P425/2
MATHEMATICS
Paper 1
April 1996
3 hours

1. Solve $3(3^{2x}) + 2(3^x) - 1 = 0$

Solution:

$$(3^{2x}) + 2\,(3x) - 1 = 0$$

We make 3^x a factor

$$3\,(3^x)^2 + 2\,(3^x) - 1 = 0$$

Let $3^x = p$

So $3p^2 + 2p - 1 = 0$

Solving for p

$$3p^2 + 3p - p - 1 = 0$$

$$3p\,(p + 1) - 1(p + 1) = 0$$

$$(p + 1)(3p - 1) = 0$$

$p + 1 = 0$ OR $3p - 1 = 0$

$p = -1$ OR $p = 1/3 = 3^{-1}$

If $p = -1$ If $p = 1/3$

$3^x = -1$ $3^x = 3^{-1}$

Has no solution $\therefore x = -1.$

Therefore the value of x is **-1.**

2. Express as equivalent fraction with a rational denominator

$$\frac{\sqrt{2}}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$$

Solution:

$$\frac{\sqrt{2}}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$$

Rationalize the denominator,

$$\frac{\sqrt{2}\left[\left(\sqrt{2}+\sqrt{3}\right)+\sqrt{5}\right]}{\left[\left(\sqrt{2}+\sqrt{3}\right)-\sqrt{5}\right]\left[\left(\sqrt{2}+\sqrt{3}\right)+\sqrt{5}\right]}$$

This simplifies to:

$$\frac{2\sqrt{6}+6+\sqrt{60}}{12}$$

$$= \frac{2\sqrt{6}+6+\sqrt{4\times 15}}{12}$$

$$= \frac{2\sqrt{6}+6+2\sqrt{15}}{12}$$

$$= \frac{\sqrt{6}+3+\sqrt{15}}{6}$$

3.Solve the inequality

$$\frac{x-1}{x-2}>\frac{x-2}{x+3}$$

Solution:

. We solve the inequality

$$\frac{x-1}{x-2} > \frac{x-2}{x+3}$$

$$\frac{(x-1)}{(x-2)}-\frac{(x-2)}{(x+3)} > 0$$

$$\frac{(x-1)(x+3)-(x-2)^2}{(x-2)(x+3)} > 0$$

$$\frac{[x^2+2x-3]-[x^2-4x+4]}{(x-2)(x+3)} > 0$$

$$\frac{6x-7}{(x-2)(x+3)} > 0$$

Let $f(x) = \dfrac{6x-7}{(x-2)(x+3)}$

We check for f (x) > 0

Use the inequality table below:

	x<-3	-3<x<7/6	7/6<x< 2	X > 2
(6x – 7)	-	-	+	+
(x- 2)	-	-	-	+
(x + 3)	-	+	+	+
(x-2)(x+3)	+	-	-	+
f(x)	-	+	-	+

Hence the solution for

$\dfrac{x-1}{x-2} > \dfrac{x-2}{x+3}$ is :

-3 < x < 7/6 or x > 2 # .

4. Find how many terms of the series

1$+\dfrac{1}{5}+\dfrac{1}{5^2}+\dfrac{1}{5^3}+---$ **must be taken so that the sum will differ from the sum to infinity by less than** 10^{-6}**.**

Solution:

We have the series.

$$1+\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+\ldots\ldots\ldots$$

This is a geometrical progression its sum to infinity is given by:

$$S_\infty = \frac{a}{1-r}$$

Where $a = 1$

and $r = 1/5$

$$\therefore S_\infty = \frac{1}{1-\frac{1}{5}} = \frac{5}{4}$$

If n is the number of terms whose sum will differ from S_∞ by less than 10^{-6}, then

$$S_n = a\frac{(1-r^n)}{1-r}$$

$$= \frac{\left[1-\left(\frac{1}{5}\right)^n\right]}{1-\frac{1}{5}}$$

$$= \frac{5\left(1-\frac{1}{5^n}\right)}{4}$$

Now

$$S_\infty > S_n$$

Such that

$$S_\infty - S_n < 10^{-6}$$

$$\frac{5}{4}-\frac{5}{4}\left(1-\frac{1}{5^n}\right) < 10^{-6}$$

$$5[1-\left(1-\frac{1}{5^n}\right)] < 4\times10^{-6}$$

$$5\left(\frac{1}{5^n}\right) \quad < \quad 4\times10^{-6}$$

$$5^{(1-n)} \quad < \quad 4\times10^{-6}$$

Taking Logarithms to the base of 10.

$$\text{Log}_{10}5^{(1-n)} \quad < \quad \log_{10}(4\times10^{-6})$$

$$(1-n)\log_{10}5 \quad < \quad (\log_{10}4 + \log_{10}10^{-6})$$

This simplifies to:

$$n \quad > \quad \frac{6 + \log_{10}5 - \log_{10}4}{\log_{10}5}$$

$$n \quad > \quad 8.722$$

So $\mathbf{n = 9}$.

5. Solve the simultaneous equations

2x - 5y + 2z = 14

9x +3y - 4z = 13

7x +3y - 2z = 3

Solution:

$2x - 5y + 2z = 14$ .... (1)

$9x + 3y - 4z = 13$ .... (2)

$7x + 3y - 2z = 3$ .... (3)

2 Eqn (1) + Eqn (2)

$$\begin{array}{l} 4x - 10y + 4z = 28 \\ \underline{9x + 3y - 4z \quad = 13} \\ 13x - 7y \qquad = 41 \quad \text{...........(4)} \end{array}$$

Eqn (1) + Eqn (3)

$$\begin{array}{l} 2x - 5y + 2z = 14 \\ \underline{7x + 3y - 2z = 14} \\ 9x - 2y \qquad = 17 \quad \text{........(5)} \end{array}$$

2Eqn (4) – 7 Eqn (5)

$$26x - 14y) \quad = \quad 82$$

$$63x - 14y \quad = \quad 119$$

$$-37x = -37$$

$$x = \frac{-37}{-37} = 1$$

Substituting for x in Eqn (4)

$$13 - 7y = 41$$
$$-7y = 28$$
$$y = -4$$

Substituting for x and y in Eqn (1)

$$2 + 20 + 2z = 14$$
$$2z = 14 - 22$$
$$2z = -8$$
$$z = -4$$

$\therefore$ x = 1 y = -4 and z = -4

6. Find the orthocenter (the point of intersection of the altitudes)of the triangle with vertices at A (-2,1), B(3,4)
and C(-6,-1).

Solution:

The orthocenter O, is the circumcenter of the circle that passes through the vertices A, B and C of triangle ABC with center (-g, -f).

By using the general equation of the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$
$$-4g + 2f + c = -5 \quad \ldots\ldots\ldots \quad (1)$$

Substituting for (3, - 4), coordinates of B,

$$9 + 16 + 6g - 8f + c = 0$$
$$6g - 8f + c = -25 \ldots\ldots\ldots \quad (2)$$

Substituting for (-6, -1), coordinates of C,

$$36 + 1 - 12g - 2f + c = 0$$
$$-12g - 2f + c = -37 \quad \ldots\ldots \ (3)$$

Eqn (1) – Eqn (2)

$$-10g + 10f = 20$$
$$-g + f = 2 \quad \ldots\ldots (4)$$

Eqn (1) – Eqn (2)

$$8g + 4f = 32$$
$$2g + f = 8 \quad \ldots\ldots.(5)$$

Eqn (4) – Eqn (5)

$$-3g = -6$$
$$g = 6/3 = 2$$

Substituting for g in Eqn (4)

$$-2 + f = 2$$
$$f = 4$$

Hence the orthocenter is O (-2, 4)

7.. Differentiate with respect to x, expressing your results as simply as possible.

$$\sin^{-1}\left[\frac{3+5\cos x}{5+3\cos x}\right]$$

Solution:

We differentiate

$$\sin^{-1}\left[\frac{3+5\cos x}{5+3\cos x}\right]$$

Let $\sin^{-1}\left[\frac{3+5\cos x}{5+3\cos x}\right] = y$

$$\Rightarrow \sin y = \frac{3+5\cos x}{5+3\cos y} \quad (1)$$

Differentiating

$$\cos y\frac{dy}{dx} = \frac{(5+3\cos x)\cdot -5\sin x - (3+5\cos x)\cdot -3^{\sin x}}{25+30\cos x+9\cos^2 x.}$$

$$\cos y \frac{dy}{dx} = \frac{-16\sin x}{25 + 30\cos x + 9\cos^2 x} \text{......(2)}$$

From (1)

$$\sin^2 y = \left(\frac{3 + 5\cos x}{5 + 3\cos x}\right)^2$$

$$1 - \cos^2 y = \left(\frac{3 + 5\cos x}{5 + 3\cos x}\right)^2$$

$$\cos y = \frac{\sqrt{(5 + 3\cos x)^2 - (3 + 5\cos x)^2}}{(5 + 3\cos x)^2}$$

$$= \frac{1}{(5 + 3\cos x)}\sqrt{16 - 16\cos^2 x}$$

$$= \frac{1}{(5 + 3\cos x)}\sqrt[4]{[1 - \cos^2 x}$$

$$= \frac{4}{5 + 3\cos x}\sqrt{\sin^2 x}$$

$$\cos y = \frac{4\sin x}{5 + 3\cos x}$$

$$\therefore \frac{4\sin x}{5 + 3\cos x} \frac{dy}{dx} = \frac{-16\sin x}{(5 + 3\cos x)^2}$$

$$\frac{dy}{dx} = \frac{-16\sin x}{(5 + 3\cos x)^2} \times \frac{(5 + 3\cos x)}{4sx.}$$

$$\therefore \frac{dy}{dx} = \frac{-4}{5 + 3\cos x}$$

8. Evaluate

$$\int_0^{\frac{1}{2}\pi} \sin 2x \cos x \, \mathbf{dx}$$

Solution:

