Catam Core Projects
Computational Projects Manual
(July 2019 Edition)

21/12/2019

- Ordinary Differential Equations
- Programs

1 Ordinary Differential Equations

1.0.1 Applying Forward Euler, AB2, RK4 methods to f: Programming Task

Programs using these methods are listed on page 16-17, named fEuler(h,Xmax), AB2(h,Xmax), RK4(h,Xmax) respectively. Here we use h = 0.0001 and Xmax = 10.

1.1 Question 1

- For h = 0.5,

variables/n	0	1	2	3	4	5	6
x_n	0	0.5	1	1.5	2	2.5	3
Y_n	0	3	-14.8445	80.2799	-431.0676	2315.9	-12441.7
$y_e(x_n)$	0	0.3496	0.1350	0.0498	0.0183	0.0067	0.0025
E_n	0	2.6504	-14.9795	80.2301	-431.0859	2315.8933	-12441.7025

Let $E_n = Ce^{\gamma x_n}$ for C constant. Then

$$\frac{|E_6|}{|E_5|} = \frac{12441.7025}{2315.8933} \Rightarrow e^{0.5\gamma} = 5.37 \Rightarrow \gamma \approx 3.36.$$

- For h = 0.375,

variables/n	0	2	3	5	6	8
x_n	0	0.75	1.125	1.875	2.25	3
Y_n	0	-7.4058	29.5168	444.4196	-1726.9	-26078.3
$y_e(x_n)$	0	0.2207	0.1053	0.0260	0.0111	0.0025
E_n	0	-7.6265	29.4115	444.3936	-1726.9111	-26078.3025

- For h = 0.25,

variables/n	0	3	6	7	9	12
x_n	0	0.75	1.5	1.75	2.25	3
Y_n	0	6.6434	-90.7083	219.1398	1277.1	-17969.6
$y_e(x_n)$	0	0.2207	0.0498	0.0302	0.0111	0.0025
E_n	0	6.4227	-90.7581	219.1096	1277.0889	-17969.6025

- For h = 0.125,

variables/n	0	4	9	14	19	24
x_n	0	0.5	1.125	1.75	2.375	3
Y_n	0	0.0159	0.4173	-0.2817	0.3209	-0.3097
$y_e(x_n)$	0	0.3496	0.1053	0.0302	0.0087	0.0025
E_n	0	-0.3337	0.312	-0.3119	0.3122	-0.3122

- For h = 0.1,

variables/n	0	5	10	15	20	30
x_n	0	0.5	1	1.5	2	3
Y_n	0	0.3909	0.1223	0.0528	0.0178	0.0025
$y_e(x_n)$	0	0.3496	0.1350	0.0498	0.0183	0.0025
E_n	0	0.0413	-0.0127	0.003	-0.0005	0

- For h = 0.05,

variables/n	0	15	25	30	45	60
x_n	0	0.75	1.25	1.5	2.25	3
Y_n	0	0.22	0.0821	0.0498	0.0111	0.0025
$y_e(x_n)$	0	0.2207	0.0820	0.0498	0.0111	0.0025
E_n	0	-0.0007	0.0001	0	0	0

The tables above show that as h becomes smaller, E_n oscillates more slightly around 0. The growth rate also decreases (from above 1 to below 1) and finally becomes 0, up to a possible round-off error.

1.2 Question 2

i) To solve

$$Y_{n+1} = Y_n + h[-12Y_n + 9(e^{-2h})^n + 4Y_{n-1} - 3(e^{-2h})^{n-1}],$$

first solve the homogeneous difference equation

$$Y_{n+1} + (12h - 1)Y_n - 4hY_{n-1} = 0 \text{ (n} \geq 1).$$

Let $Y_n = k^n$. Then

$$k^{n+1} + (12h - 1)k^n - 4hk^{n-1} = 0$$

$$\Rightarrow k_1 = \frac{(1-12h)+\sqrt{144h^2-8h+1}}{2}, k_2 = \frac{(1-12h)-\sqrt{144h^2-8h+1}}{2}.$$

So the complementary solution of the equation is

$$Y_{c_n} = A(\frac{(1-12h)+\sqrt{144h^2-8h+1}}{2})^n + B(\frac{(1-12h)-\sqrt{144h^2-8h+1}}{2})^n,$$

A, B constants. To find the particular solution, first rearrange the equation such that

$$Y_{n+1} + (12h-1)Y_n - 4hY_{n-1} = f(n)$$

where $f(n) = 9h(e^{-2h})^n - 3h(e^{-2h})^{n-1}$. Hence try $\mathrm{Y}_{p_n} = \mathrm{C}(e^{-2h})^n$, C constant.

$$\Rightarrow C(e^{-2h})^{n+1} + (12h-1)C(e^{-2h})^n - 4hC(e^{-2h})^{n-1} = 9h(e^{-2h})^n - 3h(e^{-2h})^{n-1}$$

$$\Rightarrow C = \frac{9h-3he^{2h}}{e^{-2h}+12h-1-4he^{2h}}.$$

Hence the general solution is

$$Y_n = A(\frac{(1-12h)+\sqrt{144h^2-8h+1}}{2})^n + B(\frac{(1-12h)-\sqrt{144h^2-8h+1}}{2})^n + \frac{9h-3he^{2h}}{e^{-2h}+12h-1-4he^{2h}} \times e^{-2hn}.$$

