10

Calculus : Integration

Integration or anti-differentiation

You have probably wondered if it is possible to find a function given its derivative.

For example, if $\frac{dy}{dx} = x^2 + 2x - 3$, is it possible to find y?

We can think of this as a process of anti-differentiation, the inverse operation to differentiation, though it is actually called integration, and the result an integral.

The principle is easily found for single terms. We remember that if we differentiate x^n we get nx^{n-1}. So if we differentiate x^{n+1} we get $(n + 1)x^n$. and if we differentiate $\frac{1}{n+1}x^{n+1}$ we get x^n.

Hence if $\frac{dy}{dx} = x^n$, $y = \frac{x^{n+1}}{n+1}$ i.e., the integral of x^n (with respect to x) is $\frac{x^{n+1}}{n+1}$. $(n \neq -1)$.

For example, if $\frac{dy}{dx} = x^5$, $\quad y = \frac{x^6}{6}$;

If $\frac{dy}{dx} = 3x^2$, $\quad y = \frac{3x^3}{3} = x^3$;

If $\frac{dy}{dx} = 6$, $\quad y = 6x$;

If $\frac{dy}{dx} = \frac{1}{x^2} = x^{-2}$, $\quad y = \frac{x^{-1}}{-1} = -\frac{1}{x}$.

Check each of these results by differentiating y with respect to x,

However, there is one important point to notice before proceeding further. If we differentiate $x^3 - x + 5$, $x^3 - x - 5$, $x^3 - x$ with respect to x we obtain $3x^2 - 1$ in each case.

On integrating $3x^2 - 1$ the constant term cannot be recovered, without further information. To show that there is a constant tem' in the integral, we add an arbitrary constant c (which may be zero).

Hence if
$$\frac{dy}{dx} = ax^n$$
$$y = \frac{ax^{n+1}}{n+1} + c. \quad (n \neq -1),$$

This is known as the indefinite integral of ax^n, and the constant c should always be added. We shall discuss this matter further below.

Our working rule is: increase the index of the term by 1 and divide by the new index, leaving coefficients as they are, and add an arbitrary constant. The result can always be tested by differentiation.

Example 1

Integrate with respect to x

(a) 5; $\quad$ (b) $x^{\frac{3}{2}}$; $\quad$ (c) $4\sqrt{x}$ $\quad$ (d) $\frac{1}{\sqrt{x}}$.

(a) If $\frac{dy}{dx} = 5$

Then $y = 5x + c$. (In using the rule, 5 can be thought of as $5x^o$).

(b) If $\frac{dy}{dx} = x^{\frac{3}{2}}$,

Then $y = \dfrac{x^{\frac{5}{2}}}{\frac{5}{2}}$

$= \frac{2}{5}x^{\frac{5}{2}} + c.$

(c) If $\frac{dy}{dx} = 4\sqrt{x} = 4x^{\frac{1}{2}}$,

then $y = \dfrac{4x^{\frac{3}{2}}}{\frac{3}{2}}$

$= \frac{8}{3}x^{\frac{3}{2}} + c.$

(d) If $\frac{dy}{dx} = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$,

Then $y = \dfrac{x^{\frac{1}{2}}}{\frac{1}{2}}$

$= 2x^{\frac{1}{2}} + c$ or $2\sqrt{x} + c.$

If $\frac{dy}{dx}$ is given as a polynomial, integrate term by term. In this way it is possible to integrate $\frac{x^4 + x - 3}{x^3}$ as $x + \frac{1}{x^2} - \frac{3}{x^3}$ but not $\frac{x^4 + x - 3}{x+1}$; $(2x + 3)^2$ can be integrated if expanded first.

Example 2

If $\frac{dy}{dx} = 3x^3 - 4x^2 + 5x - 1 + \frac{1}{x^2}$, find y.

$$y = \frac{3x^4}{4} - \frac{4x^3}{3} + \frac{5x^2}{2} - x + \frac{x^{-1}}{-1} + c$$

$$= \frac{3x^4}{4} - \frac{4x^3}{3} + \frac{5x^2}{2} - x - \frac{1}{x} + c.$$

Note: There is one important exception. If $\frac{dy}{dx} = \frac{1}{x} = x^{-1}$ what is y?

If we use the rule, then $y = \frac{x^0}{0}$ which is not defined. Hence $\frac{1}{x}$ cannot be integrated by this method. There is an integral, a surprising one. The integral is a logarithm function but the work is too advanced for **subsidiary level**!

Summarizing, If $\frac{dy}{dx} = ax^n$,

$y = \frac{ax^{n+1}}{n+1} + c$ Provided $n \neq -1$.

Exercise 10.1

Integrate with respect to x, simplifying your results where appropriate. Check the first ten by differentiation.

1.	x^2	2.	x^3	3.	$2x^4$	4.	$3x$
5.	$4x^5$	6.	8	7.	x^7	8.	$\frac{1}{x^3}$
9.	$2\sqrt{x}$	10	$4x^{\frac{1}{2}}$	11.	$x^{\frac{2}{3}}$	12.	$x^{-\frac{1}{2}}$
13.	1	14.	$x^{\frac{3}{4}}$	15.	$\frac{x^6}{2}$	16.	$\frac{1}{\sqrt[3]{x}}$

17. $x^3 + x^2 + x + 1$ 18. $3x^4 - x + 2$ 19. $x - \sqrt{x}$

20.	$\frac{x^4 + x + 2}{x^3}$	21.	$\frac{3x^3 + x - 2}{2x^3}$	22.	$(x + 3)^2$	23.	$(x - 1)^2$
24.	$(x - \frac{1}{x})^2$	25.	$(1 + x^2)^2$	26.	$(x - 1)^3$	27.	$\frac{(x+3)^2}{2x^4}$.

Exercise 10. 1

1.	$x^3/3$	**2.**	$x^4/4$	**3.**	$2x^5/5$	**4.**	$3x^2/2$
5.	$2x^6/3$	**6.**	$8x$	**7.**	$x^8/8$	**8.**	$-\frac{1}{2}x^{-2}$
9.	$4x^{\frac{3}{2}}/3$	**15.**	$x^7/14$	**16.**	$3x^{\frac{3}{2}}/2$		

17. $x^4/4 + x^3/3 + x^2/2 + x$

18.	$3x^5/5 - x^2/2 + 2x$	**19.**	$x^2/2 - 2x^{\frac{3}{2}}/3$
20.	$x^2/2 - 1/x - 1/x^2$	**21.**	$3x/2 - 1/2x + 1/x^2$
22.	$x^2/3 + 3x^2 + 9x$	**23.**	$x^3/3 - x^2 + x$
24.	$x^3/3 - 2x - 1/x$	**25.**	$x^5/5 + 2x^3/3 + x$
26.	$x^4/4 - x^3 + 3x^2/2 - x$	**27.**	$-3/2x^3 - 3/2x^2 - 1/2x$

The arbitrary constant

If we differentiate $y = 3x^2 + 2x + 5$ we obtain $\frac{dy}{dx} = 6x + 2$.

On integrating $\frac{dy}{dx}$ we must write $y = 3x^2 + 2x + c$.

Without further information the actual solution could be for example,

$y = 3x^2 + 2x + 5$

or $y = 3x^2 + 2x$

or $y = 3x^2 + 2x - 3$ **(fig 10.1)**.

Each of the curves shown is the graph of a solution of the differential equation $\frac{dy}{dx} = 6x + 2$. They are identical in position, which depends on the shape (all parabolas) and differ only value of the constant term. For any particular value of x, $\frac{dy}{dx}$ is the same for all the curves, so the tangents at 5 corresponding points are parallel, i.e. curves are parallel.

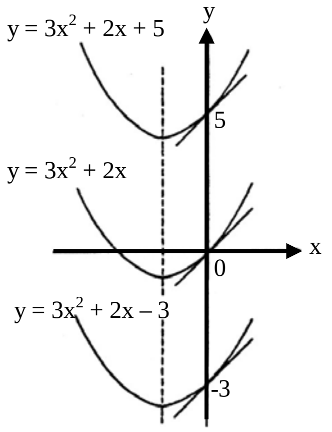


Fig 10.1

Summarizing, if $\frac{dy}{dx}$ = 6x + 2, the indefinite integral or general solution is

$$y = 3x^2 + 2x + c$$

which represents a family of parallel curves. c is an arbitrary constant. To find the actual value of c, thus identifying a particular member of the family, a pair of values of x and y (i.e., one point on the curve) must be given. See Example 4 below.

Notation for integration

If $\frac{dy}{dx}$ = 6x + 2, then we write y = $\int (6x + 2)\,dx$ (read 'integral (6x + 2)dx').

$\int$ is the sign of integration or the integral sign and $\int$ and dx must both be written. The function to be integrated, called the integrand, is placed between them. dx is written to show that the integrand is to be integrated with respect to x.

So if $\frac{dy}{dx} = f(x)$,

$y = \int f(x)dx + c$ where c is any constant.

Example 3

$$\int (x^2 + 3x + 4)dx = \frac{x^3}{3} + \frac{3x^2}{2} + 4x + c;$$

$$\int (s^2 + 4s)ds = \frac{s^3}{3} + 2s^2 + c;$$

$$\int \frac{t^3 + 3t^2 - 1}{t^2}\,dt = \int (t + 3 - \frac{1}{t^2})\,dt$$

$$= \frac{t^2}{2} + 3t + \frac{1}{t} + c.$$

Example 4

If $\frac{dy}{dx}$ = 6x + 2, find y given that y = 3 when x = 1.

$y = \int (6x + 2)dx$

$= 3x^2 + 2x + c.$

Substitute the given information.

Then $3 = 3 + 2c$

Giving $c = -2.$

Hence $y = 3x^2 + 2x - 2.$

Example 5

A particle moves in a straight line such that its acceleration after time t s is a ms^{-2} where $a = 2t^2 + t$. if its initial velocity was 3 ms^{-1} find an expression for s, the distance (in m) traveled from the start in t s.

$$a = \frac{dv}{dt}$$
$$= 2t^2 + t.$$

Hence $$v = \int (2t^2 + t)\,dt$$
$$= \frac{2t^3}{3} + \frac{t^2}{2} + c.$$

When t = 0, v = 3 which gives c = 3.

Hence $$v = \frac{2t^3}{3} + \frac{t^2}{2} + 3.$$

Now $$v = \frac{ds}{dt}$$

and hence $$s = \int\left(\frac{2t^3}{3} + \frac{t^2}{3} + 3\right)dt$$
$$= \frac{2t^4}{12} + \frac{t^3}{6} + 3t + c.$$

(A different arbitrary constant, through the same letter is used).
When t = 0, s = 0 which gives c = 0.

Hence $s = \frac{t^4}{6} + \frac{t^3}{6} + 3t$ which is the expression required.

Integration of trigonometrical functions

If $y = \sin x$, $\frac{dy}{dx} = \cos x$.

Hence $\int \cos x\,dx = \sin x + c$ (x is in radians).

If $y = \cos x$, $\frac{dy}{dx} = -\sin x$.

Hence $\int \sin x\,dx = -\cos x + c$ (x is in radians).

If $y = \tan x$, $\frac{dy}{dx} = \sec^2 x$.

Hence $\int \sec^2 x\,dx = \tan x + c$ (x is in radians).

Further, if $y = \sin ax$, $\frac{dy}{dx} = a\cos ax$, where a is a constant.

Hence $\int \cos ax\,dx = \frac{1}{a}\sin ax + c.$

Similarly $\int \sin ax\,dx = -\frac{1}{a}\cos ax + c.$

And $\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + c.$

Example 6

$$\int \sin 3x\,dx = -\frac{1}{3}\cos 3x + c;$$

$$\int \left(\cos 2x - \sin\frac{x}{2}\right)dx = \frac{1}{2}\sin 2x + 2\cos\frac{x}{2} + c;$$

$\int \sec^2 4\theta \, d\theta = \frac{1}{4}\tan 4\theta + c.$

Exercise 10.2

Find

1. $\int x\,dx$ **2.** $\int 3\,dx$ **3.** $\int (x^2 + 1)dx$

4. $\int \sin 2x\,dx$ **5.** $\int (x + 3)^2\,dx$

6. $\int \frac{dx}{x^2}$ (abbreviated form of $\int \frac{1}{x^2}dx$) **7.** $\int \cos 5x\,dx$

8. $\int \sqrt{x}\,dx$ **9.** $\int (x - \frac{1}{x})^2\,dx$ **10.** $\int \sec^2 \frac{x}{2}\,dx$

11. $\int (\sqrt{x} - \frac{1}{\sqrt{x}})dx$ **12.** $\int \frac{x^2+1}{x^2}\,dx$

13. $\int (\cos 2x - \sin 4x)\,dx$ **14.** $\int \frac{x^3 - x^2 + 1}{x^2}\,dx$ **15.** $\int x(x - 3)\,dx$

16. $\int (x + 1)(x - 2)dx$ **17.** $\int (3t + 4t^2)\,dt$ **18.** $\int (x + \cos \frac{x}{3})dx$

19. $\int (t^3 - t)\,dt$ **20.** $\int \sec^2 (\frac{2\theta}{3})\,d\theta$ **21.** $\int (\sqrt{x} - \frac{1}{x})^2\,dx$

22. $\int (\cos x + \cos 2x + \cos 3x)\,dx$ **23.** $\int \frac{2(x+3)}{x^3}\,dx.$

24. A curve is given by the differential equation $\frac{dy}{dx} = x + 2$, and it passes through the point (2, 0). Find its equation and sketch the curve.

25. If a curve is given by $\frac{dy}{dx} = 2x + 1$ and passes through the point (1, 2), find its equation and sketch the curve.

26. The rate of change of a quantity A is given by $\frac{dA}{dt} = t^2 - 1$. If $A = \frac{4}{3}$ when t = 1 find A in terms of t.

27. The velocity of a particle moving in a straight line at time t s is given by $v = 2t^2 - 3t$. Find an expression for the distance (s m) traveled, if s = 0 when t = 0.

28. A particle starts from rest at a point O and moves in a straight line in such a way that its velocity, v ms^{-1}, after time t s, is given by $v = 12t - 3t^2$, until it comes to rest again at A after 4 s. Calculate
(a) the distance OA;
(b) the greatest velocity of the particle.

29. If $\frac{dy}{d\theta} = \frac{1}{\theta^2} + \frac{1}{2}\cos 2\theta$, find y if y = 0 when $\theta = \frac{\pi}{2}$.

30. A particle is moving in a straight line and at time t s its acceleration is $(6 - kt)\ ms^{-2}$ where k is a constant. When t = 9, the acceleration of the particle is zero and its velocity is 30 ms^{-1}. Find the numerical value of the velocity when t = 0 and the distance between its' positions when t = 0 and t = 9.

31. **(i)** A curve passes through the point (0, 1) and is such that at every point of the curve $\frac{dy}{dx} = x^2$. Sketch the curve.

(ii) A particle is given an initial velocity of 12 ms^{-1} and travels in a straight line so that its retardation after t s is equal to 6t ms^{-2} until it comes to rest. If the particle then remains stationary, calcu1ate the distance traveled.

32. If $\frac{d^2y}{dx^2} = 6x - 4$ find $\frac{dy}{dx}$ given that $\frac{dy}{dx} = 3$ when $x = 0$. If also $y = 0$ when $x = 0$ find y.

33. Integrate with respect to x:

(a) $\sin^2 x$ (Use the double angle formula $\cos 2x = 1 - 2\sin^2 x$ in the from $\sin^2 x = \frac{1-2\cos 2x}{2}$ and now integrate).

(b) $\cos^2 2x$ (Use the double angle formula for cos 4x).

Exercise 10. 2

1. $x^2/2$ **2.** $3x$ **3.** $x^3/3 + x$
4. $-\frac{1}{2}\cos 2x$ **5.** $x^3/3 + 3x^2 + 9x$ **6.** $-1/x$
7. $\sin 5x/5$ **8.** $2x^{\frac{3}{2}}/3$ **9.** $x^3/3 - 2x - 1/x$
10. $2\tan x/2$ **11.** $2x^{\frac{3}{2}}/3 - 2x^{\frac{1}{2}}$ **12.** $x - 1/x$
13. $\frac{1}{2}\sin 2x + \frac{1}{4}\cos 4x$ **14.** $x^2/2 - x - 1/x$
15. $x^3/3 - 3x^2/2$ **16.** $x^3/3 - x^3/2 - 2x$ **17.** $3t^2/2 + 4t^3/3$
18. $x^2/2 + 3\sin x/3$ **19.** $t^4/4 - t^2/2$ **20.** $\frac{3}{2}\tan 2\theta/3$
21. $x^2/2 - 4x^{\frac{3}{2}} - 1/x$ **22.** $\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x$
23. $-2/x - 3/x^2$ **24.** $y = x^2/2 + 2x - 6$
25. $y = x^2 + x$ **26.** $A = t^3/3 - t + 2$
27. $s = 2t^3/3 - 3t^2/2$ **28.** (a) 32 m (b) 12 ms^{-1}
29. $y = -1/\theta + \frac{1}{4}\sin 2\theta + 2\pi$ **30.** 3; 189 m
31. (ii) 16 m **32.** $y = x^3 - 2x^2 + 3x$
33. (a) $x/2 - (\sin 2x)/4$ (b) $x/2 + (\sin 4x)/8$

Application of integration (1): Areas

Suppose $y = f(x)$ is the equation of a curve. We assume for the moment that the portion of the curve between the ordinates $x = a$ and $x = b$ ($b > a$) lies entirely above the x-axis, i.e. $y > 0$ **(fig 10.2)**. We also assume that the curve is 'continuous', i.e. that there are no breaks or gaps in it.

We now find a method of calculating the area enclosed by the curve, the x-axis and **Fig 10.2** the ordinates at A and B, i.e. the area ABCD. You will appreciate that up to now only areas which could be dissected into triangles or trapezium could be found by calculation, other shapes being found by approximate methods. Our new method is therefore very important, as it will apply to areas such as ABCD, bounded partly by a curve.

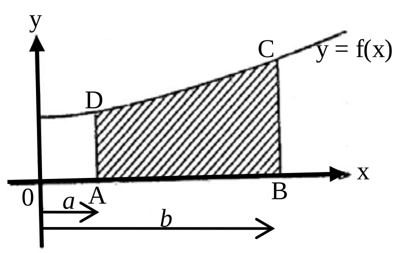


Fig 10.2

Let P be a variable point on the x-axis A and B where OP = x **(fig 10.3)**. Draw the ordinate PQ (length y) and let the shaded area APQD = A. A is thus a function of x and when x = a, A = 0. Now take an increment δx in x and the area A is increased by an amount δA, i.e. the portion PRSQ. RS is $y + \delta y$.

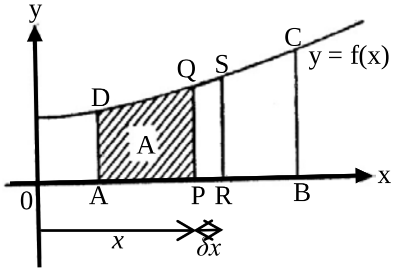


Fig 10.3

Now from **fig 10.4** it is seen that area PRTQ < ΔA < area PRSU where QT, US are parallel to the x-axis.

fig 10.4

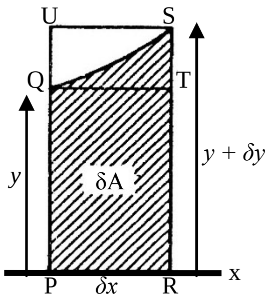


Fig 10.4

$\therefore \quad y \times \delta x < \Delta A < (y + \delta y) \times \delta x$

or $\quad y < \dfrac{\delta A}{\delta x} < (y + \delta y).$

If now $\delta x \to 0$, $\delta y \to 0$

and $\dfrac{\delta A}{\delta x} \to \dfrac{dA}{dx}$ and hence $\dfrac{dA}{dx} = y$, as the right hand term of the above inequality tends to y.

Therefore $\quad A = \int y\, dx + c$

$\quad = \int f(x)\, dx + c.$

The value of c can be found from the fact that when x = 0, A = 0. We then have A expressed as a function of x and obtain the area ABCD of **fig 10.2**.

Note: If the curve has a negative gradient in the range considered **(fig 10.5)** the above must be modified as follows.

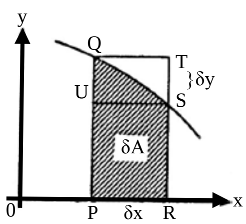


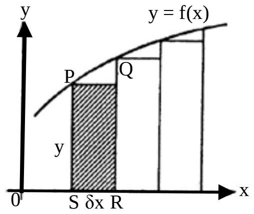
Fig 10.5

The inequality will now be area PRTQ > ΔA > area PRSU

Or $\quad y\,\delta x > \Delta A > (y + \delta y)\,\delta x$

i.e. $\quad y > \dfrac{\delta A}{\delta x} > y + \delta y$ and in the limit the same result is obtained.

If the curve contains a turning point in the range, further modification could easily be devised.
The origin of the sign $\int$ may be of interest and will give some idea of an alternative approach to the question of area calculation. P is a point (x, y) on the curve y = f(x) and PQRS is a rectangle whose side SR is δx **(fig 10.6)**. The area under the curve will contain a series of such rectangles, of area y.δx.



(fig 10.6)

Then the area under the curve will be approximately the sum of the areas of these rectangles, i.e. sum ($y\delta x$) for the range considered.
As $\delta x \longrightarrow 0$, the hunt of this sum (assuming it exists and can be found) will be the actua1 area under the curve. The initials S of sum is then written in the form $\int$ and δx is written as dx, to show that the limit has been taken.

Example 7
Find the area bounded by the curve $y = x^2 + 3$, the x-axis and the coordinates x = 1, x = 3. **(fig 10.7)**.

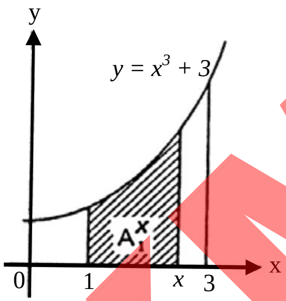


Fig 10.7

By the above, A $= \int y\,dx$

$= \int (x^2 + 3)\,dx$

$= \dfrac{x^3}{3} + 3x + c.$

When x = 1, A = 0.

Hence 0 $= \dfrac{1}{3} + 3 + c,$

giving c $= -3\dfrac{1}{3}.$

Therefore $A_1^x = \dfrac{x^3}{3} + 3x - \dfrac{10}{3}$, A_1^x meaning the area from 1 to x.

Now put x = 3.

Then A_1^3 = the required area

$= \dfrac{27}{3} + 9 - \dfrac{10}{3}$

$= \dfrac{44}{3}$ square units.

The definite integral

We can now generalize the above process, introducing a very important technique.
Consider the curve y = f(x), y > 0 in the range of x = a to x = b **(fig 10.8)**. Then the area A between the curve and the x-axis is given by

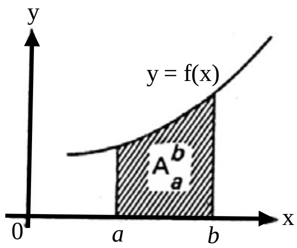


Fig 10.8

A = $\int f(x)dx + c$

= g(x) + c say.

When x = a, the area = 0.

Thus 0 = g(a) + c

or c = - g(a)

Then A_a^x = g(x) – g(a)

= (value of integral when x = b) – (value of integral when x = a)

Which is written $\int_a^b f(x)\,dx$. This is called the definite integral of f(x) with respect to x between the limits a (the lower limit) and b (the upper limit). It is a function of a and b. the arbitrary constant c disappears in the subtraction.

Hence if y = f(x) is the equation of a curve, the area between the curve, the x-axis and the coordinates x = a, x = b (b > a) is given by $\int_a^b y\,dx = \int_a^b f(x)\,dx$. In the next section we examine some complications which may arise in finding areas.

The actual technique in evaluating definite integrals is shown in the following examples.

Example 8

Evaluate $\int_1^3 (x^2 - 1)\,dx$.

$$\int_1^3 (x^2 - 1)\,dx = \left[\frac{x^3}{3} - x\right]_1^3$$

Square brackets round the indefinite integrals but omit the arbitrary constant.

$$= \left(\frac{27}{3} - 3\right) - \left(\frac{1}{3} - 1\right)$$

$$= 6 + \frac{2}{3}$$

$$= 6\frac{2}{3}$$

Example 9

Find the value of $\int_{-2}^1 (3t - 2)\,dt$.

$$\int_{-2}^1 (3t - 2)\,dt = \left[\frac{3x^2}{2} - 2t\right]_{-2}^1$$

$$= \left(\frac{3\times(1)^2}{2} - 2\right) - \left(\frac{3\times(-2)^2}{2} - 2(-2)\right)$$

$$= -\frac{1}{2} - 10$$

$$= -10\frac{1}{2}.$$

Example 10

Find $\int_0^{\pi/4} (\cos 4x + \sin 2x)\, dx$.

$$\text{Integral} = \left[\frac{\sin 4x}{4} - \frac{\cos 2x}{2}\right]_0^{\pi/4}$$

$$= \left(\frac{\sin\frac{4\pi}{4}}{4} - \frac{\cos\frac{2\pi}{4}}{2}\right) - \left(\frac{\sin 0}{4} - \frac{\cos 0}{2}\right)$$

$$= (0 - 0) - (0 - \tfrac{1}{2})$$

$$= \frac{1}{2}.$$

Exercise 10.3

Evaluate

1. $\int_0^1 dx$
2. $\int_0^3 x\,dx$
3. $\int_0^2 x^3\,dx$
4. $\int_1^2 (2x - 1)\,dx$
5. $\int_0^{\pi} \cos x\, dx$
6. $\int_0^{\pi/2} \sin 2x\, dx$
7. $\int_{-1}^1 2x^3\,dx$
8. $\int_2^3 \frac{1}{x^2}\,dx$
9. $\int_0^2 (x+1)^2\,dx$
10. $\int_{-2}^{-1} (x^2 + x - 1)\,dx$
11. $\int_0^{\pi/4} \sec^2 x\, dx$
12. $\int_{-1}^0 x(x-1)\,dx$
13. $\int_0^t (x^2 - 3)\,dx$
14. $\int_0^1 (s^2 + 3x - 2)\,ds$
15. $\int_0^{\pi} (\sin 2\theta - \cos\theta)\,d\theta$.

Exercise 10. 3

1.	**1**	**2.**	**2**	**3.**	**4**	**4.**	**2**
5.	**0**	**6.**	**1**	**7.**	**0**	**8.**	$\frac{1}{6}$
9.	**8** $\frac{2}{3}$	**10.**	$-\frac{1}{6}$	**11.**	**1**	**12.**	$\frac{2}{6}$
13.	$t^3/3 - 3t$	**14.**	$-\frac{1}{6}$	**15.**	**0**		

Further notes on areas

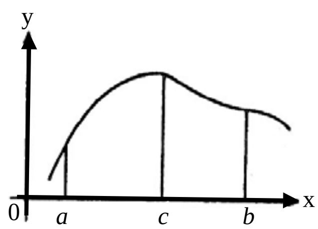


Fig 10.9

1. From fig 18.9 it is clear that $c\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$.

2. If y is negative in the range a to b, then the value obtained from the integral $\int_a^b f(x)dx$ will also be negative **(fig 10.10)** as dx is essentially positive. Thus the numerical value of the shown shaded will be $-\int_a^b f(x)dx$.

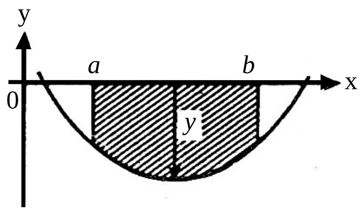


Fig 10.10

3. If the range includes both positive and negative values of y **(fig 10.11)** the total area must be found in two parts and will be.

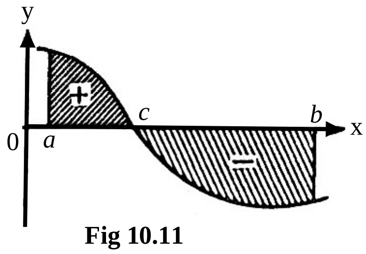


Fig 10.11

Important note: $\int_a^b f(x)dx$ is the value of the definite integral between the limits of a and b but it is NOT necessary the correct value for the area under the curve y = f(x) from a to b. it is the algebraic sum of the areas above and below the x-axis. It is wise to sketch a graph before integrating when finding an area.

4. The area between a curve and the y-axis and the lines y = a, y = b **(fig 10.12)** will be $\int_a^b xdy$. This can be proved in a manner as before.

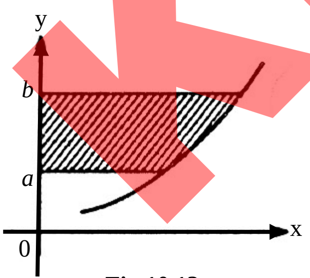


Fig 10.12

5. The area between any two curves y = f(x) and y = g(x) is easily found if the points of intersection or the limits are known **(fig 10.13)**.

The area below y = g(x) is $\int_a^b g(x)dx$.

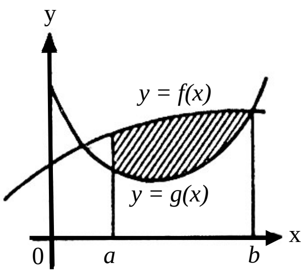


Fig 10.13

Hence the enclosed area (shown shaded) is the difference between two areas above, i.e.

$\int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b [f(x) - g(x)]dx$ assuming $f(x) > g(x)$.

A sketch should always be made to show the relative positions o f the curves.

Example 11

Find the area between the curve $y = x^2$, the x-axis and the coordinates $x = 0$ and $x = 2$.

