THE CALDERON-ZYGMUND THEOREM AND ITS GENERALISATION

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Contents

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Preface

This report is about harmonic analysis such as Calderon-Zygmund theory, Hilbert transform, singular integral and its application. Main reference is Fourier Analysis (Duoandikoetxea, J., and Zuazo, J. D. 2001,American Mathematical Soc).

Chapter 1 reviews fundamental findings in Fourier Analysis, which serve as the foundation for subsequent conclusions. The majority of the material presented herein is derived from MA5206 Graduate Analysis I course notes taught by Professor An Xingliang, spanning the first five weeks of instruction.

Chapter 2 and Chapter 3 offer a comprehensive treatment of two operators which are basic to this field: the Hardy-Littlewood maximal function and the Hilbert transform. Despite predating the techniques of Calderon-Zygmund Decomposition, we analyze these operators from their vantage point in the present study.

Chapter 4 introduced Integral theory. Techniques involved in Chapter 2 and 3 enable similar studies of Hilbert Transform to higher dimensional spaces. And these studies have many practical applications such as singular Integral theory.

As for application, Calderón-Zygmund operator is one of the most important theories in harmonic analysis, Fourier analysis and singular integral operators. It gives the L_p boundedness of a class of singular integral operators. In second-order elliptic equations, the most classic application of Calderon-Zygmund operator is in $W^{2,p}$ estimate. $W^{2,p}$ estimate refers to estimating the L_p norm of the second weak derivative of u for a second-order elliptic equation Lu = f, where L is a second-order elliptic differential operator, u is an unknown function and f is a known function. This estimate can be obtained through Calderon-Zygmund operator. Specifically, we can construct a new equation by multiplying the original equation by a weight function. Then we can use Calderon-Zygmund operator to obtain the L_p norm estimate of the solution of this new equation, and thus obtain the L_p norm estimate of the second weak derivative of the solution of the original equation.

The Calderon-Zygmund decomposition method is a powerful mathematical tool that can be used to decompose a function into a good and a bad part. This method has many applications in various fields such as signal processing, image analysis, and finance. In this report, we describe the implementation of this method in Python and provide examples to demonstrate its effectiveness. I provide examples from signal processing and finance. I analyzed the Don Jones Index Monthly data from 2008 May to 2023 April using this method. The results show that the method is a useful tool for analyzing complex financial data. Future work will involve exploring the application of this method in other fields and extending the implementation to handle more complex data.

To enhance clarity and distinguish between the original proof and the supplementary materials, the additional proofs and related background knowledge that have been introduced by the author have been highlighted in a different color. This was done with the intent of providing greater transparency and making it easier for readers to identify the original content from the author's contributions.

Another example where Calderon-Zygmund decomposition is widely used is

Here are some highlights:

Theorem 2.4 has been refined and expanded to incorporate additional details in the proof of the main theorem. Specifically, the original proof has been polished to enhance clarity and rigor, and application of Minkowski's Inequality have been pointed out to provide a more comprehensive argument.

Theorem 2.7 can be used to show pointwise convergence almost everywhere for $f \in L^p$. And is used to prove the weak (1,1) inequality for the maximal function. The property of $\limsup$ and triangle inequality were added, the construction of f_n was derived to o facilitate a deeper understanding.

Theorem 2.12 (Marcinkiewicz Interpolation) has been enhanced with explanation of $f_1 \in L^{p_1}$ *and* $f_0 \in L^{p_0}$. Additional detailed calculations have been included for a more accessible reading experience.

In section 2.6, additional detailed prove have been included to enhance the clarity and coherence of the presented lemma prove. Sharper conclusion of Corollary 2.28 (Lebesgue Differentiation Theorem) is not a immediately result of the corollary so additional materials from *Real Analysis: Measure Theory, Integration, and Hilbert Spaces* have been included.

In section 3.1, a step-by-step derivation of the relevant calculations have been included in the section on the Poisson kernel and harmonic extension.

Section 3.2 now includes an additional definition of the operator convolution, which aims to provide clarity to the proof.

The theorems of M. Riesz and Kolmogorov (Theorem 3.4) have been rewrote to better illustrate the application of Calderon-Zygmund decomposition.

At last part of section 3.3, an exercise proposed by the author of the textbook has been rigorously proved. In the Schwartz class it is straightforward to characterize the functions whose Hilbert transforms are integrable. By establishing this result, a deeper understanding of the Schwartz class and Hilbert transforms can be attained, and the practical applications of these principles can be better appreciated.

In section 3.5, an exhaustive verification making use of Plancherel Theorem have been included for a better understanding of T_m's math properties.

In section 4.2, a simple computation which shows that The homogeneity of (4,1) distribution is left by the author and this verification is added to better appreciate the underlying mathematical principles.

Proof of Corollary 4.9,4.10,4.12-4.14 were not covered in the textbook, and additional verification have been included to enhance a better understanding of Singular Integrals and its mathematical properties.

1. Basic Fourier Analysis

1.1. Basic results.

Definition 1.1. (Schwartz function) A function $f : \mathbb{R} \mapsto \mathbb{C}$ is said to be Schwartz iff

$$(\forall x \in \mathbb{R})(\forall \alpha, \beta \in \mathbb{N}) \left(\sup_{x\in\mathbb{R}} |x^\alpha D^\beta f(x)| < \infty \right).$$

Definition 1.2. (Fourier transform) Given a Schwartz function $f : \mathbb{R} \mapsto \mathbb{C}$. We define its Fourier transform by

$$\hat{f}(\rho) := \int_{\mathbb{R}} f(x) e^{-i\rho x} \, dx$$

for $\rho \in \mathbb{R}$.

Definition 1.3. (Inverse Fourier transform) Let $\hat{f}(\rho)$ be the Fourier transform of $f(x)$. We have

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\rho x} \hat{f}(\rho) \, d\rho.$$

If we define $\check{\hat{f}}(\rho)(x) = \displaystyle\int_{\mathbb{R}} e^{i\rho x} \hat{f}(\rho) \, d\rho$, then we can write

$$f(x) = \frac{1}{2\pi} \check{\hat{f}}(\rho)(x).$$

Lemma 1.4. *(Leibniz integral rule, wiki) Let $f(x,t)$ be a function such that $f(x,t)$ and its partial derivative $f_x(x,t)$ is continuous in t and x in some region of the xt-plane, including $a(x) \le t \le b(x), x_0 \le x \le x_1$. Suppose $a(x)$ and $b(x)$ are both continuous and both have continuous derivatives for $x_0 \le x \le x_1$. Then, for $x_0 \le x \le x_1$, we have*

$$\frac{d}{dx} \left(\int_{a(x)}^{b(x)} f(x,t) \, dt \right) = f(x, b(x)) \frac{d}{dx} b(x) - f(x, a(x)) \frac{d}{dx} a(x) + \int_{a(x)}^{b(x)} f_x(x,t) \, dt.$$

Proof. Skipped as for now. □

Corollary 1.5. *With a similar setting in lemma 1.4, for all $a, b \in \mathbb{R}$, we have*

$$\frac{d}{dx} \int_a^b f(x,t) \, dt = \int_a^b f_x(x,t) \, dt.$$

Proof. By lemma 1.4, taking $a(x) = a, b(x) = b$ will give us the desired result. $\square$

Remark 1.6. From now onwards, without specification, we assume $f : \mathbb{R} \mapsto \mathbb{C}$ is Schwartz.

Proposition 1.7. *Define $g : \mathbb{R} \mapsto \mathbb{C}$ by setting $g(x) = xf(x)$. We have*

(1) *g is Schwartz;*
(2) $\hat{g}(\rho) = i\frac{d}{d\rho}\hat{f}(\rho)$.

Proof.

(1): It follows definition 1.1.

(2): We have

$$\begin{aligned}
\hat{g}(\rho) &= \int_{\mathbb{R}} g(x)e^{-i\rho x}\ dx \\
&= \int_{\mathbb{R}} xf(x)e^{-i\rho x}\ dx \\
&= \frac{1}{-i}\int_{\mathbb{R}} f(x)\left(-ixe^{-i\rho x}\right)\ dx \\
&= i\int_{\mathbb{R}} f(x)\left(\frac{d}{d\rho}e^{-i\rho x}\right)\ dx \\
&= i\frac{d}{d\rho}\int_{\mathbb{R}} f(x)e^{-i\rho x}\ dx \\
&= i\frac{d}{d\rho}\hat{f}(\rho).
\end{aligned}$$

$\square$

Proposition 1.8. *Define $h : \mathbb{R} \mapsto \mathbb{C}$ by setting $h(x) = f'(x)$. We have*

(1) *h is Schwartz;*
(2) $\hat{h}(\rho) = i\rho\hat{f}(\rho)$.

Proof.

(1): It follows definition 1.1.

(2): We have

$$\begin{aligned}
\hat{h}(\rho) &= \int_{\mathbb{R}} f'(x)e^{-i\rho x}\ dx \\
&= \left[f(x)e^{-i\rho x}\right]_{-\infty}^{\infty} + i\rho \int_{\mathbb{R}} f(x)e^{-i\rho x}\ dx \\
&= 0 + i\rho\hat{f}(\rho) \\
&= i\rho\hat{f}(\rho).
\end{aligned}$$

Here, we subtly use (1) $f(\pm\infty) = 0$ (otherwise, $\sup_{x\in\mathbb{R}} |xf(x)| \not< \infty$, contradicting the fact that f is Schwartz) and (2) $\|e^{-i\rho x}\| \leq \sqrt{2}$. ☐

Definition 1.9. (Convolution) The convolution of g and h is defined by

$$(g * h)(x) = \int_{\mathbb{R}} g(\tau)h(x-\tau)\ d\tau = \int_{\mathbb{R}} h(\tau)g(x-\tau)\ d\tau$$

Proposition 1.10. *(Convolution theorem)* $\hat{F}\hat{g} = \widehat{(F * g)}$.

Theorem 1.11. *(Plancherel theorem) We have* $\displaystyle\int_{-\infty}^{\infty} |f(x)|^2\ dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\rho)|^2\ d\rho.$

Proof. Skipped as for now. ☐

Remark 1.12. In quantum mechanics, ρ is called the moment variable and x the position variable.

Definition 1.13. (Expected value) Let $f(x) \in \mathbb{R}$ be the wave function satisfying

$$\int_{\mathbb{R}} |f(x)|^2\ dx = 1.$$

The expected value of the square of the position is defined by

$$(\bar{x})^2 = \int_{\mathbb{R}} |xf(x)|^2\ dx,$$

and similarly, the expected value of the square of the position of momentum is defined by

$$(\bar{\rho})^2 = \frac{1}{2\pi}\int_{\mathbb{R}} |\rho\hat{f}(\rho)|^2 d\rho.$$

Lemma 1.14. *(Cauchy-Schwarz inequality) We have*

$$\left|\int_{\mathbb{R}} f \cdot g \; dx\right| \leq \left(\int_{\mathbb{R}} f^2 \; dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} g^2 \; dx\right)^{\frac{1}{2}}.$$

Lemma 1.15. *(Holder's inequality) Given $p \geq 1$, we define p^* to be its conjugate satisfying $\frac{1}{p} + \frac{1}{p'} = 1$. If $f \in L^p(\mathbb{R}^n), g \in L^{p'}(\mathbb{R}^n)$, we have*

$$\|fg\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^{p'}(\mathbb{R}^n)}.$$

Proof. Skipped as for now. □

Proposition 1.16. *(Uncertainty principle) $\bar{x} \cdot \bar{\rho} \geq \dfrac{1}{2}$.*

Proof. By Holder's inequality, we have

$$\left|\int_{\mathbb{R}} x f(x) f'(x) \; dx\right| \leq \left(\int_{\mathbb{R}} |xf(x)|^2 \; dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |f'(x)|^2 \; dx\right)^{\frac{1}{2}} = \bar{x} \left(\int_{\mathbb{R}} |f'(x)|^2 \; dx\right)^{\frac{1}{2}}.$$

By Plancherel theorem, we have

$$\begin{aligned}
\int_{\mathbb{R}} |f'(x)|^2 \; dx &= \frac{1}{2\pi} \int_{\mathbb{R}} |(\hat{f'})(\rho)|^2 \; d\rho \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} |i\rho \hat{f}(\rho)|^2 \; d\rho \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} |\rho \hat{f}(\rho)|^2 \; d\rho \\
&= (\bar{\rho})^2,
\end{aligned}$$

which then implies that

$$\left|\int_{\mathbb{R}} x f(x) f'(x) \; dx\right| \leq \bar{x}\bar{\rho}.$$

For the LHS, we have

$$\begin{aligned} I &= \int_{\mathbb{R}} xf(x)f'(x)\ dx = \left[x\big(f(x)\big)^2\right]_{-\infty}^{\infty} - \int_{\mathbb{R}} [f(x) + xf'(x)]f(x)\ dx \\ I &= 0 - \int_{\mathbb{R}} [f(x)]^2\ dx - I \\ 2I &= -1 \\ I &= -\frac{1}{2}, \end{aligned}$$

which implies that

$$LHS = |I| = \frac{1}{2} \leq \bar{x}\bar{\rho} = RHS.$$

We therefore conclude the uncertainty principle. □

1.2. **Using Fourier transform to solve some linear PDEs.**

Lemma 1.17. $\widehat{e^{-\frac{x^2}{2}}} = \sqrt{2\pi}e^{-\frac{k^2}{2}}$.

Proof. Let $h(x) = \dfrac{-x^2}{2}$. We have

$$h'(x) = -xe^{\frac{-x^2}{2}} = -xh(x). \tag{$*$}$$

By doing Fourier transform for $(*)$, we have

$$\begin{aligned} ik\hat{h}(k) = -\left[i\frac{d}{dk}\hat{h}(k)\right] &\implies \frac{d}{dk}\hat{h}(k) + k\hat{h}(k) = 0 \\ &\implies e^{\frac{k^2}{2}}\hat{h}(k) = C \\ &\implies \hat{h}(k) = Ce^{-\frac{k^2}{2}} \end{aligned}$$

Since $\hat{h}(0) = \int_{\mathbb{R}} h(x)\ dx = \int_{\mathbb{R}} e^{-\frac{x^2}{2}}\ dx$ □

Lemma 1.18. $\widehat{f(ax)} = \dfrac{1}{|a|}\hat{f}\left(\dfrac{k}{a}\right)$.

Question. Solve $u_t - u_{xx} = 0$ with $u(0,x) = g(x)$. Here, $u : \mathbb{R}^2 \mapsto \mathbb{R}$.

Proof. Taking Fourier transform of the equation with respect to x, we have

$$\begin{aligned} u_t - u_{xx} = 0 &\implies \widehat{u}_t - \widehat{u_{xx}} = \widehat{0} \\ &\implies \int_{\mathbb{R}} u_t e^{-i\rho x}\, dx - (i\rho)^2 \widehat{u} = 0 \\ &\implies (\widehat{u})_t + \rho^2 \widehat{u} = 0 \\ &\implies \widehat{u}(t,\rho) = Ce^{-\rho^2 t}. \end{aligned}$$

From the initial condition, we have $u(0,x) = g(x) \implies \widehat{u}(0,x) = \widehat{g}(x) \implies C = \widehat{g}(x)$, which leads to

$$\widehat{u}(t,\rho) = \hat{g}(x) e^{-\rho^2 t}.$$

By inverse Fourier transform, we have

$$\begin{aligned} u(t,\rho) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\rho x} \hat{u}(t,\rho)\, d\rho \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{u}(t,\rho)\, d\rho \end{aligned}$$

□

1.3. **Generalised Fourier transform.**

Definition 1.19. Let $x, \rho \in \mathbb{R}^n$. For $f(x) \in L^1(\mathbb{R}^n)$. Then, its Fourier transform $\widehat{f}(\rho)$ is defined by

$$\widehat{f}(\rho) = \int_{\mathbb{R}^n} e^{-ix\cdot\rho} f(x)\, dx.$$

Lemma 1.20. *Let $f(x) \in L^1(\mathbb{R}^n)$. We have*

(1) $f(x) = \dfrac{1}{(2\pi)^n} \displaystyle\int_{\mathbb{R}^n} e^{ix\cdot\rho} \hat{f}(\rho)\, d\rho$;
(2) $\widehat{fg}(\rho) = \hat{f}(\rho) * \hat{g}(\rho)$;
(3) $\widehat{f * g}(\rho) = \hat{f} \cdot \hat{g}(\rho)$.

1.4. **Schrodinger's equation.**

Definition 1.21. Let $\psi(t,x) : \mathbb{R}^1_t \times \mathbb{R}^n_x \mapsto \mathbb{C}$. The Schrodinger's equation is

$$-i\psi_t(t,x) + \frac{1}{2}\triangle\psi(t,x) = 0$$

with condition $\psi(0,x) = \psi_0(x) \in L^2(x)$.

Definition 1.22. ($L^p, W^{k,p}$ space) Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ and $p \geq 1$. We define

$$f \in L^p(\mathbb{R}^n) \iff \left(\int_{\mathbb{R}^n} |f|^p \; dx_1 dx_2 \ldots dx_n\right)^{\frac{1}{p}} < +\infty$$

$$f \in W^{k,p}(\mathbb{R}^n) \iff \sum_{s=0}^{k} \left(\int_{\mathbb{R}^n} |\nabla^s f|^p \; dx_1 dx_2 \ldots dx_n\right)^{\frac{1}{p}} < +\infty$$

for all $k \in \mathbb{Z}^+$. Specifically, we define $W^{s,2} = H^s$ for all $s \in \mathbb{N}$.

Lemma 1.23. *(Sobolev's lemma) If $s > k + \frac{n}{2}$, $k \in \mathbb{N}$, then*

$$\sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{L^\infty(\mathbb{R}^n)} \leq C_s \|f\|_{H^S(\mathbb{R}^n)} .$$

Furthermore, C_s is independent of f and only depends on s and n.

Definition 1.24. (Sobolev conjugate) If $1 \leq p < n$, the Sobolev conjugate of p is

$$p^* := \frac{np}{n-p}.$$

Theorem 1.25. *(Gagliardo-Nirenberg-Sobolev inequality) Assume $1 \leq p < n$. There exists a constant C, depending only on p and n such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C \|Du\|_{L^p(\mathbb{R}^n)} .$$

Theorem 1.26. *(Estimate for $W^{1,p}$) Let Ω be a bounded open subset of $\mathbb{R}^n$, and suppose $\partial\Omega$ is C^1. Assume $1 \leq p < n$ and $u \in W^{1,p}(\Omega)$. Then, $u \in L^{p^*}$ with the estimate*

$$\|u\|_{L^{p^*}(\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)} ,$$

where C is a constant depending on only p, n, and Ω.

Definition 1.27. (Distributional derivative) Given $k \in \mathbb{Z}^+$, the distributional derivative of order k of $f \in L^1_{loc}(\mathbb{R})$ is the linear functional

$$\Lambda_{D^k f}(\phi) := (-1)^k \int_{\mathbb{R}} f(x) D^k \phi(x)\ dx.$$

Definition 1.28. (Weak derivative) If there exists $g \in L^1_{loc}(\mathbb{R})$ such that $\Lambda_g = \Lambda_{D^k f}$, namely,

$$\int_{\mathbb{R}} g(x)\phi(x) = (-1)^k \int_{\mathbb{R}} f(x) D^k \phi(x)\ dx$$

for all $\phi \in C_c^\infty(\mathbb{R})$, then we call g the weak derivative of order k of f.

Definition 1.29. (Distribution) A distribution on the open set $\Omega \subseteq \mathbb{R}^n$ is a linear functional $\Lambda : C_c^\infty(\Omega) \mapsto \mathbb{R}$ such that for every compact $K \subseteq \Omega$, there exists an integer $N \geq 0$ and a constant C such that

$$|\Lambda(\phi)| \leq C \left\|\phi\right\|_{C^N}$$

for every $\phi \in C_c^\infty$ with $\mathrm{Supp}(\phi) \subseteq K$.

Definition 1.30. Let Λ be a distribution and α be a multi-index. We define another distribution $D^\alpha \Lambda$ by setting

$$D^\alpha \Lambda(\phi) := (-1)^{|\alpha|} \Lambda(D^\alpha \phi).$$

Lemma 1.31. *Let K be any compact set in Ω and let ϕ be a test function with support contained in K. Then, we have*

$$|D^\alpha \Lambda(\phi)| = |\Lambda(D^\alpha \phi)| \leq C \left\|D^\alpha \phi\right\|_{C^N} \leq C \left\|\phi\right\|_{C^{N+|\alpha|}}.$$

Definition 1.32. (Weak derivative) Let $f \in L^1_{loc}(\Omega)$ and Λ_f be the corresponding distribution. Given a multi-index α, if there exists $g \in L^1_{loc}(\Omega)$ such that $D^\alpha \Lambda_f = \Lambda_g$, then we say that g is the weak α-th derivative of f and write $g = D^\alpha f$.

Lemma 1.33. *(Uniqueness of weak derivative) If the weak α-th derivative of $f \in L^1_{loc}(\Omega)$ exists, then it is unique.*

Lemma 1.34. *Assume that $f \in L^1_{loc}(\Omega)$ has weak derivative $D^\alpha f$ for every $|\alpha| \leq k$. Then, for every pair of multi-indices α, β with $|\alpha| + |\beta| \leq k$, one has*

$$D^\alpha(D^\beta f) = D^\beta(D^\alpha f) = D^{\alpha+\beta} f.$$

Definition 1.35. (Sobolev space) Let $\Omega \subseteq \mathbb{R}^n$ be open, $p \in [1, +\infty]$, and $k \in \mathbb{Z}_0^+$. The Sobolev space $W^{k,p}(\Omega)$ is the space of all locally summable functions $u : \Omega \mapsto \mathbb{R}$ such that, for every multi-index α with $|\alpha| \leq k$, the weak derivative $D^\alpha u$ exists and belongs to $L^p(\Omega)$. Moreover, we define

$$\|u\|_{W^{k,p}(\Omega)} := \left(\sum_{|a| \leq k} \int_\Omega |D^\alpha u|^p \, dx \right)^{\frac{1}{p}} \quad \text{if } 1 \leq p < \infty;$$
$$\|u\|_{W^{k,\infty}(\Omega)} := \sum_{|a| \leq k} \operatorname{esssup}_{x \in \Omega} |D^\alpha u| \quad \text{if } p = \infty.$$

Definition 1.36. $W_0^{k,p} \subseteq W^{k,p}$ is defined as the closure of $\mathbf{C}_c^\infty(\Omega)$ in $W^{k,p}$. We have

$$u \in W_0^{k,p} \iff \exists \{u_n\}_{n \in \mathbb{Z}^+} [\|u - u_n\|_{W^{k,p}} \to 0].$$

Definition 1.37. $W^{k,p}_{loc}(\Omega)$ denotes the space of $f(n)$ which are locally summable in $W^{k,p}$.

Remark 1.38. For $p = 2$, $H^k(\Omega) = W^{k,p}(\Omega)$ is a Hilbert space endowed with an inner product

$$\langle u, v \rangle_{H^k} = \sum_{|\alpha| \leq k} \int_\Omega D^\alpha u D^\alpha v \, dv.$$

Lemma 1.39. *(Convergence of weak derivatives) Consider a sequence of functions $f_n \in L^1_{loc}(\Omega)$. For a fixed multi-index α, assume that f_n admits the weak derivative $g_n = D^\alpha f_n$. If $f_n \to f$ and $g_n \to g$ in $L^1_{loc}(\Omega)$, then $g = D^\alpha f$.*

Theorem 1.40. *(Basic properties of Sobolev spaces) We have*

1. *Each Sobolev space $W^{k,p}(\Omega)$ is a Banach space;*
2. *The space $W_0^{k,p}(\Omega)$ is a closed subspace of $W^{k,p}(\Omega)$, which implies that $W_0^{k,p}(\Omega)$ is a Banach space with the same norm;*
3. *The space $H^k(\Omega)$ and $H_0^k(\Omega)$ are Hilbert spaces.*

Definition 1.41. (Holder condition) We say a real or complex-valued function f on $\mathbb{R}^n$ satisfies a Holder condition/ is Holder continuous iff

$$(\exists C \in \mathbb{R}_0^+)(\exists \alpha > 0)(\forall x, y \in \mathrm{dom}(f))[|f(x) - f(y) \leq C|x-y|^\alpha].$$

Definition 1.42. (Holder norm) We define

$$\|u\|_{C^{0,r}(\mathbb{R}^n)} := \sup_{x \neq y} \left| \frac{|u(x) - u(y)|}{|x-y|^r} \right| + \|u\|_{C^0(\mathbb{R}^n)}.$$

Lemma 1.43. *If u is Holder continuous, then u is uniformly continuous.*

Theorem 1.44. *(Morrey's inequality) Assume $n < p \leq \infty$. Then there exists a constant C, depending only on n and p, such that*

$$\|u\|_{C^{0,r}(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(\mathbb{R}^n)}$$

for all $u \in C^1(\mathbb{R}^n)$, where $r = 1 - \dfrac{n}{p}$.

Lemma 1.45. *For any ball $B(x,r) \subseteq \mathbb{R}^n$, there exists a constant C, depending only on n, such that*

$$⨍_{B(x,r)} |u(y) - u(x)| \leq C \int_{B(x,r)} \frac{|Du(y)|}{|y-x|^{n-1}} \, dy.$$

Definition 1.46. Let X and Y be Banach spaces, $X \subseteq Y$. We say that X is compactly embedded in Y, written $X \subset\subset Y$, provided

(1) $\|x\|_Y \leq C \|x\|_X$ for some constant C;
(2) each bounded sequence in X is precompact in Y, i.e., if $|u_n|_{n \in \mathbb{Z}^+}$ is a bounded sequence in X, there exists a subsequence $|u_{n_i}|_{i \in \mathbb{Z}^+}$ which converges in Y.

Theorem 1.47. *Let $f : U \mapsto \mathbb{R}$ and $f \in L^1_{loc}$. We define $f^\varepsilon = \eta_\varepsilon * f \in U_\varepsilon$.*

(1) *$f^\varepsilon \in C^\infty(U_\varepsilon)$;*
(2) *$f^\varepsilon \to f$ almost everywhere as $\varepsilon \to 0$;*
(3) *if $f \in C(U)$, then $f^\varepsilon \to f$ uniformly on compact subsets of U;*
(4) *if $1 \leq p < \infty$ and $f \in L^p_{loc}(U)$, then $f^\varepsilon \to f$ in $L^p_{loc}(U)$.*

Definition 1.48. (Equicontinuous) Let X, Y be two metric spaces and $C(X, Y)$ be the collection of all functions from X to Y. Let $\mathcal{F} \subseteq C(X, Y)$. We say that $\mathcal{F}$ is equicontinuous at a point $x_0 \in X$ iff $(\forall \varepsilon \in \mathbb{R}^+)(\exists \delta \in \mathbb{R}^+)(\forall f \in \mathcal{F})(\forall x \in \mathrm{dom}(f))(d(x, x_0) < \delta \implies d(f(x_0), f(x)) < \varepsilon)$.

Definition 1.49. (Uniformly equicontinuous) Let X, Y be two metric spaces and $C(X, Y)$ be the collection of all functions from X to Y. Let $\mathcal{F} \subseteq C(X, Y)$. We say that $\mathcal{F}$ is equicontinuous at a point $x_0 \in X$ iff $(\forall \varepsilon \in \mathbb{R}^+)(\exists \delta \in \mathbb{R}^+)(\forall f \in \mathcal{F})(\forall x, y \in \mathrm{dom}(f))(d(x, y) < \delta \implies d(f(x), f(y)) < \varepsilon)$.

Theorem 1.50. *(Arzela-Ascoli theorem) Consider a sequence of real-valued continuous function $\{f_n\}_{n \in \mathbb{Z}^+}$ defined on a closed and bounded interval of $\mathbb{R}$. If the sequence is uniformly bounded and uniformly equicontinuous, then there exists a subsequence $\{f_{n_i}\}_{i \in \mathbb{Z}^+}$ that converges uniformly.*

Theorem 1.51. *(Rellich-Kondrachov compactness theorem) Assume U is a bounded open subset of $\mathbb{R}^n$, and ∂U is C^1. Suppose $1 \leq p < n$. Then, $W^{1,p}(U) \subset\subset L^q(U)$ for each $1 \leq q < p^*$.*

Notation 1.52. $(u)_U = \fint_U u \, dy =$ average of u over U.

