MEAN CURVATURE FLOW

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Contents

1. Mean curvature flow

Let $M_0 \subset \mathbb{R}^{n+1}$ be a smooth n-dimensional hypersurface without boundary, given by an immersion $X_0 : M^n \to \mathbb{R}^{n+1}$, where M^n is an abstract smooth manifold. We consider the family of embeddings $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ with

$$X(p, 0) = X_0(p)$$

for all $p \in M^n$ and

$$\partial_t X(p,t) = \mathbf{H}(p,t) = -H(p,t)\boldsymbol{\nu}(p,t) = \Delta_{M_t} X(p,t) \tag{MCF}$$

for all $(p,t) \in M^n \times [0,T)$. We abbreviate $M_t := X(M^n, t)$. In the following, we will write $\Delta := \Delta_{M_t}$ and $\nabla := \nabla^{M_t}$.

Example 1.1 (Shrinking spheres and cylinders)**.**

(i) Let $M_t = \mathbb{S}^n_{r(t)}$, then (MCF) reduces to an ODE for the radius, namely

$$r' = -\frac{n}{r}\,.$$

The solution with $r(0) = R$ is

$$r(t) = \sqrt{R^2 - 2nt}\,,$$

for $t \in (-\infty, R^2/2n)$.

(ii) The shrinking cylinders $M_t = \mathbb{S}^{n-m}_{r(t)} \times \mathbb{R}^m$ with $r(t) = \sqrt{R^2 - 2(n-m)t}$ exist for $t \in (-\infty, R^2/2(n-m))$.

(iii) For $n = 1$ the so-called grim reaper is given by $M_t = \operatorname{graph}(u_t)$, where $u(x,t) = t - \log\cos x$ with $x \in (-\pi, \pi)$.

Remark 1.2 (Normal motion and tangential diffeomorphisms)**.** See [Eck04, Remark 2.2(3)]. We will often consider smoothly embedded hypersurfaces M_t satisfying

$$(\partial_t x)^\perp = \langle \partial_t x, \boldsymbol{\nu}(x)\rangle \boldsymbol{\nu}(x) = \mathbf{H}(x)$$

for $x \in M_t$, where $\perp$ denotes the projection onto the normal space of M_t. This equation is equivalent to (MCF) up to diffeomorphisms tangent to M_t. Indeed, let

$\tilde{X}(\,\cdot\,,t) : M^n \to \mathbb{R}^{n+1}$ with $M_t = \tilde{X}(M^n,t)$ be a family of embeddings satisfying the equation

$$\Big(\partial_t \tilde{X}(q,t)\Big)^{\perp} = \tilde{\mathbf{H}}(q,t) := \mathbf{H}\Big(\tilde{X}(q,t)\Big)$$

for $q \in M^n$, where $\perp$ denotes the projection onto the normal space of $\tilde{X}(M^n,t)$. Let $\phi_t = (\,\cdot\,,t)$ be a family of diffeomorphisms of M^n satisfying

$$D^{\mathbb{R}^{n+1}}\tilde{X}(\phi(p,t),t)\partial_t\phi(p,t) = -\left(\partial_t \tilde{X}(\phi(p,t),t)\right)^{\top},$$

where $\top$ denotes projection onto the tangent space of $\tilde{X}(M^n,t)$. The local existence of such a family is guaranteed by the assumptions on $\tilde{X}$. If we set

$$X(p,t) = \tilde{X}(\phi(p,t),t)$$

then $M_t = X(M^n,t) = \tilde{X}(M^n,t)$, and

$$\partial_t X(p,t) = \partial_t \tilde{X}(p,t) + D^{\mathbb{R}^{n+1}}\tilde{X}(\phi(p,t),t)\partial_t\phi(p,t) = \Big(\partial_t\tilde{X}(q,t)\Big)^{\perp} = \mathbf{H}(X(p,t))\,.$$

The previous remark results in the following theorem.

Theorem 1.3 (see [Sch17a, Theorem 10.6])**.** *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a solution to* (MCF)*, that is $\langle \partial_t X, \boldsymbol{\nu}\rangle = -H$. Let $R \in O(n+1)$ be an orthonormal map and $\phi : M^n \times [0,T) \to M^n$ smooth. so that $\phi(\,\cdot\,,t)$ is a diffeomorphism. Then $\tilde{X}(p,t) := RX(\phi(p,t),t)$ evolves by*

$$\Big\langle \partial_t \tilde{X}(p,t), \tilde{\boldsymbol{\nu}}(p,t)\Big\rangle = -\tilde{H}(p,t)\,,$$

where $\tilde{H}(p,t) = H(\phi(p,t),t)$ for all $p \in M^n$ and $t \in [0,T)$.

Lemma 1.4 (Evolution equations)**.** *Let $(M_t)_{t\in[0,T)}$ evolve by* (MCF)*. Then,*

$$\begin{aligned}
\partial_t g_{ij} &= -Hh_{ij}\,,\\
\partial_t d\mu_t^n &= -H^2\, d\mu_t^n\,,\\
\partial_t h_{ij} &= \nabla_i\nabla_j H - Hg^{km}h_{ik}h_{jm}\,,\\
\partial_t H &= \Delta H + H|A|^2\,,\\
\partial_t |A|^2 &= \Delta|A|^2 - |\nabla A|^2 + 2|A|^4\,,\\
\partial_t|\nabla^m A|^2 &\le \Delta|\nabla^m A|^2 - 2|\nabla^{m+1}A|^2\\
&\quad + C(m,n)\sum_{i+j+k=m}|\nabla^m A|\cdot|\nabla^i A|\cdot|\nabla^j A|\cdot|\nabla^k A|
\end{aligned}$$

for all $t \in [0,T)$.

Proof. See e. g. [Sch18, Section 3]. □

Corollary 1.5. *The mean curvature flow is the negative L^2 gradient flow for the surface area functional.*

Proof. For arbitrary normal speeds $\partial_t X = -F\boldsymbol{\nu}$, we have that $\partial_t g_{ij} = -2Fh_{ij}$ and

$$\frac{d}{dt}\int_{M_t} d\mu_t^n = -\int_{M_t} FH d\mu_t^n \ge -\left(\int_{M_t} F^2 d\mu_t^n\right)^{1/2}\left(\int_{M_t} H^2 d\mu_t^n\right)^{1/2}$$

with equality if and only if $F = H$. □

Theorem 1.6 (Huisken, [Hui84, Corollary 3.6(ii)])**.** *Let $(M_t)_{t\in[0,T)}$ be a family of closed hypersurfaces moving by* (MCF)*. Assume $M_0 = X_0(M)$ closed and mean convex, i.e. $H \ge 0$. Then $H > 0$ for all $t \in (0,T)$.*

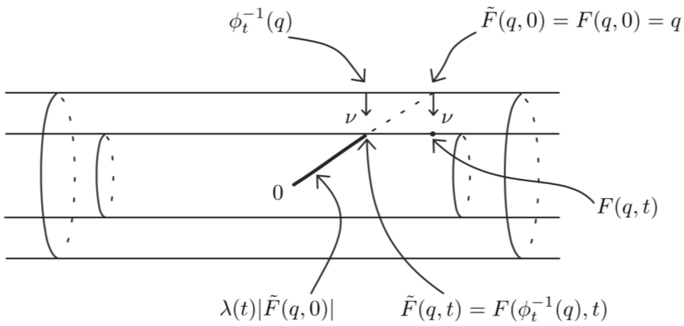

FIGURE 1. Radial and normal motion of a cylinder

Proof. See [Sch17d, Theorem 2.1.2]. That $H \geq 0$ for $t \geq 0$ follows from the evolution equation of H and the parabolic maximum principle, Theorem C.5. Assume that $H(p_0, t_0) = 0$ for some $t_0 > 0$. The strong maximum principle then implies that $H = 0$ for all (p,t) and $0 \leq t \leq t_0$. But this is impossible since any closed hypersurface in $\mathbb{R}^{n+1}$ has points where $\lambda_1 > 0$. $\square$

2. Homothetically shrinking solutions

Definition 2.1 (Homothetically shrinking solutions, Brakke [Bra78, Appendix C])**.** Let $\lambda : [t_0, T] \to \mathbb{R}_+$ be smooth and decreasing, $\lambda(t_0) = 1$ and $\lambda(T) = 0$. Let $x_0 \in \mathbb{R}^{n+1}$. A homothetically shrinking solution $X : M^n \times [t_0, T) \to \mathbb{R}^{n+1}$ to (MCF) satisfies

$$M_t = \lambda(t)(M_0 - x_0) + x_0$$

for all $t \in [t_0, T)$. This describes solutions of (MCF) which move by scaling about x_0.

