BINOMIAL THEOREM

Objectives of the topic:

- Create rows of Pascal's triangle
- Compute factorial values
- Compute binomial co-efficient by the formula
- Expand powers of binomial by Pascal's triangle and by binomial theorem
- Approximate numbers using binomial expansion

Pascal's Triangle

$$\begin{array}{ccccccccccccccc} &&&&&&&1&&&&&&& \\ &&&&&&1&&1&&&&&& \\ &&&&&1&&2&&1&&&&& \\ &&&&1&&3&&3&&1&&&& \\ &&&1&&4&&6&&4&&1&&& \\ &&1&&5&&10&&10&&5&&1&& \\ &1&&6&&15&&20&&15&&6&&1& \\ 1&&7&&21&&35&&35&&21&&7&&1 \end{array}$$

We can use Pascal triangle to expand expressions of the form $(a + b)^n$

$$(a+b)^1 = a+b$$
$$(a+b)^2 = a^2+2ab+b^2$$
$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$

Pascal's triangle helps us to calculate the powers of a binomial $(a + b)^n$ without actually multiplying it

$$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$$

Note

The literal factors are all combination of a and b where the sum of the components of the power is 4 a^4, a^3b, a^2b^2, b^4

The degree of each term is 4. The first term is actually a^4b^0 which is $a^4(1)$.

Thus, to expand $(a+b)^5$ we would anticipate the following terms in which the sum of all the components of the powers is 5.

$$?\ a^5 + \ ?a^4b + ?\ a^3b^2 + ?a^2b^3 + ?ab^4 + ?b^5$$

The question is what are the co-efficients. We can obtain the co-efficients from the Pascal's triangle above (line five above)

Example

Use Pascal's triangle to expand the following.

a) $(x+2)^5$
b) $(2x-3)^3$
c) $\left(\frac{1}{x}+2x^2\right)^4$
d) $(x^2-1)^4$
e) $(2x-3y)^5$
f) $(2-3x)^6$

Solution

Consider the Pascal's triangle

$$\begin{array}{ccccccccccccc} &&&&&&1&&&&&& \\ &&&&&1&&1&&&&& \\ &&&&1&&2&&1&&&& \\ &&&1&&3&&3&&1&&& \\ &&1&&4&&6&&4&&1&& \\ &1&&5&&10&&10&&5&&1& \\ 1&&6&&15&&20&&15&&6&&1 \end{array}$$

We can use Pascal's triangle to find $(x + 2)^5$

$$=?\,x^5+?\,(x)^4(2)+?\,x^3(2)^2+?\,x^2(2)^3+?\,x(2)^4+?\,x^5$$

We can obtain the coefficients from the Pascal's triangle above (line 6).

$$\textbf{(a)}(x+2)^5 = 1(x^5)+5(x^4)(2)+10(x^3)(2)^2 + 10(x^2)(2)^3+5(x)(2)^4+1(2)^5$$
$$(x+2)^5 = x^5+10x^4+40x^3+80x^2+80x+32$$

(b) $(2x-3)^3 = (2x)^3 + 3(2x)^2(-3) + 3(2x)(-3)^2 + (-3)^3$

$$= 8x^3 + -36x^2 + 54x - 27$$
$$= 8x^3 - 36x^2 + 54x - 27$$

(c) $\left(\frac{1}{x}+2x^2\right)^4$

$$= \left(\frac{1}{x}\right)^4 + 4\left(\frac{1}{x}\right)^3(2x^2) + 6\left(\frac{1}{x}\right)^2(2x^2)^2 + 4\left(\frac{1}{x}\right)(2x^2)^3 + (2x^2)^4$$

$$= \frac{1}{x^4} + \frac{8}{x} + 24x^2 + 32x^5 + 16x^8$$

$$(x^2-1)^4 = (x^2)^4 + 4(x^2)^3(-1) + 6(x^2)^2(-1)^2 + 4(x^2)(-1)^3 + (-1)^4$$

$$= x^8 + -4x^6 + 6x^4 - 4x^2 + 1$$
$$= x^8 - 4x^6 + 6x^4 - 4x^2 + 1$$

$$(2x-3y)^5 = (2x)^5 + 5(2x)^4(-3y) + 10(2x)^3(-3y)^2 + 10(2x)^2(-3y)^3 + 5(2x)(-3y)^4 + (-3y)^5$$

$$= 32x^5 + -240x^4y + 720x^3y - 1080x^2y^3 + 810xy^4 - 243y^5$$

$$= 32x^5 - 240x^4y + 720x^3y - 1080x^2y^3 + 810xy^4 - 243y^5$$

$$(2-3x)^6 = (2)^6 + 6(2)^5(-3x) + 15(2^4)(-3x)^2 + 20(2^3)(-3x)^3 + 15(2^2)(-3x)^4 + 6(2)(-3x)^5 + (-3x)^6$$

$= 64 - 576x + 2160x^2 - 4320x^3 + 4860x^4 + -2916x^5 + 729x^6$

We have seen that Pascal's triangle can be used to expand $(a + b)^n$ for the known value of n where n is a positive integer.

However, as n becomes large it becomes difficult to determine the co-efficient of a triangle. Imagine a task to expand $(a + b)^{10000}$.

This is so tedious yet indeed, we may not require all the terms of the expansion but just few. in the above case, we can use binomial theorem.

Binomial Theorem

It states that if n is a positive integer then

$$(a+b)^n = a^n + n\,a^{n-1}b + \frac{n(n-1)a^{(n-2)}b^2}{2!} + \cdots + b^n$$

It is also stated as

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}b^n$$

An important particular case is when $(a = 1)$ and $(b = x)$ giving

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots x^n$$

The binomial expansion discussed up to now is for the case when the exponent is positive

For the case when the number n is not positive, the binomial expansion $(1 + x)^n$ is valid when $-1 < x < 1$ **OR** $|x| < 1$.

