离散数学及其应用 第6、8章 组合

6. Counting

Counting problems arise throughout mathematics and computer science. In CS, we need to measure the time complexity of given algorithms, and know the number of passwords of a computer system and so on.

6.1 The Basics of Counting

6.1.1 Basic Counting Principles

`The Product Rule`

Suppose that a procedure can be broken down into two tasks. If there are n_1 ways to do the first task and n_2 ways to do the second after the first task has been done, then there are $n_1 n_2$ ways to complete the procedure.

In terms of sets, if $A_1, A_2, ..., A_m$ are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements of each set. The task of choosing an element in the Cartesian product $A_1 \times A_2 \times \cdots \times A_m$ is done by choosing an element in A_1 , an element in A_2 , ... , and an element in A_m . By the product rule, it follows that $|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \times |A_2| \times \cdots \times |A_m|$

`The Sum Rule`

If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the tasks.

In terms of sets, $|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$ where $A_i \cap A_j = \emptyset$ for all $i \neq j$.

`The Subtraction Rule`

If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task in $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways. The subtraction rule is also known as the principle of inclusion–exclusion.

`The Division Rule`

There are $\dfrac{n}{d}$ ways to do a task if it can be done using a procedure that can be carried out in n ways, and for every way w , exactly d of the n ways correspond to way w .

Restated in terms of sets: If the finite set A is the union of n pairwise disjoint subsets each with d elements, then $n = \dfrac{|A|}{d}$.

In terms of functions: If f is a function from A to B , where both are finite sets, and for every value $y \in B$ there are exactly d values $x \in A$ such that $f(x) = y$, then

$$|B| = \frac{|A|}{d} \ .$$

e.g.

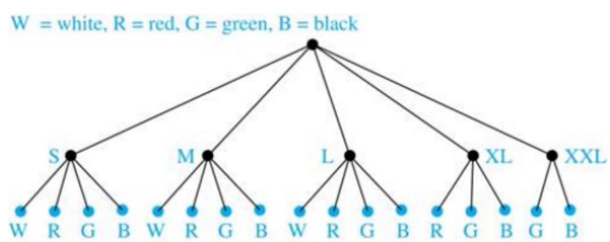
How many ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left and right neighbor?

Solution:

Number the seats around the table from 1 to 4 proceeding clockwise. There are four ways to select the person for seat 1, three for seat 2, two for seat 3, and one way for seat 4. Thus there are 4! = 24 ways to order the four people. But since two seatings are the same when each person has the same left and right neighbor, for every choice for seat 1, we get the same seating. Therefore, by the division rule, there are 24/4 = 6 different seating arrangements.

6.1.2 Tree Diagrams

We can solve many counting problems through the use of tree diagrams, where a branch represents a possible choice and the leaves represent possible outcomes.



6.2 The Pigeonhole Principle

The Pigeonhole Principle (Also called Dirichlet Drawer Principle)

If k is a positive integer and $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Proof:

We use a proof by contraposition. Suppose none of the k boxes has more than one object. Then the total number of objects would be at most k . This contradicts the statement that we have $k+1$ objects.

e.g.

In a party of 2 or more people, there are 2 people with the same number of friends in the party, assuming you can't be your own friend and that friendship is mutual.

Pigeons : the n people (with $n > 1$).

Pigeonholes: the possible number of friends, i.e. the set $\{0, 1, 2, 3, \ldots, n-1\}$

🌰e.g.

Show that for every integer n there is a multiple of n that has only 0s and 1s in its decimal expansion.

Solution:

Let n be a positive integer. Consider the $n+1$ integers 1, 11, 111, ... , 11...1 (where the last has $(n+1)$ 1s). There are n possible remainders when an integer is divided by n .

By the pigeonhole principle, when each of the $n+1$ integers is divided by n , at least two must have the same remainder. Subtract the smaller from the larger and the result is a multiple of n that has only 0s and 1s in its decimal expansion.

📋The Generalized Pigeonhole Principle

If N objects are placed into k boxes, then there is at least one box containing at least $\lceil \frac{N}{k} \rceil$ objects.

Proof:

Suppose that none of the boxes contains more than $\lceil \frac{N}{k} \rceil - 1$ objects. Then, the total number of objects is at most $k(\lceil \frac{N}{k} \rceil - 1) < k(\frac{N}{k} + 1 - 1) = N$.

🌰e.g.

Show that among any $n+1$ positive integers not exceeding $2n$ there must be an integer that divides one of the other integers.

Solution:

Let $n+1$ positive integers be $a_1, a_2, \ldots, a_{n+1} (1 \le a_i \le 2n)$

Write $a_i (i=1,2,\ldots,n+1)$ as $2^{k_i} q_i$**, where k_i is a nonnegative integer and q_i is odd positive integers less than $2n$.**

Since there are only n odd positive integers less than $2n$, by the pigeonhole principle it follows that there exist integers i and j such that $q_i = q_j = q$,

then $a_i = 2^{k_i} q$ and $a_j = 2^{k_j} q$

It follows that if $a_i < a_j$, then $a_i \mid a_j$, while if $a_j < a_i$, then $a_j \mid a_i$.

e.g.

