P425/1
PURE MATHEMATICS
Paper 1
Nov./Dec. 2023
3 hours

UGANDA NATIONAL EXAMINATIONS BOARD

Uganda Advanced Certificate of Education

PURE MATHEMATICS

Paper 1

3 hours

INSTRUCTIONS TO CANDIDATES:

Answer **all** *the* **eight** *questions in section* **A** *and any* **five** *from section* **B**.

Any additional question(s) answered will **not** *be marked.*

All *necessary working* **must** *be shown clearly.*

Begin each answer on a fresh sheet of paper.

Graph paper is provided.

Silent, non-programmable scientific calculators and mathematical tables with a list of formulae may be used.

SECTION A (40 MARKS)

Answer **all** *the questions in this section.*

1. Prove by induction that $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$. *(05 marks)*

2. If a line $y = mx + c$ is a tangent to the curve $4x^2 + 3y^2 = 12$, show that $c^2 = 4 + 3m^2$. *(05 marks)*

3. Given that $y = e^x \cos 3x$, show that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = 0$. *(05 marks)*

4. Find the angle between the line $r = \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 12 \\ 4 \end{pmatrix}$ and the plane $-x + 2y + 2z - 66 = 0$. *(05 marks)*

5. Solve the inequality $\frac{7-2x}{(x+1)(x-2)} > 0$. *(05 marks)*

6. Evaluate $\int_0^{\pi/3} (1+\cos 3y)^2 \, dy$. *(05 marks)*

7. Express $2\sin\theta + 3\cos\theta$ in the form $R\sin(\theta + \alpha)$. *(05 marks)*

8. Use Maclaurin's theorem to expand $\ln(2+x)$, in ascending powers of x as far as the term in x^2. *(05 marks)*

SECTION B (60 MARKS)

Answer any **five** *questions from this section.* **All** *questions carry equal marks.*

9. (a) Solve the equation $Z^3 - 7Z^2 + 19Z - 13 = 0$. *(06 marks)*

(b) Find the fourth roots of $8(-\sqrt{3}+i)$. *(06 marks)*

10. Express $f(x) = \dfrac{3x^3 + 2x^2 - 3x + 1}{x(1-x)}$ in partial fractions.

Hence find $\int f(x)\,dx$. *(12 marks)*

11. A point E has coordinates $(2, 0, -1)$. A line through E and parallel to the line whose equation is $\dfrac{x}{-2} = y = \dfrac{z+1}{2}$, meets a plane $x + 2y - 2z = 8$ at a point B. A perpendicular line from E meets the plane at a point C.

Determine the coordinates of;

(a) B. *(07 marks)*

(b) C. *(05 marks)*

12. (a) Four different Mathematics books and six other different books are to be arranged on a shelf. In how many ways can the Mathematics books be arranged on the shelf? *(02 marks)*

(b) On a certain day, Fatuma drunk 6 bottles of the 9 bottles of soda available. On the next day she drunk 5 bottles of the 7 bottles of soda available. In how many ways could she have chosen the bottles of soda to drink in the two days? *(03 marks)*

(c) Given that ${}^{20}C_r = {}^{20}C_{r-2}$, find the value of r. *(07 marks)*

13. (a) A curve is given by the parametric equations $x = t^2 - 3$, $y = t\,(t^2 - 3)$.

Find the Cartesian equation of the curve. *(04 marks)*

(b) A point P is such that its distance from the origin is five times its distance from $(12, 0)$.

(i) Show that the locus of P is a circle.

(ii) Determine the coordinates of the centre of the circle and its radius. *(08 marks)*

14. Given the curve $y = \dfrac{1}{4x^2 - 1}$, determine the;

(a) coordinates of the turning points of the curve. *(03 marks)*

(b) equation of the asymptotes.
Hence sketch the curve. *(09 marks)*

15. (a) Show that $\tan 3\theta = \dfrac{\tan\theta\left(3 - \tan^2\theta\right)}{\left(1 - 3\tan^2\theta\right)}$. *(05 marks)*

(b) Solve the equation $\cos 4x + \cos 6x + \cos 2x = 0$ for $0^\circ \leq x \leq 180^\circ$. *(07 marks)*

16. The rate at which a body cools is proportional to the amount by which its temperature exceeds that of its surroundings. The body is placed in a room of temperature 25 °C. After 6 minutes the temperature of the body dropped from 90 °C to 60 °C.