Now apply initial conditions

$$Y_0 = 0, Y_1 = 6h$$

and find out

$$A = \frac{-6he^{-2h} - 12h^2e^{2h} + 3h + 36h^2 - 3he^{2h} - (9h-3he^{2h})\sqrt{144h^2-8h+1}}{(2e^{-2h}+24h-2-8he^{2h})\sqrt{144h^2-8h+1}},$$

$$B = \frac{6he^{-2h} + 12h^2e^{2h} - 3h - 36h^2 + 3he^{2h} - (9h-3he^{2h})\sqrt{144h^2-8h+1}}{(2e^{-2h}+24h-2-8he^{2h})\sqrt{144h^2-8h+1}}.$$

ii) As n changes continuously among integers, $\mathrm{y}_e(\mathrm{x}_n)$ and Y_{p_n} changes continuously. However, for $\mathrm{Y}_{c_n} =$

$$A(\underbrace{\frac{(1-12h)+\sqrt{(1-12h)^2+16h}}{2}})^n+B(\underbrace{\frac{(1-12h)-\sqrt{(1-12h)^2+16h}}{2}})^n,$$

figure in the first bracket is greater than 0 and is less than 0 in the second bracket. When n changes from odd to even, Y_{c_n} decreases by a considerable amount, and hence E_n changes from positive to negative (especially when h is large and n_{max} is small). Similarly in the case when n changes from even to odd. Hence comes the instability.

As n $\rightarrow \infty$,

iii) Use Taylor expansions for e^{2h} and $\sqrt{144h^2-8h+1}$ up to h, find that as h $\rightarrow$ 0, $n \rightarrow \infty$, x = nh fixed,

$$A \rightarrow \frac{-6h\times 1+3h-3h-(9h-3h)\times 1}{(2(1-2h)+24h-2-8h\times 1)\times 1}=\frac{-12h}{12h}=-1;$$

$$B \rightarrow \frac{6h\times 1-3h+3h-(9h-3h)\times 1}{(2(1-2h)+24h-2-8h\times 1)\times 1}=\frac{0}{12h}=0;$$

$$(\frac{(1-12h)+\sqrt{144h^2-8h+1}}{2})^n \rightarrow (\frac{(1-12h)+(1-4h)}{2})^n$$

$$=((1+(-8h))^{\frac{1}{h}})^{nh} \rightarrow e^{-8x}\ by\ L'Hopital's\ rule;$$

$$\frac{9h-3he^{2h}}{e^{-2h}+12h-1-4he^{2h}}\times e^{-2hn} \rightarrow \frac{9h-3h}{1-2h+12h-1-4h}\times e^{-2hn}=e^{-2x}.$$

Hence $Y_n \rightarrow e^{-2x}-e^{-8x}$, i.e. converges to the solution (7).

1.3 Question 3

See Tbl.1 and Fig.1 below.

/methods		Euler	AB2	RK4
n/variables	x_n	Y_n	Y_n	Y_n
0	0	0	0	0
1	0.08	0.48	0.48	0.3241
2	0.16	0.5818	0.3927	0.4473
3	0.24	0.5580	0.4876	0.4716
4	0.32	0.4979	0.4164	0.4496
5	0.4	0.4323	0.4038	0.4083
6	0.48	0.3713	0.3464	0.3613
7	0.56	0.3175	0.3109	0.3149
8	0.64	0.2709	0.2663	0.2720
9	0.72	0.2310	0.2320	0.2338
10	0.8	0.1969	0.1984	0.2002
11	0.88	0.1678	0.1707	0.1712
12	0.96	0.1430	0.1457	0.1462
13	1.04	0.1218	0.1247	0.1247
14	1.12	0.1038	0.1064	0.1063
15	1.2	0.0885	0.0908	0.0907
16	1.28	0.0754	0.0774	0.0773
17	1.36	0.0642	0.0661	0.0659
18	1.44	0.0547	0.0563	0.0561
19	1.52	0.0467	0.0480	0.0478
20	1.6	0.0398	0.0409	0.0408
21	1.68	0.0339	0.0349	0.0347
22	1.76	0.0289	0.0297	0.0296
23	1.84	0.0246	0.0253	0.0252
24	1.92	0.0210	0.0216	0.0215
25	2	0.0179	0.0184	0.0183

