BAHATI ABDULKARIM

A'LEVEL APPLIED MATHEMATICS

PROJECTILES

SUITABLE FOR S.5 AND S.6

Definition of a projectile

Contents

- **Definition of a projectile**
- Horizontal projection
- Projection at an angle
- Release at an angle from a given height
- Examination-style questions

Definition of a projectile

- A **projectile** is an object that is launched into the air and is then acted on only by gravity.
- When modelling projectiles, objects are modelled as particles and factors such as air resistance and spin are ignored.
- The path that a projectile takes is two-dimensional and is known as its **trajectory**.
- The shape of a projectile's trajectory is **parabolic**.

Projectile motion

The main factors that determine the motion of a projectile are:

its initial velocity u,

its angle of projection θ,

its point of release, and gravity

Examples of projectiles

- An arrow fired at an archery target
- A golf ball hit off the tee
- A stone kicked off the top of a cliff
- A tennis ball served horizontally
- A cricketer hitting a ball
- A child throwing a ball at a wall
- A gun firing a shell
- A cricket bowler releasing a ball

Horizontal projection

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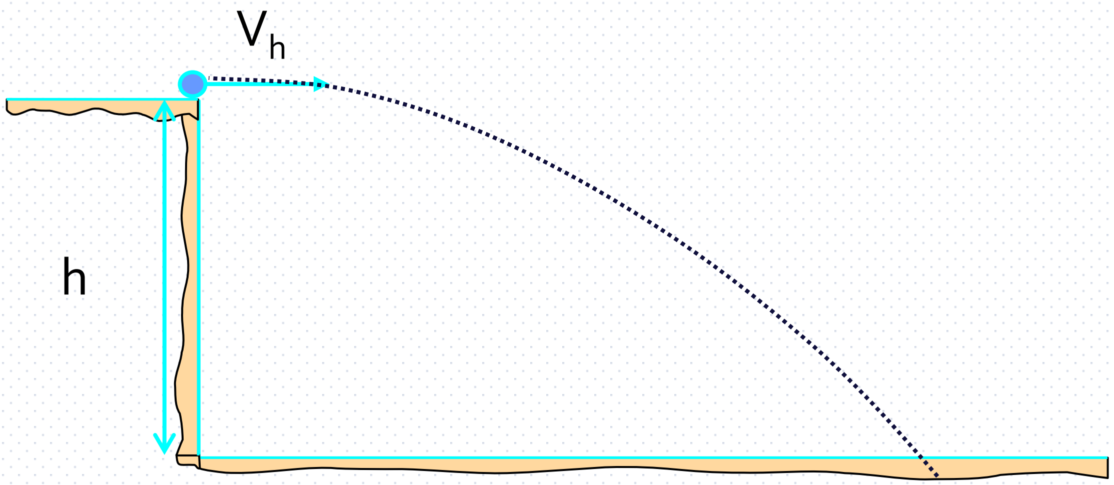
Horizontal projection



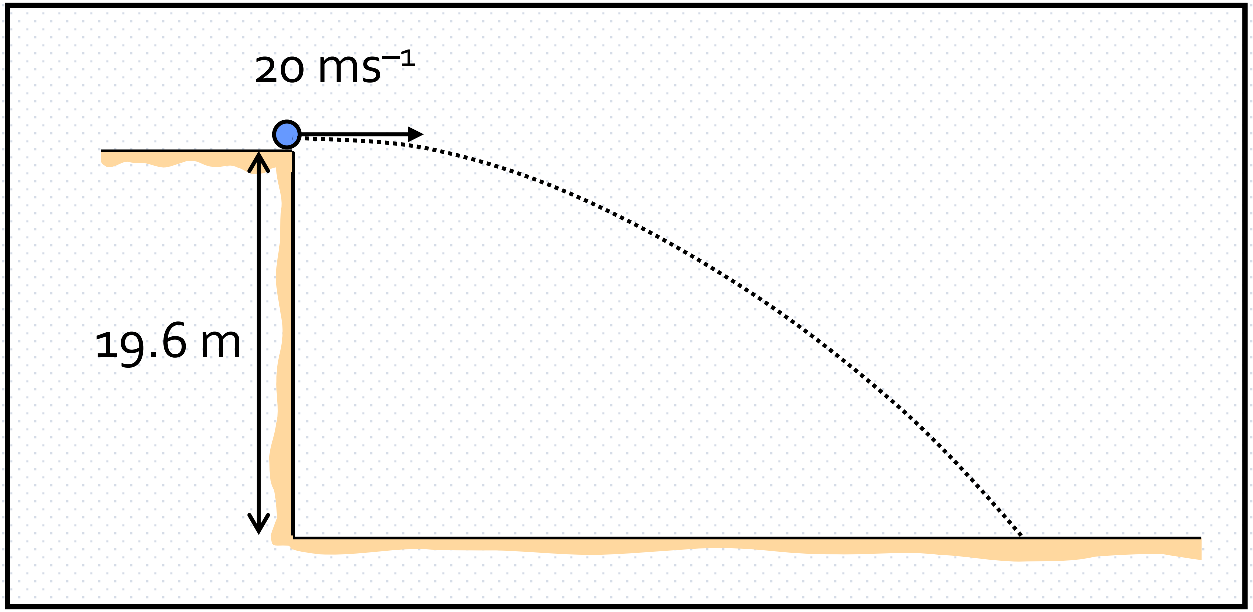
When a particle is projected horizontally, we model it as having zero acceleration (and hence a constant velocity) in the horizontal direction. In this situation, the particle has zero initial velocity and a constant acceleration of g downward in the vertical direction.

Horizontal projection question 1

A particle is projected horizontally with a velocity of 20 ms^{-1} from a height of 19.6 m. Find:

a) How long the particle takes to reach the ground.

b) The horizontal distance travelled.



Horizontal projection question 1

a) Using $s = ut + \quad at^2 \downarrow$ with $s = 19.6$, $u = 0$ and $a = g$ gives,

$$19.6 = 0 + \frac{1}{2} \times 9.8 \times t^2$$

$$4.9t^2 = 19.6$$

$$t^2 = 4$$

$$\therefore \quad t = \pm 2$$

The particle takes **2 seconds** to reach the ground.

