Numerical Linear Algebra - Sheet 10

to be handed in until January 17, 2024, 11am.

Problem 1. Show how GMRES (Algorithm 3.4.42 in the lecture notes) and Arnoldi with modified Gram-Schmidt (Algorithm 3.4.10 in the lecture notes) will converge on the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ when

$$\mathbf{A} = \begin{pmatrix} & & & & 1 \\ 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and $\mathbf{x}_0 = \mathbf{0}$.

Problem 2. Let $\mathbf{A}$ and $\mathbf{B}$ be symmetric positive-definite matrices in $\mathbb{R}^{n\times n}$. Let there be constants c_1, c_2 such that

$$c_1 \mathbf{x}^T \mathbf{A}\mathbf{x} \leq \mathbf{x}^T \mathbf{B}\mathbf{x} \leq c_2 \mathbf{x}^T \mathbf{A}\mathbf{x}.$$

Derive an estimate for the convergence of the conjugate gradient iteration for $\mathbf{A}\mathbf{x} = \mathbf{b}$ with preconditioner $\mathbf{B}^{-1}$ depending on c_1 and c_2.

Hints: Lemma 3.4.68 in the lecture notes, and Thereom 2.2.17 (Courant-Fischer min-max theorem) (note, that the definition of the Rayleigh quotient is independent of the choice of the inner product).

Problem 3. Consider again matrix $\mathbf{T}_\alpha$ as introduced in Problem 4 on sheet Sheet 6.

(a) Implement a method that computes the smallest eigenvalue of the matrix $\mathbf{T}_\alpha$ using the inverse iteration (Algorithm 2.6.1 in the lecture notes). You may reuse your code from Problem 5 on sheet Sheet 6. Do not calculate the inverse explicitly for solving the appearing linear system, but use the conjugate gradient iteration (Algorithm 3.4.28 in the lecture notes) for this matter.

(b) Use your implementation to calculate the smallest eigenvalue of $\mathbf{T}_\alpha$ with $\alpha = 2$ and $n = 20$.