A Combinatorial Proof of Wigner's Semi-Circle Law

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Some History

The study of random matrices started in the middle of the 20th century by physicists who were interested in understanding the statistical properties of complex quantum systems.

Eugene Wigner, hungarian physicist, made the first contributions to the field by introducing random matrices to model the behavior of nuclear energy levels.

The Semi-Circle Law, which is the goal of this seminar, was first observed by Wigner in 1955.

Some History

Wigner and his wife, Mary (1963).

Wigner Matrices

Our construction of Wigner matrices will be as foloows:

Let Y_n be a $n \times n$ symmetric matrix such that its entries are Gaussian random variables satisfying the following conditions:

- Y_{ii}'s are iid
- Y_{ij}'s $(i \neq j)$ are also iid, with a possibly different distribution from the diagonal terms.
- $E[Y_{ii}] = 0 = E[Y_{ij}]$
- The k-th moment of the entries is always finite. Ie, $\forall k \in \mathbb{N}$, we have
$$\max\{E[Y_{11}^k], E[Y_{12}^k]\} < \infty$$
- To avoid the degenerate case in which Y_n is diagonal, we demand that $E[Y_{12}^2] > 0$

Wigner Matrices

Remark:If $E[Y_{12}^2] = 1$, the matrices Y_n belong to the Gaussian Orthogonal Ensemble (GOE), which has nice universality properties, meaning that many statistical properties of the eigenvalues and eigenvectors are shared with other random matrix ensembles under certain conditions...

Note that as $n \to \infty$, the spectrum of the Y_n's may contain larger and larger eigenvalues. So we normalize them, giving rise to our definition of Wigner Matrices:

DEF A Sequence of Wigner Matrices is

$$X_n := \frac{1}{\sqrt{n}} Y_n$$

Where the Y_n's are as above.

The Semi-Circle Law

Three versions:

Version 1 Let X_n be a sequence of Wigner Matrices and $I \subset \mathbb{R}$ be a real interval. Consider the sequence of random variables defined as:

$$E_n(I) := \frac{\#\{\sigma(X_n) \cap I\}}{n}$$

Then $E_n(I) \xrightarrow{\mathrm{P}} \sigma_t(I)$ as $n \to \infty$.

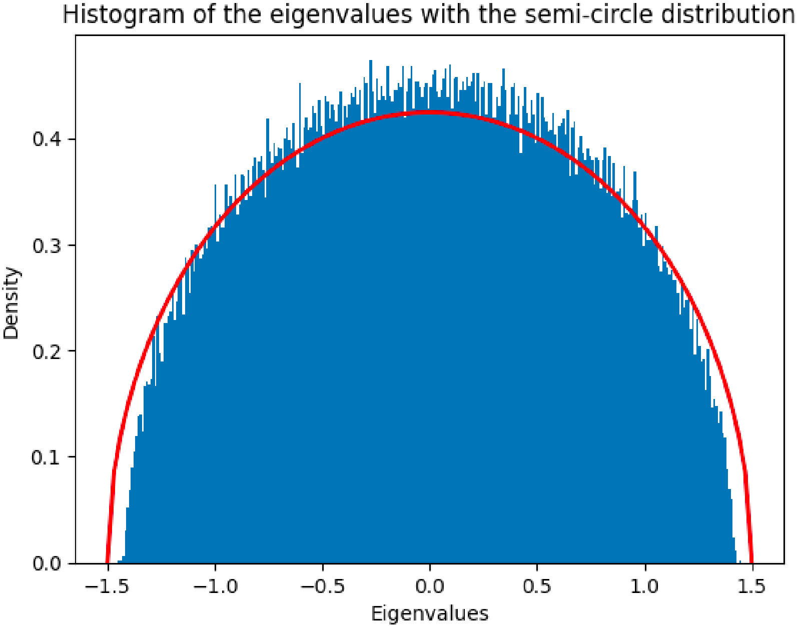
Where $\sigma_t(dx) = \frac{1}{2\pi t}\sqrt{4t - x^2}_+ dx$ and $t = E[Y_{12}^2]$

In other words, Given $I \subset \mathbb{R}$ and $\epsilon > 0$, we have

$$\lim_{n\to\infty} \mathrm{P}(|E_n(I) - \sigma_t(I)| > \epsilon) = 0$$

Remark: The parameter $t > 0$ (ie, the variance) "deforms" the graph of the semi-circle into a semi-elipse...