(a) Form a differential equation for the rate of cooling of the body. *(07 marks)*

(b) Find the time it takes for the body to cool from 40 °C to 30 °C. *(05 marks)*

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1. $\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + \dots + n^2$

$1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$

for $n = 1$

L.H.S $\qquad$ R.H.S.

$1^2 = 1 \qquad \frac{1(1+1)(2\times1+1)}{6} = \frac{6}{6} = 1$

It is true for $n=1$ B1

Assume it is true $n = k$

$1^2 + 2^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$ B1

for $n = k+1$.

then $1^2 + 2^2 + \dots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$ B1

$= \frac{k+1}{6}(2k^2 + 7k + 6)$ M1

$= \frac{k+1}{6}((k+2)(2k+3))$ M1 (factorise)

$= \frac{(k+1)(k+2)(2k+3)}{6}$. A1 (final form)

so it holds true for $n = k+1$

then it true for all integral values of n. B1 concl (05)

2. $y = mx + c$

$y^2 = m^2x^2 + 2mcx + c^2$

$4x^2 + 3(m^2x^2 + 2mcx + c^2) = 12$. M1 (Subst.)

$4x^2 + 3m^2x^2 + 6mcx + 3c^2 = 12$.

$4x^2 + 3m^2x^2 + 6mcx + 3c^2 - 12 = 0$ B1 (collecting like terms)

for tangency.

$b^2 = 4ac$.

$(6mc)^2 = 4(4+3m^2)(3c^2-12)$ M1 (for condition)

$36m^2c^2 = 4(4+3m^2)(3c^2-12)$

$9m^2c^2 = 12c^2 - 48 + 9m^2c^2 - 36m^2$

$12c^2 = 48 + 36m^2$ A1 (collecting like terms)

$c = 4 + 3m^2$ As required B1

3. $y = e^x \cos 3x$

$\frac{dy}{dx} = e^x \cos 3x - 3e^x \sin 3x$ M1

$\frac{dy}{dx} = y - 3e^x \sin 3x$

$-3e^{x}\cos 3x = \frac{dy}{dx} - y$ B1 (expression)

$\frac{d^2y}{dx^2} = \frac{dy}{dx} - 3e^x \sin 3x - 9e^x \cos 3x$ M1 (for 2nd diff)

$\frac{d^2y}{dx^2} = \frac{dy}{dx} + \frac{dy}{dx} - y - 9y$ B1 (subst)

$\frac{d^2y}{dx^2} = 2\frac{dy}{dx} - 10y$

$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = 0$ B1 (Simplification as required)

(05)

4. $d = \begin{pmatrix} 3 \\ 12 \\ 4 \end{pmatrix}$ $\underset{\sim}{n} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$

$$\theta = \sin^{-1}\left[\frac{\begin{pmatrix} 3 \\ 12 \\ 4 \end{pmatrix}\cdot\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}}{\sqrt{(9+144+16)(1+144)}}\right]$$

M1 (formula) M1 d.n
M1
M1 (|d||n|)

$\theta = \sin^{-1}\left(\frac{29}{39}\right)$ M1

$\theta = 48.0°$ B1

(5)

5. $\frac{7-2x}{(x+1)(x-2)} > 0$

Critical value.

