Supplementary Material for “Data-Driven Distributionally Robust Mixed-Integer Control through Lifted Control Policy”

A. Proof of Theorem 1

Proof. According to Proposition 2.(i), for any $(\bar{\pi}^u, \bar{\pi}^\gamma) \in \mathfrak{R}^u \times \mathfrak{R}^\gamma$ feasible for problem (DRMIC), there exists a $\boldsymbol{w}$ such that $\bar{\pi}^\gamma \in \Pi^\gamma(\boldsymbol{w})$.

Therefore, by the absolute continuity of the cost functions c_t, it remains to prove that there exists a sequence of continuous controllers $\pi_n^u \in \Pi_*^u$ such that $(\pi_n^u, \bar{\pi}^\gamma)$ are feasible for problem (DRMIC) and satisfy $\lim_{n\to\infty} d(\pi_n^u, \bar{\pi}^u) = 0$.

Since the set of feasible continuous controllers $\mathfrak{A}(\bar{\pi}^\gamma)$ has non-empty interior, there exists a $\pi_0^u \in \mathfrak{R}^u$ such that $\{\pi^u \in \mathfrak{R}^u | d(\pi^u, \pi_0^u) < \rho\} \subset \mathfrak{A}(\bar{\pi}^\gamma)$. Let $d_0 = d(\bar{\pi}^u, \pi_0^u)$. Consider the set

$$\mathcal{B}_n = \left\{ \pi^u \in \mathfrak{R}^u \,\middle|\, d\left(\pi^u, \frac{2^n - 1}{2^n}\bar{\pi}^u + \frac{1}{2^n}\pi_0^u\right) < \frac{\rho}{2^n} \right\}. \tag{A.1}$$

Since $\mathcal{B}_n$ is open in $\mathfrak{R}^u$ and Π_*^u is dense in $\mathfrak{R}^u$ by Proposition 2.(ii), then $\Pi_*^u \cap \mathcal{B}_n$ is dense in $\mathfrak{R}^u \cap \mathcal{B}_n$. Therefore, we can choose π_n^u from $\mathcal{B}_n$ such that $\pi_n^u \in \Pi_*^u$.

By this selection, we have

$$\begin{aligned}
& d(\pi_n^u, \bar{\pi}^u) && \text{(A.2)}\\
& \le d\left(\pi_n^u, \frac{2^n - 1}{2^n}\bar{\pi}^u + \frac{1}{2^n}\pi_0^u\right) + d\left(\frac{2^n - 1}{2^n}\bar{\pi}^u + \frac{1}{2^n}\pi_0^u, \bar{\pi}^u\right) && \text{(A.3)}\\
& < \frac{\rho}{2^n} + d\left(\frac{1}{2^n}\pi_0^u, \frac{1}{2^n}\bar{\pi}^u\right) = \frac{\rho}{2^n} + \frac{1}{2^n} d(\pi_0^u, \bar{\pi}^u) = \frac{\rho + d_0}{2^n}. && \text{(A.4)}
\end{aligned}$$

Therefore, the second condition $\lim_{n\to\infty} d(\pi_n^u, \bar{\pi}^u) = 0$ is satisfied.

To prove that $\pi_n^u \in \mathfrak{A}(\pi^\gamma)$, consider $\hat{\pi}_u = 2^n\pi_n^u - (2^n - 1)\bar{\pi}^u \in \mathfrak{R}^u$. Notice that

$$\begin{aligned}
d(\hat{\pi}_u, \pi_0^u) &= d(2^n\pi_n^u - (2^n - 1)\bar{\pi}^u, \pi_0^u) && \text{(A.5)}\\
&= d(2^n\pi_n^u, (2^n - 1)\bar{\pi}^u + \pi_0^u) && \text{(A.6)}\\
&= 2^n d(\pi_n^u, (2^n - 1)/2^n\bar{\pi}^u + 1/2^n\pi_0^u) < 2^n \times \rho/2^n = \rho. && \text{(A.7)}
\end{aligned}$$

Therefore, $\hat{\pi}_u \in \{\pi^u \in \mathfrak{R}^u | d(\pi^u, \pi_0^u) < \rho\} \subset \mathfrak{A}(\bar{\pi}^\gamma)$.