Remark 2.2. See [Eck04, Examples 2.3(4)]. We can make the separation of variables ansatz

$$\tilde X(q,t) = \lambda(t)\tilde X(q,t_0)$$

for a family of embeddings $\tilde X : M^n \times [t_0, T) \to \mathbb{R}^{n+1}$ with $M_t = \tilde X(M^n, t)$ satisfying the evolution equation

$$\left(\partial_t \tilde X(q,t)\right)^{\perp} = \left\langle \partial_t \tilde X(q,t), \boldsymbol{\nu}(q,t)\right\rangle = \tilde{\mathbf{H}}(q,t)$$

for $q \in M^n$. In Remark 1.2, we saw that there are tangential diffeomorphisms $\phi_t : M^n \to M^n$, $t \in [t_0, T)$, with

$$\tilde X(q,t) = X\big(\phi_t^{-1}(q), t\big)$$

for $q \in M^n$, where the embeddings $X(\,\cdot\,, t) : M^n \to \mathbb{R}^{n+1}$ satisfy (MCF). This says that, up to tangential diffeomorphisms, the radial or homothetic motion of the hypersurfaces M_t (described by $\tilde X$) is equivalent to their normal motion along the mean curvature vector (described by X). For the shrinking sphere solution these two agree, but for the shrinking cylinder they differ, see Figure 1. Since the mean curvature of the embeddings scales with factor $1/\lambda(t)$ we deduce

$$\partial_t\lambda(t)\left(\tilde X(q,t_0)\right)^{\perp} = \left(\partial_t \tilde X(q,t)\right)^{\perp} = \tilde{\mathbf{H}}(q,t) = \frac{1}{\lambda(t)}\tilde{\mathbf{H}}(q,t_0)$$

for $q \in M^n$. From this we infer that

$$\alpha \equiv 2\lambda(t)\partial_t\lambda(t) = \partial_t\lambda^2(t)$$

is independent of t. We therefore obtain under the assumption $\lambda(t_0) = 1$ that

$$\lambda(t) = \sqrt{1 + \alpha(t - t_0)}$$

for all t satisfying $t > t_0 - 1/\alpha$. Hence

$$\mathbf{H}(p,t) = \alpha \frac{\langle X(p,t), \boldsymbol{\nu}(p,t)\rangle}{2\lambda^2(t)}$$

for $(p,t) \in M^n \times (-\infty, T)$, where $T = t_0 - 1/\alpha$. This describes expanding homothetic solutions about 0 for $\alpha > 0$ and contracting homothetic solutions about 0 for $\alpha < 0$. Let us concentrate on $\alpha < 0$. If we set $\lambda(T) = 0$ for $T > t_0$, which requires the hypersurface to disappear at time T, then $\alpha = -1/(T - t_0)$ and thus

$$\lambda(t) = \sqrt{\frac{T-t}{T-t_0}}$$

and

$$\mathbf{H}(p,t) = \frac{\langle X(p,t), \boldsymbol{\nu}(p,t)\rangle}{2(T-t)}$$

for $(p,t) \in M^n \times (-\infty, T)$.

Lemma 2.3. *Let $(M_t)_{t\in(-\infty,0)}$ be an ancient solution of MCF. Then*

$$H(x) = \frac{\langle x, \boldsymbol{\nu}(x)\rangle}{-2t}$$

for all $x \in M_t$ and $t < 0$ if and only if $M_t = \sqrt{-t}M_{-1}$ for all $t < 0$.

Proof. Let $M_t = \sqrt{-t}M_{-1}$ for all $t < 0$. Then $H(x) = \langle x, \boldsymbol{\nu}(x)\rangle/(-2t)$ for all $x \in M_t$ and $t < 0$ follows by Remark 2.2.

On the other hand, let $H(x) = \langle x, \boldsymbol{\nu}(x)\rangle/(-2t)$ for all $x \in M_t$ and $t < 0$. Then

$$\langle \Delta_{M_t} X(p,t), \boldsymbol{\nu}(p,t)\rangle = -H(p,t) = -\frac{\langle X(p,t), \boldsymbol{\nu}(p,t)\rangle}{-2t}$$

and thus up to tangential motion $X(p,t) = \sqrt{-2t}X(p,t_0\rangle$. □

2.1. Hypersurfaces.

Theorem 2.4 (Huisken, [Hui90, Theorem 4.1] and [Hui93])**.** *Let $M \subset \mathbb{R}^{n+1}$ be a smooth, complete, embedded, mean convex hypersurface such that $H(x) = \langle x, \boldsymbol{\nu}\rangle/2$ at every $x \in M$ and there exists a constant $C > 0$ such that $|A| + |\nabla A| \le C$ and $\mu^n(M \cap B_R) \le Ce^R$, for every ball of radius $R > 0$ in $\mathbb{R}^{n+1}$. Then, up to a rotation in $\mathbb{R}^{n+1}$, M is of the form $\mathbb{S}^m_{\sqrt{2m}} \times \mathbb{R}^{n-m}$ for $m = 0, 1, \ldots, n$.*

Proof. See [Man11, Proposition 3.4.1]. We scale M by the factor 1/2 so that $H(x) = \langle x, \boldsymbol{\nu}(x)\rangle$ at every $x \in M$. By covariant differentiation of the equation $H = \langle x, \boldsymbol{\nu}\rangle$ in an orthonormal frame $\{\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_n\}$ on M we get by the Weingarten equations $\nabla_i \boldsymbol{\nu} = \partial_i \boldsymbol{\nu} = h_i^j \partial_j x$ that

$$\nabla_j H = \langle x, \nabla_j \boldsymbol{\nu}\rangle = \langle x, \partial_k x\rangle h_j^k$$

and by the Gauss equations $\nabla_i \nabla_j x = -h_{ij}\boldsymbol{\nu}$ and Codazzi equations $\nabla_k h_{ij} = \nabla_k h_{ji} = \nabla_j h_{ik}$ at one fixed point where the Christoffel symbols vanish, that

$$\begin{aligned}
\nabla_i \nabla_j H &= g_{ik} h_j^k + \langle x, \nabla_i \nabla_k x\rangle h_j^k + \langle x, \partial_k K\rangle \nabla_i h_j^k \\
&= h_{ij} + \langle x, \boldsymbol{\nu}\rangle h_{ik} h_j^k + \langle x, \partial_k x\rangle g^{kl} \nabla_i h_{jl} \\
&= h_{ij} - H h_{ik} h_j^k + \langle x, \partial_k x\rangle g^{kl} \nabla_l h_{ij} \\
&= h_{ij} - H h_{ik} h_j^k + \langle x, \nabla h_{ij}\rangle\,. \qquad (1)
\end{aligned}$$

Contracting with g^{ij} we have

$$\Delta H = H\left(1 - |A|^2\right) + \langle x, \nabla H\rangle\,. \qquad (2)$$

From equation (2) and the strong maximum principle for elliptic equations, Theorem C.1, we see that, since M satisfies $H \geq 0$ by assumption and

$$\Delta H \leq H + \langle x, \nabla H \rangle$$

we must either have that $H = 0$ or $H > 0$ on all M. Contracting (1) with h^{ij}, we have

$$h^{ij}\nabla_i\nabla_j H = |A|^2 - H\,\mathrm{tr}\big(A^3\big) + \frac{\langle x, \nabla |A|^2 \rangle}{2}\,,$$

which implies, by Simons' identity

$$\Delta h_{ij} = \nabla_i\nabla_j H + H h_{ik}h^k_j - |A|^2 h_{ij}$$

that

$$\begin{aligned}
\Delta|A|^2 &= \Delta(h^{ij}h_{ij}) = h^{ij}\Delta h_{ij} + 2g^{mn}\nabla_m h^{ij}\nabla_n h_{ij} + h_{ij}\Delta h^{ij} \\
&= h^{ij}\Delta h_{ij} + 2g^{mn}g^{ki}g^{lj}\nabla_m h_{kl}\nabla_n h_{ij} + h_{ij}g^{ki}g^{jl}\Delta h_{kl} \\
&= 2h^{ij}\left(\nabla_i\nabla_j H + H h_{ik}h^k_j - |A|^2 h_{ij}\right) + 2g^{mn}\nabla_m h^i_l \nabla_n h^l_i \\
&= 2|A|^2 - 2H\,\mathrm{tr}\big(A^3\big) + \langle x, \nabla|A|^2\rangle + 2H\,\mathrm{tr}\big(A^3\big) - 2|A|^4 + 2|\nabla A|^2 \\
&= 2|A|^2(1 - |A|^2) + \langle x, \nabla|A|^2\rangle + 2|\nabla A|^2\,.
\end{aligned}$$

Assume that $H = 0$. As M is complete and x is a tangent vector field on M by the equation $\langle x, \boldsymbol{\nu}\rangle = 0$, for every point $x \in M$ there is a unique solution of the ODE

$$\gamma'(s) = x(\gamma(s)) = \gamma(s)$$

passing through x and contained in M for every $s \in \mathbb{R}$, but such solution is simply the line in $\mathbb{R}^{n+1}$ passing through x and the origin. Thus, M has to be a cone and being smooth the only possibility is a hyperplane through the origin of $\mathbb{R}^{n+1}$.

Assume that $H > 0$ everywhere (so dividing by H and $|A|$ is allowed). For $R > 0$, define

$$\eta_R = \boldsymbol{\nu}_{\partial(M\cap B_R(0))}$$

to be the outward unit conormal to $M \cap B_R(0)$ along $\partial(M \cap B_R(0))$, which is a smooth boundary for almost every $R > 0$ (by Sard's theorem, see homework or Corollary B.3). Then, supposing that R belongs to the set $\mathcal{R} \subset \mathbb{R}^+$ of the regular values of the function $|\cdot|$ restricted to $M \subset \mathbb{R}^{n+1}$, from equation (2) and the divergence theorem, Theorem A.2, we compute

$$\begin{aligned}
\varepsilon_R &= \int_{\partial(M\cap B_R(0))} |A|\langle \nabla H, \eta_R\rangle \exp\left(-\frac{R^2}{2}\right) d\mu^{n-1} \\
&= \int_{M\cap B_R(0)} |A|\Delta H \exp\left(-\frac{|x|^2}{2}\right) + \left\langle \nabla\left(|A| \exp\left(-\frac{|x|^2}{2}\right)\right), \nabla H \right\rangle d\mu^n \\
&= \int_{M\cap B_R(0)} \left(|A|H\left(1 - |A|^2\right) + |A|\langle x, \nabla H\rangle\right) \exp\left(-\frac{|x|^2}{2}\right) d\mu^n \\
&\quad + \int_{M\cap B_R(0)} \left(\frac{1}{2|A|}\langle \nabla|A|^2, \nabla H\rangle - |A|\langle x, \nabla H\rangle\right) \exp\left(-\frac{|x|^2}{2}\right) d\mu^n \\
&= \int_{M\cap B_R(0)} \left(|A|H\left(1 - |A|^2\right) + \frac{1}{2|A|}\langle \nabla|A|^2, \nabla H\rangle\right) \exp\left(-\frac{|x|^2}{2}\right) d\mu^n
\end{aligned}$$