$x=-1,\ x=2,\ x=3.5$ B1 (critical values)

Testing

	$x<-1$	$-1<x<2$	$2<x<3.5$	$x>3.5$
$\frac{7-2x}{(x+1)(x-2)}$	+	−	+	−

M1 (ranges) M1 (signs)

The solution set is $x<-1$ and $2<x<3.5$

M1 for each range correct

6. $\int_0^{\pi/3} (1+\cos 3y)^2 dy$

$= \int_0^{\pi/3} (1+2\cos 3y + \cos^2 3y)\, dy$. M1 (Expansion)

$= \int_0^{\pi/3} (1+2\cos 3y + \frac{1}{2} + \frac{1}{2}\cos 6y)\, dy$. M1 (dividing 4s for ...)

$= \int_0^{\pi/3} (\frac{3}{2} + 2\cos 3y + \frac{1}{2}\cos 6y)\, dy$

$= \left[\frac{3}{2}y + \frac{2}{3}\sin 3y + \frac{1}{12}\sin 6y\right]_0^{\pi/3}$ A1 (inte. of addition)

$= \left(\frac{\pi}{2} + 0 + 0\right) - (0)$ M1 (substitution)

$= \pi/2$ B1 1.570796. (accuracy - leave it ...)

7 $2\sin\theta + 3\cos\theta = R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$.

$\Rightarrow R\sin\alpha = 3$

$R\cos\alpha = 2$. B1 (comparing coeff.)

$\tan\alpha = 3/2$. M1

$\alpha = \tan^{-1}(3/2) = 56.3^\circ$ A1

$R^2(\sin^2\alpha + \cos^2\alpha) = 9+4$

$R^2 = 13$.

$R = \sqrt{13}$ A1

α and R are similarly figured hence the M mark

$\therefore 2\sin\theta + 3\cos\theta = \sqrt{13}\sin(\theta + 56.3^\circ)$ B1

8 Let $f(x) = \ln(2+x)$

$f(0) = \ln 2$ B1

$f'(x) = \frac{1}{2+x}$ M1

N.B. quote $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0)$ before subst.

$f'(0) = 1/2$. A1

$f''(x) = \frac{-1}{(2+x)^2}$

$f''(0) = -1/4$ A1

Similarly i.e $\frac{d}{dx}$ ok

$\therefore \ln(2+x) = \ln 2 + \frac{1}{2}x - \frac{\frac{1}{4}x^2}{2!}$

$= \ln 2 + \frac{1}{2}x - \frac{x^2}{8} + \ldots$ B1

9. $z^3 - 7z^2 + 19z - 13 = 0$

Testing for $z = 1$ — Trial and Error method

$(1)^3 - 7(1)^2 + 19(1) - 13$

$= 0$ — hence B1

Hence $z = 1$ is a root and $(z-1)$ is a factor.

$$
\begin{array}{r|l}
 & z^2 - 6z + 13 \\
\hline
z-1 & z^3 - 7z^2 + 19z - 13 \quad M1 \\
 & -(z^3 - z^2) \\
 & -6z^2 + 19z - 13 \\
 & -(-6z^2 - 6z) \\
 & 13z - 13 \\
 & 13z - 13 \\
 & 0
\end{array}
$$

$\therefore (z-1)(z^2 - 6z + 13) = 0$ B1

Either $z - 1 = 0$, $z = 1$

or $z^2 - 6z + 13 = 0$

$z = \dfrac{6 \pm \sqrt{36 - 52}}{2}$ M1

$z = \dfrac{6 \pm 4i}{2} = 3 \pm 2i$

$\therefore$ roots are $z = 3 + 2i$ B1, $z = 3 - 2i$ B1 and $z = 1$

(b) Let $z = -\sqrt{3} + i$

$|z| = \sqrt{(\sqrt{3})^2 + (1)^2} = 2$

$Arg(z) = 180° - \tan^{-1}(1/\sqrt{3}) = 150°$ B1

$z = 2[\cos(150° + 360n) + i\sin(150° + 360n)]$

Let $P = 8(-\sqrt{3} + i)$

$P = 16[\cos(150 + 360n) + i\sin(150 + 360n)]$

$P_n^{1/4} = 2\left[\cos\left(\frac{150 + 360n}{4}\right) + i\sin\left(\frac{150 + 360n}{4}\right)\right]$ M1 $n = 0, 1, 2, 3$.

