Catam Project Report
PartII Additional Projects
(July 2020 Edition)

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10/03/2021

- Differential Equations for Nonlinear Oscillators
- Programs

1 Differential Equations for Nonlinear Oscillators

Set $\dot{x} = y$, we have from forced Duffing equation in PartI

$$\dot{x} = y,$$
$$\dot{y} = -ay + x - x^3 + bcos(t).$$

1.1 Question 1

1.1.1 Solving the forced Duffing equation: Programming Task

With $\mathbf{x} = (x, \dot{x}) = (x, y), \mathbf{f}(\mathbf{x}, t) = (y, -ay + x - x^3 + bcos(t))$, a program using the fourth order Runge-Kutta method is listed on page 15, named **RK4(a,b,x0,y0,h,tmax)**. Here we define h the step size and tmax the end point of the iterative process. Other quantities used the same as in the setup. h should be kept small in order to simulate the behavior.

Integrating the system from five initial conditions (i.c.s) (1,1),(0,1.1),(0.1,0),(-2,1),(1,-1) with a=0.12, b=0 gives five solutions in the $\dot{x}$-x plane as shown in Fig.1.

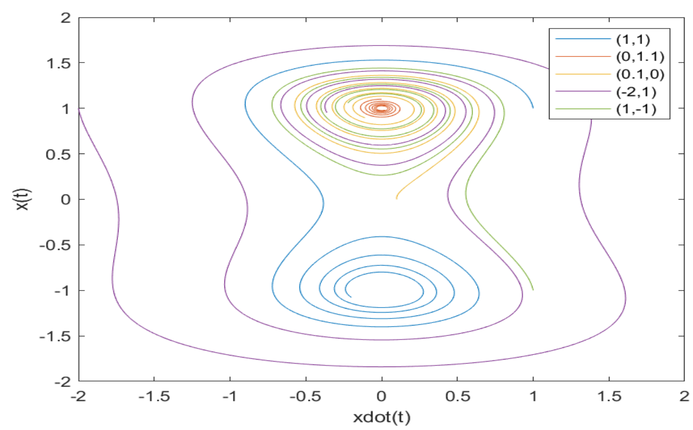

Figure 1: Solutions of the forced Duffing equation

1.2 Question 2

Running the program with $b = 0$ at $a = -0.12, a = 0$ and $a = 0.12$ gives results in Fig.2.

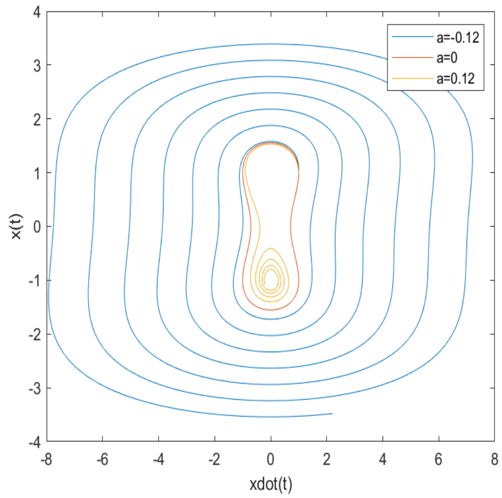

Figure 2: $a = -0.12, a = 0, a = 0.12$ with $b = 0$ and i.c.s (1,1)

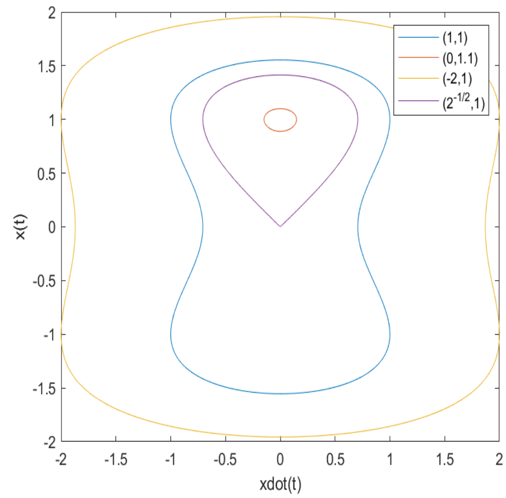

Figure 3: $a = b = 0$ with different i.c.s

when a=−0.12, b=0, starting at (1,1), trace spirals outwards around all 3 fixed points (0,0), (0,1), (0,-1) and diverges to infinity. when a=0.12, b=0, trace spirals inwards and converges to one of the fixed points (0,1), (0,-1). This can be checked by calculating the Jacobian

$$J=\begin{pmatrix} 0 & 1 \\ -3x^2 & -a \end{pmatrix},$$

which has eigenvalues $\lambda = (-a \pm \sqrt{a^2 + 4(1 - 3x^2)})/2$ implying unstable foci at (0,1) and (0,-1) when a=−0.12, while stable foci at these points when a=0.12.

when a=b=0, trace form a periodic orbit (PO) as can be seen in Fig.2-3. This is also confirmed by the Jacobian whose eigenvalues are pure imaginary at (0,1), (0,-1) implying centres. (0,0) saddle by checking signs of λ's (one +, one −). The system is **Hamiltonian** with $\dot{x} = y = \partial H/\partial y$, $\dot{y} = -\partial H/\partial x$ where $H = (2y^2 - 2x^2 + x^4)/4$. Trace is either a PO or a homoclinic orbit (2 homoclinic orbits start and end at saddle (0,0) by symmetry).

1.3 Question 3

When a=0.15, b=0.3, two stable solutions with i.c.s are shown in Fig.4-5.

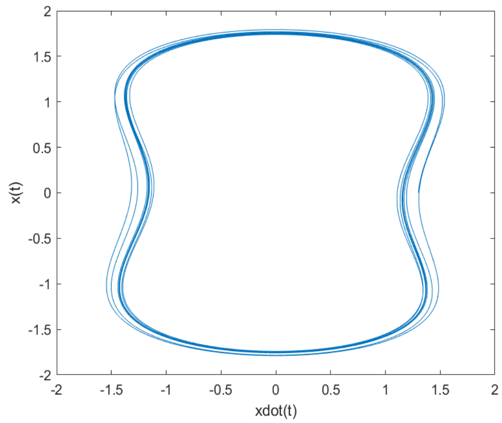

Figure 4: Approx. PO with i.c.s (1.3,0)

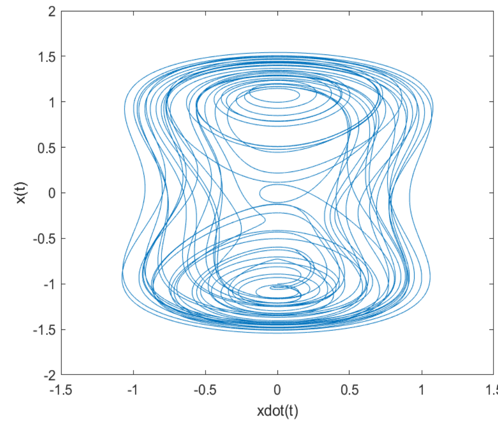

Figure 5: Strange attractor with i.c.s (0,-1)

1.4 Question 4

1.4.1 Solving the forced Duffing equation (Points Plot): Programming Task

An assistant program using RK4 with only points at $t = 2n\pi$ plotted for the solution is listed on page 16, named **RK4Q4(a,b,x0,y0,h,d,tmax)**. Here $d = 2\pi$ represents the interval of plotting. Other quantities used the same as in the setup. Results in Fig.6 (with original trajectories also presented for comparison.)

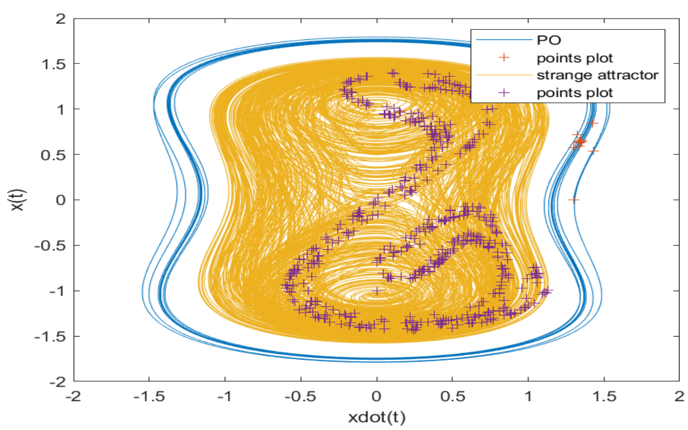

Figure 6: Points plot for PO and strange attractor at $t = 2n\pi$

While trajectories outline the whole picture better, points at $t = 2n\pi$ reveal the period of the PO by the way they accumulating on the stable solution - they distribute unevenly and most of them are located at one particular segment of the trace. Points plot also reveals the chaotic behavior of the solution, in the case of the strange attractor (see Fig.6).

