CheatSheets For Uvic University Courses

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Table 1: Time log for ELEC 360 — Assignment 2

Week of Oct 8	Week of Oct 15	Week of Oct 22	Week of Nov 5
5 hours Prepared for ELEC 360 Lab and completed it	2 hours working on lab report	3 hours spent on preparing for lab	3 hours doing the prelab and and l
3 hours of studying for quizzes	3 hours studying for midterm	2 hours studying midterm solutions	2 hours reviewing for qu
1 hour creating notes	2 hours preparing for midterm	2 hours preparing for quizzes	3 hours solving questions of as

This is my list of cheat-sheets created over the course of my computer engineering degree at Uvic. The reason I decided to create a web-page, is because I was making too much cheat-sheet and it was hard to find them all, perhaps I will modify the css, to create something that's easier to print, although it is not bad looking at the moment.

See **??**.

List of Figures

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Todo list

Contents

Chapter 1

Todo List

Make a sketch of the structure of a trebuchet. And add more figures

See KaTeX documentation , check if I can emulate KaTex in HTML. (https://khan.github.io/KaTeX/function-support.html)

Globally this can be set using Short note with prepend: the prepend, caption option for the package. Below is the effect of the option shown

. A very long and tedious note that cannot be on one line in the list of todos. using the code:

. caption option for the package. Below is the effect of the option shown.

Chapter 2

ELEC 320: Tough Class

Solving Abrupt Silicon PN Junction Question

1. Calculate V_{bi}.
2. Look up D_p on the diffusion coefficient chart. 2.3
3. Calculate the diffusion length: $L_p = \sqrt{D_p \tau_p}$ (for p^+n) - or - $L_n = \sqrt{D_n \tau_n}$ (for pn^+)
4. If: $L_p > x_p$ (for p^+n) - or - $L_n > x_n$ (for pn^+), the diode is short-base.

Bipolar Junction Transistor

$\beta_F = \frac{\alpha_F}{1-\alpha_F}$; $\beta_{dc} = \frac{\alpha_{dc}}{1-\alpha_{dc}}$	Gain
$\alpha_F = \gamma_F \alpha_T$; $\alpha_{dc} = \gamma \alpha_T$ $\alpha_R = \gamma_R \alpha_T$	Base Transport Factor (BTF)
$\alpha_{T(npn)} = \dfrac{I_{Cn}}{I_{En}}$; $\alpha_{T(pnp)} = \dfrac{I_{Cp}}{I_{EP}}$	BTF (≈ 0.999)
$\alpha_T = 1 - \frac{x_B^2}{2D_n\tau_n} = 1 - \frac{x_B^2}{2Ln^2}$	BTF (D_n. Fig 3,5)
$I_{pE} = \frac{-qA_E n_i^2 D_p}{N_{dE} x_E} \exp\left(\frac{qV_{BE}}{k_B T} - 1\right)$	Short Emitter $L_{pE} = \sqrt{D_{pE}\tau_n} > x_E$
$I_{pE} = \frac{-qA_E n_i^2 D_p}{N_{dE} L_p} \exp\left(\frac{qV_{BE}}{k_B T} - 1\right)$	Long Emitter $L_{pE} = \sqrt{D_{pE}\tau_n} < x_E$

$\gamma_F = \left[1 + \frac{x_B N_{aB} D_{pE}}{x_E N_{dE} D_{nB}}\right]^{-1}$ $\gamma_R = \left[1 + \frac{x_B N_{aB} D_{pC}}{x_E N_{dC} D_{nB}}\right]^{-1}$ = Short Emitter Forward and Reverse Emitter Injection (REI) Efficiency (γ_R swap roles, E & C)

$\gamma_F = \left[1 + \frac{x_B N_{aB} D_{pE}}{L_{PE} N_{dE} D_{nB}}\right]^{-1}$ $\gamma_R = \left[1 + \frac{x_B N_{aB} D_{pC}}{L_{PC} N_{dC} D_{nB}}\right]^{-1}$ = Long Emitter Forward and (REI) Efficiency (for γ_R swap roles of E & C)

$\gamma_{(npn)} = \dfrac{I_{En}}{I_E} = \dfrac{|I_{En}|}{|I_{En}| + |I_{Ep}|}$ Emitter injection Efficiency

$\gamma_{(pnp)} = \dfrac{I_{Ep}}{I_E} = \dfrac{|I_{Ep}|}{|I_{En}| + |I_{Ep}|}$

$I_E = I_{Ep} + I_{En}$ $I_C = I_{Cp} + I_{Cn}$ $I_B = \dfrac{I_C - I_{CE0}}{\beta_{dc}}$

$I_C = \alpha_{dc} I_E + I_{CB0}$ $I_C = \beta I_B + I_{CE0}$ $I_{Cn} \approx I_{BC0}$

$I_{Cn} \approx I_{BC0}$ Collector Reverse Saturation Current

$I_{Cn} \approx I_{BC0}$ Emitter-Collector Saturation Current

Electron Current Density (ECD) - constant base doping

$J_n = \frac{qD_n n_i^2}{x_B N_{aB}} \left[\exp\left(\frac{qV_{BC}}{k_B T}\right) - \exp\left(\frac{qV_{BE}}{k_B T}\right)\right]$ (A/cm^2)

ECD - non-constant base doping (A/cm^2)

$J_n = J_0 \left[\exp\left(\frac{qV_{BC}}{k_B T}\right) - \exp\left(\frac{qV_{BE}}{k_B T}\right)\right]$

$J_0 = \frac{q^2 n_i^2 \tilde{D}_n}{Q_B}$, $\tilde{D}_n = \text{avg}(D_n)$

Collector Current Density (under active bias)

$J_C \approx J_0 \exp\left(\frac{qV_{BE}}{k_B T}\right)$

Recombination of excess minority carriers in the base

$I_{rB} = \frac{qA_g n_i^2 x_B}{2N_{aB}\tau_n} \left[\exp\left(\frac{qV_{BE}}{k_B T}\right) - 1\right]$

Collector-Emitter Breakdown Voltage in terms of the Collector-Base Breakdown. Note that $m \approx 4$

$BV_{CE0} = \frac{BV_{CB0}}{\beta^{1/m}}$

Finding β Of a BJT

1. Look-up D_{pE} amd D_{pC} on chart
2. Find $L_{pE} = \sqrt{D_{pE}\tau_{pE}}$ and $L_{pC} = \sqrt{D_{pC}\tau_{pC}}$
3. Check if emitter is long or short $L_{pE} > x_B \rightarrow$ long emitter or $L_{pE} < x_B \rightarrow$ short emitter.
4. Find γ_F, (Short Emitter Forward Efficiency)
5. Find α_T (Base Transport Factor)
6. Find α_F (Base Transport Factor)
7. Find β_F (Current Gain)

Designing an Prototype NPN Structure for an Amplifier

1. Assume these doping levels:

$N_{dC} = 10^{16}\text{cm}^{-3}$ and $N_{aB} = 5\times10^{16}\text{cm}^{-3}$

2. Calculate $V_{bi} = \frac{k_BT}{q}\ln\left[\frac{N_dN_a}{n_i^2}\right]$

3. Use V_a, desired punch through voltage. $x_B = \left(\frac{N_{aB}}{N_{dC}}\right)^{-1}\left[\frac{2\epsilon_s}{q}\left(\frac{1}{N_{aB}}+\frac{1}{N_{dC}}\right)(V_A - V_{bi})\right]^{1/2}$

4. Calculate (shown as a design parameter) $x_{p0} = \left(\frac{N_{aB}}{N_{dC}}\right)^{-1}\left[\frac{2\epsilon_s}{q}\left(\frac{1}{N_{aB}}+\frac{1}{N_{dC}}\right)(V_{bi})\right]^{1/2}$

5. Calculate $\alpha_T = 1 - \frac{x_B^2}{2D_n\tau_n}$ use hole curve ($D_n = 23\text{cm}^2s^{-1}$) for doping levels above.

6. With these doping levels $\alpha_t \cong 1$. Since $\alpha_T \cong 1$, assume $\alpha_F = \gamma_F$.

7. Find the ratio $= \frac{N_{dE}}{D_{pE}} = \underbrace{\left(\frac{x_BN_{aB}}{x_ED_{nB}}\right)\left(\frac{1}{\gamma_F}-1\right)}_{ratio}$

8. Using $D_{pE} = \frac{N_{dE}}{ratio}$, find a good value for N_{dE} that allows you to look up D_{pE} on the diffusion chart. Use the curve for holes.

Ebers-Moll Equations

$I_E = I_F - \alpha_R I_R$	Emitter Current
$I_C = \alpha_F I_F - I_R$	Collector Current
$I_B = I_E - I_C$	
$I_B = (1-\alpha_F)I_F + (1-\alpha_R)I_R$	Base Current
$I_{F0} = qA\left[\frac{D_En_{E0}}{L_e}+\frac{D_Bp_{B0}}{W}\right]$	Forward Coefficient
$I_F = I_{F0}\left[e^{(eV_{eB}/k_BT)}-1\right]$	For Curr Component
$I_{R0} = qA\left[\frac{D_EN_{c0}}{L_C}+\frac{D_Bp_{B0}}{W}\right]$	Reverse Coefficient
$I_R = I_{R0}\left[e^{(qV_{cB}/k_BT)}-1\right]$	Rev Cur Component
$\alpha_F I_{F0} = \alpha_R I_{R0} = I_S$	
$\frac{I_{F0}}{I_{R0}} = \frac{\alpha_R}{\alpha_F}$	reciprocity Relation.
$\beta_f = \frac{\alpha_F}{1-\alpha_F}$	Normal Forward β

$\alpha_R I_R = \frac{qAD_Bp_{B0}}{W}\left[e^{(qV_{cB}/k_BT)-1}\right]$ Ebers-Moll Eqns

$\alpha_R I_R = \frac{qAD_Bp_{B0}}{W}\left[e^{(qV_{cB}/k_BT)-1}\right]$ Vol III - 47.

Transit Time and Frequency Response

τ_1	$=$ r_eC_{jE}	Emitter-Base Capacitance Charging Time
τ_2	$=$ r_CC_{jC}	Collector Capacitance Charging Time
τ_B	$=$ $\frac{x_B^2}{2D_{nB}}$	Base Transit Time
$\tau_C = \frac{x_{dc}}{v_{sat}}$		Collector Depletion Region Transit Time (x_{dc}:Depletion region width of collector) (v_{sat}: Saturation velocity, $\approx 10^7$ m/s)

$\tau_{EC} = \tau_1 + \tau_2 + \tau_B + \tau_C$ $\tau_{EC} = \tau_C + \tau_B + \tau_E$ Emitter to Collector Transit Time. $f_T = \frac{1}{2\pi\tau_{EC}}$ Cut-off Frequency.

Amplification: For amplifying BJTs, the thickness and resistivity of the collector are both large. Th is results in an increased breakdown voltage and reduces the early effect.

Switching: For switching BJT's, saturation (On-State) resistance must be minimized, which requires a very thin collector layer with a resistivity of a few tenths of an Ω-cm.

The Early Effect results in an increase in I_C due to base-width modulation when V_{CB} is increased.

Finding V_T: Using substrate resistivity $\rho \rightarrow N_A\&N_D$ (Vol I - pg 71) 2. Calculate ϕ_p 3. Calculate Q_f from given data Q_f/q 4. Calculate C_{ox} 5. Find ϕ_{MS} (depends on gate material, use Vol I - pg 96. 6. Calculate V_{FB} 7. Calculate V_T.

MOSFET's

$K_s = 11.8$	Dielectric Constant of Si (at 300 K)
$K_o = 3.9$	Dielectric Constant of SiO_2 (at 300K)
$\epsilon_s = K_s\epsilon_0 = 1.1045\times10^{-12}\frac{F}{cm}$	Permittivity of Si (at 300K)
$\epsilon_{ox} = K_0\epsilon_0 = 345.15\times10^{-15}\frac{F}{cm}$	Permittivity SiO_2 (300K)

P-type Si MOS Structure → N-channel Device

N-type Si MOS Structure → P-channel Device

Ref Voltage rel to the semicond doping concent. $\phi_{F(p-type)} = \phi_p = \frac{kT}{q}\ln\left(\frac{N_A}{n_i}\right)$

$\phi_{F(n-type)} = \phi_p = -\frac{kT}{q}\ln\left(\frac{N_D}{n_i}\right)$ Semicond Surf Pot at Depletion-Inversion Transition Point $\phi_S = 2\phi_F$

Flat-band voltage (voltage that produces flat energy bands in the oxide and silicon)

$V_{FB} = \phi_{MS} - \frac{Q_f}{C_{ox}}$ With a charge on the oxide layer.

$V_{FB} = \phi_M - \phi_S = \phi_{MS}$ oxide layer free of charge.

$W_T = \left[\frac{2K_s\epsilon_0}{qN_A}(2\phi_p)\right]^{1/2}$	Depletion Width Iv p 43
$\phi(x) = \frac{1}{q}[E_f - E_i(x)]$	Potential in Silicon
$\phi_S = \phi(0) = \frac{1}{q}(E_f - E_i(0))$	Surface Potential
$C_{ox} = \frac{\epsilon_{ox}}{x_{ox}}$ $\dot{A} = 10^{-10}m$	Oxide Layer Capacitance
$Q_{d(max)} = \sqrt{4k_s\epsilon_0 qN_A\lvert\phi_P\rvert}$	Space Charge Density (max)

$V_T = V_{FB} + 2|\phi_p| + \frac{|Q_{d(max)}|}{C_{ox}}$, Threshold Voltage(T.V.)

$V_T = 2\phi_F - \frac{K_sx_{ox}}{K_{ox}}\left[\frac{4qN_A}{K_S\epsilon_0}\phi_F\right]$

$\Delta V_G = V_T' - V_T$ (Threshold Adjustment) $\Delta V_G = \frac{-Q_l}{C_{ox}}$ $Q_l = \pm qN_l \rightarrow N_l = \pm\frac{Q_l}{q}$ V_T' Un-adjusted T.V. V_T: Adjusted T.V. N_l: # of implanted ions Q_l: Implant-related charge/cm^2 Donor{+} or Acceptor {-}

$I_D = \frac{Z\bar{\mu}_nC_{ox}}{L}\left[(V_G - V_T)V_D - \frac{V_D^2}{2}\right]$ Square-law theory

Z: Width of MOSFET $\bar{\mu}_n$ Effe hole mobil Vol IV. Pg 73

Long Channel MOSFET Equation Bulk Charge Factor (α) $I_D = \mu C_{ox}\frac{W}{L}\left[\left(V_G - V_T - \frac{1}{2}V_D\right)V_D\right]$

$I_D = \mu C_{ox}\frac{W}{2\alpha L}\left[\left(V_G - V_T - \frac{1}{2}V_D\right)V_D\right]$

Channel Carrier Velocity (Using Long-Channel Theory) $V = \dfrac{\mu_nC_{ox}\left[\left(V_G - V_T - \frac{1}{2}V_D\right)V_D\right]}{Q_NL}$

Note: The saturation velocity of carriers in Silicon is: $v_{sat} \approx 10^7$ cm/s. If this equation yields a velocity $v > v_{sat}$, long channel theory does not apply in this situation. $Q_{n(source)} = C_{ox}(V_G - V_T)$, $Q_{n(drain)} = C_{ox}(V_G - V_D - V_T)$.

Drain Saturation Voltage $V_{D(sat)} = V_G - V_T$ $V_{D(sat)} = \frac{V_G - V_T}{\alpha}$.

g_d, g_m Small Signal Parameters and Conductance

$f_{max} = \frac{g_m}{2\pi C_{ox}} = \frac{\bar{\mu}_n V_D}{2\pi L^2}$ Cutoff f $V_D \leq V_{D(sat)}$

Using $I_{D(sat)} = \mu_n C_{ox} \frac{W}{2L}(V_G - V_T)^2$

Channel Dimensions $\frac{W}{L} = \frac{2I_{D(sat)}}{\mu_n C_{ox}(V_G - V_T)^2}$

MOSFET Integrated Circuit Applications

$V_T = V_{FB} + 2|\phi_p| + \dfrac{Q_{d(max)}}{\epsilon_{ox}}(d_1 + d_2) + \dfrac{Q_{fg}}{\epsilon_{ox}} d_1$ Charge stored on a floating gate memory cell

$|Q_{fg}| = \frac{\epsilon_{ox}}{d_1}\left[V_T - V_{FB} - 2\phi_P - \frac{Q_{d(max)}}{\epsilon_{ox}}(d_1 + d_2)\right]$, Q_{fg} = Floating Gate Charge Density $V_{FB} = \phi_{MS}$

Step 1: Find V_{bi} (n-well to source/drain junction) using N_d from, the n-well and N_a from the p-channel source/drain.

Step 2: Find x_d using the doping levels and V_{bi}.

Step 3: Find V_{bi} (n-well to p-substrate junction) using N_d from the n-well and N_a from the p-substrate.

Step 4: Find x_n using V_{bi} and $V_a = V_{DD}$, and the same doping level as step 3.