Notation 1.53. $(u)_{x,r} = \fint_{B(x,r)} u \, dy$.

Theorem 1.54. *(Poincare's inequality) Let U be a bounded, connected, open subset of $\mathbb{R}^n$ with a C^1 boundary ∂U. Assume $1 \leq p \leq \infty$. Then, there exists a constant C, depending only on n, p, and U such that*

$$\|u - (u)_U\|_{L^p(U)} \leq C \|Du\|_{L^p(U)}$$

for each function $u \in W^{1,p}(U)$.

Theorem 1.55. *(Poincare's inequality for a ball) Assume $1 \leq p \leq \infty$. Then, there exists a constant C, depending only on n and p, such that*

$$\|u - (u)_{x,r}\|_{L^p(B(x,r))} \leq Cr \|Du\|_{L^p(B(x,r))}.$$

Corollary 1.56. $\fint_{B(x,r)} |u - (u)_{x,r}| \, dy \leq \|Du\|_{L^n(\mathbb{R}^n)}$.

Definition 1.57. (Bounded mean oscillation) If u satisfies $\fint_{B(x,r)} |u-(u)_{x,r}|\, dy \leq \|Du\|_{L^n(\mathbb{R}^n)}$, then we say $u \in BMO(\mathbb{R}^n)$. Moreover, we define

$$[u]_{BMO(\mathbb{R}^n)} := \sup_{B(x,r)\subseteq\mathbb{R}^n} \left\{ \fint_{B(x,r)} |u-(u)_{x,r}|\, dy \right\}.$$

Lemma 1.58. (general Sobolev embeddings) *We have*

1. $W^{2,k}(U) \subset W^{1,q}(U) \subset L^p(U)$*;*
2. *If* $k - \dfrac{n}{p} < 0$*, then* $W^{k,p}(U) \subseteq L^q(U)$ *with* $\dfrac{1}{q} = \dfrac{1}{p} - \dfrac{k}{n} = \dfrac{1}{n}\left(\dfrac{n}{p} - k\right)$.

2. The Hardy-Littlewood Maximal Function

2.1. Approximations of the identity.

Definition 2.1. (Schwartz function) A function $f : \mathbb{R}^n \mapsto \mathbb{C}$ is said to be Schwartz iff

$$(\forall x \in \mathbb{R})(\forall \alpha, \beta \in \mathbb{N}^n)\left(\sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta f(x)| = p_{\alpha,\beta}(f) < \infty\right).$$

Notation 2.2. Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz class, and S' be the space of bounded linear functionals on $\mathcal{S}$.

Theorem 2.3. *(Minkowski's integral inequality) Let (S_1, μ_1) and (S_2, μ_2) be two σ-finite measure spaces and $F : S_1 \times S_2 \mapsto \mathbb{R}$ is measurable. Then, Minkowski's integral inequality is*

$$\left[\int_{S_2}\left|\int_{S_1} F(x,y)\mu_1(dx)\right|^p \mu_2(dy)\right]^{\frac{1}{p}} \leq \int_{S_1}\left(\int_{S_2} |F(x,y)|^p \mu_2(dy)\right)^{\frac{1}{p}} \mu_1(dx).$$

Let ϕ be an integrable function on R^n such that $\int \phi = 1$, and for t¿0 define $\phi_t(x) = t^{-n}\phi(t^{-1}x)$. As $t \to 0$,ϕ_t converges in S' *to* δ, the Dirac measure at the origin: if $g \in S$ then

$$\phi_t(g) = \int_{R^n} t^{-n}\phi(t^{-1}x)g(x)dx = \int_{R^n} \phi(x)g(tx)dx$$

Here let $y = t^{-1}x \Rightarrow \phi_t(g) \int_{R^n} t^{-n}\phi(t^{-1}x)g(x)dx = \int_{R^n} t^{-n}\phi(y)g(ty)dx = \int_{R^n} \phi(y)g(ty)dy$ and so by the dominated convergence theorem,

$$\lim_{t\to 0} \phi_t(g) = g(0) = \delta(g)$$

$$\begin{aligned}
&\lim_{t\to 0}\phi_t(g)\\
&\quad = \lim_{t\to 0}\int_{R^n} t^{-n}\phi(t^{-1}x)g(x)dx\\
&\quad = \lim_{t\to 0}\int_{R^n}\phi(x)g(tx)dx\\
&\quad = \int_{R^n}\lim_{t\to 0}\phi(x)g(tx)dx\\
&\quad = \int_{R^n}\phi(x)g(0)dx\\
&\quad = g(0)\int_{R^n}\phi(x)dx \quad (\int_{R^n}\phi(x)dx = 1)\\
&\quad = g(0)
\end{aligned}$$

Since $\delta * g = g$, for $g \in S$, we have pointwise limit

$$\lim_{t\to 0}\phi_t * g = g(x)$$

$$\begin{aligned}
&\lim_{t\to 0}\phi_t * g\\
&\quad = \lim_{t\to 0}\int_{R^n}\phi_t(y)g(x-y)dy\\
&\quad = \lim_{t\to 0}\int_{R^n} t^{-n}\phi(t^{-1}y)g(x-y)dy \quad y' = t^{-1}y\\
&\quad = \lim_{t\to 0}\int_{R^n}\phi(y')g(x-ty')dy'\\
&\quad = \int_{R^n}\phi(y')\lim_{t\to 0}g(x-ty')dy'\\
&\quad = \int_{R^n}\phi(y')g(x)dy' \quad (\int_{R^n}\phi(x)dx = 1)\\
&\quad = g(x)
\end{aligned}$$

Theorem 2.4. *Let $\{\delta_t : t > 0\}$ be an approximation of the identity. Then*

$$\lim_{t\to 0} \|\phi_t * f - f\|_p = 0$$

if $f \in L^p$, $1 \le p < \infty$, and uniformly (i.e., when $p = \infty$) if $f \in C_0(\mathbb{R}^n)$.

Because ϕ has integral 1,

$$\begin{aligned}
&\phi_t * f(x) - f(x) \\
&\qquad = \int_{R^n} \phi_t(y) f(x-y)dy - f(x) \\
&\qquad = \int_{R^n} t^{-n}\phi(t^{-1}y) f(x-y)dy - f(x) \quad \textit{let new } y = t^{-1}y \\
&\qquad = \int_{R^n} \phi(y) f(x-ty)dy - f(x) \\
&\qquad = \int_{R^n} \phi(y) f(x-ty)dy - \int_{R^n} \phi(y)dy f(x) \quad (\int_{R^n} \phi(x)dx = 1) \\
&\qquad = \int_{R^n} \phi(y)[f(x-ty) - f(x)]dy
\end{aligned}$$

Given $\epsilon > 0$, choose $\delta > 0$ such that if $|h| < \delta$,

$$\|f(\cdot + h) - f(\cdot)\|_p < \frac{\epsilon}{2\|\phi\|_1} \tag{1}$$

as $t \to 0$ $\quad \|f(\cdot + h) - f(\cdot)\|_p \to 0$, equivalently $t \to 0$ $\quad f(\cdot + h) = f(\cdot)$ $a.e.$ This δ depends on f. For this fixed δ, if t is sufficiently small then

$$\int_{|y|>\delta/t} |\phi(y)|dy \le \frac{\epsilon}{4\|f\|_p} \quad \textit{here } \|f\|_p \neq 0 \tag{2}$$

Therefore, by Minkowski's inequality,

$$
\begin{aligned}
&\|\phi_t * f - f\|_p \\
&= \left\| \int_{R^n} \phi(y)[f(x-ty) - f(x)]dy \right\|_p \\
&\leq \left\| \int_{R^n} |\phi(y)||f(\cdot - ty) - f(\cdot)|dy \right\|_p \\
&= \| \| |\phi(y)||f(\cdot - ty) - f(\cdot)| \|_1 \|_p \\
&\leq \left\| \| |\phi(y)||f(\cdot - ty) - f(\cdot)| \|_p \right\|_1 \quad (Minkowski'sinequality) \\
&= \int_R \| |\phi(y)||f(\cdot - ty) - f(\cdot)| \|_p \, dy \\
&= \int_{|y|<\delta/t} \| |\phi(y)||f(\cdot - ty) - f(\cdot)| \|_p \, dy + \int_{|y|\geq\delta/t} \| |\phi(y)||f(\cdot - ty) - f(\cdot)| \|_p \, dy \\
&= \int_{|y|<\delta/t} |\phi(y)| \, \|f(\cdot - ty) - f(\cdot)\|_p \, dy + \int_{|y|\geq\delta/t} |\phi(y)| \, \|f(\cdot - ty) - f(\cdot)\|_p \, dy \\
&\leq \int_{|y|<\delta/t} \frac{\epsilon}{2\|\phi\|_1} dy + \int_{|y|\geq\delta/t} |\phi(y)| \, \|f(\cdot - ty) - f(\cdot)\|_p \, dy \quad by(1) \\
&as \, \|f(\cdot - ty) - f(\cdot)\|_p \leq 2\|f\|_p \;\; just \; triangle \; inequality \\
&\leq \int_{|y|<\delta/t} |\phi(y)| \frac{\epsilon}{2\|\phi\|_1} dy + 2\|f\|_p \int_{|y|\geq\delta/t} |\phi(y)| dy \\
&\leq \int_{|y|<\delta/t} |\phi(y)| \frac{\epsilon}{2\|\phi\|_1} dy + 2\|f\|_p \frac{\epsilon}{4\|f\|_p} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}
$$

Therefore, given $\epsilon > 0$, we will have $\|\phi_t * f - f\|_p < \epsilon$. Equivalently, $\lim_{t\to 0} \|\phi_t * f - f\|_p = 0$ Consequently, we can find a sequence of t_k, depending on f, such that $t_k \to 0$ and

$$lim_{k\to\infty} \phi_{t_k} * f(x) = f(x) \quad a.e.$$

2.2. **Weak-type inequalities and almost everywhere convergence.**

Definition 2.5. (Weak (p,q)) Let (X,μ) and (Y,ν) be measure spaces, and T be an operator from $L^p(X,\mu)$ into the space of measurable functions from Y to $\mathbb{C}$. We say that T is weak (p,q), $q<\infty$, if

$$\nu(\{y\in Y : |Tf(y)|>\lambda\}) \leq \left(\frac{C\left\|f\right\|_p}{\lambda}\right)^q.$$

We say that T is weak (p,∞) if it is a bounded operator from $L^p(X,\mu)$ to $L^\infty(Y,\nu)$.

Definition 2.6. (Strong (p,q)) We say that T is strong (p,q) if it is bounded from $L^p(X,\mu)$ to $L^q(Y,\nu)$.

If T is Strong (p,q) then it is Weak (p,q): Let E_λ=$y\in Y : |Tf(y)|>\lambda$, then
$\mu(E_\lambda)=\int_{E_\lambda} d\mu \leq \int_{E_\lambda}\left|\frac{Tf(x)}{\lambda}\right|^q d\mu \quad as\ |Tf(y)|>\lambda\ for\ every\ element\ in\ E_\lambda$
$\mu(E_\lambda)\leq \int_{E_\lambda}\left|\frac{Tf(x)}{\lambda}\right|^q d\mu \leq \frac{\|Tf\|_q^q}{\lambda^q} \leq \left(\frac{C\|f\|_p}{\lambda}\right)^q \quad Strong(p,q)\ implies\ \left\|Tf\right\|_q \leq C\left\|f\right\|_p$

Theorem 2.7. *Let $\{T_t\}$ be a family of linear operators on $L^p(X,\mu)$ and define*

$$T^*f(x)=\sup_t |T_tf(x)|.$$

If T^* is weak (p,q), then the set

$$\{f\in L^p(X,\mu) : \lim_{t\to t_0} T_tf(x) = f(x) a.e.\}$$

is closed in $L^p(X,\mu)$.

Goal: To prove a set is close we just need to prove that it contains all its limit points. And if $(X,\mu)=(Y,\nu)$, and T is the identity, the weak (p,p) inequality is the classical Chebyshev inequality

$$\mu(\{x \in X : |f(x)| > \lambda\}) \leq \frac{1}{\lambda^p} \int_{|f|>\lambda} |f|^p d\mu = \left(\frac{\|f\|_p}{\lambda} \right)^p$$

as $\lim_{n\to\infty} sup(a_n + b_n) \leq \lim_{n\to\infty} sup a_n + \lim_{n\to\infty} sup b_n$

With the goal to prove all the limits in the set, we construct a sequence of f_n that converges to f in $L^p(X, \mu)$ norm and $f_n$belongs to the set, i.e.,$T_t f(x) = f(x) a.e.$ then we have

$$\begin{aligned}
&\limsup_{t\to t_0} |T_t(f - f_n)(x)| \\
&\qquad = \limsup_{t\to t_0} |T_t(f - f_n)(x) - (f - f_n)(x) + (f - f_n)(x)| \\
&\qquad \leq \limsup_{t\to t_0} |T_t(f - f_n)(x) - (f - f_n)(x)| + \limsup_{t\to t_0} |(f - f_n)(x)|
\end{aligned}$$

then we have

$$\begin{aligned}
&\mu(\{x \in X : \limsup_{t\to t_0}(T_t(f - f_n)(x)) > \lambda\}) \\
&\qquad \leq \mu(x \in X : \{\limsup_{t\to t_0} |T_t(f - f_n)(x) - (f - f_n)(x)| > \lambda\}) \\
&\qquad + \mu(x \in X : \{\limsup_{t\to t_0} |(f - f_n)(x)| > \lambda\}) \\
&\qquad (by\ definition\ this\ term\ is\ 0) \\
&\qquad = \mu(\{x \in X : \limsup_{t\to t_0} |T_t(f - f_n)(x) - (f - f_n)(x)| > \lambda\}) \\
&\qquad \leq \mu(\{x \in X : \limsup_{t\to t_0} |T_t(f - f_n)(x)| > \lambda/2\}) \\
&\qquad + \mu(\{x \in X : |(f - f_n)(x)| > \lambda/2\}) \\
&\qquad \leq \left(\frac{C \|f - f_n\|_p}{\lambda} \right)^q \quad (weak\ property\ of\ T*) \\
&\qquad + \left(\frac{2}{\lambda} \|f - f_n\|_p \right)^p \quad by\ Chebyshev\ inequality
\end{aligned}$$

as $n \to \infty$, we will have $\|f - f_n\| \to 0$

Therefore,

$\mu(\{x \in X : \limsup_{t\to t_0}(T_t(f-f_n)(x)) > 0\})$
$\leq \sum_{k=1}^{\infty} \mu(\{x \in X : \limsup_{t\to t_0}(T_t(f-f_n)(x)) > 1/k\}) \leq \sum_{k=1}^{\infty} \left(Ck \left\|f-f_n\right\|_p\right)^q + (2k \left\|f-f_n\right\|_p)^p$
$= 0$

Notes. We can use Theorem 2.1 to construct the sequence $f_n(x)$

By the same technique, we can also prove that the set

$$\{f \in L^p(X,\mu) : \lim_{t\to t_0} T_t f(x) exists a.e.\}$$

is closed in $L^p(X,\mu)$. Which is suficces to show that

$$\{f \in L^p(X,\mu) : \limsup_{t\to t_0} T_t f(x) - \liminf_{t\to t_0} T_t f(x) = 0 \ a.e.\}$$

Definition 2.8. T^* is called the maximal operator associated with the family $\{T_i\}$.

2.3. The Marcinkiewicz interpolation theorem.

Definition 2.9. (Distribution) Let (X,μ) be a measurable space, and $f : X \mapsto \mathbb{C}$ a measurable function. We call the function $a_f : (0,\infty) \mapsto [0,\infty]$, given by

$$a_f(\lambda) = \mu(\{x \in X : |f(x)| > \lambda\}),$$

the distribution function of f (associated with μ).

Proposition 2.10. *Let $\phi : [0,\infty) \mapsto [0,\infty)$ be differentiable, increasing, and such that $\phi(0) = 0$. Then,*

$$\int_X \phi(|f(x)|)\ d\mu = \int_0^{\infty} \phi'(\lambda) a_f(\lambda)\ d\lambda.$$

$$\int_0^\infty \phi'(\lambda) a_f(\lambda)\ d\lambda$$

$$\begin{aligned}
&= \int_0^\infty a_f(\lambda)\phi'(\lambda) d\lambda \\
&= \int_0^\infty \int_X 1 d\mu \phi'(\lambda) d\lambda \\
&= \int_X \int_0^{|f(x)|} \phi'(\lambda) d\lambda \\
&= \int_X \int_0^{|f(x)|} d\phi(\lambda) \\
&= \int_X \phi(\lambda)\Big|_0^{|f(x)|} \\
&= \int_X \phi(|f(x)|) - \phi(0) d\mu \quad \phi(0) = 0\ by\ definition \\
&= \int_X \phi(|f(x)|) d\mu
\end{aligned}$$

Particularly, if $\phi(\lambda) = \lambda^p$ then $\int_X |f(x)|^p d\mu = \int_0^\infty \lambda^{p-1} p a_f(\lambda) d\lambda$ thus

$$\|f(x)\|_p^p = p \int_0^\infty \lambda^{p-1} a_f(\lambda) \quad (*)$$

Definition 2.11. (Sublinear) An operator T from a vector space of measurable functions and measurable functions is sublinear if

$$|T(f_0 + f_1)(x)| \leq |Tf_0(x)| + |Tf_1(x)|,$$

$$|T(\lambda f)| = |\lambda||Tf|, \quad \lambda \in \mathbb{C}.$$

Theorem 2.12. *(Marcinkiewicz interpolation) Let (X, μ) and (Y, ν) be measure spaces, $1 \leq p_0 < p_1 \leq \infty$, and G be a sublinear operator from $L^{p_0}(X, \mu) + L^{p_1}(X, \mu)$ to the measurable functions on Y that is weak (p_0, p_0) and (p_1, p_1). Then T is strong (p, p) for $p_0 < p < p_1$.*

Given $f \in L^p$, for any λ we can decmpose f as $f_1 + f_2$,here

$$f_0 = f\chi\{x : |f(x)| > c\lambda\}$$

$$f_1 = f\chi\{x : |f(x)| \leq c\lambda\}$$

Then $\|f_0\|_{p_0}^{p_0} = \int |f_0|^{p_0} \leq \int |f|^{p_0} = \int |f|^{p_0-p}|f|^p \leq (c\lambda)^{p_0-p} \int |f|^p$
similarly, $\|f_1\|_{p_1}^{p_1} = \int |f_1|^{p_1} \leq \int |f|^{p_1} = \int |f|^{p_1-p}|f|^p \leq (c\lambda)^{p_1-p} \int |f|^p$

Therefore,

$$f_0 \in L^{p_0}(\mu) \quad and \quad f_1 \in L^{p_1}(\mu)$$

As $a_f(\lambda) = \nu(\{y \in Y : |Tf(y)| > \lambda\})$, from the sub-linear property of T

$$|Tf| \leq |Tf_1| + |Tf_0|$$

we have

$$a_{Tf}(\lambda) \leq a_{Tf_0}(\lambda) + a_{Tf_1}(\lambda)$$

We need to prove that T is strong in (p,p), which means we need to control a_{Tf}, and by the above inequality, we just need to control $a_{Tf_0}(\lambda) + a_{Tf_1}(\lambda)$. And $1 \leq p_0 < p1 \leq \infty$, we consider two cases depending on the value of p_1
Case 1: $p_1 = \infty$ Then from the weak inequality, we can find $A_1\ with\ \|Tg\|_\infty \leq A_1 \|g\|_\infty$, since T is weak-type (p_1, p_1), then $a_{Tf_1}(\lambda/2) \leq (\frac{1}{\lambda/2} \|Tf_1\|)^{p_0} \leq (\frac{A_1}{\lambda/2} \|f_1\|_{p_1})^{p_0} = 0$as $\|f_1\|_\infty = 0$ because of the range of it. Besides, by the weak(p_0, p_0) inequality,

$$a_{Tf_0}(\lambda/2) \leq (\frac{A_0}{\lambda/2} \|f_0\|_{p_0})^{p_0}$$

Hence,

$$
\begin{aligned}
&\|Tf\|_p^p \\
&\quad = p\int_0^\infty \lambda^{p-1} a_{Tf}(\lambda)d\lambda \\
&\quad \leq p\int_0^\infty \lambda^{p-1} a_{Tf_0}(\lambda)d\lambda + p\int_0^\infty \lambda^{p-1} a_{Tf_1}(\lambda)d\lambda \\
&\quad = p\int_0^\infty \lambda^{p-1} a_{Tf_0}(\lambda)d\lambda \\
&\quad \leq p\int_0^\infty \lambda^{p-1} (\frac{A_0}{\lambda/2}\left\|f_0\right\|_{p_0})^{p_0} d\lambda \\
&\quad = p\int_0^\infty \lambda^{p-1-p_0}(2A_0)^{p_0}\int_{\{x:|f(x)|>c\lambda\}} |f(x)|^{p_0} d\mu \; d\lambda \\
&\quad = p(2A_0)^{p_0}\int_X |f(x)|^{p_0}\int_0^{|f(x)|/c} \lambda^{p-1-p_0} d\lambda \; d\mu \\
&\quad = p(2A_0)^{p_0}\int_X |f(x)|^{p_0}\frac{1}{p-p_0}\left(\frac{|f(x)|}{c}\right)^{p-p_0} d\mu \\
&\quad = p(2A_0)^{p_0}\frac{1}{p-p_0}\left(\frac{1}{c}\right)^{p-p_0}\int_X |f(x)|^p d\mu \\
&\quad = p(2A_0)^{p_0}\frac{1}{p-p_0}\left(\frac{1}{c}\right)^{p-p_0}\|f\|_p^p \\
&\quad = \frac{p}{p-p_0}(2A_0)^{p_0}(2A_1)^{p-p_0}\|f\|_p^p \\
&\quad \frac{p}{p-p_0}(2A_0)^{p_0}(2A_1)^{p-p_0} \; \textit{are constant}
\end{aligned}
$$

Case 2: $p_1 < \infty$ Then from the weak inequality, we can have

$$a_{Tf_i}(\lambda/2) \leq (\frac{A_i}{\lambda/2}\left\|f_i\right\|_{p_i})^{p_i} \quad i = 0,1$$

Hence,

$$
\begin{aligned}
&\|Tf\|_p^p\\
&\quad= p\int_0^\infty \lambda^{p-1}a_{Tf}(\lambda)d\lambda\\
&\quad\le p\int_0^\infty \lambda^{p-1}a_{Tf_0}(\lambda)d\lambda + p\int_0^\infty \lambda^{p-1}a_{Tf_1}(\lambda)d\lambda\\
&\quad\le p\int_0^\infty \lambda^{p-1}(\frac{A_0}{\lambda/2}\left\|f_0\right\|_{p_0})^{p_0}d\lambda + p\int_0^\infty \lambda^{p-1}(\frac{A_1}{\lambda/2}\left\|f_1\right\|_{p_1})^{p_1}d\lambda\\
&\quad= p\int_0^\infty \lambda^{p-1-p_0}(2A_0)^{p_0}\int_{\{x:|f(x)|>c\lambda\}}|f(x)|^{p_0}d\mu\, d\lambda\\
&\qquad+ p\int_0^\infty \lambda^{p-1-p_1}(2A_1)^{p_1}\int_{\{x:|f(x)|\le c\lambda\}}|f(x)|^{p_1}d\mu\, d\lambda\\
&\quad= p(2A_0)^{p_0}\int_X |f(x)|^{p_0}\int_0^{|f(x)|/c}\lambda^{p-1-p_0}d\lambda\, d\mu\\
&\qquad+ p(2A_1)^{p_1}\int_X |f(x)|^{p_1}\int_{|f(x)|/c}^{\infty}\lambda^{p-1-p_1}d\lambda\, d\mu\\
&\quad= p(2A_0)^{p_0}\int_X |f(x)|^{p_0}\frac{1}{p-p_0}\left(\frac{|f(x)|}{c}\right)^{p-p_0}d\mu\\
&\qquad+ p(2A_1)^{p_1}\int_X |f(x)|^{p_1}\frac{1}{p-p_1}\left(\frac{|f(x)|}{c}\right)^{p-p_1}d\mu\\
&\quad= p(2A_0)^{p_0}\frac{1}{p-p_0}\left(\frac{1}{c}\right)^{p-p_0}\int_X |f(x)|^p d\mu\\
&\qquad+ p(2A_1)^{p_1}\frac{1}{p_1-p}\left(\frac{1}{c}\right)^{p-p_1}\int_X |f(x)|^p d\mu\\
&\quad= p(2A_0)^{p_0}\frac{1}{p-p_0}\left(\frac{1}{c}\right)^{p-p_0}\|f\|_p^p + p(2A_1)^{p_1}\frac{1}{p_1-p}\left(\frac{1}{c}\right)^{p_1-p}\|f\|_p^p\\
&\quad= \left(\frac{p}{p-p_0}\frac{(2A_0)^{p_0}}{c^{p-p_0}} + \frac{p}{p_1-p}\frac{(2A_1)^{p_1}}{c^{p-p_1}}\right)\|f\|_p^p
\end{aligned}
$$

to make $\left(\frac{p}{p-p_0}\frac{(2A_0)^{p_0}}{c^{p-p_0}} + \frac{p}{p_1-p}\frac{(2A_1)^{p_1}}{c^{p-p_1}}\right)$ *as a constant*
we let $t = (2A_0c)_0^p = (2A_1c)_1^p$
where $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_0}$

$t^{\frac{1}{p}} = t^{\frac{\theta}{p_1}+\frac{1-\theta}{p_0}} = (t^{\frac{1}{p_1}})^{\theta}(t^{\frac{1}{p_0}})^{1-\theta} = (2A_1c)^{\theta}(2A_0c)^{1-\theta}, \quad 0<\theta<1$

$therefore \left(\frac{p}{p-p_0}\frac{(2A_0)^{p_0}}{c^{p-p_0}} + \frac{p}{p_1-p}\frac{(2A_1)^{p_1}}{c^{p-p_1}}\right) = 2p^{1/p}\left(\frac{1}{p-p_0}+\frac{1}{p_1-p}\right)(A_1)^{\theta}(A_0)^{1-\theta}$

We now have

$$\|Tf\|_p^p \leq 2p^{1/p}\left(\frac{1}{p-p_0}+\frac{1}{p_1-p}\right)(A_1)^{\theta}(A_0)^{1-\theta}\,\|f\|_p$$

2.4. **The Hardy-Littlewood maximal function.**

Definition 2.13. (The Hardy-Littlewood maximal function of a locally integrable function f on R^n) Let $b_r = B(0,r)$be the Euclidean ball of radius r centered at the origin, The Hardy-Littlewood maximal function of a locally integrable function f on R^n is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r|}\int_{B_r} |f(x-y)|dy$$

Definition 2.14. (The Hardy-Littlewood maximal function with cubes)
Let $Q_r$be the cube $[-r,r]^n$, define

$$M'f(x) = \sup_{r>0} \frac{1}{(2r)^n}\int_{Q_r} |f(x-y)|dy$$

Proposition 2.15. *When n=1, M and M' coincide; if $n>1$, there exist constants c_n and C_n, depending only on n, such that*

$$c_nM'f(x) \leq Mf(x) \leq C_nM'f(x)$$

This inequality indicate that the two operators M and M' are essentially interchangeable, and the choice depends on the circumstances.

Definition 2.16. (The Hardy-Littlewood maximal function genneral cases)

$$M''f(x) = \sup_{x\in Q} \frac{1}{Q}\int_{Q} |f(y)|dy$$

Theorem 2.17. *The operator is weak (1,1) and strong (p,p) $1<p\leq\infty$.*

The same result is true for M' and M" as the Proposition 2.15 implies.