We evaluate

$$\int_0^{\pi/2} \sin 2x \cos x \, dx$$

By factor formula

$$\sin P + \sin Q = 2 \sin \frac{(P+Q)}{2} \cos \frac{(P-Q)}{2}$$

Compare with $\sin 2x \cos x$

Such that

$$\frac{P+Q}{2} = 2x$$

$$\Rightarrow \quad P + Q = 4x \quad \ldots\ldots\ldots\ldots(1)$$

and

$$\frac{P-Q}{2} = x$$

$$\Rightarrow \quad P - Q = 2x \quad \ldots\ldots\ldots(2)$$

Adding (1) and (2)

$$2P = 6x$$

$$\therefore \quad P = 3x$$

Also $Q = 4x - P$

$$= 4x - 3x = x$$

Hence $\sin 3x + \sin x = 2\sin 2x \cos x$

Or $\sin 2x \cos x = \frac{1}{2}\sin 3x + \frac{1}{2}\sin x$

So $\int_0^{\pi/2} \sin 2x \cos x \, dx = \frac{1}{2}\int_0^{\pi/2} \sin 3x + \frac{1}{2}\int_0^{\pi/2} \sin x \, dx$

$$= \frac{1}{2}\left[\frac{-1}{3}\cos 3x\right]_0^{\pi/2} + \frac{1}{2}\left[-\cos x\right]_0^{\pi/2}$$

$$= \quad \frac{1}{6}[\cos 3x]_{\pi/2}^{0} + \frac{1}{2}[\cos x]_{\pi/2}^{2}$$

$$= \quad \frac{1}{6}\left[\cos - \cos\frac{3\pi}{2}\right] + \frac{1}{2}\left[\cos o - \cos\frac{\pi}{2}\right]$$

$$= \quad \frac{1}{6}\ [1 - 0] + \frac{1}{2}\ [1 - 0]$$

$$= \quad \frac{1}{6} + \frac{1}{2} \quad = \quad \frac{1+3}{6} \quad = \frac{2}{3}$$

$$\therefore \int_0^{\pi/2} \mathbf{sin2xcosxdx} \quad = \quad \mathbf{2/3} \quad \#$$

SECTION B

9.(a) Find x if $\log_x 8 - \log_{x^2} 16 = 1$

(b)The sum of p terms of an arithmetic progression is q , and the sum of q terms is p , find the sum of p + q terms.

Solution:

(a) We find x from.

$\log_x 8 - \log_{x^2} 16$

We know: $\log_a b \quad = \quad \dfrac{\log_c b}{\log_c a}$

So $_{\text{we}}$ have

$$\frac{\log_x 8}{\log_x x} - \frac{\log_x 16}{\log_x x^2} \quad = \quad 1$$

$$\frac{\log_x 8}{1} - \frac{\log_x 16}{2\log_x x} \quad = \quad 1$$

$$\log_x 8 - \frac{\log_x 16}{2} = 1$$

$$2\log_x 8^2 \;. - \log_x 16 = 2$$

$$\log_x\left(\frac{64}{16}\right) = 2$$

$$\log_x 4 = 2$$

$$x^2 = 4$$

$$\mathbf{x = 2} \qquad \#$$

(b)The sum of n terms of an arithmetic progression S_n is

$$S_n = \frac{n}{2}\{2a + (n-1)d\ )$$

Let α = 1^{st} term of the A.P and

d = common difference

Now sum to n terms of the A.P is given by

$$S_n = \frac{n}{2}(2\alpha + (n-1)d)$$

$$\Rightarrow \quad S_p = \frac{p}{2}(2\alpha + (p-1)\ d)$$

But $S_p = q$

$$\Rightarrow \frac{p}{2}[2\alpha + (p-1)\ d] = q$$

$$2\alpha + (p-1)\ d = \frac{2q}{p} \ldots\ldots\ldots\ldots(1)$$

Also $\quad S_q = \frac{q}{2}(2\alpha + (q-1)d)$

But $\quad S_q = P$

$$\Rightarrow \frac{q}{2}[2\alpha + (q-1)d = \frac{2p}{q} \ldots\ldots\ldots\ldots(2)$$

Eqn(1) – Eqn (2)

$$(p-q)d = \frac{2q}{P} - \frac{2p}{q}$$

$$= \frac{2q^2 - 2p^2}{pq}$$

$$d = \frac{2q^2 - 2p^2}{pq(p-q)}$$

$$= \frac{2(p^2 - q^2)}{pq(p-q)}$$

$$= \frac{2(p-q)(p+q)}{pq(p-q)}$$

$$= \frac{2(p+q)}{pq}$$

Substituting for d in Eqn (2)

$$2\alpha + (p-1)d = \frac{2q}{P}$$

$$2\alpha + (p-1)\left[\frac{-2(p+q)}{pq}\right] = \frac{2q}{P}$$

$$2\alpha = \frac{2q}{P} + \frac{2(p-q)(p+q)}{pq}$$

$$\alpha = \frac{q^2 + (p-q)(p+q)}{pq}$$

Sum of (p + q) terms is given as

$$S_{(p+q)} = \frac{(p+q)}{2}[2\alpha + (p+q-1)d]$$

$$= \frac{(p+q)}{2}\left[2\left(\frac{q^2 + (p-1)(p+q)}{pq}\right) + (p+q-1)\left(\frac{-2(p+q}{pq}\right)\right]$$

$$= \left(\frac{p+q}{2}\right)$$

$$\left[2\left(\frac{(q^2 + p^2 + pq - p - q)}{pq}\right) - 2\left(\frac{(q^2 + pq + pq + q^2 - p - q)}{pq}\right)\right]$$

$$= \left(\frac{p+q}{2}\right)\left[\frac{-2pq}{pq}\right]$$

$= \quad -(p + q)$

Hence the sum of (p + q) terms = -(p + q)

10. (a) Given that z = $\sqrt{3} + i$,find the modulus and argument of

(i) z^2

(ii) $\frac{1}{z}$

(iii) Show in an Argand diagram the points representing complex numbers z, z^2 and $\frac{1}{z}$

(b) In an Argand diagram, p represents a complex number z such that

$$2|z - 2| = |z - 6i|$$

Show that p lies on a circle,

Find (i) the radius of this circle

(ii) The complex number represented by its centre.

Solution:

10(a) We have

$$2\,|z - 2| = |z - 6i|$$

Let $z = x + i y$.

$$2\left|(x + I y) - 2\right| = \left|(x + i y) - 6 i\right|$$

$$2\left|(x - 2) + i y\right| = \left|x + i(y - 6)\right|$$

$$2\sqrt{(x-2)^2 + y^2} = \sqrt{x^2 + (y-6)^2}$$

Squaring both sides

$$4\left[(x-2)^2 + y^2\right] = x^2 + (y-6)^2$$

$$4\left[x^2 - 4x + 4 + y^2\right] = x^2 + y^2 - 12y + 36$$

$$4x^2 - 16x + 16 + 4y^2 - y^2 - x^2 + 12y - 36 = 0$$

$$3x^2 + 3y^2 - 16x + 12y = 20$$

Dividing by 3

$$x^2 - \frac{16}{3}x + y^2 + 4y = \frac{20}{3}$$

$$\left(x - \frac{8}{3}\right)^2 - \frac{64}{9} + (y+2)^2 - 4 = \frac{20}{3}$$

$$\left(x - \frac{8}{3}\right)^2 + (y + 2)^2 = \frac{69}{9} + \frac{4}{1} + \frac{20}{3}$$

$$= \frac{64 + 36 + 60}{9}$$

$$= \frac{160}{9}$$

$$\therefore \left(x - \frac{8}{3}\right)^2 + (y + 2)^2 = \frac{160}{9}$$

This equation is of the form

$(x - a)^2 + (y - b)^2 = r^2$

Where (a , 3) and r represent the centre of the circle respectively

(i) the radius of the circle

$$= \sqrt{\frac{160}{9}} = 4.229 \text{ (2 dec . places)}$$

(ii) the center is $\frac{8}{3}$ - 2 i # .

