Numerical Linear Algebra - Sheet 12

to be handed in until February 22, 2025, 2pm.

Problem 1. Let $\mathbf{A}$ and $\mathbf{B}$ be symmetric positive-definite matrices in $\mathbb{R}^{n\times n}$. Let there be constants c_1, c_2 auch that

$$c_1\mathbf{x}^T\mathbf{A}\mathbf{x} \leq \mathbf{x}^T\mathbf{B}\mathbf{x} \leq c_2\mathbf{x}^T\mathbf{A}\mathbf{x}.$$

Derive an estimate for the convergence of the conjugate gradient iteration for $\mathbf{A}\mathbf{x} = \mathbf{b}$ with preconditioner $\mathbf{B}^{-1}$ depending on c_1 and c_2.

Hints: Lemma 3.4.72 in the lecture notes and Theorem 2.2.17 (note that the definition of the Rayleigh quotient is independent of the choice of inner product).

Problem 2. Develop a modified version of the non-Hermitian Lanczos algorithm that produces a sequence of vectors $\mathbf{v}_i$,$\mathbf{w}_i$ that are such that each $\mathbf{v}_i$ is orthogonal to every $\mathbf{w}_j$ with $j \neq i$ and $\|\mathbf{v}_i\| = \|\mathbf{w}_i\| = 1$ for all i.

Problem 3. Find an example of a matrix with a real spectrum for which the QR method will not converge to an upper triangular matrix.