Proof: It is immediate from the definition that

$$\|Mf\|_\infty \leq \|f\|_\infty$$

$$\|f\|_\infty = \inf\{C \in \mathbb{R}_{\geq 0} : |f(x)| \leq C \text{ for almost every } x\} = \begin{cases} \text{esssup}\,|f| & \text{if } 0 < \mu(S), \\ 0 & \text{if } 0 = \mu(S). \end{cases}$$

As the S has non-zero measure for Mf,

$$\|Mf\|_\infty = \text{esssup}\,|Mf| = \text{esssup}\, \underset{r>0}{sup} \frac{1}{|B_r|} \int_{B_r} |f(x-y)|dy$$

$$\text{esssup}\, \underset{r>0}{sup} \frac{1}{|B_r|} \int_{B_r} |f(x-y)|dy \leq \text{esssup}\, \underset{r>0}{sup} \frac{1}{|B_r|} \int_{B_r} \text{esssup}\,|f(y)|dy = \text{esssup}\,|f(y)| = \|f\|_\infty$$

Then by the Marcinkiewicz interpolation therom, to prove Theorem 2.5 it will be suffice to prove that M is weak (1,1). Here we prove this whne n=1, and the general case will be proved iin section 2.6. In the one-demisional case we need the folling covering lemma.

Lemma 2.18. *Let $\{I_\alpha\}_{\alpha\in A}$ be a collection of intervals in $\mathbb{R}$ and let K be a compact set contained in their union. Then there exists a finite subcollection I_j such that*

$$K \subset \bigcup_j I_j, \quad \textit{and} \quad \sum_j \chi_{I_j}(x) \leq 2$$

If there exist three I_j such that $\chi_{I_j}(x) = 1$, Which means that these three intervals all contain x. Assume that these three intervals are $(a_i, b_i),\ i = 1, 2, 3$

Without loss of generality, assume $a_1 < a_2 < a_3 < x$

Case 1: $x < b_1 < b_2 < b_3$

just with the interval (a_1, b_1) will satisfy all conditions.

Case 2: $x < b_1 < b_3 < b_2$

just with the interval $(a_1, b_1), (a_2, b_2)$ will satisfy all conditions.

case 3: $x < b_2 < b_1 < b_3$

just with the interval $(a_1, b_1), (a_3, b_3)$ will satisfy all conditions.

case 4: $x < b_2 < b_3 < b_1$

just with the interval (a_1, b_1) will satisfy all conditions.

case 5: $x < b_3 < b_1 < b_2$

just with the interval $(a_1, b_1), (a_2, b_2)$ will satisfy all conditions.

case 6: $x < b_3 < b_2 < b_1$

just with the interval (a_1, b_1) will satisfy all conditions.

Proof of Theorem 2.17 when n=1 Let $E_\lambda = \{x \in \mathbb{R} : Mf(x) > \lambda\}$
If x $\in E_\lambda$ then there exists an interval I_x centered at x such that

$$\frac{1}{|I_x|}\int_{I_x} |f| > \lambda$$

Let $K \subset E_\lambda$ be compact. Then $K \subset \cup I_x$, by lemma 2.18 there exists a finite collection$\{I_j\}$ of intervals such that $K \subset \bigcup_j I_j, \quad and \quad \sum_j \chi_{I_j}(x) \leq 2.$

$$|K| \leq \sum_j |I_j| \leq \sum_j \frac{1}{\lambda}\int_{I_j} |f| \leq \frac{1}{\lambda}\sum_j \int_{I_j} |f| \leq \frac{1}{\lambda}\int_R \sum_j \chi_{I_j}|f| \leq \frac{2}{\lambda}\|f\|_1$$

As this inequality is valid for any compact set $K \subset E_\lambda$, the weak (1,1) inequality for M follows immediately.

$$\mu\{x : |Tf| > \lambda\} \leq \frac{c}{\lambda}\|f\|_1$$

This covering lemma is not valid in dimensions greater than 1. So in section 2.6 will use another method to prove the general case.

Proposition 2.19. *Let ϕbe a function which is positive, radical, decreasing(as a function on $(0, \infty)$) and integrable. Then*

$$\sup_{t>0} |\phi_t * f(x)| \leq \|\phi\|_1 Mf(x)$$

Assume in addition by the given hypotheses that ϕ is a simple fucntion, it can be written as

$$\phi(x) = \sum_j a_j \chi_{B_{r_j}}(x)$$

with $a_j > 0$, then

$$\phi * f(x) = \sum_j a_j \chi_{B_{r_j}}(x) * f(x) = \sum_j a_j |B_{r_j}| \frac{1}{|B_{r_j}|} \chi_{B_{r_j}}(x) * f(x)$$

$$\frac{1}{|B_{r_j}|} \chi_{B_{r_j}}(x) * f(x) = \frac{1}{|B_{r_j}|} \int_R \chi_{B_{r_j}}(x) f(x-\tau) d\tau \leq \sup_{t>0} \frac{1}{|B_{r_j}|} \int_R \chi_{B_{r_j}}(x) f(x-\tau) d\tau = Mf(x)$$

$$\phi * f(x) \leq Mf(x) \sum_j a_j |B_{r_j}|$$

As $\|\phi\|_1 = \sum a_j |B_{r_j}|$

$$\phi * f(x) \leq Mf(x) \sum_j a_j |B_{r_j}| = \|\phi\|_1 Mf(x)$$

Any arbitrary ϕ satisfying the hypotheses can be approximated by a sequence of simple functions which increase to this function monotonically. Any dilation ϕ_t is another positive, radical, decreasing function with the same integral, and it will satisfy the same inequality. Thus lead to the conclusion.

Corollary 2.20. *If $|\phi(x)| \leq \varphi(x)$ almost everywhere, where φ is positive, radical, decreasing and integrable, then the maximal function $sup_t|\phi_t * f(x)|$is weak(1,1) and string (p,p), $1 < p \leq \infty$*

Weak(1,1):

$sup_t|\phi_t * f(x)| \leq sup_t \Big| |\phi_t| * |f(x)| \Big| \leq sup_t \Big| |\varphi_t| * |f(x)| \Big| \leq \|\varphi\|_1 M|f(x)| \leq \|\varphi\|_1 Mf(x)$

$Therefore,\ \mu\{|sup_t|\phi_t * f(x)| > \lambda\} \leq \mu\{\|\varphi\|_1 Mf(x) > \lambda\} = \mu\{Mf(x) > \frac{\lambda}{\|\varphi\|_1}\} \leq \left(\frac{\|\varphi\|_1}{\lambda} \|f\|_1\right)^1$

Strong(p,p):

As M is strong(p,p) and the function is bounded by Mf(x), we can derive that it is strong(p,p)

2.5. **The dyadic maximal function.**

Definition 2.21. Dyadic cubes In $\mathbb{R}^n$define the unit cube, open on the right, to be set $[0,1)^n$, and let $\mathcal{Q}_0$be the collection of cubes in $\mathbb{R}^n$ which are congruent to $[0,1)^n$ and whose vertices lie on the lattice $\mathbb{Z}^n$ If dilate this family of cubes by a factor of 2^{-k}, then it is the collection $\mathcal{Q}_k, k \in \mathbb{Z}$
The cubes in $\cup_k \mathcal{Q}_k$ are called the dyadic cubes.

Proposition 2.22. *From this construction, there are three properties:*
(1) Given $x \in \mathbb{R}^n$, there is a unique cube in each family $\mathcal{Q}_k, k \in \mathbb{Z}$ which contains it
(2) any two dyadic cubes are either disjoint or one is wholly contained in the other
(3) a dyadic cube in $\mathcal{Q}_k$ is contained in a unique cube of each family $\mathcal{Q}_j, j < k$, and contains 2^n dyadic cubes of $\mathcal{Q}_{k+1}$

Proof of three properties:
(1) is immediately from the construction that any given k, $\mathcal{Q}_k$ is the whole space and all different cubes in this collection are disjoint as the cubes are right open.
(2) $\forall x, y \in \cup_k \mathcal{Q}_k \quad ((x \cap y = \emptyset) \vee (x \subset y) \vee (y \subset x))$
Proof: Fix $x, y \in \cup_k \mathcal{Q}_k$ Then $\exists i, j\ s.t.\ x \in \mathcal{Q}_i\ and\ y \in \mathcal{Q}_j$
case 1:$i = j \quad x\ and\ y\ are\ different\ so\ disjoint$
case 2: $i \neq j\ Suppose\ i > j,\ \exists z \in \mathcal{Q}_i, y \subseteq z$, Then it goes back to case 1 to discuss whether x and z are disjoint or the same (3) use property1 and property 2 can derive that $\forall k \forall q \in \mathcal{Q}_k (\forall j < k)(\exists q' \in \mathcal{Q}_j\ s.t.\ q \subseteq q'$
And $\mathcal{Q}_k\ should\ be\ [0, 2^{-k})^n, \mathcal{Q}_{k+1}\ should\ be\ [0, 2^{-k-1})^n$

Definition 2.23. Given $f \in L^1_{loc}(\mathbb{R}^n)$,define

$$E_k f(x) = \sum_{Q \in \mathcal{Q}_k} \left[\left(\frac{1}{|Q|} \int_Q f \right) \chi_Q(x) \right]$$

If Ω is the union of cubes in $\mathcal{Q}_k$, then

$$\int_\Omega E_k f = \int_\Omega f$$

$$\begin{aligned}
&\int_\Omega E_k f \\
&= \sum_{P \in \mathcal{Q}_k} \int_P E_k f \quad (P \textit{ are dyadic cubes in the level } k) \\
&= \sum_{P \in \mathcal{Q}_k} \int_P \sum_{Q \in \mathcal{Q}_k} \left[\left(\frac{1}{|Q|} \int_Q f \right) \chi_Q(x) \right] dx \\
&= \sum_{P \in \mathcal{Q}_k, P = Q} \int_P \left[\left(\frac{1}{|Q|} \int_Q f \right) \chi_Q(x) \right] dx + \sum_{P \in \mathcal{Q}_k, P \neq Q} \int_P \left[\left(\frac{1}{|Q|} \int_Q f \right) \chi_Q(x) \right] dx \\
&= \sum_{P \in \mathcal{Q}_k} \int_P \left(\frac{1}{|P|} \int_P f \right) dx \\
&= \sum_{P \in \mathcal{Q}_k} \int_P f dx \\
&= \int_\Omega f
\end{aligned}$$

Definition 2.24. Dyadic maximal function :

$$M_d f(x) = sup_k |E_k f(x)|$$

Theorem 2.25.

(1) The dyadic maximal function is weak (1,1)

(2) If $f \in L^1_{loc}(\mathbb{R}^n), \lim_{k\to\infty} E_k f(x) = f(x) \quad a.e.$

To prove that dyadic maximal function is weak (1,1) $\iff \mu\{x \in \mathbb{R}^n : M_d f(x) > \lambda\} \leq \frac{1}{\lambda}\|f\|_1$

Now let $\{x \in \mathbb{R}^n : M_d f(x) > \lambda\} = \cup_k \Omega_k$

where

$$\Omega_k = \{x \in \mathbb{R}^n : E_k f(x) > \lambda \; and \; E_j f(x) \leq \lambda \; if \; j < k\}$$

Since $f \in L^1, E_k f(x) \to 0 \; as \; k \to -\infty$, therefore for a fixed $\lambda > 0$, the set $\{x \in R^n : M_d f(x) > \lambda\}$ has a lower bound. And for any $j < k$, $E_j f(x) \leq \lambda$.

Thus Ω_k exists and

$$\{x \in \mathbb{R}^n : M_d f(x) > \lambda\} = \cup_k \Omega_k$$

Claim: The set Ω_k are disjoint and each one can be written as the union of cubes in $\mathcal{Q}_k$

Proof: If $x \in \Omega_k \; and \; x \in \Omega_{k'}$

$E_k f(x) > \lambda \; and \; E_j f(x) \leq \lambda \; if \; j < k$

$E_{k'} f(x) > \lambda \; and \; E_j f(x) \leq \lambda \; if \; j < k'$

Therefore, k=k'.The set Ω_k are disjoint.

Hence,

$$|\{x \in \mathbb{R}^n : M_d f(x) > \lambda\}| = |\cup_k \Omega_k| = \sum_k |\Omega_k| \leq \sum_k \frac{1}{\lambda}\int E_k f \quad (\forall x \in \Omega_k \quad E_k f \geq \lambda)$$

$$|\{x \in \mathbb{R}^n : M_d f(x) > \lambda\}| \leq \sum_k \frac{1}{\lambda}\int_{\Omega_k} E_k f = \sum_k \frac{1}{\lambda}\int_{\Omega_k} f \leq \frac{1}{\lambda}\|f\|_1$$

And if f is continuous, it is true that the limit exists. By Theorem 2.7 we can use a family of linear operators such that this limit holds for $f \in L^1$. If $f \in L^1_{loc}$, $f\chi_Q \in L^1 \; \forall Q \in \mathcal{Q}_0$,thus this limit holds for almost every $x \in Q$, thus this limit holds for almost every$x \in \mathbb{R}^n$

Notes: This prove uses a decomposition of $\mathbb{R}^n$

It is called **Calderon-Zygmund decomposition**

Theorem 2.26. *Given a function f which is integrable and non-gative, and given a positive number λ, there exists a sequence $\{Q_j\}$ of disjoint dyadic cubes such that*

$$(1) f(x) \leq \lambda \; for \; almost \; every \; x \notin \cup_j Q_j$$

$$(2) \Big| \cup_j Q_j \Big| \leq \frac{1}{\lambda} \|f\|_1$$

$$(3) \lambda < \frac{1}{|Q_j|} \int_{Q_j} f \leq 2^n \lambda$$

This theorem is talking about a function which has good part that is bounded by λ, as we can see in part (1)

And other bad part that can not be directly bounded is not very large and those cubes can also be bounded in a way.

Similarly in the above theorem, form Ω_k such that $\{x \in \mathbb{R}^n : M_d f(x) > \lambda\} = \cup_k \Omega_k$

where

$$\Omega_k = \{x \in \mathbb{R}^n : E_k f(x) > \lambda \; and \; E_j f(x) \leq \lambda \; if \; j < k\}$$

And each of it can be decomposed into disjoint dyadic cubes in $\mathcal{Q}_k$, thus all these dyadic cubes form the family $\{Q_j\}$

$$\cup_k \Omega_k = \cup_j Q_j$$

If $x \notin \cup_j Q_j$, there does not exist any $k \in \mathbb{R}$ such that $E_k f(x) > \lambda$,so the limit $\lim_{k\to\infty} E_k f(x) \leq \lambda$ and by theroem 2.25,

$$f \in L^1_{loc}(\mathbb{R}^n), \lim_{k\to\infty} E_k f(x) = f(x) \quad a.e.$$

almost every $x \notin \cup_j Q_j, f(x) \leq \lambda$.

Part(2) is the weak (1,1) inequality of Theorem 2.25.

Part(3) : As $x \in \cup_j Q_j, E_k f(x) > \lambda \; for \; the \; k \; that \; Q_j \in \mathcal{Q}_k$.

Thus

$$\lambda < \frac{1}{|Q_j|} \int_{Q_j} f$$

And if we assign $\tilde{Q}_j$ is the cube that has twice long as the sides of Q_j, then by the definition of Ω_k, this $\tilde{Q}_j$ should have property

$$\frac{1}{|\tilde{Q}_j|}\int_{\tilde{Q}_j} f \leq \lambda$$

Therefore,

$$\frac{1}{|Q_j|}\int_{Q_j} f \leq \frac{1}{|Q_j|}\int_{\tilde{Q}_j} f = \frac{|\tilde{Q}_j|}{|Q_j|}\frac{1}{|\tilde{Q}_j|}\int_{\tilde{Q}_j} f = 2^n \frac{1}{|\tilde{Q}_j|}\int_{\tilde{Q}_j} f \leq 2^n\lambda$$

2.6. **The weak $(1,1)$ inequality for the maximal function.**
To prove theorem 2.17

The operator M is weak $(1,1)$ and strong (p,p) $1 < p \leq \infty$.

Though we have already proved it is weak(1,1) when n=1, now we need to prove that it is weak (1,1) in general cases. And to prove

$$|Mf(x) > \lambda| \leq \frac{C}{\lambda}\|f\|_1$$

$$here\ we\ use |\cdot| to\ refer\ to\ the\ measure\ and\ C\ is\ a\ constant$$

Lemma 2.27. *If f is a non-negative function, then*

$$|\{x \in \mathbb{R}^n : M'f(x) > 4^n\lambda\}| \leq 2^n|\{x \in \mathbb{R}^n : M_d f(x) > \lambda\}|$$

As given this lemma and we already proved by the Theorem 2.25 that the dyadic maximal function is weak(1,1).

$$|\{x \in \mathbb{R}^n : M'f(x) > \lambda\}| \leq 2^n|\{x \in \mathbb{R}^n : M_d f(x) > 4^{-n}\lambda\}| \leq \frac{1}{4^{-n}\lambda}\|f\|_1 \cdot 2^n = \frac{8^n}{\lambda}\|f\|_1$$

Thus we can prove that M is weak(1,1).

As previous proof, form the decomposition $\{x \in \mathbb{R}^n : M_d f(x) > \lambda\} = \cup_j Q_j$
where

$$\Omega_k = \{x \in \mathbb{R}^n : E_k f(x) > \lambda \; and \; E_j f(x) \leq \lambda \; if \; j < k\}$$

and Q_j are sequence of disjoint cubes that form the Ω_k $\forall k \in R$ Let $2Q_j$ be the cube with the same center as Q_j and whose slides are twice as long. Then if we prove

$$\{x \in \mathbb{R}^n : M' f(x) > 4^n \lambda\} \subset_j \cup_j 2Q_j$$

We will have $|\{x \in \mathbb{R}^n : M' f(x) > \lambda\}| \leq |\cup_j 2Q_j| = 2^n |\cup_j Q_j| = |\{x \in \mathbb{R}^n : M_d f(x) > \lambda\}|$
So it suffice to show that $\{x \in \mathbb{R}^n : M' f(x) > 4^n \lambda\} \subset_j \cup_j 2Q_j$
And the reason why we need to construct $2Q_j$ like this is that we can prove that for any $x \notin \cup_j 2Q_j, \; M' f(x) \leq 4^n \lambda$
Fix $x \notin \cup_j 2Q_j$,and let Q be any cube that centered at x, the goal is to prove

$$M' f(x) \leq 4^n \lambda$$

which will be suffice to prove that for any Q centered at x, $\frac{1}{|Q|} \int_Q f \leq 4^n \lambda$. Let l denote the side length of Q, and there exists a $k \in F$ such that $2^{k-1} \leq l < 2^k$Then Q intersects with m dyadic cubes in $\mathcal{Q}_{-k}, \; m \leq 2^n$.Denote these dyadic cubes as $R_1, R_2, \ldots, R_m$.
Claim: None of these dyadic cubes is contained in $\cup_j Q_j$
Proof by contradiction: if $\exists Q_j \; s.t. \; R_i \in Q_j$, we can draw the $2Q_j$ with the same center as Q_j, but side length twice as long. From the property (2) of dyadic cubes, and $R_i \in Q_j, \; Q_j$can have the same level as R_i and they are both dyadic cubes so they can be the same dyadic cube. Now $2Q_j$ has length of 2^{k+1} which is larger then twice side length of Q and this$2Q_j$ contains x. Contradiction!
Hence every average value of f insideR_i is not larger than λ.

$$\frac{1}{|Q|} \int_Q f = \frac{1}{|Q|} \sum_{i=1}^{m} \int_{Q \cap R_i} f$$

$$\int_{Q \cap R_i} f \leq \int_{R_i} f \leq \frac{2^{kn}}{|R_i|} \int_{R_i} f \; |R_i| = 2^{kn} \; side \; length \; of \; R_i \; is \; 2^k$$

$$\frac{1}{|Q|}\sum_{i=1}^{m}\int_{Q\cap R_i} f \le \frac{1}{|Q|}\sum_{i=1}^{m}\frac{2^{kn}}{|R_i|}\int_{R_i} f \le \frac{2^{kn}}{|Q|}\sum_{i=1}^{m}\frac{1}{|R_i|}\int_{R_i} f \le 2^n\sum_{i=1}^{m}\frac{1}{|R_i|}\int_{R_i} f \le 2^n\sum_{i=1}^{m}\lambda$$

$$2^n\sum_{i=1}^{m}\lambda = 2^n m\lambda \le 2^n\cdot 2^n\lambda = 4^n\lambda$$

Combine all above we have

$$\frac{1}{|Q|}\int_Q f \le 4^n\lambda$$

Q.E.D.

Corollary 2.28. *(Lebesgue Differentiation Theorem) If* $f \in L^1_{loc}(\mathbb{R}^n)$ *then*

$$lim_{r\to 0^+}\frac{1}{|B_r|}\int_{B_r} f(x-y)dy = f(x)\ a.e.$$

This is a continuous analog of the Theorem 2.25 second part

$$If\ f \in L^1_{loc}(\mathbb{R}^n), \lim_{k\to\infty} E_k f(x) = f(x) \quad a.e.$$

Proof: Recall Theorem 2.7:

Let $\{T_t\}$ be a family of linear operators on $L^p(X,\mu)$ and define $T^*f(x) = \sup_t |T_t f(x)|$. If T^* is weak (p,q), then the set $\{f \in L^p(X,\mu) : \lim_{t\to t_0} T_t f(x) = f(x) a.e.\}$ is closed in $L^p(X,\mu)$.

And M is weak (1,1), M is the T^* in the theorem and the family of linear operators are

$T_r f(x) = \frac{1}{|Br|}\int_{B_r} f(x-y)dy = f(x)$

Thus the set $\{f \in L^1(X,\mu) : \lim_{r\to 0+} T_r f(x) = f(x) a.e.\}$ is closed in $L^1(X,\mu)$.

If $f \in L^1_{loc}(\mathbb{R}^n), f \in the\ set,\ lim_{r\to 0^+}\frac{1}{|B_r|}\int_{B_r} f(x-y)dy = f(x)\ a.e.$

From this corollary 2.28, $|f(x)| \le Mf(x)\ a.e.$ The same is valid if we replace M by $M'\ or M''$

This corollary 2.28 could be made sharper:

$$lim_{r\to 0^+}\frac{1}{|B_r|}\left|\int_{B_r} f(x-y)dy - f(x)\right| = 0\ a.e.$$

As

$$\frac{1}{|B_r|}\left|\int_{B_r} f(x-y)dy\right| \leq Mf(x)$$

$$\frac{1}{|B_r|}|f(x)|dy = |f(x)|$$

$$\frac{1}{|B_r|}\left|\int_{B_r} f(x-y)dy - f(x)\right| \leq \frac{1}{|B_r|}\left|\int_{B_r} f(x-y)dy\right| + \frac{1}{|B_r|}|f(x)|dy \leq Mf(x) + |f(x)|$$

Here is a proof from *Real Analysis: Measure Theory, Integration, and Hilbert Spaces*

From the corollary 2.28, for each rational r, there exists a setE_r of measure zero, such that

$$lim_{r\to 0^+}\frac{1}{|B_r|}\int_{B_r} |f(x-y)-r|dy = |f(x)-r| \; \forall x \notin E_r$$

And let $E = \cup E_r, \forall r \in \mathbb{Q}$ measure of E is 0.

Now suppose $\hat{x} \notin E$ and $f(\hat{x})$ is finite, then for any given ϵ, there exists r such that $|f(\hat{x})-r| < \epsilon$

As

$$|f(\hat{x}-y) - f(\hat{x})| \leq |f(\hat{x}-y)-r| + |f(\hat{x})-r|$$

Thus,

$$\frac{1}{|B_r|}\int_{B_r} |f(\hat{x}-y) - f(\hat{x})|dy \leq \frac{1}{|B_r|}\int_{B_r} |f(\hat{x}-y)-r|dy + |f(\hat{x})-r|$$

Add limit to both sides:

$$lim_{r\to 0^+}\frac{1}{|B_r|}\left|\int_{B_r} f(x-y)dy - f(x)\right| \leq lim_{r\to 0^+}\frac{1}{|B_r|}\int_{B_r} |f(\hat{x}-y)-r|dy + |f(\hat{x})-r| = 2|f(\hat{x})-r|$$

Thus we have

$$lim_{r\to 0^+}\frac{1}{|B_r|}\left|\int_{B_r} f(x-y)dy - f(x)\right| \leq 2\epsilon$$

Therefore, this $\hat{x}$ is an Lebesgue point.

The stronger assertion is that the right hand side tends to zero for almost every point x.

The points x for which this is true are called the **Lebesgue points** of f.

If x is a Lebesgue point and if $\{B_j\}$ is a sequence of balls such that

$$B_1 \supset B_2 \supset B_3 \supset \ldots$$

and $\cap B_j = \{x\}$, the center of balls need not be x. And then

$$\lim_{j\to\infty} \frac{1}{|B_j|} \int_{B_j} f = f(x)$$

Proof: As $B_j \subset B(x, 2r_j)$,where r_j is the side length of B_j. Then these balls $B(x, 2r_j)$ are the sequence of balls in the stronger aversion

$$lim_{r\to 0^+} \frac{1}{|B_r|} \left| \int_{B_r} f(x-y)dy - f(x) \right| = 0 \; a.e.$$

A similar argument shows that the set of Lebesgue points of f does not change if we take cubes instead of balls. The weak (1,1) inequality for M is a substitute for the strong (1,1) inequality, which is false. In fact, it never holds, as the following result shows.

Proposition 2.29. *If $f \in L_1$ and is not identically 0, then $Mf \notin L_1$.*

The proof is simple: since f is not identically 0, there exists$R > 0$ such that

$$\int_{B_R} |f| \geq \epsilon > 0$$

Now if $|x| > R, B_R \subset B(x, 2|x|)$, so

$$Mf(x) \geq \frac{1}{|2x|^n} \int_{B_R} |f| \geq \frac{\epsilon}{2^n |x|^n}$$

Therefore, $Mf \notin L_1$.

As

$$Mf(x) \geq \frac{\epsilon}{2^n |x|^n}$$

And the level is n so the power of $\frac{1}{|x|}$ should have more than n. An easy example to prove

$$\int_{\mathbb{R}^n} \frac{1}{|x|^n} dx_1 \dots dx_n = \int \int_R \frac{1}{|x|^n} |x|^{n-1} d|x| d\Omega = \int \int_R \frac{1}{|x|} d|x| d\Omega = \int log(|x|) \Big|_{x=0}^{\infty} d\Omega$$

This is not integrable.

Theorem 2.30. *If B is a bounded subset of $\mathbb{R}^n$, then*

$$\int_B Mf \leq 2|B| + C\int_{\mathbb{R}^n} |f|log^+|f|$$

where $log^+t = max(logt, 0)$

Recall

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r|}\int_{B_r} |f(x-y)|dy$$

$$\int_B Mf \leq \int_0^\infty |\{x \in B : Mf(x) > 2\lambda\}|d(2\lambda) = 2\int_0^\infty |\{x \in B : Mf(x) > 2\lambda\}|d\lambda$$

$\int_0^\infty |\{x \in B : Mf(x) > 2\lambda\}|d\lambda = \int_0^1 |\{x \in B : Mf(x) > 2\lambda\}|d\lambda$
$+ \int_1^\infty |\{x \in B : Mf(x) > 2\lambda\}|d\lambda$

$$\int_0^1 |\{x \in B : Mf(x) > 2\lambda\}|d\lambda \leq \int_0^1 |\{x \in B\}|d\lambda = |B|$$

Decompose f as $f_1 + f_2$, where $f_1 = f\chi_{\{x:|f(x)|>\lambda\}}$and $f_2 = f\chi_{\{x:|f(x)|\leq\lambda\}}$
As M is a sublinear operator,

$$Mf \leq Mf_1 + Mf_2$$

Besides

$$Mf_1 > Mf_2 \,\forall x, \quad 2\lambda < Mf(x) \leq Mf_1 + Mf_2 \leq 2Mf_1$$

Therefore,

$$\{x \in B : Mf(x) > 2\lambda\} \subset \{x \in B : Mf_1(x) > \lambda\}$$

Thus,

$$\int_1^\infty |\{x \in B : Mf(x) > 2\lambda\}|d\lambda \leq \int_1^\infty |\{x \in B : Mf_1(x) > \lambda\}|d\lambda$$

By the weak (1,1) of M,

$$|\{x \in B : Mf_1(x) > \lambda\}| \leq \frac{C}{\lambda}\|f_1\|_1 = \frac{C}{\lambda}\int_{\{x:|f(x)|>\lambda\}} |f(x)|dx$$

$$\begin{aligned}
&\int_1^\infty |\{x \in B : Mf_1(x) > \lambda\}| d\lambda \\
&\qquad \leq \int_1^\infty \frac{C}{\lambda}\int_{\{x:|f(x)|>\lambda\}} |f(x)|dxd\lambda \\
&\qquad = C\int_1^\infty \frac{1}{\lambda}\int_{\{x:|f(x)|>\lambda\}} |f(x)|dxd\lambda \\
&\qquad = C\int_{\mathbb{R}^n} |f(x)| \int_1^{max(|f(x)|,1)} \frac{1}{\lambda} d\lambda\, dx \\
&\qquad = C\int_{\mathbb{R}^n} |f(x)| log^+ |f(x)| dx
\end{aligned}$$

Therefore,

$$\int_B Mf \leq 2\int_0^\infty |\{x \in B : Mf(x) > 2\lambda\}| d\lambda \leq 2|B| + 2C\int_{\mathbb{R}^n} |f(x)| log^+ |f(x)| dx$$

Q.E.D.