During 11 weeks football games will be held at least 1 game a day, but at most 12 games be arranged each week. Show that there must be a period of some number of consecutive days during which exactly 21 games must be played.

Solution:

Let x_i be the number of football games helded on the i–th day.

Let a_i be $\sum_{k=1}^{i} x_k$, where $1 \le a_1 < a_2 < \ldots < a_{77} \le 12 \times 11 = 132$

Let c_i be $a_i + 21$, so $1 < c_1 < c_2 < \ldots < c_{77} \le 132 + 21 = 153$

$A = \{a_1, a_2, \ldots, a_{77}, c_1, c_2, \ldots, c_{77}\}$, $B = \{1, 2, \ldots, 153\}$

Since $|A| = 154 > |B| = 153$, there exist integers $i \ne j$ that $a_i = c_j = a_j + 21$

So $a_i - a_j = x_i + x_{i+1} + \ldots + x_{j+1} = 21$

e.g.

Every sequence of $n^2 + 1$ distinct integers contains a subsequence of length $n + 1$ that is either strictly increasing or strictly decreasing.

Proof:

Let the sequence be $a_1, a_2, \ldots, a_{n^2+1}$
Associate (x_k, y_k) **to the term** a_k **, where** x_k **is the length of the longest increasing subsequence starting at** a_k **,** y_k **…**

Suppose that there is no increasing or decreasing subsequence of length n**+1.Then**

$$1 \le x_k \le n \qquad 1 \le y_k \le n$$

Hence there are $n \times n = n^2$ **pairs (** x_k, y_k **),**

Since there are n^2+1 a_k **, By the pigeonhole principle, it follows that there exist terms** a_i, a_j $(1 \le i < j \le n^2+1)$
such that $(x_i, y_i) = (x_j, j_j)$

🌰e.g.

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6.3 Permutations and Combinations

💡这一节基本是高中排列组合的知识。

6.3.1 Basic Concepts

`Permutation` An ordered arrangement of the elements of a set.

`r-Permutation` An ordered arrangement of r elements of a set

📋The number of r-permutations of a set with n distinct elements is

$$P(n,r) = n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!}$$

Specially, $P(n,0)=1$, $P(n,n)=n!$.

`r-Combination` An unordered selection of r elements of a set. An r-combination is simply a subset of a set with r elements.

📋The number of r-combination of a set with n elements, where n is a positive integer and r is an integer with $0 \le r \le n$, denoted by $C(n,r) = \binom{n}{r}$, equals to

$$\frac{n!}{r!(n-r)!}$$

📋Combination Corollary

Let n and r be nonnegative integers with $r \leq n$. Then $C(n,r) = C(n,n-r)$

6.3.2 Combinatorial Proof

A `combinatorial proof` of an identity:

- `Double Counting Proofs` Uses counting arguments to prove that both sides of the identity count the same objects but in different ways.
- `Bijective Proofs` Show that there is a bijection between the sets of objects counted by the two sides of the identity.

6.4 Binomial Coefficients

💡二项式定理也是高中就学了的。

A `binomial expression` is the sum of two terms, such as $x+y$. (More generally, these

terms can be products of constants and variables.)

📋The Binomial Theorem

Let x and y be varaibles, and let n be a nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j \ .$$

Proof:

We can use counting principles to find the coefficients in the expansion of $(x+y)^n$. To form the term $x^{n-j}y^j$, it is necessary to choose $(n-j)$ x s and (j) y s from the n sums.

📋Corrolaries for the Binomial Theorem

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \ , \ \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0 \ , \ \sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$$

📋Pascal's Identity

Let n and k be positive integers with $k \leq n$. Then $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.

Proof:

We construct a subset of size k form a set $A = \{x, a_1, a_2, ..., a_n\}$ with $(n+1)$ elements. The total will include:

All of the subsets from the set of size n which do not contain the element x , $C(n,k)$, plus the subsets of size $(k-1)$ with the element x added $C(n,k-1)$.

📋Pascal's Triangle

(a) (b)

The nth row in the triangle consists of the binomial coefficients $\binom{n}{k}, k = 0, 1, \ldots, n$.

By Pascal's identity, adding two adjacent bionomial coefficients results is the binomial coefficient in the next row between these two coefficients.

📋Vandermonde's Identity

Let m , n and r be nonnegative integer with r not exceeding either m or n .

Then $\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$.

📋If n is a nonnegative integer. Then $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$.

Proof:

We use Vandermonde's Identity with $m = n = r$ to obtain

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{n-k}\binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k}^2 .$$

📋Let n and r be nonnegative integer with $r \le n$, then $\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$.