Step 5: The minimum required n-well depth to prevent punchthrough at this voltage is: $d_{n-well} = d_{p-channelsrc/drn} + x_{d(step2)} + x_{n(step4)}$

$V_{bi} = \frac{k_B T}{q} \ln\left(\frac{N_d N_a}{n_i^2}\right)$

$x_d = x_n + x_p = \left[\frac{2\epsilon_s}{q}\left(\frac{1}{N_a} + \frac{1}{N_d}\right)(V_a - V_{bi})\right]^{1/2}$,

$x_n = \left\{\frac{2K_s\epsilon_0}{q}(V_{bi} + V_a)\left[\frac{N_a}{N_d(N_a + N_d)}\right]\right\}^{1/2}$ Note: The p-channel source/drain depth $d_{p-channelsrc/drn}$ should be given in the question.

CMOS Well-Depth Design: Finding minimum well depth to prevent vertical punch through.

CMOS Structures P-Well: The substrate is N-Type. The N-Channel device is built into a P-Type well within the parent N-Type substrate. The P-channel device is built directly on the substrate.

N-Well: The substrate is P-Type. The N-channel device is built directly on the substrate, while the P-channel device is built into a N-type well within the parent P-Type substrate.

$g_d = \frac{Z\bar{\mu}_n C_o}{L}(V_G - V_T)$ $(V_D = 0)$

Practise Test 2 1) Which of the following can reduce the base transit time? c) Short base width.

2) Design the doping levels and dimensions of a silicon npn bipolar transistor such that the dc current gain is 320 and the Gummel Number is 10^{12}cm^{-2}. Assume that $\tau_n = 10^{-7}s$ in the base, $\tau_p = 10^{-8}s$ in the collector.

$GN = Q_B = \int_0^{x_B} N_{aB}(x)dx$.

$\phi(S) = 4.05 - (4.05 + E_g/2 + E_f - E_i)$

Table 3.1 MOSFET Small Signal Parameters.*

	Below pinch-off ($V_D \le V_{Dsat}$)	Post pinch-off ($V_D > V_{Dsat}$)
Square law	$g_d = \frac{Z\bar{\mu}_n C_o}{L}(V_G - V_T - V_D)$	$g_d = 0$
Bulk charge	$g_d = \frac{Z\bar{\mu}_n C_o}{L}[V_G - V_T - V_D - V_W(\sqrt{1 + V_D/2\phi_F} - 1)]$	$g_d = 0$
Square law	$g_m = \frac{Z\bar{\mu}_n C_o}{L}V_D$	$g_m = \frac{Z\bar{\mu}_n C_o}{L}(V_G - V_T)$
Bulk charge	$g_m = \frac{Z\bar{\mu}_n C_o}{L}V_D$	$g_m = \frac{Z\bar{\mu}_n C_o}{L}V_{Dsat}$ with V_{Dsat} per Eq. (3.27)

*Entries in the table were obtained by direct differentiation of Eqs. (3.15), (3.20), and (3.26). The variation of $\bar{\mu}_n$ with V_G was neglected in establishing the g_m expressions.

Figure 2.1: Vol IV Mosfet table

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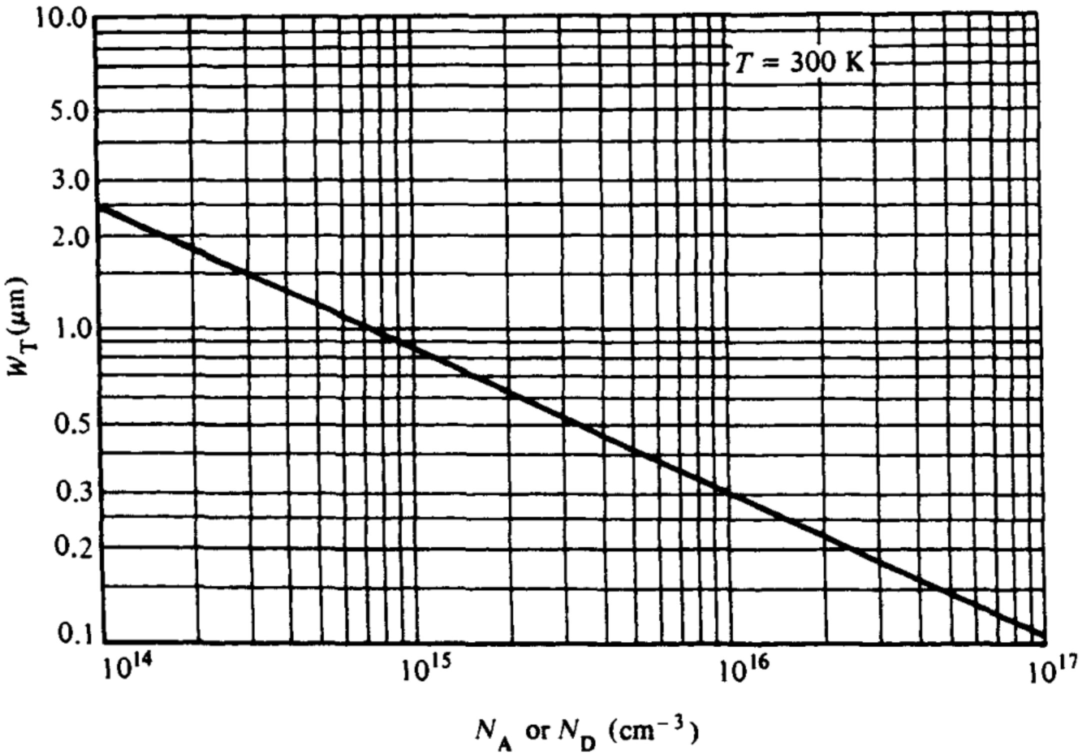

Figure 2.2: Doping dependence of the maximum equilibrium depletion width inside silicon devices maintained at 300 K.

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Table 12.1 | Notation used in the analysis of the bipolar transistor

Notation	Definition
For both the npn and pnp transistors	
N_E, N_B, N_C	Doping concentrations in the emitter, base, and collector
x_E, x_B, x_C	Widths of neutral emitter, base, and collector regions
D_E, D_B, D_C	*Minority carrier* diffusion coefficients in emitter, base, and collector regions
L_E, L_B, L_C	*Minority carrier* diffusion lengths in emitter, base, and collector regions
τ_{E0}, τ_{B0}, τ_{C0}	*Minority carrier* lifetimes in emitter, base, and collector regions
For the npn	
p_{E0}, n_{B0}, p_{C0}	Thermal-equilibrium *minority carrier* hole, electron, and hole concentrations in the emitter, base, and collector
$p_E(x')$, $n_B(x)$, $p_C(x'')$	Total *minority carrier* hole, electron, and hole concentrations in the emitter, base, and collector
$\delta p_E(x')$, $\delta n_B(x)$, $\delta p_C(x'')$	Excess *minority carrier* hole, electron, and hole concentrations in the emitter, base, and collector
For the pnp	
n_{E0}, p_{B0}, n_{C0}	Thermal-equilibrium *minority carrier* electron, hole, and electron concentrations in the emitter, base, and collector
$n_E(x')$, $p_B(x)$, $n_C(x'')$	Total *minority carrier* electron, hole, and electron concentrations in the emitter, base, and collector
$\delta n_E(x')$, $\delta p_B(x)$, $\delta n_C(x'')$	Excess *minority carrier* electron, hole, and electron concentrations in the emitter, base, and collector

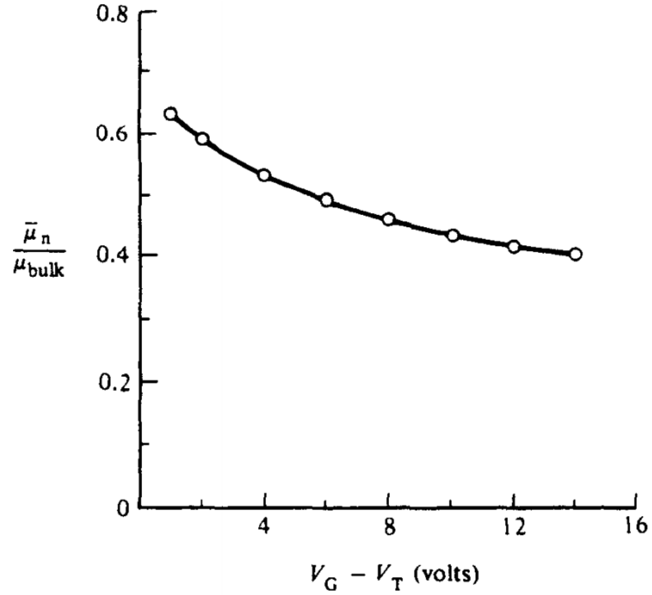

Figure 2.3: Vol IV Mosfet Gate voltages

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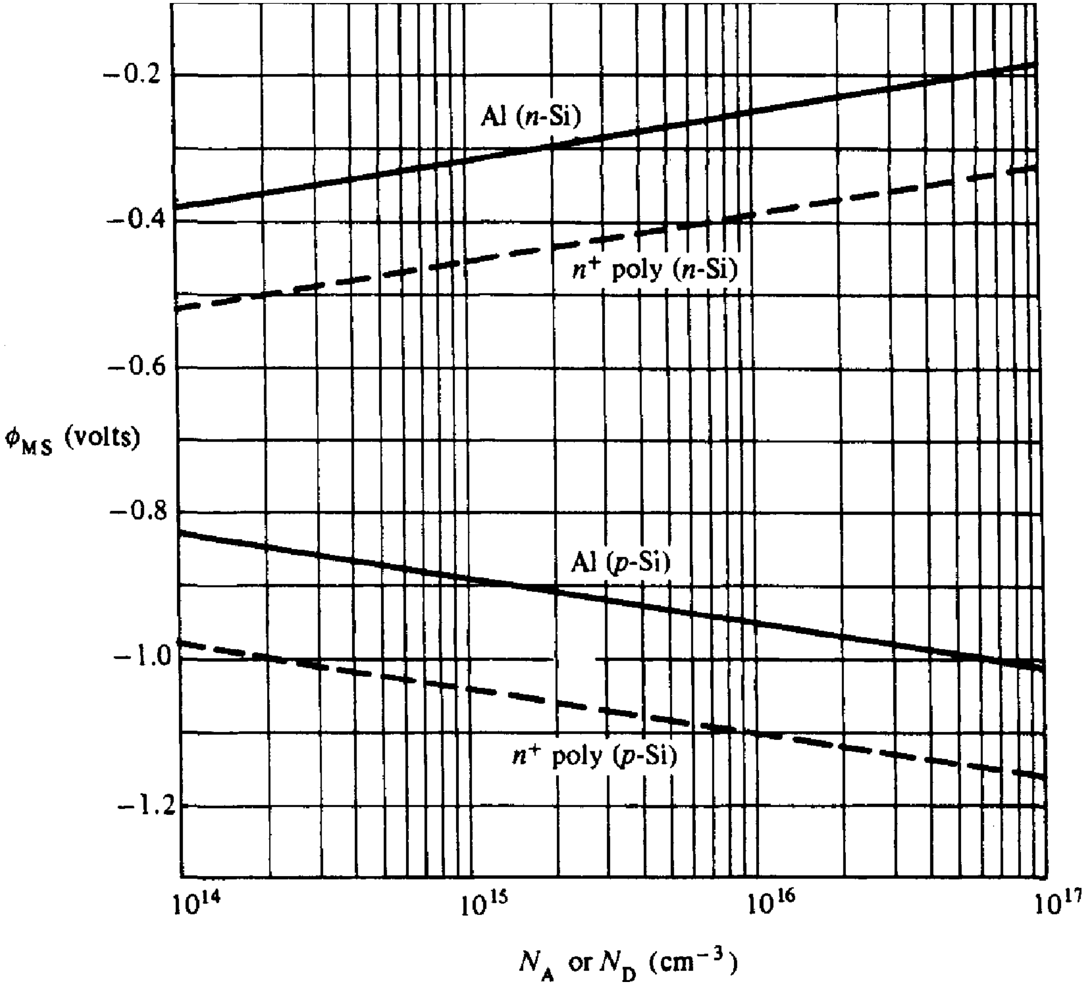

Figure 2.4: Workfunction difference as a function of a n- and p-type dopant conecntration in n^+ poly-Si-gate and Al-gate $SiO_2 - Si$ structures. ($T = 300K$. $\phi'_M - \chi' = -0.18\ eV$ for the n' poly-Si-gate structure; $\phi'_M - \chi' = 0.03eV$ for the Al-gate structure.)

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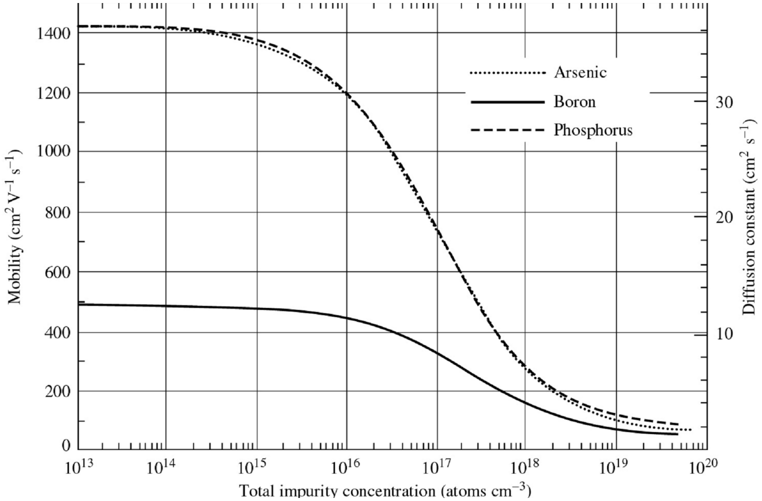


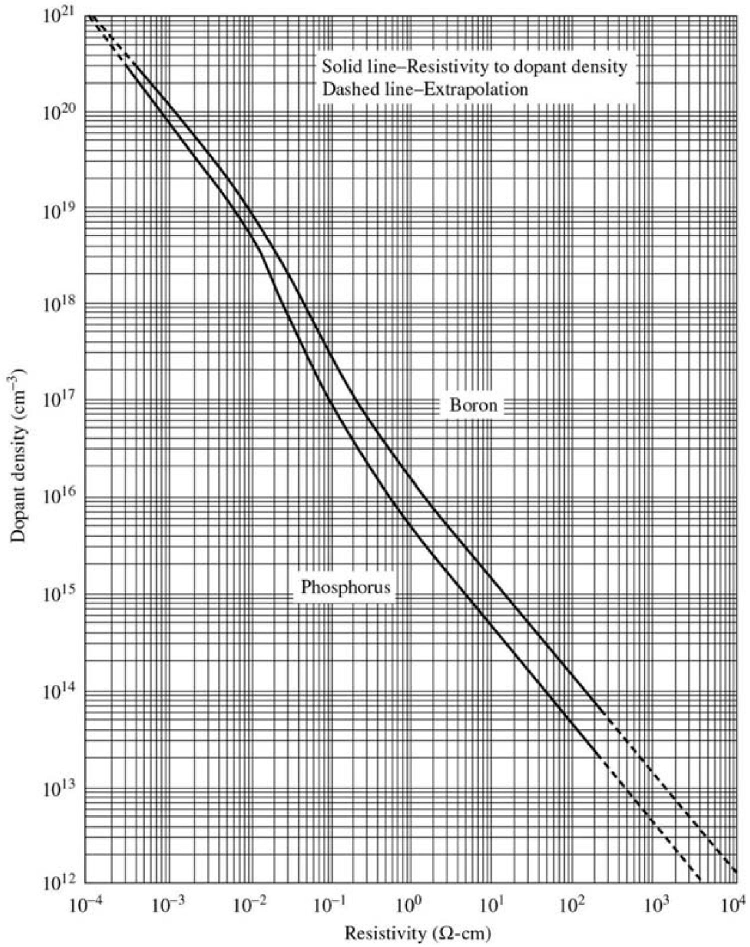
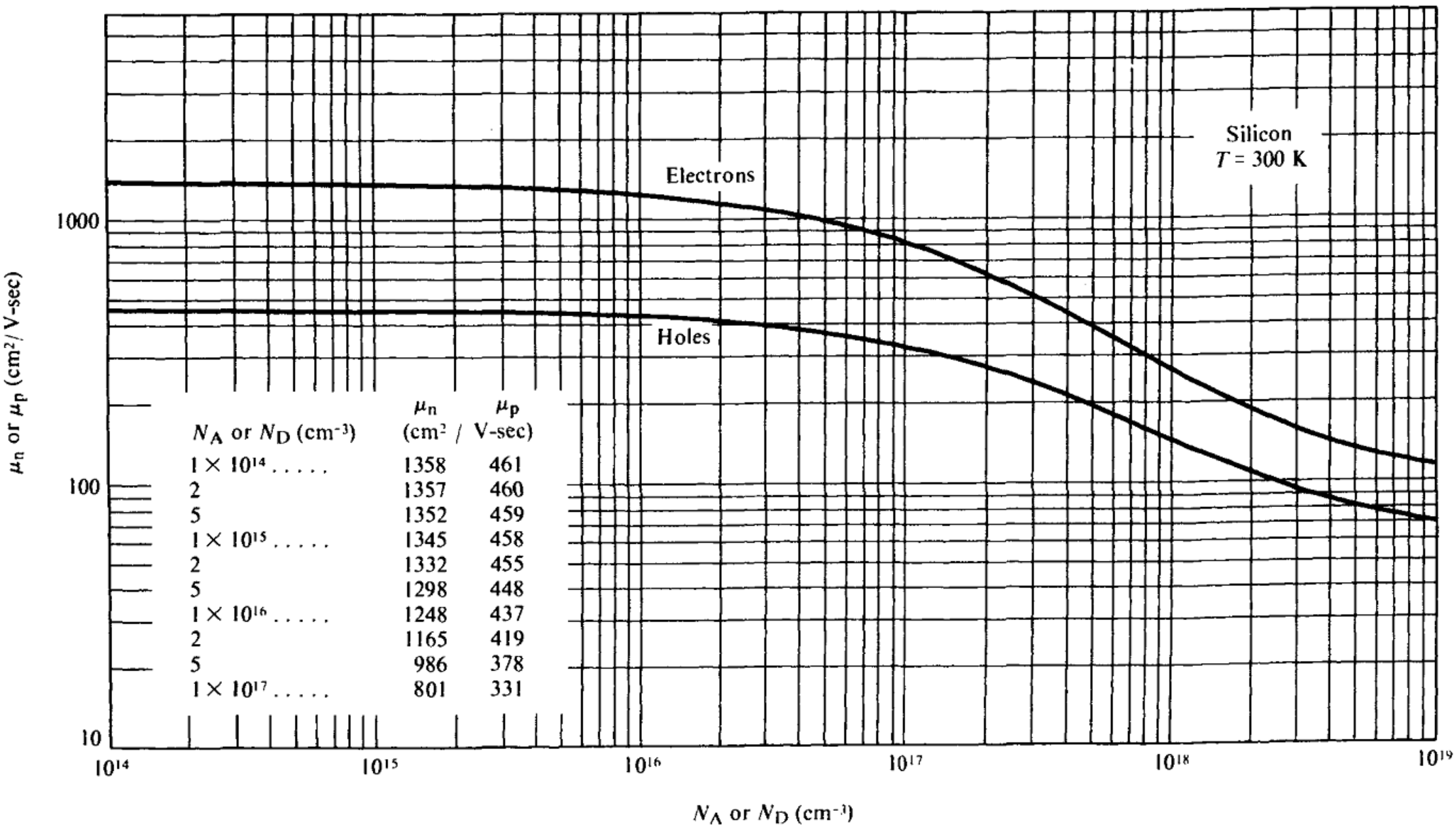


