Numerical Linear Algebra - Sheet 3

to be handed in until November 8, 2023, 11am.

Normal and Hermitian Matrices

Problem 1. Consider a Hermitian matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$ and a unitary linear operator $\mathbf{Q} \in \mathbb{C}^m \to \mathbb{C}^n$, $m < n$. Prove, that an eigenvalue $\lambda_k(\mathbf{B})$ of the matrix $\mathbf{B} = \mathbf{Q}^*\mathbf{A}\mathbf{Q} \in \mathbb{C}^{m\times m}$ is either equal to 0, or the following estimate holds

$$|\lambda_{\min}(\mathbf{A})| \leq |\lambda_k(\mathbf{B})| \leq |\lambda_{\max}(\mathbf{A})|,$$

where $\lambda_{\min}(\mathbf{A})$ and $\lambda_{\max}(\mathbf{A})$ denote the smallest and largest eigenvalues of $\mathbf{A}$ measured by their magnitude.

Problem 2. Show that a normal triangular matrix is diagonal. *Hint:* look at the norms of $\mathbf{A}\mathbf{e}_i$ and $\mathbf{A}^*\mathbf{e}_i$.

Well-posedness of EVP

Problem 3. Consider the matrix $M = \begin{pmatrix} \eta & 1 \\ \eta & \eta \end{pmatrix}$ with $|\eta| << 1$.
Explain why the problem of finding eigenvectors is *not* well-posed in this example.

Problem 4. Consider the following matrix

$$\mathbf{A} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}^T \begin{pmatrix} 1 & \\ & c \end{pmatrix} \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

with parameters $\varphi \in [0, 2\pi]$ and $c \in (0, 1)$.

1. Compute the eigenvalues and eigenvectors of $\mathbf{A}$.
2. (Programming) Write a program which computes the sequence $\mathbf{x}^{(n)} \in \mathbb{R}^2$ defined as

$$\mathbf{x}^{(n)} = \mathbf{A}\mathbf{x}^{(n-1)},$$
$$\mathbf{x}^{(0)} = \mathbf{x}^*,$$

   for $\mathbf{x}^* = (1,\ 0)^T$, $c = 0.1$, and $\varphi = \frac{\pi}{4}$. Try playing with different values of those parameters.
3. Is there a limit of $\mathbf{x}^{(n)}$? What is about the case $c = 1$?
4. Compute the limit: $\lim_{n\to\infty} \mathbf{A}^n$.

Vector Iterations: Simple Iterations

Problem 5. A diagonalizable real matrix $\mathbf{A}$ has the following spectrum:

$$\sigma(\mathbf{A}) = \{-2,\ 1-2i,\ 1+2i,\ 1,\ -i,\ i,\ 2\}.$$

Consider using the inverse power method (*vector iteration*) to compute its eigenvalues.

(a) Find a set of all shift parameters for which the inverse power method may *not* converge. Draw a sketch.

(b) For every *real* eigenvalue find a *real* range of shifts that, if used in the inverse power method, will reduce the error of approximation of the eigenvalue by a factor of 10 in each iteration.

Problem 6. Propose shift parameters that will allow you to compute *all* eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 100 & 15 & 3 \\ 15 & 20 & 5 \\ 3 & 5 & 65 \end{pmatrix}.$$

Prove that your choice is correct. *Hint:* Gershgorin circle theorem

Problem 7 (Programming)**.** Write a program that computes all eigenvectors and eigenvalues of the matrix

$$\mathbf{A}_\varepsilon = \begin{pmatrix} 100 & 15 & 3 & 0 & 0 & \varepsilon \\ 15 & 20 & 5 & 0 & \varepsilon & 0 \\ 3 & 5 & 65 & \varepsilon & 0 & 0 \\ 0 & 0 & \varepsilon & 110 & 20 & 5 \\ 0 & \varepsilon & 0 & 20 & 80 & 4 \\ \varepsilon & 0 & 0 & 5 & 4 & 30 \end{pmatrix}$$

using the shifted (inverse) power method by observing the following steps:

(a) Consider the matrix $\mathbf{A}_0$ with $\varepsilon = 0$ and examine the structure of the eigenvalue problem.

(b) Compute all eigenvalues and eigenvectors of $\mathbf{A}_0$.

(c) Does the algorithm work, if you use Gerschgorin for the computation of all six eigenvalues?

(d) Can you use part (a) to get something better?