b) Horizontal distance travelled

= horizontal component of velocity × time of flight

$= 20 \times 2$

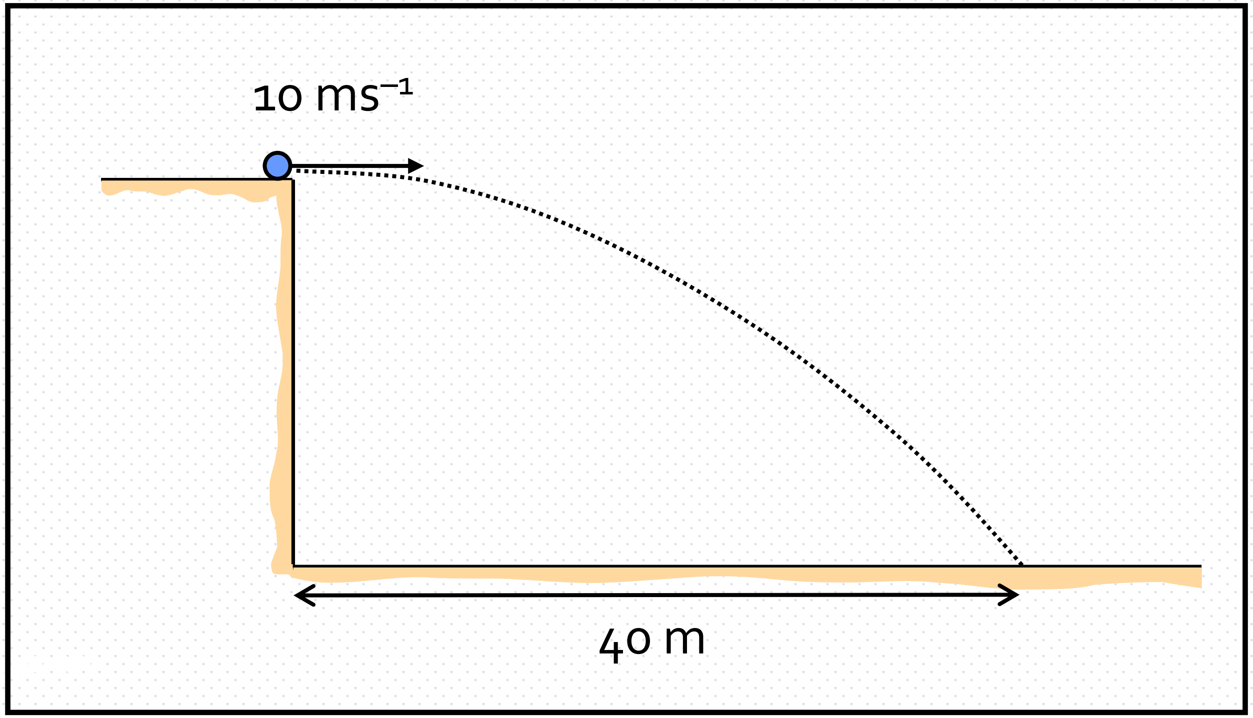
$= 40$

The particle travels **40 m** horizontally.

Horizontal projection question 2

A particle is projected horizontally with velocity 10 ms^{-1}. It travels 40 m horizontally before reaching the ground.

Find the height from which the particle is projected.



Horizontal projection question 2

The time of flight = horizontal distance travelled ÷ horizontal component of velocity

$\therefore$ Time of flight = 40 ÷ 10

= 4

The time of flight of the particle is 4 seconds.

Using $s = ut + \ \ at^2 \downarrow$ with $t = 4$, $u = 0$ and $a = g$ gives,

$$s = 0 + \quad \times 9.8 \times 4^2$$

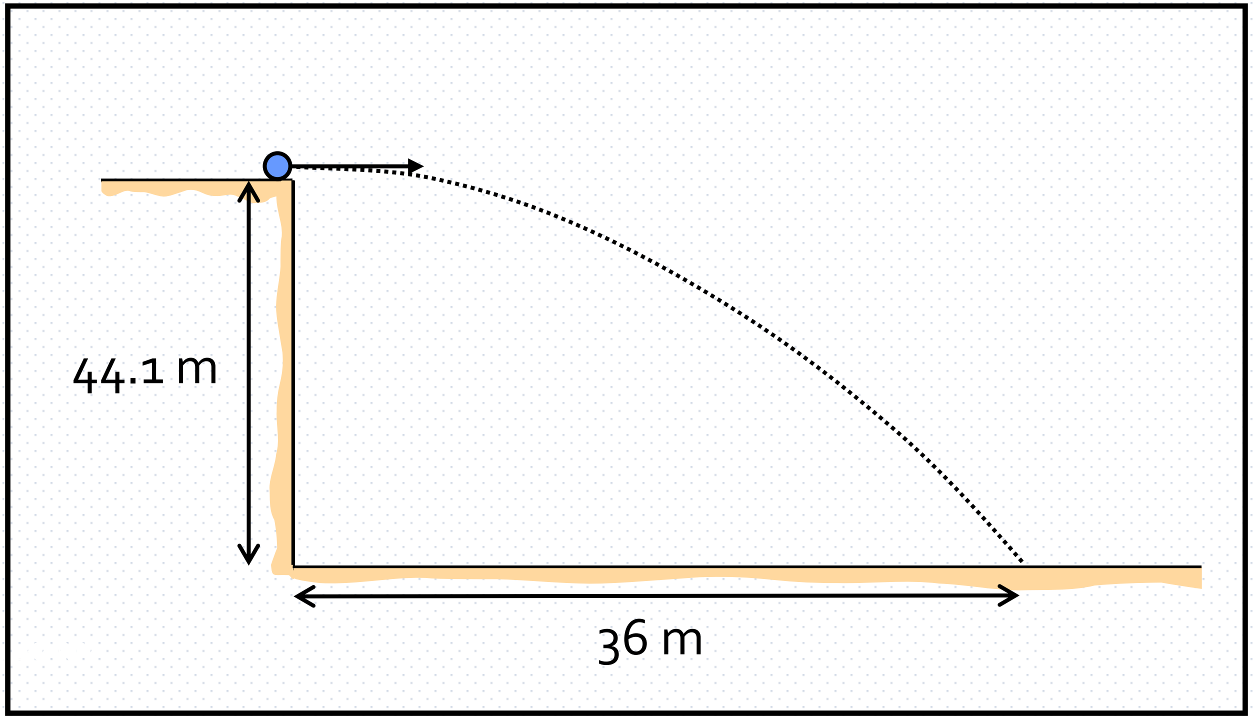
$$\therefore s = 78.4$$

The particle is projected from a height of **78.4 m**.

Horizontal projection question 3

A particle is projected horizontally from a height of 44.1 m. It travels 36 m vertically before hitting the ground.

Find the velocity with which the particle is projected.



Horizontal projection question 3

Using $s = ut + \quad at^2 \downarrow$ with $s = 44.1$, $u = 0$ and $a = g$ gives,

$$44.1 = 0 + \tfrac{1}{2} \times 9.8 \times t^2$$

$$4.9t^2 = 44.1$$

$$t^2 = 9$$

$$\therefore \quad t = \pm 3$$

The particle takes 3 seconds to reach the ground.