3. The Hilbert Transform

3.1. The conjugate Poisson kernel. Given a function f in S(R), its harmonic extension to the upper half-plane is given by $u(x,t) = P_t * f(x)$,where P_t is the Poisson kernel:

$$P_t(x) = \frac{1}{\pi}\frac{t}{x^2+t^2} \; when\; n = 1$$

$$P_t(x) = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}\frac{t}{(|x|^2+t^2)^{\frac{n+1}{2}}} \; when\; n > 1$$

To prove that

$$\int_R \hat{f}(\xi)e^{-2\pi|\xi|t}e^{2\pi i\times\xi}d\xi = \int_R f(x-\tau)P_t(\tau)d\tau$$

we know

$$z = x + it \quad P_t(x) = \frac{1}{\pi}\frac{t}{t^2+x^2} \text{ (Poisson kernel)}$$

and

$$(1) \int_0^\infty e^{2\pi isz}d\xi = \frac{i}{2\pi z} \text{ if } lm(z) > 0 (t > 0).$$

claim:

$$(2) \int_R e^{-2\pi|\xi|t}e^{2\pi i\xi x}d\xi = \frac{1}{\pi}\frac{t}{t^2+x^2}$$

$$\begin{aligned} proof: \int_0^\infty e^{-2\pi\xi t}e^{2\pi i\xi x}d\xi + \int_{-\infty}^0 e^{2\pi\xi t}e^{2\pi i\xi x}d\xi \\ = \int_0^\infty e^{2\pi iz\xi}d\xi + \int_{-\infty}^0 e^{2\pi i\bar{z}\xi}d\xi \\ = \frac{i}{2\pi z} - \frac{i}{2\pi\bar{z}} \\ = \frac{i}{2\pi}\left(\frac{1}{z} - \frac{1}{\bar{z}}\right) = \frac{i}{2\pi}\frac{-2ti}{t^2+x^2} = \frac{t}{t^2+x^2}\cdot\frac{1}{\pi} \end{aligned}$$

and $\hat{f}(\xi) = \int_R f(\tau)e^{-2\pi i\xi\cdot\tau}d\tau$. Let $\Phi(\tau,\xi) = f(\tau)e^{-2\pi i\xi\cdot\tau}e^{-2\pi|\xi|t}e^{2\pi ix\xi}$
$f \in S(\mathbb{R})$, Φ is integrable over R^2.

By Fubini's Theorem.

$$\int_R\left(\int_R \Phi(\tau,\xi)d\tau\right)d\xi = \int_R\left(\int_R \Phi(\tau,\xi)d\xi\right)d\tau$$

$$\text{LHS } = \int_R \left(\int_R f(\tau) e^{-2\pi i \xi \tau} e^{-2\pi|\xi|} e^{2\pi i x \xi} d\tau \right) d\xi$$
$$= \int_R \left(\int_R f(\tau) e^{-2i\xi\cdot\tau} d\tau \right) e^{-2\pi|\xi|t} e^{2\pi i x \xi} d\xi$$
$$= \int_R \hat{f}(\xi) e^{-2\pi|\xi|t} e^{2\pi i x \xi} d\xi$$
$$\text{RHS } = \int_R \left(\int_R f(\tau) e^{-2\pi i \xi} e^{-2\pi|\xi|t} e^{2\pi i \times \xi} d\xi \right) d\tau$$
$$= \int_R f(\tau) \left(\int_R e^{-2\pi|\xi|t} e^{2\pi i \xi(x-\tau)} d\xi \right) d\tau$$
$$= \int_R f(\tau) P_t(x-\tau) d\tau$$
$$= f * P_t(x)$$
$$= P_t * f(x)$$
$$= \int_R f(x-\tau) \cdot P_t(\tau) d\tau$$
$$u(x,t) = P_t * f(x) = \int f(x-\tau) P_t(\tau) d\tau$$
$$u(x,t) = \int_{\mathbb{R}} e^{-2\pi t \cdot |\xi|} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi$$

---

let $z = x + it$,

$$u(z) = \int_0^\infty \hat{f}(\xi) e^{2\pi i z \cdot \xi} d\xi + \int_{-\infty}^0 \hat{f}(\xi) e^{2\pi i \bar{z} \cdot \xi} d\xi$$

Define

$$iv(z) = \int_0^\infty \hat{f}(\xi) e^{2\pi i z \cdot \xi} d\xi - \int_{-\infty}^0 \hat{f}(\xi) e^{2\pi i \bar{z} \cdot \xi} d\xi$$

then v is also harmonic in $\mathbb{R}^2_+$ and both u and v are real if f is. Furthermore, $u + iv$ is analytic, so v is the harmonic conjugate of u.

Clearly, v can also be written as

$$v(z) = \int_{\mathbb{R}} -i \cdot sgn(\xi) e^{-2\pi t \cdot |\xi|} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi$$

is equivalent to

$$v(x,t) = Q_t * f(x),$$

where

$$\hat{Q}_t(\xi) = -i \cdot sgn(\xi) e^{-2\pi t \cdot |\xi|}$$

invert the Fourier transform we get

$$Q_t(x) = \frac{1}{\pi}\frac{x}{x^2+t^2} \quad whereas\ P_t(x) = \frac{1}{\pi}\frac{t}{x^2+t^2}$$

This Q_t is the conjugate Poisson kernel. Immediately verify $Q(x,t) = Q_t(x)$ is a harmonic function in the upper half-plane and the conjugate of the Poisson kernel $P_t(x)$. More precisely,

$$P_t(x) + iQ_t(x) = \frac{1}{\pi}\frac{t+ix}{x^2+t^2} = \frac{i}{\pi z}$$

which is analytic in $lm(z) = t > 0$ In Chapter 2 we studied the limit as $t \to 0\ of\ u(x,t)$ using the fact that $\{P_t\}$ is an approximation of the identity. We would like to do the same for v(x, t), but we immediately run into an obstacle: $\{Q_t\}$is not an approximation of the identity and, in fact, Qt is not integrable for any $t > 0$. Formally,

$$\lim_{t\to 0} Q_t(x) = \frac{1}{\pi x}$$

this is not even locally integrable, so it is impossible to define its convolution with smooth functions.

3.2. **The principal value of l/x.**

Definition 3.1. a tempered distribution called the principal value of l/x,abbreviated p. v. l/x,

$$p.v.\frac{1}{x}(\phi) = \lim_{\epsilon\to 0}\int_{|x|>\epsilon}\frac{\phi(x)}{x}dx,\ \phi \in S$$

rewrite it as

$$p.v.\frac{1}{x}(\phi) = \int_{|x|<1}\frac{\phi(x)-\phi(0)}{x}dx + \int_{|x|>1}\frac{\phi(x)}{x}dx$$

$$
\begin{aligned}
&p.v.\frac{1}{x}(\phi)\\
&\qquad = \lim_{\epsilon\to 0}\int_{|x|>\epsilon}\frac{\phi(x)}{x}dx\\
&\qquad = \lim_{\epsilon\to 0}\int_{\epsilon<|x|<1}\frac{\phi(x)}{x}dx + \int_{1<|x|}\frac{\phi(x)}{x}dx\\
&\qquad = \lim_{\epsilon\to 0}\int_{\epsilon<|x|<1}\frac{\phi(x)}{x}dx + \int_{1<|x|}\frac{\phi(x)}{x}dx
\end{aligned}
$$

$\lim_{\epsilon\to 0}\int_{\epsilon<|x|<1}\frac{\phi(0)}{x}dx = \lim_{\epsilon\to 0}\int_{\epsilon<x<1}\frac{\phi(0)}{x}dx + \lim_{\epsilon\to 0}\int_{-1<x<-\epsilon}\frac{\phi(0)}{x}dx$
$= \lim_{\epsilon\to 0}\phi(0)\left(log(x)\Big|_{\epsilon}^{1} - log(-x)\Big|_{-1}^{-\epsilon}\right) = 0$
Then

$$\lim_{\epsilon\to 0}\int_{\epsilon<|x|<1}\frac{\phi(x)}{x}dx = \int_{|x|<1}\frac{\phi(x)-\phi(0)}{x}dx$$

Therefore,

$$p.v.\frac{1}{x}(\phi) = \int_{|x|<1}\frac{\phi(x)-\phi(0)}{x}dx + \int_{|x|>1}\frac{\phi(x)x}{x^2}dx$$

Therefore,

$$\left|p.v.\frac{1}{x}(\phi)\right| \le C(\|\phi'\|_\infty + \|x\phi\|_\infty)$$

Proposition 3.2. *In S', $\lim_{t\to 0} Q_t = \frac{1}{\pi}\ p.v.\ \frac{1}{x}$*

For each $\epsilon > 0$, the function $\psi_\epsilon(x) = x^{-1}\chi_{\{|x|>\epsilon\}}$ are bounded and define tempered distributions. It follows at once from the definition that in S'

$$\lim_{\epsilon\to 0}\psi_\epsilon = \ p.v.\ \frac{1}{x}$$

It will be suffice to show that in S'

$$\lim_{t\to 0}(Q_t - \frac{1}{\pi}\psi_t) = 0$$

Fix $\phi \in S$;Then

$$\begin{aligned}
&(\pi Q_t - \psi_t)(\phi) \\
&= \int_{\mathbb{R}} \frac{x\phi(x)}{t^2+x^2} - \int_{|x|>t} \frac{\phi(x)}{x} dx \\
&= \int_{|x|\le t} \frac{x\phi(x)}{t^2+x^2} dx + \int_{|x|>t} \left(\frac{x\phi(x)}{t^2+x^2} - \frac{\phi(x)}{x} \right) dx \\
&= \int_{|y|\le 1} \frac{y\phi(ty)}{1+y^2} dy + \int_{|y|>1} \left(\frac{y}{1+y^2} - \frac{1}{y} \right) \phi(ty) dy \quad (y = \frac{x}{t}) \\
&= \int_{|y|\le 1} \frac{y\phi(ty)}{1+y^2} dy - \int_{|y|>1} \frac{\phi(ty)}{(1+y^2)y} dy
\end{aligned}$$

Take the limit with $t \to 0$ and apply the dominated convergence theorem,

$$\begin{aligned}
&= \int_{|y|\le 1} \lim_{t\to 0} \frac{y\phi(ty)}{1+y^2} dy - \int_{|y|>1} \lim_{t\to 0} \frac{\phi(ty)}{(1+y^2)y} dy \\
&= \int_{|y|\le 1} \frac{y\phi(0)}{1+y^2} dy - \int_{|y|>1} \frac{\phi(0)}{(1+y^2)y} dy \\
&= 0
\end{aligned}$$

we get two integrals of odd functions on symmetric domains. Hence, the limit equals 0. It converges to 0 in S'

Here we have the definition for operator convolution with function:

For $F_n \to F$ in $S' \Leftrightarrow \forall f \in S, \langle F_n, f\rangle \to \langle F, f\rangle$ Here $F \in S', F$ is an operator For $\psi \in C_c^\infty$,

$$\begin{aligned}
&F * \psi(x) = \int F(y)\psi(x-y)dy \\
&= \left\langle F, \tau_x\tilde{\psi}\right\rangle \\
&\text{where } \tau_x = C_c^\infty \to C_e^\infty \quad \tau_x f(y) = f(y-x) \\
&\sim: C_c^\infty \to C_c^\infty \quad \tilde{f}(y) = f(-y) \\
&\left(\tau_x\tilde{\psi}\right)(y) = \tau_x(\tilde{\psi}(y)) = \tau_x\psi(-y) = \psi(-(y-x)) = \psi(x-y)
\end{aligned}$$

Then $\lim_{t\to 0} Q_t = p\cdot v\cdot \frac{1}{x}$, these two operators are equal, as for convolution

$$\begin{aligned}
&\lim_{t\to 0} Q_t * f = p\cdot v\cdot \frac{1}{x} * f = p\cdot v\cdot \frac{1}{x}\left(\tau_x f(y)\right) = p\cdot v\cdot \frac{1}{x}(f(x-y)) \\
\Rightarrow &\lim_{t\to 0} Q_t * f = \lim_{t\to 0} \frac{1}{\pi}\int_{|y|>\epsilon} \frac{f(x-y)}{y}dy
\end{aligned}$$

As a consequence of this proposition

$$\lim_{t\to 0} Q_t * f(x) = \left(\frac{1}{\pi}\, p.v.\, \frac{1}{x}\right) * f(x) = \lim_{\epsilon\to 0}\frac{1}{\pi}\int_{|y|>\epsilon}\frac{f(x-y)}{y}dy$$

By the continuity of the Fourier transform on S',

$$\widehat{\left(\frac{1}{\pi}\, p.v.\,\right)}(\xi) = -i\cdot sgn(\xi)$$

Definition 3.3. Given a function $f \in S$, define **Hilbert transform** by anyone of the following equivalent expressions:

$$\begin{aligned}
&Hf = \lim_{t\to 0} Q_t * f \\
&Hf = \frac{1}{\pi}\, p.v.\, \frac{1}{x} * f(x) \\
&\widehat{(Hf)}(\xi) = -i\cdot sgn(\xi)\hat{f}(\xi)
\end{aligned}$$

The third expression allows to define the Hilbert transform of functions in $L^2(\mathbb{R})$; it satisfies

$$\|Hf\|_2 = \|f\|_2$$
$$H(Hf) = -f$$
$$\int Hf \cdot g = -\int f \cdot Hg$$

$$\widehat{H(Hf)} = \widehat{H(-ign(\xi)f)} = (-i\,\mathrm{sgn}(\xi))^2 \hat{f} = -\hat{f}$$
$$\Rightarrow H(Hf) = -f$$

As

$$\langle f, g\rangle = \int f \cdot \bar{g}$$
$$\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle$$
$$\text{Then } \langle Hf, g\rangle = \langle \hat{H}f, \widehat{g}\rangle = \langle -i \cdot \mathrm{sgn}(\xi)\hat{f}, \hat{g}\rangle$$
$$= -i \cdot \mathrm{sgn}(\xi)\langle \hat{f}, \hat{g}\rangle$$
$$= -i \cdot \mathrm{sgn}(\xi)(\overline{-i \cdot \mathrm{sgn}(())})^{-1}\langle \hat{f}, \widehat{Hg}\rangle$$
$$= -\langle \hat{f}, \widehat{Hg}\rangle$$
$$= -\langle f, Hg\rangle$$

3.3. **The theorems of M. Riesz and Kolmogorov.** The Hilbert transform, now defined for functions in S or $L^2(\mathbb{R})$, can be extended to functions in $L^P, 1 < p < \infty$,

Theorem 3.4. *For $f \in S(\mathbb{R})$ the following assertions are true:*
(1) (Kolmogorov) H is weak (1,1)

$$|\{x \in \mathbb{R} : |Hf(x)| > \lambda\}| \leq \frac{c}{\lambda}\|f\|_1$$

(2) (M. Riesz) H is strong (p,p),$1 < p < \infty$

$$\|Hf\|_p \leq C_p \|f\|_p$$

Proof for (1) Kolmogov:

Fix $\lambda > 0$ and f non-negative. Form the Calderon-Zygmund decomposition of f at height λ, this yields a sequence of disjoint intervals $\{I_j\}$such that

$$(a) f(x) \leq \lambda \ for \ a.e. \ x \notin \Omega = \cup_j I_j$$

$$(b) |\Omega| \leq \frac{1}{\lambda} \|f\|_1$$

$$(c) \lambda < \frac{1}{|I_j|} \int_{I_j} f \leq 2\lambda \ as \ n = 1$$

Align with this decomposition of $\mathbb{R}$, we now decompose f as the sum of two functions, g and b, defined by

$$g(x) = \begin{cases} f(x) & if \ x \notin \Omega \\ \frac{1}{|I_j|} \int_{I_j} f & if \ x \in \Omega \end{cases}$$

$$b(x) = \sum_j b_j(x)$$

where

$$b_j(x) = \left(f(x) - \frac{1}{|I_j|} \int_{I_j} f \right) \chi_{I_j}(x)$$

Validation:

If $x \notin \Omega$,g(x) = f(x) and $b_j(x) = \left(f(x) - \frac{1}{|I_j|} \int_{I_j} f \right) \chi_{I_j}(x) = 0 \ \forall j$, thus f(x) = g(x) +b(x).

If $x \in \Omega$, g(x) = $\frac{1}{|I_{j_x}|} \int_{I_{j_x}} f$ and $b_j(x) = \left(f(x) - \frac{1}{|I_j|} \int_{I_j} f \right) \chi_{I_j}(x) = 0 \ \forall j \neq j_x \ where x \in I_{j_x}$ and $b_{j_x}(x) = \left(f(x) - \frac{1}{|I_{j_x}|} \int_{I_{j_x}} f \right) \chi_{I_{j_x}}(x) = f(x) - \frac{1}{|I_{j_x}|} \int_{I_{j_x}} f \quad for \ x \in I_{j_x}$

thus $b(x) = \sum_j b_j(x) = b_{i_x}(x) = f(x) - \frac{1}{|I_{j_x}|} \int_{I_{j_x}} f$ then f(x) = g(x) +b(x)

And by (a) and (c) , we have $g(x) \leq 2\lambda$ *a.e.*
and b_j is supported on I_j and has zero integral:

$$\int_{I_j} b_j(x) = \int_{I_j}\left(f(x) - \frac{1}{|I_j|}\int_{I_j} f\right)\chi_{I_j}(x)dx = \int_{I_j}\left(f(x) - \frac{1}{|I_j|}\int_{I_j} f\right)dx = 0$$

Here $Hf = H(g+b) = Hg + Hb$,

$$|\{x \in \mathbb{R} : |Hf(x)| > \lambda\}| \leq |\{x \in \mathbb{R} : |Hg(x)| > \lambda/2\}| + |\{x \in \mathbb{R} : |Hb(x)| > \lambda/2\}|$$

Firt term can be estimated by

$$\begin{aligned}
&|\{x \in \mathbb{R} : |Hg(x)| > \lambda/2\}| \\
&\qquad \leq \left(\frac{2}{\lambda}\right)^2 \int_{\mathbb{R}} |Hg(x)|^2 dx \quad \textit{Chebshev Inequality} \\
&\qquad = \left(\frac{4}{\lambda^2}\right) \int_{\mathbb{R}} g(x)^2 dx \qquad as\ \|Hg\|_2 = \|g\| \\
&\qquad \leq \frac{8}{\lambda}\int_{\mathbb{R}} g(x)dx \qquad as\ g(x) \leq 2\lambda \\
&\qquad = \frac{8}{\lambda}\int_{\mathbb{R}} f(x)dx \qquad as\ \|f\|_1 = \|g\|_1 + \|b\|_1 = \|g\|_1
\end{aligned}$$

Now we estimate the second term: Let $2I_j$ be the interval with the same center as I_j and twice the length, and let $\Omega^* = \cup_j 2I_j$ Then $|\Omega^*| \leq 2|\Omega|$ and by the (b) of this decomposition that $|\Omega| \leq \frac{1}{\lambda}\|f\|_1$ we have $|\Omega^*| \leq 2\frac{1}{\lambda}\|f\|_1$ Then

$$\begin{aligned}
&|\{x \in \mathbb{R} : |Hb(x)| > \lambda/2\}| \\
&\qquad \leq |\Omega^*| + |\{x \notin \Omega^* : |Hb(x)| > \lambda/2\}| \\
&\qquad \leq 2\frac{1}{\lambda}\|f\|_1 + |\{x \notin \Omega^* : |Hb(x)| > \lambda/2\}| \\
&\qquad \leq 2\frac{1}{\lambda}\|f\|_1 + \int_{\mathbb{R}/\Omega^*} |Hb(x)|dx
\end{aligned}$$

If the sum is finite, immediately $|Hb(x)| < \sum_j |Hb_j(x)|$ almost everywhere.
If the sum is not finite, as $\sum b_j$ and $\sum Hb_j$ converges to b and Hb in L^2

To show that H is weak(1,1), it will suffice to show that

$$\sum_j \int_{\mathbb{R}/2I_j} |Hb_j(x)|dx \leq C\|f\|_1$$

Though $b_j \notin S$,when $x \notin 2I_j$,

$$Hb_j(x) = \int_{I_j} \frac{b_j(y)}{x-y}dy$$

This formula is still valid. Denote the center of I_j as c_j

$$\begin{aligned}
&\int_{\mathbb{R}/2I_j} |Hb_j(x)|dx \\
&\qquad = \int_{\mathbb{R}/2I_j} \left| \int_{I_j} \frac{b_j(y)}{x-y}dy \right| dx \\
&\qquad = \int_{\mathbb{R}/2I_j} \left| \int_{I_j} b_j(y)\Big(\frac{1}{x-y} - \frac{1}{x-c_j}\Big)dy \right| dx \quad as \int b_j(y)dy = 0 \\
&\qquad = \int_{\mathbb{R}/2I_j} \left| \int_{I_j} b_j(y)\frac{y-c_j}{(x-y)(x-c_j)}dy \right| dx \\
&\qquad \leq \int_{\mathbb{R}/2I_j} \int_{I_j} \left| b_j(y)\frac{y-c_j}{(x-y)(x-c_j)} \right| dydx \\
&\qquad = \int_{I_j} |b_j(y)| \int_{\mathbb{R}/2I_j} \left| \frac{y-c_j}{(x-y)(x-c_j)} \right| dxdy
\end{aligned}$$

As

$$|y-c_j| \leq \frac{1}{2}|I_j| \leq |x-y|$$

And

$$|x-c_j| = |x-y+y-c_j| \leq |x-y| + |y-c_j| \leq 2|x-y|$$

thus

$$\frac{|y-c_j|}{|x-y|} \leq \frac{|I_j|}{|x-c_j|}$$

$$\begin{aligned}
&\int_{\mathbb{R}/2I_j} |Hb_j(x)|dx \\
&\qquad \le \int_{I_j} |b_j(y)| \int_{\mathbb{R}/2I_j} \left| \frac{y-c_j}{(x-y)(x-c_j)} \right| dxdy \\
&\qquad \le \int_{I_j} |b_j(y)| \int_{\mathbb{R}/2I_j} \frac{|I_j|}{|x-c_j|^2} dxdy
\end{aligned}$$

$$\begin{aligned}
&\int_{\mathbb{R}/2I_j} \frac{|I_j|}{|x-c_j|^2} dx \\
&\qquad = \int_{|I_j|}^{\infty} y^{-2} dy + \int_{-\infty}^{-|I_j|} y^{-2} dy \\
&\qquad = -|I_j| \Big(\frac{1}{y} \Big|_{|I_j|}^{\infty} + \frac{1}{y} \Big|_{-\infty}^{-|I_j|} \Big) \\
&\qquad = -|I_j| \left(0 - \frac{1}{|I_j|} + \frac{1}{-|I_j|} - 0 \right) \\
&\qquad = 2
\end{aligned}$$

$$\begin{aligned}
&\int_{\mathbb{R}/2I_j} |Hb_j(x)|dx \\
&\qquad le \int_{I_j} |b_j(y)| \int_{\mathbb{R}/2I_j} \frac{|I_j|}{|x-c_j|^2} dxdy \\
&\qquad = \int_{I_j} |b_j(y)| 2dy
\end{aligned}$$

And

$$\begin{aligned}
&\int_{I_j} 2|b_j(y)|dy \\
&\qquad = \int_{I_j} 2\left|(f(y) - \frac{1}{|I_j|}\int_{I_j} f)\chi_{I_j}(y)\right|dy \\
&\qquad = \int_{I_j} 2\left|f(y) - \frac{1}{|I_j|}\int_{I_j} f\right|dy \\
&\qquad \leq \int_{I_j} 2|f(y)|dy + \int_{I_j} 2\left|\frac{1}{|I_j|}\int_{I_j} f\right|dy \\
&\qquad \leq 4\|f\|_1
\end{aligned}$$

Therefore,

$$\sum_j \int_{\mathbb{R}/2I_j} |Hb_j(x)|dx \leq \sum_j \int_{I_j} 2|b_j(y)|dy \leq 4\|f\|_1$$

Here proof of the weak (1,1) inequality is for non-negative j, but this is sufficient since an arbitrary real function can be decomposed into its positive and negative parts, and a complex function into its real and imaginary parts.

Proof for (2): Since H is weak (1,1) and strong (2,2),as $\|Hf\|_2 = \|f\|_2$, by the Marcinkiewicz interpolation theorem we have the strong (p,p) inequality for$1 < p < 2$

If $p > 2$ apply $\int Hf \cdot g = -\int f \cdot Hg$

and the result for $p < 2$: (here $p > 2$ *and* $\frac{1}{p} + \frac{1}{p'} = 1$ thus $1 < p' < 2$)

From the strong(p', p'), we have

$$|\{x \in \mathbb{R} : |Hg(x)| > \lambda\}| \leq C_{p'}\|g\|_{p'}$$

$$\|Hg(x)\|_{p'} = lim_{\lambda\to 0}|\{x \in \mathbb{R} : |Hg(x)| > \lambda\}| \leq C_{p'}\|g\|_{p'}$$

And provided that $f \in L_p(R^n)$ for each $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$

$$\|f\|_{L^p} = \sup\left\{\left|\int_{\mathbf{R}^n} f(x)g(x)\ dx\right| : \|g\|_q \leq 1\right\} = \sup\left\{\int_{\mathbf{R}^n} |f(x)g(x)|\ dx : \|g\|_q \leq 1\right\}$$

Proof need or not? https://math.stackexchange.com/questions/2594571/f-lp-sup-left-int-bf-rnfxgxdx-g-lq-le1-righ

$$
\begin{aligned}
&\|Hf\|_p \\
&\quad = sup\left\{\left|\int_R Hf \cdot g\right| : \|g\|_{p'} \leq 1\right\} \\
&\quad = sup\left\{\left|\int_R f \cdot Hg\right| : \|g\|_{p'} \leq 1\right\} \\
&\quad \leq \|f\|_p \, sup\{\|Hg\|_{p'} : \|g\|_{p'} \leq 1\} \\
&\quad \leq C_{p'} \|f\|_p \|g\|_{p'} \ where \ (\|g\|_{p'} \leq 1) \\
&\quad \leq C_{p'} \|f\|_p
\end{aligned}
$$

The functions g and b in the proof of the first part of this theorem are traditionally referred to as the good and bad parts of f.

It follows from these proofs that the constants in the strong (p,p) and (p',p') inequalities coincide and tend to infinity as p tends to 1 or infinity. More precisely,

$$C_p = O(p) \quad \text{as} \quad p \to \infty, \quad \text{and} \quad C_p = O\left((p-1)^{-1}\right) \quad \text{as} \quad p \to 1.$$

By using the inequalities in Theorem 3.4 we can extend the Hilbert transform to functions in $L^p, 1 \leq p < \infty$. If $f \in L^1$ and $\{f_n\}$ is a sequence of functions in $\mathcal{S}$ that converges to f in L^1 (i.e. $\lim \|f_n - f\|_1 = 0$), then by the weak $(1,1)$ inequality the sequence $\{Hf_n\}$ is a Cauchy sequence in measure: for any $\epsilon > 0$,

$$\lim_{m,n\to\infty} |\{x \in \mathbb{R} : |(Hf_n - Hf_m)(x)| > \epsilon\}| = 0.$$

$$|\{x \in R : |Hf(x)| > \lambda\}| \leqslant \frac{c}{\lambda}\|f\|_1$$

$$\Rightarrow |\{x \in R : |H(f_n - f_m)(x)| > \epsilon\}| \leqslant \frac{c}{\epsilon}\|f_n - f_m\|_1$$

Weak $(1,1)$ inequality H is a linear operator:

$$|\{x \in R : |(Hf_n + Hf_m)(x)| > \epsilon\}| = |\{x \in R : |H(f_n - f_m)(x)| > \in\}|$$

$$\leqslant \frac{c}{\epsilon}\|f_n - f_m\|_1$$

As $\{f_n\}$ is a sequence of Cauchy sequence

$$\forall \frac{\epsilon^2}{c} > 0 \quad \exists N \in \mathbb{N} \text{ s.t.} \qquad N < n, m$$

$$\|f_n - f_m\|_1 < \frac{\epsilon^2}{c}$$

then

$$|\{x \in R : (Hf_n - Hf_m)(x) > \epsilon\}| \leq \frac{c}{\epsilon}\|f_n - f_m\|_1 < \epsilon$$

Therefore, it converges in measure to a measurable function which we define to be the Hilbert transform of f.