Figure 2.5: Boron Chart



N_A or N_D (cm^{-3})	μ_n (cm^2 /	μ_p V-sec)
1×10^{14}	1358	461
2	1357	460
5	1352	459
1×10^{15}	1345	458
2	1332	455
5	1298	448
1×10^{16}	1248	437
2	1165	419
5	986	378
1×10^{17}	801	331

Fig. 3.5 Room-temperature carrier mobilities in silicon as a function of the dopant concentration. μ_n is the electron mobility; μ_p is the hole mobility.

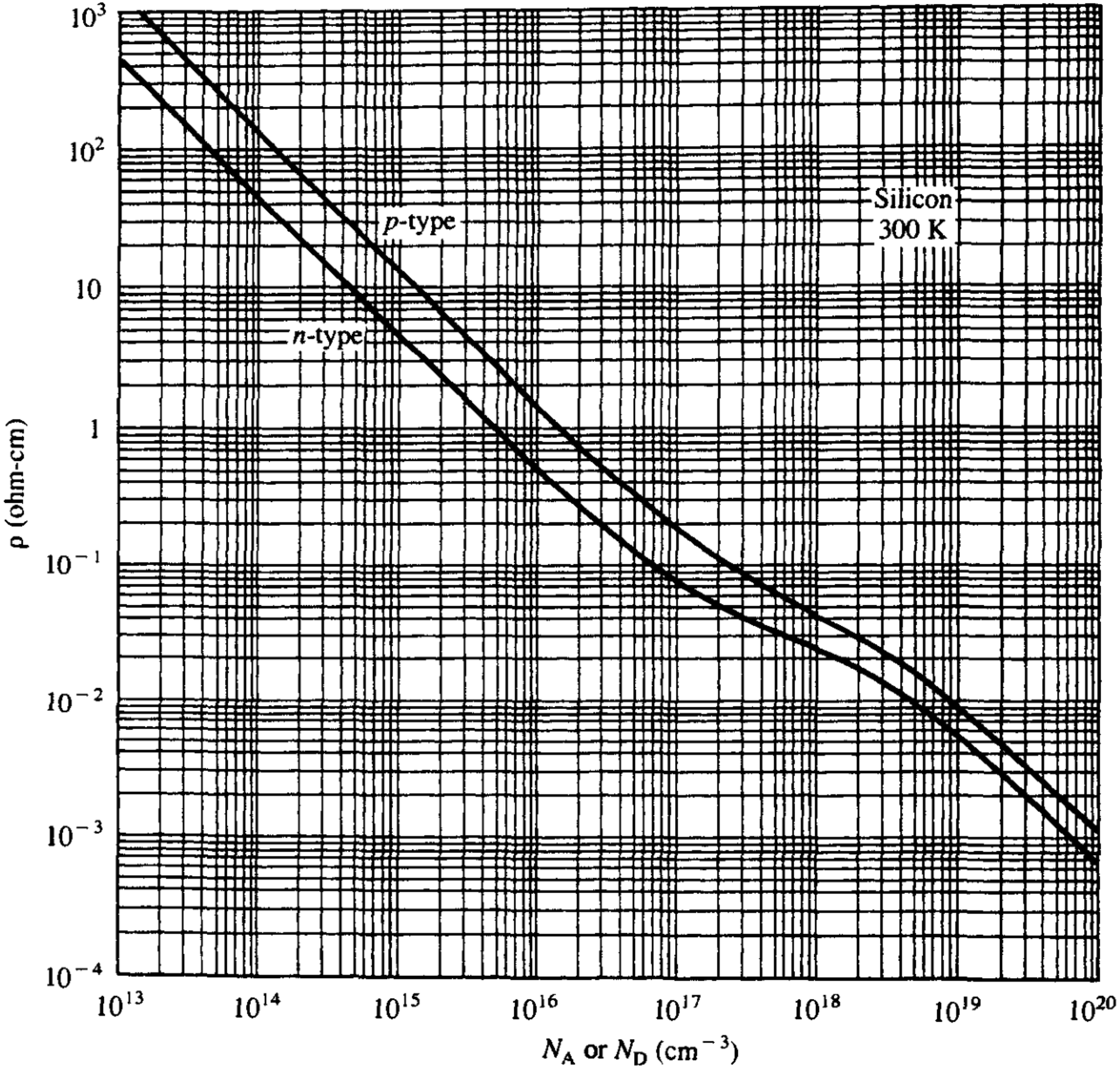


Fig. 3.7 Si resistivity versus impurity concentration at 300 K.

Chapter 3

CENG242Cheat

Future Work

Algorithm 3.1 Euclid's algorithm

```
procedure EUCLID(a, b)                        ▷ The g.c.d. of a and b
    r ← a mod b
    while r ≠ 0 do                            ▷ We have the answer if r is 0
        a ← b
        b ← r
        r ← a mod b
    end while
    return b                                  ▷ The gcd is b
end procedure
```

Basic Graph Definitions

- A graph $G = (V, E)$ consists of a finite set of *vertices* V and a finite set of *edges* E.
  - *Directed graphs*: E is a set of ordered pairs of vertices (u, v) where $u, v \in V$
  - *Undirected graph*: E is a set of unordered pairs of vertices $\{u, v\}$ where $u, v \in V$
- Edge (u, v) is *incident* to u and v

Algorithm 3.2 Merge Sort

```
function Merge(A, p, q, r)                    ▷ Where A - array, p - left, q - middle, r - right
    n_1 = q − p + 1
    n_2 = r − q
    Let L[1 . . . n_1 + 1] and R[1 . . . n_2 + 1] be new arrays
    for i = 1 to n_1 do
        L[i] = A[p + i − 1]
    end for
    for j = 1 to n_2 do
        R[i] = A[q + j]
    end for
    L[n_1 + 1] = ∞
    R[n_2 + 1] = ∞
    i = 1
    j = 1
    for k = p to r do
        if L[i] < R[j] then
            A[k] = L[i]
            i = i + 1
        else if L[i] > R[j] then
            A[k] = R[j]
            j = j + 1
        else
            A[k] = −∞                          ▷ We mark the duplicates with the largest negative integer
            j = j + 1
        end if
    end for
end function
```

- *Degree* of vertex in undirected graph is the number of edges incident to it.
- *In (out) degree* of a vertex in directed graph is the number of edges entering (leaving) it.
- A *path* from u_1 to u_2 is a sequence of vertices $< u_1{=}v_0, v_1, v_2, \cdots, v_k{=}u_2 >$ such that $(v_i, v_{i+1}) \in E$ (or $\{v_i, v_{i+1}\} \in E$)
  - We say that u_2 is *reachable* from u_1
  - The *length* of the path is k
  - It is a *cycle* if $v_0 = v_k$
- An undirected graph is *connected* / if every pair of vertices are connected by a path
  - The *connected components* are the equivalence classes of the vertices under the "reachability" relation. (All connected pair of vertices are in the same connected component).
- A directed graph is *strongly connected* if every pair of vertices are reachable from each other
  - The *strongly connected components* are the equivalence classes of the vertices under the "mutual reachability" relation.

- Graphs appear all over the place in all kinds of applications, e.g:
  - Trees ($|E| = |V| - 1$)
  - Connectivity/dependencies (house building plans, WWW-page connections = internet graph)
- Often the edges (u, v) in a graph have weights $w(u, v)$, e.g.
  - Road networks (distances)
  - Cable networks (capacity)

Representation

- *Adjacency-list* representation:
  - Array of $|V|$ list of edges incident to each vertex.

    Examples:

  - Note: For undirected graphs, every edge is stored twice.
  - If graph is weighted, a weight is stored with each edge.
- *Adjacency-matrix* representation:

- $|V| \times |V|$ matrix A where

$$a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

  Examples:

  - Note: For undirected graphs, the adjacency matrix is symmetric along the main diagonal ($A^T = A$).
  - If graph is weighted, weights are stored instead of one's.

- Comparison of matrix and list representation:

Adjacency list	Adjacency matrix
$O(\|V\| + \|E\|)$ space	$O(\|V\|^2)$ space
Good if graph *sparse* ($\|E\| << \|V\|^2$)	Good if graph *dense* ($\|E\| \approx \|V\|^2$)
No quick access to (u, v)	$O(1)$ access to (u, v)

- We will use adjacency list representation unless stated otherwise ($O(|V| + |E|)$ space).

Graph traversal

- There are two standard (and simple) ways of traversing all vertices/edges in a graph in a systematic way
  - Breadth-first
  - Depth-first
- We can use them in many fundamental algorithms, e.g finding cycles, connected components, ...

Breadth-first search (BFS)

- Main idea:
  - Start at some source vertex s and visit,
  - All vertices at distance 1,
  - Followed by all vertices at distance 2,
  - Followed by all vertices at distance 3,

  $\vdots$

- BFS corresponds to computing *shortest path* distance (number of edges) from s to all other vertices.
- To control progress of our BFS algorithm, we think about *coloring* each vertex
  - *White* before we start,
  - *Gray* after we visit the vertex but before we have visited all its adjacent vertices,
  - *Black* after we have visited the vertex and all its adjacent vertices (all adjacent vertices are gray).
- We use a queue Q to hold all gray vertices—vertices we have seen but are still not done with.
- We remember from which vertex a given vertex v is colored gray – i.e. the node that discovered v first; this is called parent$[v]$.
- Algorithm:

```
BFS(s)
      color[s] = gray
      d[s] = 0
      ENQUEUE(Q, s)
      WHILE Q not empty DO
           DEQUEUE(Q, u)
           FOR (u, v) ∈ E DO
                IF color[v] = white THEN
                    color[v] = gray
                    d[v] = d[u] + 1
                    parent[v] = u
                    ENQUEUE(Q, v)
                FI
                color[u] = black
      OD
```

- Algorithm runs in $O(|V| + |E|)$ time
- Example (for directed graph):

- Note:
  - parent$[v]$ forms a tree; *BFS-tree.*
  - $d[v]$ contains length of shortest path from s to v. (Prove by induction)
  - We can use parent$[v]$ to find the shortest path from s to a given vertex.

- If graph is not connected we have to try to start the traversal at all nodes.

```
FOR each vertex u ∈ V DO
    IF color[u] = white THEN BFS(u)
OD
```

  - Note: We can use algorithm to compute connected components in $O(|V| + |E|)$ time.

Depth-first search (DFS)

- If we use stack instead of queue Q we get another traversal order; depth-first
  - We go "as deep as possible",
  - Go back until we find unexplored adjacent vertex,
  - Go as deep as possible,

    $\vdots$

- Often we are interested in "start time" and "finish time" of vertex u
  - *Start time* (d[u]): indicates at what "time" vertex is first visited.
  - *Finish time* (f[u]): indicates at what "time" all adjacent vertices have been visited.
- We can write DFS iteratively using the same algorithm as for BFS but with a STACK instead of a QUEUE, or, we can write a recursive DFS procedure
  - We will color a vertex gray when we first meet it and black when we finish processing all adjacent vertices.
- Algorithm:

```
DFS(u)
      color[u] = gray
      d[u] = time
      time = time + 1
      FOR (u, v) ∈ E DO
           IF color[v] = white THEN
                parent[v] = u
                DFS(v)
           FI
      OD
      color[u] = black
      f[u] = time
      time = time + 1
```

- Algorithm runs in $O(|V| + |E|)$ time
  - As before we can extend algorithm to unconnected graphs and we can use it to detect cycles in $O(|V| + |E|)$ time.
- As previously parent$[v]$ forms a tree; *DFS-tree*
  - Note: If u is descendent of v in DFS-tree then $d[v] < d[u] < f[u] < f[v]$

Topological sorting

- Definition: Topological sorting of *directed acyclic graph* $G = (V, E)$ is a linear ordering of vertices V such that $(u, v) \in E \Rightarrow u$ appear before v in ordering.
- Topological ordering can be used in scheduling:
  - Example: Dressing (arrow implies "must come before")

    We want to compute order in which to get dressed. One possibility:

    The given order is one possible topological order.
- Algorithm: Topological order just reverse DFS finish time ($\Rightarrow O(|V| + |E|)$ running time).
- Correctness: $(u, v) \in E \Leftrightarrow f(v) < f(u)$
  - Proof: When (u, v) is explored by DFS algorithm, v must be white or black (gray $\Rightarrow$ cycle).

    * v white: v visited and finished before u is finished $\Rightarrow f(v) < f(u)$
    * v black: v already finished $\Rightarrow f(v) < f(u)$

- Alternative algorithm: Count in-degree of each vertex and repeatedly number and remove in-degree 0 vertex and its outgoing edges: Homework.

Chapter 4

ELEC 220 CheatSheet

Ch.1 DC Conduction

σ = conductivity (S/m) and ρ = resistivity (Ωm). $\sigma = 1/\ \rho$.