Horizontal component of velocity

= horizontal distance travelled ÷ time of flight

$= 36 \div 3$

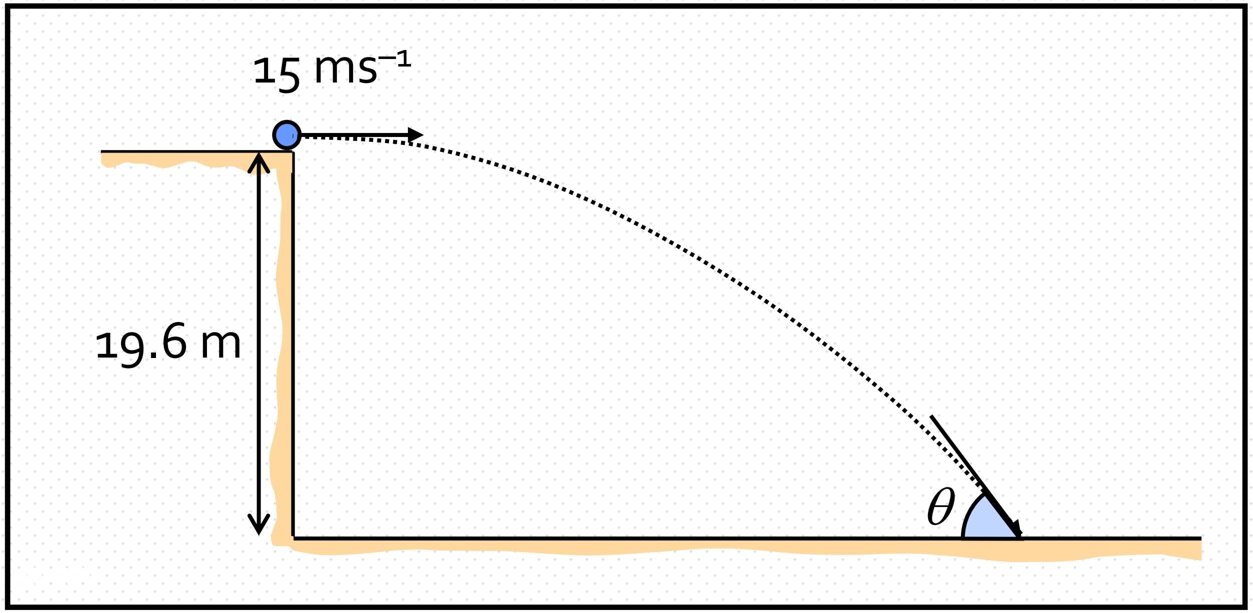
$= 12$

The particle is projected with a velocity of **$12\ \text{ms}^{-1}$**.

Horizontal projection question 4

A particle is projected horizontally with a velocity of 15 ms^{-1} from a height of 19.6 m.

Find the velocity with which the particle hits the ground and the angle it makes with the horizontal at this time.



Horizontal projection question 4

Using $v^2 = u^2 + 2as$ with $s = 19.6$, $u = 0$ and $a = g$ gives,

$$v^2 = 2 \times 9.8 \times 19.6$$

$$\therefore \quad v = 19.6$$

Horizontal component of velocity = 15 ms^{-1}

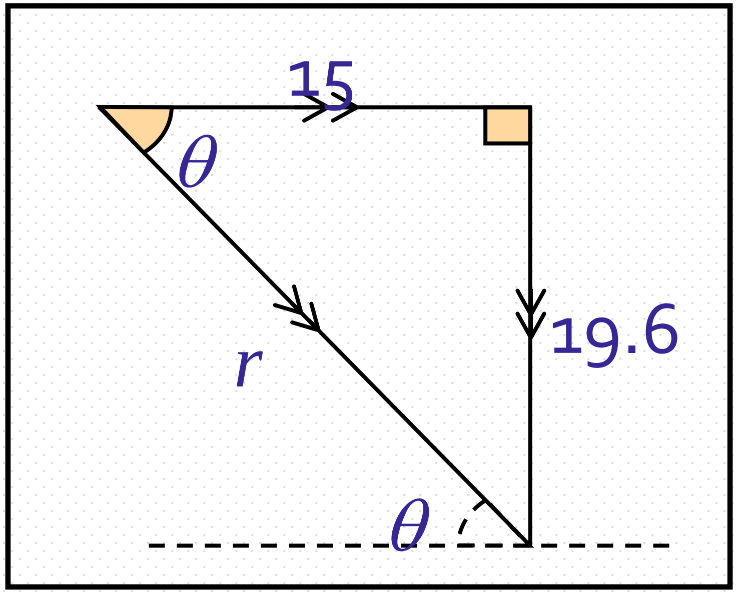
This remains constant throughout the time of flight.

Therefore the horizontal component of the final velocity is 15 ms^{-1} and the vertical component is 19.6 ms^{-1}.

To find the angle the final velocity makes with the horizontal when the particle hits the ground, the horizontal and vertical components of velocity need to be combined.

Horizontal projection question 4

Let the velocity with which the particle hits the ground be r and the angle it makes with the horizontal be θ.



$$r = \sqrt{15^2 + 19.6^2}$$

$$= 24.7 \text{ (to 3 s.f.)}$$

$$\theta = \tan^{-1}\left(\frac{19.6}{15}\right)$$

$$= 52.6° \text{ (to 3 s.f.)}$$

Therefore the particle hits the ground with a velocity of **24.7 ms^{-1}** at an angle of **52.6°** to the horizontal.

Projection at an angle

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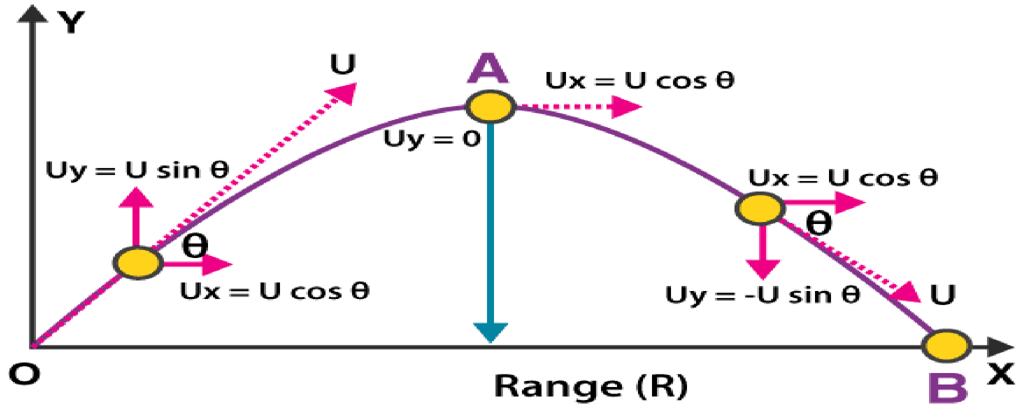
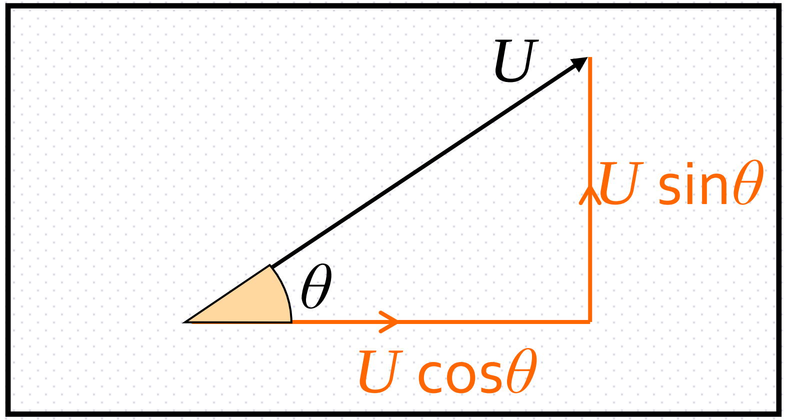
Projection at an angle

The velocity of a particle launched with a velocity U at an angle θ to the horizontal can be split into horizontal and vertical components as follows:





So, initially, when t = 0, we have,

Horizontally: $U_x = U \cos\theta$

Vertically: $U_y = U \sin\theta$

Projection at an angle

At time t, the velocity of the particle can be found using
$v = u + at$ in the horizontal and vertical directions.

Horizontally: $v_x = U \cos\theta$

Vertically: $v_y = U \sin\theta - gt$

The horizontal component of velocity is constant throughout since there is no force acting in that direction.

The vertical component of velocity changes continuously because of the weight of the particle acting downwards.

At time t, the displacement of the particle can be found using
$s = ut + \quad at^2$ in the horizontal and vertical directions.