Table 1: Y_ns applying 3 methods

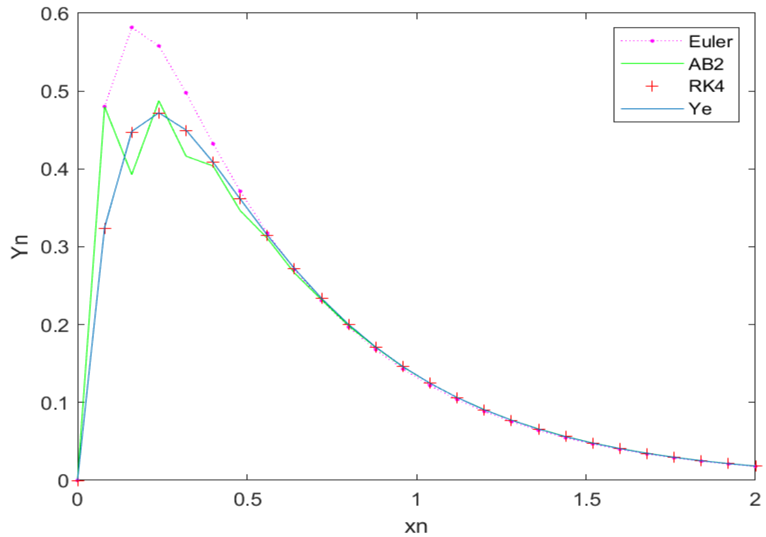

Figure 1: Y_ns and y_e applying 3 methods against x_n

1.4 Question 4

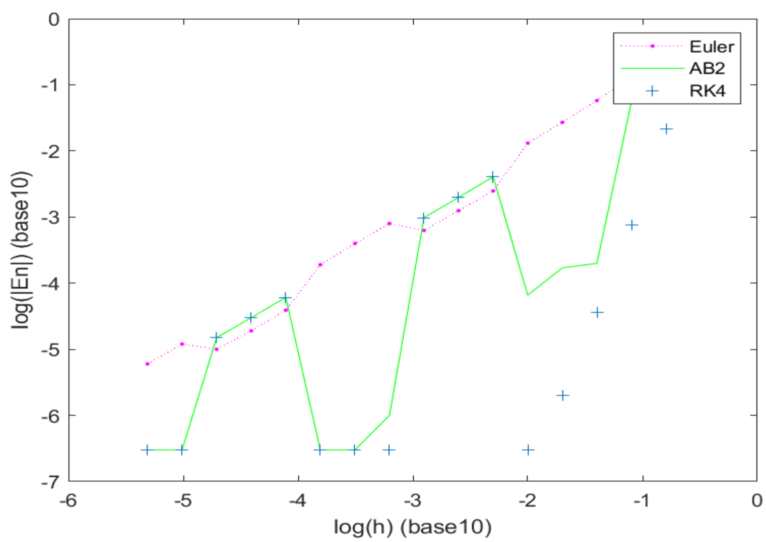

Figure 2: $\log_{10}(|E_n|)$ against $\log_{10}(h), h = 0.16/2^k$, k = 0,1,...,15

/methods		Euler	AB2	RK4
h/variables	x_n	E_n	E_n	E_n
0.16	0.16	5.11889×10^{-1}	5.11888×10^{-1}	-2.1516×10^{-2}
0.16/2		1.33717×10^{-1}	-5.5368×10^{-2}	-7.65×10^{-4}
$0.16/2^2$		5.7656×10^{-2}	-1.99×10^{-4}	-3.6×10^{-5}
$0.16/2^3$		2.7035×10^{-2}	-1.7×10^{-4}	-2×10^{-6}
$0.16/2^4$		1.3116×10^{-2}	-6.6×10^{-5}	3×10^{-7}
$0.16/2^5$		2.483×10^{-3}	-4.062×10^{-3}	-4.049×10^{-3}
$0.16/2^6$		1.248×10^{-3}	-1.982×10^{-3}	-1.977×10^{-3}
$0.16/2^7$		6.26×10^{-4}	-9.78×10^{-4}	-9.77×10^{-4}
$0.16/2^8$		7.98×10^{-4}	-1×10^{-6}	3×10^{-7}
$0.16/2^9$		3.98×10^{-4}	3×10^{-7}	3×10^{-7}
$0.16/2^{10}$		1.99×10^{-4}	3×10^{-7}	3×10^{-7}
$0.16/2^{11}$		3.9×10^{-5}	-6.1×10^{-5}	-6.1×10^{-5}
$0.16/2^{12}$		1.9×10^{-5}	-3×10^{-5}	-3×10^{-5}
$0.16/2^{13}$		1×10^{-5}	-1.5×10^{-5}	-1.5×10^{-5}
$0.16/2^{14}$		1.2×10^{-5}	3×10^{-7}	3×10^{-7}
$0.16/2^{15}$		6×10^{-6}	3×10^{-7}	3×10^{-7}

Table 2: E_ns applying 3 methods at $x_n = 0.16$

Results shown in Fig.2 and Tbl.2. We can conclude from Fig.2 that

- as h decreases towards zero ($\log_{10}(h)$ more and more negative in Fig.2), magnitude of the global error $|E_n|$ decreases in general as well ($log_{10}(|E_n|)$ more and more negative in Fig.2) for all three methods.

- the plots are also in consistency with the theoretical accuracy of each method: $|E_n|$ under Euler method is rather stable while under AB2 and RK4 method fluctuate a lot. Meanwhile, RK4 is the most accurate method among three as expected, whose global error is closest to 0, i.e. $\log_{10}(|E_n|)$ most negative, as shown in Fig.2. AB2 comes second and Euler third.

1.5 Question 5

In order to solve

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + p^2(1+x)^{-\alpha} y = 0,$$

begin by setting 1 + x = $\mathrm{e}^{z(x)}$. Then $\mathrm{d}z/\mathrm{d}x = e^{-z}$. The original equation becomes

$$-e^{-2z}y' + e^{-2z}y'' + p^2 e^{-2z} y = 0,$$

where y' and y'' are w.r.t. z. Try y = $\mathrm{e}^{\lambda z}$ and find

$$\lambda_{1,2} = \frac{1 \pm \sqrt{1-4p^2}}{2}.$$

y hence has formula based on the range value of general p:

- $|p| < 1/2$,

$$y = Ae^{\lambda_1 z} + Be^{\lambda_2 z} \Rightarrow y = A(1+x)^{\lambda_1} + B(1+x)^{\lambda_2}.$$

- $|p| = 1/2$, then we have a repeated root $\lambda_1 = \lambda_2 = 1/2$.

