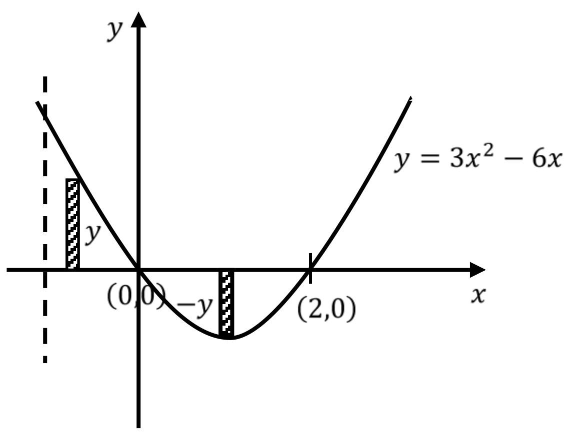
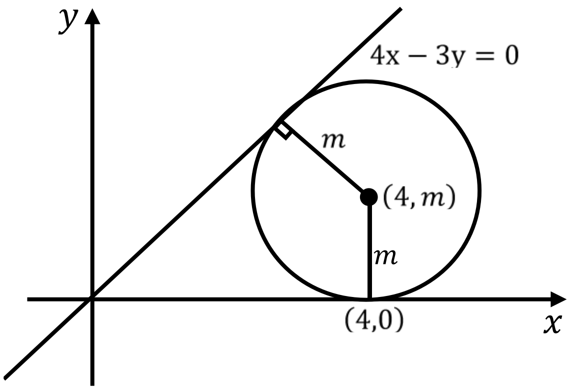
PROPOSED
MARKING GUIDE
P425/1
PURE MATHEMATICS 2023

NO	SOLUTION	Mks	Comment
1	$\begin{pmatrix} 2 & -1 & 3 \\ 1 & 4 & -1 \\ 3 & 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 14 \\ -5 \\ 17 \end{pmatrix}$ $\begin{array}{c} \begin{pmatrix} 2 & -1 & 3 & \vdots & 14 \\ 1 & 4 & -1 & \vdots & -5 \\ 3 & 1 & 4 & \vdots & 17 \end{pmatrix} \end{array}\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$ $\begin{pmatrix} 2 & -1 & 3 & \vdots & 14 \\ 1 & 4 & -1 & \vdots & -5 \\ 3 & 1 & 4 & \vdots & 17 \end{pmatrix} \xrightarrow{\substack{R_1' = R_1 - 2R_2 \\ R_3' = 2R_2 - R_3}} \begin{pmatrix} 2 & -1 & 3 & \vdots & 14 \\ 0 & -9 & 5 & \vdots & 24 \\ 0 & 11 & -7 & \vdots & -32 \end{pmatrix}$ $\xrightarrow{R_3'' = 11R_2' + 9R_3'} \begin{pmatrix} 2 & -1 & 3 & \vdots & 14 \\ 0 & -9 & 5 & \vdots & 24 \\ 0 & 0 & -8 & \vdots & -24 \end{pmatrix}$ $-8z = -24 \qquad \therefore z = 3$ $-9y + 5z = 24$ $-9y + 15 = 24$ $-9y = 9 \qquad \therefore y = -1$ $2x - y + 3z = 14$ $2x + 1 + 9 = 14$ $2x = 4 \qquad \therefore x = 2$ $\therefore x = 2, y = -1, z = 3$		
		05	
2	L.H.S $= \frac{1}{2}(\sin 8A + \sin 2A) - \frac{1}{2}(\sin 8A - \sin 6A)$ $= \frac{1}{2}[\sin 8A + \sin 2A - \sin 8A + \sin 6A]$ $= \frac{1}{2}(\sin 6A + \sin 2A)$		

	$= \frac{1}{2} \times 2 \sin 4A \cos 2A$ $= \sin 4A \cos 2A$ =R.H.S		
		05	
3	Intercepts $x, y = 0$ $0 = 3x^2 - 6x$ $0 = 3x(x - 2)$ $x = 0$ or $x = 2$ $\quad\quad \therefore (0,0)$ and $(2,0)$  $A = \int_{-1}^{0} y\, dx + \int_{0}^{2} -y\, dx$ $A = \int_{-1}^{0} (3x^2 - 6x)\, dx + \int_{0}^{2} (6x - 3x^2)\, dx$ $A = [x^3 - 3x^2]_{-1}^{0} + [3x^2 - x^3]_{0}^{2}$ $A = [0 - (-1 - 3)] + [(12 - 8) - 0]$ $A = 4 + 4$ $A = 8$ sq.units		
		05	
4	Let $\boldsymbol{c}$ be a vector perpendicular to $\boldsymbol{a}$ and $\boldsymbol{b}$ $\boldsymbol{c} = \boldsymbol{a} \Lambda \boldsymbol{b}$ $\boldsymbol{c} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{vmatrix}$		