(10) (a) We are given that

$$Z = \sqrt{3} + i$$

(i) Now modulus of z^2 is

$$|z^2| = |z|^2$$

$$= \left|\sqrt{3} + i\right|^2$$

$$= \left(\sqrt{\left(\sqrt{3}\right)^2 + 1^2}\right)^2$$

$$= \left(\sqrt{3+1}\right)^2 = \left(\sqrt{4}\right)^2 = 2^2$$

$$\therefore \quad |z^2| = 4 \quad \#$$

Argument of z^2 is

$$\operatorname{Arg} z^2 = 2 \arg z$$

$$\arg z = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \pi/6$$

$$\therefore \arg z^2 = 2 \arg z$$

$$= 2 \times \pi/6 = \pi/3 \ \#$$

(ii) Modulus of 1/z is :

$$\left|\frac{1}{z}\right| = \frac{1}{|z|}$$

$$= \frac{1}{\left|\sqrt{3} + i\right|} = \frac{1}{\sqrt{\left(\sqrt{3}\right)^2 + 1^2}}$$

$$= \frac{1}{\sqrt{4}} = ½$$

$$\therefore \left|\frac{1}{z}\right| = ½ \quad \#$$

$$\text{Arg}\left(\frac{1}{z}\right) = \arg(z^{-1})$$

$$= -\arg z$$

$$\text{But } \arg z = \frac{11}{6}$$

$$\therefore \text{Arg}\left(\frac{1}{z}\right) = \frac{11}{6} \ \# .$$

(iii) We represent on an Argand diagram

the points representing z , z^2 and $\frac{1}{z}$.

$$z = \sqrt{3} + i$$

$$z^2 = \left(\sqrt{3}+i\right)^2 = \left(\sqrt{3}+i\right)\left(\sqrt{3}+i\right)$$

$$= 3 + \sqrt{3}i + \sqrt{3}i + i^2$$

$$= 2 + 2\sqrt{3}\,i$$

$$\frac{1}{z}. = \frac{1}{\left(\sqrt{3}+i\right)\left(\sqrt{3}-i\right)}\left(\sqrt{3}-i\right)$$

$$= \frac{1}{\sqrt{3}+i}$$

11. (a) Find the equation of the circle circumscribing the triangle whose vertices are A(1,3),B(4,-5) and C(9,-1). Find also its centre and radius.

(b) If the tangent, to this circle at A(1,3) meets the x -axis at P(h, o) and the y-axis at Q (0,k) , find the values of h and k.

Solution:

(11)(a) we find the equation of the circle circumscribing the triangle whose vertices are A(1,3) , B(4,-5) and C(9,-1)

Suppose that the circle has the equation

$$x^2 + y^2 + 2gx + 2fy + C = 0$$

Since A,B,C Lie on the circle the coordinates of the three points satisfy the equation

A(1,3) $\Rightarrow 1 + 9 + 2g + 6f + c = 0$

$$2g + 6f + c = -10 \ldots\ldots(1)$$

B(4,-5) $\Rightarrow 16 + 25 + 8g - 10f + c = 0$

$$8g - 10f + c = -41 \ldots(2)$$

C(9,-1) $\Rightarrow 81 + 1 + 18g - 2f + c = 0$

$$18g - 2f + c = -82 \ldots(3)$$

From equation (1) and (2)

$$6g - 16f = -31 \ldots.(4)$$

From equations (2) and (3)

$$10g + 8f = -41 \ldots.(5)$$

$$\Rightarrow 20g + 16f = -81 \ldots.x$$

Solving equation (4) and x

$$26g = -113$$

$$g = \frac{-113}{26}$$

Now, $10g + 8f = -41$

$$\Rightarrow 10x - \frac{113}{26} + 8f = -41$$

$$\Rightarrow 8f = -41 + \frac{1130}{26} = \frac{64}{26}$$

$$f = \frac{8}{26}$$

$$8g - 10f + c = -41$$

$$8\times\left(\frac{-113}{26}\right) - 10\times\frac{8}{26} + c = -41$$

$$c = \frac{904}{26} + \frac{80}{26} - 41$$

$$= \frac{82}{26}$$

$$\therefore \quad c = \frac{-41}{13}$$

The equation of the circle is:

$$x^2 + y^2 + 2x\left(\frac{-113}{26}\right)x + 2\times\frac{8}{26}y + \left(\frac{-41}{13}\right) = 0$$

The equation is

$$26x^2 + 26y^2 - 226x + 16y - 82 = 0$$

Or $13x^2 + 13y^2 - 113x + 8y - 41 = 0$

Conventionally, the center of the circle is at (-g, $^-$ f) and the radius is $\sqrt{g^2 + f^2 - C}$

Hence the center of our circle is

$$\left(\frac{113}{26}, -\frac{8}{26}\right) \text{ or } \left(\frac{113}{26}, \frac{-4}{13}\right)$$

$$\text{radius} = \sqrt{\left(\frac{113}{26}\right)^2 + \left(\frac{4}{13}\right)^2 - \left(-\frac{41}{13}\right)}$$

$$= 4.7 \text{ units}$$

(b) The gradient S_1 of the lines PA is

$$\frac{3-0}{1-h} = \frac{3}{1-h} = S_1$$

The gradient S_2 of the line QA is

$$\frac{3-k}{1-0} = \frac{3-k}{1} = S_2$$

Note that $S_1 = S_2$

$$\Rightarrow \frac{3}{1-h} = \frac{3-k}{1}$$

Using the equation of the circle, we find the gradient at A(1,3)

$$\Rightarrow 13x^2 + 13y^2 - 113x + 8y - 41 = 0$$

$$26x + 26y\frac{dy}{dx} - 113 + 8\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}[26y + 8] = 113 - 26x$$

$$\frac{dy}{dx} = \frac{113-26x}{26y+8}$$

$$= \frac{113-26\times 1}{26\times 3+8}$$

$$= \frac{87}{86}$$

$$\frac{3}{1-h} = \frac{3-k}{1} = \frac{87}{86}$$

From which $3-k = \frac{87}{86}$

$$\Rightarrow k = \frac{117}{86}$$

And $\frac{3}{1-h} = \frac{87}{86}$

$$\Rightarrow 258 = 87 - 87h$$

$$87h = {}^{-}171$$

$$\therefore \qquad h \qquad = \qquad \frac{^{-}171}{87}$$

$$\text{Hence} \qquad h \qquad = \qquad \frac{^{-}171}{87}$$

$$\text{And} \qquad k \qquad = \qquad \frac{117}{86} \qquad \#$$

12.(a) Given that 7 tanθ + cot θ = sec θ, derive a quadratic equation for sinθ. Hence, or otherwise, find all values of θin the interval

$0^0 \leq \theta \leq 180^0$ which satisfy the given equation, giving your answer to the nearest 0.1^0, where necessary.

(b)The acute angles A and B are such that

$\cos A = \frac{1}{2}, \quad \sin B = \frac{1}{3}.$

Show, without the use of tables or a calculator, that

$$\tan(A+B) \quad = \quad \frac{9\sqrt{3}+8\sqrt{3}}{5}$$

Solution:

12(a) We have 7 tan θ + cot θ = 5 sec θ

Multiply through by sinθcosθ

$$\sin\theta\cos\theta\left\{7\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right\} = \frac{5}{\cos\theta}\times\sin\theta\cos\theta$$

$$7\sin^2\theta\cos^2 \quad = \quad 5\sin\theta$$

$$7\sin^2\theta + (1\sin^2\theta) \quad = \quad 5\sin\theta$$

$$6\sin^2\theta - 5\sin\theta \quad = \quad 0$$

Is a quadratic equation for sin θ

Solving

$$6\sin^2\theta - 3\sin\theta - 2\sin\theta + 1 = 0$$

$$3\sin\theta(2\sin\theta - 1) - 1(2\sin\theta - 1) = 0$$

$$(2\sin\theta - 1)(3\sin\theta - 1) = 0$$

$2\sin\theta - 1 = 0$ or $3\sin\theta - 1 = 0$

$\sin\theta = \frac{1}{2}$ or $\sin\theta = \frac{1}{3}$

$\theta = 30^0, 150^0$ $\theta = 19.5^0, 160.5^0$

(b) $\cos A = \frac{1}{2}$ $\sin B = \frac{1}{3}$

We show that

$$\tan(A+B) = \frac{9\sqrt{3} + 8\sqrt{3}}{5}$$

Write as

$$\frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$$

$$= \frac{\sin A\cos B + \frac{1}{2}\times\frac{1}{3}}{\frac{1}{2}\cos B - \frac{1}{3}\sin A}$$

But $\sin A = \sqrt{1-\cos^2 A}$

$$= \sqrt{1-\frac{1}{4}}$$

$$= \frac{\sqrt{3}}{2}$$

And $\cos B = \sqrt{1-\sin^2 B}$

$$= \sqrt{1-\frac{1}{9}}$$

$$= \frac{\sqrt{8}}{3} = \frac{2\sqrt{2}}{3}$$

We then have

$$\frac{\sin(A+B)}{\cos(A+B)} = \frac{\sqrt{\frac{3}{2}}\times\frac{2\sqrt{2}}{3}+\frac{1}{6}}{\frac{1}{2}\times\frac{2\sqrt{2}}{3}-\frac{1}{3}\times\sqrt{\frac{3}{2}}}$$

$$= \frac{\left(2\sqrt{6}+1\right)/6}{\left(2\sqrt{2}-\sqrt{3}\right)/6}$$

$$= \frac{\left(\sqrt{6}+1\right)\left(2\sqrt{2}+\sqrt{3}\right)}{\left(2\sqrt{2}-\sqrt{3}\right)\left(2\sqrt{2}+\sqrt{3}\right)}$$

$$= \frac{4\sqrt{12}+2\sqrt{18}+2\sqrt{2}+\sqrt{3}}{8-3}$$

$$= \frac{8\sqrt{3}+6\sqrt{2}+2\sqrt{2}+\sqrt{3}}{5}$$

$$= \frac{9\sqrt{3}+8\sqrt{2}}{5}$$

13. (a) Prove that

$(\sin 2\theta \quad \sin\theta)(1+2\cos\theta) = \sin 3\theta$.