$$y = (C + Dz)e^{1/2z} \Rightarrow y = (C + Dln(1+x))(1+x)^{1/2}.$$

- $|p| > 1/2$, then we have complex λ_1 and λ_2 equal to $(1 \pm \mathrm{i}\sqrt{4p^2-1})/2$.

$$y = e^{1/2z}[Pcos(\frac{\sqrt{4p^2-1}}{2}z) + Qsin(\frac{\sqrt{4p^2-1}}{2}z)]$$
$$\Rightarrow y = (1+x)^{1/2}[Pcos(\frac{\sqrt{4p^2-1}}{2}ln(1+x)) + Qsin(\frac{\sqrt{4p^2-1}}{2}ln(1+x))].$$

Finally insert the i.c.s(16) y = 0 and $\mathrm{d}y/\mathrm{d}x = 1$ at x = 0 to obtain:

$$y_e(x) = \begin{cases} \frac{1}{\sqrt{1-4p^2}}[(1+x)^{\frac{1+\sqrt{1-4p^2}}{2}} - (1+x)^{\frac{1-\sqrt{1-4p^2}}{2}}] & |p| < 1/2 \\ (1+x)^{1/2} ln(1+x) & |p| = 1/2 \\ \frac{2}{\sqrt{4p^2-1}}(1+x)^{1/2} sin(\frac{\sqrt{4p^2-1}}{2}ln(1+x)) & |p| > 1/2 \end{cases}.$$

Instead with i.c.s(15b), we have

- $|p| < 1/2$,

$$y(0) = A + B = 0,\ y(1) = A(2^{\lambda_1} - 2^{\lambda_2}) = 0\ \Rightarrow A = B = 0.$$

- $|p| = 1/2$,

$$y(0) = C = 0,\ y(1) = D\sqrt{2}\ ln(2) = 0\ \Rightarrow C = D = 0.$$

- $|p| > 1/2$,

$$y(0) = P = 0,\ y(1) = \sqrt{2}Qsin(\frac{\sqrt{4p^2-1}}{2}ln(2)) = 0$$

$$\Rightarrow P = 0,\ \frac{\sqrt{4p^2-1}}{2}ln(2) = k\pi,\ k \in Z \Rightarrow p^{(k)} = \pm\sqrt{\frac{1}{4} + (\frac{k\pi}{ln(2)})^2},\ k \in Z\backslash\{0\}.$$

I.e. in the cases $|p| < 1/2$ and $|p| = 1/2$, we have the trivial solution y $\equiv 0$. In the case $|p| > 1/2$, the more general non-trivial solution will be the **eigenfunctions** $y^{(k)}(x) = Q'sin(\frac{\sqrt{4(p^{(k)})^2-1}}{2}ln(1+x))$with corresponding **eigenvalues** $p^{(k)} = \pm\sqrt{\frac{1}{4} + (\frac{k\pi}{ln(2)})^2},\ k \in Z\backslash\{0\}$ (since $|p^{(k)}| > 1/2$). Hence the smallest non-negative eigenvalue p is $p = p^1 = \sqrt{\frac{1}{4} + (\frac{\pi}{ln(2)})^2}$.

1.5.1 Apply RK4 Method to the Solutions of Equations(18): Programming Task

A program using this method is listed on page 19, named Q5(p,a,h,Xmax).

1.6 Question 6

Use $\alpha = 2$, p = 4, h = $0.1/2^k$, k = 0, 1,..., 12 and Xmax = 1 in the algorithm above. For p = 4 > 1/2, calculate $y_e(1) \approx 0.1357$. For p = 5 > 1/2, calculate $y_e(1) \approx -0.0858$. Results shown in Tbl.3.

/p		4		5	
h(=1/n)/variables	x_n	Y_n	E_n	Y_n	E_n
1/10		0.4429	0.3072	0.3088	0.3946
$1/(10\times2)$		0.3105	0.1748	0.1180	0.2038
$1/(10\times2^2)$		0.2200	0.0843	0.0008	0.0866
$1/(10\times2^3)$		0.1688	0.0331	-0.0572	0.0286
$1/(10\times2^4)$		0.1519	0.0162	-0.0726	0.0132
$1/(10\times2^5)$		0.1457	0.0100	-0.0772	0.0086
$1/(10\times2^6)$		0.1407	0.0050	-0.0816	0.0042
$1/(10\times2^7)$	1	0.1377	0.0020	-0.0843	0.0015
$1/(10\times2^8)$		0.1367	0.0010	-0.0851	0.0007
$1/(10\times2^9)$		0.1363	0.0006	-0.0853	0.0005
$1/(10\times2^{10})$		0.1360	0.0003	-0.0856	0.0002
$1/(10\times2^{11})$		0.1359	0.0002	-0.0857	0.0001
$1/(10\times2^{12})$		0.1358	0.0001	-0.0858	0

Table 3: Y_ns and E_ns against h=1/n using RK4 at $x_n = 1$

Both errors decrease (while remain positive) as h decreases (that is, n increases) as expected. For larger p (here 5 compared to 4), the numerical solution Y_n converges more quickly to the exact solution y_e, as can be seen from Tbl.2.

1.6.1 Estimate Eigenvalues Using the False Position Method: Programming Task

A program using this method is listed on page 19-21, named fp(p1,p2,h,e). e stands for the acceptable error.

1.7 Question 7

Run the program fp(p1,p2,a,h,e) with p1 = 4, p2 = 5, a = 2. Refer to Tbl.3 and find that

- To choose e, first change the preferences in Matlab such that more than 4 decimals will be presented for each p_s.