	$\boldsymbol{c} = \begin{vmatrix} 1 & -2 \\ 2 & 2 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} 2 & -2 \\ 1 & 2 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \boldsymbol{k}$ $\boldsymbol{c} = 6\boldsymbol{i} - 6\boldsymbol{j} + 3\boldsymbol{k}$ ⇨ Unit vector $= \frac{6\boldsymbol{i}-6\boldsymbol{j}+3\boldsymbol{k}}{\sqrt{6^2+(-6)^2+3^2}} = \frac{1}{3}(2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k})$		
		05	
5	 $m = \frac{\lvert 4(4)-3(m)\rvert}{\sqrt{4^2+(-3)^2}}$ $5m = 16 - 3m$ $8m = 16 \qquad \therefore m = 2$ Radius, $r = 2$ and centre, C$(4,2)$ **Equation:** $(x-4)^2 + (y-2)^2 = 2^2$ $x^2 - 8x + 4 + y^2 - 4y + 4 = 4$ $\therefore x^2 + y^2 - 8x - 4y + 4 = 0$		
		05	
6	$x(2+i) + y(2-i) = 7i - 2$ $2x + xi + 2y - yi = 7i - 2$ $2x + 2y + (x-y)i = 7i - 2$ Equating the components; Real; $2x + 2y = -2; x + y = -1$ ……………(i) Imaginary; $x - y = 7$ ……………(ii) $(i) + (ii); 2x = 6 \qquad \therefore x = 3$		

	From (i); $3+y=-1$ $\therefore y=-4$ ⇨ $x+iy=3-4i$ $\lvert 3-4i\rvert=\sqrt{3^2+(-4)^2}=5$ units		
		05	
7	Let $y=x\sin x$ $y+\delta y=(x+\delta x)\sin(x+\delta x)$ $y+\delta y=(x+\delta x)(\sin x\cos\delta x+\cos x\sin\delta x)$ For small angles in radians; $\cos\delta x\approx 1, \sin\delta x\approx\delta x$ $y+\delta y=(x+\delta x)(\sin x+\delta x\cos x)$ $y+\delta y=x\sin x+x\delta x\cos x+\delta x\sin x+(\delta x)^2\cos x$ $\delta y=x\delta x\cos x+\delta x\sin x+(\delta x)^2\cos x$ $\frac{\delta y}{\delta x}=x\cos x+\sin x+\delta x\cos x$ As $\delta x\to 0$; $\frac{\delta y}{\delta x}\to\frac{dy}{dx}$ $\therefore \frac{dy}{dx}=x\cos x+\sin x$		
		05	
8	At $(4,3)$ $3=5t-7$ $5t=10$ $\therefore t=2$ $x=t^2, y=5t-7$ $\frac{dx}{dt}=2t$; $\frac{dy}{dt}=5$ $\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}$ $\frac{dy}{dx}=5\times\frac{1}{2t}$ But $t=2$; $\frac{dy}{dx}=\frac{5}{4}$ Eqn: $\frac{y-3}{x-4}=\frac{5}{4}$ $4y-12=5x-20$		