and similarly

$$\begin{aligned}
\delta_R = &\int_{\partial(M\cap B_R(0))} \frac{H}{|A|}\langle \nabla |A|^2, \eta_R\rangle \exp\left(-\frac{R^2}{2}\right)\, d\mu^{n-1}\\
= &\int_{M\cap B_R(0)} \frac{H}{|A|}\Delta |A|^2 \exp\left(-\frac{|x|^2}{2}\right) + \left\langle \nabla\left(\frac{H}{|A|}\exp\left(-\frac{|x|^2}{2}\right)\right), \nabla |A|^2\right\rangle\, d\mu^n\\
= &\int_{M\cap B_R(0)} \left(2|A|H\left(1-|A|^2\right) + \frac{2H|\nabla A|^2}{|A|}\right.\\
&\qquad\qquad \left. + \frac{H}{|A|}\langle x, \nabla|A|^2\rangle\right)\exp\left(-\frac{|x|^2}{2}\right)\, d\mu^n\\
&+ \int_{M\cap B_R(0)} \left(\frac{\langle \nabla H, \nabla |A|^2\rangle}{|A|} - \frac{H|\nabla|A|^2|^2}{2|A|^3} - \frac{H}{|A|}\langle x, \nabla |A|^2\rangle\right)\exp\left(-\frac{|x|^2}{2}\right)\, d\mu^n\\
= &\int_{M\cap B_R(0)} \left(2|A|H\left(1-|A|^2\right) + \frac{2H|\nabla A|^2}{|A|} + \frac{\langle \nabla H, \nabla |A|^2\rangle}{|A|}\right.\\
&\qquad\qquad \left. - \frac{H|\nabla|A|^2|^2}{2|A|^3}\right)\exp\left(-\frac{|x|^2}{2}\right)\, d\mu^n\,.
\end{aligned}$$

Hence,

$$\begin{aligned}
\sigma_R &= 2\delta_R - 4\varepsilon_R\\
&= \int_{M\cap B_R(0)} \left(\frac{4H|\nabla A|^2}{|A|} - \frac{H|\nabla|A|^2|^2}{|A|^3}\right)\exp\left(-\frac{|x|^2}{2}\right)\, d\mu^n\\
&= \int_{M\cap B_R(0)} \left(4|A|^2|\nabla A|^2 - \left|\nabla |A|^2\right|^2\right)\frac{H}{|A|^3}\exp\left(-\frac{|x|^2}{2}\right)\, d\mu^n\,.
\end{aligned}$$

As we have

$$4|A|^2|\nabla A|^2 \geq |\nabla |A|^2|^2$$

the quantity σ_R is nonnegative and nondecreasing in R. If now we show that

$$\liminf_{R\to\infty} \sigma_R = 0$$

we can conclude that, at every point of M,

$$4|A|^2|\nabla A|^2 = |\nabla |A|^2|^2\,. \tag{3}$$

Getting back to the definitions of ε_R and δ_R, we have

$$\begin{aligned}
|\sigma_R| = &\left| -2\int_{\partial(M\cap B_R(0))} \frac{H}{|A|}\langle \nabla |A|^2, \eta\rangle \exp\left(-\frac{R^2}{2}\right)\, d\mu^{n-1}\right.\\
&\left. +4\int_{\partial(M\cap B_R(0))} |A|\langle \nabla H, \eta\rangle \exp\left(-\frac{R^2}{2}\right)\, d\mu^{n-1}\right|\\
\leq &\, 4\exp\left(-\frac{R^2}{2}\right)\int_{\partial(M\cap B_R(0))}\left(\frac{H}{|A|}|\nabla|A|^2| + |A||\nabla H|\right)\, d\mu^{n-1}\\
\leq &\, 8\exp\left(-\frac{R^2}{2}\right)\int_{\partial(M\cap B_R(0))}\left(H|\nabla A| + |A||\nabla H|\right)\, d\mu^{n-1}\\
\leq &\, C\exp\left(-\frac{R^2}{2}\right)\mu^{n-1}(\partial(M\cap B_R(0)))\,,
\end{aligned}$$

by the estimates on A and ∇A in the hypotheses. Assume that the lefthand side does not go to zero. That is, suppose that for every R belonging to the set $\mathcal{R}\subset\mathbb{R}^+$ (which is of full measure) and R larger than some $R_0>0$ we have

$$\mu^{n-1}(\partial(M\cap B_R(0))) \geq \delta\exp\left(\frac{R^2}{2}\right) \geq \delta R\exp\left(\frac{R^2}{4}\right)$$

for some constant $\delta > 0$. Recall the area formula and divergence theorem, Theorems A.1 and A.2. As the function

$$R \mapsto \mu^n(M \cap B_R(0))$$

is monotone and continuous from the left and actually continuous at every value $R \in \mathcal{R}$, we can differentiate it almost everywhere in $\mathbb{R}^+$ and we have, for $R_0 < r < R$,

$$\begin{aligned}
\mu^n(M \cap B_R(0)) - \mu^n(M \cap B_r(0)) &= \int_r^R \frac{d}{d\xi}\mu^n(M \cap B_\xi(0))\, d\xi \\
&= \int_r^R \int_{M\cap B_\xi(0)} \operatorname{div}_{M\cap B_\xi(0)} \eta_\xi \, d\mu^{n-1}\, d\xi \\
&= -\int_r^R \int_{M\cap B_\xi(0)} \langle \eta_\xi, \mathbf{H}_{M\cap B_\xi(0)}\rangle \, d\mu^{n-1}\, d\xi \\
&\quad + \int_r^R \int_{\partial(M\cap B_\xi(0))} \langle \eta_\xi, \eta_\xi\rangle \, d\mu^{n-1}\, d\xi \\
&= \int_r^R \int_{\partial(M\cap B_\xi(0))} d\mu^{n-1}\, d\xi \\
&\geq \delta \int_r^R \xi \exp\left(\frac{\xi^2}{4}\right) d\xi = 2\delta\left(\exp\left(\frac{R^2}{4}\right) - \exp\left(\frac{r^2}{4}\right)\right).
\end{aligned}$$

Then

$$\mu^n(M \cap B_R(0))e^{-R} \to \infty\,,$$

for $R \to \infty$, in contradiction with the hypotheses of the theorem. Hence, the

$$\liminf_{R\to\infty, R\in\mathcal{R}} \exp\left(-\frac{R^2}{2}\right) \mu^{n-1}(\partial(M \cap B_R(0))) = 0\,.$$

It follows that the same holds for $|\sigma_R|$ and equation (3) is proved. By Cauchy–Schwarz,

$$4|A|^2|\nabla A|^2 = |\nabla |A|^2|^2 = 4|A\nabla A|^2 \leq 4|A|^2|\nabla A|^2$$

or in coordinates

$$\begin{aligned}
4h^i_j h^j_i \nabla_k h^m_n \nabla^k h^n_m &= \nabla_k(h^i_j h^j_i)\nabla^k(h^m_n h^n_m) \\
&= 4h^i_j h^m_n \nabla_k h^j_i \nabla^k h^n_m \leq 4h^i_j h^j_i \nabla_k h^m_n \nabla^k h^n_m
\end{aligned}$$

with equality if and only if A and ∇A are linearly dependent, that is, at every point there exist constants c_k such that

$$\nabla_k h_{ij} = c_k h_{ij}$$

for every i, j. Contracting this equation with the metric g^{ij} and with h^{ij} we get

$$\nabla_k H = c_k H \quad \text{and} \quad \nabla_k |A|^2 = 2c_k|A|^2\,,$$

hence

$$\nabla_k \log H = c_k \quad \text{and} \quad \nabla_k \log |A|^2 = 2c_k\,.$$

This implies

$$\nabla_k \log\left(\frac{H}{|A|}\right) = 0 \quad \text{so that} \quad |A| = \alpha H$$

for some constant $\alpha > 0$. By connectedness this relation has to hold globally on M. Suppose now that at a point $|\nabla H| \neq 0$, then

$$\nabla_k h_{ij} = c_k h_{ij} = \frac{\nabla_k H}{H} h_{ij} \tag{4}$$

which is a symmetric 3-tensor by the Codazzi equations, hence

$$h_{ij}\nabla_k H = h_{ik}\nabla_j H$$

at one point, where the Christoffel symbols vanish. Computing then in normal coordinates with an orthonormal basis $\{\boldsymbol{\tau}_1, \dots, \boldsymbol{\tau}_n\}$ such that $\boldsymbol{\tau}_1 = \nabla H / |\nabla H|$, we have with $g^{ij} = \delta^{ij}$,

$$\begin{aligned}
0 &= |h_{ij}\nabla_k H - h_{ik}\nabla_j H|^2 \\
&= (h_{ij}\nabla_k H - h_{ik}\nabla_j H)g^{il}g^{jm}g^{kn}(h_{lm}\nabla_n H - h_{ln}\nabla_m H) \\
&= 2|\nabla H|^2|A|^2 - 2g^{il}g^{jm}g^{kn}h_{ij}h_{ln}\nabla_k H\nabla_m H \\
&= 2|\nabla H|^2|A|^2 - 2g^{il}h_i^m h_l^k\nabla_k H\nabla_m H \\
&= 2|\nabla H|^2|A|^2 - 2g^{il}h_i^1 h_l^1\nabla_1 H\nabla_1 H \\
&= 2|\nabla H|^2\left(|A|^2 - \sum_{i=1}^n \left(h_i^1\right)^2\right).
\end{aligned}$$

Hence, $|A|^2 = \sum_{i=1}^n \left(h_i^1\right)^2$ and

$$|A|^2 = \left(h_1^1\right)^2 + 2\sum_{i=2}^n \left(h_i^1\right)^2 + \sum_{i,j\neq 1}^n \left(h_i^j\right)^2$$

so $h_j^i = 0$ unless $i = j = 1$, which means that A has rank one. Thus, we have two possible (not mutually excluding) situations at every point of M, either A has rank one or $\nabla H = 0$.