By the linearity of the constraints of the problem (DRMIC), $\mathfrak{A}(\bar{\pi}^\gamma)$ is convex. Therefore, $\pi_n^u \in \mathfrak{A}(\bar{\pi}^\gamma)$ since $\pi_n^u = 1/2^n\hat{\pi}^u + (2^n - 1)/2^n\bar{\pi}^u$. □

B. Proof of Theorem 2

Proof. Notice that

$$\begin{aligned}
& J^*(\Pi_{L_\pi}^u(\boldsymbol{w})) - J^*(\mathfrak{R}_{L_\pi}^u) && \text{(B.1)}\\
&= \min_{\pi_2 \in \Pi_{L_\pi}^u(\boldsymbol{w}) \cap \mathfrak{C}} \max_{\mathbb{P}\in\mathcal{B}} \mathbb{E}_{\mathbb{P}} \left[\sum_{t=1}^{\mathcal{T}} c_t(\boldsymbol{x}_t^{\pi_2}, \boldsymbol{u}_t^{\pi_2}) \right] - \min_{\pi_1 \in \mathfrak{R}_{L_\pi}^u \cap \mathfrak{C}} \max_{\mathbb{P}\in\mathcal{B}} \mathbb{E}_{\mathbb{P}} \left[\sum_{t=1}^{\mathcal{T}} c_t(\boldsymbol{x}_t^{\pi_1}, \boldsymbol{u}_t^{\pi_1}) \right] && \text{(B.2)}\\
&= \max_{\pi_1 \in \mathfrak{R}_{L_\pi}^u \cap \mathfrak{C}} \min_{\pi_2 \in \Pi_{L_\pi}^u(\boldsymbol{w}) \cap \mathfrak{C}} \left\{ \max_{\mathbb{P}\in\mathcal{B}} \mathbb{E}_{\mathbb{P}} \left[\sum_{t=1}^{\mathcal{T}} c_t(\boldsymbol{x}_t^{\pi_2}, \boldsymbol{u}_t^{\pi_2}) \right] - \max_{\mathbb{P}\in\mathcal{B}} \mathbb{E}_{\mathbb{P}} \left[\sum_{t=1}^{\mathcal{T}} c_t(\boldsymbol{x}_t^{\pi_1}, \boldsymbol{u}_t^{\pi_1}) \right] \right\} && \text{(B.3)}\\
&\le \max_{\pi_1 \in \mathfrak{R}_{L_\pi}^u \cap \mathfrak{C}} \min_{\pi_2 \in \Pi_{L_\pi}^u(\boldsymbol{w}) \cap \mathfrak{C}} \max_{\mathbb{P}\in\mathcal{B}} \mathbb{E}_{\mathbb{P}} \left\{ \sum_{t=1}^{\mathcal{T}} \left[c_t(\boldsymbol{x}_t^{\pi_2}, \boldsymbol{u}_t^{\pi_2}) - c_t(\boldsymbol{x}_t^{\pi_1}, \boldsymbol{u}_t^{\pi_1}) \right] \right\}, && \text{(B.4)}
\end{aligned}$$

where $\boldsymbol{x}_t^\pi$ and $\boldsymbol{u}_t^\pi$ represent state and controller induced by π.

Inequality (B.4) is in the form of a max-min problem or a two-player game. The first player chooses a $\pi_1 \in \mathfrak{N}^u_{L_\pi} \cap \mathfrak{C}$, and then in response to π_1, the second player chooses a $\pi_2 \in \Pi^u_{L_\pi}(\boldsymbol{w}) \cap \mathfrak{C}$. By Proposition 2, both $\boldsymbol{u}_t^{\pi_1}$ and $\boldsymbol{u}_t^{\pi_2}$ are in the form of additive controllers, *i.e.*, $u^{\pi_1}_{t,j} = \sum_{t'\in[t],i\in[n_\xi]} u^{\pi_1}_{t,j,t',i}$ and $u^{\pi_2}_{t,j} = \sum_{t'\in[t],i\in[n_\xi]} u^{\pi_2}_{t,j,t',i}$. Given a control policy π_1 by the first player, we set the strategy of the second player as

$$u^{\pi_2}_{t,j,t',i}(w_{t',i,k-1}) = u^{\pi_1}_{t,j,t',i}(w_{t',i,k-1}). \tag{B.5}$$

Equation (B.5) simply means that we construct π_2 such that it equals to π_1 at all the breakpoints $\boldsymbol{w}$, and this requirement can be satisfied since each $u^{\pi_2}_{t,j,t',i}$ is a piecewise linear function and breakpoints are just $w_{t',i,k-1}$, $k \in [p_{t',i}+1]$.

