$\int_0^\infty A p^3 e^{-\frac{\beta p^2}{2m}} dp$ $\qquad z = \frac{\beta p^2}{2m}$ $\qquad p^2 = \frac{2mz}{\beta}$ $\qquad p = \sqrt{\frac{2mz}{\beta}}$

$\frac{dz}{dp} = \frac{\beta p}{m}$ $\qquad p^3 = \left(\frac{2mz}{\beta}\right)^{3/2}$

$dp = \frac{m}{\beta p} dz$

$$\int_0^\infty A\left(\frac{2mz}{\beta}\right)^{3/2} e^{-z}\frac{m}{\beta} dz \sqrt{\frac{\beta}{2mz}} = \int_0^\infty A\left(\frac{2mz}{\beta}\right)^{\frac{3}{2}}\left(\frac{2mz}{\beta}\right)^{-\frac{1}{2}}\frac{m}{\beta} e^{-z}$$

$$= \int_0^\infty A\left(\frac{2mz}{\beta}\right)\cdot\frac{m}{\beta} e^{-z} = 2A\frac{m^2}{\beta^2}\int_0^\infty e^{-z} z = 2A\frac{m^2}{\beta^2}\Gamma(2) = 2A\frac{m^2}{\beta^2}$$

А для $E = \frac{p^2}{2m}$ в двумерном случае: $A = \frac{\beta}{m}$

Получаем: $P = \frac{\pi n}{\cancel{m}}\cdot\frac{2m^{\cancel{2}}}{\beta^2}\cdot\frac{\beta}{\cancel{m}} = \frac{2\pi n \cancel{\beta}}{\beta} = 2\pi n T$

№2.

Распределение молекул по модулю скорости:

$$w_v(v) = 4\pi \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} e^{-\frac{mv^2}{2kT}} v^2$$

Среднее по модулю скорость $v_{ср} = \langle v \rangle = \sqrt{\frac{8kT}{\pi m}}$

Вероятность $v > 0,1 v_{ср}$:

$$P(v > 0,1 v_{ср}) = \int_{0,1 v_{ср}}^{\infty} 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{mv^2}{2kT}} v^2 dv =$$

$$= 4\pi \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \int_{0,1 v_{ср}}^{\infty} e^{-\frac{mv^2}{2kT}} v^2 dv$$

$$\int_{0,1 v_{ср}}^{\infty} e^{-\frac{mv^2}{2kT}} v^2 dv = \left| \begin{matrix} u = v^2 & du = 2v\,dv \\ dV = e^{-\frac{mv^2}{2kT}} dv & V = e^{-\frac{mv^2}{2kT}} \left(-\frac{kT}{mv}\right) \end{matrix} \right| =$$

$$= -\frac{kT}{mv} v^2 e^{-\frac{mv^2}{2kT}} \Bigg|_{0,1 v_{ср}}^{\infty} - \int_{0,1 v_{ср}}^{\infty} e^{-\frac{mv^2}{2kT}} \left(-\frac{kT}{mv}\right) 2v\, dv =$$

$$= -\frac{kT}{m} v\, e^{-\frac{mv^2}{2kT}} \Bigg|_{0,1 v_{ср}}^{\infty} + \frac{2kT}{m} \left[-\frac{kT}{mv} e^{-\frac{mv^2}{2kT}}\right]_{0,1 v_{ср}}^{\infty} =$$

$$= \frac{kT}{m} 0,1 v_{ср}\, e^{-0,01 \frac{m}{2kT} v_{ср}^2} + \frac{2k^2T^2}{m^2} \cdot \frac{e^{-0,01 \frac{m}{2kT} v_{ср}^2}}{0,1 v_{ср}} =$$

$$= \exp\left(-0,01 \cdot \frac{m}{2kT} \cdot \frac{8kT}{\pi m}\right) \left[0,1 \cdot \frac{kT}{m} \cdot \sqrt{\frac{8kT}{\pi m}} + \frac{2k^2T^2}{m^2} \cdot 10 \cdot \sqrt{\frac{\pi m}{8kT}}\right] =$$

