# Numerical Linear Algebra - Sheet 1

to be handed in until October 23, 2024, 11am.

Please hand in your solutions in groups of 2-4. Solutions submitted by only one person will not be graded. The solutions for non-programming problems may be either handed in online via Moodle or on paper (Mathematikon, 1st floor, letterbox to be determined). The modularities for handing in the programming problems will be announced separately. To recieve the points for the exercises, write everyones first and last name on every hand-in. Details will be discussed in the exercise group on Thursday.

If you currently don't have access to Moodle, please contact your tutor Freya in the exercise group or via Mail (freya.jensen[at]iwr.uni-heidelberg.de).

**Problem 1.** Review the following items and write down at least one of the definitions and one of the theorems with proof in detail.

- Definition of a projection
- Definition of an orthogonal projection
- Theorem: Orthogonal projection is uniquely determined by subspace

  Consider a finite-dimensional space $V$ with inner product $\langle \cdot, \cdot \rangle$ and a subspace $W \subset V$. Then, there exists a unique orthogonal projection

  $$P_W : V \to W.$$

- Theorem: Best Approximation Theorem

  Let $W$ be a subspace of $\mathbb{R}^n$, let $\mathbf{x}$ be any vector in $\mathbb{R}^n$, and let $\tilde{\mathbf{x}}$ be the orthogonal projection of $\mathbf{x}$ onto $W$. Then $\tilde{\mathbf{x}}$ is the closest point in $W$ to $\mathbf{x}$, in the sense that

  $$\|\mathbf{x} - \tilde{\mathbf{x}}\| < \|\mathbf{x} - \mathbf{w}\|$$

  for all $\mathbf{w}$ in $W$ distinct from $\mathbf{x}$.

- Theorem: Orthogonal projection in orthonormal basis

  Let $B = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_p\}$ be an orthonormal basis of a subspace $W$ of a finite-dimensional space $V$ with inner product $\langle \cdot, \cdot \rangle$. Then, the orthogonal projection $P_i$ of any vector $\mathbf{v} \in V$ onto $\mathbf{u}_i$, and the orthogonal projection $P_W$ of any vector $\mathbf{v} \in V$ onto $W$ have the following expressions, respectively:

  $$P_i(\mathbf{v}) = \langle \mathbf{v}, \mathbf{u}_i \rangle \mathbf{u}_i, \qquad i = 1, 2, ..., p,$$
  $$P_W(\mathbf{v}) = \sum_{i=1}^{p} \langle \mathbf{v}, \mathbf{u}_i \rangle \mathbf{u}_i,$$

  and

  $$\mathbf{v} = P_W(\mathbf{v}) + \mathbf{z}, \qquad \mathbf{z} \perp W.$$

- Theorem: Parseval identity

  Suppose that $W$ is a finite-dimensional linear space with inner product $\langle \cdot, \cdot \rangle$. Let $\{\mathbf{e}_i\}$, $i = 1, ..., n$ be an orthonormal basis of $W$. Then, for every $\mathbf{w} \in W$ it holds

$$\sum_{i=1}^{n} |\langle \mathbf{w}, \mathbf{e}_i \rangle|^2 = \|\mathbf{w}\|^2.$$

**Problem 2** (Programming)**.** To familiarize yourself with the C++ library Armadillo write a function `gauss_eliminiation` that performs the Gaussian elimination for a system of linear equations $Ax = b$. You should use the armadillo matrix class and vector class to store the system of linear equations, handle the result and perform the vector matrix multiplication in the end. Write a `main` function in which you test your implementation with the following system of linear equations:

$$\begin{array}{rcrcrcrcr} 2x_1 & + & 8x_2 & + & 10x_3 & + & 10x_4 & = & 0 \\ x_1 & + & 5x_2 & + & 2x_3 & + & 9x_4 & = & 1 \\ -3x_1 & - & 10x_2 & - & 21x_3 & - & 6x_4 & = & -4 \\ -2x_1 & - & 3x_2 & - & 4x_3 & - & 7x_4 & = & -1 \end{array}$$

In order to test the correctness of your result you should perform a matrix vector multiplication to see whether the original matrix $A$ multiplied with the resulting vector $x$ yields the vector $b$. We will also discuss this problem in the programming tutorial on Thursday.