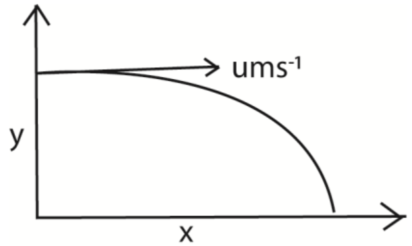
Projectile motion

This is the motion of a body which after being given an initial velocity moves freely under the influence of gravity.

Terms used in projectiles

1. ***Angle of projection:*** is the angle the initial velocity makes with the horizontal
2. ***Maximum height/ greatest height*** is the greatest height reached by projectile
3. ***Time of flight (T)*** is the time taken for projectile to complete motion. Note: the time of flight is twice the time to maximum height.
4. ***Range, R:*** is the horizontal distance covered by projectile
5. ***Maximum range (Rmax)*** is the greatest horizontal distance covered
6. ***Trajectory;*** is the path described by a projectile.

A. An object projected horizontally from a height above the ground.



Horizontal motion: Ux = u, a = 0

x = ut

Vertical motion: Uy = 0, a = $-9.8ms^{-2}$

$s = ut + ½ at^2$

$-y =- ½ gt^2$

V = u + at

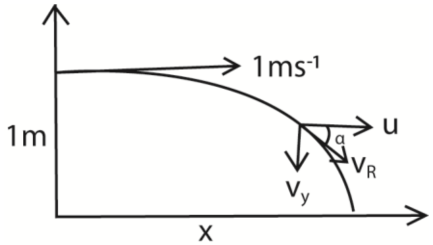
Vy = -gt

Example 1

A ball rolls of the edge of a table top 1m high above the floor with horizontal velocity 1ms-1. find:

(i) time taken to hit the floor
(ii) horizontal distance covered
(iii) the velocity when it hits the floor.

Solution



(i) vertical motion: $u = 1ms^{-1}$, $\theta = 0$

y = 1m below the point of projection

$y = \frac{1}{2} gt^2$

$-1 = \frac{1}{2}\, x - 9.8\, t^2$

t = 0.4518s

(ii) x = ut = 1 x 0.4518 = 0.4518m

(iii) $v_x = 1ms^{-1}$

$v_y = -gt = -9.8 \times 0.4518 = -4.428ms^{-1}$

$v_R = \sqrt{1^2 + (-4.428)^2} = 4.54ms^{-1}$

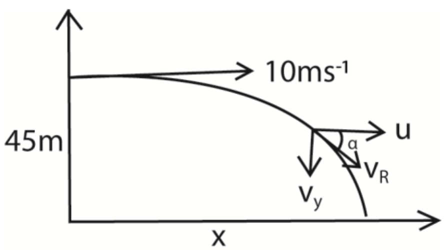
$\alpha = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{4.428}{1}\right) = 77.3^0$

Example 2

A ball is thrown forward horizontally from the top of a cliff with a velocity of $10ms^{-1}$. The height of a cliff above the ground is 45m. Calculate

(i) time to reach the ground
(ii) distance from the cliff where the ball hits the ground
(iii) velocity and direction of the ball just before it hits the ground

Solution



(ii) vertical motion: $u = 10ms^{-1}$, $\theta = 0$

y = 45m below the point of projection

$y = \frac{1}{2} gt^2$

$-45 = \frac{1}{2}\, x - 9.8\, t^2$

t = 3.03s

(ii) x = ut = 10 x 3.03 = 30.3m

(iii) $v_x = 10ms^{-1}$

$v_y = -gt = -9.8 \times 3.03 = -29.694ms^{-1}$

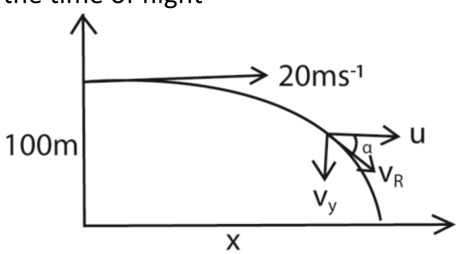
$v_R = \sqrt{10^2 + (-29.694)^2} = 31.33ms^{-1}$

$\alpha = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{29.694}{10}\right) = 71.4^0$

Example 3

An object is projected horizontally at a speed of $20ms^{-1}$ from a height of 100m. Find

(i) the time of flight



$y = \frac{1}{2} gt^2$

$-100 = \frac{1}{2}\, x - 9.8\, t^2$

$t = 4.52ms^{-1}$

(ii) the horizontal range

x = ut = 20 x 4.52 = 90.4m

(iii) its velocity on reaching the ground

v_x = 20ms^{-1}

v_y = -gt = -9.8 x 4.52 = -32.7ms^{-1}

$v_R = \sqrt{20^2 + (-32.7)^2}$ = 38 .33ms^{-1}

$\alpha = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{32.7}{20}\right)$ = 58.5^0

Example 4

At time t = 0, a particle is projected with a velocity of 3ms-1 from a point with position vector (5i + 25j)m. Find the

(i) speed and direction of the particle when t = 2s

v_x = u = 3ms^{-1}

v_y = gt = -9.8 x 2 = -19.6ms^{-1}

$v_R = \sqrt{3^2 + (-19.6)^2}$ =19.83ms^{-1}

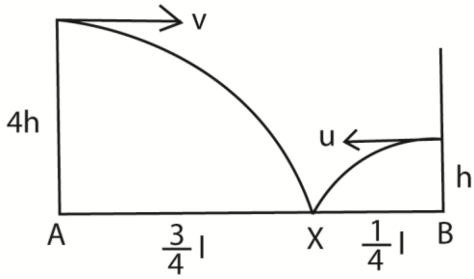
$\alpha = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{19.6}{3}\right)$ = 81.3^0

(ii) Position vector of the particle when t = 2s

$$P_{(t=t)} = \binom{x_0}{y_0} + \binom{x}{y} = \binom{5}{25} + \begin{pmatrix} ut \\ -\frac{1}{2}gt^2 \end{pmatrix} = \binom{5}{25} + \begin{pmatrix} 3\ x\ 2 \\ -\frac{1}{2}x9.8\ x\ 2^2 \end{pmatrix} = \binom{11}{5.4}m$$

Example 5

A and B are two points on a level ground. A vertical tower of height 4h has its base at A and vertical tower of height h has its bas at B. When a stone is thrown horizontally with speed v from the top the taller tower towards the smaller tower, it lands at point X where AX = $\frac{3}{4}AB$. When a stone is thrown horizontally with speed u from the top of the smaller tower towards the taller tower, it also lands at the point A. Show that 3u = 2V



For A Vertical motion: y = $\frac{1}{2}gt^2$

4h = $\frac{1}{2}gt^2$

t = $\sqrt{\frac{8h}{g}}$

Horizontal motion: x = vt

$\frac{3}{4}l = vt$

$l = \frac{4}{3}vt = \frac{4}{3}v\sqrt{\frac{8h}{g}}\ \ldots.(i)$

For B Vertical motion: y = $\frac{1}{2}gt^2$

h = $\frac{1}{2}gt^2$

t = $\sqrt{\frac{2h}{g}}$

Horizontal motion: x = ut

$\frac{1}{4}l = ut$

$l = \frac{4}{3}ut = \frac{4}{3}v\sqrt{\frac{2h}{g}}\ \ldots.(ii)$

Eqn (i) and (ii)