1.5 Question 5

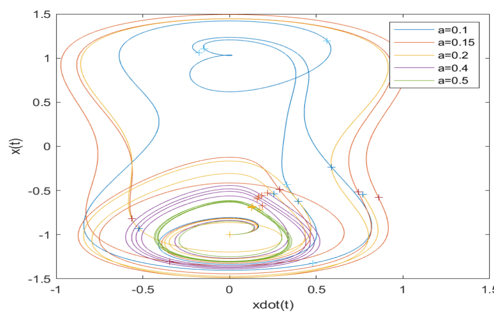

Figure 7: Vary a at (0,-1)

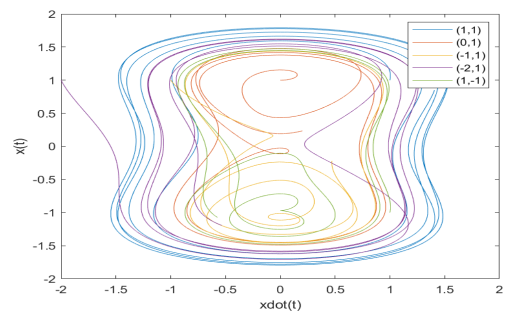

Figure 8: Vary i.c.s for a=0.15

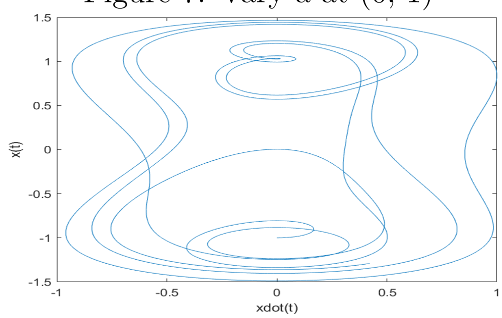

Figure 9: a=0.1

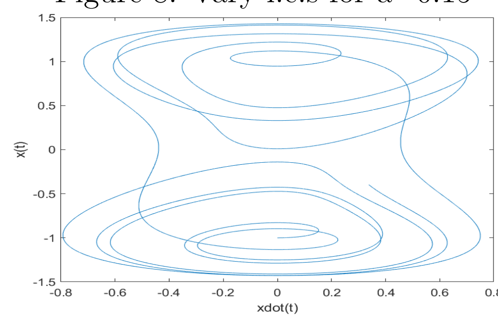

Figure 10: a=0.3

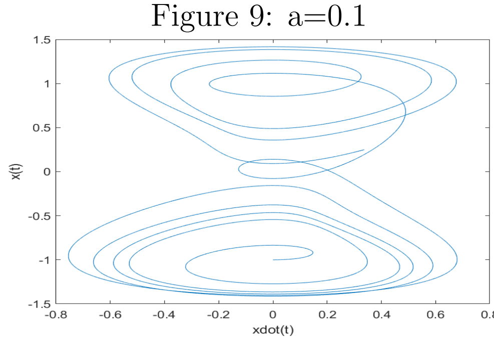

Figure 11: a=0.375

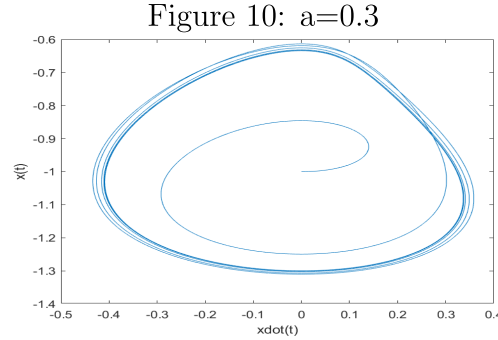

Figure 12: a=0.5

For b=0.3 fixed, integrating from $t = 0$ to $t = 10\pi$ or $t = 20\pi$ with step size $\pi/10000$,

- vary a at (0,-1) from 0.1 to 0.5, the trajectory goes from circling around 3 fixed points to circling around 1 fixed point at (0,-1) with a stretch towards (0,0)(see Fig.7)
- vary i.c.s for a =0.15, trajectory either circles around 3 fixed points (with i.c.s (1,1)) or circles around (0,1) and (0,-1) while goes near (0,0) at the start (with other i.c.s stated in Fig.8)
- there are some noteworthy features in the evolution as we change the value of a. Fix i.c.s to be (0,-1). When a=0.1(Fig.9), there is an outward spiral from (0,-1) to (0,1) to (0,-1) repeating. When a=0.3(=b)(Fig.10), same evolution but with the hourglass shape showing up. When a=0.375(Fig.11), more regular-shaped outward spiral from (0,-1) (circling half way around (0,0)) to (0,1) then to (0,-1). Process repeats giving an hourglass shape. When a=0.5(Fig.12), hourglass shape disappears and an approx. PO forms around (0,-1).

1.6 Question 6

Set $\dot{x} = y$, we have from forced Duffing equation in PartII

$$\dot{x} = y,$$
$$\dot{y} = (b - x^2)y + ax - x^3.$$

1.6.1 Solving the forced Duffing equation: Programming Task

With $\mathbf{x} = (x, \dot{x}) = (x, y)$, a program using the fourth order Runge-Kutta method is listed on page 18, named **RK4Q6(a,b,x0,y0,h,tmax)**.

Solutions plotted in Fig.13-20 with different i.c.s and different values of a, b (in each region).

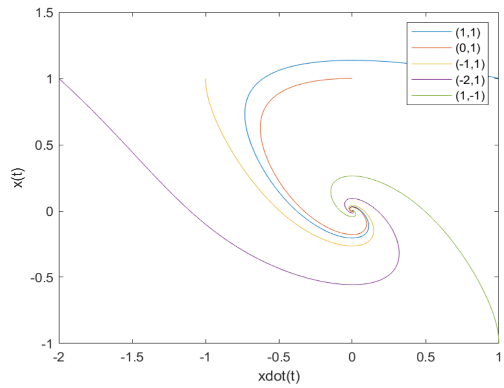

Figure 13: I, a=b=-1

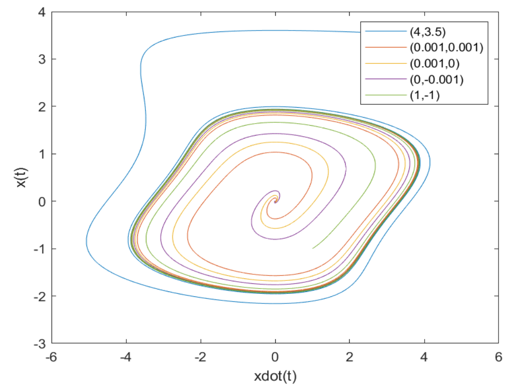

Figure 14: II, a=-1, b=1

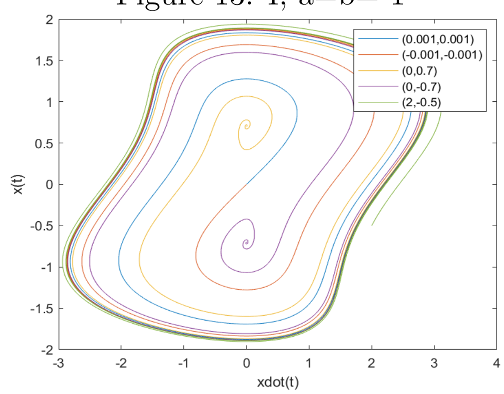

Figure 15: III, a=0.5, b=1

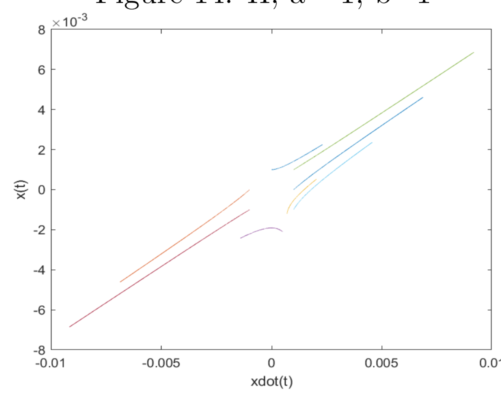

Figure 16: III, saddle at (0,0)