We next prove that the constructed π_2 is in $\Pi^u_{L_\pi}(\boldsymbol{w}) \cap \mathfrak{C}$. Notice that according to Definition 6,

$$\begin{aligned}
d(\pi_2^u, \pi_0^u) &= \sum_{t=1}^{\mathcal{T}} \sum_{j=1}^{n_u} \|u^{\pi_2}_{t,j} - u^{\pi_0}_{t,j}\|_\infty && \text{(B.6)}\\
&\le \sum_{t=1}^{\mathcal{T}} \sum_{j=1}^{n_u} \sum_{t'=1}^{t} \sum_{i=1}^{n_\xi} \|(u^{\pi_2}_{t,j,t',i} - u^{\pi_0}_{t,j,t',i})\|_\infty && \text{(B.7)}\\
&= \sum_{t=1}^{\mathcal{T}} \sum_{j=1}^{n_u} \sum_{t'=1}^{t} \sum_{i=1}^{n_\xi} \max_{k\in[p_{t',i}+1]} \left|(u^{\pi_2}_{t,j,t',i} - u^{\pi_0}_{t,j,t',i})(w_{t',i,k-1})\right| && \text{(B.8)}\\
&= \sum_{t=1}^{\mathcal{T}} \sum_{j=1}^{n_u} \sum_{t'=1}^{t} \sum_{i=1}^{n_\xi} \max_{k\in[p_{t',i}+1]} \left|(u^{\pi_1}_{t,j,t',i} - u^{\pi_0}_{t,j,t',i})(w_{t',i,k-1})\right| && \text{(B.9)}\\
&\le d(\pi_1^u, \pi_0^u) \le \rho, && \text{(B.10)}
\end{aligned}$$

where equality (B.8) holds because both $u^{\pi_2}_{t,j,t',i}$ and $u^{\pi_0}_{t,j,t',i}$ are piecewise linear functions with breakpoints $w_{t',j,k-1}$.

To verify the Lipschitz property of π_2, it is easy to see that

$$\begin{aligned}
&\max_{\xi^1,\xi^2\in\Xi_{[t]}} \left| \frac{u^{\pi_2}_{t,j}(\boldsymbol{\xi}^1) - u^{\pi_2}_{t,j}(\boldsymbol{\xi}^2)}{\|\boldsymbol{\xi}^1 - \boldsymbol{\xi}^2\|_1} \right| && \text{(B.11)}\\
=&\max_{\xi^1,\xi^2\in\Xi_{[t]}} \left| \frac{\sum_{t'=1}^{t}\sum_{i=1}^{n_\xi} u^{\pi_2}_{t,j,t',i}(\xi^1_{t',i}) - u^{\pi_2}_{t,j,t',i}(\xi^2_{t',i})}{\sum_{t'=1}^{t}\sum_{i=1}^{n_\xi} |\xi^1_{t',i} - \xi^2_{t',i}|} \right| && \text{(B.12)}
\end{aligned}$$

reaches the maximum when $\boldsymbol{\xi}^1, \boldsymbol{\xi}^2$ are at the breakpoints. Further, since $u^{\pi_2}_{t,j,t',i}$ equals $u^{\pi_1}_{t,j,t',i}$ on these breakpoints, the Lipschitz constant of π_2 is no larger than that of π_1.

Under this construction, the difference between $\boldsymbol{u}_t^{\pi_1}$ and $\boldsymbol{u}_t^{\pi_2}$ can be bounded as follows.

$$\begin{aligned}
|u^{\pi_1}_{t,j}(\boldsymbol{\xi}) - u^{\pi_2}_{t,j}(\boldsymbol{\xi})| &\le \sum_{t'=1}^{t}\sum_{i=1}^{n_\xi} |u^{\pi_1}_{t,j,t',i}(\xi_{t',i}) - u^{\pi_2}_{t,j,t',i}(\xi_{t',i})| && \text{(B.13)}\\
&\le \sum_{t'=1}^{t}\sum_{i=1}^{n_\xi} \left\{ |u^{\pi_1}_{t,j,t',i}(\xi_{t',i}) - u^{\pi_1}_{t,j,t',i}(\hat{w}_{t',i})| + |u^{\pi_2}_{t,j,t',i}(\hat{w}_{t',i}) - u^{\pi_2}_{t,j,t',i}(\xi_{t',i})| \right\} && \text{(B.14)}\\
&\le \sum_{t'=1}^{t}\sum_{i=1}^{n_\xi} 2L_\pi |\hat{w}_{t',i} - \xi_{t',i}| \le \sum_{t'=1}^{t}\sum_{i=1}^{n_\xi} 2L_\pi \epsilon(\boldsymbol{w}) \lesssim O(\epsilon(\boldsymbol{w})), && \text{(B.15)}
\end{aligned}$$

where $\hat{w}_{t',i}$ is the breakpoint nearest to $\xi_{t',i}$, *i.e.*, $\hat{w}_{t',i} = \arg\min_{w\in\{w_{t',i,k-1},k\in[p_{t',i}+1]\}} |w - \xi_{t',i}|$.