$\frac{4}{3}v\sqrt{\frac{8h}{g}} = \frac{4}{3}u\sqrt{\frac{2h}{g}}$;

v = $\frac{3}{2}u$

2v = 3u

Revision exercise 1

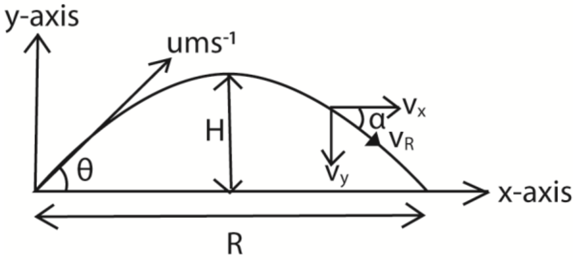
1. A pencil is accidentally knocked off the edge of a horizontal desktop. The height of the desk is 64.8cm and the pencil hits the floor a horizontal distance of 32.4cm from the edge of the desk. What was the speed of the pencil as it left the desk? [$0.9ms^{-1}$]
2. A particle is projected horizontally at $20ms^{-1}$ from a point 78.4m above a horizontal surface. Find the time taken for the particle to reach the surface and the horizontal distance travelled in that time. [4s, 80m]
3. A particle is projected horizontally with a speed of $2ms^{-1}$
   (i) Find the horizontal and vertical displacements of the particle from the point of projection, $2\frac{6}{7}$s after projection [60m, 40m below]
   (ii) Find how far the particle is then from the point of projection. [72.1m]
4. A particle is projected horizontally from a point 2.5m above the horizontal surface. The partial hits the surface at a point which is horizontally 10m from the point of projection. Find the initial speed of the particle.[$14ms^{-1}$]
5. At time t= 0, a particle is projected with a velocity of $2ims^{-1}$ from a point with position vector (10i + 150j)m. Find the
   (i) speed and direction of the particle when t = 5s [$49.04ms^{-1}$, at 87.6^0]
   (ii) position vector of the particle when t = 1s $\left[\begin{pmatrix}20\\27.5\end{pmatrix}m\right]$
6. At time t = 0, a particle is projected with a velocity of 5ims-1 from a point with position vector (20j)m. find the position vector of the particle when t = 2s. $\left[\begin{pmatrix}10\\0.4\end{pmatrix}m\right]$.
7. A batsman strikes a ball horizontally when it is 1m above the ground. The ball is caught 10cm above the grounder by a fielder standing 12m from the batsman. Find the speed with which the batsman hits the ball. [$28ms^{-1}$]
8. A darts player throws a dart horizontally with a speed of $14ms^{-1}$. The dart hits the board at a point 10cm below the level at which it is released. Find the horizontal distance travelled by the dart. [2m]
9. A tennis ball is served horizontally with initial speed of $21ms^{-1}$ from a height of 2.8m, by what distance does the ball clears a net 1m high situated 12m horizontally from the server? [20cm]
10. A fielder retrieves cricket ball and throws it horizontally with a speed of $28ms^{-1}$ to a wicket-keeper standing 12m away. If the fielder releases the ball at a height of 2m above level ground, find the height of the ball when it reaches the wicket-keeper. [110cm]
11. Initially a particle is at an origin and is projected with a velocity of ai ms^{-1}. After t second, the particle is at the point with position vector (30i – 21j)m. Find the value of t and a. [$1\frac{3}{7}, 21$]
12. Two vertical towers stand a horizontal ground level and are of height 40m and 30m. A ball is thrown horizontally from the top of the higher tower with a speed of $24.5ms^{-1}$ and just clears the short tower. Find the distance
    (i) between the two towers [35m]
    (ii) between the short and the point on the ground where the ball first lands. [35m]
13. The top of a vertical tower is 20m above ground level. When a ball is thrown horizontally from the top of this tower. By how much does the ball clears a vertical wall of height 13m situated 12m from the tower.[2m]
14. A stone is thrown horizontally with speed u from the edge of a vertical cliff of height h. The stone hits the ground at a point which is a distance d horizontally from the base of the cliff. Show that $2hu^2 = gd^2$.
15. A vertical tower stands with its base on a horizontal ground. Two particles A and B are both projected horizontally and in the same direction from the top of the tower with initial velocities

of $14ms^{-1}$ and $17.5ms^{-1}$ respectively. If A and B hit the ground at two points 10cm apart, find the height of the tower[40m]

16. O, A and B are three points with O on level ground and A and B respectively 3.6 and 40m vertically above O. A particle is projected horizontally from B with a speed of $21ms^{-1}$ and 2seconds later, a particle is projected horizontally from A with a speed of $70ms^{-1}$. Show that the two particles reach the ground at the same distance from O, find this distance.
17. An aeroplane moving horizontally at 150ms-1 releases a bomb at height of 500m. The aeroplane hits the intended target. What was the horizontal distance of aeroplane from the target when the bomb was released? [1500m]
18. A projectile is fired horizontally from the top of a cliff 250m high. The projectile lands 144m from the bottom of the cliff. Find the
    (i) initial speed of the projectile [198ms-1]
    (ii) velocity of the projectile just before it hits the ground [$210ms^{-1}$ at 19.5^0]

b. Object projected upwards from the ground at an angle to the horizontal

Suppose an object is project with velocity u at an angle θ from a horizontal ground. H and R are the maximum height reached and range respectively.



Horizontally; u_x = ucosθ, a = 0

v = u + at

v_x = ucosθ

x = ucosθt

v_y = s = ut + $\frac{1}{2}at^2$

x = ucosθt

Vertically; uy = usinθ, a = $-9.8ms^{-2}$

v = u + at

v_y = usinθ – gt

s = ut + $\frac{1}{2}at^2$

y = usinθt - $\frac{1}{2}gt^2$

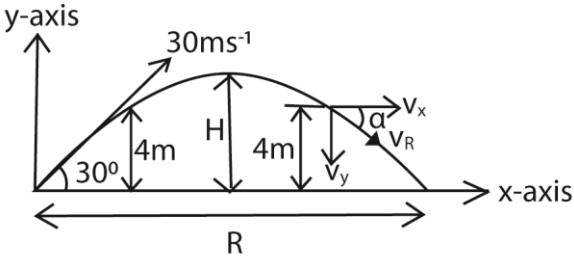
Example 6

A particle is projected with a velocity of $30ms^{-1}$ at an angle of elevation of 30^0. Find

(i) the greatest height reached

(ii) the time of flight

(iii) the velocity and direction of motion at a height of 4m on its way upwards.



