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HW 3: Logic

1. (**25 points**) A truth table for $P \vee \neg P$ is of the form:

P	$\neg P$	$P \vee \neg P$
true	false	true
false	true	true
true	true	true
false	false	false

Consider three atomic sentences A, B and C. Construct a truth table for the following in propositional logic:

(i) $(A \wedge B \wedge C)$ (ii) $(A \vee B \vee C)$

A	B	C	A ∧ B	A ∧ B ∧ C	A ∨ B	A ∨ B ∨ C
T	T	T	T	T	T	T
T	T	F	T	F	T	T
T	F	T	F	F	T	T
T	F	F	F	F	T	T
F	T	T	F	F	T	T
F	T	F	F	F	T	T
F	F	T	F	F	F	T
F	F	F	F	F	F	F

(iii) $A \Rightarrow B$

A	B	¬A	$A \Rightarrow B \equiv (\neg A \vee B)$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

(iv) $(A \wedge B) \vee C$ (v) $(\neg A) \vee (B \wedge C)$

A	B	C	$(A \wedge B)$	$(A \wedge B) \vee C$	$\neg A$	$(B \wedge C)$	$(\neg A) \vee (B \wedge C)$
T	T	T	T	T	F	T	T
T	T	F	T	T	F	F	F
T	F	T	F	T	F	F	F
T	F	F	F	F	F	F	F
F	T	T	F	T	T	T	T
F	T	F	F	F	T	F	T
F	F	T	F	T	T	F	T
F	F	F	F	F	T	F	T

2. (14 points) Inference.
(i) [7 points] Does $(A \wedge B) \models (A \Leftrightarrow B)$? Prove using a truth table.

A	B	$A \wedge B$	$A \Leftrightarrow B$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	T

According to the table I created above where $A \wedge B$ is true and $A \Leftrightarrow B$ is also true, thus we know that $(A \wedge B) \models (A \Leftrightarrow B)$ is **true**

(ii) [7 points] Does $A \Leftrightarrow B \models A \vee B$? Prove using a truth table.

A	B	$A \Leftrightarrow B$	$A \vee B$
T	F	F	T
T	T	T	T
F	F	T	F
F	T	F	T

According to the table I created above, where $A \vee B$ is true, but $A \Leftrightarrow B$ is not all true, thus we know $A \Leftrightarrow B \models A \vee B$ is **not true**.

3. (10 points) For each of the following statements, prove if it is true or false using logical equivalence rules.

(iii) $(C \vee (\neg A \wedge \neg B)) \equiv ((A \Rightarrow C) \wedge (B \Rightarrow C))$

We can simplify both sides of sentences, let's simplify the right side first:

$((A \Rightarrow C) \wedge (B \Rightarrow C)) \equiv (\neg A \vee C) \wedge (\neg B \vee C)$ —----Implication elimination

$(C \vee (\neg A \wedge \neg B)) \equiv (\neg A \vee C) \wedge (\neg B \vee C)$ —–-- Distributive law and Commutativity of V

Finally, we can rearrange and organize the both sides together:

$(\neg A \vee C) \wedge (\neg B \vee C) \equiv (\neg A \vee C) \wedge (\neg B \vee C)$

From the simplification above, we can see that the left-side sentence is similar to the right-side sentence; therefore, we proved that the **statement is true**.

(iv) $(A \vee B) \wedge \neg(A \Rightarrow B)$ is satisfiable

We can simplify the right side of the sentence to see if the given sentence is satisfiable. Here is the process:

$(A \vee B) \wedge \neg(A \Rightarrow B)$

$(A \vee B) \wedge \neg(\neg A \vee B)$ ——-----Implication elimination

$(A \vee B) \wedge \neg(\neg A) \wedge \neg B$ —------------- De Morgan

$(A \vee B) \wedge (A \wedge \neg B)$ —------------ Double-negation elimination

From my analysis, we can see that when A is true and B is false, then $(A \vee B)$ is true, and $(A \wedge \neg B)$ is also true. Therefore, $(A \vee B) \wedge (A \wedge \neg B)$ is true in this case. Thus, when A is true and B is false, the sentence **$(A \vee B) \wedge \neg(A \Rightarrow B)$ is satisfiable**.

4. (5 points) Decide if the following sentences are valid, unsatisfiable, or neither. Verify your decisions using the truth

We can start by eliminating the implications on left and right side sentences:

$((Smoke \land Heat) \Rightarrow Fire) \Leftrightarrow ((Smoke \Rightarrow Fire) \lor (Heat \Rightarrow Fire))$ —----- Original expression

$(\neg(Smoke \land Heat) \lor Fire) \Leftrightarrow ((\neg Smoke \lor Fire) \lor (\neg Heat \lor Fire))$ ------ Implication elimination (both side)

$((\neg Smoke \lor \neg Heat) \lor Fire) \Leftrightarrow ((\neg Smoke \lor Fire) \lor (\neg Heat \lor Fire))$ —-------- De Morgan (lef- side)

$((\neg Smoke \lor \neg Heat) \lor Fire) \Leftrightarrow ((\neg Smoke \lor \neg Heat) \lor Fire)$ ------ regroup and simplify the right side

From the above simplification, we can see both sides have the same expression. Since both sides are identical, the expression is a **valid statement**. It will be true regardless of the true values of Smoke, Heat, and Fire.

