$

$=4\cos 2Z 2\cos 2X \cos 2Y$

(d) $\sin^2 Y + \sin^2 Z = 1 + \cos(Y - Z)\cos X$

LHS: $\sin^2 Y + \sin^2 Z$

$= \frac{1}{2}(1 - \cos 2Y) + \frac{1}{2}(1 - \cos 2Z)$

$=\frac{1}{2}(2 - \cos 2Y - \cos 2Z)$

$=1 - \frac{1}{2}(\cos 2Y + \cos 2Z)$

$= 1 - \cos(180^0 - X)\cos(Y - Z)$

$= 1 + \cos(Y - Z)\cos X$

Example 41

(a) Factorize $\cos\theta\cos 3\theta - \cos 7\theta + \cos 9\theta$ and express it in the form $A\cos p\theta\sin q\theta\sin r\theta$ wher A, p, q and r are constants

Solution

$f(\theta)= \cos 9\theta + \cos\theta - (\cos 7\theta + \cos 3\theta)$

$= 2\cos 5\theta\cos 4\theta - 2\cos 5\theta\cos 2\theta$

$=2\cos 5\theta(\cos 4\theta - \cos 2\theta)$

$= -4\cos 5\theta(-\sin 3\theta\sin\theta)$

$=-4\cos 5\theta\sin 3\theta\sin\theta$

⇨ A = -4, p = 5, q = 3, r = 1

(b) Given that
$p = \sin\alpha + \sin\beta$
$q = \cos\alpha + \cos\beta$. Show that

$\frac{2pq}{p^2+q^2} = \sin(\alpha + \beta)$

Solution

$\frac{2pq}{p^2+q^2}$

$=\frac{2(\sin\alpha+\sin\beta)(\cos\alpha+\cos\beta)}{\sin^2\alpha+2\sin\alpha\sin\beta+\sin^2\beta+\cos^2\alpha+2\cos\alpha\cos\beta+\cos^2\beta}$

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$=\frac{2\left[2sin\frac{\alpha+\beta}{2}cos\frac{\alpha-\beta}{2}\right]\left[2cos\frac{\alpha+\beta}{2}cos\frac{\alpha-\beta}{2}\right]}{2+2(cos\alpha cos\beta+sin\alpha sin\beta}$

$=\frac{2\left[2sin\frac{\alpha+\beta}{2}cos\frac{\alpha+\beta}{2}\right]\left[2cos^2\frac{\alpha-\beta}{2}\right]}{2+2cos(\alpha-\beta)}$

$=\frac{2[\sin(\alpha+\beta)][1+\cos(\alpha-\beta)}{2[1+\cos(\alpha-\beta)]}$

= sin(α +β)

Example 42

Solve $5\cos^2 3\theta = 3(1 + \sin 3\theta)$ for $0^0 \leq \theta \leq 90^0$.

Solution

$5\cos^2 3\theta = 3(1 + \sin 3\theta)$

$5(1 - \sin^2 3\theta) = 3(1 + \sin 3\theta)$

$5 - 5\sin^2 3\theta = 3 + 3\sin 3\theta$

$5\sin^2 3\theta + 3\sin 3\theta - 0 = 0$

$(\sin 3\theta + 1)(5\sin 3\theta - 2) = 0$

$\sin 3\theta + 1 = 0$

$3\theta = \sin^{-1}(-1) = -90^0, 270^0$

Example 43

(a) solve the equation $\cos 2x = \cos^2 x - 2\sin^2 x$ for $0 \leq \theta \leq 180^0$

$\cos 2x = 4\cos^2 x - 2\sin^2$

$\cos^2 x - \sin^2 x = 4\cos^2 - 2\sin$

$3\cos^2 x - \sin^2 x =$

$4\cos^2 x - 1 = 0$

$(2\cos x + 1)(2\cos x - 1) = 0$

Either

$2\cos x + 1 = 0$

$\cos x = -\frac{1}{2}$

$x = \cos^{-1}(-\frac{1}{2}) = 120^0$

Or

$2\cos x - 1 = 0$

$\cos x = \frac{1}{2}$

$x = \cos^{-1}(\frac{1}{2}) = 60^0$

$\therefore x(60^0, 120^0)$

Alternatively

$\cos 2x = 4\cos^2 x - 2\sin^2 x$

$= \frac{4}{2}(1 + \cos 2x) - \frac{2}{2}(1 - \cos 2x)$

$= 2 + 2\cos 2x - 1 + \cos 2x$

$2\cos 2x + 1 = 0$

$\cos 2x = -\frac{1}{2}$

$2x = \cos_{-1}(-\frac{1}{2}) = 120^0, 240^0$

$x = 60^0, 120^0$

Alternatively

$\cos 2x = 4\cos^2 x - 2\sin^2 x$

$\cos^2 x - \sin^2 x = 4\cos^2 x - 2\sin^2 x$

$3\cos^2 x - \sin^2 x = 0$

$\sin^2 x = 3\cos^2 x$

$\tan^2 x = 3$

$\tan x = \pm\sqrt{3}$

Either

$\tan x = \sqrt{3}$

$x = \tan^{-1}\sqrt{3} = 60^0$

Or

$\tan x = -\sqrt{3}$

$x = \tan^{-1} -\sqrt{3} = 120^0$

Hence $x = 60^0, 120^0$

Alternatively

$\cos 2x = 4\cos^2 x - 2\sin^2 x$

$1 - 2\sin^2 x = 4(1 - \sin^2 x) - 2\sin^2 x$

$1 = 4 - 4\sin^2 x$

$4\sin^2 x = 3$

$\sin^2 x = \frac{3}{4}$

$\sin x = \pm\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$

$x = 60^0, 120^0$

Alternatively

$\cos 2x = 4\cos^2 x - 2\sin^2 x$

$1 - 2\sin^2 x = 4\cos^2 x - 2\sin^2 x$

$4\cos^2 x = 1$

$\cos x = \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2}$

$x = 60^0, 120^0$

(b) Show that if $\sin(x + \alpha) = p\sin(x - \alpha)$ then $\tan x = \left(\frac{p+1}{p-1}\right)\tan\alpha$.

Hence solve the equation $\sin(x + \alpha) = p\sin(x - \alpha)$ for $p = 2$ and $\alpha = 20^0$.

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sinxcosα + coxsinα = p(sinxcosα - coxsinα)

cosxsinα (p + 1) = sinxcosα (p - 1)

cosxsinα$\left(\frac{p+1}{P-1}\right)$ = sinxcosα

$\frac{cosxsin\alpha}{\text{sinxcos}\alpha}\left(\frac{p+1}{p-1}\right) = \frac{sinxcos\alpha}{\text{sinxcos}\alpha}$

$tanx = \left(\frac{p+1}{P-1}\right)\tan\alpha$

For sin(x + 20^0) = 2sin(x – 20^0)

tanx = $\frac{2+1}{2-1}tan20^0$ = 3tan20^0

x = $\tan^{-1}(3tan20^0)$ = 47.52^0

Example 44

Prove that in any triangle ABC,

$\frac{\sin(A-B)}{\sin(A+B)} = \frac{a^2-b^2}{c^2}$

Solution

$\frac{a^2-b^2}{c^2} = \frac{(2Rsin\ A)^2-(2Rsin\ B)^2}{(2Rsin\ C)^2}$

$= \frac{4R^2(sin^2A-sin^2B}{4R^2sin^2C}$

$= \frac{(\sin A+\sin B)(\sin A-\sin B)}{sin^2[180^0-(A+B]}$

$= \frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right).2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}{sin^2(A+B)}$

$= \frac{\sin(A+B)\sin(A-B)}{sin^2(A+B)}$

$= \frac{\sin(A-B)}{sin(A+B)}$

Inverse trigonometric functions

Note that

(a) If θ = $\cos^{-1}\left(\frac{1}{2}\right)$ then cosθ = $\frac{1}{2}$

(b) $\tan^{-1}(tan\alpha)$= tan($\tan^{-1}\alpha$) =α

(c) $\cos^{-1}[\cos(x+y)]$

= cos[$\cos^{-1}(x+y)$ = x+y

(d) $sin(\sin^{-1}\theta) = \sin^{-1}(\sin^{-1}\theta)$

To avoid errors test the values

Example 45

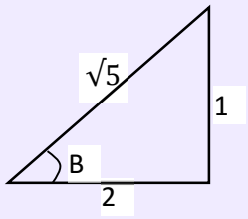
Show that

(a) $\tan^{-1}\frac{1}{3} + \sin^{-1}\frac{1}{\sqrt{5}} = \frac{\pi}{4}$

Solution

A = $\tan^{-1}\frac{1}{3}$ and B =$\sin^{-1}\frac{1}{\sqrt{5}}$

⇨ tanA = $\frac{1}{3}$ and B = $\frac{1}{\sqrt{5}}$



⇨ tan B = $\frac{1}{2}$

LHS = $\tan^{-1}\frac{1}{3} + \sin^{-1}\frac{1}{\sqrt{5}}$ = A + B

=$\tan^{-1}[\tan(A+B)]$

=$\tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{2}}{1-\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)}\right)$

=$\tan^{-1}\frac{3+3}{6-1}$

=$\tan^{-1}\frac{5}{5}$ = $\tan^{-1}1$ =$\frac{\pi}{4}$

(b) $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \frac{\pi}{4}$

Solution

Let A = $\tan^{-1}\frac{1}{3}$ and B =$\tan^{-1}\frac{1}{7}$

tanA = $\frac{1}{3}$ and tanB =$\frac{1}{7}$

LHS: $\tan^{-1}\tan(2A+B)$ but tan2A =$\frac{2\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2} = \frac{3}{4}$

$\therefore \tan^{-1}\tan(2A+B) = \frac{\frac{3}{4}+\frac{1}{7}}{1-\left(\frac{3}{4}\right)\left(\frac{1}{7}\right)}$

= $\tan^{-1}\left(\frac{21+4}{28-3}\right)$

=$\tan^{-1}1$

=$\frac{\pi}{4}$

(c) $\cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}$

Solution

Let θ= $\cos^{-1}x$; => x = cosθ = $\sin\left(\frac{\pi}{2}-\theta\right)$

sinx = $\frac{\pi}{2}-\theta$

$\therefore \cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}$

Example 46

Solve the equations

(a) $\tan^{-1}(2\theta + 1) + \tan^{-1}(2\theta - 1) = \tan^{-1}(2)$

Solution

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Let A = $\tan^{-1}(2\theta+1)$ and B =$\tan^{-1}(2\theta-1)$
⇨ tanA = $2\theta+1$ and tanB =$2\theta-1$

∴ A + B = $\tan^{-1}2$ or tan(A +B) =2

$$\frac{2\theta+1+2\theta-1}{1-(2\theta+1)(2\theta-1)}=2$$

$4\theta = 2(1 - 4\theta^2 - 1)$

$2\theta^2 + \theta - 1 = 0$

$(2\theta - 1)(\theta + 1) = 0$

$\theta = \frac{1}{2}$ or $\theta = -1$

(b) $\tan^{-1}(1+\theta) + \tan^{-1}(1-\theta) = 32$
Let A = $\tan^{-1}(1+\theta)$ and B =$\tan^{-1}(1-\theta)$
⇨ tanA = $1+\theta$ and tanB =$1-\theta$

∴ A + B = 32 or tan(A +B) = tan32

Introducing tangents

$$\frac{1+\theta+1-\theta}{1-(1+\theta)(1-\theta)}=tan32$$

$\theta^2\tan32 = 2$

$\theta = \sqrt{2cot32} = \pm1.789$

Example 47

If $x = \tan^{-1}\alpha$ and $y = \tan^{-1}\beta$;

Show that $x + y = \tan^{-1}\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)$

Solution

$\tan x = \alpha$; $\tan y = \beta$

$(x + y) = \tan[\tan^{-1}(x + y)]$

$= \tan^{-1}\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)$

Example 48

Solve the equation

$\tan^{-1}\left(\frac{1}{x-1}\right) + \tan^{-1}(x + 1) = \tan(-2)$

Solution

Let A = $\tan^{-1}\left(\frac{1}{x-1}\right)$ and B = $\tan^{-1}(x + 1)$

⇨ A + B = $\tan^{-1}(-2)$

$$\frac{\frac{1}{x-1}+(x+y)}{1-\left(\frac{1}{x-1}\right)(x+y)}=-2$$

$$\frac{1+x^2-1}{x-1-x-1}=-2$$

$\therefore x^2 = 4; x = \pm2$

Example 50

Without using tables or calculators determine the values of $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}$.

Solution

$\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}$.

$=\frac{\frac{1}{2}+\frac{1}{5}}{1-\left(\frac{1}{2}\right)\left(\frac{1}{5}\right)}+\tan^{-1}\frac{1}{8}$

$=\tan^{-1}\frac{7}{9}+\tan^{-1}\frac{1}{8}$

$=\frac{\frac{7}{9}+\frac{1}{8}}{1-\left(\frac{7}{9}\right)\left(\frac{1}{8}\right)}=\tan^{-1}\left(\frac{65}{65}\right)=\frac{\pi}{4}$

Example 51

Solve equations

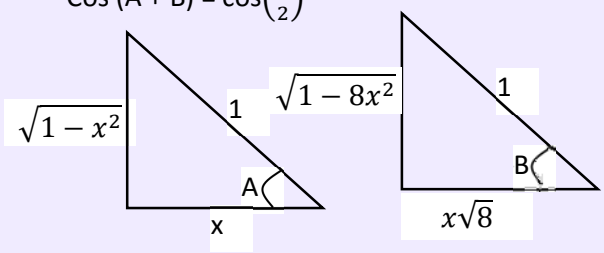
(a) $\cos^{-1}x+\cos^{-1}x\sqrt{8}=\frac{\pi}{2}$

Solution

Let A = $\cos^{-1}x$ and B =$\cos^{-1}x\sqrt{8}$

A + B = $\frac{\pi}{2}$

Cos (A + B) = $\cos\left(\frac{\pi}{2}\right)$



$$x(x\sqrt{8})-\left(\sqrt{1-x^2}\right)\left(\sqrt{1-8x^2}\right)=0$$

$$x(x\sqrt{8})=\left(\sqrt{1-x^2}\right)\left(\sqrt{1-8x^2}\right)$$

$8x^4 = (1-x^2)(1-8x^2)$

$1 - 9x^2 = 0$

$(1 - 3x)(1 + 3x) = 0$

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Either x = $\frac{1}{3}$ or x =$-\frac{1}{3}$

We discard the negative value, so the root is x = $\frac{1}{3}$

(b) $2\sin^{-1}\left(\frac{x}{2}\right) + \sin^{-1}(x\sqrt{2}) = \frac{\pi}{2}$

Solution

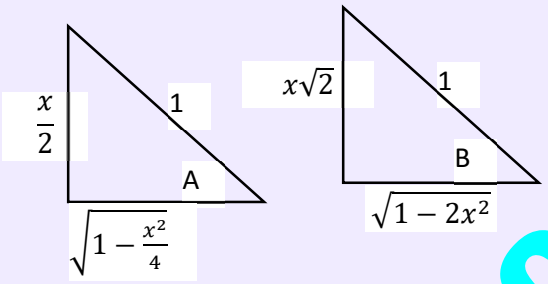
Let A = $\sin^{-1}\left(\frac{x}{2}\right)$ and B = $\sin^{-1}(x\sqrt{2})$

2A + B = $\frac{\pi}{2}$

2A = $\frac{\pi}{2} - B$

Sin(2A) = sin $\left(\frac{\pi}{2} - B\right)$

2SinAcosA = cosB



$2\left(\frac{x}{2}\right).\sqrt{1-\frac{x^2}{4}} = \sqrt{1-2x^2}$

$x.\sqrt{\frac{4-x^2}{4}} = \sqrt{1-2x^2}$

$\frac{x}{2}.\sqrt{4-x^2} = \sqrt{1-2x^2}$

$\frac{x^2}{4}.(4-x^2) = (1-2x^2)$

$X^4 - 12x^2 + 4 = 0$

$x^2 = \frac{12 \pm \sqrt{144-4(4\ x\ 1)}}{2\ x1}$

$x^2 = \frac{12 \pm \sqrt{128}}{2} = 6 \pm 4\sqrt{2}$

$x = \sqrt{6 \pm 4\sqrt{2}}$

After testing for $x = \sqrt{6+4\sqrt{2}}$ and for $x = \sqrt{6-4\sqrt{2}}$, the value that satisfies the equation is $x = \sqrt{6-4\sqrt{2}}$ = 0.5858

Hence the value of x = 0.5858

Revision exercise 6

1. If p = sinα + sin β and q = cosα + cosβ show that $\frac{p}{q} = tan\frac{\alpha+\beta}{2}$
2. (a) Prove that:
   (i) (sin2x – sinx)(1+ 2cosx) = sin3x
   (ii) cos4θ =$\frac{tan^4\theta - 6tan^2\theta + 1}{tan^4\theta + 2tan^2\theta + 1}$
   (iii) $\frac{sinx + 2sin2x + sin3x}{sinx + 2sinx + sin3x} = tan^2\frac{x}{2}$
3. Solve the equation for $0^0 \le x \le 180^0$:
   (a) sinx + sn3x + sin5x + sin7x = 0
   [x: x = 0^0, 45^0, 90^0, 135^0, 180^0]
   (b) sin7x + sinx + sin5x + sin3x = 0
   [x: x = 60^0, 180^0]
   (c) sinx + sin4x [illegible]
   [x: x= 0^0, 60[illegible] 7[illegible] 144^0, 180^0]
   (d) cos (x [illegible] 0^0) [illegible] -cos(x + 30^0) = 0
   [illegible] 0^0]
   (e) [illegible]x - sin2x = cosx
   [x: x = 0^0, 70^0, 90^0, 110^0,180^0]
   (f) sin2x +sin10x + cos4x = 0
   [x: x = 22.5^0, 35^0, 55^0, 67.5^0, 95^0, 112.5^0, 115^0, 157.5^0, 175^0]
4. Show that
   (a) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$
   (b) $2\sin^{-1}\left(\frac{1}{2}\right) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$
   (c) the positive value that satisfies the equation $\tan^{-1}3x + \tan^{-1}2x = \frac{\pi}{4}$ is $\frac{1}{6}$
   (d) $\tan^{-1}(-x) = -\tan^{-1}x$
   (e) $\cos^{-1}\left(\frac{63}{65}\right) + 2\tan^{-1}\left(\frac{1}{5}\right) = \sin^{-1}\left(\frac{3}{5}\right)$
5. Prove that
   (a) $\frac{sinA - sinB}{sinA + sinB} = tan\left(\frac{A-B}{2}\right)cot\left(\frac{A+B}{2}\right)$
   (b) Sin3x + sinx = 4sinxcos^{2}x
   (c) $\frac{sinx + sin2x + sin3x}{cosx + cos2x + cos3x} = tan2xx$
   (d) sin(A + B) – sin(A – B) =2cosAsinB
   (e) $\frac{sin5x + sinx}{sin4x + sin2x} = 2cosx - secx$
   (f) Cos3x + cosx = 4cos^{2}x – 2cosx

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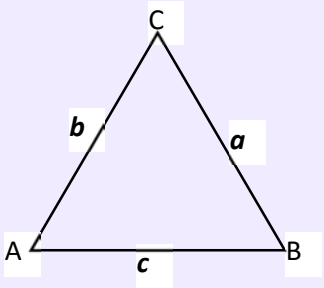
Solution to triangles

In a triangle ABC



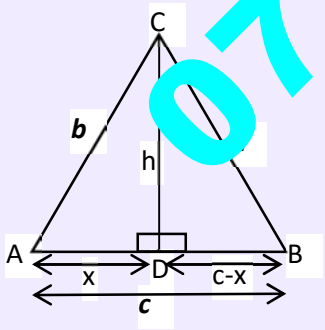
(a) Six elements are considered: three angles and three sides

Capital letters denote angles and ***small bold and italics letters*** sides

(b) The opposite side of angle A is a, of angle B is b and of angle C is c.
(c) The angle sum of a triangle is two right angles i.e. A + B + C = 180^0
(d) The sides are independent except that the sum of the two sides of the triangle should be equal to or greater than the third side

How to deal with triangles

1. The cosine rule
   (a) Given an acute angle A



From triangle
ACD: $x^2 + h^2 = \boldsymbol{b}^2$ .....(i)
BCD: $(\boldsymbol{c} - x)^2 + h^2 = \boldsymbol{a}^2$
$\boldsymbol{c}^2 - 2\boldsymbol{c}x + x^2 + h^2 = \boldsymbol{a}^2$ ..... (ii)
Substituting eqn. (i) into eqn. (ii)
$\boldsymbol{c}^2 - 2\boldsymbol{c}x + \boldsymbol{b}^2 = \boldsymbol{a}^2$
But
x = ***b***cosA
⇨ $\boldsymbol{b}^2 + \boldsymbol{c}^2 - 2\boldsymbol{bc}\cos A = a^2$
$\boldsymbol{a}^2 = \boldsymbol{b}^2 + \boldsymbol{c}^2 - 2\boldsymbol{bc}\cos A$ (1)

Similarly;

$\boldsymbol{b}^2 = \boldsymbol{a}^2 + \boldsymbol{c}^2 - 2\boldsymbol{ac}\cos B$ (2)
$\boldsymbol{c}^2 = \boldsymbol{a}^2 + \boldsymbol{b}^2 - 2\boldsymbol{ab}\cos C$ (3)

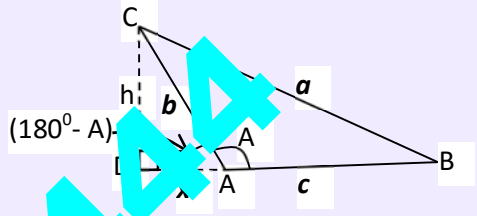
It follows that

$$cosA = \frac{b^2+c^2-a^2}{2bc}$$

$$cosB = \frac{a^2+c^2-b^2}{2ac}$$

$$cosC = \frac{a^2+b^2-c^2}{2ab}$$

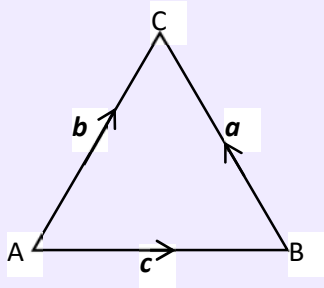
(b) Given an obtuse angle A



In triangle ABC, A is an obtuse angle and CD is the altitude.

From triangle
ACD: $x^2 + h^2 = \boldsymbol{b}^2$ .....(i)
BCD: $(\boldsymbol{c} - x)^2 + h^2 = \boldsymbol{a}^2$
$\boldsymbol{c}^2 - 2\boldsymbol{c}x + x^2 + h^2 = \boldsymbol{a}^2$ ..... (ii)
Substituting eqn. (i) into eqn. (ii)
$\boldsymbol{c}^2 - 2\boldsymbol{c}x + \boldsymbol{b}^2 = \boldsymbol{a}^2$
But
x = ***b***cos $(180^0 - A)$ = -***bc***cosA
From triangle ACD
$\boldsymbol{a}^2 = \boldsymbol{b}^2 + \boldsymbol{c}^2 - 2\boldsymbol{bc}\cos A$ as before

The cosine rule can be derived using the vector approach.



Given a triangle above with BC = ***a***, AC = ***c*** and AB = ***b***

BC = BA + AC = AC – AB

a = ***b*** – ***c***

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⇨ $\boldsymbol{a}.\boldsymbol{a} = (\boldsymbol{b} - \boldsymbol{c})(\boldsymbol{b} - \boldsymbol{c})$

$= \boldsymbol{b}.\boldsymbol{b} - 2\boldsymbol{b}.\boldsymbol{c} + \boldsymbol{c}.\boldsymbol{c}$

$= \boldsymbol{b}.\boldsymbol{b} + \boldsymbol{c}.\boldsymbol{c} - 2\boldsymbol{b}.\boldsymbol{c}$

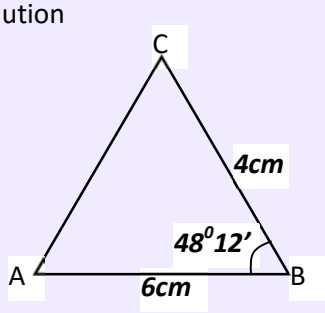
$\therefore \boldsymbol{a}^2 = \boldsymbol{b}^2 + \boldsymbol{c}^2 - 2\boldsymbol{bc}\cos A$

since $\boldsymbol{b}.\boldsymbol{c} = |bc|\cos A$

Example 52

Solve the triangle in which AB = 6cm, BC = 4cm and angle ACB = $48^0 12'$

Solution



Using: $\boldsymbol{b}^2 = \boldsymbol{a}^2 + \boldsymbol{c}^2 - 2\boldsymbol{ac}\cos B$

$= 6^2 + 4^2 - 2(6)(4)\cos 48.2^0$

1^0 (degree) = 60' (minutes)

b = 4.47cm

Using: $\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{20.0+36-16}{2(4.47)(6)}$

A =41.8^0

But A + B + C = 180^0

$41.8^0 + 48.2^0 + C = 180$

C = 90^0

∴ AC = 4.47cm, angles BAC = 41.8^0 and ACB = 90^0

Example 53

In a triangle ABC, prove that

(a) $a^2 = (b - c)^2 + 4bc\sin^2\left(\frac{A}{2}\right)$ hence that

a = (b – c)secα where $\tan\alpha = \frac{\sqrt{bc}\sin\left(\frac{A}{2}\right)}{b-c}$

From $\cos A = 1 - 2\sin^2\left(\frac{A}{2}\right)$

Substituting for cosA into the cosine formula $\boldsymbol{a}^2 = \boldsymbol{b}^2 + \boldsymbol{c}^2 - 2\boldsymbol{bc}\cos A$

$\boldsymbol{a}^2 = \boldsymbol{b}^2 + \boldsymbol{c}^2 - 2\boldsymbol{bc}[1 - 2\sin^2\left(\frac{A}{2}\right)]$

$\boldsymbol{a}^2 = \boldsymbol{b}^2 + \boldsymbol{c}^2 - 2\boldsymbol{bc} + 4\sin^2\left(\frac{A}{2}\right)$

$a^2 = (b - c)^2 + 4bc\sin^2\left(\frac{A}{2}\right)$

Hence, substituting for $\sin^2\left(\frac{A}{2}\right)$ into tanα expression we get

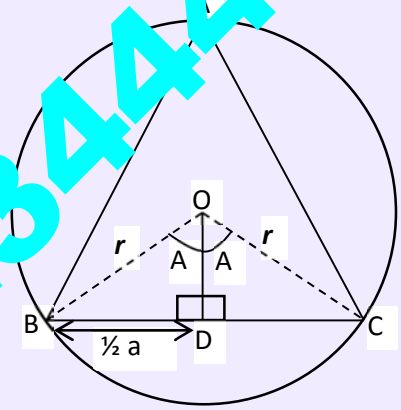
$a^2 = (b - c)^2 + (b - c)^2\tan^2\alpha$

$a^2 = (b - c)^2(1 + \tan^2\alpha)$

$a^2 = (b - c)^2\sec^2\alpha$

a = (b – c)secα

2. The Sine Rule



The figure shows a circle with centre O and radius r circumscribing triangle ABC

Angle BOC = 2A [angle subtended by the same arc at the centre of the circle is twice the angle formed at any point on the circumference]

Triangle BOC is isosceles

OD bisects angle BOC and side BC

∴BD = ½a

From triangle BOD

$\sin A = \frac{a}{2r}$ i.e. $\frac{a}{\sin A} = 2r$

if instead we consider triangles AOC and AOB, we obtain $\frac{b}{\sin B} = 2r$ and $\frac{c}{\sin C} = 2r$

In general: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

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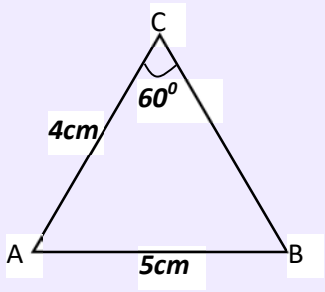
Example 54

Solve the triangle in which AB = 5cm, AC = 4cm and angle ACB = 600

Solution



Using sine rule

$\frac{b}{sinB} = \frac{c}{sinC}$ => B = $\sin^{-1}\left(\frac{bsinC}{c}\right)$

B = $\sin^{-1}\left(\frac{4}{5}sin60^0\right)$ = 43.9^0

From A + B + C+ = 180^0

A = $(180 - 60 - 43.9)^0 = 76.1^0$

Similarly a = $\frac{bsinA}{sinB} = \frac{4sin76.1^0}{sin43.9^0}$ = 5.6cm

$\therefore\ \overline{AB} = 5.6cm, B\hat{A}C = 76.1^0, A\hat{B}C = 43.9^0$

Example 55

Prove that in any triangle

$\frac{a^2-b^2}{c^2} = \frac{\sin(A-B)}{\sin(A+B)}$

Solution

From sine rule formula;
a = 2rsinA, b = 2rsinB, c = 2rsinC

By substitution

$\frac{a^2-b^2}{c^2} = \frac{(2rsinA)^2-(2rsinB)^2}{(2rsinC)^2} = \frac{sin^2A-sin^2B}{sin^2C}$

But A + B + C = 180^0

C = 180^0 – (A + B)

sinC = sin[180^0 – (A + B)] = sin (A + B)

By substitution

$\frac{a^2-b^2}{c^2} = \frac{sin^2A-sin^2B}{\sin(A+B)} = = \frac{(sinA+\sin B)(sinA-sinB}{\sin(A+B)}$

$= \frac{2sin\frac{1}{2}(A+B)cos\frac{1}{2}(A-B).2cos\frac{1}{2}(A+B)sin\frac{1}{2}(A-B)}{\sin(A+B)}$

$= \frac{2sin\frac{1}{2}(A-B)cos\frac{1}{2}(A-B)}{} = \frac{\sin(A-B)}{\sin(A+B)}$

Hence $\frac{a^2-b^2}{c^2} = \frac{\sin(A-B)}{\sin(A+B)}$

Example 56

Prove that in any triangle ABC,

$sin\frac{1}{2}(B-C) = \frac{b-c}{a}cos\frac{1}{2}A$

Solution

From sine rule formula;
a = 2rsinA, b = 2rsin[illegible] 2rsinC

By substitution

$\frac{b-c}{a} = \frac{2r[illegible]B-2[illegible]C}{2[illegible]A} = \frac{SinB-sinC}{sinA}$

A + B + C = 180^0

A = 180 – (B + C)

sinA = sin[180^0 – (B + C)] = sin(B + C)

By substitution

$\frac{b-c}{a} = \frac{SinB-sinC}{sinA} = \frac{SinB-sinC}{\sin(B+C)}$

$= \frac{2cos\frac{1}{2}(B+C)sin\frac{1}{2}(B+C)}{2cos\frac{1}{2}(B+C)cos\frac{1}{2}(B+C)}$

$= \frac{sin\frac{1}{2}(B+C)}{cos\frac{1}{2}(B+C)}$

From A + B + C = 180^0

B + C = 180^0 – A

$\frac{1}{2}(B+C) = \left(90^0 - \frac{1}{2}A\right)$

$sin\frac{1}{2}(B+C) = sin\left(90^0 - \frac{1}{2}A\right) = cos\frac{1}{2}A$

By substitution

$\frac{b-c}{a} = \frac{sin\frac{1}{2}(B+C)}{cos\frac{1}{2}A}$

$\therefore sin\frac{1}{2}(B-C) = \frac{b-c}{a}cos\frac{1}{2}A$

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3. The Tangent Rule

It states that in a triangle ABC

$$tan\frac{1}{2}(A-B) = \left(\frac{a-b}{a+b}\right)cot\frac{1}{2}C$$

$$tan\frac{1}{2}(C-A) = \left(\frac{c-a}{c+a}\right)cot\frac{1}{2}B$$

$$tan\frac{1}{2}(b-c) = \left(\frac{b-c}{b+c}\right)cot\frac{1}{2}A$$

Proof

From $\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC}$

a = 2rsinA, b = 2rsinB, c = 2rsinC

$$\frac{a-b}{a+b} = \frac{2rsinA-2rsinB}{2rsinA+2rsinB} = \frac{sinA-sinB}{sinA+sinB}$$

$$= \frac{2cos\frac{1}{2}(A+B)sin\frac{1}{2}(A-B)}{2sin\frac{1}{2}(A+B)cos\frac{1}{2}(A-B)}$$

$$= \frac{2\cos(90-\frac{1}{2}C)sin\frac{1}{2}(A-B)}{2\sin(90-\frac{1}{2}C)cos\frac{1}{2}(A-B)}$$

$$= \frac{\cos(90-\frac{1}{2}C)tan\frac{1}{2}(A-B)}{\sin\frac{1}{2}(90-C)}$$

$$= \frac{\sin\frac{1}{2}Ctan\frac{1}{2}(A-B)}{\cos\frac{1}{2}C}$$

$$\frac{a-b}{a+b} = tan\frac{1}{2}Ctan\frac{1}{2}(A-B)$$

$$\therefore tan\frac{1}{2}(A-B) = \left(\frac{a-b}{a+b}\right)cot\frac{1}{2}C$$

Example 56

Show that in a triangle PQR

$$tan\frac{1}{2}(Q-C) = \left(\frac{q-r}{q+r}\right)cot\frac{1}{2}P$$

Hence solve the triangle in which q =15.32, r = 28.6 and P = $39^0 52'$

Solution

From $\frac{p}{sinP} = \frac{q}{sinQ} = \frac{r}{sinR}$

p = 2rsinP, q = 2rsinQ, r = 2rsinR

$$\frac{q-r}{q+r} = \frac{2rsinQ-2rsinR}{2rsinQ+2rsinR} = \frac{sinQ-sinR}{sinQ+sinR}$$

$$= \frac{2cos\frac{1}{2}(Q+R)sin\frac{1}{2}(Q-R)}{2sin\frac{1}{2}(Q+R)cos\frac{1}{2}(Q-R)}$$

$$= \frac{2\cos(90-\frac{1}{2}P)sin\frac{1}{2}(Q-R)}{2\sin(90-\frac{1}{2}P)cos\frac{1}{2}(Q-R)}$$

$$= \frac{\cos(90-\frac{1}{2}P)tan\frac{1}{2}(Q-R)}{\sin\frac{1}{2}(90-P)}$$

$$= \frac{\sin\frac{1}{2}Ptan\frac{1}{2}(Q-R)}{\cos\frac{1}{2}P}$$

$$\frac{q-r}{q+r} = tan\frac{1}{2}Ptan\frac{1}{2}(Q-R)$$

$$\therefore tan\frac{1}{2}(Q-R) = \left(\frac{q-r}{q+r}\right)cot\frac{1}{2}P$$

Hence

$$tan\frac{1}{2}(Q-R) = \frac{15.32-28.6}{15.32+29.6}cot39^0 52'$$

= -0.3621

$\frac{1}{2}(Q-R) = -19.9^0$ i.e. Q – R = -39.9^0

But P + Q + R = 180

Q + R = 180 - 39.9 = 140.1^0

Solving Q = 50.15^0 and R = 89.95^0

Now p = $\frac{qsinP}{sinQ} = \frac{15.32\sin\left[39+\frac{52}{60}\right]^0}{\sin 50.15}$ = 12.79

∴p = 12.79, Q =50.15^0, R = 89.95^0

Example 57

Show that $\frac{a+b-c}{a+b+c} = tan\frac{1}{2}Atan\frac{1}{2}B$

Solution

LHS = $\frac{a+b-c}{a+b+c}$

$$=\frac{2rsinA+2rsinB-2rsinC}{2rsinA+2rsinB+2rsinC}$$

$$= \frac{sinA+sinB-sinC}{sinA+sinB+sinC}$$

$$= \frac{2sin\frac{1}{2}(A+B)cos\frac{1}{2}(A-B)-2sin\frac{1}{2}Ccos\frac{1}{2}C}{2sin\frac{1}{2}(A+B)cos\frac{1}{2}(A-B)-2sin\frac{1}{2}Ccos\frac{1}{2}C}$$

$$= \frac{2sin\left(90-\frac{1}{2}C\right)cos\frac{1}{2}(A-B)-2sin\frac{1}{2}Ccos\frac{1}{2}C}{2sin\left(90-\frac{1}{2}C\right)cos\frac{1}{2}(A-B)-2sin\frac{1}{2}Ccos\frac{1}{2}C}$$

$$= \frac{2cos\frac{1}{2}Ccos\frac{1}{2}(A-B)-2sin\frac{1}{2}Ccos\frac{1}{2}C}{2os\frac{1}{2}Ccos\frac{1}{2}(A-B)-2sin\frac{1}{2}Ccos\frac{1}{2}C}$$

$$= \frac{cos\frac{1}{2}(A-B)-sin\frac{1}{2}C}{cos\frac{1}{2}(A-B)+sin\frac{1}{2}C}$$

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$$=\frac{\cos\frac{1}{2}(A-B)-\sin\left(90-\frac{1}{2}(A+B)\right)}{\cos\frac{1}{2}(A-B)+\sin\left(90-\frac{1}{2}(A+B)\right)}$$

$$=\frac{\cos\frac{1}{2}(A-B)-\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)+\cos\frac{1}{2}(A+B)}$$

$$=\frac{-2\sin\frac{1}{2}A\sin\left(-\frac{1}{2}B\right)}{\cos\frac{1}{2}A+\cos\frac{1}{2}B}$$

$$=\tan\frac{1}{2}A\tan\frac{1}{2}B$$

Expressions for sinA, $\sin\frac{1}{2}A$ and $\cos\frac{1}{2}A$ in terms of the sides of the triangle

(a) sinA

From the identity

$\sin^2 A = 1 - \cos^2 A = (1 - \cos A)(1 + \cos A)$

$$=\left(1-\frac{b^2+c^2-a^2}{2bc}\right)\left(1+\frac{b^2+c^2-a^2}{2bc}\right)$$

$$=\left(\frac{2bc-b^2+c^2-a^2}{2bc}\right)\left(\frac{2bc+^2+c^2-a^2}{2bc}\right)$$

$$=\frac{[a^2-(b-c)^2][(b-c)^2-a^2]}{4b^2c^2}$$

$\therefore \sin^2 A = \frac{(a+c-b)(a+b-c)(b+c-a)(b+c+a)}{4b^2c^2}$

Let $s = \frac{1}{2}[perimeter\ of\ triangle]$

$$=\frac{1}{2}[a+b+c]$$

2s =[a + b + c]

a + b = 2s – c; i.e. a + b – c = 2s – c – c = 2(s – c)

a + c= 2s – b; i.e. a + c – b = 2s – b – b =2(s – b)

b + c = 2s – a; i.e. b + c – a = 2s- a – a = 2(s – a)

$\therefore \sin^2 A = \frac{2(s-b).2(s-c).2(s-a).2s}{4b^2c^2}$

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

Similarly, $\sin B = \frac{2}{ac}\sqrt{s(s-a)(s-b)(s-c)}$

$$\sin C = \frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}$$

(b) $\sin\frac{1}{2}A$ and $\cos\frac{1}{2}A$

From $\sin^2\frac{1}{2}A = \frac{1}{2}(1-\cos A)$

$$=\frac{1}{2}\left(1-\frac{b^2+c^2-a^2}{2bc}\right)$$

$$=\left(\frac{2bc-b^2+c^2-a^2}{4bc}\right)$$

$$=\left(\frac{a^2-(b-c)^2}{4bc}\right)$$

$$=\left(\frac{(a+c-b)(a+b-c)}{4bc}\right)$$

$$=\left(\frac{2(s-b).2(s-c)}{4bc}\right)$$

$$=\left(\frac{(s-b)(s-c)}{bc}\right)$$

$$\therefore \sin\frac{1}{2}A=\sqrt{\left(\frac{(s-b)(s-c)}{bc}\right)}$$

Similarly;

$$\sin\frac{1}{2}B=\sqrt{\left(\frac{(s-a)(s-c)}{ac}\right)}$$

$$\sin\frac{1}{2}C=\sqrt{\left(\frac{(s-a)(s-b)}{ab}\right)}$$

Also:

$$\cos^2\frac{1}{2}A=\frac{1}{2}(1+\cos A)$$

$$=\left(\frac{2bc+b^2+c^2-a^2}{4bc}\right)$$

$$=\left(\frac{(b+c)^2-a^2}{4bc}\right)$$

$$=\left(\frac{(b+c-a)(a+b+c)}{4bc}\right)$$

$$=\left(\frac{2(s-a).2s}{4bc}\right)$$

$$=\left(\frac{s(s-a)}{bc}\right)$$

$$\therefore \cos\frac{1}{2}A=\sqrt{\left(\frac{(s-a)}{bc}\right)}$$

Similarly;

$$\cos\frac{1}{2}B=\sqrt{\left(\frac{(s-b)}{ac}\right)}$$

$$\cos\frac{1}{2}C=\sqrt{\left(\frac{(s-c)}{ab}\right)}$$

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The expression for $tan\frac{1}{2}A$ can be deduced as follows

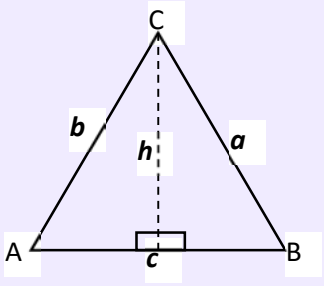
$\tan\frac{1}{2}A = \frac{sin\frac{1}{2}A}{cos\frac{1}{2}A} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

Similarly;

$\tan\frac{1}{2}B = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$

$\tan\frac{1}{2}C = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

Area of a triangle



Area, $\Delta = \frac{1}{2}(base)(perpendicular\ height)$

$= \frac{1}{2}ch$

$= \frac{1}{2}cbsinA$

Substituting for

$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$

$\Delta = \frac{1}{2}bcx\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$

$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

This a convenient form given the three sides of a triangle. The formula is called Hero's formula from the first mathematician who suggested it.

Example 58

The area of a triangle is 336m^2. The sum of the three sides is 84m and one side is 28m. Calculate the length of the remaining two sides

Solution

Given Δ = 336, a + b + c =84 and a = 28

$s = \frac{1}{2}(a+b+c) = \frac{1}{2}(84) = 42$

28 + b + c = 84

b + c = 56, or c = 56 – b

But $\Delta^2 = s(s-a)(s-b)(s-c)$

$336^2 = 42(42–28)(42– b)(42 – 56 + b)$

$b^2 – 56b + 780 = 0$

$b = \frac{56 \pm \sqrt{56^2 - 4x\ 1\ x\ 780}}{2\ x\ 1}$

b = 30 or 26

substituting for c = 56 – b

c = 26 or 30

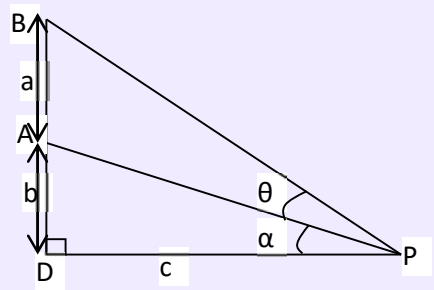
∴the remaining sides are 30m and 26m

Applications of trigonometry in finding distances and bearings

Example 59

A vertical pole BAD stands with its base D on a horizontal plane where BA = a and AD = b. A point P is situated on the horizontal plane at a distance C from D and the angle APB = θ. Prove that $\theta = \tan^{-1}\left(\frac{ac}{b^2+ab+c^2}\right)$

Solution



Let angle APD = α

For triangle APD: $\tan\alpha = \frac{b}{c}$

For triangle DPB: $\tan(\theta+\alpha) = \frac{a+b}{c}$

⇨ $\frac{tan\theta + tan\alpha}{1 - tan\theta tan\alpha} = \frac{a+b}{c}$

Substituting for tanα

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⇨ $\frac{tan\theta+\frac{b}{c}}{1-\left(\frac{b}{c}\right)tan\theta} = \frac{a+b}{c}$

$c^2tan\theta + bc = ac + bc - abtan\theta - b^2tan\theta$

$(b^2 + ab + c^2)tan\theta = ac$

$tan\theta = \frac{ac}{b^2+ab+c^2}$

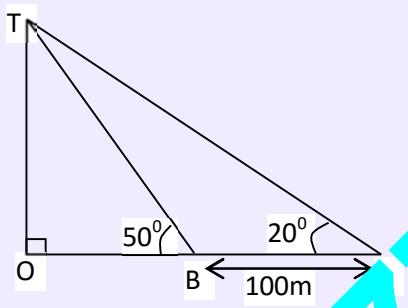
$\therefore\theta = tan^{-1}\left(\frac{ac}{b^2+ab+c^2}\right)$

Example 60

The angle of the top of a vertical tower from a point A is 20^0 and from another point B is 50^0. Given that A and B lie on the same horizontal plane in the same direction where AB = 100m. Find the height of the tower

Solution

Let OT be the height of the tower



$A\hat{T}B = 50 - 30 = 30^0$

Using sine rule

$\frac{TB}{\sin 20^0} = \frac{100}{\sin 30^0}$

$TB = \frac{100 \sin 20^0}{\sin 30^0}$

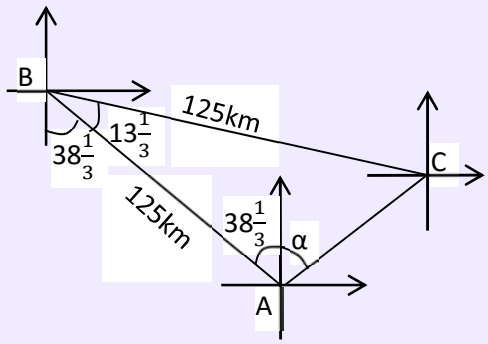
But $OT = TB\sin 50^0$

$OT = \frac{100 \sin 20^0 \sin 50^0}{\sin 30^0} = 26.2m$

Example 61

From a point A, a pilot flies in the direction $N38^0 20'W$ to point B 125km from A. He then flies in the direction $S50^0 40'E$ for 125km. He wishes to return to A from this point. How far and in what direction must he fly.

Solution



From the digram

Let $B\hat{A}C = B\hat{C}A = \theta$

⇨ $2\theta + 13\frac{1}{3}^0 = 180$

$\theta = 83\frac{1}{3}^0$

But $38\frac{1}{3}$ [illegible] = 6[illegible]

[illegible] $= 83\frac{1}{3}^0$

α = [illegible]

[illegible]om the sine rule

$\frac{AC}{sin13\frac{1}{3}} = \frac{125}{sin83\frac{1}{3}}$

AC = 29km

∴ he has to fly 29km in the direction $S45^0W$

Example 62

(a) Prove that $tan(A - B) = \frac{\tan A - tanB}{1+ tanAtanB}$

$\tan (A - B) = \frac{\sin(A-B)}{\cos(A-B)}$

$= \frac{sinA\ cosB - cosA\ sinB}{\cos AsinB + sinAsin\ B}$

Diving numerator and denominator on the R.H.S by cosAcosB

$\tan (A - B) = \frac{\frac{sinA\ cosB}{\text{cosAcosB}} - \frac{cosA\ sinB}{\text{cosAcosB}}}{\frac{\cos AcosB}{\text{cosAcosB}} + \frac{sinAsin\ B}{\text{cosAcosB}}}$

$= \frac{\tan A - tanB}{1+ tanAtanB}$

Hence show that $\frac{1-\tan 15^0}{1+\tan 15^0} = \frac{1}{\sqrt{3}}$

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$\frac{1-\tan 15^0}{1+\tan 15^0} = \frac{\tan 45^0 - tan15^0}{1+\tan 45^0 tan15^0}$

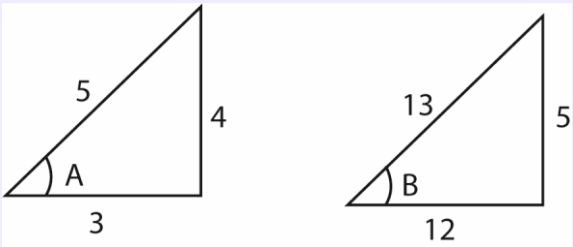
$= \tan(45^0 - 15^0)\ \tan 30^0 = \frac{1}{\sqrt{3}}$

(b) Given that cos A = $\frac{3}{5}$ and cos B = $\frac{12}{13}$ where A and B are acute, find the values of

(i) tan (A + B)

(ii) cosec (A + B)

Solution



cos A = $\frac{3}{5}$ cos B = $\frac{12}{13}$

sin A = $\frac{4}{5}$ sin B = $\frac{5}{13}$

tan A = $\frac{4}{3}$ tanB = $\frac{5}{12}$

(i) tan (A + B)=$\frac{sinA\ cosB + cosA\ sinB}{\cos A sinB - sinAsin\ B}$

$= \frac{\frac{4}{5}.\frac{12}{13} + \frac{3}{5}.\frac{5}{13}}{\frac{3}{5}.\frac{5}{13} - \frac{4}{5}.\frac{5}{13}} = 3.93$

(ii) cosec (A + B) $= \frac{1}{\sin(A+B)}$

$= \frac{1}{sin\ cos + cosAsinB}$

$= \frac{1}{\frac{4}{5}.\frac{12}{13} + \frac{3}{5}.\frac{5}{13}}$

$= 1.$

Example 63

Express cos $(\theta + 30)^0$ – cos $(\theta + 48)^0$ in the form RsinPsinQ, where R is constant.

Hence solve th3 equation

cos $(\theta + 30)^0$ – cos $(\theta + 48)^0$ = 0.2

Solution

cos $(\theta + 30)^0$ – cos $(\theta + 48)^0$

$= -2\sin\left(\frac{\theta + 30^0 + \theta + 48^0}{2}\right) \sin\left(\frac{\theta + 30^0 - \theta - 48^0}{2}\right)$

$= -2\sin(\theta + 39^0)\sin(-9^0)$

cos $(\theta + 30)^0$ – cos $(\theta + 48)^0$ = 0.

⇨ $-2\sin(\theta + 39^0)\sin(-9^0) = 0.2$

$\sin(\theta + 39^0) = 0.63925$

$\theta + 39^0 = 39.74^0$

$\theta = 0.74^0$

Example 64

Express 7cos 2θ + 6sin 2θ in form Rcos (2θ – α), where R is a constant and α is an acute angle.

7cos 2θ + 6sin 2θ ≡ Rcos (2θ – α)

7cos 2θ + 6sin 2θ ≡ Rcos 2θ cos α + Rsin2θsinα

Comparing both sides

Rcosα = 7 …… (i)

Rsinα = 6 …… (ii)

(i)2 + (ii)2 gives

$\sqrt{7^2 + 6^2} = \sqrt{85}$

From equation (i)

$\sqrt{85}\cos\alpha = 7$

$\alpha = cos^{-1}\left(\frac{7}{\sqrt{85}}\right) = 40.6^0$

Hence solve 7cos 2θ + 6sin 2θ = 5 for $0^0 \leq \theta \leq 180^0$. (07marks)

∴ 7cos 2θ + 6sin 2θ = $\sqrt{85}$ cos$(2\theta - 40.6^0)$ =5

$2\theta - 40.6 = cos^{-1}\left(\frac{5}{\sqrt{85}}\right) = 57.16^0, 302.84^0$

$\theta = 48.88^0, 171.72^0$

Revision exercise 7

1. Solve the triangles

(a) a = 17m, b = 21.42m, B = $51^0 34'$
[A =38.44^0, C = 90^0, c = 27.34m]

(b) b = 107.2m, c= 76.69m, B =$102^0 25'$
[A =33.26^0, C = 44.32^0, a = 60.21m]

(c) a = 7m, b = 3.59m, C= 47^0
[A= $103^0 2'$, B = $29^0 52'$, c= 5.25m]

(d) A = 60^0, b = 8m, C = 15
[a = 13, B= 32.2^0, C = 87.8^0]

2. Show that for all values of x

$cosx + cos\left(x + \frac{2\pi}{3}\right) + cos\left(x + \frac{3\pi}{3}\right) = 0$

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3. (a) Simplify $\frac{sin3\theta}{sin\alpha} - \frac{cos3\theta}{cos\alpha}\left[\frac{2\sin(3\theta-\alpha)}{sin2\alpha}\right]$

   (b) Express 5sinθ + 12cosθ in the form rsin(θ + α) where r and α are constant. Hence determine the minimum value of 5sinθ + 12cosθ + 7.
   [r =13, α = 67.4^0, -6]

   (c) Given that tanθ = $\frac{3}{4}$, where θ is acute, find values of tan2θ and tan$\frac{\theta}{2}$
   [tan2θ =$\frac{24}{7}$ and tan$\frac{\theta}{2}$ = $\frac{1}{3}$]

4. (a) Show that $2\tan^{-1}\left(\frac{1}{3}\right) + tan\left(\frac{1}{7}\right) = \frac{\pi}{4}$

   (b) Find x given that
   $\tan^{-1}(1+x) + \tan^{-1}(1-x) = 32$
   [x = $\pm$1.789]

   (c) Given that sinα+ sinβ = p and cosα + cos β = q
   Show that sin(α + β) = $\frac{2pq}{p^2+q^2}$

5. (a) By expressing 2sinθsin(θ + α) as a difference of cosines of two angles or otherwise, where αis constant, find the least value [minimum value = cosα – 1. It occurs when θ = $\frac{-a}{2}$]

   (b) Solve for x in the equation cosx – cos(x + 60^0) = 0.4 for 0^0 ≤ x ≤ 360^0 [x: x = 126.4^0, 3[illegible]3.6[illegible]

6. (a) Prove that in any triangle [illegible]C
   $\frac{b^2-c^2}{a^2} = \frac{\sin(B-C)}{\sin(B+C)}$

   (b) Show that for an[illegible]iso[illegible]eles triangle ABC with AB[illegible]he ba[illegible], is given by Δ =$\frac{1}{2}c\sqrt{s(s\ \ )}$ w[illegible]ere s is the perimeter of the triangle
   Given that Δ= $\sqrt{3}$ and s = 4, determine the sides of the triangle [1, 3.5, 3.5]

7. Given that $\tan^{-1}\alpha = x$ and $\tan^{-1}\beta = y$, by expressing α and β as tangents ratio of x and y and manipulating the ratios show that x + y = $\tan^{-1}\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)$
   Hence or otherwise
   (i) Solve for x in
   $\tan^{-1}\left(\frac{1}{x-1}\right) + \tan(x+1) = \tan(-2)$
   [x = $\pm$2]

   (ii) Without using tables of calculators determine the value of
   $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}\left[\frac{\pi}{4}\right]$

8. (a) Prove that $\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC} = 2R$ where ABC has all angles acute and R is the radius of the circumcircle.

   (b) From the top of a vertical cliff 10m high, the angle of depression of ship A is 10^0 and ship B is 15^0. The Bearings of A and B from the cliff are 162^0 and 202.5^0 respectively. Find the bearing of B from A [301.5^0]

9. (a) Prove that
   (sin2x –sinx)(1 + 2cosx) = sin3x

   (b) A vertical pole BAO stands with its base O on a [illegible]orizontal plane, where BA = c and A[illegible] a point P is situated on ho[illegible]nta[illegible]ane at a distance x from [illegible]angle APB = θ
   [illegible]ve [illegible]at tanθ =$\frac{cx}{x^2+b^2+bc}$
   [illegible]kes different positions on the horizontal plane, find the value of x [illegible]or which θ is greatest.
   [18^{0}26′, when x = b = c]

10. (a) Prove that sin3x = 3sinx – 4sin^{2}x.

    (b) Find all the solutions to 2sin^{2}x = 1 for 00≤ x≤ 360^0. [x =10^0, 50^0, 130^0, 170^0, 250^0, 290^0]

11. Solve cosx +$\sqrt{3}$sinx =2 for 0^0 ≤ x ≤ 360^0
    [x = 60^0]

12. From the top of a tower 12.6m high, the angles of depression of ship A and B are 12^0 and 18^0 respectively. the bearing of ship A and ship B from the tower are 148^0 and 209.5^0 respectively
    Calculate
    (i) How far the ships are from each other [53.14m]
    (ii) The bearing of ship A from ship B[108.1^0]

13. (a) Solve sin3x + $\frac{1}{2}$ = 2cos^{2}x for 0^0 ≤ x ≤ 360^0
    [x = 30^0, 60^0, 120^0, 150^0, 240^0, 300^0]

    (b) Given that in any triangle ABC, $\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}cot\left(\frac{A}{a}\right)$ solve the triangle with two sides 5 and 7 and the included angle 45^0.
    [A = 45^0, B = 89.4^0, C= 45.6^0]

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14. (a) Solve $\cot^2 x = 5(\cos cx + 1)$ for $0^0 \le x \le 360^0$ [9.6^0, 170.4^0, 270^0]
    (b) Use $\tan\frac{\theta}{2} = t$ to solve $5\sec\theta - 2\sin\theta = 2$ for $0^0 \le x \le 360^0$ [46.4^0, 270^0]
15. Given that $\sin 2x = \cos 3x$, fins the values of $\sin\theta$, $0 \le x \le \pi$ [0.309 3dp]
16. (a) Show that
    $\tan\left(\frac{A+B}{2}\right) - \tan\left(\frac{A-B}{2}\right) = \frac{2\sin B}{\cos A + \cos B}$
    (b) Find in radians the solution of the equation $\cos\theta + \sin 2\theta = \cos 3\theta$ for $0 \le \theta \le \pi$ $\left[0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\right]$
17. (a) Show that $\cot A + \tan 2A = \cot A \sec 2A$
    (b) Show that $\tan 3x = \frac{3t - t^3}{1 - 3t^2}$, where $t = \tan x$. Hence or otherwise show that $\tan^{-1}\left(\frac{\pi}{12}\right) = 2 - \sqrt{3}$
18. (a) Find all the values θ, $00 \le \theta \le 3600$, which satisfies the equation $\sin^2\theta - \sin 2\theta - 3\cos^2\theta = 0$ [$\theta = 135^0$, 315^0]
    (b) Show that $\frac{\cos x}{1 + \sin x} = \cot\left(\frac{x}{2} + 45^0\right)$. Hence or otherwise solve $\frac{\cos x}{1 + \sin x} = \frac{1}{2}$; $0^0 \le x \le 360^0$ [$x = 36.8^0$]
19. (a) Given that X, Y and Z are angles of a triangle XYZ. Prove that
    $\tan\left(\frac{X-Y}{2}\right) = \frac{x-y}{x+y}\cot\frac{Z}{2}$.
    Hence solve the triangle if [illegible] = 9[illegible]m, y = 5.7cm and Z = 57^0. [z = [illegible]cm, X= [illegible]4.4^0]
    (b) Use the substitution $t = \tan\left(\frac{\theta}{2}\right)$ to solve the equation $3\cos\theta - 5\sin\theta = -1$ for $0^0 \le \theta \le 3[illegible]^0$ [[illegible].84^0, 201.1^0]
20. Prove that
    $\tan\left(\frac{\pi}{4} + \theta\right) - \tan\left(\frac{\pi}{4} - \theta\right) = 2\tan 2\theta$
21. (a) Solve the equation $3\cos x + 4\sin x = 2$ for $0^0 \le x \le 360^0$ [$x = 119.5^0$, 346.7^0]
    (b) If A, B, C are angles of a triangle. Show that

    $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A\cos B$

22. (a) Solve $2\sin 2\theta = 3$ for $-180^0 \le x \le 180^0$ [-90^0, 48.6^0, 90^0, 131.4^0]
    (b) Solve $\sin x - \sin 4x = \sin 2x - \sin 3x$ for $-\pi \le x \le \pi$

    $\left[-\frac{\pi}{5}, -\frac{\pi}{2}, -\frac{3\pi}{5}, 0, \frac{\pi}{5}, \frac{\pi}{5}, \frac{3\pi}{5}\right]$

23. Without using tables or calculator, show that $\tan 150 = 2 - \sqrt{3}$
24. (a) Solve the equation $\cos x + \cos 2x = 1$ for $0^0 \le x \le 360^0$ [$x = 38.67^0$, 321.33^0]
    (b) (i) Prove that $\frac{\cos A + \cos B}{\sin A + \sin B} = \cot\frac{A+B}{2}$
    (ii) $\frac{\cos A + \cos B}{\sin A + \sin B} = \tan\frac{C}{2}$ where A, B and C are angles of a triangle
25. Given that $\sin(\theta - 45^0) = 3\cos(\theta + 45^0)$ show that $\tan\theta = 1$. Hence find θ if $0^0 \le \theta \le 360^0$ [45^0, 225^0]
26. (a) Use the factor formula to show that $\frac{\sin(A+2B) + \sin A}{\cos(A+2B) + \cos A} = \tan(A + B)$
    (b) Express $y = 8\cos x + 6\sin x$ in the form $R\cos(x - \alpha)$ [illegible]re R is positive and α is acut[illegible]
    Hence [illegible] the maximum and [illegible]nim[illegible] values of $\frac{1}{8\cos x + 6\sin x + 15}$
    [illegible].[illegible]04]
27. [illegible]xpr[illegible]s $\sin x + \cos x$ in the form $R\cos(x - \alpha)$. He[illegible]e, find the greatest value of [illegible]inx + cos x – 1. [0.4142]
28. (a) Solve $\cos x + \cos 3x = \cos 2x$, $0 \le x \le 360^0$ [$x = 45^0$, 60^0, 135^0, 225^0, 300^0, 315^0]
    (b) Show that $\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \frac{1 + \sin\theta}{\cos\theta}$
29. Show that $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} = \tan^{-1}\frac{7}{9}$
30. (a) Solve $3\sin x + 4\cos x = 2$ for $-180^0 \le x \le 180^0$. [-29.55^0, 103.29^0]
    (b) Show that in any triangle ABC
    $\frac{a^2 - b^2}{c^2} = \frac{\sin(A-B)}{\sin(A+B)}$
31. (a) Prove that $\frac{2\tan x}{1 + \tan^2 x} = \sin 2x$
    (b) Solve $\sin 2x = \cos x$; $0^0 \le x \le 90^0$ [$x = 30^0$, 90^0]
32. (a) Solve the equation $8\cos^4 x - 10\cos^2 x + 3 = 0$; $0^0 \le x \le 180^0$ [30^0, 45^0, 135^0, 150^0]
    (b) Prove that $\cos 4A - \cos 4B - \cos 4C = 4\sin 2B\sin 2C\cos 2A - 1$ given that A, B and C are angles of a triangle
33. Given that $\cos 2A - \cos 2B = -p$ and $\sin 2A - \sin 2B = q$, prove that $\sec(A + B) = \frac{1}{q}\sqrt{p^2 + q^2}$
34. Solve
    (a) $4\sin^2\theta - 12\sin 2\theta + 35\cos^2\theta = 0$; for $0^0 \le \theta \le 90^0$ [74.0^0]

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(b) $3\cos\theta - 2\sin\theta = 2$, for $0^0 \leq \theta \leq 360^0$
$[\theta: \theta = 22.62^0, 270.00^0]$

35. Solve the equation $\sin2\theta + \cos2\theta\cos4\theta = \cos4\theta\cos6\theta$ for $0 \leq \theta \leq \frac{\pi}{4}$. $\left[\theta = 0, \frac{3\pi}{16}\right]$

36. (a) solve the equation $\cos2x = 4\cos^2x - 2\sin^2x$ for $0 \leq \theta \leq 180^0$ $[\theta = 60^0, 120^0]$

(b) Show that if $\sin(x + \alpha) = p\sin(x - \alpha)$ then $\tan x = \left(\frac{p+1}{p-1}\right)\tan\alpha$. Hence solve the equation $\sin(x + \alpha) = p\sin(x - \alpha)$ for $p = 2$ and $\alpha = 20^0$. $[x = 47.52^0]$

37. Solve the equation
$3\tan^2\theta + 2\sec^2\theta = 2(5 - 3\tan\theta)$
for $0^0 < \theta < 180^0$ $[\theta = 38.66^0, 116.57^0]$

38. (a) Show that $\tan4\theta = \frac{4t(1-t^2)}{t^4-6t^2+1}$, where $t = \tan\theta$

(b) Solve the equation
$\sin x + \sin5x = \sin2x + \sin4x$
for $0^0 < x < 90^0$. $[x = 60^0]$

39. Solve $2\cos2\theta - 5\cos\theta = 4$
for $0^0 \leq \theta \leq 360^0$. $[\theta = 138.59^0, 221.41^0]$

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9. Complex numbers

Complex numbers

Equations $ax^2 + bx + c = 0$ where $a \neq 0$ and $b^2 < 4ac$ can only be solved by introducing $\sqrt{(-1)}$ = i or j; and this is the basis of complex numbers.

Solving equations involving complex numbers

Given $ax^2 + bx + c = 0$

$$x^2 + \left(\frac{b}{a}\right)x = \left(-\frac{c}{a}\right)$$

$$\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b^2}{4a^2} - \frac{c}{a}\right) = \frac{b^2-4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2-4ac}{4a^2}}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{4a^2}$$

If $b^2 > 4ac$; the equation has two roots which are both real.

If $b^2 = 4ac$; the equation has got two roots which are repeated and real.

If $b^2 < 4ac$; the equation has got two roots which are not real.

