Introduction to Mathematical Statistics I
Homework 4

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3.96 The telephone lines serving an airline reservation office are all busy about 60% of the time.

a If you are calling this office, what is the probability that you will complete your call on the first try? The second try? The third try?

Given: $P(\text{busy}) = 0.6$ $\quad P(\text{not busy}) = 1 - 0.6 = 0.4$

$P(\text{first try}) = 0.4$

$P(\text{secon try})$: It requires first try fails and second try successful:

$P(\text{second try}) = P(\text{busy}) * P(\text{not busy}) = 0.6 \times 0.4 = 0.24$

$P(\text{third try})$: The first two tries must busy, then third time successful: $P(\text{third try}) = P(\text{busy})^2 * P(\overline{\text{busy}}) = (0.6)^2 (0.4) = 0.144$

b If you and a friend must both complete calls to this office, what is the probability that a total of four tries will be necessary for both of you to get through?

Let Y be # of atemps until both calls are completed, then $Y \sim NB(r=2, p=0.4)$ with PMF:

$P(Y=y) = \binom{y-1}{r-1}(1-p)^{y-r} p^r$

$P(Y=4) = \binom{4-1}{2-1}(1-0.4)^{4-2}(0.4)^2 = 0.1728$

3.105 In southern California, a growing number of individuals pursuing teaching credentials are choosing paid internships over traditional student teaching programs. A group of eight candidates for three local teaching positions consisted of five who had enrolled in paid internships and three who enrolled in traditional student teaching programs. All eight candidates appear to be equally qualified, so three are randomly selected to fill the open positions. Let Y be the number of internship trained candidates who are hired.

a Does Y have a binomial or hypergeometric distribution? Why?

Since we are samplig without replacement from a finite population follows hypergeometric distribution, Because hypergeometric distribution is used when samplig without replacement, whereas the binomial distribution applys samplig with replacement. Therefor Y has a hypergeometric distribution

b Find the probability that two or more internship trained candidates are hired.

$$P(X=K) = \frac{C_k^K * C_{n-k}^{N-K}}{C_n^N} = \frac{\binom{5}{K} * \binom{3}{3-K}}{\binom{8}{3}}$$

we need to find $P(Y \geq 2) = 1 - P(Y < 2) = 1 - [P(Y=0) + P(Y=1)]$

$$P(Y=0) = \frac{\binom{5}{0} * \binom{3}{3}}{\binom{8}{3}} = \frac{1 \cdot 1}{56} \approx 0.0179$$

$$P(Y=1) = \frac{\binom{5}{1} \cdot \binom{3}{2}}{\binom{8}{3}} = \frac{3 \cdot 5}{56} \approx 0.26785$$

$$P(P \geq 2) = 1 - [P(Y=0) + P(Y=1)]$$
$$= 1 - [0.0179 + 0.26785]$$
$$= 1 - 0.28575$$
$$= 0.71425$$

c What are the mean and standard deviation of Y?

$$\mu = \frac{n \cdot K}{N} = \frac{3 \cdot 5}{8} = \frac{15}{8} = 1.875$$

$$\sigma = \sqrt{\frac{n \cdot K \cdot (N-K) \cdot (N-n)}{N^2 \cdot (N-1)}} = \sqrt{\frac{3 \cdot 5 \cdot (8-5) \cdot (8-3)}{8^2 \cdot (8-1)}} = 0.7107$$

thus the mean is 1.875 and standard deviation is 0.7107

3.121 Let Y denote a random variable that has a Poisson distribution with mean $\lambda = 2$. Find

a $P(Y = 4)$. $P(y) = \dfrac{\lambda^y}{y!} e^{-\lambda} = \dfrac{2^4 e^{-2}}{4!} = 0.09022$

b $P(Y \geq 4)$.

$$P(Y \geq 4) = 1 - (Y < 4) = 1 - [P(Y=0) + P(Y=1) + P(Y=2) + P(Y=3)]$$

$$= 1 - \left[\frac{2^0 e^{-2}}{0!} + \frac{2^1 e^{-2}}{1!} + \frac{2^2 e^{-2}}{2!} + \frac{2^3 e^{-2}}{3!}\right] \quad \left(\text{we know } P(y) = \frac{\lambda^y e^{-\lambda}}{y!}\right)$$

$$= 0.14288$$

c $P(Y < 4)$.

