THE CHAIN RULE AND RATES OF CHANGE

Math Up differentiation seminar

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(ax+b)^n \neq n(ax+b)^{n-1}$$

THE CHAIN RULE

Suppose y is a function of t and t itself is a function of x. If Δy, Δt and Δx are small increments in the variables y, t and x, then;

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta x} \frac{\Delta t}{\Delta t}$$

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta t} \cdot \frac{\Delta t}{\Delta x}$$

As $\Delta x \to 0$, $\frac{\Delta y}{\Delta x} \to \frac{dy}{dx}$, $\frac{\Delta y}{\Delta t} \to \frac{dy}{dt}$, $\frac{\Delta t}{\Delta x} \to \frac{dt}{dx}$

$$\boxed{\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}}$$

The chain rule can be extended to as many variables as possibles.

$$\frac{dv}{dx} = \frac{dv}{ds} \cdot \frac{ds}{dn} \cdot \frac{dn}{dx}$$

EXAMPLE 1

- Differentiate $(2x+3)^5$ w.r.t x

$$\frac{d}{dx}\{(2x+3)^5\} = 5(2x+3)^4 \cdot 2$$
$$= 10(2x+3)^4$$

Soln

Let $y = (2x+3)^5$

Let $t = 2x+3$

$y = t^5$

$\frac{dy}{dt} = 5t^4$

$t = 2x+3$

$\frac{dt}{dx} = 2$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$
$$= 5t^4 \cdot 2$$
$$= 10t^4$$
$$\frac{dy}{dx} = 10(2x+3)^4$$
$$\frac{d}{dx}\{(2x+3)^5\} = 10(2x+3)^4$$

EXAMPLE 2

- Find the derivative of $\frac{1}{\sqrt{3x^2+2}}$ w.r.t x

Let $y = \dfrac{1}{\sqrt{3x^2+2}}$

Let $t = 3x^2+2$

$y = \dfrac{1}{\sqrt{t}}$

$y = t^{-\frac{1}{2}}$

$\dfrac{dy}{dt} = -\dfrac{1}{2} \times t^{-\frac{3}{2}}$

$= \dfrac{-1}{2\sqrt{t^3}}$

$t = 3x^2+2$

$\dfrac{dt}{dx} = 6x$

$\dfrac{dy}{dx} = \dfrac{dy}{dt} \cdot \dfrac{dt}{dx}$

$= \dfrac{-1}{2\sqrt{t^3}} \cdot 6x$

$= \dfrac{-3x}{\sqrt{(3x^2+2)^3}}$

$\dfrac{d}{dx}\left\{(3x^2+2)^{-\frac{1}{2}}\right\} = -\dfrac{1}{2}(3x^2+2)^{-3/2} \cdot 6x$

$= -3x\,(3x^2+2)^{-3/2}$

$= \dfrac{-3x}{\sqrt{(3x^2+2)^3}}$

RATES OF CHANGE

- Rates of change in volume, area, height etc can be obtained by using chain rule. For example:
- *If **v** denotes volume and **h** denotes height of a certain shape, the rate of change in the volume of that shape is* $\frac{dv}{dt}$
- *By chain rule;*

$$\boxed{\frac{dv}{dt} = \frac{dv}{dh} \cdot \frac{dh}{dt}}$$

$$\frac{dV}{dt} = \frac{dV}{dA} \cdot \frac{dA}{dh} \cdot \frac{dh}{dt}$$

EXAMPLE 1

- A rectangle is twice as long as it is broad. Find the rate of change of its perimeter when the breadth of the rectangle is $1m$ and its area is changing at the rate $18cm^2s^{-1}$ assuming the expansion is uniform.

Soln

$\frac{dA}{dt} = 18cm^2s^{-1}$

Let the breadth be w, l be length.

