Introduction to Mathematical Statistics I
Homework 6

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5.3 Of nine executives in a business firm, four are married, three have never married, and two are divorced. Three of the executives are to be selected for promotion. Let Y_1 denote the number of married executives and Y_2 denote the number of never-married executives among the three selected for promotion. Assuming that the three are randomly selected from the nine available, find the joint probability function of Y_1 and Y_2.

Given 9 executives in total: 4 married, 3 never married, 2 divorced

- Y_1: number of married executives selected
- Y_2: number of never-married executives selected

Total outcomes $= \binom{9}{3} = 84$, the joint probability function $P(Y_1=y_1, Y_2=y_2)$ is determined under the condition that $y_1+y_2 < 3$, Because 3 executives are selected. the remain executive is neither married, nor never married.

Thus: $0 \le y_1 \le 4$ (4 married) $0 \le y_2 \le 3$ (3 never married) $Y_3 = 3 - y_1 - y_2$, $0 \le Y_3 \le 1$

we can use hypergeometric form here

$$P(x) = \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}} : P(Y_1=y_1, Y_2=y_2) = \frac{\binom{4}{y_1}\binom{3}{y_2}\binom{2}{3-y_1-y_2}}{\binom{9}{3}}$$

where $0 \le y_1$, $0 \le y_2$ and $y_1+y_2 \le 3$

5.5 Refer to Example 5.4. The joint density of Y_1, the proportion of the capacity of the tank that is stocked at the beginning of the week, and Y_2, the proportion of the capacity sold during the week, is given by

$$f(y_1, y_2) = \begin{cases} 3y_1, & 0 \le y_2 \le y_1 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

a Find $F(1/2, 1/3) = P(Y_1 \le 1/2, Y_2 \le 1/3)$.

$$F(y_1, y_2) = P(Y_1 \le y_1, Y_2 \le y_2) = \int_{-\infty}^{y_1}\int_{-\infty}^{y_2} f(t_1, t_2)\, dt_2\, dt_1 \quad \text{(joint distribution)}$$

$F(\frac{1}{2}, \frac{1}{3}) = P(Y_1 \le \frac{1}{2}, Y_2 \le \frac{1}{3}) \Rightarrow$ $y_1 \le \frac{1}{3}$, y_2 runs from $0, y_1$

$y_1 > \frac{1}{3}$, y_2 runs $[0, \frac{1}{3}]$

$$= \int_0^{\frac{1}{3}}\int_0^{y_1} 3y_1\, dy_2\, dy_1 + \int_{\frac{1}{3}}^{\frac{1}{2}}\int_0^{\frac{1}{3}} 3y_1\, dy_2\, dy_1$$

$$= \frac{1}{27} + \frac{5}{72} = \frac{23}{216} \approx 0.1065$$

b Find $P(Y_2 \le Y_1/2)$, the probability that the amount sold is less than half the amount purchased.

$$F(y_1, y_2) = P(Y_1 \le y_1, Y_2 \le y_2) = \int_{-\infty}^{y_1}\int_{-\infty}^{y_2} f(t_1, t_2)\, dt_2\, dt$$

$$P(Y_2 \le \frac{Y_1}{2}) = \int_0^1\int_0^{y/2} 3y\, dx\, dy = \frac{1}{2} = 0.5$$

5.6 Refer to Example 5.3. If a radioactive particle is randomly located in a square of unit length, a reasonable model for the joint density function for Y_1 and Y_2 is

$$f(y_1, y_2) = \begin{cases} 1, & 0 \le y_1 \le 1, 0 \le y_2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

a What is $P(Y_1 - Y_2 > .5)$?

b What is $P(Y_1 Y_2 < .5)$?