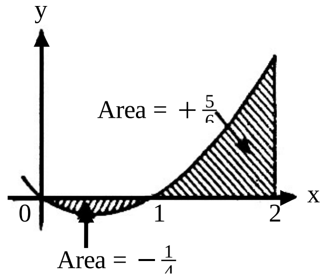


Fig 10.14

The curve crosses the x-axis where $x = 0$ and $x = 1$ **(fig 10.14)**. Hence the total area numerically

$$= \quad -\int_0^1 ydx + \int_1^2 ydx$$

$$= \quad -\int_0^1 (x^2 - x)dx + \int_1^2 (x^2 - x)dx$$

$$= \quad -\left[\frac{x^3}{3} - \frac{x^2}{2}\right]_0^1 + \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^2$$

$$= \quad -\left[\left(\tfrac{1}{3} - \tfrac{1}{2}\right) - 0\right] + -\left[\left(\tfrac{8}{3} - \tfrac{4}{2}\right) - \left(\tfrac{1}{3} - \tfrac{1}{2}\right)\right]$$

$$= \quad +\tfrac{1}{6} + \tfrac{2}{3} + \tfrac{1}{6} = 1\text{square unit.}$$

Example 12

Find the areas between the curve $y = 2x^2$ and
(i) the x-axis,
(ii) the y-axis cut off but lines parallel to the axes through the points on the curve where $x = 1$ and $x = 3$ **(fig 10.15)**.

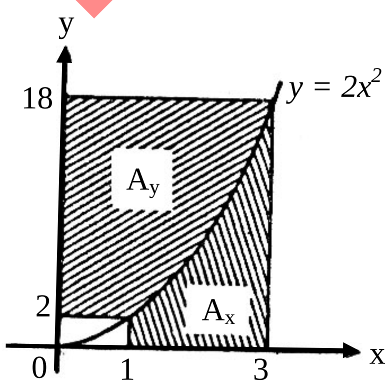


Fig 10.15

(i) A_x is required area $= \int_1^3 y\,dx$

$$= \int_1^3 2x^2\,dx$$

$$= \left[\frac{2x^3}{3}\right]_1^3$$

$$= 17\tfrac{1}{3}.$$

(ii) $A_y = \int_2^{18} x\,dy$

$$= \int_2^{18} \sqrt{\frac{y}{2}}\,dy$$

$$= \frac{1}{\sqrt{2}} \int_2^{18} \sqrt{y}\,dy$$

$$= \frac{1}{\sqrt{2}} \left[\frac{2}{3} y^{\frac{3}{2}}\right]_2^{18}$$

$$= \frac{1}{\sqrt{2}} \left[\left(\frac{2}{3} \times 18^{\frac{2}{3}}\right) - \left(\frac{2}{3} \times 2^{\frac{3}{2}}\right)\right]$$

$$= \frac{1}{\sqrt{2}} \left[\frac{2}{3} \times 27 \times 2\sqrt{2} - \frac{2}{3} \times 2\sqrt{2}\right]$$

As $18^{\frac{3}{2}} = (\sqrt{18}\,)^3$

$$= (3\sqrt{2}\,)^3$$

$$= 27 \times 2\sqrt{2}$$

$A_y = 36 - \tfrac{4}{3}$

$$= 34\tfrac{2}{3}.$$

Note: In this part the limits must be the limit s which y takes to cover the required range, and the equation of the curve must be written in the form $x = +\sqrt{\frac{y}{2}}$.

Example 13

Find the area enclosed between the curve $y = x^2 + 2$ and the line $y = 4x - 1$.

The intersections are given by

$$x^2 + 2 = 4x - 1$$

or $x^2 - 4x + 3 = 0$

Which gives $x = 1$ or 3.

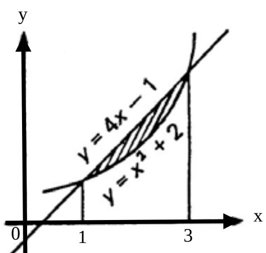


Fig 18.16

The graphs are sketched in **fig 10.16**. Between x = 1 and x = 3 the line is above the curve.

Area enclosed $= \int_1^3 (4x - 1)dx - \int_1^3 (x^2 + 2)dx$

$$= \int_1^3 (4x - 1 - x^2 - 2)dx$$

$$= \int_1^3 (4x - x^2 - 3)dx$$

$$= \left[2x^2 - \frac{x^3}{3} - 3x\right]_1^3$$

$$= (18 - 9 - 9) - (2 - \tfrac{1}{3} - 3)$$

$$= 1\tfrac{1}{3}.$$

Exercise 10.4

Find the areas between the following curves and the x-axis between the ordinates at the values given. In each case sketch the curve.

1. $y = x^2$; $x = 0, x = 3$
2. $y = 3x - 2$; $x = 3, x = 4$
3. $y = x^2 - 3x$; $x = 0, x = 3$
4. $y = x^2 - 5x + 6$; $x = 2, x = 4$
5. $y = x^3$; $x = 1, x = 3$
6. $y = \dfrac{1}{x^2}$; $x = 1, x = 3$
7. $y = (x - 1)(x - 3)$; $x = 1, x = 3$
8. $y = (1 - x)(x - 2)$: $x = 0, x = 3$
9. Sketch the curve $y = 3x - x^2$ and calculate the area between the curve and the x-axis.

10. Find the area between the curve $y = \sin x$ and the x-axis from $x = 0$ to $x = \pi$.

11. Find the area between the curve $y = \dfrac{1}{x^2}$, the y-axis and the lines $y = 4$, $y = 9$.

12. Find the area enclosed between the curves $y = 2x^2$ and $y^2 = 4x$.

13. Find the area between the x-axis and the part of the curve $y = (x - 3)(2 - x)$ which is above the x-axis.

14. Make a rough sketch of the curve $y = x(x + l)(x + 3)$ from $x = -4$ to $x = +1$. What are the slopes of the graph at the points where it closes the x-axis?
Calculate the area enclosed by the x-axis and the curve between $x = -3$ and $x = -l$.

15. Draw a rough sketch of the curve $y = x(x - 1)(x - 2)$. If this curve crosses the axis of x at O, A and B (in that order), show that the area included between the arc OA and the x-axis is equal to the area, included between the arc AB and the x-axis.

16. The curve $y = ax^2 + bx + c$ passes through the points (1, 0) and (2, 0) and its gradient at the point (2, 0) is 2. Find the numerical value of the area included between the curve and the axis of x.

17. Draw a rough sketch of the curve $y^2 = 16x$. Calculate the area enclosed by the curve and the line $x = 4$.

18. The curve $y = ax^2 + b$ passes through the points (0, k) and (h, 2k). Express a and b in terms of h and k. Show that the area bounded by the curve, the x-axis, the y-axis and the line $x = h$ is $\frac{4}{3}hk$.

19. Calculate the coordinates of the points of intersection of the line $x - y - 1 = 0$ and the curve $y = 5x - x^2 - 4$. In the same diagram sketch the line and the curve for values of x from 0 to +5. Calculate the area contained between the line and. the curve.

20. Sketch the curve whose equation is $y = (x - 1)^3$. Find the equation of the tangent to this curve at the point where $x = 3$. Calculate the area enclosed by the curve, the tangent at the point where $x = 3$ and the x-axis.

Exercise 10. 4

1.	**9**	**2.**	$8\frac{1}{2}$	**3.**	$4\frac{1}{2}$	**4.**	**1**
5.	**20**	**6.**	$\frac{2}{3}$	**7.**	$1\frac{1}{3}$	**8.**	$\frac{11}{6}$
9.	$4\frac{1}{2}$	**10.**	**2**	**11.**	**2**	**12.**	$\frac{2}{3}$
13.	$\frac{1}{6}$	**14.**	**3, -2, 6;** $\frac{8}{3}$	**15.**		**16.**	$\frac{1}{3}$
17.	$\frac{128}{3}$	**18.**	$k, k/h^2$	**19.**	**(1, 0), (3, 2),** $\frac{4}{3}$		
20.	$y = 12x - 28$; $\frac{4}{3}$						

Applications of integration (2): Volumes of revolution

A solid which has a central axis of symmetry is a solid of revolution for example, a cone, a cylinder, a flower, etc. Imagine the area under a portion AB of the curve $y = f(x)$ revolved about the x-axis through four right angles or 360^o, the x-axis acting as a kind of hinge **(fig 10.17)**. Each point of the curve describes a circle centered on the x-axis. A solid of revolution can be thought of as created this way, with two circular plane ends, cutting the x-axis at $x = a$ and $x = b$.

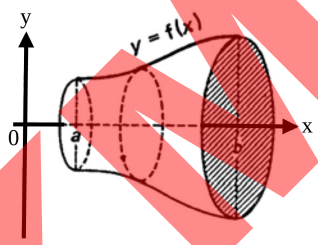


Fig 10.17

Let V be the volume of the solid from $x = a$ up to an arbitrary value of x between a and b **(fig 10.18)**. Given an increment δx in x, y takes an increment δy and V an increment δV.

Fig 18.19 shows a section through x-axis and from this it is seen that the slice δV of thickness δx is enclosed between two cylinders of outer $y + \delta y$, and inner radius y.

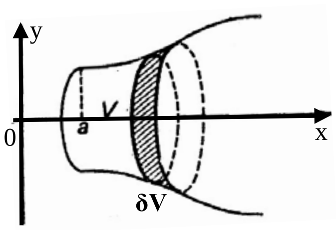


Fig 10.18

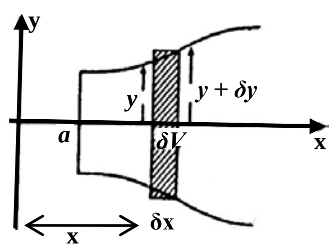


Fig 10.19

Then $\pi y^2\, \delta x < \Delta v < \pi (y + \delta y)^2\, \delta x$, with appropriate modification if the curve is falling at this point.

Then $\pi y^2 < \dfrac{\delta V}{\delta x} < \pi (y + \delta y)^2$.

Now let $\delta x \to 0$ and $\delta y \to 0$ and $\dfrac{\delta V}{\delta x} \to \dfrac{dV}{dx}$.

Hence from the above inequality, $\frac{dV}{dx} = \pi y^2$

or $V = \int_a^b \pi y^2 dx$

where y = f(x) and V is the volume e of solid generated when the curve y = f(x) between limits x = a and x = b is rotated completely around the x-axis.

Example 14

The portion of the curve $y = x^2$ between x = 0 and x = 2 is rotated completely round the x-axis. Find the volume of the solid created.

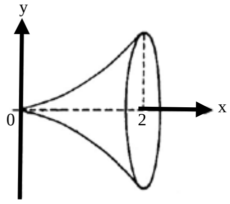


Fig 10.20

$$V = \int_0^2 \pi y^2 dx$$
$$= \int_0^2 \pi x^4 dx$$
$$= \pi\left[\frac{x^5}{5}\right]_0^2$$
$$= \frac{32\pi}{5} \ (\approx 20.1) \text{ units of volume.}$$

Similarly if a portion of the curve y = f(x) between the limits y = a and y = b is rotated completely round the y-axis, the volume of the solid generated will be given by

$V = \int_a^b \pi x^2 dy$ which can be proved in the same way.

Example 15

The part of the curve $y = x^3$ from x = 1 to x = 2 is rotated completely round the y-axis. Find the volume of the solid generated (**fig 10.21**).

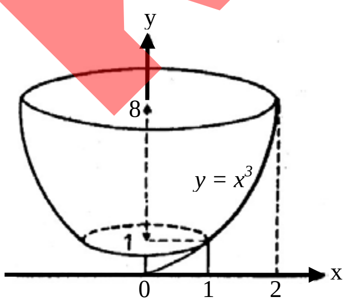


Fig 10.21

$$V = \int_1^8 \pi x^2 dy.$$

Note the limits: these are the limits of y corresponding to x = 1, x = 2. We must also express the integrand in terms of y, as we are integrating with respect to y.

$$\text{Then } V = \int_1^8 \pi x^{2\frac{2}{3}} dy$$
$$= \pi\left[\frac{3}{5}y^{\frac{5}{3}}\right]_1^8$$
$$= \pi\left(\frac{3}{5}\times 32\right) - \pi\left(\frac{3}{5}\times 1\right)$$

$$= \frac{93\pi}{5}.$$

Exercise 10.5

Leave your answers in terms of π, as in Example 15.

1. Find the volume generated by rotating the curve $y = x + 1$ from $x = 1$ to $x = 2$ completely round the x-axis.

2. The portion of the curve $y = \frac{1}{2}x^2$ from $x = 0$ to $x = 2$ is rotated about the x-axis through four right angles. Find the volume generated.

3. Sketch the curve $y = x^2 - x$. The part below the x-axis is rotated about the x-axis to form a solid of revolution. Find its volume.

4. If the part of the curve $y = x^2$ from $x = 1$ to $x = 2$ is rotated completely about the y-axis, find the volume of the solid so formed.

5. The part of the line $y = mx$ from $x = 0$ to $x = h$ is rotated about the x-axis through four right angles. Find the volume generated and hence show that the volume of a right circular cone of base radius r and height h is $\frac{1}{3}\pi r^2 h$.

6. If the area enclosed between the curves $y = x^2$ and the line $y = 2x$ is rotated around the x-axis through four right angles, find the volume of the solid generated.

7. Calculate

(a) the area bounded by the x-axis and the curve $y = x - 3\sqrt{x}$;
(b) the volume generated by revolving this area through four right angles about the x-axis. Leave this result in terms of π.

8. Find the area included between the curves $y^2 = x^3$ and $y^3 = x^2$. Find also the volume obtained by rotating this area through four right angles about the axis of x.

9. An area is bounded by the curve $y = x + \frac{3}{x}$, the x-axis and the ordinates at $x = 1$ and $x = 3$. Calculate the volume of the solid obtained by rotating this area through four right angles about the x-axis.

10. The equation $x^2 + y^2 = r^2$ represents a circle radius r, centre the origin. The quarter circle in the first quadrant is rotated completely round the x-axis to form a hemisphere. Find its volume and deduce a formula for the volume of a sphere of radius r.

11. The area contained between the curve $y^2 = x - 2$, the x-axis, the x-axis, the y-axis and the line $y = 1$ is rotated about the y-axis through four right angles. Find the volume of the solid generated.

12. Sketch the curve $y^2 = x - 1$. The area contained by this curve, the y-axis and the lines $y = \pm 2$ is completely rotated about the y-axis. Find the volume of the solid so formed.

Exercise 10. 5

1.	**$19\pi/3$**	**2.**	**$8\pi/5$**	**3.**	**$\pi/30$**	**4.**	**$15\pi/2$**
5.	**$\pi m^2h^2/3$**	**6.**	**$64\pi/15$**	**7.**	**(a) 13.5; 24.3π**		
8.	**$\frac{1}{5}$; $5\pi/28$**	**9.**	**$80\pi/3$**	**10.**	**$2\pi^3/3$**	**11.**	**$83\pi/15$**
12.	**$412\pi/15$**						

11

Compound and Multiple Angles: Trigonometry 2

Trigonometrical Identities

If A and B are two angles, it is useful to have expressions for sin (A + B), cos (A + B), etc in terms of the ratios for A and B separately. A first, suggestion might be that sin (A + B) = sinA + sinB.

Test this by taking A = 60^o, B = $30°$. Then sin(A + B) = sin 90^o = 1 but sin A + sin B = sin 60^o + sin 30^o = $\frac{\sqrt{3}}{2} + \frac{1}{2}$ which $\neq 1$.

So sin(A + B) is NOT equal to sin A + sin B. We now derive the correct formulae for compound angles A + B, A – B.

Sum of two angles (A + B)

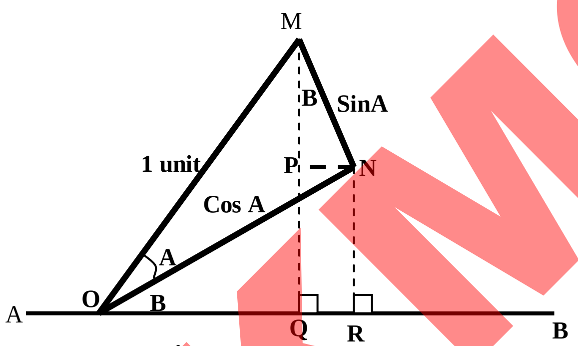


Fig. 1

A triangle MNO is right-angled at N and has NOM = A (**fig. l1.1**). The hypotenuse, OM, is of unit length, so MN = sin A and ON = cos A. The triangle is placed with its base, ON, on a horizontal line, AB. The triangle is then rotated anticlockwise about point 0, through an angle of B. Angle B = $B\hat{O}M$. Now $B\hat{O}M$ = (A + B) and it is required to find sin (A + B) and cos (A + B) in terms of sin A, cos A, sin B and cos B.

1. Drop a perpendicular MQ to AB.
2. Draw NP perpendicular to MQ, meeting it at P.
3. Drop a perpendicular NR to AB.

Now sin (A + B) = $\frac{MQ}{OM}$ = MQ (since OM is 1 unit long).

In triangle MPN, PMN = B (B = $P\hat{N}O$ = 90° - $M\hat{N}P$ = PMN)

and $M\hat{N}P$ is a right angle (PN $\perp$ PM)

$\therefore$ cos B = $\frac{MP}{MN}$

i.e. MP = MN cos B = sin A cos B.

In triangle ONR, $N\hat{R}O$ is a right angle (NR $\perp$ AB)

$\therefore$ $\frac{NR}{ON}$ = sin B

i.e. NR = ON sin B = cos A sin B but PNRQ is a rectangle (PQ and NR $\perp$ QR; NP $\perp$ MQ)

$\therefore$ NR = PQ

i.e. PQ = cos A sin B.

Now sin (A + B) = MQ = MP + PQ

= sin A cos B + cos A sin B.

$$\cos(A + B) = \frac{OQ}{OM} = OQ$$

= OR – RQ = OR – PN

= ON cosB – MN sin B

= cos A cos B – sin A sin B.

Difference of two angles (A – B)

Start with the triangle MNO again, but this time rotate it clockwise about point 0, through an angle B, as in **fig 11.2**.

Now $B\hat{O}M$ = (A – B) and it is required to find sin (A – B) and cos (A – B).

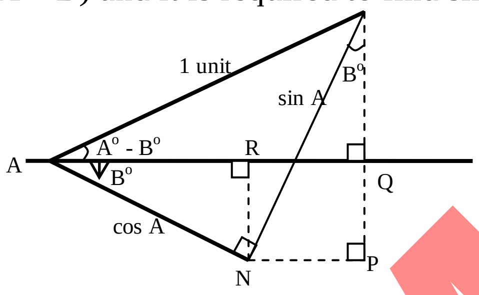


Fig 11.2

1. Draw MQ perpendicular to AB.
2. Produce MQ to P and draw NP perpendicular to MP.
3. Draw **NR** perpendicular to AB.

Now $\sin(A - B) = \frac{MQ}{OM}$ = MQ (OM is 1 unit long).

In triangle MNP, $N\hat{M}P$ = B.

Hence MP = MN cos B = sin A cos B

and NP = MN sin B = sin A sin B.

In triangle ONR, $O\hat{R}N$ is a right angle.

Hence RN = ON sin B = cos A sin B

And OR = ON cos B = cos A cos B.

Now sin (A – B) = MQ = MP – QP

= MP – RN (RN = QP: opp. sides of rectangle)

= sin A cos B – cos A sin B.

$$\cos(A - B) = \frac{OQ}{OM} = OQ$$

= OR + PQ = OR + NP (RQ = NP: opp.

= cos A cos B + sin A sin B sides of rectangle).

These results are very important and must be remembered; they are summarized below:

sin (A + B) = sin A cos B + cos A sin B

sin(A – B) = sin A cos B – cos A sin B

cos (A + B) = cos A cos B – sin A sin B

cos (A – B) = cos A cos B + sin A sin B.

These formulae are identities, as they are true for any value of angles A and B. Although a proof has only been given for acute angles, the identities can be proved true for angles in any quadrant, and of any magnitude.

The formulae can be used as an alternative way of determining the ratios of angles of any magnitude. For example:

$\sin 330^o = \sin(360^o - 30^o)$
$= \sin 360° \cos 30° - \cos 360° \sin 30^o$
$= 0 \times \cos 30^o = -\frac{1}{2}$.

$\cos 240^o = \cos(180^o + 60^o)$
$= \cos 180^o \cos 60^o - \sin 180^o \sin 60^o$
$= (-1) \times \cos 60^o - 0 \times \sin 60^o$
$= -\cos 60^o = -\frac{1}{2}$

They can also be used to find the value of negative angles. For example:

$\sin(-A) = \sin(0^o - A)$
$= \sin 0^o \cos A - \cos 0^o \sin A$
$= 0 \times \cos A - (1) \times \sin A$
$= -\sin A$.

The formulae can also be used to find the ratios of compound angles in a simple way. Care must be taken over the signs of the ratios involved.

Example 1

If $\sin A = \frac{4}{5}$ and $\cos B = \frac{12}{13}$ find the values of $\sin(A + B)$ and $\cos(A + B)$ without using tables,
(i) if A and B are both acute angles, (ii) if A is obtuse and B is acute.

If $\sin A = \frac{4}{5}$ then $\cos A = \sqrt{1-\sin^2 A}$
$= \sqrt{1-\frac{16}{25}} = \pm\frac{3}{5}$ (+ if A is acute and – if A is obtuse)

If $\cos B = \frac{12}{13}$, then $\sin B = \sqrt{1-\cos^2 B}$
$= \sqrt{1-\frac{144}{169}} = \frac{5}{13}$ (+ if B is acute).

(i) $\sin(A + B) = \sin A \cos B + \cos A \sin B$
$= \frac{4}{5} \times \frac{12}{13} + \frac{3}{5} \times \frac{5}{13} = \frac{63}{65}$.

$\cos(A + B) = \cos A \cos B - \sin A \sin B$
$= \frac{3}{5} \times \frac{12}{13} - \frac{4}{5} \times \frac{5}{13} = \frac{16}{65}$

(ii) $\sin(A + B) = \frac{4}{5} \times \frac{12}{13} + (-\frac{3}{5}) \times \frac{5}{15} = \frac{33}{65}$

$\cos(A + B) = (-\frac{3}{5}) \times \frac{12}{13} - \frac{4}{5} \times \frac{5}{15} = -\frac{56}{65}$.

If special angles are used, the answer can usually be expressed in surd form.

Example 2

Find the value of sin 15°, leaving the answer in surd form.

$\sin 15^o = \sin(45^o - 30^o) = \sin 45° \cos 30° - \cos 45° \sin 30^o$
$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}$
$= \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4}$.

The formulae can also be used to reduce certain trigonometrical expressions to a single ratio.

Example 3

Express as a single ratio: **(a)** $\cos 32^\circ \cos 48^\circ - \sin 32^\circ \sin 48^\circ$;

(b) $\dfrac{1}{\sin 46^0 \cos 44^0 + \cos 46^0 \sin 44^0}$

(a) $\cos(A + B) = \cos A \cos B - \sin A \sin B$

Put $A = 32^\circ$; $B = 48^\circ$

$\cos 32^\circ \cos 48^\circ - \sin 32^\circ \sin 48 = \cos(32^\circ + 48^\circ) = \cos 80^\circ$

(b) $\sin(A + B) = \sin A \cos B + \cos A \sin B$

Put $A = 46^\circ$; $B = 44^\circ$

$$\frac{1}{\sin 46^0 \cos 44^0 + \cos 46^0 \sin 44^0} = \frac{1}{\sin(46^0 + 44^0)}$$

$$= \frac{1}{\sin 90^0} = 1.$$

Example 4

Given that $\sin x \cos y = \frac{1}{3}$ and $\cos x \sin y = \frac{1}{6}$, find the values of $\sin(x + y)$ and $\sin(x - y)$. Hence find the values of x and y If $(x + y)$ is an acute angle.

$\sin(x + y) = \sin x \cos y + \cos x \sin y$

$= \frac{1}{3} + \frac{1}{6} = \frac{1}{2}.$

$\sin(x - y) \sin x \cos y - \cos x \sin y$

$= \frac{1}{3} - \frac{1}{6} = \frac{1}{6}.$

If $\sin(x + y) = \frac{1}{2}$ then $(x + y) = 30^\circ$

If $\sin(x - y) = \frac{1}{6} = 0.1667$, then $(x - y) = 9^\circ 36'$.

$x + y = 30^\circ$

$x - y = 9^\circ 36'$

$2x = 39^\circ 36'$

$\therefore x = 19^\circ 48'$

$y = 30^\circ - 19^\circ 48' = 10^\circ 12'.$

Exercise 11.1

1. If $\cos A = \frac{4}{5}$ and $\cos B = \frac{12}{13}$, both angles being acute, find the value of $\sin(A + B)$ without using tables.

2. Given $\cos A = 0.8$ and $\cos B = 0.6$, find, without using tables, the value of **(a)** $\sin(A - B)$; **(b)** $\cos(A - B)$; **(c)** the value of $(A + B)$ in degrees.

3. $\sin A = \frac{5}{13}$ and A is obtuse; $\cos B = \frac{4}{5}$ and B is acute. Find without using tables the value of **(a)** in $(A + B)$; **(b)** $\cos(A + B)$; (c) $\cos(A - B)$.

4. Express as single trigonometrical ratios:

(a) $\sin 36^\circ \cos 29^\circ + \cos 36 \sin 29^\circ$;

(b) $\cos 63^\circ \cos 27^\circ + \sin 63^\circ \sin 27^\circ$

(c) $\sin 120^\circ \cos 54^\circ - \cos 120^\circ \sin 54^\circ$;

(d) $\cos 23^\circ \cos 58^\circ - \sin 23^\circ \sin 58^\circ$.

5. Find the value of $\cos 75^\circ$ in surd form, using the ratios of special angles.

6. Find the value of $\cos 15^\circ$ in surd form.

7. Using the identity $\cos 90^\circ = \cos(45^\circ + 45^\circ)$, prove that $\cos 90^\circ = 0$.

(a) $\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta$; **(b)** $\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta$.

9. By using the formulae for compound angles, prove that:
(a) $\sin(180^\circ + \theta) = \sin\theta$; **(b)** $\cos(270^\circ + \theta) = \sin\theta$.

10. Final in surd form, the value of the following ratios:
(a) $\cos 10^\circ$; **(b)** $\sin 165^\circ$; **(c)** $\sin 345^\circ$.

11. Given that $\sin x \cos y = \frac{3}{4}$ and $\cos x \sin y = \frac{1}{4}$, find the values of $\sin(x + y)$ and $\sin(x - y)$. Hence, or otherwise, find the values of x and y if $(x + y) \leq 90^\circ$.

12. Given that $\cos x \cos y = \frac{2}{3}$ and $\sin x \sin y = \frac{1}{6}$, find the values of $\cos(x + y)$ and $\cos(x - y)$. Hence find the values of x and y if $(x + y) \leq 90^\circ$.

13. $\cos x \cos y = \frac{3}{4}$ and $\sin x \sin y = \frac{1}{4}$. Find he values of $\cos(x + y)$ and $\cos(x - y)$ and hence deduce the values of x and y if $(x + y) \leq 90^\circ$.

14. Solve for $0^\circ \leq x \leq 360^\circ$, the equations:
(a) $\sin 40^\circ \cos x + \cos 40^\circ \sin x = 1$;
(b) $2(\cos 50^\circ \cos x + \cos 40^\circ \sin x) = \sqrt{3}$.

Exercise 11.1(Answers)

1. $\frac{56}{65}$

2. **(a)** $\sin(A - B) = -0.28$ **(b)** $\cos(A - B) = 0.96$
(c) $A + B = 90^\circ$

3. **(a)** $\sin(A + B) = -\frac{16}{65}$ **(b)** $\cos(A + B) = -\frac{63}{65}$
(c) $\cos(A - B) = -\frac{33}{65}$

4. **(a)** $\sin 65^\circ$; **(b)** $\cos 36^\circ$; **(c)** $\sin 66^\circ$ **(d)** $\cos 81^\circ$

5. $\frac{\sqrt{2} - \sqrt{2}}{4}$ **6.** $\frac{\sqrt{6} + \sqrt{2}}{4}$

7.

8. **(a)** $\cos(60^\circ - \theta)$; **(b)** $\sin(45^\circ - \theta)$ or $(\theta + 45^\circ)$

10. **(a)** $\frac{\sqrt{2} - \sqrt{6}}{4}$; **(b)** $\frac{\sqrt{6} - \sqrt{2}}{4}$ **(c)** $-\frac{\sqrt{6} - \sqrt{2}}{4}$

11. $1, \frac{1}{2}, x = 60^\circ, y = 30^\circ$ **12.** $\frac{1}{2}, \frac{5}{6}, x = 46^\circ 47', y = 13^\circ 13'$;

13. $1, \frac{1}{2}, x = y = 30^\circ$ **14.** **(a)** 50°, **(b)** 20°, or 80°.

Tangents of compound angles

The tangents of compound angles can be deduced from the formulae for the sines and cosines of compound angles.

$$\tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)}$$

$$= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

$$= \frac{\frac{\sin A \cos B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B} - \frac{\sin A \sin B}{\cos A \cos B}}$$ (divide numerator and denominator by cosA cos B)

$$= \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Similarly

$$\tan (A - B) = \frac{\sin(A-B)}{\cos(A-B)}$$

$$= \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B + \sin A \sin B}$$ (divide numerator and denominator by cos A cos B)

$$= \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

These formulae are important and must be remembered; they are summarized below.

$$\mathbf{\tan (A + B)} = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\mathbf{\tan (A - B)} = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

The tangent formulae are used in the same way as those for sine and cosine.