In complex numbers equations whose roots are not real are considered

Hence

$$x = -\frac{b}{2a} \pm i\frac{\sqrt{b^2-4ac}}{4a^2}$$

We let x = p ± iq

Where $p = -\frac{b}{2a}$ and $q = \frac{\sqrt{b^2-4ac}}{4a^2}$

Example 1

(a) $x^2 + 4 = 0$

Solution

$x^2 = -4$

$x = \sqrt{-4} = \pm 2i$

(b) $x^2 + 2x + 5 = 0$

Solution

from $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$= \frac{-2\pm\sqrt{4-4\,(1)(5)}}{2\,(1)} = \frac{-2\pm\sqrt{-16}}{2} = \frac{-2\pm4i}{2}$

x = -1 ± 2i

(c) $2x^2 - 3x + 4 = 0$

$x = \frac{3\pm\sqrt{9-4\,(2)(4)}}{2\,(2)} = \frac{3\pm\sqrt{-25}}{4} = \frac{3\pm5i}{4}$

$x = \frac{3}{4} \pm \frac{5}{4}i$

The above simple examples reveal the following

(a) A complex number Z may be written as Z = x + iy, where x and y are real numbers. The real part of Z is x and the imaginary part is y. i.e. *x = Re(Z)* and *y = im(Z).*

(b) Solution of such equations whose roots are complex occur in pairs, called conjugates. For instance, compare Z = $\pm 2i$, -1 ± 2i and $\frac{3}{4} \pm \frac{5}{4}i$

   (i) If Z = -1 + 2i is one root of an equation, then its conjugate $\bar{Z}$ = -1 – 2i is another root.

   (ii) An equation cannot have odd number of complex roots

(c) If x + iy = 0, then x = 0 and y = 0

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This implies that two complex numbers are equal if and only if their corresponding real and imaginary parts are equal
i.e. $Z_1 = x_1 + iy_1$ and $Z_2 = x_2 + iy_2$, then
$Z_1 = Z_2$, only and only if $x_1 = x_2$ and $y_1 = y_2$.
Proof: If Z1 = Z2
$x_1 + iy_1 = x_2 + iy_2$

i.e. $(x_1 - x_2) + (y_1 - y_2) = 0$

=> $x_1 - x_2 = 0$ and $y_1 - y_2 = 0$

$x_1 = x_2$ and $y_1 = y_2$

Algebra of complex numbers

Like in the real plane, complex numbers can be subjected to four basic operations of addition, subtraction, multiplications and division.

Using $Z_1 = x_1 + iy_1$ and $Z_2 = x_2 + iy_2$

Addition of complex number

When adding two or more complex numbers, real parts are added separately and imaginary parts are added separately

$Z_1 + Z_2 = (x_1 + x_2) + i(y_1 + y_2)$

Illustration

If $Z_1 = 2 + 3i$ and $Z_2 = 1 - 2i$

$Z_1 + Z_2 = (2 + 1) + i(3 + -1)$

$= 3 + 2i$

Subtraction of complex numbers

When subtracting two or more complex numbers, real parts are subtracted from real parts and imaginary parts subtracted from imaginary parts.

$Z_1 - Z_2 = (x_1 - x_2) + i(y_1 - y_2)$

Illustration

If $Z_1 = 2 + 3i$ and $Z_2 = 1 - 2i$

$Z_1 + Z_2 = (2 - 1) + i(3 - -1)$

$= 1 + 4i$

Geometrically, the operations of addition and subtraction of complex numbers are diagonals of a parallelogram. (see the Argand diagrams)

Multiplication of complex numbers

When multiplying complex numbers the following properties should be observed

$i^2 = \sqrt{-1} \times \sqrt{-1} = -1$;

$i^3 = -1 \times i = -1$

$i^4 = i^2 \times i^2 = -1 \times -1 = 1$

$i^5 = i^3 \times i^2 = -1 \times -1 = 1$

$i^6 = i^3 \times i^3 = -1 \times -1 = i^2 = -1$

$i^8 = i^4 \times i^4 = 1 \times 1 = 1$

Now

$Z_1Z_2 = (x_1 + iy_1)(x_2 + iy_2)$

$= x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2$

$= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$

Division of complex numbers

When diving a complex number with another, the result is also a complex number. It can be done in two ways

Either:

Let $\frac{Z_1}{Z_s} = p + iq$ for some real numbers p and q.

⇨ $Z_1 = (p + iq)Z_2$

$x_1 + iy_1 = (p + iq)(x_2 + iy_2)$

$= px_2 + ipy_2 + iqx_2 + i2qy_2$

$= px_2 - qy_2 + i(py_2 + qx_2)$

Equating the corresponding real and imaginary parts of the resulting complex number

$x_1 = px_2 - qy_2$ and $x_2 = py_2 + qx_2$

Solving these equation simultaneously for p and q

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p = $\frac{x_1x_2+y_1y_2}{x_2{}^2+y_2{}^2}$ and q = $\frac{x_1x_2-y_1y_2}{x_2{}^2+y_2{}^2}$

Or:

'Realizing' the denominator by multiplying through by it conjugate

$$\frac{Z_1}{Z_s}=\frac{(x_1+iy_1)(x_2-iy_2)}{(x_2+iy_2)(x_2-iy_2)}=\frac{x_1x_2-ix_1y_2-ix_2y_1\ y_1y_2}{x_2{}^2+y_2{}^2}$$

$$=\frac{x_1x_2+y_1y_2}{x_2{}^2+y_2{}^2}+i\frac{(x_1x_2-y_1y_2)}{x_2{}^2+y_2{}^2}$$

Example 2

Given that $Z_1 = 3 - 2i$ and $Z_2 = 2 + 3i$, by expressing your answer in the form a + bi, find

(a) $Z_1 + Z_2$
Solution
$Z_1 + 3Z_2 = (3 - 2i) + 3(2 + 3i) = 9 + 7i$

(b) $2Z_1 - Z2$
Solution
$2(3 - 2i) - (2 + 3i) = 4 - 7i$

(c) If Z1 = 2 – 3i and Z2 = 4 + i, find the values of

(i) $Z_1{}^3Z_2$
Solution

$Z_1{}^3Z_2 = (2 - 3i)^3(4 + i)$

Using Pascal's triangle

$= [2^3 + 3(2)^2(-3i) + 3(2)(-3i)^2 + (-3i)^3](4+ i)$

$=(8 - 36i - 54 + 27i)(4 + i)$

$=(-46 - 9i)(4 + i)$

$= -184 - 46i - 36i + 9$

$= -175 - 82i$

(ii) $\frac{Z_2{}^1}{Z_1}$

Solutions

$$\frac{Z_2{}^1}{Z_1}=\frac{(4+i)^2}{(2-3i)}=\frac{16+8i-1}{(2-3i)}=\frac{15+8i}{(2-3i)}$$

$$=\frac{(15+8i)(2+3i)}{(2-3i)(2+3i)}=\frac{30+45i+16i-24}{4+9}$$

$$=\frac{6}{13}+i\frac{61}{13}$$

Forming quadratic equations given roots

When given roots to a complex number, the quadratic equation say Z can be obtained by finding the sum and the products of the roots

i.e. the equation is given by
Z^2 – (sun of roots)Z + product of roots

Not that if Z is a root of the equation, then its conjugate $\bar{Z}$ is also a root

Example 3

Form a quadratic equation in Z given the following roots

(a) 1 + 5i
Solution
Let Z = 1 + 5i, the Z* = 1 – 5i
Sum of roots (1 + 5i) + (1 - 5i) = 2
Product of roots (1 + 5i)(1 – 5i)= 26
Equation: $Z^2 - 2Z + 26 = 0$
Or
Let Z = 1 + 5i => Z – 1 – 5i = 0
Z* = 1 – 5i => Z – 1 + 5i = 0
⇨ (Z – 1 – 5i)(Z – 1 + 5i) = 0
$Z^2 - 2Z + 26 = 0$

(b) 4 – 3i
Solution
Let Z = 4 – 3i, the Z* = 4 + 3i
Sum of roots (4 - 3i) + (4 + 3i) = 8
Product of roots (4 - 3i)(4 - 3i)= 41
Equation: $Z^2 - 8Z + 41 = 0$
Or
Let Z = 4 - 3i => Z – 4 + 3i = 0
Z = 4 + 3i => Z – 4 - 3i = 0
⇨ (Z – 4 + 3i)(Z – 4 - 3i) = 0
$Z^2 - 8Z + 41 = 0$

Solving cubic equations

It is quite difficult to solve cubic equations, however, we use the approach of inspection, i.e. we substitute the assumed factors into the given equation. If the unction is f(x) = 0 and x – a = 0 is a root, then f(a) = 0.

We then get another equation (quadratic) using long division which can easily be solved to obtain other factors.

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Example 4

Solve the following equations

(a) $x^3 - 4x^2 + x + 26 = 0$

Solution

Let $f(x) = x^3 - 4x^2 + x + 26 = 0$

$f(1) = 2 - 4 + 1 + 26 \neq 0$; => $x - 1$ is not a factor

$f(2) = 8 - 8 + 2 + 26 \neq 0$; => $x - 2$ is not a factor

$f(-2) = -8 - 16 - 2 + 26 = 0$; $x + 2$ is a factor

Using long division

$$
\begin{array}{r}
x^2 - 6x + 13 \\
(x+2)\overline{)\,x^3 - 4x^2 + x + 26} \\
-\ \underline{x^3 + 2x^2} \\
-6x^2 + x + 26 \\
-\ \underline{-6x^2 - 12x} \\
13x + 26 \\
-\ \underline{13x + 26} \\
0 + 0
\end{array}
$$

$\therefore x^3 - 4x^2 + x + 26 = (x+2)(x^2 - 6x + 13) = 0$

Either $x + 2 = 0$; $x = -2$

Or $x^2 - 6x + 13 = 0$

Solving $x^2 - 6x + 13 = 0$

$x = \frac{6 \pm \sqrt{36 - 4(1)(13)}}{2(1)} = \frac{6 \pm \sqrt{16}\, x - 1}{2} = 3 \pm 2i$

The roots of the equation are $x = -2, 3 \pm 2i$

Alternatively

Given the equation $x^2 - 6x + 13 = 0$

Let the roots be $x \pm 2i$

Sum of roots $x + iy + x - iy = 6$

$2x = 6$; $x = 3$

Product of roots $= (x + iy)(x - iy) = 13$

$x^2 + y^2 = 13$

Substituting for x;

$9 + y^2 = 13$; $y = \pm 2$

The roots of the equation are $x = -2, 3 \pm 2i$

(b) $2x^3 - 12x^2 + 25x - 21 = 0$

Solution

Let $f(x) = 2x^3 - 12x^2 + 25x - 21 = 0$

$f(1) = 2 - 12 + 25 - 12 = -6 \neq 0$; $x - 1$ is not a factor

$f(2) = 16 - 48 + 50 - 21 = -3 \neq 0$; $x - 2$ is not a factor

$f(3) = 54 - 108 + 75 - 21 = 0$; $x - 3$ is a factor

Using long division

$$
\begin{array}{r}
2x^2 - 6x + 7 \\
(x-3)\overline{)\,2x^3 - 12x^2 + 25x - 21} \\
\underline{2x^3 - 6x^2} \\
-6x^2 + 25x - 21 \\
-\ \underline{-6x^2 + 18x} \\
7x - 21 \\
-\ \underline{7x - 21} \\
0 + 0
\end{array}
$$

$\therefore\ 2x^3 - 12x^2 + 25x - 21$

$= (x-3)(2x^2 - 6x + 7) = 0$

Either $x - 3 = 0$; $x = 3$

Or $2x^2 - 6x + 7 = 0$

$x = \frac{6 \pm \sqrt{36 - 4(2)(7)}}{2(2)} = \frac{6 \pm \sqrt{4\ x\ 5\ x - 1}}{4} = \frac{3}{2} \pm \frac{\sqrt{5}}{2}$

The roots of the equation are $x = 3, \frac{3}{2} \pm \frac{\sqrt{5}}{2}$

Alternatively

Given the equation $2x^2 - 6x + 7 = 0$

Or $x^2 - 3 + \frac{7}{2} = 0$

Let the roots be $x \pm 2i$

Sum of roots $x + iy + x - iy = 3$

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$2x = 3;\ x = \frac{3}{2}$

Product of roots = $(x + iy)(x - iy) = \frac{7}{2}$

$$x^2 + y^2 = \frac{7}{2}$$

Substituting for x;

$\frac{9}{4} + y^2 = \frac{7}{2};\ y = \frac{\pm\sqrt{5}}{2}$

The roots of the equation are $x = 3, \frac{3}{2} \pm \frac{\sqrt{5}}{2}$

Finding the square root of a complex number

Suppose that the square root of Z is a + bi

Then $a + bi = Z^{\frac{1}{2}} \Rightarrow (a + bi)^2 = Z$

Example 5

Find the square roots of

(a) -5 + 12i

Solution

Approach 1

Let $\sqrt{(-5 + 12i)} = x + iy$

By squaring both sides

$-5 + 12i = x^2 - y^2 + i2xy$

Equating corresponding parts

$x^2 - y^2 = -5$ ..... (i)

$2xy = 12$

$x = \frac{6}{y}$..... (ii)

Substituting eqn.(ii) into eqn. (i)

$\left(\frac{6}{y}\right)^2 - y^2 = -5$

$36 - y^4 = -5y^2$

Let k = y2

$k^2 - 5k - 36 = 0$

$(k - 9)(k + 4) = 0$ i.e. k = 9 or k = -4

When k = -4, $y^2 = -4$ which is inadmissible since y must be real.

When k = 9; $y^2 = 9 \Rightarrow y = \pm 3$

$\Rightarrow\ x = \frac{6}{\pm 3} = \pm 2$

$\therefore \sqrt{(-5 + 12i)} = 2 + 3i$ or $-2 - 3i \equiv \pm(2 + 3i)$

Approach 2

Let $\sqrt{(-5 + 12i)} = x + iy$

$x^2 - y^2 + i2xy = -5 + 12i$

Equating corresponding parts

$x^2 - y^2 = -5$ ..... (i)

$2xy = 12$..... (ii)

Eqn. $(i)^2$ + eqn. (ii)

$x^4 - 2x^2y^2 + y^4 = 25$

$+ \quad 4x^2y^2 = 144$

$x^4 + 2x^2y^2 + y^4 = 169$

$(x^2 + y^2)^2 = 169$

$x^2 + y^2 = 13$ .....(iii)

Eqn.(i) + eqn. (iii)

$2x^2 = 8;\ x = \pm 2$

Substituting for x in eqn. (iii)

$4 + y^2 = 13;\ y = \pm 3$

$\therefore \sqrt{(-5 + 12i)} = 2 + 3i$ or $-2 - 3i \equiv \pm(2 + 3i)$

(b) 3 + 4i

Let $\sqrt{(3 + 4i)} = x + iy$

$x^2 - y^2 + i2xy = 3 + 4i$

Equating corresponding parts

$x^2 - y^2 = 3$ ..... (i) and $2xy = 4$..... (ii)

Eqn. $(i)^2$ + eqn. $(ii)^2$

$x^4 - 2x^2y^2 + y^4 = 9$

$+ \quad 4x^2y^2 = 16$

$x^4 + 2x^2y^2 + y^4 = 25$

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$(x^2 + y^2)^2 = 25$

$x^2 + y^2 = 5$ .....(iii)

Eqn.(i) + eqn. (iii)

$2x^2 = 8$; $x = \pm 2$

Substituting for x in eqn. (iii)

$4 + y^2 = 5$; $y = \pm 1$

$\therefore \sqrt{(3+4i)} = 2 + i$ or $-2 - i \equiv \pm(2+i)$

Other equations

Example 6

Solve the simultaneous equations

$Z_1 + Z_2 = 8$

$4Z_1 - 3iZ_2 = 26 + 8i$

Solution

$Z_1 + Z_2 = 8$

$Z_1 = 8 - Z_2$ ..... (i)

$4Z_1 - 3iZ_2 = 26 + 8i$..... (ii)

Substituting eqn. (i) into eqn. (ii)

$4(8 - Z_2) + 3iZ_2 = 26 + 8i$

$32 - 4Z_2 - 3iZ_2 = 26 + 8i$

$(4 + 3i)Z_2 = 6 - 8i$

$z_2 = \frac{(6-8i)(4-3i)}{(4+3i)(4-3i)} = \frac{[illegible]50i}{[illegible]} = [illegible]-2i$

Substituting for Z_2 into (i)

$Z_1 = 8 + 2i$

$\therefore Z_1 = 8 + 2i$ and $Z_2 = -2i$

Example 7

Solve the equation 6x2 – 2(1+ 2i)x – 1 = 0

Solution

Using $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{2(1+2i) \pm \sqrt{4(1+2i)^2 - 4(6)(-1)}}{2(6)}$$

$= \frac{1+2i \pm \sqrt{(3+4i)}}{6}$

But $\sqrt{(3+4i)} = \pm(2+i)$

⇨ $x = \frac{1+2i \pm (2+i)}{6}$

Either $x = \frac{1}{2} + \frac{1}{2}i$ or $x = -\frac{1}{6} + \frac{1}{6}i$

Example 8

Solve the equation $\left(\frac{Z-1}{Z+1}\right)^2 = i$, If Z is a complex number in form a + bi

Solution

$(Z - 1)^2 = i(Z + 1)^2$

$Z^2 - 2Z + 1 = iZ^2 + 2iZ$ [illegible]

$(i - 1)Z^2 + 2(i + [illegible]) = 0$

⇨ $Z^2 - [illegible]iZ[illegible] = 0$

$Z = [illegible] \frac{[illegible]-4-4}{[illegible]} = \frac{2i \pm 2i\sqrt{2}}{2} = i \pm \sqrt{2}$

$\therefore [illegible] = (1 \pm \sqrt{2})i$

Revision exercise 1

1. Simplify each of the following and express your answer in the form a + bi
   (a) (3 + 4i) + (2 + 3i) [5 + 7i]
   (b) (3 + 4i) – (2 – 3i) [1+ 7i]
   (c) $(2i)^2$ [-4]
   (d) (2 + 3i)(2 – 3i) [13]
   (e) (3 + 4i)(1 – 2i) [11 – 2i]
2. If Z is a complex number in form (a + bi) solve
   (a) $\left(\frac{Z-1}{Z+1}\right)^2 = i$ [$Z = (1 \pm \sqrt{2})i$]
   (b) $\frac{Z}{Z+1} = 1 + 2i$ $\left[-1 + \frac{1}{2}i\right]$

(c) Express each of the following complex numbers in the form a + bi
   (a) $Z_1 = (1 - i)(1 + 2i)$ [3 + i]
   (b) $Z_2 = \frac{2+6i}{3-i}$ [2i]
   (c) $Z_3 = \frac{-4i}{1-i}$ [2 – 2i]

(d) Find the square root of
   (a) 12i – 5 [2 +3i or -2 – 3i]
   (b) 5 + 2i [±(3 +2i)]
   (c) 15 + 8i [±(4 + i)]

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(d) $7 - 24i$ [$\pm(4 - 3i)$]

(e) Given that $Z = 3 + 4i$, find the value of the expression $Z + \frac{25}{Z}$ [6]

(f) Given that the complex number Z^* and its complex conjugate satisfy the equation $ZZ^* + 2iZ = 12 + 6i$, find the possible values of Z [$3 + 3i$; $3 - i$]

(g) Solve the simultaneous equation

$Z_1 + Z_2 = 8$

$4Z_1 - 3iZ_2 = 26 + 8i$

[$Z_1 = 8 + 2i$ and $Z_2 = -2i$]

(h) Express each of the following in the form $a + bi$

(a) $\frac{2+3i}{1-i}$ $\left[-\frac{1}{2} + \frac{5}{2}i\right]$

(b) $\frac{3-i}{1+2i}$ $\left[\frac{1}{5} - \frac{7}{5}i\right]$

(c) $\frac{4}{1+i}$ [$-1 + i$]

(d) $\frac{3+4i}{3-4i} - \frac{3-4i}{4+4i}$ $\left[\frac{48i}{25}\right]$

(i) Solve the following equations

(a) $2x^2 + 32 = 0$ [$\pm 4i$]

(b) $4x^2 + 9 = 0$ $\left[\pm\frac{3}{2}i\right]$

(c) $x^2 + 2x + 5 = 0$ [$-1 \pm 2i$]

(d) $x^2 - 4x + 5 = 0$ [$2 \pm i$]

(e) $2x^2 + x + 1 = 0$ $\left[\frac{-1\pm\sqrt{71}}{4}\right]$

(j) Form quadratic equations having roots

(a) $3i, -3i$ [$x^2 + 9 = 0$]

(b) $1 + 2i, 1 - 2i$ [$x^2 - 2x + 5 = 0$]

(c) $2 + I, 2 - i$[$x^2 - 4x + 5 = 0$]

(d) $3+4i, 3 - 4i$ [$x^2 - 6x + 25 = 0$]

(k) Solve the following equations if each has at least one real root

(a) $x^3 - 7x^2 + 19x - 13 = 0$ [$1, 3 \pm 2i$]

(b) $2x^3 - 2x^2 - 3x - 2 = 0$ $\left[2, -\frac{1}{2} \pm \frac{1}{2}i\right]$

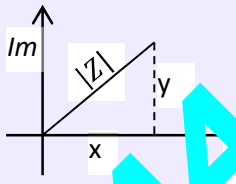
(c) $x^3 + 3x^2 + 5x + 3 = 0$ [$-1, \pm\sqrt{2}i$]

The Argand diagram

Complex numbers can be represented graphically just as coordinates are. However, instead of the x – axis and the y – axes, thereal (*Re*) and the imaginary (*Im*) exes are used respectively instead. This representation was first suggested by a mathematician named R. Argand, hence the Argand diagram. Each complex number is represented by a line of certain length in a particular direction. Thus each complex number is shown as a vector on Argand diagram.

The sum and the difference of the two complex numbers can be shown on an Argand diagram in the same way as we show vectors which are normally added or subtracted.

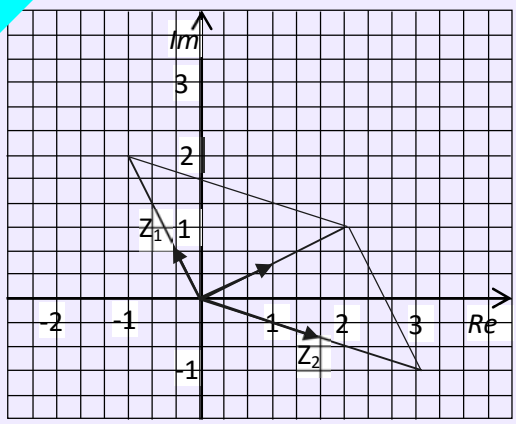
The complex number x = x + iy is represented as shown below



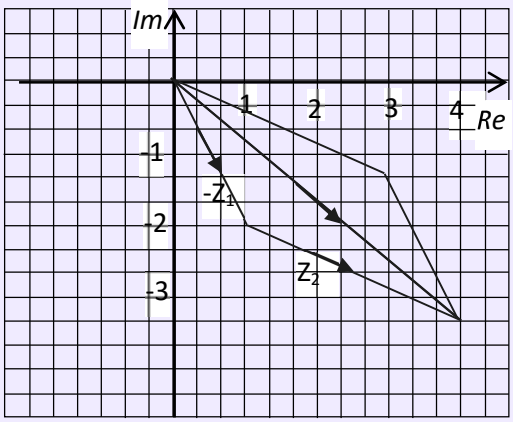
Example

Given $Z_1 = -1 + 2i$ and $Z_2 = 3 - i$ show $Z_1 + Z_2$ and $Z_2 - Z_1$ using Argand diagram

(a) $Z_1 + Z_2 = -1 + 2i + 3 - i = 2 + i$



(b) $Z_2 - Z_1 = (3 - i) - (-1 + 2i) = 4 - 3i$



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The modulus and argument of a complex number

From the above diagram

$r^2 = |Z|^2 = x^2 + y^2$

i.e. $r = |Z| = \sqrt{x^2 + y^2}$ which is modulus Z = mod(Z) and $\tan\theta = \frac{x}{y}$, i.e. $\theta = \tan^{-1}\frac{y}{x}$, which is the argument, arg(Z) or amplitude, amp(Z).

Note: for principal values, $-\pi \le Arg(Z) \le \pi$

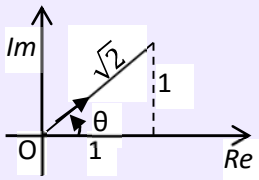
Example 9

Find the modulus and argument of the following

(a) $Z_1 = 1 + i$

Solution

$|Z_1| = \sqrt{(1^2 + 1^2)} = \sqrt{2}$

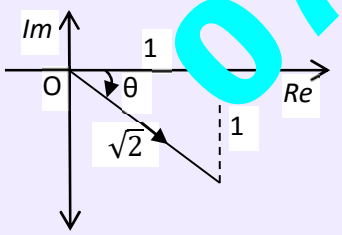


Arg $(Z_1) = \tan^{-1}\left(\frac{1}{1}\right) = 45^0 \; or \; \frac{\pi}{4}$

(b) $Z_2 = 1 - i$

Solution

$|Z_1| = \sqrt{(1^2 + (-1)^2)} = \sqrt{2}$

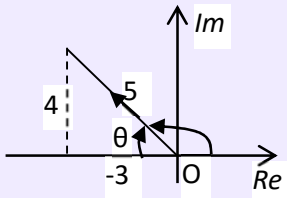


Arg$(Z_2) = \tan^{-1}(-1) = 45^0 \; or \; -\frac{\pi}{4}$

(c) $Z_3 = -3 + 4i$

Solution

$|Z_2| = \sqrt{((-3)^2 + 4^2)} = 5$

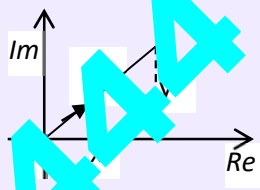


Arg$(Z_3) = \tan^{-1}\left(\frac{4}{3}\right) = 180^0 - 53.1^0 = 126.9^0$

Note: Due care must be taken in obtaining the argument of complex numbers as above example shows. Although the moduli for Z_1 and Z_2 are the same, the arguments differ because the fall in different quadrants. A sketch may be thus may be a necessary pre-requisite.

The polar (modulus – argument) form of a complex number.

Given Z = x + iy (the Cartesian form)



$x = r\cos\theta$ and $y = r\sin\theta$

$\therefore Z = x + iy = r\cos\theta + r\sin\theta$

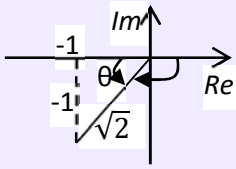
$\therefore Z = r\cos\theta + r\sin\theta$ or simply Z = rcis. This form is referred to as polar form of Z.

Example 10

Express $Z_1 = -1 - i$ and $Z_2 = -\sqrt{3} + 3i$

Solution

$(a)|Z_1| = \sqrt{((-1)^2 + (-1)^2)} = \sqrt{2}$



Arg$(Z_1) = \tan^{-1}\left(\frac{-1}{-1}\right) = 225^0 = -135^0 or -\frac{\pi}{4}$

$Z_1 = \sqrt{2}(\cos(-135^0) + isin135^0)$

$= \sqrt{2}\left(cos\frac{3\pi}{4} - isin\frac{3\pi}{4}\right)$

$(b)|Z_2| = \sqrt{\left((-\sqrt{3})^2 + 3^2\right)} = 2\sqrt{3}$

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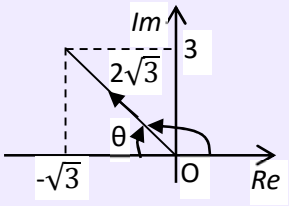

Arg(Z_2) = $\tan^{-1}\left(\frac{3}{-\sqrt{3}}\right)$= 120^0 = $\frac{2\pi}{3}$

$Z_2 = 2\sqrt{3}(\cos 120^0 + i\sin 120^0)$

$Z_2 = 2\sqrt{3}(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$

Multiplication and division of polar form

The two operations offer valuable results in complex plane

Let $|Z_1| = r_1$ and arg(Z_1) = θ_1 i.e.

$|Z_1| = r_1(cos\theta_1 + isin\theta_1)$

Let $|Z_2| = r_2$ and arg(Z_2) = θ_2 i.e.

$|Z_2| = r_2(cos\theta_2 + isin\theta_2)$

(a) Multiplication of polar form

$Z_1Z_2 = r_1r_2(cos\theta_1 + isin\theta_1)(cos\theta_2 + isin\theta_2)$

$= r_1r_2(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - \sin\theta_1\cos\theta_2)$

$= r_1r_2\{(\cos\theta_1\cos\theta_2 - \sin\theta_1\cos\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\}$

$= r_1r_2\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$

Deductions:

(i) $|Z_1Z_2| = r_1r_2 = |Z_1||Z_2|$
(ii) $arg(Z_1Z_2) = \theta_1 + \theta_1$
$= \arg(Z_1) + \arg(Z_2)$
(iii) $|Z^2| = |Z||Z| = |Z|^2$
(iv) $arg(Z^2) = \arg(Z) + \arg(Z) = 2\arg(Z)$

In general

(i) $|Z_1Z_2Z_3 \dots Z_n| = |Z_1||Z_2||Z_2| \dots |Z_n|$
(ii) $|Z^n| = |Z||Z||Z| \dots |Z| = |Z|^n$
(iii) $arg(Z_1Z_2Z_3 \dots Z_n)$

$= \arg(Z_1) + \arg(Z_2) + \arg(Z_3) \dots \arg(Z_n)$

(iv) $arg(Z^n) = n\arg(Z)$

Example 11

Given that Z_1 = 3 + 4i and Z_2 = 1 – i, find

(i) $|Z_1Z_2|$
(ii) $Arg(Z_1Z_2)$

Solution

$|Z_1| = \sqrt{3^2 + 4^2} = 5$

arg(Z_1) =$\tan^{-1}\left(\frac{4}{3}\right) = 53.1^0$

$|Z_2| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

arg(Z_2) =$\tan^{-1}\left(-\frac{1}{1}\right) = -45^0$

(i) $|Z_1Z_2| = |Z_1||Z_2| = 5\sqrt{2}$
(ii) $arg(Z_1Z_2) = \theta_1 + \theta_1$
$= 53.1^0 + (-45^0) = 8.1^0$

Example 12

Given that Z_1= 3($\cos\frac{\pi}{3}$ + i$\sin\frac{\pi}{3}$)
And Z_2 = 5($\cos\frac{\pi}{4}$ + i$\sin\frac{\pi}{4}$) find
(i) $|Z_1Z_2|$
(ii) $Arg(Z_1Z_2)$
Solution
For

Z_1= 3($\cos\frac{\pi}{3}$ + i$\sin\frac{\pi}{3}$); $|Z_1| = 3\ and\ \arg(Z_1) = \frac{\pi}{3}$

For

Z_2 = 5($\cos\frac{\pi}{4}$ + i$\sin\frac{\pi}{4}$); $|Z_2| = 5\ and\ \arg(Z_2) = \frac{\pi}{4}$

(i) $|Z_1Z_2| = |Z_1||Z_2| = 3x5 = 15$
(ii) $arg(Z_1Z_2) = \theta_1 + \theta_1 = \frac{\pi}{3} + \frac{\pi}{4} = \frac{7\pi}{12}$

(b) Division of polar form

$\frac{Z_1}{Z_2}$
$= \frac{r_1}{r_2}\left\{\frac{cos\theta_1cos\theta_2 - icos\theta_1sin\theta_2 + isincos\theta_2 - sin\theta_1sin\theta_2}{(cos^2\theta_2 + sin^2\theta_2)}\right\}$

$= \frac{r_1}{r_2}\{cos\theta_1cos\theta_2 - icos\theta_1sin\theta_2 + isincos\theta_2 - sin\theta_1sin\theta_2\}$

$= \frac{r_1}{r_2}\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$

Deduction

(a) $\frac{Z_1}{Z_2} = \frac{r_1}{r_2} = \frac{|Z_1|}{|Z_2|}$
(b) $arg\left(\frac{Z_1}{Z_2}\right) = \theta_1 - \theta_2$

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$= \arg(Z_1) - \arg(Z_2)$

Example 13

Given that $Z_1 = 3 + 4i$ and $Z_2 = 1 - i$, find

(i) $\left|\frac{Z_1}{Z_2}\right|$

(ii) $Arg\left(\frac{Z_1}{Z_2}\right)$

Solution

$|Z_1| = \sqrt{3^2 + 4^2} = 5$

$\arg(Z_1) = \tan^{-1}\left(\frac{4}{3}\right) = 53.1^0$

$|Z_2| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$\arg(Z_2) = \tan^{-1}\left(-\frac{1}{1}\right) = -45^0$

(i) $\left|\frac{Z_1}{Z_2}\right| = \frac{|Z_1|}{|Z_2|} = \frac{5}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$

(ii) $arg\left(\frac{Z_1}{Z_2}\right) = \theta_1 - \theta_1$

$= 53.1^0 - (-45^0) = 98.1^0$

Example 14

Three complex numbers are given as

$Z_1 = -1 - i$; $Z_2 = 3 - i\sqrt{3}$; $Z_3 = -1 + i\sqrt{3}$

Find

(a) $\left|\frac{Z_1}{Z_2}\right|$

(b) $arg\left(\frac{Z_2}{Z_1 Z_3}\right)$

(c) $\left|\frac{{Z_1}^2}{{Z_2}^3 Z_3}\right|$

(d) $arg\left(\frac{{Z_3}^3 {Z_2}^2}{Z_1}\right)$

Solution

$|Z_1| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

$\arg(Z_1) = \tan^{-1}\left(\frac{-1}{-1}\right) = -135^0 = -\frac{3\pi}{4}$

$|Z_2| = \sqrt{3^2 + (\sqrt{3})^2} = 2\sqrt{3}$

$\arg(Z_2) = \tan^{-1}\left(-\frac{\sqrt{3}}{3}\right) = -30^0 = -\frac{\pi}{6}$

$|Z_3| = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$

$\arg(Z_1) = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right) = 120^0 = -\frac{2\pi}{3}$

(a) $\left|\frac{Z_1}{Z_2}\right| = \frac{|Z_1|}{|Z_2|} = \frac{\sqrt{2}}{2}$

(b) $arg\left(\frac{Z_2}{Z_1 Z_3}\right) = argZ_2 - arg(Z_1 Z_3)$

$= argZ_2 - argZ_1 - argZ_3$

$= -30^0 + 135^0 - 120^0 = -15 = -\frac{\pi}{12}$

(c) $\left|\frac{{Z_1}^2}{{Z_2}^3 Z_3}\right| = \frac{|Z_1|^2}{|Z_2|^3 |Z_3|} = \frac{2}{24\sqrt{3}(2)} = \frac{\sqrt{3}}{72}$

(d) $arg\left(\frac{{Z_3}^3 {Z_2}^2}{Z_1}\right)$

$= arg({Z_3}^2) + \arg({Z_2}^2) - \arg(Z_1)$

$= 3\arg(Z_3) + 2\arg(Z_2) - \arg(Z_1)$

$= 3(120^0) - 2(30^0) - 45^0 = 75^0 = \frac{5\pi}{12}$

Note: the reader should correlate the computations of arguments Z_1Z_2 and $\frac{Z_1}{Z_2}$ with the laws of logarithms

Example 15

(a) Given that $Z_1 = 3 + 4i$, $Z_2 = 12 + 5i$ find

(i) $|Z_1 Z_2|$

(ii) $\arg(Z_1 Z_2)$

(iii) $\left|\frac{Z_1}{Z_2}\right|$

(iv) $\arg(Z_1 Z_2)$

Solution

$|Z_1| = \sqrt{(3)^2 + (4)^2} = 5$

$Arg(Z_1) = \tan^{-1}\left(\frac{4}{3}\right) = 53.1^0$

$|Z_3| = \sqrt{(12)^2 + (3)^2} = 13$

$Arg(Z_1) = \tan^{-1}\left(\frac{5}{12}\right) = 22.6^0$

(i) $|Z_1 Z_2| = |Z_1||Z_2| = 5\ x\ 13 = .65$

(ii) $\arg(Z_1 Z_2) = \arg(Z_1) + \arg(Z_2)$

$= 53.1^0 + 22.6^0 = 75.7^0$

(iii) $\left|\frac{Z_1}{Z_2}\right| = \frac{|Z_1|}{|Z_2|} = \frac{5}{13}$

(iv) $\arg\left(\frac{Z_1}{Z_2}\right) = \arg(Z_1) - \arg(Z_2)$

$= 53.1^0 - 22.6^0 = 30.5^0$

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(b) Express Z = -1 -i$\sqrt{3}$ in modulus-urgument form. Hence find $\frac{1}{Z}$ in form a + bi where a and b are real numbers

Solution

$$|Z| = \sqrt{(-1)^2 + (-\sqrt{3})^2} = 2$$

Arg(Z) =$\tan^{-1}\left(\frac{-\sqrt{3}}{-1}\right) = -120^0 = -\frac{2}{3}\pi$

$\therefore$Z =2$\left(cos\frac{2\pi}{3} - isin\frac{2\pi}{3}\right)$

Hence, $\frac{1}{Z} = Z^{-1} = 2^{-1}\left(cos\frac{2\pi}{3} - isin\frac{2\pi}{3}\right)^{-1}$

$$.= \frac{1}{2}\left(cos\left(-\frac{2\pi}{3}\right) - isin\left(-\frac{2\pi}{3}\right)\right)$$

$$= \frac{1}{2}\left(cos\left(\frac{2\pi}{3}\right) + isin\left(\frac{2\pi}{3}\right)\right)$$

$$= -\frac{1}{4} + i\frac{\sqrt{3}}{4}$$

Solving equations of higher order

When one or more roots of a complex number is(are) given, their conjugates are also the roots of the same equation

To prove that a certain root is a factor of a complex function, we either use long division to show that the quotient of division is zero or we substitute the factor given into the equation to note the remainder or we get the product of the already existing roots and then carry on long division to observe the remainder or we use the synthetic approach

Example 16

Given that (1+ 3i) and (2 – i) are roots of the equation

$aZ^4 + bZ^3 + cZ^2 + dZ + e = 0$, find

(i) The other two roots
(ii) The sum of the four roots
(iii) The product of the four roots

Solution

(i) Since 1 + 3i and 2 – I are roots of the equation, then their conjugates are also roots of the same equations. Hence the other two roots are 1 – 3i and 2 + i.

(ii) Sum of roots = (1 + 3i) + (1 – 3i) + (2 – i) + (2 + i) = 6

(iii) Products of roots
= (1 + 3i)(1 – 3i)(2 – i (2 + i) = 6
= (1 – $9i^2$)(4 – i^2)
=(1 + 9)(4 + 1)= 50

Example 17

Show that Z_1 = 2 + 3i is a root of the equation $Z^4 - 5Z^3 + 18Z^2 - 17Z + 13 = 0$

Hence find the other remaining roots

Solution

Approach

Given 2 + 3i is a root, its conjugate 2 - 3i is also a root of the equation

The equation of these two roots

Sum of roots 2 + 3i + 2 – 3i = 4

Product of roots = (2 + 3i)(2 – 3i) = 13

The equation is $Z^2 - 4Z + 13 = 0$

Using long division

Let f(Z) = $Z^4 - 5Z^3 + 18Z^2 - 17Z + 13$

$$\begin{array}{r|l} & Z^2 - Z + 1 \\ \hline Z^2 - 4Z + 13 & Z^4 - 5Z^3 + 18Z^2 - 17Z + 13 \\ & \underline{Z^4 - 4Z^3 + 13Z^2} \\ & -Z^3 + 5Z^2 - 17Z + 13 \\ & \underline{-Z^3 + 4Z^2 - 13Z} \\ & Z^2 - 4Z + 13 \\ & \underline{Z^2 - 4Z + 13} \\ & 0 + 0 + 0 \end{array}$$

Since the remainder is zero, $Z^2 - Z + 1$is also a factor of f(Z)

⇨ f(Z) = $(Z^2 - 4Z + 13)(Z^2 - Z + 1)$

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Either $Z^2 - 4Z + 13 = 0$ or $Z^2 - Z + 1 = 0$

Factors of $Z^2 - Z + 1$

$$Z = \frac{1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = \frac{1 \pm i\sqrt{3}}{2}$$

The roots are 2 ± 3i and $\frac{1 \pm i\sqrt{3}}{2}$

Approach 2

Taking Z = 2 + 3i and substituting in the given equation

LHS = $(2 + 3i)^4 - 5(2 + 3i)^3 + 18(2 + 3i)^2 - 17(2+3i) + 13 = 0$ (RHS)

Z1 = 2 + 3i is a root

This means $Z_2 = 2 - 3i$ is also a root of the equation as it is a conjugate of the given root.

Given $f(Z) = Z^4 - 5Z^3 + 18Z^2 - 17Z + 13 = 0$

Sum of roots = 5 and

Product of roots = 13

Let the other roots be α and β

Sum of the roots = 2 + 3i + 2 – 3i + α + β = 5

4 + α + β = 5

α + β = 1

α = (1 – β) ..... (i)

Product of roots = (2 + 3i)(2 – 3i)αβ = 13

13αβ = 13

αβ = 1 ..... (ii)

Substituting (i) into (ii)

(1 – β)β = 1 i.e. β2 – β + 1 = 0

$$\beta = \frac{1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = \frac{1 \pm i\sqrt{3}}{2}$$

∴ The roots are 2 ± 3i and $\frac{1 \pm i\sqrt{3}}{2}$

Approach 3: using synthetic method

2 + 3i	1	-5	18	-17	13
	0	2+3i	-15-3i	15+3i	-13
	1	-3 + 3i	3 -3i	-2+3i	0

Since the last value in the table is zero, therefore 2 + 3i is a root of the equation. Other roots can be obtained as shown above.

Illustration of the synthetic method

Procedure

(a) Write down the root and coefficients of the expression in the first row (as shown above)
(b) Write zero immediately below the first coefficient (this is the only entry in the second row that is simply written, the other are to be obtained by multiplication)
(c) At the first coefficient to zero to get the first entry in the third row
(d) Obtain the second entry in the second row multiplying 1 by 2 + 3i, then add to -5 to get the second entry in the third row i.e. -3 + 3i
(e) Repeat (d) by multiplying -3 + 3i to get the third entry in the second row, this entry is added to 18 to get to get the third row i.e. 3 – 3i. continue with this trend up to the last entry. If the last entry in the table is zero, then the value being tested is a root.

Example 18

Show that Z1 = 1 + I is a root of the equation $Z^4 - 6Z^3 + 25Z^2 - 34Z + 26 = 0$

Solution

Using synthetic division

1 + i	1	-6	25	-34	26
	0	2+i	-6-4i	21+13i	-26
	1	-5 + i	17 - 4i	-3+ 13i	0

Since the last entry in the table is zero, 1 + I is a root.

The conjugate 1 – i is also a root

Let the other roots be α and β

Sum of roots = 1 + i + 1 – i + α + β = 6

2 + α + β = 6

α + β = 4

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α = 4 – β …… (i)

Product of roots (1 + i)(1 – i)αβ = 26

2αβ = 26

Aβ = 13 …… (ii)

Substituting eqn.(i) into eqn. (ii)

(4 – β)β = 13

β^2 - 4β + 13 = 0

$\beta = \frac{4 \pm \sqrt{16-4(1)(13)}}{2(1)} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2}$

= 2 ± 3i

Hence the other roots are 2 + 3i and 2 – 3i

Revision exercise 2

1. Given Z_1 = 7 – 4i and Z_2 = -1 + 3i, express in the modulus-argument form the following
   (a) Z_1Z_2
   {r = 25.45 [cos(0.437π) + isin(0.437π)]}
   (b) $\left|\frac{Z_1}{Z_2}\right|$
   $\{p = 2.55[cos(1.233\pi) + isin(1.233\pi)]\}$
2. Find the modulus and the principal arguments of
   (a) $\frac{1-i}{1+i}\left[1, \frac{-\pi}{3}\right]$
   (b) $\frac{-1-7i}{4+3i}\left[\sqrt{2}, \frac{-3\pi}{4}\right]$
   (c) $\frac{1+i}{2-i}\left[\frac{\sqrt{10}}{5}, 12\right]$
   (d) $\frac{(3+i)^2}{1-i}\left[2\sqrt{5}, 1\right]$
3. If z_1 and z_2 are complex numbers, solve the simultaneous equation, giving your answer in the form a + bi.
   $4z_1 + 3z_2$= 23
   $z_1 + iz_2$ = 6 [2 +3i, 5 – 4i]
4. Given that 2 + I is a root of the equation z^3 – 11z + 20 = 0. Find the remaining roots [2 – i, -4]
5. Show that 2+ i is a root of the equation $2z^3 - 9z^2 + 14z - 5 = 0$. Hence find other roots [2- i, ½]
6. Find equation whose roots are -1 ± i, where i = $\sqrt{-1}$ [z^2 +2z + 2 = 0]
7. Show that z = 1 is a root of the equation $z^3 - 5z^2 + 9z - 5 = 0$ [2 – i, 2 + i]
8. The complex number satisfies $\frac{z}{z+2} = 2 - i$ find the real and imaginary parts of z, and the modulus and argument of z.
   [*Re*(z) = -3,*Im*(z) = -1, $\sqrt{10}$, -2.82rads]
9. One root of the equation z2 + az + b = 0 where a and b are real constants is 2 + 3i. Find the values of a and b [a = -4, b = 13]
10. If $z_1 = 2\left(cos\frac{2\pi}{3} + isin\frac{2\pi}{3}\right)$ and $z_2 = 6\left(cos\left(\frac{-3\pi}{4}\right) + isin\left(\frac{-3\pi}{4}\right)\right)$ find
    (a) $\left|\frac{z_1}{z_2}\right|$ $\left[\frac{1}{3}\right]$
    (b) $arg\left(\frac{z_1}{z_2}\right)$ [1
    (c) $\left|\frac{z_2}{z_1}\right|$[3
    (d) $arg\left(\frac{z_1}{z_2}\right)$ $\left[\frac{7\pi}{12}\right]$
11. If z1 and z2 are two complex numbers such that $|z_1 - z_2| = |z_1 + z_2|$. Show that the difference of their arguments is $\frac{\pi}{2}$ or $\frac{\pi}{2}$.
12. Find the modulus and argument of $\frac{(2-i)^2(3i-1}{i+3}$ [5, 0.6435 rad]
13. If $z_1 = \frac{1+7i}{1-i}$ and $z_2 = \frac{17-7i}{2+2i}$ find the moduli of
    (a) z_1 [5]
    (b) z_2 [6.5]
    (c) $z_1 + z_2$ [2.061]
    (d) z_1z_2 [32.5]

Demoivre's theorem

This states that for all rational integers

$r(\cos\theta + i\sin\theta)^n = r(\cos n\theta + i\sin n\theta)$

The theorem can be proved by induction approach or otherwise

By induction

Given that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Suppose n = 1

$\cos\theta + i\sin\theta = \cos 1\theta + i\sin 1\theta$

⇨ $\cos\theta + i\sin\theta = \cos\theta + i\sin\theta$ i.e. the proof holds for n = 1

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Suppose n = k

$(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

For n = k + 1

$(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)^1$

$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$

$= \cos k\theta\cos\theta + i\cos k\theta\sin\theta + i\sin k\theta\cos\theta + i^2\sin k\theta\sin\theta$

$= \cos k\theta\cos\theta - \sin k\theta\sin\theta + i(\cos k\theta\sin\theta + \sin k\theta\cos\theta)$

$= \cos(k+1)\theta + i\sin(k+1)\theta$

which is true when n = k + 1

Therefore the proof holds for all values of n

Using otherwise

Given $z = \cos\theta + i\sin\theta$

$Z^2 = (\cos\theta + i\sin\theta)^2$

$= \cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta$

$= \cos^2\theta - \sin^2\theta + i2\sin\theta\cos\theta$

$= \cos2\theta + i\sin2\theta$

$Z^3 = (\cos2\theta + i\sin2\theta)^2(\cos\theta + i\sin\theta)$

$= \cos2\theta\cos\theta - \sin2\theta\sin\theta + i(\sin2\theta\cos\theta + \cos2\theta\sin\theta)$

$= \cos3\theta + i\sin3\theta$

⇨ $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ hold for all positive values of n

Also, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ holds for fractional and negative indices

(a) $z^{-1} = \frac{1}{Z} = \frac{1}{\cos\theta + i\sin\theta}$

$= \left(\frac{1}{\cos\theta + i\sin\theta}\right)\left(\frac{(\cos\theta - i\sin\theta)}{(\cos\theta - i\sin\theta)}\right)$

$= \frac{(\cos\theta - i\sin\theta)}{(\cos^2\theta - i^1\sin^2\theta)} = \cos\theta - i\sin\theta$

⇨ $z^{-1} = \cos(-1\theta) + i\sin(-1\theta)$

(b) suppose n = -m

$z^{-m} = \frac{1}{Z^m} = \frac{1}{(\cos\theta + i\sin\theta)^m}$

$= \left(\frac{1}{\cos m\theta + i\sin m\theta}\right)\left(\frac{(\cos m\theta - i\sin m\theta)}{(\cos m\theta - i\sin m\theta)}\right)$

⇨ $(\cos\theta + i\sin\theta)^{-m}$

$= \cos(-m\theta) + i\sin(-m\theta)$

Hence the theorem holds for all values of n.

Applications of Demoivre's theorem

It is mainly employed in proving trigonometric functions and finding roots of complex function.

Proving identities

Example 19

By using Demoivre's theorem, show that

(a) $\tan5x = \frac{5\tan x - 10\tan^3x + \tan^5x}{1 - 10\tan^2x + \tan4x}$

Solution

$\cos5x + i\sin5x = (\cos x + i\sin x)5$

$= \cos^5x + 5i\sin x\cos^4x - 10\sin^2x\cos^3x - 10i\sin^3x\cos^2x + \sin^4x\cos x + i\sin^5x$

Equating real parts

$\cos5x = \cos^5x - 10\cos^3x\sin^2x + \cos x\sin^4x$

Equating imaginary parts

$\sin5x = 5\cos^4x\sin x - 10\cos^2x\sin^3x + \sin^5x$

$\tan5x = \frac{\sin5x}{\cos5x} = \frac{5\cos^4x\sin x - 10\cos^2x\sin^2x + \sin^5x}{\cos^5x - 10\cos^3x\sin^2x + \cos x\sin^4x}$

Dividing terms of the R.H.S by $\cos^5x$

$\tan5x = \frac{5\tan x - 10\tan^3x + \tan^5x}{1 - 10\tan^2x + \tan4x}$

(b) $\frac{\cos3x + i\sin3x}{\cos5x + i\sin5x} = \cos8x + i\sin8x$

Solution

$\frac{\cos3x + i\sin3x}{\cos5x + i\sin5x} = \frac{(\cos x + i\sin x)^3}{(\cos x + i\sin x)^5}$

$= (\cos x + i\sin x)^{3-(-5)}$

$= (\cos x + i\sin x)^8$

$= \cos8x + i\sin8x$

(c) $\frac{(\cos x + i\sin x)(\cos2x + i\sin2x)}{\cos\frac{x}{2} + i\sin\frac{x}{2}} = \cos\frac{5x}{2} + i\sin\frac{5x}{2}$

Solution

$\frac{(\cos x + i\sin x)(\cos2x + i\sin2x)}{\cos\frac{x}{2} + i\sin\frac{x}{2}}$

$= \frac{(\cos x + i\sin x)^1(\cos x + i\sin x)^2}{(\cos x + i\sin x)^{\frac{1}{2}}}$

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$=(cosx + isinx)^{1+2-\frac{1}{2}}$

$=(cosx + isinx)^{\frac{5}{2}} = cos\frac{5x}{2} + isin\frac{5x}{2}$

(d) $16\sin^5 x = \sin 5x - 5\sin 3x + 10\sin x$

Given Z = cosx + isinx

$\bar{Z}$ = cosx – isinx

(Z – $\bar{Z}$) = 2isinx

This means that (Z – $\bar{Z}$)^n = 2isin$n$x

(Z – Z*)^5 = 2isin5x = 32isin^{5}x

(Z – $\bar{Z}$)^5

= $Z^5 - 5Z^4\bar{Z} + 10Z^3\bar{Z}^2 + 10Z^2(-\bar{Z})^3 + 5Z\bar{Z}^4 - \bar{Z}^5$

= $Z^5 - 5Z^3 + 10Z - 10(\bar{Z}) + 5\bar{Z}^3 - \bar{Z}^5$

$=[Z^5 - \bar{Z}^5] - 5[Z^3 - \bar{Z}^3] + 10[Z - \bar{Z}]$

= 2isin5x – 10isin3x + 20isinx

⇨ $32\sin^5 x = 2\sin 5x - 10\sin 3x + 20\sin x$

$\therefore 16\sin^5 x = \sin 5x - 5\sin 3x + 10\sin x$

Finding roots

Suppose that z = r(cosθ + isinθ)

Then $z^n = r^n(\cos n\theta + i\sin n\theta)$

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(cos\frac{1}{n}\theta + isin\frac{1}{n}\theta\right)$$

The general expression for finding the roots of a complex function is given by

$z = r[cos(\theta + 2\pi k) + isin(\theta + 2\pi k)]$ where k = 0, 1, 2, …k and θ is usually in radians.

Now for nth root, we have

$z^n = r^n[cosn(\theta + 2\pi k) + isin\, n(\theta + 2\pi k)]$

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(cos\frac{1}{n}(\theta + 2\pi k) + isin\frac{1}{n}(\theta + 2\pi k)\right)$$

Example 20

Use Demoivre's theorem to find the square root of

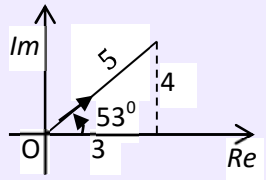
(a) 3 + 4i

Solution

$|z| = \sqrt{(9 + 16} = 5$

$\theta = \tan^{-1}\frac{4}{3} = 53.1^0 = 0.295\pi$

The principal angle lies in the first quadrant



z = $5[cos(0.295\pi + 2\pi k) + isin(0.295\pi + 2\pi k)]$

$z^{\frac{1}{2}}$

$= 5^{\frac{1}{2}}\left[cos\left(\frac{0.295\pi+2\pi k}{2}\right) + isin\left(\frac{0.295\pi+2\pi k}{2}\right)\right]$

Substituting for k = 0

$z_1^{\frac{1}{2}} = 5^{\frac{1}{2}}\left[cos\left(\frac{0.295\pi}{2}\right) + isin\left(\frac{0.295\pi}{2}\right)\right]$= -2 - i

$\therefore \sqrt{3 + 4i} =$ -2-i

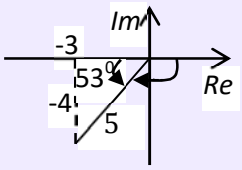
(b) -3 – 4i

Solution

$|z| = \sqrt{9 + 16} = 5$

$\theta = \tan^{-1}\frac{-4}{-3} = -127^0 = -0.7\pi$

The principal angle lies in the third quadrant



z = $5[cos(0.295\pi + 2\pi k) + isin(0.295\pi + 2\pi k)]$

$z^{\frac{1}{2}} = 5^{\frac{1}{2}}\left[cos\left(\frac{0.295\pi+2\pi k}{2}\right) + isin\left(\frac{0.295\pi+2\pi k}{2}\right)\right]$

Substituting for k = 0

$z_1^{\frac{1}{2}} = 5^{\frac{1}{2}}\left[cos\left(\frac{0.295\pi}{2}\right) + isin\left(\frac{0.295\pi}{2}\right)\right]$= -2 - i

Substituting for k = 1

$z_2 = 5^{\frac{1}{2}}\left[cos\left(\frac{0.295\pi+2\pi}{2}\right) + isin\left(\frac{0.295\pi+2\pi}{2}\right)\right]$

= ±(1-2i)

$\therefore \sqrt{-3 - 4i} = \pm(1 - 2i)$

(c) 5 – 12i

Solution

$|z| = \sqrt{(25 + 144} = 13$

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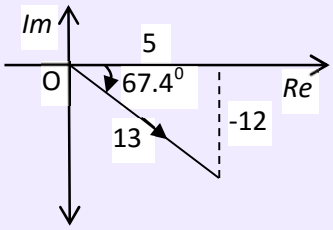
$\theta = \tan^{-1}\frac{-12}{5} = -67.4^0 = -0.374\pi$

The principal angle lies in the third quadrant



z = $13[cos(-0.374\pi + 2\pi k) + isin(-0.374\pi + 2\pi k)]$

$z^{\frac{1}{2}} = 13^{\frac{1}{2}}\left[cos\left(\frac{-0.374\pi+2\pi k}{2}\right) + isin\left(\frac{-0.374\pi+2\pi k}{2}\right)\right]$

Substituting for k = 0

$z_1 = 13^{\frac{1}{2}}\left[cos\left(\frac{-0.374\pi}{2}\right) + isin\left(\frac{-0.374\pi}{2}\right)\right]$

= 3 - 2i

Substituting for k = 1

$z_2 =$
$13^{\frac{1}{2}}\left[cos\left(\frac{-0.374\pi+2\pi}{2}\right) + isin\left(\frac{-0.374\pi+2\pi}{2}\right)\right]$

= -3 + 2i

$\therefore \sqrt{5 - 12i} = \pm(3 - 2i)$

Example 21

(a) Use roots 1, α and α^2 of unity. Hence show that $1 + \alpha + \alpha^2 = 0$

Solution

Let $\sqrt[1]{1} = z$

z = $1^{\frac{1}{3}} = (cos0^0 + isin0^0)$ (In mod-Arg form

=$[cos(0 + 2\pi k) + isin\,(0 + 2\pi k)]^{\frac{1}{3}}$, k = 0,1,2

= $cos(\frac{2\pi k}{3}) + isin\,(\frac{2\pi k}{3})$

When k = 0

$z_1 = cos0^0 + isin\,0^0 = 1$

When k = 1

$z_2 = cos(\frac{2\pi}{3}) + isin\,(\frac{2\pi}{3}) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

When k = 2

$z_3 = cos(\frac{4\pi}{3}) + isin\,(\frac{4\pi}{3}) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

∴the cube roots are 1, $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

Using otherwise

Taking

z = $1^{\frac{1}{3}}$, then $z^3 - 1 = 0$

Let f(z) = $z^3 - 1$

f(1) = $1^3 - 1 = 0$

∴ z – 1 is a factor.

Using long division

$$\begin{array}{r} z^2 + z + 1 \\ z-1\overline{)z^3 - 1} \\ \underline{z^3 - z^2} \\ z^2 - 1 \\ \underline{z^2 - 1} \end{array}$$

$z^3 - 1 = (z - 1)(z^2 + z + 1)$

Either z – 1 = 0, z = 1

Or $z^2 + z + 1 = 0$

$z = \frac{-1\pm\sqrt{1^2-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

∴the cube roots are 1, $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

Hence

Letting α = $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$, the

$\alpha^2 = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

Letting α = $-\frac{1}{2} - i\frac{\sqrt{3}}{2}$, the

$\alpha^2 = \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

Hence the roots can be written as 1, α + α^2

Adding these roots

$1 - \frac{1}{2} - i\frac{\sqrt{3}}{2} - \frac{1}{2} + i\frac{\sqrt{3}}{2} = 0$

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(b) Evaluate $(-16)^{\frac{1}{4}}$

Solution

Let z = $(-16)^{\frac{1}{4}} = 16^{\frac{1}{4}}(-1)^{\frac{1}{4}}$ = $2(-1)^{\frac{1}{4}}$

= $[cos180^0 + isin180^0]^{\frac{1}{4}}$

=2 $\left[cos\left(\frac{\pi+2\pi k}{4}\right) + isin\left(\frac{\pi+2\pi k}{4}\right)\right]$, k = 0, 1, 2, 3

When k = 0

z_1 = $[cos45^0 + isin45^0]$=$\sqrt{2} + i\sqrt{2}$

When k = 1

Z_2 = $[cos135^0 + isin135^0]$=-$\sqrt{2} + i\sqrt{2}$

When k = 2

Z_3 = $[cos225^0 + isin225^0]$=$\sqrt{2} - i\sqrt{2}$

When k = 3

Z_3 = $[cos315^0 + isin315^0]$=-$\sqrt{2} - i\sqrt{2}$

$\therefore (-16)^{\frac{1}{4}} = \pm(\sqrt{2} \pm i\sqrt{2})$

Example 23

Evaluate $(1 + \sqrt{3}i)^{\frac{2}{3}}$

Solution

$(1 + \sqrt{3}i)^{\frac{2}{3}} = \left[(1 + \sqrt{3}i)^2\right]^{\frac{1}{3}}$

$= \left[(1 + \sqrt{3}i)(1 + \sqrt{3}i)\right]^{\frac{1}{3}}$

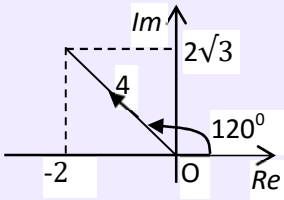
$= \left[-2 + 2\sqrt{3}i\right]^{\frac{1}{3}}$

Let z = -2 + 2$\sqrt{3}$i

$|z| = \sqrt{(-2)^2 + (2\sqrt{3})^2} = 4$

Arg(z) = $\tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = -60^0 = 120^0 = \frac{2\pi}{3}$

The principal angle lies in the second quadrant



$z = 4\left[cos\left(\frac{2\pi}{3} + 2\pi k\right) + isin\left(\frac{2\pi}{3} + 2\pi k\right)\right]$

$z^{\frac{1}{3}} = 4^{\frac{1}{3}}\left[cos\left(\frac{2\pi+6\pi k}{9}\right) + isin\left(\frac{2\pi+6\pi k}{9}\right)\right]$

When k = 0

$z_1 = 4^{\frac{1}{3}}\left[cos\left(\frac{2\pi}{9}\right) + isin\left(\frac{2\pi}{9}\right)\right]$

=1.2160 +1.0204i

When k = 1

$z_2 = 4^{\frac{1}{3}}\left[cos\left(\frac{8\pi}{9}\right) + isin\left(\frac{8\pi}{9}\right)\right]$

= -1.4917 + 0.5429i

When k = 2

$z_3 = 4^{\frac{1}{3}}\left[cos\left(\frac{[illegible]}{9}\right) + isin\left(\frac{14\pi}{9}\right)\right]$

= 0.2[illegible]1.5633i

$(1 + \sqrt{3}i)^{\frac{2}{3}}$ =1.2160 +1.0204i, -1.49[illegible] + 0.5429i, 0.2756 – 1.5633i

Example 24

Find the root of $z^4 + 4 = 0$ using Demoivre's theorem

z^4 = -4 = 4(-1 + 0i)

arg(z) =$\tan^{-1}\left(\frac{0}{-1}\right) = 180^0 = \pi$

⇨ $z^4 = 4[cos180^0 + sin180^0]$

$z = 4^{\frac{1}{4}}\left[cos\left(\frac{\pi+2\pi k}{4}\right) + sin\left(\frac{\pi+2\pi k}{4}\right)\right]$

When k = 0

$z_1 = 4^{\frac{1}{4}}\left[cos\left(\frac{\pi}{4}\right) + sin\left(\frac{\pi}{4}\right)\right]$

$= \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = 1 + i$

When k = 1

$z_2 = 4^{\frac{1}{4}}\left[cos\left(\frac{3\pi}{4}\right) + sin\left(\frac{3\pi}{4}\right)\right]$

$= \sqrt{2}\left(\frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = -1 + i$

When k = 2

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$z_5 = 4^{\frac{1}{4}}\left[cos\left(\frac{5\pi}{4}\right) + sin\left(\frac{5\pi}{4}\right)\right]$

$= \sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = -1 - i$

When k = 3

$z_4 = 4^{\frac{1}{4}}\left[cos\left(\frac{7\pi}{4}\right) + sin\left(\frac{7\pi}{4}\right)\right]$

$= \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = 1 - i$

∴ the roots of $z^4 + 4 = 0$ are 1 ± i and -1 ± i

The locus of a complex number

This is an equation of a set of point representing a variable complex number or a path traced. It may be in form of an equation of a circle or a straight line.