$l = 2w$

$P = 2(l + w)$

$P = 2(2w + w)$

$P = 6w$

$\frac{dP}{dw} = 6$

$A = 2w \times w$

$A = 2w^2$

$\frac{dA}{dw} = 4w$

$\frac{dP}{dt} = \frac{dP}{dw} \cdot \frac{dw}{dA} \cdot \frac{dA}{dt}$

$\frac{dP}{dt} = \cancel{6}^{3} \times \frac{1}{\cancel{4}_{2}w} \cdot \cancel{18}^{9}$

$\frac{dP}{dt} = \frac{27}{w}$

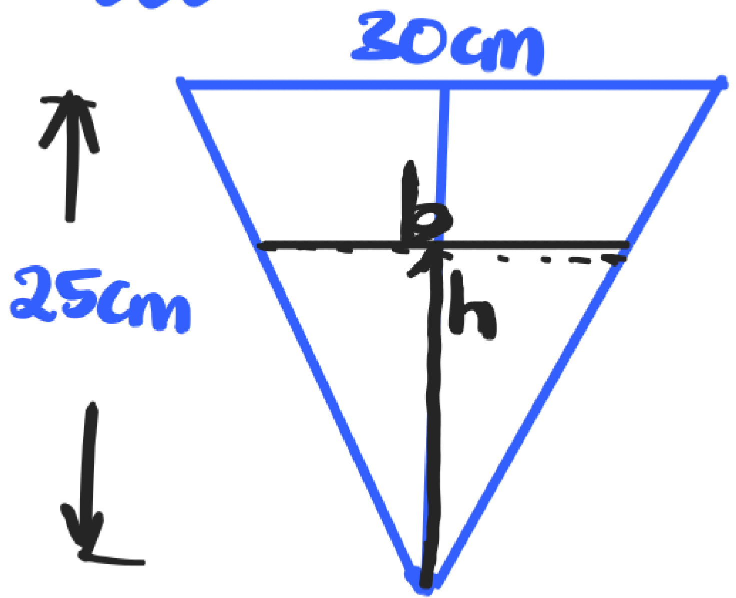
$\left.\frac{dP}{dt}\right|_{w=100} = \frac{27}{100}$

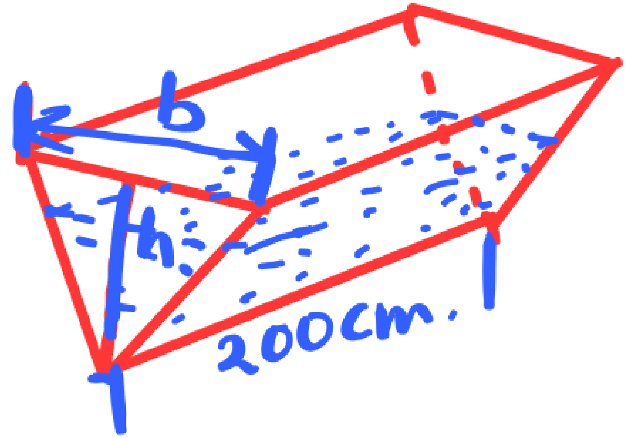
$= 0.27cms^{-1}$

EXAMPLE 2

- A horse trough has triangular cross-section of height 25 cm and base 30 cm and is 2 m long. A horse is drinking steadily and when the water level is 5 cm below the top, it is being lowered at a rate of 1 $cm \quad min^{-1}$. Find the rate of consumption in litres per minute.

$$\frac{dh}{dt} = 1\,cm\,min^{-1}$$





$$\frac{b}{h} = \frac{30}{25}$$

$$b = \frac{30h}{25} = \frac{6h}{5}$$

$$V = \frac{1}{2} b \times h \times L$$

$$= \frac{1}{2} \times \frac{6h}{5} \times h \times 200$$

$$V = 120h^2$$

$$\frac{dV}{dh} = 240h$$

$$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$$

$$= 240h \cdot 1$$

$$\frac{dV}{dt} = 240h$$

$$h = 25 - 5 = 20$$

$$\left.\frac{dV}{dt}\right|_{h=20} = 240 \times 20$$

$$= 4800 \text{ cm}^3 \text{ min}^{-1}$$

$$\left.\frac{dV}{dt}\right|_{h=20} = 4.8 \text{ litres min}^{-1}$$

EXAMPLE 3

- Water is poured into a vessel in the shape of a right circular cone of vertical angle 60°, with the axis vertical, at a rate of $8\ m^3 s^{-1}$. At what rate is the water surface rising when the depth of water is $4\ m$.