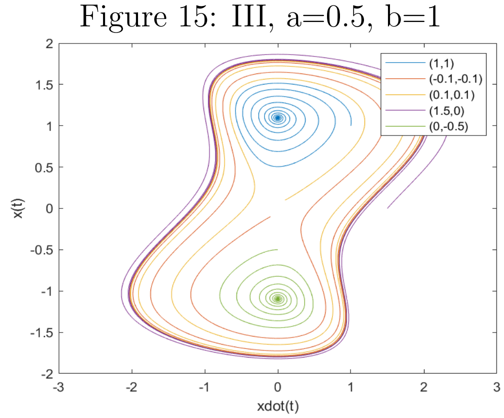

Figure 17: IV, a=1.2, b=1

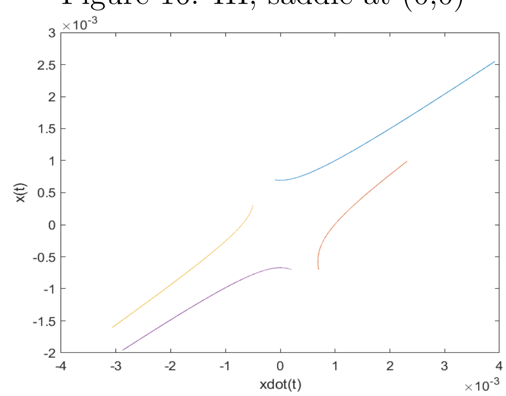

Figure 18: IV, saddle at (0,0)

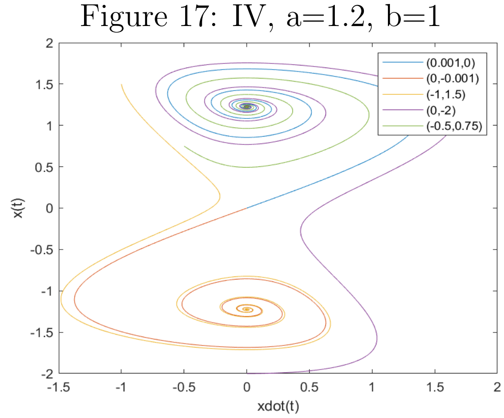

Figure 19: V, a=1.5, b-=1

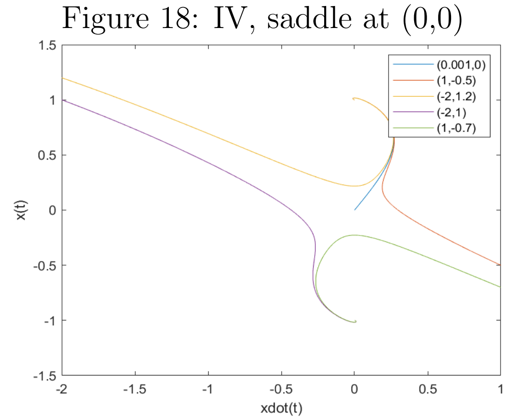

Figure 20: VI, a=1, b=-1

1.7 Question 7

By setting different values of a, b (as stated in Fig.13-20), integrating from $t = 0$ to $t = 30\pi$ with step size $\pi/10000$ over several i.c.s, look into different dynamics in region

- I. All trajectories spiral anticlockwise inwards to the fixed point (0,0), i.e., converge to (0,0), a stable focus (judging either by calculating the Jacobian or plain sight into the picture), which is asymptotically stable.

- II. All trajectories inside/outside the PO spiral outwards/inwards to the PO with inside spirals trace back to (0,0), an unstable focus.

- III. All trajectories inside/outside the PO spiral outwards/inwards to the PO with inside spirals trace back to $(0, \pm\sqrt{a})$, each being an unstable focus, or to (0,0), a saddle (by calculating the Jacobian or plain sight into Fig.16); 2 heteroclinic orbits from $(0, \pm\sqrt{a})$ to (0,0).

- IV. Trajectories around $(0, \pm\sqrt{a})$ spiral inwards to $(0, \pm\sqrt{a})$, each being a stable focus; trajectories around the saddle (0,0) (judging by calculating the Jacobian or plain sight into Fig.18) spiral outwards (around both $(0, \sqrt{a})$ and $(0, -\sqrt{a})$) to the PO; trajectories starting close to the PO (on the inside/outside) spiral outwards/inwards to the PO; 2 homoclinic orbits start and end at (0,0), circling around $(0, \sqrt{a})$ and $(0, -\sqrt{a})$ respectively.

- V. All except two trajectories spiral inwards from infinity to $(0, \pm\sqrt{a})$, each being a stable focus; 2 trajectories converge from infinity to (0,0), a saddle point; 2 heteroclinic orbits start from (0,0) and end at $(0, \pm\sqrt{a})$.

- VI. Same as in region V but with an unfolded shape towards direction of one of the eigenvectors.

When moving through the boundaries between each region, we see dynamics as in Fig.21-22 below.

- As we move from I to II, **supercritical Hopf bifurcation** occurs at the centre (0,0) - single fixed point (0,0) changes stability from stable focus (to centre instantaneously then) to unstable focus. A PO is created.

- As we move from II to III, the unstable focus at (0,0) split into 2 unstable foci at $(0, \sqrt{a})$ and $(0, -\sqrt{a})$ under **subcritical pitchfork bifurcation**. A saddle point is created at (0,0).

- As we move from III to IV, the fixed points at $(0, \sqrt{a})$ and $(0, -\sqrt{a})$ change stability from unstable foci (to centres instantaneously then) to stable foci. 2 heteroclinic orbits are created from $(0, \pm\sqrt{a})$ to (0,0).
- As we move from IV to V, PO destroyed under **supercritical Hopf bifurcation**.
- As we move from V to VI (letting c=4/7), the hourglass shape broke. Trajectories converging to $(0, \pm\sqrt{a})$ start and end at the same side of $(0, \pm\sqrt{a})$ without crossing the particular trajectory that comes from infinity to the saddle (0,0).
- As we move from VI to I, line of fixed points goes through (0,0), which changes from a saddle (to an improper node instantaneously then) into a stable focus. 2 stable foci disappeared under **supercritical pitchfork bifurcation**.

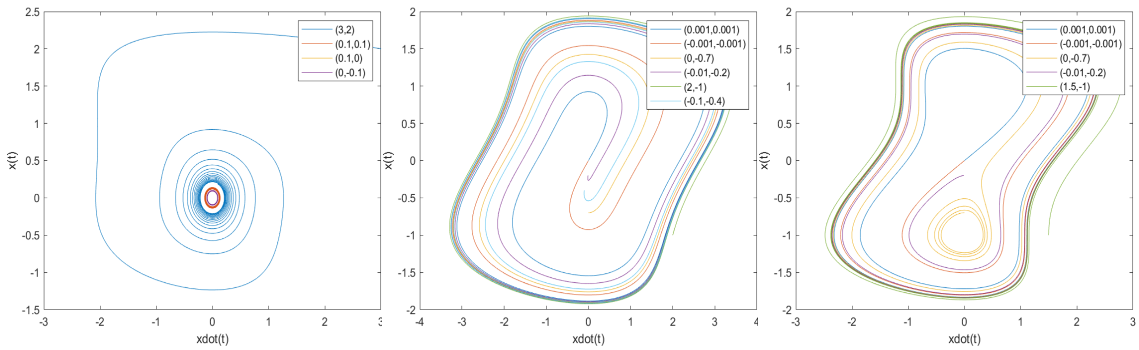

Figure 21: I-III, II-III, III-IV

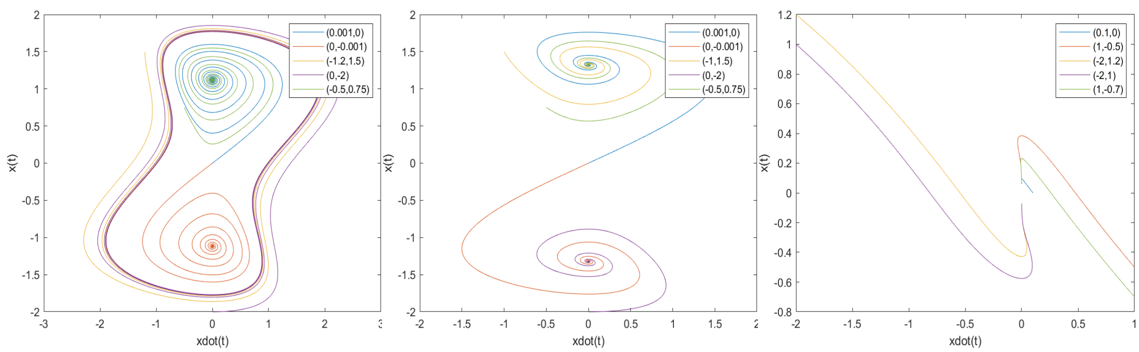

Figure 22: IV-V, V-VI, VI-I

1.8 Question 8

By writing the forced van der Pol oscillator in Lienard coordinates, we also have from the forced van der Pol oscillator in PartIII