Example 5

If $\tan A = \frac{1}{3}$ and $\tan B = \frac{3}{4}$, both A and B being acute angles, find the value of tan (A + B) without using tables.

$$\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$= \frac{\frac{1}{3} + \frac{3}{4}}{1 - \frac{1}{3} \times \frac{3}{4}} = \frac{\frac{4+9}{12}}{1 - \frac{1}{4}}$$

$$= \frac{\frac{13}{12}}{\frac{3}{4}} = \frac{13}{12} \times \frac{4}{3} = \frac{13}{9}.$$

Example 6

If $\tan (A - B) = \frac{1}{5}$, and tan A = 2, find the value of tan B.

$$\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} = \frac{1}{5}$$

$$= \frac{2 - \tan B}{1 + 2 \tan B} = \frac{1}{5}$$

i.e. $1 + 2 \tan B = 10 - 5 \tan B$

$\therefore$ $7 \tan B = 9$ i.e. $\tan B = \frac{9}{7}$.

Example 7

Express the following as a single trigonometrical ratio:

$$\frac{\sqrt{3} + \tan \theta}{1 - \sqrt{3} \tan \theta}.$$

$$\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Hence tan A = $\sqrt{3}$
i.e. A = 60^o (or 240°) and B = θ

$$\therefore \quad \frac{\sqrt{3}+\tan\theta}{1-\sqrt{3}\tan\theta} = \tan(60° + \theta) \text{ or } \tan(240° + \theta).$$

Exercise 11.2

1. Find the value of each of the following expressions in surd form:
(a) tan (45° + 30^o); **(b)** tan 15^o; **(c)** tan 105°;
(d) cot 165°.

2. If tan A = $\frac{1}{3}$ and tan B = $\frac{1}{2}$, both A and B being acute, find the value of tan (A – B) without using tables.

3. If tan A = $\frac{1}{2}$ and tan B = $-\frac{1}{4}$, A being acute and B being obtuse, find the value of tan (A + B) without using tables

4. Express the following as single ratios:

(i) $\frac{\tan\theta-\sqrt{3}}{r+\sqrt{3}\tan\theta}$; **(ii)** $\frac{1+\tan\theta}{1-\tan\theta}$.

5. Find without using tables, the value of:

(i) $\frac{\tan 20^0+\tan 25^0}{1-\tan 20^0\tan 25^0}$; **(ii)** $\frac{1+\tan 15^0}{1-\tan 15^0}$;

(iii) $\frac{\tan 75^0+\sqrt{3}}{1-\sqrt{3}\tan 75^0}$.

6. Find, without using tables, the value of tan B if tan A = $\frac{1}{4}$ and tan (A + B) = 2.

7. Find, without using tables, the value of cot A if cot B = $\frac{1}{2}$ and cot (A – B) = 4.

(Hint: tan (A – B) = $\frac{1}{\cot(A-B)}$).

8. If sin A = $\frac{3}{5}$ and cos B = $\frac{12}{13}$, where A and B are both obtuse, find, without using tables, the values of
(i) tan (A + B); **(ii)** tan (A – B).

Exercise 11.2(Answers)

1. **(a) 2 + √3** **(b) 2 - √3** **(c) –(2 + √3)**
(d) –(2 + √3)

2. $-\frac{1}{7}$ **3.** $\frac{2}{9}$

4. **(i) tan (θ – 60^o)** **(ii) tan (θ + 45^o)**

5. **(i) 1,** **(ii) √3** **(iii) -1** **6.** **tan B = $\frac{7}{6}$**

7. **cot A = $\frac{2}{9}$**

8. **tan (A + B) = $\frac{56}{33}$; tan (A – B) = $-\frac{16}{33}$**

Multiple angles

The formulae for compound angles can be used to find va1uesq$ multiple angles. In the first instance, angle B Is made equal to angle A.

Hence $\sin 2A = \sin(A + A) = \sin A\cos A + \cos A\sin A$
$= 2\sin A\cos A$

$\cos 2A = \cos(A + A) = \cos A\cos A - \sin A\sin A$
$= \cos^2 A - \sin^2 A.$

But $\cos^2 A + \sin^2 A = 1$

$\therefore \sin^2 A = 1 - \cos^2 A$

$\therefore \cos 2A = \cos^2 A - (1 - \cos^2 A)$
$= 2\cos^2 A - 1$

Also $\cos^2 A = 1 - \sin^2 A$

$\cos 2A = (1 - \sin^2 A) - \sin^2 A$
$= 1 - 2\sin^2 A$

$$\tan^2 A = \tan(A + A) = \frac{\tan A + \tan A}{1 - \tan A \tan A}$$
$$= \frac{2\tan A}{1 - \tan^2 A}$$

The formulae for multiple angles are important, and should be memorized; they are summarized next page.

$\sin 2A = 2\sin A\cos A$

$\cos 2A = \cos^2 A - \sin^2 A$
$= 2\cos^2 A - 1$
$= 1 - 2\sin^2 A$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Higher multiple angles can be found by building up on the results already found, e.g. sin 3A = sin (2A + A); the right band expression is expanded, using the compound angle formula, then sin 2A is substituted from the results above.
Similarly, cos 4A = cos (2A + 2A). In this way ratios of multiples of A can be found in terms of ratios of single angles.

Half Angles

Half angles can be substituted in the compound angle formulae:

$$\sin A = \sin\left(\frac{A}{2} + \frac{A}{2}\right) = 2\sin\frac{A}{2}\cos\frac{A}{2}.$$

Similarly, $\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$

$= 2\cos^2\frac{A}{2} - 1.$

$= 1 - 2\sin^2\frac{A}{2}$

And $\tan A = \dfrac{2\tan\frac{A}{2}}{1 - \tan^2\frac{A}{2}}$

A useful substitution in later trigonometrical work can be leaved by expressing the trigonometrical ratios In terms of the tangent of the half angle.

$\sin A = 2\sin\dfrac{A}{2}\cos\dfrac{A}{2}$ (As $\sin^2\frac{A}{2} + \cos^2\frac{A}{2} = 1$ the original expression is unaltered)

$$= \frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{\sin^2\frac{A}{2}\cos^2\frac{A}{2}}$$

$$= \frac{\frac{2\sin A/2\cos A/2}{\cos^2 A/2}}{\frac{\sin^2 A/2}{\cos^2 A/2} + \frac{\cos^2 A/2}{\cos^2 A/2}}$$ (dividing numerator and denominator by $\cos^2 \frac{A}{2}$)

$$= \frac{2\tan\frac{A}{2}}{\tan^2\frac{A}{2}+1}$$ now put $\tan\frac{A}{2} = t$.

We have $\sin A = \frac{2t}{1+t^2}$

$\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$

$= \frac{\cos^2\frac{A}{2} - \sin^2\frac{A}{2}}{\cos^2\frac{A}{2} + \sin^2\frac{A}{2}}$

$= \frac{1-t^2}{1+t^2}$ (after dividing numerator and denominator by $\cos^2\frac{A}{2}$).

$\tan A = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$

$= \frac{2t}{1-t^2}$.

To summarize the results:

$\sin A = \frac{2t}{1+t^2}$

$\cos A = \frac{1-t^2}{1+t^2}$ These are called the t formulae,

$\tan A = \frac{2t}{1-t^2}$ where $t = \tan\frac{A}{2}$

Example 8

Express sin 3A in terms of sin A.

$\sin 3A = \sin(2A + A)$
$= \sin 2A \cos A + \cos 2A \sin A$
$= (2\sin A \cos A)\cos A + (1 - 2\sin^2 A)\sin A$
$= 2\sin A\cos^2 A + (1 - 2\sin^2 A)\sin A$
$= 2\sin A(1 - \sin^2 A) + (1 - 2\sin^2 A)\sin A$
$= 2\sin A - 2\sin^3 A + \sin A - 2\sin^3 A$
$= 3\sin A - 4\sin^3 A$.

Example 9

Express simply $1 - 2\sin^2 42^o$.

Now $\cos 2A = 1 - 2\sin 2 A$; put $A = 42^o$.

$1 - 2\sin^2 42^o = \cos 84^o$.

Example 10

Evaluate, without using tables, $\frac{2\tan 30^0}{1-\tan^2 30^0}$.

Now $\quad \tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$; put $A = 30^0$.

$$\frac{2\tan 30^0}{1-\tan^2 30^0} = \tan 60^o = \sqrt{3}.$$

Example 11

If $\tan\theta = \frac{3}{4}$, find (i) $\sin 2\theta$; (ii) $\tan 2\theta$ without using tables.

Now $\quad \sin^2\theta + \cos^2\theta = 1$

i.e. $\quad \tan^2\theta + 1 = \sec^2\theta$ (dividing throughout by $\cos^2\theta$)

$$\cos^2\theta = \frac{1}{1+\tan^2\theta}$$

$$\cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}} = \frac{1}{\pm\sqrt{1+\frac{9}{16}}}$$

$$= \frac{\pm\sqrt{16}}{\sqrt{25}} = \pm\frac{4}{5}.$$

Note: If θ is in the 1st quadrant, $\cos\theta = \pm\dfrac{4}{5}$; if θ is in the 3rd quadrant, $\cos\theta = -\dfrac{4}{5}$

$$\sin\theta = \pm\sqrt{1-\cos^2\theta}$$

$$= \pm\sqrt{1-\frac{16}{25}}$$

$$= \pm\sqrt{\frac{9}{25}}$$

$$= \pm\frac{3}{5}.$$

(sin θ, similarly, can be $\pm\dfrac{3}{5}$ in the 1st quadrant, and $-\dfrac{3}{5}$ in the 3rd

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\times\tfrac{3}{5}\times\tfrac{4}{5}$$

$$= \tfrac{24}{25}.$$

(Only a positive result as, if θ is in the 1st quadrant, both $\sin\theta$ and $\cos\theta$ are positive; if θ is in the 3rd quadrant, both ratios are negative).

$$\sin 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\times\frac{3}{4}}{1-\frac{9}{16}} = \frac{\frac{6}{4}}{\frac{7}{16}}$$

$$= \tfrac{24}{7} \text{ (+ as } \tan\theta \text{ is given as } + \tfrac{4}{5}).$$

Example 12

θ is an obtuse angle and $\tan 2\theta = \frac{5}{12}$. Without using tables, find the value of

(a) $\tan\theta$; **(b)** $\cos 2\theta$ **(c)** $\cos 4\theta$.

$\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$ $\quad$ Let $\tan\theta = t$.

Then $\tan 2\theta = \dfrac{2t}{1-t^2} = \dfrac{5}{12}$

i.e. $\quad 24t = 5 - 5t^2$ which is $5t^2 + 24t - 5 = 0$

i.e. $\quad (5t - 1)(t + 5) = 0$ when $t = \frac{1}{5}$ or -5.

Since θ is obtuse, $\tan\theta = -5$.

If θ is obtuse, and $\tan 2\theta$ is positive, 2θ must lie in the 3rd quadrant.

$$\cos 2\theta = \frac{1}{-\sqrt{1+\tan^2 2\theta}} = \frac{1}{-\sqrt{1+\frac{25}{144}}}$$

$$= -\sqrt{\frac{144}{169}} = -\frac{12}{13}$$

$$\cos 4\theta = 2\cos^2 2\theta - 1$$

$$= 2 \times \left(\frac{-12}{13}\right)^2 - 1$$

$$= 2 \times \frac{144}{169} - 1 = \frac{119}{169}.$$

Exercise 11.3

1. Express more simply as a single ratio:

(a) $2\sin 40^o \cos 40^o$; **(b)** $2\cos^2 32^o - 1$;

(c) $1 - 2\sin^2 65^o$; **(d)** $\frac{2\tan 55^0}{1-\tan^2 30^0}$;

(e) $2\cos^2 2\theta - 1$.

2. Evaluate, without using trigonometrical tables:

(a) $\sin 75^o \cos 75^o$; **(b)** $\frac{2\tan 30^0}{1-\tan^2 30^0}$;

(c) $1 - 2\sin^2 15°$; **(d)** $\cos^2 30° - \sin^2 30°$;

(e) $2\cos^2 30° - 1$.

3. Find, without using tables, the value of $\sin 2\theta$ when

(a) $\sin\theta = \frac{12}{13}$; **(b)** $\sin\theta = \frac{\sqrt{3}}{2}$.

4. If $\cos 2\theta = \frac{119}{169}$, find the value of (a) $\sin\theta$; (b) $\cos\theta$ without using tables.

5. If $\tan 2\theta = \frac{120}{119}$, find, without using tables, the value of (a) $\tan\theta$; (b) $\cos\theta$; (c) $\sin\theta$;

6. If $\tan\theta = \frac{1}{2}$, find the va1ues of (a) $\tan 2\theta$; (b) $\tan 4\theta$; (c) $\tan(4\theta + 45^o)$.

7. Given $\tan 2A = -\frac{8}{15}$, find, without using tables, the following ratios, given that angle A is acute: (a) tan A; (b) sin A; (c) cos2A.

8. If tan A and tan B are the roots of the equation $x^2 + bx + c = 0$, find in terms of b and c the value of (a) tan(A + B); (b) cos (A + B).

9. θ is an acute angle and $\cos 2\theta = -\frac{3}{5}$. Without using trigonometrical tables, find (a) $\tan\theta$; (b) $\tan 4\theta$.

10. Express cos 3A in terms of cos A.

Exercise 11.3(Answers)

1.	**(a) $\sin 80^o$**	**(b) $\cos 64^o$**	**(c) $\cos 130^o$**	**(d) $\tan 110^o$**	**(e) $\cos 4\theta$**
2.	**(a) $\frac{1}{4}$**	**(b) $\sqrt{3}$**	**(c) $\sqrt{3}/2$**	**(d) $\frac{1}{2}$**	**(e) $\frac{1}{2}$**
3.	**(a) $\pm\frac{120}{169}$**	**(b) $\pm\sqrt{3}/2$**			
4.	**(a) $\sin\theta = \frac{5}{13}$**		**(b) $\cos\theta = \frac{12}{13}$**		
5.	**(a) $\tan\theta = \frac{5}{12}$**	**(b) $\cos\theta = \frac{12}{13}$**	**(c) $\sin\theta = \frac{5}{13}$**		
6.	**(a) $\frac{4}{3}$**	**(b) $-\frac{24}{7}$**	**(c) $-\frac{17}{31}$**		
7.	**(a) tan A = 4**	**(b) $\sin A = 4/\sqrt{17}$**	**(c) $\cos 2A = \frac{15}{17}$**		

8. **(a)** $\frac{-b}{1-c}$ **(b)** $\frac{\pm(1-c)}{\sqrt{1-2c+c^2-b^2}}$

9. **(a) tan θ = 2** **(b) tan 4θ =** $\frac{24}{7}$

10. **cos 3A = 4 cos^3 A – 3 cos A.**

The factor formulae

Using the compound angle formulae, the sum of two sines can be written as:

sin (A + B) + sin (A – B)

= sin A cosB + cos A sin B + sin A cos B – cosA sin B

= 2 sin A cosB.

Now let A + B = X and A – B = Y

then X + Y = 2A ∴ A = $\frac{X+Y}{2}$

and X – Y = 2B ∴ B = $\frac{X-Y}{2}$

Substituting the values of X and Y for A and B, we have:

$$\sin X + \sin Y = 2\sin\frac{X+Y}{2}\cos\frac{X-Y}{2}.$$

The formulae for the sum of two cosines can be written as:

cos (A + B) + cos(A – B)

= cosAcos B – sin A sin B + cosAcos B + sin A sin B

= 2 cos A cos B.

Substituting the values of x and y for A and B, we have:

$$\cos X + \cos Y = 2\cos\frac{X+Y}{2}\cos\frac{X-Y}{2}.$$

Example 13

Express in factor: sin 5 θ + sin θ.

The factors are obtained from the formula for the sum of two sines.

$$\sin X + \sin Y = 2\sin\frac{X+Y}{2}\cos\frac{X-Y}{2} \quad (\text{here } X = 5\theta \text{ and } Y = \theta)$$

$$\therefore \quad \sin 5\theta \sin\theta = 2\sin\frac{5\theta+\theta}{2}\cos\frac{5\theta-\theta}{2}$$

$$= 2\sin 3\theta\cos 2\theta.$$

The difference of two sines can be written as:

sin (A + B) – sin (A – B)

= sin A cos B + cos A sin B - (sinAcos B – cosA sinB)

= 2 cos A sin B.

Substituting the values of X and Y for A and B, we have:

$$\cos X - \cos Y = -2\sin\frac{X+Y}{2}\cos\frac{X-Y}{2}.$$

Note: Remember the negative sign in this formula.

Example 14

Express in factors: cos 5 θ – cot 3 θ.

$$\cos X - \cos Y = -2\sin\frac{X+Y}{2}\cos\frac{X-Y}{2}$$

$\cos 5\theta - \cos 3\theta \quad = \quad -2\sin\frac{5\theta+3\theta}{2}\cos\frac{5\theta-3\theta}{2}$

$= \quad 2\sin 4\theta \sin\theta.$

The factor formulae are important and should be summarized below:

$$\sin X + \sin Y = 2\sin\frac{X+Y}{2}\cos\frac{X-Y}{2}$$

$$\sin X - \sin Y = 2\cos\frac{X+Y}{2}\sin\frac{X-Y}{2}$$

$$\cos X + \cos Y = 2\cos\frac{X+Y}{2}\cos\frac{X-Y}{2}$$

$$\cos X - \cos Y = -2\sin\frac{X+Y}{2}\sin\frac{X-Y}{2}.$$

These formulae can also be remembered verbally, as under:

sine + sine = 2 sin ($\frac{1}{2}$ sum) cos ($\frac{1}{2}$ difference)

sine – sine = 2 cos (") sin (")

cos + cos = 2 cos (") cos (")

cos – cos = - 2 sin (") sin (").

Example 15

Express $2\sin 2\theta\cos 5\theta$ as a difference between two trigonometrical ratios.

$$\sin X - \sin Y = 2\cos\frac{X+Y}{2}\sin\frac{X-Y}{2}.$$

i.e. $\frac{X+Y}{2} = 5\theta \quad \therefore \quad X + Y = 10\theta$ .... (1)

and $\frac{X-Y}{2} = 2\theta \quad \therefore \quad X - Y = 4\theta$ .... (2)

Adding (1) and (2), $2X = 14\theta \quad \therefore \quad X = 7\theta.$

Subtracting (2) from (1), $2Y = 6\theta \quad \therefore \quad Y = 3\theta$

$\therefore \quad 2\sin 2\theta\cos 5\theta = 2\cos 5\theta\sin 2\theta$

$= \sin 7\theta - \sin 3\theta.$

Equations with multiple angles

The original equation is converted to an equation in the unit angle, using trigonometrical identities.

Example 16

Solve the equation $5\cos 2x + 9\sin x = 7$ for values of x between 0^o and $360°$.

$5\cos 2x + 9\sin x - 7 = 0$ Change to an equation in sin x.

i.e. $5(1 - 2\sin^2 x) + 9\sin x - 7 = 0,$

i.e. $10\sin^2 x - 9\sin x + 2 = 0,$

factorizing $(2\sin x - 1\ \ (5\sin x - 2) = 0,$

i.e. $\sin x = \frac{1}{2} \quad \therefore \quad x = 30^o$ or 150^o

or $\sin x = 0.4 \quad \therefore \quad x = 23^o\ 35'$ or $156^o\ 25'$

Example 17

Solve the equation $4\tan 2x + \tan x = 0$ for values of x between 0^o and 360^o inclusive.

Substitution $\frac{4\times 2\tan x}{1-\tan^2 x} + \tan x = 0,$

i.e. $8\tan x + \tan x - \tan 3x = 0,$

or $\tan^3 x - 9\tan x = 0$*

i.e. $\tan x(\tan^2 x - 9) = 0,$

either $\tan x = 0$ i.e. $x = 0^o, 180^o, 360^o$

or $\tan^2 x = 9,$

i.e. $\tan x = \pm 3$ so $x = 71^\circ 34'; 108^\circ 26'; 251^\circ 34'; 288^\circ 26'$.
Solutions are $0^\circ, 71^\circ 34', 108^\circ 26', 180^\circ, 251^\circ 34', 288^\circ 26', 360^\circ$.

Do not divide throughout by tan x otherwise the solutions of tan x = 0 will be lost. Never divide by a factor which may give solutions.

Equations of the type $a\cos\theta + b\sin\theta = c$

There are various methods of solving this common type of equation. The following examples show two methods, of which the first is more useful.

Example 18

Solve the equation $3\cos x + 4\sin x = 2$, for values of x between 0° and 360°.
The L.H.S. is converted to the form $R\cos(x - ø)$ where ø (the auxiliary angle) is acute.
Then $3\cos x + 4\sin x$
$= R(\cos x\cos ø + \sin x \sin ø)$
$= R(\cos x\cos ø + R\sin x\sin ø)$.
As this is an identity, then $R\cos ø = 3$
$R\sin ø = 4$.

Squaring and adding:

$R^2(\cos^2 ø + \sin^2 ø) = R^2(9 + 16) = 25 \quad \therefore \quad R = 5$
(R is always positive).

Dividing $\dfrac{R\sin\phi}{R\cos\phi} = \tan ø = \dfrac{4}{3} \quad \therefore \quad ø = 53^\circ 7'$

(smallest value taken for the auxiliary angle).
Hence $3\cos x + 4\sin x = 5\cos(x - ø) = 2$

i.e. $\cos(x - 53^\circ 7') = \dfrac{2}{5} = 0.4$.

Hence $x - 53^\circ 7' = 66^\circ 25'$ or $293^\circ 35'$.
The solutions are $x = 119^\circ 32'$ and $346^\circ 42'$.

Example 19

Find the values of θ, for $0 \le \theta \le 2\pi$, which satisfy the equation $\sqrt{3}\cos\theta - \sin\theta = 1$.
We convert the L.H.S., $\sqrt{3}\cos\theta - \sin\theta$ to the form $R\cos(\theta + ø)$.
We choose $\cos(\theta + ø)$ as there is a negative sign between cos θ and sin θ.
As In Example 18, R > 0 and ø (the auxiliary angle) is acute.
Now $R\cos(\theta + ø) = (R\cos ø)\cos\theta — (R\sin ø)\sin\theta$.
Compare this with $\sqrt{3}\cos\theta - 1\sin\theta$.
Then $R\cos ø = \sqrt{3}$ and $R\sin ø = 1$.
Squaring and adding: $R^2\cos^2 ø + R^2\sin^2 ø = R^2 = (\sqrt{3})^2 + (1)^2 = 4$ and $R = 2$.

Dividing $\dfrac{R\sin\phi}{R\cos\phi} = \tan ø = \dfrac{1}{\sqrt{3}}$ so $ø = \dfrac{\pi}{6}$.

Hence $\sqrt{3}\cos\theta - \sin\theta = 2\cos(\theta + \dfrac{\pi}{6}) = 1$.

$\therefore \quad \cos(\theta + \dfrac{\pi}{6}) = \dfrac{1}{2}$ which gives $\theta + \dfrac{\pi}{6} = \dfrac{\pi}{3}$ or $\dfrac{5\pi}{3}$ and $\theta = \dfrac{\pi}{6}$ or $\dfrac{3\pi}{2}$.

Example 20

An alternative method of solving the equation given in Example 18 uses the t formulae as shown below.

Take $t = \tan\dfrac{x}{2}$ and substitute for sin x and cos x.

$3 \cos x + 4 \sin x \quad = \quad 2,$

so $\quad \dfrac{3(1-t^2)}{1+t^2} + \dfrac{4\times 2t}{1+t^2} \quad = \quad 2$

which reduces to the quadratic equation

$5t^2 - 8t - 1 \quad = 0$; this is solved by the formula.

$\therefore \quad t = \dfrac{8 \pm \sqrt{84}}{10} \quad = \quad 1.717$ or -0.1165.

Hence $\quad \tan\dfrac{x}{2} \quad = \quad 1.717$ or $\tan\dfrac{x}{2} \quad = \quad -0.1165,$

i.e. $\quad \dfrac{x}{2} \quad = \quad 59^o\ 47'$ or $\tan\dfrac{x}{2} \quad = \quad 173^o\ 24'$

so $\quad x \quad = \quad 119^o\ 34'$ and $346^o\ 42'$

which are the solutions found in Example 18. (There is a slight difference due to limitations of the tables). This is a satisfactory method to use for solving an equation of the type a cos θ + b sin θ = c, but maximum and minimum values of a cos θ + b sin θ cannot be found by this method.

Example 21

Find the maximum and minimum values of 3 cos x + 4 sin x at and the values of x where they occur.

From Example 18

$3\cos x + 4\sin x = 5\cos (x - 53^o\ 7')$

Now the maximum value of $5\cos (x - 53^o\ 7')$ 5 x 1.

When $(x - 53°\ 7') = 0°$ i.e. $x = 53°\ 7'$.

The minimum value of $5 \cos (x - 53°\ 7')$

$= \quad 5 \times (-1) \quad = \quad -5$ when $(x - 53°\ 7') = 180^o$

i.e. $\quad x \quad = \quad 233^o\ 7'$.

Exercise 11.4

1. Express the following in factors:

(a) sin5θ + sin3θ; **(b)** cos5θ + cos3θ;
(c) sin2θ – sinθ; **(d)** cos5θ – cos3θ;
(e) sin3θ – sin7 θ; **(f)** cos2θ – cos8θ;

2. Express the following as the sum or difference of two ratios:

(a) 2 sin3θ cosθ; **(b)** 2 cos3A sinA;
(c) 2 cos5x cos3x; **(d)** -2 sin5θ cos3θ;
(e) -2 cos5AsinA; **(f)** 2 sin7θ sinθ.

3. Express in factors:

(a) cos2θ + cos60°; **(b)** sin4θ – sin120°;
(c) cos2θ – cos60°.

4. Express the following as the sum or difference of two ratios:

(a) 2 sin(2 θ + 3 θ°) cos (20 – 30°)
(b) 2 cos(θ + 40°)sin(θ – 40°)
(c) -2 sin(A + 62°)sin(A – 62°)

5. Express as factors:

(a) sin (θ + 20°) + sin θ; **(b)** cos(θ + 60°) — cosθ;
(c) cos 2θ + cos(90° - 3 θ); **(d)** cos(2 θ + 10°) + sin(100° – 2θ).
(Hint: cos θ = sin (90° – θ).

6. Solve the following equations for $0 \le x \le 360^o$.

(a) sin2x = tan x **(b)** sin2x = cosx
(c) cos2θ – cosθ = 0 **(d)** 3cos2x – 5 cosx + 4 = 0
(e) 4 cos2x + 2 sinx – 5 = 0 **(f)** 3tan 2x + 4 tan x = 0
(g) 12 tan2θ tanθ = 1 **(h)** tan2ytany + 3 = 0.

7. Express in the form pf R cos (θ ± ø) stating the values of R and ø:

(a) $2\cos\theta + 3\sin\theta$; **(b)** $4\cos\theta - 3\sin\theta$;
(c) $5\cos\theta - 12\sin\theta$; **(d)** $2\sqrt{2}\sin\theta + \cos\theta$.

8. State the maximum and minimum values of the functions in 7(a) - (d) giving the values of θ ($0^o < \theta < 360^o$) where they occur.

9. Use the results of 7(a) - (d) to solve the following equation for $0o < \theta < 360^o$:

(a) $2\cos\theta + 3\sin\theta = \frac{\sqrt{13}}{2}$; (b) $4\cos\theta - 3\sin\theta = 1$;

(c) $5\cos\theta - 13\sin\theta = -6.5$; (d) $2\sqrt{2}\sin\theta + \cos\theta = 3$.

10. Find the values of x, ($0 \le x \le 2\pi$) which satisfy the following equations:

(a) $\sin x + \cos x = 1$. (b) $\sqrt{3}\cos x - \sin x = \sqrt{3}$.

11. Using the t formulae, solve the following equations for values of x between 0° and 360°:

(a) $4\cos\theta - 3\sin\theta = l$; (b) $5\cos\theta + 12\sin\theta = 13$.

12. (a) Find the maximum and minimum values of

$\cos\theta - 2\sqrt{2}\sin\theta$

starting the values of θ ($0 < \theta < 360^o$) where they occur.

(b) Solve the equation $\cos\theta - 2\sqrt{2}\sin\theta = 2$ for $0 < \theta < 360°$.

13. Using the substitution $t = \tan\theta$, solve the equation $\cos 2\theta + \sin 2\theta = 1$ for $0 \le \theta \le 2\pi$

Exercise 11.4(Answers)

1. **(a) $2\sin 4\theta\cos\theta$; (b) $2\cos 4\theta\cos\theta$ (c) $2\cos 3\theta / 2\sin\theta/2$**
(d) $-2\sin 4\theta\sin\theta$; (e) $-2\cos 5\theta\sin 2\theta$ (f) $2\sin 5\theta\sin 3\theta$

2. **(a) $\sin 4\theta + \sin 2\theta$; (b) $\sin 4A = \sin 2A$**
(c) $\cos 8x + \cos 2x$; (d) $-(\sin 8\theta + \sin 2\theta)$
(e) $\sin 4A - \sin 6A$; (f) $\cos 6\theta - \cos 8\theta$

3. **(a) $2\cos(\theta + 30°)\cos(\theta - 30°)$;**
(b) $2\cos(2\theta \div 60°)\sin(2\theta - 60°)$;
(c) $-2\sin(\theta + 30°)\sin(\theta - 30°)$

4. **(a) $\sin 4\theta + \sin 60°$; (b) $\sin 2\theta - \sin 80°$;**
(c) $\cos 2A - \cos 124°$

5. **(a) $2\sin(\theta + 10°)\cos 10°$; (b) $2\sin(45° - \frac{\theta}{2})\sin(\frac{5\theta}{2} - 45°)$;**
(c) $-2\sin(\theta + 30°)\sin 30°$; (d) $2\cos 2\theta\cos 10^o$

6. **(a) 0^o ; 45° ; 135° ; 180° ; 225° ; 315° ; 360°**
(b) 30°; 90°; 150°; 270°
(c) 0°; 120°; 240°; 360°
(d) 60°; 70° 32'; 289° 28'; 300°
(e) 30°; 150°; 194° 29'; 345° 31'
(f) 0°; 57° 41′, 122° 19′; 180° ; 237°41′; 302° 19′; 360°
(g) 11° 19′; 168' 41′; 191° 19′; 348°41′
(h) 60°; 120°; 240°; 300°

7. **(a) $\sqrt{13}\cos(\theta - 56° 18')$ (b) $5\cos(\theta + 36° 52')$**
(c) $13\cos(\theta + 67° 22')$ (d) $3\cos(\theta - 70° 32')$

8.