	$\therefore 5x - 4y - 8 = 0$		
		05	
9	(a) At the point of intersection; $\begin{pmatrix}6\\-1\\2\end{pmatrix} + \lambda\begin{pmatrix}3\\-2\\-1\end{pmatrix} = \begin{pmatrix}7\\3\\-3\end{pmatrix} + \mu\begin{pmatrix}2\\1\\-3\end{pmatrix}$ $6 + 3\lambda = 7 + 2\mu$ $3\lambda - 2\mu = 1$ ……………………………….(i) $-1 - 2\lambda = 3 + \mu$ $2\lambda + \mu = -4$ …………………….(ii) $2 - \lambda = -3 - 3\mu$ $-\lambda + 3\mu = -5$ ………………….(iii) Solving (i) and (ii) simultaneously $\mu = -2, \lambda = -1$ $x = 6 - 3 = 3, y = -1 + 2 = 1, z = 2 + 1 = 3$ $(3, 1, 3)$ is the point of intersection (b) Let $\boldsymbol{d}_1 = \begin{pmatrix}3\\-2\\-1\end{pmatrix}$ and $\boldsymbol{d}_2 = \begin{pmatrix}2\\1\\-3\end{pmatrix}$ $\boldsymbol{d}_1 \cdot \boldsymbol{d}_1 = \lvert\boldsymbol{d}_1\rvert\lvert\boldsymbol{d}_2\rvert \cos\theta$ $\begin{pmatrix}3\\-2\\-1\end{pmatrix} \cdot \begin{pmatrix}2\\1\\-3\end{pmatrix} = \sqrt{3^2 + (-2)^2 + (-1)^2}\,\sqrt{2^2 + 1^2 + (-3)^2}\cos\theta$ $6 - 2 + 3 = \sqrt{14}\,\sqrt{14}\cos\theta$ $7 = 14\cos\theta$ $\cos\theta = \frac{1}{2}$ $\theta = \cos^{-1}\left(\frac{1}{2}\right) = 60^0$ (c) $\boldsymbol{n} = \boldsymbol{d}_1 \Lambda \boldsymbol{d}_2$		

	$\boldsymbol{n} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ 3 & -2 & -1 \\ 2 & 1 & -3 \end{vmatrix}$ $\boldsymbol{n} = \begin{vmatrix} -2 & -1 \\ 1 & 3 \end{vmatrix}\boldsymbol{i} - \begin{vmatrix} 3 & -1 \\ 2 & 3 \end{vmatrix}\boldsymbol{j} + \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix}\boldsymbol{k}$ $\boldsymbol{n} = -5\boldsymbol{i} - 11\boldsymbol{j} + 7\boldsymbol{k}$ Using $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -5 \\ -11 \\ 7 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -5 \\ -11 \\ 7 \end{pmatrix}$ $-5x - 11y + 7z = -15 - 11 + 21$ $\therefore 5x + 11y - 7z = 2$		
		12	
10	(a) Let α and $\alpha + 3$ be the roots Sum of roots; $\alpha + \alpha + 3 = -p$ $2\alpha + 3 = p$ $\therefore \alpha = \frac{-p-3}{2}$ Product of roots; $\alpha(\alpha + 3) = p + 9$ $\left(\frac{-p-3}{2}\right)\left(\frac{-p-3}{2} + 3\right) = p + 9$ $-\left(\frac{3+p}{2}\right)\left(\frac{3-p}{2}\right) = p + 9$ $-\left(\frac{9-p^2}{4}\right) = p + 9$ $-9 + p^2 = 4p + 36$ $p^2 - 4p - 45 = 0$ $(p + 5)(p - 9) = 0$ $p = -5$ or $p = 9$ (b) $p(x) = (x - 2)^2 Q(x) + ax + b$ When $x = 2, p(2) = 8 - 12 + 4 - 5 = -5$ $p(2) = 2a + b = -5$		

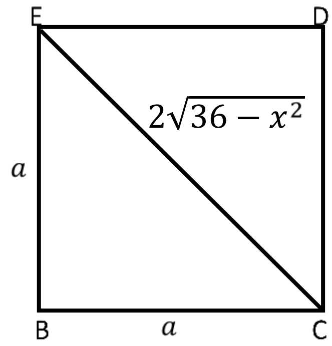
	$2a + b = -5$ ....(i) $p'(x) = 2(x-2)Q(x) + (x-2)^2Q'(x) + a$ $p'(x) = 3x^2 - 6x + 2$ When $x = 2, p'(2) = 12 - 12 + 2 = 2$ $p'(2) = a = 2 \qquad \therefore a = 2$ From (i); $4 + b = -5 \qquad \therefore b = -9$ Hence the remainder is $2x - 9$		
		12	
11	(a) $\cos(\theta + 60^0) = \sin\theta$ $\cos\theta\cos 60^0 - \sin\theta\sin 60^0 = \sin\theta$ $\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta = \sin\theta$ $\frac{1}{2}\cos\theta = \sin\theta\left(1 + \frac{\sqrt{3}}{2}\right)$ $\cos\theta = \left(2 + \sqrt{3}\right)\sin\theta$ $\tan\theta = \frac{1}{2+\sqrt{3}}$ $= \frac{2-\sqrt{3}}{2^2-\left(\sqrt{3}\right)^2}$ $= \frac{2-\sqrt{3}}{4-3}$ $= 2 - \sqrt{3}$ Hence ; $\theta = \tan^{-1}\left(2 - \sqrt{3}\right)$ $\theta = 15^0, 195^0$ (b) L.H.S $= sin^2A + sin^2B - sin^2C$ $= sin^2A + (\sin B + \sin C)(\sin B - \sin C)$ $= sin^2A + 2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right) \cdot 2\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)$ $= sin^2A + \sin(B+C)\sin(B-C)$ $B + C = 180^0 - A$		