The Semi-Circle Law



The Semi-Circle Law

Version 2 For any real continuous and bounded function f, and for any $\epsilon > 0$, we have

$$\lim_{n\to\infty} \mathrm{P}(|\frac{1}{n}\sum_{i=1}^{n} f(\lambda_i(X_n)) - \int f d\sigma_t| > \epsilon) = 0$$

By the Stone-Weierstrass Approximation Theorem, we may use linearity to claim that it suffices to prove the theorem for monomials instead of continuous functions...

Version 3 Let k be a fixed positive integer. Then, for any $\epsilon > 0$, we have

$$\lim_{n\to\infty} \mathrm{P}(|\frac{1}{n} TR(X_n^k) - \int x^k \sigma_t(dx)| > \epsilon) = 0$$

Auxiliary Results

We will focus on the third version of the theorem.

Fun Fact: The k-th moments of the semi-circle distribution are

$$\int x^k \sigma_t(dx) = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \\ t^{k/2} C_{k/2} & \text{if } k \text{ is even} \end{cases}$$

Where C_m is the m-th Catalan number!

$$C_m = \frac{1}{m+1}\binom{2m}{m}$$

Auxiliary Results

Lemma: Let W_n be a sequence of random variables such that:

- $\lim_{n\to\infty} E[W_n] = m$
- $\lim_{n\to\infty} VAR(W_n) = 0$

Then $W_n \xrightarrow{P} m$ as $n \to \infty$.

Proof: Just a corollary from Markov's Inequality.

Therefore, it is enough to show the following statements:

- $\lim_{n\to\infty} \frac{1}{n} E[TR(X_n^k)] = \begin{cases} 0 & \text{if } k \text{ is odd} \\ t^{k/2} C_{k/2} & \text{if } k \text{ is even} \end{cases}$
- $\forall k \in \mathbb{N}$, $\lim_{n\to\infty} VAR(\frac{1}{n} TR(X_n^k)) = 0$

Proof of the first statement

Observe that

$$\frac{1}{n}E[TR(X_n^k)] = \frac{1}{n\sqrt{n^k}}E[TR(Y_n^k)]$$

We want the trace, so our interest is in the diagonal entries of Y_n^k, which have the form

$$[Y_n^k]_{ii} = \sum_{1 \le i_2 \ldots i_k \le n} Y_{ii_2} Y_{i_2 i_3} \ldots Y_{i_k i}$$

Therefore,

$$E[TR(Y_n^k)] = E[\sum (Y_n^k)_{ii}] = \sum_{I \in [n]^k} E[Y_I]$$

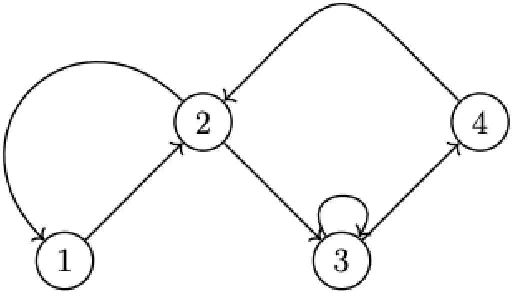
Where we define $[n] := \{1, \ldots, n\}$

The Combinatorial Approach

For $I \in [n]^k$, we can associate a graph $G_I = (V_I, E_I)$ with a circuit w_I in it as in the following example:

$$I = (1, 2, 3, 3, 4, 2) \in [4]^6 \mapsto (G_I, w_I)$$

Where $V_I = \{1, 2, 3, 4\}$, $E_I = \{\{1, 2\}, \{2, 3\}, \{3, 3\}, \{3, 4\}, \{4, 2\}\}$ and the circuit w_I is given the order of the edges in E_I:



Some Definitions

Note that G_I is always connected, and w_I encodes the number of times that an edge is crossed in the circuit.