(i) (↑) v_y = usinθ, a = $-9.8ms^{-2}$

At maximum height, v_y = 0

From $v^2 = u^2 + 2as$

H = $\frac{(30\sin 30)^2}{2\,x\,9.8} = 11{,}47m$

(ii) at time of flight, s = 0

s = ut + $\frac{1}{2}at^2$

0 = 30sin30T - $\frac{1}{2}x\ 9.8\ xT^2$

0 = (30sin30 - $\frac{1}{2}x\ 9.8\ T)T$

Either T = 0

or (30sin30 - $\frac{1}{2}x\ 9.8\ T$) = 0

T = 3.0612s

(→): u_x = ucosθ, a = 0

s = ut + $\frac{1}{2}at^2$

R = ucosθT

R = (30cos 30) x 3.0612 = 79.5329m

(↑) y = usinθt - $\frac{1}{2}gt^2$

4 = 30sin30t - $\frac{1}{2}x\ 9.8\ x\ t^2$

t = 0.30s or t = 2.76s

t = 0.30s is the correct time since it is smaller indicating that the body is moving upwards

u_x = ucosθ = 30cos30 = 25.98ms^{-1}

v_y = usinθ – gt

= 30sin30 – 9.8 x 0.30 = 12.06ms^{-1}

v = $\sqrt{(v_x)^2 + (v_y)^2}$

= $\sqrt{25.98^2 + 12.06^2}$ = 28.64ms^{-1}

direction α = $\tan^{-1}\left(\frac{12.06}{25.98}\right)$= 24.9^0

Example 7

A particle is projected from the origin at a velocity of (10i + 20j)ms^{-1}. Find the position and velocity vectors of the particle 3s after projection. (Take g = 10ms^{-2})

Solution

$$P_{t=t} = \binom{x_0}{y_0} + \binom{x}{y}$$

$$P_{(t=3)} = \binom{0}{0} + \begin{pmatrix} ucos\theta t \\ usin\theta t - \frac{1}{2}gt^2 \end{pmatrix}$$

$$P_{(t=3)} = \begin{pmatrix} 10\ x\ 3 \\ 20\ x\ 3 - \frac{1}{2}\ x9.8\ x\ 3^2 \end{pmatrix} = \binom{30}{15} m$$

$$v_{(t=t)} = \binom{v_x}{v_y} = \begin{pmatrix} ucos\theta \\ usin\theta - gt \end{pmatrix}$$

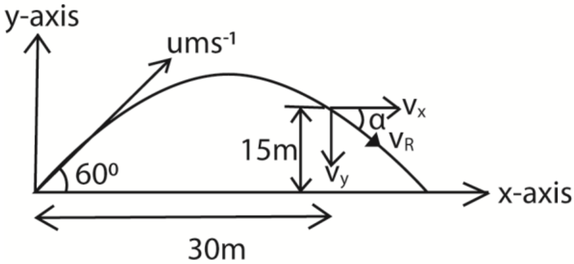
$$v_{(t=3)} = \begin{pmatrix} 10 \\ 20 - 10x\ 3 \end{pmatrix}$$

$$= \binom{10}{-10} ms^{-1}$$

Example 8

A projectile fired at an angle of 60^0 above the horizontal strikes a building 30m away at a point 15m above the point of projection. Find

(i) speed of projection (i) velocity when it strikes a building



(→) x = ucosθt

30 = utcos 60

t = $\frac{60}{u}$

(↑) y = $usin\theta t - \frac{1}{2}gt^2$

15 = usin60 x $\frac{60}{u}$ - $\frac{1}{2}\ x\ 9.8\ x\ \left(\frac{60}{u}\right)^2$

u = 21.86ms^{-1}

(ii) t = $\frac{60}{u} = \frac{60}{21.86}$ = 2.75s

u_x = ucosθ

= 21.86x cos 60 = 10.93ms^{-1}

u_y = usinθ – gt

= 21.86x sin 60 – 9.8 x 2.75

= -8.09ms^{-1}

v = $\sqrt{(v_x)^2 + (v_y)^2}$

= $\sqrt{10.93^2 + (-8.93)^2}$ = 13.60ms^{-1}

direction α = $\tan^{-1}\left(\frac{8.09}{10.93}\right)$= 36.58^0

Example 9

A football player projects a ball at a speed of $8ms^{-1}$ at an angle of 30^0 with the ground. The ball strikes the ground at a point which is level with the point of projection. After impact with the ground, the ball bounces and the horizontal component of the velocity of the ball remains the same but the vertical component is reversed in direction and halved in magnitude. The player running after the ball. Kicks it again at appoint which is at a horizontal distance 1.0m from the point where it bounced, so that the ball continues in the same direction. Find the

(a) horizontal distance between the point of projection and the point at which the ball first strikes the ground. (Take $g = 10ms^{-2}$)
(b) (i) the time interval between the ball striking the ground and the player kicking it again.
(ii) the height of the ball above the ground when it is kicked again (take $g = 10ms^{-2}$)

Solution

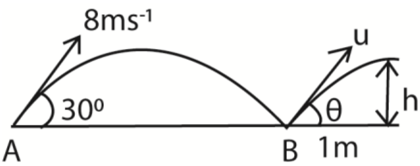
y = usinθt - $\frac{1}{2}gt^2$

At AB:

0 = 8sin30 x t - $\frac{1}{2}$ x 10 t^2

t = 0.8s

x = cosθt = 8cos30 x 0.8 = 5.543m



(b)(i) x = ucosθt

1 = 8cos (30)t

t = 0.1443s

(ii) y = usinθt - $\frac{1}{2}gt^2$

h = 4sin30 x 0.1443 - $\frac{1}{2}$ $x\ 10\ x\ (0.1443)^2$

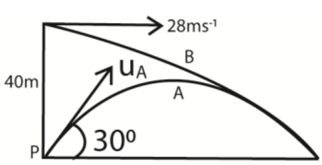
h = 0.185m

Example 10

Two objects A and B are projected simultaneously from different points. A is projected from the top of vertical cliff and A from the base. Particle B is projected horizontally with a speed $28ms^{-1}$ and A is projected at an angle θ above the horizontal. The height of the cliff is 40m and the particles hit the same point on the ground, find;

(a) time taken and the distance from P to where they hit
(b) speed and angle of projection of A

Solution



(a) For B: y = $\frac{1}{2}gt^2$

-40 = $\frac{1}{2}$ $x - 9.8t^2$

t = $\frac{20}{7}s$

x = ut = 28 x $\frac{20}{7}$ = 80m

For A: x =ucosθt

80 = $u_A$cosθ x $\frac{20}{7}$

$u_A$cosθ = 28 …….. (i)

y = usinθt - $\frac{1}{2}gt^2$

0 = $u_A$sinθ x $\frac{20}{7}$ - $\frac{1}{2}$ $x - 9.8\ x\ \left(\frac{20}{7}\right)^2$

$u_A$sinθ = 14………… (ii)

(i) ÷ (ii)

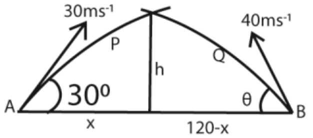
θ = $\tan^{-1}\frac{14}{28} = 26.6^0$

$u_A$sin26.6 = 14

u_A = $31.3ms^{-1}$

Example 11

A particle P is projected a point A with initial velocity 30ms-1at an angle of elevation 300 to the horizontal. At the same instant a particle Q is projected in the opposite direction with initial speed of 40ms-1 from a point at the same level with a and 120m from A. Given that the particles collide. Find (i) angle of projection of Q (ii) time when collision occur.



