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Introduction to Mathematical Statistics I
Homework 2

2.39 An experiment consists of tossing a pair of dice.

a Use the combinatorial theorems to determine the number of sample points in the sample space S.

Each die has 6 faces, so each die can land on one of 6 numbers, according to the combinatiotorial theorems when tossig two dice is product of the possible outcomes for each die

thus, $S = 6 \times 6 = 36$

b Find the probability that the sum of the numbers appearing on the dice is equal to 7.

First we have to find all possible outcomes where the sum of numbers on the two diee equals 7, thus we have outcomes as follwig (2,5), (5,2), (4,3), (3,4) (1,6) (6,1)

$$\text{probability} = \frac{\text{number of favorable outcomes}}{\text{Total number of outcomes}} = \frac{6}{36} = \frac{1}{6} \approx 0.16667$$

∴ the probability is 1/6 that the numbers appeary on the dice is equal to 7

2.58 Five cards are dealt from a standard 52-card deck. What is the probability that we draw

a 3 aces and 2 kings?

- The number of way choose 3 aces : $C^4_3 = \frac{4!}{3!(4-3)!} = 4$
- The number of way choose 2 kings: $C^4_2 = \frac{4!}{2!(4-2)!} = 6$
- The number of favorable out comes : $4 \times 6 = 24$
- The total number of possible 5 cards from 52 cards: $C^{52}_5 = 2,598,960$

$$\therefore \text{probability} = \frac{\text{number of favorable outcomes}}{\text{Total number of outcomes}} = \frac{C^4_3 \cdot C^4_2}{\binom{52}{5}} = \frac{24}{2598960}$$

b a “full house” (3 cards of one kind, 2 cards of another kind)?

- The number of ways to choose 3 cards one kind : $\binom{13}{1} \times \binom{4}{3} = 52$
- The number of ways to choose 2 cards of another kind: $\binom{12}{1}\binom{4}{2} = 72$

$$\text{probability} = \frac{\text{number of favorable outcomes}}{\text{Total number of outcomes}} = \frac{52 \times 72}{\binom{52}{5}} = \frac{3744}{2598960} \approx 0.00144$$

2.69 Prove that $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

- The left-hand side, $\binom{n+1}{k}$, represents the number of ways to choose k items from a set of $n+1$ items.
- The right-hand side, $\binom{n}{k} + \binom{n}{k-1}$ which means :

$\binom{n}{k}$: The number of ways to choose k items from the first n items

$\binom{n}{k-1}$: The number of ways to choose $k-1$ items from the first n items

- Consider a set of $n+1$ items, we can divide this set into two subsets : one with the first n items and another with the last item.
- To choose k items from the first n items without choosy the last item : This is represented by $\binom{n}{k}$.
- choose $k-1$ items from the first n items and then choose the last item : This is represented by $\binom{n}{k-1}$.

Since these two cases cover all possibilites, the total number of ways to choose k items from the entire set is sum of these two quantities. Therefore, the identity holds.

2.77 A study of the posttreatment behavior of a large number of drug abusers suggests that the likelihood of conviction within a two-year period after treatment may depend upon the offenders education. The proportions of the total number of cases falling in four education–conviction categories are shown in the following table:

	Status within 2 Years after Treatment		
Education	Convicted	Not Convicted	*Total*
10 years or more	.10	.30	.40
9 years or less	.27	.33	.60
Total	.37	.63	1.00

Suppose that a single offender is selected from the treatment program. Define the events:

A: The offender has 10 or more years of education.
B: The offender is convicted within two years after completion of treatment.

Find the following:

a $P(A)$.

$P(A) = 0.40$

b $P(B)$.

$P(b) = 0.37$

c $P(A \cap B)$

The probability that offender has 10 or more years of education and is convicted is intersection of $A \cap B$

thus $P(A \cap B) = 0.1$

d $P(A \cup B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.40 + 0.37 - 0.1 = 0.67$

e $P(\overline{A})$.

$P(\overline{A}) = 1 - P(A) = 1 - 0.40 = 0.60$

f $P(\overline{A \cup B})$.

$P(\overline{A \cup B}) = 1 - P(A \cup B) = 1 - 0.67 = 0.33$

g $P(\overline{A \cap B})$.

$P(\overline{A \cap B}) = 1 - P(A \cap B) = 1 - 0.1 = 0.9$

h $P(A|B)$.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.1}{0.37} \approx 0.27027$$

i $P(B|A)$.

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.1}{0.40} = 0.25$$

2.114 A lie detector will show a positive reading (indicate a lie) 10% of the time when a person is telling the truth and 95% of the time when the person is lying. Suppose two people are suspects in a one-person crime and (for certain) one is guilty and will lie. Assume further that the lie detector operates independently for the truthful person and the liar. What is the probability that the detector

a shows a positive reading for both suspects?