$P_0^{1/4} = 2(\cos 37.5° + i\sin 37.5°)$

$= 1.5867 + 1.2175i$ A1

for $n = 1$ $P_1^{1/4} = 2(\cos 127.5 + i\sin 127.5)$

$= -1.2175 + 1.5867i$ A1

for $n = 2$ $P_2^{1/4} = 2(\cos 217.5 + i\sin 217.5°)$

$= -1.5867 - 1.2175i$ A1

for $n = 3$ $P_3^{1/4} = 2(\cos 307.5° + i\sin 307.5°)$ A1

$= 1.2175 - 1.5867i$

10. $f(x) =$

$$\begin{array}{r} -3x-5 \\ -x^2+x \;\big)\; \overline{3x^3+2x^2-3x+1} \\ 3x^3-3x^2 \\ \hline 5x^2-3x+1 \\ 5x^2-5x+0 \\ \hline 2x+1 \end{array}$$

M1

long division

B1

$$f(x) = \frac{3x^3+2x^2-3x+1}{x(1-x)} = -3x-5+\frac{2x+1}{x(1-x)} \quad \text{B1}$$

But $\dfrac{2x+1}{x(1-x)} = \dfrac{A}{x} + \dfrac{B}{1-x}.$

$2x+1 = A(1-x) + Bx \cdot$ M1

putting $x = 0 \cdot$

$A = 1$ B1

putting $x = 1$

$3 = B \cdot$ B1

$$\therefore \frac{2x+1}{x(1-x)} = \frac{1}{x} + \frac{3}{1-x}. \quad \text{A1}$$

$$\therefore f(x) = -3x-5+\frac{1}{x}+\frac{3}{1-x} \quad \text{B1}$$

$$\int f(x)\,dx = \int\left(3x-5+\frac{1}{x}+\frac{3}{1-x}\right)dx \quad \text{M1}$$

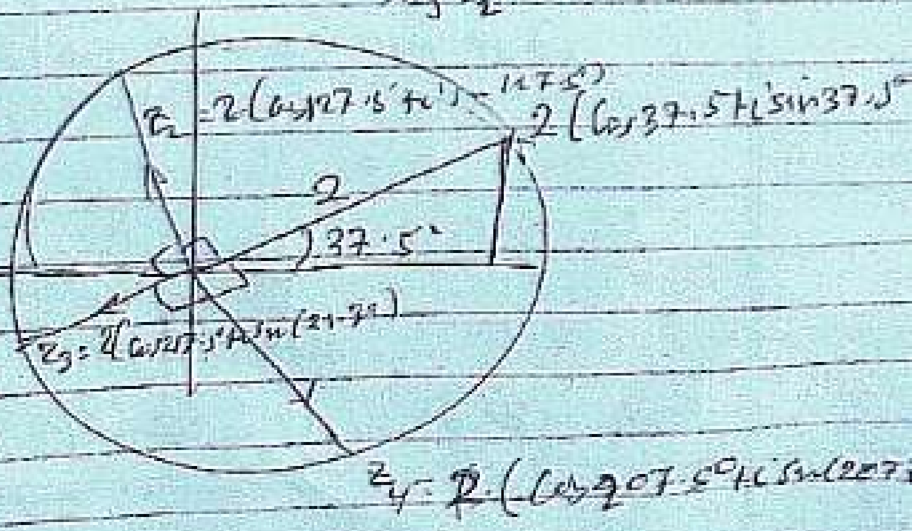
M1

$$= -\tfrac{3}{2}x^2 - 5x + \ln x - 3\ln(1-x) + c \quad \text{A1}$$

$$= -\tfrac{3}{2}x^2 - 5x + \ln\left(\frac{x}{(1-x)^3}\right) + c. \quad \text{B1} \qquad (12)$$

