Numerical Linear Algebra - Sheet 7

to be handed in until December 6, 2023, 11am.

Problem 1. Prove Theorem 3.2.15 in the Lecture Notes (Convergence of Richardson Iteration).

Problem 2 (Programming)**.** Let $\mathbf{L}_d$ be the discretization of the d-dimensional Laplace operator on the unit square by the five-point stencel with a uniform Dirichlet boundary condition to the mesh size $h = \frac{1}{n+1}$. In 1D it is given in terms of the matrix $\mathbf{T}_2$ defined in Problem 4 of Sheet 6 by

$$\mathbf{L}_1 = \frac{1}{h^2}\mathbf{T}_2 \in \mathbb{R}^{n\times n}.$$

In 2D it is given by

$$\mathbf{L}_2 = \frac{1}{h^2}\begin{pmatrix} \mathbf{D} & -\mathbb{I} & & & \\ -\mathbb{I} & \mathbf{D} & -\mathbb{I} & & \\ & \ddots & \ddots & \ddots & \\ & & -\mathbb{I} & \mathbf{D} & -\mathbb{I} \\ & & & -\mathbb{I} & \mathbf{D} \end{pmatrix} \in \mathbb{R}^{n^2\times n^2},$$

where $\mathbb{I} \in \mathbb{R}^{n\times n}$ and $\mathbf{D} \in \mathbb{R}^{n\times n}$ are defined as

$$\mathbb{I} = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{D} = \begin{pmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 4 & -1 \\ & & & -1 & 4 \end{pmatrix}.$$

Details may be found in appendix B.3 on finite difference methods in the lecture notes.

Let $\mathbf{A} = \mathbf{L}_2 \in \mathbb{R}^{n^2\times n^2}$ as defined above and $\mathbf{b} \in \mathbb{R}^{n^2}$. Write a program that solves the 2D Laplace problem

$$\mathbf{A}\mathbf{x} = \mathbf{b},$$

with the Richardson iteration by observing the following steps:

(a) Implement a function vmult that performs the matrix-vector product $\mathbf{A}\mathbf{v}$ of $\mathbf{A}$ with a given vector $\mathbf{v}$, and returns a vector $\mathbf{w}$. Do not store the system matrix in this function. The parameters should only be the vector $\mathbf{v}$ and, optionally, n.

(b) Write a function richardson_step that implements a single step of the Richardson iteration. Similar to (a) this function should take vectors $\mathbf{v}$ and $\mathbf{b}$, optionally n, and return the resulting vector $\mathbf{w}$. *Hint: The eigenvalues of* $\mathbf{L}_2$ *are given by the pairwise sums of the eigenvalues of* $\mathbf{T}_2$*, see B.3.13.*

(c) Use the null-vector $\mathbf{x}^{(0)} = (0, ..., 0)^T$ as initial guess and test your program for the constant vector $\mathbf{b} = (1, ..., 1)^T$ with $n = 20$ and $n = 100$. Observe the evolution of the residual in the 2-norm

$$r^{(k)} = \|\mathbf{A} \cdot \mathbf{x}^{(k)} - \mathbf{b}\|_2$$

for 50 steps of the Richardson iteration.

(d) Plot the obtained result vector after 1, 5, 15, 30, and 50 iterations as a two-dimensional function. *Hint: For plotting you have to invert the numbering of the vector (use, for example, np.reshape) and then plot the two dimensional grid using a numpy color map (greyscale).*

(e) Compute and discuss the observed convergence rate (see Definition 3.2.22 in the Lecture Notes).