Business Analytics

Regression Analysis

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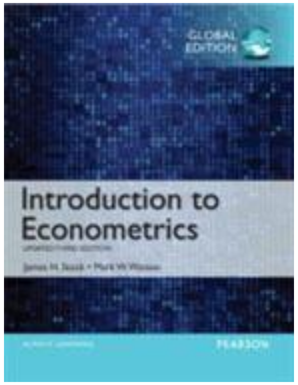
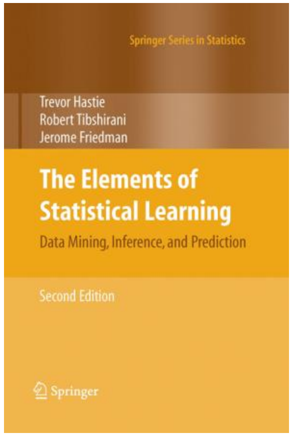
Course Content

- Introduction
- **Regression Analysis**
- Regression Diagnostics
- Logistic and Poisson Regression
- Naive Bayes and Bayesian Networks
- Decision Tree Classifiers
- Data Preparation and Causal Inference
- Model Selection and Learning Theory
- Ensemble Methods and Clustering
- High-Dimensional Problems
- Association Rules and Recommenders
- Neural Networks

Recommended Literature



- **Introduction to Econometrics**
  - Stock, James H., and Mark W. Watson
  - Chapter 2 – 7, 17, 18

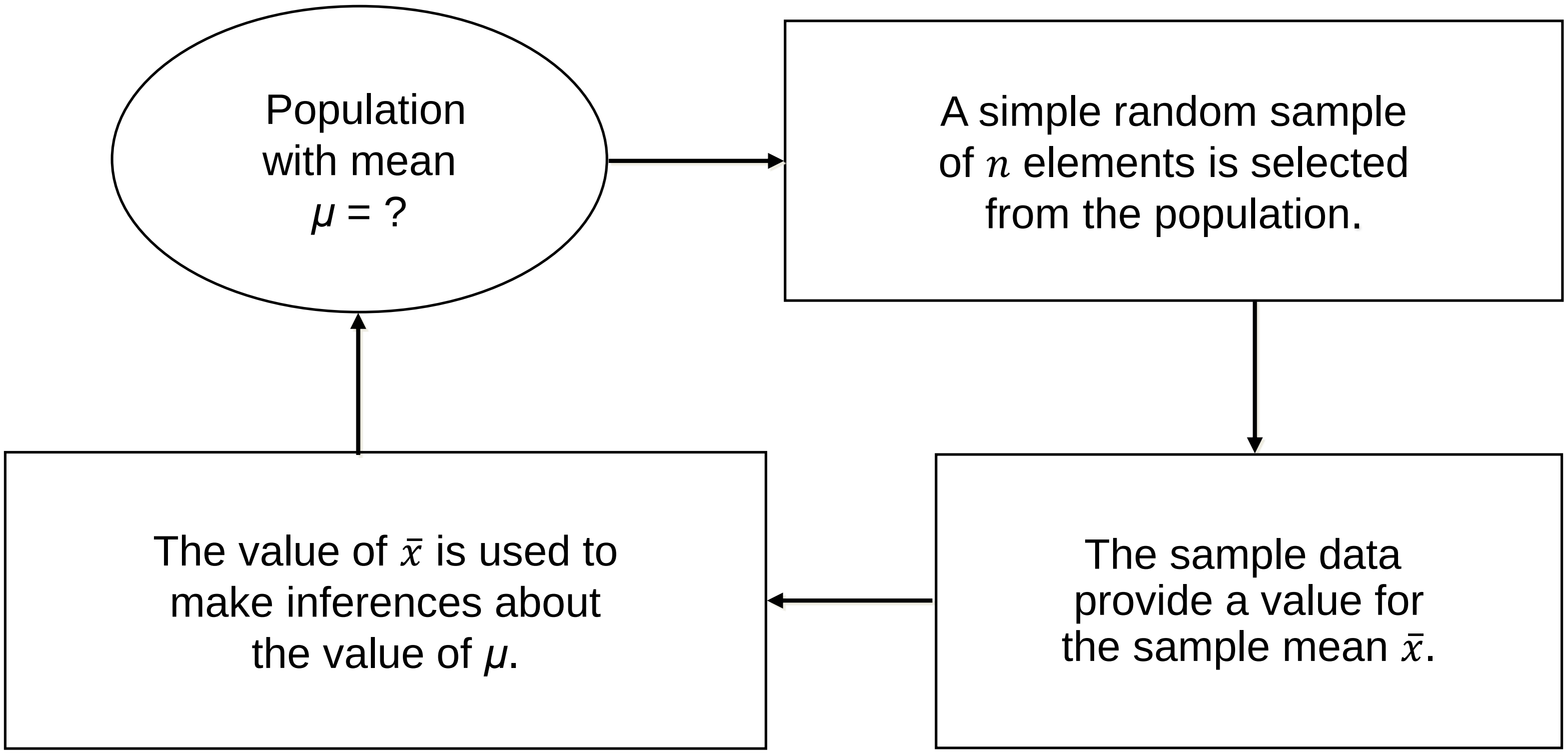


- **The Elements of Statistical Learning**
  - Trevor Hastie, Robert Tibshirani, Jerome Friedman
  - https://web.stanford.edu/~hastie/ElemStatLearn/
  - Section 3.1-3.2: Linear Methods for Regression

- **Any Introduction to Statistics**
  (e.g.: Statistical Inference by George Casella, Roger L. Berger
  or online course http://onlinestatbook.com/)

Today we revisit three important elements of statistical inference:
- Estimation, testing, regression

Statistical Estimation



Weak Law of Large Numbers

Weak Law of Large Numbers:

$$\lim_{\{n\to\infty\}} \Pr(|\bar{X}_n - \mu| > \varepsilon) = 0$$

In other words: $n \to \infty$, $\bar{X}_n \to \mu$

Chebyshev‘s inequality:
No more than a certain fraction of values can be more than a certain distance from the mean.

