Introduction to Mathematical Statistics I
Homework 5

Student: Freya Zou

4.20 If, as in Exercise 4.16, Y has density function

$$f(y) = \begin{cases} (1/2)(2-y), & 0 \le y \le 2, \\ 0, & \text{elsewhere,} \end{cases}$$

find the mean and variance of Y.

$$E[Y] = \int_{-\infty}^{\infty} y f(y)\,dy = \int_0^2 y \tfrac{1}{2}(2-y)\,dy = \frac{2}{3} \approx 0.67$$

$$\text{var}(Y) = E(Y^2) - [E(Y)]^2 \quad \text{variance property}$$

$$= \int_{-\infty}^{\infty} y^2 f(y)\,dy - [E(Y)]^2$$

$$= \int_0^2 y^2 \tfrac{1}{2}(2-y)\,dy - \boxed{\left(\frac{2}{3}\right)^2} \quad \text{from } E[Y]$$

$$= \frac{2}{3} - \frac{4}{9} = \frac{2}{9} \approx 0.222$$

4.26 If Y is a continuous random variable with mean μ and variance σ^2 and a and b are constants, use Theorem 4.5 to prove the following:

a $E(aY+b) = aE(Y)+b = a\mu+b$.

b $V(aY+b) = a^2V(Y) = a^2\sigma^2$.

a) $E(aY+b) = E(aY) + E(b)$ (the linearity of expectation $E[R_1+R_2] = E[R_1] + E[R_2]$)

we know that a, b are constants, thus we can get $E(aY) = aE(Y)$ (properties of continuous variable $E[c\,g(x)] = cE[g(x)]$)

and $E(b)=b$, thus we know $E(aY+b) = aE(Y)+b$

Let $E(Y) = \mu$, we get: $E(aY+b) = a\mu+b$

Therefore we proved $E(aY+b) = aE(Y)+b = a\mu+b$

b) By the defination of varriance: $V(x) = E(x^2) - [E(x^2)]^2$

we can subsitude: $V(aY+b) = E[(aY+b-E(aY+b))^2]$

From part (a) we've already proved $E(aY+b) = aE(Y)+b = a\mu+b$

so we know $V(aY+b) = E[(aY+b-(a\mu+b))^2]$, we can simplyey the inside of expression:

$$V(aY+b) = E[(aY-a\mu)^2].$$

Factor out a from the expression: $V(aY+b) = a^2E[(Y-\mu)^2]$

By the defination $E[(Y-\mu)^2] = V(Y) = \sigma^2$, thus,

$$V(aY+b) = a^2\sigma^2.$$

Therefore, we proved $V(aY+b) = a^2V(Y) = a^2\sigma^2$

4.42 The *median* of the distribution of a continuous random variable Y is the value $\phi_{.5}$ such that $P(Y \le \phi_{.5}) = 0.5$. What is the median of the uniform distribution on the interval (θ_1, θ_2)?

PDF of unifor distribuftion : $f(y) = \begin{cases} \frac{1}{\theta_2 - \theta_1}, & \theta_1 \le y \le \theta_2 \\ 0, & \text{otherwise} \end{cases}$

CDF $F(y)$ is given: $F(y) = P(Y \le y) = \frac{y - \theta_1}{\theta_2 - \theta_1}$, $\theta_1 \le y \le \theta_2$

To Find median, we can set $F(\phi 0.5) = 0.5$

$$F(\phi 0.5) = \frac{\phi 0.5 - \theta_1}{\theta_2 - \theta_1} = 0.5$$

$$\Rightarrow \phi 0.5 - \theta_1 = 0.5(\theta_2 - \theta_1)$$

$$\Rightarrow \phi 0.5 = 0.5(\theta_2 - \theta_1) + \theta_1$$

$$\Rightarrow \phi 0.5 = 0.5\theta_2 - 0.5\theta_1 + \theta_2$$

$$\Rightarrow \phi 0.5 = \frac{\theta_1 + \theta_2}{2}$$

Thus the median of the uniform distribution on the interval (θ_1, θ_2) is $\frac{\theta_1 + \theta_2}{2}$

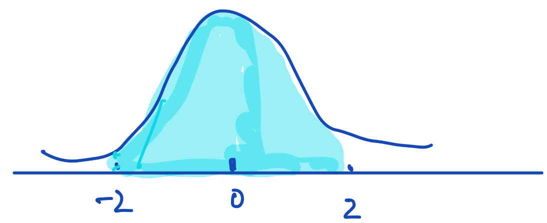
4.71 Wires manufactured for use in a computer system are specified to have resistances between .12 and .14 ohms. The actual measured resistances of the wires produced by company A have a normal probability distribution with mean .13 ohm and standard deviation .005 ohm.

a What is the probability that a randomly selected wire from company A's production will meet the specifications?

b If four of these wires are used in each computer system and all are selected from company A, what is the probability that all four in a randomly selected system will meet the specifications?

a) $x = 0.12$: $Z = \frac{x-\mu}{\sigma} = \frac{0.12-0.13}{0.005} = -2$

$x = 0.14$: $Z = \frac{x-\mu}{\sigma} = \frac{0.14-0.13}{0.005} = 2$



$\left.\begin{array}{l} P(Z \leq 2) \approx 0.9772 \\ P(Z \leq -2) \approx 0.0228 \end{array}\right\}$ area found on the table.

$$P(0.12 \leq Y \leq 0.14) = P(Z \leq 2) - P(Z \leq -2)$$
$$= 0.9772 - 0.0228 = 0.9544$$

b) we already know that a single wire meets the specification from part (a) is 0.9544 . and we kown that each system used 4 wires.

$$P(\text{all four wires meet specification}) = (0.9544)^4 \approx 0.8297$$

the probabiliy that all four in a randomly selected system will meet the specification is 82.97%

4.104 The lifetime (in hours) Y of an electronic component is a random variable with density function given by

$$f(y) = \begin{cases} \frac{1}{100}e^{-y/100}, & y > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Three of these components operate independently in a piece of equipment. The equipment fails if at least two of the components fail. Find the probability that the equipment will operate for at least 200 hours without failure.