Proof:

The left–hand side counts the bit strings of length $(n+1)$ containing $(r+1)$ 1s. We show that the right–hand side counts the same objects by considering the cases corresponding to the possible locations of the final 1 in a string with $(r+1)$ ones.

6.5 Generalized Permutations and Combinations

6.5.1 Permutations and Combinations with Repetition

📋The number of r–permutations of a set of n objects with repetition allowed is n^r .

📋There are $H_r^n = C(n-1+r, r)$ r–combination from a set with n elements when repetition of elements is allowed.

Proof:

1. The $(n-1)$ bars are used to mark off n different cells, with the ith cell contains a star for each time the ith element of the set occurs in the combination. For example, * * | * | | * | * * * means the first element occurs twice, the second element occurs once, the fourth element occurs once, and the fifth element occurs three times.
2. Each r–combination of a set with n elements when repetition is allowed can be represented by a list of $(n-1)$ bars and r stars.
3. The number of such lists is $C(n-1+r, r)$.

🌰e.g.

How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 16$ where $x_i (i = 1, 2, 3, 4)$ is nonnegative integer?

Solution:

Since a solution of this equation corresponds to a way of selecting 16 items from a set with four element, such that x_1 items of type one, x_2 items of type two, x_3 items of type three, x_4 items of type four are chosen.

Hence the number of solutions is $H_4^{16} = C(4-1+16, 16) = C(19, 3)$.

How about the number of solutions to $x_1 + x_2 + x_3 + x_4 \le 16$?

Solution:

We can introduce an auxiliary variable x_5 so that $x_1+x_2+x_3+x_4+x_5=16$.

$H_5^{16}=C(5-1+16,16)=C(20,4)$.

6.5.2 Permutations with Indistinguishable Objects

📋The number of different permutations of n objects, $A=\{n_1\cdot a_1,n_2\cdot a_2,\ldots,n_k\cdot a_k\}$, where there are n_1 indistinguishable objects of type 1, ... ,and n_k indistinguishable objects of type k, is $\dfrac{n!}{n_1!n_2!...n_k!}$.

Proof:

$$C(n,n_1)\cdot C(n-n_1,n_2)\cdot ...\cdot C(n-n_1-...-n_{k-1},n_k)$$
$$=\frac{n!}{n_1!(n-n_1)!}\cdot\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}\cdot ...\cdot\frac{(n-n_1-n_2-...-n_{k-1})!}{n_k!(n-n_1-n_2-...-n_k)!}$$
$$=\frac{n!}{n_1!n_2!...n_k!}$$

🌰e.g.

How many different strings can be made form the letters in MISSISSIPPI, using all the letters?

Solution:

$A=\{1\cdot M,4\cdot I,4\cdot S,2\cdot P\}$, so there are $\dfrac{11!}{1!\cdot 4!\cdot 4!\cdot 2!}$ different strings.

6.5.3 Distributing Objects into Boxes

Many counting problems can be solved by counting the ways objects can be placed in boxes. The objects may be either different from each other (distinguishable) or identical (indistinguishable). The boxes may be labeled (distinguishable) or unlabeled (indistinguishable).

📋The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are place into box i, i=1,2,…,k, equals $\dfrac{n!}{n_1!n_2!...n_k!}$.

It can be proved by setting up a one-to-one correspondence between the permutations in the last theorem and the ways to distribute objects counted here.

🌰e.g.

How many ways are there to distribute hands of 5 cards to each of four players from the standard deck of 52 cards?

Solution:

It is typical problem that involves distributing distinguishable objects into distinguishable boxes.

The distinguishable objects are the 52 cards.

The five distinguishable boxes are the hands of the four players and the rest of the deck.

So there are $\frac{52!}{5!\cdot 5!\cdot 5!\cdot 5!\cdot 32!}$ kinds of distribution.

📋There are $C(n+k-1,n-1)$ ways to place k indistinguishable objects into n distinguishable boxes.

Proof based on one-to-one correspondence between k-combinations from a set with n-elements when repetition is allowed and the ways to place k indistinguishable objects into n distinguishable boxes.

📋Stirling Numbers（斯特林数） of the Second Kind

$S(n,j)$, the number of ways to distribute n distinguishable objects into j indistinguishable boxes so that no boxes is empty.

- $S(r,1)=S(r,r)=1,(r\geq 1)$
- $S(r,2)=2^{r-1}-1$
- $S(r,r-1)=C(r,2)$
- $S(r+1,n)=S(r,n-1)+nS(r,n)$
- $S(n,j)$ is the number of ways to partition the set with nelements into j nonempty and disjoint subsets.
- $S(n,j)j!$ is the number of ways to distribute n distinguishable objects into j distinguishable boxes so that no boxes is empty.
- $\sum_{j=1}^{k}S(n,j)$ is the number of ways to place n distinguishable objects into k

indistinguishable boxes.