Example 25

Determine the locus of z if

(a) $|z - 4| = 3$
(b) Arg(z+2) = $\frac{\pi}{6}$

Solution

Let z = x + iy

(a) $|x + iy - 4| = 3$
$|(x - 4) + iy| = 3$
$(x - 4)^2 + y^2 = 9$
$x^2 + y^2 - 8x + 7 = 0$
The locus of z is a circle with centre (4, 0)

(b) Arg(x + iy +2) = $\frac{\pi}{6}$
Arg((x+2) + iy) = $\frac{\pi}{6}$
$\tan^{-1}\left(\frac{y}{x+2}\right) = \frac{\pi}{6}$
$\frac{y}{x+2} = tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$
$\frac{y}{x+2} = \frac{\sqrt{3}}{3}$
$y = \frac{\sqrt{3}}{3}(x + 2)$

The locus of z is a straight line

Example 26

Determine the locus of z if:

(a) $|z| = 2$ and sketch it
(b) $\arg(z - 1) = \frac{\pi}{4}$ and sketch it
(c) $2|z - 1| = |z + i|$
(d) $\arg\left(\frac{z-i}{z+1}\right) = \frac{\pi}{3}$

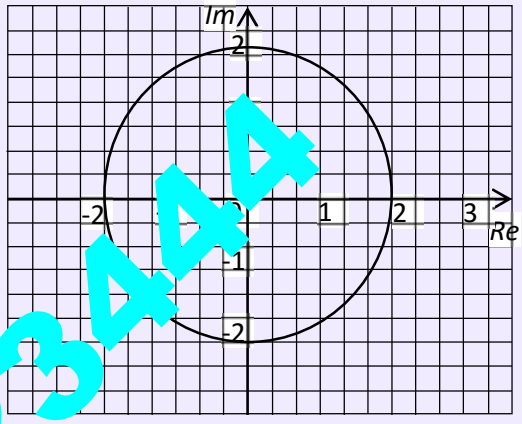
Solution

Let z = x + iy

(a) $|x + iy| = 2$ i.e. $\sqrt{x^2 + y^2} = 2$

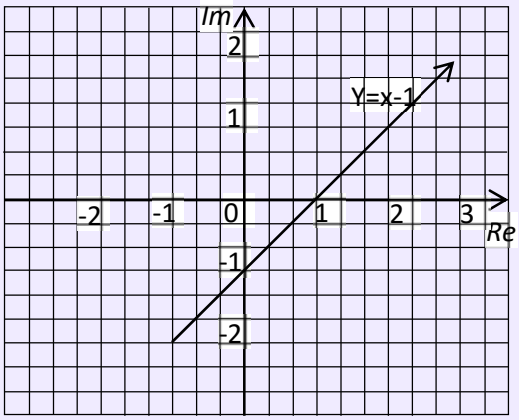
$x^2 + y^2 = 4 \; or \; (x - 0)^2 + (y - 0)^2 = 2^2$

The locus of z is a circle with centre (0, 0) and radius 2units



(b) Arg[(x – 1) + iy] = $\frac{\pi}{4}$
$\tan^{-1}\left(\frac{y}{x-1}\right) = \frac{\pi}{4}$
$\frac{y}{x-1} = tan\frac{\pi}{4} = 1$
$y = x - 1$

The locus is a straight line



(c) $2|(x - 1) + iy| = |x + i(y + 1)|$

$\left[2\sqrt{(x-1)^2 + y^2}\right]^2 = \left[\sqrt{x^2 + (y+1)^2}\right]^2$

$3x^2 + 3y^2 - 8x - 2y + 3 = 0$

$x^2 + y^2 - \frac{8}{3}x - \frac{2}{3}y + 1 = 0$

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$= \left(x - \frac{4}{3}\right)^2 + \left(y - \frac{1}{3}\right)^2 = -1 + \frac{16}{9} + \frac{1}{9}$

$= \left(x - \frac{4}{3}\right)^2 + \left(y - \frac{1}{3}\right)^2 = \frac{8}{9}$

The locus of Z is a circle with the centre $\left(\frac{4}{3}, \frac{1}{3}\right)$

and radius $\sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$

(d) $\arg\left(\frac{x+iy-i}{x+iy+1}\right) = \frac{\pi}{3}$

$\arg\left(\frac{x+i(y-1)}{(x+1)+iy}\right) = \frac{\pi}{3}$

$\arg[x + i(y-1)] - \arg[x+1) + iy] = \frac{\pi}{3}$

$\tan^{-1}\left(\frac{y-1}{x}\right) - \tan^{-1}\left(\frac{y}{x+1}\right) = \frac{\pi}{3}$

$\frac{\left(\frac{y-1}{x}\right) - \left(\frac{y}{x+1}\right)}{1 + \left(\frac{y-1}{x}\right)\left(\frac{y}{x+1}\right)} = \sqrt{3}$

$\frac{y-x-1}{x^2+y^2+x-y} = \sqrt{3}$

$\sqrt{3}x^2 + \sqrt{3}y^2 + (\sqrt{3}+1)x - (\sqrt{3}+1)y + \frac{\sqrt{3}}{3} = 0$

$x^2 + y^2 + \left(\frac{3+\sqrt{3}}{3}\right)x - \left(\frac{3+\sqrt{3}}{3}\right)y + \frac{\sqrt{3}}{3} = 0$

The locus is a circle with centre at

$\left[-\frac{1}{6}(3+\sqrt{3}) - \frac{1}{6}(3+\sqrt{3})\right]$and [illegible]lius $\sqrt{3}$[illegible]

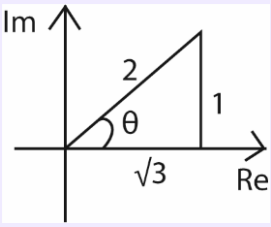
Example 27

The complex number [illegible] $\sqrt{3}$ +[illegible]

$\bar{Z}$ *is the conjugate* [illegible] *f* Z[illegible]

(a) Express Z in the modulus argument form

$|z| = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{3+1} = 2$



$\tan\theta = \frac{1}{\sqrt{3}}$

$\theta = \text{Arg}(z) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

Hence $z = 2\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$

or $z = 2[\cos 30^0 + i\sin 30^0]$

(i) On the same Argand diagram plot $\bar{Z}$ and $2\bar{Z} + 3i$

$\bar{z} = \sqrt{3} - i$

$|z| = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{3+1} = 2$

$2\bar{Z} + 3i = 2(\sqrt{3} - i) + 3i = 2\sqrt{3} + i$

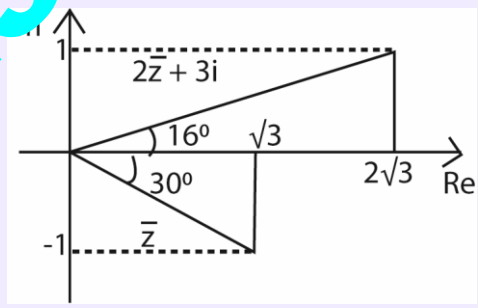
$|2\bar{Z} + 3i| = \sqrt{(2\sqrt{3})^2 + 1^2} = \sqrt{13}$

Finding $\arg(\bar{z})$:

$\text{Arg}(\bar{z}) = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -30^0$

Fin[illegible] $\arg(2\bar{Z} + 3i)$:

$\text{Arg}(2\bar{Z} + 3i) = \tan^{-1}\left(\frac{1}{2\sqrt{3}}\right) = 16.2^0$



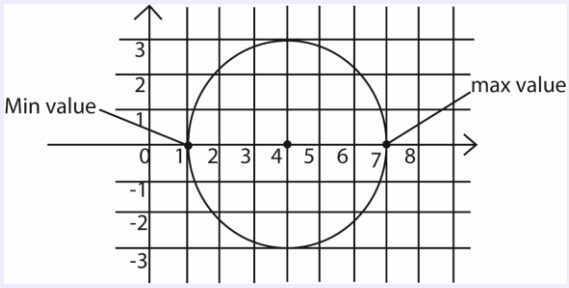
(b) What are the greatest and least values of $|Z|$ *if* $|Z - 4| \leq 3$?

$|z - 2| \leq 3$

$|x - 4 + iy| \leq 3$

$(x-4)^2 + y^2 \leq 3$

This is an equation of the circle with centre (4, 0) and radius 3.



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Greatest value of $|z| = 4 + 3 = 7$

Lowest value of $|z| = 4 - 3 = 1$

Example 28

(a) Given that the complex number Z and its conjugate $\bar{Z}$ satisfy the equation

$$Z\bar{Z} + 2iZ = 12 + 6i. \text{ Find Z.}$$

Let z = x + iy then $\bar{z} = x - iy$

(x + iy)(x – iy) +2i(x + iy) = 12 + 6i

$x^2 + y^2 - 2y + 2xi = 12 + 6i$

Equating imaginary parts

2xi = 6i

x = 3

Equating real parts

$x^2 + y^2 - 2y = 12$

By substituting for x

$9 + y^2 - 2y = 12$

$y^2 - 2y - 3 = 0$

(y – 3)(y+1) = 0

y = 3 or y = -1

when y = 3; z = 3 + 3i

when y = -1; z = 3 - i

∴ the possible values of z are 3 +3i and 3- i.

(b) One root of the equation

$Z^3 - 3Z^2 + 9Z + 13 = 0$ *is* $2 + 3i$.

Determine other roots

Solution

Given root 2 + 3i; its conjugate is also a root of the equation

The equation of these two is

z^2 – (sum of roots)z + product of roots = 0

Sum = 2 + 3i + 2 – 3i = 4

Product of roots =(2 + 3i)(2 – 3i) = 2 + 9 = 13

$$z^2 - 4z + 13 = 0$$

$$\begin{array}{r|l} & z + 1 \\ \hline z^2 - 4z + 13 & z^3 - 3z^2 - 9z + 13 \\ & -z^3 - 4z^2 - 13z \\ \hline & z^2 - 4z + 13 \\ & -z^2 - 4z + 13 \\ \hline \end{array}$$

$$(z^2 - 4z + 13)(z + 1) = 0$$

z + 1= 0

z = -1

So the roots are 2±3i and -1

Example 29

(a) If $z_1 = \frac{2i}{1+3i}$ and $z_2 = \frac{3+2i}{5}$, find $|z_1 - z_2|$

Solution

$$z_1 - z_2 = \frac{2i}{1+3i} - \frac{3+2i}{5},$$

$$= \frac{10i-(1+3i)(3+2i)}{5(1+3i)} = \frac{10i-[3+2i+9i-6]}{5(1+3i)}$$

$$= \frac{10i-11i+3}{5(1+3i)} = \frac{3-i}{5(1+3i)}$$

$$= \frac{(3-i)(1-3i)}{5(1+3i)(1-3i)} = \frac{3-9i-i-3}{5(1+9)} = \frac{-i}{5}$$

$$|z_1 - z_2| = \sqrt{0^2 - \left(-\frac{1}{5}\right)^2} = \frac{1}{5}$$

Alterative 2

$$z_1 = \frac{2i}{1+3i} = \frac{2i\,(1-3i)}{(1+3i)(1-3i)} = \frac{2i+6}{1+9}$$

$$= \frac{2i+6}{10} = \frac{3+2i}{5}$$

$$z_1 - z_2 = \frac{3+2i}{5} - \frac{3+2i}{5} = \frac{-i}{5}$$

$$|z_1 - z_2| = \sqrt{0^2 - \left(-\frac{1}{5}\right)^2} = \frac{1}{5}$$

(b) Given the complex number z = x + iy

(i) Find $\frac{z+i}{z+2}$

$$\frac{z+i}{z+2} = \frac{x+i(1+y)}{(x+2)+iy} = \frac{[x+i(1+y)][(x+2)-iy]}{[(x+2)+iy][(x+2)-iy]}$$

$$= \frac{x[(x+2)-iy]-i(1+y)[(x+2)-iy]}{(x+2)^2+y^2}$$

$$= \frac{x^2+2x-ixy+i(x+2+xy+2y)+y+y^2}{(x+2)^2+y^2}$$

$$= \frac{x^2+2x+y^2+y+i(2+x+2y)}{(x+2)^2+y^2}$$

(ii) Show that the locus of $\frac{z+i}{z+2}$ is a straight line when its imaginary part is zero. State the gradient of the line.

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If imaginary part is zero

$(2 + x + 2y) = 0$

$2y = -x - 2$

$y = -\frac{1}{2}x + 1$

Comparing with y = mx + c

The gradient = $-\frac{1}{2}$

Example 30

(a) Given that a complex number Z* and its conjugate satisfy the equation

ZZ* + 2iZ = 12 + 6i, find the possible values of Z.

Solution

Let z* = x + iy, then z = x –iy

(x + iy)(x – iy) + 2i(x – iy) + 6i = 12 + 6i

$x^2 + y^2 + 2y + 2xi = 12 + 6i$

Equating real parts

$x^2 + y^2 + 2y = 12$

Equating imaginary parts

2x = 6 => x = 3

$9 + y^2 + 2y = 12$

$y^2 + 2y - 3 = 0$

(y – 1)(y + 3) = 0 => y = 1, y = -3

When y = -3; z = 3 + 3i and

When y = 1, z = 3 –i

The possible values of z are 3 + 3i and 3 - i

(b) Find the Cartesian equation, in its simplest form of the curve described by $|z - 3 + 6i| = 2|z|$ where z is the complex number x + iy.

Hence sketch an Argand diagram the region satisfying $|z - 3 + 6i| \leq 2|z|$

Solution

Substituting z = x = iy

$|(x - 3) + i(y - 6)| = 2|x + iy|$

$(x - 3)^2 + (y + 6)^2 = 4(x^2 + y^2)$

$x^2 + y^2 - 6x + 12y + 45 = 4x^2 + 4y^2$

$x^2 + y^2 + 2x - 4y - 15 = 0$

Hence sketching the region satisfying $|z - 3 + 6i| \leq 2|z|$

i.e. $x^2 + y^2 + 2x - 4y - 15 \geq 0$

substituting (0, 0)

LHS = - 15 ; RHS = 0

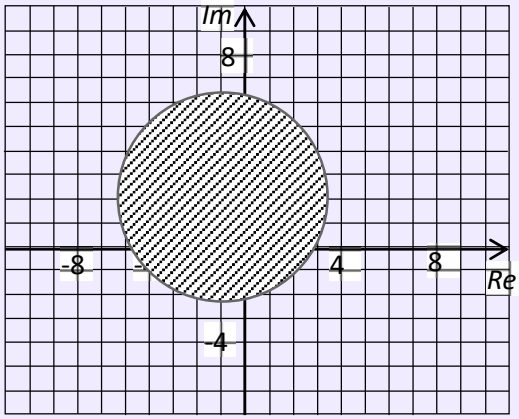
Since LHS< RHS, the point (0, 0) does not lie in the region $|z - 3 + 6i| \leq 2|z|$ therefore it lies in unwanted region.

Now $x^2 + y^2 + 2x - 4y - 15 \geq 0$

$= (x + 1)^2 + (y - 2)^2 \geq 20$

The [illegible] is a circle centre (-1, 2) and radius $\sqrt{20}$

The plot of $|z - 3 + 6i| \leq 2|z|$



Example 31

If n is a variable and z = 4n + 3i(1 – n), show that the locus of Z is a straight line. Determine the minimum value of $|z|$

Solution

Let z = x = iy

⇨ x + iy = 4n +3i(1 – n)

Equating real parts

x = 4n => n = $\frac{x}{4}$

Equating imaginary parts

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y = 3(1 – n)

⇨ y = $3\left(1-\frac{x}{4}\right)$

y =$-\frac{3x}{4}+3$ which is a straight line with gradient$-\frac{3}{4}$ and intercept 3

Given z = 4n + 3i(1 – n)

$|z|^2 = 16n^2 + 9(1-n)^2$

$|z|^2 = 25n^2 - 18n + 9$

The minimum value can be obtained by completing squares or differentiation

By completing squares

$|z|^2 = 25(n^2 - \frac{18}{25}n + \frac{9}{25})$

$= 25\left[\left(n-\frac{9}{25}\right)^2 + \frac{9}{25} - \frac{81}{625}\right]$

$= 25\left[\left(n-\frac{9}{25}\right)^2 + \frac{144}{625}\right]$

$= 25\left(n-\frac{9}{25}\right)^2 + \frac{144}{25}$

$|z|^2$ is minimum when $n = \frac{9}{25}$

$|z|^2{}_{min}$ = $\frac{144}{25}$

⇨ $|z|_{min} = \frac{12}{5}$

By differentiation

$2|z|\left|\frac{dz}{dn}\right| = 50n - 18 = 0$

n = $\frac{18}{50} = \frac{9}{25}$

$|z_{min}|^2 = 25\left(\frac{9}{25}\right)^2 - 18\left(\frac{9}{25}\right) + 9 = \frac{3600}{625}$

⇨ $|z|_{min} = \frac{60}{25} = \frac{12}{5}$

Example 32

Show that the locus of z if $\frac{z-3}{z+i}$ is wholly imaginary is a circle with centre at $\frac{3}{2}-\frac{1}{2}i$ and radius $\frac{1}{2}\sqrt{10}$

Solution

Let z = x + iy

$\frac{z-3}{z+i} = \frac{(x-3+iy)}{(x+i(y+1))}.\frac{(x-i(y+1))}{(x-i(y+1))}$

$= \frac{x(x-3)+y(y+1)+i[(xy-(x-3)(y+1)}{x^2+(y+1)^2}$

If the above expression is wholly imaginary, then the real part must be zero

i.e. $\frac{x(x-3)+y(y+1)}{x^2+(y+1)^2} = 0$

=> $x^2 + y^2 - 3x + y = 0$

$\left(x-\frac{3}{2}\right)^2 + \left(y+\frac{1}{2}\right)^2 = \frac{9}{4}+\frac{1}{4} = \frac{10}{4}$

The locus is a circle with centre with $\frac{3}{2}-\frac{1}{2}i$ and radius $\frac{1}{2}\sqrt{10}$

Revision exercise 4

1. Use Demoivre's theorem to show that
(a) cos6x = $\cos^6$x -1[illegible] $\sin^4$x + 15coxsin^{4}x – $\sin^6$x
(b) tan3x = $\frac{3t[illegible]x[illegible]n^3x}{1[illegible]an^2x}$

2. (a) E[illegible]ate $(1+i\sqrt{3})^{\frac{2}{3}}$
[illegible]0 + 1.0204i, -1.4917 +0.5429i]
(b) [illegible]nd the roots of z^4 + 4 = 0 using Demoivre's theorem
[1± i, -1± i]

3. (a) Given that the complex number Z and its conjugate $\overline{Z}$ satisfy the equation
$Z\overline{Z} - 2Z + 2\overline{Z} = 5 - 4i$. Find possible values of Z. (06marks)
[Z = $\pm 2 + 1$]
(b) Prove that if $\frac{Z-6i}{Z+8}$ is real, then the locus of the point representing the complex number Z is a straight line. (06marks) [y = $\frac{3}{4}x + 6$]

4. (a)Express 5 + 12i in polar form
[5 + 12i = 13(cos 67.38^0 + isin67.38^0)]
Hence, evaluate $\sqrt[3]{5+12i}$, giving your answer in the form a + ib where a and b are corrected to two decimal places. (12marks)

[z = 2.17 + 0.90i, -1.86 + 1.43i, -0.31 – 2.33i]

5. (a) The total impedance z in an electric circuit with two branches z_1 and z_2 is given by $\frac{1}{z} = \frac{1}{z_1} + \frac{1}{z_2}$. Given that z_1= 3 + 4i and z_2

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= 5 + 5i where i =$\sqrt{-1}$, calculate the total impedance z in form a + bi

$\left[z = \frac{275}{145} + i\frac{325}{145}\right]$

(b) If n is a variable and z = 4n + 3i(1 – n), show that the locus of Z is a straight line. Determine the minimum value of $|z|$ $\left[|z|_{min} = \frac{12}{5}\right]$

6. Given that z_1 = 3 + 4i and z_2 = -1 + 2i
   (a) Express z_1, z_2, $z_1 + z_2$ and $z_1 - z_2$ in the form r(cosθ + isinθ)
   [z_1= 5(scos53.1^0+ isin53.1^0),
   z_2 =$\sqrt{5}$(cos116.6^0 + isin116.6^0),
   $z_1 + z_2$ = 2$\sqrt{10}$(cos71.6^0 + isin71.6^0)
   $z_1 - z_2$ = 2$\sqrt{5}$(cos-153.4^0 + isin-153.4^0)]
   (b) Determine the angle between $z_1 + z_2$ and z_1- z_2 [225^0]
7. Solve the simultaneous equation
   $z_1 + z_2$ = 8
   4z_1 – 3iz_2 = 26 + 8i
   Using the values of z_1 and z_2, find the modulus and argument of $z_1 + z_2 - z_1z_2$.
   [z_1 = 8 +2i, z_2 = -2i; 6.49, 75.96^0]
8. (a) Given that z = 3 + 4i, find the value of expression z + $\frac{25}{z}$ [6]
   Given that $\left|\frac{z-1}{z+1}\right| = 2$, show that the locus of the complex number is
   $x^2 + y^2 + \frac{10x}{3} + 1 = 0$ sketch the locus
   [The centre of the circle is $\left(-\frac{5}{3}, 0\right)$ it passes through -3 but does not touch the y-axis]
9. (a)(i) Express each of the following complex numbers in the form a + bi
   z1 = (1 –i)(1 + 2i) [3 + i]
   z2 = $\frac{2+6i}{3-i}$ [2i]
   z3 = $\frac{-4i}{1-i}$ [2 – 2i]
   (b) Find the modulus and argument of $z_1z_2z_3$ [17.889, 63.4349^0]
   (c) Find the square root of 12i – 5 [1 + 3i or -2 – 3i]
10. (a) given that z = $\sqrt{3}$+ i, find the modulus and argument
   (i) z^2 $\left[4, \frac{\pi}{3}\right]$
   (ii) $\frac{1}{z}\left[\frac{1}{2}, -\frac{\pi}{6}\right]$
   (iii) Show in an Argand diagram the points representing complex numbers z, z^2 and $\frac{1}{z}$.
   (b) In an Argand diagram, P represents a complex number z such that $2|z-2| = |z-6i|$ show that P lies on a circle, find
   (i) The radius of this circle[4.2164]
   (ii) The complex number represented by it centre
   $\left[z = \frac{8}{3} - 2i\right]$
11. (a) given the complex number z_1 = 1 – i, z_2 = 7 + I represents z_1z_2 and $z_1 - z_2$ on the Argand diagram
   Determine the modulus and argument of $\frac{z_1-z_2}{z_1z_2}$ [0.63, -24.7^0]
   (b) z is a complex number of form a + bi, solve $\left(\frac{z-1}{z+1}\right)^2 = i$ [z = 1 ±i$\sqrt{2}$]xz
12. (a) show the region represented by $|z-2+i| < 1$ on an Argand diagram [it is a circle of centre(2, -1) and radius 1 unit, the circle is drawn with dotted line since the equation is less than, and the unwanted region is the area outside the circle]
   (b) Express the complex number z = 1 - i$\sqrt{3}$ in modulus argument form and hence find z^2 and $\frac{1}{z}$ in the form a + bi
   $\left[z = 2\left(cos\left(-\frac{\pi}{3}\right) + isin\left(-\frac{\pi}{3}\right)\right), z^2 = -2 - i2\sqrt{3}, \frac{1}{z} = \frac{1}{4} + i\frac{\sqrt{3}}{4}\right]$
13. (a)Given that z_1 = -i + 1, z_2 = 2 + i, and z_3 = 1 + 5i, represent z_2z_3, z_2-z_1 $\frac{1}{z_1}$and $\frac{z_2z_3}{z_2-z_1} + \frac{1}{z_1}$ o, Argand diagram
   (b) Prove that for positive integers $(cos\theta + sin\theta)^n = cosn\theta + sinn\theta$. Deduce that this formula is also true for negative values of n.
14. If z is a complex number, describe and illustrate on the Argand diagram the locus given by each of the following

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(a) $\left|\frac{z+i}{z-2}\right| = 3$

[it is a circle centre $\left(\frac{9}{4}, \frac{1}{8}\right)$ radius 0. 8385]

(ii) Arg (z + 3) = $\frac{\pi}{6}$ [is it a straight line represented by equation y = $\frac{\sqrt{3}}{3}x + 3$]

15. (a) Use Demoivre's theorem or otherwise to simplify $\frac{(cos\theta+isin\theta)(cos2\theta+isin2\theta}{cos\frac{\theta}{2}+isin\frac{\theta}{2}}$

$\left[cos\left(\frac{5\theta}{2}\right) + isin\left(\frac{5\theta}{2}\right)\right]$

(b) Express $\frac{i}{4+6i}$ in modulus-argument form.

{z = 0.1387[cos(0.187π)+ isin(0.187π)]}

(c) Solve (z + zz*)z = 5 + 2z where z* is the complex conjugate of z.
[z = 1 + 2i, z* = 1 -2i]

16. Show that 2 + I is a root of the equation $2z^3 - 9z^2 + 14 - 5 = 0$. Hence find the other roots 2 – i and ½

17. (a) find the equation whose root are -1 ± i where i = $\sqrt{-1}$ [$z^2 + 2z + 2 = 0$]

(b) Find the sum of the first 10 terms of the series 1 + 2i+ -4 – 8i + 16 + ...
[205 + 410i]

(c) Prove by induction that $(cos\theta + sin\theta)^n = cosn\theta + isinn\theta$.

18. Show that z = 1 is a root of the equation $z^3 - 5z2 + 9z - 5 = 0$
Hence solve the equation for other roots
[2 ± i]

19. (a) Use Demoivre's theorem to express tan5θ in terms of tanθ.

(b) Solve the equation $z^3 + 1 = 0$

$\left[-1, \frac{1}{2} + i\frac{\sqrt{3}}{2}, \frac{1}{2} - i\frac{\sqrt{3}}{2}\right]$

20. (a) Without using tables or calculators, simplify $\frac{\left(cos\frac{\pi}{17}+isin\frac{\pi}{17}\right)^8}{\left(cos\frac{\pi}{17}-isin\frac{\pi}{17}\right)^9}$ [-1]

(b) Given that x and y are real, find the values of x and y which satisfy the equation: $\frac{2y+4i}{2x+y} - \frac{y}{x-i} = 0$ [x = -1 when y = -2 and x = 1 when y = 2]

21. Given the complex number z = $\frac{(3i+1)(i-2)^2}{i-3}$

(i) Determine z in form a + bi where a and b are constant [-4,- 3i]

(ii) arg(z) [-143.13^0]

22. (a) Express the complex numbers z_1 =4i and z_2 = 2 – 2i in trigonometric form r(cosθ + isinθ).
[z_1 =4(cos90^0 + isin90^0),
z_2 = 2√2(cos-45^0 + isin-45^0)]
Hence or otherwise evaluate $\frac{z_1}{z_2}$ $\left[\frac{1}{2}\right]$

(b) Find the values of x and y in $\frac{x}{3+3i} - \frac{y}{(2-3i)} = \frac{(6+2i)}{(1+8i)}$ [x = 2.8, y =0.4]

23. Find the fourth roots of 4 + 3i

[±(1.4760 + 0.[illegible]71i), ±(0.2397 – 1.4760i)]

24. (a) Given that $\frac{ix}{1+iy} = \frac{3x+4i}{x+3y}$; find the values of x and y
[x = 2, y= 1.5 or x = -2 and y = -1.5]

(b) If z = x +iy, find the equation of the locus $\left|\frac{z+3}{z-1}\right| = 4$

$\left[x^2 + y^2 - \frac{38}{15}x + \frac{7}{15} = 0\right]$

25. (a) given that the complex number z and its conjugate Z*, satisfy the equation zz*+ 3z* = 34 – 12i, find the value of z.
[z = 3 - 4i and z = -6 + 4i]

(b) Find the Cartesian equation of the locus of a point P represented by equation $\left|\frac{z+3}{z+2-4i}\right| = i$
[the locus P is an equation of line 8y + 2x = 11]

26. (a) form a quadratic equation having -3 + 4i as one of its roots. [$z^2 + 6z + 25$]

(b) Given z_1 = -1 + i√3 and z_2 = -1 - i√3

(i) Express $\frac{z_1}{z_2}$ in form a + i√b, where a and b are real number
$\left[\frac{-1}{2} - i\frac{\sqrt{3}}{2}\right]$

(ii) Represent $\frac{z_1}{z_2}$ on an Argand diagram

(iii) find $\left|\frac{z_1}{z_2}\right|$ [1]

27. If z = $\frac{(2-i)(5+12i)}{(1+2i)^2}$

(a) Find the

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(i) Modulus of z [5.814]

(ii) Argument z [-86.055^0]

(b) Represent z on a complex plane

(c) Write z in the polar form

[z = 5.814(cos(-86.055^0) + isin(-86.055^0) or z = 5.814(cos (0.47π) + isin(0.47π)]

28. (a) the complex number z = $\sqrt{3} + i$. $\bar{Z}$ *is the conjugate of* Z.

(i) Express Z in the modulus argument form

[z = $2\left[\cos\left(\frac{\pi}{6}\right) + isin\left(\frac{\pi}{6}\right)\right]$ or z = $2[\cos 30^0 + isin30^0]$]

(ii) On the same Argand diagram plot $\bar{Z}$ and $2\bar{Z} + 3i$

(b) What are the greatest and least values of $|Z|$ if $|Z - 4| \leq 3$?

[Greatest value of $|z| = 7$, Lowest value of $|z| = = 1$]

29. (a) Given that the complex number Z and its conjugate $\bar{Z}$ satisfy the equation

$Z\bar{Z} + 2iZ = 12 + 6i$. Find Z.

[possible values of z are 3 +3i and [illegible]

(b)One root of the equation $Z^3 - 3Z^2 + 9Z + 13 = 0$ is $2 + 3i$. Determine other roots

[other roots are 2-3i and [illegible]]

30. (a) If $z_1 = \frac{2i}{1+3i}$ and $z_2 = \frac{[illegible]+2i}{5}$, find $|z_1 - z_2|$ $\left[\frac{1}{5}\right]$

(b) Given the complex number z = x + iy

(i) Find $\frac{z+i}{z+2}$ $\left[\frac{x^2+2x+y^2+y+i(2+x+2y)}{(x+2)^2+y^2}\right]$

(ii) Show that the locus of $\frac{z+i}{z+2}$ is a straight line when its imaginary part is zero. State the gradient of the line. [y = $-\frac{1}{2}x$ + 1]

31. (a) Given that the complex number Z and its conjugate $\bar{Z}$ satisfy the equation

$Z\bar{Z} - 2Z + 2\bar{Z} = 5 - 4i$. Find possible values of Z. [Z = $\pm 2 + 1$]

(b) Prove that if $\frac{Z-6i}{Z+[illegible]}$ is real, then the locus of the point represent[illegible] the complex number Z [illegible] straight line. [y = $\frac{3}{4}x$ + 6]

32. (a) E[illegible]p[illegible] 5 + 12i in polar form [illegible]nce, evaluate $\sqrt[3]{5 + 12i}$, giving your answer in the form a + ib where a and b are corrected to two decimal places. (12marks)

[$\sqrt[3]{5 + 12i} = \sqrt[3]{13}\left[\cos\left(\frac{67.38+2\pi k}{3}\right) + isin\left(\frac{67.38+2\pi k}{3}\right)\right]$ $taking$ $k = 0, 1, 2$;

2.17 + 0.90i; -1.86 + 1.43i; -0.31 – 2.33i]

33. Show that the modulus of $\frac{(1-i)^6}{(1+i)}$=$4\sqrt{2}$

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11. Integration

Integration (A-level)

It the reverse of differentiation.

During integration the following concepts should be considered.

(a) Polynomial functions;

$\int x^n dx = \frac{1}{n+1}x^{n+1} + 1 \ where\ n \neq -1$

i.e. increase the power by 1 and divide the term by the new power, e.g.

(i) $\int 1dx = \int (x^0)dx = x + c$

(ii) $\int xdx = \int x^1 dx$

$= \frac{1}{1+1}x^{1+1}$

$= \frac{1}{2}x^2$

(iii) $\int x^4 dx = \frac{1}{4+1}x^{4+1} = \frac{1}{4}x^5$

(iv) $\int 4x^3 dx = 4\int x^3 dx = \frac{4}{(3+1)}x^4$

(v) $\int x^{-3} = \frac{1}{-3+1}x^{-3+1} = -\frac{1}{2}x^{-2} = -\frac{1}{2x^2}$

(b) Trigonometric functions e.g.

(i) $\frac{d}{dx}(cosx) = -sinx + c$

- $\int sinx\, dx = -cosx + c$

(ii) $\frac{d}{dx}(sinx) = cosx$

- $\int cosx\, dx = sinx + c$

(iii) $\frac{d}{dx}(tanx) = sec^2 + c$

- $\int sec^2\, dx = tanx + c$

(iv) $\frac{d}{dx}(cotx) = -cosec^2 x$

- $\int cosec^2 x = -cotx + c$

Methods of integration

The choice of the method depends on judgement. Below are some of the methods:

Integration by change of variable where a derivative exist/integration by recognition or inspection

Example 1

(i) $\int x\sqrt{(x^2-2)}dx$

Solution

Let u = x^2-2

- $du = 2u\ i.e.\ xdx = \frac{1}{2}du$

$\int x\sqrt{(x^2-3)}dx = \int \sqrt{(x^2-3)}\,xdx$

$= \int \sqrt{u}.\frac{1}{2}du$

$= \frac{1}{2}\int u^{\frac{1}{2}}du$

$= \frac{1}{2}.\frac{2}{3}u^{\frac{3}{2}} + c$

$= \frac{1}{3}(x^2-3)^{\frac{3}{2}} + c$

$= \frac{1}{3}(x^2-3)\sqrt{(x^2-3)} + c$

Or let u =$\sqrt{(x^2-32)}$ => $u^2 = x^2 - 3$

2du = 2xdx i.e. xdx = udu

$\int x\sqrt{(x^2-3)}\,dx = \int \sqrt{(x^2-3)}\,xdx$

$\int u.udu = \int u^2 du$

$= \frac{1}{3}u^3 + c$

$= \frac{1}{3}(x^2-3)\sqrt{(x^2-3)} + c$

(ii) $\int xcosec^2(x^2)dx$

Solution

Let u = x^2 => du =2xdx i.e. xdx = $\frac{1}{2}du$

$\int xcosec^2(x^2)dx = \int cosec^2(x^2)xdx$

$= \frac{1}{2}\int cosec^2 udu$

$= -cotu + c$

$= -cotx^2 + c$

(iii) $\int_0^1 \frac{x^2-1}{\sqrt{(x^3-3x+5)}}dx$

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Solution

Let u = $x^3 - 3x + 5$ =>$du = (3x^2 - 3)dx$

i.e. $(3x2 - 3)dx = \frac{1}{3}du$

$\therefore \int_0^1 \frac{x^2-1}{\sqrt{(x^3-3x+5)}}dx = \frac{1}{3}\int_0^1 \frac{1}{\sqrt{u}}dx$

$= \frac{1}{3}u^{\frac{1}{2}}(2) + c$

$= \frac{2}{3}\sqrt{(x^3 - 3x + 5)}$

- $\int_0^1 \frac{x^2-1}{\sqrt{(x^3-3x+5)}}dx = \frac{2}{3}\sqrt{(x^3 - 3x + 5)}\Big|_0^1$

$= \frac{2}{3}\left(\sqrt{1-3+5} - \sqrt{5}\right)$

$= \frac{2}{3}\left(\sqrt{3} - \sqrt{5}\right) = 0.336$

(iv) $\int_0^{\frac{\pi}{4}} sec^2x\sqrt{tanx}dx$

Solution

Let u = $tanx$ => du = sec^2xdx

$\int_0^{\frac{\pi}{4}} sec^2x\sqrt{tanx}dx = \int_0^{\frac{\pi}{4}} u^{\frac{1}{2}}du$

$= \left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^{\frac{\pi}{4}}$

$= \left[\frac{2}{3}(tanx)^{\frac{3}{2}}\right]_0^{\frac{\pi}{4}}$

$= \frac{2}{3}$

(v) $\int cosx\sqrt{1 - 2sinx}dx$

Solution

Let u = 1-2sinx => du = -2cosx

i.e. cosxdx =$-\frac{1}{2}du$

$\therefore \int cosx\sqrt{1 - 2sinx}dx = -\frac{1}{2}\int u^{\frac{1}{2}}dx$

$= \frac{1}{2}.\frac{2}{3}u^{\frac{3}{2}} + c$

$= \frac{1}{3}(1 - 2sinx)^{\frac{3}{2}} + c$

$= \frac{1}{3}(1 - 2sinx)\sqrt{(1 - 2sinx)} + c$

Integration by change of variable where a derivative is given

Example 2

(a) Using the substation x =cos2θ or otherwise, prove that

$\int_0^1 \sqrt{\left(\frac{1-x}{1+x}\right)}dx = \frac{\pi}{2} - 1$

Solution

Given x= cos2θ => dx = -2sin2θdθ

Changing limits

x	θ
0	$\frac{1}{4}\pi$
1	0

$\int_0^1 \sqrt{\left(\frac{1-x}{1+x}\right)}dx = -2\int_{\frac{\pi}{4}}^0 \sqrt{\left(\frac{1-cos2\theta}{1+cos\theta}\right)}sin2\theta\, d\theta$

$= 2\int_0^{\frac{\pi}{4}} \sqrt{\left(\frac{2sin^2\theta}{2cos^2\theta}\right)}sin2\theta\, d\theta$

$= 2\int_0^{\frac{\pi}{4}} \left(\frac{sin\theta}{cos\theta}\right)2sin\theta cos\theta\, d\theta$

$= 4\int_0^{\frac{\pi}{4}} sin^2\theta d\theta$

$= \int_0^{\frac{\pi}{4}} (1 - cos2\theta)d\theta$

(double angle form

$= 2\left[\theta - \frac{1}{2}sin2\theta\right]_0^{\frac{\pi}{4}}$

$= 2\left(\frac{\pi}{4} - \frac{1}{2}\right) = \frac{\pi}{2} - 1$

) Use the substitution x = sinθ to evaluate

$\int_{\frac{1}{2}}^{\sqrt{\frac{1}{2}}} \frac{dx}{x^2(1-x^2)}$

Solution

Given x = sinθ => dx =cosθdθ

Changing limits

x	θ
$\frac{1}{2}$	$\frac{1}{6}\pi$
$\sqrt{\frac{1}{2}}$	$\frac{1}{4}\pi$

$\therefore \int_{\frac{1}{2}}^{\sqrt{\frac{1}{2}}} \frac{dx}{x^2(1-x^2)} = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{cos\theta d\theta}{sin^2\theta(1-sin^2\theta)}$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{d\theta}{sin^2\theta} = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} cosec^2\theta d\theta$

$= [-cot\theta]_{\frac{\pi}{6}}^{\frac{\pi}{4}} = [cot\theta]_{\frac{\pi}{4}}^{\frac{\pi}{6}}$

$= cot\frac{\pi}{6} - cot\frac{\pi}{4} = \sqrt{3} - 1$

Revision exercise 1

Integrate the following using the suggested substitution in each case.

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1. $\int x(x+4)^3 dx$, u = x + 4
   $\left[\frac{1}{5}(x-1)(x+4)^4 + c\right]$
2. $\int (x-4)(x-1)^3 dx$, u = x – 1
   $\left[\frac{1}{4}(4x-19)(x-1)^4 + c\right]$
3. $\int x(2x-3)^2 dx$, u = 2x – 3
   $\left[\frac{1}{16}(2x+1)(2x-3)^3 + c\right]$
4. $\int (3x+1)\,(2x-5)^2 dx$, u = 2x – 5
   $\left[\frac{1}{48}(18x+23)(2x-5)^3 + c\right]$
5. $\int \frac{x}{x+3} dx$, u= x+3 $[x - 3In(x+3) + c]$
6. $\int \frac{x}{(x+1)^2} dx$, u= x+1 $\left[\frac{1}{x+1} + In(x+1) + c\right]$
7. $\int \frac{x+1}{(2x-3)^3} dx$, u= 2x – 1 $\left[-\frac{4x+1}{8(2x-3)^2}\right]$
8. $\int \sqrt{(x+1)}\, dx$, u = x + 1
   $\left[\frac{2}{15}(3x-2)\sqrt{(x+1)^2} + c\right]$
9. $\int x\sqrt{(x-1)}\, dx, u = \sqrt{(x-1)}$
   $\left[\frac{2}{15}(3x+2)\sqrt{(x-1)^2} + c\right]$
10. $\int (x-4)\sqrt{(x+5)}$, u = x + 5
    $\left[\frac{2}{5}(x-10)\sqrt{(x+1)^3} + c\right]$
11. $\int (3x-2)\sqrt{(1-2x)}dx, u = \sqrt{(1-2x)}$
    $\left[\frac{1}{15}(7-9x)\sqrt{(1-2x)^2} + c\right]$
12. $\int \frac{x}{\sqrt{x+1}} dx$, u= x+1
    $\left[\frac{2}{3}(x-2)\sqrt{x+1} + c\right]$
13. $\int \frac{x}{\sqrt{x-3}} dx$, u= $\sqrt{x-3}$
    $\left[\frac{2}{3}(x+6)\sqrt{x-3} + c\right]$
14. $\int \frac{x-2}{\sqrt{x-4}} dx$, u= $x - 4$
    $\left[\frac{2}{3}(x+2)\sqrt{x-4} + c\right]$
15. $\int \frac{x+3}{\sqrt{5-x}} dx$, u= $\sqrt{5-x}$
    $\left[-\frac{2}{3}(x+19)\sqrt{5-x} + c\right]$
16. Use the substitution $x = \frac{1}{u}$ to evaluate $\int_1^2 \frac{dx}{x\sqrt{x^2-1}}$ $\left[\frac{\pi}{3}\right]$
17. Use the substitution u = $\sqrt{x-2}$ to evaluate $\int_3^4 \frac{3x}{\sqrt{x-2}} dx$ $[16\sqrt{2} - 4]$
18. Evaluate
    (a) $\int_3^5 x(x-3)^2 dx$ [12]
    (b) $\int_4^7 \frac{5-x}{\sqrt{x-3}} dx$ $\left[-\frac{2}{3}\right]$
    (c) $\int_1^3 (3x+1)(2-x)^4\, dx$ $\left[2\frac{4}{5}\right]$
19. By using the substitution u = $\sqrt{1+x^2}$, show that $\int_0^{\sqrt{3}} x^2\sqrt{(1+x^2)}\, dx = 3\frac{13}{15}$

Integration by change of variable where a derivative not exist

Here a term is solved by changing it to another variable

Example 3

Find

(a) $\int_5^6 x\sqrt{(x-5)}\, dx$

Solution

Let u= $\sqrt{(x-5)}$ hence $u^2 = x - 5$ => x =u² + 5

dx = du

Changing limits

x	θ
6	1
5	0

$$\therefore \int_5^6 x\sqrt{(x-5)}\, dx = \int_0^1 (u^2+5)u.2udu$$
$$= 2\int_0^1 (u^4 + 5u^2)\, du$$
$$= 2\left[\frac{1}{5}u^5 + \frac{5}{3}u^3\right]_0^1$$
$$= \frac{2}{15}(3+25) = \frac{56}{15}$$

(b) $\int \frac{x-3}{\sqrt{x+1}} dx$

Solution

Let u= $\sqrt{x+1}$ i.e.$u^2 = x + 1$

- x = u² – 1 and dx = 2udu

$$\therefore \int \frac{x-3}{\sqrt{x+1}} dx = \int \frac{(u^2-1)}{u}.2udu$$
$$= 2\int (u^2-4)du$$
$$= 2\left(\frac{1}{3}u^3 - 4u\right) + c$$
$$= \frac{2}{3}u(u^2-12) + c$$
$$= \frac{2}{3}\sqrt{(x+1)}(x+1-12) + c$$
$$= \frac{2}{3}(x-11)\sqrt{(x+1)} + c$$

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(c) $\int (2x-1)(x+2)^3\,dx$

Solution

Let u = x+2 => x = u-2 and dx = du

$$\therefore \int (2x-1)(x+2)^3\,dx$$
$$= \int [2(u-2) - 1u^3 du]$$
$$= \int (2u^4 - 5u^3)\,du$$
$$= \frac{2}{5}u^5 - \frac{5}{4}u^4 + c$$
$$= \frac{1}{20}u^4(8u-25) + c$$
$$= \frac{1}{20}(x+2)^4(8(x+2+-25) + c$$
$$= \frac{1}{20}(8x-9)(x+2)^4 + c$$

(d) $\int \frac{sin\sqrt{x}}{\sqrt{x}}dx$

Solution

Let u = $\sqrt{x}$ => $x = u^2$ and dx= 2u

$$\therefore \int \frac{sinu}{u}.2udx = 2\int sinudu$$
$$= -2\text{cosu} + \text{c}$$
$$= -2\text{cos}\sqrt{x} + \text{c}$$

Revision exercise 2

1. Integrate each of the following with resp[illegible] to x using suitable substitution
   (a) $x(x+3)^3$
   $\left[\frac{1}{20}(4x - 3x + 3^4) + c\right]$
   (b) $x\sqrt{5-x}$
   $\left[-\frac{2}{15}\left(3x + 10\sqrt{([illegible]3)^3} + [illegible]\right)\right]$
   (c) $\frac{x-3}{(x+2)^2}$ $\left[\frac{5}{[illegible]+2} + [illegible]n(x+2) + c\right]$
   (d) $\frac{x}{\sqrt{2x+1}}$ $\left[\frac{1}{[illegible]}([illegible] - 1)\sqrt{2x+1} + c\right]$
   (e) $(x-3)(5-2x)^4$
   $\left[\frac{1}{120}(31-10x)(5-2x)^5 + c\right]$
   (f) $\frac{x}{\sqrt{(x+1)^3}}$ $\left[\frac{2(x+2)}{\sqrt{x+1}} + c\right]$
   (g) $\frac{x+3}{(3-x)^2}$ $\left[\frac{6}{3-x} + In(3-x) + c\right]$
   (h) $x^2(x-1)^4$
   $\left[\frac{1}{105}(15x^2 + 5x + 1)(x-1)^5 + c\right]$
   (i) $x\sqrt{(1-x)^3}$
   $\left[-\frac{2}{35}(5x+2)\sqrt{(1-x)^5}\right]$

Integrations involving trigonometric functions

A. The double formulae, i.e.

- Cos2x = $2\cos^2 x - 1 = 1 - 2\sin^2 x$
- 1+ cos2x = $2\cos^2 x$
- 1 – cos2x = $2\sin^2 x$
- Sin2x =2sinxcos2

Example 4

Find the following integrals

(a) $\frac{tan\theta}{\sqrt{1+cos2\theta}}d\theta$

Solution

$$\frac{tan\theta}{\sqrt{1+cos2\theta}}d\theta = \int \frac{[illegible]}{\sqrt{2[illegible]}}[illegible]\theta = \int \frac{sin\theta}{\sqrt{2}(cos^2\theta)}d\theta$$

Let u = cosθ => [illegible]inθdθ

$$-[illegible] = \frac{du}{-sin\theta}$$

$$\frac{1}{[illegible]}\int \frac{[illegible]}{(cos^2[illegible])}d\theta = \frac{1}{\sqrt{2}}\int \frac{sin\theta}{u^2}.\frac{du}{-sin\theta}$$
$$= -\frac{1}{\sqrt{2}}\int \frac{du}{u^2} = -\frac{1}{\sqrt{2}}\int u^{-2}du$$
$$= -\frac{1}{\sqrt{2}}(-u^{-1}) + c$$
$$= \frac{1}{\sqrt{2}cos\theta} + c$$

(b) $\int \sqrt{1-cos4\theta}\,d\theta$

Solution

$$\int \sqrt{1-cos4\theta}\,d\theta = \int \sqrt{2sin^2 2\theta}d\theta$$
$$= \sqrt{2}\int sin2\theta d\theta$$
$$= \frac{-\sqrt{2}}{2}cos2\theta + c$$

(c) $\int sin3\theta cos3\theta d\theta = \frac{1}{2}\int 2sin3\theta cos3\theta d\theta$
$$= \frac{1}{2}\int sin6\theta$$
$$= -\frac{1}{12}cos6\theta + c$$

B. The factor formulae, i.e.

- $cos\alpha + cos\beta = 2cos\left(\frac{\alpha+\beta}{2}\right)cos\left(\frac{\alpha-\beta}{2}\right)$
- $cos\alpha - cos\beta = -2sin\left(\frac{\alpha+\beta}{2}\right)sin\left(\frac{\alpha-\beta}{2}\right)$
- $sin\alpha + sin\beta = 2sin\left(\frac{\alpha+\beta}{2}\right)cos\left(\frac{\alpha-\beta}{2}\right)$
- $sin\alpha - sin\beta = -2cos\left(\frac{\alpha+\beta}{2}\right)sin\left(\frac{\alpha-\beta}{2}\right)$

Example 5

Find the following integrals

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(a) $\int cos2\theta cos\theta d\theta$

Solution

$$\int cos2\theta cos\theta d\theta = \frac{1}{2}\int 2cos2\theta cos\theta d\theta$$
$$= \frac{1}{2}\int (cos3\theta + cos\theta)d\theta$$
$$= \frac{1}{2}\left(\frac{1}{3}sin3\theta + sin\theta\right) + c$$
$$= \frac{1}{6}sin3\theta + \frac{1}{2}sin\theta + c$$

(b) $\int sin4\theta sin3\theta d\theta$

Solution

$$\int sin4\theta sin3\theta d\theta = \frac{1}{2}\int 2sin4\theta sin3\theta d\theta$$
$$= \frac{1}{2}\int (cos7\theta - cos\theta)d\theta$$
$$= \frac{1}{2}\left(\frac{1}{7}sin7\theta - sin\theta\right) + c$$
$$= \frac{1}{14}sin7\theta + \frac{1}{2}sin\theta + c$$

(c) $\int cos3\theta sin\theta d\theta$

Solution

$$\int cos3\theta sin\theta d\theta = \frac{1}{2}\int 2cos3\theta sin\theta d\theta$$
$$= \frac{1}{2}\int (sin4\theta - sin2\theta)d\theta$$
$$= \frac{1}{2}\left(-\frac{1}{4}cos4\theta + \frac{1}{2}cos2\theta\right) + c$$
$$= \frac{1}{4}cos2\theta - \frac{1}{8}cos4\theta + c$$

(d) $\int sin\frac{3}{2}\theta cos\frac{1}{2}\theta d\theta$

Solution

$$\int sin\frac{3}{2}\theta cos\frac{1}{2}\theta d\theta = \frac{1}{2}\int 2sin\frac{3}{2}\theta cos\frac{1}{2}\theta d\theta$$
$$= \frac{1}{2}\int (sin2\theta + sin\theta)d\theta$$
$$= \frac{1}{2}\left(-\frac{1}{2}cos2\theta - cos\theta\right) + c$$
$$= -\frac{1}{4}(cos2\theta + cos\theta) + c$$

(e) $\int sin2\theta cos\theta d\theta$

Solution

$$\int sin2\theta cos\theta d\theta = \frac{1}{2}\int 2sin2\theta cos\theta d\theta$$
$$= \frac{1}{2}\int (sin3\theta + sin\theta)d\theta$$
$$= \frac{1}{2}\left(-\frac{1}{3}cos3\theta - cos\theta\right) + c$$
$$= -\frac{1}{6}(cos2\theta + 3cos\theta) + c$$

Note

(i) The integral $\int sin2\theta cos2\theta d\theta$, where the angles are the same can be solved in two ways.

Method I: double angle formula

$$\int sin2\theta cos2\theta d\theta = \frac{1}{2}\int sin4\theta d\theta$$
$$= -\frac{1}{8}cos4\theta + c$$

Method II: the factor formula

$$\int sin2\theta cos2\theta d\theta = \frac{1}{2}\int (sin4\theta + sin0)d\theta$$
$$= -\frac{1}{8}cos4\theta + c$$

(ii) The integral of $\int sin4\theta cos2\theta$ where the angles are different, use method I because method II is inapplicable.

Revision exercise 3

1. Evaluate
   (a) $\int_0^{\frac{\pi}{2}} sin2[illegible]cosx[illegible]$ $\left[\frac{2}{3}\right]$
   (b) $\int_{[illegible]}^{\frac{\pi}{[illegible]}} sin3[illegible]xdx$ [0.1083]
2. Integr[illegible]te [illegible]e following using appropriate [illegible]stit[illegible]ion.
   (a) [illegible]$xsin(x^2 - 4)\,dx$
   [illegible]$-3cos(x^2 - 4) + c]$
   (b) $\int 5xcos(5 - x^2)\,dx$
   $\left[-\frac{5}{2}sin(5 - x^2) + c\right]$
   (c) $\int 3x\sqrt{1 + x^2}\,dx$
   $\left[(1 + x^2)\sqrt{1 + x^2} + c\right]$
   (d) $\int 3x(x^2 + 6)^5 dx$ $\left[\frac{1}{4}(x^2 + 6)^6 + c\right]$
   (e) $\int \frac{x}{\sqrt{(2x^2-5)}}dx$ $\left[\frac{1}{2}\sqrt{(2x^2 - 5)}\right] + c$

Integrations of odd and even powers of trigonometric functions

(sinx, cosx, tanx, cotx, secx and cosecx)

Integration of trigonometric functions rose to odd powers

The Pythagoras theorem in trigonometry is handy namely

- $cos^2x + sin^2x = 1$
- $1 + tan^2x = sec^2x$
- $1 + cot^2x = cosec^2x$

Example 6

Integrate the following

(a) $\int cos^5xdx$

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Solution

$\int cos^5xdx = \int cos^4xcosxdx$

$= \int (1 - sin^2x)^2cosxdx$

$= \int (cosx - 2sin^2xcosx + sin^4xcosx)dx$

$= sinx - \frac{2}{3}sin^3x + \frac{1}{5}sin^5x + c$

(b) $\int cos^3 2xdx$

Solution

$\int cos^3 2xdx = \int sin^2 2xsin2xdx$

$= \int (1 - cos^2 2x)sin2xdx$

$= \int sin2x - cos^2 2csin2xdx$

$= -\frac{1}{2}cos2x + \frac{1}{2}\left(\frac{1}{3}\right)cos^3 2x + c$

$= \frac{1}{6}cos^3 2x - \frac{1}{2}cos2x + c$

(c) $\int_0^{\frac{\pi}{12}} cos^3 6xdx$

Solution

$\int_0^{\frac{\pi}{12}} cos^3 6xdx = \int_0^{\frac{\pi}{12}} (1 - sin^2 6x)cos6xdx$

$= \int_0^{\frac{\pi}{12}} (cos6x - sin^2 6xcos6x)\, dx$

$= \left[\frac{1}{6}sin6x - \frac{1}{6}\left(\frac{1}{3}\right)sin^3 6x\right]_0^{\frac{\pi}{12}}$

$= \frac{1}{6}sin\frac{\pi}{2} - \frac{1}{18}\left(sin\frac{\pi}{2}\right)^3 = \frac{1}{9}$

(d) $\int sin^3xdx$

Solution

$\int sin^3xdx = \int sinx(1 - cos^2x)dx$

$= \int (sinx - sixcosx)dx$

$= -cosx - \left(-\frac{1}{3}cos^3x\right) + c$

$= \frac{1}{3}cos^3x - cosx$

(e) $\int tan^3xdx$

Solution

$\int tan^3xdx = \int tanxtan^2x\, dx$

$= \int tanx(sec^2x - 1)dx$

$= \int tanxsec^2xdx - \int tanx\, dx$

By inspection

$\frac{d}{dx}(In(cosx) = -tanx$

$=> \int -tanxdx = In(cosx)$

Also

$\frac{d}{dx}(tan^2x) = 2tanxsec^2x$

$=> \int tanxsec^2xdx = \frac{1}{2}tan^2x$

$\therefore \int tan^3xdx = \frac{1}{2}tan^2x + In(cosx) + c$

Or

$\frac{d}{dx}In(secx) = tanx$

$=> \int tanxdx = In(secx)$

Also $\frac{d}{dx}(sec^2x) = 2tanxsec^2x$

$=> \int tanxsec^2xdx = \frac{1}{2}sec^2x$

$\therefore \int tan^3xdx = \frac{1}{2}sec^2x + In(secx) + c$

(f) $\int tan^5 2xdx$

Solution

$\int tan^5 2xdx = \int tan2xtan^4 2xdx$

$= \int tan2x(sec^2 2x - 1)^2dx$

$= \int tan2x(sec^4 2x - 2sec^2 2x) - 1)dx$

$= \int t(an2xsec^4 2x - 2tan2xsec^2 2x - tan2x)dx$

$= \frac{1}{8}sec^4 2x - \frac{1}{2}sec^2 2x + \frac{1}{2}In(sec2x) + c$

Or

$\int tan^5 2xdx = \int tan^3 2xtan^2 2xdx$

$= \int tan^3 2x(sec^2 2x - 1)dx$

$= \int tan^3 2xsec^2 2xdx - \int tan^3 2xdx$

$= \frac{1}{8}tan^4 2x - \int tan2x(sec^2 2x - 1)dx$

$= \frac{1}{8}tan^4 2x - \int tan2xsec^2 2xdx + \int tan2x\, dx$

$= \frac{1}{8}tan^4 2x - \frac{1}{4}tan^2 2x + \frac{1}{2}In(sec2x) + c$

(g) $\int cot^3xdx$

Solution

$\int cot^3xdx = \int cotxcot^2xdx$

$= \int cotx(cosec^2x - 1)dx$

$= \int cotxcosec^2xdx - \int cotxdx$

$= \frac{1}{2}cot2x - In(sinx) + c$

Note: the integration of odd powers of secx and cosec x are done using integration by parts.

Integration of trigonometric functions rose to even powers

These are worked out using double angle formulae.