$$P(Y < 4) = P(Y=0) + P(Y=1) + P(Y=2) + P(Y=3)$$

$$= \frac{2^0 e^{-2}}{0!} + \frac{2^1 e^{-2}}{1!} + \frac{2^2 e^{-2}}{2!} + \frac{2^3 e^{-2}}{3!} = 0.85712$$

d $P(Y \geq 4 | Y \geq 2)$.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = P(Y \geq 4 | Y \geq 2) = \frac{P(Y \geq 4) \cap P(Y \geq 2)}{P(Y \geq 2)}$$

$$= \frac{P(Y \geq 4)}{P(Y \geq 2)} = \frac{0.14288 \text{ (part b)}}{1 - P(Y < 2)} = \frac{0.14288}{1 - (P(Y=0) + P(Y=1)]}$$

$$= \frac{0.14288}{1 - \left[\frac{2^0 e^{-2}}{0!} + \frac{2^1 e^{-2}}{1!}\right]} = \frac{0.14288}{0.5941} = 0.240498$$

3.139 In the daily production of a certain kind of rope, the number of defects per foot Y is assumed to have a Poisson distribution with mean $\lambda = 2$. The profit per foot when the rope is sold is given by X, where $X = 50 - 2Y - Y^2$. Find the expected profit per foot.

Since $X = 50 - 2Y - Y^2$, thus $E(X) = E(50 - 2Y - Y^2)$

$\Rightarrow 50 - 2E(Y) - E(Y^2)$, now we need to find $E(Y)$ and $E(Y^2)$ for Possion distribution, given

$E(Y) = \lambda = 2$, we know that $Var(Y) = E(Y^2) - [E(Y)]^2$

so $E(Y^2) = var(Y) + [E(Y)]^2 = 2 + 2^2 = 2 + 4 = 6$

thus: $E(X) = 50 - 2 \cdot 2 - 6 = 50 - 4 - 6 = 40$

Therefore, expected profit is 40 per foot.

3.147 If Y has a geometric distribution with probability of success p, show that the moment-generating function for Y is

$$m(t) = \frac{pe^t}{1 - qe^t}, \qquad \text{where } q = 1 - p.$$

MGF: $m(t) = E(e^{tY})$

PMF for a geometric distribution is: $P(Y=k) = p \cdot q^{k-1}$, for $k = 1, 2, 3 \ldots$ where $q = 1-p$.

thus: $m(t) = E(e^{tY}) = \sum_{k=1}^{\infty} e^{tk} \cdot P(Y=k)$, we can subsitute the PMF $P(Y=k) = p \cdot q^{k-1}$ we got:

$$m(t) = \sum_{k=1}^{\infty} e^{tk} \cdot p \cdot q^{k-1}$$

factor p

$$\Rightarrow m(t) = p \sum_{k=1}^{\infty} (e^t q)^{k-1} e^t$$

thus the moment-generaty function for Y is

$$m(t) = p \sum_{k=1}^{\infty} (e^t q)^{k-1} e^t$$

3.153 Find the distributions of the random variables that have each of the following moment-generating functions:

a $m(t) = [(1/3)e^t + (2/3)]^5$.

b $m(t) = \dfrac{e^t}{2 - e^t}$.

c $m(t) = e^{2(e^t - 1)}$.

a. The MGF of binomial distribution is $m(t) = (pe^t + (1-p))^n$ compare this MGF to given part (a), we can see $n=5$ which indicates 5 trials, $p = \frac{1}{3}$ (the probability of sucess)

b. The MGF of geometric distribution with probability of success p is: $m(t) = \dfrac{pe^t}{1-(1-p)e^t}$, we can compare with the given MGF, we can write $(2-e^t) = 1-(1-p)e^t$, we can see $1-p = 1-\frac{1}{2} = \frac{1}{2}$, thus $p = \frac{1}{2}$

c. The MGF of Poisson distribution: $m(t) = e^{\lambda(e^t - 1)}$, we can tell from the given MGF, $\lambda = 2$

4.1 Let Y be a random variable with $p(y)$ given in the table below.

y	1	2	3	4
$p(y)$	.4	.3	.2	.1

a Give the distribution function, $F(y)$. Be sure to specify the value of $F(y)$ for all y, $-\infty < y < \infty$.