Scattering time Formula: $\sigma = \dfrac{e^2 N_e \tau}{m_e}$

Ch. 2 AC Conduction

$$\textbf{Skin Depth:} \quad \delta = \left(\frac{2}{\omega\mu\sigma}\right)^{1/2} \tag{4.1}$$

$$E = E_0 e^{-i(wt - z/\ \delta)} e^{-z/\ \delta} \tag{4.2}$$

$$I \sim e^{-2z/\ \delta} \tag{4.3}$$

$$I \propto |E|^2 \tag{4.4}$$

$$\omega = \frac{2\pi c}{\lambda_o} \tag{4.5}$$

$$\textbf{Plasma Frequency:} \quad \omega_p = \left(\frac{N_e e^2}{m\epsilon}\right)^{1/2} \tag{4.6}$$

$$k^2 = \omega^2\mu\epsilon - \frac{N_e e^2 \mu}{m} = \omega^2\mu\epsilon\left(1 - \frac{\omega_p^2}{\omega^2}\right) \tag{4.7}$$

Ch. 3 DC/AC Dielectrics

$$\textbf{Relative Permittivity:} \quad = \frac{C'}{C} = \frac{Q'}{Q} = \epsilon_r \tag{4.8}$$

$$PA = Q' - Q = C'V - CV = CV(\epsilon_r - 1) \tag{4.9}$$

$$P = \epsilon_0 E(\epsilon_r - 1) \ and \ \epsilon_r = 1 + \frac{P}{\epsilon_0 E} = 1 + \chi \tag{4.10}$$

$$D = \epsilon E = \epsilon_0\epsilon_r E = \epsilon_0 E + P \tag{4.11}$$

$$P = Np = Nqd \tag{4.12}$$

$$\epsilon_{=} 1 + \frac{N\alpha_e}{\epsilon_0} = 1 + \chi \tag{4.13}$$

$$\textbf{Debye Model:} \quad \epsilon_d(\omega) = \frac{\epsilon_d(0)}{1 - i\omega\tau_r} = \epsilon_d' + i\epsilon_d'' \tag{4.14}$$

$$\epsilon_d' = \frac{\epsilon_d(0)}{1 + \omega^2\tau_r^2} \quad and \quad \epsilon_d''(\omega) = \frac{\epsilon_d(0)}{1 + \omega^2\tau_r^2}\omega\tau_r \tag{4.15}$$

Different Polarization Mechanisms(Decreasing Speed)
Electronic
Ionic
Dipolar (Orientational)
Space Charge (Interfacial)
Ferroelectric

Ch. 4 AC Dielectrics Cont'd

$$\nu = \frac{c}{n} = \frac{1}{\sqrt{\epsilon_r\epsilon_o\mu_o}} \tag{4.16}$$

$$n = \sqrt{\epsilon_r} \quad \text{reflactive index} \tag{4.17}$$

$$\sin(\theta_c) = \frac{n_1}{n2} \tag{4.18}$$

$$k_{imag} = \frac{\omega\epsilon_r''}{2c\sqrt{\epsilon_r'}} = \frac{\omega}{2c}\sqrt{\epsilon_r'}\tan\delta \tag{4.19}$$

$$dB = 8.69 \times k_{imag} \tag{4.20}$$

Ch. 8 Schrodinger's Equation

Planck's constant $\hbar = \frac{h}{2\pi}$ (4.21)

$$1.6 \times 10^{-19} J = 1eV \tag{4.22}$$

$$p = \frac{h}{\lambda} \tag{4.23}$$

$$E = \frac{\hbar^2 k^2}{2m} = \frac{n^2 h^2}{8mL^2} \tag{4.24}$$

$$k = (2mE)^{1/2} h^{-1} \tag{4.25}$$

$$\int_0^L \psi^2 dz = 1 = A_n^2 \int_0^L sin^2(n\pi z/L) dz = \frac{A_n^2}{2} L \tag{4.26}$$

$$< S >= \frac{\int \psi^* S \psi dV}{\int \psi^* \psi dV} = \int \psi^* S \psi dV \tag{4.27}$$

The wavefunction y is complex valued • We interpret the absolute value of y squared (i.e., $\psi \times \psi^*$) as the probability that the particle is in a given position • This requires appropriate normalization over space so that the total probability that the particle is anywhere is 1 (i.e., the particle exists)

Ch. 12 Free Electron Theory of Metals

1D Box: $k_F = \frac{N\pi}{2L}$ (4.28)

$$E_F = \frac{\hbar^2 k_F^2}{2m} = \frac{h^2}{32m} \left(\frac{N}{L}\right)^2 \tag{4.29}$$

$$Z(E) = \frac{dN(E)}{dE} = CE^{-1/2} \tag{4.30}$$

2D Box: $k_F^2 = 2\pi \frac{N}{L^2}$ (4.31)

$$E_F = \frac{\hbar^2 k_F^2}{2m} = \frac{h^2}{4\pi m} \frac{N}{L^2} \tag{4.32}$$

$$Z(E) = \frac{dN(E)}{dE} = C \tag{4.33}$$

3D Box: $k_F^3 = 3\pi^2 \frac{N}{L^3}$ (4.34)

$$E_F = \frac{\hbar^2 k_F^2}{2m} = \frac{h^2}{2m} \left(\frac{3N}{8\pi L^3}\right)^{2/3} \tag{4.35}$$

$$Z(E) = \frac{dN(E)}{dE} = \frac{4\pi L^3 (2m)^{3/2}}{h^3} E^{1/2} = CE^{1/2} \tag{4.36}$$

Fermi Distribution

$$F(E) = 1 \; if \; E < E_F \tag{4.37}$$

$$F(E) = 0 \; if \; E > E_F \tag{4.38}$$

$$F(E) = \frac{1}{1 + e^{\frac{E - E_F}{k_B T}}} \tag{4.39}$$

$$E_{tot} = \int EZ(E)F(E)dE \tag{4.40}$$

Ch. 13 Band Theory

$$V = V_0 \cos\left(\frac{2\pi x}{a}\right) \tag{4.41}$$

$$v_g = \frac{\partial \omega}{\partial k} = \frac{1}{\hbar} \frac{\partial E}{\partial k} \tag{4.42}$$

$$a = \frac{dv_g}{dt} = \frac{1}{\hbar} \frac{\partial^2 E}{\partial k^2} \frac{dk}{dt} \tag{4.43}$$

$$F = \frac{dp}{dt} = \hbar \frac{k}{t} \tag{4.44}$$

$$m^* = \frac{F}{a} = \frac{\hbar^2}{\frac{\partial^2 E}{\partial k^2}} \tag{4.45}$$

Ch. 14 Metals and Insulators

$$\sigma = \frac{v_F^2 Z(E_F)}{3} e^2 \tau_F \tag{4.46}$$

$$\nu = \frac{E_g}{h} \tag{4.47}$$

Ch. 15 Semiconductors

Total # of Electrons in Conduction Band: $N_e = N_c \cdot e^{\left(\frac{E_c - E_F}{k_B T}\right)}$

$$N_c = 2\left[\frac{2\pi m_e^* kT}{h^2}^{3/2}\right]$$

Total # of Hole in Valence Band: $N_h = N_v \cdot e^{\left(\frac{E_F - E_v}{k_B T}\right)}$

$$N_v = 2\left[\frac{2\pi m_h^* kT}{h^2}^{3/2}\right]$$

$$E_f = E_v + \frac{E_g}{2} - \frac{1}{2}kTln\left(\frac{N_c}{N-V}\right) = E_v + \frac{E_g}{2} - \frac{3}{4}kTln\left(\frac{m_e^*}{m_h^*}\right)$$

Total Conductivity: $\sigma = e(N_e\mu_e + N_h\mu_h) = eN_e(\mu_e + \mu_h)$

Einstein's Relation(Ch.16.3 Diffusion Current) $\frac{D_h}{\mu_h} = \frac{k_b T}{e}$

N_i = Intrinsic Carrier Density. For an Intrinsic semiconductor holes = electrons $N_i^2 = N_v N_c \exp(-E_g/(kT))$.

n-type: $N_e \approx N_D \quad and \quad N_h \approx \frac{N_i^2}{N_D}$ Minority carrier in holes

p-type: $N_h \approx N_A \quad and \quad N_e \approx \frac{N_i^2}{N_A}$ Minority carrier is electrons

Chapter 5

ELEC 403 CheatSheet

Algorithm 1.1 General optimization algorithm
Step 1:
(a) Set $k = 0$ and initialize x_0
(b) Compute $F_0 = f(x_0)$
Step 2:
(a) Set $k = k + 1$
(b) Compute the changes in x_k given by column vector ∇x_k

$$\text{where} \quad \nabla x_k^T = \begin{bmatrix} \nabla x_1 & \nabla x_2 & \cdots & \nabla x_n \end{bmatrix}$$

by using an appropriate procedure.
(c) Set $x_k = x_{k-1} + \nabla x_k$
(d) Compute $F_k = f(x_k)$ and $\nabla F_k = F_{k-1} - F_k$.
Step 3:
Check if convergence has been achieved by using an appropriate criterion, e.g., by checking ∇F_k and/or ∇x_k. If this is the case, continue to Step 4; otherwise, go to Step 2.
Step 4:
(a) Output $x^* = x_k$ and $F^* = f(x^*)$.
(b) Stop

Ch.2

Gradient: $g(x) = \nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}^T$
Hessian Matrix: $H(x) = \nabla g(x) = \nabla\{\nabla^T f(x)\}$.

$$H(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial^2 x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(x)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x)}{\partial x_2^2} & \cdots & \frac{\partial f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f}{\partial x_n \partial x_1} & \frac{\partial f}{\partial x_n \partial x_2} & \cdots & \frac{\partial f}{\partial x_n \partial x_n} \end{bmatrix}$$

Taylor Series: (quad approx, linear approx): $\delta = \begin{bmatrix} \delta_1 & \delta_2 \end{bmatrix}^T$ $f(x+\delta) = f(x) + g(x)^T\delta + 12\delta^T H(x)\delta + o(||\delta||^2)$
Linear approximation: $f(x+\delta) \approx f(x) + g(x)^T\delta$ $\tilde{x} = x + \delta \quad \delta = \tilde{x} - x$ $\hat{f}(\tilde{x}) = f(x) + g(x)(\tilde{x} - x)^T + 0.5(\tilde{x} - x)^T H(x)(\tilde{x} - x)^T$ The gradient $g(x)$ and the Hessian $H(x)$ must satisfy certain conditions at a local minimizer x^*.
1. Conditions which are satisfied at a local minimizer x^*.
2. Conditions which guarantee that x^* is a local minimizer.

Definition 2.1 A point $x^* \in R$, where R is the feasible region, is said to be a weak local minimizer of f(x) if there exists a distance $\epsilon > 0$ such that $f(x) \geq f(x^*)$ (2.5) if $x \in R$ and $||x - x^*|| < \epsilon$

Definition 2.2 A point $x^* \in R$ is said to be a weak global minimizer of f(x) if $f(x) \geq f(x^*)$ (2.6) for all $x \in R$.
Definition 2.3

If Eq. (2.5) in Def. 2.1 or Eq. (2.6) in Def. 2.2 is replaced by $f(x) > f(x^*)$ (2.7) x^* is said to be a strong local (or global) minimizer. d

Definition 2.4 Let $\delta = \alpha\mathbf{d}$ be a change in x where α is a positive constant and d is a direction vector. If R is the feasible region and a constant $\hat{\alpha} > 0$ exists such that $x + \alpha d \in R$ for all α in the range $0 \leq \alpha \leq \hat{\alpha}$, then d is said to be a feasible direction at point x.

Definition 2.5
(a) Let d be an arbitrary direction vector at point x. The quadratic form $d^T H(x)d$ is said to be *positive definite, positive semidefinite, negative semidefinite, negative definite* if $d^T H(x)d > 0, \geq 0, \leq 0, < 0$, respectively, for all $d \neq 0$ at x. If $d^T H(x)d$ can assume positive as well as negative values, it is said to be indefinite.

(b) If $d^T H(x)d$ is positive definite, positive semidefinite, etc., then matrix $H(x)$ is said to be positive definite, positive semidefinite, etc.

The objective function must satisfy two sets of conditions in order to have a minimum, namely, first- and second-order conditions.

First-order necessary conditions for a minimum

(a) If $f(x) \in C^1$ and x^* is a local minimizer, then $g(x^*)^T d \geq 0$ for every feasible direction d at x^*.
(b) If x^* is located in the interior of $\mathcal{R}$ then $g(x^*) = 0$

Second-order necessary conditions for a minimum

(a) If $f(x) \in C_2$ and x^* is a local minimizer, then for every feasible direction d at x^*.
(i) $g(x^*)Td \geq 0$
(ii) If $g(x^*)^T d = 0$, then $d^T H(x^*)d \geq 0$
(b) If x^* is a local minimizer in the interior of R, then
(i) $g(x^*) = 0$
(ii) $d^T H(x^*)d \geq 0$ for all $d \neq 0$

Second-order sufficient conditions for a minimum

If $f(x) \in C_2$ and x^* is located in the interior of $\mathcal{R}$, then the conditions (a) $g(x^*) = 0$ (b) $H(x^*)$ is positive definite are sufficient for x^* to be a strong local minimizer.

Definition 2.6

A point $\bar{x} \in \mathcal{R}$, where $\mathcal{R}$ is the feasible region, is said to be a saddle point if
(a) $g(\bar{x}) = 0$
(b) point $\bar{x}$ is neither a maximizer nor a minimizer.

Stationary points can be located and classified as follows:
1. Find the points x_i at which $g(x_i) = 0$.
2. Obtain the Hessian $H(x_i)$.
3. Determine the character of $H(x_i)$ for each point x_i.
If $H(x_i)$ is positive (or negative) definite, x_i is a minimizer (or maximizer); if $H(x_i)$ is indefinite, x_i is a saddle point.

Techniques to compute Hessian P.D. , N.D.
Eigenvalues: det $(\lambda I - A) = 0$ Multiplying all eigenvalues is equal to the determinant.
The leading principal minors of a matrix A or its negative $-A$ can be used to establish whether the matrix is positive or negative definite whereas the principal minors of A or $-A$ can be used to establish whether the matrix is positive or negative semidefinite.

Theorem 2.9 Properties of matrices

(a) If **H** is positive semidefinite or positive definite, then det $\mathbf{H} \geq 0$ or > 0
(b) **H** is positive definite if and only if all its leading principal minors are positive, i.e., det $\mathbf{H_i} > 0$ for $i = 1, 2, \cdots, n$.
(c) **H** is positive semidefinite if and only if all its principal minors are nonnegative, i.e., det $(H_i^{(l)}) \geq 0$ for all possible selections of $\{l_1, l_2, \cdots, l_i\}$ for $i = 1, 2, \cdots, n$.
(d) **H** is negative definite if and only if all the leading principal minors of $-$**H** are positive, i.e., $det(-H_i) > 0$ for $i = 1, 2, \cdots, n$.
(e) **H** is negative semidefinite if and only if all the principal minors of -**H** are nonnegative, i.e., det $(-H_i^{(l)}) \geq 0$ for all possible selections of $\{l_1, l_2, \cdots, l_i\}$ for $i = 1, 2, \cdots, n$.
(f) **H** is indefinite if neither (c) nor (e) holds.

Definition 2.7

A set $\mathcal{R}_c \subset E_n$ is said to be convex if for every pair of points $x_1, x_2 \subset R_c$ and for every real number α in the range $0 < \alpha < 1$, the point $x = \alpha x_1 + (1 - \alpha)x_2$ is located in $\mathcal{R}_c$, i.e., $x \in \mathcal{R}_c$.

Definition 2.8

(a) A function $f(x)$ defined over a convex set $\mathcal{R}_c$ is said to be convex if for every pair of points $x_1, x_2 \in \mathcal{R}_c$ and every real number α in the range $0 < \alpha < 1$, the inequality $f[\alpha x_1 + (1 - \alpha)x_2] \leq \alpha f(x_1) + (1 - \alpha)f(x_2)$ holds. If $x_1 \neq x_2$ and $f[\alpha x_1 + (1 - \alpha)x_2] < \alpha f(x_1) + (1 - \alpha)f(x_2)$ then f(x) is said to be strictly convex.
(b) If $\phi(x)$ is defined over a convex set $\mathcal{R}_c$ and f(x) = -$\phi(x)$ is convex, then $\phi(x)$ is said to be concave. If f(x) is strictly convex, $\phi(x)$ is strictly concave.

Property of convex functions relating to the Hessian A function $f(x) \in C^2$is convex over a convex set $\mathcal{R}_c$ if and only if the Hessian H(x) of f(x) is positive semidefinite for $x \in \mathcal{R}_c$.

Theorem 2.15 Relation between local and global minimizers in convex functions

If $f(x)$ is a convex function defined on a convex set $\mathcal{R}_c$, then
(a) the set of points S_c where $f(x)$ is minimum is convex;
(b) any local minimizer of $f(x)$ is a global minimizer.

Ch. 4

Dichotomous Search

Two function evaluations per iteration.
A **unimodal function** on an interval has exactly one point where a maximum or minimum occurs in the interval.
Consider a unimodal function which is known to have a minimum in the interval $[x_L,\ x_U]$. This interval is said to be the range of uncertainty.
In this method, f(x) is evaluated at two points $x_a = x_1 - \epsilon/2$ and $x_b = x_1 + \epsilon/2$ where ϵ is a

small positive number. Then depending on whether $f(x_a) < f(x_b)$ or $f(x_a) > f(x_b)$, range x_L to $x_1 + \epsilon/2$ or $x_1 - \epsilon/2$ to x_U can be selected and if $f(x_a) = f(x_b)$ either will do fine. If we assume that $x_1 - x_L = x_U - x_1$, i.e., $x_1 = (x_L + x_U)/2$, the region of uncertainty is immediately reduced by half. The same procedure can be repeated for the reduced range, that is, f(x) can be evaluated at $x_2 - \epsilon/2$ and $x2 + \epsilon/2$ where x_2 is located at the center of the reduced range, and so on.