Example

Expand $\left(2 + \frac{x}{3}\right)^4$

Solution:

Using the binomial theorem;

$$(a+b)^n = a^n + n\,a^{n-1}b + \frac{n(n-1)a^{n-2}b^2}{2!} + \ldots b^n$$

$$\left(2+\tfrac{x}{3}\right)^4 = (2)^4 + 4(2)^3\left(\tfrac{x}{3}\right) + \frac{4(3)(2^2)(\frac{x}{3})^2}{2!} + \frac{4(3)(2)(2)^1(\frac{x}{3})^3}{3!} + \frac{4(3)(2)(1)(2^0)(\frac{x}{3})^4}{4!}$$

$$= 16 + \frac{32}{3}x + \frac{48x^2}{18} + \frac{48x^3}{27\times 6} + \frac{1}{81}x^4$$

$$= 16 + \frac{32}{3}x + \frac{8}{3}x^2 + \frac{8}{27}x^3 + \frac{1}{81}x^4$$

Example II

Expand $(2x - 3y)^4$

Solution

Using binomial expansion,

$$(a+b)^n = a^n + n\,a^{n-1}b + \frac{n(n-1)a^{n-2}b^2}{2!} + \cdots b^n$$

$$(2x-3y)^4 = (2x)^4 + (4)(2x)^3(-3y) + \frac{(4)(3)(2x)^2(-3y)^2}{2!} + \frac{(4)(3)(2)(2x)(-3y)^3}{3!} + (-3y)^4$$

$$= 16x^4 + -96x^3y + \frac{432}{2}x^2y^2 + -216xy^3 + 81y^4$$

$$= 16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4$$

Example III

Expand $\left(\frac{x}{2} + \frac{2}{x}\right)^3$

Solution

$$(a+b)^n = a^n + n\,a^{n-1}b + \frac{n(n-1)a^{n-2}b^2}{2!} + \ldots b^n$$

$$\left(\frac{x}{2}+\frac{2}{x}\right)^3 = \left(\frac{x}{2}\right)^3 + 3\left(\frac{x}{2}\right)^2\left(\frac{2}{x}\right) + \frac{3(2)\left(\frac{x}{2}\right)\left(\frac{2}{x}\right)^2}{2!} + \left(\frac{2}{x}\right)^3$$

$$= \frac{x^3}{8} + 3\left(\frac{x^2}{4}\right)\left(\frac{2}{x}\right) + \frac{6\left(\frac{x}{2}\right)\left(\frac{4}{x^2}\right)}{2} + \frac{8}{x^3}$$

$$= \frac{x^3}{8} + \frac{3}{2}x + \frac{6}{x^2} + \frac{8}{x^3}$$

The r^{th} term in a binomial expansion

(co-efficient of a term in a binomial expansion)

The r^{th} term of a binomial expansion is given by

$$U_{r+1} = {}^nC_r a^{(n-r)} b^r$$

$$\boxed{\mathbf{U_{r+1} = {}^nC_r a^{(n-r)} b^r}}$$

Example I

Write down the terms indicated in the expansion of the following and simplify your answers.

a) $(x+2)^8$, term in x^5

b) $(3x-2)^5$ term in x^3

c) $\left(2x - \frac{1}{2}\right)^{12}$ term in x^7

d) $(2x+y)^{11}$ term in x^3

Solution

$(x + 2)^8 = (a + b)^n$

$n = 8$, $a = x$, $b = 2$

(a) $U_{r+1} = nC_r\ (x)^{8-r}(2)^r$

$= 8\ C_r\ 2^r x^{8-r}$

$\Rightarrow 8 - r = 5$

$r = 3$

$8C_3\ x^5(2)^3 = 448x^5$

(b) $(3x-2)^5$ term in x^3

$(3x - 2)^5 = (a + b)^n$

$a = 3x$, $b = -2$

$U_{r+1} = nC_r\ a^{n-r} b^r$

$5C_r (3x)^{5-r}(-2)^r$

$5C_r 3^{5-r} x^{5-r}(-2)^r$

$5 - r = 3$

$r = 2$

$U_{(r+1)} = 5C_2 (3x)^{(5-2)}(-2)^2$

$= 5C_2 (4)(27x^3)$

$= 1080x^3$

(c) $\left(2x - \frac{1}{2}\right)^{12}$

$(2x - \frac{1}{2})^{12} = (a+b)^n$

$a = 2x,\ b = \frac{-1}{2}$

$= 12C_r\ (2x)^{12-r}\left(-\frac{1}{2}\right)^r$

$= 12C_r\ 2^{12-r}x^{12-r}\left(-\frac{1}{2}\right)^r$

$12 - r = 7$

$r = 5$

$U_6 = 12C_5\ (2)^7 x^7\left(-\frac{1}{2}\right)^5$

$= -3168x^7$

(d) $(2x+y)^{11}$ term in x^3

$U_{r+1} = nC_r\ a^{n-r} b^r$

$= 11C_r\ (2x)^{11-r}y^r$

$= 11C_r 2^{11-r} x^{11-r} y^r$

$11 - r = 3$

$r = 8$

$11C_8\ (2)^3 x^3 y^8$

$= 1320\ x^3 y^8$

Example III

Find the term independent of x in expansion of $\left(x^2 - \frac{1}{3x}\right)^9$

Solution

$\left(x^2 - \frac{1}{3x}\right)^9 = (a+b)^n$

$a = x^2,\ b = \frac{-1}{3x}$

$= 9C_r\ (x^2)^{9-r}\left(-\frac{1}{3}x\right)^r$

$= 9C_r\ x^{18-2r}\left(-\frac{1}{3}\right)^r x^{-r}$

$= 9C_r\ x^{18-3r}\left(-\frac{1}{3}\right)^r$

$18 - 3r = 0$

$r = 6$

$= {}^9C_6 x^0\left(-\frac{1}{3}\right)^6$

$= 84\left(\frac{1}{729}\right)$

$= \frac{28}{243}$

Example IV

Find the co-efficient of x in the expansion of $\left(x + \frac{2}{x^2}\right)^{10}$