$$\Pr(|X - \mu| > \varepsilon) \le \frac{Var(X)}{\varepsilon^2}$$

$$\left.\Pr(|\bar{X}_n - \mu| > \varepsilon) \le \frac{Var(\bar{X}_n)}{\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2}\right\} \to 0 \text{ with } n \to \infty$$

$$\bar{X}_n = \frac{1}{n}\Sigma_{i=1}^{n} X_i,\ Var(\bar{X}_n) = \frac{1}{n^2}\Sigma_{i=1}^{n} Var(X_i) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

Statistical Estimation

Estimate

Point estimate

- sample mean
- sample proportion

Interval estimate

- confidence interval for mean
- confidence interval for proportion

Point estimate is always within the interval estimate

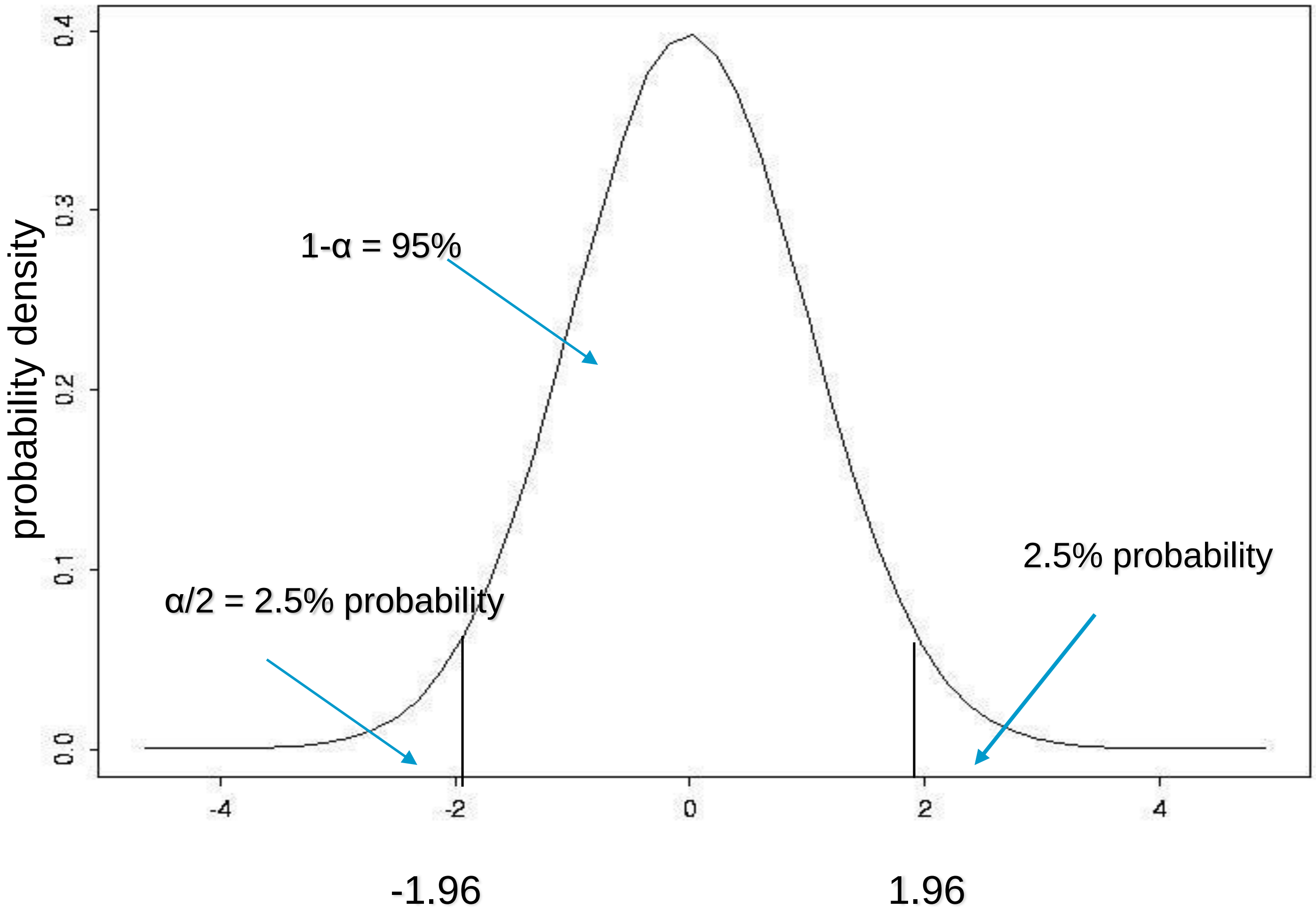
Confidence Interval (CI)

Provide us with a range of values that we believe, with a given level of confidence, contains a population parameter CI for the population means:

$$\Pr(\bar{X} - 1.96\,SD < \mu < \bar{X} + 1.96\,SD) = 0.95$$

There is a 95% chance that your interval contains μ.

Standard Normal Distribution



Example

Suppose sample of n = 100 persons

mean = 215, standard deviation = 20

95% CI = $\bar{X} \pm 1.96 * s/\sqrt{n}$

Lower Limit: 215 – 1.96*20/10

Upper Limit: 215 + 1.96*20/10

= (211, 219)

"We are 95% confident that the interval 211-219 contains μ."

Effect of Sample Size

Suppose we had only 10 observations

What happens to the confidence interval?

$$\bar{X} \pm 1.96 * \frac{s}{\sqrt{n}}$$

For $n = 100$, $215 \pm 1.96 * (20)/\sqrt{100} \approx (211{,}219)$

For $n = 10$, $215 \pm 1.96 * (20)/\sqrt{10} \approx (203{,}227)$

Larger sample size = smaller interval

Effect of Confidence Level

Suppose we use a 90% confidence level
What happens to the confidence interval?

$$\bar{X} \pm 1.645 * s / \sqrt{n}$$

90%: $215 \pm 1.645 * (20) / \sqrt{100} \approx (212, 218)$

Lower confidence level = smaller interval
(A 99% interval would use 2.58 as multiplier and
the interval would be larger)

Effect of Standard Deviation

Suppose we had a *s* of 40 (instead of 20)

What happens to the confidence interval?

