

Introduction to Mathematical Statistics I

Final

Name:

OSU-ID:

Instructions

- The exam is worth **30 points** and consists of **6 problems**.
- You **must upload** your final solutions to Canvas. The link to upload your solutions will **expire** at 9:30 PM.
- You **must show your work** and give concise, yet complete answers to receive full credit.
- You **must clearly** identify the problem you are solving in your solutions.
- You don't need to type your answers, but your solutions **must be clear and well organized**. If we cannot read it, we cannot grade it.
- The exam is **open-notes and open-book**, so you **can** use your class notes, homeworks and recitations when working on your problems.
- You are **allowed** to use a calculator, but **not software** to do your calculations. Any calculator will work, a graphic calculator is not required.
- You **must work independently**. You are not allowed to communicate with any person in any way or form to discuss exam problems.
- You are **not allowed** to use the Canvas discussion board (or any other platform) to seek and/or offer any clarifications about any questions or class materials during the exam.
- Answer the questions to the **best of your knowledge** or interpretation. No clarifications will be given during the exam.

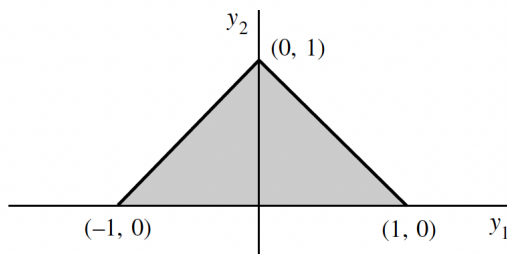
Problem 1

Determine whether the following statements are True or False. Justify your answer.

- (i) If X and Y are independent, then $E(\frac{X}{Y}) = \frac{E(X)}{E(Y)}$.
- (ii) If $X \sim \text{Normal}(1, 1)$, then the second moment of X is equal to 2.
- (iii) If X and Y satisfy $E(XY) = E(X)E(Y)$, then X and Y are independent.

Problem 2

Suppose that Y_1 and Y_2 have joint density function $f(y_1, y_2) = c$ (constant) over the region indicated in the figure:



- (i) Find the value of the constant c .
- (ii) Find $P(Y_1 - Y_2 \geq 0)$.
- (iii) Find $E(Y_1 Y_2)$.

Problem 3

The SAT and ACT college entrance exams are taken by thousands of students each year. The mathematics portions of each of these exams produce scores that are approximately normally distributed. In recent years, SAT mathematics exam scores have averaged 480 with a standard deviation of 100. The average and standard deviation for ACT mathematics scores are 18 and 6, respectively.

- (i) An engineering school sets 550 as the minimum SAT score for new students. What percentage of students will score below 550 in a typical year?
- (ii) What score should the engineering school set as a comparable standard of the ACT math test?
- (iii) A fellowship program requires the applicants to be among the top 10% of scores in the SAT exam. What should be your minimum score to be eligible for the fellowship program?

Problem 4

Let X, Y be random variables with joint pdf

$$f(x, y) = \begin{cases} 6(1 - y) & , 0 < x < y < 1 \\ 0 & , \text{otherwise} \end{cases}$$

- (i) Are X and Y independent? (Explain why or why not)
- (ii) Find $P(\frac{Y}{X} > 3)$.
- (iii) Find $E(X)$.

Problem 5

Let $N \sim \text{Binomial}(n, p)$, and let X_1, X_2, X_3, \dots be i.i.d Poisson(λ). Define

$$X = \sum_{i=1}^{N+1} X_i,$$

so that X is the sum of a random number of random variables.

- i) Find $E(X)$, the unconditional expectation of X .
- ii) Find $\text{Var}(X)$, the unconditional variance of X .

Problem 6

Suppose that Y_1, Y_2, \dots, Y_n are independent and identically distributed Bernoulli(p) random variables. That is, $P(Y_i = 1) = p$ and $P(Y_i = 0) = 1 - p$ for $i = 1, \dots, n$.

- (i) Find the moment generating function $m_{Y_1}(t)$ of Y_1 (the first observation).
- (ii) Find the moment generating function of $W = Y_1 + \dots + Y_n$
- (iii) What is the distribution of W ? Justify your answer.

Problem 7

Suppose that Y_1 and Y_2 are independent random variables with distribution $\text{Uniform}(0, 1)$. Define $X = \min\{Y_1, Y_2\}$.