$$\begin{aligned} \dot{x} &= y - a(x^3/3 - x) \\ \ddot{x} &= -x + 1 + b - a(y - a(x^3/3 - x))(x^2 - 1). \end{aligned} \tag{*}$$

with Jacobian

$$\text{J}=\begin{pmatrix} -a(x^2-1) & 1 \\ -1 + a^2(x^2-1)^2 - 2ax(y - a(x^3/3 - x)) & -a(x^2-1) \end{pmatrix}.$$

1.8.1 Solving the forced van der Pol oscillator: Programming Task

Two programs using the fourth order Runge-Kutta method in Lienard coordinates with y(t) against x(t) and x(t) against $\dot{x}(t)$ (using (*)) respectively are listed on page 22-23, named **RK4Q8L(a,b,x0,y0,h,tmax)**, **RK4Q8(a,b,x0,y0,h,tmax)**.

Since b small, by finding the fixed point $(\dot{x}, x)$ of the system (*) in the absence of b, $(-2a/3, 1)$, we perturb the original system from the fixed point. Aiming a Hopf bifurcation, start at $(-2a/3, 1)$, the trajectory spirals outwards with the external edge forming an approx. PO at the location shown in Fig.23. (Same if we start outside the PO, where the trajectory spirals inwards to the PO.)

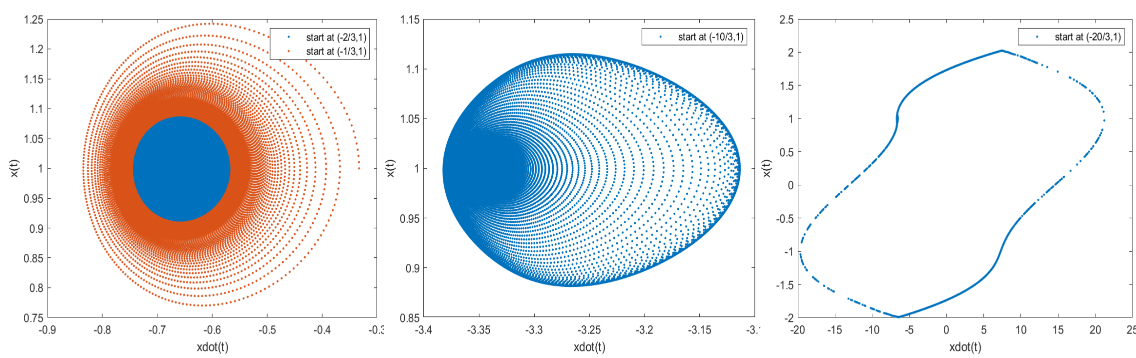

Figure 23: a=1, 5, 10

1.9 Question 9

Investigate the evolution of the PO for b $\in$ [-0.1, 0) at each of these values of a.

- When a=1, as b goes from 0 to -0.1, the PO in the $\dot{x}$-x plane has larger radius and are more stretched from a ellipse-like shape (see Fig.25).

- When a=5, as b goes from 0 to more negative values, the PO has larger radius. When b reaches -0.0012(this and below critical values all corrected to 2 significant figures), the trajectory moves away from the ellipse-like shaped region to form a much larger squished rectangular PO at some point (see Fig.26). When b reaches -0.023, the trajectory goes straight to the rectangular PO without circling around the fixed point (see Fig.27).

- When a=10, the two critical values of b change while the shape of the trajectory remains similar. The trajectory moves to the larger PO (in $\dot{x}$-direction) eventually when b reaches -0.00032 (see Fig.28) and stops lingering around the fixed point in the first place when b reaches -0.012 (see Fig.29).

- From Fig.24 it can be seen that as a increases, the trajectory gets more stretched horizontally.

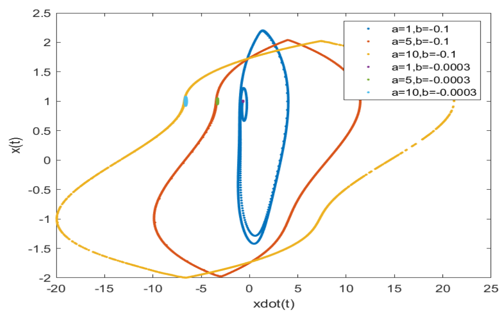

Figure 24: Various a, b's

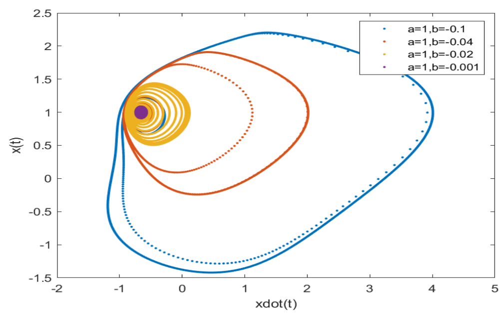

Figure 25: Various b with a=1

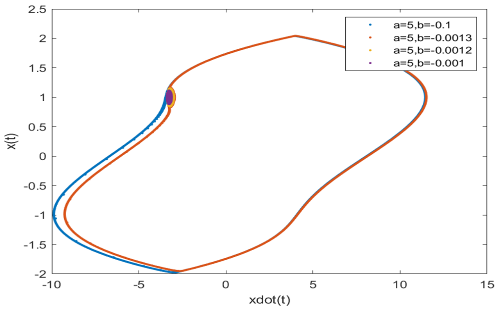

Figure 26: Various b with a=5

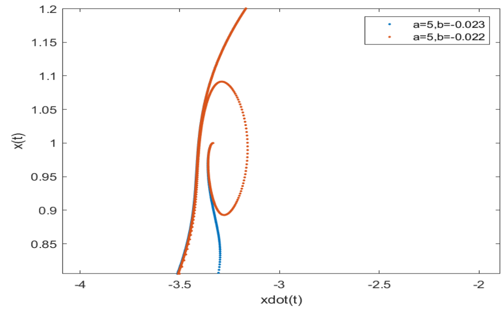

Figure 27: Various b with a=5 (zoomed)

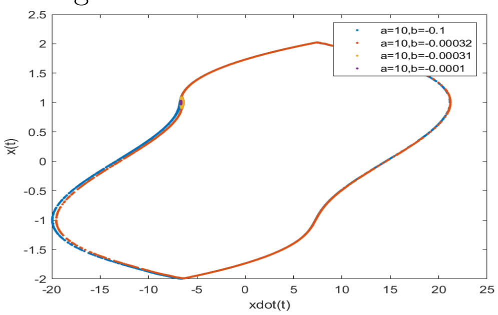

Figure 28: Various b with a=10

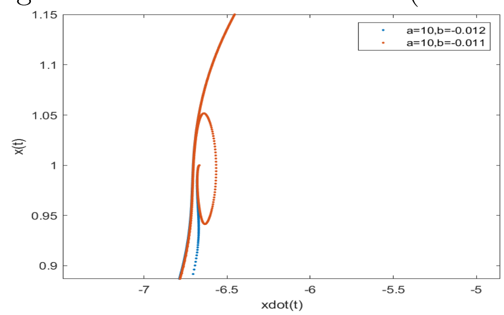

Figure 29: Various b with a=10 (zoomed)

Above critical values of b can also be deduced from the Jacobian at the fixed point,

$$J\Big|_{(a(1+b)^3/3-a(1+b),1+b)}=\begin{pmatrix} -a(2b+b^2) & 1 \\ -1+a^2(2b+b^2)^2 & -a(2b+b^2) \end{pmatrix},$$

whose determinant D=1>0 and trace T=$-2a(2b+b^2)$>0 (since a >0 and -1<b<0) with $T^2-4D=4a^2(2b+b^2)^2-4$>0 if

$$-1-\sqrt{1-1/a}<b<-1+\sqrt{1-1/a}, \tag{1}$$

and <0 if

$$-1+\sqrt{1-1/a}<b<-1+\sqrt{1+1/a}, \tag{2}$$

within the range [-0.1,0). I.e., the only fixed point of (*) $((a(1+b)^3/3-a(1+b),1+b)$ is an unstable node if $-0.1\le b<-1+\sqrt{1-1/a}$, and an unstable focus if $-1+\sqrt{1-1/a}<b$ <0. The more negative critical value is hence $-1+\sqrt{1-1/a}$. This differs from what we have obtained by numerical methods, which is due to our simplified choice of i.c.s and accumulation of numerical errors.