Check with the analytic solution found in Q5. The exact smallest positive eigenvalue $p^{(1)} = \sqrt{\frac{1}{4} + (\frac{\pi}{ln(2)})^2} \approx 4.559856$. Meanwhile $p^{(2)} = \sqrt{\frac{1}{4} + (\frac{2\pi}{ln(2)})^2} \approx 9.078500$ outside the range $4.559856 \pm 5\times10^{-6}$. Hence for $p \in [4.559851, 4.559861]$, $\phi(p) \in [-1.04705\times10^{-6}, 1.12887\times10^{-6}]$ by calculating $y_e(1)$ under $p = 4.559851, 4.559861$. Choose $\epsilon = min\,|\phi(p)| = 1.04705\times10^{-6}$.

- To choose h, refer to Tbl.3 and notice that when $p = 4$, error in Y_n is of magnitude 10^{-4}, way beneath what we require for $|\phi(p)|$. Hence choose $h = 1/(10\times2^{19})$ and check whether p_s is in the correct range. If not, choose h even smaller, since from Q6 we know that the error level goes down as h decreases. Turns out this value of h is enough.
- In addition to the two points above, we need also check and modify ϵ and h s.t. each result for eigenvalue p is finally stable and only slightly changes in the 7th or higher digits after the decimal point, that is, more precisely, within $4.559856 \pm 5\times10^{-6}$

variables/n	1	2	3	4	5	6
p_1	4	4	4	4	4	4
p_2	5	4.612568	4.565360	4.560423	4.559916	4.559864
p_s	4.612568	4.565360	4.560423	4.559916	4.559864	4.559858

Table 4: p_1, p_2 and p_s under false position method

1.8 Question 8

In order to find the approximations to the eigenvalues, we use WKB approximation mentioned in the Hint and solve. Let $y = e^{pS_0(x)+S_1(x)}$. Insert y into (15a) and get

$$pS_0'' + S_1'' + (pS_0' + S_1')^2 + p^2(1+x)^{-10} = 0$$

Then for p large, p term diminishes compared to p^2, the equation reduced to

$$p^2S_0'^2 + p^2(1+x)^{-10} = 0 \Rightarrow S_0' = \pm i(1+x)^{-5}$$

For the rest p terms, have

$$S_o'' + 2S_0'S_1' = 0 \Rightarrow S_1' = -\frac{1}{2}\frac{S_0''}{S_0'} \Rightarrow S_1 = -\frac{1}{2}ln(S_0') = \frac{5}{2}ln(1+x) + D$$

for some constant D. Because $S_0' = \pm i(1+x)^{-5}$, y has two independent solutions:

$$y = (Ae^{p\int_0^x i(1+t)^{-5}dt} + Be^{-p\int_0^x i(1+t)^{-5}dt})(1+x)^{5/2}$$
$$= (Pcos(\frac{p(1+x)^{-4}-p}{4}) + Qsin(\frac{p(1+x)^{-4}-p}{4}))(1+x)^{5/2}.$$

Insert i.c.s (15b) and find, if y not identically zero,

$$y(0) = 0 \Rightarrow P = 0;\ y(1) = 0 \Rightarrow p = \frac{64k\pi}{15}(p > 0)$$

Since this p is calculated under WKB approximation, where O(1/p) terms are lost. Hence the rather rough approximations to the five smallest positive p are when k = 1, 2, 3, 4, 5, which equal to 13.4, 26.8, 40.2, 53.6, 67.0.

To find them exactly within the error range, run the program fp, where $h = 1/(10 \times 2^{19})$ and $\epsilon = 1.04705 * 10^{-10}$ are chosen s.t. each result for eigenvalue p is finally stable and only slightly changes in the 7th or higher digits after the decimal point. Since from Tbl.3 we know that the global error decreases faster as p increases or h decreases, this previously used h and ϵ will be good enough for this problem. Also choose initial p_1s and p_2s just above or below each approximated p (e.g. by 2) as the end points of intervals. (The program fp is designed s.t. a result is only calculated when $Y_1 \times Y_2 > 0$.)

By the above process, we ensure the five smallest eigenvalues p as well as the accuracy. The approximations to the five smallest eigenvalues $p^{(k)}$ hence finally by computer are:

$$12.576619,\ 26.183026,\ 39.711997,\ 53.201075,\ 66.667579.$$

Now, calculate the coefficient Q in front of y using the energy normalization and the substitution m = $(1+x)^{-4} - 1$:

$$\int_0^1 (1+x)^{-10}p^2Q^2sin^2(\frac{p((1+x)^{-4}-1)}{4})(1+x)^5dx = 1$$
$$\Rightarrow \int_{-15/16}^0 \frac{1}{4}p^2Q^2sin^2(\frac{pm}{4})dm = 1$$
$$\Rightarrow \frac{1}{4}p^2Q^2(\frac{15}{32} - \frac{1}{p}sin(\frac{15}{32}p)) = 1$$
$$\Rightarrow Q = \frac{2}{p\sqrt{\frac{15}{32} - \frac{1}{p}sin(\frac{15}{32}p)}}.\ (w.l.o.g.\ take\ positive\ Q)$$

We get

$$y^{(k)} = \frac{2}{p^{(k)}\sqrt{\frac{15}{32} - \frac{1}{p^{(k)}}sin(\frac{15}{32}p^{(k)})}} sin(\frac{p^{(k)}(1+x)^{-4} - p^{(k)}}{4})(1+x)^{5/2}.$$

where $p^{(k)} = \frac{64k\pi}{15}$, k $\in Z$. Plots as below.