6.6 Generating Permutations and Combinations

`Lexicographic Ordering for Permutations` The permutation $a_1a_2\ldots a_n$ precedes the permutation of $b_1b_2\ldots b_n$, if for some k , with $1 \leq k \leq n$, $a_1 = b_1, a_2 = b_2, \ldots, a_{k-1} = b_{k-1}$, and $a_k < b_k$.

📋Generating All the Permutations

1. List the elements in lexicographic order.
2. Find the least permutation.
3. Find the next least permutation until the largest permutation is found.

🌰e.g.

> Algorithm of producing the $n!$ permutations of the integers $1, 2, \ldots, n$.

1. Begin with the smallest permutation in lexicographic order, namely $1, 2, 3, \ldots, n$.
2. Produce the next largest permutation.
3. Continue until all $n!$ permutations have been found.

📋Generating All the Combinations

1. A combination is just a subset, so we need to list all subsets of the finite set.
2. Use bit strings of length n to represent a subset of a set with n elements. If the subset contains the ith element, then the ith bit of the string is 1; otherwise the ith bit is 0.
3. The 2^n bit strings can be listed in order of their increasing size as integers in their binary expansions.

8. Advanced Counting Techniques

8.1 Applications of Recurrence Relations

A `recurrence relation` for the sequence $\{a_n\}$ is an equation that express a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, a_2, ..., a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integers.

$a_n = f(a_0, a_1, a_2, \dots, a_{n-1}), \ \ n \geq n_0$

A `solution` of a recurrence relation is a sequence if its terms satisfy the recurrence relation. Normally, there are many sequences which satisfy a recurrence relation. We should distinguish them by initial conditions.

The `degree` of a recurrence relation is the number of initial conditions it needs to determine a specific sequence. For example, $a_n = a_{n-1} + a_{n-8}$ is a recurrence relation of degree 8.

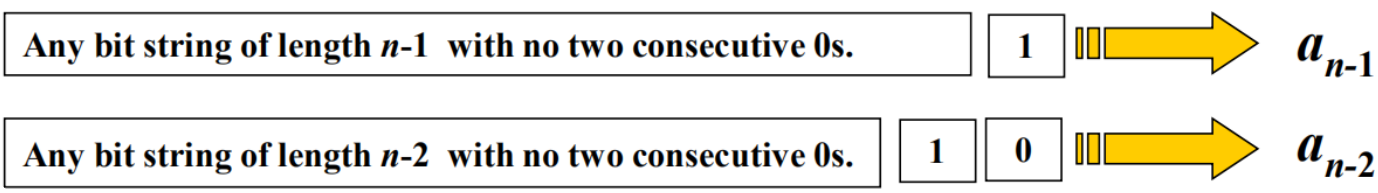
💡Many relationships are most easily described using recurrence relations.

🌰e.g.

Find a recurrence relation for the number of bit strings of length n that don't have two consecutive 0s.

Solution:

Let a_n denote the number of bit strings of length n that don't have two consecutive 0s.



Recurrence relation: $a_n = a_{n-1} + a_{n-2}$.

Initial conditions: $a_1 = 2, \ a_2 = 3$.

Note that $\{a_n\}$ satisfies the same recurrence relation as the Fibonacci sequence. Since $a_1 = f_3$ and $a_2 = f_4$, we conclude that $a_n = f_{n+2}$.

⚙️ Recurrence relations play an important role in many aspects of the study of algorithms and their complexity.

- Dynamic programming algorithm
  - An algorithm follows the dynamic programming paradigm when it recursively breaks down a problem into simpler overlapping subproblems, and computes the solution using the solutions of the subproblems.
  - Recurrence relations are used to find the overall solution from the solutions of the subproblems.

- Divide-and-conquer algorithm
    - Recurrence relations can be used to analyze the complexity of divide-and-conquer algorithms.

8.2 Solving Linear Recurrence Relations

8.2.1 Linear Homogeneous Recurrence Relations

> 线性齐次常系数递推关系

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k}$ where $c_1, c_2, ..., c_k$ are real numbers, and $c_k \neq 0$.

`Linear` : The right-hand side is a sum of the previous terms of the sequence each multiplied by a

function of n .

`Constant Coefficients` : The coefficients in the sum of the a_i s are constants, independent of n .

`Degree` : a_n is expressed in terms of the previous k terms of the sequence. By strong induction, a sequence satisfying such a recurrence relation is uniquely determined by the recurrence relation and the k initial conditions $a_0 = C_1, a_1 = C_2, \ldots, a_{k-1} = C_{k-1}$.

`Homogeneous` : Because no terms occur that are not multiples of the a_j s. Otherwise `inhomogeneous` or `nonhomogeneous` .