y = usinθt - $\frac{1}{2}gt^2$

For P

h = 30sin30 x t - $\frac{1}{2}$ $x9.8t^2$.... (i)

For Q

h = 40sinθ x t - $\frac{1}{2}$ $x9.8t^2$..... (ii)

(i) and (ii)

30sin30 x t = 40sinθ x t

θ = 24.5^0

For P

x – 30cos30t .... (iii)

For Q

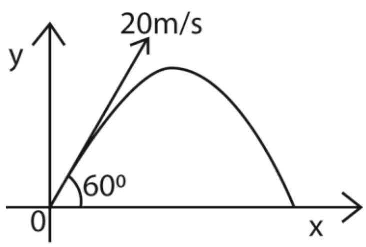
120 –x = 40cos24.5t ... (iv)

(iii)and (iv)

t = 1.9s

Example 12

A particle is projected from a point O with speed20m/s at an angle 60^0 to the horizontal. Express in vector form its velocity v and its displacement, r, from O at any time t seconds . (05marks)



At time t = 0

V = 20cos60^0i + 20sin60^0j = 10i + 10$\sqrt{3}$j

But A = -gj

At any time t,

V = $\int adt$ = -gj$\int dt$

= -gtj + c

At t = 0

10i + 10$\sqrt{3}$j = 0 + cn

⇨ At time t

V= 10i + $(10\sqrt{3} - gt)$j = 10i + $(10\sqrt{3} - 9.8t)$j

Or

V = $\begin{pmatrix} 10 \\ 10\sqrt{3} - 9.8t \end{pmatrix}$

r = $\int vdt = \int \begin{pmatrix} 10 \\ 10\sqrt{3} - 9.8t \end{pmatrix} dt$ = $\begin{pmatrix} 10t \\ 10\sqrt{3}t - 4.9t^2 \end{pmatrix}$+ c

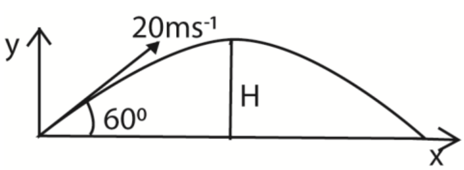
at t = 0, r = $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

⇨ $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + c$, c = $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Hence at time t, r = $\begin{pmatrix} 10t \\ 10\sqrt{3}t - 4.9t^2 \end{pmatrix}$

Example 13

A particle is projected at an angle 600 to the horizontal with velocity of 20ms^{-1}. Calculate the greatest height the particle attains. [Use g = 10ms^{-2}]



Use $v^2 = u^2 + 2as$; at maximum height, v = 0

0 = $(20 \sin 60)^2 - 2gH$

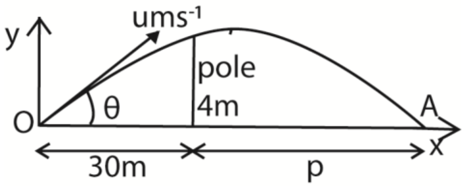
H = 15m

Examples 14

A particle is projected from ground level towards a vertical pole 4m high and 30m away from the point of projection. It just passes the pole in one second. Find

(i) its initial speed and angle of projection
(ii) the distance beyond the pole where the particle falls

Solution



(→) $x = u_x t$ but t = 1

30 = ucosθ x 1

30 = ucosθ ………………….. (i)

(↑) $y = u_y t - \frac{1}{2}gt^2$

$4 = u\sin\theta \times 1 - \frac{1}{2} x\ 9.8\ x\ 1^2$

4 = usinθ – 4.9

8.9 = usinθ ………. (ii)

(ii) ÷ (i)

$\theta = \tan^{-1}\left(\frac{8.9}{30}\right) = 16.5^0$

30 = ucos16.5

$u = 31.29ms^{-1}$

(b) At point O and A, y=0

$y = u\sin\theta t - \frac{1}{2}gt^2$

$0 = 31.29\sin16.5T - \frac{1}{2}\ x\ 9.8xT^2$

T = 1.8136s

Range, R= ucosθT

R = 31.29cos 16.6 x 1.8136 = 54.36m

p = 54.36 – 30 = 24.36m

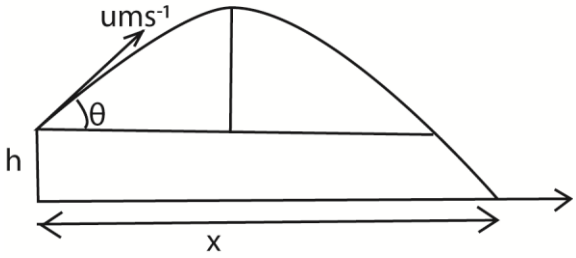
Revision exercise 2

1. A particle projected at an angle of 30^0 to the horizontal with velocity of $60ms^{-1}$. Calculate
   (a) time taken for the particle to reach maximum height [3s]
   (b) maximum height [45m]
   (c) horizontal range of the particle [312m]
2. A particle is projected from an origin O and has an initial velocity of $30\sqrt{2}$ ms^{-1} at angle 450 above the horizontal. Find the horizontal and vertical components of displacement 2s after projection. [60m, 40.4m] Hence find the distance of motion of the particle at that time [72.3m]
3. A particle projected from a point on the level ground has horizontal range of 240m and time of flight of 6s. find the magnitude and direction of velocity of projection [$50ms^{-1}$, 36.9^0]
4. A particle is projected with a velocity of $30ms^{-1}$ at an angle of 40^0 above the horizontal plane. Find
   (i) the time for which the particle is in air [3.9s]
   (ii) the horizontal distance it travels [22.9m]
5. A body is projected with a velocity of 200ms-1 at an angle of 300 above the horizontal. Calculate
   (i) time taken to reach the maximum height[10.2s]
   (ii) its velocity after 16s [$183ms^{-1}$ at 19.1^0]
6. A football is kicked from O on a level ground. 2s later the football just clears a vertical wall of height 2.4m. If O is 22m from the wall, find the velocity with which the ball is kicked. [$15.6ms^{-1}$ at 45^0 above the horizontal]

7. A particle is projected from a level ground in such away that its horizontal and vertical components of velocity are $20ms^{-1}$ and $10ms^{-1}$ respectively. find
   (a) maximum height of the particle [5.0m]
   (b) its horizontal distance from the point of projection when it returns to the ground. [40m]
   (c) the magnitude and direction of the velocity on landing [$22.4ms^{-1}$ at 26.6^0 below the horizontal]
8. A particle is projected with a speed of 25ms-1 at 300 above the horizontal. Find
   (a) time taken to reach the height point of the trajectory. [1.5s]
   (b) the magnitude and direction of velocity after 2.0s.[$22.9ms^{-1}$ at 19.1^0 below the horizontal]
9. A particle is projected from the origin at a velocity $(4i + 13j)ms^{-1}$. Find the position vector and distance of the particle in 2s after projection (take $g = 10ms^{-2}$) [(8i + 6j)m, 10m]
10. A particle is projected from the origin at a velocity of $(4i + 11j)ms^{-1}$. and passes a point P which has a position vector (8i + xj)m. Find the time taken for the particle to reach P from O and the value of x. [2s, 2.4m]
11. A Particle is projected from the origin at a velocity of $(7i + 5j)ms^{-1}$ and passes a point P which has position vector (xi – 30j)m. Find the time taken for the particle to reach P from O and the value of x [3s, 21]
12. A particle is projected from the origin at velocity of $(4i + 2j)ms^{-1}$. Find
    (a) the direction in which it is moving after 1s[63.43^0]
    (b) Two second later after launch of the first particle, a second particle is projected from the same point with a velocity $(8i – 26j)ms^{-1}$. Show that the two particles collide and find the time and position at which this occurs [t =4s, (16i – 72j)m]
13. A particle is projected at $84ms^{-1}$ to hit a point 360m away and on the same horizontal level at the point as the point of projection. Find the two possible angles of projection. [15^0, 75^0]
14. A golfer hits a golf ball at $30ms^{-1}$ and wishes it to land at a point 45m away, on the same horizontal level as the starting point. Find the two possible angles of projection. [4.7^0, 75.3^0]
15. A particle is projected from a horizontal ground and has an initial speed of $35ms^{-1}$. When the ball is travelling horizontally, it strikes a vertical wall. If the wall is 25m from the point of projection, find the two possible angles of projection. [11.8^0, 8.2^0]
16. A particle is projected from a point O and passes through a point A when the particle is travelling horizontally. If A is 10m horizontally and 8m vertically from, find the magnitude and direction of the velocity of projection. [$14.8ms^{-1}$, 58^0 above the horizontal]
17. A stone is projected at an angle of 20^0 to the horizontal and just clears a wall which is 10m high and 30m from the point of projection. Find the
    (a) speed of projection [$73.78ms^{-1}$]
    (b) angle which the stone makes with the horizontal as it clears the wall [16.9^0]
18. A particle is projected from a point on a horizontal plane and has an initial speed of 28ms-1. If the particle passes through a point above the plane, 40m horizontally and 20m vertically from the point of projection, find the possible angles of projection. [45^0, 71.6^0]
19. Two objects A and B are projected simultaneously from different points. A is projected from the top of a vertical cliff and B from the base. Particle A is projected horizontally with a speed $3u\ ms^{-1}$ and B is projected at an angle θ above the horizontal with speed $5u\ ms^{-1}$. The height of the cliff is 56m and the particles collide after 2s, find
    (a) horizontal and vertical distances from the point of collision to the base of the cliff. [42m, 36.4m]
    (b) value of angle u and θ. [$7ms^{-1}$, 53.1^0]

c. Objects projected upwards from a point above the ground at an angle to the horizontal

Suppose an object is projected with velocity u at an angle θ from a height h.



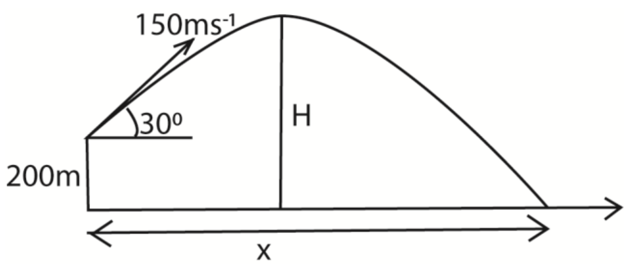
Horizontal: u_x = ucosθ, a = 0

Vertical: uy = usinθ, a = -g = -9.8

Example 15

A bullet is fired from a gun at a height of 200m with velocity150ms-1 at an angle of 300 to the horizontal. Find

(i) maximum height attained.



$v^2 = u^2 + 2as$

at maximum height, H, v =0

$0^2 = (150\sin30)^2 - 2 \times 9.8H$

H = 86.70m

(ii) Time taken for the bullet to hit the ground

y = usinθt - $\frac{1}{2}gt^2$

y = -200m since it's below the point of projection

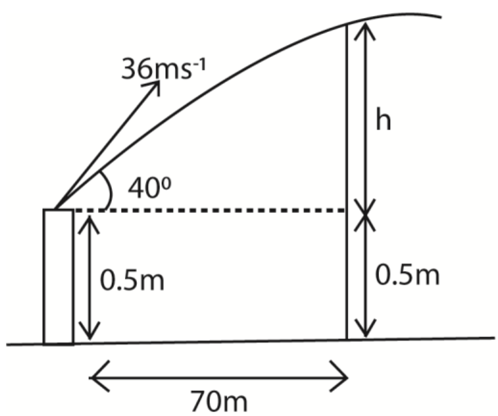
-200 = 150sin30t - $\frac{1}{2}x\ 9.8t^2$

t = 17.61s, or t = -2.32s

time taken = 17.61

Example 16

A particle I projected with a speed of 36ms^{-1} at an angle of 40^0 to the horizontal from a point 0.5m above the level ground. It just clears a wall which is 70 metres on the horizontal plane from a point of projection. Find the;



Vx = 36cos40^0

Vy = 36sin40^0

(a) (i) time taken for the particle to reach the wall.

Time take to clear the wall = time taken to cover a horizontal distance of 70m

Usin X= Vxt

t = $\frac{70}{36\sin40^0}$ = 2.5384s

(ii) height of the wall (08marks)

Using h = usinθt - $\frac{1}{2}gt^2$

=36sin40^0 x 2.5384 - $\frac{1}{2}\ x\ 9.8\ x\ (2.5384)^2$

= 27.1664 + 0.5 = 27.6664m

(a) Maximum height reached by the particle from the point of projection. (04marks)

From v^2 = u^2 +2as

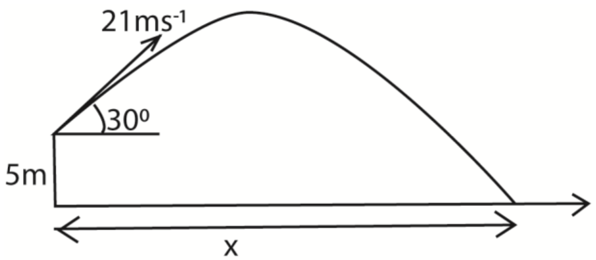
At maximum height vertical component of velocity is zero

⇨ 0 = $(36\sin 40^2)^2 - 2\ x\ 9.8H$

H = $\frac{(36\sin 40^2)^2}{2\ x\ 9.8}$ =27.32m

Example 17

A particle is projected at an angle of 30^0 with speed of 21ms^{-1}. If the point of projection is 5m above the horizontal ground, find the horizontal distance that the particle travels before hitting the ground



y = usinθt - $\frac{1}{2}gt^2$

y = -5m since it's below the point of projection

-5 = 21sin30T - $\frac{1}{2}x\ 9.8T^2$

T = 2.54s

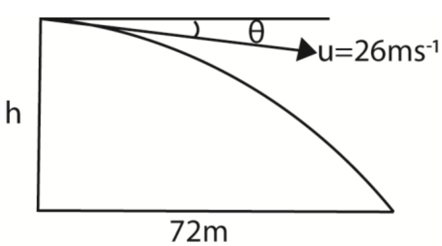
(→): x = ucosθ x T

= 21cos30 x 2.54 = 46.19m

Example 18

A stone is thrown from the edge of a vertical cliff and has initial velocity of 26ms^{-1} at an angle of $\tan^{-1}\left(\frac{5}{12}\right)$ below the horizontal. The stone hits the sea at a point level with the base of the cliff and 72m from it. Find the height of the cliff and the time for which the stone is in the air.