$$\bar{X} \pm 1.96 * s / \sqrt{n}$$

$$215 \pm 1.96 * (40) / \sqrt{100} \approx (207, 223)$$

More variation = larger interval

Estimation for Population Mean μ

Point estimate: $\bar{X} = \frac{\sum X}{n}$

Estimate of variability in population $s = \sqrt{\frac{1}{n-1}\sum_i (X_i - \bar{X})^2}$

(<u>if σ is unknown, use s</u>)

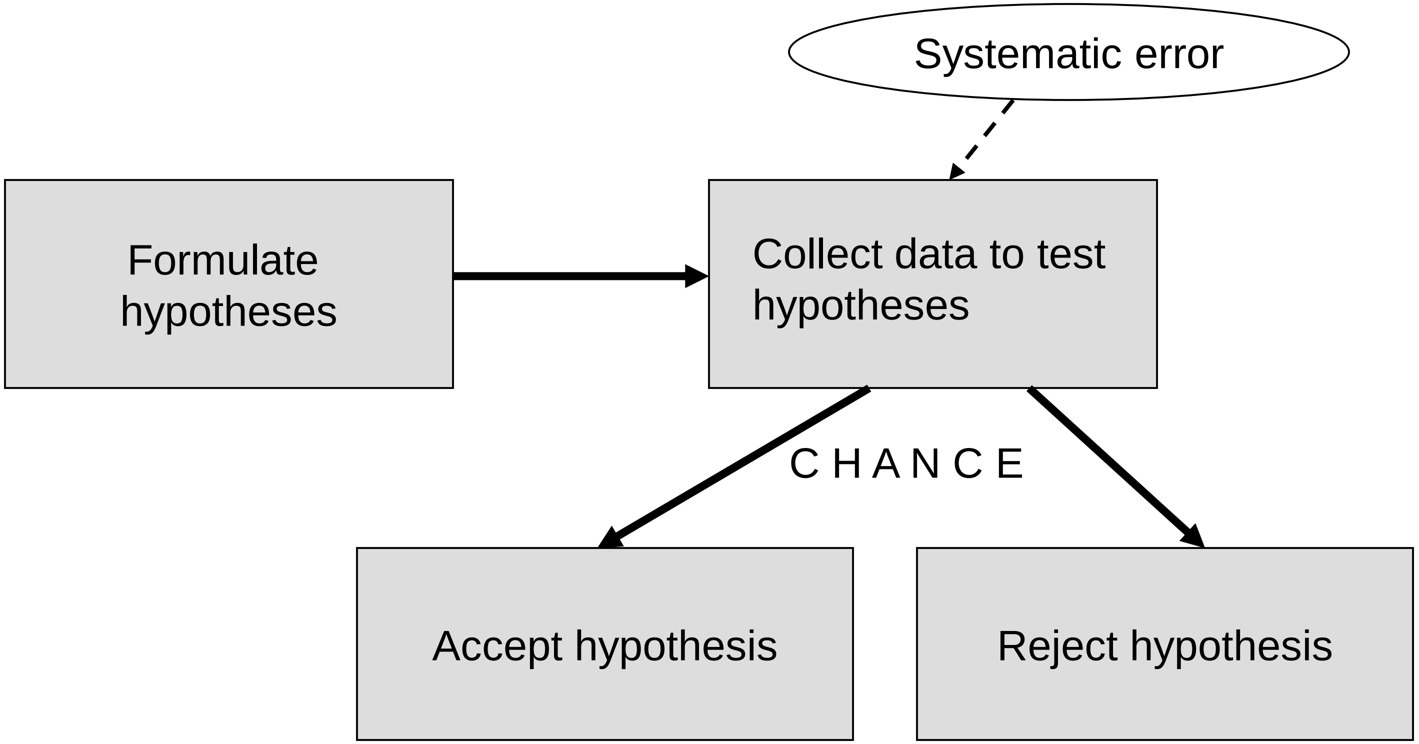
True standard deviation of sample mean $SD = \sigma/\sqrt{n}$

Standard error of sample mean $SE = s/\sqrt{n}$

95% confidence Interval $\bar{X} \pm 1.96\, SD$

, or $\bar{X} \pm 1.96\, SE$

Statistical Tests



Random error (chance) can be controlled by statistical significance or by confidence interval

Hypothesis Testing

- State null and alternative hypothesis (H_0 and H_1)
  - H_0 usually a statement of no effect or no difference between groups
- Choose α level (related to confidence level)
  - Probability of falsely rejecting H_0 (Type I error), typically 0.05 or 0.01
- Calculate test statistic, find p-value (p)
  - Measures how far data are from what you expect under null hypothesis
- State conclusion:

$p \leq \alpha$, reject H_0

$p > \alpha$, insufficient evidence to reject H_0

Hypothesis Testing

Hypothesis: A statement about parameters of population or of a model ($\mu = 200$?)

Test: Does the data agree with the hypothesis? (sample mean 220)
Simple random sample from a normal population
(or n large enough for CLT)

H_o: $\mu = \mu_o$
H_1 : $\mu \neq \mu_o$, pick α

Z-Test

Problem of interest:

- Population mean μ of a normal distribution
- known σ

Z-confidence interval: $\bar{X} \pm \mathrm{z}_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}$

Z-test: $\mathrm{z} = \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}$

$\underline{H_1}$	<u>Rejection region</u>
$\mu \neq \mu_0$	$\lvert \mathrm{z} \rvert \geq \mathrm{z}_{1-\alpha/2}$
$\mu > \mu_0$	$\mathrm{z} \geq \mathrm{z}_{1-\alpha}$
$\mu < \mu_0$	$\mathrm{z} \leq \mathrm{z}_{\alpha} \quad = -\mathrm{z}_{1-\alpha}$