	Maximum	Minimum
	(a) $\sqrt{13}$; 56^o 18′	**$-\sqrt{13}$; 236^o 18′**
	(b) 5; 323^o 8′	**-5; 143^o 8′**
	(c) 13; 292^o 38′	**-13; 112^o 38′**

	(d) 3; 70° 32′	**-3; 250° 32′**
9.	**(a) 11° 18′; 356° 18′**	**(b) 41° 36′; 244° 40′**
	(c) 52° 38′; 172° 38′	**(d) 70° 32′**
10.	**(a) 0;** $\frac{\pi}{2}$**; 2π**	**(b) 0;** $\frac{5\pi}{3}$**; 2π**
11.	**(a) 41° 36′; 244° 40′**	**(b) 67° 23′**
12.	**(a) Max 3, 289° 28′, min -3, 109° 23′**	
	(b) 241° 17′; 337° 39′	
13.	**0;** $\frac{\pi}{4}$**; π;** $\frac{5\pi}{4}$**; 2π**	

Simple identities

A trigonometrical identity is an expression that is valid for all values of the angles contained in the expression,
e.g. $\sin^2\theta + \cos^2\theta = 1$ is true for all values of θ.
The three basic identities, already found, are:

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ 1 + \tan^2\theta &= \sec^2\theta \\ 1 + \cot^2\theta &= \text{cosec}^2\theta \end{aligned}$$

These three basic identities, together with the identities for compound and multiple angles and the identities of the factor formulae, can be used to manipulate trigonometrical expressions into different forms.

Exercises on trigonometrical identities give two e4ressions which have to be proved equal; the proof depends on substituting the trigonometrical formulae (themselves identities) given so far. There is no general method of attacking such problems, but the following suggestions may be useful.

1. Multiplication, or division, by $(\sin^2\theta + \cos^2\theta)$ can always be carried out, as the expression is equal to unity, and hence does not alter the value of the original expression. Any of the alternative identities from Pythagoras' theorem can also be used.

2. Always look for $(\sin^2\theta + \cos^2\theta)$, or its alternative forms, when simplifying an expression, and replace it by unity.

3. Expressions involving tan 0 and cot 0 are often simplified by replacing tan θ by $\frac{\sin\theta}{\cos\theta}$ and cot θ by $\frac{\cos\theta}{\sin\theta}$.

4. Do not be worried by the size of a trigonometrical expression; the larger expressions usually are simplified more easily.

5. You can either prove that the left hand side (LHS) of an expression is equal to the right hand side (RHS) or RHS = LHS. Start with the more complicated expression and simplify it. Sometimes it is easier to rearrange the identity first and then prove the rearrangement (see Example 23 below).

6. Try to get the simplest or most elegant proof. The following examples will illustrate the methods.

Example 22
Prove that $\cot^2\theta + \tan\theta = \text{cosec}\theta\sec\theta$.

$$\text{LHS} = \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} = \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{1}{\sin\theta\cos\theta} \qquad (\sin^2\theta + \cos^2\theta = 1)$$

$$= \text{cosec}\theta\ \sec\theta \qquad (\text{cosec}\theta = \frac{1}{\sin\theta}, \sec\theta = \frac{1}{\cos\theta}).$$

Example 23

Prove that $\tan^2\theta + \sin^2\theta = \sec^2\theta - \cos^2\theta$.

LHS $\sec^2\theta - 1 + 1 - \cos^2\theta = \sec^2\theta - \cos^2\theta$.

Alternatively, if the original identity is rearranged as

$$\cos^2 + \sin^2\theta = \sec^2\theta - \tan^2\theta.$$

This is seen to be true, as both sides equal 1.

Hence the original identity must be true.

Identities involving compound angles

Example 24

Prove that $\cot(A - B) = \frac{1 + \cot A \cot B}{\cot b - \cot A}$.

$$\text{LHS} = \frac{\cos(A - B)}{\sin(A - B} = \frac{\cos A\cos B + \sin A\sin B}{\sin A\cos B - \cos A\sin B}$$

As the RHS begins with the figure 1, dividing by sin A sin B, numerator and denominator, should produce an expression of the correct form.

$$\text{LHS} = \frac{\frac{\cos A\cos B}{\sin A\sin B} + \frac{\sin A\sin B}{\sin A\sin B}}{\frac{\sin A\cos B}{\sin A\sin B} - \frac{\cos A\sin B}{\sin A\sin B}}$$

$$= \frac{\cot A\cot B + 1}{\cot B - \cot A}$$

$$= \text{RHS}.$$

Example 25

Prove that 2 sin A cos A cos B

$$= \sin B + \sin A\cos(A + B) + \cos A\sin(A - B).$$

Start with the RNS, which is more complicated.

RHS $=$ sinB + sinA (cosA cosB – sinAsinB)

+ cosA (sinA cosB – cosA sinB)

$=$ sinB + sinA cosA cosB – $\sin^2$Asin B

+ cos A sin A cos B – $\cos^2$A sin B

$=$ sinB (1 - $\sin^2$ A – $\cos^2$ A) + 2 sin A cos A cos B

but $\sin^2 A + \cos^2 A = 1$ $\quad\therefore\quad$ $(1 - \sin^2 A - \cos^2 A) = 0$.

$\therefore$ RHS $=$ 2 sinAcosAcosB = LHS.

Identities involving multiple angles

Example 26

Prove that $\tan 2\theta\ \text{cosec}\ \theta = \frac{1}{\cos\theta + \sin\theta} + \frac{1}{\cos\theta + \sin\theta}$.

1. Simplify RHS, which is more complicated.
2. Look for sin 2 θ and cos 2 θ to give tan 2 θ.

$$\text{RHS} = \frac{(\cos\theta - \sin\theta) + (\cos\theta + \sin\theta)}{(\cos\theta + \sin\theta)(\cos\theta - \sin\theta}$$

$= \dfrac{2\cos\theta}{\cos^2\theta - \sin^2\theta}$

$= \dfrac{2\cos\theta}{\cos 2\theta}$ (2 cos θ has to be changed to sin 2 θ)

$= \dfrac{2\cos\theta}{\cos 2\theta} \times \dfrac{\sin\theta}{\sin\theta}$

$= \dfrac{2\sin\theta\cos\theta}{\cos 2\theta \sin\theta}$

$= \dfrac{\sin 2\theta}{\cos 2\theta \sin\theta}$

$=$ tan 2 θ cosec θ = LHS.

Example 27

Prove that 2cos θ sin3 θ = sin 2θ (2 cos2 θ + 1)

RHS = 2 sin 2 θ cos 2 θ + sin 2 θ.

= sin 4 θ + sin 2 θ

= 2 sin 3θ cos θ (using formula for the sum o f two sines).

Identities involving the factor formulae

Example 28

Prove that $\dfrac{\sin\theta - \sin 2\theta + \sin 3\theta}{\cos\theta - \cos 2\theta + \cos 3\theta}$ = tan 2 θ

1. Simplify LHS which is more complicated.
2. There are no half angles in RHS, so the ratios must be paired to avoid half angles.

LHS $= \dfrac{(\sin 3\theta + \sin\theta) - \sin 2\theta}{(\cos 3\theta + \cos\theta) - \cos 2\theta}$

$= \dfrac{\sin 2\theta(2\cos\theta - 1)}{(\cos 2\theta(2\cos\theta - 1)}$

$=$ tan 2 θ = RHS.

Example 29

Prove $\dfrac{\cos 2(X+Y) + \cos 2X + \cos 2Y + 1}{\cos 2(X+Y) - \cos 2X - \cos 2Y + 1}$ = - cot X cot Y.

1. Simplify LHS, which is more complicated.
2. Simplify double compound angles and double angles.
3. Expand compound angles.

LHS $= \dfrac{[2\cos^2(X+Y) - 1] + 2\cos(X+Y)\cos(X-Y) + 1}{[2\cos^2(X+Y) - 1] - 2\cos(X+Y) - \cos(X - Y + 1}$

$= \dfrac{2\cos(X+Y)[\cos(X+Y) + \cos(X-Y)]}{2\cos(X+Y)[\cos(X+Y) - \cos(X-Y)]}$

$= \dfrac{\cos(X+Y) + \cos(Y-X)}{\cos(X+Y) - \cos(X-Y)}$ $\left(\dfrac{\text{sum of two cosines}}{\text{difference of two cosines}}\right)$

$= \dfrac{2\cos X\cos Y}{-2\sin X\sin Y}$

$=$ - cotX cotY = RHS.

Exercise 11.5

Prove the following identities:

1. $\sin^4\theta - \cos^4\theta = \sin^2\theta - \cos^2\theta$.

2. (cosec A + cot A) (cosec A – cot A) = 1.

3. $\dfrac{\operatorname{cosec}\theta + \sec\theta}{\cot\theta + \tan\theta} \quad = \quad \dfrac{\cot\theta - \tan\theta}{\operatorname{cosec}\theta - \sec\theta}$.

4. $\dfrac{1-\tan^2 x}{1+\tan^2 x} \quad = \quad 2\cos^2 x - 1$.

5. $(\tan\theta + \sec\theta)^2 \quad = \quad \dfrac{1+\sin\theta}{1-\sin\theta}$.

6. If $m = \dfrac{1-\cos A}{\sin A}$, then $\dfrac{1}{m} \quad = \quad \dfrac{1+\cos A}{\sin A}$

7. $\tan A - \tan B = \dfrac{\sin(A-B)}{\cos A\cos B}$.

8. $\sin A \sin(A-B) = \cos B - \cos A\cos(A-B)$.

9. $\cos(A+B) = \dfrac{\operatorname{cosec} A \operatorname{cosec} B - \sec A \sec B}{\sec A \sec B \operatorname{cosec} A \operatorname{cosec} B}$.

10. $\sin 2\theta(\tan\theta + \cot\theta) = 2$.

11. $\dfrac{\sin(X+Y)+\sin(X-Y)}{\cos(X-Y)-\cos(X+Y)} \quad = \quad \cot Y$.

12. $\dfrac{2\sin\theta}{\cos 3\theta} = \tan 3\theta - \tan\theta$.

13. $\tan 2A = \dfrac{2}{\cot A - \tan A}$

14. $\cos 3\theta + \sin 3\theta = (1 + 2\sin 2\theta)(\cos\theta - \sin\theta)$.

15. $\dfrac{\sin 3\theta + \sin\theta}{\cot 3\theta + \cos\theta} = \tan 2\theta$.

16. $\cos B + \sin 2B - \cos 3B = \sin 2B(2\sin B + 1)$.

17. $\dfrac{\sin 2(A+B) - \sin 2A - \sin 2B}{\sin 2(A+B) + \sin 2A + \sin 2B} = -\tan A\tan B$.

18. $\dfrac{\cos 2(C+D) + \cos 2C - \cos 2D - 1}{\sin 2(C+D) - \sin 2C - \sin 2D} = \cot D$.

12

The Solution of Triangles

A triangle possesses six elements – the three sides and the three angles. If any three elements (other than three angles) are given, the remaining three elements can be found. This is called solving the triangle. If three angles are given, an infinite number of similar triangles can be formed, so there is no definite solution of the triangle.

In solving triangles, two geometrical facts are useful. They are:

1. In any triangle, the sum of the angles is 180^0.
2. In any triangle, the longest side is opposite the greatest angle, and the shortest side is opposite the smallest angle.

Relationships between the sides and the angles must now be established.

The sine rule

In **fig 12.la**, a triangle ABC has a circumcircle around it. Consider angle A. O is the centre of the circumcircle, and BP is a diameter of the circle. PC is joined. Angle BCP is a right angle as it is the angle in a semicircle. Let the radius of the circumcircle be R.

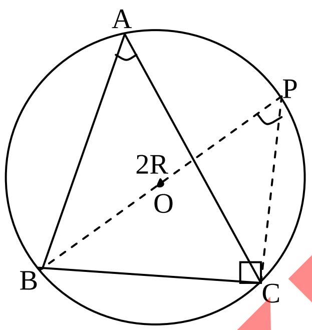


Fig 12.1a

Now $\hat{BAC} = \hat{BPC}$ (angles in same segment) $\hat{BAC}$

$\therefore \quad \text{Sin } A = \sin \hat{BPC}$

but $\sin \hat{BPC} = \dfrac{BC}{BP}$ (from triangle BPC, right angled at C)

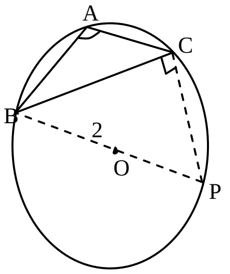
$= \dfrac{a}{2R}$ where a is the side opposite angle A

$\therefore \quad \sin A = \dfrac{a}{2R}$

i.e. $2R = \dfrac{a}{\sin A}$.

If A is an obtuse angle

Fig12.lb shows BAC as obtuse. The construction is identical with that for an acute angle in **fig 12.la**.



Fig

Now $\hat{BAC} + \hat{BPC} = 180^0$ (opp. angles of cyclic quad.)

$$\hat{BAC} = 180^0 - \hat{BPC}$$

$$\sin \hat{BAC} = \sin(180^0 - \hat{BPC})$$

$$= \sin \hat{BPC}$$

$$\sin \hat{BPC} = \frac{BC}{BP} \text{ (from triangle BPC, right-angled at C)}$$

$$= \frac{a}{2R}$$

$$\therefore \quad \sin A = \frac{a}{2R} \quad \text{i.e. } 2R = \frac{a}{\sin A}.$$

Similarly, by considering $\hat{B}$ or $\hat{C}$, it can be proved that

$$\frac{a}{\sin B} = 2R \text{ and } \frac{a}{\sin C} = 2R,$$

where b = AC, the side opposite angle B, and c = AB, the side opposite angle C.

Hence $\dfrac{a}{\sin A} = \dfrac{a}{\sin B} = \dfrac{a}{\sin C} = 2R.$

This is the sine rule and should be memorized.

Note: this rule can also be written in the form: $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$.

The cosine rule

Fig 12.2 shows a triangle ABC. Consider angle A which is acute in **fig 7 .2a** and obtuse in **fig 12.2b.** An altitude, CN, is drawn, of length h.

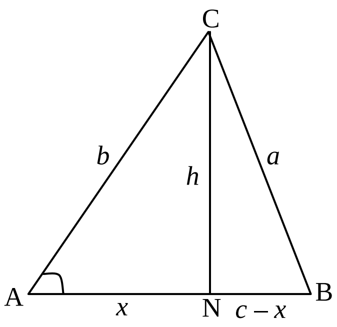


Fig 12.2a

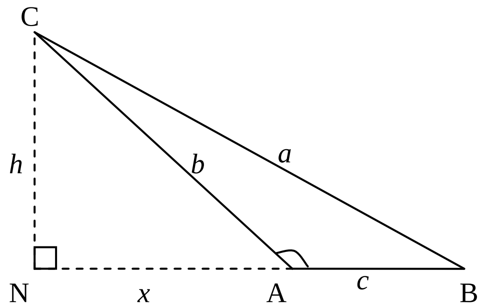


Fig 12.2b

When A acute

$$a^2 = h^2 + (c - x)^2 \quad \text{(from triangle CNB)}$$

$$= h^2 + c^2 - 2cx + x^2$$

$$b^2 = h^2 + x^2 \quad \text{(from triangle CNA)}$$

$$\therefore \quad a^2 = b^2 + c^2 - 2cx.$$

Also $\cos A = \dfrac{x}{b}$

$\therefore \quad x = b \cos A$

$\therefore \quad a^2 = b^2 + c^2 - 2bc \cos A.$

When A is obtuse

$a^2 = h^2 + (c + x)^2$ (from triangle CBN)

$= h^2 + c^2 + 2cx + x^2$

$b^2 = h^2 + x^2$ (from triangle CAN)

$\therefore \quad a^2 = b^2 + c^2 + 2cx$

Also $\cos C\hat{A}N = \dfrac{x}{b}$

$\therefore \quad x = b \cos C\hat{A}N = b \cos (180^0 - A) = -b \cos A$

$\therefore \quad a^2 = b^2 + c^2 - 2bc \cos A.$

Hence in their triangle $a^2 = b^2 + c^2 - 2bc \cos A.$

This is the cosine rule for angle A. By taking angles B and C, two similar formulae can be derived; the three formulae are:

$a^2 = b^2 + c^2 - 2bc \cos A$

$b^2 = a^2 + c^2 - 2ac \cos B$

$c^2 = a^2 + b^2 - 2ab \cos C.$

These formulae should be memorized

The solution of triangles

Case I Two sides and the included angle

Fig 12.3 shows a triangle ABC in which two sides (b and c) and the included A angle (A) are given. This triangle is solved by first applying the cosine rule; in this triangle $a^2 = b^2 + c^2 - 2bc \cos A$ from which side a is calculated. Angle B is then calculated from

$$\frac{\sin B}{b} = \frac{\sin A}{a},$$

giving $\sin B = \dfrac{b \sin A}{a}.$

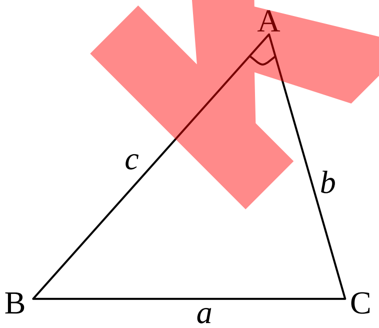


Fig 12.3

The third angle (C) can now be found by subtracting the sum of the angles A and B from 180^0.

Case II Two sides and the included angle

The triangle is solved by first applying the cosine rule in the form $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ from which angle A is calculated. Angle B (or C) can be found using the sine rule. Note that the remaining angles could be calculated by the cosine rule, but the sine rule is **(a)** easier to use, with less calculation and **(b)** more convenient when using logarithms.

In any example, always sketch the triangle and mark in the given information; next, use the appropriate formula to solve the triangle, or obtain a particular piece of information.

Example 1
In a triangle XYZ, YZ = 6.7 cm,

XY = 2.3 cm, $X\hat{Y}Z$ = 46^0 32. Calculate $X\hat{Z}Y$

First draw the triangle, marking in the information, as shown in **fig 12.4.** This is an example of two sides and the included angle being given. Now XZ must first be found, using the cosine rule, and then Z found from the sine rule.

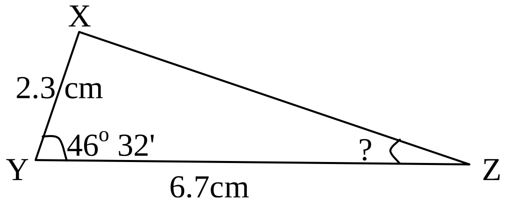


Fig 12.4

$$\begin{aligned} XZ^2 &= YZ^2 + XY^2 - 2.YZ.XY \cos X\hat{Y}Z \\ &= 6.7^2 + 2.3^2 - 2 \times 6.7 \times 2.3 \cos 46° 32' \\ &= 44.89 + 5.29 - 4.6 \times 6.7 \cos 46^0 32' \\ &= 50.18 - 21.20 \\ &= 28.98 \\ \therefore \quad XZ &= 5.383 \end{aligned}$$

$$\begin{aligned} \text{Sin } X\hat{Y}Z &= \frac{XY \sin XYZ}{XZ} \\ &= \frac{2.3 \sin 46^0\ 32'}{5.383} \\ \therefore \quad X\hat{Y}Z &= 18^0\ 04' \text{ or } 161^0\ 56' \end{aligned}$$

As XY is the smallest side of the triangle, $X\hat{Y}Z$ must be the smallest angle.

Hence $X\hat{Y}Z$ = 18^0 04'.
(**Note:** lg cos and lg sin have been used in the working).

Example 2
In the triangle PQR, QR = 4 cm, PR = 5 cm, PQ = 7 cm. calculate the size of the largest angle in the triangle.
First draw the triangle, marking in the information, as shown in **fig 12.5.** The required angle is $Q\hat{P}R$, opposite the side of 7 cm (the longest side)

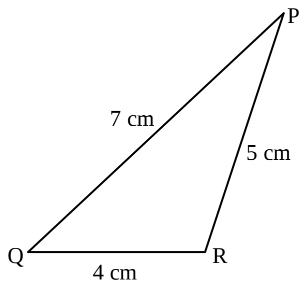


Fig 12.5

$$\cos Q\hat{P}R = \frac{QR^2 + PR^2 - PQ^2}{2QR.PR} \qquad \text{from the cosine rule}$$

$$= \frac{4^2 + 5^2 - 7^2}{2 \times 4 \times 5}$$

$$= \frac{16 + 25 - 49}{40}$$

$$= \frac{-8}{40} = -0.2000$$

$$\therefore \quad Q\hat{P}R = 180^0 - 78^0\ 28' = 101^0\ 32'.$$

Exercise 12.1

1. In a triangle PQR, PR = 40 cm, PQ = 50 cm, $Q\hat{P}R = 66°$. Calculate the length of side QR.
2. If C = 10^0 37', a = 156 m, b = 146 m, calculate the magnitude of angle B.
3. In triangle DEF, EF = 10 cm, DE = 9.87 cm, $D\hat{E}F = 29^0$ 09'. Calculate $E\hat{D}F$.
4. In triangle ABC, a = 2x, b = 3x, B = 95^0. Find c, and calculate the smallest angle.
5. In triangle ABC, side a = 6.53 cm, side b = 24 cm, C = 26^0 14'. Calculate the remaining side and angles of the triangle.
6. The sides of a triangle are 100 cm, 90 cm, and 50 cm. Calculate the magnitude of the largest angle.
7. The sides of a triangle are 7p, 8p, 5p. Calculate the size of the smallest angle of the triangle.
8. The triangle ABC has sides with the following measurements: 0.6 km, 0.9 km, 1 km. Calculate the three angles of the triangle.
9. The sides of a triangle are 0.7x, 1.5x, 1.1x. Calculate the angles of the triangle.

Exercise 12.1

1. 49.7 cm **2. $64^0 25^1$** **3. $76^0 56^1$** **4. 2.07x; $41^0 37^1$**
5. 4.5 cm; $39^0 49^1$; $66^0 3^1$ **6. $86^0 11^1$** **7. $38^0 13^1$**
8. $36^0 20^1$; $62^0 43^1$; $80^0 57^1$ **9. $25^0 51^1$; $110^0 55^1$; $43^0 14^1$**

Case III Two angles and one side

Fig 12.6 illustrates the given information. The third angle is found by subtracting the sum of the two given angles from 180^0. The sine rule is then used to determine the remaining two sides.

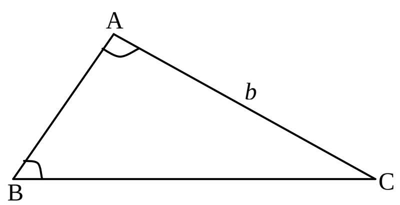


Fig 12.6

Example 3
In triangle ABC, A = 59^0, B = 39^0, a = 6.73 cm. Find the length of the smallest side.
First sketch the triangle with the given information **(fig 12.7).**

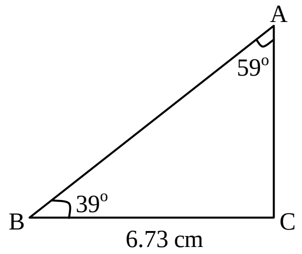


Fig 12.7

$C = 180^0 - (39^0 + 59^0) = 82^0$

Since B is the smallest angle, b is the smallest side.

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$

$$\frac{b}{\sin 39^0} = \frac{6.73}{\sin 59^0}$$

$$b = \frac{6.73 \times \sin 39^0}{\sin 59^0}$$

$$= 4.94 \text{ cm.}$$

Case IV Two sides and non-included angle

Suppose we are given the sides a, c and the angle a **(fig 12.8).**

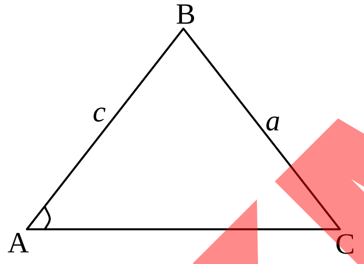


Fig 12.8

We use the sine rule to find C first. The sine rule is used again to find b, as the third angle B is now known.

Example 4

In triangle ABC, A = 30^0, c = 10 cm and a = 7.5 cm. Sole the triangle.

We use the sine rule to find C **(fig 12.9).**

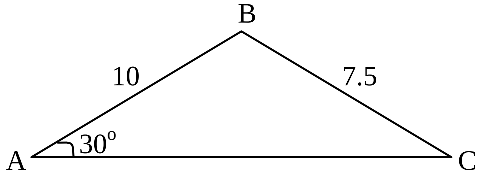


Fig 12.9

$$\frac{7.5}{\sin 30^0} = \frac{10}{\sin C}$$

or $\sin C = \frac{10 \times \sin 30^0}{7.5} = \frac{10 \times \frac{1}{2}}{7.5} = 0.6667$

giving $C = 41^0\ 49' \text{ or } 138^0\ 11'$.

We have two values for C and both e are possible. Hence there are two possible triangles ABC_1 and ABC_2 **(fig 12.10).** This is called the ambiguous case. It will usually arise if the opposite the given angle is less than the other side and the given angle is acute. (For the exception see Example 6).

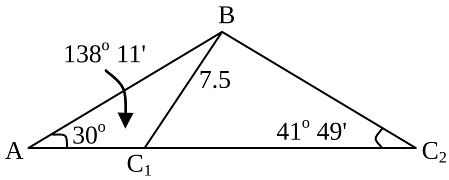


Fig 12.10

We must now solve both triangles.

Triangle ABC_1,

$B = 180^0 - (30^0 + 138^0) = 11^0\ 49'$

By the sine rule $\dfrac{AC_1}{\sin 11^0\ 49'} = \dfrac{7.5}{\sin 30^0}$;

$AC_1 = \dfrac{7.5 \times \sin 11^0\ 49'}{\sin 30^0} = \dfrac{7.5 \times 0.2048}{0.5}$

$= 3.072$ cm.

Triangle ABC_2

$B = 180^0 - (30^0 + 41^0\ 49') = 108^0\ 11'$

By the sine rule $\dfrac{AC_2}{\sin 108^0\ 11'} = \dfrac{7.5}{\sin 30^0}$;

$AC_2 = \dfrac{7.5 \times \sin 71^0\ 49'}{\sin 30^0} = \dfrac{7.5 \times 0.9501}{0.5}$

$= 14.25$ cm.

Example 5

In triangle ABC, $B = 40^0$, $c = 8$ cm and $b = 12$ cm. Solve the triangle.

This is not the ambiguous case as b is greater than c, and B is acute **(fig 12.11)**. It should also be noted that C must be less than 40^0 and hence could not be obtuse, as it would be in the ambiguous case. The solution is straight forward.

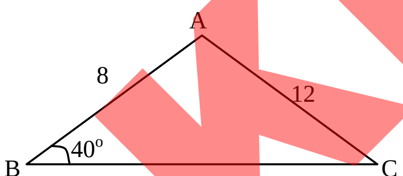


Fig 12.11

By the sine rule $\dfrac{8}{\sin C} = \dfrac{12}{\sin 40^0}$;

so $\sin C = \dfrac{8 \times \sin 40^0}{12} = 0.4285$,

and thus C = 250 22’ (the obtuse value not being possible);

the third angle A $= 180^0 - (40^0 + 25^0\ 22') = 114^0\ 38'$.

Using the sine rule again, $\dfrac{a}{\sin 114^0\ 38'} = \dfrac{12}{\sin 40^0}$;

So $a = \dfrac{12 \times \sin 65^0\ 22'}{\sin 40^0} = 16.97$ cm.

Example 6

In triangle ABC, $A = 30^0$, $c = 10$ cm and $a = 4$ cm. Solve the triangle.

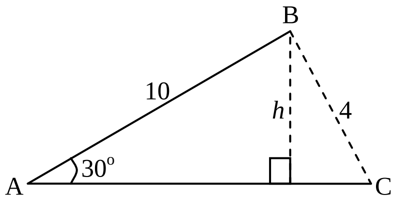


Fig 12.12

By the sine rule (fig 7.12) $\dfrac{10}{\sin C} = \dfrac{4}{\sin 30^0}$.