Example 7

Find the integrals of the following

(a) $\int cos^2xdx$

Solution

$\int cos^2xdx = \int \frac{1}{2}(1 + cos2x)dx$

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$= \int \left(\frac{1}{2} + \frac{1}{2}cos2x\right) dx$

$= \frac{x}{2} + \frac{1}{4}sin2x + c$

(b) $\int cos^4 x dx$

Solution

$\int cos^4 x dx = \int (cos^2 x)^2 dx$

$= \int \frac{1}{4}\left((1 + cos2x)\right)^2 dx$

$= \int \frac{1}{4}(1 + 2cos2x + cos^2 2x)dx$

$= \int \left(\frac{1}{4} + \frac{1}{2}cos2x\right) dx + \frac{1}{4}\int \frac{1}{2}(1 + cos4x)dx$

$= \frac{1}{4}x + \frac{1}{4}sin2x + \frac{1}{8}x + \frac{1}{32}sin4x + c$

(c) $\int sin^2 3x dx$

Solution

$\int sin^2 3x dx = \frac{1}{2}\int (1 - cos6x)dx$

$= \frac{1}{2}x - \frac{1}{12}sin6x + c$

(d) $\int tan^4 x dx$

Solution

$\int tan^4 x dx = \int tan^2 x . tan^2 x dx$

$= \int tan^2 x(sec^2 x - 1)dx$

$= \int tan^2 x sec^2 x dx - \int tan^2 x\, dx$

$= \frac{1}{3}tan^3 x - \int (sec^2 - 1)\, dx$

$= \frac{1}{3}tan^3 x - tanx + x + c$

(e) $\int sec^4 x dx$

Solution

$\int sec^4 x dx = \int sec^2 x . sec^2 x dx$

$= \int sec^2 x(tan^2 x + 1)dx$

$= \int (sec^2 x tan^2 x + sec^2 x)dx$

$= \frac{1}{3}tan^3 x + tanx + c$

(f) $\int cosec^2\left(\frac{1}{2}x\right) dx$

Solution

$\int cosec^2\left(\frac{1}{2}x\right) dx = -2cot\frac{1}{2}x + c$

(g) $\int cot^4 x dx$

Solution

$\int cot^4 x dx = \int cot^2 x . cot^2 x dx$

$= \int cot^2 x(cosec^2 x - 1)dx$

$= \int cot^2 x cosec^2 x - \int cot^2 x\, dx$

$= \frac{1}{3}cot^3 x - \int (cosec^2 x - 1)dx$

$= \frac{1}{3}cot^3 x + cotx + x + c$

Exercise 4

1. Integrate each of the following

(a) $sinxcos^5x$ $\left[-\frac{1}{6}cos^6 x + c\right]$

(b) $cos^3 4x$ $\left[\frac{1}{4}sin4x - \frac{1}{2}sin^3 4x + c\right]$

(c) $sin^3 xcos^2 x$ $\left[-\frac{1}{3}cos^3 x + \frac{1}{5}cos^5 x + c\right]$

(d) $cos^3 xsin^4 x$ $\left[\frac{1}{5}sin^5 x - \frac{1}{7}sin^7 x + c\right]$

(e) $cos^3 2xsin^2 2x$ $\left[\frac{1}{6}sin^3 2x - \frac{1}{10}sin^5 2x + c\right]$

(f) $sin2xsin^2 x$ $\left[\frac{1}{2}sin^4 + c\right]$

(g) $cos^3 xsin^3 x$ $\left[\frac{1}{4}sin^4 x - \frac{1}{6}sin^6 x + c\right]$

2. Integrate each of the following

(a) $cot^2 2x$ $\left[\frac{1}{2}x + \frac{1}{8}sin4x + c\right]$

(b) $cos^2 3x$ $\left[\frac{1}{2}x + \frac{1}{8}sin6x + c\right]$

(c) $sin^3 xcos^3 x$ $\left[\frac{1}{2}x + \frac{1}{16}sin8x + c\right]$

(d) $cos^3 x$ $\left[\frac{1}{2}x + \frac{1}{24}sin12x + c\right]$

(e) $sin^3\left(\frac{1}{2}x\right)$ $\left[\frac{1}{2}x - \frac{1}{2}sinx + c\right]$

(f) $cos^4 x$ $\left[\frac{3}{8}x + \frac{1}{4}sin2x + \frac{1}{32}sin4x + c\right]$

(g) $sin^4 2x$ $\left[\frac{3}{8}x - \frac{1}{4}sin4x + \frac{1}{64}sin8x + c\right]$

3. Integrate each of the following

(a) $cot^2 x$ $[-cotx - x + c]$

(b) $tan^2 2x$ $\left[\frac{1}{2}tan2x + c\right]$

(c) $sec^2 xtan^3 x$ $\left[\frac{1}{4}tan^4 x + c\right]$

(d) $cosec^2 2xcot^4 2x$ $\left[-\frac{1}{10}cot^5 2x\right]$

(e) $tan^3 x$ $\left[\frac{1}{2}tan^2 x + In(cosx)\right]$

(f) $cot^4 3x$ $\left[-\frac{1}{9}cot^3 3x + \frac{1}{3}cot3x + c\right]$

(g) $tan^4 5x$ $\left[\frac{1}{15}tan^3 5x - \frac{1}{5}tan5x + x + c\right]$

(h) $tan^5 2x$

$\left[\frac{1}{8}tan^4 2x - \frac{1}{4}tan^2 2x + \frac{1}{2}In(sec2x) + c\right]$

(i) $cosecxcot^3 x$ $\left[cosecx - \frac{1}{3}cosec^3 x + c\right]$

(j) $tan^5 xsecx$

$\left[secx - \frac{2}{3}sec^3 x + \frac{1}{5}sec^5 c + c\right]$

4. Find the Integral of each of the following

(a) $cosec^2 x$ $[-cotx + c]$

(b) $sec^2 3x$ $\left[\frac{1}{3}tan3x + c\right]$

(c) $cosec^2\left(\frac{1}{3}x\right)$ $\left[-3cot\left(\frac{1}{3}x\right) + c\right]$

(d) $sec^4 x$ $\left[\frac{1}{3}tan^3 x + tanx + c\right]$

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(e) $\text{cosec}^4 5x$ $\left[-\frac{1}{15}cot^3 5x - \frac{1}{5}cot 5x + c\right]$

(f) $\sec^4 3x$ $\left[\frac{1}{9}tan^3 3x + \frac{1}{3}tan 3x + c\right]$

(g) $\sec^6 x$ $\left[\frac{1}{5}tan^5 x + \frac{2}{3}tan^3 x + tanx + c\right]$

Integration involving inverse trigonometric functions

A. From $\frac{d}{dx}(sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

- $\int \frac{1}{\sqrt{1-x^2}} = sin^{-1}x + c$

This result enables the integration of the form $\int \frac{1}{\sqrt{a^2-b^2x^2}}dx$ to be workout, i.e.

$$\int \frac{1}{\sqrt{a^2-b^2x^2}}dx = \int \frac{1}{a\sqrt{1-\frac{b^2x^2}{a^2}}}dx = \int \frac{1}{a\sqrt{1-\left(\frac{bx}{a}\right)^2}}dx$$

Let $\frac{bx}{a} = sinu; dx = \frac{a}{b}cosudu$

- $$\int \frac{1}{\sqrt{a^2-b^2x^2}}dx = \int \frac{1}{a\sqrt{1-sin^2u}}.\frac{a}{b}cosudu$$
$$= \int \frac{1}{acosu}.\frac{a}{b}cosudu$$
$$= \frac{1}{b}\int du = \frac{1}{b}u + c$$
$$= \frac{1}{b}sin^{-1}\left(\frac{bx}{a}\right) + c$$

Example 8

Integrate the following

(a) $\int \frac{1}{\sqrt{4-9x^2}}dx$

Solution

$$\int \frac{1}{\sqrt{4-9x^2}}dx = \int \frac{1}{\sqrt{\left(1-\left(\frac{3x}{2}\right)^2\right)}}dx$$

Let sinu $=\frac{3x}{2}$, dx $=\frac{2}{3}cosudu$

- $$\int \frac{1}{\sqrt{4-9x^2}}dx = \int \frac{1}{2\sqrt{(1-sin^2u)}}.\frac{2}{3}cosudu$$
$$= \int \frac{1}{2cosu}.\frac{2}{3}cosudu$$
$$= \frac{1}{3}\int du = \frac{1}{3}u + c$$
$$= \frac{1}{3}sin^{-1}\left(\frac{3x}{2}\right) + c$$

(b) $\int \frac{1}{\sqrt{1-16x^2)}}dx$

Solution

$$\int \frac{1}{\sqrt{1-16x^2}}dx = \int \frac{1}{\sqrt{(1-(4x)^2)}}dx$$

Let sinu $=4x$, dx $=\frac{1}{4}cosudu$

- $$\int \frac{1}{\sqrt{(1-(4x)^2)}}dx = \int \frac{1}{\sqrt{(1-sin^2u)}}.\frac{1}{4}cosudu$$
$$= \int \frac{1}{cosu}.\frac{1}{4}cosudu$$
$$= \frac{1}{4}\int du = \frac{1}{4}u + c$$
$$= \frac{1}{4}sin^{-1}(4x) + c$$

(c) $\int \frac{4x^2}{\sqrt{1-x^6)}}dx$

Solution

$$\int \frac{4x^2}{\sqrt{1-x^6)}}dx = \int \frac{4x^2}{\sqrt{(1-(x^3)^2)}}dx$$

Let sinu $=x^3$, dx $=\frac{1}{3x^2}cosudu$

- $$\int \frac{4x^2}{\sqrt{(1-(x^3)^2)}}dx = \int \frac{4x^3}{\sqrt{(1-sin^2u)}}.\frac{1}{3x^2}cosudu$$
$$= \int \frac{1}{cosu}.\frac{4}{3}cosudu$$
$$= \frac{4}{3}\int du = \frac{4}{3}u + c$$
$$= \frac{4}{3}s\ ^{-1}(x^3) + c$$

(d) $\int \frac{1}{\sqrt{36-4x^2}}d$

Soluti

$$\int \frac{1}{\ 4x^2}dx = \int \frac{1}{6\sqrt{\left(1-\left(\frac{2x}{6}\right)^2\right)}}dx$$

Let sinu $=\frac{2x}{6} = \frac{1}{3}x$, dx $=3cosudu$

$$\int \frac{1}{\sqrt{36-4x^2}}dx = \int \frac{1}{6\sqrt{(1-sin^2u)}}.3cosudu$$
$$= \int \frac{1}{6cosu}.3cosudu$$
$$= \frac{1}{2}\int du = \frac{1}{2}u + c$$
$$= \frac{1}{3}sin^{-1}\left(\frac{x}{3}\right) + c$$

(e) $\int \frac{1}{\sqrt{25-9x^2}}dx$

Solution

$$\int \frac{1}{\sqrt{25-9x^2}}dx = \int \frac{1}{5\sqrt{\left(1-\left(\frac{3x}{5}\right)^2\right)}}dx$$

Let sinu $=\frac{3x}{5}$, dx $=\frac{5}{3}cosudu$

- $$\int \frac{1}{\sqrt{25-9x^2}}dx = \int \frac{1}{5\sqrt{(1-sin^2u)}}.\frac{5}{3}cosudu$$
$$= \int \frac{1}{5cosu}.\frac{5}{3}cosudu$$
$$= \frac{1}{3}\int du = \frac{1}{3}u + c$$
$$= \frac{1}{3}sin^{-1}\left(\frac{3x}{5}\right) + c$$

(f) $\int \frac{1}{\sqrt{3-2x-x^2}}dx$

Solution

$$3 - 2x - x^2 = -(x^2 + 2x - 3)$$

By completing squares

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$$-(x^2+2x-3) = -[(x+1)^2-3-1]$$

$$= 4-(x+1)^2$$

$$\int \frac{1}{\sqrt{3-2x-x^2}}dx = \int \frac{1}{\sqrt{4-(x+1)^2}}dx$$

$$\int \frac{1}{\sqrt{4-(x+1)^2}}dx = \int \frac{1}{2\sqrt{1-\left(\frac{x+1}{2}\right)^2}}$$

Let sinu $=\frac{x+1}{2}$, dx $=2cosudu$

- $$\int \frac{1}{\sqrt{4-(x+1)^2}}dx = \int \frac{1}{2\sqrt{(1-sin^2u)}}.2cosudu$$

$$= \int \frac{1}{2cosu}.2cosudu$$

$$= \int du = u + c$$

$$= sin^{-1}\left(\frac{x+1}{2}\right) + c$$

B. From $\frac{d}{dx}\left(tan^{-1}x\right) = \frac{1}{1+x^2}$

- $$\int \frac{1}{1+x^2}dx = tan^{-1}x + c$$

This result enables the integration of the form $\int \frac{1}{\sqrt{a^2+b^2x^2}}dx$ to be workout, i.e

$$\int \frac{1}{a^2+b^2x^2}dx = \int \frac{1}{a^2\left(1+\frac{b^2x^2}{a^2}\right)}dx$$

$$= \int \frac{1}{a^2\left(1+\left(\frac{bx}{a}\right)^2\right)}dx$$

Let $\frac{bx}{a}$= tanu, dx= $\frac{a}{b}sec^2udu$

$$=> \int \frac{1}{a^2\left(1+\left(\frac{bx}{a}\right)^2\right)}dx = \int \frac{1}{a^2(1+[illegible])}.\frac{a}{b}sec[illegible]udu$$

$$= \frac{1}{ab}\int \frac{[illegible]}{sec[illegible]}.sec^2udu$$

$$\frac{1}{ab}\int du = \frac{1}{ab}u + c$$

$$= \frac{1}{ab}tan^{-1}\left(\frac{bx}{a}\right) + c$$

Example 9

Find

(a) $\int \frac{1}{9+25x^2}dx$

Solution

Comparing $\int \frac{1}{9+25x^2}$ with $\int \frac{1}{a^2+b^2x^2}dx$

a=3 and b =5

$$\int \frac{1}{9+25x^2} = \frac{1}{3x5}\left[tan^{-1}\left(\frac{5}{3}x\right)\right] + c$$

$$= \frac{1}{15}\left[tan^{-1}\left(\frac{5}{3}x\right)\right] + c$$

(b) $\int \frac{1}{5+9x^2}dx$

Solution

Comparing $\int \frac{1}{5+9x^2}$ with $\int \frac{1}{a^2+b^2x^2}dx$

a=$\sqrt{5}$ and b =3

$$\int \frac{1}{9+25x^2} = \frac{1}{\sqrt{5}x3}\left[tan^{-1}\left(\frac{3}{\sqrt{5}}x\right)\right] + c$$

$$= \frac{1}{3\sqrt{5}}\left[tan^{-1}\left(\frac{3\sqrt{5}}{5}x\right)\right] + c$$

(c) $\int \frac{1}{1+2x+4x^2}dx$

Solution

$$1+2x+4x^2 = 4x^2+2x+1$$

$$= 4\left(x^2+\frac{1}{2}x+\frac{1}{4}\right)$$

$$= 4\left[\left(x+\frac{1}{4}\right)^2+\frac{1}{4}-\frac{1}{16}\right]$$

$$= 4\left(x+\frac{1}{4}\right)^2+\frac{3}{4}$$

$$\int \frac{1}{1+2x+4x^2dx} = \int \frac{1}{[illegible]\left(x+[illegible]\right)+\frac{3}{4}}dx = \int \frac{1}{\frac{3}{4}+4\left(x+\frac{1}{4}\right)^2}dx$$

Comp[illegible] $\frac{1}{\frac{3}{4}+4\left(x+\frac{1}{4}\right)^2}$ with $\int \frac{1}{a^2+b^2x^2}dx$

a=$\frac{\sqrt{3}}{2}$ and b =2

$$\int \frac{1}{\frac{3}{4}+4\left(x+\frac{1}{4}\right)^2} = \frac{1}{\frac{\sqrt{3}}{2}x2}\left[tan^{-1}\left(\frac{2\left(x+\frac{1}{4}\right)}{\frac{\sqrt{3}}{2}}x\right)\right] + c$$

$$= \frac{\sqrt{3}}{3}\left[tan^{-1}\left(\frac{4x+1}{\sqrt{3}}x\right)\right] + c$$

(d) $\int_0^1 \frac{1}{3x^2+6x+4}dx$

Solution

$$3x^2+6x+4 = 3\left(x^2+2x+\frac{4}{3}\right)$$

$$= 3\left[(x+1)^2+\frac{4}{3}-1\right]$$

$$= 3\left[(x+1)^2+\frac{1}{3}\right]$$

$$= 3(x+1)^2+1$$

$$\int \frac{1}{3x^2+6x+4}dx = \int \frac{1}{1+3(x+1)^2}$$

Comparing $\int \frac{1}{1+3(x+1)^2}$ with $\int \frac{1}{a^2+b^2x^2}dx$

a=1 and b =$\sqrt{3}$

$$\int \frac{1}{1+3(x+1)^2} = \frac{1}{\sqrt{3}}\left[tan^{-1}\left(\frac{\sqrt{3}(x+1)}{1}x\right)\right] + c$$

$$= \frac{1}{\sqrt{3}}\left[tan^{-1}\left(\sqrt{3}(x+1)\right)\right] + c$$

$$\therefore \int_0^1 \frac{1}{3x^2+6x+4}dx = \left[\frac{1}{\sqrt{3}}\left[tan^{-1}\left(\sqrt{3}(x+1)\right)\right]\right]_0^1$$

$$= \frac{1}{\sqrt{3}}\left[tan^{-1}(2\sqrt{3}) - tan^{-1}(\sqrt{3})\right]$$

$$= 0.44$$

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Revision exercise 5

Find

(a) $\int \frac{1}{9+x^2}dx$ $\left[\frac{1}{3}tan^{-1}\left(\frac{1}{3}x\right)+c\right]$

(b) $\int_{-2}^{2} \frac{1}{4+x^2}dx$ [0.7854]

(c) $\int_{-\sqrt{3}}^{3} \frac{1}{x^2+3}dx$ [1.833]

(d) $\int \frac{1}{4-x^2}dx$ $[sin^{-1}x+c]$

(e) $\int \frac{1}{\sqrt{16-x^2}}dx$ $\left[sin^{-1}\left(\frac{x}{4}\right)+c\right]$

(f) $\int \frac{1}{\sqrt{49-x^2}}dx$ $\left[sin^{-1}\left(\frac{x}{7}\right)+c\right]$

(g) $\int \frac{1}{\sqrt{25-4x^2}}dx$ $\left[\frac{1}{2}sin^{-1}\left(\frac{2}{4}x\right)+c\right]$

(h) $\int \frac{3}{9+x^2}dx$ $\left[tan^{-1}\left(\frac{x}{3}\right)+c\right]$

(i) $\int \frac{1}{25+x^2}dx$ $\left[\frac{1}{5}tan^{-1}\left(\frac{x}{5}\right)+c\right]$

(j) $\int \frac{2}{100+9x^2}dx$ $\left[\frac{1}{15}tan^{-1}\left(\frac{3}{10}x\right)+c\right]$

(k) $\int \frac{1}{3-2x+x^2}dx$ $\left[\frac{1}{\sqrt{2}}tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right)+c\right]$

(l) $\int \frac{1}{x^2\sqrt{4-x^2}}dx$ $\left[-\frac{1}{4}cotsin^{-1}\left(\frac{x}{2}\right)+c\right]$

Integration of exponential and logarithmic functions.

A. From $\frac{d}{dx}e^x = e^x$

- $\int e^x dx = e^x + c$

Example 10

Find

(a) $\int xe^{x^2}dx$

Solution

Let u = $3x^2$ => du = 6xdx i.e. xdx =$\frac{1}{6}du$

$\int xe^{x^2}dx = \frac{1}{6}\int e^u du = \frac{1}{6}e^u + c$

$\therefore \int xe^{x^2}dx = \frac{1}{6}e^{x^2} + c$

(b) $\int secxtanxe^{secx}dx$

Solution

Let u = secx => du=secxtanxdx

$\int secxtanxe^{secx}dx = \int e^u du = e^u + c$

$\therefore \int secxtanxe^{secx}dx = e^{secx} + c$

(c) $\int \frac{e^{cotx}}{sin^2x}dx$

Solution

Let u = cotx

- Du = -cosec^{2}x = $-\frac{1}{sin^2x}$

$\int \frac{e^{cotx}}{sin^2x}dx = -\int e^u du = -e^u + c$

$\therefore \int \frac{e^{cotx}}{sin^2x}dx = -e^{cotx} + c$

(d) $\int \frac{e^{-\frac{1}{x}}}{x^2}dx$

Solution

Let u = $-\frac{1}{x}$ => du = $\frac{1}{x^2}dx$

$\int \frac{e^{-\frac{1}{x}}}{x^2}dx = \int e^u du = -e^u + c$

$\therefore \int \frac{e^{-\frac{1}{x}}}{x^2}dx = e^{-\frac{1}{x}} + c$

B. From $\frac{d}{dx}(Inx) = \frac{1}{x}$

- $\int \frac{1}{x}dx = Inx + c$ [illegible] $InAx$

This result shows that

$\int \frac{f'(x)}{f(x)}dx$ [illegible] $= In[f(x)] + c$ **i.e.**

- $\int cot2xdx = \int \frac{cos2x}{sin2x}dx$

$= \frac{1}{2}In(sin2x) + c$

- $\int \frac{a}{b+cx}dx = \frac{a}{c}In(b+cx) + k$

Example 11

Find

(a) $\int \frac{1}{3x+4}dx$

Solution

Let u = 3x+4 => du 3dx i.e. dx=$\frac{1}{3}du$

$\therefore \int \frac{1}{3x+4}dx = \frac{1}{3}\int du = \frac{1}{3}Inu + c$

$= \frac{1}{3}In(3x+4) + c$

(b) $\int \frac{x}{1-5x^2}dx$

Solution

Let u = $1-5x^2$

=> du -10xdx i.e. dx=$-\frac{1}{10x}du$

$\therefore \int \frac{x}{1-5x^2}dx = \frac{1}{10}\int \frac{1}{u}du = \frac{1}{10}Inu + c$

$= \frac{1}{3}In(1-5x^2) + c$

(c) $\int tan^3xdx$

Solution

$\int tan^3xdx = \int tan^2xtanxdx$ (an odd power)

$= \int (sec^2x - 1)tanxdx$

$= \int sec^2xtanxdx - \int tanxdx$

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For $\int sec^2xtanxdx$

Let u = tanx, => du = sec^2xdx

$\therefore \int sec^2xtanxdx = \int udu = \frac{1}{2}tan^2x + c$

For $\int tanxdx = \int \frac{sinx}{cosx}dx$

Let u =cosx, => du = -sinxdx

$\therefore \int tanxdx = -\int \frac{1}{u}du = -Inu + c$

= -ln(cosx)+c =ln(secx)+c

$\therefore \int tan^3xdx = \frac{1}{2}tan^2x + In(secx) + c$

(d) $\int_0^1 \frac{x+1}{3+4x^2}dx$

Solution

$\int_0^1 \frac{x+1}{3+4x^2}dx = \int_0^1 \frac{1}{3+4x^2}dx + \int_0^1 \frac{1}{3+4x^2}dx$

$= \left[\frac{1}{8}In(3+4x^2 + \frac{1}{2\sqrt{3}}tan^{-1}\left(\frac{2}{\sqrt{3}}\right)\right]_0^1$

$= \frac{1}{8}In\left(\frac{7}{3}\right) + \frac{1}{2\sqrt{3}}tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$

(e) $\int_a^{2a} \frac{x^3}{x^4+a^4}dx$

Solution

$\int_a^{2a} \frac{x^3}{x^4+a^4}dx = \frac{1}{4}[In(x^4+a^4]_a^{2a}$

$= \frac{1}{4}(In17a^4 - In2$

$= \frac{1}{4}In\left(\frac{17}{2}\right)$ =0.5[illegible]5

C. From $\frac{d}{dx}a^x = a^xIna$

⇨ $\int a^xdx = \frac{1}{Ina}a^x + c$

It follows that $\int$ [illegible]^{x}a[illegible] = $\frac{2}{In2} + c$

Example 12

Integrate

(a) $\int x^2 2^{3x^2}dx$

Solution

Let u = $3x^3$, => du = $9x^2$ i.e. $x^2dx = \frac{1}{9}du$

$\int x^2 2^{3x^2}dx = \frac{1}{9}\int 2^u du = \frac{1}{9}\frac{2^u}{In2} + c$

$= \frac{1}{9}\frac{2^{3x^2}}{In2} + c$

(b) $\int cosx.5^{sinx}dx$

Solution

Let u = sinx, => du = cosxdx

$\int cosx.5^{sinx}dx = \int 5^u du = \frac{5^u}{In5} + c$

$= \frac{5^{sinx}}{In5} + c$

(c) $\int \frac{3^{cotx}}{sin^2x}dx$

Solution

Let u = cotx, => du=-$cosec^2$x

$\int \frac{3^{cotx}}{sin^2x}dx = -\int 3^{cotx}cosec^2xdx$

$= \int 3^u du = \frac{3^u}{In3} + c$

$= \frac{3^{cotx}}{In3} + c$

Revision exercise 6

1. Find the following integrals

(a) $\int e^x(3+e^x)^2dx$ $\left[\frac{1}{3}(3+e^x)^3 + c\right]$

(b) $\int 2e^x(e^x - 4$ [illegible] $\left[\frac{1}{2}(e^x-4)^4 + c\right]$

(c) $\int \frac{4e^{-2}}{(1+e^{-}$ [illegible] dx $\left[\frac{2}{1+e^{-2x}} + c\right]$

(d) $\int$ [illegible] dx $\left[-\frac{1}{3}(e^{-x}+7)^3 + c\right]$

(e) $\int$ [illegible] $+ e^x}dx$ $\left[\frac{2}{3}\sqrt{(4+e^x)^3 + c}\right]$

(f) [illegible] $^{5x}\sqrt{e^{5x}+2}\,dx$ $\left[\frac{2}{15}\sqrt{(e^{5x}+2)^3} + c\right]$

(g) $\int \frac{e^{3x}}{\sqrt{e^{3x}-1}}$ $\left[\frac{2}{3}\sqrt{e^{3x}-1} + c\right]$

(h) $\int \frac{1}{2e^x\sqrt{1-e^{-x}}}dx$ $\left[\sqrt{1-e^{-x}} + c\right]$

(i) $\int 5^xdx$ $\left[\frac{5^x}{In5} + c\right]$

(j) $\int 3^{2x}$ $\left[\frac{3^{2x}}{In9} + c\right]$

(k) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}$ $\left[2e^{\sqrt{x}} + c\right]$

(l) $\int x^2e^{x^3}$ $\left[\frac{1}{3}e^{x^3}\right]$

(m) $\int 4^xdx$ $\left[\frac{4x}{In4}\right]$

(n) $\int x10^xdx$ $\left[\frac{x10^x}{In10} - \frac{10^x}{(In10)^2} + c\right]$

2. Evaluate

(a) $\int_1^3 e^xdx$ $[e(e^2-1)]$

(b) $\int_0^3 e^{-x}dx$ $\left[1 - \frac{1}{e^3}\right]$

(c) $\int_1^2 2e^{(2x+1)}$ $[e^3(e^2-1)]$

(d) $\int_{-1}^1 2e^{(1-2x)}$ $\left[e^3 - \frac{1}{e}\right]$

(e) $\int_0^1(4xe^{x^2}+1)dx$ $[2e-1]$

Integration involving partial fractions

There are three established types of partial fractions depending on the nature of the denominator.

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A. Denominators with linear factors e.g. 3x – 1, x + 2 and 3x – 4.

Each linear factor (ax + b) in the denominator has a corresponding partial fraction of the form $\frac{c}{(ax+b)}$, where a, b and c are constants.

Example 13

(a) Express each of the following in partial fraction. Hence find the integral of each with respect to x.

(i) $\frac{x-1}{(x+1)(x-2)}$

Solution

Let $\frac{x-1}{(x+1)(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x-2)}$

Multiplying by (x+1)(x-2)

⇨ x – 1 = A(x – 2) + B(x+1)

then we find the values of A and B

Putting x = 2: 1 = 3B, => B = $\frac{1}{3}$

Putting x = -1: -2 = -3A, => A = $\frac{2}{3}$

$$\therefore \frac{x-1}{(x+1)(x-2)} = \frac{\frac{2}{3}}{(x+1)} + \frac{\frac{2}{3}}{(x-2)}$$

$$= \frac{2}{3(x+1)} + \frac{1}{3(x-2)}$$

Hence,

$$\int \frac{x-1}{(x+1)(x-2)}dx = \frac{2}{3}\int \frac{1}{(x+1)}dx + \frac{1}{3}\int \frac{1}{(x-2)}dx$$

$$= \frac{2}{3}In(x+1) + \frac{1}{3}In(x-2) + c$$

$$= \frac{2}{3}In(x+1)^2(x-2) + c$$

(ii) $\frac{1}{x^3-9x}$

Solution

$$\frac{1}{x^3-9x} = \frac{1}{x(x^2-9)} = \frac{1}{x(x-3)(x+3)}$$

⇨ $\frac{1}{x^3-9x} = \frac{A}{x} + \frac{B}{(x-3)} + \frac{C}{(x+3)}$

Multiplying through with $x(x-3)(x+3)$

1= $A(x^2-9) + B(x^2+3x) + C(x^2-3x)$

Putting x = 0; 1 = -9A => $A = -\frac{1}{9}$

Putting x = 3; 1 = 18B => $B = \frac{1}{18}$

Putting x = -3; 1 = 18C => $C = \frac{1}{18}$

⇨ $\frac{1}{x^3-9x} = -\frac{1}{9x} + \frac{1}{18(x-3)} + \frac{1}{18(x+3)}$

Hence,

$$\int \frac{1}{x^3-9x}dx$$

$$= -\frac{1}{9}\int \frac{1}{x}dx + \frac{1}{18}\int \frac{1}{(x-3)}dx + \frac{1}{18}\int \frac{1}{(x+3)}dx$$

$$= -\frac{1}{9}Inx + \frac{1}{18}In(x+3) + \frac{1}{18}In(x-3) + c$$

$$= \frac{1}{18}(In(x+3) + In(x-3) - 2Inx) + c$$

$$= \frac{1}{18}\left[In\frac{(x+3)(x-3)}{x^2}\right] + c$$

(iii) $\frac{2x+1}{(x-1)(3x^2+7x+2)}$

Solution

$$\frac{2x+1}{(x-1)(3x^2+7x+2)} = \frac{2x+1}{(x-1)(x+2)(3x+1)}$$

$$\frac{2x+1}{(x-1)(3x^2+7x+2)} = \frac{A}{(x-1)} + \frac{B}{(x+2)} + \frac{1}{(3x+1)}$$

Multiplying by $(x-1)(x+2)(3x+1)$

2x + 1= A(x+2)(3x+1)+B(x-1)(3x+1)+C(x-1)(x+2)

Putting x = 1; 3 = 12A => $A = \frac{1}{4}$

Putting x = -2; -3 = 15B => $B = -\frac{1}{5}$

Putting x = $-\frac{1}{3}$; $\frac{1}{3} = -\frac{20}{9}$C => $C = -\frac{3}{20}$

$$\therefore \frac{2x+1}{(x-1)(3x^2+7x+2)} = \frac{1}{4(x-1)} - \frac{1}{5(x+2)} - \frac{3}{20(3x+1)}$$

Hence,

$$\int \frac{2x+1}{(x-1)(3x^2+7x+2)}dx$$

$$= \frac{1}{4}\int \frac{1}{(x-1)}dx - \frac{1}{5}\int \frac{1}{(x+2)}dx - \frac{3}{20}\int \frac{1}{(3x+1)}dx$$

$$= \frac{1}{4}In(x-1) - \frac{1}{5}In(x+2) - \frac{3}{20}In(3x+1)$$

$$= \frac{1}{20}In\frac{(x-1)^5}{(x+2)^4(3x+1)^3}$$

(iv) $\frac{2x^2-x+1}{(x^2-1)(x+2)}$

Solution

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$$\frac{2x^2-x+1}{(x^2-1)(x+2)} = \frac{2x^2-x+1}{(x+1)(x-1)(x+2)}$$

⇨ $\frac{2x^2-x+1}{(x^2-1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{C}{(x+2)}$

Multiplying through by (x+1)(x-1)(x+2)

$2x^2 - x+1$= A(x-1)(x+2)+B(x+1)(x+2)+C(x+1)(x-1)

Putting x = -1; 4 = -2A => $A = -2$

Putting x = 1; 2 = 6B => $B = \frac{1}{3}$

Putting x = -2; 11 = 3C => $C = \frac{11}{3}$

$\therefore \frac{2x^2-x+1}{(x^2-1)(x+2)} = \frac{1}{3(x-1)} - \frac{2}{(x+1)} + \frac{11}{3(x+2)}$

Hence,

$\int \frac{2x^2-x+1}{(x^2-1)(x+2)} dx$

$= \frac{1}{3}\int \frac{1}{(x-1)} dx - 2\int \frac{1}{(x+1)} dx + \frac{11}{3}\int \frac{1}{(x+2)} dx$

$= \frac{1}{3} In(x-1) - 2In(x+1) + \frac{11}{3} In(x+2) + c$

(b) Evaluate $\int_1^3 \frac{x^2+1}{x^3+4x^2+3x} dx$

Solution

$\frac{x^2+1}{x^3+4x^2+3x} = \frac{x^2+1}{x(x+1)(x+3)}$

Let $\frac{x^2+1}{x^3+4x^2+3x} = \frac{A}{x} + \frac{B}{(x-3)} + \frac{C}{(x+3)}$

Multiplying with x(x – 3)(x + 3)

x^2+ 1 = A(x – 3)(x+3)+ B(x)(x+3)+C(x)(x-3)

Putting x = 0; 1 = 3A => $A = \frac{1}{3}$

Putting x = -1; 2 = -2B => $B = -1$

Putting x = -3; 10 = 6C => $C = \frac{5}{3}$

$\therefore \frac{x^2+1}{x^3+4x^2+3x} = \frac{1}{3x} - \frac{1}{(x+1)} + \frac{5}{3(x+3)}$

Hence

$\int_1^3 \frac{x^2+1}{x^3+4x^2+3x} dx$

$= \frac{1}{3}\int_1^3 \frac{1}{x} dx - \int_1^3 \frac{1}{(x+1)} dx + \frac{5}{3}\int_1^3 \frac{1}{(x+3)} dx$

$= \left[\frac{1}{3} Inx - In(x+1) + \frac{5}{3} In(x+3)\right]_1^3$

$= \left\{\frac{1}{3} In3 - In4 + \frac{5}{3} In6)\right\} - \left\{\frac{1}{3} In1 - In2 + \frac{5}{3} In4)\right\}$

$= 0.3488$

B. Denominators with linear factors
Quadratic factors

Each quadratic factors (ax^2+bx +c) has a corresponding partial fraction of the form $\frac{Ax+B}{(ax^2+bx+c)}$ where a, b, c and A and B are constants.

Example 14

(a) Express $\frac{7x^2+2x-28}{(x-6)(x^2+3x+5)}$ partial fraction.

Solution

Let $\frac{7x^2+2x-28}{(x-6)(x^2+3x+5)} = \frac{A}{x-6} + \frac{Bx+c}{x^2+3x+5}$

Multiplying through by (x – 6)(x^2+3x+5)

7x^2+2x-28= A(x^2+3x+5)+ (Bx +C)(x – 6)

Putting x = 6; 236= 59A, => A = 4

Equating coefficients of x^2

7 = A + B

7 = 4 + B; => B = 3

Equating constants

-28 = 5A – 6C

-28 = 20 – 6C

C = 8

$\therefore \frac{7x^2+2x-28}{(x-6)(x^2+3x+5)} = \frac{4}{x-6} + \frac{3x+8}{x^2+3x+5}$

(b) Find the integral of $f(x) = \frac{2x-1}{(x-1)(x^2+1)}$

Solution

Let $\frac{2x-1}{(x-1)(x^2+1)} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+1)}$

Multiplying through by $(x-1)(x^2+1)$

2x – 1 = A(x^2+1)+ (Bx + C)(x-1)

Putting x = 1; 1 = 2A => A =$\frac{1}{2}$

Putting x= 0; -1 = A –C => C = $\frac{3}{2}$

Putting x= -1; 2A + 2B – 2C => B = $-\frac{1}{2}$

$\therefore \frac{2x-1}{(x-1)(x^2+1)} = \frac{1}{2(x-1)} + \frac{-\frac{1}{2}x+\frac{3}{2}}{(x^2+1)}$

$\frac{2x-1}{(x-1)(x^2+1)} = \frac{1}{2(x-1)} + \frac{3-x}{2(x^2+1)}$

Note the values of x = 0 and x = -1 are conveniently chosen, but the constants B and C

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by expansion of the expression and equating constants, i.e.

-1 = A –C => C = $\frac{3}{2}$

2 = C – B

B = $\frac{3}{2} - 2 = -\frac{1}{2}$

Thus,

$$\int \frac{2x-1}{(x-1)(x^2+1)}dx$$

$$= \frac{1}{2}\int \frac{1}{(x-1)}dx + \frac{3}{2}\int \frac{1}{(x^2+1)}dx - \frac{1}{2}\int \frac{x}{(x^2+1)}dx$$

$$= \frac{1}{2}In(x-1) + \frac{3}{2}tan^{-1}x - \frac{1}{4}In(x^2+1) + c$$

(c) Evaluate

(i) $\int_2^3 \frac{3+3x}{x^3-1}dx$

Solution

Note memorize the identities

$$x^3 - 1 = (x-1)(x^2+x+1)$$

$$x^3 + 1 = (x-1)(x^2-x+1)$$

Then

$$\frac{3+3x}{x^3-1} = \frac{3+3x}{(x-1)(x^2+x+1)}$$

$$Let\ \frac{3+3x}{x^3-1} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+x+1)}$$

Multiplying through by (x – 1)(x[illegible]x+1)

3+3x = A(x^2+x+1) + (B[illegible]+C)([illegible] – 1)

Putting x = 1, 6 = 3A, [illegible] A = 2

By expanding and equating coefficients

x^2: A + B = 0, => B = 0 – 2 = -2

x^0: A – C = 3, => C = 2 – 3 = -1

$$\therefore \frac{3+3x}{x^3-1} = \frac{2}{(x-1)} - \frac{2x+1}{(x^2+x+1)}$$

$$\int_2^3 \frac{3+3x}{x^3-1}dx = 2\int_2^3 \frac{1}{(x-1)}dx - \int_2^3 \frac{2x+1}{(x^2+x+1)}$$

$$= [2In(x-1) - In(x^2+x+1)]_2^3$$

$$= 2In(2) + In\left(\frac{7}{13}\right)$$

$$= 0.7673$$

(ii) $\int_2^3 \frac{x^2}{x^4-1}dx$

Solution

$$\frac{x^2}{x^4-1} = \frac{x^2}{(x-1)(x+1)(x^2+1)}$$

Let $\frac{x^2}{(x-1)(x+1)(x^2+1)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{Cx+D}{(x^2+1)}$

By multiplying through by (x – 1)(x+1)(x^2+1)

x^2= A(x+1)(x^2+1)+ B(x -1)(x^2+1) + (Cx+D)(x^2-1)

By equating coefficients

x^3: A + B + C = 0 ..... (i)

x^2: A – B + D= 1 .....(ii)

x^1: A + B – C = 0 .....(iii)

x^0: A – B – D = 0 .....(iv)

Eqn. (ii) – Eqn. [illegible]

2D = 2 => D = $\frac{1}{2}$

E[illegible] + (iii)

2A + 2B = 0 ..... (v)

Eqn. (ii) + Eqn. (iv)

2A – 2B = 1.....(vi)

Eqn. (v)+ Eqn. (vi)

4A = 1 => A = $\frac{1}{4}$

Eqn. (v)

B = $-\frac{1}{4}$

Eqn. (i)

C = 0

$$\therefore \frac{x^2}{x^4-1} = \frac{1}{4(x-1)} - \frac{1}{4(x+1)} + \frac{1}{2(x^2+1)}$$

$$\int \frac{x^2}{x^4-1}dx$$

$$= \frac{1}{4}\int \frac{1}{(x-1)}dx - \frac{1}{4}\int \frac{1}{(x+1)}dx + \frac{1}{2}\int \frac{1}{(x^2+1)}dx$$

$$= \frac{1}{4}In(x-1) - \frac{1}{4}In(x+1) + \frac{1}{2}tan^{-1}x + c$$

$$\int_2^3 \frac{x^2}{x^4-1}dx$$

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$= \left[\frac{1}{4}In(x-1) - \frac{1}{4}In(x+1) + \frac{1}{2}tan^{-1}x\right]_2^3$

$= \frac{1}{4}\left[In\frac{x-1}{x+1} + 2tan^{-1}x\right]_2^3$

$= \frac{1}{4}\left\{\left[In\frac{2}{4} + 2tan^{-1}3\right] - \left[In\frac{1}{3} + 2tan^{-1}2\right]\right\}$

$= \frac{1}{4}\left[In\frac{1}{2} - In\frac{1}{3} + 2(tan^{-1}3 - tan^{-1}2)\right]$

$= \frac{1}{4}(0.405 + 0.1\pi)$

$= 0.18$

(iii) $\int_0^1 \frac{x^2+6}{(x^2+4)(x^2+9)}dx$

Solution

Let $\frac{x^2+6}{(x^2+4)(x^2+9)} = \frac{Ax+B}{(x^2+4)} + \frac{Cx+D}{(x^2+9)}$

Multiplying by $(x^2+4)(x^2+9)$

$x^2+6 = (Ax+B)(x^2+9) + (Cx+D)(x^2+4)$

$x^2+6 = (A+C)x^3+(B+D)x^2+(9A+4C)x+9B+4D$

Equating coefficients

x^3: A + C = 0 .....(i)

x^2: B+D = 1.....(ii)

x^1: 9A + 4C = 0 .....(iii)

x^0: 9B + 4D = 6 .....

Solving simultaneously

A= C= 0; B=$\frac{2}{5}$ and D=$\frac{3}{5}$

$\therefore \frac{x^2+6}{(x^2+4)(x^2+9)} = \frac{2}{5(x^2+4)} + \frac{3}{5(x^2+9)}$

$\int_0^1 \frac{x^2+6}{(x^2+4)(x^2+9)}dx$

$= \frac{2}{5}\int_0^1 \frac{1}{(x^2+4)}dx + \frac{3}{5}\int_0^1 \frac{1}{(x^2+9)}dx$

$= \frac{1}{5}\left[tan^{-1}\frac{1}{2}x + tan^{-1}\frac{1}{3}x\right]_0^1$

$= \frac{1}{5}tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}\right) = 0.1571$

C. Repeated factors

Each repeated factor $(ax^2+b)^n$ in the denominator has corresponding partial fraction of the form: $\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$, where a, b, A_i are constants (i = 1, 2, ...n)

Example 15

Express each of the follow in partial fraction and hence find their integrals.

(a) $\frac{4x-9}{(x-3)^2}$

Solution

Let $\frac{4x-9}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2}$

Multiplying through by $(x-3)^2$

4x – 9 = A(x-3) + B = Ax – 3A + B

Equating coefficients

x^1: x = 4

x^0: -3A + B = -9; B = 3

$\therefore \frac{4x-9}{(x-3)^2} = \frac{4}{x-3} + \frac{3}{(x-3)^2}$

Hence

$\int \frac{4x-9}{(x-3)^2}dx = 4\int \frac{1}{x-3}dx + 3\int (x-3)^{-2}\,dx$

$= 4In(x-3) - \frac{3}{x-3} + c$

(b) $\frac{3x-14}{x^2-8x+16}$

Solution

$\frac{3x-14}{x^2-8x+16} = \frac{3x-14}{(x-4)^2}$

Let $\frac{3x-14}{(x-4)^2} = \frac{A}{x-4} + \frac{B}{(x-4)^2}$

Multiplying through by $(x-4)^2$

3x – 14 = A(x-4) + B = Ax – 4A + B

Equating coefficients

x^1: x = 3

x^0: -4A + B = -14; B = -2

$\therefore \frac{3x-14}{(x-3)^2} = \frac{3}{x-3} - \frac{2}{(x-4)^2}$

Hence

$\int \frac{3x-14}{(x-4)^2}dx = 3\int \frac{1}{x-4}dx - 2\int (x-4)^{-2}\,dx$

$= 3In(x-4) + \frac{2}{x-4} + c$

(c) $\frac{2x^2-5x+7}{(x-2)(x-1)^2}$

Solution

Let $\frac{2x^2-5x+7}{(x-2)(x-1)^2} = \frac{A}{x-2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$

Multiplying through by $(x-2)(x-1)^2$

$2x^2-5x+7 = A(x-1)^2+B(x-2)(x-1)+C(x-2)$

Putting x = 1: 4 = -C, => C = -4

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Putting x = 2: A = 5

Putting x = 0, 7 =A – 2B – 2C; B = -2

$\therefore \frac{2x^2-5x+7}{(x-2)(x-1)^2} = \frac{5}{x-2} - \frac{2}{x-1} - \frac{4}{(x-1)^2}$

Hence

$\int \frac{2x^2-5x+7}{(x-2)(x-1)^2} dx$

$= 5\int \frac{1}{x-2} dx - 2\int \frac{1}{x-1} dx - 4\int (x-1)^{-2}\, dx$

$= 5In(x-2) - 2In(x-1) - \frac{4}{x-1} + c$

(d) $\frac{7x+2}{3x^3+x^2}$

Solution

$\frac{7x+2}{3x^3+x^2} = \frac{7x+2}{x^2(3x+1)}$

Let $\frac{7x+2}{x^2(3x+1)} = \frac{A}{(3x+1)} + \frac{B}{x} + \frac{C}{x^2}$

Multiplying through by $x^2(3x+1)$

$7x+ 2 = Ax^2 + Bx(3x+1)+C(3x+1)$

Putting x = 0; c = 2

Putting x = $-\frac{1}{3}$; $\frac{A}{9} = 2 - \frac{7}{3}$ => A = -3

Putting x= -1; -5 = A + 2B- 2C, =>

$\therefore \frac{7x+2}{x^2(3x+1)} = \frac{-3}{(3x+1)} + \frac{1}{x} +$

Hence

$\int \frac{7x+2}{x^2(3x+1)} dx$

$= -\int \frac{3}{(3x+1)} dx + \int \frac{1}{x} dx + 2\int x^{-2} dx$

$= -In(3x+1) + Inx - \frac{2}{x} + c$

$= In\frac{x}{3x+3} - \frac{2}{x} + c$In

Integration of improper fractions

Improper fractions are those whose index of the numerator is equal to or greater than that of the denominators.

They are first changed to proper fraction by long division or otherwise, before being integrated.

Example 16

(a) Express $\frac{5x^2-71}{(x+5)(x-4)}$ in partial fractions.

Hence find $\int \frac{5x^2-71}{(x+5)(x-5)} dx$

Solution

$\frac{5x^2-71}{(x+5)(x-4)} = \frac{5x^2-71}{x^2+x-20}$

Using long division

```
        5
x²+x-20)5x²+0x-71
      - 5x²+5x-100
        -5 +29
```

=> $\frac{5x^2-71}{(x+5)(x-4)} = 5 + \frac{-5x+29}{x^2+x-20}$

Let $\frac{-5x+29}{(x+5)(x-4)} = \frac{A}{x+5} + \frac{B}{x-4}$

Multiplying through by (x+5)(x-4)

-5x + 29 = A(x - 4) + B(x + 5)

Putting x = 4; B = 1

Putting x = -5; A = -6

$\therefore \frac{-5x+29}{(x+5)(x-4)} = \frac{-6}{x+5} + \frac{1}{x-4}$

Hence

$\int \frac{5x^2-71}{(x+5)(x-4)} dx$

$= 5\int dx - 6\int \frac{1}{x+5} dx + \int \frac{1}{x-4} dx$

$= 5x - 6In(x+5) + In(x-4) + c$

(b) Evaluate $\int_0^1 \frac{3-2x}{1+x} dx$

Solution

$\frac{3-2x}{1+x} = \frac{-2x+3}{x+1}$

Using long division

```
      -2
x+1)-2x+3
    -2x -2
        5
```

$\therefore \frac{3-2x}{1+x} = -2 + \frac{5}{x+1}$

Hence

$\int_0^1 \frac{3-2x}{1+x} dx = -2\int_0^1 dx + 5\int_0^1 \frac{1}{x+1} dx$

$= [-2x + 5In(x+1)]_0^1$

= -2+ 5In2

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= 1.4657

Revision exercise 7

1. Express the following into partial fraction

(a) $\frac{8x}{x^2-4x-12}$ $\left[\frac{6}{x-6}+\frac{2}{x+2}\right]$

(b) $\frac{x^4-x^3+x^2+1}{x^3+x}$ $\left[x-1+\frac{1}{x}+\frac{x-1}{x^2+1}\right]$

(c) $\frac{5x-1}{2x^2+x}-10$ $\left[\frac{3}{2x+5}+\frac{1}{x-2}\right]$

(d) $\frac{2x^2-7x+1}{(2x+1)(2x-1)(x-2)}$

$\left[\frac{1}{2x+1}+\frac{2}{3(2x-1)}-\frac{1}{3(x-2)}\right]$

(e) $\frac{6x+7}{(x^2+2)(x+3)}$ $\left[\frac{x+3}{x^2+2}-\frac{1}{x+3}\right]$

(f) $\frac{5x+7}{(x+1)^2(x+2)}$ $\left[\frac{3}{x+1}+\frac{2}{(x+1)^2}-\frac{3}{x+2}\right]$

(g) $\frac{2x^3+3x^2-x-4}{x^2(x+1)}$ $\left[2+\frac{3}{x}+\frac{4}{x^2}-\frac{2}{x+1}\right]$

2. Find

(a) $\int\frac{x^2}{x^4-1}dx$ $\left[\frac{1}{4}In\left(\frac{x-1}{x+1}\right)+tan^{-1}x+c\right]$

(b) $\int\frac{x^2-4}{(x+1)^2(x-5)}dx$

$\left[\frac{5}{12}In(x+1)-\frac{1}{2(x+1)}+\frac{7}{12}In(x-5)\right]$

(c) $\int\frac{3x^2+x+1}{(x-2)(x+1)^3}dx$

$\left[\frac{5}{9}In(x-2)-\frac{5}{9}In(x+1)-\frac{4}{3(x+1)}+\frac{1}{[illegible]+1)}\right]$

(d) $\int\frac{x^4-x^3+x^2+1}{x^3+x}dx$

$\left[\frac{x^2}{2}-x+Inx+tan^{-1}x-\frac{1}{[illegible]}In(1+[illegible]x^2)+c\right]$

(e) $\int\frac{5x-1}{2x^2+x-10}dx$

$\left[\frac{3}{2}In(2x+5)+[illegible]n([illegible]-2)+c\right]$

(f) $\int\frac{x^2-9x+2}{(x11)(x-1)(x-2)}d[illegible]$

$[2In(x+1)+3In(x-1)-4In(x-2)]+c$

(g) $\int\frac{9x+7}{(2x^2+3)(x+2)}dx$

$\left[\frac{1}{2}In(2x^2+3)+\frac{5}{\sqrt{10}}tan^{-1}\left(\sqrt{\frac{2}{3}}x\right)-In(x+2)+c\right]$

(h) $\int\frac{7+5x-6x^2}{(2x+1)^2(x+2)}dx$

$\left[\frac{3}{2}In(2x+1)-\frac{1}{2x+1}-3In(x+2)+c\right]$

(i) $\int\frac{x^2+7x-14}{(x+5)(x-3)}dx$

$[x+3In(x+5)+2In(x-3)+c]$

3. Evaluate

(a) $\int_0^2\frac{3x^4+7x^3+8x^2+53-186}{(x+4)(x^2+9)}dx$ [-4.5489]

(b) $\int_2^3\frac{x^2}{x^4-1}dx$ [0.18]

(c) $\int_1^3\frac{x^2+1}{x^3+4x^2+3x}dx$ [0.3489]

(d) $\int_0^1\frac{x^3}{x^2+1}dx$ [0.1535]

(e) $\int_6^7\frac{x^2-4}{(x+1)^2(x-5)}dx$ [0.4689]

(f) $\int_3^4\frac{3x^2+x+1}{(x-2)(x+1)^3}dx$ [0.3165]

(g) $\int_0^2\frac{8x}{x^2-4x-12}dx$ [1.05]

Integration by parts

This stems from differentiating the product of a function, y = uv,

$\frac{d}{dx}(uv)=v\frac{du}{dx}+u\frac{dv}{dx}$

$u\frac{dv}{dx}=\frac{d}{dx}(uv)-v\frac{du}{dx}$

$\int u\frac{dv}{dx}dx=\int\frac{d}{dx}(uv)dx-\int v\frac{du}{dx}dx$

$\int u\frac{dv}{dx}dx=uv-\int v\frac{du}{dx}dx$

[illegible] sin[illegible]y $\int udv=uv-\int vdu$

[illegible] function chosen as u should be easily differentiated whereas the other function chosen as v should be easily integrated.

The above expression of the integration by parts can be summarized by using a technique of integration by parts

This is summarized in the table below

Sign	Differentiate	Integrates
+	u_1	$\frac{dv}{dx}$
-	u_2	v_1
+	u_3	v_2
-	u_4	v_3

NB: the signs change as +, -, + etc.

The u function is differentiated until a zero value is obtained otherwise we continue with differentiation.

The integral of the function is equal to the sum of result shown in the table above.

Integration by parts is applied in the following areas:

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A. Integration products of polynomials by parts

Example 17

(a) Find

(i) $\int x(x+2)^3 dx$

Solution

Let u = x and $\frac{dv}{dx} = (x+2)^3$

$\frac{du}{dx} = 1$; v = $\frac{1}{4}(x+2)^4$

From $\int u dv = uv - \int v du$

$\int x(x+2)^3 dx$

$=\frac{1}{4}x(x+2)^4 - \int 1.\frac{1}{4}(x+2)^4 dx$

$=\frac{1}{4}x(x+2)^4 - \frac{1}{4}\int (x+2)^4 dx$

$=\frac{1}{4}x(x+2)^4 - \frac{1}{20}(x+2)^5 + c$

$=\frac{1}{2o}(x+2)^4(5x-x-2) + c$

$=\frac{1}{2o}(x+2)^4(4x-2) + c$

$=\frac{1}{1o}(x+2)^4(x-1) + c$

Or by using basic techniques

Sign	Differentiate	Integrates
+	x	$(x+2)^3$
-	1	$\frac{1}{4}(x+2)^4$
+	0	$\frac{1}{20}(x+2)^5$

$\int x(x+2)^3 dx = \frac{1}{4}x(x+2)^4 - \frac{1}{20}(x+2)^5 + c$

$=\frac{1}{1o}(x+2)^4(x-1) + c$

$\therefore \int x(x+2)^3 dx = \frac{1}{1o}(x+2)^4(x-1) + c$

(ii) $\int (x+3)(x-4)^5 dx$

Solution

Let u = (x+3) and $\frac{dv}{dx} = (x-4)^5$

$\frac{du}{dx} = 1$; v = $\frac{1}{6}(x-4)^6$

$\int (x+3)(x-4)^5 dx$

$=\frac{1}{6}(x+3)(x-4)^6 - \frac{1}{6}\int 1.(x-4)^6 dx$

$=\frac{1}{6}(x+3)(x-4)^6 - \frac{1}{6}\int (x-4)^6 dx$

$=\frac{1}{6}(x+3)(x-4)^6 - \frac{1}{42}(x-4)^7 + c$

$=\frac{1}{42}(x-4)^6((7(x+3)-x+4) + c$

$=\frac{1}{42}(x-4)^6(6x+25) + c$

Sign	Differentiate	Integrates
+	x+3	$(x-4)^5$
-	1	$\frac{1}{6}(x+2)^6$
+	0	$\frac{1}{42}(x+2)^7$

$\int (x+3)(x-4)^5 dx$

$=(x+3)(x-4)^5 - \frac{1}{42}x(x+2)^7 + c$

$\therefore \int (x+3)(x-4)^5 dx$

$=\frac{1}{42}(x-4)^6(6x+25) + c$

(iii) $\int \frac{3x-4}{(x+2)^4} dx$

Solution

$\int \frac{3x-4}{(x+2)^4} dx = \int (3x-4)(x+2)^{-4} dx$

Let u =(3x - 4) and $\frac{dv}{dx} = (x+2)^{-4}$

$\frac{du}{dx} = 3$ $v = -\frac{1}{3}(x+2)^{-3}$

$\int \frac{3x-4}{(x+2)^4} dx$

$= -\frac{1}{3}(3x-4)(x+2)^{-3} - \int 3.-\frac{1}{3}(x+2)^{-3} dx$

$= -\frac{1}{3}(3x-4)(x+2)^{-3} + \int (x+2)^{-3} dx$

$= -\frac{1}{3}(3x-4)(x+2)^{-3} - \frac{1}{2}(x+2)^{-2} + c$

$= \frac{4-3x}{3(x+2)^3} - \frac{1}{2(x+2)^2} + c$

$= \frac{2(4-3x)-3(x+2)}{6(x+2)^3} + c = \frac{2-9x}{6(x+2)^3} + c$

$\therefore \int \frac{3x-4}{(x+2)^4} dx = \frac{2(4-3x)-3(x+2)}{6(x+2)^3} + c = \frac{2-9x}{6(x+2)^3} + c$

Sign	Differentiate	Integrates
+	3x-4	$(x-4)^{-4}$
-	3	$\frac{1}{3}(x-4)^{-3}$
+	0	$\frac{1}{6}(x-4)^{-2}$

$\int \frac{3x-4}{(x+2)^4} dx$

$= -\frac{1}{3}(3x-4)(x-4)^{-3} - \frac{1}{2}(x-4)^{-2} + c$

$= \frac{2-9x}{6(x+2)^3} + c$

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(b) Evaluate

(i) $\int_0^2 x(x-3)^2 dx$

Solution

Sign	Differentiate	Integrates
+	x	$(x-3)^2$
-	1	$\frac{1}{3}(x-3)^3$
+	0	$\frac{1}{12}(x-3)^4$

$\int x(x-3)^2 dx = \frac{1}{3}x(x-3)^3 - \frac{1}{12}(x-3)^4$+c

$= \frac{1}{12}(x-3)^3(4x-x+3)+c$

$= \frac{1}{12}(x-3)^3(3x+3)+c$

$= \frac{1}{4}(x-3)^3(x+1)+c$

⇨ $\int_0^2 x(x-3)^2 dx = \left[\frac{1}{4}(x-3)^3(x+1)\right]_0^2$

$= \frac{-3}{4} - \frac{-27}{4} = \frac{24}{4} = 6$

(i) $\int_3^6 \frac{x}{\sqrt{x-2}} dx$

Solution

Sign	Differentiate	Integrates
+	x	$(x-2)^{\frac{1}{2}}$
-	1	$2(x-2)^{\frac{1}{2}}$
+	0	$\frac{4}{3}(x-2)^{\frac{3}{2}}$

$\int \frac{x}{\sqrt{x-2}} dx = 2x\,(x-2)^{\frac{1}{2}} - \frac{4}{3}(x-2)^{\frac{3}{2}}$+c

$= \frac{2}{3}(x-2)^{\frac{1}{2}}[3x-2(x-2)]+c$

$= \frac{2}{3}(x-2)^{\frac{1}{2}}(x+4)+c$

⇨ $\int_3^6 \frac{x}{\sqrt{x-2}} dx = \left[\frac{2}{3}(x-2)^{\frac{1}{2}}(x+4)\right]_3^6$

$= \left[\frac{2}{3}(6-2)^{\frac{1}{2}}(6+4)\right] - \left[\frac{2}{3}(3-2)^{\frac{1}{2}}(3+4)\right]$

$= \frac{2}{3}(20-7) = \frac{26}{3} = 8\frac{2}{3}$

Revision exercise 8

1. Integrate

(a) $\int (x-1)(x+2)^2$dx

$\left[\frac{1}{4}(x-2)(x+2)^3+c\right]$

(b) $\int (3x-1)(2x+3)^2$dx

$\left[\frac{1}{48}(18x-17)(2x+3)^3+c\right]$

(c) $\int (2-5x)(4-x)^4$dx

$\left[\frac{1}{30}(25x+8)(4-x)^5\right]$

(d) $\int \frac{x-2}{(2x-3)^2}$dx

$\left[\frac{1}{4}In(2x-3)+\frac{1}{4(2x-3)}+c\right]$

(e) $\int \frac{x+4}{\sqrt{3x-2}}$dx

$\left[\frac{2}{27}(3x+40)\sqrt{3x-2}+c\right]$

(f) $\int \frac{3x+1}{\sqrt{1-2x}}$dx

$In\left(\frac{2-x}{5-x}\right)+\frac{1}{5-x}+c$

2. Evaluate

(a) $\int_{-1}^1 x^2(x+3)^3 dx$ $\left[\frac{108}{5}\right]$

(b) $\int_3^6 \frac{x^2}{\sqrt{1-2x}} dx$ $\left[\frac{586}{15}\right]$

B. Integration prod[illegible] of polynomials and circular/trigonom[illegible] functions by parts

Example 18

(a) Find

(i) $\int x\sin x dx$

Solution

Let u = x and $\frac{dv}{dx} = sinx$

$\frac{du}{dx} = 1$, v = -cosx

$\int xcos\,xdx = -xcos\,x - \int 1.-cosx$

$= -xcos\,x + sinx + c$

Or: by using basic technique

Sign	Differentiate	Integrates
+	x	$sinx$
-	1	$-cosx$
+	0	$-sinx$

$\int xcos\,xdx = -xcos\,x + sinx + c$

(ii) $\int x^2 cosxdx$

Solution

Let u = x^2 and $\frac{dv}{dx} = cosx$

$\frac{du}{dx} = 2x$, v = sinx

$\int x^2 cosxdx = x^2 sinx - 2\int xsinxdx$+c

Let u = x and $\frac{dv}{dx} = sinx$

$\frac{du}{dx} = 1$, v = -cosx

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$\int x^2 cosxdx$

$= x^2 sinx - 2[-xcosx - \int -cosxdx] + c$

$= x^2 sinx - 2[-xcosx + \int cosxdx] + c$

$= x^2 sinx - 2[-xcosx + sinx] + c$

$= x^2 sinx + 2xcosx - 2sinx + c$

Or using basic technique

Sign	Differentiate	Integrates
+	x^2	$cosx$
-	2x	$sinx$
+	2	$-cosx$
-	0	$-sinx$

$\int x^2 cosxdx$ =x^2sins+ 2xcosx – 2sinx + c

(iii) $\int x^2 sin^2 xdx$

Solution

Let u = x^2 and $\frac{dv}{dx} = sin^2 x = \frac{1}{2}(1 - cos2x)$

$\frac{du}{dx} = 2x$, v = $\frac{1}{2}\left(x - \frac{1}{2}sin2x\right)$

$\int x^2 sin^2 xdx$

$=\frac{1}{2}x^2\left(x - \frac{1}{2}sin2x\right) - \int 2x.\frac{1}{2}\left(x - \frac{1}{2}sin2x\right)dx$

$=\frac{1}{2}x^2\left(x - \frac{1}{2}sin2x\right) - \int x^2 dx + \frac{1}{2}\int xsin2xdx$

$=\frac{1}{2}x^2\left(x - \frac{1}{2}sin2x\right) - \frac{1}{3}x^3 + \frac{1}{2}\int xsin2xdx$

Let u = x and $\frac{dv}{dx} = sin2x$

$\frac{du}{dx} = 1$and v = -$\frac{1}{2}$cos2x

$\int xsin2xdx$ =-$\frac{1}{2}$xcos2x+$\frac{1}{4}$sin2x+c

Substituting for$\int xsin2xdx$

$\int x^2 sin^2 xdx$

$=\frac{1}{2}x^2\left(x - \frac{1}{2}sin2x\right) - \frac{1}{3}x^3 + \frac{1}{2}\left[-\frac{1}{2}\text{xcos2x} + \frac{1}{4}\text{sin2x}\right]$

$=\frac{1}{2}x^2\left(x - \frac{1}{2}sin2x\right) - \frac{1}{3}x^3 - \frac{1}{4}\text{xcos2x} + \frac{1}{8}\text{sin2x} + c$

$= \frac{1}{6}x^3 - \frac{1}{4}x^2 sin2x - \frac{1}{4}xcos2x + \frac{1}{8}sin2x + c$

Or by using basic technique

Sign	Differentiate	Integrates
+	x^2	$sin^2 x = \frac{1}{2}(1 - cos2x)$
-	2x	$\frac{1}{2}x - \frac{1}{4}sins2x$
+	2	$\frac{1}{4}x^2 + \frac{1}{8}cos2x$
-	0	$\frac{1}{12}x^3 + \frac{1}{16}sin2x$

$\int x^2 sin^2 xdx$

$= \frac{1}{2}x^2\left(x - \frac{1}{2}sin2x\right) - 2x\left(\frac{1}{4}x^2 + \frac{1}{8}cos2x\right) + 2\left(\frac{1}{12}x^3 + \frac{1}{16}sin2x\right)$

$= \frac{1}{6}x^3 - \frac{1}{4}x^2 sin2x - \frac{1}{4}xcos2x + \frac{1}{8}sin2x + c$

(iv) $\int xcos^2 xdx$

Solution

$\int xcos^2 xdx$

Let u = x and $\frac{dv}{dx} = cos^2 x$= $\frac{1}{2} + \frac{1}{2}cos2x$

$\frac{du}{dx} = 1$, v =$\frac{1}{2}\left(x + \frac{1}{2}sin2x\right)$

$\int xcos^2 xdx$

$= \frac{1}{2}x\left(x + \frac{1}{2}sin2x\right) - \int 1.\frac{1}{2}\left(x + \frac{1}{2}sin2x\right)$ dx

$= \frac{1}{2}x\left(x + \frac{1}{2}sin2x\right) - \frac{1}{2}\int xdx - \frac{1}{4}\int sin2x\text{dx}$

$= \frac{1}{2}x^2 + \frac{1}{4}sin2x - \frac{1}{4}x^2 + \frac{1}{8}cos2x + c$

$= \frac{1}{4}x^2 + \frac{1}{4}sin2x + \frac{1}{8}cos2x + c$

(b) Evaluate

(i) $\int_0^{\frac{\pi}{2}} x^2 cosxdx$

Solution

$\int_0^{\frac{\pi}{2}} x^2 cosxdx$

$=[x^2 sinx + 2xcosx - 2sinx]_0^{\frac{\pi}{2}}$

$=\left[\frac{\pi^2}{4}sin\frac{\pi}{2} + 2.\frac{\pi}{2}cos\frac{\pi}{2} - 2sin\frac{\pi}{2}\right] - 0$

$=\left(\frac{\pi^2}{4} - 2\right) = 0.4674$

(ii) $\int_0^{\frac{\pi}{4}} xtan^2 xdx$

Solution

Let u = x and $\frac{dv}{dx} = tan^2 x = sec^2 x - 1$

$\frac{du}{dx} = 1$; v = tanx - x

$\int xtan^2 xdx$=xtanx - x^2-$\int(tanx - x)dx$

=xtanx - x^2+lncosx+$\frac{1}{2}x^2 + c$

=xtanx +lncosx$-\frac{1}{2}x^2 + c$

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Or by using basic technique

Sign	Differentiate	Integrates
+	x	$tan^2x = sec^2x - 1$
-	1	tanx – x
+	0	-Incosx$-\frac{1}{2}x^2 +$

$\int xtan^2xdx$=xtanx - x^2+Incosx+$\frac{1}{2}x^2 + c$

=xtanx +Incosx$-\frac{1}{2}x^2 + c$

Hence;

$$\int_0^{\frac{\pi}{4}} xtan^2xdx = \left[\text{xtanx } + \text{Incosx} - \frac{1}{2}x^2\right]_0^{\frac{\pi}{4}}$$

$$= \left[\frac{\pi}{4}\tan\frac{\pi}{4} + \text{Incos}\frac{\pi}{4} - \frac{1}{2}\left(\frac{\pi}{4}\right)^2\right] - 0$$

$$= 0.1304$$

Revision Exercise 9

1. Integrate each of the following
   (a) $\int xsin2xdx$
   $\left[-\frac{x}{2}cos2x + \frac{1}{4}sin2x + c\right]$
   (b) $\int x^2sinxdx$
   $[-x^2cosx + 2xsinx + 2cosx + c]$
   (c) $\int (x+1)^2sinxdx$
   $[(1 - 2x - x^2)cosx = 2(x+1)s$[illegible]$]$
   (d) $\int x^2sinxcosxdx$
   $\left[\frac{1}{8}cos2x(1 - 2x^2) + \frac{1}{4}x$[illegible]$2x +$ [illegible]$\right]$
   (e) $\int x^3 \cos x^2\, dx$
   $\left[\frac{1}{2}x^2 \sin x^2 + \frac{1}{2}c$[illegible]$x$ [illegible] $+ c\right]$
   (f) $\int (xcosx)^2d$[illegible]
   $\left[\frac{1}{6}x^3 + \frac{1}{8}(2$[illegible] $+ \frac{1}{4}xsin2x + c\right]$
2. Evaluate
   (a) $\int_0^{\pi} x^2sinxdx$ [5.8696]
   (b) $\int_0^{\pi} x^2cos2xdx$ [0.0584]
   (c) $\int_0^{\frac{\pi}{4}} xtan^2xdx$ [0.1304]

C. Integration products of polynomials and exponential functions by parts

Examples 19

(a) Find
(i) $\int xe^xdx$

Solution

Let u = x and $\frac{dv}{dx} = e^x$

$\frac{du}{dx} = 1$; v =e^x

$\int xe^xdx = xe^x - \int 1.e^xdx$

$= xe^x - e^x + c$

Or by using basic technique

Sign	Differentiate	Integrates
+	x	e^x
-	1	e^x
+	0	e^x

$\int xe^xdx = xe^x - e^x + c$

(ii) $\int xe^{-x}dx$

Solution

Let u = x and $\frac{dv}{dx} = e^{-}$[illegible]

$\frac{du}{dx} = 1$; v =$-e^{-}$[illegible]

$\int xe^xdx$ [illegible] $^{-x} - \int 1.-e^{-x}dx$

[illegible] $-xe^{-x} + \int e^{-x}dx$

[illegible] $= -xe^x - e^x + c$

[illegible] using basic technique

Sign	Differentiate	Integrates
+	x	e^{-x}
-	1	$-e^{-x}$
+	0	e^x

$\int xe^xdx = -xe^x - e^x + c$

(iii) $\int xe^{3x}dx$

Solution

Let u = x and $\frac{dv}{dx} = e^{3x}$

$\frac{du}{dx} = 1$; v =$\frac{1}{3}e^{3x}$

$\int xe^{3x}dx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + c$

Or by using basic technique

Sign	Differentiate	Integrates
+	x	e^{3x}
-	1	$\frac{1}{3}e^{3x}$
+	0	$\frac{1}{9}e^{3x}$

$\int xe^xdx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + c$

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(b) Find

(i) $\int x.2^x dx$

Solution

Let u = x and $\frac{dv}{dx} = 2^x$

$\frac{du}{dx} = 1$; v = $\frac{2^x}{In2}$

$\int x.2^x dx = \frac{x.2^x}{In2} - \frac{1}{In2}\int 2^x dx$

$= \frac{x.2^x}{In2} - \frac{1}{In2}\left(\frac{2^x}{In2}\right) + c$

$= \frac{2x}{In2}(x-1) + c$

(ii) $\int 3^{\sqrt{(2x-1)}} dx$

Solution

Let p = $\sqrt{(2x-1)}$, p^2 = 2x – 1

2pdp = 2dx

pdp = dx

⇨ $\int 3^{\sqrt{(2x-1)}} dx = \int 3^p . pdp$

Let u =p and $\frac{dv}{dp} = 3^p$

$\frac{du}{dp} = 1$, v = $\frac{3^p}{In3}$

$\int 3^p . pdp = \frac{3^p.p}{In3} - \frac{1}{In3}\int 3^p$

$= \frac{3^p.p}{In3} - \frac{1}{In3}\left(\frac{3^p}{}\right) +$

$\therefore \int 3^{\sqrt{(2x-1)}} dx$

$= \frac{\sqrt{(2x-1)}3^{\sqrt{(2x-1)}}}{In3} - \frac{1}{In3}\left(\frac{3^{\sqrt{(2x-1)}}}{In3}\right) + c$

$= \frac{3^{\sqrt{(2x-1)}}}{In3}\left(\sqrt{(2x-1)} - \frac{1}{In3}\right) + c$

(c) Evaluate

(i) $\int_0^1 xe^{-x} dx$

Solution

$\int_0^1 xe^{-x} dx = [-xe^{-x} - e^{-x}]_0^1$

$= (-e^{-1} - e^{-1}) - (0 - e^0)$

$= -2e^{-1} + 1$

$= 1 - \frac{2}{e}$

$= 0.2642$

(ii) $\int_0^1 xe^{3x} dx$

Solution

$\int_0^1 xe^{3x} dx = \left[\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x}\right]_0^1$

$= \left[\frac{1}{3}e^3 - \frac{1}{9}e^3\right] - \left[0 - \frac{1}{9}e^0\right]$

$= \frac{2}{9}e^3 + \frac{1}{9} = 4.5746$

Revision exercise 10

1. Integrate each of the following with respect to x

(a) xe^{3x} $\left[\frac{e^{3x}}{9}(3x-1) + c\right]$

(b) $x^2 e^x$ $[e^x(x^2 - 2x + 2) + c]$

(c) $x e^{x^2}$ $\left[\frac{e^{x^2}}{2}(x^2 - 1) + c\right]$

(d) $x e^{-2x}$ $\left[-\frac{e^{-2}}{4}(2x^2 + 2x + 1) + c\right]$

(e) $\frac{x^2}{e^{-x^3}}$ $\left[-\frac{1}{3}e^{-x^3} + c\right]$

(f) $e^x(3 + e^x)^2$ $\left[\frac{1}{3}(3 + e^x)^3 + c\right]$

2. Evaluate each of the following

(a) $\int_0^1 x^2 e^{2x} dx$ [1.5973]