- (i) Derive the pdf of X . (You need to show all the steps to obtain the pdf and not just writing the final formula)
- (ii) Find $E(X)$.
- (iii) Find $P(X > 0.5)$.

Problem 8

The lifetime (in hours) Y of an electronic component is a random variable with pdf

$$f(y) = \begin{cases} \frac{1}{200}e^{-y/200} & , 0 < y < \infty \\ 0 & , \text{otherwise} \end{cases}$$

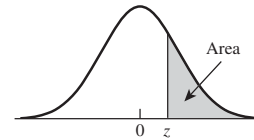
Suppose that four of these components operate independently in a certain equipment, and that the equipment fails if at least two of these components fail.

Find the probability that the equipment will operate for at least 400 hours without failure.

Distribution	Probability Function	Mean	Variance	Moment-Generating Function
Binomial	$p(y) = \binom{n}{y} p^y (1-p)^{n-y};$ $y = 0, 1, \dots, n$	np	$np(1-p)$	$[pe^t + (1-p)]^n$
Geometric	$p(y) = p(1-p)^{y-1};$ $y = 1, 2, \dots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{pe^t}{1-(1-p)e^t}$
Hypergeometric	$p(y) = \frac{\binom{r}{y} \binom{N-r}{n-y}}{\binom{N}{n}};$ $y = 0, 1, \dots, n$ if $n \leq r$, $y = 0, 1, \dots, r$ if $n > r$	$\frac{nr}{N}$	$n \left(\frac{r}{N} \right) \left(\frac{N-r}{N} \right) \left(\frac{N-n}{N-1} \right)$	does not exist in closed form
Poisson	$p(y) = \frac{\lambda^y e^{-\lambda}}{y!};$ $y = 0, 1, 2, \dots$	λ	λ	$\exp[\lambda(e^t - 1)]$
Negative binomial	$p(y) = \binom{y-1}{r-1} p^r (1-p)^{y-r};$ $y = r, r+1, \dots$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$	$\left[\frac{pe^t}{1-(1-p)e^t} \right]^r$

Distribution	Probability Function	Mean	Variance	Moment-Generating Function
Uniform	$f(y) = \frac{1}{\theta_2 - \theta_1}; \theta_1 \leq y \leq \theta_2$	$\frac{\theta_1 + \theta_2}{2}$	$\frac{(\theta_2 - \theta_1)^2}{12}$	$\frac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_2 - \theta_1)}$
Normal	$f(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp \left[-\left(\frac{1}{2\sigma^2} \right) (y - \mu)^2 \right]$ $-\infty < y < +\infty$	μ	σ^2	$\exp \left(\mu t + \frac{t^2 \sigma^2}{2} \right)$
Exponential	$f(y) = \frac{1}{\beta} e^{-y/\beta}; \quad \beta > 0$ $0 < y < \infty$	β	β^2	$(1 - \beta t)^{-1}$
Gamma	$f(y) = \left[\frac{1}{\Gamma(\alpha)\beta^\alpha} \right] y^{\alpha-1} e^{-y/\beta};$ $0 < y < \infty$	$\alpha\beta$	$\alpha\beta^2$	$(1 - \beta t)^{-\alpha}$
Chi-square	$f(y) = \frac{(y)^{(v/2)-1} e^{-y/2}}{2^{v/2} \Gamma(v/2)};$ $y > 0$	v	$2v$	$(1 - 2t)^{-v/2}$
Beta	$f(y) = \left[\frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \right] y^{\alpha-1} (1-y)^{\beta-1};$ $0 < y < 1$	$\frac{\alpha}{\alpha + \beta}$	$\frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$	does not exist in closed form

Table 4 Normal Curve Areas
 Standard normal probability in right-hand tail
 (for negative values of z , areas are found by symmetry)



z	Second decimal place of z									
	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641
0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0722	.0708	.0694	.0681
1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
1.8	.0359	.0352	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
2.9	.0019	.0018	.0017	.0017	.0016	.0016	.0015	.0015	.0014	.0014
3.0	.00135									
3.5	.000 233									
4.0	.000 031 7									
4.5	.000 003 40									
5.0	.000 000 287									

From R. E. Walpole, *Introduction to Statistics* (New York: Macmillan, 1968).