So $\sin C = \dfrac{10 \times \sin 30^0}{4} = \dfrac{10 \times \frac{1}{2}}{4} = 1.25.$

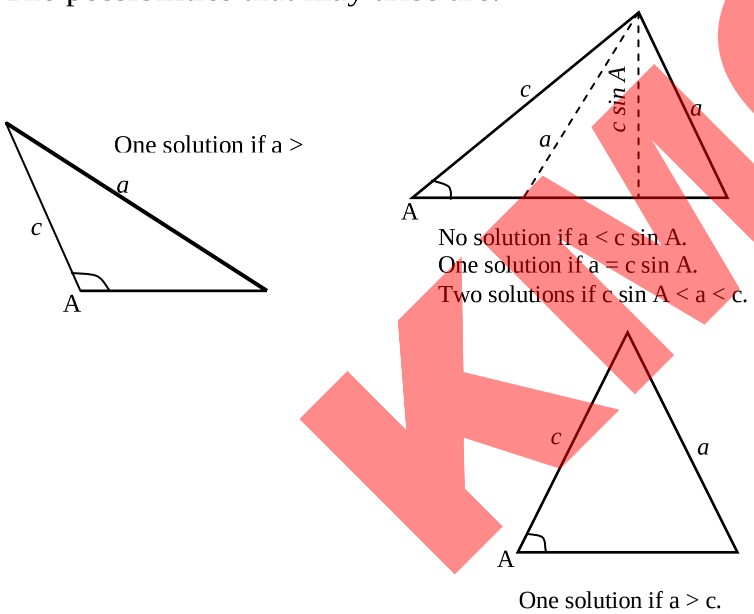
But this is impossible as sin C cannot exceed 1. Hence the triangle cannot be drawn and there is no solution.

If h is the length of the perpendicular from B to AC, $h = 10 \times \sin 30^0 = 5$, which is greater than BC. This confirms there is no solution.

In this example, A is acute and less than c. If a is less than c sin A, no triangle can be drawn and there is no solution.

Summary of Case IV

The possibilities that may arise are:



Exercise 12.2

1. In triangle ABC, A = 77^0 43’, B = 60^0 45’, b = 4.31 cm. Find the length of the longest side.

2. In triangle DEF, D = 74^0, E = 42^0 16’, DE = 2400 m. Solve the triangle to find the missing elements.

3. In triangle ABC, B = 122^0 44’, C = 15^0 22’, c = 25 cm. Solve the triangle.

4. In the triangle ABC, B = 31^0 19’, C = 109^0 51’, a = 28 m. Solve the triangle.

5. Examine the data for the following triangles and state whether the ambiguous case is involved or not, giving your reasons.

(a) Triangle ABC: B = 63^0 17’, c = 14.2 cm, b = l0.1cm.
(b) Triangle ABC: C = 27^0 38’, a = 7.9 cm, c = 11.2 cm.
(c) Triangle PQR: Q = 101^0 33’ , p = 1.3 cm, q = 4.2 cm.
(d) Triangle XYZ: Y = 69^0 23’, x = 8.2 cm, y = 9.5 cm.
(e) Triangle PQR: P = 48^0 32’, q = 5.3 m, p = 4.1 m.

6. In the triangle ABC, B = 28^0 05’, a = 0.6 cm, b = 0.35 cm. Solve the triangle completely, giving both solutions if ambiguous.

7. In the triangle ABC, B = 42^0 16’, b = 1.8 cm, c = 2.4 cm. Solve the triangle for all possible solutions.
8. Solve completely the triangle in which A = 128^0 17’, a = 24.4 cm, c = 15.6cm.
9. Solve completely the triangle in which B = 69^0 14’, b = 10.0 cm, c = 8.5cm.
10. Give all possible solutions for the triangle in which B = 65^0 08’, a = 31.2 m, b = 29.2 m.
11. Solve completely the triangle in which B = 45^0 07’, a = 89.2 m, b = 67.4 m.
12. Solve completely the triangle in which C = 95^0 44’, a = 50 cm, c = 90 cm.
13. In the triangle ABC, A = 25^0 25’, a = 8 mm, b = 5mm. Solve the triangle.
14. Solve the triangle ABC, where A = 140^0, b = 2.4 cm, c = 4.5 cm.

Exercise 12.2

1. 4.83 cm
2. F = $63^0 41^1$; EF = 2572 m; DF = 1800 m
3. A = $41^0 54^1$; a = 6.3 cm; b = 7.93cm
4. A = $38^0 50^1$; b = 23.2 m ; c = 42cm
5. (a) Ambiguous case; b < c;
(b) Not the ambiguous case; c > a;
(c) Not the ambiguous case; Q is not obtuse;
(d) Not the ambiguous case; y > x;
(e) Ambiguous case; p < q
6. A = $53^0 49^1$; C = $98^0 05.5^1$; c = 0.736 cm
A = $126^0 11^1$; C = $25^0 43.5^1$; c = 0.323 cm
7. A = 74^0; C = $63^0 44^1$; a = 2.57 cm
A = $21^0 28^1$; C = $116^0 16^1$; a = 0.979 cm
8. B = $21^0 36^1$; C = $30^0 07^1$; b = 11.4(5) cm
9. A = $58^0 08^1$; C = $52^0 38^1$; a = 9.08 cm
10. A = $75^0 45^1$; C = $39^0 07^1$; c = 20.3 m
A = $104^0 15^1$; C = $10^0 37^1$; c = 5.93 m
11. A = $69^0 39^1$; C = $65^0 14^1$; c = 86.4 m
A = $110^0 21^1$; C = $24^0 32^1$; c = 39.5 m
12. A = $33^0 34^1$; B = $50^0 42^1$; b = 70 cm
13. B = $15^0 34^1$; C = $139^0 1^1$; c = 12.23 mm
14. a = 6.52 cm C = $26^0 20^1$; B = $13^0 20^1$

The area of a triangle

The area of a triangle is found from the formula $\frac{1}{2}$ (base) x (altitude). The base and altitude of a triangle are shown in **fig 12.14**.

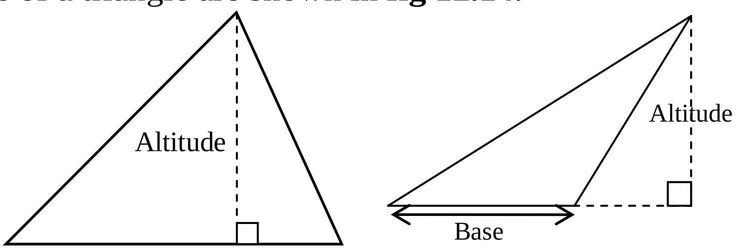


Fig 12.14

In **fig 12.15**, a triangle BC has a perpendicular drawn from A to the side BC.

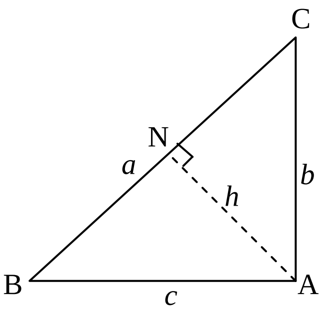


Fig 12.15

If h is the length of the perpendicular, then h = b sin C or c sin B from the triangles CAN, ABN.

Now area of the triangle $= \frac{1}{2}ah$

$= \frac{1}{2}ab\sin C$

$= \frac{1}{2}ac\sin B$

Similarly it can be shown that the area equals $\frac{1}{2}bc\sin A$. The area of a triangle is usually represented by the **symbol** Δ.

Hero's formula

In a triangle ABC, where the sides are a, b, c, the perimeter is (a + b + c).
Let the perimeter be 2s, i.e. 2s = a + b + c; s is called the semi perimeter.

Now (i) $a + b - c = a + b + c - 2c = 2s = 2c = 2(s - c)$;
(ii) $a + c - b = a + b + c - 2b = 2s = 2b = 2(s - b)$;
(iii) $b + c - a = a + b + c - 2a = 2s = 2a = 2(s - a)$.

In a triangle ABC, $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ and from Pythagoras' theorem

$\sin^2 A = 1 - \cos^2 A$.

$$\begin{aligned}
\therefore \quad \sin^2 A &= (1 - \cos A)(1 + \cos A) \\
&= \left[1 - \frac{b^2 + c^2 - a^2}{2bc}\right] \times \left[1 + \frac{b^2 + c^2 - a^2}{2bc}\right] \\
&= \frac{2bc - (b^2 + c^2 - a^2)}{2bc} \times \frac{2bc + (b^2 + c^2 - a^2)}{2bc} \\
&= \frac{a^2 - (b - c)^2}{2bc} \times \frac{(b + c)^2 - a^2}{2bc} \\
&= \frac{(a - b + c)(a + b - c)}{2bc} \times \frac{(b + c - a)(b + c + a)}{2bc} \\
&= \frac{2(s - b).2(s - c).\,2(s - a) : 2s}{4b^2c^2}.
\end{aligned}$$

$$\therefore \quad \sin A = \frac{2}{bc}\sqrt{s(-a)(s - b)(s - c)};$$

But $\Delta = \frac{1}{2}bc\sin A$;

$\therefore \quad \Delta = \sqrt{s(-a)(s - b)(s - c)}$ (Hero's formula).

Using Hero's formula, the area of a triangle can be found from the three sides.

$\Delta = \frac{1}{2} \times (\text{base}) \times (\text{altitude})$

$= \frac{1}{2}bc\sin A$

$= \frac{1}{2}ac\sin B$

$$= \tfrac{1}{2}ab\ \sin C$$
$$= \sqrt{s(-a)\,(s-b)\,(s-c)}\,.$$

Example 7

The sides of a triangle are: a = 12.7 cm, b = 13.9 cm, c = 8.6 cm. Calculate the height of the perpendicular from A to side BC.

First the area of triangle must be found, using Hero's formula.

$$s = \tfrac{1}{2}(12.7 + 13.9 + 8.6) = 17.6$$
$$s - a = 17.6 - 12.7 = 4.9$$
$$s - b = 17.6 - 13.9 = 3.7$$
$$s - c = 17.6 - 8.6 = 9.0$$
$$\therefore\ \Delta = \sqrt{17.6 \times 4.9 \times 3.7 \times 9}$$
$$\text{Now}\ \Delta = \tfrac{1}{2}ah$$

where h is the length of the perpendicular.

$$\therefore\ h = \frac{\Delta}{\frac{a}{2}} = \frac{\Delta}{6.35}$$
$$\approx 8.44 \text{ cm.}$$

Example 8

The area of triangle ABC is $20\sqrt{3}$ cm2. A = 60^0 and B = 8 cm. Find the length of side a.

$$\Delta = \tfrac{1}{2}bc\ \sin A;$$
$$c = \frac{2\Delta}{b\sin A} = \frac{2 \times 20\sqrt{3}}{8 \times \frac{\sqrt{3}}{2}} = 10 \text{ cm.}$$
$$\text{Now}\ a^2 = b^2 + c^2 - 2bc\cos A \qquad (\cos A = \cos 60^0 = \tfrac{1}{2})$$
$$= 64 + 100 - 2 \times 8 \times 10 \times \tfrac{1}{2} = 164 - 80$$
$$= 84$$
$$\therefore\ a = 9.17 \text{ cm.}$$

(**Note:** Calculation by logarithms is unnecessary, as only simple figures are involved in the working.)

Example 9

A triangle has the following measurements: s – a = 5 cm, s – b = 3 cm, s – c = 2 cm. Calculate the area of the triangle.

Add all three measurements:

$$(s-a) + (a-b) + (s-c) = 3s - (a + b + c)$$
$$= 5 + 3 + 2 = 10 \text{ cm}$$
$$= s \text{ since } (a + b + c) = 2s;$$
$$\Delta = \sqrt{s(s-a)\,(s-b)\,(s-c)}$$
$$= \sqrt{10 \times 5 \times 3 \times 2}$$
$$= \sqrt{300} = 10\sqrt{3} \approx 17.3 \text{ cm}^2.$$

Exercise 12.3

1. In triangle PQR, P = 30^0, PR = 8 cm, PQ = 35 cm. Find the area of the triangle.
2. In triangle ABC, A = 47^0 36', b = 4.8 cm, c = 69 cm. Find its area.
3. The sides of a triangle are 4 cm, 5 cm, and 7 cm long. Calculate its area to 2 sig. figs.
4. The sides of a triangle are 31.2 cm, 29.2 cm, 18.8 cm. Calculate the area of the triangle to 3 sig. figs.

5. In triangle ABC, a = 5 cm, b = 4 cm, and the area of the triangle is 86 cm^2. Calculate the size of angle C, given that it is obtuse.
6. The sides of a triangle are 6 cm, 15 cm and 19cm respectively. Calculate the altitude of the triangle, taking the longest side as the base. (Answer to 3 sig. figs.)
7. A triangle has the following measurements: s – a = 1.5 cm, s – b = 1.8 cm, s – c = 2.7 cm. Calculate its area correct to 3 sig. figs.
8. The sides of a triangle are respectively 4p, 7p and 5p units, and the area is 245 square units. Calculate the value of p.
9. The triangle ABC has an area of $\frac{25\sqrt{3}}{2}$ cm^2, $C = 60^0$, and a = 5 cm.

   Find angles A and B.
10. In triangle ABC, $B = 62^0\ 13'$, c = 30 cm, and the area is 53.1 cm^2 calculate side a.
11. Triangle ABC has $C = 40^0$, a = 160x cm, b = 100x cm, and area = 1290 cm^2. Calculate the value of x.
12. The following measurements are given for a triangle ABC: s = 30 cm, s – a = 8.4 cm, s – b = 15.6 cm. Calculate the area of the triangle.

Exercise 12.3

1. 1 cm^2	**2. 12.2 cm^2**	**3. 9.80 cm^2**	**4. 268 cm^2**
5. $120^0 41^1$	**6. 3.94 cm**	**7. 6.61 cm^2**	**8. p = 5**
9. A = 30^0; B = 90^0	**10. a = 4 cm**	**11. x = $\frac{1}{2}$**	**12. 154 cm^2**

13

Complex numbers:

Definitions:

A complex number, z, is a number of the form z = x + iy
Where x and y are real numbers and $i = \sqrt{-1}$

x is called the real part of z; y the imaginary part.

Since $i = \sqrt{-1}$
$i^2 = -1$,
$i^3 = -i$,
$i^4 = 1$, ...

The modulus of z is $|z| = \sqrt{x^2 + y^2}$.

The argument of z is $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$ where $-180^o < \arg(z) \leq 180^o$.

The conjugate of z, denoted by z* or $\overline{Z}$, is x –iy.

$z_1 = a + ib$ and $z_2 = c + id$ are equal if and only if a = c and b = d, i.e. if the real parts are equal and the imaginary parts are equal.
z = x+ iy is zero if and only if x = 0 and y = 0

Example 1

For the complex number $z = \frac{\sqrt{3}}{2} + \frac{1}{2}i$, find:

(a) $|z|$,
(b) arg z,
(c) z*.

(a) $|z| = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = 1$

(b) $\arg z = \tan^{-1}\left(\frac{1/2}{\sqrt{3}/2}\right)$

$= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$= \frac{\pi}{6}$

(c) $z^* = \frac{\sqrt{3}}{2} - \frac{1}{2}i$

Example 2

Find the real values of x and y if (x -1) + i(y -2) = 0.

If (x -1) + i(y -2) = 0

Then (x -1) = 0 and (y - 2) = 0
S0 x =1 and y = 2.

Operations

Let $z_1 = a + ib$

And $z_2 = c + id.$

Addition: $z_1 + z_2 = (a + ib) + (c + id)$
$= (a + c) + i(b + d)$

Subtraction: $z_1 - z_2 = (a + ib) - (c + id)$
$= (a - c) + i(b - d)$

Multiplication: $z_1 z_2 = (a + ib)(c + id)$
$= ac + i^2bd + iad + ibc$
$= (ac - bd) + i(ad + bc)$

Division: $z_1 \div z_2 = \frac{(a+ib)}{(c+id)}$

$$= \frac{(a+ib)(c-id)}{(c+id)(c-id)}$$

$$= \left(\frac{ac+bd}{c^2+d^2}\right) + i\left(\frac{bc-ad}{c^2+d^2}\right)$$

Example 3

If P = -2 + 3i and q = 1 + 2i, express as complex numbers in the form x + iy,

(a) p + q, (b) p - q,

(c) pq, (d) p ÷ q.

(a) $p + q = (-2 + 3i) + (1 + 2i)$
$= -1 + 5i$

(b) $p - q = (-2 + 3i) - (1 + 2i)$
$= -3 + i.$

(c) $pq = (-2 + 3i)(1 + 2i)$
$= -2 + 6i^2 + 3i - 4i$
$= -2 - 6 - i$
$= -8 - i.$

(d) $p \div q = \frac{(-2+3i)}{(1+2i)}$

$$= \frac{(-2+3i)(1-2i)}{(1+2i)(1-2i)}$$

$$= \left(\frac{-2+6}{1+4}\right) + i\left(\frac{3--4}{1+4}\right)$$

$$= \frac{4}{5} + \frac{7}{5}i$$

Operations with the conjugate

Addition: $z + z^* = (x + iy) + (x - iy)$
$= 2x$

Subtraction: $z - z^* = (x + iy) - (x - iy)$
$= 2iy$

Multiplication: $zz^* = (x + iy)(x - iy)$
$= x^2 + y^2$

Division; $\dfrac{z}{z^*} = \dfrac{(x+iy)}{(x+iy)}$

$$= \frac{(x+iy)(x+iy)}{(x+iy)(x+iy)}$$

$$= \frac{(x^2-y^2)}{(x^2+y^2)} + i\left(\frac{2xy}{x^2+y^2}\right)$$

Example 4

If z = 3 + 4i, evaluate:

(a) z + z* (b) z-z*

(c) zz* (d) z ÷z*.

(a) z + z* = (3 + 4i) + (3 – 4i)
= 6 .

(b) z-z* = (3 + 4i) –(3-4i)
= 8i.

(c) zz* = (3 + 4i)(3 -4i)
= 9 + 16
= 25.

(d) Z ÷z* $= \dfrac{3+4i}{3-4i}$

$$= \frac{(3+4i)(3+4i)}{(3-4i)(3+4i)}$$

$$= \left(\frac{-7}{25}\right) + \left(\frac{24}{25}\right)i$$

Roots of equations

If the complex number p + iq is a root of a polynomial equation with real coefficients then its conjugate, p-iq is also a root.

Example 5

If (2 + 3i) is a root of a quadratic equation with real coefficients, find the equation.

Since (2 + 3i) is a root, (2 – 3i) is the other root.

The required equation is

[x – (2 + 3i)][x-(2-3i)] = 0

i.e x^2 – [(2 + 3i) + (2 -3i)]x + (2 + 3i)(2 – 3i) =0

i.e $x^2 - 4x + 13 = 0$

Example 6

(a) Given that z_1 = 2 -3i and z_2 = 3 + 4i find

(i) z_1z_2,

(ii) $\dfrac{z_1}{z_2}$, in the form p+ iq where p and q are real.

(iii) Given that 2 + 3i is a root of the equation $z^3 - 6z^2 + 21z - 26 = 0$, find the other two roots.

(a)

(i) z_1z_2 = (2 -3i)(3+4i)
= (6 + 12) + (-9+8)

$$= \quad 18 - i$$

(ii) $$\frac{z_1}{z_2} = \frac{2-3i}{3+4i}$$

$$= \frac{(2-3i)(3-4i)}{(3+4i)(3-4i)}$$

$$= \frac{(6-12)+i(-9-8)}{9+16}$$

$$= -\frac{6}{25} - \frac{17}{25}i$$

(b) $z^3 - 6z^2 + 21z - 26 = 0$

We are given that 2 + 3i is a root of this equation. Since the coefficients of the equation are real, 2-3i is also a root.
Hence $z - (2 + 3i)$ and $z - (2 - 3i)$ are factors of the equation.
The product of these factors is

$$[z - (2 + 3i)][z - (2-3i)] = z^2 - 4z + 13$$

Dividing the LHS of the original equation by $z^2 - 4z + 13$ gives $z - 2$.
Hence $z^3 - 6z^2 + 21z - 26 = 0$ can be written as

$$(z - 2)[z - (2 + 3i)][z-(2-3i)] = 0,$$

Giving the other two required roots as z = 2 and z = 2-3i.

Example 7

(i) Express the square roots of -2i in the form ± (a + ib) where a and b are real numbers.
(ii) Solve the equation $z^2 - 3(1 + i)z + 5i = 0$
Giving your answers in the form a + ib.

Hence or otherwise solve the equation.

$$z^2 - 3(1 - i)z + 5i = 0$$

(i) Let $(a + ib)^2 = -2i$.
Work out $(a+ib)^2$.
Equate real and imaginary parts.
Find a and b.
Hence roots are ± (a + ib).

(ii) Let z = a + ib.
Work out LHS.
Equate real and imaginary parts.
Find a and b.
Hence roots are a + ib.

Second equation is obtained from first by replacing i by –i.
Hence the roots are a –ib.

Exercise 13.1

1. Express (6 + 5i)(7 + 2i) in the form a + ib. Write down (6-5i)(7-2i) in a similar form. Hence find the prime factors of $32^2 + 47^2$.

2. Expand $z = (1 + ic)^6$ in powers of c and find the five real finite values of c for which z is real.

3. If (1 + i) z-iw + I iz + (1 –i)w -3i =6, find the complex numbers z, w, expressing each in the form a + bi where a, b are real.

4(a) Express $\frac{-1+i\sqrt{3}}{-1-i\sqrt{3}}$ in the form a + ib, where a and b are real numbers.

(b) Find the quadratic equation whose roots are -3 + 4i and -3 - 4i, expressing your answer in the form $x^2 + px + q = 0$, where p and q are real numbers.

5. Find the real values of a and b such that $(a + ib)^2 = i$. Hence, or otherwise, solve the equation $z^2 + 2zi + 1 - i = 0$, giving your solutions in the form z = p + iq.

6. Let z = x + iy be any non-zero complex number.

Express $\frac{1}{z}$ in the form u + iv.

Given that $z + \frac{1}{z} = k$ with k real, prove that either y = 0 or $x^2 + y^2 = 1$. Show
(i) that if y = 0 then $|k| \geq 2$,
(ii) that if $x^2 + y^2 = 1$ then $|k| \leq 2$.

7 (a) Given that z = x + iy, where x and y are real numbers, find z^2 in terms of x and y. Hence, or otherwise, find both square roots of i.

(b) One root of a quadratic equation with real coefficients is (7-24i)/5. State the other root of this equation, and find the equation in its simplest form.

8 The roots of the quadratic equation $z^2 + pz + q = 0$ are 1 + i and 4 + 3i. Find the complex numbers p and q. It is given that 1 + i is also a root of the equation
$z^2 + (a + 2i)z + 5 + ib = 0$, where b are real.
Determine the values of a and b.

9. Obtain quadratic function: $f(z) = z^2 + az + b$, where a and b are real constants such that f(-1-2i) =0.

10. Show that 1 + I is a root of the equation $x^4 + 3x^2 - 6x + 10 = 0$. Hence write down one quadratic factor of $x^4 + 3x^2 - 6x + 10$, and find all the roots of the equation.

11. Given that a = 1+3i is a root of the equation $z^2 - (p + 2i)z + q(1 + i) = 0$, and that p and q are real, determine p, q and the root of the equation.

12. Given that $(x + iy)^2 = a + ib$, where x,y,a,b are real prove that

$4x^4 - 4ax^2 - b^2 = 0$. Hence, or otherwise, find the values of $(5+12i)^{\frac{1}{2}}$. What are the values of $(5-12i)^{\frac{1}{2}}$?
Solve the equation $z^2 - (7+4i)z + (7 + 11i) = 0$.
State the roots of $z^2 - (7 - 4i)z + (7 - 11i) = 0$.
[Give all your answers in the form u + iv where u, v are real.]

13. In the quadratic equation $x^2 + (p + iq)x + 3i = 0$, p and q are real. Given that the sum of the squares of the roots is 8, find all possible pairs of values of p and q.

Argand diagram

Any complex number z = x + iy may be represented on an Argand diagram by either
(a) the point P(x, y),

or (b) the position vector $\overrightarrow{OP}$.

The modulus of z, $|z|$, is the length of OP.
The argument of z, arg z, is the angle θ between OP and the positive real axis, where $-\pi < \theta \le \pi$.

Imaginary axis

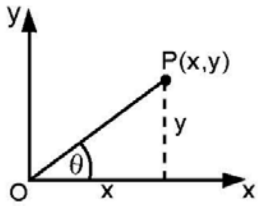


Figure 1 real axis

$|z| = \sqrt{(x^2 + y^2)}$ arg z = θ

$= \tan^{-1}\left(\dfrac{y}{x}\right)$

Polar form (also called modulus-argument form)

The polar form of a complex number is z = r (cos θ),
where r = OP and θ = x OP.

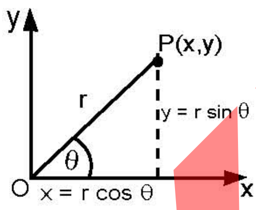


Figure 2

$|z| = r$, where $r \ge 0$.
arg z $= \theta$, where $-\pi < \theta \le \pi$.

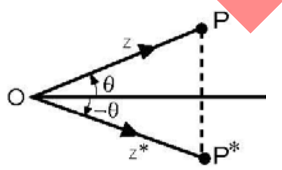


Figure 3

$Z^* = r(\cos\theta - i\sin\theta)$
$= r(\cos(-\theta) + i\sin(-\theta))$
$|z^*| = r$ and arg $z^* = -\theta$

Example 8

Express the complex number $\sqrt{3} - i$ in polar from and illustrate it on an Argand diagram.

Let $\sqrt{3} - i = r(\cos\theta + i\sin\theta)$.

$r = |z|$
$= \sqrt{((\sqrt{3})^2 + (-1)^2)} = 2$

$$\theta = \arg z = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = -\frac{\pi}{6}$$

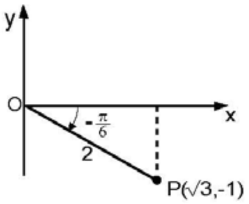


Figure 4

So, in polar form $\sqrt{3} - i$ is $2(\cos(-\pi/6))$

Multiplication and division in polar form

Let $z_1 = r_1(\cos\theta + i\sin\theta)$
and $z_2 = r_2(\cos\phi + i\sin\phi)$.

Multiplication

$$z_1z_2 = r_1r_2[\cos(\theta + \phi) + i\sin(\theta + \phi)]$$
$$|z_1z_2| = |z_1||z_2|$$
and $\arg(z_1z_2) = \arg z_1 + \arg z_2$

Division: $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta - \phi) + i\sin(\theta - \phi)]$

$$\left|\frac{z_1}{z_2}\right| = \left|\frac{z_1}{z_2}\right|$$

And $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

Example 9

If $z_1 = 4(\cos\pi/3 + i\sin\pi/3)$ and $z_2 = 2(\cos\pi/6 + i\sin\pi/6)$, evaluate:
(a) z_1z_2 and
(b) $z_1 \div z_2$.

(a) $z_1z_2 = 4(\cos\pi/3 + i\sin\pi/3) \times 2(\cos\pi/6 + i\sin\pi/6)$
$= 8[\cos(\pi/3+\pi/6) + i\sin(\pi/3 + \pi/6)]$
$= 8(\cos\pi/2 + i\sin\pi/2)$

(b) $z_1 \div z_2 = \frac{4(\cos\pi/3 + i\sin\pi/3)}{2(\cos\pi/6 + i\sin\pi/6)}$
$= 2[\cos\pi/3 - \pi/6) + i\sin(\pi/3+\pi/6)]$
$= 2(\cos\pi/6 + i\sin\pi/6)$

Geometric representation of operations

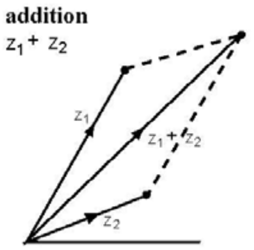


Figure 5

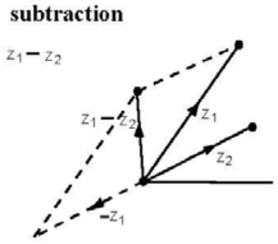


Figure 6

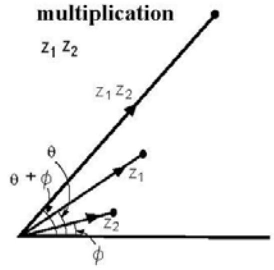


Figure 7

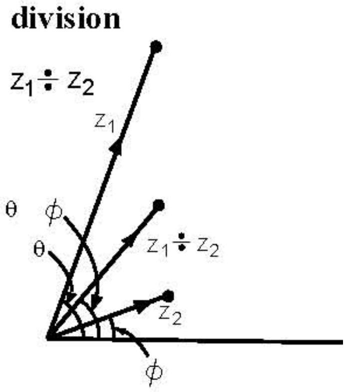


Figure 8

Loci

If z is a variable complex number, represented by the position vector $\overrightarrow{OZ}$, then the locus of Z under certain condition the be sketched. Four common loci are illustrated below.

The locus of Z when $|z| = a$ is a crcle, centre O radius a.

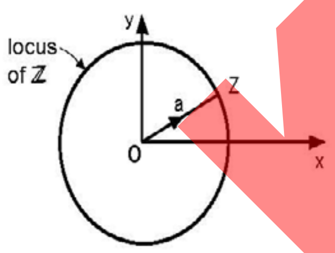


Figure 9

The locus of Z when $|z - p| = a$, where p is a fixed complex number, is a circle, centre P, radius.

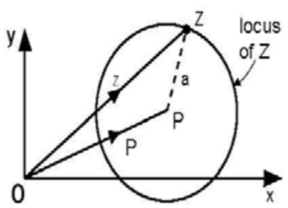


Figure 10

The locus of Z when arg z = α, ($-\pi < \alpha \leq \pi$) is a half line from 0, at an angle α with the real axis.

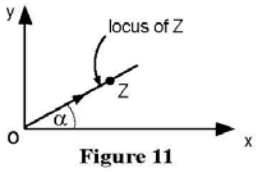


Figure 11

The locus of Z when arg(z –p) =arg q, where p and q are fixed complex numbers, is a half line from P, parallel to OQ.