Algorithm 4.1 Fibonacci
Computing n $= I_n = \frac{I_1}{F_n}$, function evaluations = n-1
Step 1
Input $x_{L,1}, x_{U,1}$, and n.
Step 2
Compute $F_1, F_2, \cdots, F_n$ using Eq. (4.4).
Step 3
Assign $I_1 = x_{U,1} - x_{L,1}$ and compute

$$I_2 = \frac{F_{n-1}}{F_n} I_1 (\text{see Eq. (4.6)})$$
$$x_{a,1} = x_{U,1} - I_2, \quad x_{b,1} = x_{L,1} + I_2$$
$$f_{a,1} = f(x_{a,1}), \quad f_{b,1} = f(x_{b,1})$$

Set k = 1.
Step 4
Compute I_{k+2} using Eq. (4.6). If $f_{a,k} \geq f_{b,k}$, then update Eqs. (4.7) to (4.12) using

$$x_{L,k+1} = x_{a,k}$$
$$x_{U,k+1} = x_{U,k}$$
$$x_{a,k+1} = x_{b,k}$$
$$x_{b,k+1} = x_{L,k+1} + I_{k+1}$$
$$f_{a,k+1} = f_{b,k}$$
$$f_{b,k+1} = f(x_{b,k+1})$$

using Otherwise, if $f_{a,k} < f_{b,k}$, update information using Eqs. (4.13) to (4.18) using

$$x_{L,k+1} = x_{L,k}$$
$$x_{U,k+1} = x_{b,k}$$
$$x_{a,k+1} = x_{U,k+1} - I_{k+2}$$
$$x_{b,k+1} = x_{a,k}$$
$$f_{a,k+1} = f(x_{a,k+1})$$
$$f_{b,k+1} = f_{a,k}$$

Step 5
If $k = n - 2$ or $x_{a,k+1} > x_{b,k+1}$, output $x^* = x_{a,k+1}$ and $f^* = f(x^*)$, and stop. Otherwise, set $k = k + 1$ and repeat from Step 4. The condition $x_{a,k+1} > x_{b,k+1}$ implies that $x_{a,k+1} \approx x_{b,k+1}$ within the precision of the computer used, as was stated earlier, or that there is an error in the algorithm. It is thus used as an alternative stopping criterion.

Algorithm 4.2 Golden-section search
(function evaluations = k+1) and Golden Ratio: $K = \dfrac{1+\sqrt{5}}{2}$ $\Lambda_{GS} = I_n = \frac{I_1}{K_{n-1}}$ $\quad \Lambda_F = I_n = \frac{I_1}{F_n} \approx \frac{\sqrt{5}}{K^{n+1}} I_1$
$\frac{I_k}{I_{k+1}} = \frac{I_{k+1}}{I_{k+2}} = \frac{I_{k+2}}{I_{k+3}} = \cdots = K$
Step 1
Input $x_{L,1}, x_{U,1}$, and ϵ.
Step 2
Assign $I_1 = x_{U,1} - x_{L,1}, K = 1.618034$ and compute

$$I_2 = I_1/K$$
$$x_{a,1} = x_{U,1} - I_2, \quad x_{b,1} = x_{L,1} + I_2$$
$$f_{a,1} = f(x_{a,1}), \quad f_{b,1} = f(x_{b,1})$$

Set $k = 1$.
Step 3
Compute $I_{k+2} = I_{k+1}/K$
If $f_{a,k} \geq f_{b,k}$, then update $x_{L,k+1}, x_{U,k+1}, x_{a,k+1}, x_{b,k+1}, f_{a,k+1}$, and $f_{b,k+1}$ as

$$x_{L,k+1} = x_{a,k}$$
$$x_{U,k+1} = x_{U,k}$$
$$x_{a,k+1} = x_{b,k}$$
$$x_{b,k+1} = x_{L,k+1} + I_{k+1}$$
$$f_{a,k+1} = f_{b,k}$$
$$f_{b,k+1} = f(x_{b,k+1})$$

Or use using Eqs. (4.7) to (4.12). Otherwise if $f_{a,k} < f_{b,k}$, then update $x_{L,k+1}, x_{U,k+1}, x_{a,k+1}, x_{b,k+1}, f_{a,k+1}$, and $f_{b,k+1}$ as

$$x_{L,k+1} = x_{L,k}$$
$$x_{U,k+1} = x_{b,k}$$
$$x_{a,k+1} = x_{U,k+1} - I_{k+2}$$
$$x_{b,k+1} = x_{a,k}$$
$$f_{a,k+1} = f(x_{a,k+1})$$
$$f_{b,k+1} = f_{a,k}$$

Otherwise, if $f_{a,k} < f_{b,k}$, update information using Eqs. (4.13) to (4.18).
Step 4
If $I_k < \epsilon$ or $x_{a,k+1} > x_{b,k+1}$, then do:
If $f_{a,k+1} > f_{b,k+1}$, compute $x^* = 0.5(x_{b,k+1} + x_{U,k+1})$
If $f_{a,k+1} = f_{b,k+1}$, compute $x^* = 0.5(x_{a,k+1} + x_{b,k+1})$
If $f_{a,k+1} < f_{b,k+1}$, compute $x^* = 0.5(x_{L,k+1} + x_{a,k+1})$
Compute $f^* = f(x^*)$.
Output x^* and f^*, and stop.
Step 5
Set $k = k + 1$ and repeat from Step 3.

Equations 4.4, 4.6, (4.7-4.12) and (4.13-4.18)
$F_k = F_{k-1} + F_{k-2} \quad$ for $k \geq 2$ (4.4)
$I_{k+2} = \frac{F_{n-k-1}}{F_{n-k}} I_{k+1}$(4.6)

If $f_{a,k} > f_{b,k}$, then x^* is in interval $[x_{a,k}, x_{U,k}]$ and so the new bounds of $x^* \rightarrow x_{L,k+1} = x_{a,k}$(4.7) $\quad x_{U,k+1} = x_{U,k}$(4.8) Similarly, the two interior points of the

new interval, namely, $x_{a,k+1}$ and $x_{b,k+1}$ will be $x_{b,k}$ and $x_{L,k+1} + I_{k+2}$, respectively. We can thus assign $x_{a,k+1} = x_{b,k}$ (4.9) $x_{b,k+1} = x_{L,k+1} + I_{k+2}$ (4.10) as illustrated in Fig. 4.5.
The value $f_{b,k}$ is retained as the value of f(x) at $x_{a,k+1}$, and the value of f(x) at $x_{b,k+1}$ is calculated, i.e., $f_{a,k+1} = f_{b,k}$ (4.11) $f_{b,k+1} = f(x_{b,k+1})$ (4.12)
On the other hand, if $f_{a,k} < f_{b,k}$, then x^* is in interval $[x_{L,k}, x_{b,k}]$. In this case, we assign $x_{L,k+1} = x_{L,k}$ (4.13) $x_{U,k+1} = x_{b,k}$ (4.14) $x_{a,k+1} = x_{U,k+1} - I_{k+2}$ (4.15) $x_{b,k+1} = x_{a,k}$ (4.16) $f_{b,k+1} = f_{a,k}$ (4.17) and calculate $f_{a,k+1} = f(x_{a,k+1})$ (4.18)

Algorithm 4.6 Inexact line search
Step 1:
Input x_k, d_k, and compute g_k.
Initialize algorithm parameters ρ, σ, τ, and χ.
Set $\alpha_L = 0$ and $\alpha_U = 10^{99}$.
Step 2:
Compute $f_L = f(x_k + \alpha_L dk)$.
Compute $f'_L = g(x_k + \alpha_L dk)^T dk$.
Step 3:
Estimate α_0.
Step 4:
Compute $f_0 = f(x_k + \alpha_0 dk)$.
Step 5 (Interpolation)
If $f_0 > f_L + \rho(\alpha_0 - \alpha_L) f'_L$, then do:
a. If $\alpha_0 < \alpha_U$, then set $\alpha_U = \alpha_0$.
b. Compute $\breve{\alpha}_0$ using the interpolation formula Eq. (4.57).

$$\breve{\alpha}_0 = \alpha_L + \frac{(\alpha_0 - \alpha_L)^2 f'_L}{2[f_L - f_0 + (\alpha_0 - \alpha_L) f'_L]}$$

c. If $\breve{\alpha}_0 < \alpha_L + \tau(\alpha_U - \alpha_L)$ then set $\breve{\alpha}_0 = \alpha_L + \tau(\alpha_U - \alpha_L)$.
d. If $\breve{\alpha}_0 > \alpha_U - \tau(\alpha_U - \alpha_L)$ then set $\breve{\alpha}_0 = \alpha_U - \tau(\alpha_U - \alpha_L)$.
e. Set $\alpha_0 = \breve{\alpha}_0$ and go to Step 4.
Step 6
Compute $f'_0 = g(x_k + \alpha_0 dk)^T dk$.
Step 7 (Extrapolation)
If $f'_0 < \alpha f'_L$, then do:
a. Compute $\nabla\alpha_0 = (\alpha_0 - \alpha_L) f'_0 / (f'_L - f'_0)$ (see Eq. (4.58)).

$$\breve{\alpha}_0 = \alpha_0 + (\alpha_0 - \alpha_L) f'_0 (f'_L - f'_0) \quad (Eq.(4.58))$$

b. If $\nabla\alpha_0 < \tau(\alpha_0 - \alpha_L)$, then set $\nabla\alpha_0 = \tau(\alpha_0 - \alpha_L)$.
c. If $\nabla\alpha_0 > \chi(\alpha_0 - \alpha_L)$, then set $\nabla\alpha_0 = \chi(\alpha_0 - \alpha_L)$.
d. Compute $\breve{\alpha}_0 = \alpha_0 + \nabla\alpha_0$.
e. Set $\alpha_L = \alpha_0, \alpha_0 = \breve{\alpha}_0, f_L = f_0, f'_L = f'_0$, and go to Step 4.
Step 8
Output α_0 and $f_0 = f(x_k + \alpha_0 dk)$, and stop.

Ch. 5

Standard form: $f(x) = \frac{1}{2} x^T H x + x^T g(x) + C$
Rate of Convergence $\beta = (1 - r^2)/(1 + r^2)$, where r is the smallest eigenvalue divided by the biggest eigenvalue.

$$H^{-1} = \begin{bmatrix} a & c \\ c & b \end{bmatrix}^{-1} = \frac{1}{ab - c^2} \begin{bmatrix} b & -c \\ -c & a \end{bmatrix} \quad ab - c^2 \neq 0$$

$[f(x_k) - f(x^*)] \leq \left(\frac{1-r}{1+r}\right)^2 [f(x_k) - f(x^*)]$
Algorithm 5.1 Steepest-descent algorithm
Step 1
Input x_0 and initialize the tolerance ϵ.
Set $k = 0$.
Step 2
Calculate gradient g_k and set $d_k = -g_k$.
Step 3
Find α_k, the value of α that minimizes $f(x_k + \alpha d_k)$, using a line search (Algorithm Inexact Line Search. 4.6).
Step 4
Set $x_{k+1} = x_k + \alpha_k d_k$ and calculate $f_{k+1} = f(x_{k+1})$.
Step 5
If $||\alpha_k d_k|| < \epsilon$, then do:
Output $x^* = x_{k+1}$ and $f(x^*) = f_{k+1}$, and stop.
Otherwise, set $k = k + 1$ and repeat from Step 2.

Algorithm 5.3 Basic Newton algorithm
Step 1
Input x_0 and initialize the tolerance ϵ
Set $k = 0$.
Step 2
Compute g_k and H_k.
If H_k is not positive definite, force it to become positive definite.
Step 3
Compute H_k^{-1} and $d_k = -H_k^{-1} g_k$
Step 4
Find α_k , the value of α that minimizes $f(x + \alpha d_k)$, using a line search.
Step 5
Set $x_{k+1} = x_k + \alpha_k d_k$
Compute $f_{k+1} = f(x_{k+1})$.
Step 6
If $||\alpha_k d_k|| < \epsilon$, then do:
Output $x^* = x_{k+1}$ and $f(x^*) = f(x_{k+1})$, and stop
Otherwise, set $k = k + 1$ and repeat from Step 2.
Algorithm 5.5 Gauss — Newton Algorithm
$f = \begin{bmatrix} f_1(x) & f_2(x) & \cdots f_m(x) \end{bmatrix}^T$, J = Jacobian
$F(x) = \sum_{p=1}^{m} f_p(x)^2 = f^T f$

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Step 1
Input x_0 and initialize the tolerance ϵ.
Set $k = 0$.
Step 2
Compute $f_{pk} = f_p(x_k)$ for $p = 1, 2, \cdots, m$ and F_k.
Step 3

Compute $J_k, g_k = 2J_k^T f_k$, and $H_k = 2J_k^T J_k$.
Step 4
$d_k = -H_k^{-1} g_k$
Step 5
Find α_k, the value of α that minimizes $f(x_k + \alpha d_k)$.
Step 6
Set $x_{k+1} = x_k + \alpha_k d_k$.
Compute $f_{p,k+1}$ for $p = 1, 2, \cdots, m$ and F_{k+1}.
Step 7
If $|F_{k+1} - F_k| < \epsilon$, then do:
Output $x^* = x_{k+1}$, $f_{p,k+1}(x^*)$ for $p = 1, 2, \cdots, m$, and F_{k+1}.
Stop.
Otherwise, set $k = k + 1$ and repeat from Step 3.

Ch. 7

Problems with Rank-one Method

1. positive definite S_k may not yield positive definite S_{k+1}
2. denominator in correction formula may approach zero

THE DFP and BFGS are implementing of the basic algorithms Quasi Newton (7.2) with changes to the updating function. $d_k = -S_k g_k$ and $f(x_k + \alpha d_k) \rightarrow$ $\alpha_k = \frac{g_k^T S_k g_k}{g_k^T S_k H S_k g_k}$

Convergence equation: $\beta = \left(\frac{1-r}{1+r}\right)^2 f(x_{k+1}) - f(x^*) \leq \left(\frac{1-r}{1+r}\right)^2 [f(x_k) - f(x^*)]$

BFGS and then DFP properties
For convex quadratic functions (BFGS)

- — $Sk + 1$ becomes identical to H^{-1} for $k = n - 1$.
- — Directions $\delta_0, \delta_1, \cdots, \delta_{n-1}$ form a conjugate set.
- — S_{k+1} is positive definite if S_k is positive definite.
- — $\delta_k^T \gamma_k = \delta_k^T g_{k+1} - \delta_T^k g_k > 0$ applies.

For DFP (from textbook)

1. If S_k is PD, then the matrix S_{k+1} generated by DFP is also PD.
2. Directions $\delta_0, \delta_1, \cdots, \delta_{n-1}$ form a conjugate set.

Algorithm 7.2 adjusted for DFP/BFGS
Step 1
Input x_0 and initialize the tolerance ϵ.
Set $k = 0$ and $S_0 = I_n$.
Compute g_0.
Step 2
Set $d_k = -S_k g_k$
Find α_k, the value of α that minimizes $f(x_k + \alpha d_k)$, using a line search
Set $\delta_k = \alpha_k d_k$ and $x_{k+1} = x_k + \delta_k$
Step 3
If $||\delta_k|| < \epsilon$, output $x^* = x_{k+1}$ and $f(x^*) = f(x_{k+1})$, and stop
Step 4
Compute g_{k+1} and set $\gamma_k = g_{k+1} - g_k$
Compute S_{k+1} using appropriate formula.

Basic/ Rank One: $S_{k+1} = S_k + \frac{(\delta_k - S_k\gamma_k)(\delta_k - S_k\gamma}{\gamma_k^T(\delta_k - S_k\gamma_k)}$

DFP: $S_{k+1} = S_k + \frac{\delta_k\delta_k^T}{\delta_k^T\gamma_k} - \frac{S_k\gamma_k\gamma_k^T S_k}{\gamma_k^T S_k \gamma_k}$

BFGS: $S_{k+1} = S_k + \left(1 + \frac{\gamma_k^T S_k \gamma_k}{\gamma_k^T \delta_k}\right)\frac{\delta_k\delta_k^T}{\gamma_k^T\delta_k} - \frac{(\delta_k\gamma_k^T S_k}{\gamma_k}$

Set $k = k + 1$ and repeat from Step 2.

- — remember that $\left(1 + \frac{\gamma_k^T S_k \gamma_k}{\gamma_k^T \delta_k}\right)$ is a single number
- — $\delta_k \delta_k^T$ is a matrix.
- — Focus on minimization, $\max[f(x)] = -\min[-f(x)]$
- — Hessian is positive semidefinite for concave functions.