Solution

$$\left(x+\frac{2}{x^2}\right)^{10} = (a+b)^n$$

$$a = x,\ b = \frac{2}{x^2},\ n = 10$$

$$U_{r+1} = nC_r\ a^{n-r}b^r$$

$$\Rightarrow 10C_r x^{10-r}\left(\frac{2}{x^2}\right)^r$$

$$= 10C_r\ x^{10-r}(2)^r(x^{-2})^r$$

$$= 10C_r x^{10-r}(2)^r x^{-2r}$$

$$= 10C_r\ 2^r x^{10-3r}$$

$$\Rightarrow\ 10 - 3r = 1$$

$$r = 3$$

$$= 10C_3 2^3 x$$

$$= 960x$$

The coefficient is 960

Example V

Find the co-efficient of the term in x^6 in the expansion $\left(x-\frac{2}{x}\right)^8$

Solution

$$\left(x-\frac{2}{x}\right)^8 = (a+b)^n$$

$$a = x,\ b = \frac{-2}{x}$$

$$U_{r+1} = nC_r\ a^{n-r}b^r$$

$$= 8C_r\ x^{8-r}\ \left(-\frac{2}{x}\right)^r$$

$$= 8C_r\ x^{8-r}\left(\frac{(-2)^r}{x^r}\right)$$

$$= 8C_r\ x^{8-r}(-2)^r x^{-r}$$

$$= 8C_r\ (-2)^r\ x^{8-2r}$$

$$8 - 2r = 6$$

$$2 = 2r$$

$$r = 1$$

$$8C_1(-2)^1 x^6$$

$$= -16x^6$$

The coefficient is -16

Validity of a Binomial Expansion

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \dots$$

When n is not positive the binomial theorem is valid for $-1 < x < 1$ or when $|x| < 1$

Example I

State what values of *x* for which the following expansions are valid:

a) $\left(1-\frac{x}{2}\right)^{-5}$

b) $(1+2x)^{\frac{1}{2}}$

c) $\left(\frac{1}{(2+x)^2}\right)$

d) $\left(1+\frac{1}{x}\right)^{\frac{1}{2}}$

Solution

$(1+x)^n$ is valid when $|x|<1$

So $\left(1-\frac{x}{2}\right)^{-5}$ is valid $\left|-\frac{x}{2}\right| < 1$

$\Rightarrow \pm\left(-\frac{x}{2}\right) < 1$

$-\frac{x}{2} < 1$

$-x < 2$

$-x < 2$

$x > -2$

$-\left(-\frac{x}{2}\right) < 1$

$\frac{x}{2} < 1$

$x < 2$

$\Rightarrow -2 < x < 2$

$\left(1-\frac{x}{2}\right)^{-5}$ is valid when $-2 < x < 2$

(b) $(1+2x)^{\frac{1}{2}}$

$(1+x)^n$ is valid when $|x| < 1$

$|2x| < 1$

$\pm(2x) < 1$

$x < \frac{1}{2}$

$-2x < 1$

$x > -\frac{1}{2}$

$\frac{-1}{2} < x < \frac{1}{2}$

$\Rightarrow (1+2x)^{\frac{1}{2}}$ is valid when $-\frac{1}{2} < x < \frac{1}{2}$

(c) $\frac{1}{(2+x)^2} = (2+x)^{-2}$

$$= 2^{-2}\left(1+\frac{x}{2}\right)^{-2}$$

it is valid when $\left|\frac{x}{2}\right| < 1$

$\pm\left(\frac{x}{2}\right) < 1$

$x < 2$

$-\frac{x}{2} < 1$

$x > -2$

$-2 < x < 2$

$$\Rightarrow \frac{1}{(2+x)^2} \text{ is valid} -2 < x < 2$$

(d) $\left(1+\frac{1}{x}\right)^{\frac{1}{2}}$, for validity $\left|\frac{1}{x}\right| < 1$

$\frac{1}{|x|} < 1,\ \ 1 < |x|$

$|x| > 1$ for expansion $\left(1+\frac{1}{x}\right)^{\frac{1}{2}}$ to be valid

More examples on Binomial Expansion

Example I (UNEB Question)

Expand $(1-x)^{\frac{1}{3}}$ as far as the term in x^3. Hence evaluate $\sqrt[3]{24}$

Solution

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \ldots$$

$$(1-x)^{1/3} = 1 + \frac{1}{3}(-x) + \frac{1/3(1/3-1)(-x)^2}{2!} + \frac{1/3(1/3-1)(1/3-2)(-x)^3}{3!} + \ldots$$

$$(1-x)^{\frac{1}{3}} = 1 + \frac{1}{3}(-x) + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)(-x)^2}{2!} + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)(-x)^3}{3!} \ldots$$

$$(1-x)^{\frac{1}{3}} = 1 - \frac{1}{3}x - \frac{1}{9}x^2 - \frac{5}{81}x^3 + \cdots$$

$$\sqrt[3]{24} = \sqrt[3]{27-3} = (27-3)^{\frac{1}{3}}$$

$$(27-3)^{\frac{1}{3}} = 27^{\frac{1}{3}}\left(1-\frac{3}{27}\right)^{\frac{1}{3}} = 3\left(1-\frac{1}{9}\right)^{\frac{1}{3}}$$

Comparing $\left(1-\frac{1}{9}\right)^{\frac{1}{3}}$ with $(1-x)^{\frac{1}{3}} \Rightarrow x = \frac{1}{9}$

$$\left(1-\frac{1}{9}\right)^{\frac{1}{3}} = 1 - \frac{1}{3}\left(\frac{1}{9}\right) - \frac{1}{9}\left(\frac{1}{9}\right)^2 - \frac{5}{81}\left(\frac{1}{9}\right)^3 + \cdots$$

$$\left(1-\frac{1}{9}\right)^{\frac{1}{3}} \approx 0.961506545$$

$$(27)^{\frac{1}{3}} \approx 3\left(1-\frac{1}{9}\right)^{\frac{1}{3}}$$

$$= 3(0.961506545)$$

$$= 2.8845$$

$$\Rightarrow \sqrt[3]{24} = 2.8845$$

Example II

Determine the expansion of $\left(1-\frac{x}{3}\right)^{\frac{1}{2}}$ as far as the term in x^3 Hence evaluate $\sqrt{8}$