DEF For any $e \in E_I$, $w_I(e)$ is the number of times that the edge e appears in w_I. If $e \notin w_I$, we say that $w_I(e) = 0$.

In the previous example, we have $w_I(\{1,2\}) = 2$...

$$\Rightarrow Y_I = \prod_{1 \le i \le j \le n} Y_{ij}^{w_I(\{i,j\})}$$

Now, we classify the edges as loops, denoted by E_I^α (if it connects a vertex to it self) or bridges, denoted by E_I^β(otherwise). Then, by independence, we get

$$E(Y_I) = \prod_{1 \le i \le j \le n} E[Y_{ij}^{w_I(\{i,j\})}] = \prod_{e \in E_I^\alpha} E[Y_{11}^{w_I(e)}] \prod_{e \in E_I^\beta} E[Y_{12}^{w_I(e)}]$$

Some Definitions

DEF Let $\mathfrak{G}_k$ be the set of all pairs (G, w), where G is a connected graph with no more than k vertices and w is a closed path in G of length k that passes through all the edges.

Remark:It makes difference to specify where the path begins!

DEF Given $(G, w) \in \mathfrak{G}_k$, we define

$$\Pi(G, w) := \prod_{e \in E_I^{\alpha}} E[Y_{11}^{w_I(e)}] \prod_{e \in E_I^{\beta}} E[Y_{12}^{w_I(e)}]$$

Reformulating The Problem

With that in mind, we can rewrite

$$E[TR(Y_n^k)] = \sum_{I\in[n]^k} \Pi(G_I, w_I) = \sum_{(G,w)\in\mathfrak{G}_k} \sum_{I\in[n]^k} \Pi(G,w)\chi_{\{(G,w)=(G_I,w_I)\}}$$

$$= \sum_{(G,w)\in\mathfrak{G}_k} \Pi(G,w)\#\{I\in[n]^k | (G_I,w_I) = (G,w)\}$$

Remark:The map $I \mapsto (G_I, w_I)$ is not injective! That's why we must count the size of its pre-images... Ie, to obtain $\sum E[Y_I]$, have to make a weighted sum of the $\Pi(G,w)$'s.

Putting it all together

$$\frac{1}{n}E[TR(X_n^k)] = \sum_{(G,w)\in\mathfrak{G}_k} \Pi(G,w)\frac{\#\{I\in[n]^k | (G_I,w_I) = (G,w)\}}{n^{\frac{k}{2}+1}}$$

Counting pre-images

Given (G, w), the size of the set $\{I \in [n]^k | (G_I, w_I) = (G, w)\}$ depends on the value of n.

Lemma:Let $|G|$ denote the cardinality of the vertices of G. Then we have

$$\#\{I \in [n]^k | (G_I, w_I) = (G, w)\} = n(n-1)(n-2)\dots(n-|G|+1)$$

Therefore,
$\frac{1}{n}E[TR(X_n^k)] = \sum_{(G,w)\in\mathfrak{G}} \Pi(G, w)\frac{n(n-1)(n-2)\dots(n-|G|+1)}{n^{\frac{k}{2}+1}}$.

Counting pre-images

Remark:Since for large values of n we have $(n-c) \approx n$ for any constant c, then

$$n(n-1)(n-2)\dots(n-|G|+1) \approx n^{|G|}$$

Remark:By hypothesis, we may throw away all the edges that are passed through just once:

$$E[Y_{ij}^{w(e)}] = E[Y_{ij}] = 0$$

Since we are considering just (G, w)'s such that $w(e) \geq 2 \forall e \in E_G$, we get that $\# E_G \leq k/2$.