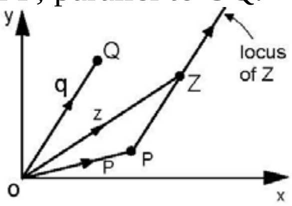


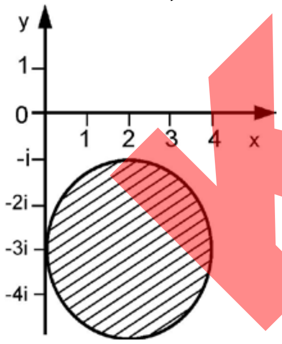
Figure 12

Example 10

(a) Indicate on an Argand the region in which z lies if $|z-2+3i| \leq 2$.

(b) If the real part of $\frac{z+1}{z-1}$ is zero, show that the locus of the point representing z in the Argand plane is a circle and write down its centre and radius.

(a) $|z - 2+3i| \leq 2$ can be re-written as $|z - (2+3i)| \leq 2$.
This says that the distance between the fixed point 2-3i and the variable point z in the Argand plane must always be less than or equal to 2. i.e the circular disc, centre 2 – 3i and radius 2, shown in the Argand diagram.



(b) Let z = x + iy,

$$\text{Then } \frac{z+1}{z-1} = \frac{x+iy+1}{x+iy-1}$$

$$= \frac{[(x+1)+iy][x-1)-iy]}{[(x-1)+iy][(x-1)-iy]}$$

$$= \frac{(x^2-1+y^2)+i(-2y)+i(-2y)}{(x-1)^2+y^2}$$

If the real part of $\frac{z+1}{z-1} = 0$,

Then $x^2 - 1 + y^2 = 0$

$\Rightarrow x^2 + y^2 = 1$.

Hence the locus of z is a circle, centre (0, 0), radius 1.

Example 11

Express the complex numbers $z = \sqrt{2} + i\sqrt{2}$ and $w = -3 + i^3\sqrt{3}$ in modulus-argument form and hence write down the modulus and argument of each of the following:

(i) $\frac{1}{z}$

(ii) zw

(iii) $\frac{z}{w}$. Show in an Argand diagram the points representing the complex numbers $\frac{1}{z}$, zw, $\frac{z}{w}$.

Let $z = r(\cos\theta + i\sin\theta)$
$= \sqrt{2} + i\sqrt{2}$,

And find r and θ.
Hence z can be written in modulus-argument form.
Do the same for $w = -3 + i3\sqrt{3}$.

(i) $\frac{1}{z} = \frac{1}{r}(\cos\theta - i\sin\theta)$

(ii) To find zw, multiply the moduli and add the arguments.

(iii) To find $\frac{z}{w}$, divide the moduli and subtract the arguments.

Having written down (i), (ii), the points representing these complex numbers can easily be shown in an Argand diagram.

Exercise 13.2

1 (a) The complex number $z_1 = 2i$. Find the values of a and b such that: $(a + ib)^2 = z_1$.
If these two resulting complex numbers are z_2 and z_3, express z_1, z_2 and z_3 in modulus-argument form and display all three on the same Argand diagram.

(b) The complex number $z_4 = \sqrt{3} + i$. Find $(z_4)^2$.
Express z_4 and $(z_4)^2$ in modulus-argument form and display them on the same Argand diagram. Deduce a further complex number z_5 such that: $(z_5)^2 = (z_4)^2$.

2. Express $\frac{1}{x + i\sqrt{3}}$ in the form $r(\cos\theta + i\sin\theta)$ where $r > 0$ and $-\pi < \theta \le \pi$.

3. Given that $z = \sqrt{3} + i$, find the modulus and argument of (a) z^2, (b) $\frac{1}{z}$.
Show in an Argand diagram the points representing the complex numbers z, z^2 and $\frac{1}{z}$.

4. You are given that $z = \cos\theta + i\sin\theta$ $(0 < \theta < \frac{1}{2}\pi)$. Draw and Argand diagram to illustrate the relative positions of the points representing z, $z + 1$, -1. Hence, or otherwise,

(a) Determine the modulus and argument of each of these three complex numbers;

(b) Prove that the real part of $\frac{z-1}{z+1}$ is zero.

14

Vectors:

Representation

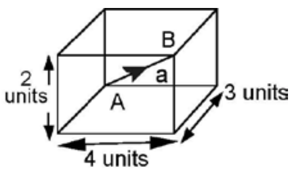
A vector has magnitude and direction. In print a vector is denoted by bold type e.g. **a**, or by two capital letters and an arrow, e.g. $\overrightarrow{AB}$.

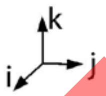
In 2-dimensions, the vector **a** can be represented by

$$a = \begin{pmatrix} x \\ y \end{pmatrix} \text{ or } a = (x\mathbf{i} + y\mathbf{j})$$

Where $i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $j = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are called base vectors.

In 3-dimensions, $a = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ or $a = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$





$$a = \overline{AB} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$$

or $(3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k})$

Base vectors in 3-dimensions:

$$i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ j = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ k = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Definitions

The **magnitude** of **a,** $|a|$, is $\sqrt{(x^2 + y^2)}$ in 2-d
And $\sqrt{(x^2 + y^2 + z^2)}$ in 3-d.

A **unit vector** has magnitude 1. $\hat{a}$ is the unit vector in the direction of a.
The **zero vector**, 0, is any vector with zero magnitude.
The inverse of **a** is –**a**
Two vectors $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ are **equal**, if and only if x = a, y = b and z = c.

Example

If a = 5**i** – s**j** -2**k** and b = t **i** + 2 **j** – u **k** are equal vectors, find
(a) s, t and u,
(b) |a|

(a) Since **a** = **b**,
then t = 5,
s = -2,
u = 2

(b) $|a| = \sqrt{[5^2 + 2^2 + (-2)^2]}$
$= \sqrt{33}$

Addition and subtraction

The triangle law is used to add and subtract vectors.
Addition:
a + **b** = **c**

Addition is commutative.
i.e. **a** + **b** = **b** + **a**

And associative,
i.e. (**a** + **b**) + **c** = **a** + (**b** + **c**)

Subtraction
a - **b** = **a** + (-**b**)

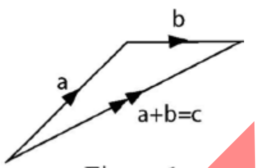


Figure 1

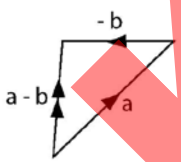


Figure 2

Example 1

Given $a = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $b = \begin{pmatrix} -1 \\ 5 \\ -3 \end{pmatrix}$, find

(a) **a** + **b**
(b) **a** – **b**

(a) $\mathbf{a} + \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \begin{pmatrix} -1 \\ 5 \\ -3 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}$

(b) $\mathbf{a}-\mathbf{b} = \begin{pmatrix}2\\-1\\3\end{pmatrix} - \begin{pmatrix}-1\\5\\-3\end{pmatrix}$

$= \begin{pmatrix}3\\-6\\6\end{pmatrix}$

Multiplication

A **scalar** is a real number, it has only magnitude. If k is a scalar, then k**a** is a vector parallel to **a** but with k times the magnitude.
If k>0, then k**a** is in the same direction as **a**.
If k<0, then k**a** is in the opposite direction to **a**.

Multiplication by a scalar is distributive over vector addition, i.e k(**a** + **b**) = k**a** +k**b**.
Example

Solve the vector equation $s\begin{pmatrix}-2\\1\end{pmatrix} + t\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-5\\1\end{pmatrix}$

$$S\begin{pmatrix}-2\\1\end{pmatrix} + t\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-5\\1\end{pmatrix}$$

$$\Rightarrow \left.\begin{array}{r}-2s+t=-5\\ s+t=\ \ 1\end{array}\right\} \Rightarrow s=2, t=-1.$$

Position vectors.

The position of a point P(x, y) in the plane can be given by the vector

$\overrightarrow{OP} = r$

$= \begin{pmatrix}x\\y\end{pmatrix}$ or (xi + yi).

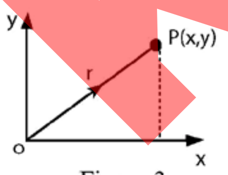


Figure 3

(i) The 3-dimensional position vector $\overrightarrow{OQ}$ can be written as

$\overrightarrow{OQ} = \mathbf{q} = \begin{pmatrix}2\\5\\3\end{pmatrix}$ Or (2**i** + 5**j** + 3**k**).

In 3-dimensions, $r = \begin{pmatrix}x\\y\\z\end{pmatrix}$ or (x **i** + y **j** + z **k**).

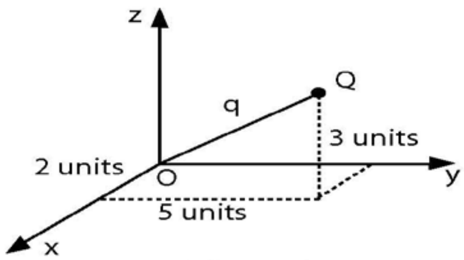


Figure 4

Ratio theorem

If C divides AB internally in the ratio λ:μ, then

$$C = \frac{\lambda b + \mu a}{\lambda + \mu}$$

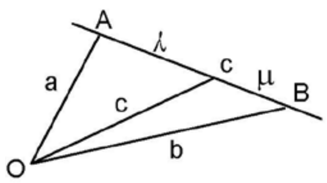


Figure 5

If the division is external, then $c = \dfrac{\lambda b + \mu a}{\lambda + \mu}$

Example

If **a** = (2**i** + 3**j**) and **b** = (8 **i** + 9 **j**) are the position vectors of A and B, find the position vector, c, of C which divides AB internally in the ratio 1:2.

$$\begin{aligned} c &= \frac{1(8i+9j)+2(2i+3j)}{1+2} \\ &= \frac{12i+15j}{3} \\ &= 4\mathbf{i} + 5\mathbf{j}. \end{aligned}$$

Example 2

In the diagram, ST=2TQ, $\overrightarrow{PQ}$ = a, $\overrightarrow{SK}$ = and $\overrightarrow{SP}$ =b.

(a) Find in terms of a and b:

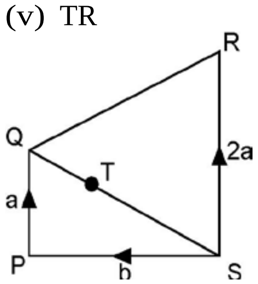
(i) $\overrightarrow{SQ}$

(ii) $\overrightarrow{TQ}$

(iii) $\overrightarrow{RQ}$

(iv) $\overrightarrow{PT}$

(v) $\overrightarrow{TR}$



(b) What do your answers to (iv) and (v) tell you about the points P, T, R?

(a) (i) $\overrightarrow{SQ} = \overrightarrow{SP} + \overrightarrow{PQ}$

$= \mathbf{b} + \mathbf{a}$ (or $\mathbf{a} + \mathbf{b}$ by commutativity)

(ii) $\overrightarrow{TQ} = \frac{1}{3}\overrightarrow{SQ}$

$= \frac{1}{3}(\mathbf{a} + \mathbf{b})$

(iii) $\overrightarrow{RQ} = \overrightarrow{RS} + \overrightarrow{SQ}$

$= -2\mathbf{a} + (\mathbf{a} + \mathbf{b})$

$= \mathbf{b} - \mathbf{a}$

(iv) $\overrightarrow{PT} = \overrightarrow{PS} + \overrightarrow{ST}$

$= -\mathbf{b} + \frac{2}{3}(\mathbf{a} - \mathbf{b}) = \frac{2}{3}\mathbf{a} - \frac{1}{3}\mathbf{b}$

$= \frac{1}{3}(2\mathbf{a} - \mathbf{b})$

(v) $\overrightarrow{TR} = \overrightarrow{TS} + \overrightarrow{SR}$

$= -\frac{4}{3}\mathbf{a} - \frac{2}{3}\mathbf{b}$

$= \frac{2}{3}(2\mathbf{a} - \mathbf{b})$.

(b) Since $\overrightarrow{PT} = \frac{1}{3}(2\mathbf{a} - \mathbf{b})$ and $\overrightarrow{PT}$ and $\overrightarrow{TR}$ are both multiples of the same vector $(2\mathbf{a} - \mathbf{b})$. Hence PT and TR are parallel and T is common to both lines, so, P, T, R lie on the same line, i.e. they are collinear.

Example 3

(a) Given that a = 3i – 4j and b = -5i + 12j, find

(i) (3a + b) .b

(ii) The angle between a and b.

(b) use vector method to show that the points A(3,1), B(4,4) and C(2,3) form a right angled triangle.

Solution:

(a) Given that $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j}$ and $\mathbf{b} = -5\mathbf{i} + 12\mathbf{j}$

(i) We are required to find (3a + b) .b

Now $3\mathbf{a} + \mathbf{b} = 3(3\mathbf{i} - 4\mathbf{j}) - 5\mathbf{i} + 12\mathbf{j}$

$= 9\mathbf{i} - 12\mathbf{j} + -5\mathbf{i} + 12\mathbf{j}$

$= 9\mathbf{i} - 5\mathbf{i} - 12\mathbf{j} + 12\mathbf{j}$

$= 4\mathbf{i} + 0\mathbf{j}$

$3\mathbf{a} + \mathbf{b} = 4\mathbf{i}$

So $(3\mathbf{a} + \mathbf{b}).\mathbf{b} = (4\mathbf{i})(-5\mathbf{i} + 12\mathbf{j})$

$= (4\mathbf{i} \cdot 5\mathbf{i}) + (4\mathbf{i} \cdot 12\mathbf{j})$, by distributive law of the scalar product

$= -20\mathbf{i}.\mathbf{i} + 48\mathbf{i}.\mathbf{j}$

$= -20 \times 1\ 48 \times 0$, sine $= \mathbf{i}.\mathbf{i} = 1$ and $\mathbf{i}.\mathbf{j} = 0$

$= -20 + 0$

$= -20$

$(3\mathbf{a} + \mathbf{b}) \cdot \mathbf{b} \quad = \quad -20$

Or simply $(3\mathbf{a} + \mathbf{b}) \cdot \mathbf{b} = (4\mathbf{i}) \cdot (-5\mathbf{i} + 12\mathbf{j})$

$= -20 + 0$

$= -20,$

(ii) By scalar (dot) product,

$\mathbf{a.b} = |a| \times |b| \times \cos\theta$, where θ is the angle between the two vectors, a and b

Now $\mathbf{a} . \mathbf{b} = (3\mathbf{i} - 4\mathbf{j}) . (5\mathbf{i} + 12\mathbf{j})$

$= 3\mathbf{i}. (-5\mathbf{i} + 12\mathbf{j}) - 4\mathbf{j}. (-5\mathbf{i} + 12\mathbf{j}),$

by distributive law of the scalar product.

$= -15\mathbf{i} . \mathbf{i} + 36\mathbf{i.j} + 20\mathbf{j} .\mathbf{i} - 48\mathbf{j.j},$ on applying the law again.

$= -15 \times 1 + 36 \times 0 + 20 \times 0 - 48 \times 1$, since $\mathbf{i.i} = \mathbf{j.j} = 1$ and $\mathbf{i.j} = \mathbf{j.i} = 0$

$= -15 - 48$

$\mathbf{a.b} = -63$

or simply $\mathbf{a.b} = (3\mathbf{i} - 4\mathbf{j}). (-5\mathbf{i} + 12\mathbf{j})$

$= (3 \times -5) - (4 \times 12)$

$= -15 - 48$

$= -63$

Also $|a| = \sqrt{3^2 + (-4)^2}$

$= \sqrt{9+16}$

$= \sqrt{25}$

$|a| = 5$

And $|b| = \sqrt{(-5)^2 + 12^2}$

$= \sqrt{(25+144)}$

$= \sqrt{169}$

$|b| = 13$

Substituting for $\mathbf{a} . \mathbf{b}$, $|a|$ and $|b|$,

$\Rightarrow -63 = 5 \times 13 \times \cos\theta$

$\Rightarrow \cos\theta = \dfrac{-63}{5 \times 13}$

$\cos\theta = -0.9692.$

$\theta = \cos^{-1}(-0.9692)$

$\theta = 165.7^0$

Is the angle (obtuse angle)

(b) The position vectors of the points A (3, 1), B (4, 4) and C (2, 3) written as column vectors are $\mathbf{OA} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\mathbf{OB} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$ and $\mathbf{OC} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ respectively, where O is the origin.

Now $\mathbf{AB} = \mathbf{OB} - \mathbf{OA}$

$= \begin{pmatrix} 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\mathbf{AB} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\mathbf{AC} = \mathbf{OC} - \mathbf{OA}$

$= \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$$\mathbf{AC} = \begin{pmatrix}-1\\2\end{pmatrix}$$

$$\mathbf{BC} = \mathbf{OC} - \mathbf{OB}$$

$$= \begin{pmatrix}2\\3\end{pmatrix} - \begin{pmatrix}4\\4\end{pmatrix}$$

$$\mathbf{BC} = \begin{pmatrix}-2\\-1\end{pmatrix}$$

Note:
For any two vectors **a** and **b** to be perpendicular, their scalar, their scalar product must be zero. i.e $\mathbf{a} \cdot \mathbf{b} = |a| \times |b| \times 0$, since $\cos 90^0 = 0$

$\mathbf{a} \cdot \mathbf{b} = 0$ is the condition for perpendicularity of any two vectors **a** and **b**
Thus for the points A, B and C to form a right-angled triangle the scalar product of one pair of any two of the vectors AB, AC and BC

Now $\mathbf{AB} \cdot \mathbf{AC} = \begin{pmatrix}1\\3\end{pmatrix} \cdot \begin{pmatrix}-1\\2\end{pmatrix}$

$= (1 \times -1) + (3 \times 2)$
$= -1 + 6$

$\mathbf{AB} \cdot \mathbf{AC} = 5 \neq 0$

The vectors **AB** and **AC** are not perpendicular.

$\mathbf{AB} \cdot \mathbf{BC} = \begin{pmatrix}1\\3\end{pmatrix} \cdot \begin{pmatrix}-2\\-1\end{pmatrix}$

$= (1 \times -2) + (3 \times -1)$
$= -2 + -3$

$\mathbf{AB} \cdot \mathbf{BC} = -5 \neq 0$

The vectors **AB** and **BC** are not perpendicular.

$\mathbf{AC} \cdot \mathbf{BC} = \begin{pmatrix}-1\\2\end{pmatrix} \cdot \begin{pmatrix}-2\\-1\end{pmatrix}$

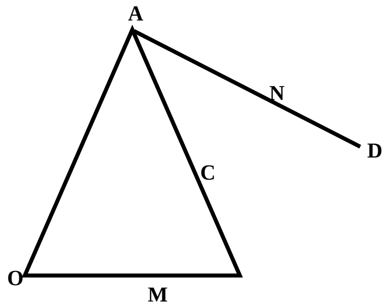
$= (-1 \times -2) + (2 \times -1)$
$= 2 - 2$

$\mathbf{AC} \cdot \mathbf{BC} = 0$

The vectors **AC** and **BC** are perpendicular i.e $\angle ACB = 90^0$ and so the points A, B, and form a right-angled triangle.

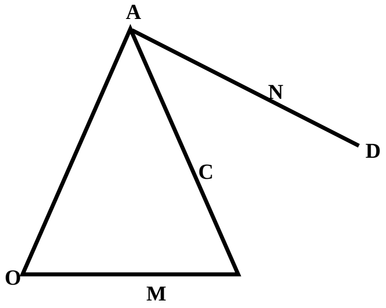
Example 4

The diagram below shows the points A and B with position vectors a and b respectively. The point M divides $\overline{OB}$ such that OM : MB = 3:1, C divides AB in the ratio 3: 1 and N is the mid-point of $\overline{AD}$. Also 3 AB = 4 CD.



(a) Find the vectors OC, OD and BN in terms of a and b.
(b) Show that MN is parallel to OA. State the ratio MN : OA.

Solution



OC = **OA** + **AC** (see vector diagram)
But AC : CB = 3:1 (total ratio = 3 + 1 = 4)

$\Rightarrow$ **AC** = $\frac{3}{4}$**AB**

And **AB** = **-OA** + **OB**
AB = **OB** – **OA**

So **AC** = $\frac{3}{4}$(**OB** – **OA**)

OC = **OA** + $\frac{3}{4}$(**OB** – **OA**)

= **OA** + $\frac{3}{4}$**OB** - $\frac{3}{4}$**OA**

= $\frac{1}{4}$**OA** + $\frac{3}{4}$**OB**

= $\frac{1}{4}$ (**OA** + 3**OB**) since **OA** = **a** and **OB** = **b**

= $\frac{1}{4}$(**a** + **b**);

OD = **OA** + **AC** + **CD** (see diagram)
But given that 3**AB** = 4**CD**

$\Rightarrow$ **CD** = $\frac{3}{4}$**AB**

CD = $\frac{3}{4}$ (**OB** – **OA**)

Also **AC** = $\frac{3}{4}$(**OB** – **OA**), from above

So **OD** = **OA** + $\frac{3}{4}$(**OB** – **OA**) + $\frac{3}{4}$(**OB** – **OA**)

$= \frac{4OA + 3OB - 3OA + 3OB - OA}{4}$

$= \frac{6OB - 2OA}{4}$

$= \frac{3OB - OA}{2}$

$= \frac{1}{2}(3\mathbf{OB} - \mathbf{OA})$

$\mathbf{OD} = \frac{1}{2}(3\mathbf{b} - \mathbf{a})$

$\mathbf{BN} = \mathbf{BA} + \mathbf{AN}$ (see diagram)

But $\mathbf{BA} = \mathbf{OA} - \mathbf{OB}$

$= \mathbf{a} - \mathbf{b}$

And $\mathbf{AN} = \frac{1}{2}\mathbf{AD}$ since N is mid-point of $\overline{AD}$.

Also $\mathbf{AD} = \mathbf{OD} - \mathbf{OA}$

$= \frac{1}{2}(3\mathbf{b} - \mathbf{a}) - \mathbf{a}$, since $\mathbf{OD} = \frac{1}{2}(3\mathbf{b} - \mathbf{a})$ from above

$= \frac{3b - a - 2a}{2}$

$= \frac{3b - 3a}{2}$

$\mathbf{AD} = \frac{3}{2}(\mathbf{b} - \mathbf{a})$

$\Rightarrow \mathbf{AN} = \frac{3}{4}(\mathbf{b} - \mathbf{a})$

$\Rightarrow \mathbf{BN} = \mathbf{a} - \mathbf{b} + \frac{3}{4}(\mathbf{b} - \mathbf{a})$

$= \frac{4a - 4b + 3b - 3a}{4}$

$= \frac{a - b}{4}$

$\mathbf{BN} = \frac{1}{4}(\mathbf{a} - \mathbf{b})$

(b) $\mathbf{MN} = \mathbf{MB} + \mathbf{BN}$ (see diagram)

But **OM**: **MB** = 3:1 (total ratio = 3 + 1 = 4)

$\Rightarrow \mathbf{MB} = \frac{1}{4}\mathbf{OB}$

$= \frac{1}{4}\mathbf{b}$

Also $\mathbf{BN} = \frac{1}{4}(\mathbf{a} - \mathbf{b})$, from above

So $\mathbf{MN} = \frac{1}{4}\mathbf{b} + \frac{1}{4}(\mathbf{a} - \mathbf{b})$

$= \frac{1}{4}\mathbf{b} + \frac{1}{4}\mathbf{a} - \frac{1}{4}\mathbf{b}$

$\mathbf{MN} = \frac{1}{4}\mathbf{a}$

but $\mathbf{OA} = \mathbf{a}$.

This means that $\mathbf{MN} = \frac{1}{4}\mathbf{OA}$

MN is a scalar multiple of **OA** and so **MN** is parallel to **OA**

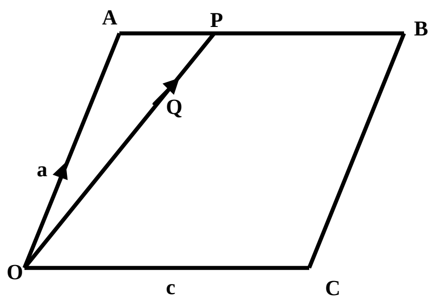
Now $\quad \mathbf{MN} = \frac{1}{4}\mathbf{OA}$

$\Rightarrow \quad \frac{MN}{OA} = \frac{1}{4}$

$\mathbf{MN : OA} = 1 : 4$

Example 5

(a) Given that **a** = 3**i** + 2**j** and **b** = **-i** + λ**j**, find the value of λ such that vectors **a** and **b** are at right angles.
(b) The diagram below shows a parallelogram OABC. **OA** = **a** and **OC** = **c** divides $\overline{AB}$ in the ratio 1:3 and Q divides $\overline{OP}$ in the ratio 4:1.



(i) Find vectors **QB** in terms of vectors **a** and **c**.
(ii) Show that the point Q lies on $\overline{AC}$.

Solution:

(a) **a** = 3**i** + 2**j**, **b** = **-i** + λ**j**

Using the scalar product (dot product).

$\mathbf{a.b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$

If **a** and **b** are at right angle, then $\theta = 90^0$

$\Rightarrow \quad \cos\theta = \cos 90^0 = 0$

$\Rightarrow \quad \mathbf{a.b} = |\mathbf{a}|\,|\mathbf{b}| \times 0$

i.e $\quad \mathbf{a\,.\,b} = 0$

on substitution,

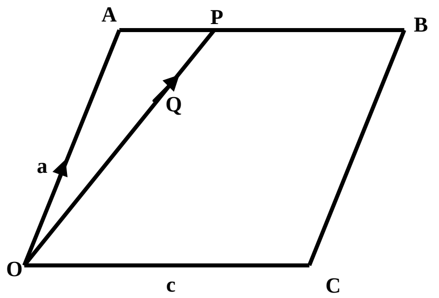
$\Rightarrow \quad (3\mathbf{i} + 2\mathbf{j}).(-\mathbf{i} + \lambda\mathbf{j}) = 0$

$-3 + 2\lambda = 0.$

$2\lambda = 3$

$\lambda = \frac{3}{2} = 1.5$

Therefore $\quad \lambda = 1.5$



(b)

Given **OA** = **a**, **OQ** : **QP** = 4 : 1

(i) **QB** = **QP** + **PB** ....(i) (see diagram)

But **OQ** : **QP** = 4 : 1 (total ratio = 5)

$\Rightarrow$ **QP** = $\frac{1}{5}$**QP** (see diagram)

And **OP** = **OA** + **AP** (see diagram)

Where **AP**:**PB** = 1:3 (total ratio = 4)

so **AP** = $\frac{1}{4}$**OC**, since **AB** = **OC** as OABC is a parallelogram.

$\Rightarrow$ **OP** = **OA** + $\frac{1}{4}$ **OC**

OP = $\mathbf{a} + \frac{1}{4}\mathbf{c}$

= $\frac{1}{5}(\mathbf{a} + \frac{1}{4}\mathbf{c})$

QP = $\frac{1}{20}(4\mathbf{a} + \mathbf{c})$

Now we also find PB:

From AP : PB = 1 : 3,

PB = $\frac{3}{4}$**AB** (since total ratio = 4 which correspond to AB: also see diagram).

$\Rightarrow$ **PB** = $\frac{3}{4}$**OC**, as **AB** = **OC** (opposite sides / vectors of a parallelogram).

Substituting for **QP** and **PB** in (i) we have:

QB = $\frac{1}{20}(4\mathbf{a} + \mathbf{c}) + \frac{3}{4}\mathbf{c}$

= $\frac{4a + c + 15c}{20}$

= $\frac{4(a + 14c)}{20}$

QB = $\frac{1}{5}(\mathbf{a} + 4\mathbf{c})$

(ii) **AQ** = **OQ** – **OA**

But **OQ** = $\frac{4}{5}$**OP** (since OQ:QP = 4:1)

and **OP** = $\mathbf{a} + \frac{1}{4}\mathbf{c}$, from b(ii) above

$$\Rightarrow \quad \mathbf{OQ} = \frac{4}{5}\left(a+\frac{1}{4}\right)$$

$$\mathbf{OQ} = \frac{4a}{5} + \frac{1c}{5}$$

So $$\mathbf{AQ} = \frac{4a}{5} + \frac{1}{5}c - a, \qquad \text{as} \qquad \mathbf{OA} = \mathbf{a}$$

$$= \frac{4a+c-5a}{5}$$

$$\mathbf{AQ} = \frac{1}{5}(c-a) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(*)$$

Also $\mathbf{AC} = \mathbf{OC} - \mathbf{OA}$

$\mathbf{AC} = \mathbf{c} - \mathbf{a}$, as $\mathbf{OC} = \mathbf{c}$ and $\mathbf{OA} = \mathbf{a}$

Substituting for ($\mathbf{c} - \mathbf{a}$) in equation $(*)$

$$\Rightarrow \quad \mathbf{AQ} = \frac{1}{5}\mathbf{AC}$$

But if any two vectors **a** and **b** are such that $\mathbf{a} = \kappa\mathbf{b}$ where k is a scalar, i.e **a** is a scalar multiple of **b**, then the two vectors are parallel.
So since **AQ** is a scalar multiple of **AC**, then **AQ** is parallel to **AC**. But since both vectors contain a common point, A, then the points A, Q and C are collinear (i.e on straight line)
Hence the point Q is on $\overline{AC}$.

Example 6

The position vectors of the point, P, Q and R are $\begin{pmatrix}4\\2\end{pmatrix}$, $\begin{pmatrix}1\\3\end{pmatrix}$ and $\begin{pmatrix}1\\-4\end{pmatrix}$ respectively.