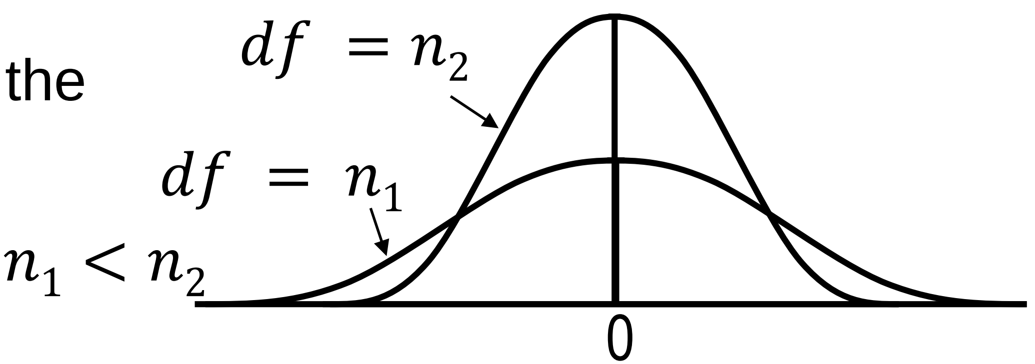
Student t-Distribution: Test Statistic for a Normal Mean μ with unknown σ

$$t(df = n - 1) = \frac{\bar{X} - \mu}{s/\sqrt{n}}$$

When the population is normally distributed, the statistic t is *Student t* distributed.

The "degrees of freedom (df)", a function of the sample size, determines how spread the distribution is (compared to the normal distribution)

The t distribution is bell-shaped, and symmetric around zero.

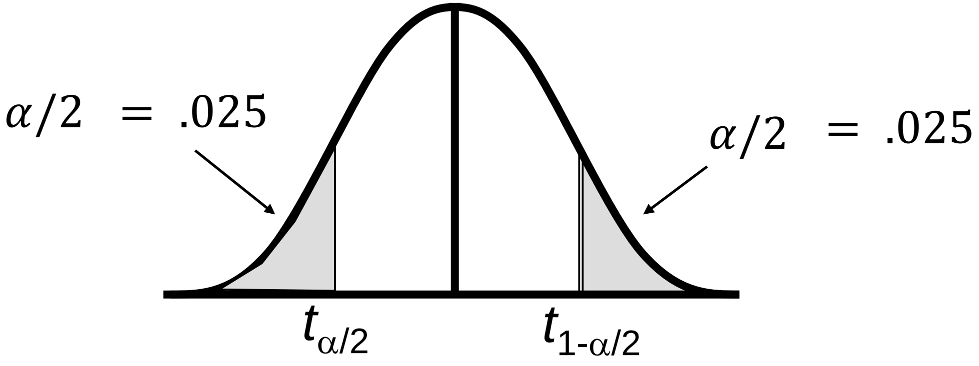


CI and 2-Sided Tests

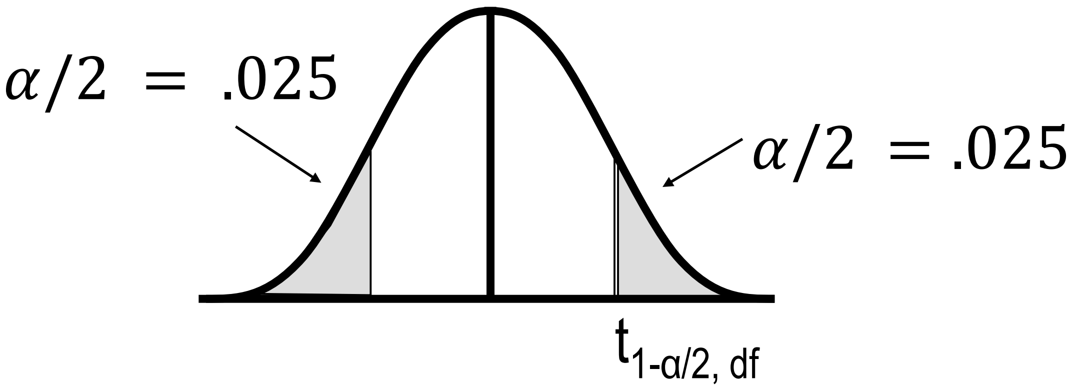
- A level α 2-sided test rejects $\mathrm{H}_0: \mu = \mu_0$ exactly when the value μ_0 falls outside a level $1 - \alpha$ confidence interval for μ.
- Calculate $1 - \alpha$ level confidence interval, then
  - if μ_0 within the interval, do not reject the null hypothesis,

$$|\mathrm{t}| < \mathrm{t}_{1-\alpha/2}$$

  - otherwise, $|\mathrm{t}| \geq \mathrm{t}_{1-\alpha/2}$ => reject the null hypothesis.



Student t-Distribution for α=0.05



Degrees of Freedom	$t_{.9}$	$t_{.95}$	$t_{.975}$	$t_{.99}$	$t_{.995}$
1	3.078	6.314	12.706	31.821	63.657
2	1.886	2.92	4.303	6.965	9.925
.	.	.	.	.	.
.	.	.	.	.	.
24	.	1.711	2.064	2.492	.
.	.	.	.	.	.
.	.	.	.	.	.
200	1.286	1.653	1.972	2.345	2.601
∞	1.282	1.645	1.96	2.326	2.576

t-distribution critical values

Possible Results of Tests

What we decide

Reality	Reject null	Fail to reject null
Null true	Type I Error (α) (false positive)	Correct
Null false	Correct	Type II Error (β) (false negative)

Type I error - You reject the null hypothesis when the null hypothesis is actually true.

Type II error - You fail to reject the null hypothesis when the the alternative hypothesis is true.

t-Tests

Formula is slightly different for each:

- *Single sample:*
  - tests whether a sample mean is significantly different from a pre-existing value
- *Paired samples:*
  - tests the relationship between 2 linked samples, e.g. means obtained in 2 conditions by a single group of participants
- *Independent samples:*
  - tests the relationship between 2 independent populations

The Paired t -Test with 2 Paired Samples

Null hypothesis: $\mathrm{H}_0\!: \mu_d = \mu_1 - \mu_2 = \Delta_0$

Test statistic: $t = \frac{\bar{d} - \Delta_0}{s/\sqrt{n}}$

H_1	Rejection region
$\mu_d \neq \Delta_0$	$\lvert t \rvert \geq t_{1-\alpha/2,\, n-1}$
$\mu_d > \Delta_0$	$t \geq t_{1-\alpha,\, n-1}$
$\mu_d < \Delta_0$	$t \leq t_{\alpha,\, n-1} = -t_{1-\alpha,\, n-1}$

Observations are dependent, e.g., pre and post test, left and right eyes, brother-sister pairs

The Paired t -Test with 2 Paired Samples

Subjects: random sample of 25 students from TUM

Mean grades of the students on two subsequent exams A and B

Is there a significant difference between the two exams?