	$\sin(B+C) = \sin(180^0 - A) = \sin A$ $= sin^2 A + \sin A \sin(B-C)$ $= \sin A\,[\sin A + \sin(B-C)]$ $= \sin A\,[\sin(B+C) + \sin(B-C)]$ $= \sin A \cdot 2\sin B \cos C$ $= 2\sin A \sin B \cos C$		
		12	
12	(a) Let $y = \tan x \Leftrightarrow \frac{dy}{dx} = sec^2 x$ Set $x = 45^0, y = \tan 45^0 = 1$ $y + \Delta y = \tan(x + \Delta x)$ $1 + \Delta y = \tan(45^0 + \Delta x)$ => $\Delta x = 1 = \frac{\pi}{180}$ rads $\Delta y \approx \frac{dy}{dx} \cdot \Delta x$ $\Delta y \approx (sec^2 x)\Delta x$ $\Delta y \approx sec^2 45^0 \times \frac{\pi}{180}$ $\Delta y \approx \frac{\pi}{90}$ $\therefore \tan 46^0 = 1 + \Delta y$ $\approx 1 + \frac{\pi}{90}$ ≈ 1.034906585 ≈ 1.0349 (4dps) (b) $\int_4^5 \frac{x^3}{x^2-9}\,dx$ $\begin{array}{r\|l} & x \\ \hline (x^2-9) & x^3 \\ - & x^3 - 9x \\ \hline & 9x \end{array}$ $\frac{x^3}{x^2-9} = x + \frac{9x}{x^2-9}$		

	$\Rightarrow \int_4^5 \frac{x^3}{x^2-9}\,dx = \int_4^5 x\,dx + 9\int_4^5 \frac{x}{x^2-9}\,dx$ $= \left.\frac{x^2}{2}\right]_4^5 + \left[\frac{9}{2}\ln(x^2-9)\right]_4^5$ $= \frac{1}{2}(25-16) + \frac{9}{2}[\ln 16 - \ln 7]$ $= 8.220053579$ ≈ 8.2201 (4dps)		
		12	
13	(a) Series $= (3+4)+(9+8)+(27+12)+\ldots$ $= (3+9+27+\cdots)+(4+8+12+\cdots)$ $= \frac{a(r^n-1)}{r-1} + \frac{1}{2}n[2a+(n-1)d]$ $= \frac{2(3^{20}-1)}{3-1} + \frac{1}{2}(20)[8+4\times 19]$ $= 3{,}486{,}785{,}240$ (b) $(1-4x)^{1/2} = 1 + \frac{1}{2}(-4x) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)(-4x)^2}{2!} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-4x)^3}{3!} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{4}\right)(-4x)^4}{4!} + \cdots$ $(1-4x)^{1/2} = 1 - 2x - 2x^2 - 4x^3 - 5x^4 + \cdots$ Range; $\lvert 4x\rvert < 1$ $\quad -\frac{1}{4} < x < \frac{1}{4}$ $\sqrt{15} = (16-1)^{1/2}$ $= 4\left(1-\frac{1}{16}\right)^{1/2}$ $\Rightarrow 4x = \frac{1}{16}$ $\quad \therefore x = \frac{1}{64}$ $\Rightarrow \sqrt{15} \approx 4\left[1 - 2\left(\frac{1}{64}\right) - 2\left(\frac{1}{64}\right)^2 - 4\left(\frac{1}{64}\right)^3 - 5\left(\frac{1}{64}\right)^4\right]$ ≈ 3.872984648 ≈ 3.8730 (4dps)		
		12	
14	(a) Volume, $v = \frac{1}{3}\times$base are $\times$height		

Height $= (6 + x)$

$CP = \sqrt{36 - x^2}, CE = 2\sqrt{36 - x^2}$



From the diagram;