(b) A vertical pole BAO stands with its base O on a horizontal plane, where BA = c and AO = b, A point P is situated on the horizontal plane at distance x from O and the angle APB = θ.

(i) Prove that

$$\tan\theta = \frac{cx}{x^2+b^2+bc}$$

(ii) As p takes different positions on the horizontal plane, find the value of x for which θ is greatest.

Solution:

(13)(a) We prove that

$(\sin 2\theta - \sin\theta)(1+2\cos\theta) = \sin 3\theta$

L.H.S = $\sin 2\theta + 2\sin 2\theta\cos\theta - \sin\theta - 2\sin\theta\cos\theta$

$= \sin 2\theta + 2\sin 2\theta\cos\theta - \sin\theta - \sin 2\theta$

$= (2\sin 2\theta\cos\theta) - \sin\theta$

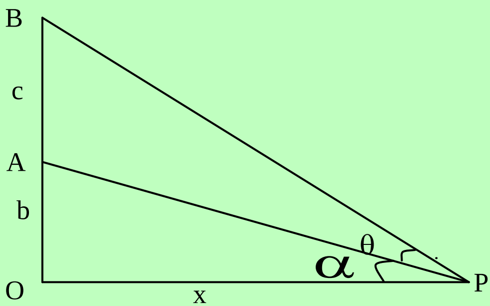
[By factor formula $2\sin 2\theta\cos\theta = \sin 3\theta + \sin\theta$]

$= (\sin 3\theta + \sin\theta) - \sin\theta$

$= \sin 3\theta + \sin\theta - \sin\theta$

$= \sin 3\theta$ = R.H.S

(b)



Let angle OPA $= \alpha$

Now $\tan(\theta + \alpha) = \dfrac{OB}{OP}$

$= \dfrac{b + C}{x}$

Expanding

$$\frac{\tan\theta + \tan\alpha}{1 - \tan\theta\tan\alpha} = \frac{b + c}{x}$$

$x\tan\theta \quad x\tan\alpha \quad = (b + c) - (b + c)\tan\theta\tan\alpha$

$x\tan\theta \quad (b + c)\tan\theta\tan\alpha = (b + c) - x\tan\alpha$

$\tan\theta[x + (b + c)\tan\alpha] = (b + c) - x\tan\alpha$

$$\tan\theta = \frac{(b+c) - x\tan\alpha}{x + (b+c)\tan\alpha}$$

But $\tan\alpha = \dfrac{b}{x}$

[From diagram]

$$\Rightarrow \quad \tan\theta = \frac{(b+c) - x \times \frac{b}{x}}{x + (b+c)\times\frac{b}{x}}$$

$$= \frac{b + c - b}{\left[x^2 + b(b+c)\right]/x}$$

$$\therefore \quad \tan\theta = \frac{cx}{x^2 + b^2 + bc} \qquad \#$$

As required.

14. A curve is given by

$$y = \frac{(x-1)(x-9)}{(x+1)(x+9)}$$

(i) Determine the turning points of the curve.
(ii) Determine the equations of the asymptotes of the curve.
(iii) Sketch the curve
Solution:

14. (i) $y = \dfrac{(x-1)(x-9)}{(x+1)(x+9)} = \dfrac{x^2 - 10x + 9}{x^2 + 10x + 9}$

$$\frac{dy}{dx} = \frac{(x^2+10x+9)(2x-10)-(x^2-10+9)(2x+10)}{(x^2+10x+9)^2}$$

But at turning points, $\frac{dy}{dx} = 0$

$\Rightarrow (2x - 10)(x^2 + 10x + 9) - (2x + 10)(x^2 - 10x + 9) = 0$

By opening brackets and simplifying, we have $20x^2 - 180 = 0$

$$x^2 = \frac{180}{20}$$

$$x = \pm\sqrt{9}$$

$$= \pm 3$$

When x = 3, y = $\frac{(3-1)(3-9)}{(-3+1)(-3+12)}$

$$= \frac{(2)(-12)}{-2\times 6} = -\frac{1}{4}$$

Hence (x, y) = $\left[-3, -\frac{1}{4}\right]$

When x = -3, y = $\frac{(-3-1)(-3-9)}{(-3+1)(-3+9)}$

$$= \frac{(-4)(-12)}{(-2)(6)} = -4$$

Hence (x, y) = (-3, -4)

Finding the nature of the turning points

$$\frac{dy}{dx} = \frac{20x^2-180}{(x^2+10x+9)^2}$$

$$\frac{d^2y}{dx^2}$$

$$= \frac{(x^2+10x+9)(40x)-(20x^2-180)[2(x^2+10x+9)(2x+10)]}{(x^2+10x+9)^3}$$

$$= \frac{40x(x^2+10x+9)-2(2x+10)(20x^2-180)}{(x^2+10x+9)^2}$$

At x = 3

$$\frac{d^2y}{dx^2} = \frac{120(9+30+9)-2(6+10)(180-180)}{(9+30+9)^3}$$

$$= \frac{120\times 48}{48^3}$$

$$= \frac{120}{48\times 48\times 48}$$

This is a positive value.

Hence the point (x, y) = $\left(3, -\frac{1}{4}\right)$ is a minimum point.

When x = -3,

$$\frac{d^2y}{dx^2} = \frac{-120(9+30+9)-2(-6+10)(180-180)}{(9+30+9)^3}$$

$$= \frac{-120\times -30}{-30^3}$$

$$= \frac{3600}{-30^3}$$

This is negative hence the point (-3, -4) is a maximum.

(ii) *Asymptotes*

For vertical asymptotes,

(x + 1) (x + 9) = 0

Either x + 1 = 0

x = -1

Or x + 9 = 0

x = -9

For horizontal asymptotes,

$$y = \frac{x^2 - 10x + 9}{x^2 + 10 \times 9}$$

Dividing numerator and denominator on the L.H.S, by x^2

$$y = \frac{1 - {}^{10}\!/\!_{x} + {}^{9}\!/\!_{x^2}}{1 + {}^{10}\!/\!_{x} + {}^{9}\!/\!_{x^2}}$$

As $x \to \pm\infty$, $y \to 1$

Since ${}^{10}\!/\!_{x} \to 0$ and ${}^{9}\!/\!_{x^2} \to 0$

Hence the horizontal asymptote is **y = 1**

(iii) *Finding the intercepts*

When x = 0, $y = {}^{9}\!/\!_{9} = 1$

Hence (x, y) = (0,1)

When y = 0, Then (x-1)(x-9) = 0

Either x – 1 = 0,
x = 1
Or x – 9 = 0
x = 9

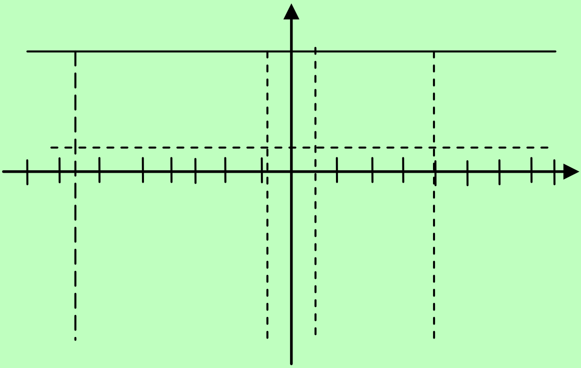
Hence (x, y) = (1, 0) and (9, 0)

Investigating the regions where the curve lies:

Since the factors in the equation are x – 1, x – 9, x + 1 and x + 9, the critical values of x are x = -9, -1, 1 and 9.

	x<9	-9<x<1	-1<x<1	1<x<9	x>9
x-1	-	-	-	+	+
x-9	-	-	-	-	+
x+1	-	-	+	+	+
x+9	-	+	+	+	+
$y=\frac{(x-1)(x-9)}{(x+1)(x+9)}$	+	-	+	-	+

Graph of $y = \dfrac{(x-1)(x-9)}{(x+1)(x+9)}$



15. Find the general solution of the equation

$$x\frac{dy}{dx} - 2y = (x-2)e^x.$$

(b) The rate of cooling of a body is given by the equation

$$\frac{dT}{dt} = -k(T-10)$$

where T is the temperature in degrees centigrade, k is a constant , and t is the time in minutes.

When t = 0, T = 90 and when t = 5, T = 60.

Find T when t = 10.

Solution:

15(a) We find the general solution of

$$x\frac{dy}{dx} - 2y = (x-2)\,e^x$$

Dividing through by x

$$\frac{dy}{dx}-\frac{2}{x}y = \left(\frac{x-2}{x}\right)e^{x}$$

This is of form $\frac{dy}{dx} + py = Q$

Thus requiring us to find an integrating factor to make it exact.