Chapter 6

ELEC 360 CheatSheet

Some keywords used can be found in the glossary section including:

- Partial Fractions, open-loop control system, Single Input Single Output, Multiple Input Multiple Output, and linear time-invariant
- Control System, closed-loop control system, DC Motors, and Op Amps

LAPLACE TRANSFORMS

Final Value Theorem

In Control Engineering, the Final Value Theorem is used most frequently to determine the steady-state value of a system. The real part of the poles of the function must be < 0.

$$\lim_{t\to\infty} x(t) = \lim_{t\to 0} sX(s) \tag{6.1}$$

Initial Value Theorem

$$\lim_{t\to 0} x(t) = \lim_{t\to\infty} sX(s) \tag{6.2}$$

SOLUTION OF LINEAR DIFFERENTIAL EQUATION

$$A\ddot{y} + B\dot{y} + Cy = u\left(t\right)$$

$$\text{with initial conditions } \dot{y}\left(0\right) \ \ and\ y(0)$$

$$\text{is solved by constructing the equation}$$

$$A\left[s^2Y\left(s\right) - sy\left(0\right) - \dot{y}\left(0\right)\right] + B\left[\mathrm{s}Y\left(s\right) - y\left(0\right)\right] + CY\left(s\right)$$

$$that\ is\ to\ say:$$

$$\mathcal{L}\left\{\ddot{y}\right\} = s^2Y\left(s\right) - sy\left(0\right) - \dot{y}(0)$$

and

$$\mathcal{L}\left\{\dot{y}\right\} = sY\left(s\right) - y\left(0\right)$$

STATESPACE REPRESENTATIONS

To generate from a differential equation:

Use the **closed loop transfer function**

$$\frac{Y\left(s\right)}{U\left(s\right)} = \frac{s + A}{s^3 + \mathrm{B}s^2 + s + A}$$

Separate and take the inverse Laplace transform

$$s^3Y\left(s\right) + \mathrm{B}s^2Y\left(s\right) + sY\left(s\right) + AY\left(s\right) = sU\left(s\right) + AU\left(s\right)$$

$$= \dddot{y} + B\ddot{y} + \dot{y} + Ay = \dot{u} + Au$$

Then define state variables:

$x_1 = y \; \dot{x_1} = \dot{y} = x_2$

$x_2 = \dot{y} \; \dot{x_2} = \ddot{y} = x_3$

$x_3 = \ddot{y} \; \dot{x_3} = \dddot{y} = \dot{u} + Au - B\ddot{y} - \dot{y} - Ay$

Then construct the state space matrix

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -A & -1 & -B \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} u$$

and

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \beta_0 u$$

β values can be calculated as follows:

$$\beta_0 = b_0$$

$$\beta_1 = b_1 - a_1\beta_0$$

$$\beta_2 = b_2 - a_1\beta_1 - a_2\beta_0$$

$$\beta_3 = b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0$$

Where the values for a_x and $b_x$come from:

$$\dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3 y = b_0\dddot{u} + b_1\ddot{u} + b_2\dot{u} + b_3 u$$

$$\ddot{x} = (F - c\dot{x_1} - kx)/m$$

where x=position,$\dot{x}$=speed/velocity,$\ddot{x}$=acceleration c = damping constant, m = mass, F = force k = spring constant,

To perform inverse (find transfer function from states-pace model)

$$G(s) = d + c(sI - A)^{-1}b$$

where

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} u$$

And

$$y = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [d]u$$

The inverse of a square 2x2 matrix is found by:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The inverse of a square 3x3 matrix is found by:

For a 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

the matrix inverse is:

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} & \begin{bmatrix} a_{13} & a_{12} \\ a_{33} & a_{32} \end{bmatrix} & \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix} \\ \begin{bmatrix} a_{23} & a_{21} \\ a_{33} & a_{31} \end{bmatrix} & \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} & \begin{bmatrix} a_{13} & a_{11} \\ a_{23} & a_{21} \end{bmatrix} \\ \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} & \begin{bmatrix} a_{12} & a_{11} \\ a_{32} & a_{31} \end{bmatrix} & \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{bmatrix}$$

Solving a three order polynomial, without a fancy calculator

$$x = \left[q + \left[q^2 + (r - p^2)^3\right]^{1/2}\right]^{1/3}$$
$$\left[q - \left[q^2 + (r - p^2)^3\right]^{1/2}\right]^{1/3} + p$$

Where

$$p = \frac{-b}{3a}, \quad q = p^3 + \frac{bc - 3ad}{3a^2}, \quad r = \frac{c}{3a}$$

SECOND ORDER SYSTEMS

$$G(s) = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$K = \omega_n^2; \quad T = 2\zeta\omega_n = 2\sigma; \quad \zeta = \frac{T}{2\sqrt{K}}; \quad \omega_d = \omega_n\sqrt{1}$$

$$\zeta = damping\ ratio; \sigma = real\ part\ of\ root;$$

$$\omega_d = damped\ natural\ frequency$$

$$omega_n = undamped\ natural\ frequency$$

$$Undamped: \ \zeta = 0;$$

$$Critically\ Damped: \zeta = 1$$

$$Over\ Damped:\ \zeta > 1$$

Imaginary axis:

Frequency of oscillations

Real axis:

Decay time

UNIT STEP RESPONSE OF A 2$^{\mathrm{ND}}$ ORDER UNDAMPED SYSTEM

t_d = delay time - to reach 50% of $c(\infty)$for the first time. t_r = rise time - time to reach 100 % of $c(\infty)$ for first time . t_p = peak time - time to reach first peak. t_s = settling time - time to reach & stay within 2% or 5% M_p = maximum overshoot.

$$t_r = \frac{1}{\omega_d}\left(-\frac{\omega_d}{\sigma}\right); \quad t_p = \frac{\pi}{\omega_d}\backslash n$$

$$M_p = e^{-\frac{\zeta\omega_n\pi}{\omega_d}} = e^{-\frac{\eta\pi}{\sqrt{1-\zeta^2}}} = e^{-\frac{\sigma\pi}{\omega_d}}$$

$$t_s = \frac{4}{\sigma} = \frac{4}{\zeta\omega_n}\ (2\%\ band)$$

$$t_s = \frac{3}{\sigma} = \frac{3}{\zeta\omega_n}\ (5\%\ band)$$

Dominant poles are the ones closest to the imaginary axis

ROUTH-HURWITZ STABILITY TEST

$$a_0 s^n + a_1 s^{n-1} + \cdots + an - 1s + a_n = 0$$

row n	a_0	a_2	a_4	$\cdots$
row n-1	a_1	a_3	a_5	$\cdots$
row n-2	b_1	b_2	b_3	$\cdots$
row n-3	c_1	c_2	c_3	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
row 2	$*$	$*$		$\cdots$
row 1	$*$			$\cdots$
row 0	$*$			$\cdots$

$$b_1 = -\frac{det\begin{bmatrix} a_0 & a_2 \\ a_1 & a_3 \end{bmatrix}}{a_1} = \frac{a_1a_2 - a_0a_3}{a_1}$$

$$b_2 = -\frac{det\begin{bmatrix} a_0 & a_4 \\ a_1 & a_5 \end{bmatrix}}{a_1} = \frac{a_1a_4 - a_0a_5}{a_1}$$

$$b_3 = -\frac{det\begin{bmatrix} a_0 & a_6 \\ a_1 & a_7 \end{bmatrix}}{a_1} = \frac{a_1a_6 - a_0a_7}{a_1}$$

$$c_1 = -\frac{det\begin{bmatrix} a_1 & a_3 \\ b_1 & b_2 \end{bmatrix}}{b_1} = \frac{b_1a_3 - a_1b_2}{b_1}$$

$$c_2 = -\frac{det\begin{bmatrix} a_1 & a_5 \\ b_1 & b_3 \end{bmatrix}}{b_1} = \frac{b_1a_5 - a_1b_3}{b_1}$$

$$c_3 = -\frac{det\begin{bmatrix} a_1 & a_7 \\ b_1 & b_4 \end{bmatrix}}{b_1} = \frac{b_1a_7 - a_1b_4}{b_1}$$

STEADY STATE ERROR ANALYSIS

$$K_p = G(s)\,H(s)$$

$$K_v = \mathrm{sG}(s); \quad K_v = s\,(\mathrm{KG}(s))$$

$$K_a = s^2G(s); \quad K_a = s^2\,(\mathrm{KG}(s))$$

The type of system is determined by the number of poles at the origin. For example:

ROOT LOCUS

Root Locus presents the poles of the closed loop system when the gain K changes from zero to infinity.

Construction of the Root Locus

Open loop transfer function

$$\mathrm{KH}(s)\,G(s) = K\frac{B(s)}{A(s)}$$

m: the order of the **open-loop** numerator polynomial

n: the order of the **open-loop** denominator polynomial

Rule 1: number of branches equals the number of poles of the open-loop transfer function

Rule 2: If the total number of poles and zeros of the open-loop system to the right of the s-point on the real axis is odd, then this point lies on the locus.

Rule 3: The locus starting point (K=0) are at the open-loop poles and the locus ending points (K=∞) are at the open loop zeros and n-m branches terminate at infinity.

Rule 4: Slope of asymptotes of root locus as 's' approaches infinity

Rule 5: Abscissa of the intersection between asymptotes of root locus and real-axis.

Rule 6: Break-away and break-in points. From the characteristic equation

$$f(s) = A(s) + KB(s) = 0 \quad and \quad K = -\frac{A(s)}{B(s)}$$

The break-away and break-in points can be found from

$$\frac{\mathrm{dK}}{\mathrm{ds}} = -\frac{A'(s)\,B(s) - A(s)\,B'(s)}{B^2(s)} = 0$$

Rule 7: Angle of departure from complex poles or zeros. Subtract from 180° the sum of all angles from all other zeros and poles of the open-loop system to the complex pole (or zero) with appropriate signs.

Rule 8: Imaginary-axis crossing points. Use Ruth-Hurwitz table to find value of K where system becomes unstable.

BODE DIAGRAMS

1. Gain Factor K:

Horizontal straight line at magnitude: $20\log(K)$dB

Phase is zero.

2. Integral or derivative factors

$(\mathbf{j}\omega)^{\pm\mathbf{1}}$

$$(\mathrm{j}\omega)^{-1} \to 20\log\left|\frac{1}{\mathrm{j}\omega}\right| = -20\log\omega$$

Magnitude: strait line with slope -20 dB/decade

Phase: -90°

$$(\mathrm{j}\omega) \to 20\log|\mathrm{j}\omega| = 20\log\omega$$

Magnitude: straight line with slope 20dB/decade

Phase: +90°

3. First Order Factors

$(\mathbf{1} + \mathbf{j}\omega\mathbf{T})^{\pm\mathbf{1}}$

$$(1 + j\omega T)^{-1} \to 20\log\left|\frac{1}{1 + j\omega T}\right| = -20\log\sqrt{1 + \omega^2 T^2}\ [dB] \tag{6.3}$$

Approximation for Magnitude:

$$For\ \omega\ between\ 0\ and\ \frac{1}{T} \to 0dB$$

$$For\ \omega \gg \frac{1}{T} \to -20dB/decade$$

Phase:

$$\omega = 0 \to \varphi = 0$$

$$\omega = \frac{1}{T} \to \varphi = -45$$

$$\omega = \infty \to \varphi = -90$$

$(\mathbf{1} + \mathbf{j}\omega\mathbf{T})^{+\mathbf{1}}$

4. Quadratic Factors

$$G(\mathrm{j}\omega) = \frac{1}{1 + 2\zeta\left(\frac{\omega}{\omega_n}\right) + \left(\frac{\mathrm{j}\omega}{\omega_n}\right)^2} \quad ; \quad 0 < \zeta < 1$$

Approximation for magnitude:

$$\omega \ll \omega_n \to 0dB$$

$$\omega \gg \omega_n \to -20\log\left(\frac{\omega^2}{\omega_n^2}\right) = -40\log\left(\frac{\omega}{\omega_n}\right)\mathrm{dB}$$

$$Phase:$$

$$\omega = 0 \to \varphi = 0$$

$$\frac{\omega}{\omega_n} = 1 \to \varphi = -90\backslash$$

$$\omega = \infty \to \varphi = -180$$

$$Resonant\ Frequency:$$

$$\omega_r = \omega_n\sqrt{1 - 2\zeta^2}\ \ for\ 0 < \zeta < 0.707$$

$$Resonant\ Peak\ Value:$$

$$M_r = |G(\mathrm{j}\omega)|_{\max} = \frac{1}{2\zeta\sqrt{1 - \zeta^2}}\ for\ 0 < \zeta < 0.707$$

Consider

$$G_1(s) = \frac{1}{1 + Ts}\ G_2(s) = \frac{1}{1 - Ts}\ G_3(s) = \frac{1}{Ts - 1}$$

Then...

$$|G_1(j\omega)| = |G_2(j\omega)| = |G_3(j\omega)|$$

And...

$$\angle G_2\,(\mathrm{j}\omega) = -\angle G_1\,(\mathrm{j}\omega)\; and$$
$$\angle G_3\,(\mathrm{j}\omega) = 180 - \angle G_1\,(\mathrm{j}\omega)$$

$$+90\; and\; \angle G_3\,(\mathrm{j}\omega)\;\; goes\;\; from - 180\; to - 90$$

Generate based on the Bode Plot.

The Nyquist Stability Criterion: relates the stability of the closed loop system to the frequency response of the open loop system.

$$Z = N + P$$

Z: Number of zeros of $(1+H(S)G(s))$ in the right half plane = number of unstable poles of the closed-loop system.

N: Number of clockwise encirclements of the point $-1 + j0$.

P: Number of poles of $G(s)H(s)$ in the right half plane.

If the plot makes a counter-clockwise encirclement of the $-1 + j0$ point then N becomes -1.

If Z = 0 the closed loop system is stable. If Z > 0 the closed loop system has Z unstable poles. If Z < 0 a mistake has been made and the calculations need to be rechecked.

PHASE AND GAIN MARGINS

A measure for relative stability of the closed-loop system is how close $G(j\omega)$, the frequency response of the open-loop system, comes to the point $-1 + j0$. This is represented by the phase and gain margins.

Phase Margin: The amount of additional phase lag at the Gain Crossover Frequency ω_0 required to bring the system to the verge of instability.

Gain crossover frequency: ω_0 for which $|G\,(j\omega_0)| = 1$

Phase margin: $\gamma = 180 + \angle G\,(j\omega_0) = 180 + \phi$

Gain Margin: The reciprocal of the magnitude $|G(j\omega_1)|$ at the Phase crossover frequency ω_1 required to bring the system to the verge of instability.

Phase crossover frequency: $\omega_1\; where\; \angle G\,(j\omega_1) = -180$

Gain margin:

$$K_g = \frac{1}{|G(j\omega_1)|}$$

$$K_g = -20\log|G\,(j\omega_1)|$$

$$K_g\; in\; dB > 0 = stable$$

for minimum phase systems.

$$K_g in\; dB < 0$$

= unstable for minimum phase systems.

Minimum phase systems: all poles and zeros are in the left half plane.

If the open-loop system is minimum phase and has both phase and gain margins positive then the closed-loop system is stable.

For good relative stability both margins are required to be positive.

Good values for minimum phase system are:

Phase Margin: 30°-60°

Gain Margin: above 6dB

Chapter 7

ELEC 370 Cheatsheet

MAGNETIC CIRCUITS

$$\mu = \ \mu_o\mu_r \text{ where } \mu_o = 4\pi \times 10^{-7}\ [\frac{H}{m}]$$

Field Intensity: $H = \ \frac{\mathrm{Ni}}{l_c}\ [\frac{\mathrm{At}}{m}]$

Flux Density: $B = \ \mu H\ [\frac{\mathrm{Wb}}{m^2}]$

Reluctance: $\mathcal{R} = \frac{l_c}{\mu A}\ \left[\frac{\mathrm{At}}{\mathrm{Wb}}\right]$

Flux: $\phi = B \times A = \ \frac{\mathcal{F}}{\mathcal{R}}\ [Wb]$

Flux: $\phi = \ \frac{\mu\mathrm{NiA}}{l_c}\ [Wb]$

Induced EMF: $\varepsilon = N \times \frac{\mathrm{d}\phi}{\mathrm{dt}}\ \ [V]$

Flux linkage: $\lambda = N\phi\ \ [\mathrm{Wbt}]$

Eddy current loss: $P_e = \ K_e\ f^2B_m^2\ \left[\frac{\mathrm{Wb}}{\mathrm{kg}}\right]$

$$B_m = \ B_{\max} \text{ of core}$$

Inductance: $L = \ \frac{\mathrm{N}\phi}{I} = \ \frac{N^2}{\mathcal{R}} = \ \frac{N^2\mu\mathrm{A}}{l_c}\ [H]$

Leakage inductance: $L_l = \ \frac{N\phi_l}{I}\ [H]$

Energy: $\varepsilon = \ \frac{Li^2}{2} = \ \frac{\mathcal{R}\phi^2}{2}\ [J]$

TRANSFORMERS

Turns ratio: $\frac{V_1}{V_2} = \frac{i_1}{i_2} = \frac{N_1}{N_2} = k$

Load impedance as seen from primary:

$Z'_L = (\frac{N_1}{N_2})^2 \times Z_L\ [\Omega]$

Peak flux: ϕ_m

Peak voltage: $V_{\mathrm{pk}} = \varepsilon_{11p} = N_1\theta_m\omega \quad [V]$