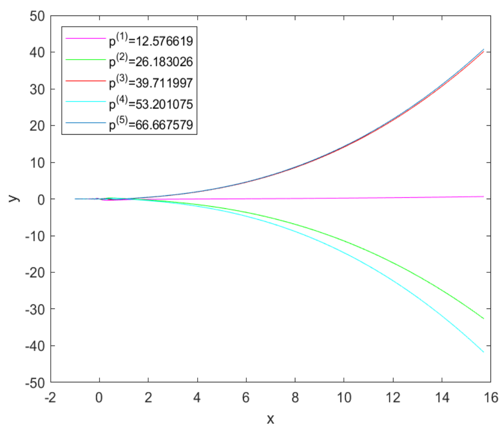

Figure 3: $y^{(k)}$ with k = 1,2,3,4,5 under corresponding $p^{(k)}$

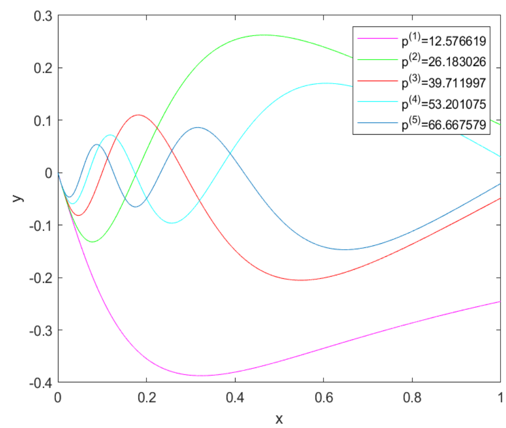

Figure 4: Enlarged over [0, 1]

y fluctuates along x-axis with larger and larger amplitude and the zeros are spaced farther and farther apart. Both measures go to infinity when x is beyond a certain value c, here approximately 1.5. Among different $y^{(k)}$s, the less the corresponding $p^{(k)}$ is, the larger the amplitude and smaller the frequency are below c. Above c, however, y with smaller p has greater absolute value at each point x.

These can be deduced **mathematically** from the approximate formula above. $y^{(k)} = 0$ requires $sin(\frac{p^{(k)}(1+x)^{-4} - p^{(k)}}{4}) = 0$ (besides $x = -1$), i.e.

$$\frac{64k\pi}{15}\frac{((1+x)^{-4} - 1)}{4} = n\pi \Rightarrow x_n = (\frac{15n}{16k} + 1)^4 - 1$$
$$\Rightarrow \frac{d(x_{n+1} - x_n)}{dk} = \frac{1}{k^2}((\frac{15n}{16k} + 1)^3\frac{15n}{4} - (\frac{15(n+1)}{16k} + 1)^3\frac{15(n+1)}{4}) < 0.$$

Hence we have checked that for fixed n, distance between two zeros decreases as k (equivalently $p^{(k)}$) increases. In terms of magnitude (A_k), also deduce from the approximate formula:

$$\frac{A_{k+1}}{A_k} = \frac{p^{(k)}\sqrt{\frac{15}{32} - \frac{1}{p^{(k)}} sin(\frac{15}{32}p^{(k)})}}{p^{(k+1)}\sqrt{\frac{15}{32} - \frac{1}{p^{(k+1)}} sin(\frac{15}{32}p^{(k+1)})}} = \frac{k}{k+1} < 1.$$

Hence we have checked that amplitude of $y^{(k)}$ (local maximum and minimun) decreases as $p^{(k)}$ increases. As x continues to increase to infinity, $sin(\frac{p^{(k)}(1+x)^{-4}-p^{(k)}}{4}) \to sin(-\frac{p^{(k)}}{4})$ nearly a constant. Hence the term $(1+x)^{5/2}$ dominates and $y^{(k)}$ goes to posi- or negative infinity according to the value of $p^{(k)}$ which decides the sign of $sin(-\frac{p^{(k)}}{4})$, as suggested by Fig.3.

On the other hand, also see that the original equation (15a) can be rearranged to

$$\frac{d^2y}{dx^2} = -\omega^2(x)y$$

where $\omega = 2\pi f = p^{(k)}(1+x)^{-5}$ is the angular frequency of a wave. Hence from a **physical** standpoint, as $p^{(k)}$ increases, ω and so f (frequency) decreases, hence less the distance between two zeros. The typical solution to the differential equation is sinusoidal if ω is constant, hence we expect a sine-like appearance at some point for the actual curve.

2 Programs

Note: Some informal programs (running just for comparison, or combining pictures, etc.) have 'return' added after excessively long texts for clarity, which is needed to be removed before tested. This situation will not happen, however, in a formal programming task that is specifically asked by the core projects.