Use: $\quad I.F = e^{\int p\,dx}$

Thus $\quad I.F = e^{\int 2/x\,dx} = e^{-2\log_e x}$

$$= e^{\log_e x^{-2}}$$

$$I.F = x^{-2}$$

$$\therefore x^{-2}\frac{dy}{dx} - \frac{2x^{-2}}{x}y = x^{-2}\left(\frac{x-2}{x}\right)e^{x}$$

$$\frac{d}{dx}\left\{x^{-2}y\right\} = \left(\frac{x-2}{x^{3}}\right)e^{x}$$

$$\frac{y}{x^{2}} = \int\left(\frac{x-2}{x^{3}}\right)e^{x}\,dx.$$

$$= \int x^{-2}e^{x}\,dx - 2\int x^{-3}e^{x}\,dx.$$

Note: You can see that there exists a special relationship between

$$\left\{x^{-2}e^{x} \quad \text{and} \quad -2x^{-3}e^{x}\right\}$$

i.e. $\frac{d}{dx}\left\{x^{-2}e^{x}\right\} = x^{-2}e^{x} + e^{x} - 2x^{-3}$

$= x^{-2}e^{x} - 2x^{-3}e^{x}$

By method of inspection, therefore, we may prudently ascertain that

$\int x^{-2}e^{x}dx - 2\int x^{-3}e^{x}\,dx$ is the same as

$\int\left(x^{-3}e^{x} - 2x^{-3}e^{x}\right)dx = x^{-2}e^{x} + c.$

Thus $\frac{y}{x^{2}} = x^{-2}e^{x} + c$

Or $y = e^{x} + cx^{2}$ #

is the general solution.

(b) Rate of cooling is given by

$$\frac{dT}{dt} = -k(T-10)$$

T - Temperature in ^{o}C

K - Constant

and T – time (in minutes)

we are given

when $t = 0$, $T = 90$ and

when $t = 5$, $T = 60$

Asked to find T when $t = 10$.

Separating variables and integrating

$$\int\frac{dT}{T-10} = -\int k\,dt$$

$$\log_{e}(T-10) = -Kt + A$$

A is constant.

To find A, use $t = 0$, $T = 90$

$$\Rightarrow \log e(90-10). \quad = \quad -k\times 0 + A$$

$$\therefore \quad A \quad = \quad \log_e 80$$

Hence $\log_e (T-10) \quad = \quad kt + \log_e 80$

Or $\log_e \left(\frac{T-10}{80}\right) \quad = \quad kt$

$$\frac{T-10}{80} \quad = \quad e^{-kt}$$

$$T - 10 \quad = \quad 80e^{-kt}$$

$$\therefore \quad T \quad = \quad 10 + 80e^{-kt.}$$

To find k, use $t = 5$, $T = 60$

$$\Rightarrow 60 \quad = \quad 10 + 80e^{5k}$$

$$80e^{-5k} = \quad 50$$

$$e^{-5k} \quad = \quad \frac{5}{8}$$

$$\Rightarrow \quad \left(e^{-k}\right)^5 \quad = \quad \frac{5}{8}$$

from which k can be got.

When $t = 10$, use the general equation

$$T \quad = \quad 10 + 80e^{-kt.}$$

$$= \quad 10 + 80e^{-k\times 10} \qquad (t=10)$$

$$= \quad 10 + 80e^{-10k}$$

$$T \quad = \quad 10 + 80\left(e^{-5k}\right)^2$$

But $e^{-5k} \quad = \quad \dfrac{5}{8}$

$\therefore \quad T \quad = \quad 10 + 80\left(\frac{5}{8}\right)^2$

$\quad\quad\quad = \quad 41.25$

Hence **T** **=** $\mathbf{41.25^0 C}$ when t = 10.

16. (a) In the triangle ABC p is the point on BC such that Bp : pc = $\lambda : \mu$.
Show that $(\lambda+\mu)$ AP = λAC + μAB.

(b) The non -collinear points A, B and C have position vectors a, b and c respectively with respect to an origin O. The point M on AC is such that AM: MC = 2:1 and the point N on AB is such that AN: NB = 2:1

(i) Show that $\mathbf{BM} = \frac{1}{3}\mathbf{a} - \mathbf{b} + \frac{2}{3}\mathbf{c}$

and find a similar expression for **CN**.

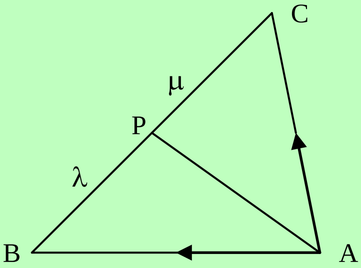
(ii) The lines BM and CN intersect at L .

Given that **BL** = r**BM** and **CL** = s**CN**, where r and s are scalars, express BL and CL in terms of r, s, a, b and c.

(iii) Hence, by using triangle BLC, or otherwise, find r and s.

Solution

16.



$$AP = AB + BP$$

$$\overline{AP} = \overline{AB} + \frac{\lambda}{\mu+\lambda}\overline{BC}$$

But $\overline{BC} = \overline{BA} + \overline{AC}$

$$\Rightarrow \quad \overline{AP} = \overline{AB} - \frac{\lambda}{\mu+\lambda}(\overline{BA} + \overline{AC})$$

$$\overline{AP} = \overline{AB} + \overline{BA}\left(\frac{\lambda}{\mu+\lambda}\right) + \overline{AC}\left(\frac{\lambda}{\mu+\lambda}\right)$$

$$\overline{AP} = \overline{AB} - \left(\frac{\lambda}{\mu+\lambda}\right)\overline{AB} + \overline{AC}\left(\frac{\lambda}{\mu+\lambda}\right)$$

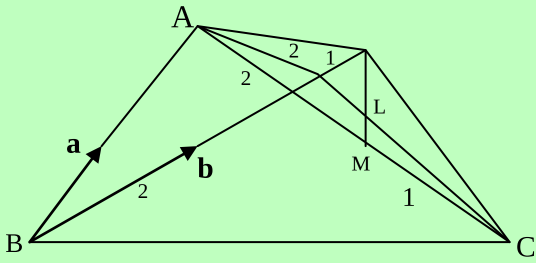
$$\overline{AP} = \left(1 - \frac{\lambda}{\mu+\lambda}\right)\overline{AB} + \left(\frac{\lambda}{\mu+\lambda}\right)\overline{AC}$$

$$(\mu+\lambda)\overline{AP} = (\lambda + \mu - \lambda)\overline{AB} + \lambda\overline{AC}$$

$$(\mu+\lambda)\overline{AP} = \mu\overline{AB} + \lambda\overline{AC}$$

Hence $(\lambda + \mu)\overline{AP} = \lambda\overline{AC} + \mu\overline{AB}$

(b) (i)



$$\overline{BM} = \overline{BA} + \overline{AM}$$

$$= \overline{BO} + \overline{OA} + \frac{3}{B}\overline{AC}$$

$$= \mathbf{b} + \mathbf{a} + \frac{2}{3}(\overline{AO} + \overline{OC})$$

$$= \mathbf{-b} + \frac{2}{3}\mathbf{a} + \frac{2}{3}\mathbf{c}$$

$$= \frac{1}{3}\mathbf{a} \mathbf{- b} + \frac{2}{3}\mathbf{c}$$

$$\overline{CN} = \overline{CA} + \overline{AN}$$

$$= \mathbf{-c} + \mathbf{a} + \frac{2}{3}\overline{AB}$$

$$= \mathbf{-c} + \mathbf{a} + \frac{2}{3}(b\mathbf{-}a)$$

$$= \mathbf{-c} + \mathbf{a} + \frac{2}{3}\mathbf{b} \mathbf{-} \frac{2}{3}\mathbf{a}$$

$$= \frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{b} - \mathbf{c}$$

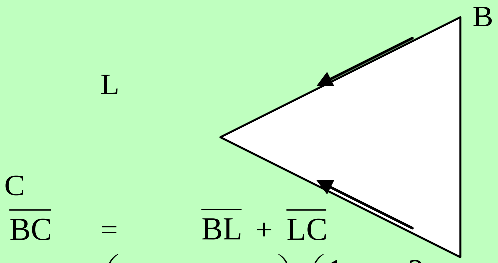
(ii) $\overline{BL} = r\,\overline{BM}$

$$= r\left[\frac{a}{3} - b + \frac{2c}{3}\right]$$

$$= \frac{r}{3}\mathbf{a} - r\mathbf{b} + \frac{2r}{3}\mathbf{c}$$

$$\overline{CL} = s\left(\frac{1}{3}a + \frac{2}{3}b - c\right)$$

$$= \frac{1}{3}sa + \frac{2}{3}sb - sc$$



$$\overline{BC} = \overline{BL} + \overline{LC}$$

$$= \left(\frac{ra}{3} - rb + 2rc\right)\left(\frac{1}{3}sa + \frac{2}{3}sb - sc\right)$$

$$= \left(\frac{r}{3} - \frac{1}{3}s\right)\mathbf{a} + \left(-r - \frac{2}{3s}\right)\mathbf{b} + \left(\frac{2r}{3} + s\right)\mathbf{c}$$

Also $\overline{BC} = \overline{BO} + \overline{OC}$

$= -\mathbf{b} + \mathbf{c}$

$= (0)\,\mathbf{a} + -\mathbf{b} + \mathbf{c}$

By equating the corresponding unit vectors;

For a,

$$\frac{r}{3} - \frac{1}{3}s = 0$$

$$r = s$$

For b,

$$-r - \frac{2}{3}s = -1$$

$$-3r - 2s = -1 \;\ldots\ldots\ldots\ldots \text{(i)}$$

For c,

$$\frac{2r}{3} + s = 1$$

$$2r + 3s = 3 \;\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

2 Eqn (i) + 3 Eqn (ii)

$$-6r - 4s = -6$$

$$6r + 9s = 9$$

$$5s = 3$$

$$s = \frac{3}{5}$$

Since r = s

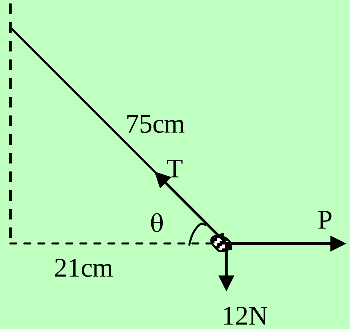
$$\Rightarrow r = {}^{3}\!/_{5}$$

Hence $r = {}^{3}\!/_{5}$ and $s = {}^{3}\!/_{5}$

P425/2
MATHEMATICS
Paper 2
April 1996
3Hours
SECTION A.