$$= \exp\left(-\frac{0,04}{\pi}\right) \left[0,1 \sqrt{\frac{8}{\pi}} \left(\frac{kT}{m}\right)^{3/2} + 20 \sqrt{\frac{\pi}{8}} \left(\frac{kT}{m}\right)^{\frac{3}{2}}\right]$$

$$P(v > 0,1 v_{ср}) = 4\pi \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \left(\frac{kT}{m}\right)^{-\frac{3}{2}} \cdot \left(\frac{kT}{m}\right)^{\frac{3}{2}} \exp\left(-\frac{0,04}{\pi}\right) \left[0,1 \sqrt{\frac{8}{\pi}} + 20 \sqrt{\frac{\pi}{8}}\right]$$

$$P(v > 0,1 v_{ср}) = 4\pi \left(\frac{1}{2\pi}\right)^{3/2} \exp\left(-\frac{0,04}{\pi}\right) \left[0,1 \sqrt{\frac{8}{\pi}} + 20 \sqrt{\frac{\pi}{8}}\right] = 0,99991962 78$$

Соотв. $P(v < 0,1 v_{ср}) = 1 - P(v > 0,1 V_{ср}) = 1 - 0,9999196278 \approx$

$\approx 8 \cdot 10^{-5}$

Ответ: $P(v < 0,1 v_{ср}) \approx 8 \cdot 10^{-5}$

№3. Дисперсия $\mathcal{E}=pc$

Число частиц в единичном объеме, имеющих импульс в интервале d^2p в среднем равно: $dn = nAe^{-\beta cp}d^2p$

$dj_x = v_x dn = c\frac{p_x}{p}dn$, с учетом $v_x \approx c\frac{p_x}{p}$

$dN = dj_x\, dl\, d\tau$ - число частиц, которое столкнется с dl за время $d\tau$

$dF = 2p_x\frac{dN}{d\tau}$; $dP = \frac{dF}{dl} = 2p_x dj_x$

Инт. по всем импульсам с $p_x>0$:

$$P = \int\limits_{p_x>0} 2p_x dj_x = 2nc\int\limits_{p_x>0} A\frac{p_x^2}{p}e^{-\beta cp}d^2p = \quad ; \quad d^2p = p\,d\varphi\,dp, \quad p_x = p\sin\varphi$$

$$= 2nc\int\limits_{\pi} A\cdot\frac{p^2\sin^2\varphi}{p}\cdot e^{-\beta cp}p\,d\varphi\,dp =$$

$$= 2nc\int\limits_0 \sin^2\varphi\,d\varphi\int\limits_0^\infty Ap^2e^{-\beta cp}dp = 2nc\frac{\pi}{2}\int\limits_0^\infty Ap^2e^{-\beta cp}dp =$$

$$= \pi nc\langle p\rangle = \pi n\langle\mathcal{E}\rangle$$

Средняя энергия для двухмерного случая:

$$\langle\mathcal{E}\rangle = \frac{\int\limits_0^\infty cp\,e^{-\beta cp}p\,dp}{\int\limits_0^\infty e^{-\beta cp}p\,dp} = \frac{1}{\beta}\frac{\int\limits_0^\infty e^{-z}z^2dz}{\int\limits_0^\infty e^{-z}z\,dz} = kT\cdot\frac{\Gamma(3)}{\Gamma(2)} = 2kT$$

Получаем $P = n\pi\langle\mathcal{E}\rangle = 2n\pi kT$

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Дисперсия $E = \frac{p^2}{2m}$

$dn = nAe^{-\frac{\beta p^2}{2m}}d^2p$

$dj_x = v_x dn = \frac{p_x}{m}dn$; $dN = dj_x\,dl\,d\tau$; $dF = 2p_x\frac{dN}{d\tau}$; $dP = \frac{dF}{dl} = \frac{2p_x^2}{m}d$

$$P = \int\limits_{p_x>0} 2p_x^2/m\; nAe^{-\frac{\beta p^2}{2m}}d^2p = \frac{2n}{m}\int\limits_{p_x>0} p_x^2Ae^{-\frac{\beta p^2}{2m}}d^2p = \frac{2n}{m}\int p^2\sin^2\varphi\, A\, e^{-\frac{\beta p^2}{2m}}p\,dp\,d\varphi$$

$$= \frac{2n}{m}\int\limits_0^\pi \sin^2\varphi\,d\varphi\int\limits_0^\infty Ap^3e^{-\frac{\beta p^2}{2m}}dp = \frac{\pi n}{m}\int\limits_0^\infty Ap^3e^{-\frac{\beta p^2}{2m}}dp.$$