Solution

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \ldots$$

$$\left(1-\frac{x}{3}\right)^{\frac{1}{2}} = 1 + \frac{1}{2}\left(-\frac{x}{3}\right) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{x}{3}\right)^2}{2!} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{x}{3}\right)^3}{3!} + \cdots$$

$$\left(1-\frac{x}{3}\right) = 1 - \frac{x}{6} - \frac{x^2}{72} - \frac{1}{432}x^3 + \cdots$$

$$\sqrt{8} = (8)^{\frac{1}{2}} = (9-1)^{\frac{1}{2}}$$

$$= 9^{\frac{1}{2}}\left(1-\frac{1}{9}\right)^{\frac{1}{2}}$$

$$= 3\left(1-\frac{1}{9}\right)^{\frac{1}{2}}$$

Comparing $\left(1-\frac{1}{9}\right)^{\frac{1}{2}}$ with $\left(1-\frac{x}{3}\right)^{\frac{1}{2}}$

$$\Rightarrow \frac{x}{3} = \frac{1}{9},\ x = \frac{1}{3}$$

$$\Rightarrow \left(1-\frac{1}{9}\right)^{\frac{1}{2}} = 1 - \frac{\frac{1}{3}}{6} - \frac{\left(\frac{1}{3}\right)^2}{72} - \frac{1}{432}\left(\frac{1}{3}\right)^3 + \cdots$$

$$\sqrt{8} = 3\left(1-\frac{1}{9}\right)^{\frac{1}{2}}$$

$$\sqrt{8} = 3\left(1 - \frac{1}{18} - \frac{1}{648} - \frac{1}{11664} + \cdots \ldots\ldots\ldots\right)$$

$$\sqrt{8} = 2.8284$$

Example III

Expand $\left(1+\frac{x}{3}\right)^{15}$ up to and including the term in x^3.

Solution

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \ldots$$

$$\left(1+\frac{x}{3}\right)^{15} = 1 + 15\left(\frac{x}{3}\right) + \frac{15(14)\left(\frac{x}{3}\right)^2}{2!} + \frac{(15)(14)(13)\left(\frac{x}{3}\right)}{3!} + \cdots$$

$$\left(1+\frac{x}{3}\right)^{15}=1+\frac{15x}{3}+\frac{35x^2}{3}+\frac{455x^3}{27}+\cdots$$

Example IV

Expand $(1+2x)^{\frac{1}{2}}$ as far as the term inx^4.

Solution

$$(1+2x)^{\frac{1}{2}}$$

$$(1+x)^n=1+nx+\frac{n(n-1)x^2}{2!}+\frac{n(n-1)(n-2)}{3!}x^3+\ldots$$

$$(1+2x)^{\frac{1}{2}}=1+\frac{1}{2}(2x)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(2x)^2}{2!}+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(2x)^3}{3!}+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(2x)^4}{4!}+\cdots$$

$$=1+x-\frac{1}{2}x^2+\frac{1}{2}x^3-\frac{5}{8}x^4+\cdots$$

$$(1+2x)^{\frac{1}{2}}=1+x-\frac{1}{2}x^2+\frac{1}{2}x^3-\frac{5}{8}x^4+\cdots$$

Example V

Expand $\left(1-\frac{x}{2}\right)^{-5}$ as far as the fourth term

Solution

$$(1+x)^n=1+nx+\frac{n(n-1)x^2}{2!}+\frac{n(n-1)(n-2)x^3}{3!}+\ldots$$

$$\left(1-\frac{x}{2}\right)^n=1+nx+\frac{n(n-1)x^2}{2!}+\frac{n(n-1)(n-2)x^3}{3!}+\cdots$$

$$\left(1-\frac{x}{2}\right)^{-5}=1+(-5)\left(-\frac{x}{2}\right)+\frac{(-5)(-6)\left(-\frac{x}{2}\right)^2}{2!}+\frac{(-5)(-6)(-7)\left(-\frac{x}{2}\right)^3}{3!}+\cdots$$

$$\left(1-\frac{x}{2}\right)^{-5}=1+\frac{5}{2}x+\frac{15}{4}x^2+\frac{35}{8}x^3+\cdots$$

Example VI

Expand $(3+x)^{-1/2}$ as far as the third term

Solution

$$(3+x)^{-1/2}=\left[3\left(1+\frac{x}{3}\right)\right]^{-1/2}$$

$$=3^{-1/2}\left(1+\frac{x}{3}\right)^{-1/2}$$

$$(3+x)^{-\frac{1}{2}}=3^{(-1/2)\left(1+\frac{x}{3}\right)^{\frac{-1}{2}}}$$

$$=\frac{1}{3^{\frac{1}{2}}}\left(1+\frac{x}{3}\right)^{-\frac{1}{2}}$$

But $\left(1+\frac{x}{3}\right)^{-\frac{1}{2}}$

$$=1+\left(-\frac{1}{2}\right)\left(\frac{x}{3}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{x}{3}\right)^2}{2!}+\cdots$$

$$=1-\frac{x}{6}+\frac{x^2}{24}+\cdots$$

$$(3+x)^{-\frac{1}{2}}=3^{-\frac{1}{2}}\left(1+\frac{x}{3}\right)^{-\frac{1}{2}}$$

$$=\frac{1}{\sqrt{3}}\left(1-\frac{x}{6}+\frac{x^2}{24}+\cdots\right)$$

$$=\frac{1}{\sqrt{3}}-\frac{1}{6\sqrt{3}}x+\frac{1}{24\sqrt{3}}x^2+\cdots$$

Example VII

Find the expansion of $\left(1+\frac{1}{x}\right)^{1/2}$ as far as the term x^{-3}

Solution:

$$(1+x)^n=1+nx+\frac{n(n-1)x^2}{2!}+\frac{n(n-1)(n-2)x^3}{3!}+\ldots$$

$$\left(1+1/x\right)^{1/2}=1+\frac{1}{2}\left(\frac{1}{x}\right)+\frac{\frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{1}{x}\right)^2}{2!}+\frac{\frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{1}{x}\right)^3}{3!}+\ldots$$

$$\left(1+\frac{1}{x}\right)^{\frac{1}{2}}=1+\frac{1}{2x}-\frac{1}{8x^2}+\frac{1}{16x^3}+\cdots$$

Example VIII

Expand $(1-x)^3(2+x)^6$ up to the term in x^2.