To bound the the difference between $\boldsymbol{x}_t^{\pi_1}$ and $\boldsymbol{x}_t^{\pi_2}$, note that

$$\|\boldsymbol{x}_t^{\pi_1}(\boldsymbol{\xi}) - \boldsymbol{x}_t^{\pi_2}(\boldsymbol{\xi})\|_1 = \|\boldsymbol{A}_t(\boldsymbol{x}^{\pi_1}_{t-1} - \boldsymbol{x}^{\pi_2}_{t-1}) + \boldsymbol{B}_t(\boldsymbol{u}^{\pi_1}_{t-1} - \boldsymbol{u}^{\pi_2}_{t-1})\|_1. \tag{B.16}$$

Therefore, by induction, the difference between $\boldsymbol{x}_t^{\pi_1}$ and $\boldsymbol{x}_t^{\pi_2}$ is also in the order of $O(\epsilon(\boldsymbol{w}))$.

Since c_t is Lipschitz continuous, we have

$$|c_t(x_t^{\pi_1}, u_t^{\pi_1}) - c_t(x_t^{\pi_2}, u_t^{\pi_2})| \leq L_c \left(\left\|x_t^{\pi_1} - x_t^{\pi_2}\right\| + \left\|u_t^{\pi_1} - u_t^{\pi_2}\right\|\right) \tag{B.17}$$

$$= L_c \left(\sum_{j=1}^{n_x} |x_{t,j}^{\pi_1} - x_{t,j}^{\pi_2}| + \sum_{j=1}^{n_u} |u_{t,j}^{\pi_1} - u_{t,j}^{\pi_2}| \right) \lesssim O(\epsilon(\boldsymbol{w})). \tag{B.18}$$

By the above observations, (B.4) can be further bounded by

$$(B.4) \lesssim \max_{\pi_1 \in \mathfrak{R}_{L_\pi}^u \cap \mathfrak{C}} \max_{\mathbb{P} \in \mathscr{B}} \mathbb{E}_{\mathbb{P}}[O(\epsilon(\boldsymbol{w}))] \lesssim O(\epsilon(\boldsymbol{w})). \tag{B.19}$$

To prove the tightness of this bound, we need to construct a concrete example whose non-asymptotic performance is lower bounded by $\Omega(\epsilon(\boldsymbol{w}))$. To this aim, we consider a single-horizon problem where the state x, continuous control u, and disturbance ξ are of dimension one. Support $\Xi = [0,1]$ and the breakpoints $\boldsymbol{w}$ segment the support into equal lengths of ϵ, so $\epsilon(\boldsymbol{w}) = \epsilon$. The given control policy is set to $\pi_0^u(\xi) = 0, \forall \xi \in \Xi$. The ambiguity set $\mathscr{B}$ contains only one distribution $\hat{P} = 1/4\delta_0 + 1/4\delta_\epsilon + 1/2\delta_{\epsilon/2}$, *i.e.*, $\mathscr{B} = \{\hat{P}\}$, which corresponds to the Wasserstein ambiguity set centered at $\hat{P}$ with radius 0. The state transition function is given by $x = \xi$, and the cost function is given by

$$c(x,u) = \frac{L_c}{L_\pi + 1} |u - L_\pi |x - \epsilon/2||, \tag{B.20}$$

whose Lipschitz constant is

$$\frac{L_c}{L_\pi + 1} \max\{L_\pi, 1\}, \tag{B.21}$$

which is less than L_c.

The optimal control u_{opt} of this problem is a L_π Lipschitz piecewise linear one as follows.

$$u_{\text{opt}}(\xi) = \begin{cases} L_\pi|\xi - \epsilon/2|, \xi \in [0,\epsilon] \\ L_\pi \epsilon/2 \qquad , \xi \in (\epsilon, 1]. \end{cases} \tag{B.22}$$

Since $u_{\text{opt}} \in \mathfrak{R}_{L_\pi}^u$, we have $J^*(\mathfrak{R}_{L_\pi}^u) = 0$.