(b) $\int_0^1 (x-1)e^x dx$ [2]

D. Integration products of polynomials and inverse trigonometric functions by parts

Example 20

(a) Find

(i) $\int sin^{-1}x dx$

Solution

$\int sin^{-1}x dx = \int 1. sin^{-1}x dx$

Let u=$sin^{-1}x$ and $\frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$; v = x

$\int sin^{-1}x dx = xsin^{-1}x - \int \frac{x}{\sqrt{1-x^2}} dx$

For $\int \frac{x}{\sqrt{1-x^2}} dx$

Let u = 1 – x^2

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Du= -2x

$-\frac{1}{2x}du = dx$

$\int \frac{x}{\sqrt{1-x^2}}dx = \int \frac{x}{u^{\frac{1}{2}}}.-\frac{1}{2x}du$

$= -\frac{1}{2}\int u^{-\frac{1}{2}}du$

$= -\frac{1}{2}\left[2u^{\frac{1}{2}} + c\right] = -u^{\frac{1}{2}} + c$

By substitution

$\int sin^{-1}xdx = xsin^{-1}x + u^{\frac{1}{2}} + c$

$\therefore \int sin^{-1}xdx = xsin^{-1}x + \sqrt{1-x^2} + c$

(ii) $\int cos^{-1}\left(\frac{x}{a}\right)dx$

Solution

$\int cos^{-1}\left(\frac{x}{a}\right)dx = \int 1.cos^{-1}\left(\frac{x}{a}\right)dx$

Let u = $cos^{-1}\left(\frac{x}{a}\right)$ and $\frac{dv}{dx} = 1$

$\frac{du}{dx} = -\frac{1}{\sqrt{a^2-x^2}}$

$\int cos^{-1}\left(\frac{x}{a}\right)dx = xcos^{-1}\left(\frac{x}{a}\right) + \int \frac{1}{\sqrt{a^2}}$ dx

For $\int \frac{x}{\sqrt{a^2-x^2}}dx$

Let u = $a^2 - x^2$

Du= -2x

$-\frac{1}{2x}du = dx$

$\int \frac{x}{\sqrt{a^2-x^2}}dx = \int \frac{x}{u^{\frac{1}{2}}}.-\frac{1}{2x}du$

$= -\frac{1}{2}\int u^{-\frac{1}{2}}du$

$= -\frac{1}{2}\left[2u^{\frac{1}{2}} + c\right] = -u^{\frac{1}{2}} + c$

By substitution

$\int cos^{-1}\left(\frac{x}{a}\right)dx = xcos^{-1}x + u^{\frac{1}{2}} + c$

$\therefore \int cos^{-1}\left(\frac{x}{a}\right)dx = xcos^{-1}x + \sqrt{a^2-x^2} + c$

(iii) $\int xtan^{-1}x\,dx$

Solution

Let u = $tan^{-1}x$ and $\frac{dv}{dx} = x$

$\frac{du}{dx} = \frac{1}{1+x^2}$; v = $\frac{1}{2}x^2$

$\int xtan^{-1}x\,dx = \frac{1}{2}x^2tan^{-1}x - \frac{1}{2}\int \frac{x^2}{1+x^2}$dx

For $\int \frac{x^2}{1+x^2}$dx $= \int \left(1 - \frac{1}{1+x^2}\right)dx$

$= \int dx - \int \frac{1}{1+x^2}dx$

$= x - tan^{-1}x + c$

By substitution

$\int xtan^{-1}x\,dx$

$= \frac{1}{2}x^2tan^{-1}x - \frac{1}{2}[x - tan^{-1}x] + c$

$= \frac{1}{2}x^2tan^{-1}x - \frac{1}{2}x + \frac{1}{2}tan^{-1}x + c$

$= \frac{1}{2}[(x^2+1)tan^{-1}x - x] + c$

(b) Evaluate $\int_0^1 xsin^{-1}xdx$

Solution

Let u = sin^{-1}x and $\frac{dv}{dx} = x$

$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$; v= $\frac{1}{2}x^2$

$\int xsin^{-1}xdx = \frac{1}{2}x^2sin^{-1}x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}dx$

For $\int \frac{x^2}{\sqrt{1-x^2}}dx$

Let x = sinθ => dx = cosθdθ

$\int \frac{x^2}{\sqrt{1-x^2}}dx = \int \frac{sin^2\theta}{\sqrt{1-sin^2\theta}}.cos\theta d\theta$

$= \int \frac{sin^2\theta}{cos\theta}.cos\theta d\theta$

$= \int sin^2\theta\,d\theta$

$= \frac{1}{2}\int(1 - cos2\theta)\,d\theta$

$= \frac{1}{2}\theta - \frac{1}{4}sin2\theta + c$

$= \frac{1}{2}\theta - \frac{1}{4}(2sin\theta cos\theta) + c$

$= \frac{1}{2}\theta - \frac{1}{2}(sin\theta cos\theta) + c$

$= \frac{1}{2}sin^{-1}x - \frac{1}{2}x\sqrt{1-x^2} + c$

$\therefore \int xsin^{-1}xdx$

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$= \frac{1}{2}x^2 sin^{-1}x - \frac{1}{4}sin^{-1}x + \frac{1}{4}x\sqrt{1-x^2}$

$\int_0^1 xsin^{-1}xdx$

$= \left[\frac{1}{2}x^2 sin^{-1}x - \frac{1}{4}sin^{-1}x + \frac{1}{4}x\sqrt{1-x^2}\right]_0^1$

$= \left[\frac{1}{2}.1.sin^{-1}1 - \frac{1}{4}sin^{-1}(1)\right] - (0)$

$= \frac{\pi}{4} - \frac{\pi}{8} = \frac{\pi}{8}$

Revision exercise 11

1. Find the following integrals

(a) $\int tan^{-1}3xdx$

$\left[xtan^{-1}x - \frac{1}{6}In(1+9x^2) + c\right]$

(b) $\int \frac{xsin^{-1}x}{\sqrt{1-x^2}}dx$

$\left[xsin^{-1}x + \sqrt{1-x^{2+c}}\right]$

(c) $\int sec^{-1}x\,dx$

$\left[xsec^{-1}x - In\left(x+\sqrt{x^2-1}\right) + c\right]$

(d) $\int cot^{-1}xdx$

$\left[xcot^{-1}x + \frac{1}{2}In(1+x^2) + c\right]$

2. Evaluate

(a) $\int_0^1 sin^{-1}xdx$ $\left[\frac{\pi}{2} - 1\right]$

(b) $\int_0^1 cos^{-1}xdx$ $[1]$

E. Integration products of polynomials and logarithmic functions by parts

Example 21

(a) Integrate

(i) $\int Inx^2dx$

Solution

$\int Inx^2dx = \int 1.Inx^2dx$

Let u = Inx^2, $\frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{2x}{x^2} = \frac{2}{x}$; v = x

$\int Inx^2dx = xInx^2 - 2\int x.\frac{1}{x}dx$

$= xInx^2 - 2x + c$

$= 2xInx - 2x + c$

$\therefore \int Inx^2dx = 2xInx - 2x + c$

(ii) $\int xIn(x^2-1)dx$

Solution

Let u = $In(x^2-1)$ and $\frac{dv}{dx} = x$

$\frac{du}{dx} = \frac{2x}{x^2-1}$; $v = \frac{1}{2}x^2$

$\int xIn(x^2-1)dx$

$= \frac{1}{2}x^2In(x^2-1) - \int \frac{1}{2}x^2.\frac{2x}{x^2-1}dx$

$= \frac{1}{2}x^2In(x^2-1) - \int \frac{x^3}{x^2-1}dx$

For $\int \frac{x^3}{x^2-1}dx$

By using long division

$\frac{x^3}{x^2-1} = x + \frac{x}{x^2-1}$

⇨ $\int \frac{x^3}{x^2-1}dx = \int xdx + \int \frac{x}{x^2-1}dx$

$= \frac{1}{2}x^2 + \frac{1}{2}In(x^2-1) + c$

$\therefore \int xIn(x^2-1)dx$

$= \frac{1}{2}x^2In(x^2-1) - \frac{1}{2}x^2 - \frac{1}{2}In(x^2-1) + c$

(iii) $\int x^{-3}Inxdx$

Solution

Let u = Inx and $\frac{dv}{dx} = x^{-3}$

$\frac{du}{dx} = \frac{1}{x}$ and v = $-\frac{1}{2}x^{-2}$

$\int x^{-3}Inxdx = -\frac{1}{2}x^{-2}Inx + \frac{1}{2}\int \frac{1}{x}.x^{-2}$ dx

$= -\frac{1}{2}x^{-2}Inx + \frac{1}{2}\int x^{-3}\,dx$

$= -\frac{1}{2}x^{-2}Inx - \frac{1}{4}x^{-2} + c$

$= -\frac{1}{4}x^{-2}(Inx + 1) + c$

(b) Evaluate $\int_1^{10} x\log_{10}xdx$

Solution

Changing from base 10 to base e

$\log_{10}x = \frac{Ine}{In10}$

$\int_1^{10} x\log_{10}xdx = \frac{1}{In10}\int_1^{10} xInxdx$

Let u= Inx; $\frac{dv}{dx} = x$

$\frac{du}{dx} = \frac{1}{x}; v = \frac{1}{2}x^2$

$$\frac{1}{In10}\int_1^{10} xInxdx = \frac{1}{In10}\left[\frac{1}{2}x^2Inx - \frac{1}{4}x^2\right]_1^{10}$$

$$= \frac{1}{In10}\left[(50In10 - 25) - \frac{1}{4}\right]$$

$$= \frac{1}{In10}\left[50In10 - \frac{99}{4}\right] \quad = 50 - \frac{99}{4In10}$$

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Revision exercise 12

1. Integrate each of the following
   (a) $xInx$ $\left[\frac{x^2}{4}(2Inx-1)+c\right]$
   (b) x^2Inx $\left[\frac{x^2}{9}(3Inx-1)+c\right]$
   (c) $\sqrt{x}Inx$ $\left[\frac{2}{9}\sqrt{x^3}(3Inx-2)+c\right]$
   (d) $(Inx)^2$ $[x(2-2Inx+(Inx)^2)+c]$
   (e) $\frac{Inx}{x^2}$ $\left[-\frac{1}{x}(Inx+1)+c\right]$
   (f) $3^x x$ $\left[\frac{3^x}{(In3)^2}(xIn3-1)+c\right]$
   (g) $x(Inx)^2$
   $\left[\frac{1}{4}x^2(1-2Inx+2(Inx)^2)+c\right]$
2. Evaluate the following
   (a) $\int_2^4 x^3Inxdx$ [70.9503]
   (b) $\int_2^4 (x-1)In(2x)dx$ [1.0794]
   (c) $\int_1^4 \frac{Inx}{x^2}dx$ [0.4034]

F. Integration of products of exponential and trigonometric functions by parts

Example 22

(a) Find
(i) $\int e^{-x}sinxdx$

Solution

Taking I = $\int e^{-x}sinxdx$

Let u = e^{-x}, $\frac{dv}{dx}=sinx$

$\frac{du}{dx}=-e^{-x}$; v = -cos

$=> I=-e^{-x}cosx-\int -e^{-x}.-cosxdx$

$I=-e^{-x}cosx-\int e^{-x}.cosxdx$ ...(*)

For $\int e^{-x}.cosxdx$

Let u = e^{-x}, $\frac{dv}{dx}=cosx$

$\frac{du}{dx}=-e^{-x}$; v = sinx

$\int e^{-x}.cosxdx=e^{-x}sinx-\int -e^{-x}sinx$

$=e^{-x}sinx+I$ .....(**)

Substituting for (**) in equation (*)

$I=-e^{-x}cosx-e^{-x}sinx-I$

$2I=-e^{-x}cosx-e^{-x}sinx-I+A$

$I=-\frac{1}{2}e^{-x}(cosx+sinx)+c$

$\therefore \int e^{-x}sinxdx=-\frac{1}{2}e^{-x}(cosx+sinx)+c$

Or by using basic technique

sign	Differentiate	integrate
+	e^{-x}	sinx
-	$-e^{-x}$	-cosx
+	e^{-x}	-sinx

$I=-e^{-x}cosx-e^{-x}sinx-\int e^{-x}sinx$

$I=-e^{-x}cosx-e^{-x}sinx-I$

$2I=-e^{-x}cosx-e^{-x}sinx+A$

$I=-\frac{1}{2}e^{-x}(cosx+sinx)+c$

$\therefore \int e^{-x}sinxdx=-\frac{1}{2}e^{-x}(cosx+sinx)+c$

(ii) $\int e^{2x}cos3xdx$

Solution

Taking I = $\int e^{2x}cos3xdx$

Let u = e^{2x}, $\frac{dv}{dx}=cos3x$

$\frac{du}{dx}=2e^{2x}$; v = $\frac{1}{3}$sin3x

$I=\frac{1}{3}e^{2x}sin3x-\int 2e^{2x}.\frac{1}{3}sin3xdx$

$I=\frac{1}{3}e^{2x}$sin3x$-\frac{2}{3}\int e^{2x}$sin3xdx ... (*)

For $\int e^{2x}$sin3xdx

Let u = e^{2x}, $\frac{dv}{dx}=sin3x$

$\frac{du}{dx}=2e^{2x}$; v =- $\frac{1}{3}$cos3x

$\int e^{2x}$ sin3xdx

$=-\frac{1}{3}e^{2x}$cos3x$-\int 2e^{2x}.-\frac{1}{3}$cos3xdx

$=-\frac{1}{3}e^{2x}$cos3x$+\frac{2}{3}\int e^{2x}$cos3xdx

$=-\frac{1}{3}e^{2x}$cos3x$+\frac{2}{3}I$ .....(**)

Substituting (**) into (*)

$I=\frac{1}{3}e^{2x}$sin3x$-\frac{2}{3}\left(-\frac{1}{3}e^{2x}\text{cos3x}+\frac{2}{3}I\right)$

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$= \frac{1}{3}e^{2x}\sin 3x + \frac{2}{9}e^{2x}\cos 3x + \frac{4}{9}I + c$

$\frac{13}{9}I = \frac{1}{3}e^{2x}\sin 3x + \frac{2}{9}e^{2x}\cos 3x + A$

$I = \frac{3}{13}e^{2x}\sin 3x + \frac{2}{13}e^{2x}\cos 3x + c$

$I = \frac{1}{13}e^{2x}(3sin3x + 2cos3x) + c$

$\therefore \int e^{2x}cos3xdx = \frac{1}{13}e^{2x}(3sin3x + 2cos3x) + c$

(iii) $\int e^{3x}sin2xdx$

Solution

Taking I = $\int e^{3x}sin2xdx$

Let u = e^{3x}, $\frac{dv}{dx} = sin2x$

$\frac{du}{dx} = 3e^{3x}$; v = $-\frac{1}{2}\cos 2x$

$I = -\frac{1}{2}e^{3x}cos2x - \int 3e^{3x}.-\frac{1}{2}cos2xdx$

$I = -\frac{1}{2}e^{3x}\cos 2x + \frac{3}{2}\int e^{3x}\cos 2xdx$ ... (*)

For $\int e^{3x}\cos 2xdx$

Let u = e^{3x}, $\frac{dv}{dx} = cos2x$

$\frac{du}{dx} = 3e^{3x}$; v = $\frac{1}{2}\sin 2x$

$\int e^{3x}\cos 2xdx$

$= \frac{1}{2}e^{3x}\sin 2x - \int 3e^{3x}.\frac{1}{2}\sin 2xdx$

$= \frac{1}{2}e^{3x}\sin 2x - \frac{3}{2}\int e^{3x}\sin 2xdx$

$= \frac{1}{2}e^{3x}\sin 2x - \frac{3}{2}I$ .....(**)

Substituting (**) into (*)

$I = -\frac{1}{2}e^{3x}cos2x + \frac{3}{2}\left(\frac{1}{2}e^{3x}\sin 2x - \frac{3}{2}I\right)$

$= -\frac{1}{2}e^{3x}cos2x + \frac{3}{4}e^{3x}\sin 2x - \frac{9}{4}I + c$

$\frac{13}{4}I = -\frac{1}{2}e^{3x}cos2x + \frac{3}{4}e^{3x}\sin 2x + A$

$I = -\frac{2}{13}e^{3x}\cos 2x + \frac{3}{13}e^{3x}\sin 2x + c$

$I = \frac{1}{13}e^{3x}(3sin2x - 2cos2x) + c$

$\therefore \int e^{3x}sin2xdx = \frac{1}{13}e^{2x}(3sin2x - 2cos3x) + c$

Or using basic technique

sign	Differentiate	integrate
+	e^{3x}	sin2x
-	$3e^{3x}$	$-\frac{1}{2}$cos2x
+	$9e^{3x}$	$-\frac{1}{4}$sinx

$I = -\frac{1}{2}e^{3x}cos2x + \frac{3}{4}e^{3x}\sin 2x - \frac{9}{4}I + c$

$\frac{13}{4}I = -\frac{1}{2}e^{3x}cos2x + \frac{3}{4}e^{3x}\sin 2x + A$

$I = -\frac{2}{13}e^{3x}\cos 2x + \frac{3}{13}e^{3x}\sin 2x + c$

$I = \frac{1}{13}e^{3x}(3sin2x - 2cos2x) + c$

$\therefore \int e^{3x}sin2xdx = \frac{1}{13}e^{2x}(3sin2x - 2cos3x) + c$

(b) Evaluate $\int_0^{\infty} e^{-2x}sin3xdx$

Solution

Taking I = $\int e^{-2x}sin3xdx$

Let u = e^{-2x}, $\frac{dv}{dx} = sin3x$

$\frac{du}{dx} = -2e^{-2x}$; v = $-\frac{1}{3}\cos 3x$

$I = -\frac{1}{3}e^{-2x}\cos 3x - \int 2e^{2x}.\frac{1}{3}\sin 3xdx$

$I = \frac{1}{3}e^{3x}\sin 3x - \frac{2}{3}\int e^{-2x}\cos 3xdx$ ..... (*)

For $\int e^{-2x}\cos 3xdx$

Let u = e^{-2x}, $\frac{dv}{dx} = cosx$

$\frac{du}{dx} = -2e^{2x}$; v = $\frac{1}{3}\sin 3x$

$\int e^{-2x}\cos 3xdx$

$= -\frac{1}{3}e^{-2x}\sin 3x + \frac{2}{3}\int e^{2x}\sin s3xdx$

$= -\frac{1}{3}e^{-2x}\sin 3x + \frac{2}{3}I$ .....(**)

Substituting (**) into (*)

$I = -\frac{1}{3}e^{-2x}\cos 3x - \frac{2}{3}\left(\frac{1}{3}e^{-2x}\cos 3x + \frac{2}{3}I\right)$

$= -\frac{1}{3}e^{-2x}\cos 3x - \frac{2}{9}e^{-2x}\sin 3x - \frac{4}{9}I + c$

$\frac{13}{9}I = -\frac{1}{3}e^{-2x}\text{scos}3x - \frac{2}{9}e^{-2x}\sin 3x + c$

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$I = -\frac{1}{13}e^{-2x}(3cos3x + 2sin3x) + c$

$\therefore \int e^{-2x}sin3xdx$

$= -\frac{1}{13}e^{-2x}(3cos3x + 2sin3x) + c$

⇨ $\int_0^{\infty} e^{-2x}sin3xdx$

$=\left[-\frac{1}{13}e^{-2x}(3cos3x + 2sin3x)\right]_0^{\infty}$

$= \frac{3}{13}$ since $e^{\infty} = 0$

Revision exercise 13

Integrate each of the following with respect to x

(a) $e^x cosx$ $\left[\frac{1}{2}e^x(sinx + cosx) + c\right]$

(b) $e^x sinxx$ $\left[\frac{1}{2}e^x(sinx - cosx) + c\right]$

(c) $e^{ax}cosbx$ $\left[\frac{e^{ax}}{b^2+a^2}(acosbx + bsinbx) + c\right]$

(d) $e^{3x}sin2x$ $\left[\frac{1}{13}e^{3x}(3sin2x - 2cos2x) + c\right]$

G. Integration of products of trigonometric functions by parts

A student should take note of the following

(i) $\int tanxdx = In(\sec x)$+ c

Proof

$\frac{d}{dx}In(sex)dx = \frac{secxtanx}{secx} = t[illegible]x$

Hence$\int tanxdx = In(sec[illegible])$+ [illegible]

(ii) $\int cosecxdx = -In(c[illegible]x + c[illegible]x) + c$

Proof

$\frac{d}{dx}In(cosecx + cotx)dx = \frac{-cosecxcotx-cosec^2x}{cosecx+cotx}$

$= \frac{-cosecx(cotx-cosecx)}{cosecx+cotx}$

$= -$cosecx

$\therefore \int cosecxdx = -In(cosecx + cotx) + c$

(iii) $\int secxdx = In(secx + tanx) + c$

Proof

$\frac{d}{dx}In(secx + tanx)dx = \frac{secxtanx+sec^2x}{secx+tanx}$

$= \frac{secx(In(secx+tanx))}{secx+tanx}$

=secx

$\therefore \int secxdx = In(secx + tanx) + c$

(iv) $\int cotxdx = In(sinx) + c$

Proof

$\frac{d}{dx}In(sinx) = \frac{cosx}{sinx} = cotx$

$\therefore \int cotxdx = In(sinx) + c$

Example 22

(a) Find

(i) $\int sec^3xdx$

Solution

Taking $I = \int sec^3xdx = \int secxsec^2xdx$

Let u = secx and $\frac{dv}{dx} = sec^2x$

$\frac{du}{dx} = secxtanx$; v = tanx

$I = secxtanx-\int(sec[illegible]tanxdx$

$= secxtanx-[illegible]tan^2xdx$

$= secx[illegible]\int secx(sec^2x - 1)dx$

$= [illegible]xtanx-\int sec^3xdx + \int secx\,dx$

$= [illegible]tanx-$ I + In(secx+tanx)+c

$secxtanx$ + In(secx+tanx)+c

$I = \frac{1}{2}[secxtanx + \text{In}(\text{secx} + \text{tanx})] + c$

$\therefore \int sec^3xdx$

$= \frac{1}{2}[secxtanx + \text{In}(\text{secx} + \text{tanx})] + c$

(ii) $\int cosec^3xdx$

Taking $I = \int cosec^3xdx = \int cosecxcosec^2xdx$

Let u = cosecx and $\frac{dv}{dx} = cosec^2x$

$\frac{du}{dx} = cosecxcotx$; v = -cotx

$I = -cotxcosecx-\int(secxcotx)cotxdx$

$= -cosecxcotx-\int secxcot^2xdx$

$= -cosecxcotx-\int cosecx(cosec^2x - 1)dx$

$= -cosecxtanx-\int cosec^3xdx + \int cosecx\,dx$

$= -cotxcosecx-I + \int cosecx\,dx$+A

$2I = -cotxcosex) - In(cosecx + cotx)$+c

$I = -\frac{1}{2}[cotxcosecx + In(cosecx + cotx)] + c$

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$\therefore \int cosec^3 x dx$

$= -\frac{1}{2}[cotxcosecx + In(cosecx + cotx)] + c$

(b) Show that $\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} sec^3 xtanx = \frac{8}{27}(9 - \sqrt{3})$

Solution

$\int sec^3 xtanxdx = \int sec^2 xsecxtanx$

Let u $= sec^2 x$ and $\frac{dv}{dx} = secxtanx$

$\frac{du}{dx} = 2sec^2 x; v = secx$

$\int sec^3 xtanxdx = sec^3 x - 2\int sec^3 xtanxdx$

$I - sec^3 x - 2I + c$

$3I = sec^3 x$

$I = \frac{1}{3}sec^3 x + c$

$\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} sec^3 xtanx = \left[\frac{1}{3}sec^3 x\right]_{-\frac{\pi}{6}}^{\frac{\pi}{3}}$

$= \frac{1}{3}\left[sec^3\left(\frac{\pi}{3}\right) - sec^3\left(-\frac{\pi}{6}\right)\right]$

$= \frac{1}{3}\left[8 - \frac{8}{3\sqrt{3}}\right] = \frac{1}{3}\left[8 - \frac{8}{9}\right]$

$= \frac{8}{3}\left[1 - \frac{\sqrt{3}}{9}\right]$

$= \frac{8}{27}[9 - \sqrt{3}]$

Revision exercise 1

Integrate each of the following with respect to x

1. $sec^3 x$
   $\left[\frac{1}{2}[secxtanx + \text{In}(\text{secx} + \text{tanx})] + c\right]$
2. $cosec^3 x$
   $\left[-\frac{1}{2}[cotxcosecx + In(cosecx + cotx)] + c\right]$
3. $sec^3 xtanx$
   $\left[\frac{1}{3}sec^3 x\right]$

Integration using t- substitution

Case 1

We know that if t = $tan\frac{1}{2}\theta$, then

$\sin\theta = \frac{2t}{1+t^2}$ and

$cos\theta = \frac{1-t^2}{1+t^2}$

Generally

If t = $tan\frac{1}{2}k\theta$, then

$\sin k\theta = \frac{2t}{1+t^2}$ and

$coskθ = \frac{1-t^2}{1+t^2}$

Example 23

Find

(a) $\int cosecxdx$

Solution

Let t = tan

$dt = sec^2 \frac{1}{2}xdx$

$dt = (1 + t^2)dx$

$= \frac{2}{1+t^2}dt$

$\int cosecxdx = \int \frac{1}{sinx}dx = \int \frac{1+t^2}{2t}.\frac{2}{1+t^2}dt$

$= \int \frac{1}{t}dt = Int + c$

$\therefore \int cosecxdx = In\left(tan\frac{1}{2}x\right) + c$

(b) $\int secxdx$

Solution

Let t = $\tan\frac{1}{2}x$

$dt = sec^2\frac{1}{2}xdx$

$2dt = (1 + t^2)dx$

$dx = \frac{2}{1+t^2}dt$

$\int secxdx = \int \frac{1}{cosx}dx = \int \frac{1+t^2}{1-t^2}.\frac{2}{1+t^2}dt$

$= \int \frac{2}{1-t^2}dt = \int \frac{2}{(1+t)(1-t)}dt$

Let $\frac{2}{(1+t)(1-t)} = \frac{A}{1+t} + \frac{B}{1-t}$

2= A(1 – t) + B(1 + t)

Putting t = 1, B = 1

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Putting t = -1; A= 1

=>$\frac{2}{(1+t)(1-t)} = \frac{1}{1+t} + \frac{1}{1-t}$

$\int \frac{2}{1-t^2} dt = \int \frac{1}{1+t} dt + \int \frac{1}{1-t} dt$

$= In(1+t) - In(1-t) + c = In\left(\frac{1+t}{1-t}\right) + c$

$\therefore \int secxdx = In\left(\frac{1+tan\frac{1}{2}x}{1-tan\frac{1}{2}x}\right) + c$

(c) $\int sec3xdx$

Solution

Let t = $tan\frac{1}{2}(3x) = \tan\frac{3}{2}x$

$2dt = 3(1+t^2)dx$

$dx = \frac{2}{3(1+)} dt$

$\int sec3xdx = \int \frac{1}{cos3x} dx = \int \frac{1+t^2}{1-t^2} \cdot \frac{2}{3(1+t^2)} dt$

$= \frac{2}{3} \int \frac{1}{1-t^2} dt = \frac{1}{3} \int \frac{2}{1-t^2} dt$

Let$\frac{2}{(1+t)(1-t)} = \frac{A}{1+t} + \frac{B}{1-t}$

2= A(1 – t) + B(1 + t)

Putting t = 1, B = 1

Putting t = -1; A= 1

=>$\frac{2}{(1+t)(1-t)} = \frac{1}{1+t} + \frac{1}{1-t}$

$\int \frac{2}{1-t^2} dt = \int \frac{1}{1+t} dt + \int \frac{1}{1-t} dt$

$= In(1+t) - In(1-t) + c = In\left(\frac{1+t}{1-t}\right) + c$

$\therefore \int sec3xdx = \frac{1}{3} In\left(\frac{1+tan\frac{1}{2}x}{1-tan\frac{1}{2}x}\right) + c$

(d) $\int \frac{1}{3-2cosx} dx$

Solution

Let t = $tan\frac{1}{2}x$

$dt = sec^2\frac{1}{2}xdx$

$2dt = (1+t^2)dx$

$dx = \frac{2}{1+t^2} dt$

$\int \frac{1}{3-2cosx} dx = \int \frac{1}{3-2\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2}{1+t^2} dt = 2\int \frac{1}{1+5t^2} dt$

$= \frac{2}{\sqrt{5}} tan^{-1}(\sqrt{5}t) + c = \frac{2\sqrt{5}}{5} tan^{-1}(\sqrt{5}t) + c$

$\therefore \int \frac{1}{3-2cosx} dx = \frac{2\sqrt{5}}{5} tan^{-1}\left(\sqrt{5}tan\frac{1}{2}x\right) + c$

(e) $\int \frac{2}{3sin2x+4} dx$

Solution

Let t = $tanx$

$dt = sec^2xdx$

$dt = (1+t^2)dx$

$dx = \frac{1}{1+t^2} dt$

$\int \frac{2}{3sin2x+4} dx = \int \frac{2}{3\left(\frac{2t}{1+t^2}\right)+4} \cdot \frac{1}{1+t^2} dt$

$\int \frac{1}{2t^2+3t+2} dt = \int \frac{1}{\frac{7}{8}+2\left(t+\frac{3}{4}\right)^2} dt$

$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{8}}{\sqrt{7}} tan^{-1} \frac{\sqrt{2}\left(t+\frac{3}{4}\right)}{\sqrt{\left(\frac{7}{8}\right)}} + c$

$= \frac{2}{\sqrt{7}} tan^{-1} \frac{(4t+3)}{\sqrt{7}} + c$

$\therefore \int \frac{2}{3sin2x+4} dx = \frac{2\sqrt{7}}{7} tan^{-1}\left(\frac{4tanx+3}{\sqrt{7}}\right) + c$

(f) $\int \frac{2}{3+5cos\frac{1}{2}x}$

Solution

Let t = $tan\frac{1}{4}x$

$dt = \frac{1}{4} sec^2\frac{1}{4}xdx$

$dt = \frac{1}{4}(1+t^2)dx$

$dx = \frac{4}{1+t^2} dt$

(g) $\int \frac{2}{3+5cos\frac{1}{2}x} = \int \frac{2}{3+5\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{4}{1+t^2} dt$

$= \int \frac{2}{4-t^2} dt = \int \frac{2}{(2+t)(2-t)} dt$

Let $\frac{2}{(2+t)(2-t)} = \frac{A}{(2+t)} + \frac{B}{2-t}$

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2= A(2 – t)+ B(2 + t)

Putting t = 2; B= $\frac{1}{2}$

Putting t = -2; A= $\frac{1}{2}$

$$\int \frac{2}{3+5cos\frac{1}{2}x}dx = \frac{1}{2}\int \frac{1}{2+t}dt + \frac{1}{2}\int \frac{1}{2-t}dt$$

$$= \frac{1}{2}In(2+t) - \frac{1}{2}In(2-t) + c$$

$$= \frac{1}{2}In\left(\frac{2+t}{2-t}\right) + c$$

$$\therefore \int \frac{2}{3+5cos\frac{1}{2}x}dx = \frac{1}{2}In\left(\frac{2+tan\frac{1}{4}x}{2-tan\frac{1}{4}x}\right) + c$$

Case II

When integrating fractional trigonometric functions containing the square of sinx, cosx, etc.

We use the

t-substitution, t = tanx

For sin^2kx or cos^2kx, we use t = tanx

Example 24

Find the integrals of the following

(a) $\int \frac{1}{4sin^2x-9cos^2x}dx$

Solution

Dividing numerator and denominator by cos^2x

$$\int \frac{1}{4sin^2x-9cos^2x}dx = \int \frac{sec^2x}{4tan^2x-9}dx$$

$$= \int \frac{1+tan^2x}{4tan^2x-9}dx$$

Let t = tanx

$$dt = sec^2xdx = (1+t^2)dx$$

$$dx = \frac{dt}{(1+t^2)}$$

$$\int \frac{1+tan^2x}{4tan^2x-9}dx = \int \frac{1+t^2}{4t^2-9}.\frac{dt}{(1+t^2)}$$

$$= \int \frac{1}{(2t+3)(2t-3)}dt$$

Let $\frac{1}{(2t+3)(2t-3)} = \frac{A}{(2t+3)} + \frac{B}{(2t-3)}$

1 = A(2t – 3) + B(2t + 3)

Putting t = $\frac{3}{2}$; B = $\frac{1}{6}$

Putting t = $-\frac{3}{2}$; A = $-\frac{1}{6}$

⇨ $\int \frac{1}{(2t+3)(2t-3)}dt$

$$= \frac{1}{6}\int \frac{B}{(2t-3)}dt + \frac{1}{6}\int \frac{1}{(2t+3)}dt$$

$$= \frac{1}{6}.\frac{1}{2}In(2t-3) - \frac{1}{6}.\frac{1}{2}In(2t+3) + c$$

$$= \frac{1}{12}In\left(\frac{2t-3}{2t+3}\right) + c$$

$$\int \frac{1}{4sin^2x-9cos^2x}dx = \frac{1}{12}In\left(\frac{2tanx-3}{2tanx+3}\right) + c$$

(b) $\int \frac{1}{3+4sin^2 5x}dx$

Solution

Dividing by the numerator and denominator $cos^2 5x$

$$\int \frac{1}{3+4sin^2 5x}dx = \int \frac{sec^2 5x}{3sec^2 5x-4}dx$$

$$= \int \frac{1+tan^2 5x}{3+7tan^2 5x}dx$$

Let t = tan5x

$$dt = sec^2 5xdx = 5(1+t^2)dx$$

$$dx = \frac{dt}{5(1+t^2)}$$

$$\int \frac{1}{3+4sin^2 5x}dx = \int \left(\frac{1+t^2}{3+7t^2}\right).\frac{dt}{5(1+t^2)}$$

$$= \frac{1}{5}\int \frac{1}{3+7t^2}dt$$

$$= \frac{1}{5}.\frac{1}{\sqrt{7}}.\frac{1}{\sqrt{3}}tan^{-1}\left(\frac{\sqrt{7}}{\sqrt{3}}t\right) + c$$

$$= \frac{1}{5\sqrt{21}}tan^{-1}\left(\frac{\sqrt{7}}{\sqrt{3}}tan5x\right) + c$$

$$\therefore \int \frac{1}{3+4sin^2 5x}dx = \frac{\sqrt{21}}{105}tan^{-1}\left(\frac{\sqrt{21}}{3}tan5x\right) + c$$

(c) $\int \frac{sin^2 3x}{1+cos^2 3x}dx$

Solution

Dividing numerator and denominator by $cos^2 3x$

$$\int \frac{sin^2 3x}{1+cos^2 3x}dx = \int \frac{tan^2 3x}{sec^2 3x+1}dx = \int \frac{tan^2 3x}{2+tan^2 3x}dx$$

Let t = tan3x

$$dt = 3sec^2 3xdx = 3(1+t^2)dx$$

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$dx = \frac{dt}{3(1+t^2)}$

$\int \frac{tan^2 3x}{2+tan^2 3x} dx = \int \frac{t^2}{2+t^2} \cdot \frac{dt}{3(1+t^2)}$

$= \frac{1}{3} \int \frac{t^2}{(2+t^2)(1+t^2)} dt$

Let $\frac{t^2}{(2+t^2)(1+t^2)} = \frac{Ax+B}{(2+t^2)} + \frac{Cx+D}{(1+t^2)}$

By equating coefficients and solving simultaneously

A = 2, C = -1, B = D = 0

$\int \frac{sin^2 3x}{1+cos^2 3x} dx = \int \frac{2}{(2+t^2)} dt - \int \frac{1}{(1+t^2)} dt$

$= \frac{1}{3}\left[\frac{2}{\sqrt{2}} tan^{-1}\left(\frac{t}{\sqrt{2}}\right) - tan^{-1} t\right] + c$

$= \frac{1}{3}\left[\frac{2}{\sqrt{2}} tan^{-1}\left(\frac{tan3x}{\sqrt{2}}\right) - tan^{-1}(tan3x)\right] + c$

$= \frac{\sqrt{2}}{3} tan^{-1}\left(\frac{tan3x}{\sqrt{2}}\right) - \frac{1}{3} tan^{-1}(tan3x) + c$

(d) $\int \frac{1}{cos2x - 3sin^2 x} dx$

Solution

$\int \frac{1}{cos2x - 3sin^2 x} dx = \int \frac{1}{1-5sin^2 x} dx$

Dividing the numerator and denominator by cos_2x

$\int \frac{1}{sec^2 - 5tan^2 x} dx$

Let t = tanx

$dt = sec^2 x dx = (1+t^2) dx$

$dx = \frac{dt}{(1+t^2)}$

$\int \frac{1}{1-5sin^2 x} dx = \int \frac{1}{1-5t^2} \frac{dt}{(1+t^2)}$

$= \int \frac{1}{1-4t^2} dt = \int \frac{1}{(1+2t)(1-2t)} dt$

Let $\frac{1}{(1+2t)(1-2t)} = \frac{A}{1+2t} + \frac{B}{1-2t}$

1 = A(1 – 2t) + B (1 + 2t)

Putting t = $\frac{1}{2}$; B = $\frac{1}{2}$

Putting t = $-\frac{1}{2}$; A = $\frac{1}{2}$

$\int \frac{1}{(1+2t)(1-2t)} dt = \frac{1}{2} \int \frac{dt}{1+2t} + \frac{1}{2} \int \frac{dt}{1-2t}$

$= \frac{1}{2}\left[\frac{1}{2} In(1+2t) - \frac{1}{2} In(1-2t)\right] + c$

$= \frac{1}{4} In\left(\frac{1+2t}{1-2t}\right) + c$

$\therefore \int \frac{1}{cos2x - 3sin^2 x} dx = \frac{1}{4} In\left(\frac{1+2tanx}{1-2tanx}\right) + c$

Revision exercise 14

1. Integrate the following

(a) $\int \frac{4}{3+5sinx} dx$ $\left[\frac{3tan\frac{1}{2}x+1}{tan\frac{1}{2}x+3} + c\right]$

(b) $\int \frac{1}{4+5cosx} dx$ $\left[\frac{1}{3} In\left(\frac{3+tan\frac{1}{2}x}{3-tan\frac{1}{2}x}\right) + c\right]$

(c) $\int \frac{1}{1+5sin2x} dx$ $\left[-\frac{1}{1+tanx} + c\right]$

(d) $\int \frac{4}{5+3cos\frac{1}{2}x} dx$ $\left[tan^{-1}\left(\frac{1}{2} tan\frac{1}{4}x\right) + c\right]$

(e) $\int \frac{4}{2+sin\frac{1}{2}x} dx$

$\left[\frac{8\sqrt{3}}{9} tan^{-1}\left(\frac{2tan\frac{1}{4}x+1}{\sqrt{3}}\right) + c\right]$

2. Integrate each of the following

(a) $\int \frac{1}{1+2sin^2 x} dx$

$\left[\frac{\sqrt{3}}{3} tan^{-1}(\sqrt{3}tan[illegible] + c\right]$

(b) $\int \frac{1}{1-10sin^2 x} dx$

$\left[\frac{1}{6} In(1+3[illegible] - \frac{1}{6} In(1-3tanx)\right] + c$

(c) $\int \frac{sin[illegible]}{1+co[illegible]^2 x}$

$[\sqrt{2}tan[illegible]^{-1}\left(\frac{\sqrt{2}}{2} tanx\right) - x + c]$

(d) $\int \frac{4}{[illegible]x+9sin^2 x} dx$

$[\frac{4}{3} tan^{-1}(3tanx) + c]$

(e) $\int \frac{1+sinx}{cos^2 x} dx$

$[tanx + secx + c]$

(f) $\int \frac{1}{1+tanx} dx$

$\left[\frac{1}{2}x + \frac{1}{2} In(cosx + sinx) + c\right]$

3. Evaluate

(a) $\int_0^{\frac{\pi}{2}} \frac{3}{1+sinx} dx$ [3]

(b) $\int_0^{\frac{2\pi}{3}} \frac{3}{5+4cosx} dx$ $\left[\frac{1}{3}\pi\right]$

(c) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{3}{4+5cosx} dx$ $[2In2]$

(d) $\int_0^{\frac{\pi}{2}} \frac{5}{3sinx+4cosx} dx$ $[In6]$

Integration of special cases involving splitting the numerator

Case 1

When a fractional integrand with quadratic denominator expressed in the form of $\frac{f(x)}{g(x)}$ is such that g(x) cannot be factorized or written in simple partial fractions, it is normally very useful to express it as a fraction by splitting the numerator.

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i.e. Numerator = A(derivative of denominator + B

Example 25

Find the integral of each of the following

(a) $\int \frac{2x-1}{4x^2+3}dx$

Solution

Numerator=$A\left[\frac{d}{dx}(4x^2+3)\right]+B$

2x – 1 =A(8x)+B

Putting x = 0, B= -1

Putting x = 1, A = $\frac{1}{4}$

$\int \frac{2x-1}{4x^2+3}dx = \frac{1}{4}\int \frac{8x}{4x^2+3}dx - \int \frac{1}{4x^2+3}dx$

$= \frac{1}{4}In(4x^2+3) - \frac{\sqrt{3}}{6}tan^{-1}\left(\frac{2\sqrt{3}}{3}x\right)+c$

(b) $\int \frac{2x+3}{x^2+2x+10}dx$

Solution

Numerator=$A\left[\frac{d}{dx}(x^2+2x+10)\right]+B$

2x +3 =A(2x+2)+B

Putting x = -1, B= 1

Putting x = 0, A = 1

$\int \frac{2x+3}{x^2+2x+10}dx$

$= \frac{1}{4}\int \frac{2x+2}{x^2+2x+10}dx + \int \frac{1}{x^2+2x+10}dx$

$= In(x^2+2x+10) + \int \frac{1}{9+(x+1)^2}dx$

$= In(x^2+2x+10) + \frac{1}{3}tan^{-1}\left(\frac{x+1}{3}\right)+c$

(c) $\int \frac{x}{x^2+3x+5}dx$

Solution

Numerator=$A\left[\frac{d}{dx}(x^2+3x+5)\right]+B$

x =A(2x+3)+B

Putting x = $-\frac{3}{2}$, B= $-\frac{3}{2}$

Putting x = 0, A = $-\frac{1}{2}$

$\int \frac{x}{x^2+3x+5}dx$

$= \frac{1}{2}\int \frac{2x+3}{x^2+3x+5}dx - \frac{3}{2}\int \frac{1}{x^2+3x+5}dx$

$= \frac{1}{2}In(x^2+3x+5) - \frac{3}{2}\int \frac{1}{\frac{11}{4}+\left(x+\frac{3}{2}\right)^2}dx$

$= \frac{1}{2}In(x^2+3x+5) - \frac{3}{\sqrt{11}}tan^{-1}\left(\frac{2x+3}{\sqrt{11}}\right)+c$

(d) $\int \frac{1-2x}{9-(x+2)^2}dx$

Solution

$\int \frac{1-2x}{\sqrt{9-(x+2)^2}}dx$

$= \int \frac{1}{\sqrt{9-(x+2)^2}}dx - \int \frac{2x}{\sqrt{9-(x+2)^2}}dx$

$= sin^{-1}\left(\frac{x+2}{3}\right) - \int \frac{2x}{\sqrt{9-(x+2)^2}}dx$

For $\int \frac{2x}{\sqrt{9-(x+2)^2}}dx$

Let $sinu = \frac{x+2}{3}$

3sinu= x+2

3cosudu=dx

$\int \frac{2x}{\sqrt{9-(x+2)^2}}dx = \int \frac{2(3sinu-2)}{\sqrt{9-9sin^2u}}.3cosudu$

$= \int \frac{6sinu-4}{3\sqrt{(1-sin^2u)}}.3cosudu$

$= \int (6sinu-4)du$

$= -6cosu - 4u + c$

$= -6\sqrt{1-\left(\frac{x+2}{3}\right)^2} - 2sin^{-1}\left(\frac{x+2}{3}\right)+c$

Substituting for $\int \frac{2x}{\sqrt{9-(x+2)^2}}dx$

$\int \frac{1-2x}{\sqrt{9-(x+2)^2}}dx$

$= sin^{-1}\left(\frac{x+2}{3}\right) + -6\sqrt{1-\left(\frac{x+2}{3}\right)^2} - 4sin^{-1}\left(\frac{x+2}{3}\right)+c$

$= 5sin^{-1}\left(\frac{x+2}{3}\right) + 6\sqrt{1-\left(\frac{x+2}{3}\right)^2}+c$

Case II

When finding the integral of fractional trigonometric function expressed in the form $\int \frac{acosx+bsinx}{c\ cosx+dsinx}$, a, b, c and d are constants, we split the numerator as:

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Numerator = A(derivative of denominator)+ (denominator)

Example 26

1. Find

(a) $\int \frac{2cosx+9sinx}{3cosx+sinx}dx$

Solution

Let 2cosx+ 9sinx =A$\frac{d}{dx}$(3cosx+sinx)+B(3cosx+sinx)

2cosx+ 9sinx =A(-3sinx+cosx)+B(3cosx+sinx)

2cosx+ 9sinx =(A+3B)cosx +(-3A+B)sinx

Equating coefficients:

For cosx: A+3B = 2 ..... (i)

For sinx: -3A+B = 9.....(ii)

Solving Eqn. (i) and Eqn. (ii) simultaneously

A= $-\frac{5}{2}$ and B = $\frac{3}{2}$

⇨ $\int \frac{2cosx+9sinx}{3cosx+sinx}dx$

$=-\frac{5}{2}\int \frac{-3sinx+cosx}{3cosx+sinx}dx+\frac{3}{2}\int \frac{3cosx+sinx}{3cosx+sinx}$

$=-\frac{5}{2}$ln(3cosx+sinx)+ $\frac{3}{2}x+c$

(b) $\int \frac{3sinx}{4cosx-sinx}dx$

Solution

Let 3sinx = A$\frac{d}{dx}$(4cosx–sinx) + B(4cosx–sinx)

3sinx = A(-4sinx–cosx) + B(4cosx – sinx)

3sinx = (-A+B)cosx +(-4A-B)sinx

Equating coefficients

For cosx: -A + 4B = 0 ..... (i)

For sin x: -4A- B = 3 ..... (ii)

Solving Eqn. (i) and Eqn. (ii) simultaneously

A= $-\frac{12}{17}$ and B= $-\frac{3}{17}$

$\int \frac{3sinx}{4cosx-sinx}dx$

$=-\frac{12}{17}\int \frac{-4sinx-cosx}{4cosx-sinx}dx-\frac{3}{17}\int \frac{4cosx-sinx}{4cosx-sinx}dx$

$=-\frac{12}{17}In(4cosx-sinx)-\frac{3}{17}x$+c

2. Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{cosx-sinx}{3cosx+2sinx}dx$

Solution

Let 3sinx = A$\frac{d}{dx}$(3cosx+2sinx) + B(3cosx+2sinx)

cosx-sinx = A(-3sinx+2cosx) + B (3cosx +2sinx)

cosx- sinx = (2A+3B)cosx +(-3A+2B)sinx

Equating coefficients

For cosx: 2A + 3B = 1 ..... (i)

For sin x: -3A+2B = -1 ..... (ii)

Solving Eqn. (i) and Eqn. (ii) simultaneously

A= $\frac{5}{13}$ and B= $-\frac{1}{13}$

$\int \frac{cosx-sinx}{3cosx+2sinx}dx$

$=\frac{5}{13}\int \frac{3cosx+2sinx}{3cosx+2sinx}dx+\frac{1}{13}\int \frac{3cosx+2sinx}{3cosx+2sinx}dx$

$=\frac{5}{13}In(3cosx+2sinx)+\frac{1}{13}x+c$

$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{cosx-sinx}{3cosx+2sinx}dx$

$=\left[\frac{5}{13}In(3cosx+2sinx)+\frac{1}{13}x+c\right]_{\frac{1}{6}\pi}^{\frac{1}{2}\pi}$

$=\left[\frac{5}{13}In\left(3cos\frac{\pi}{2}+2sin\frac{\pi}{2}\right)+\frac{1}{13}.\frac{\pi}{2}\right]-\left[\frac{5}{13}In\left(3cos\frac{\pi}{6}+2sin\frac{\pi}{6}\right)+\frac{1}{13}.\frac{\pi}{6}\right]$

$=\left[\frac{5}{13}In2+\frac{\pi}{26}\right]-\left[\frac{5}{13}In\frac{2+\sqrt{3}}{2}+\frac{\pi}{78}\right]$

$=\frac{5}{13}In\left(\frac{4}{2+3\sqrt{3}}\right)+\frac{\pi}{39}$

Revision exercise 15

1. Integrate each of the following

(a) $\int \frac{x+2}{x^2+2x+4}dx$

$\left[\frac{1}{2}In(x^2+2x+4)+\frac{\sqrt{3}}{3}tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right)+c\right]$

(b) $\int \frac{x}{x^2-x+3}dx$

$\left[\frac{1}{2}In(x^2-x+3)+\frac{2}{\sqrt{11}}tan^{-1}\left(\frac{2x-1}{\sqrt{11}}\right)+c\right]$

(c) $\int \frac{2(x+1)}{x^2+4x+8}dx$

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$\left[In(x^2+4x+8) - tan^{-1}\left(\frac{x+2}{2}\right) + c\right]$

(d) $\int \frac{5x+7}{x^2+4x+8}dx$

$\left[\frac{5}{2}In(x^2+4x+8) - \frac{3}{2}tan^{-1}\left(\frac{x+2}{2}\right) + c\right]$

2. Integrate the following

(a) $\int \frac{cosx-2sinx}{3cosx+4sinx}dx$

$\left[\frac{2}{5}In(4sinx+3cox) - \frac{1}{5}x + c\right]$

(b) $\int \frac{cosx}{2cosx-sinx}dx$

$\left[-\frac{1}{5}In(2cosx-sinx) + \frac{2}{5}x + c\right]$

(c) $\int \frac{cosx}{cosx-2sinx}dx$

$\left[-\frac{2}{5}In(cosx-2sinx) - \frac{1}{5}x + c\right]$

(d) $\int \frac{2cosx+sinx}{4cosx+3sinx}dx$

$\left[-\frac{14}{15}In(4cosx+3sinx) - \frac{11}{5}x + c\right]$

Revision exercise 16: general topical revision questions

1. Find

(a) $\int sinxdx$ $[xsin^{-1}x + \sqrt{1-x^2} + c]$

(b) $\int xsec^2xdx$ $[xtan$ [illegible] $Incos$ [illegible] $+ c]$

(c) $\int \frac{x^2}{\sqrt{1-x^2}}dx$ $\left[\sqrt{1-x^2}\left(\frac{-2-x^2}{3}\right) + c\right]$

(d) $\int In(x^2 - 4)dx$

$\left[xIn(x^2-4) - 2x + 2\left(In\frac{x+2}{x-2}\right) + c\right]$

(e) $\int \frac{dx}{3-2cosx}dx$ $\left[\frac{2}{\sqrt{5}}tan^{-1}\left(\sqrt{5}tan\frac{x}{2}\right) + c\right]$

(f) $\int 3^{\sqrt{2x-1}}dx$

$\left[\frac{3^{\sqrt{2x-1}}}{In3}\left(\sqrt{2x-1} - \frac{1}{In3}\right) + c\right]$

(g) $\int sin^2xdx$ $\left[\frac{1}{2}\left(x - \frac{1}{2}sin2x\right) + c\right]$

(h) $\int tan^3xdx$ $\left[\frac{1}{2}tan^2x - Incosx + c\right]$

(i) $\int \frac{4x^2}{\sqrt{1-x^6}}dx$ $\left[\frac{4}{3}sin^{-1}(x^3) + c\right]$

(j) $\int \frac{x^2}{x^4-1}dx$ $\left[\frac{1}{4}In\left(\frac{x-1}{x+1}\right) + \frac{1}{2}tan^{-1}x + c\right]$

(k) $\int \frac{2x}{\sqrt{x^2+4}}dx$ $\left[2\sqrt{x^2+4} + c\right]$

(l) $\int xInxdx$ $\left[\frac{x^2}{2}Inx - \frac{x^2}{4} + c\right]$

(m) $\int x^3e^{x^4}dx$ $\left[\frac{1}{4}e^{x^4} + c\right]$

(n) $\int \frac{1}{1+sin^2x}$ $\left[\frac{\sqrt{2}}{2}tan^{-1}(\sqrt{2}tanx) + c\right]$

(o) $\int Inxdx$ $[x(Inx-1) + c]$

(p) $\int x^2sin2xdx$

$\left[-\frac{1}{2}x^2cos2x + \frac{1}{2}xsin2x + \frac{1}{4}cos2x + c\right]$

(q) $\int Inx^2dx$ $[2x(Inx-1) + c]$

(r) $\int \frac{dx}{e^x-1}$ $[In(1-e^{-x}) + c]$

(s) $\int \frac{x^2}{(1+x^2)^{\frac{1}{2}}}dx$ $\left[\frac{1}{3}(1+x^2)^{\frac{1}{2}}(x^2-2) + c\right]$

(t) $\int \frac{dx}{1-cosx}$ $\left[-cot\left(\frac{x}{2}\right) + c\right]$

(u) $\int \frac{x^4-x^3+x^2+1}{x^3+x}dx$

$\left[\frac{x^2}{2} - x + Inx$ [illegible] ta [illegible] $x - \frac{1}{2}In(1+x^2) + c\right]$

(v) $\int$ [illegible] $^{-1}2$ [illegible] dx $\left[\left(\frac{sin^{-1}2x}{2}\right)^2 + c\right]$

(w) $\int$ [illegible] $(1-x^2)^{\frac{1}{2}}dx$ $\left[\frac{1}{3}(1-x^2)^{\frac{3}{2}} + c\right]$

(x) [illegible] $\frac{+\sqrt{x}}{2\sqrt{x}}$dx $\left[\sqrt{x} + \frac{x}{2} + c\right]$

(y) $\int x^2e^xdx$ $[x^2e^x - 2xe^x + 2e^x + c]$

(z) $\int \frac{dx}{x^2\sqrt{(25-x^2)}}dx$ $\left[-\frac{1}{25}\left(\frac{5\sqrt{25-x^2}}{x^2}\right) + C\right]$

2. Evaluate

(a) $\int_0^{\frac{\pi}{2}} xcos^2xdx$ [0.3669]

(b) $\int_1^{\sqrt{3}}(x+tanx)dx$ [1.0003]

(c) $\int_0^{\frac{\pi}{2}} sin2xcosxdx$ $\left[\frac{2}{3}\right]$

(d) $\int_1^3 \frac{x^2+1}{x^3+4x^2+3x}dx$ [0.3489]

(e) $\int_0^1 \frac{x}{\sqrt{1+x}}dx$ [0.3905]

(f) $\int_0^{\frac{\pi}{2}} \frac{cosx}{1+sin^2x}dx$ [0.7854]

(g) $\int_0^{\frac{\pi}{6}} sinxsin3xdx$ [0.1083]

(h) $\int_0^1 \frac{x^3}{x^2+1}dx$ [0.15345]

(i) $\int_0^{\sqrt{\frac{\pi}{2}}} 2xcosx^2dx$ [1]

(j) $\int_0^2 \frac{8x}{x^2-4x-12}dx$ [1.05]

(k) $\int_0^{\frac{\pi}{2}} \frac{dx}{1+sinx+cosx}$ $[In2]$

(l) $\int_0^{\frac{\pi}{2}} sin2xcosxdx$ $\left[\frac{2}{3}\right]$

(m) $\int_4^6 \frac{dx}{x^2-2x-3}$ [0.1905]

(n) $\int_0^{\frac{\pi}{2}} xsin^22xdx$ $\left[\frac{\pi^2}{16}\right]$

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(o) $\int_1^{\sqrt{3}} \frac{x^2}{\sqrt{x^4-x^2}}dx$ [2]

(p) $\int_1^3 \frac{3x^2+4x+1}{x^3+2x^2+x}dx$ $[In12]$

(q) $\int_0^{\frac{\pi}{2}} x^2 sinxdx$ $[\pi - 2]$

(r) $\int_0^1 xe^{2x}dx$ [2.0973]

(s) $\int_{\frac{\pi}{3}}^{\pi} xsinxdx$ [2.7992]

(t) $\int_{\frac{1}{2}}^1 10x\sqrt{(1-x^2)}\,dx$ [2.165]

(u) $\int_0^{\frac{\pi}{2}} sin5xcos3xdx$ $\left[\frac{1}{2}\right]$

(v) $\int_0^{\frac{\sqrt{3}}{2}} \frac{dx}{9+4x^2}dx$ $\left[\frac{\pi}{36}\right]$

3. Show that

(a) $\int_0^1 \frac{x^2+6}{(x^2+4)(x^2+9)} = \frac{\pi}{20}$

(b) $\int_0^{\frac{\pi}{2}} xtan^2xdx = \frac{1}{32}(8\pi - \pi^2 - 16\log_e 2)$

(c) $\int_2^4 xInxdx = 14In2 - 3$

(d)

4. Given that

$\frac{3x^3+2x^2-6x-2}{(x^2+x-2)(x^2-2)} = \frac{1}{x+2} + \frac{B}{x-1} + \frac{Cx+D}{x^2-2}$

Determine the values of A, B, C, D

Hence evaluate $\int_3^4 \frac{3x^3+2x^2-6x-2}{(x^2+x-2)(x^2-2)}$dx

$[A = B = C = 1, D = 0; 2.4770]$

5. Use the substitution of x = $\frac{1}{u}$ to evaluate

$\int_1^2 \frac{dx}{x\sqrt{x^2-1}}$ $\left[\frac{\pi}{3}\right]$

6. Express $\frac{x^3-3}{(x-2)(x^2+1)}$ as partial fractions

$\left[\frac{x^3-3}{(x-2)(x^2+1)} = 1 + \frac{1}{x-2} + \frac{x+1}{x^2+1}\right]$

Hence find $\int \frac{x^3-3}{(x-2)(x^2+1)}dx$

$\left[x + In(x-2) + \frac{1}{2}In(x^2+1) + tan^{-1}x + c\right]$

7. Express f(x) = $\frac{2x^2-x+14}{(4x^2-1)(x+3)}$ in partial fraction

$\left[\frac{2x^2-x+14}{(4x^2-1)(x+3)} = \frac{-3}{2x+1} + \frac{2}{2x-1} + \frac{1}{x+3}\right]$

Hence evaluate $\int_1^3 f(x)dx$ [0.7440]

8. Using the substitution 2x+1 = u, find

$\int_0^1 \frac{xdx}{(2x+1)^2}$ $\left[\frac{1}{18}\right]$

9. Express

(a) f(x) = $\frac{6x}{(x-2)(x+4)^2}$ in partial fraction

$\left[\frac{6x}{(x-2)(x+4)^2} = \frac{1}{3(x-2)} - \frac{1}{3(x+4)} + \frac{4}{(x+4)^2}\right]$

Hence evaluate $\int f(x)dx$

$\left[\frac{1}{3}In\left(\frac{x-2}{x+4}\right) - \frac{4}{(x+4)} + c\right]$

(b) f(x) = $\frac{3x^2+x+1}{(x-2)(x+1)^2}$ in partial fraction

$\frac{3x^2+x+1}{(x-2)(x+1)^2}$

$= \frac{5}{9(x-2)} - \frac{5}{9(x+2)} + \frac{4}{3(x+1)^2} - \frac{1}{(x+1)^3}$

Hence evaluate

$\int_3^4 \frac{3x^2+x+1}{(x-2)(x+1)^2}dx$ [0.317]

(c)

10. Using the substitution x = 3sinθ, evaluate

(a) $\int_0^3 \sqrt{\left(\frac{3+x}{3-x}\right)}dx$ [7.7125]

(b) $\int_0^{\pi} \frac{dx}{3+5cosx}$ [0.2747]

(e)

11. Use t= tan$\frac{1}{2}$ to evaluate

(a) $\int_0^{\frac{\pi}{2}} \frac{dx}{cos}$ [0.6755]

(b)

12. Given that $\int_0^a (x^2+2x-6)\,dx = 0$, find the value of a [a=-6]

13. Use the substitution $x^2 = \theta$ to find

$\int \frac{x}{1+\cos x^2}dx$ $\left[\frac{1-3x}{3(x+1)^{\frac{5}{3}}(x-1)^{\frac{4}{5}}}\right]$

14. Resolve $y = \frac{x^3+5x^2-6x+6}{(x-1)^2(x^2+2)}$ into partial fraction

$\left[\frac{x^3+5x^2-6x+6}{(x-1)^2(x^2+2)} \equiv \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{4}{(x^2+2)}\right]$

Hence find $\int ydx$

$\left[In(x-1) + \frac{-2}{(x-1)} + \frac{4}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + c\right]$

15. Express f(x) = $\frac{1}{x^2(x-1)}$ in partial fraction

$\left[\frac{1}{x^2(x-1)} = -\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-1}\right]$

Hence evaluate $\int_2^3 f(x)dx$ [0.12102]

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Application of integration

Like differentiation, integration has a wide spectrum of application, some of which are discussed below

Acceleration, velocity, displacement

Given the acceleration, a, of a particle, its velocity, v and displacement, s can be computed as long as the initial values are known.

Acceleration, a = $\frac{dv}{dt}$ => v = $\int adt$

Also, velocity v = $\frac{ds}{dt}$ => s = $\int vdt$

Example 27

The acceleration of a particle after t seconds is given by a = 5 + t.

If initially, the particle is moving at 1ms^{-1}, find the velocity after 2s and the distance it would have covered by then

Given $\frac{dv}{dt} = 5 + t$

⇨ $dv = (5 + t)dt$

$v = 5t + \frac{1}{2}t^2 + c$

Whet t = 0, v = 1, =>c = 1

$\therefore v = 5t + \frac{1}{2}t^2 + 1$

When t = 2s

$v = 5(2) + \frac{1}{2}(2)^2 + 1 = 13$ms^{-1}.

And $\frac{ds}{dt} = 5t + \frac{1}{2}t^2 + 1$

$ds = \left(5t + \frac{1}{2}t^2 + 1\right)dt$

s = $\frac{5}{2}t^2 + \frac{1}{6}t^3 + t + c$

when t = 0, s= 0 => c = 0

∴s = $\frac{5}{2}t^2 + \frac{1}{6}t^3 + t$

At t =2s

s = $\frac{5}{2}(2)^2 + \frac{1}{6}(2)^3 + 2$=13$\frac{1}{3}$m

Example 28

A particle with a velocity (2i+3j)ms^{-1} is accelerated uniformly at the rate of (3ti – 2j)ms^{-1} from the origin. Find

(i) The speed reached by the particle at t = 4s.

Solution

Given a= 3ti – 2j

v = $\int adt = \int(3\text{ti} - 2\text{j})dt$

$= \frac{3}{2}t^2 i - 2tj + c$

At t= 0, 2i+3j

c =2i+3j

By substitution

$v = \left(\frac{3}{2}t^2 + 2\right)i + (-2t + 3)j$

At t = 4s

$v = \left(\frac{3}{2}(4)^2 + 2\right)i + (-2(4) + 3)j$

= (26i – 5j)ms^{-1}

Speed = $|v| = \sqrt{26^2 + (-5)^2}$ =26.5ms^{-1}

(ii) The distance travelled by the particle after 2s

Solution

r = $\int vdt$

r= $\int\left(\left(\frac{3}{2}(4)^2 + 2\right)i + (-2(4) + 3)j\right)dt$

$= \left(\frac{3}{6}t^3 + 2t\right)i + \left(\frac{-2}{2}t^2 + 3t\right)j + c$

At t = 0, r = 0; => c = 0

$\therefore r = \left(\frac{3}{6}t^3 + 2t\right)i + \left(\frac{-2}{2}t^2 + 3t\right)j$

At t = 2

$\therefore r = \left(\frac{8}{2} + 4\right)i + (-4 + 6)j$

$|r| = \sqrt{8^2 + 2^2}$ =8.25m

Hence the distance = 8.25m

Example 29

A particle has initial position of (7i+5j)m. the particle moves with constant velocity of (ai+bi)ms-1 and 3s later its position is (10i – j)m. fins the values of a and b.