Take g = 10ms^{-2}.



y = xtanθ - $\frac{gx^2(1+\tan^2\theta)}{2u^2}$

h = 72 x$\left(-\frac{5}{12}\right)$ = $\frac{10\ x\ 72^2\left[1+\left(\left(-\frac{5}{12}\right)\right)^2\right]}{2\ x\ 26^2}$

= -75m

h= 75m below the point of projection

x = ucosθt

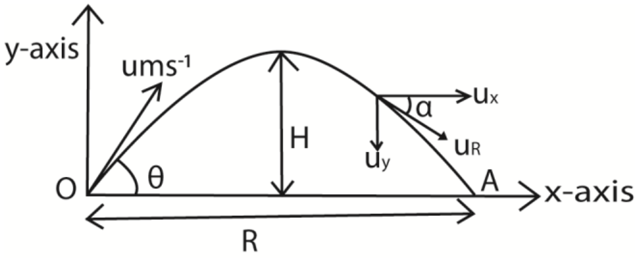
72 = 26$\left[\cos\left(-\tan^{-1}\frac{5}{12}\right)\right]t$; t =3s

Revision exercise 3

1. A stone is thrown from the edge of a vertical cliff with velocity of 50ms^{-1} at an angle of $\tan^{-1}\frac{7}{24}$ above the horizontal. The stone strikes the sea at a point 240m from the foot of the cliff. Find the time for which the stone is in air and the height of the cliff. [5s, 52.5m]
2. A particle is projected with a velocity of 10ms-1 at an angle of 45^0 to the horizontal; it hits the ground at a point which is 3m below the point of projection. Find the time for which it is in the air and the horizontal distance covered the particle in this time. [1.76s, 12.42m]
3. A batsman hits a ball with velocity of 17ms^{-1} at an angle $\tan^{-1}\frac{3}{4}$ above the horizontal, the ball initially being60cm above the level ground. The ball is caught by a fielder standing 28m from the batsman. Find the time taken for the ball to reach the fielder and the height above the ground at which he takes the catch. [2s, 2m]
4. A stone is thrown from the edge of a vertical cliff 70m high at an angle of 300 below the horizontal. the stone hits the sea at a point level with the base of the cliff and 20m from it. Find the initial speed of the stone and the direction it is moving when it hits the sea. [6.69mms^{-1}, 81.2^0]
5. A vertical tower stands on a level ground. A stone is thrown from the top of the tower and has an initial velocity of 24.5ms^{-1} at an angle of $\tan^{-1}\frac{4}{3}$ above the horizontal. The stone strikes the ground at a point 73.5m from the foot of the tower. Find the time taken for the stone to reach the ground and the height of the tower. [5s, 24.5m]
6. A stone is thrown from the top of a vertical cliff, 100mabove sea level. The initial velocity of the stone is 13ms-1 at an angle of elevation of$\tan^{-1}\frac{5}{12}$. Find the time taken for the stone to reach the sea and its horizontal distance from the cliff at that time. Take g = 10ms^{-2}. [5s. 60m]
7. A golfer hits a golf ball with a velocity of 30ms^{-1} at an angle of $\tan^{-1}\frac{4}{3}$ above the horizontal. The ball lands on green 5m below the level from which it was struck. Find the horizontal distance travelled by the ball. Take g = 10ms^{-2}. [90m]
8. A pebble is thrown from the top of a cliff at a speed of 10ms^{-1} and at 30^0 above the horizontal. It hits the sea below the cliff 6s later. Find
   (a) the height of the cliff [150m]
   (b) the distance from the base of the cliff at which the pebble falls into the sea. [52m]
9. An arrow is fired from a point at a height 1.5m above the horizontal. It has a velocity of 12ms^{-1} at an angle 30^0 above the horizontal. The arrow hits the target at a height of 1m above the horizontal ground, find
   (i) time taken for the arrow to hit the target [1.3s]
   (ii) horizontal distance between where the arrow is fired and the target. [13.51m]
   (iii) speed of the arrow when the arrow hit the target [12.39ms^{-1}]

Standard equations of the projectile

Suppose an object is projected with velocity u at an angle θ from a horizontal ground



(a) Maximum height (greatest height), H

For vertical motion, at maximum height v = 0,

a = -g = -9,8ms^{-2}

$v^2 = u^2 + 2as$

$0 = (u\sin\theta)^2 - 2gH$

$H = \frac{(u\sin\theta)^2}{2g}$

(b) Time to reach maximum height

(↑): $v = u_y + at$, at maximum height, v = 0

0 = usinµ – gt

$t = \frac{u\sin\theta}{g}$

(c) Time of flight, T

(↑): $s_y = u_y t - \frac{1}{2}gt^2$

At A $s_y = 0$

$0 = u\sin\theta T - \frac{1}{2}gT^2$

$T = \frac{2u\sin\theta}{g}$

(d) Range

(→) x = ucosθt

R = ucosθT

$= u\cos\theta.\frac{2u\sin\theta}{g}$

$= \frac{2u^2\cos\theta\sin\theta}{g}$

$R = \frac{2u^2\sin2\theta}{g}$

(e) Maximum range

for maximum range sin2θ = 1

$2\theta = \sin^{-1} 1$

2θ =90

$R_{max} = \frac{2u^2\sin90}{g} = \frac{u^2}{g}$

(f) Equation of trajectory

A trajectory is expresses zin terms of horizontal distance x and vertical distance y

(→): x = ucosθt

$t = \frac{x}{u\cos\theta}$ .... (i)

(↑): $y = u\sin\theta t - \frac{1}{2}gt^2$ .... (ii)

putting (i) into (ii)

$y = u\sin\theta\left(\frac{x}{u\cos\theta}\right) - \frac{1}{2}g\left(\frac{x}{u\cos\theta}\right)^2$

$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$

$= x\tan\theta - \frac{gx^2\sec^2\theta}{2u^2}$

$y = x\tan\theta - \frac{gx^2(1+\tan^2\theta)}{2u^2}$

Example 19

A ball is projected from the horizontal ground and has an initial velocity of $20ms^{-1}$ at an angle of elevation$\tan^{-1}\frac{7}{24}$. When the ball is travelling horizontally it strike a vertical wall. How high above the ground does the impact occur.