(a) $P(Y_1 - Y_2 > .5) \Rightarrow P(Y_1 > .5 + Y_2$ and given Y_1 and Y_2 need to be in $[0,1]$, when $Y_2 \le 0.5$, $Y_1 \in [Y_2 + 0.5, 1]$

when $Y_2 > 0.5$, $Y_1 \notin [0, 0.5]$

$$F(y_1, y_2) = P(Y_1 \le y_1, Y_2 \le y_2) = \int_{-\infty}^{y_1}\int_{-\infty}^{y_2} f(t_1, t_2)\, dt_2\, dt_1$$

$$P(Y_1 - Y_2 > 0.5) = \int_0^{0.5}\int_{y_2+0.5}^{1} 1\, dy_1\, dy_2 = 0.125$$

b) $P(Y_1 Y_2 < .5)$

The condition $Y_1 Y_2 < 0.5$ defines a region below the hyperbola $Y_1 Y_2 = 0.5$, restricted to the unit square $0 \le Y_1, Y_2 \le 1$.
For a Fixed Y_1, Y_2 ranges from 0 to $\frac{0.5}{Y_1}$, as long as $Y_1 \ge 0.5$.

$$F(y_1, y_2) = P(Y_1 \le y_1, Y_2 \le y_2) = \int_{-\infty}^{y_1}\int_{-\infty}^{y_2} f(t_1, t_2)\, dt_2\, dt_1$$

$$P\left(\frac{0.5}{y_2} < y_1 < 1, y_2 > 0.5\right) = P\left(y_1 < \frac{0.5}{y_2}\right) = 1 - \int_{0.5}^{1}\int_{0.5/y_2}^{1} 1\, dy_1\, dy_2$$

$$= 1 - \int_{0.5}^{1} [y_1]_{0.5/y_2}^{1}\, dy_2$$

$$= 1 - \int_{0.5}^{1} \left(1 - \frac{0.5}{y_2}\right) dy_2$$

$$= 0.8466$$

5.9 Let Y_1 and Y_2 have the joint probability density function given by

$$f(y_1, y_2) = \begin{cases} k(1-y_2), & 0 \le y_1 \le y_2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

a Find the value of k that makes this a probability density function.

b Find $P(Y_1 \le 3/4, Y_2 \ge 1/2)$.

(a) we know that $\sum_{y_1,y_2} P(y_1, y_2) = 1$, the intergal over the entire region has to equal 1.

$$\int_0^1 \int_0^{y_2} K(1-y_2)\,dy_1\,dy_2 = 1$$

$$\Rightarrow \int_0^1 k y_2(-y_2+1)\,dy_2 = \frac{1}{6}k = 1$$

$$\Rightarrow k = 6$$

(b) 1: when $1/2 \le Y_2 \le 3/4$: Y_1 ranges from 0 to Y_2, Because $Y_1 \le Y_2$ and $Y_1 \le 3/4$ is satisfied $Y_2 \le 3/4$

2: when $3/4 < Y_2 \le 1$: Y_1 still go up to 3/4, Because $Y_1 \le 3/4$

$$F(y_1, y_2) = P(Y_1 \le y_1, Y_2 \le y_2) = \int_{-\infty}^{y_1} \int_{-\infty}^{y_2} f(t_1, t_2)\,dt_2\,dt_1$$

First Intergal $\{1/2 \le Y_2 \le 3/4)$: $\int_{1/2}^{3/4} \int_0^{y_2} 6(1-y_2)\,dy_1\,dy_2$

Second Intergral $\{3/4 < Y_2 \le 1)$: $\int_{3/4}^{1} \int_0^{3/4} 6(1-y_2)\,dy_1\,dy_2$

Thus: $P(Y_1 \le 3/4, Y_2 \ge \frac{1}{2}) = \int_{\frac{1}{2}}^{3/4} \int_0^{y_2} 6(1-y_2)\,dy_1\,dy_2 + \int_{3/4}^{1} \int_0^{3/4} 6(1-y_2)\,dy_1\,dy_2$

$$= \frac{11}{32} + \frac{9}{64} = \frac{31}{64} \approx 0.484375$$

5.23 In Example 5.4 and Exercise 5.5, we considered the joint density of Y_1, the proportion of the capacity of the tank that is stocked at the beginning of the week, and Y_2, the proportion of the capacity sold during the week, given by

$$f(y_1, y_2) = \begin{cases} 3y_1, & 0 \le y_2 \le y_1 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

a Find the marginal density function for Y_2.

b For what values of y_2 is the conditional density $f(y_1|y_2)$ defined?

c What is the probability that more than half a tank is sold given that three-fourths of a tank is stocked?