Solution

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Given v = ai +bj

r =$\int vdt = \int(ai + bi)dt + c$

$= ati + btj + c$

at t = 0; r = c =(7i+5j)m

$\therefore r = (at + 7)i + (bt + 5)j$

After 3s

$10i - j = (3a + 7)i + (3b + 5)j$

Equating corresponding vectors

For i: 10 = 3a+7 => a = 1

For j: -1 = 3b +5 => b = -2

$\therefore a = 1 \; and \; b = -2$

Example 30

A particle of mass 2kg, initially at rest at (0, 0, 0) is acted on by a force $\begin{pmatrix} 2t \\ 2t \\ 4t \end{pmatrix} N$. Find

(i) its acceleration at time t

from F = Ma

$\begin{pmatrix} 2t \\ 2t \\ 4t \end{pmatrix} = 2a => a = \begin{pmatrix} t \\ t \\ 2t \end{pmatrix}$

(ii) its velocity after 3s

velocity v =$\int adt = \int \begin{pmatrix} t \\ t \\ 2t \end{pmatrix} dt$

v=$\begin{pmatrix} \frac{t^2}{2} \\ \frac{t^2}{2} \\ t^2 \end{pmatrix} + c$

at t = 0, v = 0 => c= 0

∴v=$\begin{pmatrix} \frac{t^2}{2} \\ \frac{t^2}{2} \\ t^2 \end{pmatrix}$

At t = 3s

$v = \frac{9}{2}i + \frac{9}{2}j + 9k$

(iii) the distance of the particle travelled after 3s.

r= $\int vdt$ =$\int \left(\frac{t^2}{2}i + \frac{t^2}{2}j + t^2k\right) dt$

= $\left(\frac{t^3}{6}i + \frac{t^3}{6}j + \frac{1}{3}t^3k\right) + c$

At t = 0, r = 0 => c = 0

$\therefore r = \left(\frac{t^3}{6}i + \frac{t^3}{6}j + \frac{1}{3}t^3k\right)$

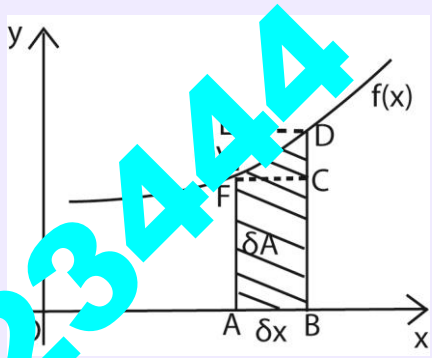
At t = 3

$r = \left(\frac{3^3}{6}i + \frac{3^3}{6}j + \frac{1}{3}.3^3k\right) = \left(\frac{9}{2}i + \frac{9}{2}j + 9k\right)$

$|r| = \sqrt{\left(\frac{9}{2}\right)^2 + \left(\frac{9}{2}\right)^2 + 9^2}$ =11.02m

Area under a curve

If the area under the curve y = f(x) for α≤ x ≤β is required, a small strip can be used for analysis



Suppose the shaded region is δA, the area of the shaded strip lies between areas of the rectangles ABCF and AVDE.

i.e. Area of ABCF ≤ δA ≤ ABDE.

$y\delta x \leq \delta A \leq (y + \delta y)dx$

Dividing by δx

$y \leq \frac{\delta A}{\delta x} \leq (y + \delta y)$

$\lim_{\delta x \to 0} \frac{\delta A}{\delta x} \to \frac{dA}{dx}$ and $\delta y \to 0$

Hence $\frac{dA}{dx} = y$

Integrating both sides with respect to x

$\int \frac{dA}{dx} dx = \int ydx$

Now for the interval $\alpha \leq x \leq \beta$

$A = \int_\alpha^\beta ydx \; Or \;\; A = \int_\alpha^\beta f(x)dx$

Note: when finding the area under the curve, it is advisable that you sketch the curve first in order to establish the required region.

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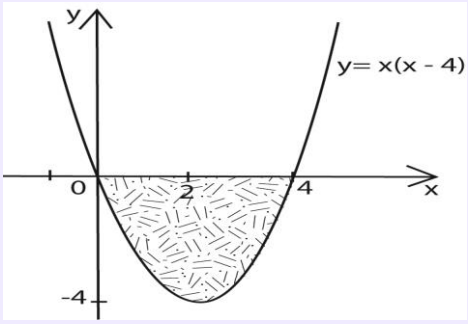
Area between the curve and the x-axis

Example 31

(i) Find the area enclosed by y = x(x – 4) and x-axis

Solution

By sketching the graph y = x(x – 4) with the x-axis we have



Area required = $\int_0^4 x(x-4)dx$

$=\int_0^4 x^2 - 4xdx$

$=\left[\frac{x^3}{3} - 2x^2\right]_0^4$

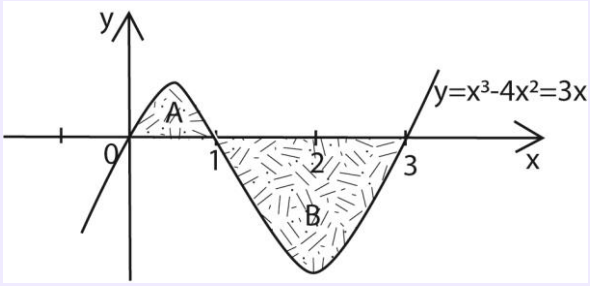
$=\frac{64}{3} - 32 = \frac{-32}{3}$

Hence the area under the curve is $\frac{-32}{3}$ $sq.units$ (- sign indicates that the area is below the x-axis).

(ii) Find the area enclosed by the curve y = $x^3 - 4x^2 + 3x$ and the x-axis from x = 0 and x = 3

Solution

By sketching the graph y = $x^3 - 4x^2 + 3x$ with the x-axis we have



Required area = A+ B

Area A =$\int_0^1 (x^3 - 4x^2 + 3x)\, dx$

$=\left[\frac{x^4}{4} - \frac{4x^3}{3} + \frac{3x^2}{2}\right]_0^1$

$=\left(\frac{1}{4} - \frac{4}{3} + \frac{3}{2}\right) - (0) = \frac{5}{12}$

Area B =$\int_1^3 (x^3 - 4x^2 + 3x)\, dx$

$=\left[\frac{x^4}{4} - \frac{4x^3}{3} + \frac{3x^2}{2}\right]_1^3$

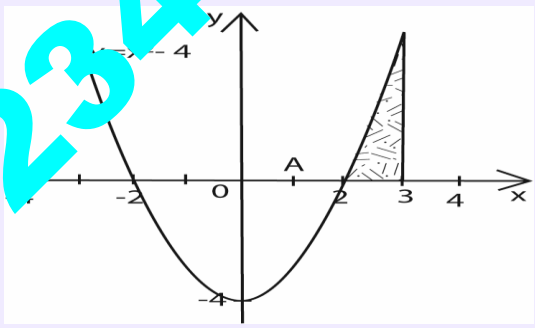
$=\left(\frac{81}{4} - 36 + \frac{27}{2}\right) - \left(\frac{1}{4} - \frac{4}{3} + \frac{3}{2}\right) = -\frac{8}{3}$

Area=$\frac{5}{12} + \frac{8}{3} = \frac{37}{12} sq.units$

(iii) Find the area between y=x^2 – 4, the x-axis and line x = 3.

Solution

By sketching the graph of y=x^2 – 4 with the x-axis, we have



Required $= \int_2^3 (x^2 - 4)\, dx$

$= \frac{1}{3}[x^3 - 4x]_2^3$

$= \frac{7}{3}$ sq. units

Area between the curve and the y-axis

This involves finding the area under the curve with respect to y or by subtracting the area under the curve with the x-axis from the rectangle (s) formed.

Example 32

Find the area enclosed by the curve y = x^2- 4 and the y = x2 -4 and y-axis between

(i) y = -4 and y = 0

Solution

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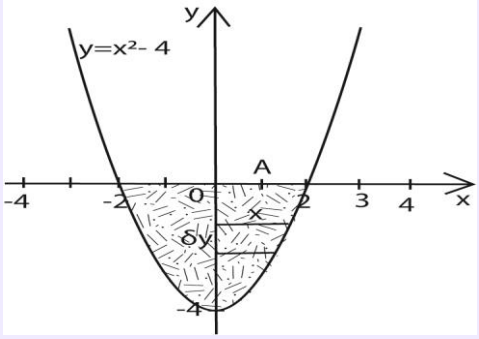


1st Approach

Integrating with respect to x

Required area $= \int_{-2}^{2}(x^2 - 4)dx$

$= \frac{1}{3}[x^3 - 4x]_{-2}^{2}$

$= \left(\frac{8}{3} - 8\right) - \left(\frac{-8}{3} + 8\right)$

$= \left(\frac{16}{3} - 16\right)$sq. units

$= \frac{-32}{3}$

Hence the required area is $\frac{32}{3} sq.units$

2nd approach

y = $x^2 - 4$

x = $(y + 4)^{\frac{1}{2}}$

Required area = $2\int_{-1}^{0} xdy$

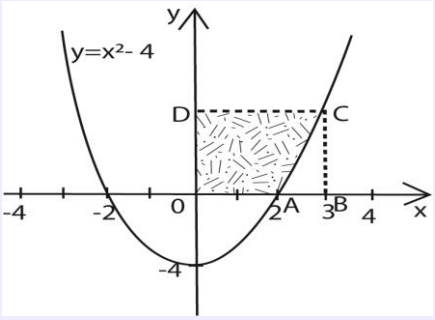
= $2\int_{-4}^{0}(y + 4)^{\frac{1}{2}}dy$

= $2\left[\frac{2}{3}(y + 4)^{\frac{3}{2}}\right]_{-4}^{0}$

= $2\frac{2}{3}[(8) - (0)]$

= $\frac{32}{3}$ sq. units

(ii) y = 0 and y = 5



1st approach

Required area= 2 x shaded region

Required area = $2\int_{0}^{5} xdy$

= $2\int_{0}^{5}(y + 4)^{\frac{1}{2}}dy$

= $2\left[\frac{2}{3}(y + 4)^{\frac{3}{2}}\right]_{0}^{5}$

= $2\frac{2}{3}[(2) - (8)]$

= $\frac{76}{3}$ sq. units

2nd approach

Required area = 2 x shaded area

= [area of OBCD – area of ABC]

= $2[(3 \text{ x } 5) - \int_{2}^{3}(x^2 - 4)dx]$

=2[15– $\frac{7}{3}$]=$\frac{76}{3}$sq.units

Area between two curves

Suppose we want to find the area between two intersecting functions f(x) and g(x), required it to

(i) find the point of intersection of the functions
(ii) sketch the functions f(x) and g(x)

Note if f(x) is above g(x), then the required area

$= \int f(x)dx - \int g(x)dx$

Example 33

Find the area enclosed between the curves

(a) y = $x^2 - 4$ and y = $4 - x^2$

Solution

Finding the points of intersection

$x^2 - 4 = 4 - x^2$

$2x^2 = 8$

x = 2 or x = 2

when x = 2, y = 0

when x = -2, y = 0

The sketch of the functions:

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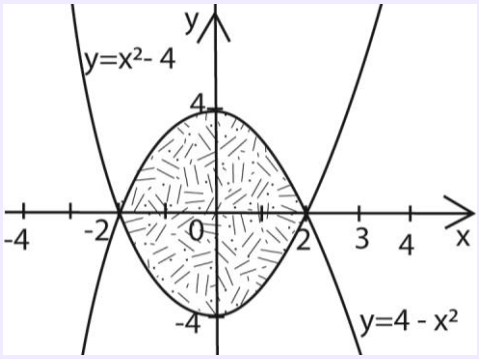


Required area

$= \int_{-2}^{2}[(4-x^2)-(x^2-4)]dx$

$= \int_{-2}^{2}(8-2x^2)dx$

$= \left[8x - \frac{2x^3}{3}\right]_{-2}^{2}$

$= \left(16 - \frac{16}{3}\right) - \left(-16 + \frac{16}{3}\right)$

$= \frac{32}{3} + \frac{32}{3} = \frac{64}{3}$ sq. units

(b) $y = 2x^2+7x+3$ and $y=9+4x-x^2$

Solution

Finding the points of intersection

$2x^2 + 7x + 3 = 9 + 4x - x^2$

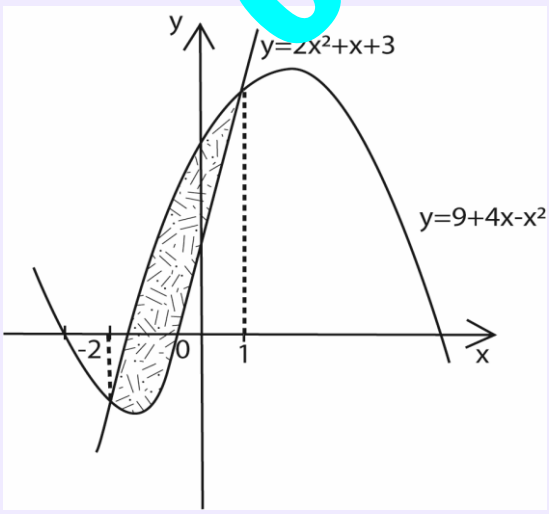
$3x^2 + 3x - 6 = 0$

$(x+2)(x-1) = 0$

x = -2 of x = 1

When x = -2, y = -3

When x = 1, y = 12



Required area

$= \int_{-2}^{1}[(9+4x-x^2)-(2x^2+7x+3)]\,dx$

$= \int_{-2}^{1}(6-3x-3x^2)\,dx$

$= \left[6x - \frac{3x^2}{2} - x^3\right]_{-2}^{1}$

$= \left(6 - \frac{3}{2} - 1\right) - (-12 - 6 + 8)$

$= 13.5 sq.units$

Example 34

Find the area enclosed between the curve $y = x^2 - x - 3$ and the line 2x +1

Solution

Finding the points of intersection

$x^2 - x - 3 = 2x +$

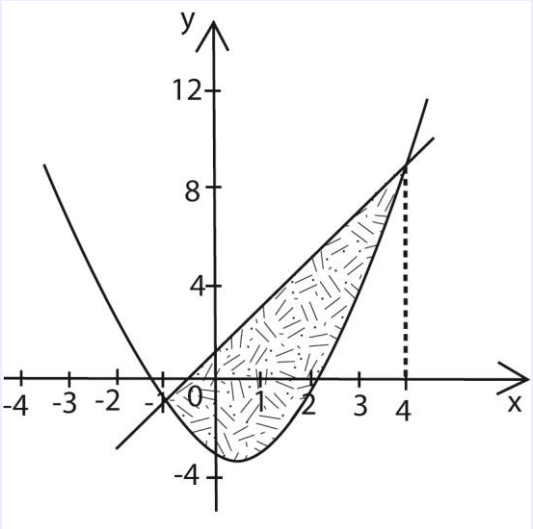
$x^2 - 3x - 4 =$

$(x$ [illegible] $)($ [illegible] $) = 0$

$=$ -1 [illegible] $x = 4$

[illegible]en x = -1, y = -1

When x= 4, y = 9



Area required

$= \int_{-1}^{4}[(2x+1)-(x^2-x-3)]\,dx$

$= \int_{-1}^{4}(4+3x-x^2)dx$

$= \left[4x + \frac{3}{2}x^2 - \frac{1}{3}x^3\right]_{-1}^{4}$

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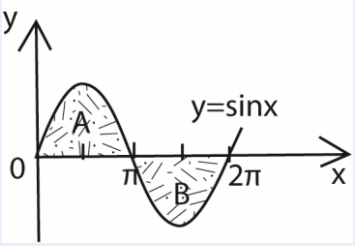
$=\left(16+24-\frac{64}{3}\right)-\left(-4+\frac{3}{3}+\frac{1}{2}\right)$

=20.83 sq.units

Example 35

Find the area enclosed by the curve y = sinx and the x-axis between x = 0 and x = 2π.

Solution



Required area = A + B

$=\int_0^{\pi} sinxdx+\int_{\pi}^{2\pi} sinxdx$

$=[-cosx]_0^{\pi}+[-cosx]_{\pi}^{2\pi}$

$=-(-cos\pi-cos0)-(-cos2\pi-cos\pi)$

=-(-1 – 1)- (-1 – 1)

= 2 +2 = 4sq. units

Volume of a solid of revolution

A solid of revolution is formed when a given area rotates about a fixed axix. Due to the way in which it is formed, it is referred to as solid of revolution.

These bodies have always got axes of symmetry.

The solids formed is subdivided into small cylindrical disks of thickness δx and height y.

Volume of each disk = $\pi y^2 dx$

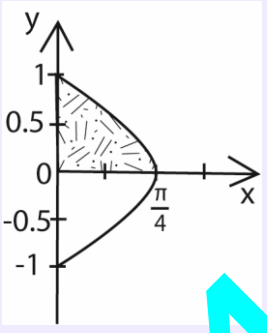
Thereforethe colume of the whole solid of revolution is obtained by rotating through one revolution about the x-axis, the region bounded by the curve y = f(x) and the linex x = a and x =b is given by v = $\int_a^b \pi y^2 dx$

If the rotation is about the y-axis, the volume is given by v =$\int_a^b \pi x^2 dy$

Example 36

(a) Find the volume of revolution when the portion of the curve y = cos2x for 0≤x≤$\frac{\pi}{4}$ is rotated through four right angles about the x-axis.

Solution



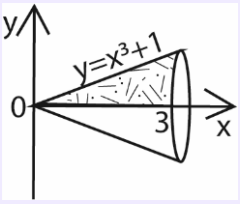
$V=\pi\int_0^{\frac{\pi}{2}} y^2 dx=\pi\int_0^{\frac{\pi}{2}} cos^2 2xdx$

$=\pi\int_0^{\frac{\pi}{2}}(1+cos4x)dx$

$=\frac{\pi}{2}\left[x+\frac{1}{4}sin4x\right]_0^{\frac{\pi}{2}}$

$=\frac{1}{4}\pi^2$ cubic units

(b) Find the volume of the area bounded by the curve y = x^3 + 1, the x-axis and limits x = 0 and x = 3 when rotated through four right angles about the x-axis.



$V=\pi\int_0^3 y^2 dx=\pi\int_0^3(x^3+1)^2 dx$

$=\pi\int_0^3(x^6+2x^3+1)^2 dx$

$=\pi\left[\frac{x^7}{7}+\frac{x^4}{2}+x\right]_0^3$

$=\pi\left(\frac{3^7}{7}+\frac{3^4}{2}+3\right)-(0)$

$=1118.25$ cubic units.

Rotation the area enclosed between two curves

If we have two curves y_1 and y_2 that enclose some area between a and b as shown below

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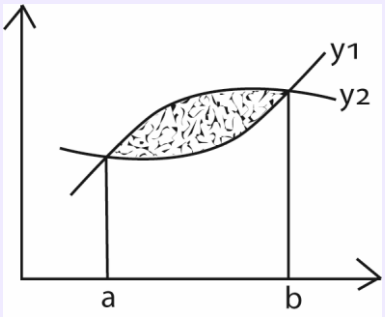


Now if we rotate this area about the x-axis the volume of the solid formed is given by

$$v = \pi \int_a^b [(y_2)^2 - (y_1)^2]\, dx$$

Example 36

(a)A cup is madeby rotating the area between $y = x^2$ and $y = x+1$ with $x \geq 0$ about the x-axis. Find the volume of the material needed to make the cup.

Solution

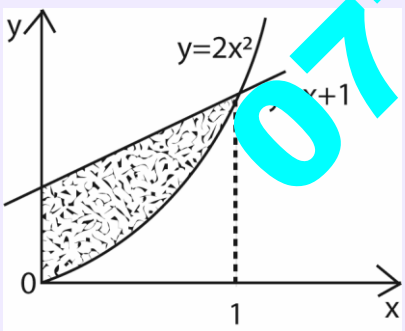
Finding the points of intersection

$2x^2 = x + 1$

$2x^2 - x + 1 = 0$

$(2x+1)(x - 1) = 0$

x =1 since we only need to cons[illegible] x≥ 0.



$V = \pi \int_0^1 [(y+1)^2 - (2x^2)^2] dx$

$= \pi \int_0^1 (x^2 + 2x + 1 - 4x^4) dx$

$= \pi \left[\frac{x^3}{3} + x^2 + x - \frac{4x^5}{5}\right]_0^1$

$= \pi \left(\frac{1}{3} + 1 + 1 - \frac{4}{5}\right) - 0$

$= \frac{23}{15}\pi$ units cubed

Example 37

Find the volume of revolution when the portion of the area between the curves $y = x^2$ and $x = y^2$ is rotated through 360^0 about the x-axis.

Solution

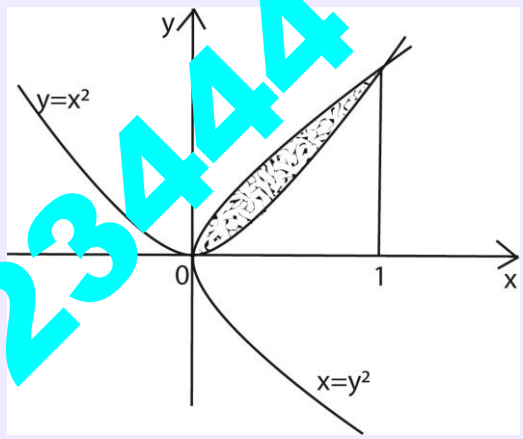
Points of intersection

$x^2 = x^{\frac{1}{2}} => x^4 = x$

$x^4 - x = 0$

$x(x^3 - 1) = 0$

Either x = 0 or x = 1



The volumeof revolution

$= \pi \int_0^1 \left[\left(\sqrt{x}\right)^2 - (x^2)^2\right] dx$

$= \pi \int_0^1 (x - x^4) dx$

$= \pi \left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1$

$= \pi \left[\left(\frac{1}{2} - \frac{1}{5}\right) - 0\right]$

$= \frac{3}{10}\pi$

Example 38

Find the volume generated when the area enclosed by the curve $y = 4 - x^2$ and the line $y = 4 - 2x$ is rotated through 2π.

Solution

Finding the points of intersection

$4 - 2x = 4 - x^2$

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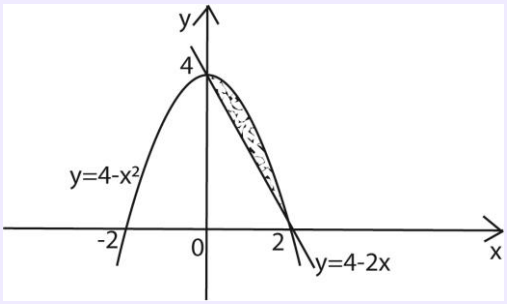
$x^2 - 2x = 0$

$x(x - 2) = 0$

Either x = 0 or x = 2

When x = 0, y = 4

When x = 2, y = 0



Required volume

$=\pi \int_0^2 [(4 - x^2)^2 - (4 - 2x)^2]dx$

$= \pi \int_0^2 [(16 - 8x^2 + x^4) - (16 - 8x - 4x^2)]dx$

$=\pi \int_0^2 (x^4 - 4x^2 + 8x)\, dx$

$=\pi \left[\frac{x^5}{5} - \frac{4x^3}{3} + 4x^2\right]_0^2$

$=\frac{176}{15}\pi = 36.86$cubic units

Example 39

(a) Sketch the curve y = x^3 – 8 (08marks)

$y = x^3 - 8$

Intercepts

When x = 0, y = -8

When y = 0, x = 2

(x, y) = (2, 0)

Turning point: $\frac{dy}{dx} = 3x^2$

3x2 = 0

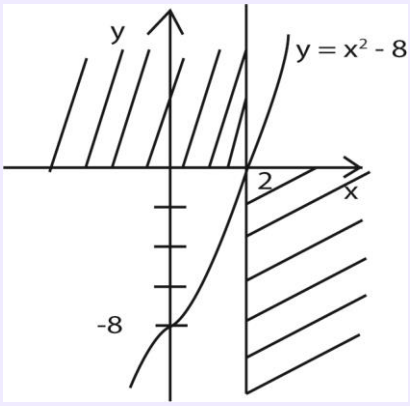
x = 0

$\frac{d^2y}{dx^2} = 6x$

$\frac{d^2y}{dx^2} = 0, x = 0$

Point of reflection= (0, 8)

	x < 2	x > 2
y	-	+



(b) The area enclosed by the curve in (a), the y-axis and x-axis is rotate about the line y = 0 through 360^0. Determine the volume of the solid generated. ([illegible] marks)

$V = \pi \int_0^2 y^2 dx$

$= \pi \int_0^2 ([illegible])^2 dx$

$= [illegible]^2 ([illegible] - 16x^3 + 64)dx$

$= [illegible] - 4x^4 + 64x\Big]_0^2$

$[illegible]\pi \left(\frac{128}{7} - 64 + 128\right)$

$= \frac{576\pi}{7}$ =250.5082 units^3

The mean value theorem for integrals

If f(x) is a continuous function on the closed interval [a, b], then there exist a number c in the closed interval such that

Area of the rectangle = f(c).(b-a)

But area under the curve between a and b

$=\int_a^b f(x)dx$

Equating the two

$\int_a^b f(x)dx - f(c).(b - a)$

Dividing both sides by (b – a)

$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$

Where f(c) is the height of the rectangle

This height is the average value of the function over the interval in the question.

Hence the mean value of f(x) over a closed interval (a, b) is given by

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M.V $=\frac{1}{b-a}\int_a^b f(x)dx$

Example 40

Find the mean value of $y= x^2 + 2$ for x = 1 and x = 4.

Solution

M.V $=\frac{1}{4-1}\int_1^4 (x^2+2)dx$

$=\frac{1}{3}\int_1^4 (x^2+2)dx$

$=\frac{1}{3}\left[\frac{x^3}{3}+2x\right]_1^4$

$=\frac{1}{3}\left[\left(\frac{64}{3}+8\right)-\left(\frac{1}{3}+2\right)\right]=9$

Example 41

Find the mean value of

$y = \frac{1}{1+sin^2\theta}$ for $0 \le \theta \le \frac{\pi}{4}$

Solution

M.V $=\frac{1}{\frac{\pi}{4}-0}\int_0^{\frac{\pi}{4}}\frac{1}{1+sin^2\theta}dx$

$=\frac{4}{\pi}\int_0^{\frac{\pi}{4}}\frac{sec^2\theta}{sec^2\theta+tan^2\theta}dx$

$=\frac{4}{\pi}\int_0^{\frac{\pi}{4}}\frac{1+tan^2\theta}{1+2tan^2\theta}dx$

Let t = tan θ => dt = $sec^2\theta d\theta = (1+t^2)d\theta$

$d\theta = \frac{dt}{1+t^2}$

Changing limits

When θ = 0, t = 0 and when $\theta=\frac{\pi}{4}$, t = 1

$\therefore M.V = \frac{\pi}{4}\int_0^1 \frac{1+t^2}{1+2t^2}.\frac{dt}{1+t^2}$

$= \frac{\pi}{4}\int_0^1 \frac{1}{1+2t^2}dt$

$= \frac{\pi}{4}\left[\frac{1}{\sqrt{2}}tan^{-1}\sqrt{2t}\right]_0^1$

$= \frac{2\sqrt{2}}{\pi}tan^{-1}\sqrt{2}$

= 0.86

Example 42

Find the mean value of y = x(4 – x)in the interval where y ≥ 0.

Solution

Given y ≥ 0 => x(4 –x)≥0 (positive)

The solution is 0 ≤ x ≤ 4

M.V $=\frac{1}{4-0}\int_0^4 x(4-x)dx = \frac{1}{4}\int_0^4(4x-x^2)dx$

$=\frac{1}{4}\left[2x^2-\frac{x^3}{3}\right]_0^4=\frac{8}{3}$

Revision exercise 17

1. Find the volume generated in each case whenthe area enclosed by the curve $y = x^2 - 6x + 18$ and the line y = 10 is rotated about
   (i) [illegible] = [illegible] [1541π units^3]
   (ii) [illegible] axix, [256π units^3]
2. Find the volume generated when the area enclosed by the curve y = x4 from y = 3 and y = 6 is rotated about the y-axis [6.33π units^3]
3. The displacement x of a particle at time t is given by x = sint. Find the mean value of its velocity over the interval $0 < t < \frac{\pi}{2}$
   (i) with respect to t [0.637ms^{-1}]
   (ii) with respect to displacement x [0.785ms^{-1}]
4. (a) Determine the equation of the normal to the curve $y = \frac{1}{x}$ and the point x = 2. Find the coordinates of the other point where the normal meets the curve again.
   [2y – 8x+15 =0; $\left(-\frac{1}{8},-8\right)$
   (b) Find the area of the region bounded by the curve $y =\frac{1}{x(2x+1)}$, the x-axis and the lines x = 1 and x = 2. $\left(In\left(\frac{6}{5}\right)\right)$
5. A shell is formed by rotating the portion of the parabola y2= 4x for which 0 ≤ x ≤ 1 through two right angles about its axis. Find
   (i) the volume of the solid formed [2π]
   (ii) the area of the base of the solid formed [4π units^2]

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6. Show that the tangents at (-1,3) and (1,5)on the curve y = $2x^2 + x + 2$ passes through the origin. Find the area enclosed between the curve and these two tangents $\left[\frac{4}{3}\right]$

7. Sketch the curve $y = x - \frac{8}{x^2}$ for x > 0, showing any a symptotes. Find the area enclosed by the x-axis, the line x = 4 and the curve $x - \frac{8}{x^2}$. [10 sq. units]
   If this area is now rotated about the x-axis through 3600, determine the volume of the solid generated, correct to 3 significant figures. [42.1 cubic units]

8. Show that the tangents to the curve $4 - 2x - 2x^2$ at points 9-1, 4) and $\left(\frac{1}{2}, 2\frac{1}{2}\right)$respectively passes through the point $\left(-\frac{1}{4}, 5\frac{1}{2}\right)$. Calculate the area of the curve enclosed between the curve and the x-axis. [9sq.units]

9. (i) find the Cartesian equation of the curve given parametrically by
   x = $\frac{1+t}{1-t}$, y = $\frac{2t^2}{1-t}$ $\left[y = \frac{(x-1)^2}{x+1}\right]$
   (ii) sketch the curve
   (iii) find the area enclosed between the curve and the line y = 1 [1.955sq. units]

10. Given the curve y = sin3x, find the
    (a)(i) the value of $\frac{dy}{dx}$ at the point $\left(\frac{\pi}{2}, 0\right)$
    (ii) equation of the tangent to the curve at this point [y= 3x+π)

    (b) (i) sketch the curve y = sin3x

    (ii) Calculate the area bounded by the tangent in (a)(i) above, the curve and y-axis

    [0.9783sq. units]

11. A hemisphericalbowl of internal radius r is fixed with its rim horizontal and contains a liquid to the depth h. show by integration that the volume of the liquid in the bowl is $\frac{1}{3}\pi h^2(3r - h)$

12. Find the volume of the solid of revolution formed by rotating the area enclosed by the curve y = x^2(1+x), the x-axis, the lines x = 2 and x = 4 through four right anglesabout the x-axis. [31.033π cubic units]

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13. Curve sketching

Curve sketching (A-level)

The procedures of curve sketching depend on the nature of the curve to be sketched

Graphs of y = f(x) (Non rational functions)

For any graph of the form y = f(x) where f(x) is not linear, some or all the following steps are followed.

(a) Determine if the curve is symmetrical about either or both axes of coordinates.
- Symmetry about the x-axis occurs if the equation contains only even powers of y. here equation will be uncharged when (-y) is substituted for y. this applies to graphs of the type y^2=f(x)
- Symmetry about the y-axis occurs if the equation contains only even powers of x. Here the equation will be uncharged when (-x) is substituted for x. Here the graph is said to even i.e. f(x) = f(-x). For example the graph of y = x^2 [illegible] if there are odd powers of x and y then there will be no symmetry.

(b) Determine if there is symmetry about the origin. Here symmetry occurs when a change in the sign of x causes a change in the sign of y without altering its numerical value.

(c) Find the intercepts i.e. the curve cuts the x-axis at a point when y = 0 and cuts the y-axis at the point when x = 0.

(d) The curve passes through the origin if (x, y)= (0, 0)

The behaviour of the neighbour of the curve through the origin is studied by considering the ratio of $\frac{y}{x}$.

- If this ratio is small, the curve keeps close to x-axis near the origin.
- If the ratio is unity, the direction of the curve bisects the angle between the axes.
- If the ratio is large, the curve keeps near the y-axis.

Or:

We consider the behaviour of $\frac{dy}{dx}$ near the origin.

- If $\frac{dy}{dx}$ is very small, then the curve lies near the x-axis.
- If $\frac{dy}{dx}$ is large, then the curve lies near the y-axis.
- If $\frac{dy}{dx}$ is near unity, then the direction of the curve (tangent at origin) bisects the angle between the axes.

(e) Examine the behaviour of the function as x→±∞ and y →±∞ (if any)

(f) Find the turning points and their nature as well as points if inflexion (if any)
Use the second derivative
- For min point, $\frac{d^2y}{dx^2} = +ve$
- For max point, $\frac{d^2y}{dx^2} = -ve$
- Point of inflexion, $\frac{d^2y}{dx^2} = 0$

Example 1

(a) Sketch the graph of y= 5 + 4x – x^2.
Steps taken
- Finding intercepts
  x – intercept; y = 0
  0 = 5 + 4x – x^2.
  5 + 5x –x – x^2 = 0
  5(1 + x) – x(1 + x) =0
  (5 – x)(1 + x)

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Either 5 – x = 0; x = 5

Or 1 + x = 0; x = -1

Hence the curve cuts the x-axis at point (-1, 0) and (0, 5)

y – intercept, when x = 0, y = 5

hence the curve cuts the y-axis at point (0, 5)

- As x→+∞, y →-∞ and x→-∞, y →+∞
- Finding turning point

$\frac{dy}{dx} = 4 - 2x$

At turning point $\frac{dy}{dx} = 0$

2x – 4 = 0; x = 2

When x = 2; y= 5 + 4(2) – $(2)^2$= 9

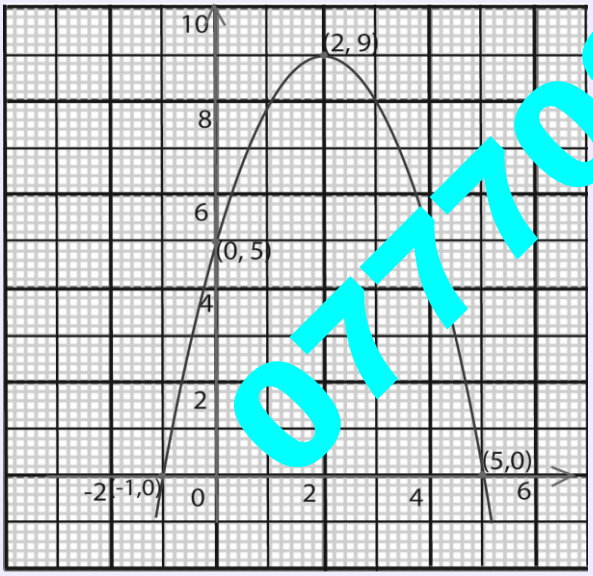
Hence turning point = (2, 9)

Finding the nature of turning point

$\frac{dy}{dx} = 4 - 2x$

$\frac{d^2y}{dx^2} = -2$

Since $\frac{d^2y}{dx^2} < 0$, hence the turning point is maximum.



(b) Sketch the curve y = $x^3 - x^2 - 5x + 6$

Steps taken

For y – intercept; x = 0, y = 0

Hence the y – intercept is (0, 6)

For x – intercept, y = 0

$x^3 - x^2 - 5x + 6 = 0$

error approach is used to find the first factor i.e. (x-2), then other factor is found by long division

$$
\begin{array}{r}
x^2 + x - 3 \\
(x-2)\overline{)x^3 - x^2 - 5x + 6} \\
-\ \underline{x^3 - 2x^2} \\
x^2 - 5x + 6 \\
-\ \underline{x^2 - 2x} \\
3x + 6 \\
-\ \underline{3x + 6} \\
0 + 0
\end{array}
$$

⇨ $x^3 - x^2 - 5x + 6 = (x-2)(x^2 + x - 3) = 0$

Solving $x^2 + x - 3 = 0$

$x = \frac{-1 \pm \sqrt{1+12}}{2} = \frac{-1 \pm 3.6}{2}$

x = 1.3, -2.6

Hence the x- intercepts are (2, 0), (1.3, 0) and (-2.3, 0)

Finding turning points

$y = x^3 - x^2 - 5x + 6$

$\frac{dy}{dx} = 3x^2$-2x-5

At turning point $\frac{dy}{dx} = 0$

$3x^2$-2x-5 = (3x – 5)(x+1) = 0

Either 3x -5 = 0 x = $\frac{5}{3}$

Or x + 1 = 0; x =-1

When x = $\frac{5}{3}$;

$y = \left(\frac{5}{3}\right)^3 - \left(\frac{5}{3}\right)^2 - 5\left(\frac{5}{3}\right) + 6 = \frac{-13}{27}$

When x = -1

$y = (-1)^3 - (-1)^2 - 5(-1) + 6 = 9$

Hence turning points are $\left(\frac{5}{3}; \frac{-13}{27}\right)$ and(-1, 9)

Finding the nature of turning points

$\frac{dy}{dx} = 3x^2$-2x-5

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$\frac{d^2y}{dx^2} = 6x - 2$

For$\left(\frac{5}{3}; \frac{-13}{27}\right)$

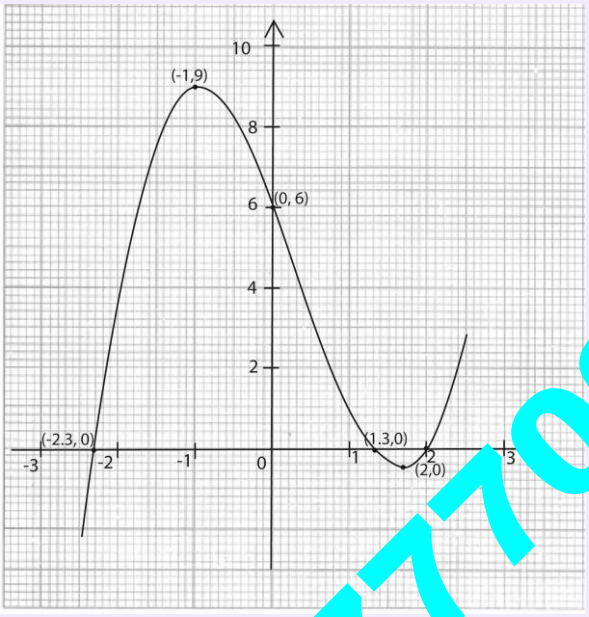
$\frac{d^2y}{dx^2} = 6\left(\frac{5}{3}\right) - 2 = 8\ (> 0)$

$\therefore \left(\frac{5}{3}; \frac{-13}{27}\right)$is minimum

For (-1, 9)

$\frac{d^2y}{dx^2} = 6(-1) - 2 = -8\ (< 0)$

$\therefore (-1; 9)$is maximum



(c) sketch the curve y = x^3 –

y = $x^3 - 8$

Intercepts

When x = 0, y = -8

When y = 0, x = 2

(x, y) = (2, 0)

Turning point: $\frac{dy}{dx} = 3x^2$

$3x^2 = 0$

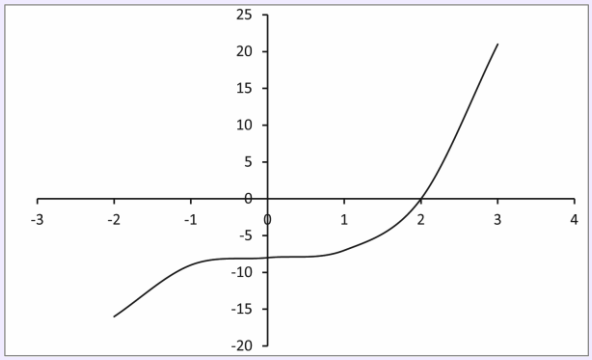
x = 0

$\frac{d^2y}{dx^2} = 6x$

$\frac{d^2y}{dx^2} = 0, x = 0$

Point of reflection= (0, 8)

	x < 2	x > 2
y	-	+



(d) Sketch the curve $y = x^2(x - 4)$

Steps taken

- Finding the intercepts

y – intercept, (0,0)

hence y – intercept is (0, 0)

For x – intercept y = 0

=>$x^2(x - 4) = 0$

Either x = 0 or x = 4

Hence x intercept are (0, 0) and (4, 0)

As x→+∞, y →+∞ and x→-∞, y →-∞

- Finding turning point(s)

$y = x^2(x - 4) = x^3 - 4x^2$

$\frac{dy}{dx} = 3x^2 - 8x$

At turning point, $\frac{dy}{dx} = 0$

⇨ $3x^2 - 8x = x(3x - 8) = 0$

Either x = 0

Or x = $\frac{8}{3}$

When x = 0; y=0

When x =$\frac{8}{3}$; =$3\left(\frac{8}{3}\right)^2 - 8\left(\frac{8}{3}\right) = \frac{-256}{27}$

Hence turning points are (0,0) and$\left(\frac{8}{3}, \frac{-256}{27}\right)$

- Finding the nature of turning points

$\frac{dy}{dx} = 3x^2 - 8x$

$\frac{d^2y}{dx^2} = 6x - 8$

For (0, 0); $\frac{d^2y}{dx^2} = 6(0) - 8 = -8\ (< 0)$

Hence (0, 0) is maximum

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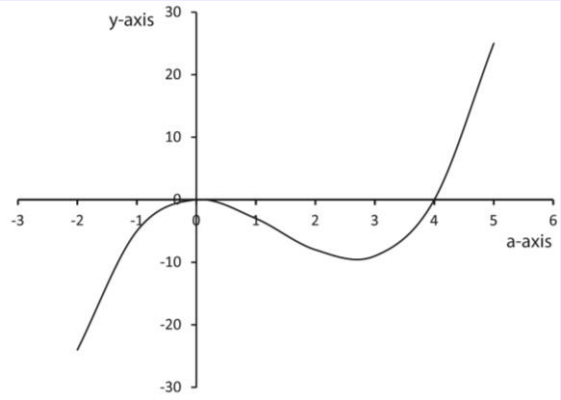
For $\left(\frac{8}{3}, \frac{-256}{27}\right); \frac{d^2y}{dx^2} = 6\left(\frac{8}{3}\right) - 8 = 8\ (> 0)$

Hence $\frac{8}{3}$ is minimum



Graphs of rational functions

Rational functions are fractions expressed in the form $y = \frac{f(x)}{g(x)}$.

The basic principles followed when sketching rational curves

(a) Determine if the curve is symmetrical about either or both axes of coordinates.

(b) Find the intercepts on both axes.

(c) Examine the behaviour of the curve as it tends to infinity.

(d) Find the turning points and their nature

(e) Determine the possible asymptotes of the curve

- Vertical asymptote is the value of x which make(s)y tend to infinity. Here we equate the denominator of the function to zero
- Horizontal asymptote is the value of x which make(s)x tend to infinity. Here we divide terms of the numerator and denominators by x with the highest power.
  Alternatively; when finding the horizontal asymptote, we re-arrange the equation and solve for x or make x the subject and then observe the limits, i.e. x→∞, see how y behaves
- Slating asymptotes; this only occurs if horizontal asymptote does not exist and the frictions is improper. Here we divide the terms of the numerator by those of the denominator an y equated to the quotient becomes the asymptote, i.e. asymptote is the y-quotient.

(f) Determine the region where the curve exists/does not exist. This is done by finding a quadratic equation in x such that for real values of x; $b^2>4ac$

Example 2

(a) Sketch the graph of $y = \frac{(x-1)(x+2)}{(x-2)(x+1)}$

Solution

Steps taken

- Finding intercepts

  For y-intercept: x= 0, $y = \frac{(-1)(2)}{(-2)(+1)} = 1$

  Hence the y-intercept = (0, 1)

  For x-intercept y= 0, $\frac{(x-1)(x+2)}{(x-2)(x+1)} = 0$;

  x = 1 or x = -2

  Hence the x-intercept are (1, 0) and (-2, 0)
- Finding turning points

  $y = \frac{(-1)(2)}{(-2)(+1)} = \frac{x^2+x-2}{x^2-x-2}$

  $\frac{dy}{dx} = \frac{(x^2-x-2)(2x+1)-(x^2+x-2)(2x-1)}{(x^2-x-2)^2}$

  At turning point, $\frac{dy}{dx} = 0$

  ⇨ $\frac{(x^2-x-2)(2x+1)-(x^2+x-2)(2x-1)}{(x^2-x-2)^2} = 0$

  $(2x^3 - x^2 - 5x - 2)$

  $-(2x^3 + x2 - 5x + 2) = 0$

  $2x^2 + 4 = 0$

  $x^2 + 2 = 0$

  There is no real value of x, hence there is no turning points.
- Finding asymptotes;

  Vertical asymptote

  $(x - 2)(x + 1) = 0$

  Either (x – 2) = 0; x = 2

  Or (x + 1) = 0; x = -1

  Hence the vertical asymptotes are x = 2 and x = 1

  Horizontal asymptotes

  $y = \frac{x^2+x-2}{x^2-x-2}$

  Dividing terms on the LHS by x^2

  $y = \frac{1+\frac{1}{x}-\frac{2}{x^2}}{1-\frac{1}{x}-\frac{2}{x^2}}$

  As x →∞, y→1

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Hence the horizontal asymptote y = 1

- Finding the regions where the curve does not exist

$y = \frac{x^2+x-2}{x^2-x-2}$

$y(x^2 - x - 2) = x^2 + x - 2$

$y(x^2 - x - 2) - x^2 - x + 2 = 0$

$(y-1)x^2 + (-y-1)x + (2-2y) = 0$

For real value of x, $b^2 > 4ac$

⇨ $(-y-1)^2 > 8(y-1)(1-y)$

$(y+1)^2 + 8(y-1)^2 > 0$

$9y^2 - 6y + 9 > 0$

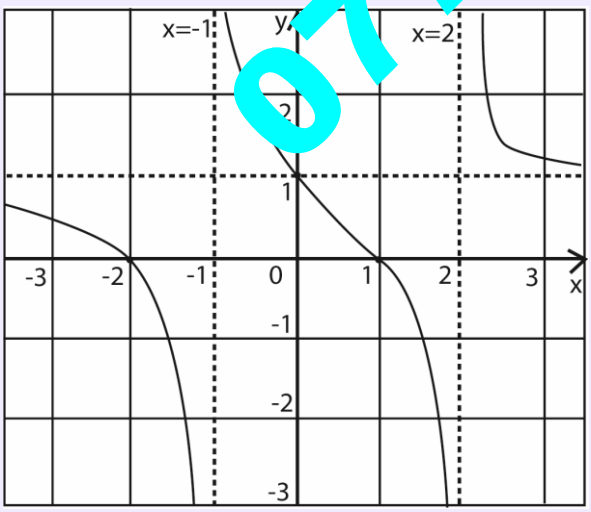
There is no real value of y which means that there is no restriction on y

- Determining the sign of the function throughout its domain. The function will only change sign where the curve cuts the x-axis and vertical asymptotes

The critical value are -, -1, 1, 2

	X<-2	-2<x<-1	-1<x<1	1<x<2	x>2
x-1	-	-	-	+	+
x+2	-	+	+	+	+
x-2	-	-	-	-	[illegible]
X+1	-	-	+	+	+
y	+	-	+	-	[illegible]

Graph of $y = \frac{(x-1)(x+2)}{(x-2)(x+1)}$



(b) Sketch the graph

Sketch the graph of $y = \frac{x(x-2)}{x+1}$

Steps taken

- Finding the intercepts

For y – intercept; x = 0, and y =0

Hence the y-intercept is (0, 0)

For x- intercept y= 0

⇨ $\frac{x(x-2)}{x+1} = 0$

$x(x-2) = 0$

Either x = 0

Or (x-2) = 0; x = 2

Hence the x-intercepts are(0, 0) and (2, 0)

- Finding turning points

$y = \frac{x^2-2x}{x+1}$

$\frac{dy}{dx} = \frac{x^2-2x(1)-(x+1)(2x-2)}{(x+1)^2} = \frac{x^2+2x-2}{(x+1)^2}$

At turning points, $\frac{dy}{dx} = 0$

⇨ $x^2 + 2x - 2 = 0$

$x = \frac{-2\pm\sqrt{2^2-(4x1x[illegible]2)}}{[illegible]}$

$x = \frac{-2\pm\sqrt{12}}{2} = [illegible] \pm 1.732$

Wher[illegible] $-1 + 1.732 = 0.732$

$y = \frac{0.732(0.732-2)}{0.732+1} = -0.54$

W[illegible] $x = -1 - 1.732 = -2.732$

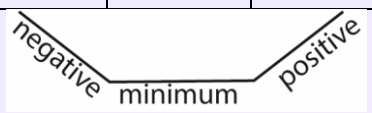
$y = \frac{-2.732(-2.732-2)}{-2.732+1} = -7.46$

Hence the turning points are (0.73, -0.54) and (-2.73, -7.46)

- Finding the nature of the turning points

For (0.73, -0.54)

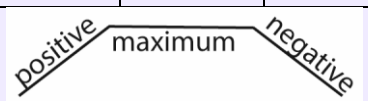
x	O	0.73	3
$\frac{dy}{dx}$	-2	0	0.25



Hence the turning point (0.73, -0.54) is minimum

For (-2.73, -7.46)

x	-3	-2.73	-2
$\frac{dy}{dx}$	0.25	0	-2



Hence the turning point (-2.73, -7.46) is maximum

- Finding asymptotes

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For vertical asymptote, the denominator = 0

⇨ x+ 1 = 0; x = -1

since the function is improper fraction, there must be slating asymptote.

Dividing the numerator by denominator;

$$\begin{array}{r} x-3 \\ x+1\overline{)\,x^2-2x} \\ -\ x^2+x \\ \hline -\ 3x-3 \\ \hline 3 \end{array}$$

The slanting asymptote is y = x – 3

X	0	3
y	-3	0

- Finding the region where the curve does not exist.

$y = \frac{x^2-2x}{x+1}$

$y(x+1) = x^2 - 2x$

$x^2 - (2+y)x - y = 0$

For real values of x, $b^2 \geq 4ac$

$(2+y)^2 > 4x\ 1\ x(-y)$

$y^2 + 8y + 4 > 0$

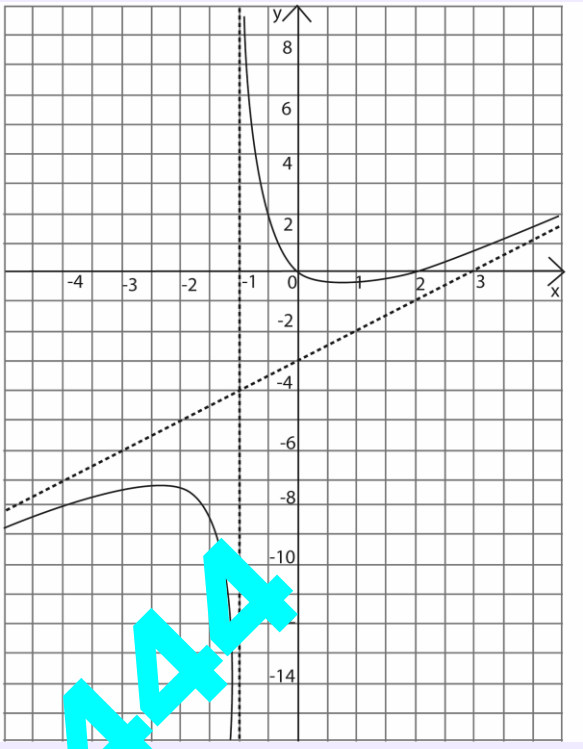
The inequality cannot be factorized therefore we may not proceed further even though there is no real value of y

- Determining the sign of the function through it domain

The critical values are -1, 0, 2

	X<-1	-1<x<0	0<x<2	x>2
x(x-1)	+	+	-	+
x+1	-	+	+	+
y	-	-	-	+

Graph of $y = \frac{x(x-1)}{x+1}$



(c) Given the curve $y = \frac{x(x-1)}{(x-2)(x+1)}$

- Finding intercept

For y-intercept x= 0; y= 0

Hence y-intercept = (0, 0)

For x-intercept y = 0

⇨ x(x-1) = 0

Either x = 0

Or x -1 =0; x = -1

Hence x-intercept as are (0, 0) and (1, 0)

- Finding turning points

$y = \frac{x^2-x}{x^2-x-2}$

$\frac{dy}{dx} = \frac{(x^2-x-2)(2x-1)-(x^2-x)(2x-1)}{(x^2-x-2)^2}$

At turning point $\frac{dy}{dx} = 0$

$(2x-1)\{(x^2-x-2)-(x^2-x)\} = 0$

(2x – 1)(-2) = 0

⇨ 2x – 1 = 0; x = $\frac{1}{2}$

When x = $\frac{1}{2}$, $y = \frac{\left(\frac{1}{2}\right)^2 - \frac{1}{2}}{\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right) - 2} = \frac{1}{9}$

Hence turning point is $\left(\frac{1}{2}, \frac{1}{9}\right)$

- Determining nature of turning point

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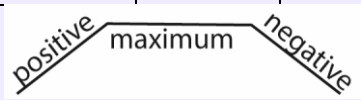
x	0	$\frac{1}{2}$	1
$\frac{dy}{dx}$	2	0	-6



Hence the turning point $\left(\frac{1}{2}, \frac{1}{9}\right)$ is maximum.

- Finding the asymptote(s)

For vertical asymptote

(x – 2)(x + 1) = 0

Either (x – 2) = 0; x = 2

Or (x + 1) =0; x = -1

For horizontal asymptote

Dividing the numerator and denominator on the RHS by x^2.

$$y = \frac{1+\frac{1}{x}}{1-\frac{1}{x}-\frac{2}{x^2}}$$

As x → ∞, y→1

- Finding the region where x does not exist

$$y = \frac{x(x-1)}{(x-2)(x+1)}$$

$$y(x-2)(x+1) = x(x-1)$$

$$y(x^2 - x - 2) = x^2 - x$$

$$yx^2 - yx - 2y - x^2 + x = 0$$

$$(y-1)x^2 + (1-y)x - 2y = 0$$

For real values of x, $b^2 \geq 4ac$

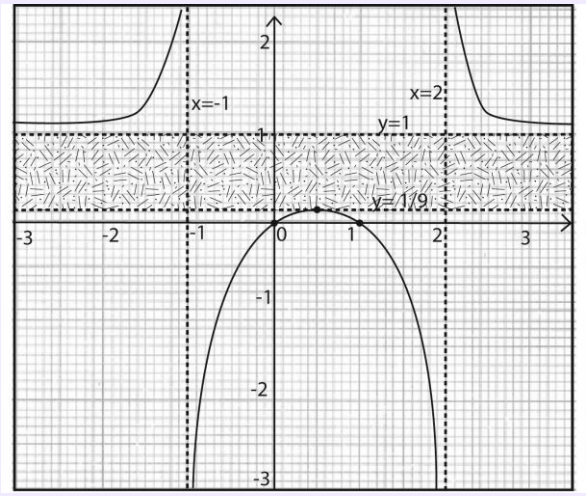
$$(1-y)^2 > -8y(y-1)$$

$$1 - 2y + y^2 + 8y(y-1) > 0$$

(9y – 1)(y – 1) = 0

	$y<\frac{1}{9}$	$\frac{1}{9} < y < 1$	Y > 1
9y – 1	-	+	+
y – 1	-	-	+
(9y – 1)(y – 1)	+	-	+

Hence the curve does not lie in the range $\frac{1}{9} < y < 1$



(d) Sketch the curve $y = \frac{x^2+4x+3}{x+2}$

- Finding the range of values over which the curve does not exist

$$y = \frac{x^2+4x+3}{x+2}$$

$$y(x+2) = x^2 + 4x + 3$$

$$x^2 + (4-y)x + (3-2y) = 0$$

For real values of x, $b^2 \geq 4ac$

$$(4-y)^2 > 4(3-2y)$$

$$16 - 8y + y^2 - 12 + 8y \geq 0$$

$$y^2 + 4 \geq 0$$

Since there are no real values of y, this means that there is no restriction on y.

- Finding intercepts

For y intercept, x= 0; $y = \frac{3}{2}$

Hence y- intercept is $\left(0, \frac{3}{2}\right)$

For x – intercepts y = 0

⇨ $\frac{x^2+4x+3}{x+2} = 0$

$$x^2 + 4x + 3 = 0$$

(x + 3)(x+ 1) = 0

Either (x + 3) = 0; x= -3

Or (x+1) = 0, x = -1

Hence x-intercepts are (-1, 0) and (-3, 0)

- Finding turning points

$$y = \frac{x^2+4x+3}{x+2}$$

$$\frac{dy}{dx} = \frac{(x+2)(2x+4)-(x^2+4x+3)(1)}{(x+2)^2}$$

At turning points, $\frac{dy}{dx} = 0$

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$x^2 + 4x + 5x = 0$

$x = \frac{-4 \pm \sqrt{4^2 - 4(1)(5)}}{2(1)}$

$x = \frac{-4 \pm \sqrt{-4}}{2}$

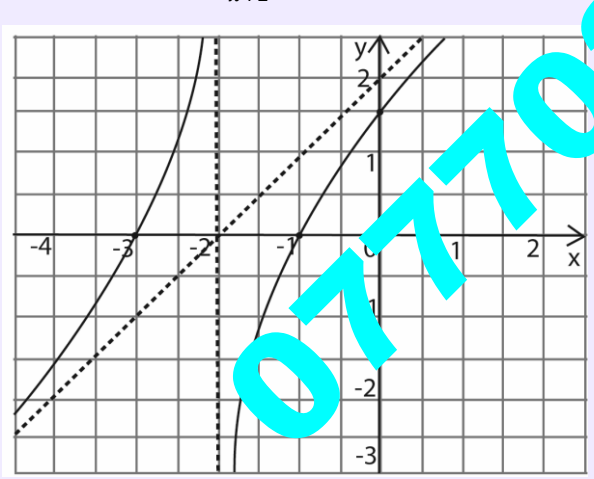
Since there is no real value of x, this means that the curve has no turning point

- Finding vertical asymptote
  (x +2) = 0
  x = -2
  Since the function is improper fraction, there could be a slating asymptote

```
          x + 2
(x+2)) x² + 4x + 3
       - x² + 2x
          2x + 3
        - 2x + 4
             -1
```

Hence the slanting asymptote is y =x + 2

A curve $y = \frac{x^2+4x+3}{x+2}$



(e) A curve is given by $y = \frac{(x-1)}{(2x-1)(x+1)}$

(i) Show that for real values of x, y cannot take on values in the interval $\left(\frac{1}{9}, 1\right)$

$y = \frac{(x-1)}{(2x-1)(x+1)}$

$y(2x-1)(x+1) = x - 1$

$y(2x^2 + x - y) = x - 1$

$2yx^2 + (y-1)x + (1-y) = 0$

For real values of x, $b^2 \geq 0$

$(y-1)^2 \geq 8y(1-y)$

$(y-1)^2 + 8y(y-1) \geq 0$

(9y – 1)(y – 1) = 0

	$y < \frac{1}{9}$	$\frac{1}{9} < y < 1$	Y > 1
9y – 1	-	+	+
y – 1	-	-	+
(9y – 1)(y – 1)	+	-	+

Hence the curve does not lie in the range $\frac{1}{9} < y < 1$

(ii) Determine the turning points of the curve

$y = \frac{(x-1)}{(2x-1)(x+1)} = \frac{(x-1)}{2x^2+x-1}$

$\frac{dy}{dx} = \frac{(2x^2+x-1)(1)-(x-1)(4x+1)}{(2x^2+x-1)^2}$

$= \frac{-2x^2+4}{(2x^2+x-1)^2}$

At turning point $\frac{dy}{dx} = 0$

⇨ $-2x^2 + 4x = 0$

$-2x(x-2) = 0$

Either 2x = 0; x= 0

Or (x - 2) = 0; x= 2

When x = 0; $y = \frac{-1}{-1} = 1$ => (x, y) =(0, 1)

When x = 2, $y = \frac{1}{3x3} = \frac{1}{9}$=> (x, y) =$\left(2, \frac{1}{9}\right)$

Determining the nature of turning points

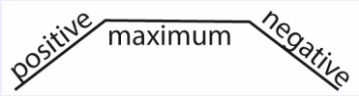
For (0,1)

x	-0.5	0	0.5
$\frac{dy}{dx}$	-2.5	0	1.5

Hence (0, 1) is minimum

For$\left(2, \frac{1}{9}\right)$

x	1	2	3
$\frac{dy}{dx}$	+0.025	0	-0.074



Hence $\left(2, \frac{1}{9}\right)$is maximum

(iii) State with reasons the asymptotes of the curve

For vertical asymptote

(2x – 1)(x + 1) = 0

Either 2x – 1 =0; x=$\frac{1}{2}$

Or (x + 1) = 0; x = -1

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For horizontal asymptotes

Dividing the numerator and denominator on the RHS by x

$$y = \frac{1-\frac{1}{x}}{2x-1-\frac{1}{x}}$$

As x→ ∞, y → 0

Hence horizontal asymptote is y = 0

(iv) Sketch the curve

Finding intercepts

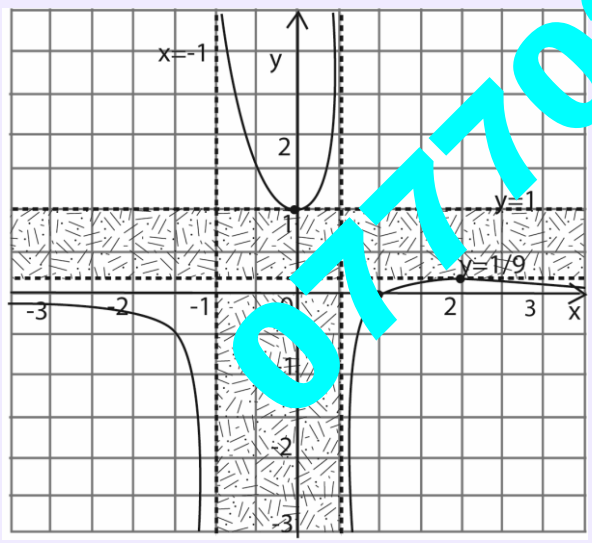
For y-intercept x = 0; y = 1

Hence y-intercept (0,1)

For x-intercept y = 0; x= 1

Hence x – intercept is(1, 0)

A graph of $y = \frac{(x-1)}{(2x-1)(x+1)}$



Sketching graphs of parametric equations

It requires eliminating parameters in the equations given and then following similar steps in above examples.

Example 3

(a) A curve is given by parametric equations

x = t + 2 and y = $\frac{t^2-t}{t+1}$

(a) Find the Cartesian equation of the curve

Solution

x = t – 2; t = x + 2

Substituting t into the equation

$$y = \frac{(x-2)^2-(x-2)}{(x-2)+1} = \frac{(x-2)(x-3)}{(x-1)}$$

Hence Cartesian equation is $y = \frac{(x-2)(x-3)}{(x-1)}$

(b) Sketch the curve

$$y = \frac{(x-2)(x-3)}{(x-1)}$$

- Finding intercepts

For y-intercept, x = 0, y = -6

For x-intercept y = 0

⇨ $(x-2)(x-3) = 0$

Either x – 2 = 0; x = 2

Or (x – 3) = 0; x = 3

Hence x- intercepts are (2, 0) and (3, 0)

- Finding the turning points

$$y = \frac{x^2-5x+6}{x-1}$$

$$\frac{dy}{dx} = \frac{(2x-5)-(x^2-5x+6)(1)}{(x-1)^2} = \frac{x^2-2x-1}{(x-1)^2}$$

At turning point $\frac{dy}{dx} = 0$

$$x^2 - 2x - 1 = 0$$

$$x = \frac{2\pm\sqrt{(-2)^2-4(1)(-1)}}{2(1)}$$

$$x = \frac{2\pm\sqrt{8}}{2} = \frac{2\pm2\sqrt{2}}{2} = 1 \pm \sqrt{2}$$

Either x = $1 = \sqrt{2}$= 2.4

Or x = $1 - \sqrt{2}$= -0.4

When x = 2.4

$$y = \frac{(2.4-2)(2.4-3)}{(2.4-1)} = -0.17$$

When x = -0.4

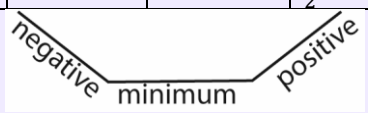
$$y = \frac{(-0.4-2)(-0.4-3)}{(-0.4-1)} = -5.83$$

Hence the turning points are (2.4, -0.17) and (-0.4, - 5.83)

- Finding the nature of turning points

For (2.4, -0.17)

X	2	2.4	3
$\frac{dy}{dx}$	-1	0	$\frac{1}{2}$



Hence the turning point (2.4, -0.17) is minimum.

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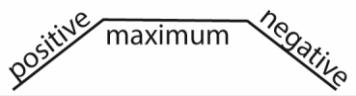
For (-0.4, -5.83)

X	-1	-0.4	0
$\frac{dy}{dx}$	$\frac{1}{2}$	0	-1



Hence the turning point (-0.4, -5.3) is maximum

- Finding asymptotes

For vertical asymptotes

x – 1 = 0, x= 1

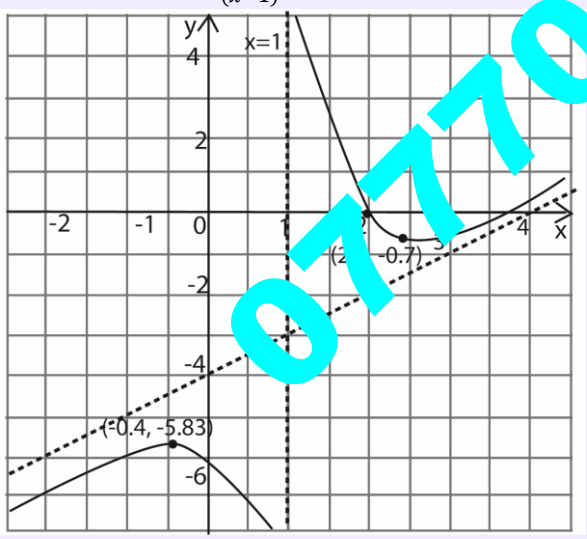
For slanting asymptote

$$\begin{array}{r|l} & x - 4 \\ \hline (x-1) & x^2 - 5x + 6 \\ & -x^2 - x \\ \hline & -4x + 6 \\ & -\ -4x + 4 \\ \hline & 2 \end{array}$$

Hence the slanting asymptote is y = x – 4

x	0	4
y	-4	0

Graph $y = \frac{(x-2)(x-3)}{(x-1)}$



(c) A curve is given by parametric equations

x = cos2θ and y =2sinθ.

(i) Find the equation of the normal to the curve at θ=$\frac{5\pi}{6}$

x = cos2θ

$\frac{dx}{d\theta} = -2sin2\theta$

y =2sinθ.

$\frac{dy}{d\theta} = 2cos\theta$

$\frac{dy}{dx} = \frac{dy}{d\theta}.\frac{d\theta}{dx} = \frac{-2cos\theta}{2sin2\theta} = -\frac{cos\theta}{sin2\theta}$

At θ=$\frac{5\pi}{6}$

$\frac{dy}{dx} = \frac{cos\left(\frac{5\pi}{6}\right)}{sin\left(\frac{5\pi}{3}\right)} = -1$

Gradient of the normal =$\frac{-1}{-1} = 1$

x = cos2θ

$x = cos\frac{5\pi}{3} = \frac{1}{2}$

$y = 2sin\frac{5\pi}{6} = 1$

Let a point (x, y) lie on the normal

⇨ $\frac{y-1}{x-\frac{1}{2}} = 1$

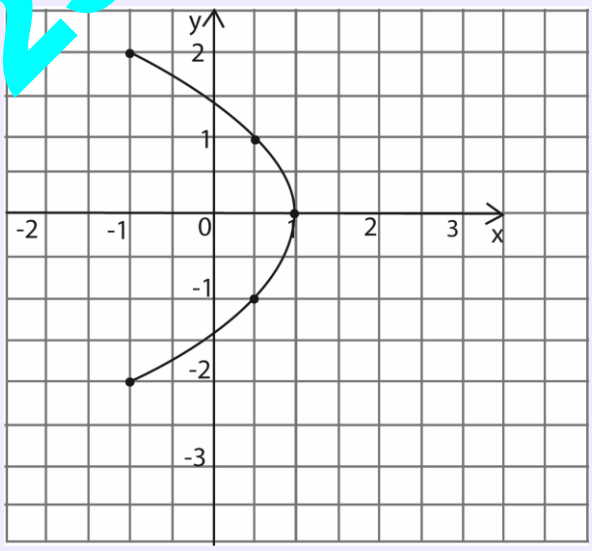
$y = x + \frac{1}{2}$

Hence the equation of the normal to the curve at $\theta = \frac{5\pi}{6}$ is $y = x + \frac{1}{2}$

(ii) Sketch the curve $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

θ	$\frac{\pi}{2}$	[illegible]	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
x=cos2θ	[illegible]	0.5	0.5	1	0.5	-0.5	-1
y=2sinθ	[illegible]	1.73	-1	0	1.73	1.73	2

A graph x = $1 - \frac{y^2}{2}$



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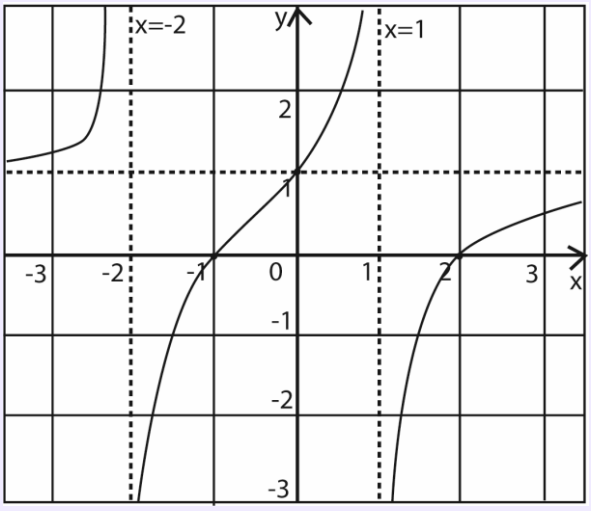
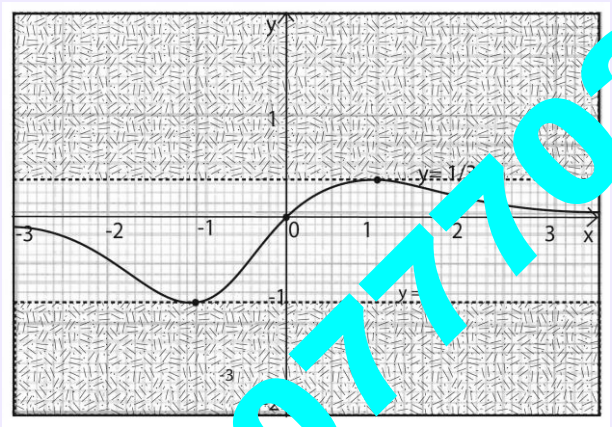
Revision question 1

1. Sketch the graph $y = \frac{(x-2)(x+1)}{(x-1)(x+2)}$

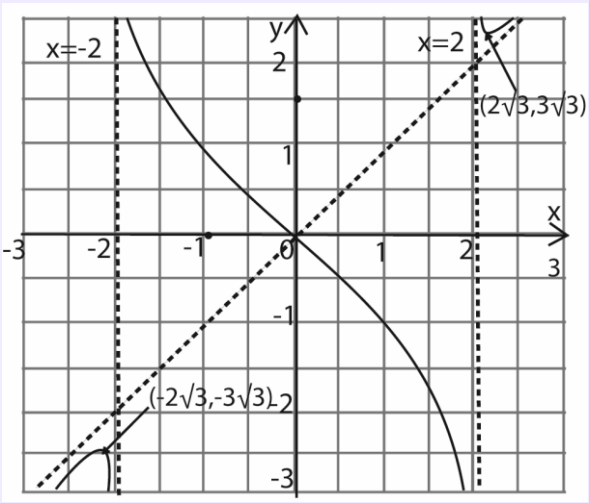
   A graph $y = \frac{(x-2)(x+1)}{(x-1)(x+2)}$



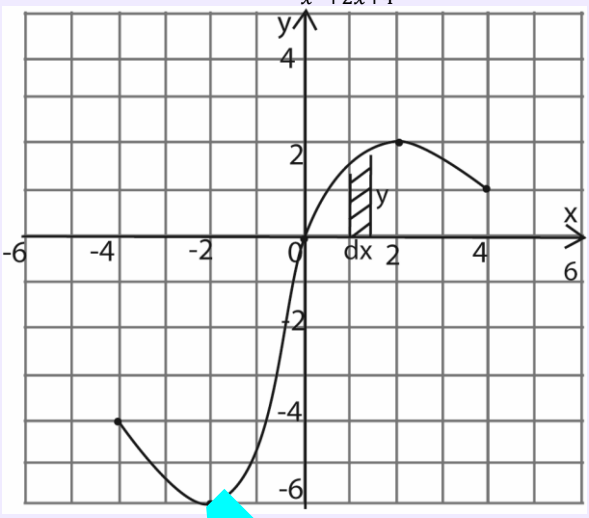
2. Sketch the graph $y = \frac{x}{x^2+x+1}$



3. Sketch the curve $y = \frac{x^2}{x^2-4}$



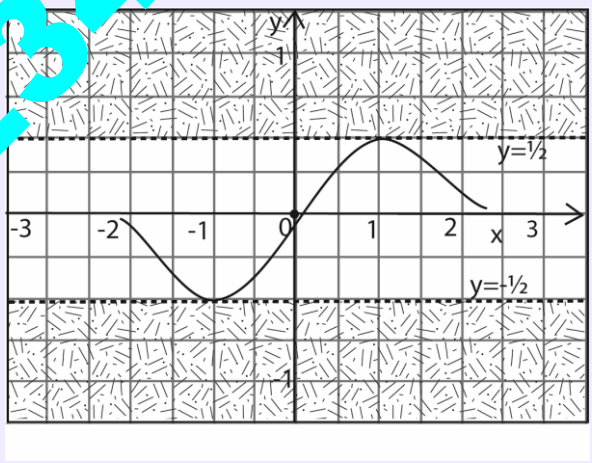
4. (a) Sketch the curve $y = \frac{12}{x^2+2x+4}$



   (b) Find the area [illegible]ed by the curve, x-axis and 0≤ [illegible]<4 [(0 [illegible] to3dp)

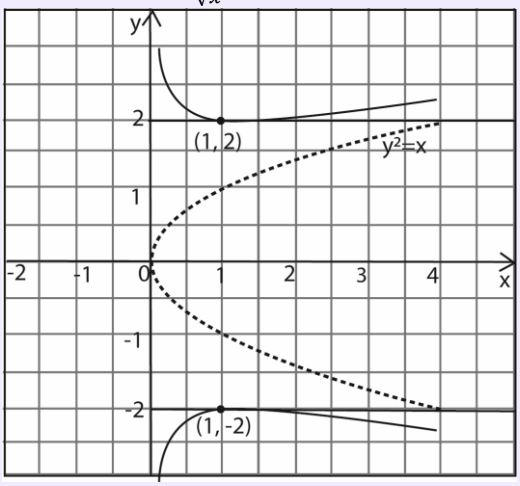
5. Determine [illegible]e [illegible]tionary points (including point[illegible] inf[illegible]ion) of the curve $y = \frac{x}{x^2+1}$.

   Sketch [illegible] [illegible]rveSketch



6. Sketch the curve give by the following parametric equations x= t^2 and $y = t + \frac{1}{t}$.

   A graph of $y = \frac{x+1}{\sqrt{x}}$



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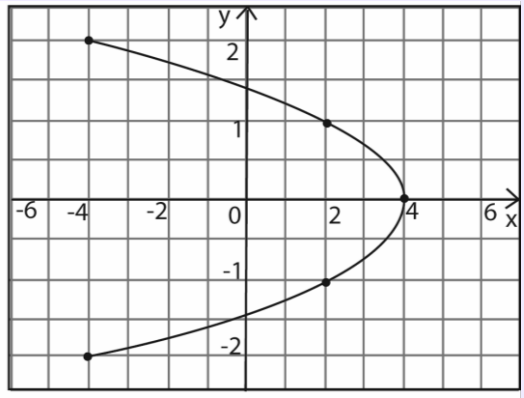
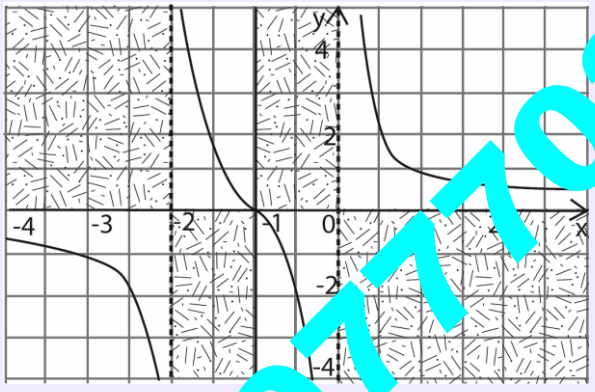
7. A curve is given by the parametric equations x= 4cos2t and y = 2sint
   (i) Find the equation of the normal to the curve at t = $\frac{5\pi}{6}$ [y = 4x – 7]
   (ii) Sketch the curve for $-\frac{\pi}{2} < t < \frac{\pi}{2}$

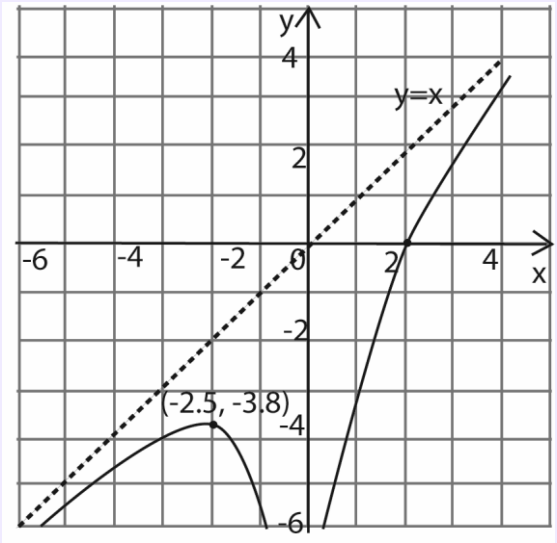


   (iii) Find the area enclosed by the curve and the y-axis [7.543 units (3d,p)]

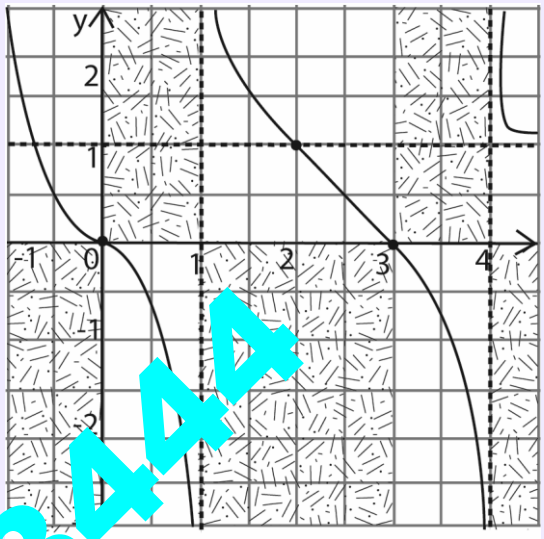
8. Sketch the curve $y = \frac{x+1}{x^2+2x}$



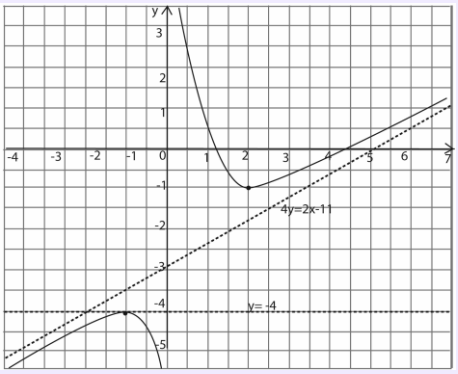
9. Sketch the curve $y = x - \frac{8}{x^2}$



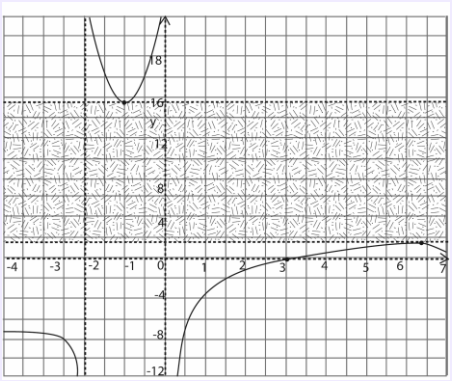
10. Given the curve $y = \frac{x(x-3)}{(x-1)(x-4)}$
    (i) Show that the curve does not have turning points $\left[\frac{dy}{dx} = 0; has\ no\ roots\right)$
    (ii) Find the equations of asymptotes. Hence sketch the graph



11. Determining the nature of the turning points of the curve $y = \frac{x^2-6x+5}{2x+1}$, sketch the graph of the curve for x= -2 to x=7. Show any asymptotes.



12. Sketch the curve $y = \frac{4(x-3)}{x(x+2)}$



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14. Approximations

Approximations

The Maclaurin's Theorem

The polynomial of Maclaurin's series of any infinitely differentiable function, f(x)m whose value and all values of all its derivatives, exist at x = 0

$$f(x) = f(0)x + \frac{f\prime(0)}{2!}x^2 + \frac{f\prime\prime(0)}{3!}x^3 + \cdots$$

Maclaurin's series of sinx

Let $f(x) = sinx \Rightarrow f(0) = \sin(0) = 0$

$f'(x) = cosx \Rightarrow f'(0) = cos(0) = 1$

$f''(x) = -sinx \Rightarrow f''(0) - sin(0) = 0$

$f'''(x) = -cosx \Rightarrow f'''(0) = -cos($ =

$f^{iv}(x) = sinx \Rightarrow f'''(0) = sin($) = –

Note that the fourth derivative takes us back to the starting point. So the values repeat in a cycle of four as 0, 1, 0, -1; 0, 1, 0, -1; etc.

By substitution, we have

$$sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

The Maclaurin's series of sin x is valid for all values of x.

Maclaurin series of cos x

$$cosx = \frac{d}{dx}(sinx)$$

$$= \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{3!} - \frac{x^6}{6!} + \cdots$$

The Maclaurin's series of cos x is valid for all values of x.

Maclaurin's series of e^x

Let f(x) = e^x => f(0) = e^0=1

f'(x) = e^x => f'(0) = e^0=1

f''(x) = e^x => f''(0) = e^0=1

f'''(x) = e^x => f'''(0) = e^0= etc.

Here we see that the function and all its derivatives are the same, so these values repeat themselves infinitely at 1, 1, 1,1, etc. by substitution we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots$$

The Maclaurin's series of e^x is valid for all values of x

Maclaurin series of Inx

Let f(x) = Inx => f(0) =In(0) = ?

f'(x) =$\frac{1}{x}$ => f'(0) = $\frac{1}{0}$=?

f''(x) = $-\frac{1}{x^2}$ => f''(0) = $\frac{1}{0^2}$=?

Here we notice that neither the function nor any of the derivatives exist as x= 0, so there is no polynomial Maclaurin's expansion of natural logarithm, Inx.

Maclaurin series of In(1+x)

Let f(x) = In(1+x) => f(0) =In(1+0) = 0

f'(x) =$-\frac{1}{1+x}$ => f'(0) = $-\frac{1}{1+0}$= 1

f''(x) = $\frac{-1}{(1+x)^2}$ => f''(0) = $\frac{1}{(1+0)^2}$ =-1

f'''(x) = $\frac{2}{(1+x)^3}$ => f'''(0) = $\frac{2}{(1+0)^3}$ =2

f^{iv}(x) = $-\frac{3x2}{(1+x)^4}$ => f^{iv} (0) = $\frac{-3x2}{(1+0)^4}$ =-3x2 etc

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by substitution we have

$$In(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$$

The Maclaurin's series of In(1+x) is valid for -1< x ≤ 1

Summary

f(x)	Expansion	Validity
e^x	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$	for all x
e^{-x}	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$	for all x
sinx	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	for all x
cosx	$1 - \frac{x^2}{3!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	for all x
$tan^{-1}x$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	for -1< x ≤ 1
In(1+x)	$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	for -1< x ≤ 1
$(1+x)^k$	$1 + kx - \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$	for -1< x ≤ 1
$\frac{1}{1-x}$	1+ x + x^2 + x^3 + ……	for -1< x ≤ 1
$\frac{1}{(1-x)^2}$	1+ 2x + 3x^2 + 4x^3 + ……	for -1< x ≤ 1

Note the validity of **Maclaurin series** is arrived at by using ratio test theorem whose derivation is outside the scope of our coverage

Answering questions

The questions usually require to produce Maclaurin's series of a function, specifies nth term and then its application

Examples

1. Find the Maclaurin series for $(1+x)^{-1}$ as far as x^4.
   Deduce the Maclaurin series for
   (i) In(1+x)
   (ii) In(1-x)
   (iii) $In\left(\frac{1-x}{1+x}\right)$

   Solution

   Maclaurin expansion series is given by
   $$f(x) = f(0) + f'^{(0)}x + \frac{f''(0)}{2!} + \frac{f'''(0)}{3!} + \ldots$$

   Let f(x) = $(1+x)^{-1}$ => f(0)= $(1+0)^{-1}$ = 0

   f'(x) = $-1(1+x)^{-2}$ => f'(0) = $-1(1+0)^{-2}$=-1

   f''(x) = $2(1+x)^{-3}$ => f'(0) = $2(1+0)^{-3}$= 2

   f'''(x) = $-6(1+x)^{-4}$ => f'''(0) = $-6(1+0)^{-4}$= -6

   f^{iv}(x) = $14(1+x)^{-5}$ => f^{iv}(0) = $24(1+0)^{-5}$= 24

   By substitution we have

   $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4$

   (i) We know the $\int \frac{dx}{1+x}$=In(1+x)
   ⇨ In(1+x)=$\int(1 - x + x^2 - x^3 + x^4)dx$
   $=x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$

   This valid for-1 < x ≤ 1

   (ii) Replacing x by –x in (i)

   In(1-x) =$-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5}$

   This valid for-1 < x ≤ 1

   (iii) Subtracting (ii) from (i)
   In(1+x) – In(1-x)= $2x + \frac{2x^3}{3} + \frac{2x^5}{5}$

2. Find the Maclaurin series for (1+x)-1 as far as x^4.
   Deduce the Maclaurin series for

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(i) $\frac{1}{1+x^2}$ as far as x^6.

(ii) $tan^{-1}x$ as far as x^7.

Show that $tan^{-1}\left(\frac{1}{2}\right) + tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{4}$

Solution

Maclaurin expansion series is given by

$f(x) = f(0) + f'^{(0)}x + \frac{f''(0)}{2!} + \frac{f'''(0)}{3!} + \ldots$

Let $f(x) = (1+x)^{-1} \Rightarrow f(0) = (1+0)^{-1} = 0$

$f'(x) = -1(1+x)^{-2} \Rightarrow f'(0) = -1(1+0)^{-2} = -1$

$f''(x) = 2(1+x)^{-3} \Rightarrow f'(0) = 2(1+0)^{-3} = 2$

$f'''(x) = -6(1+x)^{-4} \Rightarrow f'''(0) = -6(1+0)^{-4} = -6$

$f^{iv}(x) = 14(1+x)^{-5} \Rightarrow f^{iv}(0) = 24(1+0)^{-5} = 24$

By substitution we have

$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4$

(i) Replacing x by x^2 gives

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6$$

(ii) We know that $\int \frac{dx}{1+x^2} = tan^{-1}x$

$\Rightarrow \quad tan^{-1}x = \int (1 - x^2 + x^4 - x^6)\,dx$

$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$

We also know that

$tan^{-1}A + tan^{-1}B = tan^{-1}\frac{A+B}{1-A.B}$

$\Rightarrow \quad tan^{-1}\left(\frac{1}{2}\right) + tan^{-1}\left(\frac{1}{3}\right) = tan^{-1}\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}.\frac{1}{3}}$

$= tan^{-1}(1) = \frac{\pi}{4}$

3. Use Maclaurin theorem to expand e^x up to the term x^4, use your expansion to evaluate e correct to 4 decimal places.

Let $f(x) = e^x \Rightarrow f(0) = e^0 = 1$

$f'(x) = e^x \Rightarrow f'(0) = e^0 = 1$

$f''(x) = e^x \Rightarrow f''(0) = e^0 = 1$

$f'''(x) = e^x \Rightarrow f'''(0) = e^0 = 1$ etc.

by substitution we have

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

Evaluating e

$e = e^1$, substituting for x = 1

$e^1 = 1 + (1) + \frac{(1)^2}{2!} + \frac{(1)^3}{3!} + \frac{(1)^4}{4!}$

$= 2 + \frac{1}{6} + \frac{1}{24} = \frac{65}{24} = 2.7083\ (4d.p)$

4. Expand $\sqrt{\left(\frac{1+2x}{1-x}\right)}$ up to the term x^2. Hence find the value of $\sqrt{\left(\frac{1.04}{0.98}\right)}$ to four significant figures. (12marks)

$$\sqrt{\left(\frac{1+2x}{1-x}\right)} = (1+2x)^{\frac{1}{2}}(1-x)^{\frac{-1}{2}}$$

Using $(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \cdots$

$\sqrt{\left(\frac{1+2x}{1-x}\right)} = \left(1 + x - \frac{x^2}{2}\right)\left(1 + \frac{1}{2}x + \frac{3}{8}x^2\right)$

$= 1 - \frac{x^2}{2} + \frac{3}{8}x^2 + x + \frac{1}{2}x^2 - \frac{1}{2}x^2$

$= 1 + \frac{3}{2}x + \frac{3}{8}x^2$

$\therefore \sqrt{\left(\frac{1+2x}{1-x}\right)} \approx 1 + \frac{3}{2}x + \frac{3}{8}x^2$

Substituting for x = 0.02

$\sqrt{\left(\frac{1.04}{0.98}\right)} = \sqrt{\frac{1+2(0.02)}{1-0.02}}$

$= 1 + \frac{3}{2}(0.02) + \frac{3}{8}(0.02)^2$

$= 1.030$

5. Obtain the first two non-zero terms of Maclaurin's series for sec x

$f(x) = f(0) + f'^{(0)}x + \frac{f''(0)}{2!} + \frac{f'''(0)}{3!} + \ldots$

$f(x) = \sec x \Rightarrow f(0) = sec0 = 1$

$f'(x) = secxtanx \Rightarrow f'(0) = sec0tan0 = 0$

$f''(x) = secxsec^2x + tanxsecxtanx$

$\Rightarrow \quad f''(0) = sec0sec^20 + tan0sec0tan0 = 1 + 0 = 1$

Hence the first two non-zero terms of Maclaurin series of $sec\ x = 1 + \frac{x^2}{2}$

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Revision exercise

1. Use Maclaurin theorem to expand the following up to
   (i) $In\left(\frac{1+x}{1-x}\right)$ up to x^3. Hence, find the approximation of In2 correct to 3 significant figure

   $\left[In\left(\frac{1+x}{1-x}\right) = 2x + \frac{2}{3}x^3; 0.691\right]$

   (ii) $e^{-x}sinx$
   $\left[x - x^2 + \frac{1}{3}x^3\right]$

   (iii) $In\sqrt{\left(\frac{1+sinx}{1-sinx}\right)}$ $\left[2x + \frac{x^3}{6}\right]$

   (iv) $In(1 + sinx)$ $\left[x - \frac{x^2}{2} + \frac{x^3}{6}\right]$

   (v) $In(1 + x)^2$ $\left[2x - x^2 + \frac{2x^3}{3} - \frac{x^4}{2}\right]$

   (vi) $\frac{1}{\sqrt{(1+x)}}$ $\left[1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16}\right]$

2. Given $y = tan^{-1}\sqrt{1-x}$ show that
   (i) $(2-x)\frac{dy}{dx} + \frac{1}{2\sqrt{(1-x)}} = 0$
   (ii) $(2-x)\frac{d^2y}{dx^2} - \frac{dy}{dx} + \frac{1}{4}(1-x)^{-\frac{1}{2}} = 0$

3. Use Maclaurin theorem to show that
   (i) $\frac{cosx}{1-x^2} = 1 + \frac{1}{2}x^2 + \frac{11}{24}x^4$
   (ii) $e^{-x}sinx = \frac{x}{3}(x^2 - 3x + 3)$. Hence evaluate $e^{-x}sin\frac{\pi}{3}$ to 4d.p [0.3334]

4. Given that $y = e^{tan^{-1}x}$, show that $(1+x^2)\frac{d^2y}{dx^2} + (2x-1)\frac{dy}{dx} = 0$. Hence or otherwise, determine the first four non-zero terms of the Maclaurin expansion of y
   $\left[1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right]$

5. Given that y = $In\left\{e^x\left(\frac{x-2}{x+2}\right)^{\frac{3}{4}}\right\}$, show that
   $\frac{dy}{dx} = \frac{x^2-1}{x^2-4}$

6. Use Maclaurin's theorem to express In(sinx+cosx) as a power series up to the term x^2.[$x - x^2$]

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(b) Evaluate $\int_0^{\pi/6} \sin x \sin 3x \, dx$ ***(4 marks)***

(c) (using the substitution x = 3 sin θ, evaluate

$\int_o^3 \sqrt{\left(\frac{3+x}{3-x}\right)} dx$ ***(6marks)***

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2002

SECTION A (40 *marks*)

*Attempt **all** questions in this section.*

1. Solve the equation 2cosθ - cosecθ = 0;
$0^o < \theta < 270^o$ ***(05 marks)***

2. The vertices of a triangle are P (2,-1,5), Q (7, 1,-3) and R (13,-2, 0). Show that ∠PQR = 90^0. Find the coordinates of S if PQRS is a rectangle ***(05 marks)***

3. Show that 2 + i is a root of the equation $2z^3 - 9z^2 + 14z - 5 = 0$.
hence find the other roots ***(06 marks)***

4. The points R(2,0) and P(3,0) lie on the x-axis and Q (0,y) lies on the y-axis.
The perpendicular from the origin to RQ meets PQ at point S(X, - Y)
Determine the locus of S in terms of X and Y
(05 marks)

5. If $y = \sqrt{(5x^2+3)}$,

show that $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 5$ ***(04 marks)***

6. Given that $\log_3 x = p$ and $\log_6 x = q$, show that

$\log_6 3 = \frac{q}{p-q}$

(05 marks)

7.Given that $y = \text{In}\left(1 - \frac{1}{u}\right)^{1/2}$,

$2u = \left(x + \frac{1}{x}\right)$ 'show that

$\frac{dy}{dx} = \frac{(x+1)}{(x-1)(x^2+1)}$

8. Evaluate $\int_0^1 \frac{x^3}{x^2+1} dx$

SECTION B (60 marks)

9. (a) The tenth value of an arithmetic progression (AP) is 29 and the fifteenth term is 44.
(i) Find the value of the common difference and the first term.
(ii)Hence find the first 60 terms ***(07 marks)***
(b)A cab le 10m long is divided into ten pieces whose lengths are in a geometrical progression. The length of the longest piece is 8 times the length of the shortest piece. Calculate to the nearest centimeter the length of the third piece. ***(5 marks)***

10. P is a variable point given by the parametric equations

$x = \frac{a}{2}(t + 1/t)$; $y = \frac{b}{2}(t - 1/t)$.

Show that the locus of P is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

State the asymptotes. Determine the coordinates of the points where the tangent from p meets the asymptotes.

11.(a) Find the equation of the perpendicular line from point

$\mathbf{A} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$ to the line $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$.

What is the distance from **A** to **r**?***(08 marks)***

(b) Find the angle contained between line OR and the x- y plane, where

$OR = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ ***(04 marks)***

12.(a) Show that

$\frac{\sin 3\theta \sin 6\theta + \sin\theta \sin 2\theta}{\sin 3\theta \cos 6\theta + \sin\theta \cos 2\theta} = \tan 5\theta$

(b) Express 4cosθ - 5sinθ in the form Rcos(θ + β), where R is a constant and β an acute angle.
(i) Determine the maximum value of the expression and the value of θ for which it occurs
(ii) Solve the equation
4cosθ - 5sinθ = 2.2, for $0^0 < \theta < 360^0$
(07marks)

13…(a) Find the equation whose roots are – 1 ± i , where, i = √- 1.
(b) Find the sum of the first 10 terms of the series 1+ 2i- 4 -8i + 16+ …, in the form a + bi , where a and b are constants and I = √- 1 ***(05 marks)***
(c) Prove by induction that
$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$
(05 marks)

14.(a) use $t = \tan\frac{\theta}{2}$ to evaluate

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$\int_0^{\pi/2} \frac{d\theta}{3-\cos\theta}$. ***(05 marks)***

(b) Integrate the following with respect to x:

(i) In x (ii) $x^2 \sin 2x$

(07 marks)

15. Given the curve $y = \frac{x(x-3)}{(x-1)(x-4)}$,

(i) show that the curve does not have turning points.

(ii) Find the equations of the asymptotes. Hence sketch the curve.***(12 marks)***

16. (i) The volume of a water reservoir is gene rated by rotating the curve $y = kx^2$ about the y- axis. Show that when the central depth of the water in the reservoir is h meters, the surface area a is proportional to *h* and the volume *v* is proportional to h^2

(06 marks)

(ii) If the rate of loss of water from the reservoir due to evaporation is λ**A** m^2 per day, obtain a differential equation for h after t days.

Hence deduce that the depth of water decreases at a constant rate.

(ii) Given that λ = ½, determine how it will take for the depth of water to decrease from 20m to 2 m

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SECTION A

1. Show that z = 1 is a root of the equation $z^3 - 5z^2 + 9z - 5 = 0$.

Hence solve the equation for the other roots. ***(5 marks)***

2. Given the position vectors OA = (3, -2, 5) and OB = (9,1,-1) find the position vector of point C , such that C divides AB internally in the ratio 5:-3.

(5 marks)

3. Solve the equation cos2θ +cosθ = 0:$0^0 \le 180^0$ ***(5 marks)***

4. Find $\int x \ln x dx$ x ***(5 marks)***

5. Solve for x in the equation

$\log_4 (6-x) = \log_2 x$ ***(5 marks)***

6. If $y = e^{-t}\cos(t + \beta)$,show that

$\frac{d^2y}{dt^2} + \frac{2dy}{dt} + 2y = 0$ ***(5 marks)***

7. The points A (2,1), P(α, β) and points B(1, 2) lie in the same plane. PA meets the a-axis at the point (h,o and PB meets the y-axis at the point (o, k).

Find h and k in terms of α and β ***(5 marks)***

8. Determine $\frac{d}{dx}\left[\ln \frac{x}{\sqrt{(1+x^2)}}\right]$ when $x = 2$

(5 marks)

SECTION B

9(a) Show that cot A + tan2A = cot A sec 2A. ***(4 marks)***

(b) Show that $\tan\theta = 3t - t^3 / 1-3t^2$, where t = tanθ. Hence or otherwise show that

$\tan\frac{\pi}{12} = 2-\sqrt{3}$

(8 marks)

10 (a) Given the inequalities y > x - 5 and $0 < {}^6/_x$, illustrate graphically by shading out the unwanted regions.

(6 marks)

(b) Solve the simultaneous equation the equation of the tangents at the point (-3, 3) to the ellipse

$\frac{x^2}{16} + \frac{y^2}{9} = 1$ ***(7 marks)***

15.(a) Find $\int_x$ ***(03 marks)*** ***(3 marks)***

(b) Use the substitution t = tan x to ***(03 marks)***

d $\int \frac{1}{1+\sin^2 x} dx$ ***(9marks)***

1. (a) Solve the differential equation $\frac{dR}{dt} = e^{2t} + t$, ,

given that R (0) = 3.

(b) The acceleration of a particle after time t seconds is given by a = 5 + $\cos {}^1/_2$ t . if initially the particle is moving at l ms^{-1} , find its velocity after 2π seconds and the distance it would have covered by then.

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2005

SECTION A (40 Marks)

Attempt all questions in this section

1. Given the complex number

$z = \frac{(3i+1)(i-2)^2}{i-3}$ determine

(i) z in the form a +ib, where a and b are constant

(ii) arg (z) ***(5 marks)***

2. Find $\int_0^{\sqrt{\frac{\pi}{2}}} 2x\cos(x^2)dx$ ***(5marks)***

3. Solve the inequality $(0.6)^{-2x} < 3.6$, correct to **2** decimal places ***(5 marks)***

4. Given that the vectors a**i** – 2**j** + **k**

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and a**j** – 4**k** are perpendicular, find the values of a
(5 marks)

5. A spherical balloon is inflated such that the rate at which its radius is increasing is $0.5cm\ s^{-1}$.Find the rate at which
(i) the volume is increasing at the instant r = 5.0 cm
(ii) its surface area is increasing when r = 8.5 cm
(5 marks)

6. Solve the equation $2\sin 2\theta + 3\cos\theta = 0$,
$0^0 \le \theta \le 360^0$ ***(5 marks)***

7. Sketch the parabola $y^2 = 12(x-4)$.
State the focus and equation of the directrix.
(5 marks)

8. Determine $\frac{d}{dx}\left\{\ln\left(\frac{x}{\sqrt{1+x^2}}\right)\right\}$,
when x = 2 ***(5 marks)***

SECTION B (60 MARKS)
Attempt five questions from this section. All questions carry equal marks

9 (a) Given that X,Y and Z are angles of a triangle XYZ. Prove that

$$\tan(Z-Y) = \frac{x-y}{x+y}\cot\frac{Z}{2}$$

Hence solve the triangle if x = 9cm, y = 5.7cm and Z = 57^0 ***(7 marks)***

(b) Use the substitution $t = \tan\frac{\theta}{2}$ to solve the equation
$3\cos\theta - 5\sin\theta = -1$ for $0^0 < \theta < 360^0$ ***(5 marks)***

10(i) Determine the coordinates of the point of intersection of the line
$\frac{x+1}{2} = \frac{y-3}{5} = \frac{z+2}{-1}$ and
the plane x + y =12
(ii)Find the angle between the line
$\frac{x+1}{2} = \frac{y-3}{5} = \frac{z+1}{-1}$ and the plane
x + y + z = 12 ***(12 marks)***

11(a)Determine the Binomial expansion of $\left(1+\frac{x}{2}\right)^4$. Hence evaluate $(2.1)^4$ correct to **2** decimal places ***(6marks)***
(b). A geometric progression (GP) has a common ratio r <1, $u_1 = 15$ and $S\infty = 22.5$,where $S\infty$ is its sum to infinity and u_1, the first term. Find the
(i) value of **r**
(ii) ratio of $\mathbf{u_2:u_3}$
(6marks)

12 (a) Find the equation of a circle which passes through the points (5,7), (1,3)and (2,).
(6 marks)
(b) (i) If x = 0 and y = 0 are tangents to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$,
show that $c = g^2 = f^2$
(ii)Given that the line 3x - 4y +6 = 0 is also a tangent to the circle in (b) (i) above , Determine the equation of the circle lying in the first quadrant
(6 marks)

13. Given the curve y = sin 3x find the
(a) (i)value of $\frac{dy}{dx}$ at the point $\left(\frac{\pi}{3},0\right)$
(ii)equation of the tangent to the curve at this point
(b) (i) sketch the curve y = sin 3x
(ii) calculate the area bounded by the tangent in (a) (i) above the curve and the y-axis
(12 marks)

14 (a) Solve the equation
$\frac{4x-3y}{4} = \frac{2y-x}{3} = \frac{2z+x}{2}$ and
6x + 6y + 2z = ***(12 marks)***
(b) Given the polynomial
f(x) = Q(x)g(x) + R(x), where Q(x) is the quotient , g(x) = (x-α)(x-β) and R(x) the remainder , show that
$$R(x) = \frac{(\alpha-\beta)f(\alpha) + (\alpha - x)f(\beta)}{\alpha-\beta}$$
when f(x) is divided by g(x)
Hence find the remainder when f(x) is divided by $x^2 - 9$, given that f(x) divided by x-3 is 2 and when divided by x +3 is -3
(7 marks)

15. Express $\frac{3x^2+x+1}{(x-2)(x+1)^3}$ into partial fraction
Hence evaluate $\int_3^4 \frac{3x^2+x=1}{(x-2(x+1)^3}dx$ give your answer correct to **3** decimal places ***(12marks)***

16 (a) Solve the differential equation
$\frac{1}{x}\frac{dy}{dx} = \sin x \sec^2 3y$ ***(4 marks)***
(b) A hot body at a temperature of 100^0C is placed in a room of temperature 20^0C. Ten minutes later; its temperature is 60^0C.
(i) Write down a differential equation to represent the rate of change of temperature, θ of the body with time t
(ii) Determine the temperature of the body after a further ten minutes. ***(8marks)***

P425/1
MATHEMATICS
PAPER 1

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NOV./DEC. 2006
3 hours
SECTION A

1. Prove $\tan\left(\frac{\pi}{4}+\theta\right) - \tan\left(\frac{\pi}{4}-\theta\right) = 2\tan 2\theta$

2. Differentiate
$\frac{3x+4}{\sqrt{2x^2+3x-2}}$ with respect to x

3. Show that when the quadratic expressions $x^2 + bx + c = 0$ and $x^2 + px + q = 0$ have a common root, then, $(c-q)^2 = (b-p)(cp-bq)$

4. Prove that y = -3x + 6 is a tangent to the rectangular hyperbola whose parametric co-ordinates are of the form $\left(3t, \frac{\sqrt{3}}{t}\right)$

5. Find the point of intersection of the plane 11x – 3y + 7z = 8 and the line
$r = \begin{pmatrix}-3\\1\\5\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\-2\end{pmatrix}$, where λ is a scalar.

6. A group of nine has to be selected from ten boys and four girls or four boys and five girls. How many different groups can be chosen?

7. Solve the differential equation
$\frac{dy}{dx} + 3y + 7z = e^{2x}$ given that when x = 0, y = 1
(05marks)

8. Evaluate $\int_0^2 \frac{8x}{x^2-4x-12}\,dx$ correct to 2 decimal places.

SECTION B

9. (a) Express the complex numbers $z_1 = 4i$ and $z_2 = 2 - 2i$ in the trigonometric form
r(Cosθ + I sin θ).
Hence or otherwise evaluate $\frac{z_1}{z_2^2}$

(b) Find the values of x and y in
$$\frac{x}{2+3i} - \frac{y}{3-2i} = \frac{6+2i}{1+8i}$$

10. (a) Differentiate from first principles
$y = \frac{x}{x^2+1}$ with respect to x. (05 marks)

(b)(i) Determine the turning points of the curve $y = x^2(x-4)$.
(ii) Sketch the curve in (i) above for $-2 \le x \le 5$.
(iii) Find the area enclosed by the curve above the x – axis. ***(07 marks)***

11. (a) Given the vectors a = 3i – 2j + k and b = i – 2j + 2k,
(i) the acute angle between the vectors.
(ii) vector **c** such it is perpendicular to both vectors **a** and **b.**

(b) Given that OA = a and OB = b, point R is on $\overline{OB}$ such that $\overline{OR}:\overline{RB}$ = 4:1.point P is on $\overline{BP}:\overline{PA} = 2:3$ and when $\overline{RP}$ and $\overline{OA}$ are both produced they meet at point Q
Find:
(i) **OR** and **OP** in terms of **a** and **b.**
(ii) OQ in terms of **a.**

12. (a) Solve the equation 3 cosx + 4 sinx = 2 for $0 \le x \le 360^0$. ***(06 marks)***

(b) If A, B, C are angles of a triangle, show that Cos2A + Cos2B + Cos2C = -1 – 4 Cos A CosB CosC. ***(06 marks)***

13. (a) Form the equation of a circle that passes through the points
A (-1, 4), B (2, 5) and C (0, 1). ***(07 marks)***

(b) The line x + y = c is a tangent to the circle $x^2 + y^2 - 4y + 2 = 0$. Find the coordinates of the points of contact of the tangent for each value of c.

14.(a) Find the first three terms of the expansion of $\frac{1}{1+x}$, using Maclaurin's theorem.

15. (a) Expand $(a + b)^4$. Hence find $(1.996)^4$, correct to 3 decimal places.

(b) A credit society gives out a compound interest of 4.5% per annum. Muggaga deposits shs 300,000 at the beginning of each year.
How much money will he have at the end 4 years, if there are no withdrawals during this period?

16. Find:
(a) $\int \ln x^2\,dx$.
(b) $\int \frac{dx}{e^x - 1}$

2007 PAPER ONE
SECTION A

1 The 5th term of an article progression (A.P.) is 12 and the sum of the first 5 terms is 80. Determine the first term and the common difference ***(05 marks)***

2 Given that $\int_o^a (x^2 + 2x - 6)\,dx = 0$, find the value of a. ***(05 marks)***

3. Solve the equation $\log_2 x - \log_x 8 = 2$

4. Show that
$\frac{\sin\theta - 2\sin 2\theta + \sin 3\theta}{\sin\theta + 2\sin 2\theta + \sin 3\theta} = \tan^2\frac{\theta}{2}$ ***(05 marks)***

5 A point P has coordinates (1, -2, 3) and a certain plane has the equation x + 2y + 2z = 8. The line through P parallel to the line
$\frac{x}{3} = \frac{y+1}{-1} = z + 1$ meets the plane at a point Q.

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Find the co-ordinates of Q. ***(05 marks)***

6 . A hemispherical bowl of internal radius, r is fixed with its rim horizontal and contains a liquid to a depth, h. Show by integration that the volume of the liquid in the bowl is

$\left(\frac{1}{3}\right)\pi h^2(3r - h)$

7. Find the locus of the point p(x, y) which moves such that its distance from the point S (-3, 0) is equal to its distance from a fixed line x = 3. ***(05 marks)***

8 Differentiate: $\log_e \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}$ .Simplify your answer. ***(05 marks)***

SECTION B

9 (a) The function $f(x) = x^3 + px^2 - 5x + q$ has a factor $(x - 2)$ and has a value of 5 when x = -3. Find p and q. (04 marks)

(b) The roots of the equation $ax^2 + bx + c = 0$ are α and β. Form the equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$

(c) Simplify: $\frac{\sqrt{3-2}}{\left(\sqrt[2]{3+3}\right)}$ in form of $p + q\sqrt{3}$ where p, q are rational numbers. (03 marks)

10. Sketch the curve: $y = \frac{4(x-3)}{x(x+2)}$

11(i) Show that the equation of the tan[illegible] to [illegible] hyperbola (a secθ b tanθ) is bx – ay sinθ – ab cosθ = 0.

(ii) Find the equations of the tangents t[illegible] $\frac{x^2}{4} - \frac{y^2}{9} = 0$, at the points [illegible] θ = 45° and where θ = 135°.

(iii) Find the asymptotes

12 (a) Solve 2 sin 2x = 3 cos x, for $-180 \le x \le 180^0$.

(b) Solve sinx – sin 4x = sin2x – sin3x for $-\pi \le x \le \pi$

14 (a) What is the smallest number of terms of Geometric Progression (G.P) 5, 10, 20, … that can give a sum greater than 500,000? (04 marks)

(b) Prove by induction $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$

(c) Solve simultaneously $a^3 + b^3 = 26$ and a + b = 2 ***(04 marks)***

15 Given that the position vectors of A, B and C are

$OA = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$, $OB = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ and $OC = \begin{pmatrix} 7 \\ -2 \\ 2 \end{pmatrix}$,

(i) Prove that A, B and C are collinear.

(ii) Find the acute angle between OA and OB.

(iii) If OABD is a parallelogram, find the position vectors of E and F such that E divides DA in the ratio 1:2 and E it externally in the ratio 1:2

16 (a) Given x = r cos θ and y = r sin θ,

show that $\frac{d^2y}{dx^2} = -\frac{1}{r}\operatorname{cosec}^3\theta$.

(b) Solve: $\frac{dy}{dx} + 2y\tan x = \cos^2 x$ given that y = 2, when x = 0

PAPE[illegible] ONE 2008

SE[illegible]N A

1.Find the fourth [illegible] of [illegible] + 3i

2.Without using [illegible] or calculators, show that [illegible] 15 [illegible] $2 - \sqrt{3}$

3.Ev[illegible]ate [illegible] $\sin 2\theta c \cos d\theta$ 0

Give[illegible]e vectors a = **i** − 3**j** +3**k** and b = **i** − 3**j** + 2**k**, find the:

([illegible]cute angle between vectors a and b

(ii)equation of the plane containing and b.

5.Given the points O (0,0) and P(4,2), A is the locus of the points such that OA:AP = 1:2.

Q is the mid point of AP. Find the locus of Q in its simplest form.

6.Given that a and b are the roots of the equation $\alpha^2 + px + q = 0$, express $(\alpha^2 - \beta^2)$ and $(\alpha^3 - \beta^3)$ in terms of p and q.

7.Differentiate $\tan^{-1}\left[\frac{x^2}{2} + 2x^3\right]$ with respect $+_o x$.

8. Find the volume of the solid of revolution formed by rotating the area enclosed by the curve y = x(1+x), the x-axis, the lines x = 2 and x = 3 through four right angles about the x-axis.

SECTION B:

9.A circle cuts the y-axis at two points A and B. It touches the x-axis at a distance 4 units from the origin and distance AB is 6 units.

A is the point (0,1)

(a) equation of the circle

(b) equation of the tangents to the circle at A and B.

10. (a) Solve the equation cos x + cos 2x = 1 for values of x from 0° to 360° inclusive.

(b) (i) Prove that

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$$\frac{\cos A + \cos B}{\sin A + \sin B} = \cos\frac{A+B}{2}$$

(ii) Deduce that $\frac{\cos A + \cos B}{\sin A + \sin B} = \tan\left(\frac{c}{2}\right)$

Where A, B and C are angles of a triangle.

11. (a) Give that $y = \frac{1+\sin^2 x}{\cos^2 x + 1}$, show that

$$\frac{dy}{dx} = \frac{3\sin 2x}{(\cos^2 x + 1)^2}$$

Hence, find $\frac{dy}{dx}$ when $x = \frac{2\pi}{3}$

(b)A curve is represented by the parametric equations x = 3t and $y = \frac{4}{t^2+1}$. Find the general equation of the length to the curve in terms of x, y and t. hence determine the equation of the tangent at the point (3, 2).

12.The position vectors of points A and B are OA = 2i – 4j – k and OB = 5i – 2j + 3k respectively. The line AB is produced to meet the plane 2x + 6y – 3z = -5 at a point c

Find the;

(a)coordinate of c

(b)angle between AB and the plane

13.(a)Use partial fractions to evaluate

$$\int_{0}^{6} \frac{dx}{4x^2 - 2x - 3}$$

(b)Evaluate $\int_{0}^{\frac{\pi}{2}} x\sin 2^2 x dx$

14.On the same axes sketch the curves $f(x) = x^2(x + 2)$ and $g(x) = \frac{1}{f(x)}$

Show the asymptotes and turning points.

15 (a) Find the binominal expansion of $\left(1 - \frac{x}{x}\right)^2$.

Use your expansion to estimate (0.875)5 to four decimal places.

(b)A financial credit society gives a 2% compound interest for annum to its members. If Ochola deposits shs 100, 000 at the beginning of every year starting with 2004, how much would he collect at the end of 2008 if there are no withdraws within thus period?

16.(a) Solve the differential equation:

$$x\frac{dy}{dx} - y = x^3 e^{x2}$$

(b)The number of car accidents x in a year on a high way was found to approximate the differential equation $\frac{dy}{dx} = kx$, where t is the time in years and k a constant. At the beginning of 2000 the number of recorded accidents was 50.

If the number of accidents increased to 60 at the beginning of 2003, estimate the number that was expected at the beginning of 2005.

P425/1
PURE MATHEMATICS
2009

SECTION A

1.. Solve the simultaneous equations;

$$\begin{aligned} p + 2q - r &= -1 \\ 3p - q + 2r &= 16 \\ 2p + 3q + r &= 3. \end{aligned}$$

2.Give that $\sin(\theta - 45^0) = \cos(\theta + 45^0)$

Show that $\tan\theta = 1$.

Hence find θ if $0^0 \leq \theta = 560^0$.

3. Differentiate e^{ax2}

4.If $y = \frac{3-2x}{4+x^2}$ find the range of possible values of y for real x

5.The points P(2, 3), Q. (-11, 8) and R. (-4, -5) we vertices of a parallelogram PQRS which has PR as a diagonal. Find the co-ordinates of vertex S.

6.Find $\int \frac{dx}{1-\cos x}$

7.Find the equation of a line through the point (1,3,2-2) and perpendicular to the plane whose equation is

4x + 3y – 2z – 16 + 0

8. Solve the differential equation

$$x(1-y)\frac{dy}{dx} + y = 0$$

Or SECTION B

9.(a) By using the binorminal theorem expand $(8 - 24x)^{2/3}$ as far the 4th term.

Hence evaluate $4^{2/3}$ to one decimal place.

(b) Find the coefficient of x in the expansion of $\left(x + \frac{2}{x^2}\right)^{10}$

10.(a) Defferentiate In $(1- 2x^2)^{-1/2}$

With respect to x.

(b) Integrate $\frac{4x - x^3 + x^2 + 1}{x^3 + x}$ with respect to x, we have; x.

11. (a) use the factor formula to show that

$$\frac{\sin(A+2B) + \sin A}{\cos(A+2B)\cos A} = \tan(A+B)$$

(b) Express y = 8cos x + 6 sin x in the formula R cos (x – x) where R is positive and x is acute.

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Hence find the maximum and minimum values of

$$\frac{1}{8\cos x + 6\sin x + 15}$$

12(a) Given that

$$\frac{ix}{1+iy} = \frac{3x+i4}{x+3y},$$ find the values of x and y,

(07 marks)

b) if z = x + iy, find the equation of the locus

$$\left|\frac{z+3}{z-1}\right| = 4.$$ ***(05 marks)***

13.(a)Find the angle between the planes $x - 2y + z = 0$ and $x - y = 1$ ***(04 marks)***

(b) Two lines are given by the parametric equations:
$-i + 2j + k + t(i - 2j + 3k)$ and
$-3i + pj + 7k + 5(i - j + 2k)$.
If the lines intersect, find the
(i)values of t, s and p
(ii) the co-ordinates of the points of intersection.
(08) marks)

14.(a)Use Maclaurin's theorem m to expand

$$\frac{1}{\sqrt{(1+x)}}$$ up to the term in x^3.

(b) Given that $e^x = \tan 2y$ show that

$$\frac{d^2y}{dx^2} = \frac{e^x - e^{3x}}{2(1+e^2x)}$$

15(a) Find the equation of the tangent and normal the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$ at the point P ($2\cos\theta$, $\sin\theta$).

(b) If the tangent in (9) cuts the y axis at point A and the x –axis at point C, find the co-ordinates of the points A, B and C.

16.In a certain process the rate of production of yeast is Rx grammar per minute, where x grammar is the amount produced and R = 0.003.

(a) show that the amount of yeast is doubled in about 230 minutes.

b) If in addition yeast is removed at a contact rate of m grammas per minute, find the

(i) amount of yeast at time t minutes, given that when t = 0, x = p grammas

ii) Value of m if p= 20, 000 grammas and the supply of yeast is exhausted in 100 minutes.

PURE MATHEMATICS
NOV. / DEC. 2010
Paper 1
SECTION A: (40 MARKS)
Answer all questions in this section.

1.Solve the inequality $\frac{3x-1}{x+2} \geq 2$. ***(05 marks)***

2. The points *A* and *B* lie on the positive sides of the x – axis and y- axis respectively. If the length of *AB* is 5 units and angle *OAB* is θ, where *O* is the origin, find the equation of the line *AB*. *(Leave θ in your answer)*
(05 marks)

3.Solve the simultaneous equations:
$2x - y + 3z = 10$,
$x + 2y - 5z = -9$,
$5y + y + 4z = 11$. ***(05 marks)***

4.Show from first principles that

$$\frac{d}{dx}(\tan x) = \sec^2 x.$$ ***(05 marks)***

5.Given the points *A*(-3, 3, 4), *B*(5, 7, 2) and *C*(1, 1, 4), find the vector equation of a line which joins the mid-points of *AB* and *AC*. ***(05 marks)***

6.Evaluate $\int_1^{\sqrt{5}} \frac{x^2}{\sqrt{x^4 - x^2}}dx$ ***(05 marks***

7.Express sin x + cos x in the form Rcos (x – a). Hence find the greatest value of sin x + cos x – 1. ***(05 marks)***

Solve the differential equation:

$$\frac{dy}{dx} = e^x$$ ***(05 marks)***

SECTION B: (60 MARKS)
Answer any five questions from this section. All questions carry equal marks.

9.(a)Expand $\sqrt{\left(\frac{1+x}{1-x}\right)}$ in ascending powers of x to a term in x^2.***(04 marks)***

(b)(i)Using the expansion of $(1 + x)^{½}$ up to the term in x^3,
Find the value of $\sqrt{1.08}$ to 4 decimal places.***(04 marks)***

(ii)Express $\sqrt{1.08}$ in the form $\frac{a}{b}\sqrt{c}$.

Hence evaluate $\sqrt{3}$
Correct to 3 significant figures. ***(04 marks)***

10.(a)Use the substitution $x^2 = \theta$ to find

$$\int \frac{x}{1+\cos x^2}dx.$$ ***(06 marks)***

(b)Given that $y = 3\sqrt{\frac{x-1}{(x^2-1)^2}}$, show that

$$\frac{dy}{dx} = \frac{1-3x}{3(x+1)^{5/3}(x-1)^{4/3}}$$ ***(06 marks)***

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11.(a) The equation of the Plane R is $\boldsymbol{r}$. $\begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} =$ 16, where $\boldsymbol{r}$ is the position vector of R. Find the perpendicular distance of the plane from the origin. ***(04 marks)***

(b)Find the Cartesian equation of the plane through the points P(1,0, -2) and Q(3, -1, 1) parallel to the line with a vector equation

$\boldsymbol{r} = 3i + (2a - 1)j + (5 - a)k$. ***(08 marks)***

12.(a)Find the gradient of the curve $y = x^2 - 25 \log_{10}x$. at the point when $x = 10$.

(Give your answer correct to 3 significant figures.) ***(05 marks)***

(b)Use Maclaurin's theorem to express In (sin x + cos x) as a power series up to the term in x^2. ***(07 marks)***

13.(a) (i)Find the co-ordinates of the points where the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$ cuts the axes. ***(04 marks)***

(ii)Express the given equation in a(i) above, in it's polar form. ***(03 marks)***

(b) If the line $y = mx + c$ is a tangent to the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$,

Show that $c^2 = 4m^2 + 9$. ***(05 marks)***

14.(a)Given that the complex number Z and its conjugate $\overline{Z}$, satisfy the equation

$Z\overline{Z} + 3\overline{Z} = 34 - 12i$, find the values of Z. ***(07 marks)***

(b)Find the Cartesian equation of the locus of a point P represented by the equation

$\left|\frac{Z+3}{Z+2-4i}\right| = 1$ ***(07 marks)***

15.(a)Solve cos$x$ + cos3x = cos2x for all values of x between 0^0 and 360^0. ***(05 marks)***

(b)Show that $\tan\left(\frac{\pi}{4}+\frac{A}{2}\right) = \frac{1+\sin A}{\cos A}$ ***(07 marks)***

16.Bacteria in a culture increase at a rate proportional to the number of bacteria present. If the number increases from 1000 to 2000 in one hour,

(a)how many bacteria will be present after 1½ hours? ***(09 marks)***

(b)how long will it take for the number of bacteria in the culture ***(03 marks)***

END.

P425/1 PAPER 1

PURE MATHEMATICS
Nov.Dec.2011. 3 hours.
SECTION A

1. Solve the equation $\log_{25} 4x^2 = \log_5 (3 - x^2)$ ***(05 marks)***

2. Find the equation of a line through the point (2,3) and perpendicular to the line $x + 2y + 5 = 0$. ***(05 marks)***

3. Evaluate $\int_1^3 \left(\frac{3x^2+4x+1}{x^3+2x^2+x}\right) dx$. ***(05 marks)***

4. A committee of 4 men and 3 women is to be formed from 10 men and 8 women. In how many ways can the committee be formed? ***(05 marks)***

5. Show that: $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right) = \tan^{-1}\left(\frac{7}{9}\right)$ ***(05 marks)***

6. Given that $R=q\sqrt{(10-q^2)}$, find

(a) $\frac{dR}{dq}$, ***(05 marks)***

(b) the value of q when R is maximum. ***(05 marks)***

7. Show that points A ,B and C with position vectors $3\boldsymbol{i} + 5\boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{i} + 7\boldsymbol{j} + 4\boldsymbol{k}$ *and* $11\boldsymbol{i} + 4\boldsymbol{j} + 5\boldsymbol{k}$ respectively, are vertices of a triangle. ***(05 marks)***

(a) Form a differential equation by eliminating the constants a and b from x= a cost t +b sin t. ***(04 marks)***

(b) State the order of the differential equation formed in (a) above. ***(01 mark)***

SECTION B: (60 MARKS)

9. (a) The first term of an Arithmetic progression (A.P) is ½. The sixth term of the A.P is Fourth term. Find the common difference of the A.P.

(b) The roots of a quadratic equation $x^2+px + q =0$ are α and β .

Show the quadratic equation whose roots are $\alpha^2 - q\alpha$ and $\beta^2 - q\beta$ is given by $x^2 - (P^2 + pq - 2q)\, x + q^2 (q + p\ 1) = 0$. ***(07 marks)***

10. (a) Form a quadratic equation having -3 +4$\boldsymbol{i}$ as one of it's roots. ***(05 marks)***

(b) Given that $Z_1 = -1 + i\sqrt{3}$ and $Z_2 = -1\ i\sqrt{3}$,

(i) express $\frac{Z_1}{Z_2}$ on an Argand diagram.

(ii) find $\frac{Z_1}{Z_2}$ ***(07 marks)***

11. In the diagram below; the curve $y = 6 - x^2$ meets the line y = 2 at A and B, and the x- axis at C and D.

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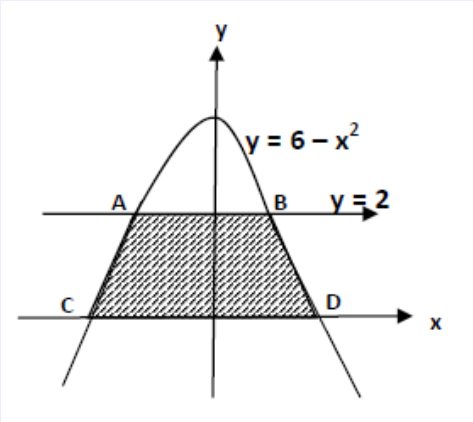

Find the
(a) coordinates of A,B,C and D. ***(04 marks)***
(b) area of the shaded region, correct to **one** decimal place. ***(08 marks)***

12. (a) Find the angle between the lines
$x = \frac{y-1}{2} = \frac{z-2}{3}$ and $\frac{x}{2} = \frac{y+1}{3} = \frac{z+2}{4}$ ***(06 marks)***
(b) Find in vector form the equation of the line of intersection of the two planes,
$2x + 3y - z = 4$ and $x - y + 2z = 5$ ***(06 marks)***

13. (a) Find the equation of the tangent to the parabola $y^2 = \frac{x}{16}$ at the point $\left(t^2, \frac{t}{4}\right)$. ***(05 marks)***
(b) if the tangents to the parabola in (a) above at the points $P\left(P^2, \frac{P}{4}\right)$ and $Q\left(Q^2, \frac{Q}{4}\right)$ meet on the line $y = 2$. ***(07 marks)***

14. (a) Solve $3\sin x + 4\cos x = 2$ for $-180^0 \le x \le 180^0$. ***(04 marks)***
(b) Show that in any triangle ABC, $\frac{a^2 - b^2}{c^2} = \frac{\sin(A-B)}{\sin(A+B)}$. ***(04 marks)***

15. (a) Differentiate the following with respect to *x:*
(i) $(x+1)^{½}(x+2)^2$
(ii) $\frac{2x^2 + 3x}{(x-4)^2}$ ***(06 marks)***
(b) The base radius of a right circular cone increases and the volume changes by 2%. If the height of the cone remains constant, find the percentage increase in the circumference of the base. ***(06 marks)***

16. (a) Solve the differential equation.
$\frac{dy}{dx} = \frac{\sin^2 x}{y^2}$, given that $y = 1$ when $x = 0$. ***(05 marks)***
(b) It is observed that the rate at which a body cools is proportional to the amount by which it's temperature exceeds that of it's surroundings. A body at 78^0C is placed in a room at 20^0C and after 5 minutes the body has cooled to 65^0C. What will be It's temperature after a further 5 minutes? ***(07 marks)***

P425/1 , PAPER 1
PURE MATHEMATICS
Nov.Dec.2012. 3 hours.
SECTION A

1. Solve the simultaneous equations.
$3x - y + z = 3$,
$x - 2y + 4z = 3$,
$2x + 3y - z = 4$. ***(05 marks)***

2. (a) prove that $\frac{2\tan\theta}{1+\tan^2\theta} = \sin 2\theta$.
(b) solve $\sin 2\theta = \cos\theta$ for $0^0 \le \theta \le 90^0$ ***(05 marks)***

3. Differentiate $\frac{3x+1}{\sqrt{(x^2+1)}}$ with respect to x. ***(05 marks)***

4. [illegible] passes through the points A(4,6,3 and B(1,3,3.
(a) Find the vector equation of the line.
(b) Show that the point C(2,4,3) lies on the line in (a0 above. ***(05 marks)***

5. The sum of the first n terms of the geometric progression (G.P.) is $\frac{4}{3}(4^n - 1)$. Find n^{th} term as an intergral power of 2. ***(05 marks).***

6. The line y=mx + c is a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ when $c = = \sqrt{(a^2m^2 + b^2)}$. Find the equation of the tangents to the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$ from the point $(0, \sqrt{5})$ ***(05 marks).***

7. Using a suitable substitution, find $\int \frac{\sin^{-1} 2x}{\sqrt{(1-4x^2)}} dx$ ***(05 marks)***

8. Find the equation of the normal to the curve $x^2y + 3y^2 - 4x - 12 = 0$ at the point (0,2). ***(05 marks)***

SECTION B: (60 MARKS)

9. If $z = \frac{(2-i)(5+12i)}{(1+2i)^2}$;
(a) find the:
(i) modulus of z,
(ii)argument of z. ***(08 marks)***
(b) represent z on a complex plane. ***(02 marks)***
(c) write z in the polar form. ***(02 marks)***

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10. (a) Solve the equation $8\cos^4 x - 10\cos^2 x + 3 = 0$ for x, in the range $0^0 \le x \le 180^0$ x. ***(07 marks)***

(b) prove that $\cos 4A - \cos AB - \cos 4C = 4\sin 2B \sin 2C \cos 2A - 1$ given that *A*, *B* and *C* are angles of a triangle. ***(05 marks)***

11.(a) Find the derivatives with respect to x of the following functions:

(i) $\dfrac{\cos 2x}{1+\sin 2x}$,

(ii) *In* (sec x + tan x). ***(06 marks)***

(b) $\int_0^{\frac{\pi}{2}} x^2 \sin x \, dx$.