If $\ker A \equiv \emptyset$ on M, A must have rank at least two as we assumed $n \geq 2$, then we have $\nabla H = 0$ which implies $\nabla A = 0$ and

$$h_{ij} = Hh_{ik}h_j^k = Hh_{ik}g^{kl}h_{lj}$$

by equation (1). This means that for an eigenvalue λ_m with eigenvector ξ_m,

$$h_{ij}\xi_m^j = Hh_{ik}g^{kl}h_{lj}\xi_m^j = Hh_{ik}g^{kl}\lambda_m g_{lj}\xi_m^j = \lambda_m Hh_{ij}\xi_m^j$$

so that all the eigenvalues of A are 0 or $1/H$. As the kernel is empty

$$H = \sum_{i=1}^n \lambda_m = \frac{n}{H}$$

so that

$$H = \sqrt{n} \quad \text{and} \quad h_{ij} = \frac{g_{ij}}{\sqrt{n}}.$$

Then, the complete hypersurface M has to be the sphere $\mathbb{S}^n_{\sqrt{n}}$, indeed we compute

$$\begin{aligned}
\Delta|x|^2 = \Delta|x|^2 &= 2\nabla\langle x, \nabla x\rangle = 2n + 2\langle x, \Delta x\rangle \\
&= 2n - 2H\langle x, \boldsymbol{\nu}\rangle = 2n - 2H^2 = 0,
\end{aligned}$$

by means of the structural equation $H = \langle x, \boldsymbol{\nu}\rangle$. Hence, $|x|^2$ is a harmonic function on M. Looking at the point of M of minimum distance from the origin, by the strong maximum principle for elliptic equations, Theorem C.1, it must be constant on M and $M = \mathbb{S}^n_{\sqrt{n}}$.

Let now $\ker A(x) \neq \emptyset$ at some point $x \in M$ and let $v_1(x), \dots, v_{n-m}(x) \in T_xM \subset \mathbb{R}^{n+1}$ be a family of unit orthonormal tangent vectors spanning $\ker A(x)$, where $\dim\ker A(x) = (n-m)$, that is $h_{ij}(x)v_k^j(x) = 0$ for $k = 1, \dots, n-m$. By (4), the geodesic $\gamma(s)$ from $x \in M$ (M is complete) with initial velocity $\partial_s\gamma(0) = v_k(x)$ satisfies

$$\nabla_{\partial_s\gamma}(h_{ij}\partial_s\gamma^j) = \frac{\langle\nabla H, \partial_s\gamma\rangle}{H}h_{ij}\partial_s\gamma^j$$

hence, by Gronwalls lemma there holds

$$h_{ij}(\gamma(s))\partial_s\gamma^j(s) = h_{ij}(\gamma(0))\partial_s\gamma^j(0)\exp\left(\int_0^s \frac{\langle\nabla H, \partial_\sigma\gamma\rangle}{H}\,d\sigma\right) = 0$$

for every $s \in \mathbb{R}$. Since γ is a geodesic in M, $\partial_s^2\gamma(s) \in (T_{\gamma(s)}M)^\perp$, that is, the normal to the curve in $\mathbb{R}^{n+1}$ is also the normal to M, then letting κ be the curvature of γ in $\mathbb{R}^{n+1}$, we have

$$\kappa = -\langle \boldsymbol{\nu}_M, \partial_s^2\gamma\rangle = h_{ij}\partial_s\gamma^i\partial_s\gamma^j = 0\,,$$

thus γ is a straight line in $\mathbb{R}^{n+1}$. Then all the $(n-m)$-dimensional affine subspace $x + \ker A(x) \subset \mathbb{R}^{n+1}$ is contained in M. Let now $\sigma(s)$ be a geodesic from x to another point y parametrized by arclength and extend by parallel transport the vectors $v_k(x)$, $k = 1, \dots, n-m$, along σ, then

$$\nabla_{\partial_s\sigma}(h_{ij}v_k^j) = \frac{\langle \nabla H, \partial_s\sigma\rangle}{H} h_{ij}v_k^j$$

and again by Gronwalls lemma it follows that $h_{ij}v_k^j(s) = 0$ for every $s \in \mathbb{R}$ and $k = 1, \dots, n-m$, in particular $v_k(y) \in \ker A(y)$. Hence, $\dim\ker A \equiv n-m$ on M with $0 < m < n$ (as A is never zero) and all the affine $(n-m)$-dimensional subspaces $x + S(x) \subset \mathbb{R}^{n+1}$ are contained in M. Moreover, as $h_{ij}v_k^j = 0$ along the geodesic σ, we have

$$D^{\mathbb{R}^{n+1}}_{\partial_s\sigma} v_k = \nabla_{\partial_s\sigma}v_k + \langle \nabla_{\partial_s\sigma}v_k, \boldsymbol{\nu}_M\rangle\boldsymbol{\nu}_M = -h_{ij}v_k^j\partial_s\sigma^i\boldsymbol{\nu}_M = 0\,,$$

so the extended vectors v_k are constant in $\mathbb{R}^{n+1}$, which means that the parallel extension is independent of the geodesic σ, that the subspaces $\ker A(x)$ are all a common $(n-m)$-dimensional vector subspace of $\mathbb{R}^{n+1}$ and

$$M = M + \ker A \subset \mathbb{R}^{n+1}\,.$$

By Sards theorem, Corollary B.3, there exists a vector $y \in M$ such that

$$N = M \cap \big(y + (\ker A)^\perp\big)$$

is a smooth, complete m-dimensional submanifold of $\mathbb{R}^{n+1}$. Moreover,

$$M = N \times \ker A\,.$$

This implies that

$$L = M \cap (\ker A)^\perp$$

is a smooth, complete m-dimensional submanifold of $(\ker A)^\perp = \mathbb{R}^{m+1}$ with

$$M = L \times \ker A\,.$$

Moreover, as $\ker A$ is in the tangent space to every point of L, the normal $\boldsymbol{\nu}_M$ to M at a point of L stays in $(\ker A)^\perp$ so it must coincide with the normal $\boldsymbol{\nu}_L$ to L in $(\ker A)^\perp$, then a simple computation shows that the mean curvature H_M of M at the points of L is equal to the mean curvature H_L of L as a hypersurface of $(\ker A)^\perp = \mathbb{R}^{m+1}$. This shows that L is a hypersurface in $\mathbb{R}^{m+1}$ satisfying $H_L(z) = \langle z, \boldsymbol{\nu}_L(z)\rangle$ for every $z \in L$. Finally, as by construction the second fundamental form of L has empty kernel, by the previous discussion we have $L = \mathbb{S}^m_{\sqrt{m}}$ and $M = \mathbb{S}^m_{\sqrt{m}} \times \mathbb{R}^{n-m}$ which proves the claim. □

Theorem 2.5 (Colding–Minicozzi, [CM12, Theorem 10.1])**.** *If M^n, for $n \geq 2$, is an embedded hypersurface in $\mathbb{R}^{n+1}$, with non-negative mean curvature, satisfying $H = \langle x, \boldsymbol{\nu}\rangle/2$, then M^n is of the form $\mathbb{S}^m_{\sqrt{2m}} \times \mathbb{R}^{n-m}$ for $m = 0, 1, \dots, n$.*

2.2. **Curves.**

Theorem 2.6 (Abresch–Langer, [AL86])**.** *Let $\Sigma \subset \mathbb{R}^2$ be a smooth, complete, embedded curve satisfying $\kappa(x) = \langle x, \boldsymbol{\nu}(x)\rangle/2$ for every $x \in \Sigma$. Then Σ is either the line through the origin or the $\mathbb{S}^1_{\sqrt{2}}$.*

Proof. □

Appendix A. Hypersurfaces in $\mathbb{R}^{n+1}$

A topological space is called Hausdorff space if for any two distinct points there exists a neighbourhood of each which is disjoint from the neighbourhood of the other. A topological space M^n is called locally Euclidean of dimension n, if M^n can be covered with open sets where every set is homeomorphic to an open subset of $\mathbb{R}^n$. A pair (U, φ), where $U \subset M^n$ is open and $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$ is a homeomorphism, is called chard of M^n. A collection A of chards is called atlas of M^n if

$$M^n \subset \bigcup_{(U,\varphi)\in A} U \,.$$

Two chards (U, φ) and (V, ψ) are called C^k-compatible, $k \geq 1$, if

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$$

is a C^k-diffeomorphism. An atlas is called of class C^k, if each of its chards are C^k-compatible. If A is a C^k atlas, there exists exactly one maximal C^k atlas A_0 with $A \subset A_0$; it contains all chards which are C^k compatible with the chards of A. A differentiable (C^k) structure on M^n is a maximal C^k atlas on M^n. A local Euclidean Hausdorff space with a differentiable structure is called differentiable manifold.

Let N^{n+m} be a differentiable manifold. A subset $M^n \subset N^{n+m}$, $n, m \geq 1$, is called n-dimensional C^k-submanifold of N^{n+m} if for every $x \in M^n$ there exists an open neighbourhood $U \subset N^{n+m}$ and a C^k diffeomorphism $\varphi : U \to \varphi(U) \subset \mathbb{R}^{n+m}$ with

$$\varphi(U \cap M) = \varphi(U) \cap (\mathbb{R}^n \times \{0_{\mathbb{R}^m}\}) \,.$$

Such an M^n owns a C^k atlas, that is

$$A := \{(U \cap M, \varphi|_{U\cap M}) \mid \text{ where } (U, \varphi) \text{ as above}\} \,.$$

Then, M^n is locally Euclidean of dimension m and

$$(\psi|_{V\cap M}) \circ (\varphi|_{U\cap M})^{-1} = \psi \circ \varphi^{-1}|_{(\mathbb{R}^n\times\{0\})\cap\varphi(U\cap V)} \in C^k$$

for two diffeomorphisms ψ and φ.

