Numerical Linear Algebra - Sheet 7

to be handed in until December 6, 2023, 11am.

Problem 1. Prove Theorem 3.2.15 in the Lecture Notes (Convergence of Richardson Iteration).

Problem 2. Gegeben seien die Zahlenfolgen

h	a_h	b_h	c_h
1/2	1.07627	1.70051	0.429204
1/4	0.604185	1.71382	0.00455975
1/8	0.320317	1.71716	1.68691e-05
1/16	0.164945	1.71800	1.62880e-08
1/32	0.0836993	1.71821	3.96572e-12
1/64	0.0421601	1.71826	2.22045e-16
1/128	0.0211582	1.71828	

Die **Konvergenzordnung** einer Folge x_h sei die größte Zahl ϱ so dass $x_h = \mathcal{O}(h^\varrho)$ gilt. Sie kann berechnet werden als

$$\varrho = \frac{1}{\log 2} \lim_{h \to 0} \log \left| \frac{x_h}{x_{\frac{h}{2}}} \right| .$$

1. Bestimmen Sie eine Approximation der Konvergenzordnung ϱ für a_h. Welche Zeilen der Tabelle benutzen sie dazu am besten? Wie verifizieren Sie Ihr Ergebnis?

2. Sei $b = \lim_{h\to 0} b_h$. Bestimmen Sie ohne b zu kennen die "intrinsische" Konvergenzordnung der Folge $b - b_h$. Nutzen Sie dazu die Darstellung $b - b_h = b - b_{h/2} + b_{h/2} - b_h$, um die Formel

   $$\varrho \approx \frac{1}{\log 2} \log \left| \frac{b_h - b_{\frac{h}{2}}}{b_{\frac{h}{2}} - b_{\frac{h}{4}}} \right|$$

   zu rechtfertigen.

3. Kommentieren Sie die Frage der Konvergenzordnung der Folge c_h

Problem 3 (Programming)**.** Let $\mathbf{L}_d$ be the discretization of the d-dimensional Laplace operator on the unit square by the five-point stencel with a uniform Dirichlet boundary condition to the mesh size $h = \frac{1}{n+1}$. In 1D it is given in terms of the matrix $\mathbf{T}_2$ defined in Problem 4 of Sheet 6 by

$$\mathbf{L}_1 = \frac{1}{h^2}\mathbf{T}_2 \in \mathbb{R}^{n\times n}.$$

In 2D it is given by

$$\mathbf{L}_2 = \frac{1}{h^2} \begin{pmatrix} \mathbf{D} & -\mathbb{I} & & & \\ -\mathbb{I} & \mathbf{D} & -\mathbb{I} & & \\ & \ddots & \ddots & \ddots & \\ & & -\mathbb{I} & \mathbf{D} & -\mathbb{I} \\ & & & -\mathbb{I} & \mathbf{D} \end{pmatrix} \in \mathbb{R}^{n^2 \times n^2},$$

where $\mathbb{I} \in \mathbb{R}^{n \times n}$ and $\mathbf{D} \in \mathbb{R}^{n \times n}$ are defined as

$$\mathbb{I} = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{D} = \begin{pmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 4 & -1 \\ & & & -1 & 4 \end{pmatrix}.$$

Let $\mathbf{A} = \mathbf{L}_2 \in \mathbb{R}^{n^2 \times n^2}$ as defined above and $\mathbf{b} = (1, ..., 1)^T$. Write a program that solves the 2D Laplace problem

$$\mathbf{A}\mathbf{x} = \mathbf{b},$$

with the Richardson iteration by observing the following steps:

(a) Implement a function vmult that performs the matrix-vector product $\mathbf{A} \cdot \mathbf{v}$ of $\mathbf{A}$ with a given vector $\mathbf{v}$, and returns a vector $\mathbf{w}$. Do not explicitly form the system matrix in this function. The parameters should only be the vector $\mathbf{v}$ and, optionally, n.

(b) Write a function richardson_step that implements a single step of the Richardson iteration. As in (a) this function should take a vector $\mathbf{v}$, optionally n, and return the resulting vector $\mathbf{w}$. What is the optimal choice for the relaxation parameter ω_k? The eigenvalues of $\mathbf{L}_2$ are given by the pairwise sums of the eigenvalues of $\mathbf{T}_2$.

(c) Use the null-vector $\mathbf{x}^{(0)} = (0, ..., 0)^T$ as initial guess and test your program with $n = 20$ and $n = 100$. Observe the evolution of the residual in the 2-norm

$$r^{(k)} = \|\mathbf{A} \cdot \mathbf{x}^{(k)} - \mathbf{b}\|_2$$

for 50 steps of the Richardson iteration. Plot the obtained result vector after 1, 5, 15, 30, and 50 iterations. *Hint: For plotting you have to invert the numbering of the vector (use, for example, np.reshape) and then plot the two dimensional grid using a numpy color map (greyscale). Details may be found in appendix B.3 on finite difference methods in the lecture notes.*