For controller $u \in \Pi_{L_\pi}^u(\boldsymbol{w})$, only the first segment $[0,\epsilon]$ affects the performance, so we have

$$J^*(\Pi_{L_\pi}^u(\boldsymbol{w})) - J^*(\mathfrak{R}_{L_\pi}^u) = J^*(\Pi_{L_\pi}^u(\boldsymbol{w})) = \max_{u(0),u(\epsilon)} \mathbb{E}_{\hat{P}}[c(x,u)] \tag{B.23}$$

$$= \frac{L_c}{L_\pi + 1} \max_{u(0),u(\epsilon)} \left\{ \frac{1}{4}\left|u(0) - \frac{\epsilon L_\pi}{2}\right| + \frac{1}{4}\left|u(\epsilon) - \frac{\epsilon L_\pi}{2}\right| + \frac{1}{2}\left|\frac{u(0)+u(\epsilon)}{2}\right| \right\} \tag{B.24}$$

$$\geq \frac{L_c}{L_\pi + 1} \max_{u(0),u(\epsilon)} \left\{ \frac{1}{4}\left|u(0) - \frac{\epsilon L_\pi}{2} + u(\epsilon) - \frac{\epsilon L_\pi}{2} - 2\frac{u(0)+u(\epsilon)}{2}\right| \right\} = \frac{L_c}{L_\pi + 1}\frac{L_\pi}{4}\epsilon \gtrsim \Omega(\epsilon), \tag{B.25}$$

where the last equality in Equation (B.23) holds because u is linear on $[0,\epsilon]$, so it is determined by the endpoint value $u(0)$ and $u(\epsilon)$.

Therefore, the $O(\epsilon(\boldsymbol{w}))$ bound is tight. □

C. Superiority to Feng et al. (2021)

In Feng et al. (2021), the Wasserstein ambiguity set $\mathscr{B}$ (54) is relaxed to $\mathscr{B}_{\text{Lifted}}$ as follows.

$$\mathscr{B}_{\text{Lifted}} = \left\{ \mathbb{Q} \in \mathscr{P}(\text{conv}(\Xi^*)) \middle| d_W\left(\mathbb{Q}, \widehat{\mathbb{P}}_{\text{Lifted}}\right) \leq \theta^* \right\}, \tag{C.1}$$

$$\theta^* = \sup_{\mathbb{P}\in\mathscr{B}} d_W(\mathbb{P}\circ G^{-1}, \widehat{\mathbb{P}}_{\text{Lifted}}), \tag{C.2}$$

where $\widehat{\mathbb{P}}_{\text{Lifted}} = \sum_{s=1}^{N} \delta_{G(\widehat{\xi}^s)}/N$ and $\mathbb{P}\circ G^{-1}$ is the push-forward probability measure by lifting function G.

By (C.1), the lifted ambiguity set $\mathscr{B}_{\text{Lifted}}$ 'contains' the standard Wasserstein ambiguity set $\mathscr{B}$. Hence it can be viewed as a conservative approximation of $\mathscr{B}$.

Based on $\mathscr{B}_{\text{Lifted}}$, Feng et al. (2021) resorts to optimizing the following objective.

$$\min_{\boldsymbol{Y}_t,\boldsymbol{y}_t^0,\boldsymbol{Z}_t,\boldsymbol{z}_t^0} \quad \max_{\mathbb{P}^*\in\mathscr{B}_{\text{Lifted}}} \mathbb{E}_{\boldsymbol{\xi}^*\sim\mathbb{P}^*}\left[\max_{k\in[K]}\left\{\boldsymbol{d}_k^T\boldsymbol{\xi}^* + r_k\right\}\right]. \tag{C.3}$$

We prove in Corollary C.1 that (C.3) is a relaxation of (39), which implies that by directly optimizing over (39) our method is always better than the method in Feng et al. (2021).

Corollary C.1. *For any $\boldsymbol{Y}_t$, $\boldsymbol{y}_t^0$, $\boldsymbol{Z}_t$, and $\boldsymbol{z}_t^0$, it holds that*

$$\max_{\mathbb{P}^*\in\mathscr{B}_{Lifted}} \mathbb{E}_{\boldsymbol{\xi}^*\sim\mathbb{P}^*}\left[\max_{k\in[K]}\left\{\boldsymbol{d}_k^T\boldsymbol{\xi}^* + r_k\right\}\right] \geq \max_{\mathbb{P}\in\mathscr{B}} \mathbb{E}_{\boldsymbol{\xi}\sim\mathbb{P}}\left[\max_{k\in[K]}\left\{\boldsymbol{d}_k^T G(\boldsymbol{\xi}) + r_k\right\}\right]. \tag{C.4}$$

Proof. $\forall \mathbb{P}\in\mathscr{B}$, we have

$$\mathbb{E}_{\boldsymbol{\xi}\sim\mathbb{P}}\left[\max_{k\in[K]}\left\{\boldsymbol{d}_k^T G(\boldsymbol{\xi}) + r_k\right\}\right] = \mathbb{E}_{\boldsymbol{\xi}^*\sim\mathbb{P}_G}\left[\max_{k\in[K]}\left\{\boldsymbol{d}_k^T\boldsymbol{\xi}^* + r_k\right\}\right] \leq \max_{\mathbb{P}^*\in\mathscr{B}_{\text{Lifted}}} \mathbb{E}_{\boldsymbol{\xi}^*\sim\mathbb{P}^*}\left[\max_{k\in[K]}\left\{\boldsymbol{d}_k^T\boldsymbol{\xi}^* + r_k\right\}\right]. \tag{C.5}$$