Counting pre-images

Lemma:Let $G = (V, E)$ be a finite connected graph. Then

$$|G| = \#V =\leq \#E + 1$$

(With equality if and only if G is a Tree). Corollary:$|G| \leq \frac{k}{2} + 1$.

- If k is odd:

$$|G| \leq \frac{k+1}{2} \Rightarrow \frac{n(n-1)(n-2)\ldots(n-|G|+1)}{n^{\frac{k}{2}+1}} = \mathcal{O}(\frac{1}{\sqrt{n}})$$

Since $\Pi(G, w)$ does not change with n and all the sums are finite, we obtain

$$\lim_{n\to\infty} \frac{1}{n} E[TR(X_n^k)] = 0$$

As we wanted, for odd values of k :)

Counting pre-images

- If k is even:

  If $|G| \leq k/2$, we fall in the situation where everything converges to zero. So we must control the cases in which $w \geq 2$ and $|G| = \frac{k}{2} + 1$.

  Proposition:Let $(G, w) \in \mathfrak{G}_k$, with k even and $w \geq 2$. Then:
  - If there exists a loop in G, then $|G| \leq k/2$
  - If $\exists e \in E_G$ such that $w(e) \geq 3$, then $|G| \leq k/2$

  Proof: If G has loops, then it is not a tree. So by the previous lemma, $|G| < \#E + 1$. But $w \geq 2 \Rightarrow \#E \leq k/2$.

  $$\Rightarrow |G| < \frac{k}{2} + 1 \Rightarrow |G| \leq k/2.$$

Counting pre-images

$w(e) \geq 3 \Rightarrow$ The sum of the values of w over $E_G - \{e\}$ is $\leq k-3$ (The sum over all edges is k...)

$w \geq 2 \Rightarrow \#\{E_G - \{e\}\} \leq \frac{k-3}{2}$.

$\Rightarrow \#E \leq \frac{k-3}{2} + 1 = \frac{k-1}{2}$

Once again, by the lemma, we get $|G| \in \mathbb{Z} \leq \frac{k+1}{2}$, but $k+1$ is odd! Therefore $|G| \leq k/2$.

Remark:If k is even, most of the terms of the trace go to zero as $n \to \infty$. The only exception is the case which $w(e) = 2 \forall e (\Rightarrow \#E = k/2)$. Ie, when G is a tree!

Proof of the first statement

We learned that
$\frac{1}{n}E[TR(X_n^k)] = \sum_{|G|=\frac{k}{2}+1} \Pi(G,w)\frac{n\ldots(n-|G|+1)}{n^{\frac{k}{2}+1}} + \mathcal{O}_k$. Where $\mathcal{O}_k$ denotes the terms that go to zero as n approaches infinity.

Since $|G| = \frac{k}{2}+1$, $n\ldots(n-|G|+1) \approx n^{\frac{k}{2}+1}$.

$$\Rightarrow \frac{n\ldots(n-|G|+1)}{n^{\frac{k}{2}+1}} \approx 1$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n}E[TR(X_n^k)] = \sum_{(G,w)\in\bar{\mathfrak{G}}_k} \Pi(G,w)$$

Where
$\bar{\mathfrak{G}}_k := \{(G,w) \in \mathfrak{G}_k)|w = 2 \text{ for all edges, and } |G| = \frac{k}{2}+1\}$

Proof of the first statement

The terms of the desired sum have the form

$$\Pi(G, w) = \prod_{e \in E^\beta} E[Y_{12}^{w(e)}] = \prod_{e \in E^\beta} E[Y_{12}^2] = t^{\#E} = t^{k/2}$$

$$\Rightarrow \lim_{n \to \infty} \frac{1}{n} E[TR(X_n^k)] = t^{k/2} \# \bar{\mathfrak{G}}_k$$

Finally, it remains to show that $\# \bar{\mathfrak{G}}_k = C_{k/2}$. We shall do this associating each pair in $\bar{\mathfrak{G}}_k$ to a Dyck path...