RMS voltage: $V_{\mathrm{RMS}} = \frac{V_{\mathrm{peak}}}{\sqrt{2}} \quad [V]$

$$L_{l1} = \frac{N_1\phi_{l1}}{i_1} \ \ ; \ \ L_{l2} = \frac{N_2\phi_{l2}}{i_2}$$

$$i'_m = i_1 - \frac{N_2}{N_1}i_2 \ \ ; \ \ i''_m = i_2 - \frac{N_1}{N_2}i_1$$

$$L'_m = \frac{N_1\varnothing_m}{i'_m} \ \ ; \ \ L''_m = \frac{N_1\varnothing_m}{i''_m}$$

$$\frac{L'_m}{L''_m} = \left(\frac{N_1}{N_2}\right)^2$$

$$X'_{l2} = \left(\frac{N_1}{N_2}\right)^2 X_{l2} \ \ ; \ \ R'_2 = \left(\frac{N_1}{N_2}\right)^2 R_2$$

$$X''_{l1} = \left(\frac{N_2}{N_1}\right)^2 X_{l1} \quad ; \quad R''_1 = \left(\frac{N_2}{N_1}\right)^2 R_1$$

$$X''_m = \left(\frac{N_2}{N_1}\right)^2 X'_m \quad ; \quad R''_c = \left(\frac{N_2}{N_1}\right)^2 R'_c$$

$$\overline{\varepsilon''_1} = \frac{N_2}{N_1}\overline{\varepsilon_1} \quad ; \quad \overline{V''_1} = \frac{N_2}{N_1}\overline{V_1}$$

$$\overline{I''_1} = \frac{N_1}{N_2}\overline{I_1} \quad ; \quad \overline{I''_m} = \frac{N_1}{N_2}\overline{I_m}$$

$$P_c = \frac{V_1^2}{R'_c} \quad ; \quad P_w = I_1^2 R_1 + I_2^2 R_2$$

$$\eta = \left(\frac{V_2 I_2 \cos\theta_2}{V_2 I_2 \cos\theta_2 + P_c + P_w}\right) * 100\%$$

Regulation: $\frac{|\overline{V_{\mathrm{NL}}}| - |\overline{V_{\mathrm{FL}}}|}{|\overline{V_{\mathrm{FL}}}|}$

OPEN CIRCUIT TEST

$$\cos\theta_{\mathrm{OC}} = \frac{P_{\mathrm{OC}}}{V_{\mathrm{OC}} I_{\mathrm{OC}}}$$

$$R'_c = \frac{V_{\mathrm{OC}}}{I_{\mathrm{OC}}\cos\theta_{\mathrm{OC}}} \quad ; \quad X'_m = \frac{V_{\mathrm{OC}}}{I_{\mathrm{OC}}\sin\theta_{\mathrm{OC}}}$$

SHORT CIRCUIT TEST

$$|Z_{\mathrm{SC}}| = \frac{V_{\mathrm{SC}}}{I_{\mathrm{SC}}} = \sqrt{(R_1 + k^2 R_2)^2 + (X_{l1} + k^2 X_{l2})^2}$$

$$P_{\mathrm{SC}} = I_{\mathrm{SC}}^2 (R_1 + k^2 R_2)$$

$$R'_{\mathrm{eq}} = R_1 + R'_2 = R_1 + k^2 R_2 = \frac{P_{\mathrm{SC}}}{I_{\mathrm{SC}}^2}$$

$$X'_{\mathrm{leq}} = \sqrt{|Z_{\mathrm{SC}}|^2 - R'^2_{\mathrm{eq}}} = X_{l1} + X'_{l2} = X_{l1} + k^2 X_{l2}$$

Assume: $R_1 = R'_2$ & $X_{l1} = X'_{l2}$

PER-UNIT VALUES

$$I_{\mathrm{BASE}} = \frac{S_{\mathrm{BASE}}}{V_{\mathrm{BASE}}} \quad [\textit{per winding}]$$

$$R_{\mathrm{BASE}} = X_{\mathrm{BASE}} = Z_{\mathrm{BASE}} = \frac{V_{\mathrm{BASE}}}{I_{\mathrm{BASE}}}$$

$$P_{\mathrm{BASE}} = Q_{\mathrm{BASE}} = |S_{\mathrm{BASE}}| = V_{\mathrm{BASE}} I_{\mathrm{BASE}}$$

$$P.U. = \frac{\text{actual amount}}{\text{base amount}}$$

AUTO-TRANSFORMER

$$I_x = I_2 - I_1 \quad ; \quad \mathrm{CU}_{\mathrm{RATIO}} = 1 - \frac{N_2}{N_1}$$

DC-MACHINES

Lossless Machine: $vi = T_e \omega_m \quad ; \quad F = i(l \times B)$

DC GENERATORS

LAP WINDING

$$a = P\ ;\quad \varepsilon_a = \frac{\phi Z\Omega}{2\pi}$$

WAVE WINDING

$$a = 2\ ;\quad \varepsilon_a = \frac{\phi Z\Omega P}{4\pi}$$

Average Flux density per pole: $B_a = \frac{\phi \mathrm{P}}{\pi l_a D}$

$$\varepsilon_{a1} = \phi P N\ \text{(per coil)}\ ;\quad \varepsilon_a = \frac{\phi Z\Omega P}{2\pi a}$$

$$K = \frac{\mathrm{ZP}}{2\pi a}$$

$$T_d = K\phi I_a$$

Air gap power: $P_{\text{ag}} = \varepsilon_a I_a = T_d\Omega \quad [W]$

Where: $\phi = \frac{\text{flux}}{\text{pole}}$

$$Z = total\ armature\ conductors$$

$$= \#slots \frac{\#conductors}{\text{slot}}$$

$$P = \#generator\ poles\ (always\ even)$$

$$\Omega = \frac{2\pi N}{60} = angular\ velocity\ \left[\frac{\text{rads}}{s}\right]$$

$$N = armature\ speed\ [rpm]$$

$$a = \#parallel\ paths\ in\ armature\ winding$$

SEPERATLY EXCITED

$$v_f = R_f i_f + L_f \frac{di_f}{\mathrm{dt}}\ ;\quad V_f = I_f R_f$$

$$v_t = K\phi\Omega - L_a \frac{di_a}{\mathrm{dt}} - i_a R_a$$

$$V_t = K\phi\Omega - i_a R_a$$

$$T_{\text{shaft}} = K\phi i_a + J\frac{d\Omega}{\mathrm{dt}} + T_{\text{loss}}$$

$$T_{\text{shaft}} = K\phi i_a + T_{\text{loss}}$$

$$I_f = \frac{V_{\text{fs}}}{R_e + R_f}\ ;\quad I_l = I_a$$

$$I_l = \frac{V_t}{R_L} = \frac{\varepsilon_a}{R_a + R_L} = \frac{\mathrm{K}\phi\Omega}{R_a + R_L}$$

$$\varepsilon_a = V_t + I_a R_a = K\phi\Omega$$

$$\mathrm{K}\phi \text{ depends on } I_f$$

$$V_t = \varepsilon_a - I_a R_a = \varepsilon_a - I_L R_a$$

SHUNT

No Load:

$$\varepsilon_a = I_f(R_a + R_e + R_f)$$

$$V_t = I_f\left(R_e + R_f\right) \cong \varepsilon_a = f(I_f)$$

Loaded:

$$V_t = I_L R_L = \varepsilon_a - I_a R_a$$

$$I_a = I_L + I_f$$

$$I_L = \frac{P_L}{I_L} \quad ; \quad I_f = \frac{V_t}{R_f}$$

Voltage will not build if:

1. There is no residual magnetism present
2. Field connected opposes permanent magnetism
3. $R_f > R_{\text{critical}}$

$$\varepsilon_a = V_t + I_a R_a = I_f\left(R_e + R_f\right) + I_a R_a$$

SERIES

$$I_a = I_L = I_f$$

$$V_t = \varepsilon_a - I_L(R_a + R_s)$$

EFFICIENCY

$$P_{\text{in}} = T_{\text{applied}}\Omega + V_f I_f$$

$$P_{\text{out}} = V_t I_L$$

$$\eta = \frac{P_{\text{out}}}{P_{\text{in}}} = \frac{V_t I_L}{T_{\text{applied}}\Omega + V_f I_f}$$

LOSSES

$$P_{STRAY\_LOSS}$$

$$P_{MECHANICAL\_LOSS} = winding\ \&\ friction$$

$$P_{MAGNETIG\_LOSS} = core\ losses$$

$$P_{ELECTRICAL\_LOSS} = I_a^2 R_a + I_f^2 R_f$$

$$P_{ARMATURE\_LOSS} = \varepsilon_a I_a = T_d \Omega$$

$$P_{ARMATURE\_CU\_LOSS} = P_a = I_a^2 R_a$$

$$P_{SHUNT\_FIELD\_CU\_LOSS} = P_f = V_t I_f$$

$$P_{BRUSH\_LOSS} = V_{\text{BD}} I_a$$

$$P_{\text{ROT}\_LOSS} = P_{\text{CORE}} + P_{\text{MECH}} = E_a I_a = (V_t - I_a R_a) I_a$$

$$P_{\text{ROT}} - \text{calculated at no load conditions}$$

DC MOTORS

SEPARATELY EXCITED

$$v_f = R_f i_f + L_f \frac{di_f}{\mathrm{dt}}\ ;\ V_f = I_f R_f$$

$$v_t = K\phi\Omega + L_a \frac{di_a}{\mathrm{dt}} + i_a R_a$$

$$V_t = \varepsilon_a + I_a R_a = K\phi\Omega + I_a R_a$$

$$T_{\mathrm{LOAD}} = K\phi i_a - J\frac{d\Omega}{\mathrm{dt}} - T_{\mathrm{LOSS}} = K\phi i_a - T_{\mathrm{LOSS}}$$

$$\varepsilon_a = K\phi\Omega = f\left(I_f\right)|_{\Omega=\Omega_{\mathrm{RATED}}}$$

$$T = T_{\mathrm{internal}} = K\phi I_a$$

SPEED CONTROL

$$\Omega = \frac{V_t - I_a R_a}{\mathrm{K}\phi} = \frac{V_t}{\mathrm{K}\phi} - \frac{TR_a}{(\mathrm{K}\phi)^2}$$

$$\Omega = \frac{V_t}{\mathrm{K}\phi} - \frac{T(R_a + R_d)}{(\mathrm{K}\phi)^2}$$

$$T = T_{\mathrm{LOSS}} + T_{\mathrm{LOAD}}$$

SHUNT

$$V_t = I_f\left(R_e + R_f\right) = \varepsilon_a + I_a R_a = K\phi\Omega + i_a R_a$$

$$T_{\mathrm{LOAD}} = K\phi I_a - T_{\mathrm{LOSS}}$$

$$\varepsilon_a = K\phi\Omega = f\left(I_f\right)|_{\Omega=\Omega_R}$$

$$T = T_{\mathrm{INTERNAL}} = K\phi I_a$$

$$\Omega = \frac{V_t}{\mathrm{K}\phi} - \frac{TR_a}{(\mathrm{K}\phi)^2}$$

$$\Omega = \frac{V_t}{\mathrm{K}\phi} - \frac{T(R_a + R_d)}{(\mathrm{K}\phi)^2}$$

BLOCKED ROTOR

$$R_a = \frac{V_a}{I_a}\ \ ;\ \ R_f = \frac{V_f}{I_f}$$

SERIES

$$T = K\phi I_a = \mathrm{K}\phi I_L$$

$$\varepsilon_a = K\phi\Omega = f(I_f)|_{\Omega=\Omega_R}$$

$$V_t = \varepsilon_a + I_a(R_a + R_s) = K\phi\Omega + I_a(R_a + R_s)$$

$$T_{\mathrm{LOAD}} = K\phi I_L - T_{\mathrm{LOSS}}$$

$$T = T_{\mathrm{LOSS}} + T_{\mathrm{LOAD}}$$

$$\Omega = \frac{V_t}{\mathrm{K}\phi} - \frac{T(R_a + R_s)}{(\mathrm{K}\phi)^2}$$

$$for\ liner\ range\ of\ mag\ curve:\ I_L < I_{a(RATED)}$$

$$K\varnothing = K_f I_L^2$$

$$K\phi = \sqrt{K_f T}$$

$$T = K_f I_L^2$$

$$T_{\text{DEVELOPED}} = \frac{P_{\text{DEVELOPED}}}{\Omega}$$

INDUCTION MOTORS

$$P = 2n\ ; where\ P = \#poles,\ n = \#stator\ slots\ or\ poles\ or\ phases$$

Synchronous Speed: $N_s = \frac{120f}{P}$

$$\omega_s = \frac{P}{2} \times \frac{2\pi N_s}{60}$$

$$\%slip = \left(\frac{N_s - N}{N_s}\right) \times 100\%$$

Rotor speed: $N = (1-s)N_s$

$$N_r = sN_s$$

$$X_{2BR} = \omega_s L_s\ ; blocked\ rotor\ leakage\ L$$

$$X_2 = sX_{2BR}$$

$$\varepsilon_{2BR} = 4.44 f \phi_m N_t\ ; where:\ N_t = \#rotor\ tuns,\ \phi_m = \max \text{flux}$$

$$\varepsilon_2 = s\varepsilon_{2BR}$$

$$I_2 = \frac{\varepsilon_{2BR}}{\frac{R_2}{s} + jX_{2BR}} = \frac{s\varepsilon_{2BR}}{R_2 + jsX_{2BR}}$$

$$\frac{R_2}{s} = R_2 + \frac{R_2}{s}(1-s)$$

$$R'_2 = \left(\frac{N_1}{N_2}\right)^2 R'_2\ \ ;\ \ X'_{2BR} = \left(\frac{N_1}{N_2}\right)^2 X_{2BR}$$

EQUIVALENT CIRCUIT PARAMETERS

$$P_{OC-3\phi} = P_{\text{total}} = W_1 + W_2$$

$$P_{OC-1\phi} = \frac{P_{OC-3\phi}}{3}$$

$$P_{OC-1\phi} = \frac{P_{OC-3\phi} - 3I_{\text{OC}}^2 R_1 - P_{\text{mech}}}{3}$$

$$P_{NL-1\phi} = P_{OC-1\phi -} I_{\text{OC}}^2 R_1 - P_{mech-loss-1\phi}$$

$$R_c = \frac{V_{\text{OC}}^2}{P_{NL-1\phi}}\ \ ;\ \ V_{OC-1\phi} = \frac{V_{\text{NL}}}{\sqrt{3}}$$

Where: $V_{\text{OC}} = V\ line\ to\ neutral = V_{\text{RATED}}$

$$\cos\theta_{\text{OC}} = \frac{P_{NL-1\phi}}{V_{\text{OC}} I_{\text{OC}}}$$

$$X_m = \frac{V_{\text{OC}}}{I_{\text{OC}}\sin\theta_{\text{OC}}} = \frac{V_{\text{OC}}^2}{\sqrt{V_{\text{OC}}^2 I_{\text{OC}}^2 - P_{NL-1\phi}^2}}$$

$$k = \frac{N_{\text{stator}}}{N_{\text{rotor}}}$$

WOUND ROTOR

$$k=\sqrt{\frac{V_{\mathrm{ss}}V_{\mathrm{sm}}}{V_{rm}V_{\mathrm{rs}}}}\ ;where$$

$$V_{\mathrm{sm}}=measured\ V_{\mathrm{stator}}\text{for } V_{\mathrm{rotor}}=V_{\mathrm{rs}}$$

$$V_{\mathrm{rm}}=measured\ V_{\mathrm{rotor}}\text{for } V_{\mathrm{stator}}=V_{\mathrm{ss}}$$

BLOCKED ROTOR TEST

$$R_{\mathrm{eq}}=R_1+R'_2=R_1+k^2R_2=\frac{P_{SC-1\phi}}{I_{\mathrm{SC}}^2}$$

$$|Z_{\mathrm{SC}}|=\frac{V_{\mathrm{SC}}}{I_{\mathrm{SC}}}\quad;\quad V_{SC-1\phi}=\frac{V_{\mathrm{SC}}}{\sqrt{3}}$$

$$X_{\mathrm{eq}}=X_1+k^2X_{2BR}=\sqrt{|Z_{\mathrm{SC}}|^2-{R'}_{\mathrm{eq}}^2}$$

$$X_1=X'_{2BR}\cong\frac{X_{\mathrm{eq}}}{2}$$

$Y-Connected:\ R_1=\frac{R_m}{2}$

$$\Delta-Connected: R_1=\frac{3R_m}{2}$$

$$P_{CU-LOSS}=3I_1^2R_1$$

$$P_{AG-3\phi}=P_{i-3\phi}-P_{CU-LOSS}-P_{CORE-LOSS}=3{I'}_2^2\frac{R'_2}{s}$$

$$P_{d-3\phi}=3{I'}_2^2R'_2\frac{(1-s)}{s}=P_{ag-3\phi}(1-s)$$

$$P_{o-3\phi}=P_{d-3\phi}-P_{\mathrm{MECH}}-P_{CORE-LOSS}$$

$$\frac{P_{R-CU-LOSS}}{P_{d-MECH}}=\frac{s}{1-s}$$

SPEED TORQUE CHARACTERISTIC

$$T_e=\frac{P_{d-3\phi}}{\Omega_m}\ ;\ \Omega_m=\Omega_s(1-s)$$

$$T_e=\frac{3}{\Omega_s}\times\frac{{I'}_2^2R'_2}{s}\quad;\quad T_{e-start}=\frac{3}{\Omega_s}\times{I'}_2^2R'_2$$

$$I'_2=\frac{V_1}{\left(R_1+\frac{R'_2}{s}\right)+j\left(X_1+X_{2BR}\right)}$$

$$T_{e-normal}=\frac{3V_1^2s}{\Omega_sR'_2}$$

THEVENIN EQUIVALENT

$$V_{\mathrm{TH}}=\frac{jX_mV_1}{R_1+j\left(X_1+X_m\right)}$$

$$Z_{\mathrm{TH}}=R_{\mathrm{TH}}+jX_{\mathrm{TH}}=\frac{jX_m\left(R_1+jX_1\right)}{R_1+j\left(X_1+X_m\right)}$$

$$T_{e-start}=\frac{3V_1^2R'_2}{\Omega_s\left(X_1+X'_{2BR}\right)^2}$$

$$s_m=\frac{R'_2}{\sqrt{R_1^2+\left(X_1+X'_{2BR}\right)^2}}$$

$$T_{e-max}=\frac{3V_1^2}{2\Omega_s\left(R_1+\sqrt{R_1^2+\left(X_1+X'_{2BR}\right)^2}\right)}$$