If $f \in L^p, 1 < p < \infty$, and $\{f_n\}$ is a sequence of functions in $\mathcal{S}$ that converges to f in L^p, by the strong (p,p) inequality, $\{Hf_n\}$ is a Cauchy sequence in L^p, so it converges to a function in L^p which we call the Hilbert transform of f.

In either case, a subsequence of $\{Hf_n\}$, depending on f, converges pointwise almost everywhere to Hf as defined.

The strong (p,p) inequality is false if $p = 1$ or $p = \infty$; this can easily be seen if we let $f = \chi_{[0,1]}$. Then

$$Hf(x) = \frac{1}{\pi}\log\left|\frac{x}{x-1}\right|$$

and Hf is neither integrable nor bounded.

In the Schwartz class it is straightforward to characterize the functions whose Hilbert transforms are integrable: for $\phi \in \mathcal{S}, H\phi \in L^1$ if and only if $\int \phi = 0$.

(1) $\int \phi = 0$

$$\begin{aligned}
\|H\phi\|_1 &= \int_R |H\phi| dx \\
&= \lim_{\epsilon \to 0} \int_R \left| \int_{|y|>\epsilon} \frac{1}{\pi} \frac{\phi(x-y)}{y} dy \right| dx \\
&\leq \lim_{\epsilon \to 0} \int_R \int_{|y|>\epsilon} \frac{1}{\pi} \frac{|\phi(x-y)|}{|y|} dy dx \\
&= \lim_{t \to 0} \int_{|y|>\epsilon} \frac{1}{\pi} \cdot \frac{1}{|y|} \int_R |\phi(x-y)| dx dy
\end{aligned}$$

Here we have $\int \phi = 0 \Rightarrow \int_R |\phi(x-y)| dx = 0$ for any fixed y

then $\|H\phi\|_1 \leqslant \lim_{\epsilon \to 0} \int_{|y|>\epsilon} \frac{1}{\pi} \frac{1}{|y|} \left(\int \phi\right) dy = 0$ thus $H\phi \in L^1$

(2) $H\phi \in L^1$ then $H\phi$ also bounded.

$$\begin{aligned}
\int_R H\phi \, dx &= \int_R \lim_{\epsilon \to 0} \int_{|y|>\epsilon} \frac{1}{\pi} \frac{\phi(x-y)}{y} dy dx \\
&= \lim_{\epsilon \to 0} \int_{|y|>\epsilon} \int_R \frac{1}{\pi} \frac{\phi(x-y)}{y} dx dy \\
&= \lim_{\epsilon \to 0} \int_{|y|>\epsilon} \frac{\int \phi}{\pi y} dy \\
&= \int \phi \lim_{\epsilon \to 0} \int_{|y|>\epsilon} \frac{1}{\pi} \cdot \frac{1}{y} dy
\end{aligned}$$

$\lim_{\epsilon \to 0} \int_{|y|>\epsilon} \frac{1}{y} dy$ is not bounded, while $\|H\phi\|_1$ is bounded.

Then we must have $\int \phi = 0$

3.4. **Truncated integrals and pointwise convergence.** For $\epsilon > 0$, the functions $y^{-1}\chi_{\{|y|>\epsilon\}}$ belong to $L^q(\mathbb{R}), 1 < q \leq \infty$, so the functions

$$H_\epsilon f(x) = \frac{1}{\pi} \int_{|y|>\epsilon} \frac{f(x-y)}{y} dy$$

are well defined if $f \in L^p, p \geq 1$. Moreover, H_ϵ satisfies weak $(1,1)$ and strong (p,p) estimates like those in Theorem 3.2 with constants that are uniformly bounded for all ϵ. To see this, we first note that

$$\begin{aligned}
\left(\frac{1}{y}\chi_{\{|y|>\epsilon\}}\right)^{-}(\xi) &= \lim_{N\to\infty} \int_{\epsilon<|y|<N} \frac{e^{-2\pi i y\xi}}{y} dy \\
&= \lim_{N\to\infty} \int_{\epsilon<|y|<N} -i\frac{\sin(2\pi y\xi)}{y} dy \\
&= -2i\,\mathrm{sgn}(\xi) \lim_{N\to\infty} \int_{2\pi\epsilon|\xi|}^{2\pi N|\xi|} \frac{\sin(t)}{t} dt.
\end{aligned}$$

This is uniformly bounded, so the strong $(2,2)$ inequality holds with constant independent of ϵ. We can now prove the weak $(1,1)$ inequality exactly as in Theorem 3.4, and the strong (p,p) inequalities follow by interpolation and duality.

If we fix $f \in L^p, 1 \leq p < \infty$, then the sequence $\{H_\epsilon f\}$ converges to Hf as defined above in L^p norm if $p > 1$ and in measure if $p = 1$. To see this, fix a sequence $\{f_n\}$ converging to f in L^p. Then

$$Hf = \lim_{n\to\infty} Hf_n = \lim_{n\to\infty}\lim_{\epsilon\to 0} H_\epsilon f_n = \lim_{\epsilon\to 0}\lim_{n\to\infty} H_\epsilon f_n = \lim_{\epsilon\to 0} H_\epsilon f$$

the second and third equalities hold because of the corresponding (uniform) (p,p) inequality.

We now want to show that the same equality holds pointwise almost everywhere.

Theorem 3.5. *Given $f \in L^p, 1 \leq p < \infty$, then*

$$Hf(x) = \lim_{\epsilon\to 0} H_\epsilon f(x) \qquad a.e.\ x \in \mathbb{R}.$$

Since we know that above Theorem holds for some subsequence $\{H_{\epsilon_k} f\}$, we only need to show that $\lim H_\epsilon f(x)$ exists for almost every x. By Theorem 2.7 (and the remarks following it) it will suffice to show that the maximal operator

$$H^* f(x) = \sup_{\epsilon>0} |H_\epsilon f(x)|$$

is weak (p, p). This however, follows from the next result.

Theorem 3.6. *H^* is strong $(p, p), 1 < p < \infty$, and weak $(1, 1)$.*

To prove this we need a lemma which is referred to as Cotlar's inequality.

Lemma 3.7. *If $f \in \mathcal{S}$ then $H^* f(x) \leq M(Hf)(x) + CMf(x)$.*

Proof. It will suffice to prove this inequality for each H_ϵ with a constant independent of ϵ.

Fix a function $\phi \in \mathcal{S}(\mathbb{R})$ which is non-negative, even, decreasing on $(0, \infty)$, supported on $\{x \in \mathbb{R} : |x| \leq 1/2\}$ and has integral 1. Let $\phi_\epsilon(x) = \epsilon^{-1}\phi(x/\epsilon)$. Then

$$\frac{1}{y}\chi_{\{|y|>\epsilon\}} = \left(\phi_\epsilon * \text{p.v} \cdot \frac{1}{x}\right)(y) + \left[\frac{1}{y}\chi_{\{|y|>\epsilon\}} - \left(\phi_\epsilon * \text{p} \cdot \text{v} \cdot \frac{1}{x}\right)(y)\right],$$

and the convolution of the first term on the right-hand side with f is dominated by $M(Hf)(x)$ (cf. Proposition 2.7). It will suffice to find the pointwise estimate for the second term when $\epsilon = 1$ since it follows for any other ϵ by dilation.

If $|y| > 1$ then

$$\begin{aligned}\left|\frac{1}{y} - \int_{|x|<1/2} \frac{\phi(x)}{y-x} dx\right| &= \left|\int_{|x|<1/2} \phi(x)\left(\frac{1}{y} - \frac{1}{y-x}\right) dx\right| \\ &\leq \int_{|x|<1/2} \frac{\phi(x)|x|}{|y||y-x|} dx \\ &\leq \frac{C}{y^2}\end{aligned}$$

if $|y| < 1$ then

$$\left| -\lim_{\delta\to 0}\int_{|x|>\delta}\frac{\phi(y-x)}{x}dx\right| \leq \left|\int_{|x|<2}\frac{\phi(y-x)-\phi(y)}{x}dx\right| \leq C.$$

Hence,

$$\mid \frac{1}{y}\chi_{\{|y|>\epsilon\}} - \left(\phi_\epsilon * \text{ p.v. } \frac{1}{x}\right)(y) \mid\leq \frac{C}{1+y^2}$$

and by Proposition ? the convolution of the right-hand term with f is dominated by $Mf(x)$.

Proof of Theorem 3.6. Since both the maximal function and the Hilbert transform are strong $(p,p), 1<p<\infty$, it follows at once from Lemma 3.7 that H^* is strong (p,p).

To show that H^* is weak $(1,1)$, we argue initially as in the proof of Theorem 3.4. We may assume that $f\geq 0$. Now fix $\lambda>0$ and form the Calderón-Zygmund decomposition of f at height λ. Then we can write f as

$$f=g+b=g+\sum_j b_j$$

The part of the argument involving g proceeds as in Theorem 3.4 using the fact that H^* is strong $(2,2)$. Therefore, the problem reduces to showing that

$$|\{x\notin\Omega^*: H^*b(x)>\lambda\}| \leq \frac{C}{\lambda}\|b\|_1.$$

Fix $x\notin\Omega^*, \epsilon>0$ and b_j with support I_j. Then one of the following holds:

(1) $(x-\epsilon, x+\epsilon)\cap I_j = I_j$,

(2) $(x-\epsilon, x+\epsilon)\cap I_j = \emptyset$,

(3) $x-\epsilon\in I_j$ or $x+\epsilon\in I_j$.

In the first case, $H_\epsilon b_j(x)=0$. In the second, $H_\epsilon b_j(x) = Hb_j(x)$; hence, if we let c_j denote the center of I_j, since b_j has zero average,

$$|H_\epsilon b_j(x)| \leq \int_{I_j}\left|\frac{1}{x-y}-\frac{1}{x-c_j}\right||b_j(y)|\,dy \leq \frac{|I_j|}{|x-c_j|^2}\left\|b_j\right\|_1.$$

In the third case, since $x \notin \Omega^*, I_j \subset (x-3\epsilon, x+3\epsilon)$, and for all $y \in I_j$, $|x-y| > \epsilon/3$. Therefore,

$$|H_\epsilon b_j(x)| \leq \int_{I_j} \frac{|b_j(y)|}{|x-y|} dy \leq \frac{3}{\epsilon} \int_{x-3\epsilon}^{x+3\epsilon} |b_j(y)|\, dy.$$

If we sum over all j 's we get

$$\begin{aligned} |H_\epsilon b(x)| &\leq \sum_j \frac{|I_j|}{|x-c_j|^2} \|b_j\|_1 + \frac{3}{\epsilon} \int_{x-3\epsilon}^{x+3\epsilon} |b(y)| dy \\ &\leq \sum_j \frac{|I_j|}{|x-c_j|^2} \|b_j\|_1 + CMb(x). \end{aligned}$$

It follows from this that

$$\begin{aligned} |\{x \notin \Omega^* : H^* b(x) > \lambda\}| \leq & \left| \left\{ x \notin \Omega^* : \sum_j \frac{|I_j|}{|x-c_j|^2} \|b_j\|_1 > \lambda/2 \right\} \right| \\ & + |\{x \in \mathbb{R} : Mb(x) > \lambda/2C\}| \\ \leq & \frac{2}{\lambda} \|b_j\|_1 \sum_j \int_{\mathbb{R} \backslash 2I_j} \frac{|I_j|}{|x-c_j|^2} dx + \frac{C'}{\lambda} \|b\|_1 \\ \leq & \frac{C''}{\lambda} \|b\|_1. \end{aligned}$$

3.5. **Multipliers.**

Given a function $m \in L^\infty(\mathbb{R}^n)$, we define a bounded operator T_m on $L^2(\mathbb{R}^n)$ by

$$\widehat{(T_m f)}(\xi) = m(\xi) \hat{f}(\xi)$$

Theorem 3.8. *(Plancherel theorem) if f(x) is a function on the real line, and $\widehat{f}(\xi)$is its frequency spectrum, then*

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\widehat{f}(\xi)|^2 d\xi$$

Then, we have

$$\begin{aligned}\int_R |Tmf(x)|^2 dx &= \int_R \left|(Tmf)^\wedge(\xi)\right|^2 d\xi \\ &= \int_R |m(\xi)\hat{f}(\xi)|^2 d\xi \\ &\leqslant \|m(\xi)\|_\infty^2 \int_R |\hat{f}(\xi)|^2 d\xi \\ &= \|m(\xi)\|_\infty^2 \int_R |f(x)|^2 dx \\ &= \|m(\xi)\|_\infty^2 \cdot \|f\|_2^2\end{aligned}$$

By the Plancherel theorem, $T_m f$ is well defined if $f \in L^2$ and

$$\|T_m f\|_2 \leq \|m\|_\infty \|f\|_2$$

Proposition 3.9. *the operator norm of T_m is $\|m\|_\infty$*

---

fix $\epsilon > 0$ and let A be a measurable subset of $\{x \in \mathbb{R}^n : |m(x)| > \|m\|_\infty - \epsilon\}$ whose measure is finite and positive, and let f be the L^2 function such that $\hat{f} = \chi_A$. Then

$$\|T_m f\|_2 > (\|m\|_\infty - \epsilon)\, \|f\|_2$$

$$A \subset \{x \in R^n : |m(x)| > \|m\|_\infty - \epsilon\}$$

Then $\forall x \in A,\ |m(x)| > \|m\|_\infty - \epsilon$

$$
\begin{aligned}
\|T_m f\|_2 = \left(\int_{R^n} |T_m f(x)|^2 dx\right)^{\frac{1}{2}} &= \left(\int_{R^n} \left|(T_m f)^\wedge(\xi)\right|^2 d\xi\right)^{\frac{1}{2}} \\
&= \left(\int_{R^n} |m(\xi)\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}} \\
&= \left(\int_{A} |m(\xi)f(\xi)|^2 d\xi\right)^{\frac{1}{2}} \\
&> (\|m\|_\infty - \epsilon)\left(\int_{A} |\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}} \\
&= (\|m\|_\infty - \epsilon)\,\|f\|_2
\end{aligned}
$$

Take the limit of ϵ,

$$\lim_{\epsilon\to 0}(\|m\|_\infty - \epsilon)\,\|f\|_2 \leqslant \|T_m f\|_2 \leqslant \|m\|_\infty \|f\|_2$$

Thus operator norm of $\mathrm{T_m}$ is $\|m\|_\infty$

We say that m is the multiplier of the operator T_m, though occasionally we will refer to the operator itself as a multiplier. When T_m can be extended to a bounded operator on L^p we say that m is a multiplier on L^p.

For instance, the multiplier of the Hilbert transform, $m(\xi) = -i\,\mathrm{sgn}(\xi)$, is a multiplier on L^p. More generally, given $a, b \in \mathbb{R}, a < b$, define $m_{a,b}(\xi) = \chi_{(a,b)}(\xi)$. Let $S_{a,b}$ be the operator associated with this multiplier:

$$(S_{a,b}f)^\wedge(\xi) = \chi_{(a,b)}(\xi)\hat{f}(\xi)$$

We have the equivalent expression

$$S_{a,b} = \frac{i}{2}\left(M_a H M_{-a} - M_b H M_{-b}\right)$$

where M_a is the operator given by pointwise multiplication by $e^{2\pi i a x}$,

$$M_a f(x) = e^{2\pi i a x} f(x).$$

Here we prove a property of Fourier Transform.

$$\begin{aligned}
&(\tau_h f)^{\wedge}(\xi) = \hat{f}(\xi) e^{2\pi i h \cdot \xi}, \text{ where } \tau_h f(x) = f(x+h) \\
&\left(f e^{2\pi i h \cdot x}\right)^{\wedge}(\xi) = \hat{f}(\xi - h)
\end{aligned}$$

$$\hat{f}(\xi) = \int_R f(x) e^{-2\pi i x \cdot \xi} dx$$

(1) $\hat{f}(\xi) \cdot e^{2\pi i h \xi} = \int_R f(x) e^{-2\pi i \xi (x-h)} dx = \int_R f(x-h+h) e^{-2\pi i \xi (x-h)} d(x-h)$

$$= \int_R f(y) e^{-2\pi i \xi y} dy = (\tau_n f)^{\wedge}(\xi)$$

(2) $\left(f e^{2\pi i h x}\right)^{\wedge}(\xi) = \int_R f(x) e^{2\pi i h x} \cdot e^{-2\pi i \xi x} dx$

$$= \int_R f(x) e^{-2\pi i (\xi - h) x} dx = \hat{f}(\xi - h)$$

then define $m \in L^P\left(R^n\right), 1 \leqslant P \leqslant +\infty$,

$$\begin{aligned}
(i M_a H M_{-a} f)^{\wedge}(\xi) &= m(\xi) \hat{f}(\xi) \\
M - a f = \quad & e^{-2\pi i a x} f(x) \Rightarrow (M - a f)^{\wedge}(\xi) = \hat{f}(\xi + a) \\
(H M - a f)^{\wedge}(\xi) &= -i \operatorname{sgn}(\xi) (M - a f)^{\wedge}(\xi) = -i \operatorname{sgn}(\xi) \hat{f}(\xi + a) \\
(M a H M - a f)^{\wedge}(\xi) &= (H M - a f)^{\wedge}(\xi - a) = -i \operatorname{sgn}(\xi - a) \hat{f}(\xi) \\
(i M a H M - a f)^{\wedge}(\xi) &= i(-i \operatorname{sgn}(\xi - a) \hat{f}(\xi)) = \operatorname{sgn}(\xi - a) \hat{f}(\xi)
\end{aligned}$$

Therefore, $m(\xi) = \operatorname{sgn}(\xi - a)$ Similarly, we have the multiplier of $i M_b H M_b$ is $\operatorname{sgn}(\xi - b)$ Here $M_a f = e^{2\pi i a x} f(x)$ and $(M_a f)^{\wedge}(\xi) = \hat{f}(\xi - a) \Rightarrow \|M_a f\|_p \leqslant \|f\|_p$ M_a is bounded on $L^p, 1 \leq p \leq +\infty$ with norm 1 . Therefore, $S_{a,b} = \frac{i}{2}\left(M_a H M_{-a} - M_b H M_{-b}\right), H$ is strong (p,p). we can derive $S_{a,b}$ is bounded on L^p

By making the obvious changes in the argument we see that this is still true if $a = -\infty$ or $b = \infty$. Hence, we have the following result.

Proposition 3.10. *There exists a constant $C_p, 1 < p < \infty$, such that for all a and $b, -\infty \leq a < b \leq \infty$,*

$$\|S_{a,b}f\|_p \leq C_p \|f\|_p$$

For an application of this result, let $a = -R, b = R$. Then $S_{a,b}$ is the partial sum operator S_R introduced in Chapter 1: $S_R f = D_R * f$, where D_R is the Dirichlet kernel. Hence,

$$\|S_R f\|_p \leq C_p \|f\|_p$$

with a constant independent of R. This yields the following corollary.

Corollary 3.11. *If $f \in L^p(\mathbb{R}), 1 < p < \infty$, then*

$$\lim_{R\to\infty} \|S_R f - f\|_p = 0.$$

When $p = 1$ we do not have convergence in norm but only in measure:

$$\lim_{R\to\infty} |\{x \in \mathbb{R} : |S_R f(x) - f(x)| > \epsilon\}| = 0.$$

From this result and from Corollary above we see that there exists a sequence $\{S_{R_k} f(x)\}$ that converges to $f(x)$ for almost every x, but the sequence depends on f.

Given a family of uniformly bounded operators on L^p, any convex combination of them is also bounded.

Corollary 3.12. *If m is a function of bounded variation on $\mathbb{R}$, then m is a multiplier on $L^p, 1 < p < \infty$.*

Proof. Since m is of bounded variation, the limit of $m(t)$ as $t \to -\infty$ exists, so by adding a constant to m if necessary we may assume that this limit equals 0.

$$\lim_{t \to -\infty} m(t) + c = 0 \; as \; c = -\lim_{t \to -\infty} m(t)$$

Furthermore, we may assume m is normalized so that it is right continuous at each $x \in \mathbb{R}$. Let dm be the Lebesgue-Stieltjes measure associated with m; then

$$m(\xi) = \int_{-\infty}^{\xi} dm(t) = \int_{\mathbb{R}} \chi_{(-\infty,\xi)}(t) dm(t) = \int_{\mathbb{R}} \chi_{(t,\infty)}(\xi) dm(t).$$

Therefore,

$$(T_m f)^{\wedge}(\xi) = \int_{\mathbb{R}} \chi_{(t,\infty)}(\xi) \hat{f}(\xi) dm(t)$$

and as

$$\widehat{(T_m f)}(\xi) = m(\xi) \hat{f}(\xi)$$

we have

$$T_m f(x) = \int_{\mathbb{R}} S_{t,\infty} f(x) dm(t)$$

By Minkowski's inequality

$$\|T_m f\|_p \le \int_{\mathbb{R}} \|S_{t,\infty} f\|_p \, |dm|(t) \le C_p \|f\|_p \int_{\mathbb{R}} |dm|(t)$$

the integral of $|dm|$ is the total variation of m and by assumption this is finite.

Notes: Given a multiplier, we can construct others from it by translation, dilation and rotation.

Proposition 3.13. *If m is a multiplier on $L^p(\mathbb{R}^n)$, then the functions defined by $m(\xi + a), a \in \mathbb{R}^n, m(\lambda\xi), \lambda > 0,$ and $m(\rho\xi), \rho \in O(n)$ (orthogonal transformations), are multipliers of bounded operators on L^p with the same norm as T_m.*

The proof of this result follows at once from properties of the Fourier transform. We have proved that

$$(\tau_h f)^\wedge(\xi) = \hat{f}(\xi)e^{2\pi i h\cdot\xi}, \text{ where } \tau_h f(x) = f(x+h)$$
$$\left(fe^{2\pi i h\cdot x}\right)^\wedge(\xi) = \hat{f}(\xi - h)$$

(2) if $\rho \in O_n$ (an orthogonal transformation) then $(f(\rho\cdot))^\wedge(\xi) = \hat{f}(\rho\xi)$

$$\hat{f}(\rho\xi) = \int_R f(x)e^{-2xix\cdot\rho\xi}dx = \int_R f(\rho x)e^{-2\pi i(\rho x)\xi}d(\rho x) = (f(\rho\cdot))^\wedge(\xi)$$

(3) if $g(x) = \lambda^{-n} f\left(\lambda^{-1}x\right)$, then $\hat{g}(\xi) = \hat{f}(\lambda\xi)$

$$\begin{aligned}\hat{g}(\xi) &= \int_{R^n} g(x)e^{-2\pi i x\cdot\xi}dx = \int_{R^n} \lambda^{-n} f\left(\lambda^{-1}x\right)e^{-2\pi i x\cdot\xi}dx \\ &= \int_{R^n} f(y)e^{-2\pi i\lambda y\cdot\xi}dy \quad \left(y = \lambda^{-1}x\right) \\ &= \hat{f}(\lambda\xi)\end{aligned}$$

First, $(\mathrm{T}_m f)^\wedge(\xi) = m(\xi)\hat{f}(\xi)$ then $m(\xi+a)\hat{f}(\xi) = m(\xi+a)\hat{f}(\xi+a)e^{2\pi i(-a)\cdot\xi}$ then $(T'_m f)^\wedge(\xi) = m(\xi+a)\hat{f}(\xi) = (T_m f)^\wedge(\xi+a)e^{2\pi i(-a)\xi} = (T_m f)^\wedge(\xi)$ Second, $m(P\xi)\hat{f}(\xi) = m(P\xi)\hat{f}(\rho\xi)\cdot\frac{\hat{f}(\xi)}{(f(\rho\cdot))^\wedge(\xi)} = (Tmf)^\wedge(\rho\xi)\cdot\frac{\hat{f}(\xi)}{(f(\rho\cdot))^\wedge(\xi)}$

$$\|m(\rho\cdot)\|_\infty = \|m\|_\infty$$

Third,

$$\begin{aligned}m(\lambda\xi)\hat{f}(\xi) &= m(\lambda\xi)\hat{f}(\lambda\xi)\cdot\frac{\hat{f}(\xi)}{\hat{g}(\xi)} \quad \text{where } g(x) = \lambda^{-n}f\left(\lambda^{-1}x\right) \\ &= (\mathrm{Tm}\, f)^\wedge(\lambda\xi)\cdot\frac{\hat{f}(\xi)}{\hat{g}(\xi)}\end{aligned}$$
$$\|m(\lambda\cdot)\|_\infty = \|m\|_\infty$$

If m is a multiplier on $L^p(\mathbb{R})$, then the function on $\mathbb{R}^n$ given by $\tilde{m}(\xi) = m(\xi_1)$ is a multiplier on $L^p(\mathbb{R}^n)$. Actually, if T_m is the one-dimensional operator associated with m, then for f defined on $\mathbb{R}^n$,

$$T_{\tilde{m}} f(x) = T_m f(\cdot, x_2, \ldots, x_n)(x_1)$$

Then by Fubini's theorem and the boundedness of T_m on $L^p(\mathbb{R})$,

$$\begin{aligned}\int_{\mathbb{R}^n} |T_{\tilde{m}} f(x)|^p \, dx &= \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} |T_m f(\cdot, x_2, \ldots, x_n)(x_1)|^p \, dx_1 \right) dx_2 \cdots dx_n \\ &\leq C \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} |f(x_1, \ldots, x_n)|^p \, dx_1 \cdots dx_n.\end{aligned}$$

If $m = \chi_{(0,\infty)}$, then m is a multiplier on $L^p(\mathbb{R}), 1 < p < \infty$. Hence, by the preceding argument the characteristic function of the half-space $\{\xi \in \mathbb{R}^n : \xi_1 > 0\}$ will be a multiplier on $L^p(\mathbb{R}^n)$. Further, the same will be true for the characteristic function of any half-space (since it can be gotten from the one above by a rotation and translation). The characteristic function of a convex polyhedron with N faces can be written as the product of N characteristic functions of halfspaces, so it is also a multiplier of $L^p(\mathbb{R}^n), 1 < p < \infty$. This fact has the following consequence.

Corollary 3.14. *If $P \subset \mathbb{R}^n$ is a convex polyhedron that contains the origin, then*

$$\lim_{\lambda \to \infty} \|S_{\lambda P} f - f\|_p = 0, \quad 1 < p < \infty$$

where $S_{\lambda P}$is the operator whose multiplier is the characteristic function of

$$\lambda P = \{\lambda P = \lambda x : x \in P\}$$

4. Singular Integrals

4.1. Definition and examples.

Definition 4.1. The singular integrals we are interested in are operators of the form

$$Tf(x) = \lim_{\epsilon \to 0} \int_{|y|>\epsilon} \frac{\Omega(y')}{|y|^n} f(x-y) dy$$

where Ω is defined on the unit sphere in $\mathbb{R}^n$, S^{n-1}, is integrable with zero average and where $y' = y/|y|$.

With these hypotheses, (4.1) is defined for Schwartz functions since it is the convolution of f with the tempered distribution .