Horizontally: $x = Ut \cos\theta$

Vertically: $y = Ut \sin\theta - \quad gt^2$

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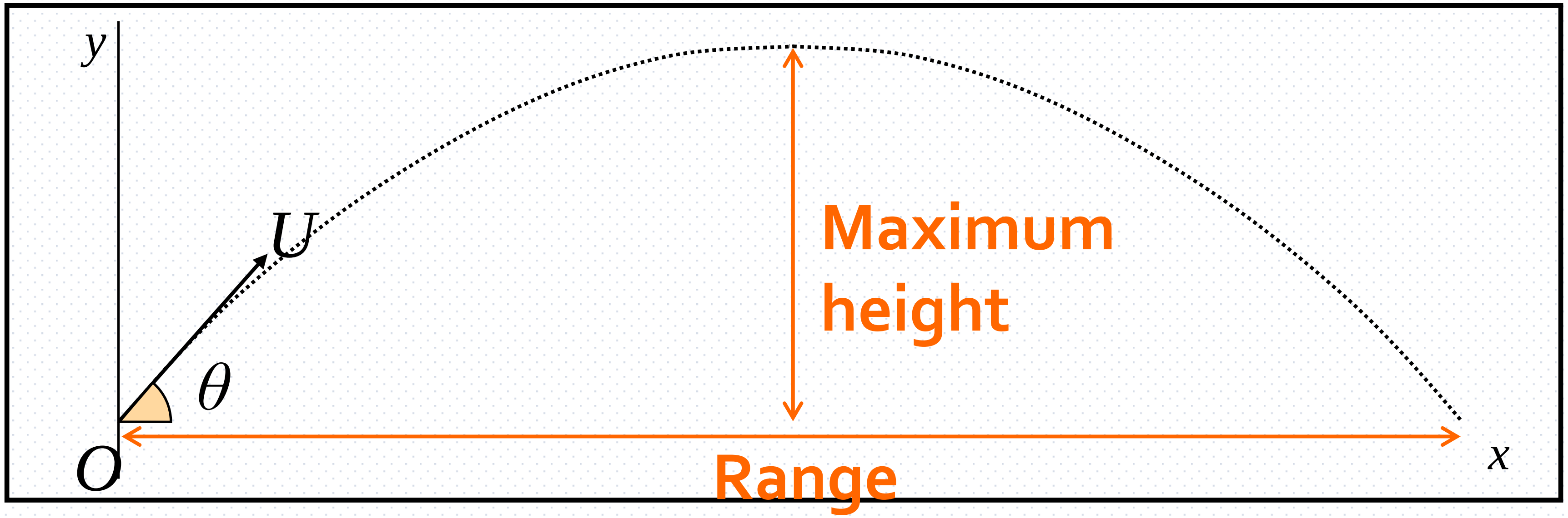
Maximum height, range and time of flight

Suppose a particle is initially launched from O with a velocity U and at an angle θ to the horizontal.

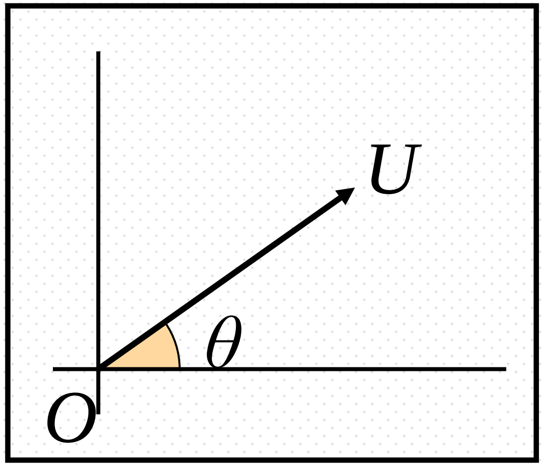


The main things that can be calculated are:

- The maximum height reached by the particle
- The range of the flight
- The total time that the particle is in flight

Maximum height

Assuming O is at ground level, a projectile reaches maximum height, H, when its vertical component of velocity is 0.



Using $v^2 = u^2 + 2as \uparrow$,

$$u = U \sin\theta$$

$$v = 0$$

$$a = -g$$

$$s = H$$

So,

$$0^2 = U^2\sin^2\theta - 2gH$$

$$2gH = U^2\sin^2\theta$$

$$\boxed{H = \frac{U^2\sin^2\theta}{2g}}$$

Time of flight

The path of a projectile is symmetrical on horizontal ground.

This means that if we find the time taken to reach the maximum height, the total time that the particle is in flight will be double that amount.

Using $v = u + at \uparrow$,

$$u = U \sin \theta$$

$$v = 0$$

$$a = -g$$

$$0 = U\sin\theta - gt$$

The time to reach maximum height is therefore $\dfrac{U\sin\theta}{g}$.

$$\therefore \quad \text{The time of flight} = \frac{2U\sin\theta}{g}$$

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Range of flight

We have just shown that the time of flight for a particle launched from O with a velocity U and at an angle θ to the horizontal is:

$$t = \frac{2U\sin\theta}{g}$$

The range of the flight, R, is given by the horizontal velocity × the time of flight.

So,

$$R = \frac{U\cos\theta \times 2U\sin\theta}{g}$$

$$\boxed{R = \frac{2U^2\sin\theta\cos\theta}{g}}$$

Projection at an angle question 1

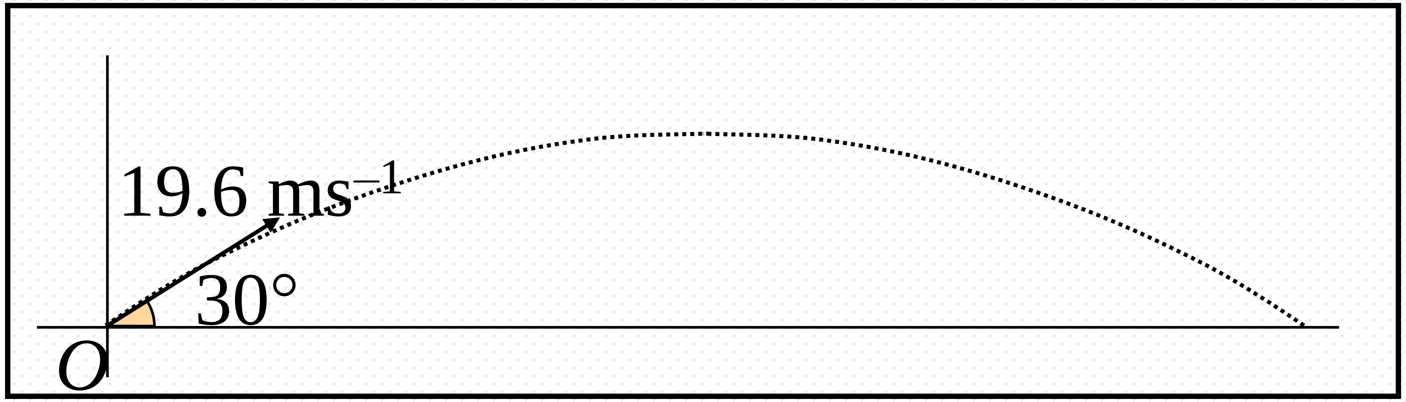
A particle is projected with a velocity of 19.6 ms^{-1} at an angle of 30° to the horizontal. Find:

a) The time taken for the particle to reach its maximum height.

b) The maximum height reached by the particle.

c) The total time of the flight.

d) The horizontal range of the particle.