A topological manifold with boundary is a Hausdorff space in which every point has a neighborhood homeomorphic to an open subset of the Euclidean half-space $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$. The boundary ∂M^n of M^n is the set of all points $p \in M^n$ such that $(\varphi(p))^n = 0$ for all chards (U, φ) of M^n. If M^n is a manifold with boundary, then the interior $\operatorname{int} M^n = M^n \setminus \partial M^n$ is a manifold (without boundary) of dimension n and boundary ∂M^n is a manifold (without boundary) of dimension $n - 1$.

Let M^n be an abstract, smooth, compact, n-dimensional manifold without boundary and X a smooth immersion ($\operatorname{rank} DX \equiv n$) with

$$X : M^n \to \mathbb{R}^{n+m} \,.$$

We call $M := X(M^n)$ a hypersurface in $\mathbb{R}^{n+m}$. For all $p \in M^n$ and $v, w \in T_pM^n$, the embedding X induces an isomorphism

$$dX_p : T_pM^n \to T_{X(p)}M \,,$$

and the first fundamental form or metric $g_p : T_pM^n \times T_pM^n \to \mathbb{R}$ with

$$g_p(v, w) := \langle dX_p(v), dX_p(w)\rangle_{\mathbb{R}^{n+m}} \,.$$

Let $(U_i, \varphi_i)_{i\in I}$ be an atlas of M^n and

$$\partial_i = \frac{\partial}{\partial p_i} = d\varphi^{-1}(e_i) \in TM^n$$

then the matrix entries of the metric are

$$g_{ij} = g(\partial_i, \partial_j) = \langle dX(\partial_i), dX(\partial_j)\rangle_{\mathbb{R}^{n+m}} = \langle \partial_i X, \partial_j X\rangle_{\mathbb{R}^{n+m}} = \delta_{\alpha\beta}\partial_i X^\alpha \partial_j X^\beta$$

for $1 \le \alpha, \beta \le n+m$. We define by $(g^{ij})_{ij}$ the coordinate dependent inverse of the matrix $(g_{ij})_{ij}$ and the measure

$$d\mu^n = \sqrt{\det(g_{ij})}\, dp\,.$$

Observe that

$$\partial_k g_{ij} = \langle \partial_k\partial_i X, \partial_j X\rangle + \langle \partial_i X, \partial_k\partial_j X\rangle$$

and

$$\partial_k g^{ij} = -g^{pi}g^{qj}\partial_k g_{pq}\,.$$

The corresponding Levi–Cevita connection on M^n is given by

$$\nabla_v w = dX^{-1}\Big(\big(D_{dX(v)}dX(w)\big)^\top\Big)\,.$$

Here D is the standard connection in $\mathbb{R}^{n+m}$, and $^\top$ denotes the tangential component with respect to M, that is the orthogonal projection onto $dX(p)(T_pM^n) = T_{X(p)}M$. The connection can be evaluated in coordinates in terms of the Christoffel symbols Γ^k_{ij} defined by

$$\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k\,,$$

where Γ^k_{ij} is explicitly given by We define the Christoffel symbols by

$$\Gamma^k_{ij} := g^{kl}\langle \partial_i\partial_j X, \partial_l X\rangle\,.$$

Here and in the following, we sum over repeated indices. Then,

$$\Gamma^k_{ij}\partial_k X = \langle \partial_i\partial_j X, \partial_l X\rangle \partial_l X\,.$$

At a fixed point, we can choose a coordinate system such that $\Gamma^k_{ij} = 0$. We calculate

$$0 = \partial_k \delta^i_j = \partial_k(g^{il}g_{jl}) = g^{il}\partial_k g_{jl} + g_{jl}\partial_k g^{il}\,,$$

so that

$$\begin{aligned}\partial_k g^{ij} &= -g^{il}g^{jm}\partial_k g_{lm} = -g^{il}g^{jm}\partial_k\langle \partial_l X, \partial_m X\rangle \\ &= -g^{il}g^{jm}(\langle \partial_k\partial_l X, \partial_m X\rangle + \langle \partial_l X, \partial_k\partial_m X\rangle) = -g^{il}\Gamma^j_{kl} - g^{jm}\Gamma^i_{km}\,.\end{aligned}$$

Being in a Levi–Cevita connection the Li bracket $[\,\cdot\,,\,\cdot\,]$ is given by

$$[v,w] = \nabla_v w - \nabla_w v = \big(v(\mu^k) - w(\lambda^k)\big)\,\partial_k\,.$$

The tangential gradient of a function $f \in C^1(M)$ is given by

$$\nabla^M f = g^{ij}\partial_i f \partial_j\,.$$

The tangential divergence $\mathrm{div}_M : T_pM^n \to \mathbb{R}$ is given by

$$\mathrm{div}_M\, v = g^{ij}\langle \partial_i v, \partial_j X\rangle_{\mathbb{R}^{n+m}}\,.$$

For the embedding vector X, we therefore have

$$\mathrm{div}_M\, X = g^{ij}\langle \partial_i X, \partial_j X\rangle_{\mathbb{R}^{n+m}} = g^{ij}g_{ij} = n\,.$$

For $\omega = df = \frac{\partial f}{\partial p_i}dp^i$, we obtain the Hessian of the function f

$$(\mathrm{Hess}_M\, f)(v,w) := \left(\nabla^2 f\right)(v,w) = (\nabla_v f)(w)\,,$$

or in coordinates

$$\nabla_i\nabla_j f = (\mathrm{Hess}_M\, f)(\partial_i, \partial_j) = \partial_i\partial_j f - \Gamma^k_{ij}\partial_k f\,.$$

The Laplace–Beltrami operator $\Delta_M : C^2(M^n) \to C^0(M^n)$ is defined as

$$\Delta_M f := \frac{1}{\sqrt{\det g_{kl}}}\partial_j\Big(\sqrt{\det g_{kl}}\, g^{ij}\partial_j f\Big) = \mathrm{div}_M(\nabla^M f) = g^{ij}\nabla_i\nabla_j f\,.$$

We define the second fundamental form $\mathbf{A}_p : T_pM^n \times T_pM^n \to \big(T_{X(p)}M\big)^{\perp}$ by

$$\begin{aligned}\mathbf{A}_p(v,w) &:= -\sum_{k=1}^m \big\langle D_{dX_p(v)} dX_p(w), \boldsymbol{\nu}_k(p)\big\rangle\, \boldsymbol{\nu}_k(p)\\ &= \sum_{k=1}^m \big\langle dX_p(w), D_{dX_p(v)}\boldsymbol{\nu}_k(p)\big\rangle\, \boldsymbol{\nu}_k(p)\,,\end{aligned}$$

where $\{\boldsymbol{\nu}_k\}_{1\le k\le m}$ is an orthonormal frame for $(TM)^{\perp}$. In coordinates $\{p_i\}_{1\le i\le n}$,

$$\mathbf{A}_{ij} := \mathbf{A}_p(\partial_i,\partial_j) = \sum_{k=1}^m \langle \partial_i X, \partial_j \boldsymbol{\nu}_k\rangle\, \boldsymbol{\nu}_k\,.$$

The mean curvature vector $\mathbf{H} : M \to (TM)^{\perp}$ is the trace of the second fundamental form

$$\mathbf{H} := -g^{ij}\mathbf{A}_{ij} = -g^{ij}\sum_{k=1}^m \langle \partial_i X, \partial_j \boldsymbol{\nu}_k\rangle\, \boldsymbol{\nu}_k = -\sum_{k=1}^m \mathrm{div}(\boldsymbol{\nu}_k)\boldsymbol{\nu}_k\,.$$

We calculate that

$$\begin{aligned}\Delta_M X &= g^{ij}\left(\partial_i\partial_j X - \Gamma_{ij}^k\partial_k X\right) = g^{ij}\sum_{k=1}^m \langle \partial_i\partial_j X, \boldsymbol{\nu}_k\rangle\, \boldsymbol{\nu}_k\\ &= -g^{ij}\sum_{k=1}^m \langle \partial_i X, \partial_j\boldsymbol{\nu}_k\rangle\, \boldsymbol{\nu}_k = \mathbf{H}\,.\end{aligned}$$

For a submanifold Σ of M, the mean curvature vector is given by

$$\mathbf{H}_\Sigma = -\sum_{k=1}^m \mathrm{div}_\Sigma(\boldsymbol{\nu}_k)\boldsymbol{\nu}_k - \mathrm{div}_\Sigma(\boldsymbol{\nu}_\Sigma)\boldsymbol{\nu}_\Sigma\,,$$

where $\boldsymbol{\nu}_\Sigma$ is the unit co-normal of Σ. Since $\boldsymbol{\nu}_\Sigma$ tangential to M,

$$\langle \mathbf{H}_\Sigma, \boldsymbol{\nu}_\Sigma\rangle = -\,\mathrm{div}_\Sigma\, \boldsymbol{\nu}_\Sigma$$

and on Σ,

$$\begin{aligned}\Delta_\Sigma X &= g_\Sigma^{ij}\left(\partial_i\partial_j X - \Gamma_{ij}^k\partial_k X\right)\\ &= \sum_{k=1}^m g_\Sigma^{ij}\langle \partial_i\partial_j X, \boldsymbol{\nu}_k\rangle\, \boldsymbol{\nu}_k + g_\Sigma^{ij}\langle \partial_i\partial_j X, \boldsymbol{\nu}_\Sigma\rangle\, \boldsymbol{\nu}_\Sigma = \mathbf{H}_\Sigma\,.\end{aligned}$$