1. One end of a light inextensible string of length 75cm is fixed to a point on a vertical pole. A particle of weight 12N is attached to the other end of the string. The particle is held 21cm away from the pole by horizontal force. Find the magnitude of this force and the tension in the string.

Solution:



Let P = the magnitude of the force

θ = angle between the horizontal and the string

T = tension in the string

The particle is in equilibrium.

Resolving vertically;

$$T \sin \theta = 12 \qquad (1)$$

$$T \times \frac{24}{25} = 12$$

$$\therefore \quad T = 12.5 \text{ N.}$$

Resolving horizontally;

$$P = T \cos \theta$$

$$= 12.5 \times \frac{7}{25} = 3.5\text{N.}$$

Therefore the horizontal force is **3.5N** and the tension is **12.5N.**

2. A particle with position vector 40**i** +10**j** +20**k** moves with constant speed 5ms^{-1} in the direction of the vector 4**i** + 7**j** + 4**k**. Find its distance from the origin after 9 seconds.

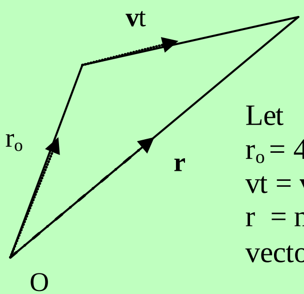
Solution:

Position vector of particle = 40**i** + 10**j**+ 20**k**

Speed of particle = 5m/s

Direction of motion = 4**i** +7**j**+4**k**.

Time taken = 9 seconds.



Let

$r_o = 40i+10j+20k$

vt = **v**×9

r = new position vector

We need $|r|$.

$$\text{Now } v = \left\{\frac{4i+7j+4k}{\sqrt{4^2+7^2+4^2}}\right\}\times 5$$

$$= \frac{20}{9}i + \frac{35}{9}j + \frac{20}{9}k$$

So $\mathbf{r} = \mathbf{r}_o + \mathbf{v}t$

$$= \begin{pmatrix} 40 \\ 10 \\ 20 \end{pmatrix} + 9\begin{pmatrix} 20/9 \\ 35/9 \\ 20/9 \end{pmatrix}$$

$$= \begin{pmatrix} 40 \\ 10 \\ 20 \end{pmatrix} + \begin{pmatrix} 20 \\ 35 \\ 20 \end{pmatrix}$$

$$= \begin{pmatrix} 60 \\ 45 \\ 40 \end{pmatrix}$$

$= 60i + 45j + 40k.$

Distance $= |r|$

$= \sqrt{60^2 + 45^2 + 40^2}$

$= 85$ m

Thus, the distance is **85 m.**

3. A cyclist travel 1.25km as he accelerates uniformly at a rate of Q ms^{-1} from a speed of $15km^{-1}$. Find the value of Q .

Solution:

Given S = 1.25km

= 1250m

u = $15kmh^{-1}$

= $25/6\ ms^{-1}$

Note: *It is very difficult to find Q unless either final velocity V or time t is given.*

Now taking an assumption that t = 12sec.

Then by using

$s = ut + \frac{1}{2}at^2$

$1250 = \frac{25}{5} \times 12 + \frac{1}{2} \times Q \times 144$

$1250 = 50 + 72Q$

$72Q = 1200$

$\mathbf{Q = 16.7ms^{-2}}$

4. In an experiment the following observations were recorded

T : 0 12 20 30

θ : 6.6 2.9 -0.1 -2.9

Use linear interpolation to find

(i) θ when T =16

(ii) T when θ = -1

Solution:

Using

	A	B	C
T	12	16	20
θ	2.91	θ	-0.1

The gradient of AB = gradient of AC

$$\frac{\theta - 2.9}{16 - 12} = \frac{0.1 - 2.9}{20 - 12}$$

$$\frac{\theta - 2.9}{4} = \frac{-3}{8}$$

$$8\theta = 11.2$$

$$\theta = 1.4$$

(ii) Also

	A	B	C
T	20	T	30

θ	-0.1	-1	-2.9

Gradient of AB = gradient of AC

$$\frac{-1-0.1}{T-20} = \frac{0.1-2.9}{20-12}$$

$$\frac{-0.9}{T-20} = \frac{-2.8}{10}$$

$$T-20 = \frac{0.9}{0.28}$$

$$T = 20 + \frac{0.9}{0.28}$$

= 23.21 (2dp)

5. A balanced coin is tossed three times and the number of times X a 'Head' appear is recorded. Complete the following table.

n	0	1	2	3
Event	(TTT)		HHT,HTT,THH	
P(X=n)	1/8			

Determine the average of the expected number of heads to appear.

Solution:

n	0	1	2	3
Event	TTT	HTT THT TTH	HHT THH HTH	HHH
P(X = n)	1/8	3/8	3/8	1/8

Expected number of heads to appear

= E(X)

$= \sum n\, P(X = n)$

$= (0 \times {}^{1}/_{8}) + (1\times 3/8) + (2 \times 3/8) + (3 \times 1/8)$

$= 0 + \frac{3}{8} + \frac{6}{8} + \frac{3}{8}$

$= \frac{12}{8} = \frac{3}{2} = 1.5$

E (X) = 1.5

Thus the expected number of heads to appear is 1.5

6. In a certain year in the mid-1980s the production of tea in the common wealth as per the following countries was as shown below.

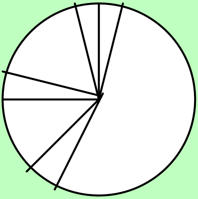
Country	production of tea in millions of kg
Bangladesh	41
India	635
Indonesia	108
Kenya	140
Malawi	40
Sri Lanka	212
Tanzania	17
Uganda	7

Give a pie chart representation of the data.

Solution:

We draw the pie chart for the information given.

Note: the data should be transformed into degrees where 360^0 = Total production



7. A bicycle dealer imports 40% and 60% of spare parts from countries A and B respectively. The percentages of parts produced defective in the countries are 0.3% and 0.5% respectively. A spare part is drawn at random from a sample of parts imported from A and B. Find the probability that
(i) it is defective and is from country B
(ii) it is defective.

Solution:

Defining the events

A – Imports from country A

B imports from country B

D imported spare parts (detective).

Given:

$$P(A) = \frac{40}{100} = \frac{4}{10}$$

$$P(B) = \frac{60}{100} = \frac{6}{10}$$

$$P(D/A) = \frac{0.3}{100}$$

$$= \frac{3}{1000}$$

$$P(D/B) = \frac{0.5}{100}$$

$$= \frac{5}{1000}$$

Asked

(i) $P(D \cap A) = P(D/A) \times P(A)$

$$= \frac{3}{1000} \times \frac{4}{10}$$

$$= \frac{12}{10,000}$$

$$= \frac{3}{2500}$$

(ii) $P(D) = P(D \cap A) + P(D \cap B)$

$$= \frac{3}{2500} + P(B) \times P(D/B)$$

$$= \frac{3}{2500} + \frac{6}{10} \times \frac{5}{1000}$$

$$= \frac{6+15}{5000} = \frac{21}{5000}$$

$\therefore P(D) = 0.0042 \neq$

8. A population consists of 15 numbers, 2,4,7,3,5,6,3,6,10,7,8,9,3,4,3

Find

(i) The mode

(ii) The median

(iii) The mean and standard deviation of the population.

Solution:

The population in order is:

2,3,3,3,3) ,4,4) ,5(6,6,(7,7,8,9,10.

(i) The mode = 3

(ii) The median = 5

(iii)Let y represent the population.

y	2	3	3	3	3	4	4	5
y^2	4	9	9	9	9	16	16	25

6	6	7	7	8	9	10
36	36	49	49	64	81	50

$$\sum y = 80$$

$$\sum y^2. = 512$$

$$\text{Mean} = \frac{\sum y}{n} = \frac{80}{15}$$

$$= 5.33\bar{3}$$

$$\text{Standard deviation} = \sqrt{\frac{\sum y^2}{n} - -\left(\frac{\sum y}{n}\right)^2}$$

$$= \sqrt{\left(\frac{512}{15}\right) - \left(\frac{80}{15}\right)^2}$$

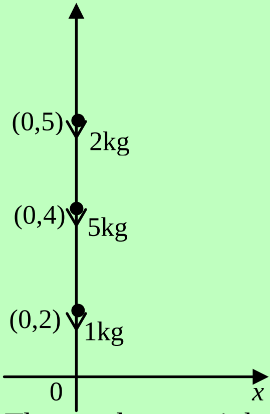
$$= 2.385 \quad \#$$

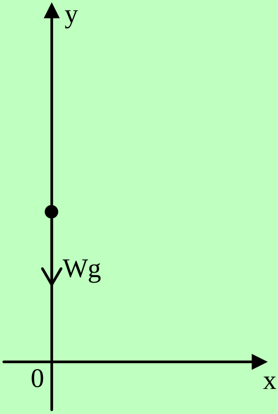
SECTION B.