DEF For any even k, the set of Dyck paths of length k is $\mathfrak{D}_k := \{(d_1, \ldots, d_k) \in \{-1, 1\}^k | \sum_{i=1}^{j} d_i \geq 0 \forall j \in \{1, \ldots, k\}$ and $\sum_{i=1}^{k} d_i = 0\}$.

Remark(!):$\# \mathfrak{D}_k = C_{k/2}$

Proof of the first statement

The last step in this proof consists in showing that

$$\#\bar{\mathfrak{G}}_k = \#\mathfrak{D}_k$$

There is a bijection $d : \bar{\mathfrak{G}}_k \to \mathfrak{D}_k$ defined as follows:

Given a pair (G, w), we construct a sequence of 1's and -1's walking on G as w tells us to do so.

If its the first time that w passes through the i-th edge, then the i-th term of the sequence is 1 (and -1, otherwise).

Similarly, we can construct a pair (G, w) form a sequence $(d_1, \ldots, d_k)$ starting with a vertex v_i, and then linking it to an other v_{i+1} if $d_i = 1$ (or linking it to its previous vertex if $d_i = -1$).

Proof of the second statement

First, we rewrite the variance:

$$VAR(\frac{1}{n}TR(X_n^k)) = \frac{1}{n^{k+2}}(E[(TR(Y_n^k))^2] - (E[TR(Y_n^k])^2)$$

$$= \frac{1}{n^{k+2}} \sum_{I,J\in[n]^k} (E[Y_I Y_J] - E[Y_I]E[Y_J]) = \frac{1}{n^{k+2}} \sum_{I,J\in[n]^k} COV(Y_I, Y_J)$$

As before, for every pair $(I, J) \in [n]^k \times [n]^k$, we associate a graph G_{I*J}, that is the join of G_I and G_J (may not be connected!) and two circuits w_I and w_J.

Remark: If $E_I \cap E_J = \emptyset$, then $COV(Y_I, Y_J) = 0$, by independence. Thus, words must overlap!

DEF For an edge e in G_{I*J}, define $w_{I*J}(e) := w_I(e) + w_J(e)$.

Proof of the second statement

DEF The set of triples (G, w_1, w_2) where $G = (V, E)$ is a connected graph with $\#V \leq 2k$ and w_1 and w_2 are closed circuits of length k such that their union passes through all edges of G is denoted by $\mathfrak{G}_{k,k}$.

Analogously to the previous proof, set
$\Pi(G_{I*J}, w_I, w_J) := COV(Y_I, Y_J)$...

The variance can now be written as:

$$\sum_{(G,w_1,w_2)\in\mathfrak{G}_{k,k}} \Pi(G, w_1, w_2) \frac{\#\{(I,J)\in[n]^k\times[n]^k|(G,w_1,w_2)=(G_{I*J},w_I,w_J)\}}{n^{k+2}}$$

Once again, we have that

$$\#\{(I,J) \in [n]^k \times [n]^k|(G, w_1, w_2) = (G_{I*J}, w_I, w_J)\} =$$
$$n(n-1)\ldots(n-|G|+1) \approx n^{|G|}$$

Proof of the second statement

As before, we can forget about triples that have an edge crossed only once:

$$w_{I*J}(e) \geq 2 \ \forall e \in E_G$$

Therefore, since $\sum_{e \in E} w(e) = k$, we have that $\#E_{I*J} \leq k$.

And by the previous lemma, $|G| \leq k+1$.

$$\Rightarrow VAR(\frac{1}{n} TR(X_n^k)) \approx \sum_{(G,w_1,w_2) \in \mathfrak{G}_{k,k}} \Pi(G, w_1, w_2) \frac{n^{k+1}}{n^{k+2}} = \mathcal{O}(1/n)$$

Remark: In fact, it can be shown that $VAR(\frac{1}{n} TR(X_n^k)) = \mathcal{O}(\frac{1}{n^2})$...

Thank you! :)