Definition 4.2. It is the convolution of f with the tempered distribution $p.v.\Omega(x')/|x|^n$.

$$p.v.\Omega(x')/|x|^n = \int_{|x|<1} \frac{\Omega(x')}{|x|^n}[\phi(x) - \phi(0)]dx + \int_{|x|>1} \frac{\Omega(x')}{|x|^n}\phi(x)dx$$

---

defined by

$$\begin{aligned}
&\text{p. v. } \frac{\Omega(x')}{|x|^n}(\phi) = \lim_{\epsilon \to 0} \int_{|x|>\epsilon} \frac{\Omega(x')}{|x|^n}\phi(x)dx \\
&= \lim_{\epsilon \to 0} \int_{\epsilon<|x|<1} \frac{\Omega(x')}{|x|^n}\phi(x)dx + \int_{|x|>1} \frac{\Omega(x')}{|x|^n}\phi(x)dx \\
&= \int_{|x|<1} \frac{\Omega(x')}{|x|^n}[\phi(x) - \phi(0)]dx + \int_{|x|>1} \frac{\Omega(x')}{|x|^n}\phi(x)dx.
\end{aligned}$$

The last equality follows since Ω has zero average.

$$\phi(0) \lim_{\epsilon \to 0} \int_{\epsilon<|x|<1} \phi(0) \frac{\Omega(x')}{|x|^n} dx = \frac{\Omega(x')}{|x|^n}\phi(0) = 0$$

Since $\phi \in \mathcal{S}$, both integrals converge.

Proposition 4.3. *A necessary condition for the limit in definition 4.1 or above convolution to exist is that Ω have zero average on S^{n-1}.*

Necessary condition: Limit exists $\Rightarrow \Omega$ have zero average on S^{n-1}. To prove this, we need to find a function which has limit in (4.1), and impliy that Ω must have zero average on S^{n-1}. Construct $f \in S\left(R^{n}\right)$ be such that $f(x)=1 \quad \forall|x| \leqslant 2$. Then for $|x|<1$,

$$
\begin{aligned}
Tf(x) &= \int_{|y|>|} \frac{\Omega\left(y'\right)}{|y|^{n}} f(x-y) dy+\lim_{\epsilon \rightarrow 0} \int_{\epsilon\langle|y|<|} \frac{\Omega\left(y'\right)}{|y|^{n}} f(x-y) dy \\
&= \int_{|y|>|} \frac{\Omega\left(y'\right)}{|y|^{n}} f(x-y) dy+\lim_{t \rightarrow 0} \int_{\epsilon|y|<1} \frac{\Omega\left(y'\right)}{|y|^{n}} dy
\end{aligned}
$$

The first item always converges as $f \in S\left(R^{n}\right)$ and $\frac{\Omega(y')}{|y|^{n}}$ should converge. The second item

$$
\lim_{\epsilon \rightarrow 0} \int_{\epsilon<|y|<1} \frac{\Omega\left(y'\right)}{|y|^{n}} dy=\lim_{\epsilon \rightarrow 0} \int_{\xi^{n-1}} \Omega\left(y'\right) d\sigma\left(y'\right) \cdot \log \left(\frac{1}{\epsilon}\right)
$$

This is finite only if the integral of Ω on S^{n-1} is zero $\Leftrightarrow \int_{S^{n-1}} \Omega\left(y'\right) d\sigma\left(y'\right)=0$

When $n=1$ the unit sphere reduces to two points, 1 and -1 , and Ω must take opposite values on them. Thus on the real line, any operator of the type in (4.1) is a multiple of the Hilbert transform.

In higher dimensions we consider two examples given by A. P. Calderón and A. Zygmund. First, let $f\left(x_{1}, x_{2}\right)$ be the density of a mass distribution in the plane. Then its Newtonian potential in the half-space $\mathbb{R}_{+}^{3}$ is

$$
u\left(x_{1}, x_{2}, x_{3}\right)=\int_{\mathbb{R}^{2}} \frac{f\left(y_{1}, y_{2}\right)}{\left(x_{1}-y_{1}\right)^{2}+\left(x_{2}-y_{2}\right)^{2}+x_{3}^{2}} dy_{1} dy_{2}.
$$

The strength of the gravitational field is gotten by taking partial derivatives of the potential. The component in the x_{3} direction is equivalent to a multiple of the Poisson integral which we have already considered. The other two are similar to one another; for example, for the first we formally have that

$$\lim_{x_3 \to 0} \frac{\partial u}{\partial x_1} (x_1, x_2, x_3) = -\int_{\mathbb{R}^2} \frac{f (y_1, y_2) (x_1 - y_1)}{\left[(x_1 - y_1)^2 + (x_2 - y_2)^2 \right]^{3/2}} dy_1 dy_2.$$

This integral does not converge in general, but it exists as a principal value if f is smooth and is in fact the value of the limit of $\partial u / \partial x_1$. It corresponds to a singular integral of the form (4.1) in $\mathbb{R}^2$ with $\Omega (x') = -x_1/|x|$. (In polar coordinates this becomes $\cos(\theta)$.)

The second example is gotten from the logarithmic potential associated with a mass distribution $f (x_1, x_2)$ in the plane (i.e. the solution of the equation $\Delta u = f$) :

$$u (x_1, x_2) = \int_{\mathbb{R}^2} f (y_1, y_2) \log \left(\frac{1}{\left[(x_1 - y_1)^2 + (x_2 - y_2)^2 \right]^{1/2}} \right) dy_1 dy_2.$$

This integral converges absolutely if, for example, f has compact support, and partial derivatives can be taken under the integral sign. However, this is not possible for second derivatives: one can show that $\partial^2 u / \partial x_1 \partial x_2$ is given by an operator of the form (4.1) with $\Omega (x') = 2x_1 x_2 / |x|^2$.

4.2. **The Fourier transform of the kernel.**

Definition 4.4. A function f is homogeneous of degree a if for any $x \in \mathbb{R}^n$ and any $\lambda > 0$,

$$f(\lambda x) = \lambda^a f(x).$$

Proposition 4.5. *Given any function ϕ, define $\phi_\lambda(x) = \lambda^{-n} \phi (\lambda^{-1} x)$; then*

$$\int_{\mathbb{R}^n} f(x) \phi_\lambda(x) dx = \lambda^a \int_{\mathbb{R}^n} f(x) \phi(x) dx$$

Definition 4.6. (The homogeneity of a distribution) A distribution T is homogeneous of degree a if for every test function ϕ it satisfies

$$T (\phi_\lambda) = \lambda^a T(\phi)$$

The distribution given by (4.2) is homogeneous of degree $-n$:

$$\begin{aligned}
&\text{p. v. } \frac{\Omega\left(x'\right)}{|x|^n}(\phi_\lambda) \\
&\qquad = \lim_{\epsilon\to 0}\int_{|x|>\epsilon}\frac{\Omega\left(x'\right)}{|x|^n}\phi_\lambda(x)dx \\
&\qquad = \lambda^{-n}\lim_{\epsilon\to 0}\int_{|x|>\epsilon}\frac{\Omega\left(x'\right)}{|x|^n}\phi_\lambda(x)dx\ byproposition 4.5 \\
&\qquad = \lambda^{-n}\text{ p. v. }\frac{\Omega\left(x'\right)}{|x|^n}(\phi)
\end{aligned}$$

Proposition 4.7. *If T is a tempered distribution which is homogeneous of degree a, then its Fourier transform is homogeneous of degree $-n-a$.*

Proof: If $\phi \in \mathcal{S}$ then

$$\hat{T}\left(\phi_\lambda\right) = T(\hat{\phi}(\lambda\cdot)) = \lambda^{-n}T\left(\hat{\phi}_{\lambda^{-1}}\right) = \lambda^{-n-a}T(\hat{\phi}) = \lambda^{-n-a}\hat{T}(\phi).$$

As a corollary to this proposition we can calculate the Fourier transform of $f(x) = |x|^{-a}$ if $n/2 < a < n$. For in this case f is the sum of an L^1 function (its restriction to $\{|x| < 1\}$) and an L^2 function, so by Proposition 4.7, $\hat{f}$ is a homogeneous function of degree $a-n$. Hence, since $\hat{f}$ is rotationally invariant, $\hat{f}(\xi) = c_{a,n}|\xi|^{a-n}$. We calculate $c_{a,n}$

First proof:

$$\hat{f}(\xi) = e^{-\pi|\xi|^2}\ if\ f(x) = e^{-\pi x^2}$$

$$f(x) = e^{-\pi x^2}$$
$$\hat{f}(\xi) = \int_R f(x) e^{-2\pi i x\xi} dx$$
$$= \int_R e^{-\pi x^2} e^{-2\pi i x \xi} dx$$
$$= \int_R e^{-\pi(x^2+2ix\xi)} dx$$
$$= \int_R e^{-\pi(x+i\xi)^2} dx e^{\pi i^2 \xi^2}$$
$$= e^{\pi i^2 \xi^2}$$

As
$$\int f\hat{g} = \int \hat{f} g$$
we have
$$\int_{\mathbb{R}^n} e^{-\pi|x|^2}|x|^{-a} dx = c_{a,n} \int_{\mathbb{R}^n} e^{-\pi|x|^2}|x|^{a-n} dx$$
where $f(x) = |x|^{-a}$ and $g(x) = e^{-\pi|\xi|^2}$ since
$$\int_0^\infty e^{-\pi r^2} r^b dr = \frac{1}{2}\pi^{-\frac{1+b}{2}}\Gamma\left(\frac{1+b}{2}\right)$$
Left side can be seen as $Gamma(\frac{b+1}{2}, \pi)$ distribution without the normalization term.

Therefore,
$$\left(|x|^{-a}\right)^{\wedge}(\xi) = \frac{\pi^{a-\frac{n}{2}}\Gamma\left(\frac{n-a}{2}\right)}{\Gamma\left(\frac{a}{2}\right)}|\xi|^{a-n}.$$

Actually, this formula holds for all $a, 0 < a < n$.

For $0 < a < n/2$ it follows from the inversion formula
$$f(x) = \int_{R^n} \hat{f}(\xi) e^{2\pi i x \xi} d\xi$$

Further, since $|x|^{-a}$ tends to $|x|^{n/2}$ as $a \to n/2$, and since the Fourier transform is continuous, (4.3) holds with $a = n/2$.

Theorem 4.8. *If Ω is an integrable function on S^{n-1} with zero average, then the Fourier transform of p.v. $\Omega\left(x'\right)/|x|^n$ is a homogeneous function of degree 0 given by*

$$m(\xi)=\int_{S^{n-1}} \Omega(u)\left[\log \left(\frac{1}{\left|u \cdot \xi'\right|}\right)-i \frac{\pi}{2} \operatorname{sgn}\left(u \cdot \xi'\right)\right] d \sigma(u)$$

Proof. By Proposition 4.7 the Fourier transform is homogeneous of degree 0 ; therefore, we may now assume that $|\xi|=1$. Since Ω has zero average,

$$\begin{aligned} m(\xi) & =\lim _{\epsilon \rightarrow 0} \int_{\epsilon<|y|<1 / \epsilon} \frac{\Omega\left(y'\right)}{|y|^n} e^{-2 \pi i y \cdot \xi} d y \\ & =\lim _{\epsilon \rightarrow 0} \int_{S^{n-1}} \Omega(u)\left[\int_\epsilon^1\left(e^{-2 \pi i r u \cdot \xi}-1\right) \frac{d r}{r}+\int_1^{1 / \epsilon} e^{-2 \pi i r u \cdot \xi} \frac{d r}{r}\right] d \sigma(u) .\end{aligned}$$

Thus $m(\xi)=I_1-i I_2$, where

$$\begin{aligned} I_1=\lim _{\epsilon \rightarrow 0} \int_{S^{n-1}} \Omega(u) & {\left[\int_\epsilon^1(\cos (2 \pi r u \cdot \xi)-1) \frac{d r}{r}\right.} \\ & \left.+\int_1^{1 / \epsilon} \cos (2 \pi r u \cdot \xi) \frac{d r}{r}\right] d \sigma(u), \\ I_2=\lim _{\epsilon \rightarrow 0} \int_{S^{n-1}} & \Omega(u)\left[\int_\epsilon^{1 / \epsilon} \sin (2 \pi r u \cdot \xi) \frac{d r}{r}\right] d \sigma(u) .\end{aligned}$$

By the dominated convergence theorem we can exchange the limit and outer integral. In each inner integral make the change of variables $s=2 \pi r|u \cdot \xi|$. We may assume that $u \cdot \xi \neq 0$, so in I_2 we get

$$\int_{2 \pi|u \cdot \xi| \epsilon}^{2 \pi|u \cdot \xi| / \epsilon} \sin (s) \operatorname{sgn}(u \cdot \xi) \frac{d s}{s}$$

As $\epsilon \rightarrow 0$ this becomes

$$\operatorname{sgn}(u \cdot \xi) \int_0^{\infty} \frac{\sin (s)}{s} d s=\frac{\pi}{2} \operatorname{sgn}(u \cdot \xi)$$

After the change of variables in I_1 we get

$$\int_{2\pi|u\cdot\xi|\epsilon}^{1}(\cos(s)-1)\frac{ds}{s}+\int_{1}^{2\pi|u\cdot\xi|/\epsilon}\cos(s)\frac{ds}{s}-\int_{1}^{2\pi|u\cdot\xi|}\frac{ds}{s}$$

as $\epsilon\to 0$ this becomes

$$\int_{0}^{1}(\cos(s)-1)\frac{ds}{s}+\int_{1}^{\infty}\frac{\cos(s)}{s}ds-\log|2\pi|-\log|u\cdot\xi|$$

If we integrate against Ω, which has zero average on S^{n-1}, the constant terms disappear and we get the desired formula for m.

In formula (4.8) the factor multiplied against Ω has two terms: the first is even and its contribution is zero if Ω is odd; it is not bounded but any power of it is integrable. The second is odd and its contribution is zero if Ω is even; further, it is bounded. Since any function Ω on S^{n-1} can be decomposed into its even and odd parts,

$$\Omega_e(u)=\frac{1}{2}(\Omega(u)+\Omega(-u)),\quad \Omega_o(u)=\frac{1}{2}(\Omega(u)-\Omega(-u)),$$

we immediately get the following corollary.

Corollary 4.9. *Given a function Ω with zero average on S^{n-1}, suppose that $\Omega_o\in L^1\left(S^{n-1}\right)$ and $\Omega_e\in L^q\left(S^{n-1}\right)$ for some $q>1$. Then the Fourier transform of p.v. $\Omega\left(x'\right)/|x|^n$ is bounded.*

From theorem 4.8 one can find an integrable function Ω such that m is not bounded.

Nevertheless, in Corollary 4.9 we can substitute the weaker hypothesis $\Omega_e\in L\log L\left(S^{n-1}\right)$, that is,

$$\int_{S^{n-1}}|\Omega_e(u)|\log^{+}|\Omega_e(u)|\,d\sigma(u)<\infty$$

(Recall that $\log^{+}t=\max(0,\log t)$.) The sufficiency of this condition follows from the inequality

$$AB \leq A \log A + e^B, \quad A \geq 1, B \geq 0.$$

Proof:

$$\begin{aligned}
&let\ f(A,B) = A \log A + e^B - AB, \quad A \geq 1, B \geq 0 \\
&\frac{\partial f}{\partial A} = 1 + logA - B \\
&\frac{\partial f}{\partial B} = e^B - A \\
&f(A,B) \geq f(A, logA) = A \log A + logA - Alog(A) = A \geq 1
\end{aligned}$$

For in the region $D = \{u \in S^{n-1} : |\Omega(u)| \geq 1\}$,

$$\begin{aligned}
&\left| \int_D \Omega(u) \log\left(\frac{1}{|u \cdot \xi'|}\right) d\sigma(u) \right| \\
&\quad \leq \int_D 2|\Omega(u)| \log(2|\Omega(u)|) d\sigma(u) + \int_D |u \cdot \xi'|^{-1/2} d\sigma(u).
\end{aligned}$$

Ω is bounded in the complement of D, and so the integral is finite.

4.3. **The method of rotations.** Corollary 4.9 together with the Plancherel theorem gives sufficient conditions for operators T as in (4.1) to be bounded on L^2. In this and the following section we develop techniques due to Calderón and Zygmund which will let us prove they are bounded on $L^p, 1 < p < \infty$.

Let T be a one-dimensional operator which is bounded on $L^p(\mathbb{R})$ and let $u \in S^{n-1}$. Starting from T we can define a bounded operator T_u on $\mathbb{R}^n$ as follows: let $L_u = \{\lambda u : \lambda \in \mathbb{R}\}$ and let $L_u^\perp$ be its orthogonal complement in $\mathbb{R}^n$. Then given any $x \in \mathbb{R}^n$, there exists a unique $x_1 \in \mathbb{R}$ and $\bar{x} \in L_u^\perp$ such that $x = x_1 u + \bar{x}$. Now define $T_u f(x)$ to be the value at x_1 of the image under

T of the one-dimensional function $f(\cdot u+\bar{x})$. If C_p is the norm of T in $L^p(\mathbb{R})$, then by Fubini's theorem,

$$\begin{aligned}\int_{\mathbb{R}^n}|T_u f(x)|^p\,dx &= \int_{L_u^{\perp}}\int_{\mathbb{R}}|\,T\left(f(\cdot u+\bar{x})\right)(x_1)|^p\,dx_1 dx\\ &\leq C_p^p\int_{L^{\perp}}\int_{\mathbb{R}}|f(\cdot u+\bar{x})\,(x_1)|^p\,dx_1 dx\\ &= C_p^p\int_{\mathbb{R}^n}|f(x)|^p dx.\end{aligned}$$

Operators gotten in this way include the directional Hardy-Littlewood maximal function,

$$M_u f(x)=\sup_{h>0}\frac{1}{2h}\int_{-h}^{h}|f(x-tu)|dt$$

and the directional Hilbert transform,

$$H_u f(x)=\frac{1}{\pi}\lim_{\epsilon\to 0}\int_{|t|>\epsilon}f(x-tu)\frac{dt}{t}.$$

Since the operators T_u obtained from the operator T are uniformly bounded on $L^p\left(\mathbb{R}^n\right)$, any convex combination of them is also a bounded operator. Hence, the next result is an immediate consequence of Minkowski's integral inequality.

Proposition 4.10. *Given a one-dimensional operator T which is bounded on $L^p(\mathbb{R})$ with norm C_p, let T_u be the directional operators defined from T. Then for any $\Omega\in L^1\left(S^{n-1}\right)$, the operator T_Ω defined by*

$$T_\Omega f(x)=\int_{S^{n-1}}\Omega(u)T_u f(x)d\sigma(u)$$

is bounded on $L^p\left(\mathbb{R}^n\right)$ with norm at most $C_p\|\Omega\|_1$.

Proof:

$$
\begin{aligned}
&\int_{\mathbb{R}^n} |T_\Omega f(x)|^p dx \\
&= \int_{\mathbb{R}^n} \left| \int_{S^{n-1}} \Omega(u) T_u f(x) d\sigma(u) \right|^p dx \\
&\leq \|\Omega\|_1^p \int_{\mathbb{R}^n} \int_{S^{n-1}} \left|^p T_u f(x) d\sigma(u) \right| dx \\
&\leq \|\Omega\|_1^p \int_{\mathbb{R}^n} \int_{S^{n-1}} \left| T_u f(x) \right| d\sigma(u) dx \\
&\leq \|\Omega\|_1^p \int_{\mathbb{R}^n} \int_{S^{n-1}} |\Omega(u) T_u f(x)| d\sigma(u) dx \\
&\leq \|\Omega\|_1^p C_p^p \|f\|_p^p
\end{aligned}
$$

Since T_u is bounded by C_p :

$$\|T_u f(x)\|_p \leq C_p \|f\|_p$$

By using this result we can pass from a one-dimensional result to one in higher dimensions. Its most common application is in the method of rotations, which uses integration in polar coordinates to get directional operators in its radial part.

For example, given $\Omega \in L^1\left(S^{n-1}\right)$, define the "rough" maximal function

Definition 4.11. "rough" maximal function:

$$M_\Omega f(x) = \sup_{R>0} \frac{1}{|B(0,R)|} \int_{B(0,R)} |\Omega\left(y'\right)| \, |f(x-y)| dy.$$

If we rewrite this integral in polar coordinates we get

$$M_\Omega f(x) = \sup_{R>0} \frac{1}{|B(0,1)| R^n} \int_{S^{n-1}} |\Omega(u)| \int_0^R |f(x-ru)| r^{n-1} dr d\sigma(u)$$

$$\leq \frac{1}{|B(0,1)|} \int_{S^{n-1}} |\Omega(u)| M_u f(x) d\sigma(u).$$

Corollary 4.12. *If $\Omega \in L^1\left(S^{n-1}\right)$ then M_Ω is bounded on $L^p\left(\mathbb{R}^n\right), 1 < p \leq \infty$*

If let $\Omega(u) = 1$ for all u, then M_Ω becomes the Hardy-Littlewood maximal function in $\mathbb{R}^n$:

$$M_\Omega f(x) = \sup_{R>0} \frac{1}{|B(0,R)|} \int_{B(0,R)} |f(x-y)| dy.$$

thus the method of rotations shows it is bounded on $L^p\left(\mathbb{R}^n\right), p > 1$, starting from the one-dimensional case.

Further, note that the L^p constant we get is $O(n)$ since

$$\frac{|S^{n-1}|}{|B(0,1)|} = n$$

On the other hand, the method of rotations does not give us the weak $(1,1)$ result.

Apply Proposition 4.10 to operators of the form (4.1) when Ω is odd. In fact, if we fix a Schwartz function f, then

$$\begin{aligned} Tf(x) &= \lim_{\epsilon \to 0} \int_{S^{n-1}} \Omega(u) \int_\epsilon^\infty f(x-ru) \frac{dr}{r} d\sigma(u) \\ &= \lim_{\epsilon \to 0} \frac{1}{2} \int_{S^{n-1}} \Omega(u) \int_{|r|>\epsilon} f(x-ru) \frac{dr}{r} d\sigma(u); \end{aligned}$$

since Ω has zero average, we can argue as we did in (4.2) to get

$$\begin{aligned} &= \frac{1}{2} \lim_{\epsilon \to 0} \int_{S^{n-1}} \Omega(u) \int_{\epsilon<|r|<1} (f(x-ru) - f(x)) \frac{dr}{r} d\sigma(u) \\ &\quad + \frac{1}{2} \int_{S^{n-1}} \Omega(u) \int_{|r|>1} f(x-ru) \frac{dr}{r} d\sigma(u). \end{aligned}$$

Because $f \in \mathcal{S}$, the inner integral is uniformly bounded, so we can apply the dominated convergence theorem to get

$$= \frac{\pi}{2} \int_{S^{n-1}} \Omega(u) H_u f(x) d\sigma(u).$$

Since the Hilbert transform is strong $(p, p), 1 < p < \infty$, we have :

Corollary 4.13. *If Ω is an odd integrable function on S^{n-1}, then the operator T defined by (4.1) is bounded on $L^p(\mathbb{R}^n), 1 < p < \infty$.*

Corollary 4.13 defined Tf for $f \in L^p$ as a limit in L^p norm.

However, as with the Hilbert transform, we can show that (4.1) holds for almost every x.

Claim:The maximal operator associated with the singular integral,

$$T^* f(x) = \sup_{\epsilon > 0} \left| \int_{|y|>\epsilon} \frac{\Omega(y')}{|y|^n} f(x-y) dy \right|,$$

satisfies

$$T^* f(x) \leq \frac{\pi}{2} \int_{S^{n-1}} |\Omega(u)| H_u^* f(x) d\sigma(u),$$

where H_u^* is the directional operator defined from the maximal Hilbert transform. Therefore,

Proof:

$$\begin{aligned} T^* f(x) &= \sup_{\epsilon > 0} \left| \int_{|y|>\epsilon} \frac{\Omega(y')}{|y|^n} f(x-y) dy \right| \\ &= \sup_{\epsilon > 0} \left| \int_{S^{n-1}} \Omega(u) \int_{\epsilon}^{\infty} f(x-ru) \frac{dr}{r} d\sigma(u) \right| \\ &= \sup_{\epsilon > 0} \frac{1}{2} \left| \int_{S^{n-1}} \Omega(u) \int_{|r|>\epsilon} f(x-ru) \frac{dr}{r} d\sigma(u) \right|; \end{aligned}$$

since Ω has zero average

$$\leq \frac{1}{2}\sup_{\epsilon>0}\left|\int_{S^{n-1}}\Omega(u)\int_{\epsilon<|r|<1}(f(x-ru)-f(x))\frac{dr}{r}d\sigma(u)\right|$$
$$+\frac{1}{2}\sup_{\epsilon>0}\left|\int_{S^{n-1}}\Omega(u)\int_{|r|>1}f(x-ru)\frac{dr}{r}d\sigma(u)\right|.$$

apply the dominated convergence theorem to get

$$=\frac{\pi}{2}\int_{S^{n-1}}|\Omega(u)|H_u^*f(x)d\sigma(u).$$

Corollary 4.14. *With the same hypotheses as before, the operator T^* defined by (4.6) is strong $(p,p), 1<p<\infty$. In particular, given $f\in L^p$, the limit (4.1) holds for almost every $x\in\mathbb{R}^n$.*

$$T^*f(x)=\sup_{\epsilon>0}\left|\int_{|y|>\epsilon}\frac{\Omega(y')}{|y|^n}f(x-y)dy\right|\leq\frac{\pi}{2}\int_{S^{n-1}}|\Omega(u)|H^*uf(x)d\sigma(u)$$

As H_u^* is strong (p, p), we have T^* is strong(p,p)

Apply Theorem 2.7: Let$\{T_t\}$ be a family of linear operators on $L^p(X,\mu)$ and define

$$T^*f(x)=\sup_t|T_tf(x)|.$$

If T^* is weak (p,q), then the set

$$\{f\in L^p(X,\mu):\lim_{t\to t_0}T_tf(x)=f(x)a.e.\}$$

is closed in $L^p(X,\mu)$.

Here: T^* is strong (p, p) implies T^* is weak (p,p),

then the set $\left\{f\in L^P(x,\mu):\lim_{t\to\epsilon_0}\left|\int_{|y|>\epsilon_0}\frac{\Omega(y')}{|y|^n}f(x-y)dy\right|=f(x)\text{ a.e. }\right\}$ is closed in $L^P(x,\mu)$,

thus the limit holds for almost every $x\in R^n$

An important family of operators with odd kernels consists of the Riesz transforms,

$$R_j f(x) = c_n \text{ p.v. } \int_{\mathbb{R}^n} \frac{y_j}{|y|^{n+1}} f(x-y)dy, \quad 1 \le j \le n,$$

where

$$c_n = \Gamma\left(\frac{n+1}{2}\right)\pi^{-\frac{n+1}{2}}.$$

The constant is fixed so that for $f \in L^2$,

$$(R_j f)^{\imath}(\xi) = -i\frac{\xi_j}{|\xi|}\hat{f}(\xi)$$

this in turn implies that

$$\sum_{j=1}^{n} R_j^2 = -I,$$

where I is the identity operator in L^2. Since the Schwartz functions are dense in $L^p, p > 1$, this identity in fact holds in every L^p space.

To prove

$$(R_j f)^{\wedge}(\xi) = -i\frac{\xi j}{|\xi|}\hat{f}(\xi)$$

First method:

$$\begin{aligned} m(\xi) &= c_n \cdot \lim_{\epsilon \to 0} \int_{\epsilon \in |yk/1)} \frac{y_j}{1yn^{n+1}} e^{-2\pi i\xi\xi} dy \\ &= c_n \cdot \lim_{\epsilon \to 0} \int_{s^{n-1}} u_j \left[\int_{\epsilon}^{1} \left(e^{-2\pi iru\cdot\xi} - 1\right)\frac{dr}{r} + \int_{1}^{y/\epsilon} e^{-2\pi iru\cdot\xi}\frac{dr}{r}\right] d\sigma(u) \end{aligned}$$

Thus by Theorem 4.8.

$$
\begin{aligned}
m(\xi) &= \int_{S^{n-1}} u_j \left[\log\left(\frac{1}{|u\cdot\xi'|}\right) - i\frac{\pi}{2}\operatorname{sgn}(u\cdot\xi')\right] d\sigma(u) \\
&= -i\frac{\xi_j}{|\xi|}
\end{aligned}
$$

Another method:

With equality in the sense of distributions,

$$\frac{\partial}{\partial x_j}|x|^{-n+1} = (1-n)\ \text{p. v.}\ \frac{x_j}{|x|^{n+1}}$$

since

$$
\begin{aligned}
&\left(|x|^{-a}\right)^{\wedge}(\xi) = \frac{\pi^{a-\frac{n}{2}}\Gamma\left(\frac{n-a}{2}\right)}{\Gamma\left(\frac{a}{2}\right)}|\xi|^{a-n} \\
&\Rightarrow \left(|x|^{-n+1}\right)^{\wedge}(\xi) = \frac{\pi^{\frac{n}{2}-1}\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}|\xi|^{-1} \qquad (\text{where } a = n-1)
\end{aligned}
$$

we have

$$
\begin{aligned}
\left(\text{p.v. } \frac{x_j}{|x|^{n+1}}\right)^{-}(\xi) &= \frac{1}{1-n}\left(\frac{\partial}{\partial x_j}|x|^{-n+1}\right)^{-}(\xi) \\
&= \frac{2\pi i\xi_j}{1-n}\left(|x|^{-n+1}\right)^{-}(\xi) \\
&= \frac{2\pi i\xi_j}{1-n}\frac{\pi^{\frac{n}{2}-1}\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}|\xi|^{-1} \\
&= -i\frac{\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}\cdot\frac{\xi_j}{|\xi|}
\end{aligned}
$$

4.4. **Singular integrals with even kernel.** If the function Ω in the singular integral (4.1) is even, then the method of rotations does not apply since we cannot represent the singular integral in terms of the Hilbert transform. However, we ought to be able to argue as follows: by $\sum_{j=1}^{n} R_j^2 = -I$,

$$Tf = -\sum_{j=1}^{n} R_j^2(Tf) = -\sum_{j=1}^{n} R_j\left(R_j T\right) f$$

and the operator R_jT is odd since it is the composition of an odd and even operator.