Solution

$(1-x)^3(2+x)^6$

$$=(1+3(^-x)+\frac{3(2)(-x)^2}{2!}+\ldots)(2^6+6(2)^5x+\frac{6(5)(2^4)x^2}{2!}+\ldots)$$

$=(1-3x+3x^2\ \ldots)(64+192x+240x^2+\cdots)$

$= 64 + 192x + 240x^2 - 192x - 576x^2 + 192x^2 + \ldots$

$= 64 - 144x^2$

Example IX

Expand $\dfrac{(1+x)^2}{(1-\frac{x}{2})^3}$ up to and including the term in x^2.

Solution

$$\frac{(1+x)^2}{\left(1-\frac{x}{2}\right)^3} = (1+x)^2\left(1-\frac{x}{2}\right)^{-3}$$

$$= (1+2x+x^2)\left(1+(-3)(\tfrac{-x}{2})+\frac{(-3)(-4)(\tfrac{-x}{2})^2}{2!}+\ldots\right)$$

$$\left(1 + 2x + x^2\right)\left(1+\tfrac{3}{2}x+\tfrac{3}{2}x^2\ldots\right)$$

$$1+\frac{3x}{2}+\frac{3x^2}{2}+2x+3x^2+x^2\ldots.$$

$$1+\frac{7x}{2}+\frac{11x^2}{2}+\ldots$$

$$=1+\frac{7}{2}x+\frac{11}{2}x^2+\ldots$$

Example X

Show that, if x is small enough for its cube and higher powers to be neglected,

$$\sqrt{\frac{1-x}{1+x}}=1-x+\frac{1}{2}x^2+\ldots$$

By putting $x=\frac{1}{8}$, show that $\sqrt{7}=2\frac{83}{128}$

Solution

$$\sqrt{\frac{1-x}{1+x}}=\sqrt{\frac{(1-x)(1-x)}{(1+x)(1-x)}}$$

$$=\frac{1-x}{\sqrt{1-x^2}}$$

$$=(1-x)(1-x^2)^{-\frac{1}{2}}$$

From

$$(1+x)^n = 1+nx+\frac{n(n-1)x^2}{2!}+\cdots$$

$$(1-x^2)^{-\frac{1}{2}} = 1+\left(-\frac{1}{2}\right)(-x^2)+\cdots$$

$$(1-x^2)^{-\frac{1}{2}} = 1+\frac{1}{2}x^2+\cdots$$

$$(1-x)(1-x^2)^{-\frac{1}{2}} = (1-x)\left(1+\frac{1}{2}x^2+\cdots\right)$$

$$=1+\frac{1}{2}x^2-x-\frac{1}{2}x^3+\cdots$$

$$=1-x+\frac{1}{2}x^2+\cdots$$

$$\Rightarrow \sqrt{\frac{1-x}{1+x}} = 1-x+\frac{x^2}{2}+\cdots$$

By putting $x=\frac{1}{8}$

$$\sqrt{\frac{1-\frac{1}{8}}{1+\frac{1}{8}}} = 1-\frac{1}{8}+\frac{\left(\frac{1}{8}\right)^2}{2}+\cdots$$

$$\sqrt{\frac{7/8}{9/8}}=1-\frac{1}{8}+\frac{1}{128}+\ldots$$

$$\Rightarrow \frac{\sqrt{7}}{3}=\frac{128-16+1}{128}+\ldots$$

$$\sqrt{7}=\frac{113}{128}\times 3$$

$$\sqrt{7}=\frac{339}{128}$$

$$\sqrt{7}=2\frac{83}{128}$$

Example XI

Use Binomial expansion to expand $\sqrt{\frac{1+2x}{1-2x}}$ up to and including the term in x^3. Hence find $\sqrt{\frac{1.02}{0.98}}$ to 4 decimal places. Hence deduce the square root of 51

Solution

$$\sqrt{\frac{(1+2x)(1+2x)}{(1-2x)(1+2x)}}=\frac{1+2x}{\sqrt{1-4x^2}}$$

$$=(1+2x)(1-4x^2)^{-\frac{1}{2}}$$

$$\text{From } (1+x)^n = 1+nx+\frac{n(n-1)x^2}{2!}+\cdots$$

$$(1-4x^2)^{-\frac{1}{2}} = 1+\frac{-1}{2}(-4x^2)+\cdots$$

$$\Rightarrow (1+2x)(1-4x^2)^{-\frac{1}{2}}=(1+2x)(1+2x^2+\ldots)$$

$$= 1 + 2x^2 + 2x + 4x^3 + \ldots$$

$$\Rightarrow \sqrt{\frac{1+2x}{1-2x}} = 1+2x+2x^2+4x^3+\cdots$$

$1+2x=1.02$

$x=0.01$

$$\sqrt{\frac{1.02}{0.98}} = 1+2(0.01)+2(0.01)^2+4(0.01)^3+\cdots$$

$$\sqrt{\frac{102}{98}} = 1.020204$$

$$\frac{\sqrt{2}\times\sqrt{51}}{\sqrt{2}\times\sqrt{49}} = 1.020204$$

$$\frac{\sqrt{51}}{7} = 1.020204$$

$$\sqrt{51} = (7 \times 1.020204)$$

$$\sqrt{51} = 7.1414$$

Example XII

Given that, the first three terms of the expansion in ascending powers of x of $(1 + x + x^2)^n$ are the same as the first three terms in the expansion of $\left(\frac{1+ax}{1-3ax}\right)^3$, find the values of a and n.

Solution

$(1 + x + x^2)^n$ from

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \cdots$$

$$(1 + x + x^2)^n = 1 + n(x + x^2) + \frac{n(n-1)(x+x^2)^2}{2!} + \ldots$$

$$= 1 + nx + nx^2 + \frac{n(n-1)x^2}{2} + \cdots$$

$$= 1 + nx + \left(n + \tfrac{n(n-1)}{2}\right)x^2 + \cdots \ldots\ldots (1)$$

$$\left(\frac{1+ax}{1-3ax}\right)^3 = (1+ax)^3(1-3ax)^{-3}$$

$$(1+ax)^3 = 1 + 3(ax) + \frac{3(2)(ax)^2}{2!} + \cdots$$

$$= 1 + 3ax + \frac{6a^2x^2}{2} + \cdots$$

$$(1+ax)^3 = 1 + 3ax + 3a^2x^2 + \cdots$$

$$(1-3ax)^{-3} = 1 + (-3)(-3ax) + \frac{(-3)(-4)(-3ax)^2}{2} + \cdots$$

$$= 1 + 9ax + 54a^2x^2 + \cdots$$

$$(1+ax)^3(1-3ax)^{-3}$$

$$= (1 + 3ax + 3a^2x^2 + \cdots)(1 + 9ax + 54a^2x^2 + \cdots)$$

$$= 1 + 9ax + 54a^2x^2 + 3ax + 27a^2x^2 + 3a^2x^2 + \cdots$$

$$= 1 + 12ax + 84a^2x^2 + \cdots (2)$$

Comparing Eqn (1) and Eqn (2);

$\Rightarrow n = 12a$

$$n + \frac{n(n-1)}{2} = 84a^2$$

$$12a + \frac{12a(12a-1)}{2} = 84a^2$$

$$24a + 144a^2 - 12a = 168a^2$$

$$12a - 24a^2 = 0$$

$$12a(1-2a) = 0$$

$$a = 0 \quad a = \frac{1}{2}$$

$$n = 12a$$

$$\Rightarrow \quad n = 12 \times \frac{1}{2} = 6$$

Example XIII

Find the first three terms of the expansion $(4+x)^{-\frac{1}{2}}$ in ascending powers of x. Deduce the approximate value of $\frac{1}{\sqrt{4.16}}$

Solution

$$(4+x)^{-\frac{1}{2}} = 4^{-\frac{1}{2}}\left(1 + \frac{x}{4}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{2}\left(1 + \frac{x}{4}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{2}\left[1 + (\tfrac{-1}{2})(\tfrac{x}{4}) + \frac{(\tfrac{-1}{2})(\tfrac{-3}{2})(\tfrac{x}{4})^2}{2} + \ldots\right]$$

$$= \frac{1}{2}\left(1 - \frac{1}{8}x - \frac{3}{128}x^2 + \cdots\right)$$

$$(4+x)^{-\frac{1}{2}} = \left(\frac{1}{2} - \frac{1}{16}x - \frac{3}{256}x^2 + \cdots\right)$$

$$\frac{1}{\sqrt{4.16}} = \frac{1}{\sqrt{4+0.16}} = \frac{1}{2} - \frac{1}{16} \times 0.16 + \frac{3}{256} \times 0.16^2$$

$$\frac{1}{\sqrt{4.16}} \approx 0.4903$$

Example XIV

Write down the expansions in ascending powers of x up to the term in x^2.

(a) $(1+x)^{\frac{1}{2}}$

(b) $(1-x)^{\frac{-1}{2}}$

Hence or otherwise Expand $\sqrt{\frac{1+x}{1-x}}$ in ascending power of x up to the term in x^2. By substituting $x = \frac{1}{10}$, use your expansion to find the $\sqrt{11}$.

Solution

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}(x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)x^2}{2} + \cdots$$

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots$$

$$(1-x)^{-\frac{1}{2}} = 1 + \left(\frac{-1}{2}\right)(-x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-x)^2}{2} + \cdots$$

$$= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \cdots$$

$$\sqrt{\frac{1+x}{1-x}} = \frac{(1+x)^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}} = (1+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$$

$$= (1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots)(1 + \frac{1}{2}x + \frac{3}{8}x^2 + \cdots)$$

$$= 1 + \tfrac{1}{2}x + \tfrac{3}{8}x^2 + \tfrac{1}{2}x + \tfrac{1}{4}x^2 - \tfrac{1}{8}x^2 \ldots$$

$$= 1 + x - \tfrac{1}{2}x^2 + \ldots$$

$$\Rightarrow \sqrt{\frac{1+x}{1-x}} = 1 + x - \frac{1}{2}x^2$$

When $x = \frac{1}{10}$

$$\sqrt{\frac{1+\frac{1}{10}}{1-\frac{1}{10}}} = 1 + \frac{1}{10} - \frac{1}{2}\left(\frac{1}{10}\right)^2 + \cdots$$

$$\frac{\sqrt{11}}{\sqrt{9}} = 1 + \frac{1}{10} - \frac{1}{200} + \cdots$$

$$\sqrt{11} = 3\left(1 + \frac{1}{10} - \frac{1}{200} + \cdots\right)$$

Example **(UNEB Question)**

(a) By using the binomial theorem, expand $(8-24x)^{2/3}$ as far as the 4th term. Hence evaluate $4^{2/3}$ to one decimal place.

b) Find the coefficient of x in the expansion of $\left(x + \frac{2}{x^2}\right)^{10}$

Solution

a) By using binomial distribution theorem,

$$(1+y)^n = 1 + n(y) + \frac{n(n-1)}{2!}y^2 + \frac{n(n-2)}{3!}y^3 + \ldots$$

Now $(8-24x)^{2/3} = 8^{2/3}(1-3x)^{2/3} = 4(1-3x)^{2/3}$

Here $n = 2/3$ and $y = -3x$

By substitution,

$$4(1-3x)^{2/3} = 4\left[1 + \frac{2}{3}(-3x) + \left(\frac{2}{3}\right)\left(\frac{-1}{3}\right)\frac{(-3x)^2}{2} + \left(\frac{2}{3}\right)\left(\frac{-1}{3}\right)\left(\frac{-4}{3}\right)\frac{(-3x)^3}{6}\right]$$

$$= 4\left[1 - 2x - x^2 - \frac{4x^3}{3}\right]$$

$$= 4 - 8x - 4x^2 - \frac{16}{3}x^3$$

Evaluating $4^{2/3}$:

$$4^{2/3} = (8-4)^{2/3}$$

$$= 8^{2/3}\left(1 - \frac{1}{2}\right)^{2/3} = 4\left(1 - \frac{1}{2}\right)^{2/3}$$

Comparing $4(1-3x)^{2/3}$ with $4\left(1 - \frac{1}{2}\right)^{2/3}$

$$\Rightarrow 3x = \frac{1}{2}$$

$$x = \frac{1}{6}$$

$$\Rightarrow 4\left(1 - \frac{1}{2}\right)^{2/3} = 4 - 8\left(\frac{1}{6}\right) - 4\left(\frac{1}{36}\right) - \frac{16}{3}\left(\frac{1}{216}\right)$$

$$= 2.5 \quad (1dp)$$

b) When expanding $\left(x + \frac{2}{x^2}\right)^{10}$, we could use Pascal's triangle to get the coefficient of x, but this may seem tedious, so by using binomial expansion

$$\left(x + \frac{2}{x^3}\right)^{10} = x^{10}\left(1 + \frac{2}{x^3}\right)^{10}$$

Here, $y = \frac{2}{x^3}$ $n = 10$

$$\Rightarrow x^{10}\left(1 + \frac{2}{x^3}\right)^{10} = x^{10}\left[1 + \frac{20}{x^3} + \frac{90}{2}\left(\frac{4}{x^6}\right) + \frac{10\times9\times8}{6}\times\frac{8}{x^9} + \ldots\right]$$

$$x^{10}\left(1 + \frac{2}{x^3}\right)^{10} = x^{10} + 20x^7 + 18x^4 + 960x + \ldots$$

The coefficient of x is 960

OR By using the expansion of

$$(a+b)^n = a^n + n^{n-1}b + \frac{n(n-1)a^{n-2}b^2}{2!} + \frac{n(n-1)(n-2)a^{n-3}b^3}{3!} + \ldots + b^n$$

$$\left(x + \frac{2}{x^2}\right)^{10} = x^{10} + 10x^9\left(\frac{2}{x^2}\right) + \frac{10\times9\times x^8}{2}\times\left(\frac{2}{x^2}\right) + \frac{10\times9\times8\times x^7}{6}\times\left(\frac{2}{x^2}\right)^3 + \ldots$$

$$= x^{10} + 20x^2 + 180x^4 + 960x$$

The coefficient of x is 960

OR We could handle it by using direct approach

The term in $10_{C_3}\, x^7\left(\frac{2}{x^2}\right)^3$ is the one needed which is expanded as,

$$10_{C_3}\, x^7\left(\frac{2}{x^2}\right)^3 = 10_{C_3} \times x^7 \times \frac{8}{x^6} = 10_{C_3} \times 8x$$

The coefficient = $10_{C_3} \times 8$

= 120 × 8

= 960

Revision Exercise

1. Write down and simplify the terms indicated in the expression of the following in ascending powers of x.
   (a) $(1 + x)^9$, 4^{th} term
   (b) $\left(2 - x/2\right)^{12}$, 4^{th} term
   (c) $(3 + x)^7$, 5^{th} term
   (d) $(x + 1)^{20}$, 3^{rd} term.
2. Expand $(1 + \frac{3}{2}x - x^2)^5$ in ascending powers of x as far as the term in x^4.
3. Use the Binomial theorem to expand:
   (a) $(x + y)^4$
   (b) $(a - b)^7$
   (c) $(2 + p^2)^6$
   (d) (2h – k)5
   (e) $\left(x + \frac{1}{x}\right)^3$
   (f) $\left(z + \frac{1}{2z}\right)^8$
4. Expand the following using the Binomial theorem
   (a) $(1 + 3x)^4$
   (b) $(2x + y)^4$
   (c) $(2 - 3x)^6$
5. Write down and simplify the coefficients of the terms indicated in the expansions of the following
   (a) $\left(\frac{1}{2}t + \frac{1}{2}\right)^{10}$, term in t^4
   (b) $\left(4 + \frac{3}{4}x\right)^6$, term in x^3
   (c) $(2x - 3)^7$, term in x^5
6. Expand $\dfrac{7 + x}{(1 + x)(1 + x^2)}$ in ascending powers of x as far as the term in x^4.
7. Expand and simplify $\left(2x + \dfrac{1}{x^2}\right)^5 + \left(2x - \dfrac{1}{x^2}\right)^5$
8. Use the Binomial theorem to expand $(1 + x)^{12}$ in ascending powers of x up to and including the term in x^3.
9. Write down the coefficients of the terms indicated in the expansions of the following in ascending powers of x
   (a) $(1 + x)^{16}$, 3^{rd} term
   (b) $(2 - x)^{20}$, 18^{th} term
   (c) $(3 + 2x)^6$, 4^{th} term
   (d) $\left(2 + \frac{3}{2}x\right)^8$, 5^{th} term
10. If x is so small that x^3 and higher powers can be neglected, show that $\left(1 - \frac{3}{2}x\right)^5\left(2 + 3x\right)^6 = 64 + 96x - 720x^2$
11. The coefficient of x^3 in the expansion of $(1 + x)^2$ is four times the coefficient of x^2. Find the value of n.
12. In the Binomial expansion of $\left(1 - \frac{1}{3}x\right)^n$, the fourth and fifth terms are equal. find the value of n
13. The coefficient of x^5 in the expansion of $(1 + 5x)^8$ is equal to the coefficient of x^4 in the expansion of $(a + 5x)^7$. Find the value of a.
14. If the first three terms of the expansion of $(1 + ax)^n$ in ascending powers of x are $1 - 4x + 7x^2$, find n and a.
15. Use the expansion of $(a + b)^4$ to evaluate $(1.03)^4$ correct to 2 decimal places.
16. If x is so small to allow any term in x^5 or higher powers of x to be neglected, show that $(1 + x)^6(1 - 2x^3)^{10} \approx 1 + 6x + 15x^2 - 105x^4$.
17. When $(1 + ax)^n$ is expanded in ascending powers of x, the expansion is $1 + 2x + \dfrac{15x^2}{8} + \ldots$. Find the values of n and a.
18. When $(1 + ax)^{10}$ is expanded in ascending powers of x, the series expansion is $A + Bx + Cx^2 + 15x^3 + \ldots$. Find the values of a, A, B and C.
19. Find the ratio of the term in x^7 to the term in x^8 in the expansion of $\left(1 - \frac{1}{3}x\right)^n\left(3x + \frac{2}{3}\right)^7$
20. Expand the following in ascending powers of x as far as the terms in x^3 and state the values of x for which the expansions are valid.
    (a) $(1 + x)^{-2}$
    (b) $(1 + x)^{1/3}$
    (c) $(1 + x)^{3/2}$
    (d) $(1 + \frac{x}{2})^{-3}$
21. Write down and simplify the term independent of x in the expansion of $\left(3x^2 - \frac{1}{2x}\right)^9$ which is the numerically greatest term in this expansion when $x = \dfrac{1}{2}$.
22. In the binomial expansion of $(1 + x)^{n+1}$, n being an integer greater than 2, the coefficient of x^4 is six times the coefficient of x^2 in the expansion of $(1 + x)^{n-1}$. Determine the value of n.
23. Find the value of n for which the coefficients of x, x^2 and x^3 in the expansion of $(1 + x)^n$ are in arithmetical progression.