Using vector methods, find the:
(i) largest angle enclosed by P, Q and R.
(ii) area of the triangle PQR.

Solution:

$\mathbf{OP} = \begin{pmatrix}4\\2\end{pmatrix}$, $\mathbf{OQ} = \begin{pmatrix}1\\3\end{pmatrix}$, $\mathbf{OR} = \begin{pmatrix}1\\-4\end{pmatrix}$ where O is the origin

$$\Rightarrow \quad \mathbf{PQ} = \mathbf{OQ} - \mathbf{OP}$$

$$= \begin{pmatrix}-1\\3\end{pmatrix} - \begin{pmatrix}4\\2\end{pmatrix}$$

$$\mathbf{PQ} = \begin{pmatrix}-5\\1\end{pmatrix}$$

$$\mathbf{PR} = \mathbf{OR} - \mathbf{OP}$$

$$= \begin{pmatrix}1\\-4\end{pmatrix} - \begin{pmatrix}4\\2\end{pmatrix}$$

$$\mathbf{PR} = \begin{pmatrix}-3\\-6\end{pmatrix}$$

$$\mathbf{QR} = \mathbf{OR} - \mathbf{OQ}$$

$$= \begin{pmatrix}1\\-4\end{pmatrix} - \begin{pmatrix}-1\\3\end{pmatrix}$$

$$\mathbf{QR} = \begin{pmatrix}2\\-7\end{pmatrix}$$

Let $\quad \mathbf{a} = \mathbf{PQ} = \begin{pmatrix} -5 \\ 1 \end{pmatrix},$

$\quad\quad \mathbf{b} = \mathbf{PR} = \begin{pmatrix} -3 \\ -6 \end{pmatrix}$

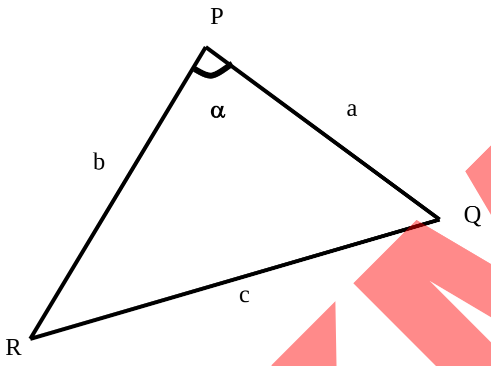
and $\quad \mathbf{c} = \mathbf{QR} = \begin{pmatrix} 2 \\ -7 \end{pmatrix}$

Now $|a| = \sqrt{(-5)^2 + 1^2} = \sqrt{26}$

$|b| = \sqrt{(-3)^2 \ ) + (-6)^2} = \sqrt{45}$

$|c| = \sqrt{2^2 + (-7)^2} = \sqrt{53}$

In any triangle, the largest angle is the angle opposite (corresponding to) the longest side. It can be seen from above that of the three vectors, **a**, **b** and **c**, the length of vectors c i.e c constitutes the longest side of triangle PQR; and so the largest angle enclosed between sides whose length are a and b.



Let this angle be α

By dot product a.b $=$ ab cos α

$\begin{pmatrix} -5 \\ 1 \end{pmatrix} . \begin{pmatrix} -3 \\ -6 \end{pmatrix} = \sqrt{26} \text{ x } \sqrt{45} \text{ x } \cos\alpha$

$15 + -6 = \sqrt{26} \text{ x } \sqrt{45} \text{ x } \cos\alpha$

$9 = \sqrt{26} \text{ x } \sqrt{45} \text{ x } \cos\alpha$

$\cos\alpha = \overline{\sqrt{26} \text{ x } \sqrt{45}}$

$\cos\alpha = 0.2631$

$\alpha = \cos^{-1}(0.2631)$

$\alpha = 74.7^0,$

Example 7

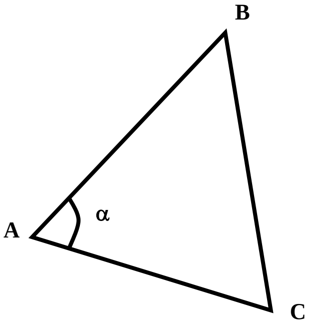
The position vectors of points A, B and C are **a** = -3**i** – **j**, **b** = 2**i** + 3**j** and **c** = 3**i** – 2**j** respectively.

(i) Find size of angle BAC,

(ii) Calculate the area of triangle ABC.

Solution:

(i) **OA** = **a** = -3**i** – **j**, **OB** = **b** = 2**i** + 3**j**, **OC** = **c** = 3**i** – 2**j**, where O is the origin.



AB = **OB** – **OA**
= 2**i** + 3**j** + 3**i** + **j**
AB = 5**i** + 4**j**

AC = **OC** – **OA**
= 3**i** – 2**j** – (-3**i** – **j**)
AC = 6**i** – **j**
If $\angle BAC$ = α, then by scalar product,
AB . **AC** = $|AB| \text{ x } |AC| \cos\alpha$
But .**AC** = (5**i** + 4**j**) . (6**i** – **j**)
= 30 – 4
= 26

$|AB| = \sqrt{5^2 + 4^2}$

$|AB| = \sqrt{41}$

Also $|AC| = \sqrt{6^2 + (-1)^2}$

$|AC| = \sqrt{37}$

$\Rightarrow$ 26 = $\sqrt{41} \text{ x } \sqrt{37} \text{ x } \cos\alpha$

$\cos\alpha = \dfrac{26}{\sqrt{41} \text{ x } \sqrt{37}}$

$\cos\alpha = 0.6675$

$\alpha = \cos^{-1}(0.6675)$

Angle BAC is 48.1^0

(ii) Area of Δ ABC $= \frac{1}{2} \text{ x } |AC| \text{ x } |AB| \text{ x } \sin\alpha$

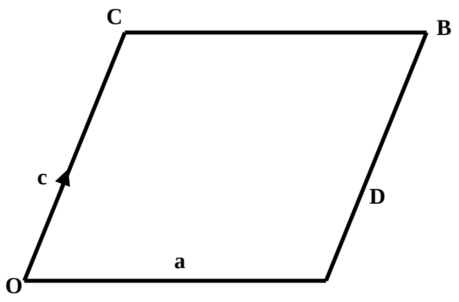
$= \frac{1}{2} \text{ x } \sqrt{37} \text{ x } \sqrt{41} \text{ x } \sin 48.1^0$,

The area of triangle ABC is 14.496 units^2

Example 8

(a) In a parallelogram OABC, **OA** = **a** and **OC** = **c**. If D lies on AB such that **AD** : **DB** = 2:3, express OD and D in terms of **a** and **c**.
(b) The points A, B , C and D have position vectors -2**i** + 3**j**, 3**i** + 8**j**, 7**i** + 6**j** and 7**i** – 4**j** respectively. Show that **AC** is perpendicular to **BD**.

Solution



OA = **a**, **OC** = **c**, **AD** : **OB** = 2 : 3

OD = **OA** + **AD** (seediagram)
But **AD** : **DB** = 2 :3 (total ratio = 2 + 3 = 5)

$\Rightarrow$ **AD** = $\frac{2}{3}$**AB**

Now **AB** = **OC** (opposite sides (vectors) of a parallelogram are equal)

$\Rightarrow$ **AD** = $\frac{2}{3}$**OC**

So **OD** = **OA** + $\frac{2}{3}$**OC**

= **a** + $\frac{2}{5}$**c**

OD = $\frac{1}{5}$(5**a** + 2**c**)

= **DA** + **AO** + **OC** (see diagram)

DC = - **AD** + -**OA** + **OC**

= -$\frac{2}{5}$**c** + -**a** + **c** as $\frac{2}{5}$**OC** from above

= $\frac{3}{5}$**c** – **a**

DC = $\frac{1}{5}$(3**c** – 5**a**).

OR:

DC = **DB** + **BC** see diagram
DC = **DB** + -**BC**

But **DB** = $\frac{3}{5}$**AB**, and **AD**: **DB** = 2 : 3

DB = $\frac{3}{5}$**OC**, as **AB** = **OC**

Also **CB** = **OA**

$\Rightarrow$ **DC** = $\frac{3}{5}$**OC** - **OA**

= $\frac{3}{5}$**c** – **a**

DC = $\frac{1}{5}$(3**c** – 5**a**), as before

(b) **OA** = -2**i** + 3**j**, **OB** = 3**i** + 8**j**
OC = 7**i** + 6**j**, **OD** = 7**i** – 4**j**, where O is the origin
Now **AC** = **OC** – **OA**

$$\begin{aligned} &= 7\mathbf{i} + 6\mathbf{j} - (-2\mathbf{i} + 3\mathbf{j}) \\ &= 7\mathbf{i} + 6\mathbf{j} + 2\mathbf{i} - 3\mathbf{j} \\ \mathbf{AC} &= 9\mathbf{i} + 3\mathbf{j} \end{aligned}$$

Also

$$\begin{aligned} \mathbf{BD} &= \mathbf{OD} - \mathbf{OB} \\ &= 7\mathbf{i} - 4\mathbf{j} - (3\mathbf{i} + 8\mathbf{j}) \\ &= 7\mathbf{i} - 4\mathbf{j} - 3\mathbf{i} - 8\mathbf{j} \\ \mathbf{BD} &= 4\mathbf{i} - 12\mathbf{j} \end{aligned}$$

Now let θ be the angle between AC and BD.
By scalar product,

$$\mathbf{AC.\ BD} = |AC| \times |BD| \times \cos\theta$$

$$\Rightarrow \cos\theta = \frac{AC.BD}{|AC| x |BD|}$$

Where

$$\begin{aligned} \mathbf{AC.BD} &= (9i + 3j) . (4i - 12j) \\ &= 36 - 36 \\ &= 0 \end{aligned}$$

$$\Rightarrow \quad \cos\theta = \frac{0}{|AC| x |BD|}$$

$$\cos\theta = 0$$

$$\theta = 90^0$$

The angle between **AC** and **BD** is 90^0
Hence **AC** is perpendicular to **BD**

Example 9

Vectors $\mathbf{OA} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, $\mathbf{OC} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$ and the points D (4, 1) and B (x , y) lie in the same plane.

If $\mathbf{AB} = \frac{1}{2}\mathbf{AC}$, find

(i) the values of x and y,
(ii) $|BC|$,
(iii) the angle between **AD** and **BC**,

Solution:

$\mathbf{OA} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, $\mathbf{OC} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$, $\mathbf{OD} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, $\mathbf{OB} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{aligned} \mathbf{AB} &= \mathbf{OB} - \mathbf{OA} \\ &= \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \end{pmatrix} \\ \mathbf{AB} &= \begin{pmatrix} x-2 \\ y-4 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{AC} &= \mathbf{OC} - \mathbf{OA} \\ &= \begin{pmatrix} 6 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \end{pmatrix} \\ \mathbf{AC} &= \begin{pmatrix} 4 \\ 0 \end{pmatrix} \end{aligned}$$

Now if **AB** $= \frac{1}{2}$**AC**,

$\Rightarrow \begin{pmatrix} x-2 \\ y-2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 4 \\ 0 \end{pmatrix}$, on substitution

$\begin{pmatrix} x-2 \\ y-4 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$\Rightarrow \quad x - 2 = 2$ ....(i)

And $\quad y - 4 = 0$ ....(ii)

From (i) $\Rightarrow x = 2 + 2$

$x = 4$

From (ii) $\Rightarrow y = 4$

$x = 4$ and $y = 4$

(ii) So **OB** $= \begin{pmatrix} 4 \\ 4 \end{pmatrix}$

But **BC** = **OC** – **OB**

$\Rightarrow$ **BC** $= \begin{pmatrix} 6 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ 4 \end{pmatrix}$

BC $= \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$\Rightarrow \quad |BC| = \sqrt{2^2 + 0^2}$

$= \sqrt{4}$

$|BC| = 2$units

(iii) **AD** = **OD** – **OA**

$= \begin{pmatrix} 4 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

AD $= \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

and BC $= \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, from (i) above

Now by scalar products,

AD.BC $= |AD| \times |BC| \times \cos\theta$, where θ is the angle between AD and BC

But **AD. BC** $= \begin{pmatrix} 2 \\ -3 \end{pmatrix} . \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$= 4 + 0$

$= 4$

and $|AD| = \sqrt{2^2 + (-3)^2}$

$= \sqrt{13}$ units

$|BC| = 2$ units, from (ii) above

$\Rightarrow \quad 4 = \sqrt{13} \times 2 \times \cos\theta$

$$\cos\theta = \frac{2}{\sqrt{13 \times 2}}$$
$$= \frac{4}{\sqrt{13}}$$
$$\theta = \cos^{-1}(0.5547)$$
$$\theta = 56.3^0, \text{ 1 dec.pl.Cal}$$

The angle between AD and BC is 56.3^0

Example 10

Given that $\mathbf{a} = \begin{pmatrix} \frac{5}{3} \\ \frac{11}{3} \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} \frac{4}{3} \\ \frac{7}{3} \end{pmatrix}$, find the

(i) angle between $\mathbf{a} + \mathbf{b}$ and $2\mathbf{a} - 3\mathbf{b}$,
(ii) area of triangle POQ, if $\mathbf{OP} = \mathbf{a} + \mathbf{b}$ and $\mathbf{OQ} = 2\mathbf{a} - 3\mathbf{b}$

Solution:
(i)

$$\mathbf{a} = \begin{pmatrix} \frac{5}{3} \\ \frac{11}{3} \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} \frac{4}{3} \\ \frac{7}{3} \end{pmatrix}$$

$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} \frac{5}{3} \\ \frac{11}{3} \end{pmatrix} + \begin{pmatrix} \frac{4}{3} \\ \frac{7}{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{9}{3} \\ \frac{18}{3} \end{pmatrix}$$

$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

$$2\mathbf{a} - 3\mathbf{b} = 2\begin{pmatrix} \frac{5}{3} \\ \frac{11}{3} \end{pmatrix} - 3\begin{pmatrix} \frac{4}{3} \\ \frac{7}{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{10}{3} \\ \frac{22}{3} \end{pmatrix} - \begin{pmatrix} \frac{12}{3} \\ \frac{21}{3} \end{pmatrix}$$

$$\mathbf{2a - 3b} = \begin{pmatrix} \frac{-2}{3} \\ \frac{1}{3} \end{pmatrix}$$

Let the angle between $\mathbf{a} + \mathbf{b}$ and $2\mathbf{a} - 3\mathbf{b}$ be α.
By dot product,

$$(\mathbf{a} + \mathbf{b}).(2\mathbf{a} - 3\mathbf{b}) = |a - b| \text{ x } |2a - 3b| \text{ x } \cos\alpha$$

But $(\mathbf{a} + \mathbf{b})\ .\ 2\mathbf{a} - 3\mathbf{b}\ ) = \begin{pmatrix} 3 \\ 2 \end{pmatrix} . \begin{pmatrix} -2/3 \\ 1/3 \end{pmatrix}$

$$= \left(3 \text{ x} \frac{-2}{3}\right) + \left(6 \text{ x} \frac{1}{3}\right)$$

$$= -2 + 2$$

$$= 0$$

$$\Rightarrow \quad 0 = |a - b| \text{ x } |2a - 3b| \text{ x } \cos\alpha$$

$$\Rightarrow \quad \cos\alpha = \frac{0}{|a + b| \text{ x} |2a - 3b|}$$

$$\cos\alpha = 0$$

$$\alpha = \cos^{-1} 0$$

$$\alpha = 90^0.$$

The angle between (**a** + **b**) and 2**a** – 3**b**) is 90^0

$$\text{Area of } \Delta \text{ POQ} = \frac{1}{2} \text{ x } |OQ| \text{ x } |OP|$$

But $|OQ| = |2a - 3b|$

$$= \sqrt{\left(\frac{-2}{3}\right)^2 + \left(\frac{1}{3}\right)2}$$

$$= \sqrt{\frac{4}{9} + \frac{1}{9}}$$

$$= \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3}$$

And $|OP| = |a + b|$

$$= \sqrt{3^2 + 6^2}$$

$$= \sqrt{9 + 36} \qquad = \sqrt{45}$$

$$= \sqrt{9 \text{ x } 5}$$

$$= \sqrt{9} \text{ x } \sqrt{5}$$

$$= 3\sqrt{5} \text{ units}$$

$$\Rightarrow \text{area of } \Delta \text{ POQ} = \frac{1}{2} \text{ x } \frac{\sqrt{5}}{3} \text{ x } 3\sqrt{5}$$

$$= \frac{(\sqrt{5})^2}{2}$$

$$\text{Area of } \Delta \text{ POQ} = 2.5 \text{ units}^2.$$

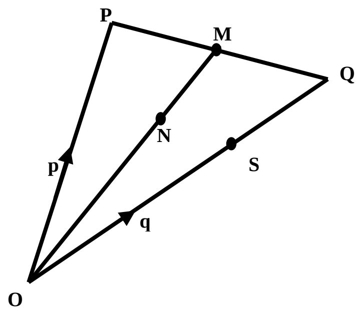
Example 11

M is the mid-point of $\overline{PQ}$ in the triangle OPQ. If **OP** = **p** and **OQ** = **q** find in terms of the vectors **p** and **q**, the vectors **PQ**, **PM** and **OM**. N is a point on $\overline{OM}$ such that **ON** : **NM** = 2 : 1

Express **ON** and **PN** in terms of **p** and **q**. Given that S is a mid-point of $\overline{OQ}$, use vector methods to show that N lies on $\overline{PS}$ and hence determine the ratio $\overline{PN}$: $\overline{NS}$

Solution:

(i)



$\mathbf{PQ} = -\mathbf{OP} + \mathbf{OQ}$

$= \mathbf{OQ} - \mathbf{OP}$

$\mathbf{PQ} = \mathbf{q} - \mathbf{p}$

$\mathbf{PM} = \frac{1}{2}\mathbf{PQ}$, since M is the mid-point of $\overline{PQ}$

$\mathbf{PM} = \frac{1}{2}(\mathbf{q} - \mathbf{p})$

$\mathbf{OM} = \mathbf{OP} + \mathbf{PM}$ (see diagram)

$= \mathbf{p} + \frac{1}{2}\mathbf{q} - \frac{1}{2}\mathbf{p}$

$= \frac{1}{2}(\mathbf{p} + \mathbf{q})$

$\mathbf{ON} : \mathbf{NM} = 2:1$ (total ratio = 2 + 1 =3)

$\Rightarrow \mathbf{ON} = \frac{2}{3}\mathbf{OM}$

$= \frac{2}{3} \text{ x } \frac{1}{2}(\mathbf{p} + \mathbf{q})$

$\mathbf{ON} = \frac{1}{3}(\mathbf{p} + \mathbf{q})$

$\mathbf{PN} = \mathbf{PO} + \mathbf{ON}$ (see diagram)

$= -\mathbf{OP} + \mathbf{ON}$

$= \mathbf{ON} - \mathbf{OP}$

$= \frac{1}{3}(\mathbf{p} + \mathbf{q}) - \mathbf{p}$

$= \frac{1}{3}\mathbf{P} + \frac{1}{3}\mathbf{q} - \mathbf{p}$

$= \frac{1}{3}\mathbf{q} - \frac{2}{3}\mathbf{p}$

$PN = \frac{1}{3}(\mathbf{q} - 2\mathbf{p})$

Now $NS = \mathbf{NO} + \mathbf{OS}$ (See diagram)

$= -\mathbf{ON} + \mathbf{OS}$

$= \mathbf{OS} - \mathbf{ON}$

But $\mathbf{OS} = \frac{1}{2}\mathbf{OQ}$, since S is the mid –point of $\overline{OQ}$

$\Rightarrow \mathbf{OS} = \frac{1}{2}\mathbf{q}$

Also $\mathbf{ON} = \frac{1}{3}(\mathbf{p} + \mathbf{q})$ from above

$$\Rightarrow \quad \mathbf{NS} = \frac{1}{2}\mathbf{q} - \frac{1}{3}(\mathbf{p} + \mathbf{q})$$
$$= \frac{1}{2}\mathbf{q} - \frac{1}{3}\mathbf{p} - \frac{1}{3}\mathbf{q}$$
$$= \frac{3q - 2p - 2q}{6}$$
$$\mathbf{NS} = \frac{1}{6}(\mathbf{q} - 2\mathbf{p})$$

But $\mathbf{PN} = \frac{1}{3}(\mathbf{q} - 2\mathbf{p})$

$$\text{So} \quad \frac{PN}{NS} = \frac{\frac{1}{3}(q-2p)}{\frac{1}{6}(q-2P)}$$
$$= \frac{1}{3} \times \frac{6}{1}$$
$$\frac{PN}{NS} = 2$$
$$\mathbf{PN} = 2\mathbf{NS}$$

Since **PN** is a scalar multiple of **NS**, then **PN** is parallel to **NS**.
But since both vectors contain a common point N, then the points P, N and S are collinear.
(lie on a straight line).
So N lies on $\overline{PS}$ as required.

$$\text{Now} \quad \frac{PN}{NS} = \frac{2}{1}$$

So PN : NS = 2:1

Hence $\overline{PN} : \overline{NS} = 2 : 1$

or $\overline{PN} : \overline{SN} = 2 : 1$

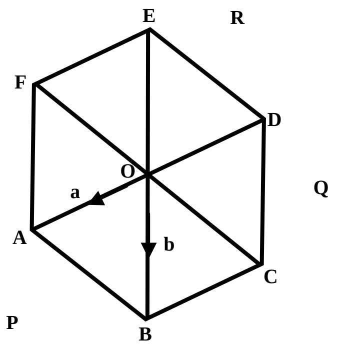
Example 12

In a regular hexagon ABCDEF with centre O, vectors **OA** = **a** and **OB** = **b**
The points, P, Q and R outside the hexagon are given by **OP** = **a** + **b**, **OQ** = **b** -2**a** and **QR** = **a** – 2**b**. Sketch the positions of these vectors in relation to the hexagon.

(i) express in terms of **a** and **b** the vectors **CB**, **AF**, **FP** and **DB**.
(ii) Find the vectors **OR**, **RF**, **FP** and **PQ**
(iii) Show that the points P,O and R are collinear.

Solution

(i)



Since ABCDEF is a regular hexagon, all the six triangles that form it are equilateral and congruent triangles.
CB = **OA** Since $\left(\overline{CB} = \overline{OA}\right.$, for equilateral triangles and also have same direction i.e parallel)

$\mathbf{CB} = \mathbf{a}$
$\mathbf{AF} = \mathbf{BO}$ (for similar reason as above)
$\quad\; = -\mathbf{OB}$
$\mathbf{AF} = -\mathbf{b}$
$\mathbf{DE} = \mathbf{BA}$ (see diagram)

But $\mathbf{BA} = \mathbf{BO} + \mathbf{OA}$
$\quad\; = -\mathbf{BO} + \mathbf{OA}$
$\quad\; = \mathbf{OA} - \mathbf{OB}$
$\Rightarrow \mathbf{BA} = \mathbf{a} - \mathbf{b}$

$\mathbf{DE} = \mathbf{a} - \mathbf{b}$
$\mathbf{DB} = \mathbf{DC} + \mathbf{CB}$

But $\mathbf{DC} = \mathbf{OB}$
$\quad\; = \mathbf{b}$

$\mathbf{FP} = \mathbf{FO} + \mathbf{OP}$ (see diagram)
$\quad\; = -\mathbf{OF} + \mathbf{OP}$
$\quad\; = \mathbf{OP} - \mathbf{OF}$

But $\mathbf{OP} = \mathbf{a} + \mathbf{b}$
And $\mathbf{OF} = (\mathbf{a} - \mathbf{b})$

$\Rightarrow \mathbf{FP} = \mathbf{a} + \mathbf{b} - (\mathbf{a} - \mathbf{b})$
$\quad\; = \mathbf{a} + \mathbf{b} - \mathbf{a} + \mathbf{b}$
$\mathbf{FP} = 2\mathbf{b}$
$\mathbf{PQ} = \mathbf{PO} + \mathbf{OQ}$ see diagram
$\quad\; = -\mathbf{OP} + \mathbf{OQ}$
$\quad\; = \mathbf{OQ} - \mathbf{OP}$

But $\mathbf{OQ} = \mathbf{b} - 2\mathbf{a}$
And $\mathbf{OP} = \mathbf{a} + \mathbf{b}$

$\Rightarrow \mathbf{PQ} = (\mathbf{b} - 2\mathbf{a}) - (\mathbf{a} + \mathbf{b})$
$\quad\; = \mathbf{b} - 2\mathbf{a} - \mathbf{a} - \mathbf{b}$
$\mathbf{PQ} = -3\mathbf{a}$

(ii) $\mathbf{PO} = -\mathbf{OP}$
$\mathbf{OP} = -(\mathbf{a} + \mathbf{b})$, from (ii) above.
So $\mathbf{PO} = \mathbf{OR}$
Since **PO** is a scalar multiple of **OR**, the **PO** is parallel to **OR**; but since the two vectors contain a common point O, then the points P, O and R are collinear.

Example 13

(a) (i) Find the lengths of the vectors $\mathbf{a} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 15 \\ 8 \end{pmatrix}$

(ii) By dot product method obtain the angle between the vectors **a** and **b**

(b) Given that vector $\mathbf{c} = \begin{pmatrix} -8 \\ \kappa \end{pmatrix}$ is perpendicular to vector **b**, determine the value of κ

Solution:

(a) (i) $\mathbf{a} = \begin{pmatrix} 6 \\ -8 \end{pmatrix}$

$\Rightarrow \quad |a| = \sqrt{6^2 + (-8)^2}$

$= \sqrt{100}$

$|a| = 10$ units

The length of **a** is 10 units

$\mathbf{b} = \begin{pmatrix} 15 \\ 8 \end{pmatrix}$

$\Rightarrow \quad |b| = \sqrt{15^2 + 8^2}$

$= \sqrt{289}$

$|b| = 17$ units

The length of **b** is 17 units

(ii) By dot product, a.b $= |a| \text{ x } |b| \text{ x } \cos\theta$, where θ is angle between a and b.

$\begin{pmatrix} 6 \\ -8 \end{pmatrix} . \begin{pmatrix} 15 \\ 8 \end{pmatrix} = 10 \text{ x } 17 \text{ x } \cos\theta$

$90 - 64 = 170 \cos\theta$

$26 = 170 \cos\theta$

$\cos\theta = \frac{26}{170} = 0.1529$

$\Rightarrow \quad \theta = \cos^{-1}(0.1529) = 81.2^0$

The angle between a and b is 81.2^0 (1 dec.pl.Cal)

(b) $\mathbf{c} = \begin{pmatrix} -8 \\ \kappa \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 15 \\ 8 \end{pmatrix}$

If **c** is perpendicular to **b**, then the their scalar (dot) product is zero

$\Rightarrow \begin{pmatrix} -8 \\ \kappa \end{pmatrix} . \begin{pmatrix} 15 \\ 8 \end{pmatrix} = 0$

$120 + 8\kappa = 0$

$8\kappa = 120$

$\kappa = \frac{120}{8}$

$\kappa = 15.$

Example 14

Given the position vectors $A = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$, $B = \begin{pmatrix} -3.5 \\ 9 \end{pmatrix}$ and $C = \begin{pmatrix} -4.5 \\ 13 \end{pmatrix}$ show that the points A, B and C are collinear.

(i) Find the ratio of $\overline{AC}$ to $\overline{AB}$.

(ii) Determine the position vector of D such that D divides $\overline{AC}$ in the ratio 3:2

Solution:

$\mathbf{OA} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$, $\mathbf{OB} = \begin{pmatrix} -3.5 \\ 9 \end{pmatrix}$, $\mathbf{OC} = \begin{pmatrix} -4.5 \\ 13 \end{pmatrix}$ where O is the origin.

$$\begin{aligned}
\mathbf{AB} &= \mathbf{OB} - \mathbf{OA} \\
&= \begin{pmatrix} -3.5 \\ 9 \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \end{pmatrix} \\
\mathbf{AB} &= \begin{pmatrix} -1.5 \\ 6 \end{pmatrix} \\
\mathbf{AB} &= 1.5 \begin{pmatrix} -1 \\ 4 \end{pmatrix} \qquad \ldots\ldots\ldots\ldots\text{(i)}
\end{aligned}$$

$$\begin{aligned}
\mathbf{AB} &= \mathbf{OC} - \mathbf{OA} \\
&= \begin{pmatrix} -4.5 \\ 13 \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \end{pmatrix} \\
&= \begin{pmatrix} -2.5 \\ 10 \end{pmatrix} \\
\mathbf{AC} &= 2.5 \begin{pmatrix} -1 \\ 4 \end{pmatrix} \qquad \ldots\ldots\text{(ii)}
\end{aligned}$$

Now for simplicity, let $\mathbf{p} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$

From (i), $\Rightarrow \mathbf{AB} = 1.5\mathbf{p}$ ……….(iii)

From (ii) $\Rightarrow \mathbf{AC} = 2.5\mathbf{p}$ ……….(iv)

Diving (iv) by (iii) we have

$$\begin{aligned}
\frac{AC}{AB} &= \frac{2.5P}{1.5P} \\
&= \frac{2.5}{1.5} \\
&= \frac{25}{15} \\
\frac{AC}{AB} &= \frac{5}{3} \\
\Rightarrow \quad \mathbf{AC} &= \frac{5}{3}\mathbf{AB},
\end{aligned}$$

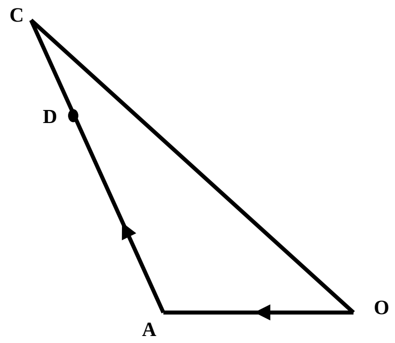
which is of form m = kn, where k is a scalar

AC is parallel to **AB**.