1.10 Question 10

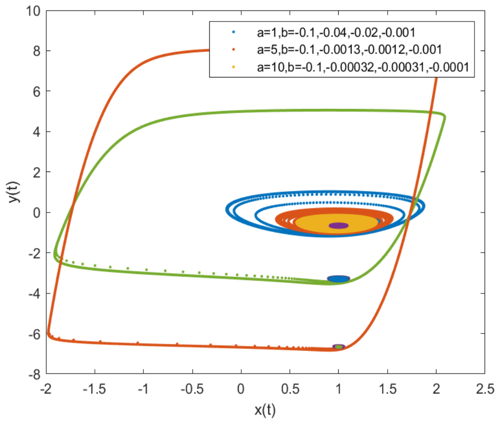

Figure 30: Various a, b's

Run the program in Q8. The trajectories in the x-y plane are similar to those ones in the $\dot{x}$-x plane, but have more regular shape for various a, b's and the relative stretching is not changed (see Fig.30).

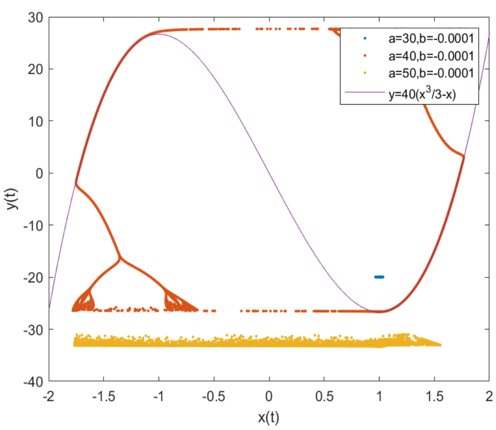

Figure 31: b=-0.0001 with large a

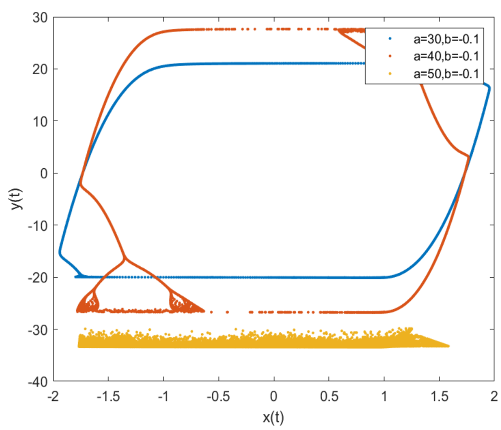

Figure 32: b=-0.1 with large a

In particular, when a is large, $\dot{x}$ dominates almost the whole x-y plane, hence the very straight horizontal line with less points accumulated on the PO. However, near $y = a(x^3/3 - x)$, $\dot{x} \approx 0$ - on either (horizontal) side of the curve, large $\dot{x}$ pushes the PO to coincide with the curve, where points are relatively dense. Start from four corners where two of them are the local maximum and minimum, the other two are the intersections of the straight horizontal lines and y,

$$\begin{aligned} &y = a(x^3/3 - x) \;\Leftrightarrow\; y' = a(x^2 - 1) = 0 \quad at \;\; x = \pm 1 \\ &y|_{\pm 1} = a(x^3/3 - x) \;\Leftrightarrow\; x = \mp 2. \end{aligned}$$

Hence at (-1,2a/3), (1,-2a/3) and (-2,-2a/3), (2,2a/3), PO leaves and approaches $y = a(x^3/3 - x)$ (see Fig.31).

For large a (e.g. a=30,40), the system goes into 'chaos' on the corner of the original PO (see Fig.31-32) where $\dot{x}$ becomes small and dominated by $\dot{y}$. The period T are mainly contributed by the parts of the PO coincided with $y = a(x^3/3 - x)$, i.e.,

$$T \approx 2 \int_{(-2,-2a/3)}^{(-1,2a/3)} dt = 2 \int_{-2}^{-1} \frac{1}{\dot{y}} \frac{dy}{dx} dx \approx 2 \int_{-2}^{-1} -a(x+1) dx = a$$

for b small. This results in scattered points on the horizontal part of the PO, making it less obvious.

For even larger a (e.g. a=50), it is difficult for the computer to capture its feature only through RK4. This is partially due to accumulation of rounding errors during integration. One essential cause is the non-linearity of the oscillator. As a increases, the nonlinear term $a(x^2 - 1)\dot{x}$ dominates, making the RK4 method unstable in terms of each step during the evolution, resulting chaotic behavior on the corner of the trajectory.

2 Programs

Note: Some programs listed on this pdf have $'return'$ added after excessively long texts, which needs to be removed before testing.

2.1 Question 1

2.1.1 RK4(a,b,x0,y0,h,tmax)

```
R=linspace(0,1,5000);
function RK4(a,b,x0,y0,h,tmax)
% RK4 solving Q1
% with x:=X, xdot:=Y,
% ai coefficients of xdot, bi coefficients of xdotdot
% e.g. RK4(-0.12,0,1,1.2,0.02,25),RK4(0.15,0.3,-2,-2,pi/10000,pi)
i=1; t=0; X=zeros(1,round(tmax/h)+1); Y=X; X(1)=x0; Y(1)=y0;
while t<=tmax
    a1=Y(i);
    b1=-a*Y(i)+X(i)-X(i)^3+b*cos(t);
    a2=Y(i)+h*b1/2;
    b2=-a*(Y(i)+h*b1/2)+(X(i)+h*a1/2)-(X(i)+h*a1/2)^3+b*cos(t+h/2);
    a3=Y(i)+h*b2/2;
    b3=-a*(Y(i)+h*b2/2)+(X(i)+h*a2/2)-(X(i)+h*a2/2)^3+b*cos(t+h/2);
    a4=Y(i)+h*b3;
    b4=-a*(Y(i)+h*b3)+(X(i)+h*a3)-(X(i)+h*a3)^3+b*cos(t+h);
    X(i+1)=X(i)+h*(a1+2*a2+2*a3+a4)/6;
    Y(i+1)=Y(i)+h*(b1+2*b2+2*b3+b4)/6;
    t=t+h;
    i=i+1;
end
plot(Y,X)
xlabel('xdot(t)'); ylabel('x(t)')
end
```

2.1.2 Q1 Plots

```
RK4(0.12,0,1,1,0.02,25); hold on;
RK4(0.12,0,1.1,0,0.02,25); hold on;
```

```
RK4(0.12,0,0,0.1,0.02,25); hold on;
RK4(0.12,0,1,-2,0.02,25); hold on;
RK4(0.12,0,-1,1,0.02,25); hold on;
legend('(1,1)','(0,1.1)','(0.1,0)','(-2,1)','(1,-1)')
```

2.2 Question 2

2.2.1 Q2 Plots

```
RK4(-0.12,0,1,1,0.02,25); hold on;
RK4(0,0,1,1,0.02,25); hold on;
RK4(0.12,0,1,1,0.02,25)
legend('a=-0.12','a=0','a=0.12')

RK4(0,0,1,1,0.02,25); hold on;
RK4(0,0,1.1,0,0.02,25); hold on;
RK4(0,0,1,-2,0.02,25);hold on;
RK4(0,0,1,2^(-1/2),0.02,25)
legend('(1,1)','(0,1.1)','(-2,1)','(2^{-1/2},1)')
```