1. for $y<1$: $F(y)=0$ for $1\le y<2$: $F(y)=P(Y=1)=0.4$

for $2\le y<3$: $F(y)=P(Y=1)+P(Y=2)=0.4+0.3=0.7$

for $3\le y<4$: $F(y)=P(Y=1)+P(Y=2)+P(Y=3)=0.4+0.3+0.2=0.9$

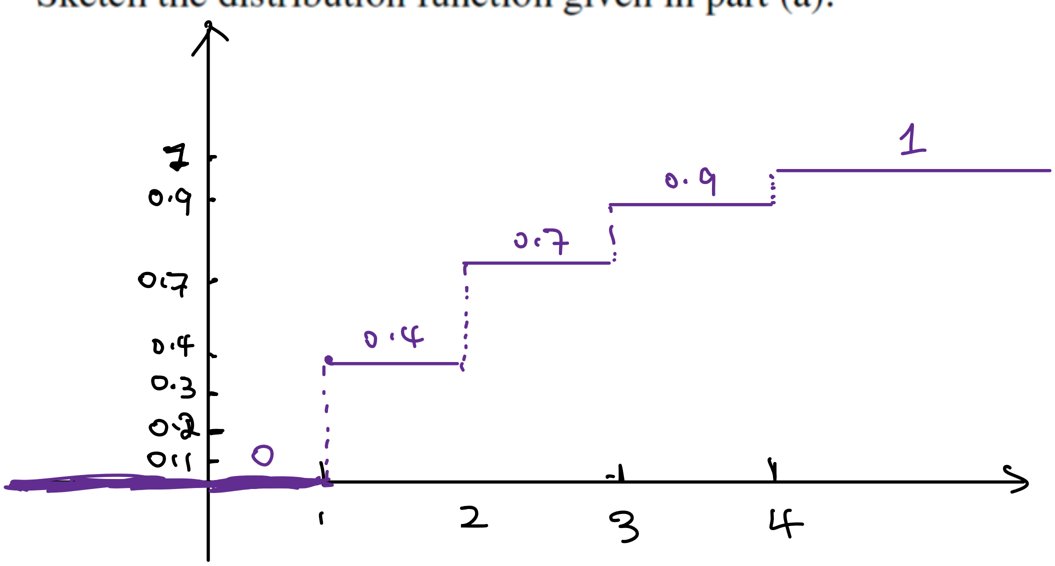
For $y\geq 4$: $F(y)=P(y=1)+P(y=2)+P(y=3)+P(y=4)$

$= 0.4+0.3+0.2+0.1$

$= 1$

$$\therefore F(y)=\begin{cases} 0, & y<1 \\ 0.4, & 1\le y\le 2 \\ 0.7, & 2\le y<3 \\ 0.9, & 3\le y<4 \\ 1, & y\geq 4 \end{cases}$$

b Sketch the distribution function given in part (a).



4.8 Suppose that Y has density function

$$f(y) = \begin{cases} ky(1-y), & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

a Find the value of k that makes $f(y)$ a probability density function.

$$\int_{-\infty}^{\infty} f(y)\,dy = 1 = \int_0^1 ky(1-y)\,dy = k\int_0^1 y(1-y)\,dy$$

$$\Rightarrow = k\int_0^1 (y-y^2)\,dy = k\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1$$

$$\Rightarrow = k\cdot\frac{3-2}{6} = \frac{k}{6} = 1$$

$$\Rightarrow k = 6$$

b Find $P(.4 \le Y \le 1)$.

we know $k=6$ from part (b)

$$f(y) = \begin{cases} 6y(1-y), & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

$$P(.4 \le y \le 1) = \int_{0.4}^{1} 6y(1-y)\,dy$$

$$= 6\left(\int_{0.4}^{1} y\,dy - \int_{0.4}^{1} y^2\,dy\right)$$

$$= 6(0.42 - 0.312)$$

$$= 0.648$$

c Find $P(.4 \le Y < 1)$.

For a continuous random variable, the probability of taking on any exact value like $(Y=1)=0$. Because the probability is defined area under the curve of the PDF, and the area at a single point is 0, thus we know $P(0.4 \le Y \le 1)$ and $P(0.4 \le Y < 1)$ both has same result 0.648

d Find $P(Y \le .4|Y \le .8)$.

$$P(Y\le 0.4 \mid Y\le 0.8) = P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(Y\le 0.4)}{P(Y\le 0.8)}$$

$$\Rightarrow P(Y\le 0.4) = \int_0^{0.4} 6y(1-y)\,dy$$

$$= 6\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^{0.4} = 6(0.08-0.0213) = 0.352$$

$$P(y\le 0.8) = \int_0^{0.8} 6y(1-y)\,dy = 6\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^{0.8}$$

$$\Rightarrow 6(0.32-0.1707) = 0.8959$$

thus $P(Y\le 0.4 \mid Y\le 0.8) = \dfrac{0.352}{0.8959} \approx 0.3931$

e Find $P(Y < .4|Y < .8)$.

This is similar to part(d), as we explained in part(e). in continous distribution, the probability at a single point is 0. thus we know the result

$$P(Y\le 0.4 \mid Y\le 0.8) = P(Y\le 0.4 \mid Y\le 0.8) = 0.3931$$

thus $P(Y<0.4 \mid Y<.8) = 0.3931$