12. Triangle *OAB* has ***OA*** = **a** and ***OB*** =**b**. C is a point on OA such that $OC = \dfrac{2a}{3}$. D is the midpoint of AB. When ***CD*** is produced it meets *OB* produced at E, such that ***DE*** = n***CD*** and ***BE*** = *kb*.
Express ***DE*** in terms of
(a) n, ***a*** and ***b,*** ***(06 marks)***
(b)k, ***a*** and ***b,***
Hence find the values of *n* and *k*. ***(06 marks)***

13. (a)Find the equation of the locus of the point whish moves such that it's distance from D(4,5) is thrice the distance from it's origin. ***(05 marks)***
(b) The line y = mx intersects the curve $y=2x^2 - x$ at the points A and B. Find the equation of the locus the point *P* which divides *AB* in the ratio 2:5 ***(05 marks)***

14. (a)On the same axes sketch the curves y= (x + 2) and y =x(4-x). ***(06 marks)***
(b) Find the area enclosed by two curves in (a). ***(03 marks)***
(c)Determine the volume of the solid generated when the area enclosed by the two curves in (a) is rotated about the x-axis. ***(03 marks)***

15. solve for x in the following equations:
(a) $9^x - 3^{(x+1)} = 10$; ***(06 marks)***

(b) $\log_4 x^2 - 6\log_x 4 - 1 = 0$. ***(06 marks)***

16. A 3.00pm, the temperature of a hot metal was 80^0C and that of the surroundings 20^0C. At 3.03pm the temperature of the metal had dropped to 42^0C. The rate of cooling of the metal was directly proportional to the difference between it's temperature and that of the surroundings.
(a) (i) Write a differential equation to represent the rate of cooling of the metal.
(ii) Solve the differential equation using the given conditions. ***(09 marks)***
(b) Find the temperature of the metal at 3:05 pm. ***(03 marks)***

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UACE MATHEMATICS PAST PAPER SOLUTIONS {2013-2020}

UACE MATHEMATICS PAPER 1 2013 guide

SECTION A (40 marks)

Answer all questions in this section

1. Solve $\log_x 5 + 4\log_5 x = 4$
2. In a Geometric Progression (G.P), the difference between the first and the second term is 156. The difference between the seventh and the fourth is 1404 Find the possible values of the common ratio.
3. Given that r= 3cosθis an equation of a circle, find its cartesian form.
4. The position vector of point A is 2i + 3j + k, B is 5i + k and of C is i + 2j + 12k. Show that ABC is a triangle.
5. Solve $5\cos^2 3\theta = 3(1 + \sin 3\theta)$ for $0^0 \leq \theta \leq 90$
6. If y = (x – 0.5) e^{2x}, find $\frac{dy}{dx}$.
   Hence determine $\int_0^1 xe^{2x}dx$
7. The region bounded by the curve y = cosx, the y-axis and x-axis from x =0 to x = $\frac{\pi}{2}$ is rotated about x-axis. Find the volume of the solid formed.
8. Solve $(1 - x^2)\frac{dy}{dx} - xy = 0$, given that y = 1 when x = 0.

SECTION B

Answer any five questions from this section. All questions carry equal marks

9. (a) the complex number z = $\sqrt{3} + i$. $\bar{Z}$ *is the conjugate of* Z.
   (i) Express Z in the modulus argument form
   (ii) On the same Argand diagram plot $\bar{Z}$ and $2\bar{Z} + 3i$

   (b) What are the greatest and least values of $|Z|$ if $|Z - 4| \leq 3$?

10. Given the equation $x^3 + x - 10 = 0$.
    (a) Show that x = 2 is a root of the equation
    (b) Deduce the values of α + β and αβ where α and β are roots of the equation.
    Hence form a quadratic equation whose roots are α^2 and β^2.
11. (a) Find the point of intersection of the lines $\frac{x-3}{4} = \frac{y-7}{4} = \frac{z+3}{-5}$ and $\frac{x-8}{7} = \frac{y-4}{1} = \frac{z-5}{3}$
    (c) The equations of a line and a plane are $\frac{x-2}{1} = \frac{y-2}{2} = \frac{z+3}{2}$ and 2x + y + 4z = 9resectively. P is a point on the line where x = 3, N is the foot of the perpendicular from P to the plane. Find the coordinates of N.

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12. (a) Find the equation of the tangent to the hyperbola whose points are of the parametric form (2t, $\frac{2}{t}$).
    (b)(i) Find the equations of the tangents in (a), which are parallel to y + 4x = 0
    (ii) Determine the distance between the tangents in (i)
13. A curve has the equation $y = \frac{2}{1+x^2}$.
    (a) Determine the nature of the turning point on the curve.
    (b) Find the equation of the asymptote. Hence sketch the curve.
14. (a) Prove that tan(A – B) = $\frac{\tan A - tanB}{1+ tanAtanB}$
    Hence show that $\frac{1-\tan 15^0}{1+\tan 15^0} = \frac{1}{\sqrt{3}}$
    (b) Given that cos A = $\frac{3}{5}$ and cos B = $\frac{12}{13}$ where A and B are acute, find the values of
    (i) tan (A + B)
    (ii) cosec (A + B)
15. Resolve $y = \frac{x^3+5x^2-6x+6}{(x-1)^2(x^2+2)}$ into partial fraction
    Hence find $\int ydx$ and $\frac{dy}{dx}$.
16. The differential equation $\frac{dp}{dt} = kp(c-p)$ shows the rate at which information flows in a student population c. p represents the number who have heard the information in t days and k is a constant.
    (a) Solve the differential equation.
    (b) A school has a population of 1000 students. Initially 20 students had heard the information. A day later, 50 students had heard the information. How many students heard the information by the tenth day?

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Solutions

1. Solve $\log_x 5 + 4\log_5 x = 4$
   Expressing terms on LHS to $\log_5$.
   $\frac{\log_5 5}{\log_5 x} + 4\log_5 x = 4$
   $\frac{1}{\log_5 x} + 4\log_5 x = 4$
   Let $\log_5 x = y$
   $\frac{1}{y} + 4y = 4$
   $4y^2 - 4y + 1 = 0$
   (2y – 1)(2y – 1) = 0
   2y = 1
   y = $\frac{1}{2}$
   ⇨ $\log_5 x = \frac{1}{2}$
   x = $5^{\frac{1}{2}} = \sqrt{5}$
2. In a Geometric Progression (G.P), the difference between the fifth and the second term is 156. The difference between the seventh and the fourth is 1404 Find the possible values of the common ratio.
   $U_5 - U_2 = 156$
   $ar^4 - ar = 156$
   $ar(r^3 - 1) = 156$ ..... (i)
   $U_7 - U_4 = 156$

$Ar^6 - ar^3 = 1404$

$ar^3(r^3 - 1) = 156$ ..... (ii)

Eqn. (ii) ÷ eqn. (i)

$\frac{ar^3(r^3-1)}{ar(r^3-1)} = \frac{1404}{156}$

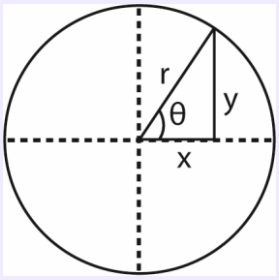
r2 = 9

r = ± 3

∴ r = 3 and r = -3

3. Given that r= 3cosθ is an equation of a circle, find its Cartesian form.

Method I: from the polar coordinates



$x^2 + y^2 = r^2$

x = rcosθ

y = rsinθ

cosθ = $\frac{x}{r}$

but r = 3cosθ

$r^2 = 3x$

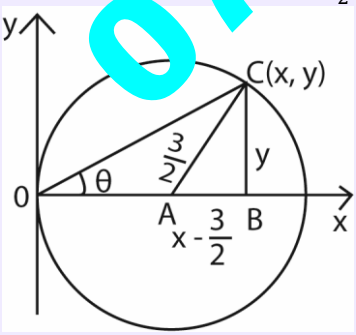
$x^2 + y^2 = 3x$

∴ $x^2 + y^2 - 3x = 0$

Method II: the Cartesian equation of a circle in polar form of radius a and centre (a, 0) is given by r = 2acosθ

⇨ 2a = 3; a = $\frac{3}{2}$

So the radius of the circle is $\frac{3}{2}$



$AC^2 = AD^2 + DC^2$

$\left(\frac{3}{2}\right)^2 = \left(x - \frac{3}{2}\right)^2 + y^2$

This is an equation of a circle in Cartesian form whose centre is $(\frac{3}{2}, 0)$ and radius $\frac{3}{2}$

4. The position vector of point A is 2i + 3j + k, of B is 5j + 4k and of C is i + 2j + 12k. Show that ABC is a triangle.

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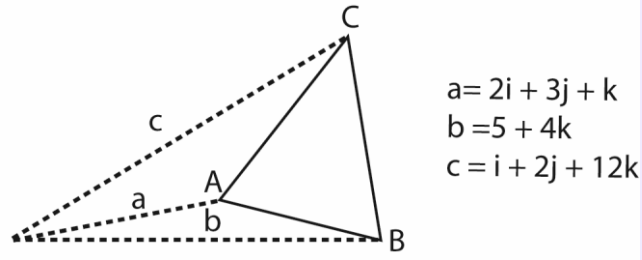

Two conditions must b fulfilled:

1st condition

For a triangle to be, $\overline{AB} + \overline{BC} + \overline{CA} = 0$

$\overline{AB} + \overline{BC} + \overline{CA}$ = (OB – OA) + (OC –OB) + (OA – OC)

$$= \begin{pmatrix}0\\5\\4\end{pmatrix} - \begin{pmatrix}2\\3\\1\end{pmatrix} + \begin{pmatrix}1\\2\\12\end{pmatrix} - \begin{pmatrix}0\\5\\4\end{pmatrix} + \begin{pmatrix}2\\3\\1\end{pmatrix} - \begin{pmatrix}1\\2\\12\end{pmatrix}$$

$$= \begin{pmatrix}-2\\2\\3\end{pmatrix} + \begin{pmatrix}1\\-3\\8\end{pmatrix} + \begin{pmatrix}1\\1\\-11\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Second condition

We work out for any angle and if it is not 0^0 or 180^0, then we conclude that ABC is a triangle

Now finding angle A

From dot product of vectors

AB.AC = $|AB||AC|cosA$

cosA = $\frac{AB.AC}{|AB||AC|}$

AB.AC = (-2i + 2j +3k).(-i + 2j + 11k)

= 2 – 2 + 33 = 33

$|AB| = \sqrt{(-2)^2 + 2^2 + 3^2} = \sqrt{4+4+9} = \sqrt{17}$

$|AC| = \sqrt{(-1)^2 + 1^2 + 11^2} = \sqrt{1+1+121} = \sqrt{123}$

A = $\cos^{-1}\left(\frac{33}{\sqrt{17\ x\ 123}}\right)$ = 43.8^0

Since A is not 0^0 or 180^0, hence ABC is a triangle

NB. The above two conditions **must** be clearly shown in order for the candidate to get all the marks.

5. Solve $5\cos^2 3\theta = 3(1 + \sin 3\theta)$ for $0^0 \leq \theta \leq 90^0$.

$5\cos^2 3\theta = 3(1 + \sin 3\theta)$

$5(1 - \sin^2 3\theta) = 3(1 + \sin 3\theta)$

$5 - 5\sin^2 3\theta = 3 + 3\sin 3\theta$

$5\sin^2 3\theta + 3\sin 3\theta - 0 = 0$

$(\sin 3\theta + 1)(5\sin 3\theta - 2) = 0$

$\sin 3\theta + 1 = 0$

$3\theta = \sin^{-1}(-1) = -90^0, 270^0$

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$\theta = -30^0, 90^0$

$5\sin 3\theta - 2 = 0$

$\sin 3\theta = \frac{2}{5}$

$3\theta = \sin^{-1}(\frac{2}{5}) = 23.578^0, 156.422^0$

$\theta = 7.859^0, 52.141^0$

Hence $\theta = (7.859^0, 52.141^0, 90^0)$

6. If $y = (x - 0.5)\, e^{2x}$, find $\frac{dy}{dx}$.

Hence determine $\int_0^1 xe^{2x}dx$

Using product rule

$y = (x - 0.5)\, e^{2x}$

Let $u = (x - 0.5)$ and $v = e^{2x}$

$\frac{du}{du} = 1 \; and \; \frac{dv}{dx} = 2e^{2x}$

$\frac{dy}{dx} = u\frac{dv}{x} + v\frac{du}{dx}$

$= (x - 0.5)(2\, e^{2x}) + e^{2x}(1)$

$= e^{2x}(2(x - 0.5) + 1)$

$= 2x\, e^{2x}$

Or

By introducing In on both sides;

$y = (x - 0.5)\, e^{2x}$

$\text{In}y = \text{In}[(x - 0.5)\, e^{2x}] = \text{In}(x\text{-}0.5) + \text{In}e^{2x} = \text{In}(x - 0.5) + 2x$

$\frac{1}{y}dy = \left(\frac{1}{x-0.5} + 2\right)dx$

$\frac{dy}{dx} = \left(\frac{1}{x-0.5} + 2\right)y$

$= \left(\frac{1}{x-0.5} + 2\right)(x - 0.5)e^{2}$

$= \left(\frac{1+2x-1}{x-0.5}\right)(x - 0.5)$

$= 2xe^{2x}$

Now $\frac{d}{dx}(x - 0.5)\, e^{2x} = 2xe^{2x}$

$\frac{1}{2}\int d\,(x - 0.5)e^{2x} = \int xe^{2x}dx$

$\int_0^1 xe^{2x}dx = \frac{1}{2}[(x - 0.5)e^{2x}]_0^1$

$= \frac{1}{2}[(1 - 0.5)e^2 - (0 - 0.5)e^0]$

$= \frac{1}{2}\left[\frac{e^2}{2} + \frac{1}{2}\right]$

$= \frac{e^2}{4} + \frac{1}{4} = 2.0973$

7. The region bounded by the curve y = cosx, the y-axis and x-axis from x =0 to x = $\frac{\pi}{2}$ is rotated about x-axis. Find the volume of the solid formed.

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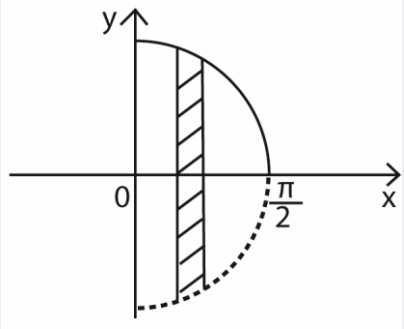

Volume formed = $\pi \int_0^{\frac{\pi}{2}} y^2 dx$

$= \pi \int_0^{\frac{\pi}{2}} cos^2 x dx$

$= \frac{\pi}{2} \int_0^{\frac{\pi}{2}} (1 + cos2x) dx$

$= \frac{\pi}{2} \left[x + \frac{1}{2} sin2x\right]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2} \left[\left(\frac{\pi}{2} + \frac{1}{2} sin\pi\right) - \left(0 + \frac{1}{2} sin0\right)\right]$

$= \frac{\pi}{2}\ x\ \frac{\pi}{2} = \frac{\pi^2}{4}\ cubic\ units$

8. Solve $(1 - x^2)\frac{dy}{dx} - xy^2 = 0$, given that y = 1 when x = 0.

$(1 - x^2)\frac{dy}{dx} - xy^2 = 0$

$(1 - x^2)\frac{dy}{dx} = xy^2$

By separation of variables

$\frac{dy}{y^2} = \frac{x}{1 - x^2} dx$

$\int y^{-2} dy = \int \frac{x}{1 - x^2} dx$

$\frac{y^{-1}}{-1} = \frac{1}{2} In(1 - x^2) + c$

$-\frac{1}{y} = -\frac{1}{2} In(1 - x^2) + c$

$\frac{1}{y} = In(1 - x^2)^{\frac{1}{2}} + c$

Substituting y = 1 and x = 0

1 =ln(1) + c

c = 1

Hence the equation is $\frac{1}{y} = In(1 - x^2)^{\frac{1}{2}} + 1$ or y = $\frac{1}{In(1 - x^2)^{\frac{1}{2}} + 1}$

SECTION B

Answer any five questions from this section. All questions carry equal marks

9. (a) The complex number z = $\sqrt{3} + i$. $\bar{Z}$ *is the conjugate of* Z.

(i) Express Z in the modulus argument form

$|z| = \sqrt{\left(\sqrt{3}\right)^2 + 1^2}$

$= \sqrt{3 + 1} = 2$

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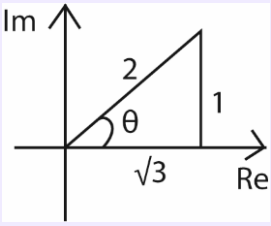


$\tan\theta = \frac{1}{\sqrt{3}}$

$\theta = \text{Arg}(z) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

Hence $z = 2\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$ or $z = 2[\cos 30^0 + i\sin 30^0]$

(ii) On the same Argand diagram plot $\bar{Z}$ and $2\bar{Z} + 3i$

$\bar{z} = \sqrt{3} - i$

$|z| = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2}$

$= \sqrt{3+1} = 2$

$2\bar{Z} + 3i = 2\left(\sqrt{3} - i\right) + 3i = 2\sqrt{3} + i$

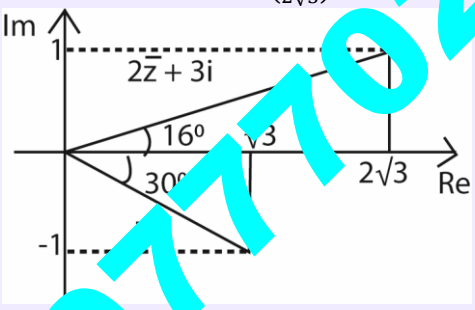
$|2\bar{Z} + 3i| = \sqrt{\left(2\sqrt{3}\right)^2 + 1^2} = \sqrt{13}$

Finding $\arg(\bar{z})$:

$\text{Arg}(\bar{z}) = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -30^0$

Finding $\arg(2\bar{Z} + 3i)$:

$\text{Arg}(2\bar{Z} + 3i) = \tan^{-1}\left(\frac{1}{2\sqrt{3}}\right) = 16.2^0$



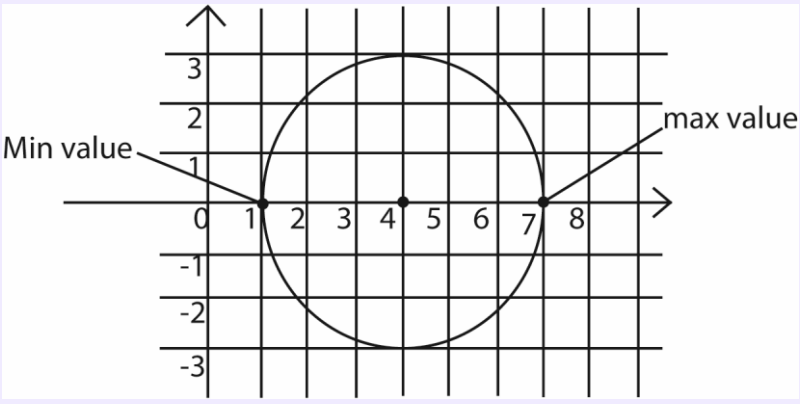
(c) What are the greatest and least values of $|Z|$ if $|Z - 4| \leq 3$?

$|z - 2| \leq$

$|x - 4 + iy| \leq 3$

$(x-4)^2 + y^2 \leq 3$

This is an equation of the circle with centre (4, 0) and radius 3.



Greatest value of $|z| = 4 + 3 = 7$

Lowest value of $|z| = 4 - 3 = 1$

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10. Given the equation $x^3 + x - 10 = 0$.

(a) Show that x = 2 is a root of the equation

Let $f(x) = x^3 + x - 10$

Substituting for x = 2

$f(2) = 2^3 + x - 10$

$= 8 + 2 - 10$

$= 10 - 10 = 0$

Hence x = 2 is a root of $x^3 + x - 10 = 0$

Or

If x = 2 is a root, then (x-2) should be a factor

$$\begin{array}{r|l} & x^2 + 2x + 5 \\ \hline (x-2) & x^3 + 2x - 10 \\ & x^3 - 2x \\ \hline & 2x^2 + x - 10 \\ & 2x^2 - 4x \\ \hline & 5x - 10 \\ & 5x - 10 \\ \hline & 0 + 0 \end{array}$$

Since the remainder is zero, hence X = 2 is a factor.

(b) Deduce the values of α + β and αβ where α and β are roots of the equation. Hence form a quadratic equation whose roots are α^2 and β^2.

⇨ $x^3 + x - 10 = (x-2)(x^2 + 2x + 5)$

Either x – 2 = 0

Or $(x^2 + 2x + 5) = 0$

$\alpha + \beta = -2$

$\alpha\beta = 5$

Sum of roots $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= (-2)^2 - 2(5) = 4 - 10 = -6$

Product= $\alpha^2\beta^2 = (\alpha\beta)^2 = 5^2 = 25$

The equation become

$x^2 - (-6)x + 25 = 0$

$x^2 + 6x + 25 = 0$

11. (a) Find the point of intersection of the lines $\frac{x-3}{4} = \frac{y-7}{4} = \frac{z+3}{-5}$ and $\frac{x-8}{7} = \frac{y-4}{1} = \frac{z-5}{3}$

Let $\frac{x-3}{4} = \frac{y-7}{4} = \frac{z+3}{-5} = \mu$

⇨ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 4 \\ -5 \end{pmatrix}$ ..... (i)

And

$\frac{x-8}{7} = \frac{y-4}{1} = \frac{z-5}{3}$

⇨ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix}$ ..... (ii)

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Equating eqn. (i) and eqn. (ii)

$$\begin{pmatrix}5\\7\\-3\end{pmatrix}+\mu\begin{pmatrix}4\\4\\-5\end{pmatrix}=\begin{pmatrix}8\\4\\5\end{pmatrix}+\lambda\begin{pmatrix}7\\1\\3\end{pmatrix}$$

Equating corresponding unit vectors

5 + 4μ = 8 + 7λ

4μ - 7λ = 3…… (iii)

7 + 4μ = 4 + λ

4μ - λ = -3…… (iv)

Eqn. (iii) – eqn.(iv)

-6λ = 6

λ = -1

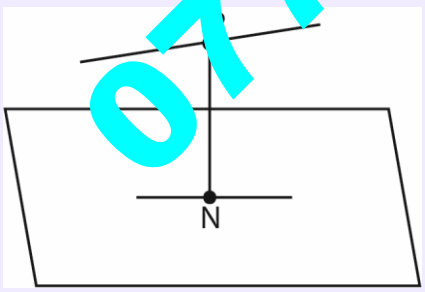
Substituting λ in eqn. (ii)

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}8\\4\\5\end{pmatrix}+-1\begin{pmatrix}7\\1\\3\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}8-7\\4-1\\5-3\end{pmatrix}=\begin{pmatrix}1\\3\\2\end{pmatrix}$$

∴ (x, y, z) = (1, 3,2)

(b) The equations of a line and a plane are $\frac{x-2}{1}=\frac{y-2}{2}=\frac{z+3}{2}$ and 2x + y + 4z = 9resectively. P is a point on the line where x = 3, N is the foot of the perpendicular from P to the plane. Find the coordinates of



Line equation

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\2\end{pmatrix}$$

$x=2+\lambda$

When x = 3

3 = 2 + λ; λ = 1

⇨ OP = $\begin{pmatrix}2\\2\\3\end{pmatrix}+\begin{pmatrix}1\\2\\2\end{pmatrix}=\begin{pmatrix}3\\4\\5\end{pmatrix}$

∴ P(3, 4, 5)

Plane equation : 2x + y + 4z = 9

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$r\begin{pmatrix}2\\1\\4\end{pmatrix} = 9$

$\therefore n = \begin{pmatrix}2\\1\\4\end{pmatrix}$

NP = n

NP = OP – ON

ON = OP – NP

$= \begin{pmatrix}3\\4\\5\end{pmatrix} - \begin{pmatrix}2\\1\\4\end{pmatrix} = \begin{pmatrix}1\\3\\1\end{pmatrix}$

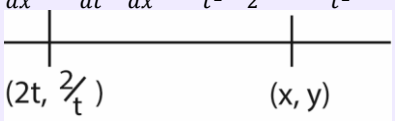
∴ N(1, 3, 1)

12. (a) Find the equation of the tangent to the hyperbola whose points are of the parametric form (2t, $\frac{2}{t}$).

x = 2t, y = $\frac{2}{t} = 2y^{-1}$

$\frac{dx}{dt} = 2, \frac{dy}{dt} = -\frac{2}{t^2}$

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = -\frac{2}{t^2} \cdot \frac{1}{2} = -\frac{1}{t^2}$



Gradient = $\frac{y-\frac{2}{t}}{x-2t}$

But gradient = $-\frac{1}{t^2}$

=> $\frac{y-\frac{2}{t}}{x-2t} = -\frac{1}{t^2}$

$t^2\left(y - \frac{2}{t}\right) = -(x - 2t)$

$t^2y + x - 4t = 0$

(b)(i) Find the equations of the tangents in (a), which are parallel to y + 4x = 0

$t^2y + x - 4t = 0$

$y = -\frac{1}{t^2}x + \frac{4}{t}$

$gradient = -\frac{1}{t^2}$

For y + 4x = 0

y = -4x

gradient = -4

But parallel lines have equal gradient

⇨ $-\frac{1}{t^2} = -4; t^2 = \frac{1}{4}$ and $t = \pm\frac{1}{2}$

Substituting for t = $\frac{1}{2}$

$y = -\frac{1}{\left(\frac{1}{2}\right)^2}x + \frac{4}{\left(\frac{1}{2}\right)}$

$y = -4x + 8$

Substituting for t = $-\frac{1}{2}$

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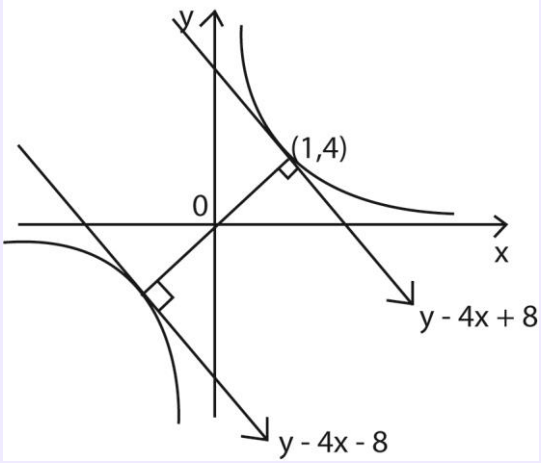
$y = -\frac{1}{\left(-\frac{1}{2}\right)^2}x + \frac{4}{\left(-\frac{1}{2}\right)}$

$y = -4x - 8$

(ii) Determine the distance between the tangents in (i)

By the nature of the parametric points in the form (2t, $\frac{2}{t}$), this is a rectangular hyperbola

Substituting for t = $\pm\frac{1}{2}$, the points become (1, 4) and (-1, -4)



The distance between two tangents = perpendicular distance between them

Using d = $\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$

We use either y = -4x + 8

Considering y = -4x + 8 or y + 4x – 8 = 0, a = 4, b = 1 c = -8

Substituting for (x, y) = (-1, -4)

d = $\left|\frac{4(-1)+1(-4)-8}{\sqrt{1^2+4^2}}\right| = \frac{16}{\sqrt{17}}$ = 3.8

NB – distance between the tangents is the perpendicular distance between the tangents

- Candidates or learners should not use the general formula of finding the distance between two points.

13. A curve has the equation $y = \frac{2}{1+x^2}$.

(a) Determine the nature of the turning point on the curve.

$y = \frac{2}{1+x^2} = 2(1 + x^2)^{-1}$

$\frac{dy}{dx} = -2(1 + x^2)^{-2}.2x = \frac{-4x}{(1+x^2)^2}$

Or

$y = \frac{2}{1+x^2}$

$Iny = In2 - In\,(1 + x^2)$

$\frac{1}{y}dy = \left(0 - \frac{2x}{1+x^2}\right)dx$

$\frac{dy}{dx} = \frac{-2x}{1+x^2}.\frac{2}{1+x^2} = \frac{-4x}{(1+x^2)^2}$

At turning points, $\frac{dy}{dx} = 0$

⇨ $\frac{-4x}{(1+x^2)^2} = 0$

$x = 0$

y = $\frac{2}{1+0} = 2$

Hence the turning points is (x, y) = (0, 2)

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Finding the nature of the turning point

X	-1	0	1
$\frac{dy}{dx}$	+1	0	-1

Max

Hence the nature of the turning point is maximum

(b) Find the equation of the asymptote. Hence sketch the curve.

$y = \frac{2}{1+x^2}$

For vertical asymptote

$1 + x^2 = 0$

$x^2 = -1$

$x = \sqrt{-1}$ which does not exist

Or

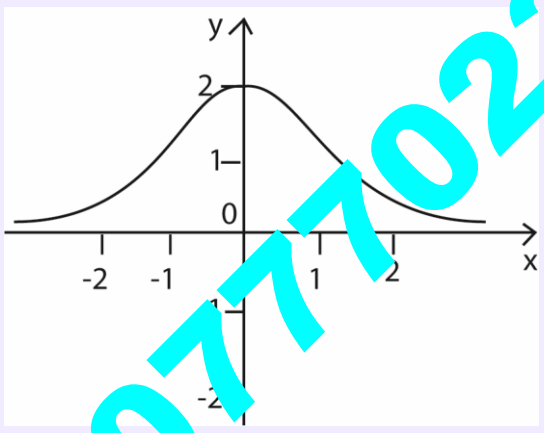
Since $1 + x^2$ cannot be factorized, there is no vertical asymptot

For horizontal asymptote

$y = \frac{2}{1+x^2}$

As $x \to \infty$, $y \to 0$

Hence the horizontal asymptote is y = 0



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NB.

- This is not an open question, therefore candidates should not waster time finding the region where the curve is confined.
- Shading is not very necessary for this question

14. (a) Prove that $\tan(A - B) = \frac{\tan A - tanB}{1 + tanAtanB}$

$\tan(A - B) = \frac{\sin(A-B)}{\cos(A-B)}$

$= \frac{sinA\ cosB - cosA\ sinB}{\cos AsinB + sinAsin\ B}$

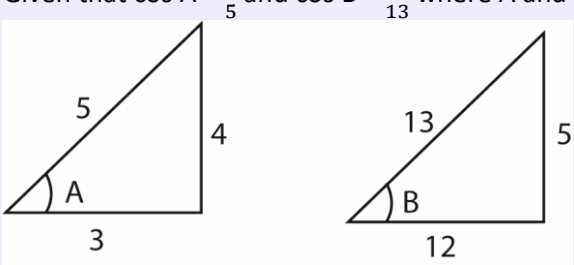
Diving numerator and denominator on the R.H.S by cosAcosB

$\tan(A - B) = \frac{\frac{sinA\ cosB}{cosAcosB} - \frac{cosA\ sinB}{cosAcosB}}{\frac{\cos AcosB}{cosAcosB} + \frac{sinAsin\ B}{cosAcosB}} = \frac{\tan A - tanB}{1 + tanAtanB}$

Hence show that $\frac{1-\tan 15^0}{1+\tan 15^0} = \frac{1}{\sqrt{3}}$

$\frac{1-\tan 15^0}{1+\tan 15^0} = \frac{\tan 45^0 - tan15^0}{1 + \tan 45^0 tan15^0} = \tan(45^0 - 15^0)\ \tan 30^0 = \frac{1}{\sqrt{3}}$

(b) Given that cos A = $\frac{3}{5}$ and cos B = $\frac{12}{13}$ where A and B are acute, find the values of



cos A = $\frac{3}{5}$ cos B = $\frac{12}{13}$

sin A = $\frac{4}{5}$ sin B = $\frac{5}{13}$

tan A = $\frac{4}{3}$ tanB = $\frac{5}{12}$

(i) tan (A + B)

tan (A + B) = $\frac{sinA\ cosB+cosA\ sinB}{\cos AsinB-sinAsin\ B}$

= $\frac{\frac{4}{5}\cdot\frac{12}{13}+\frac{3}{5}\cdot\frac{5}{13}}{\frac{3}{5}\cdot\frac{12}{13}-\frac{4}{5}\cdot\frac{5}{13}}$ = 3.9375

(ii) cosec (A + B)

cosec (A + B) = $\frac{1}{\sin(A+B}$ = $\frac{1}{sinAcosB+cosAsinB}$ = $\frac{1}{\frac{4}{5}\cdot\frac{12}{13}+\frac{3}{5}\cdot\frac{5}{13}}$ = 1.0317

15. Resolve $y = \frac{x^3+5x^2-6x+6}{(x-1)^2(x^2+2)}$ into partial fraction

Let $\frac{x^3+5x^2-6x+6}{(x-1)^2(x^2+2)} \equiv \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{Cx+D}{(x^2+2)}$

$x^3+5x^2-6x+6 \equiv A(x-1)(x^2+2)+B(x^2+2)+(Cx+D)(x-1)^2$

Taking x = 1

1 + 5 – 6 + 6 = 0 + 3B

3B = 6; B = 2

Taking x = 0

6 = -2A + 2B +D

6 = -2A + 4 + D

2 = -2A + D ..... (i)

Equating coefficients of x^3

1 = A + C ..... (ii)

Equating coefficients of x^2

5 = -A + B – 2C + D

5 = -A + 2 -2C + D

3 + A + 2C = D

Substituting for D into equation (i)

2 = -2A + 3 + A + 2C

-1= -A + 2C ..... (iii)

Eqn (ii)+ eqn. (iii)

0 = 3C; C = 0

Substituting for c into eqn. (i)

1 = A + 0

A = 1

Substituting for A into eqn. (i)

2 = -2 + D; D = 4

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⇨ $\frac{x^3+5x^2-6x+6}{(x-1)^2(x^2+2)} \equiv \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{4}{(x^2+2)}$

Hence find $\int ydx$ and $\frac{dy}{dx}$.

$\frac{x^3+5x^2-6x+6}{(x-1)^2(x^2+2)} \equiv \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{4}{(x^2+2)}$

$\int ydx = \int \left(\frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{4}{(x^2+2)}\right) dx$

= ln(x -1) + $\int \frac{2}{(x-1)^2} + \int \frac{4}{(x^2+2)}$

= ln(x – 1) + $\frac{-2}{(x-1)} + \int \frac{4}{(x^2+2)}$

Now $\int \frac{1}{a^2+b^2x^2} = \frac{1}{ab}\tan^{-1}\left(\frac{bx}{a}\right) + c$

Comparing $\frac{1}{2+x^2}$ with $\frac{1}{a^2+b^2x^2}$

a = $\sqrt{2}$ and b = 1

$\int \frac{4}{(x^2+2)} = \frac{4}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)$+ C

$\int ydx = In(x-1) + \frac{-2}{(x-1)} + \frac{4}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)$+ C

Finding $\frac{dy}{dx}$

y = $\frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{4}{(x^2+2)}$

= $(x-1)^{-1} + 2(x-1)^{-2} + 4(x^2+2)^{-1}$

$\frac{dy}{dx} = -(x-1)^{-2} \pm 4(x-1)^{-3} + -4(x^2+2)^{-2}\,2x$

$= \frac{-1}{(x-1)^2} - \frac{4}{(x-1)^3} - \frac{8x}{(x^2+2)^2}$

16. The differential equation $\frac{dp}{dt} = kp(c-p)$ shows the rate at which information flows in a student population c. p represents the number who have heard the information in t days and k is a constant.

(a) Solve the differential equation.

$\frac{dp}{dt} = kp(c-p)$

Separating variables

$\frac{dp}{p(c-p)} = dt$

$\int \frac{dp}{p(c-p)} = \int dt$

$\int \frac{dp}{p(c-p)}$ = kt + a where a is a constant

By partial fractions

$\frac{1}{p(c-p)} \equiv \frac{A}{p} + \frac{B}{c-p}$

$1 \equiv A(c-p) + B(p)$

$1 \equiv Ac - Ap + Bp$

$1 \equiv Ac + (B-A)p$

Equating constants

1 =Ac

A = $\frac{1}{c}$

Equating coefficient of p

0 = B – A

A = B = $\frac{1}{c}$

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⇨ $\int \frac{dp}{p(c-p)} = \frac{1}{c}\int \frac{1}{p}dp + \frac{1}{c}\int \frac{1}{c-p}dp$

$= \frac{1}{c} Inp - \frac{1}{c} In(c-p)$

$= \frac{1}{c} In \frac{p}{(c-p)}$

$\therefore \frac{1}{c} In \frac{p}{(c-p)} = kt + a$

(b) A school has a population of 1000 students. Initially 20 students had heard the information. A day later, 50 students had heard the information. How many students heard the information by the tenth day?

Given c =1000, at t=0, p= 20

By substitution, we have

$\frac{1}{1000} In \frac{20}{1000-20} = 0 + a$

$a = \frac{1}{1000} In \frac{20}{980} = \frac{1}{1000} In \frac{1}{49}$

⇨ $\frac{1}{1000} In \frac{p}{(1000-p)} = kt + \frac{1}{1000} In \frac{1}{49}$

After t = 1, p = 50; by substitution, we have

$\frac{1}{1000} In \frac{50}{(1000-50)} = k(1) + \frac{1}{1000} In \frac{1}{49}$

k $= \frac{1}{1000} In \frac{50}{950} - \frac{1}{1000} In \frac{1}{49} = \frac{1}{1000} In \frac{1}{19} \div \frac{1}{49} = \frac{1}{1000} In$

⇨ $\frac{1}{1000} In \frac{p}{(1000-p)} = \left(\frac{1}{1000} In \frac{49}{19}\right) t + \frac{1}{1000} In \frac{1}{49}$

$In \frac{p}{(1000-p)} = \left[In \left(\frac{49}{19}\right)\right] t + In \frac{1}{49}$

Note: by 10th day is the same as [illegible] 9da

Substituting for t = 9

$In \frac{p}{(1000-p)} = \left[In \left(\frac{49}{19}\right)\right] [illegible] In \frac{}{49}$

$p = 990.3835$

Number of studen[illegible] w[illegible] heard the information by the 10th day is 990

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UACE MATHEMATICS PAPER 1 2014

SECTION A (40 marks)

Answer all questions in this section

1. Solve the simultaneous equation
   X – 2y -2z = 0
   2x + 3y + z = 1
   3x –y – 3z = 3(05marks)
2. A focal chord PQ, to the hyperbola $y^2 = 4x$, has gradient m = 1. Find the coordinates of the mid-point of PQ. (05marks)
3. Given that cos 2A - cos 2B = -p and sin2A - sin2B = q, prove that sec (A + B) = $\frac{1}{q}\sqrt{p^2 + q^2}$ (05marks)
4. Differentiate$\log_5\left(\frac{e^{tanx}}{sin^2x}\right)$.
5. Find the equation of a line through S(1, 0, 2) and T(3, 2, 1) in the form r = a +λb. (05marks)
6. Solve the equation $\sqrt{2x+1} - \sqrt{x+1} = \sqrt{x-2}$ (05marks)
7. Find $\int x(1-x^2)^{\frac{1}{2}}$ (05marks)
8. A cylinder has radius r and height 8r. The radius increases from 4cm to 44.1cm. Find the approximate increase in the volume. (use π= 3.14) (05marks)

SECTION B (60MARKS)

Answer any five questions from this section. All questions carry equal marks)

9. (a) Given that the complex number Z and its conjugate $\bar{Z}$ satisfy the equation
   $Z\bar{Z} + 2iZ = 12 + 6i$. Find Z. (07marks)
   (b)One root of the equation $Z^3 - 3Z^2 + 9Z + 13 = 0$ is $2 + 3i$. Determine other roots (05marks)
10. A circle is described by the equation $x^2 + y^2 - 4x - 8y + 16 = 0$. A line given by the equation y = 2(x – 1) cuts the circle at points A and B. A point (x, y) moves such that its distance from the midpoint of AB is equal to its distance from the centre of the circle.
    (a) Calculate the coordinates of A and B. (04 marks)
    (b) Determine the centre and radius of the circle (04marks)
    (c) Find locus of P. (04marks)
11. (a) differentiate y = $\cot^{-1}(Inx)$ with respect to x (06marks)
    (b) Evaluate $\int_{\frac{\pi}{3}}^{\pi} xsinxdx$ (06marks)

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12. (a) Find the Cartesian equation of the plane through the points whose position vectors are 2i + 2j + 3k, 3i + j + 2k and -2j + 4k. (06marks)

    (b) Determine the angle between the plane in (a) and the line $\frac{x-2}{2} = \frac{y}{-4} = z - 5$. (06marks)

13. (a) Find the three terms of the expansion $(2 - x)^6$ and use it to find $(1.998)^6$ correct to two decimal places (06 marks)

    (b) Expand $(1 - 3x + 2x^2)^5$ in ascending power of x as far as the x^2 term (06marks)

14. (a) Find the equation of a normal to curve whose parametric equation are $x = b\sec^2\theta$, $y = b\tan^2\theta$ (06marks)

    (b) The area enclosed by the curve $x^2 + y^2 = a^2$, the y-axis and the line $y = \frac{1}{2}a$ is rotated through 90^0 about the y-axis. Find the volume of the solid generated. (06marks)

15. Solve

    (a) $4\sin^2\theta - 12\sin2\theta + 35\cos^2\theta = 0$; for $0^0 \leq \theta \leq 90^0$ (06marks)

    (b) $3\cos\theta - 2\sin\theta = 2$, for $0^0 \leq \theta \leq 360^0$ (06marks)

    (c)

16. A substance loses mass at a rate which is proportional to the amount present at time t.

    (a) Form a differential equation connecting M, t and proportionality constant k. (02marks)

    (b) If the initial mass of the substance is M_0, show that $M = M_0e^{-kt}$. (05marks)

    (c) Given that half of the substance is lost in 1600years determine the number of years 15g of the substance would take to reduce to 13.6g

Solutions

SECTION A (40 marks)

Answer all questions in this section

1. Solve the simultaneous equation

   x – 2y - 2z = 0

   2x + 3y + z = 1

   3x –y – 3z = 3 (05marks)

   X – 2y -2z = 0 ..... (i)

   2x + 3y + z = 1 ..... (ii)

   3x –y – 3z = 3 .....(iii)

   Eqn. (i) + 2eqn. (ii)

   4x + 4y = 2 .....(iv)

   3Eqn. (ii) + eqn. (iii)

   9x + 8y = 6 ..... (v)

   2eqn. (iv) – eqn. (v)

   x = -2

   Substituting x = -2 into eqn. (iv)

   5(-2) + 4y = 2

   y = 3

   Substituting x = -2 and y = 3 into eqn. (ii)

   2(-2) + 3(3) + z =2

   z = -4

   ∴ x = -2, y = 3 and z = -4

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Alternatively: using row reduction to echelon form

Expressing the equation in matrix form

$$\begin{pmatrix} 1 & -2 & -2 \\ 2 & 3 & 1 \\ 3 & -1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

The augmented matrix is

$$\begin{pmatrix} 1 & -2 & -2 : 0 \\ 2 & 3 & 1 \ : 1 \\ 3 & -1 & -3 : 3 \end{pmatrix}$$

Transforming augmented matrix to a unity triangular matrix

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\begin{pmatrix} 1 & -2 & -2 : 0 \\ 2 & 3 & 1 \ : 1 \\ 3 & -1 & -3 : 3 \end{pmatrix}\begin{matrix} \rightarrow R_1 = R_1 \\ \rightarrow 2R_1 - R_2 = R_2 \\ \rightarrow 3R_1 - R_3 = R_3 \end{matrix}\begin{pmatrix} 1 & -2 & -2 : 0 \\ 0 & -7 & -5 : -1 \\ 0 & -5 & -3 : -3 \end{pmatrix}$$

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\begin{pmatrix} 1 & -2 & -2 : 0 \\ 0 & -7 & -5 : -1 \\ 0 & -5 & -3 : -3 \end{pmatrix}\begin{matrix} \rightarrow R_1 = R_1 \\ \rightarrow 2R_1 - R_2 = R_2 \\ \rightarrow 3R_1 - R_3 = R_3 \end{matrix}\begin{pmatrix} 1 & -2 & -2 : 0 \\ 0 & 1 & \frac{5}{7} : 1/7 \\ 0 & 0 & -4 : 16 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & \frac{5}{7} \\ 0 & 0 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1/7 \\ 16 \end{pmatrix}$$

-4z = 16

z = -4

y + $\frac{5}{7}z = 1/7$

y + $\frac{5}{7}x - 4 = 1/7$

y = 3

x – 2y – 2z = 0

x -2(3) – 2(-4) = 0

x = -2

∴ x = -2, y = 3 and z =

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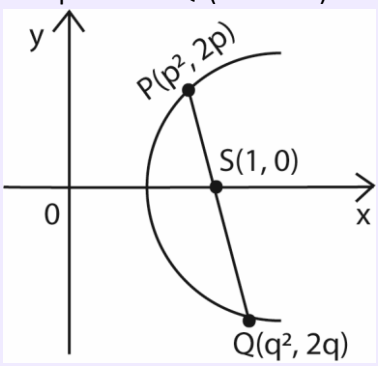
2. A focal chord PQ, to the hyperbola $y^2 = 4x$, has a gradient m = 1. Find the coordinates of the mid-point of PQ. (05marks)



Grad of $\overline{PS} = grad\ of\ \overline{SP} = grad\ of\ \overline{PQ} = 1$

$\frac{2p-0}{p^2-1} = 1$

$p^2 - 2p - 1 = 0$

$p = \frac{2\pm\sqrt{2^2-4(1)(-1)}}{2(1)} = \frac{2\pm\sqrt{4+4}}{2} = 1\pm\sqrt{2}$

$=> P\left[\left(1+\sqrt{2}\right)^2, \left(2+2\sqrt{2}\right)\right], Q\left[\left(1-\sqrt{2}\right)^2, \left(2-2\sqrt{2}\right)\right],$

Let M(x, y) be the coordinates of the mid-point of PQ.

$x = \frac{1}{2}\left[\left(1+\sqrt{2}\right)^2, \left(2+2\sqrt{2}\right)\right] = 3$

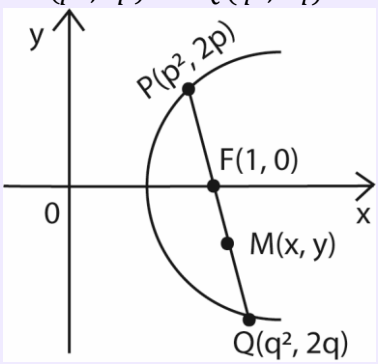
$y = \frac{1}{2}\left[\left(1-\sqrt{2}\right)^2, \left(2-2\sqrt{2}\right)\right] = 2$

∴M(3, 2)

Alternatively

From $y^2 = 4x$,; a =1

$\therefore P(p^2, 2p) and\ Q(q^2, 2q)$



From focal chord, pq = 1

Gradient = $\frac{2q-2p}{q^2-p^2}$ = 1

= $\frac{2(q-p)}{(q+p)(q-p)}$ =

⇨ q + p = 2

M($\frac{p^2+q^2}{2}$, $p+$

x = $\frac{p^2+q^2}{2}$ = $\frac{1}{2}(p+q)^2 - 2pq]$ = $\frac{1}{2}[(-2)^2 - 2(1)]$= $\frac{1}{2}(4+2) = 3$

y = p + q = 2

∴M(3, 2)

3. Given that cos 2A - cos 2B = -p and sin2A – sin2B = q, prove that sec (A + B) = $\frac{1}{q}\sqrt{p^2+q^2}$ (05marks)

cos 2A - cos 2B = -2sin$\frac{1}{2}(2A+2B)sin\frac{1}{2}(2A-2B)$

-p = -2sin (A + B)sin (A-B)

p = 2sin (A + B)sin (A- B) ..... (i)

sin2A – sin2B = -2cos$\frac{1}{2}(2A+2B)sin\frac{1}{2}(2A-2B)$

q = 2cos (A + B)sin (A- B) ..... (ii)

Eqn. (i) ÷ eqn. (ii)

$\frac{p}{q} = \tan(A+B)$

$\frac{p^2}{q^2} = tan^2\ (A+B) = sec^2\ (A+B) - 1$

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$sec^2(A+B) = \frac{p^2}{q^2} + 1 = \frac{p^2+q^2}{q^2} = \frac{1}{q^2}(p^2 + q^2)$

$\sec(A+B) = \frac{1}{q}\sqrt{(p^2 + q^2)}$

4. Differentiate$\log_5\left(\frac{e^{tanx}}{sin^2x}\right)$.

Let y =$\log_5\left(\frac{e^{tanx}}{sin^2x}\right)$

$5^y = \frac{e^{tanx}}{sin^2x}$

$yIn5 = Ine^{tanx} - Insin^2x$

$yIn5 = tanx - Insin^2x$

$\frac{dy}{dx}In5 = sec^2x - \frac{\frac{d}{dx}(sin^2x)}{sin^2x}$

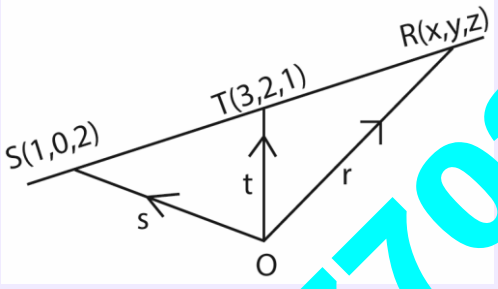
$= sec^2x - \frac{2sinxcosx}{sin^2x}$

$= sec^2x - \frac{2cosx}{\sin x}$

$= sec^2x - 2cotx$

$\frac{dy}{dx} = \frac{1}{In5}(sec^2x - 2cotx)$

5. Find the equation of a line through S(1, 0, 2) and T(3, 2, 1), the form r = a +λb. (05marks)



SR = λST

OR –OS =λ(OT – OS)

$OR - \begin{pmatrix}1\\0\\2\end{pmatrix} = \lambda\begin{pmatrix}2\\2\\1\end{pmatrix} - \begin{pmatrix}1\\0\\2\end{pmatrix}$

$r - \begin{pmatrix}1\\0\\2\end{pmatrix} = \lambda\begin{pmatrix}2\\2\\-1\end{pmatrix}$

Substituting for r= OR = $\begin{pmatrix}x\\y\\z\end{pmatrix}$

$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\0\\2\end{pmatrix} + \lambda\begin{pmatrix}2\\2\\-1\end{pmatrix}$

$\frac{x-1}{2} = \frac{y}{2} = \frac{z-2}{-1} = \lambda$

Hence the Cartesian equation is$\frac{x-1}{2} = \frac{y}{2} = \frac{z-2}{-1}$

6. Solve the equation $\sqrt{2x+3} - \sqrt{x+1} = \sqrt{x-2}$ (05marks)

$\sqrt{2x+3} - \sqrt{x+1} = \sqrt{x-2}$

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Squaring both sides

$\left(\sqrt{2x+3}-\sqrt{x+1}\right)^2=\left(\sqrt{x-2}\right)^2$

$2x+3-2\sqrt{(2x+3)(x+1)}$+x +1 = x – 2

$2x+6-2\sqrt{(2x+3)(x+1)}$ = 0

$x+3=\sqrt{(2x+3)(x+1)}$

Squaring both side

$(x+3)^2=\left(\sqrt{(2x+3)(x+1)}\right)^2$

$x^2+6x+9=2x^2+5x+3$

$x^2-x-6=0$

(x -3)(x+2) = 0

x = 3 or x = -2

Verification

Taking x= 3

L.H.S = $\sqrt{6+3}-\sqrt{3+1}$ = $\sqrt{9}-\sqrt{4}$ = 3 -2 = 1

R.H.S =$\sqrt{3-2}$ = $\sqrt{1}$ = 1

Here x = 3 satisfies the equation

Taking x = -2

L.H.S = $\sqrt{-2\,x\,2+3}-\sqrt{-2+1}$ = $\sqrt{-1}-\sqrt{-1}=0$

R.H.S = $\sqrt{-2-2}=\sqrt{-4}$

L.H.S ≠ R.H.S

∴ x = 3 is the only solution

7. Find $\int x(1-x^2)^{\frac{1}{2}}dx$ (05marks)

$\int x(1-x^2)^{\frac{1}{2}}dx$

Let u = 1-x^2

du = -2xdx

$-\frac{1}{2}du=xdx$

$=>\int x(1-x^2)^{\frac{1}{2}}dx=\int u^{\frac{1}{2}}.-\frac{1}{2}du$

$=-\frac{1}{2}\int u^{\frac{1}{2}}du$

$=\frac{1}{3}u^{\frac{3}{2}}+c$

$=\frac{1}{3}(1-x^2)^{\frac{3}{2}}+c$

8. A cylinder has radius r and height 8r. The radius increases from 4cm to 44.1cm. Find the approximate increase n the volume. (use π= 3.14) (05marks)

v =πr^2h = $\pi r^2(8r)=8\pi r^3$

$\frac{dv}{dr}=24\pi r^2$

But $\frac{\Delta v}{\Delta r}=\frac{dv}{dr}$

$\Delta v=\frac{dv}{dr}\Delta r$

$\Delta r=4.1-4.0=0.1cm$

$=>\ \Delta v=24(3.14)(4)^2(0.1)$ = 120.576cm^3

∴ Δv = 120.576cm^3

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SECTION B (60MARKS)

Answer any five questions from this section. All questions carry equal marks)

9. (a) Given that the complex number Z and its conjugate $\bar{Z}$ satisfy the equation $Z\bar{Z} + 2iZ = 12 + 6i$. Find Z. (07marks)

Let z = x + iy then $\bar{z} = x - iy$

(x + iy)(x – iy) +2i(x + iy) = 12 + 6i

$x^2 + y^2 - 2y + 2xi = 12 + 6i$

Equating imaginary parts

2xi = 6i

x = 3

Equating real parts

$x^2 + y^2 - 2y = 12$

By substituting for x

$9 + y^2 - 2y = 12$

$y^2 - 2y - 3 = 0$

(y – 3)(y+1) = 0

y = 3 or y = -1

when y = 3; z = 3 + 3i

when y = -1; z = 3 - i

Hence the possible values of z are 3 +3i and 3-

(b)One root of the equation $Z^3 - 3Z^2 + 9Z - 13 = 0$ is $2 + 3i$. Determine other roots (05marks)

Given root 2 + 3i; its conjugate is also a root of the equation

The equation of these two is

$z^2 - (sum\ of\ roots)z + product\ of\ root = 0$

Sum = 2 + 3i + 2 – 3i = 4

Product of roots =(2 + 3i)(2 – 3i) = 2 + 9 = 13

$z^2 - 4z + 13 = 0$

$$
\begin{array}{r|l}
 & z + 1 \\
\hline
z^2 - 4z + 13 & z^3 - 3z^2 - 9z + 13 \\
 & -\ z^3 - 4z^2 - 13z \\
\hline
 & z^2 - 4z + 13 \\
 & -z^2 - 4z + 13 \\
\hline
 & 0 \quad 0 \quad 0
\end{array}
$$

⇨ $(z^2 - 4z + 13)(z + 1) = 0$

z + 1= 0

z = -1

So the other roots are 2-3i and -1

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10. A circle is described by the equation $x^2 + y^2 - 4x - 8y + 16 = 0$. A line given by the equation $y = 2(x - 1)$ cuts the circle at points A and B. A point (x, y) moves such that its distance from the midpoint of AB is equal to its distance from the centre of the circle.

(a) Calculate the coordinates of A and B. (04 marks)

Substituting for $y = 2(x-1)$ into the equation $x^2 + y^2 - 4x - 8y + 16 = 0$

$x^2 + (2(x-1))^2 - 4x - 8y + 16 = 0$

$5x^2 - 28x + 36 = 0$

$(x - 2)(5x - 18) = 0$

$x - 2 = 0$

$x = 2$

and $5x - 18 = 0$

$x = \frac{18}{5} = 3.6$

taking $x = 2$

$y = 2(2 - 1)$; $y = 2$

taking $x = 3.6$

$y = 2(3.6 - 1)$; $y = 5.2$

Hence A(2,2) and B(3.6, 5.2)

(b) Determine the centre and radius of the circle (04marks)

$x^2 + y^2 - 4x - 8y + 16 = 0$

$x^2 - 4x + y^2 - 8y + 16 = 0$

$(x - 2)^2 + (y - 4)^2 = 4 + 16 - 16$

$(x - 2)^2 + (x - 4)^2 = 4$

$x - 2 = 0$

$x = 2$

$y - 4 = 0$

$y = 4$

Centre of the circle = (2, 4)

Radius = $\sqrt{4} = 2$

Alternatively

Comparing the general equation of a circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ with $x^2 + y^2 - 4x - 8y + 16 = 0$

⇨ $2g = -4$

$g = -2$

$2f = -8$

$f = -4$

Centre of the circle is (-g, -f)

Hence centre of the circle = (2, 4)

Radius = $+\sqrt{g^2 + f^2 - c}$

$= +\sqrt{(-2)^2 + (-4)^2 - 16} = \sqrt{4} = 2$

(c) Find locus of P. (04marks)

A(2, 2) and B(3.6, 5.2)

Coordinates of mid-point of AB

$= E\left(\frac{2+3.6}{2}, \frac{2+5.2}{2}\right)$

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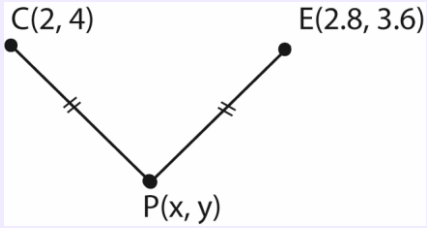

$\overline{PC} = \overline{PE}$ and $\overline{PC}^2 = \overline{PE}^2$

$(x-2)^2 + (y-4)^2 = (x-2.8)^2 + (y-3.6)^2$

$2x - y = 1$ or y = 2x -1

11. (a) Differentiate y = $\cot^{-1}(Inx)$ with respect to x (06marks)

y = $\cot^{-1}(Inx)$

coty = Inx

$\frac{d}{dx}(coty) = \frac{d}{dx}(Inx)$

$-sec^2y\frac{dy}{x} = \frac{1}{x}$

$\frac{dy}{dx} = \frac{1}{x[1+(In\,x)^2]}$

(b) Evaluate $\int_{\frac{\pi}{3}}^{\pi} xsinxdx$ (06marks)

Let u = x and $\frac{dv}{dx} = sinx$

$\frac{du}{dx} = 1, v = -cosx$

$Using\ \int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}dx$

$\int xsinxdx = xcosx - \int 1 - cosxdx$

= -xcosx + sinx + c

Or Using basic technique

Sign	Differentiate	Integrate
+	X	sinx
-	1	-cosx
+	0	-sinx

⇨ $\int_{\frac{\pi}{3}}^{\pi} xsinxdx = [-xcosx + snx]_{\frac{\pi}{3}}^{\pi}$

$= (-\pi cos\pi + sin\pi) - \left(-\frac{\pi}{s}cos\frac{\pi}{3} + sin\frac{\pi}{3}\right)$

$=\pi - \left(-\frac{\pi}{6} + \frac{\sqrt{3}}{2}\right) = \frac{7\pi}{6} - \frac{\sqrt{3}}{2}$ = 2.7992(4D)

12. (a) Find the Cartesian equation of the plane through the points whose position vectors are 2i + 2j + 3k, 3i + j + 2k and -2j + 4k. (06marks)

Let OA = 2i + 2j + 3k

OB = 3i + j + 2k

OC = -2j + 4k

Let R be the general point in the plane

Then AR = μ(AB) + λAC

OR = OA + μ(OB – OA) + λ(OC – OA)

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\2\\3\end{pmatrix} + \mu\left[\begin{pmatrix}3\\1\\2\end{pmatrix} - \begin{pmatrix}2\\2\\3\end{pmatrix} + \lambda\left[\begin{pmatrix}0\\-2\\4\end{pmatrix} - \begin{pmatrix}2\\2\\3\end{pmatrix}\right]\right]$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\2\\3\end{pmatrix} + \mu\begin{pmatrix}1\\-1\\-1\end{pmatrix} + \lambda\begin{pmatrix}-2\\-4\\1\end{pmatrix}$$

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$x = 2 + \mu - 2\lambda$ ..... (i)

$y = 2 - \mu - 4\lambda$ ..... (ii)

$z = 3 - \mu + \lambda$ .....(iii)

$Eqn\ (i) + eqn.\ (ii)$

$(x + y) = 4 - 6\lambda$ .....(iv)

$Eqn.\ (i) and\ eqn.\ (iii)$

x + z = 5 – λ

λ = -x –z + 5

Substituting for λ into eqn. (iv)

x + y =4 -6(-x –z + 5)

5x –y + 6z -26 = 0

Alternatively

Let the equation of the plane be ax + by + cz = d

Substituting point (2, 2, 3) in equation

2a + 2b +3c = d ..... (i)

Substituting point (3, 1, 2) in equation

3a + b + 2c = d ..... (ii)

Substituting point (0, -2, 4) in equation

-2b + 4c =d ..... (iii)

We have to solve for a, b, c and d

3Eqn.(i) – 2Eqn. (ii)

6a + 6b + 9c = 3d

\- 6a + 2b + 4c = 2d

4b + 5c = d..... (iv)

2eqn. (iii) + eqn. (iv)

-4b + 8c = 2d

+4b + 4c = d

13c = 3d

c = $\frac{3}{13}$d

From eqn. (iv)

4b + $\frac{15}{13}d$ = [illegible] = d - [illegible] d = $\frac{-1}{26}d$

From eqn. (i)

2a - $\frac{2}{26}d$ + $\frac{9}{13}d$ =d

2a = d + $\frac{2}{26}d$ -$\frac{9}{13}d$ = $\frac{10}{26}d$

a = $\frac{5}{26}d$

Substituting for a, b, c in the equation of the plane

$\frac{5}{26}dx - \frac{1}{26}dy + \frac{3}{13}d = d$

$Multiplying\ through\ by\ \frac{26}{d}$

$5x - y + 6z = 26$

(b) Determine the angle between the plane in (a) and the line $\frac{x-2}{2} = \frac{y}{-4} = z - 5$. (06marks)

Let n = normal vector to the plane

b = parallel vector to the plane

⇨ b = 2i – 4j + k

n =5i –j + 6k

Let θ = angle between the line and the plane

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b.n = $|b||n|\sin\theta$

$$\begin{pmatrix}2\\-4\\1\end{pmatrix}.\begin{pmatrix}5\\-1\\6\end{pmatrix} = (\sqrt{2^2+(-4)^2+1^2}.\sqrt{5^2+(-1)^2+6^2})\sin\theta$$

$10+4+6 = (\sqrt{21}.\sqrt{62})sin\theta = \sqrt{1302}sin\theta$

$sin\theta = \frac{20}{\sqrt{1302}}$; θ = 33.66^0 (2D)

13. (a) Find the three terms of the expansion $(2-x)^6$ and use it to find $(1.998)^6$ correct to two decimal places (06 marks)

$(2-x)^6 = 2^6 + \binom{6}{1}2^5(-x)^1 + \binom{6}{2}4(-x)^2$

$= 64 - 192x + 240x^2$

$(1.998)^2 = (2-0.002)^2$

$= 64 - 192(0.002) + 240(0.002)^2$

$= 64 - 0.384 + 0.00096$

$= 63.61696$

$= 63.62\ (2D)$

(b) Expand $(1-3x+2x^2)^5$ in ascending power of x as far as the x^2 term. (06marks)

$(1-3x+2x^2)^5 = [1-(-3x+2x^2)]^5$

$= 1 + \binom{5}{1}(-3x+2x^2)^1 + \binom{5}{2}(-3x+2x^2)^2$

$= 1 + 5(-3x+2x^2)^1 + 10(-3x+2x^2)^2$

$= 1 - 15x + 10x^2 + 10(9x^2 - 12x^3 + 4x^4)$

$= 1 - 15x + 10x^2 + 90x^2 - 120x^3 + 40x^4$

$= 1 - 15x + 100x^2$

14. (a) Find the equation of a normal to a curve whose parametric equation are $x = b\sec^2\theta$, $y = b\tan^2\theta$ (06marks)

$x = b\sec^2\theta$

$\frac{dx}{d\theta} = bsec^2\theta tan\theta$

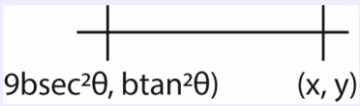
$y = b\tan^2\theta$

$\frac{dy}{d\theta} = bsec^2\theta tan\theta$

$\frac{dy}{dx} = \frac{dy}{d\theta}.\frac{d\theta}{dx} = \frac{bsec^2\theta tan\theta}{bsec^2\theta tan\theta} = 1$

Gradient of the normal = -1

Let point (x, y) lie on the normal



$\frac{y-\text{btan}^2\theta}{x-\text{bsec}^2\theta} = -1$

$y - \text{btan}^2\theta = -x + \text{bsec}^2\theta$

$(x + y) = b(\tan^2\theta + \text{bsec}^2\theta)$

(b) The area enclosed by the curve $x^2 + y^2 = a^2$, the y-axis and the line $y = \frac{1}{2}a$ is rotated through 90^0 about the y-axis. Find the volume of the solid generated. (06marks)

Note

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