Since each componentes lifetime Y follows an exponential distribution with a mean of 100 hours, thus $\lambda = \frac{1}{100}$

$P(\text{Single component lasts more than 200 hours}) = P(Y>200) = e^{-200/100} = e^{-2} \approx 0.135$

thus $P(\text{single component fails within 200 hours}) = 1-(Y>200) = 1-0.135 = 0.865$

Let X be the number of componentes that fail before 200 hours. X follows a binomial distribution with parameters $n=3$, failure $P = 0.135$

we have to find the probabiliy that the equipment will operate for at least 200 hours which means we have to find $P(X \geq 2)$

$P(X \geq 2) = P(X=2) + P(X=3)$ binmial distribution $P(x) = \binom{n}{x} p^x q^{n-x}$

$= \binom{3}{2} p^2 (1-p)^{3-2} + \binom{3}{3} p^3 (1-p)^{3-3}$

$= \binom{3}{2} (0.135)^2 (0.865) + \binom{3}{3} (0.135)^3 (0.865)^0$

$= 0.0474 + 0.0025$

$= 0.0499$

Therefore, the probability that the equipment will operates for at least 200 hours without failers is 0.0499

4.129 During an eight-hour shift, the proportion of time Y that a sheet-metal stamping machine is down for maintenance or repairs has a beta distribution with $\alpha = 1$ and $\beta = 2$. That is,

$$f(y) = \begin{cases} 2(1-y), & 0 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

The cost (in hundreds of dollars) of this downtime, due to lost production and cost of maintenance and repair, is given by $C = 10 + 20Y + 4Y^2$. Find the mean and variance of C.

$E[Y] = \frac{\alpha}{\alpha+\beta} = \frac{1}{2+1} = \frac{1}{3}$ $\quad Var(Y) = \frac{\alpha\beta}{(\beta+\alpha)^2(\alpha+\beta+1)} = \frac{2}{9 \cdot 4} = \frac{1}{18}$

$E(Y^2) = var(Y) + [E(Y)]^2 = 1/6$

we need to find $E(C) = E[10 + 20Y + 4Y^2]$

$= E[10] + E[20Y] + E[4Y^2]$ (linearity of expectation)

$= 10 + 20E[Y] + 4E[Y^2]$ $(E(c) = C)$

$= 10 + 20 \cdot \frac{1}{3} + 4 [E[Y^2] = var(Y) + (E[Y])^2]$ — (from above)

$= 10 + \frac{20}{3} + 4 \cdot \frac{1}{6} = \frac{52}{3} \approx 17.33$ (in hundreds of dollars)

$var(C) = E(C^2) - (E(C))^2$

$C^2 = (10 + 20Y + 4Y^2)(10 + 20Y + 4Y^2) = 16Y^4 + 160Y^3 + 480Y^2 + 400Y + 100$

$E(C^2) = E(16Y^4 + 160Y^3 + 480Y^2 + 400Y + 100)$

$= 16E(Y^4) + 160E(Y^3) + 480E(Y^2) + 400E[Y] + 100$ (Linearity of expectation)

$= 16 \cdot \int_0^1 y^4 2(1-y)dy + 160 \cdot \int_0^1 y^3 2(1-y)dy + 480 \cdot \frac{1}{6} + 400(\frac{1}{3}) + 100$ $\boxed{E(c) = \int_{-\infty}^{\infty} c \cdot f(y)dy}$

$= 1562/5 = 330.4$

$Var(C) = E(C^2) - [E(C)]^2 = 330.4 - \left(\frac{52}{3}\right)^2 \approx 29.9556$

4.145 A random variable Y has the density function

$$f(y) = \begin{cases} e^{y}, & y < 0, \\ 0, & \text{elsewhere.} \end{cases}$$

a Find $E(e^{3Y/2})$.

b Find the moment-generating function for Y.

c Find $V(Y)$.

a) $E(y) = \int_{-\infty}^{\infty} y f(y)\,dy = E(e^{3y/2}) = \int_{-\infty}^{0} e^{3y/2} \cdot e^{y}\,dy = \frac{2}{5} = 0.4$

b) $m_y t = E(e^{ty})$

$m_y(t) = \int_{-\infty}^{0} e^{ty} f(y)\,dy = \int_{-\infty}^{0} e^{ty} \cdot e^{y}\,dy = \int_{-\infty}^{0} e^{y(t+1)}\,dy$

$\Rightarrow \left[\frac{e^{y(t+1)}}{t+1} \right]_{-\infty}^{0} = \frac{1}{t+1} \left(e^{0} - \lim_{y \to -\infty} e^{y(t+1)} \right)$

For convergence. we need $t > -1$:

thus, $M_y(t) = \frac{1}{t+1}$, $t > -1$

c) $V(Y) = E(Y^2) - (E(Y)^2)$, we have to find $E(Y)$ and $E(Y^2)$

$E(Y) = \int_{-\infty}^{0} y f(y)\,dy = \int_{-\infty}^{0} y e^{y}\,dy = 0 - \int_{-\infty}^{0} e^{y}\,dy = -1$

$E(Y^2) = \int_{-\infty}^{0} y^2 f(y)\,dy = \int_{-\infty}^{0} y^2 e^{y}\,dy = 2$

$V(Y) = E(Y^2) - (E(Y)^2) = 2 - (-1)^2 = 1$