For $m = 1$,

$$\mathbf{A}(v,w) = A(v,w)\boldsymbol{\nu}\,,$$

where $\boldsymbol{\nu}$ is the outward pointing unit normal to M and $A : TM^n \times TM^n \to \mathbb{R}$ is given by

$$A(v,w) = -\big\langle D_{dX(v)}dX(w), \boldsymbol{\nu}\big\rangle = \big\langle dX(w), D_{dX(v)}\boldsymbol{\nu}\big\rangle\,.$$

where $\boldsymbol{\nu}$ is the outward pointing unit normal to M. In coordinates,

$$h_{ij} := A(\partial_i,\partial_j) = -\langle \partial_i\partial_j X, \boldsymbol{\nu}\rangle = \langle \partial_i X, \partial_j\boldsymbol{\nu}\rangle\,.$$

Let $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ be the eigenvalues of A, that is

$$h_{ij}\xi_k^i\xi_k^j = \lambda_k g_{ij}$$

for eigenvectors $\xi_k \in TM$ and $k = 1,\ldots,n$. The Weingarten operator $S : TM^n \to TM^n$ is given by

$$S(v) := dX^{-1}\big(D_{dX(v)}\boldsymbol{\nu}\big)$$

so that

$$A(v,w) = g(v, S(w))\,,$$

where in coordinates,

$$h^i_j := g^{ik} h_{kj}$$

and the Weingarten equations by

$$\partial_i \boldsymbol{\nu} = h^j_i \partial_j X\,. \tag{5}$$

The norm of the second fundamental form is given by

$$|A|^2 = g^{ik} g^{lj} h_{kl} h_{ij} = h^{ij} h_{ij}\,,$$

and the mean curvature vector is given by

$$\mathbf{H} = -g^{ij} h_{ij} \boldsymbol{\nu} = -H\boldsymbol{\nu}\,,$$

where we define the mean curvature H of M as the trace of the second fundamental form with

$$H = g^{ij} h_{ij} = \mathrm{div}_M\, \boldsymbol{\nu}\,.$$

We have the Gauss formula

$$\nabla_i \nabla_j X = \partial_i \partial_j X - \Gamma^k_{ij} \partial_k X = -h_{ij} \boldsymbol{\nu} \tag{6}$$

which as before leads to $\Delta_M X = \mathbf{H}$. More useful identities are the Codazzi equations in $\mathbb{R}^{n+1}$

$$\nabla_k h_{ij} - \nabla_j h_{ik} = \Gamma^l_{ij} h_{lk} - \Gamma^l_{ik} h_{lj} \tag{7}$$

and Simons' identity

$$\Delta h_{ij} = \nabla_i \nabla_j H + H h_{ik} h^k_j - |A|^2 h_{ij}\,. \tag{8}$$

We define the Riemannian curvature tensor as

$$R^k_{lij} := \nabla_i \Gamma^k_{jl} - \nabla_j \Gamma^k_{il} + \Gamma^k_{im} \Gamma^m_{jl} - \Gamma^k_{jm} \Gamma^m_{il}\,.$$

Moreover, we set

$$R_{klij} := g^{kr} R^r_{lij}$$

and define the Ricci tensor by

$$R_{ik} := R_{ijkl} g^{jl}$$

and the scalar curvature by

$$R := R_{ij} g^{ij}\,.$$

The Gauss equation

$$R_{ijkl} = h_{ik} h_{jl} - h_{il} h_{jk} \tag{9}$$

Theorem A.1 (First variation of the area formula, see [Sim83, p. 51])**.** *Let $M \subset \mathbb{R}^{n+1}$ be a smooth, compact, n-dimensional hypersurface with boundary. Let $U \subset \mathbb{R}^{n+1}$ be a open and bounded such that $M \subset U$. Let $\phi : U \times (-1,1) \to U$ be a one-parameter family of $C^{2;1}$-diffeomorphisms. Set $M_t := \phi(M,t)$ and $v(p) := \partial_t \phi(p,0)$. Then*

$$\partial_t|_{t=0} \mu^n(M_t) = \int_M \mathrm{div}_M\, v\, d\mu^n\,.$$

Theorem A.2 (Divergence theorem, see [Sim83, p. 43], [DHTK10, p. 304], [Eck04, p. 116])**.** *Let $M \subset \mathbb{R}^{n+1}$ be a smooth, compact, n-dimensional manifold with boundary. Let v be a C^1-vector field on M. Then*

$$\int_M \mathrm{div}_M\, v\, d\mu^n = -\int_M \langle v, \mathbf{H}_M \rangle_{\mathbb{R}^{n+1}}\, d\mu^n + \int_{\partial M} \langle v, \boldsymbol{\nu}_{\partial M} \rangle_{\mathbb{R}^{n+1}}\, d\mu^{n-1}\,.$$

Proof. □

Theorem A.3 (Rademachers theorem, see [Fed69, Theorem 3.1.6]). *Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^m$ be Lipschitz continuous. Then f is differentiable almost everywhere in U.*

Appendix B. Sard's theorem

Section copied from [Sch05, Section 3]. See also [BJ73].

Definition B.1. Let $f : M \to N$ differentiable. A point $p \in M$ is called regular, if the differential of f in p is surjektiv. A point $q \in N$ is called regular value, if $f^{-1}(q)$ consists of regular points. Non-regular points or values are called singular or critical.

We want to prove the following theorem.

Theorem B.2 (Sard's theorem). *Let M^m and N^n be differentiable manifolds with a countable basis of their topology. The critical set S of a C^k function $f : M \to N$ consists of those points at which the differential $df : TM \to TN$ has rank less than n as a linear transformation. If $k \geq \max\{n - m + 1, 1\}$, then the image of S has Lebesgue measure zero as a subset of N.*

Corollary B.3. *Let M^m be a differentiable manifold and $f : M^m \to \mathbb{R}^n$ a diffenrentiable. Then $f^{-1}(x) \subset M^m$ is a differentiable submanifold of co-dimension n for almost every $x \in \mathbb{R}^n$.*

Remark B.4. The set $f^{-1}(x)$ can be empty. Sard's theorem also holds for maps $f : \mathbb{R}^n \to \mathbb{R}^p$, $f \in C^k$ with $k > \max\{n - p, 0\}$ and manifolds with according dimensions.

Definition B.5. A subset $C \subset \mathbb{R}^n$ is of measure zero, if for every $\varepsilon > 0$ there exists a sequence $(W_i)_{i\in\mathbb{N}}$ of cubes in $\mathbb{R}^n$ with

$$C \subset \bigcup_{i\in\mathbb{N}} W_i \quad \text{and} \quad \sum_{i\in\mathbb{N}} |W_i| < \varepsilon\,.$$

Remark B.6. (i) The countable set of zero sets is again a zero set.
(ii) One obtains an equivalent definition for open oder closed cubes or balls.

Lemma B.7. *Let $U \subset \mathbb{R}^m$ be open and $C \subset U$ of measure zero. Let $f : U \to \mathbb{R}^m$ be Lipschitz. Then $f(C)$ has measure zero.*

Proof. Exercise. □

Definition B.8. A subset C of a differentiable manifold has measure zero, if for every chard $h : U \to U' \subset \mathbb{R}^m$ the set $h(C \cap U) \subset \mathbb{R}^m$ is of measure zero.

Remark B.9. The assumption of differentiability is important here, since zero sets are not necessarily maintained under homeomorphisms. Since a manifold owns a countable basis of the topologie, there exists an atlas with countably many chards. It is sufficient to apply the definition for such chards. Well-definedness follows, since zero sets are maintained under differentiable chard changes and countable unions.

Lemma B.10. *An open covering of the interval $[0, 1]$ by subintervals contains a countable cover $[0, 1] = \bigcup_{j=1}^{k} I_j$ with $\sum_{j=1}^{k} |I_j| \leq 2$.*

Proof. Due to the compactness, there exists a finite subcover. Choose one where no interval can be left out without loosing the covering property. Let the intervals I_j, $j = 1, \ldots, k$ be numbered so that with $I_j = (a_j, b_j)$ always holds $a_j < a_{j+1}$,

$j = 1, \dots, k-1$. Minimality and covering property imply $a_i < a_{i+1} < b_i < a_{i+2}$. So that

$$\sum_i (bi - a_i) = \sum_i (a_{i+1} - a_i) + \sum_i (b_i - a_{i+1})$$
$$< \sum_i (a_{i+1} - a_i) + \sum_i (a_{i+1} - a_{i+1}) \leq 2\,,$$

where we used that we have telecope sums in the end. $\square$

Theorem B.11 (Fubini). *Let $\mathbb{R}^{n-1}_t := \{x \in \mathbb{R}^n \,|\, x^n = t\} \subset \mathbb{R}^n$. Let $C \subset \mathbb{R}^n$ be compact and $C_: = C \cap \mathbb{R}^{n-1}_t$ be of measure zero in $\mathbb{R}^{n-1}_t \cong \mathbb{R}^{n-1}$ for all $t \in \mathbb{R}$. Then C is of measure zero in $\mathbb{R}^n$.*

Proof. Since the property of being of measure zero is maintained under countable unions, we can assume that $C \subset \mathbb{R}^{n-1} \times [0,1]$. For $t \in [0,1]$, C_t is of measure zero in $\mathbb{R}^{n-1} \times \{t\}$. Let $\varepsilon > 0$ and W^i_t be a cover of C_t by open cubes with $\sum_i |W^i_t| < \varepsilon$. Define $W_t := \bigcup_i W^i_t$ identify these with subsets of $\mathbb{R}^{n-1}$. The function $|x^n - t|$ is for fixed $t \in [0,1]$ on C continuous, vanishes exactly on C_t und attains a positive minimum in the compact set $C \setminus (W_t \times [0,1])$, which we call α. It follows

$$\{x \in C : |x^n - t| < \alpha\} \subset W_t \times I^\alpha_t\,,$$

where $I^\alpha_t = (t - \alpha, t + \alpha)$ and $\bigcup_t I^\alpha_t = [0,1]$. Choose a subcover of $[0,1]$ among the intervals I^α_t with $\sum_{t_i} |I^\alpha_{t_i}| \leq 2$. Observe that $\alpha = \alpha(t_i)$. It holds

$$C \subset \bigcup_{t_j, i} W^i_{t_j} \times I^\alpha_{t_j}\,,$$

where i is the index of the cube and we take the union over cuboids. Moreover,

$$\sum_{t_j, i} |W^i_{t_j} \times I^\alpha_{t_j}| \leq 2\varepsilon\,.$$

Sending $\varepsilon \to 0$ yields the lemma. $\square$

Remark B.12. The requirement that C is compact, can be weakened as follows: C is a countable union of compact sets, that each suffice the assumptions of the theorem. This is fulfilled by closed and open sets (which cannot be zero sets), for images of these set under continuous maps, countable union und finite intersections of these.

