Numerical Linear Algebra - Sheet 9

to be handed in until December 18, 2024, 11am.

Problem 1. Consider a matrix of the form

$$\mathbf{A} = \mathbb{I} + \alpha \mathbf{B},$$

where $\mathbf{B}$ is skew symmetric (real), i.e., such that $\mathbf{B}^T = -\mathbf{B}$.

(a) Show that $\langle \mathbf{A}\mathbf{x}, \mathbf{x}\rangle / \langle \mathbf{x}, \mathbf{x}\rangle = 1$ for all $\mathbf{x} \neq 0$.

(b) Consider the Arnoldi process for $\mathbf{A}$. Show that the resulting Hessenberg matrix will have the following tridiagonal form:

$$\mathbf{H}_m = \begin{pmatrix} 1 & -\eta_2 & & & \\ \eta_2 & 1 & -\eta_3 & & \\ & & \cdots & & \\ & & \eta_{m-1} & 1 & -\eta_m \\ & & & \eta_m & 1 \end{pmatrix}.$$

(c) Using the result of part (b), explain why the CG algorithm applied as is to a linear system with the matrix $\mathbf{A}$, which is nonsymmetric, will still yield residual vectors that are orthogonal to each other.

Problem 2. Consider the CG Method described in Algorithm 3.4.31. Assume, in line 4, $\langle \mathbf{A}\mathbf{p}_m, \mathbf{p}_m\rangle = 0$ and thus the algorithm terminates there. Why is the Krylov space K_m invariant? Why have we found the exact solution then?

Problem 3. Consider again matrix $\mathbf{T}_\alpha$ as introduced in Problem 3 on sheet Sheet 7.

(a) Implement a method that computes the smallest eigenvalue of the matrix $\mathbf{L}_2$ using the inverse iteration (Algorithm 2.6.2 in the lecture notes). Do not calculate the inverse explicitly for solving the appearing linear system, but use the conjugate gradient iteration (Algorithm 3.4.31 in the lecture notes) for this matter.

(b) Use your implementation to calculate the smallest eigenvalue of $\mathbf{L}_2$ with $n = 20$.