2.3 Question 3

2.3.1 Q3 Plots

```
RK4(0.15,0.3,0,1.3,pi/10000,100*pi) %Q3a approximate PO
RK4(0.15,0.3,-1,0,pi/10000,100*pi) %Q3b strange attractor
```

2.4 Question 4

2.4.1 RK4Q4(a,b,x0,y0,h,d,tmax)

```
function RK4Q4(a,b,x0,y0,h,d,tmax)
% RK4 solving Q4
% with x:=X, xdot:=Y,
% h:=step width, d:=step width of dot plots
% ai coefficients of xdot, bi coefficients of xdotdot
% e.g. RK4Q4(0.15,0.3,0,1.3,pi/20000,2*pi,20*pi),
```

```
%         RK4Q4(0.15,0.3,-1,0,pi/20000,2*pi,20*pi)
i=1; t=0; X=zeros(1,round(tmax/h)+1); Y=X; X(1)=x0; Y(1)=y0;
while t<=tmax
    a1=Y(i);
    b1=-a*Y(i)+X(i)-X(i)^3+b*cos(t);
    a2=Y(i)+h*b1/2;
    b2=-a*(Y(i)+h*b1/2)+(X(i)+h*a1/2)-(X(i)+h*a1/2)^3+b*cos(t+h/2);
    a3=Y(i)+h*b2/2;
    b3=-a*(Y(i)+h*b2/2)+(X(i)+h*a2/2)-(X(i)+h*a2/2)^3+b*cos(t+h/2);
    a4=Y(i)+h*b3;
    b4=-a*(Y(i)+h*b3)+(X(i)+h*a3)-(X(i)+h*a3)^3+b*cos(t+h);
    X(i+1)=X(i)+h*(a1+2*a2+2*a3+a4)/6;
    Y(i+1)=Y(i)+h*(b1+2*b2+2*b3+b4)/6;
    t=t+h;
    i=i+1;
end
XX=zeros(1,round(tmax/d)+1); YY=XX; k=1;
for j=1:(round(tmax/d)+1)
    XX(j)=X(k); YY(j)=Y(k);
    k=k+round(d/h);
end
plot(YY,XX,'+')
xlabel('xdot(t)'); ylabel('x(t)')
end
```

2.4.2 Q4 Plots

```
RK4(0.15,0.3,0,1.3,pi/20000,1000*pi);hold on;
RK4Q4(0.15,0.3,0,1.3,pi/20000,2*pi,1000*pi);hold on;
RK4(0.15,0.3,-1,0,pi/20000,1000*pi);hold on;
RK4Q4(0.15,0.3,-1,0,pi/20000,2*pi,1000*pi)
legend('PO','points plot','strange attractor','points plot')
```

2.5 Question 5

2.5.1 Q5 Plots

```
%including strange attractor; vary a at (0,-1); trajectories& points
RK4(0.1,0.3,-1,0,pi/10000,10*pi);hold on;
RK4(0.15,0.3,-1,0,pi/10000,10*pi);hold on;
RK4(0.2,0.3,-1,0,pi/10000,10*pi);hold on;
RK4(0.4,0.3,-1,0,pi/10000,10*pi);hold on;
RK4(0.5,0.3,-1,0,pi/10000,10*pi);hold on;
RK4Q4(0.1,0.3,-1,0,pi/10000,2*pi,10*pi);hold on;
RK4Q4(0.15,0.3,-1,0,pi/10000,2*pi,10*pi);hold on;
RK4Q4(0.2,0.3,-1,0,pi/10000,2*pi,10*pi);hold on;
RK4Q4(0.4,0.3,-1,0,pi/10000,2*pi,10*pi);hold on;
RK4Q4(0.5,0.3,-1,0,pi/10000,2*pi,10*pi)
legend('a=0.1','a=0.15','a=0.2','a=0.4','a=0.5')

%strange attractor; vary ic; a=0.15
RK4(0.15,0.3,1,1,pi/10000,10*pi);hold on;
RK4(0.15,0.3,1,0,pi/10000,10*pi);hold on;
RK4(0.15,0.3,1,-1,pi/10000,10*pi);hold on;
RK4(0.15,0.3,1,-2,pi/10000,10*pi);hold on;
RK4(0.15,0.3,-1,1,pi/10000,10*pi)
legend('(1,1)','(0,1)','(-1,1)','(-2,1)','(1,-1)')

%interesting pictures
RK4(0.1,0.3,-1,0,pi/10000,20*pi)%a=0.1
RK4(0.3,0.3,-1,0,pi/10000,20*pi)%a=0.3
RK4(0.375,0.3,-1,0,pi/10000,20*pi)%a=0.375
RK4(0.5,0.3,-1,0,pi/10000,20*pi)%a=0.5
```

2.6 Question 6

2.6.1 RK4Q6(a,b,x0,y0,h,tmax)

```
function RK4Q6(a,b,x0,y0,h,tmax)
% RK4 solving Q6
% with x:=X, xdot:=Y,
% ai coefficients of xdot, bi coefficients of xdotdot
% e.g. RK4Q6(-1,-1,0,1.3,pi/10000,30*pi),RK4Q6(-1,-1,-1,0,pi/10000,30*pi)
i=1; t=0; X=zeros(1,round(tmax/h)+1); Y=X; X(1)=x0; Y(1)=y0;
while t<=tmax
```

```
    a1=Y(i);
    b1=(b-X(i)^2)*Y(i)+a*X(i)-X(i)^3;
    a2=Y(i)+h*b1/2;
    b2=(b-(X(i)+h*a1/2)^2)*(Y(i)+h*b1/2)+a*(X(i)+h*a1/2)-(X(i)+h*a1/2)^3;
    a3=Y(i)+h*b2/2;
    b3=(b-(X(i)+h*a2/2)^2)*(Y(i)+h*b2/2)+a*(X(i)+h*a2/2)-(X(i)+h*a2/2)^3;
    a4=Y(i)+h*b3;
    b4=(b-(X(i)+h*a3)^2)*(Y(i)+h*b3)+a*(X(i)+h*a3)-(X(i)+h*a3)^3;
    X(i+1)=X(i)+h*(a1+2*a2+2*a3+a4)/6;
    Y(i+1)=Y(i)+h*(b1+2*b2+2*b3+b4)/6;
    t=t+h;
    i=i+1;
end
plot(Y,X)
xlabel('xdot(t)');  ylabel('x(t)')
end
```