Proof of Theorem B.2. After introducing maps it is sufficient to show: Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}^p$ smooth and $D \subset U$ be the set of critical points of f, then $f(D) \subset \mathbb{R}^p$ has measure zero.

We prove by induction over n. In case $n = 0$, $\mathbb{R}^n$ is a point. So, $f(U)$ is at most a point and has measure zero. Assume the claim is true for the case $n-1$. We proof the case n. Let $D_i \subset U$ be the set of all points points, in which the partial derivative of order $\leq i$ vanish. We obtain the decreasing sequence of relatively closed sets

$$D \supset D_1 \supset D_2 \supset \dots\,.$$

We claim that

(i) $f(D \setminus D_1)$ is of measure zero,
(ii) $f(D_i \setminus D_{i+1})$ is of measure zero,
(iii) for k big enough, $f(D_k)$ is of measure zero.

We observe, that (iii) is neccessary, since also the points, in which all derivatives vanish, can be captured. By Remark B.12, all sets occuring in (i)–(iii) can be used. Moreover, it is sufficient to prove that every point in $D \setminus D_1$ resp. $D_i \setminus D_{i+1}$ resp. D_k has a neighbourhood V, so that $f(V \cap (D \setminus D_1))$ resp. $f(V \cap (D_i \setminus D_{i+1}))$ resp. $f(V \cap D_k)$ are of measure zero. The claim then follows, since the countable union of zero set is again a zero set.

Proof of (i): Assume, that $p \geq 2$, since for $p = 1$ we already have $D = D_1$. Let $x_0 \in D \setminus D_1$. Since $x_0 \notin D_1$, there exists a partial derivative that is not vanishing in x_0, w.l.o.g. $\partial_1 f \neq 0$. Define $h : U \to \mathbb{R}^n$ by

$$h : x = (x^1, \dots, x^n) \mapsto (f^1(x), x^2, \dots, x^n) .$$

Then h is not singular in x_0. Hence there exists a neighbourhood V of x_0, so that $h : V \to h(V) = V'$ is a diffeomorphism. Define $g := f \circ h^{-1}$. In a neighbourhood of $h(x)$, g is of the form

$$g : (z^1, \dots, z^n) \mapsto (z^1, g^2(z), \dots, g^n(z)) .$$

The hyperplane $\{z \,|\, z^1 = t\}$ is (locally) mapped into the hyperplane $\{y \,|\, y^1 = t\}$. Define

$$g_t : \{t\} \times \mathbb{R}^{n-1} \cap V' \to \{t\} \times \mathbb{R}^{p-1}$$

als restriction of g. We have

$$Dg_t = \begin{pmatrix} 1 & 0 \\ ? & Dg \end{pmatrix} .$$

Hence a point in $(\{t\} \times \mathbb{R}^{n-1}) \cap V'$ is critical for g if and only if it is for g_t. By the induction assumption the set of critical values of g_t is of measure zero in $\{t\} \times \mathbb{R}^{p-1}$. Since g maps entsprechende hyperplanes onto itself, the set of critical values of g also has a intersection of measure zero with the hyperplane $\{y \,|\, y^1 = t\}$. By Fubini, Theorem B.11, the critical values of g have measure zero. Since f and g only differ by an diffeomorphism, also the criticalen values of f have measure zero. This holds locally, as long as $\partial_1 f \neq 0$. This proves (i).

Proof of (ii): We argument similarly as in the proof of (i). Let $x_0 \in D_k \setminus D_{k+1}$. Then there exist a non-vanishing $(k+1)$-st derivative, w.l.o.g.

$$\frac{\partial^{k+1} f^1}{\partial x^1 \partial x^{\nu_1} \dots \partial x^{\nu_k}}(x_0) \notin 0 .$$

Assume, that this holds in a neighbourhood V of x_0. Define $w : V \to \mathbb{R}$ by

$$w := \frac{\partial^k f^1}{\partial x^{\nu_1} \dots \partial x^{\nu_k}}(x_0) \neq 0 .$$

It holds $w(x) = 0$, $\frac{\partial}{\partial x^1} w(x) \neq 0$. The map

$$h : x \to (w(x), x^2, \dots, x^n)$$

defines a diffeomorphism $h : V \to V' = h(V)$. w and therefore all k-th derivatives of f^1 vanish at most for $x = x_0$. Hence

$$h(D_k \cap V) \subset \{0\} \times \mathbb{R}^{n-1} \subset \mathbb{R}^n .$$

Define

$$g : f \circ h^{-1} : V' \to \mathbb{R}^p$$

and

$$g_0 : \{0\} \times \mathbb{R}^{n-1} \cap V' \to \mathbb{R}^p .$$

By the induction assumption, the set of critical values of g_0 has measure zero. Let $x \in h(D_k \cap V)$. Then all derivatives of g up to order k vanish there. Since $h(D_k \cap V) \subset \{0\} \times \mathbb{R}^{k-1}$, g_0 is defined there and has vanishing derivatives up to order k. In particular, all first derivatives vanish there as well and thus we are dealing with critical points of g_0. Hence

$$g_0 \circ h(D_k \cap V) = g \circ h(D_k \cap V) = f(D_k \cap V)$$

has measure zero.

Proof of (iii): The set U ist countable union of cubes. Let $W \subset U$ be a cube with side length $a \leq 1$ and let $k > n-1$. It is sufficient to show, that $f(W \cap D_k)$ is of measure zero. By Taylor it holds that

$$f(x+h) = f(x) + R(x,h)$$

with

$$|R(x,h)| \leq c|h|^{k+1}$$

for $x \in D_k \cap W$ and $x + h \in W$, where the constant c only depends on f and W. We devide W in r^n cubes with side length a/r, $r \in \mathbb{N}$. If W_1 is a cube of this partitioning, which contains a point $x \in D_k$, then every other point in W_1 can be described as $x + h$ with $|h| \leq \sqrt{n}a/r$. Hence with Taylor

$$|f(x+h) - f(x)| \leq c\left(\frac{\sqrt{n}a}{r}\right)^{k+1} .$$

So that $f(W_1)$ is contained in a cube with side length

$$c(n)\left(\frac{\sqrt{n}a}{r}\right)^{k+1} .$$

There are at most r^n such cubes with points in D_k. The summed up volumes of the images of these cubes in $\mathbb{R}^p$ are at most

$$c(n)^p\left(\frac{\sqrt{n}a}{r}\right)^{p(k+1)} r^n = cr^{n-p(k+1)} .$$

Since $n - p(k+1) < 0$, this will get arbitrary small for $r \to \infty$. $\square$

Corollary B.13 (Brown)**.** *Let M and N be (finite dimensional) manifolds. Let $f : M \to N$ be a differentiable (C^∞-)maps. Then all the regular values of f lay dense in N.*

We want to derive Brouwer's fixed point theorem from Sard's theorem.

Definition B.14. Let $A \subset B$. A retraction is a continuous map $f : B \to A$, so that $f|_A = id$, that is, $f(x) = x$ for all $x \in A$.

Theorem B.15. *There exists no retraction of $\overline{B_1(0)} \subset \mathbb{R}^n$ on $\mathbb{S}^{n-1}$.*

Proof. We prove the claim by contradiction. Let $f : \overline{B_1(0)} \to \mathbb{S}^{n-1}$ be a retraction. Show at first, that then there also exists a C^∞-retraction of $\overline{B_1(0)}$ on $\mathbb{S}^{n-1}$: We find a retraction g, that is close to $\partial B_1(0)$ of the class C^∞, e.g.,

$$g(x) = \begin{cases} f\left(\frac{x}{|x|}\right) & \text{for } \frac{1}{2} \leq |x| \leq 1 \\ f(2x) & \text{for } 0 \leq |x| \leq \frac{1}{2} . \end{cases}$$

Mollification in the interior gives a C^∞-retraction. Hence we may assume that $f \in C^\infty\left(\overline{B_1(0)}, \mathbb{S}^{n-1}\right)$. By Corollary B.13 there exists a regular value $y \in \mathbb{S}^{n-1}$ of f. Hence the compact set $f^{-1}(y)$ is a one-dimensional submanifold (first in $B_1(0)$, but since we can mollify f, also up to the boundary, since f is after construction constant on radial line segments close to $\mathbb{S}^{n-1}$). Hence $f^{-1}(y)$ is a one-dimensional manifold with boundary in $\overline{B_1(0)}$, whose boundary is a subset of $\mathbb{S}^{n-1} = \partial B_1$. It holds that $y \in f^{-1}(y)$, since f is a retraction. Let V be the component of $f^{-1}(y)$ that contains y. Then V is a one-dimensional compact connected manifold and thus diffeomorph to a closed interval. Then y is the one boundary point of V. Let z be the other, which as well lays on $\partial B_1(0)$. It follows that $z = f(z)$ in contradiction to $y, z \in f^{-1}(y)$. $\square$

Theorem B.16 (Brouwer's fixed point theorem)**.** *Let $f : \overline{B_1(0)} \to \overline{B_1(0)}$ be continuous. Then f has one fixed point, that is, there exists $x \in \overline{B_1(0)}$ with $f(x) = x$.*

Proof. If $f(x) \neq x$ for all $x \in \overline{B_1(0)}$, we define $g(x)$ to be the intersection of a line with $\mathbb{S}^{n-1}$ beginning in $f(x)$ through x . As constructed g is a retraction of $\overline{B_1(0)}$ on $\mathbb{S}^{n-1}$. □

Appendix C. Maximum principles

We will need the following maximum principles.