$\arg z_2 = 90+37.5 = 127.5°$

Alt! $\dfrac{360-90}{4}$



11 (a)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

Equation of the line through E.

$$\underset{\sim}{r} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \quad B1$$

$$\text{Point } \underset{\sim}{B} = \begin{pmatrix} 2-2\mu \\ \mu \\ -1+2\mu \end{pmatrix} \quad B1$$

$$(2-2\mu) + 2\mu + 2 - 4\mu = 8 \quad M1$$

$$-4\mu = 4$$

$$\mu = -1 \quad A1$$

$$\text{Point } \underset{\sim}{B} = \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix} \quad M1$$

Coordinates of B is $(4, -1, -3)$ A1

(b) line through E perpendicular to the plane.

$x + 2y - 2z = 8$

$$\underset{\sim}{r} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \quad B1$$

$$\text{Point } \underset{\sim}{C} = \begin{pmatrix} 2+\beta \\ 2\beta \\ -1-2\beta \end{pmatrix} \quad B1$$

$x = 2 + \beta$
$y = 2\beta$ B1
$z = -1 - 2\beta$

$$2 + \beta + 4\beta + 2 + 4\beta = 8$$

Sub. $x + 2y - 2z = 8$
$2 + \beta + 4\beta + 2(1+2\beta) = 8$

$$9\beta = 4 \quad M1$$

$$\beta = \frac{4}{9} \quad A1$$

$9\beta = 4, \beta = \frac{4}{9}$

$$\text{Point } \underset{\sim}{C} = \begin{pmatrix} \frac{22}{9} \\ 8/9 \\ -17/9 \end{pmatrix} \quad M1$$

Coordinates of C is $\left(\frac{22}{9}, 8/9, -17/9\right)$ B1

2.(a) No of ways $= {}^{10}P_4 = 5040$ ways. M1 A1

$= 10!$

(b) Number of ways $= {}^{9}C_6 + {}^{7}C_5$ $6!$ M1

$= 84 + 21$ M1

$= 105$ ways. A1

(c) $${}^{20}C_r = {}^{20}C_{r-2}$$

$$\frac{20!}{(20-r)!r!} = \frac{20!}{(20-r+2)!(r-2)!}$$ M1

$(22-r)!(r-2)! = (20-r)!r!$ M1

M1 M1

$(22-r)(21-r)(20-r)!(r-2)! = (20-r)(r-1)(r-2)!r$

$(22-r)(21-r) = r(r-1)$

$(22-r)(21-r) = r^2 - r$ M1

$462 - 22r - 21r + r^2 = r^2 - r$

$42r = 462$ M1

$r = 11$ A1

13. (a) $y = tx$ B1 (subst)

$t = \frac{y}{x}$

$x = \frac{y^2}{x^2} - 3$ M1 Simplify

$x^3 = y^2 - 3x^2$ M1.

$y^2 = x^3 + 3x^2$ A1

(b) Let $P(x, y)$

$$\left((x-0)^2 + (y-0)^2\right)^{\frac{1}{2}} = 5\left((x-12)^2 + (y-0)^2\right)^{\frac{1}{2}} \text{ B1}$$

$$x^2 + y^2 = 25(x^2 - 24x + 144 + y^2) \text{ M1}$$

$$x^2 + y^2 = 25x^2 + 600x + 3600 + 25y^2 \text{ M1}$$

$$24x^2 + 24y^2 - 600x + 3600 = 0.$$

$$x^2 + y^2 - 25x + 150 = 0 \quad \text{A1}$$

Hence locus is a circle.

