Numerical Linear Algebra - Sheet 8

to be handed in until December 13, 2023, 11am.

Problem 1. Let $\mathbf{A}\mathbf{x} = \mathbf{b}$ be a linear system with a symmetric, positive definite matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ that has extremal eigenvalues $\lambda_{\min}$ and $\lambda_{\max}$ and spectral condition number $\operatorname{cond}(\mathbf{A})$. Consider the sequence $\left\{\mathbf{x}^{(k)}\right\}$ of one-dimensional projection processes with $K = L = \operatorname{span}\{\mathbf{e}_i\}$, where $\mathbf{e}_i$ denotes the i-th unit vector in $\mathbb{R}^n$. The index i is selected at each step k to be the index of a component of largest absolute value in the current residual vector $\mathbf{r}^{(k)} = \mathbf{b} - \mathbf{A}\mathbf{x}^{(k)}$.

(a) For $\mathbf{d}^{(k)} = \mathbf{A}^{-1}\mathbf{b} - \mathbf{x}^{(k)}$ show that

$$\|\mathbf{d}^{(k+1)}\|_{\mathbf{A}} \leq \left(1 - \frac{1}{n\operatorname{cond}(\mathbf{A})}\right)^{\frac{1}{2}} \|\mathbf{d}^{(k)}\|_{\mathbf{A}}.$$

Hint: You can use the expression

$$\left\langle \mathbf{A}\mathbf{d}^{(k+1)}, \mathbf{d}^{(k+1)} \right\rangle = \left\langle \mathbf{A}\mathbf{d}^{(k)}, \mathbf{d}^{(k)} \right\rangle - \frac{\left\langle \mathbf{r}^{(k)}, \mathbf{e}_i \right\rangle^2}{a_{ii}},$$

as well as the inequality $|\mathbf{e}_i^T\mathbf{r}^{(k)}| \geq n^{-1/2}\|\mathbf{r}^{(k)}\|_2$. a_{ii} *denotes the i-th diagonal element of A.*

(b) Does (a) prove that the algorithm converges?

Problem 2. Consider the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$, where $\mathbf{A}$ is a symmetric positive definite matrix. We define a projection method which uses a two-dimensional space at each step. At a given step, take $L = K = \operatorname{span}\{\mathbf{r}, \mathbf{A}\mathbf{r}\}$, where $\mathbf{r} = \mathbf{b} - \mathbf{A}\mathbf{x}$ is the current residual.

(a) For a basis of K use the vector $\mathbf{r}$ and the vector $\mathbf{p}$ obtained by orthogonalizing $\mathbf{A}\mathbf{r}$ against $\mathbf{r}$ with respect to the $\mathbf{A}$-inner product. Give the formula for computing $\mathbf{p}$ (no need to normalize the resulting vector).

(b) Write the algorithm for performing the projection method described above.

(c) Can you explain, not compute, why the algorithm converges for any initial guess $\mathbf{x}_0$? *Hint: Exploit the convergence results for one-dimensional projection techniques.*

Problem 3 (Programming).

(a) Implement the steepest decent method (Algorithm 3.3.13 in the lecture notes).

(b) Use your implementation of the steepest decent method to solve the 2D Laplace problem

$$\mathbf{L}_2\mathbf{x} = \mathbf{b}$$

as in Problem 2 on Sheet 7 with right hand side vector $\mathbf{b} = (1, ..., 1)^T$ and initial guess $\mathbf{x}^{(0)} = (0, ..., 0)^T$ with $n = 50$ and $n = 100$.

(c) Compute and discuss the observed convergence rate (see 3.2.21/Definition 3.2.22 in the Lecture Notes).

Problem 4 (Programming)**.**

(a) Implement the minimal residual method (Algorithm 3.3.20 in the lecture notes).

(b) Use your implementation of the steepest decent method to solve the 2D Laplace problem

$$\mathbf{L}_2\mathbf{x} = \mathbf{b}$$

as in Problem 2 on Sheet 7 with right hand side vector $\mathbf{b} = (1, ..., 1)^T$ and initial guess $\mathbf{x}^{(0)} = (0, ..., 0)^T$ with $n = 50$ and $n = 100$.

(c) Compute and discuss the observed convergence rate (see 3.2.21/Definition 3.2.22 in the Lecture Notes). Compare it with the convergence rate observed in Problem 3.