$\frac{dV}{dt} = 8m^3 s^{-1}$

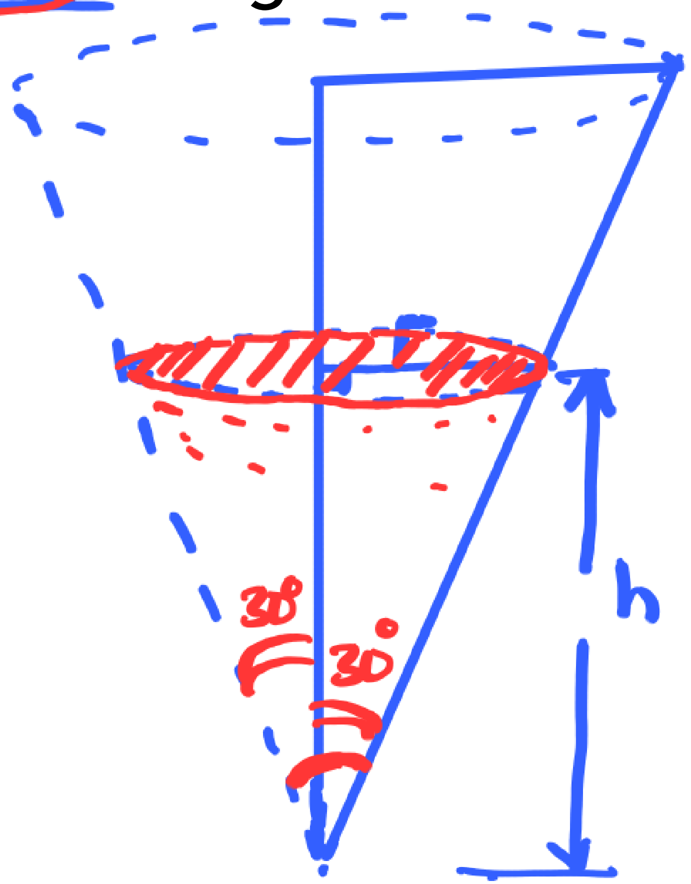
$\frac{dA}{dt}$

$\tan 30° = \frac{r}{h}$

$\frac{1}{\sqrt{3}} = \frac{r}{h}$

$r = \frac{h}{\sqrt{3}}$

$r = \frac{h\sqrt{3}}{3}$



$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi \left(\frac{h}{3}\sqrt{3}\right)^2 \cdot h$$

$$V = \frac{1}{9}\pi h^3$$

$$\frac{dV}{dh} = \frac{1}{3}\pi h^2$$

$$A = \pi r^2$$

$$A = \pi \times \left(\frac{h}{3}\sqrt{3}\right)^2$$

$$A = \pi \frac{h^2}{9} \times 3$$

$$A = \frac{\pi h^2}{3}$$

$$\frac{dA}{dh} = \frac{2\pi h}{3}$$

$$\frac{dA}{dt} = \frac{dA}{dh} \cdot \frac{dh}{dV} \cdot \frac{dV}{dt}$$

$$= \frac{2\pi h}{\cancel{3}} \cdot \frac{\cancel{3}}{\pi h^2} \cdot 8$$

$$= \frac{16}{h}$$

$$\left.\frac{dA}{dt}\right|_{h=4} = \frac{16}{4}$$

$$= 4\,\text{m}^2\,\text{s}^{-1}$$