Given Probabilities: $P(\text{Positive}|\text{truth}) = 0.1$

$P(\text{Positive}|\text{false}) = 0.95$, Also they are independent

thus we can get:

$P(\text{Positive for both}) = P(\text{Positive}|\text{truth}) \times P(\text{Positive}|\text{false})$

$= 0.1 \times 0.95 = 0.095$

b shows a positive reading for the guilty suspect and a negative reading for the innocent suspect?

we need to find: $P[(\text{Positive}|\text{False}) \cap (\text{Negative}|\text{true})]$

we already know that $P(\text{Positive}|\text{False}) = 0.95$

we can get $P(\text{negative}|\text{true}) = 1 - P(\text{Positive}|\text{truth}) = 1 - 0.1 = 0.9$

since independent:

$P[(\text{Positive}|\text{False}) \cap (\text{Negative}|\text{true})] = P(\text{positive}|\text{False}) * P(\text{Negative}|\text{true})$

$= 0.95 \times 0.9 = 0.855$

c is completely wrong—that is, that it gives a positive reading for the innocent suspect and a negative reading for the guilty?

guilty: $P(\text{Positive}|\text{true}) = 1 - P(\text{Positive}|\text{False}) = 1 - 0.95 = 0.05$

innocent: $P(\text{negative}|\text{False}) = 1 - P(\text{negative}|\text{true}) = 1 - 0.9 = 0.1$

since event are idependent: $P(\text{positive for innocent and negative for guilty})$

$= P(\text{positive}|\text{true}) * P(\text{negative}|\text{False}) = 0.05 \times 0.1 = 0.005$

so, the probability of the detector is completely wrong is 0.5%, thus that is extremely unlikely but not impossible.

d gives a positive reading for either or both of the two suspects?

we can interprete a positve for either or both of two supects which means we need to find probability that at least one is positive.

from part (a), (b) (c) we know:

$P(\text{negative} \mid \text{true}) = 0.9$, $P(\text{negative} \mid \text{false}) = 0.05$

$P(\text{negative for both}) = P(\text{negative} \mid \text{true}) * P(\text{negative} \mid \text{false})$

$= 0.9 * 0.05 = 0.045$

Thus, $P(\text{positive for either or both}) = 1 - 0.045 = 0.955$

2.175 Three events, A, B, and C, are said to be mutually independent if

$$P(A \cap B) = P(A) \times P(B), \qquad P(B \cap C) = P(B) \times P(C),$$
$$P(A \cap C) = P(A) \times P(C), \qquad P(A \cap B \cap C) = P(A) \times P(B) \times P(C).$$

Suppose that a balanced coin is independently tossed two times. Define the following events:

A: Head appears on the first toss.
B: Head appears on the second toss.
C: Both tosses yield the same outcome.

Are A, B, and C mutually independent?

$S = \{HH, HT, TH, TT\}$

coin is fair

event are position independent

$P(A) = P(HH \text{ or } HT) = P(HH) + P(HT) = P(H)P(H) + P(H)P(T) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$

$P(B) = P(HH \text{ or } TH) = P(HH) + P(TH) = P(H)P(H) + P(T)P(H) = \frac{1}{2}$

$P(C) = \frac{1}{2}$, This's because there are 4 outcomes

(HT, TH, HH, TT)

If we want to check A, B and C are mutually independence, we need to check $P(A \cap B \cap C) \neq P(A) * P(B) * P(C)$

Since A, B, C are independent, we can chek if pairwis independent

$P(A \cap B) = P(A) * P(B) = \frac{1}{2} * \frac{1}{2} = \frac{1}{4}$ → head on first and second toss

$P(A \cap C) = P(A) * P(C) = \frac{1}{2} * \frac{1}{2} = \frac{1}{4}$ → head on frist toss and both yeid on same outcome

$P(B \cap C) = P(B) * P(C) = \frac{1}{2} * \frac{1}{2} = \frac{1}{4}$ → second toss is head and both yeis on same outcome

thus we know $P(A \cap B \cap C) = \frac{1}{4}$

But $P(A) * P(B) * P(C) = \frac{1}{2} * \frac{1}{2} * \frac{1}{2} = \frac{1}{8}$

Because the triple intersection $P(A \cap B \cap C) = \frac{1}{4}$ is not equal $P(A) * P(B) * P(C) = \frac{1}{8}$, thus event A, B, C are not mutually independent

3.6 Five balls, numbered 1, 2, 3, 4, and 5, are placed in an urn. Two balls are randomly selected from the five, and their numbers noted. Find the probability distribution for the following:

a The *largest* of the two sampled numbers

$S = \binom{5}{2} = 10$ combinations are: (2,3) (2,4), (2,5), (3,4) (1,2) (1,3), (1,4), (1,5) (3,5), (4,5)

The probability distribution for the largest of two sample number are: P(largest = 2): P = 1/10 ; P(largest = 3): P = 2/10 = 1/5

P(largest = 4): P = 3/10 ; P(largest = 5): P = 4/10 = 2/5

b The *sum* of the two sampled numbers

possible outcomes are: 3, 4, 5, 6, 7, 8, 9, so we can get follow

P(sum = 3): (1,2) → P = 1/10,

P(sum = 4): (1,3) → P = 1/10,

P(sum = 5): (1,4), (2,3) → $P = \frac{2}{10} = \frac{1}{5}$

P(sum = 6): (1,5), (2,4) → $P = \frac{1}{5}$

P(sum = 7): (2,5), (3,4) → $P = \frac{2}{10} = \frac{1}{5}$

P(sum = 8): (3,5) → $P = \frac{1}{10}$

P(sum = 9): (4,5) → $P = \frac{1}{10}$