$\overline{BC}^2 + \overline{BE}^2 = \overline{EC}^2$

$a^2 + a^2 = \left(2\sqrt{36 - x^2}\right)^2$

$2a^2 = 4(36 - x^2)$

$a^2 = 2(36 - x^2)$

$a = \sqrt{2}\,(36 - x^2)^{1/2}$

$v = \frac{1}{3} \times \sqrt{2}\,(36 - x^2)^{1/2} \times \sqrt{2}\,(36 - x^2)^{1/2} \times (6 + x)$

$v = \frac{2}{3}(36 - x^2)(6 + x)$

$v = \frac{2}{3}(6 - x)(6 + x)(6 + x)$

$\therefore v = \frac{2}{3}(6 + x)^2(6 - x)$

(b) $v = \frac{2}{3}[-x^3 - 6x^2 + 36x + 216]$

$\frac{dv}{dx} = \frac{2}{3}[-3x^2 - 12x + 36]$

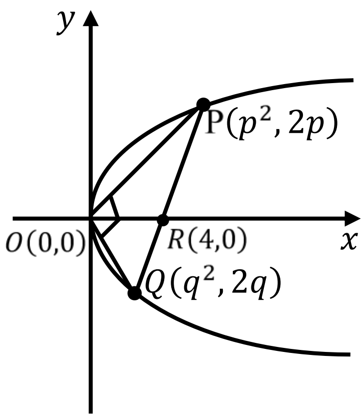
$= 2(-x^2 - 4x + 12)$

For maximum volume, $\frac{dv}{dx} = 0$

$-x^2 - 4x + 12 = 0$

$x^2 + 4x - 12 = 0$

$(x - 2)(x + 6) = 0$

	$x = 2$ or $x = -6$ ⇨ Volume, $v = \frac{2}{3}(8)^2 \times 4 = 170\frac{2}{3}\ cm^3$ or $\frac{512}{3}\ cm^3$ or $170.6667\ cm^3$		
		12	
15	(a)  Gradient of PQ $= \frac{2p-2q}{p^2-q^2}$ $= \frac{2(p-q)}{(p-q)(p+q)}$ $= \frac{2}{p+q}$ Equation; $\frac{y-2p}{x-p^2} = \frac{2}{p+q}$ $(p+q)y - 2p(p+q) = 2x - 2p^2$ $(p+q)y - 2p^2 - 2pq = 2x - 2p^2$ $(p+q)y - 2pq = 2x$ $\therefore 2x - (p+q)y + 2pq = 0$ (b) At R(4,0); $2(8) + 2pq = 0$ $2pq = -8$ $\quad\quad \therefore pq = -4$ (i) For PQ to make an angle of 90^0 at O(0,0), $pq = -4$		

	Gradient of OP $= \frac{2p}{p^2} = \frac{2}{p}$ Gradient of OQ $= \frac{2q}{q^2} = \frac{2}{q}$ $M_{OP} \times M_{OQ} = -1$ $\frac{2}{p} \times \frac{2}{q} = -1$ $\therefore pq = -4$ (ii) Mid-point of PQ; $x = \frac{p^2+q^2}{2}, y = \frac{2p+2q}{2} = p + q$ $2x = (p+q)^2 - 2pq$ $2x = y^2 - 2(-4)$ $\therefore y^2 = 2(x-4)$		
		12	
16	(a) $\frac{dx}{dt} \propto (80 - x)$ $\frac{dx}{dt} = -k(80 - x)$ $\int \frac{dx}{80-x} = \int -k\, dt$ $-\ln(80 - x) = -kt + c$ When $t = 0, x = 0$ $-\ln 80 = c$ $-\ln(80 - x) = -kt - \ln 80$ $\ln 80 - \ln(80 - x) = -kt$ $\ln\left(\frac{80}{80-x}\right) = -kt$ When $t = 1, x = 40$ $\ln\left(\frac{80}{80-40}\right) = -k$ $\qquad\qquad \therefore k = -\ln 2$ $\therefore \ln\left(\frac{80}{80-x}\right) = t \ln 2$ (b) (i) When $t = 2, x = ?$		

	$\ln\left(\frac{80}{80-x}\right) = 2\ln 2$ $\frac{80}{80-x} = 4$ $320 - 4x = 80$ $4x = 240$ $\therefore x = 60$ elephants (ii) when $x = 75, t = ?$ $\ln\left(\frac{80}{80-75}\right) = t\ln 2$ $\ln 16 = \ln 2^t$ $2^t = 2^4$ $\therefore t = 4$ months Number of elephants killed per day $= \frac{75}{120} = 0.625 \approx 1$ Meaning that on some days they would not kill an elephant.		
		12	