9.(a) Find the position of the centre of gravity of three particles of masses 1kg,5kg and 2kg which lie on the y-axis at points (0,2), (0,4) and (0,5) respectively.

(b) The area enclosed by the curve $y = x^2$ and the lines y = 0, x = 2 and x = 4 lying in the first quadrant is rotated about the x-axis through one revolution. Find the coordinates of the centre of gravity of the uniform solid so formed.

Solution:





The resultant weight w will lie on the y – axis say pt $(o, \bar{y})$.

Vertically:

$$1g + 5g + 2g = W_g$$

$$\Rightarrow \quad W = 8kg$$

Taking moments about point O:

$$(g\times2) + (5g\times4) + (2g\times5) = Wg \times \bar{y}$$

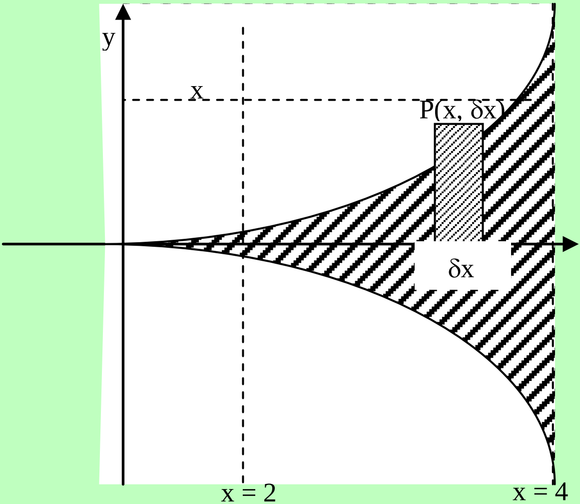
$$32g = Wg\,\bar{y}$$

$$\Rightarrow \quad \bar{y} = 32/W$$

$$= 32/8 = 4$$

$\therefore$ The position of the centre of gravity is (0, 4)

b)



Let the position of the C.g be ($\bar{x}$, o)

Element of volume = $\pi y^2 \delta x$

Element of mass = P. $\pi y^2 \delta x$ (volume × Density ρ)

Element of weight = P. πy^2. δx.g.(weight ×gravitational pull)

Taking moments of weight the strip about the y-axis = $\pi y^2 \delta x \rho g.x$

⇒ The moment of weight of the whole solute

$$= \quad \bar{x}\int_2^4 \pi y^2 P g \delta x$$

$$= \int_2^4 \pi y^2 Pgx\,dx$$

$$= \int_2^4 \pi \left(x^2\right)^2 xPg\,dx.$$

$$\Rightarrow \quad \bar{x}\,\pi\, {}^{p}\!/\!{}_{g} \int_2^4 x^4\,dx. = \pi\, {}^{p}\!/\!{}_{g} \int_2^4 x^5\,dx$$

$$\Rightarrow \quad \bar{x}\left[{}^{1}\!/\!{}_{5}\,x^5\right]_2^4 = {}^{1}\!/\!{}_{6}\,x^6 \Big|_2^4$$

$$\bar{x}\left[\frac{1024}{5} - \frac{32}{5}\right] = {}^{1}\!/\!{}_{6}\left[4096 - \underline{64}\right]$$

$$\Rightarrow \quad \bar{x}\left(\frac{992}{5}\right) = \frac{4032}{6}$$

$$\Rightarrow \quad \bar{x} = \frac{4032}{6} \times {}^{5}\!/\!{}_{992}$$

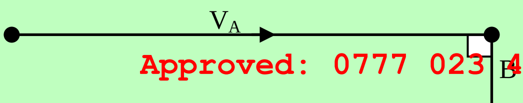
$$\bar{x} = \mathbf{3.387}$$

$\therefore$ The pt (93.4, 0) is the center of gravity.

10. **Initially two ships A and B are 65km apart with B due east of A .A is moving due east 10kmh^{-1} and B due south at 24kmh^{-1}.**
The two ships continue moving with these velocities. Find the least distance between the ships in the subsequent motion and the time taken to the nearest minute for such a situation to occur.

Solution:

Sketch:



Given AB = 65km

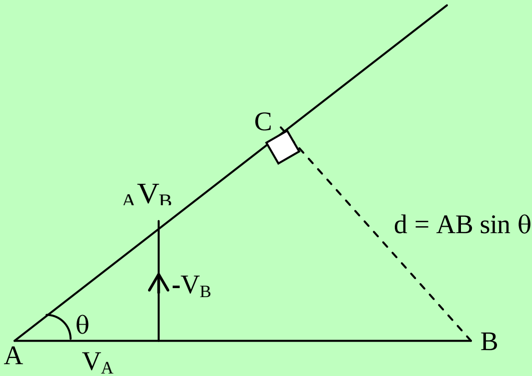
V_A = $10kmh^{-1}$ due east

V_B = $24kmh^{-1}$ due south.

Consider velocity of A relative to B ie $_AV_B$.

So $_AV_B$ = $V_A + (-V_B)$

i.e. we make B stationary so as to observe the motion of A.



The dotted line, d, represents the shortest distance (least) and must be perpendicular to

$_AV_B$!

Now $\tan\theta = \dfrac{|v_B|}{|v_A|} = \dfrac{24}{10}$

$\theta = 67.38^0$

Hence d = 65 sin 67.38^0 = 60km.
Therefore, the least distance = 60 km.
The time taken t is given by

$$\text{Time} = \frac{\text{Distance}}{\text{Speed}}$$

$$= \frac{AC}{|{}_AV_B|}$$

But AC = AB cos θ□ = 65cos 67.38^0

And ${}_AV_B$ = $\sqrt{(v_A)^2 + (v_B)^2}$

= $\sqrt{10^2 + 24^2}$

= 26

So, time = $\frac{65\cos 67.38^0}{26}$

= 0.9615 hours

= 57.69 minutes.

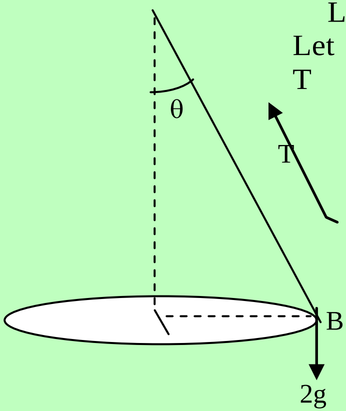
Therefore the time taken is **58 min** (to the nearest minute)

11. (a) A conical pendulum consists of a light inextensible string AB of length 50cm fixed at A and carrying a bomb of mass 2kg at B . The bomb describes a horizontal circle about the vertical through A with a constant angular speed of A radm^{-1}. Find the tension in the string.

(b) A smooth surface is inclined at 30^o to the horizontal. A body A of mass 2kg is held at rest on the surface by a light elastic string which has one end attached to A and the other to a point on the surface 1.5m away from A up a line of greatest slope. If the modulus of the string is 2g N , find it's natural length.

Solution:

(a)Resolving vertically;



Let = 0.5m

Let r = radius

T = Tension

$T\cos\theta = 2g$……… (1)

Using F = ma.

$$T\sin\theta = \frac{mr^2}{r} \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

Where v = linear velocity.

Dividing (2) by (1)

$$\tan\theta = \frac{mr^2}{r} \div \frac{2g}{1}$$

$$= \frac{mr^2}{2rg}$$

$$\tan\theta = \frac{2r^2}{2rg}$$

But $r \quad = \quad \omega r$

$$\tan\theta = \frac{(\omega r)^2}{rg} = \frac{\omega^2 r^2}{rg} = \frac{\omega^2 r}{g}$$

And $r \quad = \quad 0.5 \sin\theta$, $\omega = A$

$$\text{Tan}\theta = \frac{A^2 \times 0.5 \sin\theta}{g}$$

$$\frac{\sin\theta}{\cos\theta} = \frac{A^2 \times \sin\theta}{2g}$$

$$\cos\theta = \frac{2g}{A^2} \ldots\ldots\ldots\ldots\ldots(3)$$

Putting equation (3) in (1)

$$T\cos\theta = 2g$$

$$T \times \frac{2/g}{A^2} = 2/g$$

$$T \times \frac{\cancel{2/g}}{A^2} = \cancel{2/g}$$

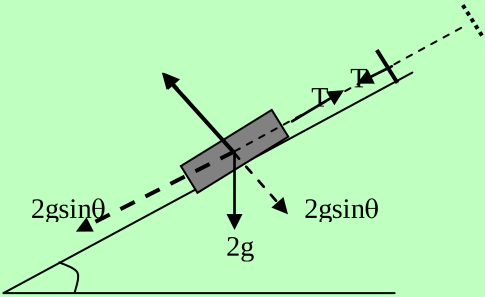
$$T = A^2$$

Therefore the tension in the string is A^2 #.

(b)Let T = Tension in string

R = Reaction (normal)

The body is in equilibrium.



Resolving parallel to the plane

$$T = 2g\sin 30^0 \ldots\ldots \quad (1)$$

Using hooks law

i.e. $T = \frac{xe}{\ell}$

Where e = extension in string

e = natural length

$$T = \frac{2g(1.5-\ell)}{\ell} \ldots.(2)$$

Equating equations (1) an d (2)

$$2g\frac{(1.5-\ell)}{\ell} = 2g\sin 30^0$$

$$\frac{1.5-\ell}{\ell} = \sin 30^0$$

$$\boldsymbol{l} = 1$$

Therefore the natural length of the string is 1m.