Projectile travel horizontally at maximum height

H = $\frac{(usin\theta)^2}{2g}$ = $\frac{20^2sin^1\left(\tan^{-1}\frac{7}{24}\right)}{2\ x\ 9.8}$ = 1.6m

Example 20

A particle is projected from a point on a horizontal ground at a speed of $84ms^{-1}$. If the particle hits a point 300m away and on the same horizontal plane as the projection, find the

(i) angle of projection

R =$\frac{2u^2sin2\theta}{g}$

360 = $\frac{2x\ 84^2\ x\ sin2\theta}{9.8}$; θ = 15^0 Or θ = 75^0

(ii) maximum height

H_1 = $\frac{(usin\theta)^2}{2g}$ = $\frac{84^2sin^2(15)}{2\ x\ 9.8}$ = 24.1m

H_2 = $\frac{(usin\theta)^2}{2g}$ = $\frac{84^2sin^2(75)}{2\ x\ 9.8}$ = 335.9m

(iii) times of flight

$T_1 = \frac{2usin\theta}{g}$ = $\frac{2x\ 84\ x\ sin15}{9.8}$ = 4.44s

$T_2 = \frac{2usin\theta}{g}$ = $\frac{2x\ 84\ x\ sin75}{9.8}$ = 16.56s

Example 21

A gun has its barrel set at an angle of elevation of 15^0. The gun fires a shell with initial speed of $210ms^{-1}$. Find the

(a) horizontal range of the shell

R =$\frac{2u^2sin2\theta}{g}$ = $\frac{2x\ 84^2\ x\ sin30}{9.8}$ = 2250m

(b) maximum range

R =$\frac{2u^2}{g}$ = R =$\frac{2\ x\ 84^2}{g}$= 4500m

Example 22

A stone thrown upwards at an angle θ to the horizontal with speed ums-1 just clears a vertical wall4m high and 10m from the point of projection when travelling horizontally. Find the angle of projection.

Solution

Projectile travel horizontally at maximum height

H = $\frac{(usin\theta)^2}{2g}$

4 = $\frac{(usin\theta)^2}{2g}$

8g = $u^2sin^2θ$ =……… (i)

Also, x = ucosθt and t = $\frac{ucos\theta}{g}$

x = ucosθ$\left(\frac{ucos\theta}{g}\right)$

10g = u^2cosθsinθ ……(ii)

(i)÷(ii)

θ = $\tan^{-1}\frac{8}{10}$ = 38.7^0

Example 23

If the horizontal range of a particle with velocity u is r, show that the greatest height H is satisfied by the equation $16gH^2 - 8Hu^2 + gR^2 = 0$

$H = \frac{(usin\theta)^2}{2g}$

$sin^2\theta = \frac{2gH}{u^2}$.......... (i)

$R = \frac{2u^2 sin2\theta}{g}$

$gR = u^2(2cos\theta sin\theta)$

$cos\theta = \frac{gR}{2u^2 sin\theta}$

$cos^2\theta = \frac{(gR)^2}{4u^4 sin^2\theta}$ .... (ii)

(i) and (ii)

$cos^2\theta = \frac{(gR)^2}{4u^4\left(\frac{2gH}{u^2}\right)}$

$cos^2\theta = \frac{gR^2}{8u^2H}$

$Cos^2\theta + sin^2\theta = 1$

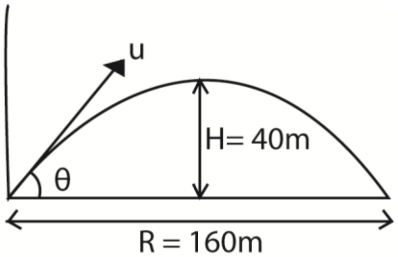
$\frac{2gH}{u^2} + \frac{gR^2}{8u^2H} = 1$

$16gH^2 - 8Hu^2 + gR^2 = 0$

Example 24

A ball is projected from point A and falls at point B which is in level with A at a distance of 160m from A. The greatest height of the ball attained is 40m. find the;

(a) angle and velocity at which the ball is projected (10marks)



Using $v^2 = u^2 + 2as$

$v^2 = u^2sin^2\theta - 2gh$

At maximum height $v_y = 0$

$0 = u^2sin^2\theta - 2gH$

$u^2 = \frac{2gH}{sin^2\theta}$ .... (i)

Range, $R = \frac{2u^2 sin\theta cos\theta}{g}$ .... (ii)

Eqn. (i) and eqn. (ii)

$R = 2\,x\,\frac{2gH}{sin^2\theta}.\frac{sin\theta cos\theta}{g} = 4Hcot\theta$

.... (iii)

Substituting for R and H in eqn. (iii)

$160 = 4 \times 40cot\theta$

$\tan\theta = 1$

$\theta = \tan^{-1} 1 = 45^0$

Substituting for θ

$u^2 = \frac{2gH}{sin^2\theta} = \frac{2\,x\,40\,x\,9.8}{sin^2 45^0}$

$u = \sqrt{\frac{2\,x\,40\,x\,9.8}{sin^2 45^0}} = 39.60ms^{-1}$

(b) time taken for the ball to attain the greatest height (02marks)

$t = \frac{usin\,\theta}{g} = \frac{39.60\,x\,\sin 45^0}{9.8} = 2.8573s$

Example 25

A boy throws a ball at an initial speed of $40ms^{-1}$ at an angle of elevation θ. Taking $g = 10ms^{-2}$, show that the times of flight corresponding to a horizontal range of 80m are positive roots of the equation $T^4 - 64T^2 + 256$.

$R = \frac{2u^2 cos\theta sin\theta}{g}$

$80 = \frac{2\,x\,40^2 cos\theta sin\theta}{10}$

$sin\theta cos\theta = 0.25$

$sin\theta = \frac{1}{4cos\theta}$ .... (i)

$x = ucos\theta t$

$80 = 40cos\theta T$

$cos\,\theta = \frac{2}{T}$ .... (ii)

but $sin^2\theta + cos^2\theta = 1$

$\left(\frac{1}{4x\frac{2}{T}}\right)^2 + \left(\frac{2}{T}\right)^2 = 1$

$\frac{T^2}{64} + \frac{4}{T^2} = 1$

$T^4 - 64T^2 + 256$

Example 26

A particle projected from a point O on a horizontal ground moves freely under gravity and it hits the ground again at A. Taking O as the origin, the equation of the path of the particle is 60y = $20\sqrt{3}$x -x^2 where x and y are measured in meters. Determine the

(a) initial speed and angle of projection

60y = $20\sqrt{3}$x -x^2

y = $\frac{\sqrt{3}}{3}$x - $\frac{x^2}{60}$

Comparing with

y = xtanθ -$\frac{gx^2(1+tan^2\theta)}{2u^2}$

θ = $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$= 30^0

$\frac{gx^2(1+tan^2\theta)}{2u^2} = \frac{x^2}{60}$

$\frac{g(1+tan^2\theta)}{2u^2} = \frac{1}{60}$

$\frac{9.8\left(1+\frac{3}{9}\right)}{2u^2} = \frac{1}{60}$; u = 19.8$ms^{-1}$

(b) Distance OA

At A, y = 0

0= $20\sqrt{3}$x -x^2

0= ($20\sqrt{3}$ – x)x

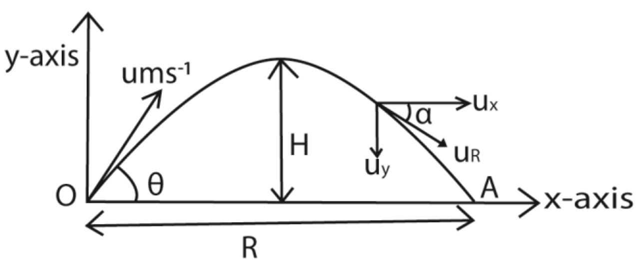
x =$20\sqrt{3}$m

Example 27

A particle is projected from a point O on a level ground with initial speed 30ms^{-1} to pass through a point which is a horizontal distance 40m from O and a distance 10 vertically above the level of O.