Projection at an angle question 1

a) Using $v = u + at \uparrow$ with $u = 19.6\ \sin 30°$, $a = -g$ and $v = 0$ gives,

$$0 = 19.6 \sin 30° - gt$$

$$0 = 9.8 - 9.8t$$

$$\therefore \quad t = 1$$

The particle takes **1 second** to reach its maximum height.

b) Using $s = \frac{1}{2}(u + v)t \uparrow$ with $u = 19.6\ \sin 30°$, $v = 0$ and $t = 1$ gives,

$$s = \tfrac{1}{2}(9.8 + 0) \times 1$$

$$\therefore \quad s = 4.9$$

Therefore the maximum height reached by the particle is **4.9 m**.

Projection at an angle question 1

c) Using the symmetry of the trajectory the total time of flight is double the time taken to reach the maximum height.

Therefore the total time of flight of the particle is **2 seconds**.

d) Horizontal distance travelled

$= \text{horizontal component of velocity} \times \text{time of flight}$

$= 19.6 \cos 30° \times 2$

$= 33.9$ (to 3 s.f.)

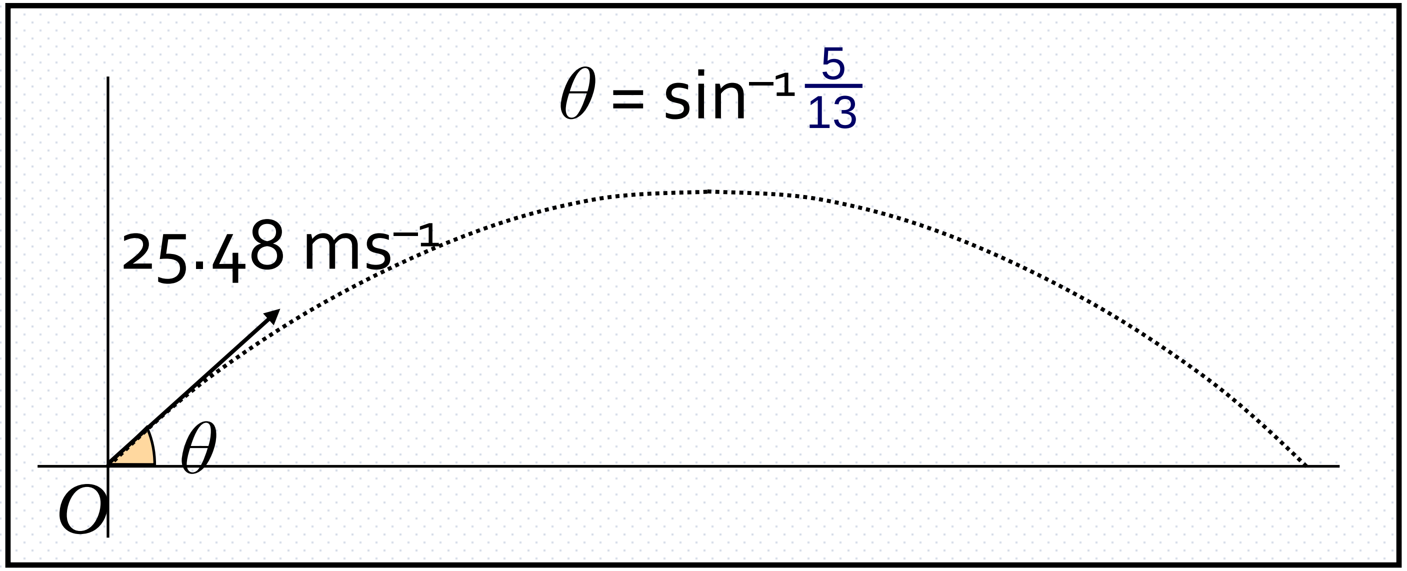
Therefore the particle travels **33.9 m** horizontally.

Projection at an angle question 2

A particle is projected with a velocity of $25.48ms^{-1}$ at an angle of $\sin^{-1}\frac{5}{13}$ to the horizontal. Find:

a) The maximum height reached by the particle

b) The horizontal range of the particle.



Projection at an angle question 2

a) Using $v^2 = u^2 + 2as \uparrow$ with $u = 25.48 \times \frac{5}{13}$, $v = 0$ and $a = -g$ gives,

$$0^2 = \left(25.48 \times \frac{5}{13}\right)^2 - 2gs$$

$$2gs = 9.8^2$$

$$s = \frac{9.8^2}{2g}$$

Therefore the maximum height reached by the particle is **4.9 m**.

Projection at an angle question 2

Using $s = ut + \quad at^2 \uparrow$ with $s = 0$, $u = 25.48 \times \frac{5}{13}$ and $a = -g$ gives,

$$0 = 9.8t - \quad gt^2$$

$$9.8t - 4.9t^2 = 0$$

$$4.9t(2 - t) = 0$$

$$\therefore \quad t = 0 \text{ or } t = 2$$

The particle is in flight for 2 seconds.

Horizontal distance travelled

= horizontal component of velocity × time of flight

$= 25.48 \times \frac{12}{13} \times 2$ (using the Pythagorean triple 5, 12, 13)

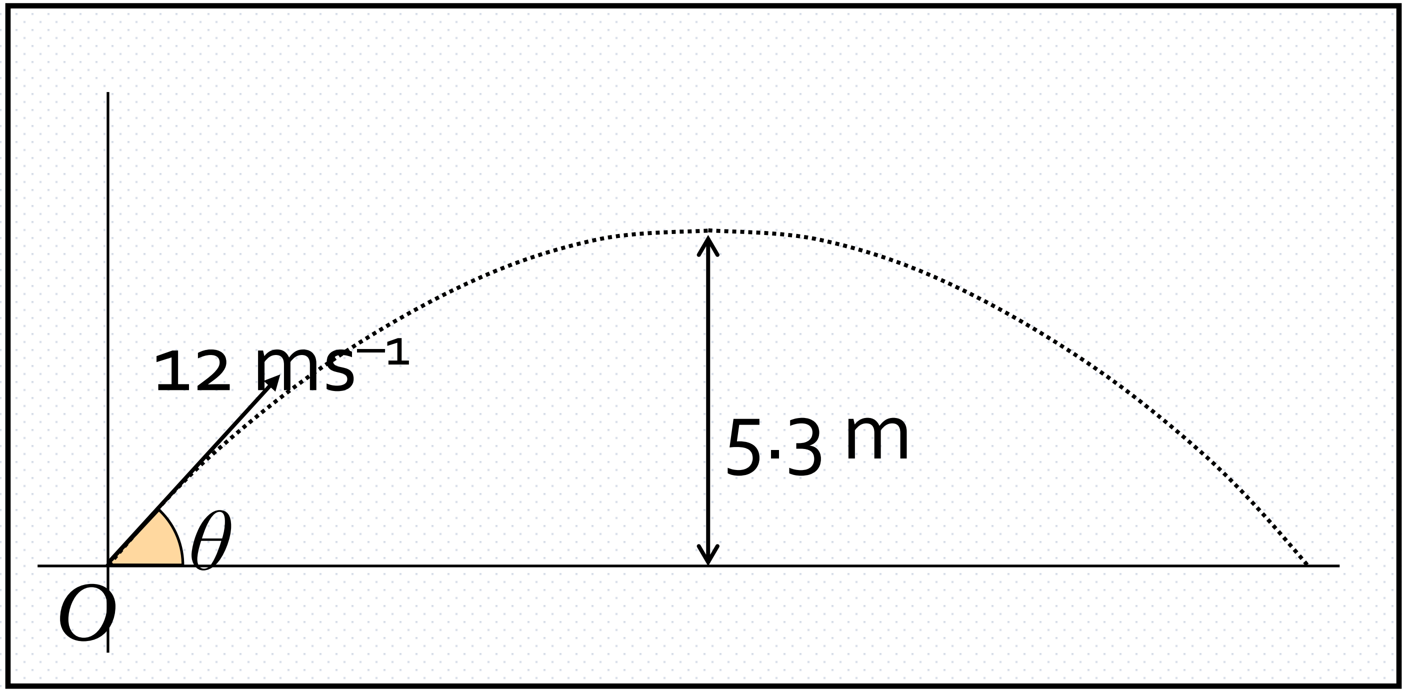
$= 47.04$

Therefore the horizontal range of the particle is **47.04 m**.

Projection AT AN ANGLE QUESTION 3

A particle is projected at a velocity of 12 ms^{-1} at an angle of $\theta°$ to the horizontal.

The maximum height reached by the particle is 5.3 m. Find θ.



Projection at an angle question 3

When the particle reaches maximum height we can use

$v^2 = u^2 + 2as\uparrow$ with $v = 0$, $u = 12\sin\theta$, $a = -g$ and $s = 5.3$ to give,

$$0^2 = (12\sin\theta)^2 - 2g \times 5.3$$

$$144\sin^2\theta = 103.88$$

$$\sin^2\theta = 0.72138\ldots$$

$$\sin\theta = 0.84934\ldots \quad \text{(ignoring the –ive solution)}$$

$$\therefore \quad \theta = 58.1 \text{ (to 3 s.f.)}$$

Therefore the particle is projected at an angle of **58.1°** to the horizontal.

Projection at an angle question 4

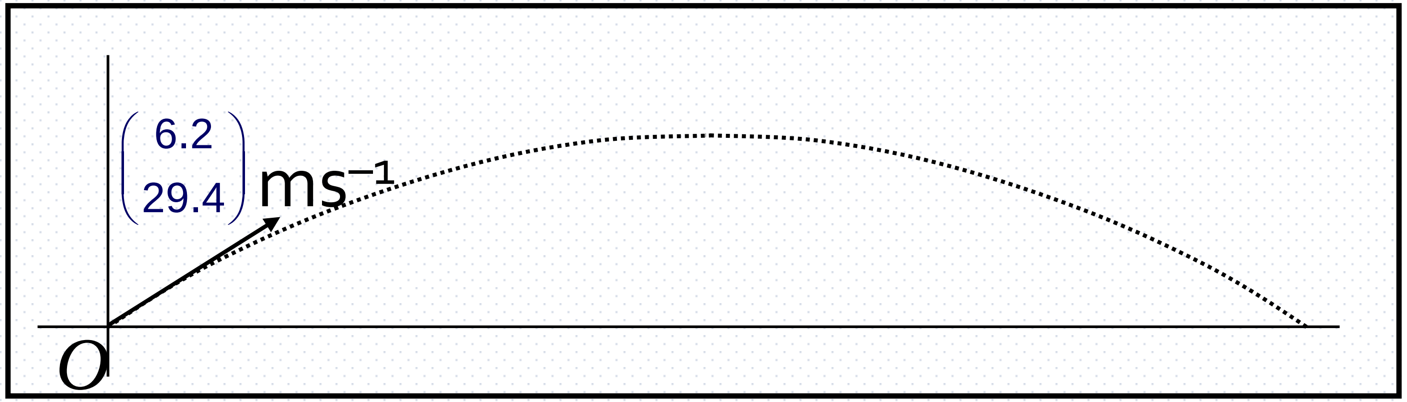
A particle is projected with a velocity of $\begin{pmatrix} 6.2 \\ 29.4 \end{pmatrix}$ ms^{-1}. Find:

a) The time taken for the particle to reach its maximum height.

b) The maximum height reached by the particle.

c) The total time of the flight.

d) The horizontal range of the particle.



Projection at an angle question 4

a) Using $v = u + at$ $\uparrow$ with $u = 29.4$, $a = -g$ and $v = 0$ gives,

$$0 = 29.4 - gt$$

$$29.4 = 9.8t$$

$$\therefore \quad t = 3$$

The particle takes **3 seconds** to reach its maximum height.

b) Using $s = \frac{1}{2}(u + v)t$ $\uparrow$ with $u = 29.4$, $v = 0$ and $t = 3$ gives,

$$s = \tfrac{1}{2}(29.4 + 0) \times 3$$

$$\therefore \quad s = 44.1$$

Therefore the maximum height reached by the particle is **44.1 m**.

Projection at an angle question 4

c) Using the symmetry of the trajectory the total time of flight is double the time taken to reach the maximum height.

Therefore the total time of flight of the particle is **6 seconds**.

d) Horizontal distance travelled

= horizontal component of velocity × time of flight

$= 6.2 \times 6$

$= 37.2$

Therefore the particle travels **37.2 m** horizontally.

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Release at an angle from a given height

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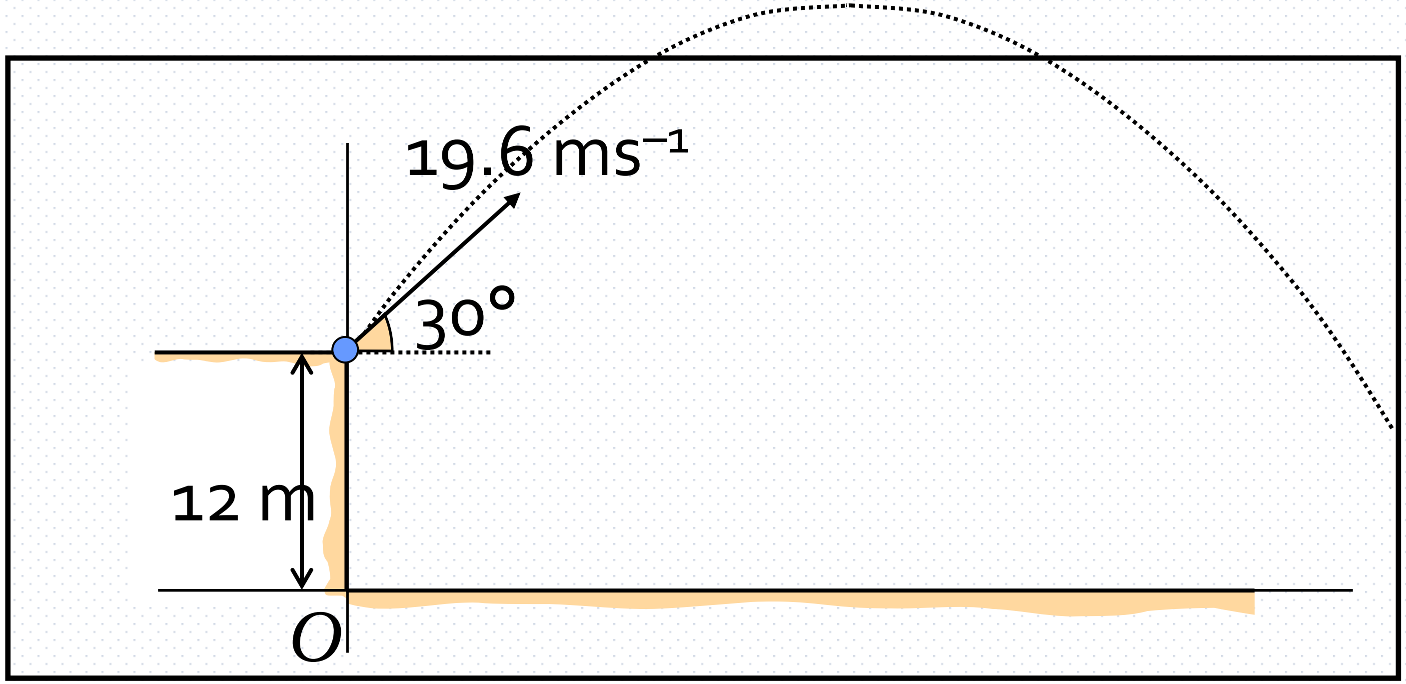
Release at an angle question 1