5. (16 points) Convert the following to CNF.

(i) S1: $A \Leftrightarrow (B \lor E)$

$(A \Rightarrow (B \lor E)) \land ((B \lor E) \Rightarrow A)$ —--- Biconditional elimination

$(\neg A \lor (B \lor E)) \land (\neg(B \lor E) \lor A)$ —-------- Implication elimination

$(\neg A \lor B \lor E) \land ((\neg B \land \neg E) \lor A)$ —-------De Morgan

$(\neg A \lor B \lor E) \land (A \lor \neg B) \land (A \lor \neg E)$ — distributive of $\lor$ over $\land$ -----------(CNF)

(ii) S2: $E \Rightarrow D$

$\neg E \lor D$ —---------------(CNF)

(iii) S3: $(C \land F) \Rightarrow \neg B$

$\neg(C \land F) \lor \neg B$

$\neg C \lor \neg F \lor \neg B$ —---------------(CNF)

(iv) S4: $E \Rightarrow B$

$\neg E \lor B$ —---------------(CNF)

(v) S5: $B \Rightarrow F$

$\neg B \lor F$ —---------------(CNF)

(vi) S6: $B \Rightarrow C$

$\neg B \lor C$ —---------------(CNF)

(vii) S7: $P \Rightarrow (Q \lor R)$

$\neg P \lor (Q \lor R)$

$\neg P \lor Q \lor R$ —---------------(CNF)

(viii) S8: $\neg(P \land Q) \Leftrightarrow S$

$(\neg(P \land Q) \Rightarrow S) \land (S \Rightarrow \neg(P \land Q))$ —--- Biconditional elimination

$(\neg(\neg(P \land Q) \lor S)) \land (\neg S \lor \neg(P \land Q))$ —-------- Implication elimination on both side

$(\neg(\neg P \lor \neg Q) \lor S) \land (\neg S \lor \neg P \lor \neg Q)$ —-------De Morgan on the right side

((P ∧ Q) ∨ S) ∧ (¬ S ∨ ¬ P ∨ ¬ Q) —------De Morgan on the left side
(P∨S)∧(Q∨S) ∧ (¬ S ∨ ¬ P ∨ ¬ Q) —------ Distributive law & CNF

6. (10 points) Using the sentences S1-S6 in Q5 in the CNF form as the knowledge base KB, prove ¬A ∧ ¬B using resolution. Hint: To prove the conjunction, we can prove each literal separately.

Given sentences Normal Form (CNF) as the knowledge base KB:
S1: (¬ A ∨ B ∨ E) ∧ (A ∨ ¬ B) ∧(A ∨ ¬ E)
S2: ¬ E ∨ D
S3: ¬ C ∨ ¬ F ∨ ¬ B
S4: ¬ E ∨ B
S5: ¬ B ∨ F
S6: ¬ B ∨ C

We have to prove KB |= ¬A ∧ ¬B

We can prove KB |= ¬A by contradiction first:
Negate the conclusion: ¬ ¬A = A
S7: A
S8: B ∨ E (S1, S7)
S9: ¬B (Resolving S7: A with S1(A ∨ ¬ B); A does not contribute here, the case simplifies directly here)
S10: ¬E (Resolving S7: A with S1(A ∨ ¬ E); A does not contribute here, the case simplifies directly here)
S11: E (S8 , S9)

According to derivation above we got **S10 : ¬ E and S11: E which is contradiction ,** hence we proved that KB |= ¬A

Now we can prove KB |= ¬B by contradiction
Negate the conclusion: ¬ ¬B = B
S7: B
S8: F (S7,S5)
S9: C (S7,S6)
S10: ¬ C (S7, S8, S3)

According to the derivation above we got S10: ¬C and 9: C which is a contradiction, hence we proved that KB |= ¬B

We prove that KB |= ¬B and KB |= ¬A are ture, thus we proved that KB |= ¬A ∧ ¬B

7. (20 points) Convert the following sentences to first-order logic sentences. We use the following predicates:

E(x) means “x is an easy course”
H(y) means “y is a happy student”
T (y, x) means “student y takes the course x”
F (x) means “x is a course with a final”

(i) “If a course is easy, some students are happy”

$$\forall x(E(x) \rightarrow \exists y(T(y,x) \wedge H(y)))$$

(ii) “If a course has a final, no students are happy”

$$\forall x(F(x) \rightarrow \forall y(T(y,x) \wedge \neg H(y)))$$

(iii) “If a course has a final, the course is not easy”

$$\forall x(F(x) \rightarrow \neg E(x))$$

(iv) “If a student is happy, then they are taking an easy course with a final”

$$\forall y(H(y) \rightarrow \exists x(T(y,x) \wedge E(x) \wedge F(x)))$$