12. A block of mass 6.5kg is projected with a velocity of $4ms^{-1}$ up a line of greatest slope of a rough plane. Calculate the initial kinetic energy of the block.
The coefficient of friction between the block and the plane is $\frac{2}{3}$ and the plane makes an angle θ with the horizontal where sin θ = 5/13. The block travels a distance d m up the plane before coming instantaneously to rest. Express in terms of d.
(i) The potential energy gained by the block in coming to rest.
(ii) The work done against friction by the block in coming to rest.
Hence calculate the value of d
(Take gravity = $10ms^{-2}$)

Solution:

Let μ = coefficient of friction.
v = initial velocity of projection.
h = height of incline.

From energy considerations as the block moves from A to B

Loss in K.E = Gain in P.E + work done against friction

Or

$$\frac{1}{2}mv^2 = mgh + \mu R \times d.$$

(i)Gain in P.E = mgh

$$= 6.5xg \times d\sin\theta$$

$$= \quad 6.5 \times g \times d \times \frac{5}{13}$$

Gain in P.E $= \quad 2.5$ gd #

but $\quad g \quad = 9.8m/s^2$

(ii) Work done against friction = μR x d

$$= \quad \mu x\ (mg \cos\theta \text{ x } d)$$

Work done against friction $= \quad \frac{2}{3} \times 6.5 \times g \times \frac{12}{13} \times d$

$$= \quad 4gd$$

(i) To find the value of d.

We saw that

$$\frac{1}{2}mv^2 = mgh + \mu R \text{ x } d$$

$$\frac{1}{2} \times 6.5 \times 4^2 = 2.5gd + 4gd$$

$$6.5 \text{ x } 16 = 5gd + 8gd$$

$$13gd = 104$$

$$gd = 8$$

But $g = 10m/s^2$ (given)

$$d = \frac{8}{10} = 0.8$$

d = 0.8 m #

13. (i) Show that the iterative formula for solving the equation $2x^2 - 6x - 3 = 0$ is

$$X_{n+-1} = \frac{2x_n^2 + 3}{4x_n + 6}$$

(ii) Show that the positive root for

$2x^2 - 6x - 3 = 0$ lies between 3 and 4. Find the root correct to 2 decimal places.

Solution:

(13)(a) Given the equation

$$2x^2 - 6x - 3 = 0$$

Let $f(x) = 2x^2 - 6x - 3 = 0$

We can use Newton's Raphsons' Formula.

Since it is also all iterative formula

$$f^1(x) = 4x - 6$$

Now $X_{n+1} = x_n - \frac{f(x_n)}{f^1(x_n)}$

$$= x_n - \frac{\{2x_n^2 - 6x_n - 3\}}{4x_n - 6}$$

$$= \frac{4x_n^2 - 6x_n - 2x_n^2 + 6x_n + 3}{4x_n - 6}$$

$$x_{n+1} = \frac{2x_n^2 + 3}{4x_n - 6}$$

is the iterative formula for solving $2x^2 - 6x - 3 = 0$.

(ii) We show that the positive root for $2x^2 - 6x - 3 = 0$ lies between 3 and 4.

Let $f(x) = 2x^2 - 6x - 3$

$f(3) = 2x(3)^2 - 6(3) - 3$

$= 18 - 18 - 3$

$= -3$ (Negative)

$f(4) = 2x(4) - 6(4) - 3$

$= 32 - 24 - 3$

$= 5$ (positive)

Since f (3) and f (4) take different signs (+ and -), it follows that a root exists.

Let $x_0 = 3$

$$x_1 = \frac{2\times 3^2+3}{4\times 3-6}$$

$$= \frac{21}{6} = 3.5$$

$$x_2 = \frac{2x_1^2+3}{4x_1-6}$$

$$= 3.4375$$

$$|x_2 - x_1| = 0.0625$$

$$x_3 = \frac{2x_2^2+3}{4x_2-6}$$

$$= 3.436491935$$

$$|x_3 - x_2| = 0.001$$

Therefore the root is **3.44** $\{2\,\text{dec.places}\}$

14. In a survey of newspaper reading habits of members of staff of a university, it is found that

80% read NEW VISION (N),
50% read MONITOR (M) and
30% read the EAST AFRICAN (E).

Further 20% read both M and N
15% read both N and E and
10% read both M and E

(a) If a member of staff is chosen at random from the university community, find the probabilities.

(i) That the member reads none of the three papers.

(ii) The member is one of those who read at least one of the three papers.

(b) Estimate the number of staff who read at least two papers if the total number is 500.
(c) What is the probability that, given that a member of staff reads at least two newspapers, he reads all the three?

Solution:

14. Given $n(g) = 100$

$n(N) = 80$

$n(M) = 50$

$n(E) = 30$

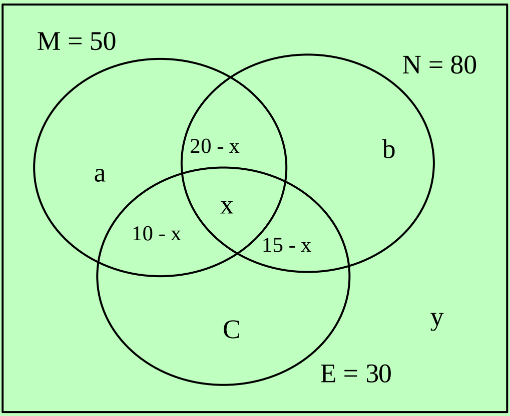
$n(N \cap E) = 15$

$n(M \cap N) = 20$

$n(M \cap E) = 10$

Let $n(M \cap N \cap E) = x$

And $n(M \cup N \cap E) = y$



n (M) only = $a = 50 - (20 - x + x + 10 - x)$

$= 50 - (30 - x)$

$= 20 + x$

$n(N)$ only $= b = 80 - (20 - x + x + 15 - x)$
$= 80 - (35 - x)$
$= 45 + x$

$n(E)$ only $= c = 30 - (10 - x + x + 15 - x)$
$= 30 - (25 - x)$
$= 5 + x$

$\Rightarrow 20 + x\ 45 + x + 5 + x + 20 - x + 10 = 100$
$- x + 15 - x + x + y = 100$
$115 + x + y = 100$

Impossible to solve!

15. A certain factory produces ball bearings. A sample of the bearing from the factory produced the following results.

Diameter of bearing in mm	frequency
91-93	4
94-96	6
97-99	34
100-102	40
103-105	13
106-108	3

(i) Determine the mean and variance of the diameter of the sample bearings.
(ii) Estimate the mean surface area of the bearings produced by the factory.

Solution:

Given the information

		f	x	fX	fx^2
91-93	90.5-93.5	4	92	368	33856
94-96	93.5-96.5	6	95	570	54150
97-99	96.5-99.5	34	98	3332	326536
100-102	99.5-102.5	40	101	4040	408040
103-105	102.5-105.5	13	104	1352	140608
106-108	105.5-108.5	3	107	321	34347

$\sum f = 10$

$\sum fx = 9983$

$\sum fx^2 = 997537$(ii)

(i)Mean Diameter $= \dfrac{\sum fx}{\sum f}$.

$= \dfrac{9983}{100}$

$= 99.83$ #

Variance $= \dfrac{\sum fx^2}{\sum f} - \left(\dfrac{\sum fx}{\sum f}\right)^2$

$= \dfrac{997537}{100} - (99.83)^2$

$= 9975.37 - 9966.0289$

$\therefore$ Variance $= 9.3411$

(ii) We estimate the mean surface area as

Area $= 4\pi r^2$

Where r = radius.

$$\text{But} \quad r = \frac{\text{diameter}}{2}$$

$$= \frac{99.83}{2}$$

$$= 49.915\text{mm}$$

$$\therefore \text{Area} = 4 \times (49.915)^2$$

Hence mean area = 31309.20mm^2

16. The following Table gives the marks obtained in Calculus, physics and Statistics by seven (7) students

Calculus 72 50 60 55 35 48 82
Physics 61 55 70 50 30 50 73
Statistics 50 40 62 70 40 40 60

Draw scatter diagrams and determine rank correlation coefficients between the performances of the students in

(i) Calculus and physics

(ii) Calculus and Statistics

Give interpretations to your results.

Solution:

Calculus (C)	72	50	60	55	35
48	82				
Physics (P)	61	55	70	50	30
50	73				
Statistics (S)	50	40	62	70	40
40	60				

Let C = Calculus

P = Physics

C	P	R_c	R_p	d	d^2
72	61	2	3	-1	1
50	55	5	4	1	1

60	70	3	2	1	1
55	50	4	5.5	-1.5	2.25
35	30	7	7	0	0.25
48	50	6	5.5	0.5	0.25
82	73	1	1	0	0
					$\sum d^2 = 5.5$

Spear man's rank correlation coefficient is given as;

$$p = 1 - \frac{6\sum d^2}{n(n^2-1)}$$

$$= 1 - \frac{6(5.5)}{7(49-1)}$$

$$= 1 - \frac{33}{336}$$

$$= 0.902$$

There is a fair or substantial positive correlation between the performance of students in Calculus and Statistics.