(a) $f_2(y_2) = \int_{-\infty}^{\infty} f(y_1, y_2)\,dy_1 = \int_{y_2}^{1} f(y_1, y_2)\,dy_1 = \int_{y_2}^{1} 3y_1\,dy_1$

$f_2(y_2) = \left[\frac{y_1^2}{2}\right]_{y_2}^{1} = 3\left(\frac{1^2}{2} - \frac{y_2^2}{2}\right) = \frac{3}{2}(1-y_2^2),\ 0 \le y_2 < 1$

(b) $f(y_1|y_2) = \frac{f(y_1, y_2)}{f_2(y_2)}$, where $f(y_1, y_2)$ in the joint desity and $f_2(y_2)$ is the marginal density. the given is $0 \le y_2 \le y_1 \le 1$.

The joint density function $f(y_1, y_2)$ is defined only when $y_2 \le y_1 \le 1$, This means that the conditional density

$f(y_1|y_2)$ is defined for: $y_2 \le y_1 \le 1$, for $y_2 \ge 0$

(c) we are asked to find $P(Y_2 > \frac{1}{2} \mid Y_1 = 3/4)$

$P(y_2|y_1) = P(Y_1 = 3/4, Y_2 > \frac{1}{2}) = \frac{P(y_1, y_2)}{P_1(y_1)} = \frac{\int_{1/2}^{3/4} f(y_1, y_2)\,dy_2}{f_{y_1}(y_1 = 3/4)}$

marginal desity for Y_1: $f_1(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2)\,dy_2 = \int_{0}^{y_1} 3y_1\,dy_2$

now, we can substitude back to the equation: $= 3y_1^2,\ 0 \le y_1 \le 1$

$P(Y_2 > \frac{1}{2} \mid Y_1 = 3/4) = \frac{\int_{1/2}^{3/4} 3y_1\,dy}{3y^2} = \frac{9/16}{3(\frac{3}{4})^2} = \frac{9}{27} = \frac{1}{3}$

5.25 Let Y_1 and Y_2 have joint density function first encountered in Exercise 5.7:

$$f(y_1, y_2) = \begin{cases} e^{-(y_1+y_2)}, & y_1 > 0, y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

a Find the marginal density functions for Y_1 and Y_2. Identify these densities as one of those studied in Chapter 4.

b What is $P(1 < Y_1 < 2.5)$? $P(1 < Y_2 < 2.5)$?

c For what values of y_2 is the conditional density $f(y_1|y_2)$ defined?

d For any $y_2 > 0$, what is the conditional density function of Y_1 given that $Y_2 = y_2$?

e For any $y_1 > 0$, what is the conditional density function of Y_2 given that $Y_1 = y_1$?

f For any $y_2 > 0$, how does the conditional density function $f(y_1|y_2)$ that you obtained in part (d) compare to the marginal density function $f_1(y_1)$ found in part (a)?

g What does your answer to part (f) imply about marginal and conditional probabilities that Y_1 falls in any interval?

(a) $f_1(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2)\,dy_2 = \int_0^{\infty} e^{-(y_1+y_2)}\,dy_2$

$= e^{-y_1}\int_0^{\infty} e^{-y_2}\,dy_2 = e^{y_1}\cdot[-e^{-y_2}]_0^{\infty}$

$= e^{-y_1}, \; y_1 > 0$

$f_2(y_2) = \int_{-\infty}^{\infty} f(y_1, y_2)\,dy_1 = \int_0^{\infty} e^{-(y_1+y_2)}\,dy_1$

$= -e^{y_2}\int_0^{\infty} e^{-y_1}\,dy_1 = e^{-y_2}\cdot[-e^{-y_1}]_0^{\infty} = e^{-y_2}, \; y_2 > 0$