A particle is projected from a height of 12 m with a velocity of 19.6 ms^{-1} at an angle of 30° to the horizontal. Find:

a) The time taken for the particle to reach the ground.

b) The maximum height above the ground reached by the particle.



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Release at an angle question 1

a) Using $s = ut + \quad at^2 \uparrow$ with $s = -12$, $u = 19.6 \sin 30^\circ$ and $a = -g$ gives,

$$-12 = 9.8t - 4.9t^2$$
$$4.9t^2 - 9.8t - 12 = 0$$
$$\therefore \quad t = 2.86 \text{ or } -0.857 \text{ (to 3 s.f.)}$$

Therefore the time taken for the particle to reach the ground is **2.86 seconds**.

b) Using $v^2 = u^2 + 2as \uparrow$ with $v = 0$, $u = 19.6 \sin 30^\circ$ and $a = -g$ gives,

$$0^2 = 9.8^2 - 2(9.8)s$$
$$2s = 9.8$$
$$s = 4.9$$

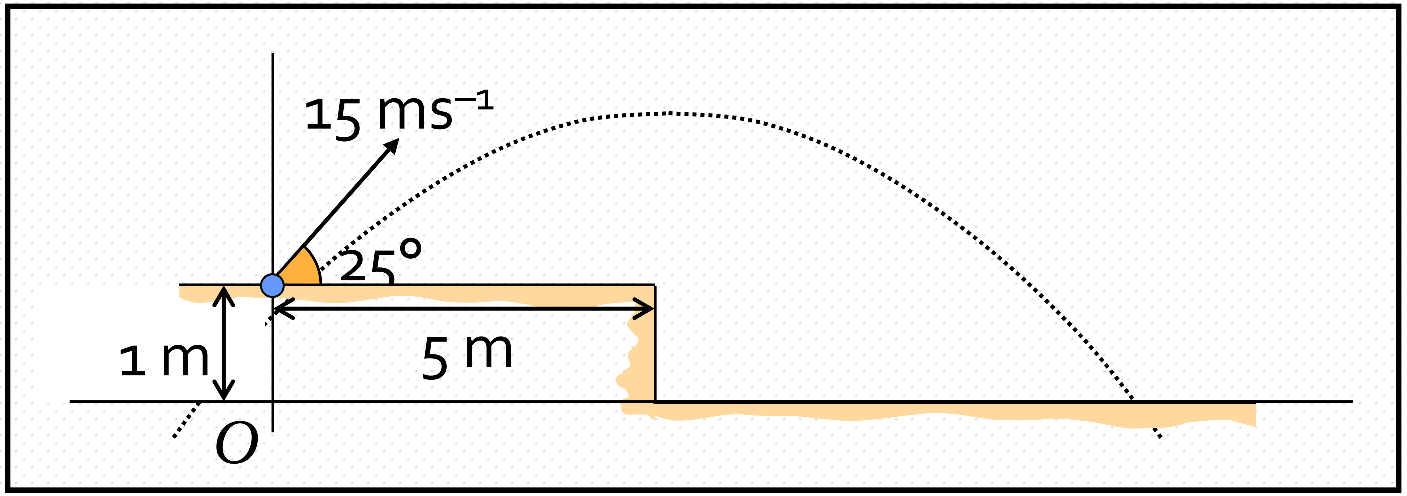
Therefore the maximum height above the ground reached by the particle is $12 + 4.9 =$ **16.9 m**.

Release at an angle question 2

A particle is projected at a velocity of 15 ms^{-1} at an angle of 25° to the horizontal. The particle is projected from the back of a raised platform that is 1m high and extends for 5 m. Find:

a) The time taken for the particle to reach the ground.

b) The velocity with which the particle hits the ground and the angle it makes with the horizontal at this time.



Release at an angle question 2

a) Using $s = ut + \quad at^2$ ↑ with $s = -1$, $u = 15\sin 25°$ and $a = -g$ gives,

$$-1 = 15t\sin 25° - 4.9t^2$$
$$4.9t^2 - (15\sin 25°)t - 1 = 0$$
$$\therefore \quad t = 1.44 \text{ or } -0.142 \text{ (to 3 s.f.)}$$

Therefore the time taken for the particle to reach the ground is **1.44 seconds**.

b) Using $v^2 = u^2 + 2as$ ↑ with $u = 15\sin 25°$, $a = -g$ and $s = -1$ gives,

$$v^2 = (15\sin 25°)^2 + 2(-9.8)(-1)$$
$$v^2 = 59.786\ldots$$
$$v = 7.73 \text{ (to 3 s.f.)}$$

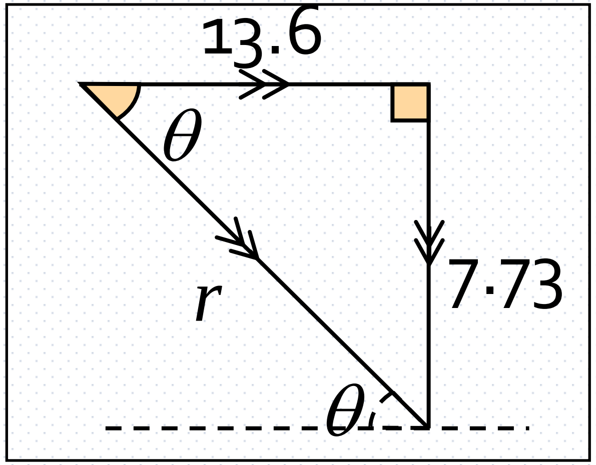
Therefore the vertical component of the velocity is $7.73\ \text{ms}^{-1}$.

Release at an angle question 2

The horizontal component of velocity = 15 cos25°

= 13.6 ms^{-1} (to 3 s.f.)

To find the velocity with which the particle hits the ground and the angle it makes with the horizontal, the horizontal and vertical components of velocity need to be combined.



$$r = \sqrt{13.6^2 + 7.73^2}$$

= 15.6 m (to 3 s.f.)

$$\theta = \tan^{-1}\left(\frac{7.73}{13.6}\right)$$

= 29.6° (to 3 s.f.)