Therefore, Corollary C.1 holds by taking maximum on both sides of (C.5). □

D. Reformulations for Variants of the Wasserstein Ambiguity Set

D.1. Mixed Wasserstein-Moment Ambiguity Set

To enhance the conventional Wasserstein ambiguity set, constructing a mixed ambiguity set has been favored in recent years, by leveraging other information to remove unrealistic probability distributions in the Wasserstein ball Zhou et al. (2021). Therefore, we consider a mixed Wasserstein-moment ambiguity set as follows.

$$\mathscr{B} = \left\{\mathbb{Q}\in\mathscr{P}(\Xi) \;\middle|\; \begin{array}{l} d_W\left(\mathbb{Q},\widehat{\mathbb{P}}\right) \leq \theta \\ \underline{\boldsymbol{\xi}} \leq \mathbb{E}_{\mathbb{Q}}[\boldsymbol{\xi}] \leq \overline{\boldsymbol{\xi}} \end{array}\right\}, \tag{D.1}$$

where $\underline{\boldsymbol{\xi}}$ and $\overline{\boldsymbol{\xi}}$ are the lower bound and upper bound of first-order moment, respectively.

The reformulation with respect to this mixed Wasserstein-moment ambiguity set is presented in Theorem D.1.

Theorem D.1. *Let $\mathscr{B}$ be the mixed Wasserstein-moment ambiguity set defined in (D.1). Then, the objective function (39) admits the following equivalent reformulation.*

$$\begin{aligned}
\min_{\lambda,\eta^s,\eta^s_{k,t,i},\underline{\boldsymbol{\beta}},\overline{\boldsymbol{\beta}},\zeta^s_{k,t,i,j},\boldsymbol{Y}_t,\boldsymbol{y}_t^0,\boldsymbol{Z}_t,\boldsymbol{z}_t^0} \quad & \lambda\theta + \frac{1}{N}\sum_{s=1}^{N}\eta^s - \underline{\boldsymbol{\beta}}^T\underline{\boldsymbol{\xi}} + \overline{\boldsymbol{\beta}}^T\overline{\boldsymbol{\xi}} \\
\text{s.t.} \quad & \eta^s \geq \sum_{t=1}^{\mathcal{T}}\sum_{i=1}^{n_\xi}\eta^s_{k,t,i} + r_k \\
& \eta^s_{k,t,i} \geq \tilde{\boldsymbol{d}}^T_{k,t,i}\boldsymbol{\varphi}^-_{t,i,j} - \zeta^s_{k,t,i,j}\left(\boldsymbol{R}_{t,i}\boldsymbol{\varphi}^-_{t,i,j} - \widehat{\xi}^s_{t,i}\right) \\
& \eta^s_{k,t,i} \geq \tilde{\boldsymbol{d}}^T_{k,t,i}\boldsymbol{\varphi}_{t,i,j-1} - \zeta^s_{k,t,i,j}\left(\boldsymbol{R}_{t,i}\boldsymbol{\varphi}_{t,i,j-1} - \widehat{\xi}^s_{t,i}\right) \\
& |\zeta^s_{k,t,i,j}| \leq \lambda, \boldsymbol{0} \leq \underline{\boldsymbol{\beta}}, \boldsymbol{0} \leq \overline{\boldsymbol{\beta}} \\
& \forall s\in[N], \forall k\in[K], \forall t\in[\mathcal{T}], \forall i\in[n_\xi], \forall j\in[p_{t,i}],
\end{aligned} \tag{D.2}$$

where $\tilde{\boldsymbol{d}}_k = \boldsymbol{d}_k + \boldsymbol{R}^T\underline{\boldsymbol{\beta}} - \boldsymbol{R}^T\overline{\boldsymbol{\beta}}$, $\boldsymbol{\varphi}_{t,i,j} = G_{t,i}(w_{t,i,j})$, and $\boldsymbol{\varphi}^-_{t,i,j} = G^-_{t,i}(w_{t,i,j})$.