(b) For exponential random variable Y_1 and Y_2 with rate $\lambda = 1$

$F_Y(y) = P(Y \leq y) = 1 - e^{-y}, \; y > 0$

$P(1 < Y_1 < 2.5) = F_Y(2.5) - F_Y(1) = (1 - e^{-2.5}) - (1 - e^{-1})$

$\approx 0.3679 - 0.0821 = 0.2858$

This result is the same for both Y_1 and Y_2 because they have identical marginal distribution. Thus $Y_1 = Y_2 = 0.2858$

(C) The conditional density $f(y_1|y_2)$ is defined $f_2(y_2) > 0$. From Part (a), $f_2(y_2) = e^{-y_2}$ for $y_2 > 0$. Thus $f(y_1|y_2)$ is defined for $y_2 > 0$.

(d) $f_1(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_2 = \int_0^{\infty} e^{-(y_1+y_2)} dy_2 = e^{-y_1}$

$f_2(y_2) = \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_1 = \int_0^{\infty} e^{-(y_1+y_2)} dy_1 = e^{-y_2}$

$$f_1(y_1 | y_2) = \frac{(y_1, y_2)}{f_2(y_2)} = \frac{e^{-(y_1+y_2)}}{e^{-y_2}}, \quad y_1 > 0, y_2 > 0$$

$$= e^{-y_1}, \quad y_1 > 0,$$

This is again exponential density function with $\lambda = 1$

(e) $$f_2(y_1 | y_2) = \frac{(y_1, y_2)}{f_1(y_1)} = \frac{e^{-(y_1+y_2)}}{e^{-y_1}} = e^{-y_2}, \quad y_2 > 0$$

(f) we note that the two functions are identical

From part (a) and (d):

$$f_1(y_1) = e^{-y_1}, \quad f_{Y_1|Y_2}(y_1 | y_2) = e^{-y_1}.$$

Thus the conditional density function $f_{Y_1|Y_2}(y_1|y_2)$ is indentical to the marginal density $f_{Y_1}(y_1)$

(g) Since $f_{Y_1|Y_2}(y_1|y_2) = f_1(y_1)$, this implies that Y_1 and Y_2 are independent random variables. For independent variables, the marginal probability of Y_1 falling in any interval regardless of the value of Y_2.

5.43 Let Y_1 and Y_2 have joint density function $f(y_1, y_2)$ and marginal densities $f_1(y_1)$ and $f_2(y_2)$, respectively. Show that Y_1 and Y_2 are independent if and only if $f(y_1|y_2) = f_1(y_1)$ for all values of y_1 and for all y_2 such that $f_2(y_2) > 0$. A completely analogous argument establishes that Y_1 and Y_2 are independent if and only if $f(y_2|y_1) = f_2(y_2)$ for all values of y_2 and for all y_1 such that $f_1(y_1) > 0$.

Two random variable Y_1 and Y_2 are independent If and only If:

$$f(y_1, y_2) = f_1(y_1) f_2(y_2), \text{ for all } y_1, y_2$$

Conditional density function: $f_1(y_1|y_2) = \dfrac{f(y_1, y_2)}{f_2(y_2)}$, $f_2(y_2) > 0$

$$f_2(y_1|y_2) = \frac{f(y_1, y_2)}{f_1(y_2)}, \quad f_1(y_1) > 0$$

- prove Y_1 and Y_2 are independent $\Leftrightarrow f_1(y_1, y_2) = f_1(y_1)$

$\Rightarrow$ Y_1 & Y_2 are independent: $f(y_1, y_2) = f_1(y_1) f_2(y_2)$

Subsitute into the definition of $f_{Y_1|Y_2}(y_1|y_2)$:

$$f_{Y_1|Y_2}(y_1|y_2) = \frac{f(y_1, y_2)}{f_{Y_2}(y_2)} = \frac{f_{Y_1}(y_1) f_{Y_2}(y_2)}{f_{Y_2}(y_2)} = f_{Y_1}(y_1)$$

thus, If Y_1 and Y_2 are independent, the conditional density function $f_{Y_1|Y_2}(y_1|y_2)$ equals marginal density $f_{Y_1}(y_1)$ for all y_1 and y_2 where $f_{Y_2}(y_2) > 0$.