📋 Solving Linear Homogeneous Recurrence Relation with Constant Coefficients

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k}$

Two key ideas to find all their solutions:

1. These recurrence relations have solutions of the form $a_n = r^n$, where r is a constant.

$$\begin{aligned} r^n - c_1 r^{n-1} - c_2 r^{n-2} - ... - c_k r^{n-k} &= 0 \\ r^{n-k}(r^k - c_1 r^{k-1} - ... - c_k) &= 0 \\ r^k - c_1 r^{k-1} - ... - c_k &= 0, (r \neq 0) \end{aligned}$$

The last equation is called the `Characteristic Equation`, and the roots of the equation is called the `Characteristic Roots`. The sequence $\{a_n\}$ with $a_n = r^n$ where $r \neq 0$ is a solution if and only if r is a characteristic root.

2. A linear combination of two solutions of a linear homogeneous recurrence relation is also a solution.

Suppose that s_n and t_n are both solutions of this recurrence relation.

Then we have

$$s_n = c_1 s_{n-1} + c_2 s_{n-2} + ... + c_k s_{n-k}$$

and

$$t_n = c_1 t_{n-1} + c_2 t_{n-2} + ... + c_k t_{n-k}$$

Now suppose that b_1 and b_2 are real numbers. Then

$$b_1 s_n + b_2 t_n = c_1(b_1 s_{n-1} + b_2 t_{n-1}) + c_2(b_1 s_{n-2} + b_2 t_{n-2}) + ... + c_k(b_1 s_{n-k} + b_2 t_{n-k})$$

This means that $b_1 s_n + b_2 t_n$ is also a solution of the same linear homogeneous recurrence relation.

📋 The Degree 2 Case

Let c_1, c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1, r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if

$a_n = b_1 r_1^n + b_2 r_2^n$ for $n = 0, 1, 2, ...$, where b_1, b_2 are constants.

Proof:

Show that if $\{a_n\}$ is a solution, then $a_n = b_1 r_1^n + b_2 r_2^n$ for some constant b_1, b_2.

The initial conditions $a_0 = C_0 = b_1 + b_2,\ a_1 = C_1 = b_1 r_1 + b_2 r_2$ hold. It will be shown that there are constants b_1, b_2 such that the sequence $\{a_n\}$ with $a_n = b_1 r_1^n + b_2 r_2^n$ satisfies these same initial conditions. This requires that:

$$b_1 = \frac{C_1 - C_0 r_2}{r_1 - r_2},\ \ b_2 = \frac{C_0 r_1 - C_1}{r_1 - r_2}$$

Hence, with these values for b_1, b_2, the sequence $\{a_n\}$ with $a_n = b_1 r_1^n + b_2 r_2^n$ satisfies the two initial conditions. We know that $\{a_n\}$ and $\{b_1 r_1^n + b_2 r_2^n\}$ are both solutions of the recurrence relation and both satisfy the initial conditions when $n = 0$ and $n = 1$.

Because there is a unique solution of a linear homogeneous recurrence relation of degree two with two initial conditions, it follows that the two solutions are the same.

📋 The Solution of the Degree 2 Case when there is a Repeated Root

Let c_1, c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0 . A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = b_1 r_0^n + b_2 n r_0^n = (b_1 + b_2 n) r_0^n,\ n = 0, 1, 2, ...$, where b_1, b_2 are constants.

📋 The General Case

Let $c_1, c_2, ..., c_k$ be real numbers. Suppose that the characteristic equation has k distinct roots $r_1, r_2, ..., r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k}$ if and only if $a_n = b_1 r_1^n + b_2 r_2^n + ... + b_k r_k^n$ for $n = 0, 1, 2, ...$ where $b_1, b_2, ..., b_k$ are constants.

The coefficients $b_1, b_2, ..., b_k$ are found by enforcing the initial conditions.

📋 The General Case with Repeated Roots Allowed

Let $c_1, c_2, ..., c_k$ be real numbers. Suppose that the characteristic equation $r_k - c_1 r^{k-1} - ... - c_k = 0$ has t distinct roots with multiplicities $m_1, m_2, ..., m_t$, respectively, so that $m_i \geq 1$ for $i = 1, 2, ..., t$ and $\sum_{j=1}^{t} m_k = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k}$ if and only if $a_n = \sum_{i=1}^{t} [(\sum_{j=0}^{m_i - 1} b_{i,j} n^j) r_i^n]$ for $n = 0, 1, 2, ...$ where $b_{i,j}$ are constants.

8.2.2 Linear Nonhomogeneous Recurrence Relation With Constant Coefficients

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k} + F(n)$, where $c_i (i = 1, 2, ..., k)$ are real number, $F(n)$ is a function not identically zero depending only on n .

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k}$ is called the **`associated homogeneous recurrence relation`** . Solution to nonhomogeneous case is sum of solution to associated homogeneous recurrence system and a `particular solution` to the nonhomogeneous case.

📋 Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

Let $\{a_n^{(p)}\}$ be a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients $a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k} + F(n)$. Then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $\{a_n^{(p)} + a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k}$.

📋 Assume a linear nonhomogeneous recurrence equation with constant coefficients with the nonlinear part $F(n)$ of the form $F(n) = (\sum_{i=0}^{t} b_i n^i) s^n$. If s is not a root of the characteristic equation of the associated homogeneous recurrence equation, there is a particular solution of the form $(\sum_{i=0}^{t} p_i n^i) s^n$. If s is a root of multiplicity m , a particular solutions is of the form $(\sum_{i=0}^{t} p_i n^i) s^n n^m$.

8.4 Generating Function

如果觉得不好理解可以看这篇文章：知 算法 | 小学生都能看懂的生成函数入门教程

The `generating function` （生成函数） for a finite sequence $a_0, a_1, a_2 ..., a_n$ of real numbers is the series $G(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n = \sum_{i=0}^{n} a_i x^i$.

The generating function for a infinite sequence $a_0, a_1, a_2 ..., a_k, ...$ of real numbers is the infinite series $G(x) = a_0 + a_1 x + a_2 x^2 + ... + a_k x^k + ... = \sum_{i=0}^{\infty} a_i x^i$.

📋 Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $g(x) = \sum_{k=0}^{\infty} b_k x^k$, then $f(x)g(x) = \sum_{i=0}^{\infty} (\sum_{j=0}^{\infty} a_j b_{i-j}) x^i$.

Proof:

$$
\begin{aligned}
LHS &= (a_0 + a_1x + a_2x^2 + ...)(b_0 + b_1x + b_2x^2 + ...) \\
&= a_0b_0 + (a_1b_0 + a_0b_1)x + (a_2b_0 + a_1b_1 + a_0b_2)x^2 + ... \\
&= RHS
\end{aligned}
$$

e.g.

Suppose that the generating function of the sequence $a_0, a_1, ..., a_n, ...$ is $G(x)$. What is the generating function for the sequence $b_k = \sum_{i=0}^{k} a_i$?

Solution:

$\{a_k\} \to G(x),\ \{b_k\} \to F(x),\ c_k = 1$

So $b_k = \sum_{i=0}^{k} a_i = \sum_{i=0}^{k} a_i c_{k-i}$. The generating function of $\{c_k\}$ is $\sum_{i=0}^{\infty} x^i = \dfrac{1}{1-x}$.

So $F(x) = G(x)\dfrac{1}{1-x}$.

The Extended Binomial Coefficient

Let u be a real number and k a nonnegative integer. Then the extended binomial coefficient is defined by $\binom{u}{k} = \begin{cases} \frac{u(u-1)(u-2)...(u-k+1)}{k!}, & \text{if k>0} \\ 1, & \text{if k=0} \end{cases}$.

Let x be a real number with $|x| < 1$ and let u be a real number. Then $(1+x)^u = \sum_{i=0}^{\infty} \binom{u}{i} x^i$

TABLE 1 Useful Generating Functions.

$G(x)$	a_k
$(1+x)^n = \sum_{k=0}^{n} C(n,k)x^k = 1 + C(n,1)x + C(n,2)x^2 + \cdots + x^n$	$C(n,k)$
$(1+ax)^n = \sum_{k=0}^{n} C(n,k)a^k x^k = 1 + C(n,1)ax + C(n,2)a^2x^2 + \cdots + a^n x^n$	$C(n,k)a^k$
$(1+x^r)^n = \sum_{k=0}^{n} C(n,k)x^{rk} = 1 + C(n,1)x^r + C(n,2)x^{2r} + \cdots + x^{rn}$	$C(n,k/r)$ if $r \mid k$; 0 otherwise
$\frac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$	1 if $k \le n$; 0 otherwise
$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$	1
$\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k = 1 + ax + a^2x^2 + \cdots$	a^k
$\frac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk} = 1 + x^r + x^{2r} + \cdots$	1 if $r \mid k$; 0 otherwise
$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1)x^k = 1 + 2x + 3x^2 + \cdots$	$k+1$
$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)x^k = 1 + C(n,1)x + C(n+1,2)x^2 + \cdots$	$C(n+k-1,k) = C(n+k-1,n-1)$
$\frac{1}{(1+x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)(-1)^k x^k = 1 - C(n,1)x + C(n+1,2)x^2 - \cdots$	$(-1)^k C(n+k-1,k) = (-1)^k C(n+k-1,n-1)$
$\frac{1}{(1-ax)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)a^k x^k = 1 + C(n,1)ax + C(n+1,2)a^2x^2 + \cdots$	$C(n+k-1,k)a^k = C(n+k-1,n-1)a^k$
$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$1/k!$
$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$(-1)^{k+1}/k$

📋 Counting Problems Using Generating Functions

This method is shown in the following examples.

e.g.

Find the number of solutions of $e_1 + e_2 + e_3 = 17$, where e_1 , e_2 , and e_3 are nonnegative integers with $2 \le e_1 \le 5$, $3 \le e_2 \le 6$, and $4 \le e_3 \le 7$.

Solution:

The number of solutions with the indicated constraints is the coefficient of x^{17} in the expansion of $(x^2 + x^3 + x^4 + x^5)(x^3 + x^4 + x^5 + x^6)(x^4 + x^5 + x^6 + x^7)$. This follows because we obtain a term equal to x^{17} in the product by picking a term in the first sum x^{e_1} , a term in the second sum x^{e_2} , and a term in the third sum x^{e_3} , where $e_1 + e_2 + e_3 = 17$. It is not hard to see that the coefficient of x^{17} in this product is 3. Hence, there are three solutions.

e.g.

Use generating functions to find the number of r−combinations from a set with n elements when repetition of elements is allowed.

Solution:

Since there are n elements in the set, each can be selected zero times, one times and so on. It follows that $G(x) = (1 + x + x^2 + ...)^n = (\dfrac{1}{1-x})^n = \dfrac{1}{(1-x)^n}$.

the number of r−combinations from a set with n elements when repetition of elements is allowed, is the coefficient ar of x^r in the expansion of $G(x)$. Since $\dfrac{1}{(1-x)^n} = \sum_{i=0}^{\infty} C(n+i-1, i)x^i$. Then the coefficient a^r equals $C(n+r-1, r)$.

e.g.

Suppose that there are $2r$ red balls, $2r$ blue balls, and $2r$ white balls. How many ways to select $3r$ balls from these balls?

Solution:

$G(x) = (1 + x + x^2 + ... + x^{2r})^3$

The coefficient a_{3r} of x^{3r} in the expansion of $G(x)$ is the solution of this problem.

So $G(x) = (\frac{1-x^{2r+1}}{1-x})^3 = \frac{1-3x^{2r+1}+3x^{4r+2}-x^{6r+3}}{(1-x)^3}$

$F(x) = \frac{1}{(1-x)^3} = (1+x+x^2+...)^3$, so $a_{3r} = C_{3r+2}^{3r} - 3C_{r+1}^{r-1}$.

e.g.

Determine the number of ways to insert tokens worth \$1,\$2 and \$5 into a vending machine to pay for an item that costs r dollars in both the case when the order in which the tokens are inserted does not matter and when the order does matter.

Solution:

(1) When the order in which the tokens are inserted does not matter:

$$G(x) = (1+x+x^2+x^3+...)(1+x^2+x^4+x^6+...)(1+x^5+x^{10}+x^{15}+...)$$

The coefficient of x^r in the expansion of $G(x)$ is the solution of this problem.

(2) When the order in which the tokens are inserted does matter:

The number of ways to insert exactly n tokens to produce a total of \$r is the coefficient of x^r in $(x+x^2+x^5)^n$. Since any number of tokens may be inserted, the number of ways to produce \$r using \$1,\$2 and \$5 tokens, is the coefficient of x^r in

$$1+(x+x^2+x^5)+(x+x^2+x^5)^2+... = \frac{1}{1-(x+x^2+x^5)} .$$

Use Generating Function to Solve Recurrence Relations

This method is shown in the following examples.

e.g.

Use generating functions to solve the recurrence relation $a_n = 2a_{n-1}+3a_{n-2}+4^n+6$ with initial conditions $a_0 = 20, a_1 = 60$.

Solution:

Multiply by x^n on both sides of $a_n = 2a_{n-1}+3a_{n-2}+4^n+6$.

$$a_n x^n = 2a_{n-1}x^n + 3a_{n-2}x^n + 4^n x^n + 6x^n$$

$$\sum_{n=2}^{\infty} a_n x^n = 2\sum_{n=2}^{\infty} a_{n-1}x^n + 3\sum_{n=2}^{\infty} a_{n-2}x^n + \sum_{n=2}^{\infty} 4^n x^n + 6\sum_{n=2}^{\infty} x^n$$

$$G(x) - a_0 - a_1 x \qquad 2x\sum_{n=1}^{\infty} a_n x^n \qquad 3x^2\sum_{n=0}^{\infty} a_n x^n \qquad \frac{1}{1-4x} - 1 - 4x \qquad 6\left(\frac{1}{1-x} - 1 - x\right)$$

$$2x(G(x) - a_0) \qquad 3x^2 G(x)$$

$$(1-2x-3x^2)G(x) = \frac{20-80x+2x^2+40x^3}{(1-4x)(1-x)}$$

$$G(x) = \frac{20-80x+2x^2+40x^3}{(1-4x)(1-x)(1+x)(1-3x)}$$

$$= \frac{16/5}{1-4x} + \frac{-3/2}{1-x} + \frac{31/20}{1+x} + \frac{67/4}{1-3x}$$

$$\frac{16}{5}\times 4^n \quad -\frac{3}{2}\times 1^n \quad \frac{31}{20}\times(-1)^n \quad \frac{67}{4}\times 3^n$$

$$a_n = \frac{16}{5}\times 4^n - \frac{2}{3} + \frac{31}{20}\times(-1)^n + \frac{67}{4}\times 3^n$$

8.5 Inclusion–Exclusion and Its Application

📋 The Formula for the Number of Elements in the Union of n Finite Sets

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{i=1}^{n} |A_i| + (-1)\sum_{1\le i<j\le n} |A_i \cap A_j| + (-1)^2 \sum_{1\le i<j<k\le n} |A_i \cap A_j \cap A_k| + \ldots + (-1)^{n-1}|A_1 \cap A_2 \cap \ldots \cap A_n|$$

There are $2^n - 1$ terms in this formula.

Proof:

An element in the union is counted exactly once by the right-hand side of the equation.

Suppose that a is an element of exactly r of the sets $A_1, A_2, ..., A_n$ where $1 \leq r \leq n$. This element is counted $C(r,1)$ times by $\sum_{i=1}^{n} |A_i|$, $C(r,2)$ times by $\sum_{1 \leq i < j \leq n} |A_i \cap A_j|$, ...

Thus it is counted exactly
$C(r,1) - C(r,2) + C(r,3) - ... + (-1)^{r-1} C(r,r) = 1 - (-1+1)^r = 1$.

📋 An Alternative Form of Inclusion–Exclusion

To solve problems that ask for the number of elements in a set that have none of n properties $P_1, P_2, ..., P_n$. Let A_i be the the subset containing the elements that have property P_i . Let $N(P_1, P_2, ..., P_n)$ be the number of elements with all properties $P_1, P_2, ..., P_n$. It follows that $N(P_1, P_2, ..., P_n) = |A_1 \cap A_2 \cap ... \cap A_n|$. Let $N(P_1', P_2', ..., P_n')$ be the number of elements with none of the properties $P_1, P_2, ..., P_n$. From the inclusion–exclusion principle, we see that:

$$N(P_1' P_2' ... P_n') = N - |A_1 \cup A_2 \cup ... \cup A_n| = N + (-1) \sum_{1 \leq i \leq n} N(P_i) +$$
$$(-1)^2 \sum_{1 \leq i < j \leq n} N(P_i P_j) + (-1)^3 \sum_{1 \leq i < j < k \leq n} N(P_i P_j P_k) + ... + (-1)^n N(P_1 P_2 ... P_n)$$

📋 The Sieve of Eratoshenes

A method to find the number of primes not exceeding a specified positive integer.

🌰 e.g.

Take 100 for example.

Solution:

A composite integer is divisible by a prime not exceeding its square root, so composite integer not exceeding 100 must have a prime factor not exceeding 10. Since the only primes less than 10 are 2,3,5,7, the primes not exceeding 100 are these four primes and the positive integers greater than 1 and not exceeding 100 that are divisible by none of 2,3,5,7.

Firstly, we pick out the composites divisible by 2 and change their color into grey. Then pick out the composites divisible by 3 from the rest numbers and so on. Then the black integers are the primes we want.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

e.g.

Let m and n be positive integers with $m \geq n$. Then, there are $n^m - C(n,1)(n-1)^m + C(n,2)(n-2)^m - ... + (-1)^{n-1}C(n,n-1)1^m$ onto functions from a set with m elements to a set with n elements.

Proof:

$$A = \{a_1, a_2, ..., a_m\}, B = \{b_1, b_2, ..., b_n\}$$

Let P_i be the property that b_i is not in the range of the function, respectively. A function is onto if and only if it has none of these properties.

`Derangement` : A derangement is a permutation of objects that leaves no object in the original position.

e.g.

21453 is a derangement of 12345 because no number is left in its original position.

📋 The number of derangements of a set with n elements can be calculated using inclusion–exclusion, and is $D_n = n![1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + ... + (-1)^n \frac{1}{n!}]$.

Proof:

Let a permutation have property P_i if it fixes element i . The number of derangements is the number of permutation having none of the properties P_i for $i = 1, 2, \ldots, n$, namely.

$$
\begin{aligned}
D_n &= N(P_1'P_2'...P_n') \\
&= N + (-1)\sum_{1\leq i\leq n} N(P_i) + (-1)^2 \sum_{1\leq i<j\leq n} N(P_iP_j) + ... + (-1)^n N(P_1P_2...P_n) \\
&= n! - C(n,1)(n-1)! + C(n,2)(n-2)! - ... + (-1)^n C(n,n)(n-n)! \\
&= n! - \frac{n!}{1!(n-1)!}(n-1)! + ... + (-1)^n \frac{n!}{n!(n-n)!}(n-n)! \\
&= n![1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + ... + (-1)^n \frac{1}{n!}]
\end{aligned}
$$