№1. $E = cp$

$p_x = p\sin\theta\cos\varphi \quad p_y = p\sin\theta\sin\varphi \quad p_z = p\cos\theta$

$d^3p = p^2\sin\theta\, dp\, d\varphi\, d\theta$

$$\omega_{p,\theta,\varphi}(p,\theta,\varphi) = \omega_{p_x}\omega_{p_y}\omega_{p_z}\left|\frac{\partial(p_x,p_y,p_z)}{\partial(p,\theta,\varphi)}\right| = A_p\cdot e^{-\beta pc}\sin\theta\, p^2$$

$$\omega_p(p) = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta\, A_p \cdot e^{-\beta pc} p^2 = 4\pi A_p \cdot e^{-\beta pc} p^2$$

$$A_p = \frac{1}{\iiint \omega_{p,\theta,\varphi}\, dp\, d\varphi\, d\theta} = \frac{1}{\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta \int_0^{\infty} e^{-\beta pc} p^2 dp} =$$

$$= \frac{1}{4\pi\cdot\frac{2}{\beta^3c^3}} = \frac{\beta^3c^3}{8\pi}$$

Получаем $\omega_p(p) = 4\pi\cdot\frac{\beta^3c^3}{8\pi}\cdot e^{-\beta pc} p^2$

Импульс с наибольшей плотностью вероятности:

$$\frac{\partial\omega_p(p)}{\partial p} = 4\pi\cdot\frac{\beta^3c^3}{8\pi}\left[e^{-\beta pc}(-\beta c)p^2 + 2p\cdot e^{-\beta pc}\right] = 0$$

$$4\pi\cdot\frac{\beta^3c^3}{8\pi}\cdot e^{-\beta pc}\cdot p\left[-\beta cp + 2\right] = 0 \Rightarrow p = \frac{2}{\beta c}$$

$$\langle p^n\rangle = \frac{\int_0^{\infty} e^{-\beta pc} p^{2+n} dp}{\int_0^{\infty} e^{-\beta pc} p^2 dp} = \frac{1}{2}\frac{\beta^3c^3}{\beta^{n+3}c^{n+3}}\cdot\Gamma(n+3) = \frac{1}{2}\frac{\Gamma(n+3)}{(\beta c)^n}$$

$$\int_0^{\infty} e^{-\beta pc} p^{2+n} dp \overset{z=\beta pc}{=} \int_0^{\infty} e^{-z}\cdot\frac{z^{n+2}}{\beta^{n+2}c^{n+2}}\frac{dz}{\beta c} = \frac{1}{\beta^{3+n}c^{3+n}}\int_0^{\infty} e^{-z} z^{n+2} dz =$$

$$= \frac{\Gamma(n+3)}{\beta^3c^3}$$

$$\int_0^{\infty} e^{-\beta pc} p^2 dp \overset{z=\beta pc}{=} \int_0^{\infty} e^{-z}\frac{z^2}{\beta^2c^2}\cdot\frac{dz}{\beta^2c^2} = \frac{1}{\beta^3c^3}\int_0^{\infty} e^{-z} z^2 dz = \frac{\Gamma(3)}{\beta^3c^3} = \frac{2}{\beta^3c^3}$$

$$\langle p\rangle = \frac{1}{2}\frac{\Gamma(4)}{\beta c} = \frac{3}{\beta c} \qquad \langle p^2\rangle = \frac{1}{2}\frac{\Gamma(5)}{\beta^2c^2} = \frac{12}{\beta^2c^2} \qquad p_{rms} = \sqrt{\langle v^2\rangle} = \frac{2\sqrt{3}}{\beta c}.$$