Numerical Linear Algebra - Sheet 6

to be handed in until November 29, 2023, 11am.

Problem 1. Prove Lemma 2.5.2:
Let $\mathbf{T} \in \mathbb{R}^{n\times n}$ be a real, symmetric, tridiagonal matrix and $\mathbf{QR} = \mathbf{T}$ its QR factorization. Then, $\tilde{\mathbf{T}} = \mathbf{RQ}$ is also symmetric and tridiagonal. Furthermore, $\mathbf{R}$ is zero except for its main and the first two upper diagonals.

The same holds for the shifted version with $\sigma \in \mathbb{R}$,

$$\mathbf{QR} = \mathbf{T} - \sigma\mathbb{I}, \qquad \tilde{\mathbf{T}} = \mathbf{RQ} + \sigma\mathbb{I}.$$

Problem 2. Prove that in case of a normal real matrix, for each complex eigenvalue pair there is a 2×2 matrix with according invariant subspace.

(a) Show that complex eigenvalues and their associated eigenvectors of a real matrix come in complex conjugate pairs.

(b) Choose real linear combinations of these vectors to obtain the 2×2 block.

Problem 3. Let $\mathbf{H}$ be a real Hessenberg matrix. Let $\mathbf{Q}_1, \mathbf{R}_1, \mathbf{Q}_2, \mathbf{R}_2$ be the matrices obtained in the explicit double-shift QR step (Algorithm 2.5.14). Let σ_1 and σ_2 are the shifts in Lemma 2.5.16. Show that the following equation in the proof of Lemma 2.5.16 holds:

$$\mathbf{Q}_1\mathbf{Q}_2\mathbf{R}_2\mathbf{R}_1 = \mathbf{M} = (\mathbf{H} - \sigma_1\mathbb{I})(\mathbf{H} - \sigma_2\mathbb{I})$$

Problem 4. Consider the $n \times n$ tridiagonal matrix

$$\mathbf{T}_\alpha = \begin{pmatrix} \alpha & -1 & & & \\ -1 & \alpha & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & \alpha & -1 \\ & & & -1 & \alpha \end{pmatrix}$$

where α is a real parameter. Verify that the eigenvalues of $\mathbf{T}_\alpha$ are given by

$$\lambda_j = \alpha - 2\cos(j\vartheta),$$

where

$$\vartheta = \frac{\pi}{n+1},$$

and the eigenvectors associated with each λ_j are given by

$$\mathbf{v}_j = \left(\sin(j\vartheta),\ \sin(2j\vartheta),\ \ldots\ ,\ \sin(nj\vartheta)\right)^T.$$

What are the conditions on α, such that the matrix $\mathbf{T}_\alpha$ becomes positive-definite?

Problem 5. Implement the inverse power method (Algorithm 2.6.1 in the lecture notes). To solve the system of linear equations use the functions `numpy.linalg.solve` and `numpy.linalg.lstsq`. Test your implementations with the matrix $\mathbf{A}_{20}$ and $\alpha = 2$ from Problem 4 and shift parameters $\sigma = \lambda_1, \lambda_2, \lambda_{10}$. Compare your results for the two different solvers.