Comparing with $x^2 + y^2 + 2gx + 2fy + c = 0$

$$2g = \frac{-25}{1}$$

$$g = -\frac{25}{2}$$

$2f = 0, \quad f = 0$

Centre is $\left(\frac{25}{2}, 0\right)$ M1 A1

Radius $= \sqrt{\left(\frac{25}{2}\right)^2 - 150}$ M1

$= \sqrt{156.25 - 150}$

$= 2.5$ units. A1

Alt: complete squares

i.e. $\left(x - \frac{25}{2}\right)^2 - \frac{625}{4} + (y-0)^2 = -150$

$\left(x - \frac{25}{2}\right)^2 + y^2 = \frac{625}{4} - 150$

4 $\frac{dy}{dx} = \frac{-8x}{(4x^2-1)^2} = 0$. M1

$-8x = 0$ ✓

$x = 0$. A1

$y = \frac{1}{4(0)^2 - 1} = -1$ ✓

Nature

L	$\frac{dy}{dx}$	R
+	0	−

maximum point.

turning point is $(0, -1)$ B1 and is a maximum point

(b) vertical asymptotes

$(2x-1)(2x+1) = 0$.

$x = \frac{1}{2}$, $x = -\frac{1}{2}$ ✓ B1 B1

horizontal asymptotes.

$4yx^2 - y = 1$

$4yx^2 - y - 1 = 0$.

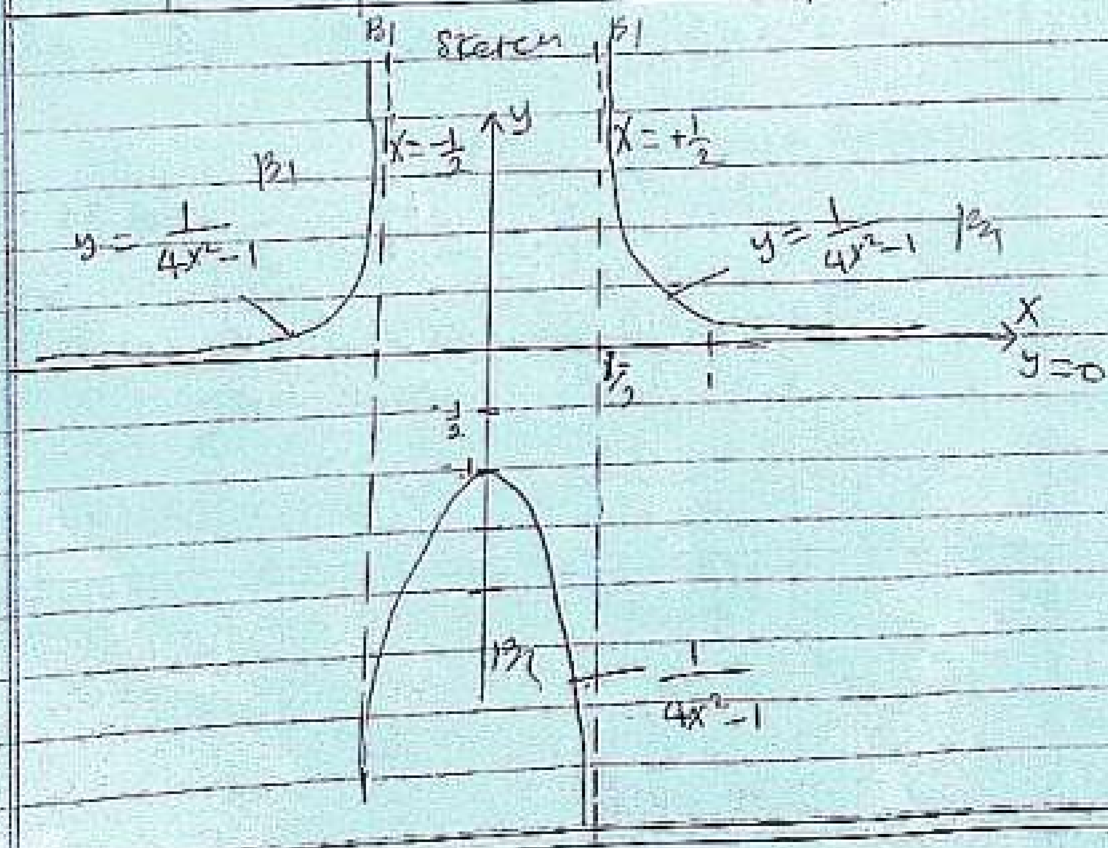
$y = 0$. ✓ B1 horizontal asymptotes.

y-intercept

$x = 0$, $y = -1$.

$(0, -1)$

	$x < -\frac{1}{2}$	$-\frac{1}{2} < x < \frac{1}{2}$	$x > \frac{1}{2}$	
$y = \frac{1}{4x^2-1}$	+		+	B1



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15 $\tan 3\theta = \tan(2\theta+\theta)$

$$= \frac{\tan 2\theta + \tan\theta}{1 - \tan 2\theta \tan\theta} \quad M1$$

or $(\cos\theta + i\sin\theta)^3$

$$= \frac{\frac{2\tan\theta}{1-\tan^2\theta} + \tan\theta}{1 - \frac{2\tan\theta}{1-\tan^2\theta}\tan\theta} \quad M1$$

$$= \frac{2\tan\theta + \tan\theta - \tan^3\theta}{1-\tan^2\theta - 2\tan^2\theta} \quad M1$$

$$= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} \quad A1$$

$$= \frac{\tan\theta(3-\tan^2\theta)}{(1-3\tan^2\theta)} \quad B1$$

(b) $\cos 4x + \cos 6x + \cos 2x = 0$

$\cos 4x + 2\cos 4x \cos 2x = 0$ M1

$\cos 4x(1 + 2\cos 2x) = 0$

$\cos 4x = 0$

$4x = \cos^{-1}(0)$ M1 $\quad 0° \le 4x \le 720°$

$4x = 90°, 270°, 450°, 630°$ A1

$x = 22.5°, 67.5°, 112.5°, 157.5°$ B1

or $2\cos 2x + 1 = 0$

$\cos 2x = -\frac{1}{2}$ M1 $\quad 0° \le 2x \le 360°$

$2x = \cos^{-1}(-\frac{1}{2})$

$2x = 120°, 240°$ A1

$x = 60°, 120°$ B1

$\therefore x = 22.5°, 60°, 67.5°, 112.5°, 120°, 157.5°$

Let θ be the temperature of the body.

16 $-\frac{d\theta}{dt} \propto (\theta - 25^\circ c)$

$\frac{d\theta}{dt} = -k(\theta - 25^\circ c)$ B1

$\int \frac{d\theta}{\theta - 25^\circ c} = \int -k\,dt$ M1/M1 Separation of variables and integration.

$\ln(\theta - 25^\circ c) = -kt + c$

$t = 0;\quad \theta = 90^\circ c$

$c = \ln 65^\circ c$ B1

$\ln(\theta - 25^\circ c) = -kt + \ln 65^\circ c$

$kt = \ln\left(\frac{65^\circ c}{\theta - 25^\circ c}\right)$ M1

$6k = \ln\left(\frac{65^\circ c}{60 - 25^\circ c}\right)$

$k = \frac{1}{6}\ln\left(\frac{13}{7}\right)$ B1

$\cancel{\frac{d\theta}{dt} = -\frac{1}{6}\ln\left(\frac{13}{7}\right)(\theta - 25^\circ c)}$

$\frac{d\theta}{dt} = \frac{1}{6}\ln\left(\frac{13}{7}\right)(\theta - 25^\circ c)$ B1

(b) $\ln\left(\frac{13}{7}\right)^{1/6} t = \ln\left(\frac{65}{\theta - 25}\right)$

when $\theta = 40^\circ c$ M1

$\ln\left(\frac{13}{7}\right)^{1/6} t = \ln\left(\frac{65}{40-25}\right)$

$t = 14.2124$ minutes. M1

$\theta = 30^\circ c$

$\ln\left(\frac{13}{7}\right)^{1/6} t = \ln\left(\frac{65}{30-25}\right)$

$t = 24.8606$ minutes B1

change in time from $40^\circ c$ to $30^\circ c$

$t = 24.8606 - 14.2124$ M1

$= 10.6482$ minutes B1