24. Express $\dfrac{2x^3}{(1+x^2)(1-x)^2}$ as a sum of three partial fractions and obtain an expansion in ascending powers of x of this expression as far as the term involving x^7.

25. If ${}^nC_r$ denotes the coefficient of x^r in the expansion of $(1 + x)^n$, prove that ${}^nC_r + 2({}^nC_{r+1}) + {}^nC_{r+2} = {}^{n+2}C_{r+2}$

26. If the coefficients of x^{r-1}, x^r, x^{r+1} in the binomial expansion of $(1 + x)^n$ are in arithmetical progression. Prove that $n^2 - n(4r + 1) + 4r^2 - 2 = 0$.

27. Show that if x is so small that x^4 and higher powers of x can be neglected, then

$$\left[(1+x/2)^3-(1+3x)^{1/2}\right]\div(1-\tfrac{5x}{6})=\frac{15x^2}{8}$$

28. (a) If $x-\dfrac{1}{4}=u$, express $x^3-\dfrac{1}{x^3}$ and $x^5-\dfrac{1}{x^5}$ in terms of u.

(b) Assuming that $(1-2kx+x^2)^{-1/2}$ may be expanded in a series of ascending powers of x, obtain the expansion as far as the term in x^3. Simplify the coefficients.

29. Write down the expansion in ascending powers of x. up to the term in x^2 of (i) $(1+x)^{1/2}$

(ii) $(1+x)^{-1/2}$ and simplify the coefficients.

Hence or otherwise, expand $\sqrt{\dfrac{1+x}{1-x}}$ in ascending powers of x up to the term in x^2. By using $x=\dfrac{1}{10}$, obtain an estimate, to three decimal places for π.

30. Find the term independent of x in the expansion of

(a) $\left(x^2-\dfrac{1}{3x}\right)^9$ (b) $\left(x-\dfrac{1}{x}\right)^8\left(x+\dfrac{1}{x^2}\right)^4$

Answers

1. (a) $84x^3$ (b) $-14080x^3$
(c) $945x^4$ (d) $190x^2$.

2. $1-\frac{15}{2}x+\frac{35}{2}x^2-\frac{15}{4}x^3-\frac{515}{16}x^4 \ldots$

3. (a) $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$
(b) $a^7 - 7a^6b + 21a^5b^2 - 35a^4b^3 + 35a^3b^4 - 21a2b^5 + 7ab^6 - b^7$
(c) $64 + 192p^2 + 240p^4 + 160p^6 + 60p^8 + 12p^{10} + p^{12}$
(d) $32h^5 - 80h^4c + 80h^3k^2 - 40h^2k^3 + 10hk^4 - k^5$
(e) $x^3 + 3x + \dfrac{3}{x} + \dfrac{1}{x^3}$
(f) $z^8-4z^6+7z^4-7z^2+\dfrac{7}{4z^2}+\dfrac{7}{6z^4}-\dfrac{1}{16z^6}+\dfrac{1}{256z^8}$

4. (a) $1 + 12x + 54x^2 + 108x^3 + 81x^4$
(b) $16x^4 + 32x^3y + 24x^2y^2 + 8xy^3 + y^4$
(c) $64 - 576x + 2160x^2 - 4320x^3 + 4860x^4 - 2916x^5 + 729x^6$

5. (a) $\dfrac{105}{512}$ (b) 540 (c) 6048

6. $7 - 6x - x^2 + 7x^4 + \ldots$.

7. $64x^5 + \dfrac{160}{x} + \dfrac{20}{x^7}$.

8. $1 + 12x + 66x^2 + 220x^3$

9. (a) 120 (b) -9120 (c) 4320 (d) 5670.

11. 14 **12.** 15 **13.** 2 **14.** 18, -½

15. 1.13 **17.** 16, ⅛ **18.** ½, 1, 5, $11\frac{1}{4}$

19. $\dfrac{8}{45x}$

20. **(a)** $1 - 2x + 3x^2 - 4x^3$, $-1 < x < 1$
(b) $1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3$, $-1 < x < 1$
(c) $1+\frac{3}{2}x+\frac{3}{8}x^2-\frac{1}{16}x^3$, $-1 < x < 1$
(d) $1-\frac{3}{2}x+\frac{3}{2}x^2-\frac{5}{15}x^3$, $-2 < x < 2$

21. $\dfrac{567}{16}$, 6th **22**. 8 **23**. 7

24. $\dfrac{1}{1+x^2}+\dfrac{1}{(1-x)^2}-\dfrac{2}{1-x}$, $2x^3 + 4x^4 + 4x^5 + 4x^6 + 6x^2$

23 (a) $u^3 + 3u$, $u^5 + 5u^3 + 5u$
(b) $1+kx+\dfrac{1}{2}(3k^2-1)x^2+\dfrac{k}{2}(5k^2-3)x^3$

29(i) $1+\dfrac{1}{2}x-\dfrac{1}{8}x^2$
(ii) $1+\frac{1}{2}x+\frac{3}{8}x^2$**,** $1 + x + \frac{1}{2}x^2$; 3.315

30(a) $\dfrac{258}{243}$ **(b)** **-**307.