2.6.2 Q6 Plots

```
%I
RK4Q6(-1,-1,1,1,pi/10000,30*pi);hold on;
RK4Q6(-1,-1,1,0,pi/10000,30*pi);hold on;
RK4Q6(-1,-1,1,-1,pi/10000,30*pi);hold on;
RK4Q6(-1,-1,1,-2,pi/10000,30*pi);hold on;
RK4Q6(-1,-1,-1,1,pi/10000,30*pi)
legend('(1,1)','(0,1)','(-1,1)','(-2,1)','(1,-1)')

%II
RK4Q6(-1,1,3.5,4,pi/10000,30*pi);hold on;
RK4Q6(-1,1,0.001,0.001,pi/10000,30*pi);hold on;
RK4Q6(-1,1,0,0.001,pi/10000,30*pi);hold on;
RK4Q6(-1,1,-0.001,0,pi/10000,30*pi);hold on;
RK4Q6(-1,1,-1,1,pi/10000,30*pi)
legend('(4,3.5)','(0.001,0.001)','(0.001,0)','(0,-0.001)','(1,-1)')

%III
RK4Q6(0.5,1,0.001,0.001,pi/10000,30*pi);hold on;
RK4Q6(0.5,1,-0.001,-0.001,pi/10000,30*pi);hold on;
```

```
RK4Q6(0.5,1,0.7,0,pi/10000,30*pi);hold on;
RK4Q6(0.5,1,-0.7,0,pi/10000,30*pi);hold on;
RK4Q6(0.5,1,-0.5,2,pi/10000,30*pi)
legend('(0.001,0.001)','(-0.001,-0.001)','(0,0.7)','(0,-0.7)','(2,-0.5)')

RK4Q6(0.5,1,0,0.001,pi/10000,pi/2);hold on;%to see saddle
RK4Q6(0.5,1,0,-0.001,pi/10000,pi/2);hold on;
RK4Q6(0.5,1,-0.0012,0.0007,pi/10000,pi/2);hold on;
RK4Q6(0.5,1,-0.0021,0.0005,pi/10000,pi/2);hold on;
RK4Q6(0.5,1,0.001,0.001,pi/10000,pi/2);hold on;
RK4Q6(0.5,1,-0.001,0.001,pi/10000,pi/2);hold on;
RK4Q6(0.5,1,-0.001,-0.001,pi/10000,pi/2);hold on;
RK4Q6(0.5,1,0.001,0,pi/10000,pi/2)

%IV
RK4Q6(1.2,1,1,1,pi/10000,30*pi);hold on;
RK4Q6(1.2,1,-0.1,-0.1,pi/10000,30*pi);hold on;
RK4Q6(1.2,1,0.1,0.1,pi/10000,30*pi);hold on;
RK4Q6(1.2,1,0,1.5,pi/10000,30*pi);hold on;
RK4Q6(1.2,1,-0.5,0,pi/10000,30*pi)
legend('(1,1)','(-0.1,-0.1)','(0.1,0.1)','(1.5,0)','(0,-0.5)')

RK4Q6(1.2,1,0.0007,-0.0001,pi/10000,pi/2);hold on;%to see saddle
RK4Q6(1.2,1,-0.0007,0.0007,pi/10000,pi/2);hold on;
RK4Q6(1.2,1,0.0003,-0.0005,pi/10000,pi/2);hold on;
RK4Q6(1.2,1,-0.0007,0.0002,pi/10000,pi/2)

%V
RK4Q6(1.5,1,0,0.001,pi/10000,30*pi);hold on;
RK4Q6(1.5,1,-0.001,0,pi/10000,30*pi);hold on;
RK4Q6(1.5,1,1.5,-1,pi/10000,30*pi);hold on;
RK4Q6(1.5,1,-2,0,pi/10000,30*pi);hold on;
RK4Q6(1.5,1,0.75,-0.5,pi/10000,30*pi)
legend('(0.001,0)','(0,-0.001)','(-1,1.5)','(0,-2)','(-0.5,0.75)')

%VI
RK4Q6(1,-1,0,0.001,pi/10000,30*pi);hold on;
RK4Q6(1,-1,-0.5,1,pi/10000,30*pi);hold on;
```

```
RK4Q6(1,-1,1.2,-2,pi/10000,30*pi);hold on;
RK4Q6(1,-1,1,-2,pi/10000,30*pi);hold on;
RK4Q6(1,-1,-0.7,1,pi/10000,30*pi)
legend('(0.001,0)','(1,-0.5)','(-2,1.2)','(-2,1)','(1,-0.7)')
```

2.7 Question 7

2.7.1 Q7 Plots

```
%I,II
RK4Q6(-1,0,2,3,pi/10000,30*pi);hold on;
RK4Q6(-1,0,0.1,0.1,pi/10000,30*pi);hold on;
RK4Q6(-1,0,0,0.1,pi/10000,30*pi);hold on;
RK4Q6(-1,0,-0.1,0,pi/10000,30*pi)
legend('(3,2)','(0.1,0.1)','(0.1,0)','(0,-0.1)')

%II,III
RK4Q6(0,1,0.001,0.001,pi/10000,30*pi);hold on;
RK4Q6(0,1,-0.001,-0.001,pi/10000,30*pi);hold on;
RK4Q6(0,1,-0.7,0,pi/10000,30*pi);hold on;
RK4Q6(0,1,-0.2,-0.01,pi/10000,30*pi);hold on;
RK4Q6(0,1,-1,2,pi/10000,30*pi);hold on;
RK4Q6(0,1,-0.4,-0.1,pi/10000,30*pi)
legend('(0.001,0.001)','(-0.001,-0.001)','(0,-0.7)','(-0.01,-0.2)',
'(2,-1)','(-0.1,-0.4)')

%III,IV
RK4Q6(1,1,0.001,0.001,pi/10000,30*pi);hold on;
RK4Q6(1,1,-0.001,-0.001,pi/10000,30*pi);hold on;
RK4Q6(1,1,-0.7,0,pi/10000,30*pi);hold on;
RK4Q6(1,1,-0.2,-0.01,pi/10000,30*pi);hold on;
RK4Q6(1,1,-1,1.5,pi/10000,30*pi)
legend('(0.001,0.001)','(-0.001,-0.001)','(0,-0.7)','(-0.01,-0.2)',
'(1.5,-1)')

%IV,V
RK4Q6(1.25,1,0,0.001,pi/10000,30*pi);hold on;
RK4Q6(1.25,1,-0.001,0,pi/10000,30*pi);hold on;
```

```
RK4Q6(1.25,1,1.5,-1.2,pi/10000,30*pi);hold on;
RK4Q6(1.25,1,-2,0,pi/10000,30*pi);hold on;
RK4Q6(1.25,1,0.75,-0.5,pi/10000,30*pi)
legend('(0.001,0)','(0,-0.001)','(-1.2,1.5)','(0,-2)','(-0.5,0.75)')

%V,VI
RK4Q6(1.75,1,0,0.001,pi/10000,30*pi);hold on;
RK4Q6(1.75,1,-0.001,0,pi/10000,30*pi);hold on;
RK4Q6(1.75,1,1.5,-1,pi/10000,30*pi);hold on;
RK4Q6(1.75,1,-2,0,pi/10000,30*pi);hold on;
RK4Q6(1.75,1,0.75,-0.5,pi/10000,30*pi)
legend('(0.001,0)','(0,-0.001)','(-1,1.5)','(0,-2)','(-0.5,0.75)')

%VI,I
RK4Q6(0,-1,0,0.1,pi/10000,30*pi);hold on;
RK4Q6(0,-1,-0.5,1,pi/10000,30*pi);hold on;
RK4Q6(0,-1,1.2,-2,pi/10000,30*pi);hold on;
RK4Q6(0,-1,1,-2,pi/10000,30*pi);hold on;
RK4Q6(0,-1,-0.7,1,pi/10000,30*pi)
legend('(0.1,0)','(1,-0.5)','(-2,1.2)','(-2,1)','(1,-0.7)')
```

2.8 Question 8

2.8.1 RK4Q8L(a,b,x0,y0,h,tmax)

```
function RK4Q8L(a,b,x0,y0,h,tmax)
% RK4 solving Q8 in Lienard coods and plot at each time step
% ai coefficients of xdot, bi coefficients of ydot
% e.g. RK4Q8L(1,-0.001,1,-2/3,pi/100,2000*pi)
i=1; t=0; X=zeros(1,round(tmax/h)+1); Y=X; X(1)=x0; Y(1)=y0;
while t<=tmax
    a1=Y(i)-a*(X(i)^3/3-X(i));
    b1=-X(i)+1+b;
    a2=Y(i)+h*b1/2-a*((X(i)+h*a1/2)^3/3-(X(i)+h*a1/2));
    b2=-(X(i)+h*a1/2)+1+b;
    a3=Y(i)+h*b2/2-a*((X(i)+h*a2/2)^3/3-(X(i)+h*a2/2));
    b3=-(X(i)+h*a2/2)+1+b;
    a4=Y(i)+h*b3-a*((X(i)+h*a3)^3/3-(X(i)+h*a3));
```

```
    b4=-(X(i)+h*a3)+1+b;
    X(i+1)=X(i)+h*(a1+2*a2+2*a3+a4)/6;
    Y(i+1)=Y(i)+h*(b1+2*b2+2*b3+b4)/6;
    t=t+h;
    i=i+1;
end
plot(X,Y,'.')
xlabel('x(t)'); ylabel('y(t)')
end
```