If we can show that R_jT has a representation of the form (4.1), then by Corollary 4.13, T is bounded on L^p.

Let Ω be an even function with zero average in $L^q\left(S^{n-1}\right)$, for some $q > 1$, and for $\epsilon > 0$ let

$$K_\epsilon(x) = \frac{\Omega\left(x'\right)}{|x|^n}\chi_{\{|x|>\epsilon\}}$$

Note that $K_\epsilon \in L^r, 1 < r \leq q$. Thus if $f \in C_c^\infty\left(\mathbb{R}^n\right)$ then by taking the Fourier transform we see that

$$R_j\left(K_\epsilon * f\right) = \left(R_j K_\epsilon\right) * f$$

Lemma 4.15. *With the preceding hypotheses, there exists a function $\widetilde{K}_j$ which is odd, homogeneous of degree $-n$ and such that*

$$\lim_{\epsilon\to 0} R_j K_\epsilon(x) = \widetilde{K}_j(x)$$

in the L^∞ norm on every compact set that does not contain the origin.

Proof. Fix $x \neq 0$ and let $0 < \epsilon < \nu < |x|/2$. Then for almost every such x, by Corollary 4.14

$$\begin{aligned} R_j K_\epsilon(x) - R_j K_\nu(x) &= c_n \lim_{\delta\to 0}\int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}}\chi_{\{|x-y|>\delta\}}\left[K_\epsilon(y) - K_\nu(y)\right] dy \\ &= c_n \int_{\epsilon<|y|<\nu} \frac{x_j - y_j}{|x-y|^{n+1}}\frac{\Omega\left(y'\right)}{|y|^n} dy; \end{aligned}$$

since Ω has zero average,

$$= c_n \int_{\epsilon<|y|<\nu} \left(\frac{x_j - y_j}{|x-y|^{n+1}} - \frac{x_j}{|x|^{n+1}} \right) \frac{\Omega(y')}{|y|^n} dy.$$

Therefore, if we apply the mean value theorem to the integrand we get

$$|R_j K_\epsilon(x) - R_j K_\nu(x)| \leq \frac{C}{|x|^{n+1}} \int_{\epsilon<|y|<\nu} \frac{|\Omega(y')|}{|y|^{n-1}} dy \leq \frac{C\|\Omega\|_1}{|x|^{n+1}} \nu$$

Hence, for any $\alpha > 0, \{R_j K_\epsilon\}$ is a Cauchy sequence in the L^∞ norm on $\{|x| > \alpha\}$. So for almost every x we can define K_j^* by

$$K_j^*(x) = \lim_{\epsilon \to 0} R_j K_\epsilon(x).$$

The function $R_j K_\epsilon$ is odd, so (by modifying K_j^* on a set of measure zero if necessary) K_j^* is also an odd function.

To find the desired function $\widetilde{K}_j$, fix $\lambda > 0$; then again for almost every x

$$\begin{aligned} R_j K_\epsilon(\lambda x) &= \lim_{\delta \to 0} c_n \int_{|\lambda x - y|>\delta} \frac{\lambda x_j - y_j}{|\lambda x - y|^{n+1}} K_\epsilon(y) dy \\ &= \lim_{\delta \to 0} c_n \int_{|x-y|>\delta/\lambda} \frac{x_j - y_j}{|x-y|^{n+1}} \lambda^{-n} K_{\epsilon/\lambda}(y) dy \\ &= \lambda^{-n} R_j K_{\epsilon/\lambda}(x). \end{aligned}$$

Hence, for almost every $x, K_j^*(\lambda x) = \lambda^{-n} K_j^*(x)$. The set of measure zero where equality does not hold depends on λ, but since K_j^* is measurable, the set

$$D = \left\{ (x, \lambda) \subset \mathbb{R}^n \times (0, \infty) : K_j^*(\lambda x) \neq \lambda^{-n} K_j^*(x) \right\}$$

has measure zero. Therefore, by Fubini's theorem there exists a sphere centered at the origin of radius ρ, S_ρ, such that $D \cap S_\rho$ has measure zero. We now define

$$\widetilde{K}_j(x) = \begin{cases} \left(\frac{\rho}{|x|}\right)^n K_j^* \left(\frac{\rho x}{|x|}\right) & \text{if } x \neq 0 \text{ and } \rho x/|x| \notin D \cap S_\rho; \\ 0 & \text{otherwise.} \end{cases}$$

This function is measurable, homogeneous of degree $-n$ (by definition) and odd. Further, $K_j^*(x) = \widetilde{K}_j(x)$ almost everywhere. To see this, let $x \neq 0$ be such that $x_0 = \rho x/|x| \notin D \cap S_\rho$. (The set of x such that $x_0 \in D \cap S_\rho$ has measure zero since $D \cap S_\rho$ has measure zero.) Then for almost every λ,

$$\widetilde{K}_j(\lambda x_0) = \lambda^{-n}\widetilde{K}_j(x_0) = \lambda^{-n}K_j^*(x_0) = K_j^*(\lambda x_0).$$

Lemma 4.16. *The kernel $\widetilde{K}_j$ defined in Lemma 4.15 satisfies*

$$\int_{S^{n-1}} \left|\widetilde{K}_j(u)\right| d\sigma(u) \leq C_q\|\Omega\|_q$$

Furthermore, if $\widetilde{K}_{j,\epsilon}(x) = \widetilde{K}_j(x)\chi_{\{|x|>\epsilon\}}$, then $\Delta_\epsilon = R_jK_\epsilon - \widetilde{K}_{j,\epsilon} \in L^1(\mathbb{R}^n)$ and $\|\Delta_\epsilon\|_1 \leq C_q'\|\Omega\|_q$.

Proof. By the homogeneity of $\widetilde{K}_j$,

$$\begin{aligned}\int_{S^{n-1}} \left|\widetilde{K}_j(u)\right| d\sigma(u) =& \frac{1}{\log 2}\int_{1<|x|<2}\left|\widetilde{K}_j(x)\right| dx\\ \leq& \frac{1}{\log 2}\int_{1<|x|<2}\left|\widetilde{K}_j(x) - R_jK_{1/2}(x)\right| dx\\ &+ \frac{1}{\log 2}\int_{1<|x|<2}\left|R_jK_{1/2}(x)\right| dx.\end{aligned}$$

Let $\nu = 1/2$ and $|x| > 1$; then if we take the limit as $\epsilon \to 0$ we get

$$\left|\widetilde{K}_j(x) - R_jK_{1/2}(x)\right| \leq \frac{C\|\Omega\|_1}{|x|^{n+1}}$$

Therefore, the first integral on the right-hand side above is bounded by $C\|\Omega\|_1 \leq C\|\Omega\|_q$. To bound the second integral, note that

$$\int_{1<|x|<2}\left|R_jK_{1/2}(x)\right| dx \leq C\left\|R_jK_{1/2}\right\|_q \leq C\left\|K_{1/2}\right\|_q \leq C\|\Omega\|_q$$

To prove the second assertion it will suffice to show that $\|\Delta_1\|_1 < \infty$ since $\Delta_\epsilon = \epsilon^{-n}\Delta_1(\epsilon^{-1}x)$. But then

$$\begin{aligned}\|\Delta_1\|_1 &= \int_{\mathbb{R}^n} \left|R_j K_1(x) - \widetilde{K}_{j,1}(x)\right| dx \\ &\leq \int_{|x|<2} |R_j K_1(x)|\, dx + \int_{1<|x|<2} \left|\widetilde{K}_j(x)\right| dx + \int_{|x|>2} |\Delta_1(x)|\, dx.\end{aligned}$$

The first integral is bounded by $C\left\|R_j K_1\right\|_q \leq C\left\|K_1\right\|_q \leq C\|\Omega\|_q$; above we showed that the second integral has the same bound.

To see that the third integral is bounded,

$$\|\Delta_1\|_1 = \int_{\mathbb{R}^n} \left|R_j K_1(x) - \widetilde{K}_{j,1}(x)\right| dx$$

Let $\mu = 1, |x| > 2$,; then if we take the limit as $\epsilon \to 0$ we get

$$\left|\widetilde{K}_j(x) - R_j K_1(x)\right| \leq \frac{C\|\Omega\|_1}{|x|^{n+1}}$$

similar to above, then we get

$|\Delta_1(x)| \leq C\|\Omega\|_1 |x|^{-n-1}$

Theorem 4.17. *Let Ω be a function on S^{n-1} with zero average such that its odd part is in $L^1\left(S^{n-1}\right)$ and its even part is in $L^q\left(S^{n-1}\right)$ for some $q > 1$. Then the singular integral T in (4.1) is bounded on $L^p\left(\mathbb{R}^n\right), 1 < p < \infty$.*

Proof. By Corollary 4.13 we may assume Ω is even. Further, by arguing as we did for the Hilbert transform, it will suffice to establish the L^p inequality for functions $f \in C_c^\infty\left(\mathbb{R}^n\right)$. For such $f, Tf = \lim_{\epsilon \to 0} K_\epsilon * f$. From

$$\sum_{j=1}^n R_j^2 = -I,$$

and

$$R_j\left(K_\epsilon * f\right) = \left(R_j K_\epsilon\right) * f$$

we have

$$K_\epsilon * f = -\sum_{j=1}^{n} R_j\left((R_j K_\epsilon) * f\right),$$

and in the notation of Lemma 4.16,

$$(R_j K_\epsilon) * f = \widetilde{K}_{j,\epsilon} * f + \Delta_\epsilon * f$$

By Lemma 4.15, $\widetilde{K}_j$ is an odd, homogeneous kernel of degree $-n$, so by Corollary 4.13 ,

$$\left\|\widetilde{K}_{j,\epsilon} * f\right\|_p \leq C \int_{S^{n-1}} \left|\widetilde{K}_j(u)\right| d\sigma(u) \|f\|_p \leq C\|\Omega\|_q \|f\|_p$$

By Lemma 4.11,

$$\|\Delta_\epsilon * f\|_p \leq \|\Delta_\epsilon\|_1 \|f\|_p \leq C\|\Omega\|_q \|f\|_p$$

If we combine these estimates and use the fact that R_j is bounded in L^p we see that

$$\|K_\epsilon * f\|_p \leq C\|\Omega\|_q \|f\|_p$$

Since the right-hand side is independent of ϵ, by Fatou's lemma we get

$$\|Tf\|_p \leq C\|\Omega\|_q \|f\|_p$$

Q.E.D.

5. Application of Calderon-Zygmund Decomposition

5.1. In second-order elliptic equations.

Definition 5.1. The Poisson equation is a partial differential equation of the form:

$$\Delta^2 u = f(x)$$

where u is an unknown scalar function of the spatial coordinates x, and f(x) is a given function.

The Poisson equation is a fundamental tool for studying a wide range of physical phenomena that involve the distribution of a scalar quantity in a region of space. Its solution can provide valuable insights into the behavior of these systems and can be used to design and optimize a variety of technological applications.

Definition 5.2. (The Newtonian potential) Let Ω be a bounded domain in $\mathbb{R}^n$ and f a function in $L^p(\Omega)$ for some $p \geqslant 1$ The Newtonian potential of f is the function $w = Nf$ defined by the convolution.

$$w(x) = \int_\Omega \Gamma(x-y) f(y) dy$$

where Γ is the fundamental solution of Laplace's equation given by

$$\Gamma(x-y) = \Gamma(|x-y|) = \begin{cases} \frac{1}{n(2-n)\omega_n} |x-y|^{2-n}, & n > 2 \\ \frac{1}{2\pi} \log |x-y|, & n = 2. \end{cases}$$

When the boundary of a domain Ω is sufficiently smooth, a function u that is twice continuously differentiable over the closure of Ω, denoted as $C^2(\bar{\Omega})$, can be expressed as the sum of a harmonic function and the Newtonian potential of its Laplacian. This observation leads to the idea that the study of Poisson's equation, given by $\Delta u = f$, can be effectively carried out by studying the Newtonian potential of the function f.

The following theory provides an estimate of the basic L^p estimates for Poisson's equation through the Newtonian potential we introduced above. This theory applies **The Calderon-Zygmund Decomposition**.

Theorem 5.3. *Let $f \in L^p(\Omega), 1 < p < \infty$, and let w be the Newtonian potential of f. Then $w \in W^{2,p}(\Omega), \Delta w = f$ a.e. and*

$$\left\|D^2 w\right\|_p \leqslant C\|f\|_p$$

where C depends only on n and p. Furthermore, when $p = 2$ we have

$$\int_{R^n} \left|D^2 w\right|^2 = \int_\Omega f^2$$

Notes:This proof makes reference to Elliptic Partial Differential Equations of Second Order (David Gilbarg , Neil S. Trudinger, 2000).

Proof: case(i) $p = 2$. If $f \in C_0^\infty(\mathbb{R}^n)$, we have $w \in C^\infty(\mathbb{R}^n)$ and $\Delta w = f$. Consequently, for any ball B_R containing the support of f

$$\int_{B_R} (\Delta w)^2 = \int_{B_R} f^2.$$

Proposition 5.4. *Let Ω be a domain for which the divergence theorem holds and let u and v be $C^2(\bar{\Omega})$ functions.*

Green's first identity:

$$\int_\Omega v\Delta u dx + \int_\Omega Du \cdot Dv dx = \int_{\partial\Omega} v\frac{\partial u}{\partial v} ds$$

Interchange u and v in above equation and subtract, then obtain Green's second identity:

$$\int_\Omega (v\Delta u - u\Delta v) dx = \int_{\partial\Omega} \left(v\frac{\partial u}{\partial v} - u\frac{\partial v}{\partial v}\right) ds.$$

Applying Green's first identity twice, we then obtain

$$\begin{aligned}\int_{B_R} \left|D^2 w\right|^2 &= \int_{B_R} \sum (D_{ij} w)^2 \\ &= \int_{B_R} f^2 + \int_{\partial B_R} Dw \cdot \frac{\partial}{\partial v} Dw.\end{aligned}$$

Take derivative of Γ:

$$D_i\Gamma(x-y)=\frac{1}{n\omega_n}\left(x_i-y_i\right)|x-y|^{-n}$$

Clearly Γ is harmonic for $x\neq y$. then derivative estimates:

$$\left|D_i\Gamma(x-y)\right|\leqslant\frac{1}{n\omega_n}|x-y|^{1-n}$$

thus we have

$$Dw=O\left(R^{1-n}\right), D^2w=O\left(R^{-n}\right)$$

uniformly on ∂B_R as $R\to\infty$, then

$$\int_{R^n}\left|D^2w\right|^2=\int_\Omega f^2$$

For fixed i,j, define the linear operator $T:L^2(\Omega)\to L^2(\Omega)$ by

$$Tf=D_{ij}w$$

Then define $\mu(t)=\mu_f(t)=|\{x\in\Omega\mid f(x)>t\}|$ we have $\mu(t)\leq t^{-2}\int_{|f|\geq t}|f|^2\leq t^{-2}\int_\Omega|f|^2$

$$\mu(t)=\mu_{Tf}(t)\leqslant\left(\frac{\|f\|_2}{t}\right)^2$$

for all $t>0$ and $f\in L^2(\Omega)$.

Proposition 5.5. *for all $t>0$ and $f\in L^2(\Omega)$*

$$\mu(t)\leqslant\frac{C\|f\|_1}{t}$$

thereby making possible the application of the Marcinkiewicz interpolation theorem.

To accomplish this, first extend f to vanish outside Ω and fix a cube $K_0\supset\Omega$, so that for fixed $t>0$

$$\int_{\boldsymbol{K}_0}f\leqslant t\left|K_0\right|.$$

The cube K_0 is now decomposed yielding a sequence of parallel subcubes $\{K_l\}_{l=1}^\infty$ such that

$$t < \frac{1}{|K_l|} \int_{h_1} |f| < 2^n t$$

and

$$|f| \leqslant t \text{ a.e. on } G = K_0 - \cup K_l.$$

The function f is split into a good part g defined by

$$g(x) = \begin{cases} f(x) & \text{for } x \in G \\ \frac{1}{|K_l|} \int_{\kappa_l} f & \text{for } x \in K_l, \quad l = 1, 2, \ldots, \end{cases}$$

and a bad part $b = f - g$.

$$|g| \leqslant 2^n t \quad \text{a.e.,}$$

$$b(x) = 0 \quad \text{for } x \in G,$$

$$\int_{\kappa_l} b = 0 \quad \text{for } l = 1, 2, \ldots$$

Since T is linear, $Tf = Tg + Tb$; and

$$\mu_{Tf}(t) \leqslant \mu_{Tg}(t/2) + \mu_{Tb}(t/2).$$

(1) Estimation of Tg : By

$$\mu(t) \leqslant \left(\frac{\|f\|_2}{t}\right)^2$$

we have that

$$\begin{aligned} \mu_{Tg}(t/2) &\leqslant \frac{4}{t^2} \int g^2 \\ &\leqslant \frac{2^{n+2}}{t} \int |g| \\ &\leqslant \frac{2^{n+2}}{t} \int |f| \end{aligned}$$

(2) Estimation of Tb :

$$b_l = b\chi_{\kappa_l} = \begin{cases} b & \text{on } K_l \\ 0 & \text{elsewhere} \end{cases}$$

and then

$$Tb = \sum_{l=1}^{\infty} Tb_l$$

Fix some l and a sequence $\{b_{lm}\} \subset C_0^{\infty}(K_l)$ converging to b_l in $L^2(\Omega)$ and satisfying

$$\int_{\kappa_l} b_{lm} = \int_{\kappa_l} b_l = 0$$

Then for $x \notin K_l$:

$$\begin{aligned} Tb_{lm}(x) &= \int_{\kappa_l} D_{ij}\Gamma(x-y)b_{lm}(y)dy \\ &= \int_{\kappa_l} \{D_{ij}\Gamma(x-y) - D_{ij}\Gamma(x-\bar{y})\} b_{lm}(y)dy, \end{aligned}$$

where $\bar{y} = \bar{y}_l$ denotes the center of K_l. Letting $\delta = \delta_l$ denote the diameter of K_l, then

$$|Tb_{lm}(x)| \leqslant C(n)\delta \left[\text{dist}\,(x, K_l)\right]^{-n-1} \int_{\kappa_l} |b_{lm}(y)|\, dy$$

Let $B_l = B_\delta(\bar{y})$*denote the concentric ball of radius*δ, by integration

$$\begin{aligned} \int_{\kappa_0 - B_l} |Tb_{lm}| &\leqslant C(n)\delta \int_{|x|\geqslant\delta/2} \frac{dx}{|x|^{n+1}} \int_{\kappa_l} |b_{lm}| \\ &\leqslant C(n) \int_{\kappa_l} |b_{lm}|\,. \end{aligned}$$

Consequently, when $m \to \infty$, define $F^* = \cup B_l, G^* = K_0 - F^*$ and summing over l, we get

$$\begin{aligned} \int_{G^*} |Tb| &\leqslant C(n) \int |b| \\ &\leqslant C(n) \int |f|, \end{aligned}$$

Therefore,

$$|\{x \in G^* \| Tb |> t/2\}| \leqslant \frac{C\|f\|_1}{t}.$$

Then

$$\begin{aligned}|F^*| &\leqslant \omega_n n^{n/2}|F| \\ &\leqslant \frac{C\|f\|_1}{t},\end{aligned}$$

and thus Proposition 5.5 holds.

(ii) By the Marcinkiewicz interpolation theorem, T is $weak(1,1)$ *and* $weak(2,2)$.Then for all $1 < p \leqslant 2$ and $f \in L^2(\Omega)$:

$$\|Tf\|_p \leqslant C\|f\|_p$$

The inequality is extended to $p > 2$ by duality. For if $f, g \in C_0^\infty(\Omega)$, then

$$\begin{aligned}\int_\Omega (Tf)g &= \int_\Omega wD_{ij}g \\ &= \iint_\Omega \Gamma(x-y)f(y)D_{ij}g(x)dxdy \\ &= \int_\Omega fTg \\ &\leqslant \|f\|_p\|Tg\|_p.\end{aligned}$$

Thus if $p > 2$, we have,

$$\begin{aligned}\|Tf\|_p &= \sup\left\{\int_\Omega (Tf)g \mid \|g\|_{p'} = 1\right\} \\ &\leqslant C\|f\|_p,\end{aligned}$$

so that T is $weak(p,p)$ for all $1 < p < \infty$. As in the case $p = 2$, we can then infer the full conclusion of Theorem by approximation.

Notation 5.6. Note that T can be defined as a bounded operator on $L^p(\Omega)$ even if Ω is unbounded, in which case the conclusion of Theorem 5.3 still holds provided $n \geqslant 3$.

The L^p estimates for solutions of Poisson's equation follow immediately from Theorem 5.3.

Theorem 5.7. *Let Ω be a domain in $\mathbb{R}^n, u \in W_0^{2,p}(\Omega), 1 < p < \infty$. Then*

$$\left\|D^2 u\right\|_p \leqslant C\|\Delta u\|_p$$

where $C = C(n,p)$. If $p = 2$

$$\left\|D^2 u\right\|_2 = \|\Delta u\|^2$$

From theorem 5.6 we will derive the interior and interior and global L^p estimates for the second derivatives of elliptic equations of the forms

$$Lu = a^{ij}(x) D_{ij} u + b^i(x) D_i u + c(x) u$$

with coefficients $a^{ij}, b^i, c,$ *where* $i, j = 1, \ldots, n$ this is defined on $\Omega \in R^n$ and a function f on Ω, a strong solution of the equation

$$Lu = f$$

is a twice weakly differentiable function on Ω and satisfying this equation almost everywhere on the domain.

Theorem 5.8. *(Interior estimates) Let Ω be an open set in $\mathbb{R}^n$ and $u \in W_{loc}^{2,p}(\Omega) \cap L^p(\Omega), 1 < p < \infty,$ a strong solution of the equation $Lu = f$ in Ω where the coefficients of L satisfy, for positive constants $\lambda, \Lambda,$*

$$\begin{aligned}
&a^{ij} \in C^0(\Omega), \quad b^i, c \in L^\infty(\Omega), \quad f \in L^p(\Omega); \\
&a^{ij}\xi_i\xi_j \geqslant \lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^n; \\
&\left|a^{ij}\right|, \left|b^i\right|, |c| \leqslant \Lambda,
\end{aligned}$$

where $i, j = 1, \ldots, n$. Then for any domain $\Omega' \subset\subset \Omega$,

$$\|u\|_{2,p;\Omega'} \leqslant C\left(\|u\|_{p;\Omega} + \|f\|_{p;\Omega}\right)$$

where C depends on $n, p, \lambda, \Lambda, \Omega', \Omega$ and the moduli of continuity of the coefficients a^{ij} on Ω'.

Theorem 5.9. *Let Ω be a $C^{1,1}$ domain in $\mathbb{R}^n$ and suppose the operator L satisfies the conditions* (9.35) *with $a^{ij} \in C^0(\bar{\Omega}), i, j = 1, \ldots, n$. Then if $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, $1 < p < \infty$, we have*

$$\|u\|_{2,p;\Omega} \leqslant C\|Lu - \sigma u\|_{p;\Omega}$$

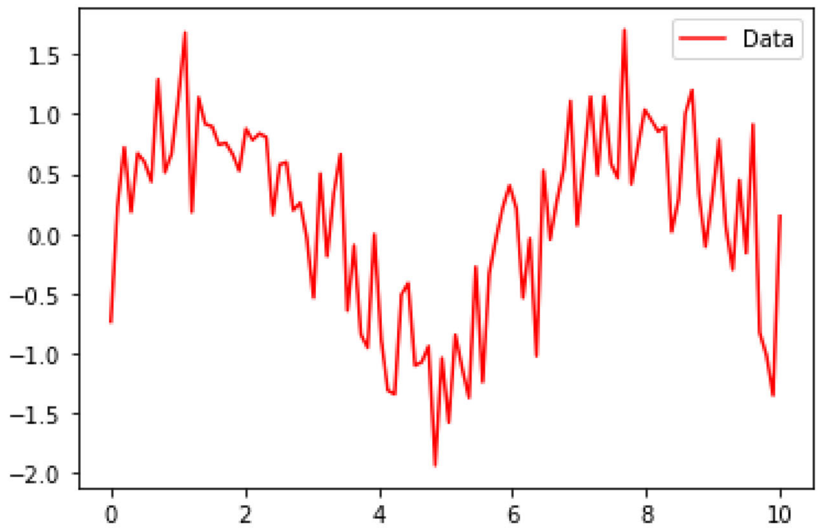
for all $\sigma \geqslant \sigma_0$, where C and σ_0 are positive constants depending only on $n, p, \lambda, \Lambda, \Omega$ and the moduli of continuity of the coefficients a^{ij}.