Theorem C.1 (Strong elliptic maximum principle)**.** *Let M be closed and $f : M \to \mathbb{R}$ satisfy*

$$-\Delta_M f + b^i \nabla_i^M f + cf \leq 0$$

for some smooth funtions b^i and $c \leq 0$. If $f \leq 0$, but $f \not\equiv 0$, then $f < 0$.

Proof. For a proof see [Eva02, §6.4, Theorem 4] or [Sch17b, Theorem 5.5] for $M^n = \mathbb{R}^n$. □

Let M^n be a smooth n-dimensional manifold with boundary whose closure is compact. Let $X : \bar{M}^n \times [0,T) \to \mathbb{R}^{n+m}$ be a family of smooth embeddings and set $M_t := X(M^n, t)$. For $f \in C^{2;1}(M^n \times [0,T))$, we define the parabolic operator

$$L(f) := \partial_t f - a^{ij} \nabla_i \nabla_j f - b^i \nabla_i f - cf \, , \tag{10}$$

where $a_{ij}, b_i, c \in L^\infty$ may depend on p, t, $(g_{kl})_{kl}$, f, ∇f, and $\nabla^2 f$, and where $(a^{ij})_{ij}$ is positive semi-definite, that is,

$$\lambda |\xi|^2 \leq a_{ij} \xi_i \xi_j \leq \Lambda |\xi|^2 \tag{11}$$

for all $\xi \in \mathbb{R}^n$. For $R > 0$, $p_0 \in M^n$ and $t_0 \in [0,T)$, define the spatial neighbourhood

$$\begin{aligned} U_R(p_0, t_0) &:= X^{-1}\big(B_R(X(p_0,t_0)) \cap M_{t_0}\big) \\ &= \big\{ p \in M^n \,\big|\, |X(p,t_0) - X(p_0,t_0)| < R \big\} \, , \end{aligned}$$

the parabolic neighbourhood

$$\begin{aligned} Q_R(p_0,t_0) &:= \big\{ (p,t) \in M^n \times \big(t_0 - R^2, t_0\big] \,\big|\, |X(p,t) - X(p_0,t)| < R \big\} \\ &= \bigcup_{t \in (t_0 - R^2, t_0]} (U_R(p_0,t) \times \{t\}) \end{aligned}$$

and, for an open set $U \subset M^n$ and $[t_1, t_0] \subset [0,T)$, the parabolic boundary

$$\mathcal{P}(U \times [t_1,t_0]) := (U \times \{t_1\}) \cup (\partial U \times (t_1, t_0]) \, .$$

Theorem C.2 (Weak maximum principle)**.** *Let $U \subset M^n$ be open and let $f \in C^{2;1}(Q) \cap C^0(\mathcal{P}Q)$ for $Q := U \times [t_1, t_0]$.*

(i) If $L(f) \geq 0$ on Q and $f \geq 0$ on $\mathcal{P}Q$. Then $f \geq 0$ in Q.
(ii) If $L(f) \leq 0$ on Q and $f \leq 0$ on $\mathcal{P}Q$. Then $f \leq 0$ in Q.

Proof. □

To prove a strong maximum principle, we need the following theorem.

Theorem C.3 (Weak Harnack inequality)**.** *Let $U_R(p_0,t_0) \times [t_1,t_0] \subset M^n \times [0,T)$ and $f \in C^{2;1}(U_R(p_0,t_0) \times [t_1,t_0]) \cap C^0(\overline{U_R(p_0,t_0) \times [t_1,t_0]})$ be a non-negative solution of*

$$L(f) \geq 0$$

in $U_R(p_0,t_0) \times [t_1,t_0]$ where there exists $\lambda, \Lambda, \nu > 0$ such that

$$\lambda g^{ij} \leq a^{ij} \leq \Lambda g^{ij} \quad \text{and} \quad \left(\frac{|b|^2}{\lambda} + |c| \right) \leq \frac{\nu \lambda}{R^2}$$

for all $t \in [t_1,t_0]$. Let

$$f \geq h > 0$$

in $U_{\varepsilon R}(p_0, t_0) \times \{t_1\}$ *for some* $0 < \varepsilon < 1/2$. *Then*

$$f \geq \varepsilon^{\kappa} \frac{h}{2}$$

in $U_{R/2}(p_0, t_0) \times \{t_0\}$ *for* $\kappa = \kappa(n, R, t_0 - t_1, \lambda, \Lambda, \nu, X_t, \frac{\partial}{\partial p_i} X_t, \frac{\partial^2}{\partial p_i \partial p_j} X_t) > 0$, *where* X_t *is the embedding of* M_t.

Proof. □

Theorem C.4 (Strong maximum principle). *Let* $U \subset M^n$ *be open,* $Q := U \times [0, T)$, *and* $f \in C^{2;1}(Q) \cap C^0(\bar{Q})$.

(i) *Let* $L(f) \geq 0$ *in* Q. *If there exists* $(p_0, t_0) \in Q \backslash \mathcal{P}Q$ *with* $f(p_0, t_0) = \min_{\bar{Q}} f$, *then* f *is constant in* $\bar{Q}$.

(ii) *Let* $L(f) \leq 0$ *in* Q. *If there exists* $(p_0, t_0) \in Q \backslash \mathcal{P}Q$ *with* $f(p_0, t_0) = \max_{\bar{Q}} f$, *then* f *is constant in* $\bar{Q}$.

Proof. □

Theorem C.5 (Strong maximum principle for parabolic equations). *Let* M *be closed and* $f : M \times [0, T) \to \mathbb{R}$ *satisfy*

$$\partial_t f \geq \Delta_{M_t} f + b^i \nabla_i^{M_t} f + cf$$

for some smooth functions b^i *and* $c \geq 0$. *If* $f(\,\cdot\,, 0) \geq 0$ *then* $\min_M f(\,\cdot\,, t) \geq \min_M f(\,\cdot\,, 0)$. *Furthermore, if* $f(p, t_0) = \min_M f(\,\cdot\,, 0)$ *for some* $p \in M$, $t > 0$, *then* $f = \min_M f(\,\cdot\,, 0)$ *for* $0 \leq t \leq t_0$.

Proof. See for example [Eva02, Chapters 6.4 and 7.1.4] or [Sch17c, Theorem 3.7] for $M^n = \mathbb{R}^n$. □

Let m_{ij} be a symmetric tensor. We say that m_{ij} is nonnegative, $m_{ij} \succ 0$, if all eigenvalues of m_{ij} are nonnegative. Let u^k be a vector field and let g_{ij}, m_{ij} and n_{ij} be symmetric tensors on a compact manifold M which may all depend on time. Assume that $n_{ij} = p(m_{ij}, g_{ij})$ is a polynomial in m_{ij} formed by contracting products of m_{ij} with itself using the metric. Furthermore, let this polynomial satisfy a null-eigenvector condition, i.e. for any null-eigenvector v of m_{ij} we have $n_{ij} v^i v^j \geq 0$. Then we have

Theorem C.6 (Weak parabolic maximum principle for symmetric 2-tensors I, Hamilton, [Ham82, Theorem 9.1]). *Suppose that on* $[0, T)$ *the evolution equation*

$$\partial_t m_{ij} = \Delta_{M_t} m_{ij} + u^k \nabla_k^{M_t} m_{ij} + n_{ij}$$

holds, where $n_{ij} = p(m_{ij}, g_{ij})$ *satisfies the null-eigenvector condition above. If* $m_{ij} \succeq 0$ *at* $t = 0$, *then it remains so on* $(0, T)$.

Proof. □

Exercise C.7. Proof Theorems C.1, C.5 and C.6 on Riemannian manifolds.

There is also a strong version of the maximum principle for tensors. We need the following lemma.

Lemma C.8 (Hamilton, [Ham86, Lemma 8.1]). *Let* V *a vector bundle over a compact manifold* M^n. *Let* v *be a smooth section of* V *satisfying*

$$\frac{\partial v}{\partial t} = \Delta_{M_t} v + \phi(v)\,.$$

Let $f(v)$ *be a convex function on the bundle invariant under parallel translation whose level curves* $f(v) \leq c$ *are preserved by the ODE*

$$\frac{dv}{dt} = \phi(v)\,.$$

Then the inequality $f(v) \leq c$ is preserved by the PDE for any constant c. Furthermore if $f(v) < c$ at one point at time $t = 0$, then $f(v) < c$ everywhere on M for all $t > 0$.

Proof. □

Theorem C.9 (Strong parabolic maximum principle for symmetric 2-tensors II, Hamilton, [Ham86, Lemma 8.2])**.** *Let M^n be closed and m_{ij} be a symmetric bilinear form of a vector bundle V over M^n, which solves*

$$\partial_t m_{ij} = \Delta_{M_t} m_{ij} + \phi_{ij} \,,$$

where ϕ_{ij} is a symmetric bilinear form, depending on m_{ij}, with the property $\phi_{ij} \succeq 0$ if $m_{ij} \succeq 0$. If $m_{ij} \succeq 0$ for $t = 0$ then $m_{ij} \succeq 0$ for all $t \leq 0$. Furthermore, for $t > 0$, the rank of the null-space of m_{ij} is constant, and the null-space is invariant under parallel transport and invariant in time.

Proof. □

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