$\Leftarrow$ If $(y_1, y_2) = f_1(y_1) f_2(y_2)$

If $f_{Y_1|Y_2}(y_1|y_2) = f_1(y_1) = \dfrac{f(y_1, y_2)}{f_2(y_2)}$ subsitiute the equality

$f_{Y_1|Y_2}(y_1|y_2) = f_1(y_1)$: $f_1(y_1) = \dfrac{f(y_1, y_2)}{f_2(y_2)} \Rightarrow f(y_1, y_2) = f_1(y_1) f(y_2)$

thus, Y_1 and Y_2 are independent.

- Analogous proof for $f_{Y_2|Y_1}(y_2|y_1) = f_2(y_2)$:

The same reasoning applies symmetrically. If $f_{Y_2|Y_1}(y_2|y_1) = f_2(y_2)$.

$f_{Y_2|Y_1}(y_2|y_1) = \dfrac{f(y_1, y_2)}{f_1(y_1)}$ and subsituting $f_{Y_2|Y_1}(y_2|y_2) = f_2(y_2)$. we get:

$$f_2(y_2) = \frac{f(y_1, y_2)}{f_1 y_1} \Rightarrow f(y_1, y_2) = f_1(y_1) f_2(y_2)$$

This prove Y_1 and Y_2 are independent.

Therefore, Y_1 and Y_2 are independent If and only if $\begin{cases} f_{Y_1|Y_2}(y_1|y_2) = f_1(y_1) \\ f_{Y_2|Y_1}(y_2|y_1) = f_2(y_2) \end{cases}$ for all y_1 and y_2

5.64 Let Y_1 and Y_2 be independent random variables that are both uniformly distributed on the interval (0, 1). Find $P(Y_1 < 2Y_2 \mid Y_1 < 3Y_2)$.

Given : Y_1 and Y_2 are independent

$Y_1, Y_2 \sim \text{Uniform}(0,1)$

1. $P(Y_1 \leq 3Y_2)$: when $Y_2 \leq \frac{1}{3}$, $Y_1 < 3Y_2 \in [0, 3Y_2]$

when $Y_2 > \frac{1}{3}$, $Y_1 < 3Y_2 \in [0, 1]$

thus: $$P(Y_1 < 3Y_2) = \int_0^{1/3} 3Y_2 \, dY_2 + \int_{1/3}^{1} 1 \, dY_2$$

$$= \left[\frac{3Y_2^2}{2}\right]_0^{1/3} + \left[Y_2\right]_0^{1/3} = \frac{1}{6} + \frac{2}{3} = \frac{5}{6}$$

2. $P(Y_1 < 2Y_2)$: when $Y_2 \leq \frac{1}{2}$, $Y_1 < 2Y_2 \in [0, 2Y_2]$

when $Y_2 > \frac{1}{2}$, $Y_1 < 2Y_2 \in [0, 1]$ because $2Y_2 > 1$

$$P(Y_1 < 2Y_2) = \int_0^{1/2} 2Y_2 \, dY_2 + \int_1^1 1 \, dY_2$$

$$= \left[Y_2^2\right]_0^{1/2} + \left[Y_2\right]_{1/2}^{1} = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$$

$$P(Y_1 < 2Y_2 \mid Y_1 < 3Y_2) = \frac{P(Y_1 < 2Y_2 \cap Y_1 < 3Y_2)}{P(Y_1 < 3Y_2)} = \frac{P(Y_1 < 2Y_2)}{P(Y_1 < 3Y_2)}$$

$$= \frac{\frac{5}{6}}{\frac{3}{4}}$$

$\frac{5}{6}$ — 1 (from above)

$\frac{3}{4}$ — 2. (from above

$$= \frac{18}{20} = \boxed{\frac{9}{10} = 0.9}$$