2.8.2 RK4Q8(a,b,x0,y0,h,tmax)

```
function RK4Q8(a,b,x0,y0,h,tmax)
% RK4 solving Q8 and plot at each time step
% ai coefficients of xdot, bi coefficients of xdotdot
% e.g. RK4Q8(1,-0.001,1,0,pi/100,2000*pi)
i=1; t=0; X=zeros(1,round(tmax/h)+1); Y=X; X(1)=x0; Y(1)=y0;
while t<=tmax
    a1=Y(i)-a*X(i)^3/3+a*X(i);
    b1=-a*(Y(i)-a*X(i)^3/3+a*X(i))*(X(i)^2-1)-X(i)+1+b;
    a2=Y(i)+h*b1/2-a*(X(i)+h*a1/2)^3/3+a*(X(i)+h*a1/2);
    b2=-a*(Y(i)+h*b1/2-a*(X(i)+h*a1/2)^3/3+a*(X(i)+h*a1/2))*
    ((X(i)+h*a1/2)^2-1)-(X(i)+h*a1/2)+1+b;
    a3=Y(i)+h*b2/2-a*(X(i)+h*a2/2)^3/3+a*(X(i)+h*a2/2);
    b3=-a*(Y(i)+h*b2/2-a*(X(i)+h*a2/2)^3/3+a*(X(i)+h*a2/2))*
    ((X(i)+h*a2/2)^2-1)-(X(i)+h*a2/2)+1+b;
    a4=Y(i)+h*b3-a*(X(i)+h*a3)^3/3+a*(X(i)+h*a3);
    b4=-a*(Y(i)+h*b3-a*(X(i)+h*a3)^3/3+a*(X(i)+h*a3))*
    ((X(i)+h*a3)^2-1)-(X(i)+h*a3)+1+b;
    X(i+1)=X(i)+h*(a1+2*a2+2*a3+a4)/6;
    Y(i+1)=Y(i)+h*(b1+2*b2+2*b3+b4)/6;
    t=t+h;
    i=i+1;
end
plot(Y,X,'.')
xlabel('xdot(t)'); ylabel('x(t)')
end
```

2.8.3 Q8 Plots

```
RK4Q8(1,-0.001,1,-2/3,pi/100,2000*pi);hold on;
RK4Q8(1,-0.001,1,-1/3,pi/100,2000*pi)%a=1 from both inside and outside PO
legend('start at (-2/3,1)','start at (-1/3,1)')

RK4Q8(5,-0.001,1,-10/3,pi/100,2000*pi)%a=5
legend('start at (-10/3,1)')

RK4Q8(10,-0.001,1,-20/3,pi/100,2000*pi)%a=10
legend('start at (-20/3,1)')
```

2.9 Question 9

2.9.1 Q9 Plots

```
%a=1,b=-0.1,-0.04,-0.02,-0.001
RK4Q8(1,-0.1,1,-2/3,pi/100,2000*pi);hold on;
RK4Q8(1,-0.04,1,-2/3,pi/100,2000*pi);hold on;
RK4Q8(1,-0.02,1,-2/3,pi/100,2000*pi);hold on;
RK4Q8(1,-0.001,1,-2/3,pi/100,2000*pi)
legend('a=1,b=-0.1','a=1,b=-0.04','a=1,b=-0.02','a=1,b=-0.001')

%a=5,b=-0.1,-0.0013,-0.0012,-0.001
RK4Q8(5,-0.1,1,-10/3,pi/100,2000*pi);hold on;
RK4Q8(5,-0.0013,1,-10/3,pi/100,2000*pi);hold on;
RK4Q8(5,-0.0012,1,-10/3,pi/100,2000*pi);hold on;
RK4Q8(5,-0.001,1,-10/3,pi/100,2000*pi)
legend('a=5,b=-0.1','a=5,b=-0.0013','a=5,b=-0.0012','a=5,b=-0.001')

RK4Q8(5,-0.023,1,-10/3,pi/100,2000*pi);hold on;%straight to PO
RK4Q8(5,-0.022,1,-10/3,pi/100,2000*pi)
legend('a=5,b=-0.023','a=5,b=-0.022')

%a=10,b=-0.1,-0.00032,-0.00031,-0.0001
RK4Q8(10,-0.1,1,-20/3,pi/100,2000*pi);hold on;
RK4Q8(10,-0.00032,1,-20/3,pi/100,2000*pi);hold on;
RK4Q8(10,-0.00031,1,-20/3,pi/100,2000*pi);hold on;
```

```
RK4Q8(10,-0.0001,1,-20/3,pi/100,2000*pi)
legend('a=10,b=-0.1','a=10,b=-0.00032','a=10,b=-0.00031','a=10,b=-0.0001')

RK4Q8(10,-0.012,1,-20/3,pi/100,2000*pi);hold on;%straight to PO
RK4Q8(10,-0.011,1,-20/3,pi/100,2000*pi)
legend('a=10,b=-0.012','a=10,b=-0.011')

%Comparison of stretching
RK4Q8(1,-0.1,1,-2/3,pi/100,2000*pi);hold on;
RK4Q8(5,-0.1,1,-10/3,pi/100,2000*pi);hold on;
RK4Q8(10,-0.1,1,-20/3,pi/100,2000*pi);hold on;
RK4Q8(1,-0.0003,1,-2/3,pi/100,2000*pi);hold on;
RK4Q8(5,-0.0003,1,-10/3,pi/100,2000*pi);hold on;
RK4Q8(10,-0.0003,1,-20/3,pi/100,2000*pi)
legend('a=1,b=-0.1','a=5,b=-0.1','a=10,b=-0.1','a=1,b=-0.0003',
'a=5,b=-0.0003','a=10,b=-0.0003')
```

2.10 Question 10

2.10.1 Q10 Plots

```
%a=1,b=-0.1,-0.04,-0.02,-0.001
RK4Q10(1,-0.1,1,-2/3,pi/100,2000*pi);hold on;
RK4Q10(1,-0.04,1,-2/3,pi/100,2000*pi);hold on;
RK4Q10(1,-0.02,1,-2/3,pi/100,2000*pi);hold on;
RK4Q10(1,-0.001,1,-2/3,pi/100,2000*pi);hold on;
RK4Q10(5,-0.1,1,-10/3,pi/100,2000*pi);hold on;
RK4Q10(5,-0.0013,1,-10/3,pi/100,2000*pi);hold on;
RK4Q10(5,-0.0012,1,-10/3,pi/100,2000*pi);hold on;
RK4Q10(5,-0.001,1,-10/3,pi/100,2000*pi);hold on;
RK4Q10(10,-0.1,1,-20/3,pi/100,2000*pi);hold on;
RK4Q10(10,-0.00032,1,-20/3,pi/100,2000*pi);hold on;
RK4Q10(10,-0.00031,1,-20/3,pi/100,2000*pi);hold on;
RK4Q10(10,-0.0001,1,-20/3,pi/100,2000*pi)
legend('a=1,b=-0.1,-0.04,-0.02,-0.001',
'a=5,b=-0.1,-0.0013,-0.0012,-0.001',
'a=10,b=-0.1,-0.00032,-0.00031,-0.0001')
```

```
%Comparison of stretching
RK4Q10(1,-0.1,1,-2/3,pi/100,2000*pi);hold on;
RK4Q10(5,-0.1,1,-10/3,pi/100,2000*pi);hold on;
RK4Q10(10,-0.1,1,-20/3,pi/100,2000*pi);hold on;
RK4Q10(1,-0.0003,1,-2/3,pi/100,2000*pi);hold on;
RK4Q10(5,-0.0003,1,-10/3,pi/100,2000*pi);hold on;
RK4Q10(10,-0.0003,1,-20/3,pi/100,2000*pi)
legend('a=1,b=-0.1','a=5,b=-0.1','a=10,b=-0.1',
'a=1,b=-0.0003','a=5,b=-0.0003','a=10,b=-0.0003')

%large a, less negative b within [-0.1,0)
RK4Q10(30,-0.0001,1,-20,pi/100,2000*pi);hold on;
RK4Q10(40,-0.0001,1,-80/3,pi/100,2000*pi);hold on;
RK4Q10(50,-0.0001,1,-100/3,pi/100,2000*pi);hold on;
X=-40:0.0001:30;
Y=40.*(X.^3./3-X);
plot(X,Y)
axis([-2,2,-40,30])
legend('a=30,b=-0.0001','a=40,b=-0.0001','a=50,b=-0.0001','y=40(x^3/3-x)')

%large a, more negative b within [-0.1,0)
RK4Q10(30,-0.1,1,-20,pi/100,2000*pi);hold on;
RK4Q10(40,-0.1,1,-80/3,pi/100,2000*pi);hold on;
RK4Q10(50,-0.1,1,-100/3,pi/100,2000*pi)
legend('a=30,b=-0.1','a=40,b=-0.1','a=50,b=-0.1')
```