Python Implementation of Calderon-Zygmund Decomposition

This function new_quad is to integrate function on the interval(a,b)

In [119…

```
import numpy as np
import matplotlib.pyplot as plt
import random

# Set a random seed
random.seed(2288)

# Generate some example data
x = np.linspace(0, 10, 100)
y = np.sin(x) + np.random.normal(scale=0.5, size=100)

def new_quad (x,y,a,b):
    n=len(x)
    s=0
    j=0
    step=x[1]-x[0]
    for i in range(n):
        if x[i]>=a:
            xmin=i
            break
    for i in range(xmin,n):
        if x[i]>=a and x[i]<=b:
            s+=y[i]
            j+=1
    return s/j


# Plot the data
plt.plot(x, y, label='Data',color="red")
plt.legend()
plt.show()
```



this function diadic_function is to find the suitable diadic cube that contain the point and larger than lambda

In [132…

```
import numpy as np

def diadic_function( i,x,y,lam):
    a=0
    if x[i]==0:
        return y[i],0,0

    b= 2**int(np.ceil(np.log2(x[i])))
    average=new_quad(x,y,a,b)
    while average <=lam:
        if x[i]<(a+b)/2:
            b=(a+b)/2
        else:
            a=(a+b)/2
        average=new_quad(x,y,a,b)
    return average,a,b


diadic_function(7,x,y,0.5)
```

Out[132]:

```
(0.6687239708682642, 0.5, 1)
```

Major function of Calderon-Zygmund decomposition:

In [135…

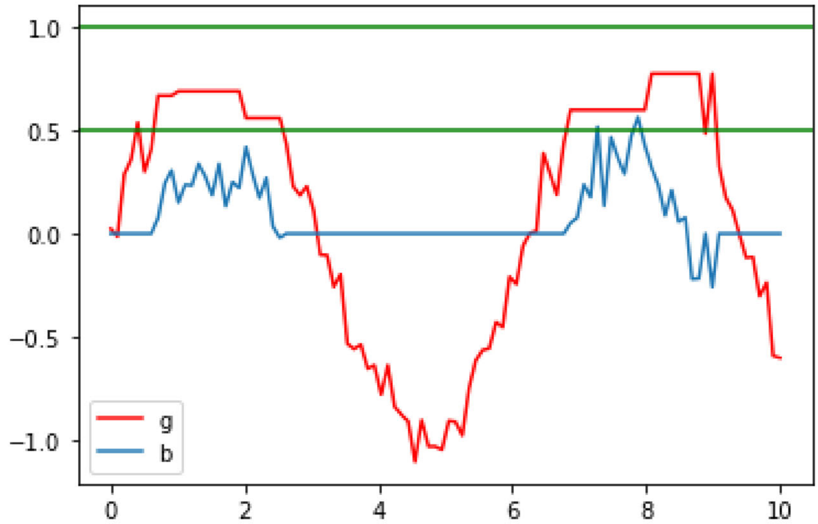
```
def Calderon_Zygmund_decomposition (x,y,lam):
    n = len(x)
    g=np.zeros(n)
    b=np.zeros(n)
    i=0
    omega=0
    while i < n:
        if y[i]<=lam:
            g[i] = y[i]
            b[i] = 0
            i=i+1
        else:
            s=0
            j=i
            a_h=np.zeros(n)
            b_h=np.zeros(n)
            while i<n and y[i]>lam  :

                g[i],a_h[i-j],b_h[i-j]=diadic_function(i,x,y,lam)
                b[i] = y[i]-g[i]
                i+=1
            omega+=max(b_h[:i-j])-min(a_h[:i-j])
    return g,b,omega
```

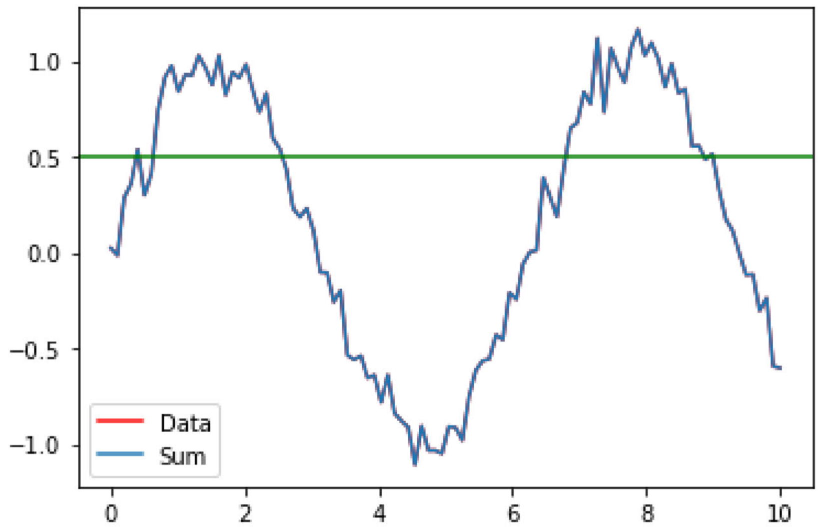
Given a sample function: y=sin(x)+random normal variable with scale of 0.5 We use Calderon-Zygmund decomposition and to verify that:

1. Good part is smaller than 2*lambda
2. Good fnction and bad function should add up to the original function
3. The measure of bad diadic cubes should be smaller than norm{f}/lambda

In [136…

```
threshold= 0.5
gx,bx,omega =   Calderon_Zygmund_decomposition (x,y,threshold)
# Plot the data and the smooth part of the decomposition
plt.plot(x, gx, label='g',color="red")
plt.plot(x, bx, label='b')
plt.axhline(y=threshold, color='green')
plt.axhline(y=2*threshold, color='green')
plt.legend()
plt.show()

new=gx+bx
# Plot the data and the smooth part of the decomposition
plt.plot(x, y, label='Data',color="red")
plt.plot(x, new, label='Sum')
plt.axhline(y=threshold, color='green')
plt.legend()
plt.show()
```





In [137…

```
# calculate the norm of f
norm_f = sum(y)

print(norm_f/threshold)
#Compare the omega and norm of f
omega<norm_f/threshold
```

```
37.46223948626364
```

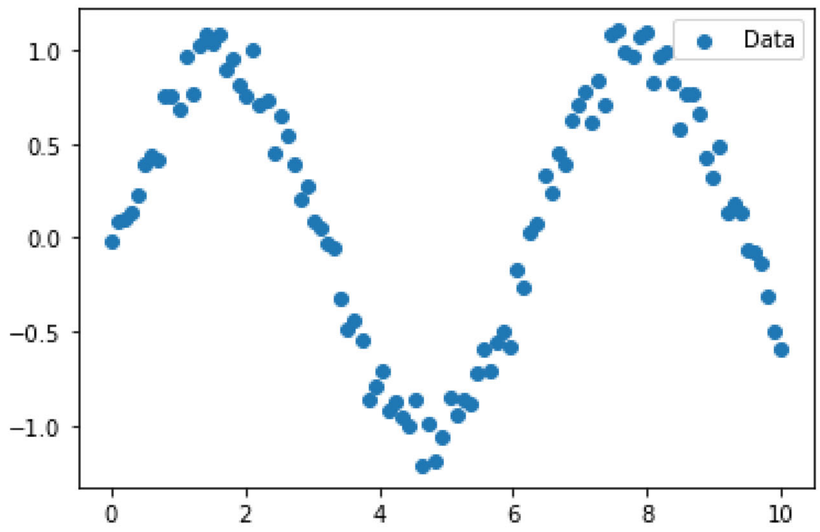
Out[137]:

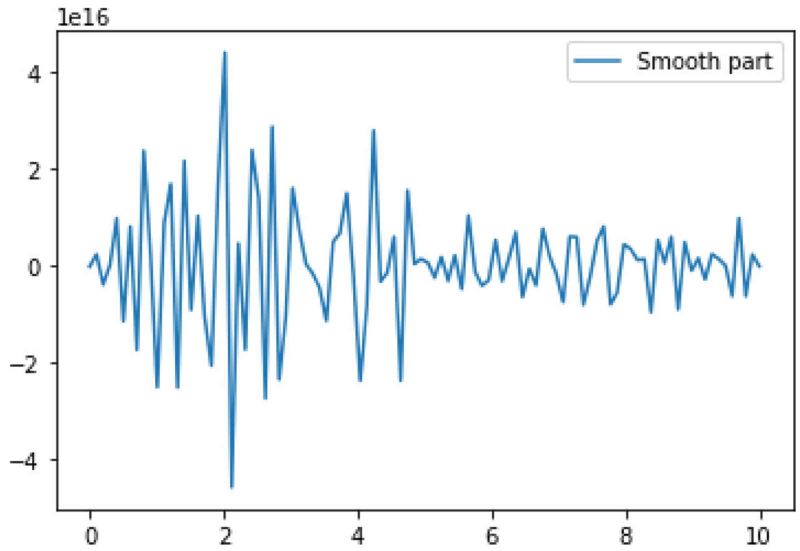
```
True
```

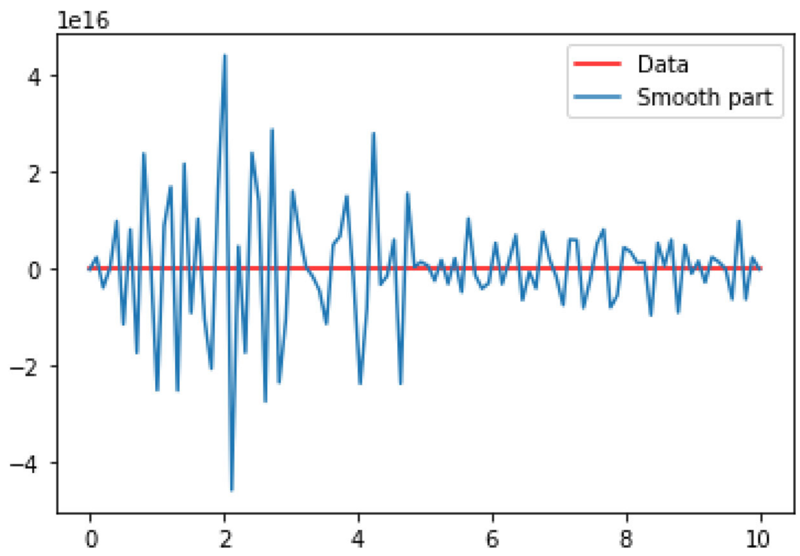
Here we create another more singular function to check

In [138…

```
import numpy as np
import matplotlib.pyplot as plt

# Generate some example data
x = np.linspace(0, 10, 100)
y = np.sin(x) + np.random.normal(scale=0.1, size=100)

# Use Gaussian Kernel to convolution
def convolution(x, y, tau):
    n = len(x)
    K = np.zeros((n, n))
    for i in range(n):
        for j in range(n):
            K[i,j] = np.exp(-np.power(x[i]-x[j], 2) / (2 * tau**2))#K(x_i,
    K_inv = np.linalg.inv(K)
    f_hat = np.dot(K_inv, y)
    e = y - np.dot(K, f_hat)
    return f_hat, e

# Apply the data
f_hat, e = convolution(x, y, 1)


plt.scatter(x, y, label='Data')
plt.legend()
plt.show()

plt.plot(x, f_hat, label='Smooth part')
plt.legend()
plt.show()


# Plot the data and the smooth part of the decomposition
plt.plot(x, y, label='Data',color="red")
plt.plot(x, f_hat, label='Smooth part')
plt.legend()
plt.show()
```



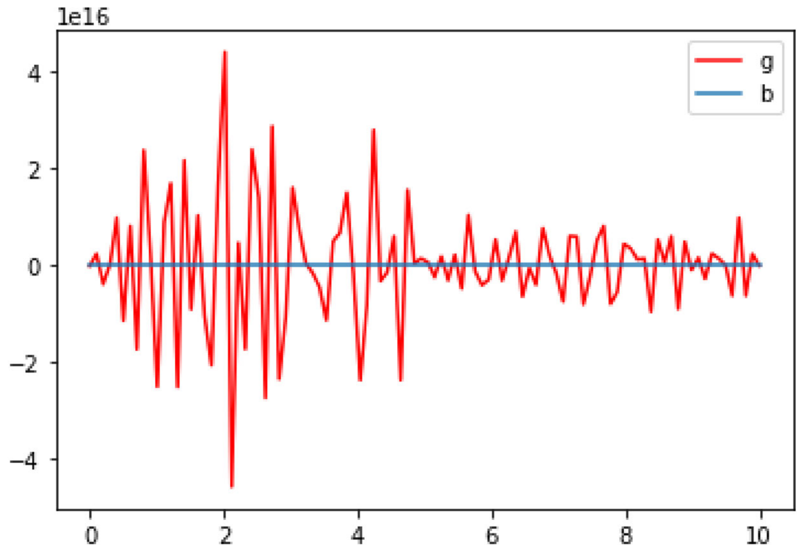


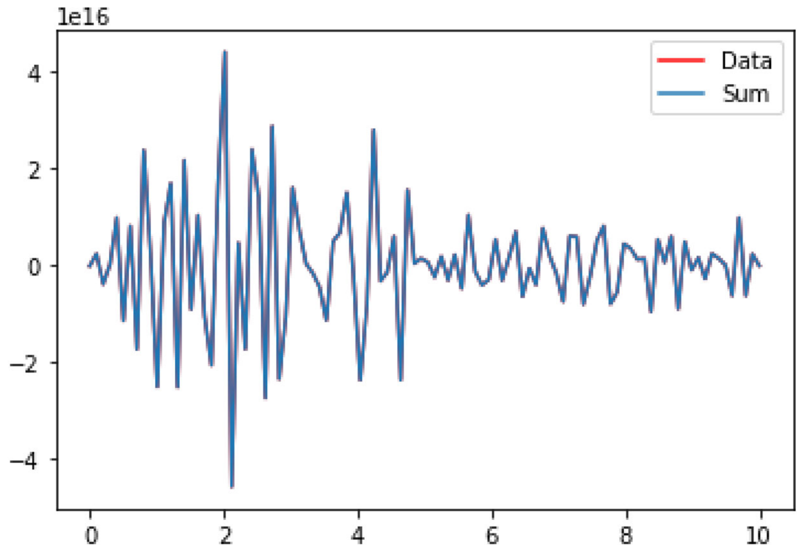


In [139…

```
threshold= 2e16
gx,bx,omega =   Calderon_Zygmund_decomposition (x,f_hat,threshold)
# Plot the data and the smooth part of the decomposition
plt.plot(x, gx, label='g',color="red")
plt.plot(x, bx, label='b')
plt.legend()
plt.show()

new=gx+bx
```

```
# Plot the data and sum of the decomposition
plt.plot(x, f_hat, label='Data',color="red")
plt.plot(x, new, label='Sum')
plt.legend()
plt.show()
```



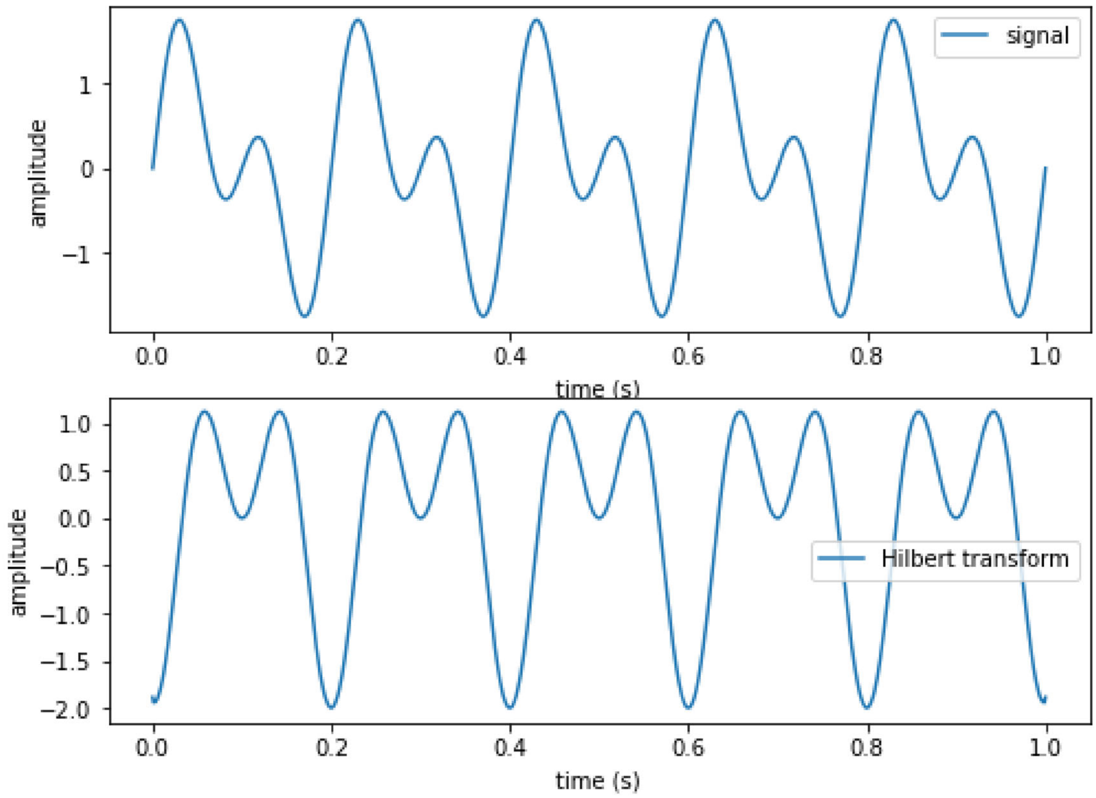


Here we have another Periodic function to decompose. We also get the Hilbert transform of this function.

In [140…

```
import numpy as np
from scipy.signal import hilbert

# Define the signal
t = np.linspace(0, 1, 1000)
x = np.sin(2 * np.pi * 5 * t) + np.sin(2 * np.pi * 10 * t)

def plot_hilbert(t,x):
    # Compute the Hilbert transform
    x_hilbert = hilbert(x)

    # Plot the original signal and its Hilbert transform
    import matplotlib.pyplot as plt

    fig, ax = plt.subplots(2, 1, figsize=(8, 6))
```

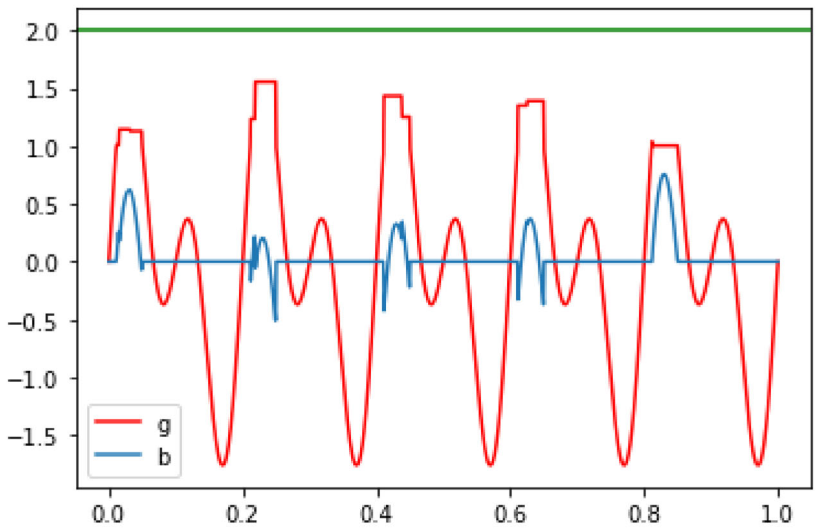
```
    ax[0].plot(t, x, label='signal')
    ax[0].set_xlabel('time (s)')
    ax[0].set_ylabel('amplitude')
    ax[0].legend()

    ax[1].plot(t, np.imag(x_hilbert), label='Hilbert transform')
    ax[1].set_xlabel('time (s)')
    ax[1].set_ylabel('amplitude')
    ax[1].legend()

    plt.show()


plot_hilbert(t,x)
```



Check Calderon-Zygmund decomposition:

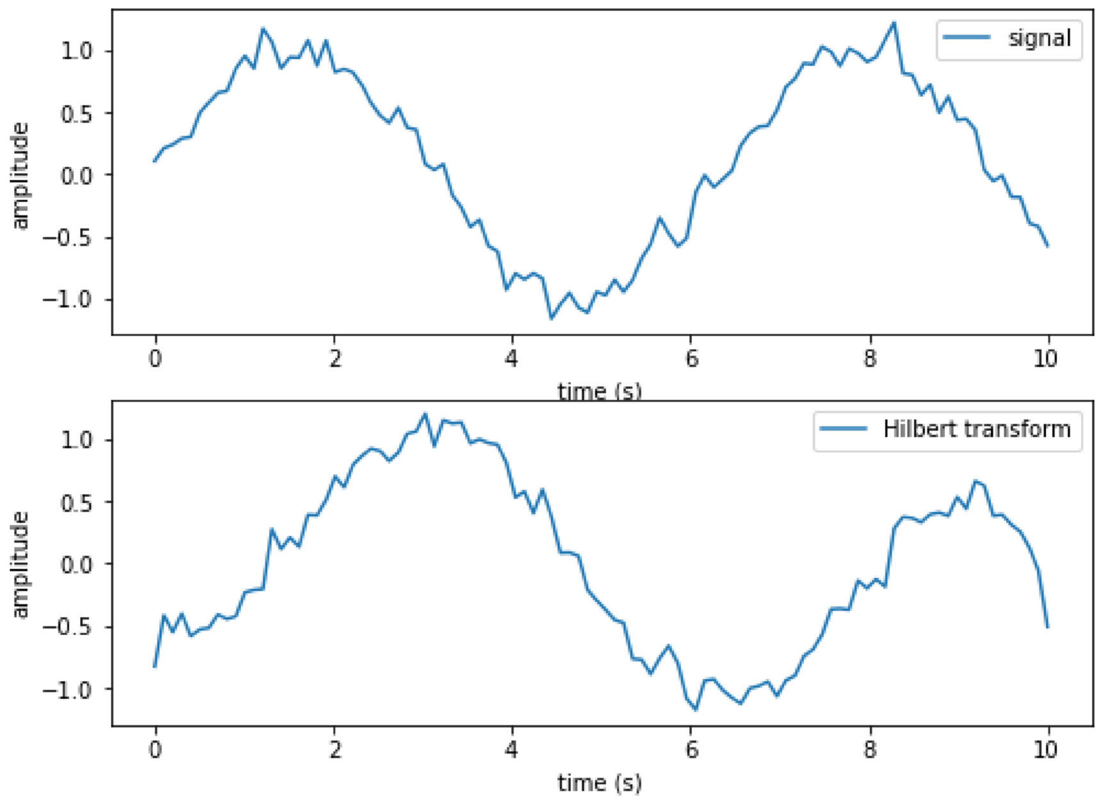
In [141…

```
threshold= 1
gx,bx,omega =   Calderon_Zygmund_decomposition (t,x,threshold)

# Plot the data and the smooth part of the decomposition
plt.plot(t, gx, label='g',color="red")
plt.plot(t, bx, label='b')
plt.axhline(y=2*threshold, color='green')
plt.legend()
plt.show()
```



Back to the first example we get the Hilbert Transform of it:

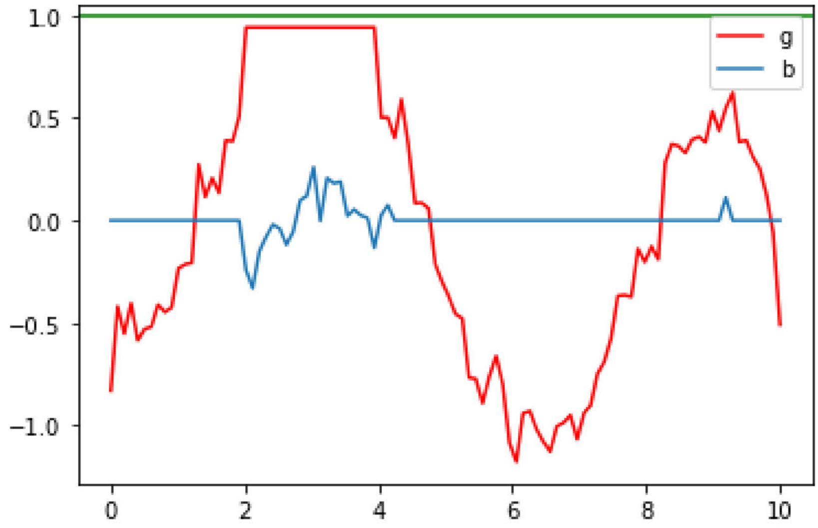
In [142…
```
# Generate some example data
x = np.linspace(0, 10, 100)
y = np.sin(x) + np.random.normal(scale=0.1, size=100)

plot_hilbert(x,y)
```



Decompose the Hf:

In [143…
```
#Hilbert transformation
y_hilbert = hilbert(y)
threshold=0.5
gx,bx,omega =   Calderon_Zygmund_decomposition (x,np.imag(y_hilbert),thresh
# Plot the data and the good&bad of the decomposition
plt.plot(x, gx, label='g',color="red")
plt.plot(x, bx, label='b')
```

```
plt.axhline(y=2*threshold, color='green')
plt.legend()
plt.show()
```

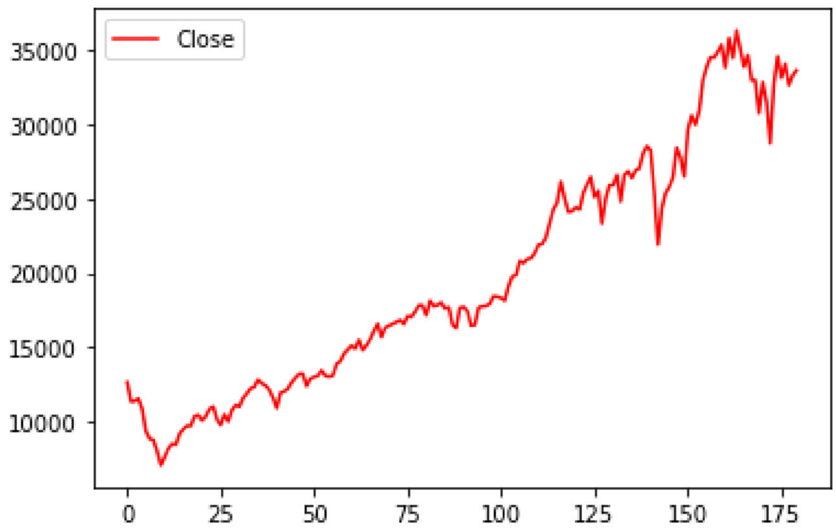


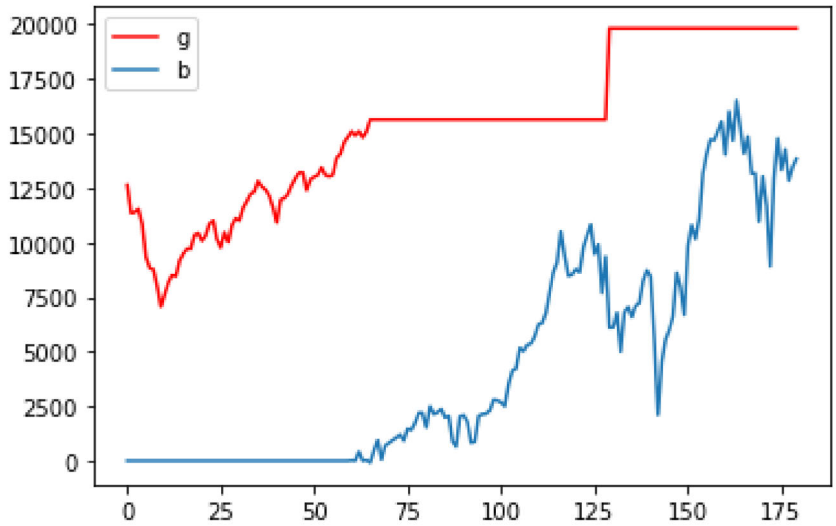
We use real life data: Dow Jones Index from 2008 May to 2023 April

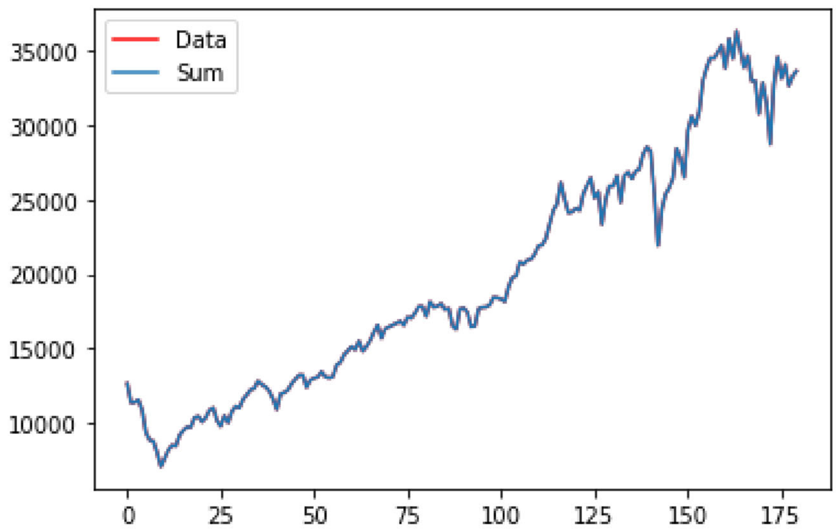
Here is the original data and we will decompose the function:

In [149…

```
import pandas as pd
#read the monthly DowJones data from 2018 May to 2023 Apr
#data source: https://stooq.com/q/d/?s=%5Edji&c=0&d1=20080516&d2=20230410&i
df = pd.read_csv('C:\\Users\\Wen Feiyang\\Desktop\\urop\\DowJones.csv')
t = df['Close'].tolist()
index = df.index.tolist()
gx ,bx ,omega =   Calderon_Zygmund_decomposition (index,t,15000)
# Plot the data
plt.plot(index,t, label='Close',color="red")
plt.legend()
plt.show()
# Plot the data and the good&bad of the decomposition
plt.plot(index, gx, label='g',color="red")
plt.plot(index, bx, label='b')
plt.legend()
plt.show()

new=gx+bx
# Plot the data and sum of the decomposition
plt.plot(index, t, label='Data',color="red")
plt.plot(index, new, label='Sum')
plt.legend()
plt.show()
```


Close
35000
30000
25000
20000
15000
10000
0
25
50
75
100
125
150
175


g
b
20000
17500
15000
12500
10000
7500
5000
2500
0
0
25
50
75
100
125
150
175


Data
Sum
35000
30000
25000
20000
15000
10000
0
25
50
75
100
125
150
175

In [ ]: