Numerical Linear Algebra - Sheet 1

to be handed in until October 25, 2023, 11am.

Please hand in your solutions in groups of two to four. The solutions for non-programming problems and the discussion of the programming problems should be handed in on paper (Mathematikon, 1st floor, letterbox 27). The code for programming problems should be handed in via Moodle by one person in your group. To recieve the points for the exercises on Müsli, write everyones first and last name on every hand-in. Details will be discussed in the exercise groups on Friday. If you currently don't have access to Moodle, please contact your tutor Laura in the exercise group (or via Müsli/Mail). If you want to get comments on your code, please also print it out and hand it in with the other solutions.

Problem 1. Review the following items and write down at least one of the definitions and one of the theorems with proof in detail.

- Definition of a projection
- Definition of an orthogonal projection
- Theorem: Orthogonal projection is uniquely determined by subspace

  Consider a finite-dimensional space V with inner product $\langle \cdot, \cdot \rangle$ and a subspace $W \subset V$. Then, there exists a unique orthogonal projection

  $$P_W : V \to W.$$

- Theorem: Best Approximation Theorem

  Let W be a subspace of $\mathbb{R}^n$, let $\mathbf{x}$ be any vector in $\mathbb{R}^n$, and let $\tilde{\mathbf{x}}$ be the orthogonal projection of $\mathbf{x}$ onto W. Then $\tilde{\mathbf{x}}$ is the closest point in W to $\mathbf{x}$, in the sense that

  $$\|\mathbf{x} - \tilde{\mathbf{x}}\| < \|\mathbf{x} - \mathbf{w}\|$$

  for all $\mathbf{w}$ in W distinct from $\mathbf{x}$.

- Theorem: Orthogonal projection in orthonormal basis

  Let $B = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_p\}$ be an orthonormal basis of a subspace W of a finite-dimensional space V with inner product $\langle \cdot, \cdot \rangle$. Then, the orthogonal projection P_i of any vector $\mathbf{v} \in V$ onto $\mathbf{u}_i$, and the orthogonal projection P_W of any vector $\mathbf{v} \in V$ onto W have the following expressions, respectively:

  $$P_i(\mathbf{v}) = \langle \mathbf{v}, \mathbf{u}_i \rangle \mathbf{u}_i, \qquad i = 1, 2, ..., p,$$
  $$P_W(\mathbf{v}) = \sum_{i=1}^{p} \langle \mathbf{v}, \mathbf{u}_i \rangle \mathbf{u}_i,$$

  and

  $$\mathbf{v} = P_W(\mathbf{v}) + \mathbf{z}, \qquad \mathbf{z} \perp W.$$

- Theorem: Parseval identity

  Suppose that W is a finite-dimensional linear space with inner product $\langle \cdot, \cdot \rangle$. Let $\{\mathbf{e}_i\}$, $i = 1, ..., n$ be an orthonormal basis of W. Then, for every $\mathbf{w} \in W$ it holds

$$\sum_{i=1}^{n} |\langle \mathbf{w}, \mathbf{e}_i \rangle|^2 = \|\mathbf{w}\|^2.$$

Problem 2 (Programming)**.** Problem 1.1.4 in the Lecture Notes.