But since both vectors **AC** and **AB** contains a common point A, then the points A, B and C are collinear.

(i) Now $\frac{AC}{AB} = \frac{5}{3}$, from above

$$\begin{aligned}
&\Rightarrow \mathbf{AC : AB} = 5 : 3 \\
&\Rightarrow \mathbf{AD : DC} = 3 : 2 \quad \text{(total ratio = 3 + 2 =5)} \\
&\Rightarrow \quad \mathbf{AD} = \frac{3}{5}\mathbf{AC}
\end{aligned}$$



(ii)

Now **AC** $= \begin{pmatrix} -2.5 \\ 10 \end{pmatrix}$, from above

$\Rightarrow$ **AD** $= \frac{3}{5}\begin{pmatrix} -2.5 \\ 10 \end{pmatrix}$

AD $= \begin{pmatrix} -1.5 \\ 6 \end{pmatrix}$

Also **OA** $= \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

OD = **OA** + **AD**

So **OD** $= \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \begin{pmatrix} -1.5 \\ 6 \end{pmatrix}$

OD $= \begin{pmatrix} -3.5 \\ 9 \end{pmatrix}$

The vector position of D, **OD** is $\begin{pmatrix} -3.5 \\ 9 \end{pmatrix}$

Exercise

1. The vector p has magnitude 7 units and bearing 052^0, and the vector q has magnitude 12 units and bearing 1630. Draw a diagram (which need not be to scale) showing **p**, **q** and the resultant **p** + **q**. Calculate, correct to one decimal place, the magnitude of **p** + **q**.

2. From an origin O the points A, C have position vectors **a**, **b**, 2**b** respectively. The points O,A, B are not collinear. The midpoint of AB is M, and the point of trisection of AC nearer to A is T. Draw a diagram to show O, A, B, C, M, T. Find, in terms **a** and **b**, the position vectors of M and T. Use your results to prove that O, M, T are collinear, and find the ratio in which M divides OT.

3. Given that **OA** = **a**, **OB** = **b**, **OP** = $\frac{4}{5}$**OA** and that Q is the midpoint of AB, express **AB** and **PQ** in terms of **a** and **b**. PQ is produced to meet OB produced at R, so that **QR** = n**PQ** and **BR** = k**b**. Express **QR**:
   (i) on terms of n, a and b;
   (ii) in terms of k, a and b. Hence find the value of n and of k.

4. The position vectors of three points A, B and C relative to an origin O are **p**, 3**q** –**p**, and 9**q** -5**p** respectively. Show that the points A, B and C lie on the same straight line, and state the ratio AB: BC. Given that OBCD is a parallelogram and that E is the point such that **DB** = $\frac{1}{3}$**DE**, find the position vectors of D and E relative to O.

5. The point A, B and C have position vectors **a**, **b** and **c** respectively referred to an origin O.
   (a) Given that the point X lies on AB produced so that AB: BX = 2:1, find x, the position vector of X, in terms of **a** and **b**.
   (b) If Y lies on BC, between B and C so that BY:YC = 1:3, find y, the position vector of Y, in terms of **b** and **c**.
   (c) Given that Z is the mid-point of AC, show that X, Y and Z are collinear.
   (d) Calculate XY:YZ.

6. O, A and B are three non-collinear points; the position vectors of A and B with respect to O are **a** and **b** respectively. M is the mid-point of OB, T is the point of trisection of AB nearer B, AMTX is a parallelogram and OX cuts AB at Y. Find, in terms of **a** and **b**, the position vectors of:
   (a) M; (b) T; (c) X; (d) Y.

7. The vertices A, B and C of a triangle have position vectors **a**, **b** and **c** respectively relative to an origin O. The point P is on BC such that BP: PC = 3:1; the point Q is on CA such that BR: AR = 2:1. The position vectors of P, Q and R are **p**, **q** and **r** respectively. Show that **q** can be expressed in terms of **p** and **r** and hence or otherwise show that P, Q and R are collinear. State the ratio of the lengths of the line segments PQ and QR.

8. The points P and Q have position vectors **p** and **q** respectively relative to an origin O, which does not lie on PQ. Three point R, S, T have respective position vectors **r** = $\frac{1}{4}$**p** + $\frac{3}{4}$**q**, **s** = 2**p** –**q**, **t** = **p** + 3**q**. Show in one diagram the positions of O, P, Q, R, S and T.

Answers

1. $|\mathbf{p}+\mathbf{q}| = 11.5$ **units**

2. $\mathbf{m} = \frac{1}{2}(\mathbf{a}+\mathbf{b})$; $\mathbf{t} = \frac{2}{3}(\mathbf{a}+\mathbf{b})$; **OM : OT = 3 : 4.**

3. **AB** = **b**- **a**, **PQ** = $\frac{1}{10}(5\mathbf{b}-3\mathbf{a})$;

(i) $\frac{n}{10}(5\mathbf{b}-3\mathbf{a})$; (ii) $\frac{1}{2}(1+2k)\mathbf{b} - \frac{1}{2}\mathbf{a}$, $n = \frac{2}{3}$, $k = \frac{1}{3}$.

4. **AB: BC = 1: 2;** $\overrightarrow{OD} = 6\mathbf{q} - 4\mathbf{p}$; $\overrightarrow{OE} = -3\mathbf{q} + 5\mathbf{p}$

5. (a) $\mathbf{x} = \frac{1}{2}(3\mathbf{b}-\mathbf{a})$; (b) $\mathbf{y} = \frac{1}{4}(\mathbf{c}+3\mathbf{b})$; (d) **XY : YZ = 1 : 1**

6. (a) $\overrightarrow{OM} = \frac{1}{2}\mathbf{b}$ (b) $\overrightarrow{OT} = \frac{1}{3}(\mathbf{a}+2\mathbf{b})$ (c) $\overrightarrow{OX} = \frac{1}{6}(8\mathbf{a}+\mathbf{b})$

(d) $\overrightarrow{OY} = \frac{1}{9}(8\mathbf{a}+\mathbf{b})$

7. $\mathbf{q} = \frac{1}{5}(\mathbf{r}+4\mathbf{q})$; **PQ: QR = 1: 4**

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Matrices:

Definitions

A matrix may be considered as a rectangular array of numbers. The entries in a matrix are called **elements**.
The **order** of a matrix is the number of rows x the number of columns.
A **row matrix** has only one row of elements.
A **column matrix** has only one column of elements.
A **square matrix** has the same number of rows as columns, i.e. its order is of the form (n x n).
Matrices are equal if and only if they are of the same order and corresponding elements are equal.
A **zero** or **null matrix**, **0**, is a matrix in which every element is zero.
The **identity** or **unit matrix**, I, is a square matrix in which each element in the leading diagonal is I and every other element is zero.

The determinant of a 2 x 2 matrix A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the number

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$= ad - bc.$$

If det A = 0, then A is called a **singular** matrix.
Every non-singular n x n matrix A has an inverse A^{-1} such that $AA^{-1} = A^{-1}A=1$.

If A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $A^{-1} = \dfrac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Operations

Matrices may be added (or subtracted) if and only if they are of the same order. Add (or subtract) corresponding elements.
Matrix addition is commutative and associative.

To multiply a matrix by a scalar, multiply each element of the matrix by the scalar.

Two matrices A and B may be multiplied together if and only if they are compatible, i.e. if the number of columns of A equals the number of row of B. Each element of AB comes from a row in 1 and columns in B.
In general, matrix multiplication is not commutative.
However, it is associative.
Examples

1. If A = $\begin{pmatrix} 2 & 3 & 5 \\ 4 & 7 & 1 \end{pmatrix}$ and B = $\begin{pmatrix} 0 & -1 & 3 \\ 8 & 2 & -5 \end{pmatrix}$, then

A + **B** = $\begin{pmatrix} 2 & 2 & 8 \\ 12 & 9 & -4 \end{pmatrix}$ and A – B = $\begin{pmatrix} 2 & 4 & 2 \\ -4 & 5 & 6 \end{pmatrix}$

2. $3\begin{pmatrix} 5 & -3 & 6 \\ -1 & 9 & 7 \end{pmatrix} = \begin{pmatrix} 15 & -9 & 18 \\ -3 & 0 & 21 \end{pmatrix}$

3. $\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}\begin{pmatrix} p & q \\ r & s \\ t & u \end{pmatrix} = \begin{pmatrix} ap+br+ct & aq+bs+cu \\ dp+er+ft & dq+es+fu \end{pmatrix}$

(2x3matrix) x (3x2 martrix) $\rightarrow$(2 x 2 matrix)

Transformations of points Transformations in the plane, other than transalations, can be produced and described using 2x2 matrices.
Any point (x,y) can be mapped to (x_1,y_1) using a 2x2 matrix M,

where $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix}$.

To find the matrix describing a given transformation,
(a) find the image of P (1,0), say P_1(a,b),
(b) find the image of Q (0,1), say Q_1(c,d),
(c) the required matrix is $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

If M is the matrix which represents a transformation in the plane and det $M \neq 0$ then M is the matrix which represents the **inverse transformation**.

If M and N are two matrices representing two transformations for which the origin is an invariant point, then NM is the matrix which represents the result **M followed by N**

Example 1

If M = $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, a reflection in y = -x, then (2, 3) is mapped to (-3, -2) by M, since

$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$

Example 2

For rotation of + 90^0 about 0,
P(1,0) $\rightarrow P_1$(0,1)
Q(0,1) $\rightarrow$ Q1(-1, 0)

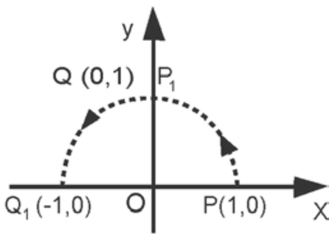

Figure 1

The matrix is M = $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Example 3

M = $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and M^{-1} = $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

M^{-1} is a rotation of -90^0 about 0

Example 4

$$NM = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

M, a reflection in y=x, followed by N, a reflection in x=0, is equivalent to a rotation of + 90^0 about 0.

Transformations of lines

The linear transformation of the plane defined by

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow T\begin{pmatrix} x \\ y \end{pmatrix},$$

Where $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $ad - bc \neq 0$, maps any line in the plane to a line in the plane.

Example 4

Find the image of y=3x under the mapping $\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$

Let any point on y =3x be (λ, 3λ), where λ is a parameter. The image of (λ, 3λ) is given by

$$\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} \lambda \\ 3\lambda \end{pmatrix} = \begin{pmatrix} 11\lambda \\ 7\lambda \end{pmatrix}$$

This is the position vector of any point on 11y =7x.
So 11y = 7x is the required image of y =3x.

Example 5

(a) A transformation T is equivalent to a shear parallel to the x-axis (the invariant line) which takes (1, 2) to (7, 2), followed by a reflection in the y = x. Find the matrix which defines T.
(b) A liner transformation P of the plane maps the points (1, 3), (-2, -3) to the point (2, 4), (-3, -11), respectively. Find the matrix of this transformation.

(a) The matrix $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ represents a shear parallel to the x-axis (the invariant line).

Since $(1,2) \rightarrow (7,2)$, we have

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1+2k \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 7 \\ 2 \end{pmatrix}$$

$$\Rightarrow \quad k = 3.$$

So $S = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ defines the shear.

The matrix R which defines reflection in y = x is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Hence, the matrix which represents the shear followed by the reflection is

$$RS = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}.$$

(a) Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the matrix which defines P.

Now $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} a+3b \\ c+3d \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$,

And $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -2 \\ -3 \end{pmatrix} = \begin{pmatrix} -2a+3b \\ -2c+3d \end{pmatrix} = \begin{pmatrix} -3 \\ -11 \end{pmatrix}$

So, $a + 3b = 2$,
And $2a + 3b = 3$,

$\Rightarrow \quad a = 1, \; b = \dfrac{1}{3}$.

Also, $c + 3d = 4$,
And $2c + 3d = 11$,

$\Rightarrow \quad c = 7, \; d = -1$.

Hence $\begin{pmatrix} 1 & 1/3 \\ 7 & -1 \end{pmatrix}$ is the matrix defines P.

Example 6

(a) Given that A = $\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$**, B =** $\begin{pmatrix} 1 & 2 \\ -1 & 5 \end{pmatrix}$ **and C = AB, find** C^{-1}

(b) Use matrix method to solves the simultaneous equations

$$3x - y = 1$$
$$2x + 3y = 19$$

Solution:

(a) Given A $= \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$, B $= \begin{pmatrix} 1 & 2 \\ -1 & 5 \end{pmatrix}$

C = AB

$$\Rightarrow \quad C = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 3x1+1x-1 & 3x2+1x5 \\ 0x1+2x-1 & 0x2+2x5 \end{pmatrix}$$

$$C = \begin{pmatrix} 2 & 11 \\ -2 & 10 \end{pmatrix}$$

$\Rightarrow$ det C $= (2 \text{ x } 10) - (11 \text{ x } -2)$

$= 20 - (-22)$
$= 42$

So $C^{-1} = \begin{pmatrix} 10 & -11 \\ 2 & 2 \end{pmatrix}$

$$C^{-1} = \begin{pmatrix} \frac{10}{42} & \frac{-11}{42} \\ \frac{2}{42} & \frac{2}{42} \end{pmatrix}$$

(b) $3x - y = 1$

$$2x + 3y = 19$$

$$\Rightarrow \begin{pmatrix} 3 & -1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 19 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 19 \end{pmatrix}$$

$$\begin{pmatrix} 11 & 0 \\ 0 & 11 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 22 \\ 55 \end{pmatrix}$$

$$\begin{pmatrix} 11x \\ 11y \end{pmatrix} = \begin{pmatrix} 22 \\ 55 \end{pmatrix}$$

$$\Rightarrow 11x = 22$$

$$\text{i.e } x = 2$$
$$\text{and } 11y = 55$$
$$\text{i.e } y = 5$$

$x = 2$ and $y = 5$

Example 7

10. A triangle OAB has vertices O(0,0), A(4,3) and B (1,3.) It is mapped onto its image oA'B' by a transformation represented by the matrix P = $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$**.**

(a) Determine the coordinates of OA' and B'

(b) The image OAB undergoes another transformation represented by the matrix Q = $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ **to give OA"B". Find the coordinates of O A"**

(c) If the area of triangle OAB is 4.5 square units, find the area of the its final image OA"B"

(d) Determine a single matrix of transformation that will map O A"B" back onto OAB.

Solution:

(a) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & A & B \\ 0 & 4 & 1 \\ 0 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 0 & A & B \\ 0+0 & 0+-3 & 0+-3 \\ 0+0 & -4+0 & 0+-3 \end{pmatrix}$

$$= \begin{pmatrix} 0 & A'' & B'' \\ 0 & -9 & -9 \\ 0 & -12 & -3 \end{pmatrix}$$

The co-ordinates of O, A" and B" are O(0,0), A"(-9, -12) and B" (-9, -3) respectively
OAB is mapped onto OA"B" by matrix P followed by Q.
So the combined matrix QP will map OAB directly onto OA"B"
Hence the inverse of this matrix i.e (QP)-1, will map OA"B" back onto OAB.

Now **P** $= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

$\Rightarrow$ **QP** $= \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0+0 & -3+0 \\ 0+-3 & 0+0 \end{pmatrix}$

QP $= \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$

Example 8

(a) Given the matrices A = $\begin{pmatrix} 5 & 1 \\ 0 & 2 \end{pmatrix}$**, B =** $\begin{pmatrix} -2 & 3 \\ 1 & 0 \end{pmatrix}$ **and C =** $\begin{pmatrix} 2 & 1 & -1 \\ 1 & 5 & 2 \end{pmatrix}$**, find**

(i) ABC,
(ii) (A + B)C.
(b) By use of matrices, solve simultaneous equations:

3x + 4y = 8
x + 2y = 3.

Solution:

(a)

Given $A = \begin{pmatrix} 5 & 1 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 3 \\ 1 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 5 & 2 \end{pmatrix}$

(i) **ABC** = (A x B) x C

Now AxB $= \begin{pmatrix} 5 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -2 & 3 \\ 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} -10+1 & 15+0 \\ 0+2 & 0+0 \end{pmatrix}$

A x B $= \begin{pmatrix} -9 & -15 \\ 2 & 0 \end{pmatrix}$

$\Rightarrow$ **ABC** $= \begin{pmatrix} -9 & -15 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 & -1 \\ 1 & 5 & 2 \end{pmatrix}$

ABC $= \begin{pmatrix} -3 & 66 & 39 \\ 4 & 2 & -2 \end{pmatrix}$

(ii) (A + B) C

A + B $= \begin{pmatrix} 5 & 1 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} -2 & 3 \\ 1 & 0 \end{pmatrix}$

A + B $= \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$

(A + B) C $= \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 & -1 \\ 1 & 5 & 2 \end{pmatrix}$

$= \begin{pmatrix} 6+4 & 3+20 & -3+8 \\ 2+2 & 1+10 & -1+4 \end{pmatrix}$

(A + B) C $= \begin{pmatrix} 10 & 23 & 5 \\ 4 & 11 & 3 \end{pmatrix}$

Note: Two matrices **P** and **Q** can be added (or subtracted) only if they are of the same order (have same number of rows and columns)
Columns of P is equal to the number of rows of Q i.e if the orders of P is a x b and the order of Q is a X c, then P Q is possible. The two matrices are then said to be incompatible.

(b) $\Rightarrow \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 3 \end{pmatrix}$

$$\begin{pmatrix} 2 & -4 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 8 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 6-4 & 8-8 \\ -3+3 & -4+6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 16-12 \\ -8+9 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2x \\ 2y \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$\Rightarrow \quad 2x = 4$

$x = 2$

Also $2y = 1$

$y = \frac{1}{2}$

$x = 2$ and $y = \frac{1}{2}$

Example 9

Given the matrix A = $\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$**, evaluate $A^2 - 2A$ Hence find the inverse of A.**

Solution:

$A = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$

$A^2 = A \times A$

$= \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1+6 & 3+3 \\ 2+1 & 6+1 \end{pmatrix}$

$A^2 = \begin{pmatrix} 7 & 6 \\ 4 & 7 \end{pmatrix}$

$2A = 2\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$

$2A = \begin{pmatrix} 2 & 6 \\ 4 & 2 \end{pmatrix}$

So $A^2 - 2A = \begin{pmatrix} 7 & 6 \\ 4 & 7 \end{pmatrix} - \begin{pmatrix} 2 & 6 \\ 4 & 2 \end{pmatrix}$

$$A^2 - 2A = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

$\Rightarrow A^2 - 2A = 5\mathbf{I}$, since $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

But A = A**I** and AA-1

So we can write the equation as follows:

$AA - 2A\mathbf{I} = 5AA^{-1}$

$\Rightarrow A(A - 2\mathbf{I}) = 5AA^{-1}$, since $\mathbf{MN} \pm \mathbf{MP}$

$= \mathbf{M}\,(\mathbf{N} \pm P)$

$A(A - 21) = 5A\ \text{-}1$

$\Rightarrow A^{-1} = \frac{1}{5}(A - 21)$

But $A = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$ and $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow A^{-1} = \frac{1}{5}$

Example 10

(a) Given the matrix M = $\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$, find the scalar λ such that the matrix (M - λ I) is singular, where I is 2 x2 identity matrix

(b) Use the matrix method to the simultaneous equations:

4x + 2y =6

3x + 5y = 5

Solution

(a)

$$M = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$$

$$I = \begin{pmatrix} 2 & 6 \\ 1 & 4 \end{pmatrix}$$

$$\Rightarrow M - \lambda\mathbf{I} = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$\Rightarrow M - \lambda I = \begin{pmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{pmatrix}$$

If (M - λ I = is a singular matrix, then

$\Rightarrow$ det (M - λI) = 0 (since a singular matrix is one whose determinant is zero).

But det (M - λ1) = (2 - λ) (4 - λ) – 3 x 1

= 8 – 2λ - 4x + λ^2 – 3

$\Rightarrow$ det (M - λI) = 5 - 6λ + λ^2

so 5 - 6λ + λ^2 = 0

$\Rightarrow$ λ^2 - 6λ + 5 = 0 on rearrangement.

(-5, -1)

$\Rightarrow \lambda^2 - 5\lambda + -\lambda + 5 \quad = \quad 0$

$(\lambda^2 - 5\lambda) + (-\lambda + 5) \quad = \quad 0$

$\lambda(\lambda - 5) + -1(\lambda - 5) \quad = \quad 0$

$(\lambda - 5)(\lambda - 1) \quad = \quad 0$

Either $\lambda - 5 = 0$ or $\lambda - 1 = 0$

$\lambda = 5$ or $\lambda = 1$

The valves of the scalar λ such that the matrix $(M - \lambda 1)$ is singular are 1 and 5

$4x + 2y = 6$

$3x + 5y = 5$

$$\begin{pmatrix} 4 & 2 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$$

Let $C = \begin{pmatrix} 4 & 2 \\ 3 & 5 \end{pmatrix}$

$\Rightarrow \quad \det C \quad = \quad (4 \times 5) - (2 \times 3) = 14$

$\text{adj}C \quad = \quad \begin{pmatrix} 5 & -2 \\ -3 & 5 \end{pmatrix}$

Inverse of C,

$C^{-1} \quad = \quad \frac{1}{\det C} \times \text{adj } C$

$$= 1 \over 14\begin{pmatrix} 5 & -2 \\ -3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{14} & \frac{-2}{14} \\ \frac{-3}{14} & \frac{4}{14} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \frac{5}{14} & \frac{-2}{14} \\ \frac{-3}{14} & \frac{4}{14} \end{pmatrix}\begin{pmatrix} 4 & 2 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{30}{14} - \frac{10}{14} \\ \frac{-18}{14} + \frac{20}{14} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{20}{14} \\ \frac{2}{14} \end{pmatrix}$$

$x = \frac{20}{14} = \frac{10}{7}$

and $\gamma = \frac{2}{14} = \frac{1}{7}$

Hence $x = \frac{10}{7}$ and $= \frac{1}{7}$

Example 11

10. Under a 2 x 2 matrix transformation the points A(-4, -2), B(-2, -2), C(-2, -4) and D(-4,-4) of a square are mapped onto the points A'(6,2), B'(4,0), C' (6, -2), and D'(8,0) respectively.
Find the:
- **(i) area of square ABCD,**
- **(ii) matrix of transformation and area of the image A'B'C'D'**
- **(iii) matrix that would map A'B'C'D' back onto ABCD, respectively.**

Solution:

(i) Side of square ABCD = $\overline{AB} = \overline{BC} = \overline{CD} = \overline{AD}$

$$= \sqrt{(-2)^2+0^2}$$

$$= \sqrt{4}$$

$$\overline{AB} = 2 \text{ units}$$

But area of a square = $(\text{sides})^2$

$\Rightarrow$ area of square ABCD $= \overline{AB}^2$

$= 22$

Area of square ABCD $= 4 \text{ units}^2$

(ii) Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the matrix of transformation

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} A & B & C & C \\ -4 & -2 & -2 & -4 \\ -2 & -2 & -4 & -4 \end{pmatrix} = \begin{pmatrix} A' & B' & C' & D' \\ 6 & 4 & 6 & 8 \\ 2 & 0 & -2 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -4a-2b & -2a-2b & -2a-4b & -4a-4b \\ -4c-2d & -2c-2d & -2c-4d & -4c-4d \end{pmatrix} = \begin{pmatrix} 6 & 4 & 6 & 8 \\ 2 & 0 & -2 & 0 \end{pmatrix}$$

Equating corresponding elements in the matrices on both sides we have.

$-4a - 2b = 6$

i.e $-2a - b = 3$ ………(i)

$-2a - 4b = 6$ ………..(ii)
$-4c - 2d = 2$
i.e $-2c - d = 1$ ………… (iii)
$-2c - 2d = 0$ …………..(iv)

Subtracting (ii) from (i)

$-2a - b - (-2a - 4b) = 3 - 6$
$-2a - b + 2a + 4b = -3$
$3b = -3$
$b = -1$

substituting for b in (i), we have

$-2a - (-1) = 3$
$-2a + 1 = 3$
$-2a = 2$
$a = -1$

subtracting (iv) from (iii)

-2c – d- (-2c – 2d) = 1 – 0
-2c – d + 2c + 2d = 1
D = 1

Substituting for a d in (iii), we have

-2c – 1 = 1
-2c = 2
c = -1
a = - 1, b = -1, c = -1, d = 1

The reader may check that these obtained values are consistent with the other equations not outlined from the matrices. $\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$

Let M = $\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$

$\Rightarrow$ det = (-1 x 1) – (-1 x -1)
= -1 -1
det = -2

But also Scale Factor (ASF) is numerically equal to the determinant of the transformation matrix.

So ASF = 2

But also ASF = $\dfrac{\text{Area of image (A'B'C'D')}}{\text{Area of object (ABCD)}}$

$\Rightarrow$ 2 = $\dfrac{\text{Area of image (A'B'C'D')}}{\text{Area of (ABCD)}}$

2 = $\dfrac{\text{Area of (A'B'C'D')}}{4}$, as area of ABCD = 4 $units^2$ from (i)abovearea of A’B’C’D’) = 2 x 4

= 8 $unit^2$

The area of (A’B’C’D’) = 8 $units^2$

OR:

Area of (A’B’C’D’) = $\overline{A'B'}$ x $\overline{B'C'}$

But $\overline{A'B'}$ = $\sqrt{(6-4)^2+(2-0)^2}$

= $\sqrt{2^2+2^2}$

$\overline{A'B'}$ = $\sqrt{8}$ units.

Also $\overline{B'C'}$ = $\sqrt{(6-4)^2+(2-0)^2}$

= $\sqrt{2^2+(-2)^2}$

$\overline{B'C'} = \sqrt{8}$ units

So area of A'B'C'D' = $\sqrt{8}$ x $\sqrt{8}$
= 8 units2, as before

Note: the matrix M is associated with a transformation which does not change the shape of the object is A'B'C'D' applied above is valid.

(iii) If M = $\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$ is the matrix that transformed ABCD onto A'B'C'D', then M^{-1}, the inverse of M will map A'B'C'D' back onto ABCD

Now det **M** = (-1 x 1) – (-1 x -1)

= - 2

adjM = $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$\Rightarrow$ $M^{-1} = \frac{1}{\det M}$ x adjM

$= \frac{1}{-2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$M^{-1} = \begin{pmatrix} \frac{-1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{pmatrix}$

The matrix that would map A'B'C'D' back onto ABCD is

$$\begin{pmatrix} \frac{-1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{pmatrix}$$

Example 12

Given that Q = $\begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix}$**, find the inverse of Q**

Hence solve the equations:
x – 2y = -4
3x + y = 9

Solution:

Q = $\begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix}$

detQ = (1 x 1) – (-2 x 3)
= 1 – (-6)
= 7

adjQ = $\begin{pmatrix} 1 & 2 \\ -3 & 1 \end{pmatrix}$

$Q^{-1} = \frac{\text{adj}Q}{\det Q}$

$= \overline{1\,7(}\begin{pmatrix} 1 & 2 \\ -3 & 1 \end{pmatrix}$

Exercise

1 (i) If $M = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$, find the values of M^2, M^3 and M^{-1}

Find x and y, given that $M\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

(ii) A transformation T is equivalent to an enlargement with centre at the origin, scale factor 2, followed by a reflection in the line $x + y = 0$. What matrix defines T? If T maps a point P onto (6, 2), what are the coordinates of P?

2 A transformation M is represented by the matrix M where $M = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$.

(i) Find the image of the point (-2, 5) under M.
(ii) Find the inverse of M.
(iii) Given that the point (11,13) is the image of the point (a, b) under M, find the value of a and of b.
(iv) Find, in terms of a, the image of the point (a, a) under M.
(v) State the equation of the invariant line under M.

*(C)

3 If $A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & -1 \\ -1 & 2 & -1 \end{pmatrix}$, find

(a) AB;
(b) a matrix X such that AX + B =A.

4 Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

(a) The plane is mapped onto itself by the map under which the point P of co-ordinates (x1, y1) is mapped to the point Q of co-ordinates (x_2, y_2), where $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = A\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$.

By considering $A\begin{pmatrix} x \\ m_1x + c \end{pmatrix}$, prove that the line $y = m_1x + c$ is mapped onto a line of slope m_2, determining m_2 in terms of m_1. Hence or otherwise determine whether any line through the origin is mapped onto itself, and find any such line.

(b) Prove that there is no non-singular matrix P such that $P^{-1}AP = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$ for real k_1, k_2.

5 The transformation with matrix T, where $T = \begin{pmatrix} 2 & 1 \\ 2 & -2 \end{pmatrix}$, maps the point (x, y) into the point (x^1, y^1) so that $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^1 \\ y^1 \end{pmatrix}$.

Find the equation of the image of the line $y = 3x$ under this transformation. Find also the equations of the lines through the origin which are turned through a right angle about the origin under this transformation.

Answers
Exercise

1. (i) $M^2 = \begin{pmatrix} 3 & -5 \\ 5 & 8 \end{pmatrix}$, $M^3 = \begin{pmatrix} 1 & -18 \\ 18 & 19 \end{pmatrix}$, $M^{-1} = \frac{1}{7}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$; $x = 2,\ y = 1$

2. (i) (13, 19), **(ii)** $\frac{1}{10}\begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix}$ **(iii)** $a = 2,\ b = 3$ **(iv)** $(5\alpha, 5\alpha)$ **(v)** $y = x$

3. (a) 1