(a) Show that there are two possible angles of projection
(b) If these angles are α and β, prove that tan (α+ β) = -4, take g = 10ms^{-2}

Solution



y = xtanθ -$\frac{gx^2(1+tan^2\theta)}{2u^2}$

10 = 40tanθ -$\frac{10\,x\,40(1+tan^2\theta)}{2\,x\,30^2}$

8tan_2θ – 36tanθ + 17 = 0

since it's quadratic equation in tanθ;

it has two roots and hence two values of θ < 90

(b) tanα + tanβ = $\frac{36}{8}$

tanα tanβ = $\frac{17}{8}$

tan(α + β) = $\frac{\tan\alpha+tan\beta}{1-\tan\alpha tan\beta}$

= $\frac{\frac{36}{8}}{1-\frac{17}{8}}$

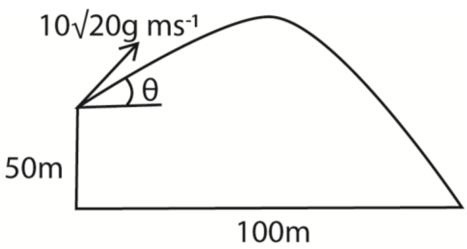
= $\frac{\frac{36}{8}}{\frac{8-17}{8}}$

=$\frac{36}{8}\ x\ \frac{8}{-9}$

tan(α + β) = -4

Example 28

A particle is projected with a speed $10\sqrt{2g}$ms-1from the top of a cliff 50m high. The particle hits the sea at a distance of 100m from the vertical through the point of projection. Show that there two possible directions of projection which are perpendicular. Determine the time taken from the point of projection in each case.



$y = x\tan\theta - \frac{gx^2(1+\tan^2\theta)}{2u^2}$

$-50 = 100\tan\theta - \frac{g \ x \ 100^2(1+\tan^2\theta)}{2x \ 100 \ x \ 2g}$

$\tan^2\theta - 4\tan\theta - 1 = 0$

$\tan\theta_1 = 2+\sqrt{5}$ and $\tan\theta_2 = 2-\sqrt{5}$

$\tan\theta_1 \tan\theta_2 = (2+\sqrt{5})(2-\sqrt{5}) = -1$

Hence they are perpendicular

For horizontal motion

x = ucosθt

$\theta_1 = \tan^{-1}(2+\sqrt{5}) = 76.72^0$

$t_1 = \frac{100}{10\sqrt{2g}\cos(76.72^0)} = 9.83s$

$\theta_2 = \tan^{-1}(2-\sqrt{5}) = -13.36^0$

$t_2 = \frac{100}{10\sqrt{2g}\cos(-13.38^0)} = 2.32s$

Revision exercise 4

1. A golfer hits a ball with a velocity of 44.1ms-1 at an angle of $\sin^{-1}\frac{3}{5}$ above the horizontal. The ball lands on the green at a point which is level with the point of projection. Find the time for which the golf ball was in air. [5.4s]
2. a tennis ball is served horizontally from a point which is 2.5m vertically above a point A. The ball first strikes the horizontal ground through A at a distance 20m from A.
   (i) show that the ball is served with speed 28ms-1
   (ii) During its flight the ball passes over a net which is horizontal distance 12m from A. Find the vertical distance of the ball above the horizontal ground at the instant when it passes over the net. [1.6m]
3. An aircraft, at a height of 180m above horizontal ground and flying horizontally with speed of $70ms^{-1}$ releases emergency supplies. If these supplies are to land at a specific point, at what horizontal distance from this point must the aircraft release them? {take $g = 10ms^{-2}$) {420m]
4. A stone is projected from top of a vertical cliff of height of h and the stone attains a maximum height (h+ b) above the ground. The stone strikes a sea at a distance, a from the foot of the cliff. Prove that the angle of elevation θ of the stone is given by $a^2\tan^2\theta - 4ab\tan\theta - 4bh = 0$
5. At time t = 0 a particle is projected from a point O on a horizontal plane with speed $14ms^{-1}$ in a direction inclined at an angle $\tan^{-1}\frac{3}{4}$ above the horizontal. The particle just clears the top of a vertical wall, the base of which is 8m from O. Find
   (a) time at which the particle passes over the wall [$\frac{5}{7}s$]
   (b) height of the wall [3.5m]
6. A particle P is projected from a point O with a speed of 60ms-1 at an angle $\cos^{-1}\frac{4}{5}$ above the horizontal. Find
   (a) time the particle takes to reach the point Q whose horizontal displacement from O is 96m. [2s]
   (b) height of Q above O [52.4m]
   (c) speed of the particle 2s after projection [$50.7ms^{-1}$]
7. A particle P is projected from a point O with a speed$50ms^{-1}$ at an angle $\sin^{-1}\frac{7}{25}$ above the horizontal. Find
   (a) height of P at the point where its horizontal displacement from O is 120m[4.375m]

(b) speed of P 2s after projection. [48.3ms^{-1}]

(c) times after projection at which P is moving at an angle of $\tan^{-1}\frac{1}{4}$ to the ground [0.204s, 2.65s]

8. A child throws a small ball from a height of 1.5m above level ground, aiming at a small target. The target is on top of a vertical pole of height 2m from the ground and horizontal displacement of the child from the pole is 6m. The initial velocity of the ball has magnitude ums^{-1} at an angle of elevation 40^0. The ball moves freely under gravity. (Take g = 10ms^{-2})
   (a) For u = 10, find the greatest height above the ground reached by the ball. [3.6m]
   (b) Calculate the value of u for which the ball hits the target[8.2]
9. A girl thrown a stone from a height of 1.5m above the ground with speed of 10ms^{-1} and hits a bottle standing on a wall 4m high and 5m from her. Take g = 10ms^{-1}.
   (a) Show that if α is the angle of projection of the stone as it leaves her hand then $1.25\tan^2\alpha - 5\tan\alpha + 3.75 = 0$
   (b) the horizontal component of the stone's velocity has to be6ms^{-1} for the bottle to be knocked off. By solving the above equation or otherwise, show that α has to be 45^0 for the bottle to be knocked off.
10. A basketball is released from player's hands with a speed of 8ms^{-1} at inclination of α^0 above the horizontal so as t land in the centre of the basket, which is 4m horizontally from the point of release and a vertical height of 0.5m above it. Take g =10ms^{-2}.
   (a) show that α satisfies the quadratic equation; $5\tan^2\alpha - 16\tan\alpha + 7 = 0$
   (b) Given that the player throws the ball at a large angle of projection find α for the ball to land in the basket. [70^0, 1.43s]