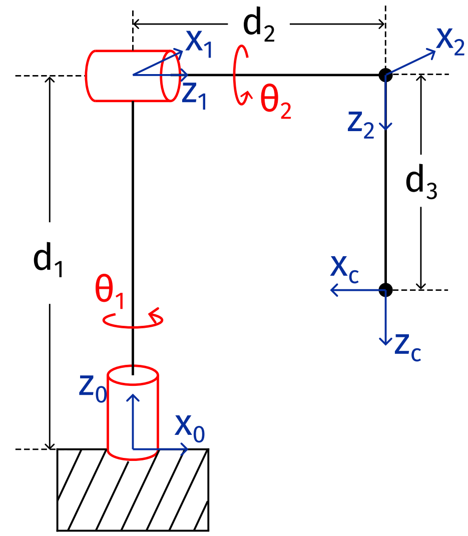
ME 4/567 – ECE 464: Robotics and Automated Systems
Spring 2025 — Project I: Kinematics of Furuta Pendulum

In this project, we will analyze the kinematics of the Furuta pendulum [1] using the software package MuJoCo [2].

The Furuta pendulum consists of three links (including the fixed/world frame $\{0\}$), connected by two revolute joints (θ_1, θ_2), which rotate in orthogonal directions, z_0 and z_1. The first joint is actuated and the second joint is passive. The end-effector of the Furuta pendulum is equipped with a camera, whose frame is denoted by $\{C\}$. The frames $\{1\}$ and $\{2\}$ are attached to the first and second links, respectively.

Let the link lengths be $d_1 = 1.0$m and $d_2 = d_3 = 0.8$m. At the home position, shown in the figure, the angles are $\theta_1 = 90°$ and $\theta_2 = 0°$. Gravity acts in the $-\boldsymbol{z}_0$ direction at $9.804 \mathrm{m\,s^{-2}}$.



Furuta Pendulum

Question 1 .... *25 points*

Resolve the position-level forward and inverse kinematics of the Furuta pendulum with the help of the coordinate systems shown as shown in the figure. Once you have the kinematic equations, answer the following questions:

(a) [5 points] If the joint angles $\boldsymbol{\theta}$ are given by $\boldsymbol{\theta} = \begin{bmatrix}\theta_1 & \theta_2\end{bmatrix}^\top = \begin{bmatrix}\pi/3 & -3\pi/7\end{bmatrix}^\top$, what is the pose of the end-effector, i.e., the frame $\{C\}$ with respect to the fixed frame $\{0\}$? Provide the orientation using both the rotation matrix and the corresponding quaternion.

(b) [5 points] At the same angles given in part (a), what is the pose of frame $\{2\}$ with respect to the fixed frame $\{0\}$? Provide the orientation using both the rotation matrix and the corresponding quaternion.

(c) [5 points] Provide the axis-angle representation of the orientation for parts (a) and (b). Use Rodrigues's formula to recover the rotation matrices you found in parts (a) and (b).

(d) [5 points] Suppose a camera is installed at the end-effector of the Furuta pendulum. Let $\{C\}$ denote the camera frame, as shown. What is the value of $\boldsymbol{\theta}$ if the $\boldsymbol{z}-$axis of the camera points in the direction $\boldsymbol{k}$, where

$$\boldsymbol{k} = -\frac{1}{2}\boldsymbol{x}_0 + \frac{1}{2}\boldsymbol{y}_0 - \frac{\sqrt{2}}{2}\boldsymbol{z}_0.$$

(e) [5 points] Repeat part (d) to find the value of $\boldsymbol{\theta}$ if the origin of frame $\{C\}$ is located at a point P, whose coordinate vector is given in frame $\{0\}$ by

$$\boldsymbol{p} = 1.07\boldsymbol{x}_0 + 0.23\boldsymbol{y}_0 + 1.486\boldsymbol{z}_0.$$

Question 2 .... *20 points*

Resolve the velocity-level forward and inverse kinematics of the Furuta pendulum and answer the following questions:

(a) [5 points] At the pose given in part (a) of Question 1, if the joint velocities are given by $\dot{\boldsymbol{\theta}} = \begin{bmatrix} 1 & 2 \end{bmatrix}^\top$, what are the body and spatial twists (angular and linear) velocity of the end-effector?

(b) [5 points] At the same pose in part (a) what are the body and spatial twists of frame $\{2\}$?

(c) [5 points] At the same pose in part (a), if the end-effector frame $\{C\}$ is moving with a body twist given by ${}^c\boldsymbol{V}_c = (\boldsymbol{\omega}_c, \boldsymbol{v}_c) = \begin{bmatrix} -1.2 & 0.487 & 0.111 & 0.39 & 0.871 & 0.39 \end{bmatrix}^\top$, what are the joint rates?

(d) [5 points] Repeat part (c) for the case where the spatial twist of the end-effector is given by ${}^0\boldsymbol{V}_c = (\boldsymbol{\omega}_c, \boldsymbol{v}_c) = \begin{bmatrix} -0.35 & -0.606 & 0.5 & -0.044 & 0.475 & 0.546 \end{bmatrix}^\top$.

Question 3 .... *35 points*

Implement the Furuta pendulum in MuJoCo and use its utility functions to verify all the kinematic equations you derived in Questions 1 and 2. Make sure that the frames that you define in MuJoCo match the ones given in the figure above.

Question 4 .... *20 points*

Have MuJoCo simulate the Furuta pendulum using the fourth-order Runge-Kutta integrator for 15s under the action of gravity and a control torque on the first link given by

$$\tau = -15 \operatorname{sgn} \dot{\theta}_2$$

and record the motion of the system. Provide the following plots:

- The height of the end-effector with respect to time.
- The linear velocity of the origin of the end-effector frame $\{C\}$ as seen from the fixed frame with respect to time.
- The spatial angular velocity of the end-effector frame $\{C\}$ plotted with respect to the spatial angular velocity of frame $\{1\}$, each as seen from the fixed frame.

Comment on the behavior of the system.

References

[1] K. Furuta, M. Yamakita, and S. Kobayashi. Swing-up control of inverted pendulum using pseudo-state feedback. *Proceedings of the Institution of Mechanical Engineers, Part I: Journal of Systems and Control Engineering*, 206(4):263–269, 1992.

[2] E. Todorov, T. Erez, and Y. Tassa. Mujoco: A physics engine for model-based control. In *2012 IEEE/RSJ International Conference on Intelligent Robots and Systems*, pages 5026–5033. IEEE, 2012. doi: 10.1109/IROS.2012.6386109.