P425/1
Pure Mathematics

ASSHU ANKOLE JOINT MOCK EXAMINATIONS
2024
Marking Guides

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SECTION A (40 marks)

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1.
$$2x + y - 3z = 7 \quad \text{--- i)}$$
$$4x - 2y + z = 15 \quad \text{--- ii)}$$
$$3x + 3y + 2z = 1 \quad \text{--- iii)}$$

from i) and ii)

$$\begin{array}{c|l} 4 & 2x + y - 3z = 7 \\ 2 & 4x - 2y + z = 15 \end{array}$$

$$- \begin{array}{|l} 8x + 4y - 12z = 28 \\ 8x - 4y + 2z = 30 \end{array}$$

$$8y - 14z = -2$$
$$4y - 7z = -1 \quad \text{--- iv)}$$

from i) and iii)

$$\begin{array}{c|l} 3 & 2x + y - 3z = 7 \\ 2 & 3x + 3y + 2z = 1 \end{array}$$

$$- \begin{array}{|l} 6x + 3y - 9z = 21 \\ 6x + 6y + 4z = 2 \end{array}$$

$$-3y - 13z = 19$$
$$3y + 13z = -19 \quad \text{--- v)}$$

from iv) and v)

$$\begin{array}{r|l} 3 & 4y - 7z = -1 \\ 4 & 3y + 13z = -19 \end{array}$$

$$- \begin{array}{|l} 12y - 21z = -3 \\ 12y + 52z = -76 \end{array}$$

$$-73z = 73$$

$$z = -1$$

from iv)

$$4y - 7z = -1$$

$$4y - 7(-1) = -1$$

$$4y + 7 = -1$$

$$y = -2$$

from i)

$$2x + y - 3z = 7$$

$$2x + (-2) - 3(-1) = 7$$

$$2x - 2 + 3 = 7$$

$$2x + 1 = 7$$

$$x = 3$$

$\therefore$ $x = 3$, $y = -2$ and $z = -1$

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2. $$\int_0^{\pi/2} \sin 3x \cos 5x \, dx$$

but $\sin 3x \cos 5x = \cos 5x \sin 3x$

$$= \frac{1}{2}(\sin 8x - \sin 2x)$$

$\therefore$ $$\int \cos 5x \sin 3x \, dx = \frac{1}{2} \int (\sin 8x - \sin 2x) \, dx$$

$$= \frac{1}{2}\left(-\frac{\cos 8x}{8} + \frac{\cos 2x}{2}\right) + c$$

$$= \frac{\cos 2x}{4} - \frac{\cos 8x}{16} + c$$

$$\Rightarrow \int_0^{\pi/2} \sin 3x \cos 5x \, dx = \left[ \frac{\cos 2x}{4} - \frac{\cos 8x}{16} \right]_0^{\pi/2}$$

$$= \left( \frac{\cos \pi}{4} - \frac{\cos 4\pi}{16} \right) - \left( \frac{\cos 0}{4} - \frac{\cos 0}{16} \right)$$

$$= \frac{-1}{4} - \frac{1}{16} - \frac{1}{4} + \frac{1}{16}$$

$$= \frac{-2}{4}$$

$$= -\tfrac{1}{2}$$

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3. $4 \cos x + 3 \cos \frac{x}{2} = 1$

$$4 \left( 2\cos^2 \frac{x}{2} - 1 \right) + 3 \cos \frac{x}{2} = 1$$

$$8 \cos^2 \frac{x}{2} + 3 \cos \frac{x}{2} - 4 - 1 = 0$$

$$8 \cos^2 \frac{x}{2} + 3 \cos \frac{x}{2} - 5 = 0$$

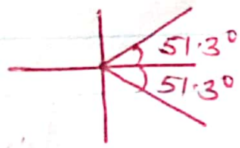
$$\cos x/2 = \frac{-3 \pm \sqrt{9 - (4 \times 8 \times -5)}}{2 \times 8}$$

$$= \frac{-3 \pm \sqrt{169}}{16}$$

$$= \frac{-3 \pm 13}{16}$$

Either $\cos \frac{x}{2} = \frac{-3 + 13}{16} = \frac{10}{16} = 5/8$

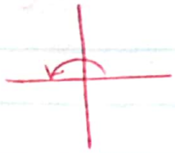
$$x/2 = \cos^{-1}(5/8)$$




$$x/2 = 51 \cdot 3^\circ$$

$$x = 102 \cdot 6^\circ$$

or $\cos\frac{x}{2} = \frac{-3-13}{16} = \frac{-16}{16} = -1$



$\frac{x}{2} = \cos^{-1}(1) = 0°$

$\frac{x}{2} = 180°$

$x = 360°$

$\therefore \quad x = 102.6°, \; 360°$

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4. $\underset{\sim}{d_1} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ and $\underset{\sim}{d_2} = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$

from $\underset{\sim}{d_1} \cdot \underset{\sim}{d_2} = |\underset{\sim}{d_1}||\underset{\sim}{d_2}| \cos\theta$

$$\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} = \left(\sqrt{2^2+(-2)^2+1^2}\right)\left(\sqrt{2^2+3^2+6^2}\right)\cos\theta$$

$$4-6+6 = \sqrt{9} \cdot \sqrt{49} \cos\theta$$

$$4 = (3\times 7)\cos\theta$$

$$\cos\theta = \frac{4}{21}$$

$$\theta = \cos^{-1}\left(\frac{4}{21}\right)$$

$$\theta = 79°$$

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5. let $y = \log_2\left(\frac{e^{\frac{2}{x^2}}}{\sin 2x}\right)$

$$\log_2\left(\frac{e^{\frac{2}{x^2}}}{\sin 2x}\right) = y$$

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$$\frac{e^{\frac{2}{x^2}}}{\sin 2x} = 2^y$$

$$y \ln 2 = \ln e^{\frac{2}{x^2}} - \ln \sin 2x$$

$$y \ln 2 = \frac{2}{x^2} \ln e - \ln \sin 2x$$

$$y \ln 2 = 2x^{-2} - \ln \sin 2x$$

$$\ln 2 \frac{dy}{dx} = -4x^{-3} - \frac{2\cos 2x}{\sin 2x}$$

$$= \frac{-4}{x^3} - 2\cot 2x$$

$$= \frac{-4 - 2x^3 \cot 2x}{x^3}$$

$$\frac{dy}{dx} = \frac{-2}{\ln 2}\left(\frac{2 + x^3 \cot 2x}{x^3}\right)$$

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6. $\log_4 x = \log_2 (3-2x)$

$$\frac{\log_2 x}{\log_2 4} = \log_2 (3-2x)$$

$$\frac{\log_2 x}{2} = \log_2 (3-2x)$$

$$\log_2 x = 2\log_2 (3-2x)$$

$$\log_2 x = \log_2 (3-2x)^2$$

$$\Rightarrow x = (3-2x)^2$$

$$x = 9 - 12x + 4x^2$$

$4x^2 - 13x + 9 = 0$

$$x = \frac{13 \pm \sqrt{169 - 4 \times 4 \times 9}}{2 \times 4}$$

$$x = \frac{13 \pm \sqrt{169 - 144}}{8}$$

$$x = \frac{13 \pm \sqrt{25}}{8}$$

$$x = \frac{13 \pm 5}{8}$$

Either $x = \frac{13+5}{8} = \frac{18}{8} = \frac{9}{4}$

or

$x = \frac{13-5}{8} = \frac{8}{8} = 1$

$\therefore x = 1$

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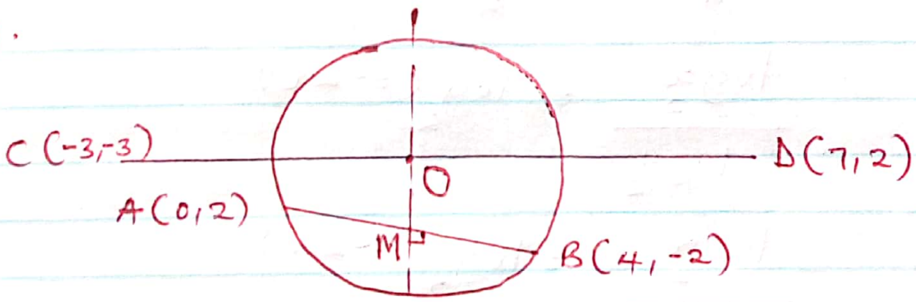
7.




Gradient of AB, $M_{AB} = \frac{4-0}{-2-2} = -1$

Gradient of the normal to AB, $M_N$

from $M_{AB} \times M_N = -1$

$-1 \times M_N = -1$

$M_N = 1$

Midpoint of AB, $M\left(\frac{4+0}{2}, \frac{-2+2}{2}\right)$

$M(2,0)$

Equation of the normal to AB

$$\frac{y-0}{x-2} = 1$$

$y = x-2$ —— i)

Equation of the ~~no~~ line CD

$$\frac{y+3}{x+3} = \frac{2+3}{7+3}$$

$$\frac{y+3}{x+3} = \frac{5}{10}$$

$$\frac{y+3}{x+3} = \frac{1}{2}$$

$2y+6 = x+3$

$2y = x-3$ —— ii)

from i)

$2(x-2) = x-3$

$2x-4 = x-3$

$x = 1$

from i) also

$y = 1-2 = -1$

The point of intersection of the normal to AB and the line CD = Centre, $O(1,-1)$ of the Circle.

Radius of the circle, $r = \overline{OB}$

$r = \sqrt{(4-1)^2+(-2-1)^2}$

$r = \sqrt{10}$ units.

from $(x-1)^2 + (y-1)^2 = (\sqrt{10})^2$

$x^2 - 2x + 1 + (y+1)^2 = 10$

$$x^2 - 2x + 1 + y^2 + 2y + 1 - 10 = 0$$
$$x^2 + 2y^2 - 2x + 2y + 2 - 10 = 0$$

$x^2 + y^2 - 2x + 2y - 8 = 0$ is the equation of the circle.

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8. $x = \sqrt{t^2+3} = (t^2+3)^{1/2}$

$$y = 3t + 4$$

$$\frac{dx}{dt} = \frac{1}{2}(t^2+3)^{-1/2} \cdot 2t$$

$$\frac{dx}{dt} = \frac{t}{(t^2+3)^{1/2}}$$

$$\frac{dy}{dt} = 3$$

$\frac{dy}{dx}$ = gradient, m of the equation of the tangent

$$m = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$m = 3 \cdot \frac{(t^2+3)^{\frac{1}{2}}}{t}$$

$$m = \frac{3(t^2+3)^{1/2}}{t}$$

At $(2,7)$,

from $y = 3t + 4$

$$7 = 3t + 4$$
$$7 - 4 = 3t$$
$$3t = 3$$
$$t = 1$$

$\therefore \quad m = \dfrac{3(1^2+3)^{1/2}}{1}$

$m = 3 \cdot 4^{1/2} = 6$

Hence the equation of the tangent at $(2,7)$ is

$$\frac{y-7}{x-2} = 6$$

$$y-7 = 6x-12$$

$$y = 6x-12+7$$

$$y = 6x-5$$

## SECTION B (60 marks)

9 i) $\int_0^1 \frac{3x+9}{x^2+5x+4}\,dx = \ln 5$

but $x^2+5x+4 = x^2+x+4x+4$

$= x(x+1)+4(x+1)$

$= (x+1)(x+4)$

$\because \frac{3x+9}{(x+1)(x+4)} = \frac{A}{x+1} + \frac{B}{x+4}$

$3x+9 = A(x+4)+B(x+1)$

$= Ax+4A+Bx+B$

$3x+9 = (A+B)x+(4A+B)$

$A+B = 3$

$A = 3-B$ —— i)

Also $4A+B = 9$

$4(3-B)+B = 9$

$12-4B+B = 9$

$-3B = -3$

$B = 1$

from i)

$A = 3-1 = 2.$

$\because \frac{3x+9}{x^2+5x+4} = \frac{2}{x+1} + \frac{1}{x+4}$

$\Rightarrow \int \frac{3x+9}{x^2+5x+4}\,dx = 2\int \frac{1}{x+1}\,dx + \int \frac{1}{x+4}\,dx$

$= 2\ln(x+1) + \ln(x+4) + c$

$= \ln(x+1)^2(x+4) + c$

$\therefore \int_0^1 \frac{3x+9}{x^2+5x+4}\,dx = \left[\ln(x+1)^2(x+4)\right]_0^1$

$$= \ln(1+1)^2(1+4) - \ln(0+1)^2(0+4)$$

$$= \ln(4\times5) - \ln(1\times4)$$

$$= \ln 20 - \ln 4$$

$$= \ln\left(\frac{20}{4}\right) = \ln 5$$

$\therefore \displaystyle\int_0^1 \frac{3x+9}{x^2+5x+4}\,dx = \ln 5$ , Hence shown.

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ii) $$\int_0^{2\pi/3} \frac{3dx}{5+4\cos x} = \pi/3$$

but $3\displaystyle\int \frac{1}{5+4\cos x}\,dx$

Let $t = \tan\frac{x}{2}$

$$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2} = \frac{1+\tan^2\frac{x}{2}}{2}$$

$$\frac{dt}{dx} = \frac{1+t^2}{2}$$

$$dx = \frac{2dt}{1+t^2}$$

and $\cos x = \dfrac{1-t^2}{1+t^2}$

$$\therefore 3\int \frac{1}{5+4\cos x}\,dx = 3\int \frac{1}{5+4\frac{(1-t^2)}{1+t^2}} \cdot \frac{2dt}{1+t^2}$$

$$= \ln(1+1)^2(1+4) - \ln(0+1)^2(0+4)$$

$$= \ln(4\times5) - \ln(1\times4)$$

$$= \ln 20 - \ln 4$$

$$= \ln\left(\frac{20}{4}\right) = \ln 5$$

$\therefore \int_0^1 \frac{3x+9}{x^2+5x+4}\,dx = \ln 5$ , Hence shown.

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ii) $$\int_0^{2\pi/3} \frac{3dx}{5+4\cos x} = \pi/3$$

but $3\int \frac{1}{5+4\cos x}\,dx$

let $t = \tan\frac{x}{2}$

$$\frac{dt}{dx} = \tfrac{1}{2}\sec^2\frac{x}{2} = \frac{1+\tan^2\frac{x}{2}}{2}$$

$$\frac{dt}{dx} = \frac{1+t^2}{2}$$

$$dx = \frac{2dt}{1+t^2}$$

and $\cos x = \dfrac{1-t^2}{1+t^2}$

$$\therefore 3\int \frac{1}{5+4\cos x}\,dx = 3\int \frac{1}{5+4\frac{(1-t^2)}{1+t^2}} \cdot \frac{2dt}{1+t^2}$$

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$$= 3\int \frac{1+t^2}{5(1+t^2)+4(1-t^2)} \cdot \frac{2\,dt}{1+t^2}$$

$$= 6\int \frac{1}{5+5t^2+4-4t^2}\,dt$$

$$= 6\int \frac{1}{9+t^2}\,dt$$

$$= 6\int \frac{1}{9\left(1+\frac{t^2}{9}\right)}\,dt$$

$$= \frac{6}{9}\int \frac{1}{\left(1+\frac{t^2}{9}\right)}\,dt$$

$$= \frac{2}{3}\int \frac{1}{\left(1+\frac{t^2}{9}\right)}\,dt$$

Let $\tan u = \frac{t}{3}$

$$\frac{\sec^2 u\,du}{dt} = \frac{1}{3}$$

$$dt = 3\sec^2 u\,du.$$

$$= \frac{2}{3}\int \frac{1}{1+\tan^2 u} \cdot 3\sec^2 u\,du$$

$$= 2\int du$$

$$= 2u + c$$

$$= 2\tan^{-1}\left(\frac{t}{3}\right) = 2\tan^{-1}\left(\frac{\tan \frac{x}{2}}{3}\right) + c.$$

$$\int_0^{2\pi/3} \frac{3\,dx}{5+4\cos x} = \left[2\tan^{-1}\left(\frac{\tan x/2}{3}\right)\right]_0^{2\pi/3}$$

$$= 2\tan^{-1}\left(\frac{\tan\frac{2\pi}{6}}{3}\right) - 2\tan^{-1}\left(\frac{\tan 0}{3}\right)$$

$$= 2\tan^{-1}\left(\frac{\tan \pi/3}{3}\right) - 2\tan^{-1}0$$

$$= 2\times \pi/6 - 0$$

$$= \pi/3$$

$$\therefore \int_0^{2\pi/3} \frac{3\,dx}{5+4\cos x} = \pi/3$$ , Hence shown

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10\. a)

$$(-1+i\sqrt{3})^8$$

but $(-1+i\sqrt{3})^8 = (-1+i\sqrt{3})^4(-1+i\sqrt{3})^4$

Also $(-1+i\sqrt{3})^4 = (-1+i\sqrt{3})^2(-1+i\sqrt{3})^2$

but $(-1+i\sqrt{3})^2 = 1-2i\sqrt{3}+i^2(\sqrt{3})^2$

$$= 1-2i\sqrt{3}-3$$
$$= -2-2i\sqrt{3}$$

$\therefore (-1+i\sqrt{3})^4 = (-2-2i\sqrt{3})(-2-2i\sqrt{3})$

$$= (-2-2\sqrt{3}i)^2$$

$$= 4+8\sqrt{3}i+(4\times 3)i^2$$
$$= 4+8\sqrt{3}i-12$$

$-8+8\sqrt{3}i$

$\Rightarrow (-1+i\sqrt{3})^8 = (-8+8\sqrt{3}i)^2$

$= 64-128\sqrt{3}i+(64\times3)i^2$

$= 64-128\sqrt{3}i-192$

$= -128-128\sqrt{3}i$

$= -128(1+\sqrt{3}i)$

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b) $|z-2| = 2|z+1-3i|$

Let $z = x+iy$

$|x+iy-2| = 2|x+iy+1-3i|$

$|(x-2)+iy| = 2|(x+1)+(y-3)i|$

$\sqrt{(x-2)^2+y^2} = 2\sqrt{(x+1)^2+(y-3)^2}$

$x^2-4x+4+y^2 = 4(x^2+2x+1+y^2-6y+9)$

$x^2-4x+4+y^2 = 4x^2+8x+40+4y^2-24y$

$3x^2+3y^2+12x-24y+36=0$

$x^2+y^2+4x-8y+12=0$

Hence the curve is a circle with;

Centre, $O(-g,-f)$

but $2g=4$, $g=2$ and $2f=-8$, $f=-4$

$\therefore O(-2,4)$

and radius, $r = \sqrt{g^2+f^2-c}$

$= \sqrt{2^2+(-4)^2-12}$

$= \sqrt{8}$ units

Intercepts,

when $x = 0$,

$y^2 - 8y + 12 = 0$

$y^2 - 2y - 6y + 12 = 0$

$y(y-2) - 6(y-2) = 0$

$(y-2)(y-6) = 0$

$y = 2$ or $y = 6$

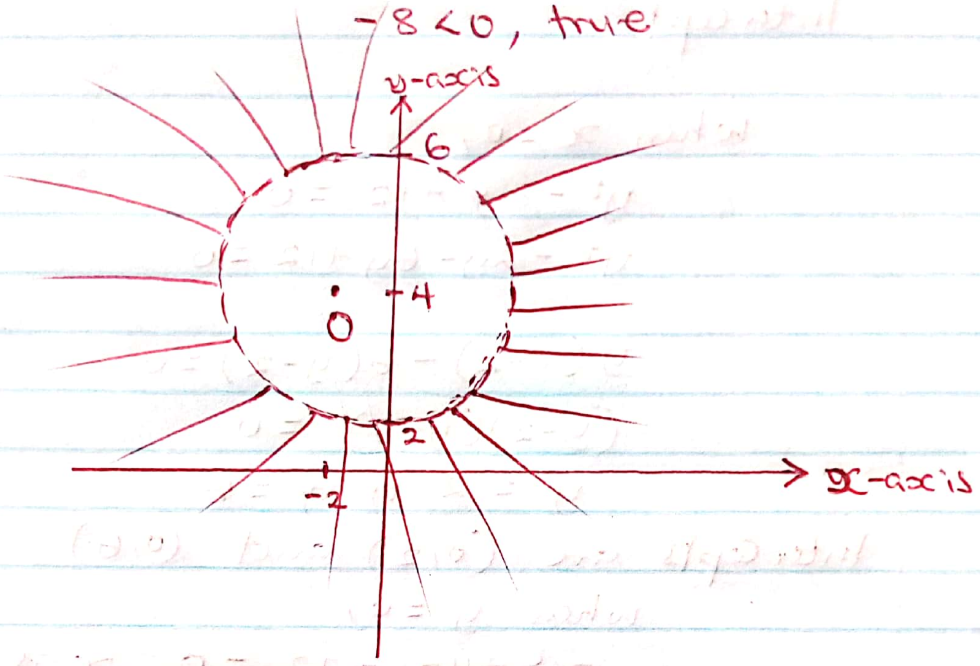
Intercepts are $(0,2)$ and $(0,6)$

when $y = 0$,

$x^2 + 4x + 12 = 0$, $x$ is undefined.

for $x^2+y^2+4x-8y+12<0$
using $O(-2,4)$ as the test point

$$(-2)^2+(4)^2+4(-2)-8(4)+12<0$$
$$4+16-8-32+12<0$$
$$-8<0, \text{ true}$$




11 a) From $\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{d}$

$$\underset{\sim}{r} = \begin{pmatrix}1\\1\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\3\\3\end{pmatrix}$$

$$\underset{\sim}{r} = (1+2\lambda)i + (1+3\lambda)j + (3+3\lambda)k$$

b)
$$\underset{\sim}{r_1} = \begin{pmatrix}2+5\lambda\\3-3\lambda\\-1+2\lambda\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2+5\lambda\\3-3\lambda\\-1+2\lambda\end{pmatrix} \text{ ——— i)}$$

Also $\underset{\sim}{r_2} = \begin{pmatrix}9-3\mu\\2+5\mu\\2-\mu\end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9-3\mu \\ 2+5\mu \\ 2-\mu \end{pmatrix} \text{ — ii)}$$

from i) and ii),

$$\begin{pmatrix} 2+5\lambda \\ 3-3\lambda \\ -1+2\lambda \end{pmatrix} = \begin{pmatrix} 9-3\mu \\ 2+5\mu \\ 2-\mu \end{pmatrix} \text{ — *}$$

$2+5\lambda = 9-3\mu$

$5\lambda + 3\mu = 7$ — iii)

$3-3\lambda = 2+5\mu.$

$3\lambda + 5\mu = +1$ — iv)

from iii) and iv)

$$\begin{array}{c|l} 3 & 5\lambda + 3\mu = 7 \\ 5 & 3\lambda + 5\mu = 1 \end{array}$$

$$\begin{array}{l} 15\lambda + 9\mu = 21 \\ - \; 15\lambda + 25\mu = 5 \end{array}$$

$-16\mu = 16$

$\mu = -1$

from iii)

$5\lambda + 3(-1) = 7$

$5\lambda = 10$

$\lambda = 2$

from *)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2+5(2) \\ 3-3(2) \\ -1+2(2) \end{pmatrix} = \begin{pmatrix} 9-3(-1) \\ 2+5(-1) \\ 2-(-1) \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ -3 \\ 3 \end{pmatrix} = \begin{pmatrix} 12 \\ -3 \\ 3 \end{pmatrix}$$

Hence the position vector of the point of intersection of $\underset{\sim}{r_1}$ and $\underset{\sim}{r_2}$ is $\begin{pmatrix} 12 \\ -3 \\ 3 \end{pmatrix}$

Direction of $\underset{\sim}{r_1}$, $\underset{\sim}{d_1} = \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix}$

Direction of $\underset{\sim}{r_2}$, $\underset{\sim}{d_2} = \begin{pmatrix} -3 \\ 5 \\ -1 \end{pmatrix}$

In vector form, $\underset{\sim}{r} = \underset{\sim}{d_1} + \lambda \underset{\sim}{d_1} + \mu \underset{\sim}{d_2}$

$$\underset{\sim}{r} = \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 5 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 + 5\lambda - 3\mu \\ -3 - 3\lambda + 5\mu \\ 2 + 2\lambda - \mu \end{pmatrix}$$

$x = 5 + 5\lambda - 3\mu$

$5\lambda - 3\mu = x - 5$ ——— i)

$y = -3 - 3\lambda + 5\mu$

$-3\lambda + 5\mu = y + 3$ ——— (ii)

$z = 2 + 2\lambda - \mu$

$2\lambda - \mu = z - 2$ ——— iii)

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from i) and ii)

$$\begin{array}{r|l} -3 & 5\lambda - 3\mu = x-5 \\ 5 & -3\lambda + 5\mu = y+3 \end{array}$$

$$- \left| \begin{array}{l} -15\lambda + 9\mu = -3x+15 \\ -15\lambda + 25\mu = 5y+15 \end{array} \right.$$

$$-16\mu = -3x-5y$$

$$\mu = \frac{3x+5y}{16}$$

from i)

$$5\lambda - 3\frac{(3x+5y)}{16} = x-5$$

$$80\lambda - 9x - 15y = 16x - 80$$

$$80\lambda = 25x + 15y - 80$$

$$16\lambda = 5x + 3y - 16$$

$$\lambda = \frac{5x+3y-16}{16}$$

from iii)

$$2\frac{(5x+3y-16)}{16} - \frac{3x+5y}{16} = z-2$$

$$10x + 6y - 32 - 3x - 5y = 16z - 32$$

$$7x + y - 16z = 0$$

12 a) Given $2x^2 - 7x + 1 = 0$

$$\alpha + \beta = \frac{7}{2}$$

$$\alpha\beta = \frac{1}{2}$$

$$\left(\sqrt{\alpha/\beta} - \sqrt{\frac{\beta}{\alpha}}\right)^2$$

$$= \frac{\alpha}{\beta} - 2\sqrt{\frac{\alpha}{\beta} \cdot \frac{\beta}{\alpha}} + \frac{\beta}{\alpha}$$

$$= \frac{\alpha}{\beta} + \frac{\beta}{\alpha} - 2$$

$$= \frac{\alpha^2 + \beta^2}{\alpha\beta} - 2 = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} - 2$$

$$= \frac{(\alpha+\beta)^2 - 2\alpha\beta - 2\alpha\beta}{\alpha\beta}$$

$$= \frac{(\alpha+\beta)^2 - 4\alpha\beta}{\alpha\beta}$$

$$= \frac{\left(\frac{7}{2}\right)^2 - \frac{4}{2}}{\frac{1}{2}}$$

$$= \left(\frac{49}{4} - 2\right) \times 2$$

$$= \frac{49 - 8}{4} \times 2$$

$$= \frac{41}{2}$$, Hence shown.

b) Given $f(x) = x^4 + ax^3 + bx^2 + cx + 4$

$f'(x) = 4x^3 + 3ax^2 + 2bx + c$

$(x-2)^2 = 0$

$x = 2$

$f(2) = 0$

$f(2) = 2^4 + a(2^3) + b(2^2) + c(2) + 4$

$0 = 16 + 8a + 4b + 2c + 4$

$8a + 4b + 2c = -20$

$4a + 2b + c = -10$ —— i)

Also $f'(2) = 0$

$f'(2) = 4(2^3) + 3a(2^2) + 2b(2) + c$

$0 = 32 + 12a + 4b + c$

$12a + 4b + c = -32$ —— ii)

Also $x - 1 = 0$

$x = 1$

$f(1) = 2$

$f(1) = 1^4 + a(1^3) + b(1^2) + c(1) + 4$

$2 = 1 + a + b + c + 4$

$a + b + c = 2 - 5$

$a + b + c = -3$ —— iii)

from iii)

$a = -3 - b - c$

from i)

$4(-3-b-c)+2b+c=-10$

$-12-4b-4c+2b+c=-10$

$-2b-3c=-10+12$

$2b+3c=-2$

$b=\dfrac{-2-3c}{2}$ —— iv)

from ii)

$12(-3-b-c)+4b+c=-32$

$-36-12b-12c+4b+c=-32$

$-8b-11c=36-32$

$8b+11c=-4$

from iv)

$\dfrac{8(-2-3c)}{2}+11c=-4$

$4(-2-3c)+11c=-4$

$-8-12c+11c=-4$

$-c=8-4$

$c=-4$

from iv)

$b=\dfrac{-2-3(-4)}{2}$

$b=5$

from $a=-3-b-c$

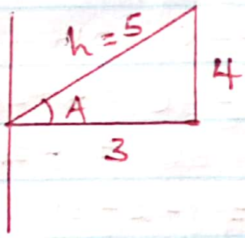
$a=-3-5-{}^{-}4$

$a=-8+4$

$a=-4$

$\therefore$ $a=-4$, $b=5$ and $c=-4$

13 a) Given $\tan A = \frac{4}{3}$



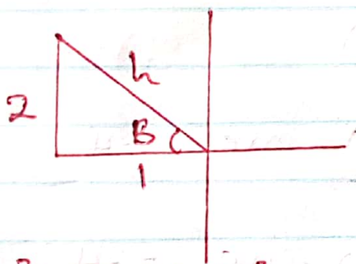

$$h^2 = 3^2 + 4^2$$

$$h^2 = 25$$

$$h = 5$$

$\sin A = \frac{4}{5}$ and $\cos A = 3/5$

Also $\tan B = -2 = \frac{2}{-1}$




$$h^2 = 2^2 + 1^2$$

$$h^2 = 5$$

$$h = \sqrt{5}$$

$\sin B = \frac{2}{\sqrt{5}}$ and $\cos B = \frac{-1}{\sqrt{5}}$

i) $\sin(A-B) = \sin A \cos B - \cos A \sin B$

$$= \left(\frac{4}{5} \cdot \frac{-1}{\sqrt{5}}\right) - \left(3/5 \cdot \frac{2}{\sqrt{5}}\right)$$

$$= \frac{-4}{5\sqrt{5}} - \frac{6}{5\sqrt{5}}$$

$$= \frac{-10}{5\sqrt{5}}$$

$$= \frac{-2}{\sqrt{5}} = \frac{-2\sqrt{5}}{5}$$

ii) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$$= \left(\frac{3}{5} \cdot \frac{-1}{\sqrt{5}}\right) - \left(\frac{4}{5} \cdot \frac{2}{\sqrt{5}}\right)$$

$$= \frac{-3}{5\sqrt{5}} - \frac{8}{5\sqrt{5}}$$

$$= \frac{-11}{5\sqrt{5}}$$

$$= \frac{-11\sqrt{5}}{25}$$

---

b)

Given $\frac{a+b-c}{a+b+c} = \tan\frac{A}{2} \tan\frac{B}{2}$

but $a = 2R\sin A$

$b = 2R\sin B$

$c = 2R\sin C$

$$\therefore \frac{a+b-c}{a+b+c} = \frac{2R\sin A + 2R\sin B - 2R\sin C}{2R\sin A + 2R\sin B + 2R\sin C}$$

$$= \frac{2R(\sin A + \sin B - \sin C)}{2R(\sin A + \sin B - \sin C)}$$

$$= \frac{\sin A + \sin B - \sin C}{\sin A + \sin B + \sin C}$$

$$= \frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) - \sin C}{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + \sin C}$$

$A+B+C = 180°$

$A+B = 180° - C$

$$\frac{A+B}{2} = 90^\circ - \frac{C}{2}$$

$$\sin\left(\frac{A+B}{2}\right) = \sin 90^\circ \cos C/_2 - \cos 90^\circ \sin C/_2$$

$$= \cos C/_2$$

$$= \frac{2\cos C/_2 \cos\left(\frac{A-B}{2}\right) - \sin C}{2\cos C/_2 \cos\left(\frac{A-B}{2}\right) + \sin C}$$

$$= \frac{2\cos C/_2 \cos\left(\frac{A-B}{2}\right) - 2\sin C/_2 \cos C/_2}{2\cos C/_2 \cos\left(\frac{A-B}{2}\right) + 2\sin C/_2 \cos C/_2}$$

$$= \frac{2\cos C/_2 \left[\cos\left(\frac{A-B}{2}\right) - \sin C/_2\right]}{2\cos C/_2 \left[\cos\left(\frac{A-B}{2}\right) + \sin C/_2\right]}$$

$$= \frac{\cos\left(\frac{A-B}{2}\right) - \sin\frac{C}{2}}{\cos\left(\frac{A-B}{2}\right) + \sin C/_2}$$

$$A+B+C = 180^\circ$$

$$C = 180^\circ - (A+B)$$

$$C/_2 = 90^\circ - \frac{(A+B)}{2}$$

$$\sin C/_2 = \sin\left(90^\circ - \frac{(A+B)}{2}\right)$$

$$= \sin 90^\circ \cos\frac{(A+B)}{2} - \cos 90^\circ \sin\frac{(A+B)}{2}$$

$$= \cos\frac{(A+B)}{2}$$

$$= \frac{\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)}{\cos\left(\frac{A-B}{2}\right) + \cos\left(\frac{A+B}{2}\right)}$$

$$= \frac{-\left[\cos\left(\frac{A+B}{2}\right) - \cos\left(\frac{A-B}{2}\right)\right]}{\cos\left(\frac{A+B}{2}\right) + \cos\left(\frac{A-B}{2}\right)}$$

$$= \frac{-\left[-2\sin\left(\frac{A/2 + B/2 + A/2 - B/2}{2}\right)\sin\left(\frac{A/2 + B/2 - A/2 + B/2}{2}\right)\right]}{2\cos\left(\frac{A/2 + B/2 + A/2 - B/2}{2}\right)\cos\left(\frac{A/2 + B/2 - A/2 + B/2}{2}\right)}$$

$$= \frac{2\sin A/2 \sin B/2}{2\cos A/2 \cos B/2}$$

$$= \frac{\sin A/2}{\cos A/2} \cdot \frac{\sin \frac{B}{2}}{\cos B/2}$$

$= \tan A/2 \tan B/2$, Hence proved.

14 a) Given $y = \dfrac{5x+3}{\sqrt{1-2x^2}}$

$$\ln y = \ln(5x+3) - \tfrac{1}{2}\ln(1-2x^2)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{5}{5x+3} - \frac{1}{2} \cdot \frac{-4x}{1-2x^2}$$

$$= \frac{5}{5x+3} + \frac{2x}{1-2x^2}$$

$$= \frac{5(1-2x^2) + 2x(5x+3)}{(5x+3)(1-2x^2)}$$

$$= \frac{5 - 10x^2 + 10x^2 + 6x}{(5x+3)(1-2x^2)}$$

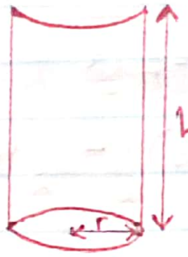
$$= \frac{5+6x}{(5x+3)(1-2x^2)}$$

$$\frac{dy}{dx} = \frac{5+6x}{(5x+3)(1-2x^2)} \cdot y$$

$$= \frac{5+6x}{(5x+3)(1-2x^2)} \cdot \frac{5x+3}{(1-2x^2)^{1/2}}$$

$$= \frac{5+6x}{(1-2x^2)^{3/2}}$$

b)




Total surface area, $S = \pi r^2 + 2\pi rh$

but Volume, $V = \pi r^2 h$

$$h = \frac{V}{\pi r^2}$$

$$\Rightarrow \quad S = \pi r^2 + 2\pi r \cdot \frac{V}{\pi r^2}$$

$$S = \pi r^2 + \frac{2V}{r}$$

$$Sr = \pi r^3 + 2V$$

$$2V = Sr - \pi r^3$$ , Hence proved.

$$V = \frac{1}{2}(Sr - \pi r^3)$$

$$\frac{dv}{dr} = \frac{1}{2}(S - 3\pi r^2)$$

At maximum Volume, $\frac{dv}{dr} = 0$

$$\frac{1}{2}(S - 3\pi r^2) = 0$$

$$S - 3\pi r^2 = 0$$

but $S = \pi r^2 + 2\pi rh$

$$\pi r^2 + 2\pi rh - 3\pi r^2 = 0$$

$$-2\pi r^2 + 2\pi rh = 0$$

$$2\pi rh = 2\pi r^2$$

$$h = r$$

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let $d$ = diameter.

At $h:d = 1:2$

$$\frac{h}{d} = \frac{1}{2}$$

but $d = 2r$

$$\frac{h}{2r} = \frac{1}{2}$$

$$2h = 2r$$

$h = r$, Hence the Volume is maximum when the ratio of $h:d = 1:2$.

15 a) $x = 4\cos\theta$

$\cos\theta = \frac{x}{4}$

and $y = 3\sin\theta$

$\sin\theta = \frac{y}{3}$

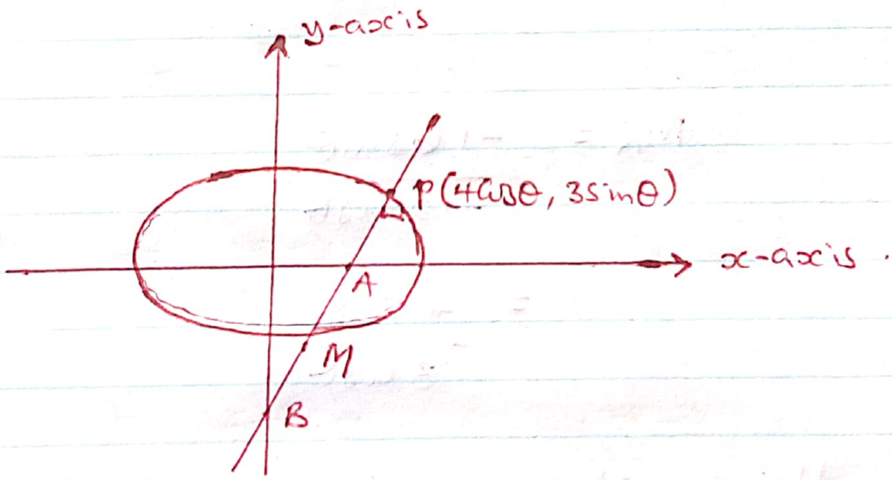
from $\sin^2\theta + \cos^2\theta = 1$

$$\left(\frac{y}{3}\right)^2 + \left(\frac{x}{4}\right)^2 = 1$$

$$\frac{y^2}{9} + \frac{x^2}{16} = 1$$

$\frac{x^2}{16} + \frac{y^2}{9} = 1$, Hence shown.

b)




Given $\frac{x^2}{16} + \frac{y^2}{9} = 1$

$$\frac{2x}{16} + \frac{2y}{9}\frac{dy}{dx} = 0$$

$$\frac{2y}{9}\frac{dy}{dx} = -\frac{x}{8}$$

$$\frac{dy}{dx} = \frac{-9x}{2y(8)} = \frac{-9x}{16y}$$

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At $p(4\cos\theta, 3\sin\theta)$

$$\frac{dy}{dx} = \frac{-9(4\cos\theta)}{16(3\sin\theta)} = \frac{-36\cos\theta}{48\sin\theta}$$

let $M_T$ be the gradient of the tangent and $M_N$ the gradient of the normal to the ellipse.

$$MT = \cancel{\frac{-12\cos\theta}{\quad}} \frac{-12\cos\theta}{16\sin\theta}$$

from $M_T \times M_N = -1$

$$\frac{-12\cos\theta}{16\sin\theta} M_N = -1$$

$$M_N = \frac{-16\sin\theta}{-12\cos\theta}$$

$$= \frac{4\sin\theta}{3\cos\theta}$$

∴ The equation of the normal is

$$\frac{y-3\sin\theta}{x-4\cos\theta} = \frac{4\sin\theta}{3\cos\theta}$$

$$3y\cos\theta - 9\sin\theta\cos\theta = 4x\sin\theta - 16\sin\theta\cos\theta$$

$$3y - 9\sin\theta = 4x\tan\theta - 16\sin\theta$$

$$3y = 4x\tan\theta - 16\sin\theta + 9\sin\theta$$

$$3y = 4x\tan\theta - 7\sin\theta$$

when $y = 0$

$$0 = 4x\tan\theta - 7\sin\theta$$

$$4x\tan\theta = 7\sin\theta$$

$$x = \frac{7\sin\theta}{4\tan\theta} = 7\sin\theta \div \frac{4\sin\theta}{\cos\theta}$$

$$= 7\sin\theta \cdot \frac{\cos\theta}{4\sin\theta}$$

$$x = \frac{7\cos\theta}{4}$$

$$A\left(\frac{7\cos\theta}{4}, 0\right)$$

when $x = 0$

$$3y = 4 \times 0 \times \tan\theta - 7\sin\theta$$

$$3y = -7\sin\theta$$

$$y = -\frac{7}{3}\sin\theta$$

$$B(0, -7/3\sin\theta)$$

$$M\left(\frac{\frac{7\cos\theta}{4} + 0}{2}, \frac{0 + -\frac{7}{3}\sin\theta}{2}\right)$$

$$M\left(\frac{7\cos\theta}{8}, \frac{-7\sin\theta}{6}\right)$$

$$x = \frac{7\cos\theta}{8} \quad \text{and} \quad y = \frac{-7\sin\theta}{6}$$

$$\cos\theta = \frac{8x}{7}, \quad \sin\theta = \frac{6y}{-7}$$

from $\cos^2\theta + \sin^2\theta = 1$

$$\left(\frac{8x}{7}\right)^2 + \left(\frac{6y}{-7}\right)^2 = 1$$

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$$\frac{64x^2}{49} + \frac{36y^2}{49} = 1$$, Hence the locus of M is an ellipse.

---

16 a) $$\frac{dy}{dx} = e^{2x+y}$$

$$\frac{dy}{dx} = e^{2x} \cdot e^{y}$$

$$\frac{dy}{e^{+y}} = e^{2x}dx$$

$$\int e^{-y}dy = \int e^{2x}dx$$

$$-e^{-y} = \frac{e^{2x}}{2} + c$$

$$-2e^{-y} = e^{2x} + 2c.$$

$$e^{-y} = \frac{-e^{2x} - 2c}{2}$$

$$-y\ln e = \ln(-e^{2x} - 2c) - \ln 2.$$

$$-y = \ln(-e^{2x} - 2c) - \ln 2.$$

$$y = -\ln(-e^{2x} - 2c) + \ln 2$$

$$y = \ln 2 - \ln(-e^{2x} - 2c)$$

$$y = \ln\left(\frac{+2}{-(e^{2x} + 2c)}\right)$$

$$\therefore y = \ln\left(\frac{-2}{e^{2x} + 2c}\right)$$

b) $\frac{dN}{dt} \propto N$

$\frac{dN}{dt} = KN$

$\int \frac{dN}{N} = \int K\,dt$

$\ln N = Kt + C$

At $t = 0$, $N = 10{,}000$

$\ln 10{,}000 = K(0) + C$

$C = \ln 10{,}000$

$\ln N = Kt + \ln 10{,}000$

At $t = 5$, $N = 20{,}000$

$\ln 20{,}000 = K(5) + \ln 10{,}000$

$\ln 20{,}000 - \ln 10{,}000 = 5K$

$\ln \left( \frac{20{,}000}{10{,}000} \right) = 5K$.

$\ln 2 = 5K$.

$K = \frac{1}{5} \ln 2$.

$\ln N = \frac{1}{5} t \ln 2 + \ln 10{,}000$

At $t = 5+5 = 10$ years, $N = ??$

$$\ln N = \frac{10}{5}\ln 2 + \ln 10{,}000$$

$$\ln N = 2\ln 2 + \ln 10{,}000$$

$$\ln N = \ln 4 + \ln 10{,}000$$

$$\ln N = \ln(4 \times 10{,}000)$$

$$\ln N = \ln(40{,}000)$$

$N = 40{,}000$ people.