Proof. The reformulation of the worst-case expectation with respect to the mixed Wasserstein-moment ambiguity set has been developed by Zhou et al. (2021) as follows.

$$\max_{\mathbb{P}\in\mathcal{B}} \mathbb{E}_{\boldsymbol{\xi}\sim\mathbb{P}}\left[\max_{k\in[K]}\left\{\boldsymbol{d}_k^T G(\boldsymbol{\xi}) + r_k\right\}\right] \tag{D.3}$$

$$= \begin{cases} \min\limits_{0\le\lambda,\boldsymbol{0}\le\underline{\boldsymbol{\beta}},\boldsymbol{0}\le\overline{\boldsymbol{\beta}}} \lambda\theta + \dfrac{1}{N}\sum\limits_{s=1}^{N}\eta^s - \underline{\boldsymbol{\beta}}^T\underline{\boldsymbol{\xi}} + \overline{\boldsymbol{\beta}}^T\overline{\boldsymbol{\xi}} \\ \text{s.t.} \quad \eta^s \ge \max\limits_{\boldsymbol{\xi}\in\Xi}\left\{\max\limits_{k\in[K]}\left\{\left(\boldsymbol{d}_k + \boldsymbol{R}^T\underline{\boldsymbol{\beta}} - \boldsymbol{R}^T\overline{\boldsymbol{\beta}}\right)^T G(\boldsymbol{\xi}) \right.\right. \\ \qquad \left.\left. + r_k\right\} - \lambda\left\|\boldsymbol{\xi} - \widehat{\boldsymbol{\xi}}^s\right\|\right\}, \forall s\in[N]. \end{cases} \tag{D.4}$$

By applying Lemma 1 to the constraint of (D.4), we can reach to the reformulation (D.2). □

D.2. Event-Wise Wasserstein Ambiguity Set

In some cases, the disturbance distribution is affected by certainty events, such as the distribution of daily average temperature depending on the season. Therefore, constructing an event-wise ambiguity set can characterize such disturbance more precisely (Chen et al., 2020).

We consider an event with L possible realizations. Conditioning on the realization of a scenario $l \in [L]$, the disturbance has support Ξ^l and N_l historical data $\widehat{\boldsymbol{\xi}}^{l,s}$, $s \in [N_l]$ is available. Based on this information, we construct the following event-wise Wasserstein ambiguity set.

$$\mathcal{B} = \left\{ \mathbb{Q} \in \mathcal{P}\left(\mathbb{R}^{\sum_{t=1}^{\mathcal{T}} n_\xi} \times [L]\right) \;\middle|\; \begin{array}{r} (\tilde{\boldsymbol{\xi}}, \tilde{l}) \sim \mathbb{Q} \\ \mathbb{Q}(\tilde{l} = l) = p_l, \forall l \in [L] \\ \mathbb{Q}\left(\tilde{\boldsymbol{\xi}} \in \Xi^l | \tilde{l} = l\right) = 1 \\ , \forall l \in [L] \\ d_W\left(\mathbb{Q}(\tilde{\boldsymbol{\xi}}|\tilde{l} = l), \widehat{\mathbb{P}}^l\right) \le \theta_l \\ , \forall l \in [L] \end{array} \right\}, \tag{D.5}$$

where $\widehat{\mathbb{P}}^l = \sum_{s=1}^{N_l} \delta_{\widehat{\boldsymbol{\xi}}^{l,s}}/N_l$ and p_l is the probability of scenario l with $\sum_{l=1}^{L} p_l = 1$.

To adapt to this event-wise disturbance distribution, we construct distinct LCPs for different scenarios. For scenario l, we leverage the lifting function G^l with breakpoints $w_{t,i,j}^l$, $j \in [p_{t,i}^l - 1]$ to construct the LCP for continuous and integer controls with optimization variables $\boldsymbol{Y}_t^l, \boldsymbol{y}_t^{l,0}, \boldsymbol{Z}_t^l, z_t^{l,0}$. Based on the event-wise LCP, the objective (39) is equivalent to

$$\min_{\boldsymbol{Y}_t^l,\boldsymbol{y}_t^{l,0},\boldsymbol{Z}_t^l,z_t^{l,0}} \max_{\mathbb{P}\in\mathcal{B}} \mathbb{E}_{(\tilde{\boldsymbol{\xi}},\tilde{l})\sim\mathbb{P}}\left[\sum_{l=1}^{L} \mathbb{1}_{\tilde{l}=l} \max_{k\in[K]}\left\{\left(\boldsymbol{d}_k^l\right)^T G^l(\boldsymbol{\xi}) + r_k^l\right\}\right] \tag{D.6}$$

$$\text{s.t.} \qquad \boldsymbol{m}_l \ge \boldsymbol{E}_l^T G^l(\boldsymbol{\xi}), \forall \boldsymbol{\xi} \in \Xi^l \tag{D.7}$$

$$\boldsymbol{Y}_t^l \in \mathbb{R}^{n_u \times n_{V_t^l}}, y_t^{l,0} \in \mathbb{R}^{n_u}, \boldsymbol{Z}_t^l \in \mathbb{Z}^{n_\gamma \times n_{Q_t^l}}, z_t^{l,0} \in \mathbb{Z}^{n_\gamma}, \tag{D.8}$$

where $\mathbb{1}$ is the indicator function and $\boldsymbol{d}_k^l, r_k^l, \boldsymbol{m}_l, \boldsymbol{E}_l$ can be computed according to (37)-(38).