2.1 Question 1

2.1.1 fEuler(h,Xmax)

```
function fEuler(h,Xmax)
%Forward Euler method solving f(x,y)=-8y+6e^(-2x).
%  e.g. fEuler(0.0001,10)
n=1; X=0; Y=0;
while X<=Xmax-h
     f=-8*Y+6*exp(-2*X);
     Y=Y+h*f, X=X+n*h,
end
fprintf('Y=%5g\n' , Y)
end
```

2.1.2 AB2(h,Xmax)

```
function AB2(h,Xmax)
%AB2 method solving f(x,y)=-8y+6e^(-2x)
%with y0=0 at x0=0.
%  e.g. AB2(0.0001,10)
n=1; X=0; Y=0;
f=-8*Y+6*exp(-2*X);
YY=Y+h*f; XX=X+n*h;
while XX<=Xmax-h
     ff=-8*YY+6*exp(-2*XX);
     f=-8*Y+6*exp(-2*X);
     YY=YY+h*(3/2*ff-1/2*f), XX=XX+n*h,
     Y=YY-h*(3/2*ff-1/2*f); X=XX-n*h;
end
```

```
fprintf('Y=%5g\n' , YY)
end
```

2.1.3 RK4(h,Xmax)

```
function RK4(h,Xmax)
%RK4 method solving f(x,y)=-8y+6e^(-2x).
%  e.g. RK4(0.0001,10)
n=1; X=0; Y=0;
while X<=Xmax-h
    f1=-8*Y+6*exp(-2*X);
    f2=-8*(Y+1/2*h*f1)+6*exp(-2*(X+1/2*h));
    f3=-8*(Y+1/2*h*f2)+6*exp(-2*(X+1/2*h));
    f4=-8*(Y+h*f3)+6*exp(-2*(X+h));
    Y=Y+1/6*h*(f1+2*f2+2*f3+f4), X=X+n*h,
end
fprintf('Y=%5g\n' , Y)
end
```

2.2 Question 3

Programs for calculating Y_e and plotting.

2.2.1 Calculate Y_e

```
function Ye
Ye=zeros(1,26); i=1;x=0
while i<=26
    Ye(1,i)=exp(-2*x)-exp(-8*x);
    i=i+1; x=x+0.08;
end
Ye
end
```

2.2.2 Plotting

```
Y1=[0 0.48 0.5818 0.5580 0.4979 0.4323 0.3713 0.3175 0.2709
0.2310 0.1969 0.1678 0.1430 0.1218 0.1038 0.0885 0.0754 0.0642
0.0547 0.0467 0.0398 0.0339 0.0289 0.0246 0.0210 0.0179];
Y2=[0 0.48 0.3927 0.4876 0.4164 0.4038 0.3464 0.3109 0.2663
0.2320 0.1984 0.1707 0.1457 0.1247 0.1064 0.0908 0.0774 0.0661
0.0563 0.0480 0.0409 0.0349 0.0297 0.0253 0.0216 0.0184];
Y3=[0 0.3241 0.4473 0.4716 0.4496 0.4083 0.3613 0.3149 0.2720
0.2338 0.2002 0.1712 0.1462 0.1247 0.1063 0.0907 0.0773 0.0659
0.0561 0.0478 0.0408 0.0347 0.0296 0.0252 0.0215 0.0183];
Ye=[0 0.3249 0.4481  0.4722 0.4500 0.4086 0.3614 0.3149 0.2721
0.2338 0.2002 0.1712 0.1461 0.1247 0.1063 0.0907 0.0773 0.0659
0.0561 0.0478 0.0408 0.0347 0.0296 0.0252 0.0215 0.0183];
X=0:0.08:2;
plot( X,Y1, 'm:.',X,Y2,'g ',X,Y3,'r+ ',X,Ye)
xlabel('xn'), ylabel('Yn')
legend('Euler' , 'AB2', 'RK4', 'Ye')
```

2.3 Question 4

Programs for plotting.

```
function IIQ4
h=[0.16 0.16/2 0.16/2^2 0.16/2^3 0.16/2^4 0.16/2^5 0.16/2^6
0.16/2^7 0.16/2^8 0.16/2^9 0.16/2^10 0.16/2^11 0.16/2^12
0.16/2^13 0.16/2^14 0.16/2^15];
H=log(h)/log(10);
e1=[0.511889 0.133717 0.057656 0.027035 0.013116 0.002483
0.001248 0.000626 0.000798 0.000398 0.00019 0.000039 0.000019
0.00001 0.000012 0.000006];
e2=[0.511888 0.055368 0.000199 0.00017 0.000066 0.004062
0.001982 0.000978 0.000001 0.0000003 0.0000003 0.000061 0.00003
0.000015 0.0000003 0.0000003];
e3=[0.021516 0.000765 0.000036 0.000002 0.0000003 0.004049
0.001977 0.000977 0.0000003 0.0000003 0.0000003 0.000061
0.00003 0.000015 0.0000003 0.0000003];
E1=log(e1)/log(10);
E2=log(e2)/log(10);
E3=log(e3)/log(10);
```

```
plot( vpa(H),vpa(E1), 'm:.',vpa(H),vpa(E2),'g ',vpa(H),vpa(E3),'+')
xlabel('log(h) (base10)'), ylabel('log(|En|) (base10)')
legend('Euler' , 'AB2', 'RK4')
end
```

2.4 Question 5

```
function Q5(p,a,h,Xmax)
% e.g. Q5(4,2,0.1,1)
X=0; Y=0; Z=1;
while X<=Xmax-h
    f1=Z;
    f2=Z+1/2*h*f1;
    f3=Z+1/2*h*f2;
    f4=Z+h*f3;
    g1=-p^2*(1+X)^(-a)*Y;
    g2=-p^2*(1+X+1/2*h)^(-a)*(Y+1/2*h*g1);
    g3=-p^2*(1+X+1/2*h)^(-a)*(Y+1/2*h*g2);
    g4=-p^2*(1+X+h)^(-a)*(Y+h*g3);
    Y=Y+1/6*h*(f1+2*f2+2*f3+f4);
    Z=Z+1/6*h*(g1+2*g2+2*g3+g4);
    X=X+h;
end
[Y Z]
end
```

2.5 Question 6

```
function fp(p1,p2,a,h,e)
% e.g. fp(4,5,2,0.1/2^19,1.04705*10^(-6))
X=0; Y1=0; Z1=1;
while X<=1-h
    f1=Z1;
    f2=Z1+1/2*h*f1;
    f3=Z1+1/2*h*f2;
    f4=Z1+h*f3;
```

```
      g1=-p1^2*(1+X)^(-a)*Y1;
      g2=-p1^2*(1+X+1/2*h)^(-a)*(Y1+1/2*h*g1);
      g3=-p1^2*(1+X+1/2*h)^(-a)*(Y1+1/2*h*g2);
      g4=-p1^2*(1+X+h)^(-a)*(Y1+h*g3);
      Y1=Y1+1/6*h*(f1+2*f2+2*f3+f4);
      Z1=Z1+1/6*h*(g1+2*g2+2*g3+g4);
      X=X+h;
end
X=0; Y2=0; Z2=1;
while X<=1-h
      f1=Z2;
      f2=Z2+1/2*h*f1;
      f3=Z2+1/2*h*f2;
      f4=Z2+h*f3;
      g1=-p2^2*(1+X)^(-a)*Y2;
      g2=-p2^2*(1+X+1/2*h)^(-a)*(Y2+1/2*h*g1);
      g3=-p2^2*(1+X+1/2*h)^(-a)*(Y2+1/2*h*g2);
      g4=-p2^2*(1+X+h)^(-a)*(Y2+h*g3);
      Y2=Y2+1/6*h*(f1+2*f2+2*f3+f4);
      Z2=Z2+1/6*h*(g1+2*g2+2*g3+g4);
      X=X+h;
end
if Y1*Y2<=0
ps=(Y2*p1-Y1*p2)/(Y2-Y1)
X=0; Ys=0; Zs=1;
while X<=1-h
      f1=Zs;
      f2=Zs+1/2*h*f1;
      f3=Zs+1/2*h*f2;
      f4=Zs+h*f3;
      g1=-ps^2*(1+X)^(-a)*Ys;
      g2=-ps^2*(1+X+1/2*h)^(-a)*(Ys+1/2*h*g1);
      g3=-ps^2*(1+X+1/2*h)^(-a)*(Ys+1/2*h*g2);
      g4=-ps^2*(1+X+h)^(-a)*(Ys+h*g3);
      Ys=Ys+1/6*h*(f1+2*f2+2*f3+f4);
      Zs=Zs+1/6*h*(g1+2*g2+2*g3+g4);
      X=X+h;
end
```

```
while abs(Ys)>=e
    if Ys*Y1>0
        p1=ps, Y1=Ys;
    else
        p2=ps, Y2=Ys;
    end
    ps=(Y2*p1-Y1*p2)/(Y2-Y1)
    X=0; Ys=0; Zs=1;
    while X<=1-h
        f1=Zs;
        f2=Zs+1/2*h*f1;
        f3=Zs+1/2*h*f2;
        f4=Zs+h*f3;
        g1=-ps^2*(1+X)^(-a)*Ys;
        g2=-ps^2*(1+X+1/2*h)^(-a)*(Ys+1/2*h*g1);
        g3=-ps^2*(1+X+1/2*h)^(-a)*(Ys+1/2*h*g2);
        g4=-ps^2*(1+X+h)^(-a)*(Ys+h*g3);
        Ys=Ys+1/6*h*(f1+2*f2+2*f3+f4);
        Zs=Zs+1/6*h*(g1+2*g2+2*g3+g4);
        X=X+h;
    end
end
else
    fprintf('Please choose another pair of p1, p2.')
end
end
```

2.6 Question 8

Programs for plotting.

```
X=-1:0.0001:5*pi;
Y1=2/(12.576619*(15/32-1/12.576619*sin(15/32*12.576619))^(1/2))
*sin(12.576619*((1+X).^(-4)-1)/4).*(1+X).^(5/2);
Y2=2/(26.183026*(15/32-1/26.183026*sin(15/32*26.183026))^(1/2))
*sin(26.183026*((1+X).^(-4)-1)/4).*(1+X).^(5/2);
Y3=2/(39.711997*(15/32-1/39.711997*sin(15/32*39.711997))^(1/2))
*sin(39.711997*((1+X).^(-4)-1)/4).*(1+X).^(5/2);
```

```
Y4=2/(53.201075*(15/32-1/53.201075*sin(15/32*53.201075))^(1/2))
*sin(53.201075*((1+X).^(-4)-1)/4).*(1+X).^(5/2);
Y5=2/(66.667579*(15/32-1/66.667579*sin(15/32*66.667579))^(1/2))
*sin(66.667579*((1+X).^(-4)-1)/4).*(1+X).^(5/2);
plot( X,Y1, 'm ', X,Y2, 'g ',X,Y3, 'r ',X,Y4, 'c ',X,Y5)
xlabel('x'), ylabel('y')
legend('p^{(1)}=12.576619' , 'p^{(2)}=26.183026', 'p^{(3)}=39.711997',
'p^{(4)}=53.201075', 'p^{(5)}=66.667579')
```