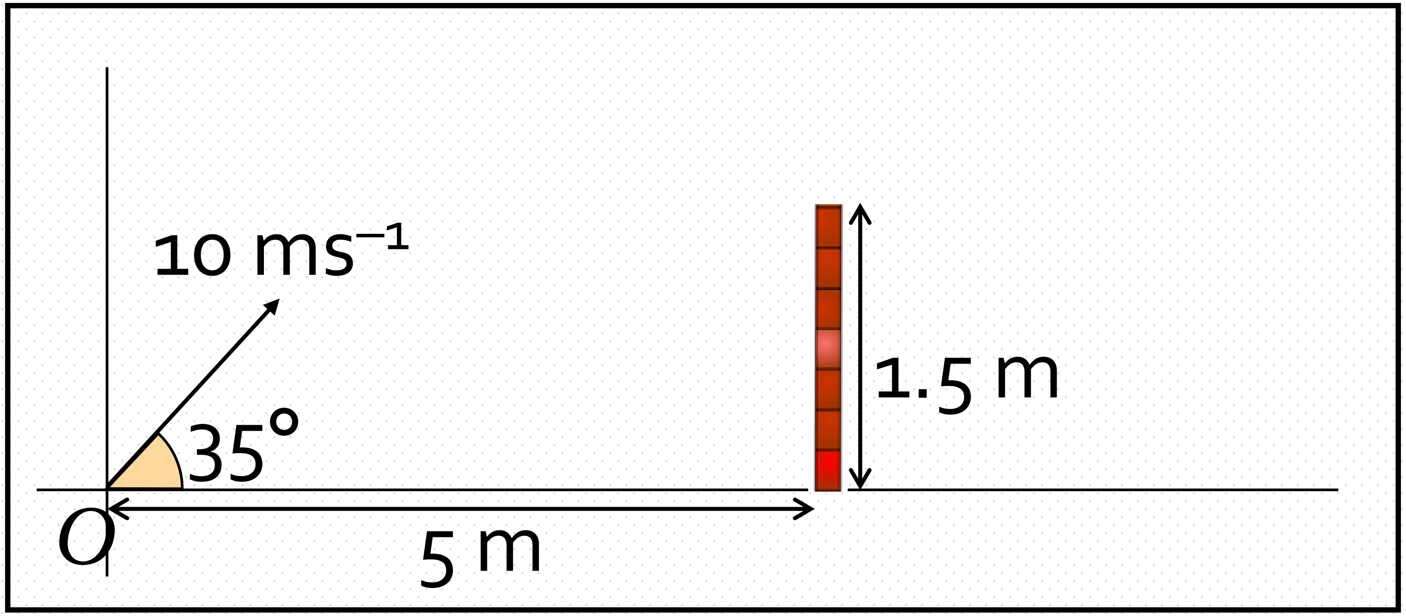
Therefore the particle hits the ground with a velocity of **15.6 ms^{-1}** at an angle of **29.6°** to the horizontal.

Clearing an obstacle

A particle is projected with a velocity of 10 ms^{-1} at an angle of 35° to the horizontal.

A wall of height 1.5 m lies 5 m horizontally from the point of projection.

Will the particle clear the wall?



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Clearing an obstacle

Time taken to travel 5 m horizontally

= 5 ÷ horizontal component of velocity

= 5 ÷ 10 cos35°

= 0.610 (to 3 s.f.)

Using $s = ut + \quad at^2$ ↑ with u = 10 sin35°, t = 0.610 and $a = -g$ gives,

s = 10 sin35° × 0.610 – 4.9 × 0.610²

s = 1.68

Therefore when the particle has travelled 5 m horizontally it is at a height of 1.68 m.

So, yes, the particle will clear a wall of height 1.68 m.

Release at an angle from a given height

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- **Examination-style questions**

Examination-style question 1

A ball is thrown from a point O with a velocity of 10 ms^{-1} at an angle of $\theta°$ to the horizontal.

At time t seconds after projection the ball is at the point (x, y).

a) i) Find expressions in terms of θ and t for x and y.

ii) By eliminating t, find the equation of the trajectory of the ball.

In the case where $\theta = 45°$, find:

b) i) the maximum height of the ball above the level of projection.

ii) the time of the flight.

Examination-style question 1

a) i) Using $s = ut + \quad at^2 \uparrow$ with $s = y$, $u = 10 \sin\theta$ and $a = -g$ gives,

$$y = 10t \sin\theta - 4.9t^2$$

Horizontal distance travelled

= horizontal component of velocity × time

= $10 \cos\theta \times t$

$\therefore \quad x = 10t \cos\theta \qquad \Rightarrow \quad t = \dfrac{x}{10 \cos\theta}$

$$y = 10\left(\frac{x}{10\cos\theta}\right)\sin\theta - 4.9\left(\frac{x^2}{100\cos^2\theta}\right)$$

$$y = x\tan\theta - \frac{0.049x^2}{\cos^2\theta}$$

Examination-style question 1

b) i) Using $v^2 = u^2 + 2as \uparrow$ with $u = 10 \sin 45°$, $v = 0$ and $a = -g$ gives,

$$0^2 = (10 \sin 45°)^2 - 2gs$$
$$2gs = 50$$
$$s = 2.55 \text{ (to 3 s.f.)}$$

Therefore the maximum height above the level of projection reached by the ball is **2.55 m**.

b) ii) Using $s = ut + \quad at^2 \uparrow$ with $s = 0$, $u = 10 \sin 45°$, and $a = -g$ gives,

$$0 = 10t \cos 45° - 4.9t^2$$
$$t(4.9t - 10\cos 45°) = 0$$
$$\therefore\ t = 0 \text{ or } t = 1.44 \text{ (to 3 s.f.)}$$

Therefore the time of flight is **1.44 seconds**.

Examination-style question 2

A golf ball is projected from the tee with a velocity of 50 ms^{-1} at an angle to the horizontal of 45°. The ball lands at a point 4 m above the level of projection.

a) Show that the equation of the path of the golf ball is $y = x - \dfrac{gx^2}{2500}$

b) Find the horizontal distance travelled by the golf ball until it first reaches the ground.

a) Using $s = ut + \frac{1}{2}at^2$ $\uparrow$ with $s = y$, $u = 50 \sin 45°$ and $a = -g$ gives,

$$y = 50t \sin 45° - \tfrac{1}{2}gt^2$$

Examination-style question 2

a) Horizontal distance travelled

= horizontal component of velocity × time

$$\therefore \quad x = 50t\cos 45° \qquad \Rightarrow \quad t = \frac{x}{50\cos 45°}$$

Substituting this into the equation in y gives

$$y = 50\left(\frac{x}{50\cos 45^\circ}\right)\sin 45° - \frac{g}{2}\left(\frac{x^2}{2500\cos^2 45^\circ}\right)$$

Using the fact that $\frac{\sin\theta}{\cos\theta} = \tan\theta$ and $\cos^2 45° = \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}$

$$y = x\tan 45^\circ - \frac{gx^2}{2\times 2500\times \frac{1}{2}}$$

$$\therefore \quad y = x - \frac{gx^2}{2500} \qquad \text{as required.}$$

Examination-style question 2

b) Using $s = ut + \quad at^2$ ↑ with $s = 4$, $u = 50 \sin 45°$ and $a = -g$ gives,

$$4 = 50t \sin 45° - 4.9t^2$$

$$4.9t^2 - 50t \sin 45° + 4 = 0$$

$$\therefore\ t = 0.115 \text{ or } t = 7.10 \text{ (to 3 s.f.)}$$

Therefore the time of flight of the golf ball is 7.10 seconds.

Horizontal distance travelled

= horizontal component of velocity × time of flight.

$= 50 \cos 45° \times 7.10$

$= 251$ (to 3 s.f.)

Therefore the horizontal range of the golf ball is **251 m**.


VectorStock