The equivalent reformulation of the objective function (D.6) is developed in Theorem D.2.

Theorem D.2. *Let $\mathcal{B}$ be the event-wise Wasserstein ambiguity set defined in (D.5). Then, the objective function (D.6)* *admits the following equivalent reformulation.*

$$
\begin{aligned}
\min_{\lambda_l,\eta^{l,s},\eta^{l,s}_{k,t,i},\zeta^{l,s}_{k,t,i,j},\boldsymbol{Y}^l_t,\boldsymbol{y}^{l,0}_t,\boldsymbol{Z}^l_t,\boldsymbol{z}^{l,0}_t} \quad & \sum_{l=1}^{L} p_l \left[\lambda_l \theta_l + \frac{1}{N_l} \sum_{s=1}^{N_l} \eta^{l,s} \right] \\
\text{s.t.} \qquad & \eta^{l,s} \geq \sum_{t=1}^{\mathcal{T}} \sum_{i=1}^{n_\xi} \eta^{l,s}_{k,t,i} + r^l_k \\
& \eta^{l,s}_{k,t,i} \geq \boldsymbol{d}^{l\ T}_{k,t,i} \boldsymbol{\varphi}^{l-}_{t,i,j} - \zeta^{l,s}_{t,i,j} \left(\boldsymbol{R}_{t,i} \boldsymbol{\varphi}^{l-}_{t,i,j} - \widehat{\xi}^{l,s}_{t,i} \right) \\
& \eta^{l,s}_{k,t,i} \geq \boldsymbol{d}^{l\ T}_{k,t,i} \boldsymbol{\varphi}^{l}_{t,i,j-1} - \zeta^{l,s}_{t,i,j} \left(\boldsymbol{R}_{t,i} \boldsymbol{\varphi}^{l}_{t,i,j-1} - \widehat{\xi}^{l,s}_{t,i} \right) \\
& |\zeta^{l,s}_{k,t,i,j}| \leq \lambda \\
& \forall l \in [L], \forall s \in [N_l], \forall k \in [K], \\
& \forall t \in [\mathcal{T}], \forall i \in [n_\xi], \forall j \in [p^l_{t,i}],
\end{aligned}
\tag{D.9}
$$

where $\boldsymbol{\varphi}^l_{t,i,j} = G^l_{t,i}(w_{t,i,j})$ *and* $\boldsymbol{\varphi}^{l-}_{t,i,j} = G^{l-}_{t,i}(w_{t,i,j})$.

Proof. Notice that the worst-case expectation is equivalent to

$$
\sum_{l=1}^{L} p_l \max_{\mathbb{P} \in \mathscr{B}_l} \mathbb{E}_{\boldsymbol{\xi} \sim \mathbb{P}} \left[\max_{k \in [K]} \left\{ \left(\boldsymbol{d}^l_k \right)^T G^l(\boldsymbol{\xi}) + r^l_k \right\} \right], \tag{D.10}
$$

where

$$
\mathscr{B}_l = \left\{ \mathbb{Q} \in \mathscr{P}(\Xi^l) \middle| d_W \left(\mathbb{Q}, \widehat{\mathbb{P}}^l \right) \leq \theta_l \right\}. \tag{D.11}
$$

Therefore, by applying Theorem 3 to the inner worst-case expectation term with respect to $\mathscr{B}_l$, the reformulation (D.9) is derived. □

References

Chen, Z., Sim, M., and Xiong, P. Robust stochastic optimization made easy with RSOME. *Manag. Sci.*, 66(8):3329–3339, 2020.

Feng, W., Feng, Y., and Zhang, Q. Multistage distributionally robust optimization for integrated production and maintenance scheduling. *AIChE Journal*, 67(9):e17329, 2021.

Zhou, Y., Wei, Z., Shahidehpour, M., and Chen, S. Distributionally robust resilient operation of integrated energy systems using moment and wasserstein metric for contingencies. *IEEE Trans. Power Syst.*, 36(4):3574–3584, 2021.