Null Hypothesis: $E(A) = E(B)$

Answer can be given based on significance testing

No.	*A*	*B*	d=*A-B*
1	3.7	3.5	0.2
2	2.2	2.3	-0.1
...			
25	4.8	4.4	0.4

$\bar{d} = 0.093$

$s = 0.150$

$n = 25$

$s/\sqrt{n} = 0.03$

$t_{0.975;24} = 2.064$

$$t = \frac{\bar{d}}{s/\sqrt{n}} = \frac{0.093}{0.03} = 3.1$$

$$p = \Pr\{|\mathrm{t}| > 3.1 | DF = 24\} = 0.005$$

The p-Value

The p-value describes the probability of having $t = 3.1$ (or larger), given the null hypothesis. The smaller the p-value, the more unlikely it is to observe the corresponding sample value (or more extreme) by chance under H_0.

```
> # R code
> t.test(1:10,y=c(7:20),alternative = c("two.sided"),
         paired = FALSE, var.equal=FALSE, conf.level = 0.95)

        Welch Two Sample t-test

data:  1:10 and c(7:20)
t = -5.4349, df = 21.982, p-value = 1.855e-05
alternative hypothesis: true difference in means is not equal to 0
95 percent confidence interval:
 -11.052802  -4.947198
sample estimates:
mean of x mean of y 
      5.5      13.5 
```

Independent Samples

2 independent samples (possibly different size and variance):

Does the amount of credit card debt differ between households in rural areas compared to households in urban areas?

Population 1	Population 2
All Rural Households	All Urban Households
m_1	m_2

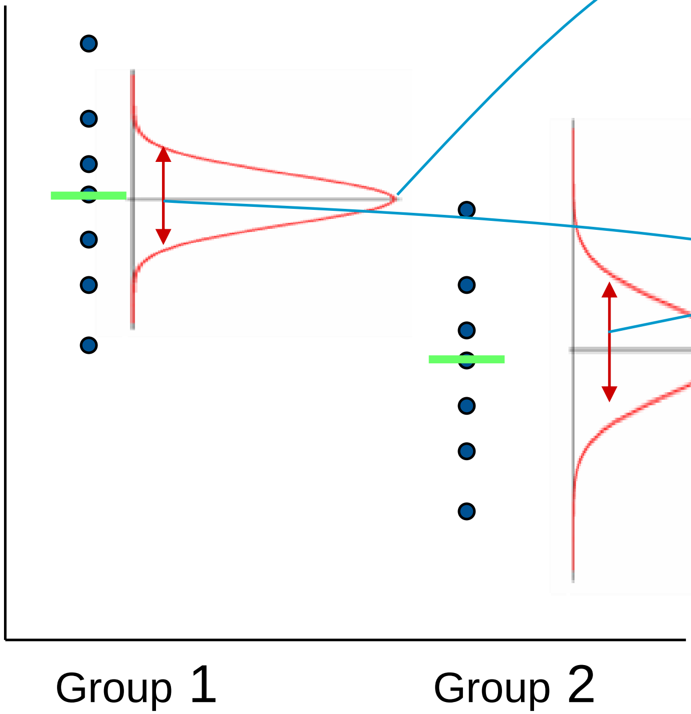
Null Hypothesis: $\mathrm{H}_0 : m_1 = m_2$

Alternate Hypothesis: $\mathrm{H}_1 : m_1 \neq m_2$

Independent Two-Sample t -Test

Two-sample unpaired t-test with (un)equal sample sizes, assuming unequal variance

Under $\mathrm{H_o}$ t follows a t-distribution with $\dfrac{(s_1{}^2/n_1 + s_2{}^2/n_2)^2}{(s_1{}^2/n_1)^2/(n_1-1) + (s_2{}^2/n_2)^2/(n_2-1)}$ degrees of freedom (df)



$$t = \frac{\bar{x}_1 - \bar{x}_2}{S_{\bar{x}_1 - \bar{x}_2}}$$

$$S_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

Independent Two-Sample t –Test: Example

Group 1	Group 2
21	22
19	25
18	27
18	24
23	26
17	24
19	28
16	26
21	30
18	28
$\bar{x} = 19$	$\bar{x} = 26$
$s = \sqrt{40}$	$s = \sqrt{50}$

$$df = 18 \ (rounded\ to\ integer)$$

$$t = \frac{\bar{x}_1 - \bar{x}_2}{S_{\bar{x}_1 - \bar{x}_2}} = \frac{19 - 26}{3} = -2.333$$

$$S_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} = \sqrt{\frac{40}{10} + \frac{50}{10}}$$

$$t_{(0.975,18)} = 2.101$$

$$|t| \geq t_{(0.975,18)}$$

→ Reject H_0 $(\mu_1 - \mu_2 = 0)$

Selected Statistical Tests

- **Parametric Tests**
  - The family of t-tests
    - Compares two sample means or tests a single sample mean
  - F-test
    - Compares the equivalence of variances of two samples
- **Non-parametric Tests**
  - Wilcoxon signed-rank test
    - Independence of two means for 2 paired i.i.d samples, when normality is not assumed.
  - Mann-Whitney-U test is used for 2 independent samples
  - Kruskal-Wallis-Test
    - Equivalence of multiple means in case of several i.i.d non-normally distributed samples
- **Tests of the Probability Distribution**
  - Kolmogorov-Smirnov and Chi-square test
    - used to determine whether two underlying probability distributions differ, or whether an underlying probability distribution differs from a hypothesized distribution

Linear Regression

- **Regressions identify relationships between dependent and independent variables**
  - Is there an association between the two variables
  - Estimation of impact of an independent variable
  - Formulation of the relation in a functional form
  - Used for numerical prediction and time series forecasting

- **Regression as an established statistical technique:**
  - Sir Francis Galton (1822-1911) studied the relationship between a father's height and the son's height

Terminology

- Data streams X and Y, forming the measurement tuples $(x_1, y_1), \ldots, (x_n, y_n)$
- x_i is the predictor (regressor, covariate, feature, independent variable)
- y_i is the response (dependent variable, outcome)
- Denote the *regression function* by: $\eta(x) = E\,(Y\,|x)$
- The linear regression model assumes a specific linear form

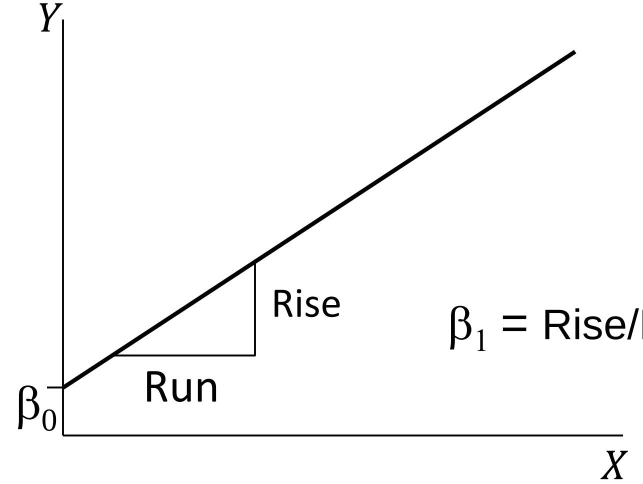
The Simple Linear Regression Model

- Linear regression is a statistical tool for numerical predictions
- The first order linear model

$$Y = \beta_0 + \beta_1 X + \varepsilon$$

β_0 and β_1 are unknown, therefore, are estimated from the data

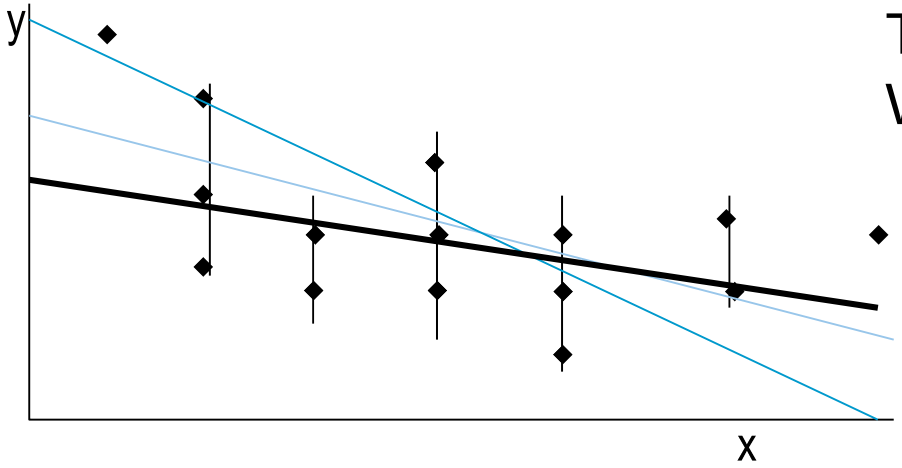
Y = respond variable
X = predictor variable
β_0= y-axis intercept
β_1 = slope of the line
ε = random error term (residual)



β_1 = Rise/Run

Estimating the Coefficients

- Coefficients are random variables
- (Ordinary Least Squares) estimates are determined by
  - drawing a sample from the population of interest
  - calculating sample statistics
  - producing a straight line that cuts into the data



The question is:
Which straight line fits best?

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$$

OLS Estimators

- Ordinary Least Squares (OLS) approach:
  - Minimize the sum of squared residuals (aka. loss function)

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$$

$$\min \sum_{\mathrm{i}} e_{\mathrm{i}}^2 = \min \sum_{\mathrm{i}} (y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_i))^2$$

$$\hat{\beta}_1 = \frac{\sum_i (x_i - \overline{x})(y_i - \overline{y})}{\sum_i (x_i - \overline{x})^2} = \frac{Cov(x, y)}{Var(x)}$$

$$\hat{\beta}_0 = \overline{y} - \hat{\beta}_1 \overline{x}$$

Example

- A car dealer wants to find the relationship between the odometer reading and the selling price of used cars.
- A random sample of 100 cars is selected, and the data recorded.
- Find the regression line.

Car	Odometer	Price
1	37388	5318
2	44758	5061
3	45833	5008
4	30862	5795
5	31705	5784
6	34010	5359
.	.	.
.	.	.
.	.	.

Independent/predictor variable x

Dependent/respond variable y

Solving by Hand

- To calculate β_0 and β_1 we can calculate several statistics first:

$$\bar{x} = 36009.45; \qquad s_x^2 = \frac{\sum (x_i - \bar{x})^2}{n-1} = 43{,}528{,}688$$

$$\bar{y} = 5411.41; \qquad \operatorname{cov}(X,Y) = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{n-1} = -1{,}356{,}256$$

where $n = 100$:

$$\hat{\beta}_1 = \frac{\operatorname{cov}(X,Y)}{s_x^2} = \frac{-1{,}356{,}256}{43{,}528{,}688} = -.0312$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} = 5411.41 - (-.0312)(36{,}009.45) = 6{,}533$$

$$\boxed{\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x = 6{,}533 - 0.0312x}$$

Residual Sum of Squares (RSS)

- This is the sum of squared differences between the points and the regression line
- It can serve as a measure of how well the line fits the data (fits well, if statistic is small)
- An unbiased estimator of the RSS of the population is given by

$$RSS = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$$

Total Deviation

- The Total Sum of Squares (TSS) is the sum of the Explained Sum of Squares (ESS) and the RSS.

$$\boxed{\sum(y-\bar{y})^2 \qquad = \sum(\hat{y}-\bar{y})^2 \qquad + \sum(y-\hat{y})^2}$$

TSS = ESS + RSS

Total deviation = explained deviation + unexplained deviation

Coefficient of Determination

- R^2 measures the proportion of the variation in y that is explained by the variation in x

$$ESS = \sum_{i=1}^{n} (\hat{y}_i - \bar{y})^2 \;, TSS = \sum_{i=1}^{n} (y_i - \bar{y})^2 = ESS + RSS$$

$$R^2 = \frac{TSS - RSS}{TSS} = 1 - \frac{RSS}{TSS} = \frac{ESS}{TSS}$$

- R^2 takes on any value between zero and one
  – $R^2 = 1$: Perfect match between the line and data points
  – $R^2 = 0$: There is no linear relationship between x and y

Testing the Coefficients

- Test the significance of the linear relationship

  $\mathrm{H}_0: \beta_1 = 0$

  $\mathrm{H}_1: \beta_1 \neq 0$

- The test statistic is

$$t = \frac{\hat{\beta}_1}{\mathrm{SE}(\hat{\beta}_1)} = \frac{\hat{\beta}_1}{\sqrt{\frac{\mathrm{RSS}}{\sum_{i=1}^{n}(x_i - \bar{x})^2} \cdot \frac{1}{n-2}}}$$

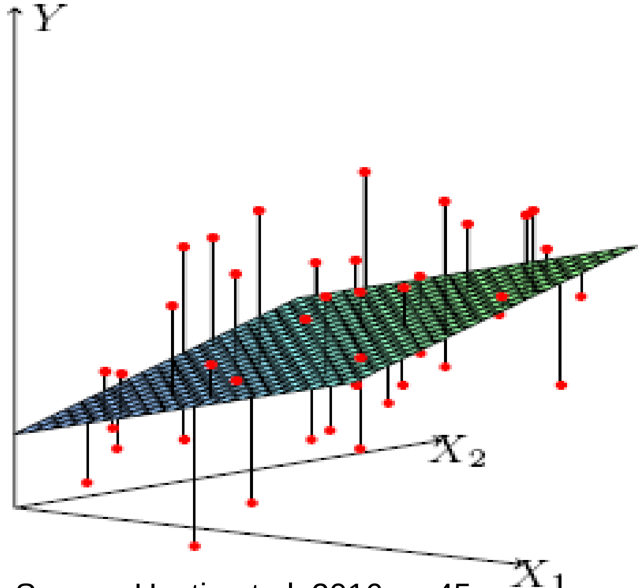
Variance of $\hat{\beta}_1$

- If $SE(\hat{\beta}_1)$ is large, then $\hat{\beta}_1$ must be large to reject H_0
- $SE(\hat{\beta}_1)$ is smaller, if the x_i are more spread out
- If the error variable is normally distributed, the statistic is a Student t distribution with $n-2$ degrees of freedom (if n is large, draw on the CLT)
- Reject H_0, if: $t < t_{\alpha/2}$ or $t > t_{1-\alpha/2}$

The Multiple Linear Regression Model

- A p-variable regression model can be expressed as a series of equations
- Equations condensed into a matrix form, give the general linear model
- β coefficients are known as partial regression coefficients
- X_1, X_2, for example,
  - X_1='years of experience'
  - X_2='age'
  - Y='salary'
- Estimated equation:

$\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X_1 + \hat{\beta}_2 X_2 = \mathbf{X}\hat{\beta}$



Source: Hastie et al. 2016, p. 45

Matrix Notation

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 1\, x_{11}\, x_{12} \dots x_{1p} \\ 1\, x_{21}\, x_{22} \dots x_{2p} \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ 1\, x_{n1}\, x_{n2} \cdots x_{np} \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_p \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix}$$

y =	$\boldsymbol{X}$	β	+ ε
$(n \times 1)$	$(n \times (p+1))$	$((p+1) \times 1)$	$(n \times 1)$

OLS Estimation

- Sample-based counter part to population regression model:

$$y = \mathbf{X}\beta + \varepsilon$$
$$y = \mathbf{X}\hat{\beta} + e$$

- OLS requires choosing values of the estimated coefficients, such that Residual Sum of Squares (RSS) is as small as possible for the sample

$$RSS = e^T e = (y - \mathbf{X}\hat{\beta})^T (y - \mathbf{X}\hat{\beta})$$

- Need to differentiate with respect to the unknown coefficients

Finding the Least Squares

X is $n \times (p+1)$, y is the N-vector of outputs

RSS(β) = $(y - \mathbf{X}\beta)^T (y - \mathbf{X}\beta)$

If X is full rank, then $X^T X$ is positive definite

$$\Rightarrow \frac{\partial \mathrm{RSS}}{\partial \beta} = -2\mathrm{X}^{\mathrm{T}}(\mathrm{y} - \mathrm{X}\beta), \qquad \frac{\partial^2 RSS}{\partial \beta \partial \beta^T} = 2X^T X$$

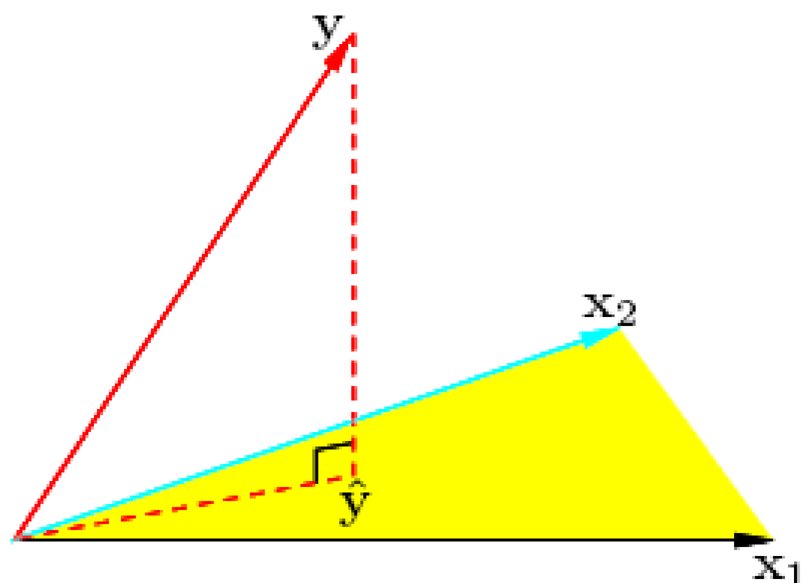
$$\frac{\partial \mathrm{RSS}}{\partial \beta} = 0 \Rightarrow \mathrm{X}^{\mathrm{T}}(\mathrm{y} - \mathrm{X}\beta) = 0$$

$$\Rightarrow \hat{\beta} = (\mathrm{X}^{\mathrm{T}}\mathrm{X})^{-1}\mathrm{X}^{\mathrm{T}}\mathrm{y} \Rightarrow \hat{\mathrm{y}} = \mathrm{X}\hat{\beta} = \underbrace{\mathrm{X}(\mathrm{X}^{\mathrm{T}}\mathrm{X})^{-1}\mathrm{X}^{\mathrm{T}}}_{\text{“Hat” or projection matrix } H}\mathrm{y}$$

Geometrical Representation

- Least square estimates in $\Re^N$
- Minimize RSS(β)=$||y - \mathbf{X}\beta||^2$, s.t. residual vector y – $\hat{y}$ is orthogonal to this subspace

Figure 3.2: *The N-dimensional geometry of least squares regression with two predictors. The outcome vector* $\mathbf{y}$ *is orthogonally projected onto the hyperplane spanned by the input vectors* $\mathbf{x}_1$ *and* $\mathbf{x}_2$*. The projection* $\hat{\mathbf{y}}$ *represents the vector of the least squares predictions*



Source: Hastie et al. 2016, p. 46

Example

y:	2.6	1.6	4.0	3.0	4.9
x:	1.2	3.0	4.5	5.8	7.2

$y = \mathbf{X}\hat{\beta} + e$

$$\begin{pmatrix} 2.6 \\ 1.6 \\ 4.0 \\ 3.0 \\ 4.9 \end{pmatrix} = \begin{pmatrix} 1 & 1.2 \\ 1 & 3.0 \\ 1 & 4.5 \\ 1 & 5.8 \\ 1 & 7.2 \end{pmatrix} \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix} + \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{pmatrix}$$

$\hat{\beta} = \left(\mathbf{X}^{\mathbf{T}}\mathbf{X}\right)^{-\mathbf{1}}\mathbf{X}^{\mathbf{T}}\mathrm{y}$

$$\left(\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1.2 & 3.0 & 4.5 & 5.8 & 7.2 \end{pmatrix} \begin{pmatrix} 1 & 1.2 \\ 1 & 3.0 \\ 1 & 4.5 \\ 1 & 5.8 \\ 1 & 7.2 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1.2 & 3.0 & 4.5 & 5.8 & 7.2 \end{pmatrix} \begin{pmatrix} 2.6 \\ 1.6 \\ 4.0 \\ 3.0 \\ 4.9 \end{pmatrix} =$$

$$\begin{pmatrix} 5 & 21.7 \\ 21.7 & 116.17 \end{pmatrix}^{-1} \begin{pmatrix} 16.1 \\ 78.6 \end{pmatrix} = \begin{pmatrix} 1.0565 & -0.1973 \\ -0.1973 & 0.0455 \end{pmatrix} \begin{pmatrix} 16.1 \\ 78.6 \end{pmatrix} = \begin{pmatrix} 1.498 \\ 0.397 \end{pmatrix}$$

Check Results in R

```
> y <- c(2.6, 1.6, 4.0, 3.0, 4.9)
> x <- c(1.2, 3.0, 4.5, 5.8, 7.2)
> mod <- lm(y ~ x)
> summary(mod)

Call:
lm(formula = y ~ x)

Residuals:
      1       2       3       4       5
 0.6259 -1.0883  0.7165 -0.7993  0.5452

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   1.4980     1.0322   1.451    0.243
x             0.3968     0.2142   1.853    0.161

Residual standard error: 1.004 on 3 degrees of freedom
Multiple R-Squared: 0.5336,     Adjusted R-squared: 0.3782
F-statistic: 3.433 on 1 and 3 DF,  p-value: 0.1610
```

1. check coefficients
2. check significance
3. check coefficient of determination

Selected Statistics

Adjusted R^2

- It represents the proportion of variability of y explained by X
  R^2 is adjusted so that models with a different number of variables can be compared

$$\bar{R}^2 = 1 - \frac{RSS/(n-p-1)}{TSS/(n-1)}$$

The F-test

- Significant F indicates a linear relationship between y and at least one of the xs: $\mathrm{H}_0\text{:}\ \beta_1 = \beta_2 \ldots \beta_p = 0$

The *t*-test of each partial regression coefficient

- Significant t indicates that the variable in question influences the response variable while controlling for other explanatory variables

Model Specification

In regression analysis the specification is the process of developing a regression model.

- This process consists of selecting an appropriate functional form for the model and choosing which variables to include.
- The model might include irrelevant variables or omit relevant variables

Non-linear models are challenging, but some nonlinear regression problems can be linearized.

- Dummy variables for discrete variables (e.g. 0/1 for gender)
- Quadratic models: $y = \beta_0 + \beta_1 x_1 + \beta_2 {x_2}^2 + \varepsilon$ use $z_2 = {x_2}^2$
- Models with interaction terms $y = \beta_0 + \beta_1 x_1 x_2$ use $z_1 = x_1 x_2$
- Exponential terms $y = \alpha x^\beta \varepsilon$ can be transformed using the logarithm to

$$\ln(y) = \ln(\alpha) + \beta \ln(x) + \ln(\varepsilon)$$

Subset Selection

- Setting: Possibly a large set of predictor variables, some irrelevant
- Goal: Fit a parsimonious model that explains variation in Y with a small set of predictors
  - Aka. subset selection or feature selection problem
- Automated procedures:
  - Best subset (among all exponentially many, computationally expensive)
  - Backward elimination (top down approach)
  - Forward selection (bottom up approach)
  - Stepwise regression (combines forward/backward)

- More in the context of the class on dimensionality reduction
  - Subset selection vs. shrinkage methods

Example: Backward Elimination

- Select a significance level to stay in the model (generally 0.05 is too low, causing too many variables to be removed)
- Fit the full model with all possible predictors
- Consider the predictor with lowest t-statistic (highest p-value).
  - If p > sign. level, remove the predictor and fit model without this variable (must re-fit model here because partial regression coefficients change)
  - If $p \leq$ sign. level, stop and keep current model
- Continue until all predictors have p-values below sign. level

- Forward selection is similar: predictors with lowest p-value are added until none is left with p > sign. level.