INDUCTION MOTOR PERFORMANCE

Neglecting mechanical and core losses:

$$P_{o-3\phi} = P_{d-3\phi} \;\; ; \;\; \eta \cong 1 - s$$

THREE PHASE THEORY

$$V_L = Line\ Voltage = line\ to\ line\ voltage$$

$$V_P = Phase\ Voltage = line\ to\ neutral\ voltage$$

$$Y - Connected$$

$$V_L = \sqrt{3} V_P$$

$$P_T = \sqrt{3} V_L I_L \cos\theta$$

$$P_P = V_P I_P \cos\theta$$

$$\Delta - Connected$$

$$I_L = \sqrt{3} I_P$$

$$V_L = V_P$$

$$P_T = \sqrt{3} V_L I_L \cos\theta$$

SYNCHRONOUS MACHINES

SYNCHRONOUS GENERATOR

Open Circuit Characteristic:

$$\text{Plot } V_{\text{OC}} \text{ vs. } I_f$$

$$I_{f0} \rightarrow E_0$$

Short Circuit Characteristic:

$$short\ armature,\ keep\ I_{f0}\ \text{constant}$$

$$\text{measure } I_a$$

$$|Z_s| = \frac{E_0}{I_a}$$

Resistance Measurement:

$$Y - connected$$

$$R_{a(\text{DC})} = \frac{R_{\text{measured}}}{2}$$

$$-connected$$

$$R_{a(\text{DC})} = \frac{3R_{\text{measured}}}{2}$$

$$R_{\text{eff}} = 1.4 R_{a(\text{DC})}$$

$$Voltage\ induced\ in\ phase\ a:$$

$$\varepsilon_a = E_m \sin(\omega t)$$

Where:

$$E_m = 2\pi f N \phi$$

$$\phi = flux\ per\ pole$$

$$\omega = 2\pi f$$

$$f = frequency\ of\ induced\ voltage$$

$$\varepsilon_b = E_m \sin(\omega t - 120)$$

$$\varepsilon_c = E_m \sin(\omega t - 240)$$

$$E_0 = K\phi f\ (rms\ or\ peak)$$

PER-PHASE

$$E_0 = V_t + I_a \left(R_a + jX_s\right)$$

$$\%VR = \frac{\left|E_0\right| - \left|V_t\right|}{\left|V_t\right|} \times 100\%$$

$$\delta = \angle between\ E_0\ \&\ V_t = power\ angle$$

$$\phi = \angle between\ I_a\ \&\ V_t$$

$$\text{gennerally } X_s \gg R_a\ \therefore\ Z_s \cong jX_s$$

Neglecting $\mathbf{R_0}$:

$$E_0 \sin\delta = X_s I_a \cos\phi$$

$$P_{d-1\phi} = V_t I_d \cos\phi = V_t \left(\frac{E_0 \sin\delta}{X_s}\right)$$

Including $\mathbf{R_0}$:

Unity PF Load:

$$E_0 = \sqrt{\left(V_t + I_a R_a\right)^2 + \left(I_a X_s\right)^2}$$

Lagging PF Load:

$$E_0 = \sqrt{\left(V_t \cos(\phi) + I_a R_a\right)^2 + \left(V_t \sin(\phi) + I_a X_s\right)^2}$$

Leading PF Load:

$$E_0 = \sqrt{\left(V_t \cos(\phi) + I_a R_a\right)^2 + \left(V_t \sin(\phi) - I_a X_s\right)^2}$$

Active and Reactive Power:

$$E_0 = V_t + I_a \left(R_a + jX_s\right) = E_0(\cos\delta + j\sin\delta)$$

$$I_a = \frac{1}{\left|Z_s\right|^2} \left[R_a \left(E_0 \cos(\delta) - V_t\right) + X_s E_0 \sin(\delta) + jR_a E_0 \sin(\delta) - jX_s \left(E_0 \cos(\delta) - V\right.\right.$$

$$where:\ \left|Z_S\right|^2 = R_a^2 + X_s^2$$

$$S = V_t I_a = P + jQ$$

$$P=\frac{1}{|Z_s|^2}\left[R_a\left(V_tE_0\cos(\delta)-V_t^2\right)+X_sV_tE_0\sin(\delta)\right]$$

$$Q=\frac{1}{|Z_s|^2}\left[-R_aV_tE_0\sin(\delta)+X_s\left(V_tE_0\cos(\delta)-V_t^2\right)\right]$$

Neglecting R_a :

$$P\cong\frac{V_tE_0\sin(\delta)}{X_s}\ \ ;\ Q\cong\frac{-V_t^2+V_tE_0\cos(\delta)}{X_s}$$

$$T\cong\frac{V_tE_0\sin(\delta)}{\Omega_sX_s}$$

SYNCHRONOUS MOTOR

$$T_d\cong\frac{3V_tE_0\sin(\delta)}{\Omega_sX_s}$$

PER-PHASE

neglecting R_a :

$$P_{d-1\phi}=\frac{V_tE_0\sin(\delta)}{X_s}=constant$$

$$I_aX_s\cos(\phi)=E_0\sin(\delta)=constant$$

SALIENT-POLE MACHINES

$$I_d=I_a\sin(\delta+\phi)$$

$$I_q=I_a\cos(\delta+\phi)$$

$$E_0=V_t\cos(\delta)+I_aX_a$$

$$P_{d-1\phi}=\frac{V_tE_0\sin(\delta)}{X_d}+\frac{V_t^2}{2}\left(\frac{1}{X_q}-\frac{1}{X_d}\right)$$

Chapter 8

CENG 355 CheatSheet

I/O

Interfacing

Memory

Principle of Locality:

Programs tend to reuse data and instructions near those they have used recently, or that were recently referenced themselves.

Temporal locality: Recently referenced items are likely to be referenced again in the near future

Spatial locality: Items with nearby addresses tend to be referenced close together in time.

Data:

— Reference array elements in succession: spatial locality

— Reference sum each iteration: temporal locality

Instructions:

— Reference instructions in sequence: spatial locality.

— Cycle through loop repeatedly: temporal locality.

Blocked Matrix Code

```
void bijk(array A, array B, array C, int n, int bsize)
{
        int i, j, k, kk, jj;
        double sum;
        int en = bsize * (n/bsize); /* Amount that fits ev

        for (i = 0; i < n; i++)
        for (j = 0; j < n; j++)
        C[i][j] = 0.0;

        for (kk = 0; kk < en; kk += bsize) {
                for (jj = 0; jj < en; jj += bsize) {
                        for (i = 0; i < n; i++) {
                                for (j = jj; j < jj + bsiz
                                        sum = C[i][j];
                                        for (k = kk; k < k
                                                sum += A[i
                                        }
                                        C[i][j] = sum;
                                }
                        }
                }
        }
}
```

Copy-paste code here to remove the line numbers.

CENG 355 Cheatsheet

DMA vs Polling

$$\text{Polling Cost} = \begin{matrix}\text{Access}\\ \text{Rate}\end{matrix} \times \left(\begin{matrix}\text{Cycles not}\\ \text{Ready}\end{matrix} \times \begin{matrix}\text{Percent}\\ \text{Active}\end{matrix} + \begin{matrix}\text{Cycles}\\ \text{Ready}\end{matrix} \left(1 - \begin{matrix}\text{Percent}\\ \text{Active}\end{matrix}\right)\right)$$

$$\text{DMA Cost} = \begin{matrix}\text{Access}\\ \text{Rate}\end{matrix} \times \begin{matrix}\text{Percent}\\ \text{Active}\end{matrix} \times \left(\begin{matrix}\text{DMA}\\ \text{Start}\end{matrix} + \begin{matrix}\text{DMA}\\ \text{End}\end{matrix} \right)$$

$$\text{Interrupt Cost} = \begin{matrix}\text{Interrupt}\\ \text{Cycles}\end{matrix} \times \begin{matrix}\text{Access}\\ \text{Rate}\end{matrix} \times \begin{matrix}\text{Percent}\\ \text{Active}\end{matrix}$$

$$\text{CPU Busy} = \left(\frac{\text{Busy Cycles per Second}}{\text{Total Cycles per Second}}\right) \times 100\,\%$$

$$\text{Access Rate} = \frac{\text{Data Transfer Rate } (^{\text{B}}/_{\text{s}})}{\text{Block Size (B)}}$$

Priority Driven Scheduling

Static Scheduling

Rate Monotonic: $T_{ik} = \dfrac{1}{P_i}$

Deadline Monotonic: $T_{ik} = \dfrac{1}{D_i}$

Dynamic Scheduling

Earliest Dealine First: $T_{ik} = \dfrac{1}{\phi_i + kP_i + D_i}$

Least Laxity First: $T_{ik} = \dfrac{1}{\phi_i + kP_i + D_i - t - \Delta C_i}$

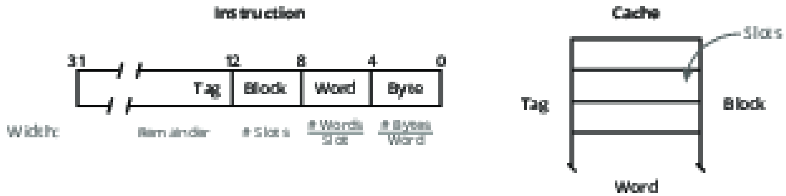
T_i	Task Priority	C_i	Worst Case Execution Time
P_i	Period	ΔC_i	Remaining Execution Time
ϕ_i	Initial Delay	D_i	Deadline

$$\text{CPU Utilization} = \sum_{i=1}^{n} \frac{C_i}{P_i} = \frac{C_1}{P_1} + \frac{C_2}{P_2} + \dots$$

Cache Mapping

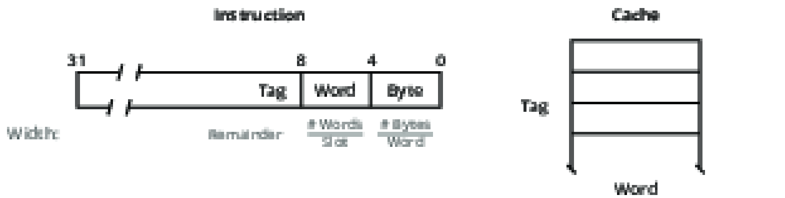
Direct Mapped

Every address has a block it belongs to which corresponds to one slot. This is equivalent to a "1-Way Set Associative" scheme.



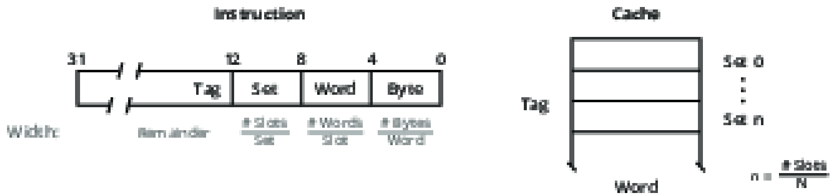
Fully Associative

Each address can go anywhere, when we're full we bump stuff (typically the least recently accessed slot). This is equivalent to a "# Slots-Way Set Associative" scheme.



N-Way Set Associative

Every address has a set it belongs. Each set contains N slots, e.g., # Sets $= \dfrac{\text{\# Slots}}{N}$.



Cache Miss and Page Faults

	CPU Cache
Cache Read Time	C_i
Cache Hit Rate	$h_i = \left(\frac{\text{Cache Hits}}{\text{Cache Misses}}\right) \times 100\,\%$
Cache Miss Penalty	$M_{C_i} = T_{C_{i+1}}$ or T_P
Average Cache Access Time	$T_{C_i} = h_i C_i + M_{C_i}\,(1 - h_i)$
	Main Memory
Memory Read Time	M_P
Page Fault Service Time	D
Page Fault Rate	$p = \left(\frac{\text{Number of Page Faults}}{\text{Total Number of Accesses}}\right) \times 100\,\%$
Average Memory Access Time	$T_P = M_P\,(1 - p) + pD$

Finite State Machines

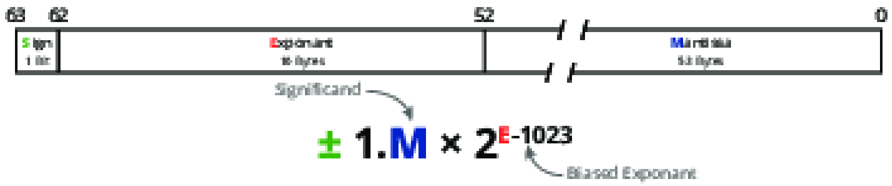
Mealy Machine

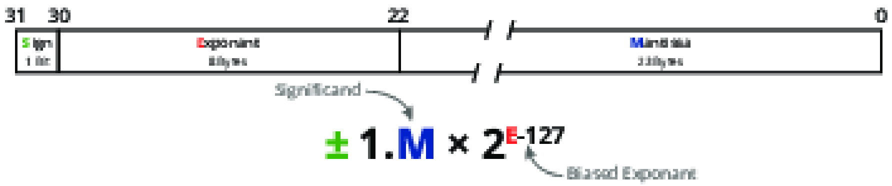
A Mealy machine is a finite-state machine whose output values are determined both by its **current state** and the **current inputs**. (This is in contrast to a Moore machine, whose output values are determined solely by its current state.)

Moore Machine

A Moore machine is a finite-state machine whose output values are determined solely by its **current state**. This is in contrast to a Mealy machine, whose output values are determined both by its current state and by the values of its inputs. The Moore machine is named after Edward F. Moore.

Floating Point





Mantissa	Exponent 0	Exponent $1 \rightarrow 254$	Exponent 255
Zero	0	Powers of Two	∞
Not Zero	$\pm 1.M \times 2^{-126}$	Ordinary Numbers	NaN

x	2^x	x	2^x	
1	2	-1	$^1/_2$	0.500000000000
2	4	-2	$^1/_4$	0.250000000000
3	8	-3	$^1/_8$	0.125000000000
4	16	-4	$^1/_{16}$	0.062500000000
5	32	-5	$^1/_{32}$	0.031250000000
6	64	-6	$^1/_{64}$	0.015625000000
7	128	-7	$^1/_{128}$	0.007812500000
8	256	-8	$^1/_{256}$	0.003906250000
9	512	-9	$^1/_{512}$	0.001953125000
10	1024	-10	$^1/_{1024}$	0.000976562500
11	2048	-11	$^1/_{2048}$	0.000488281250
12	4096	-12	$^1/_{4096}$	0.000244140625

Arithmetic

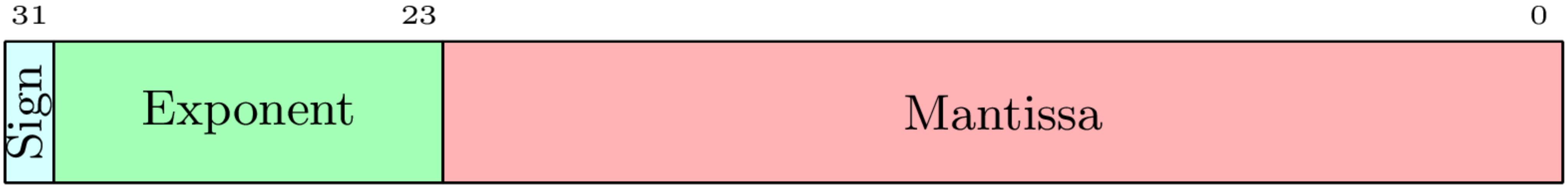


Figure 8.3: IEEE 754 floating point (correct later)

$$t_{avg} = h_1 C_1 + (1 - h_1)(h_2 C_2 + (1 - h_2)M) \tag{8.1}$$

where

h_1 is the hit rate in the L_1 caches.
h_2 is the hit rate in the L_2 cache.
C_1 is the time to access information in the L_1 caches.
C_2 is the miss penalty to transfer information from the L_2 cache to an L_1 *cache.*
M is the miss penalty to transfer information from the main memory to the L_2 cache.

Concurrency

why Fun with colour

$$\text{Amdahl's Law} = \frac{1}{f_{unenh} + f_{enh}/p}$$

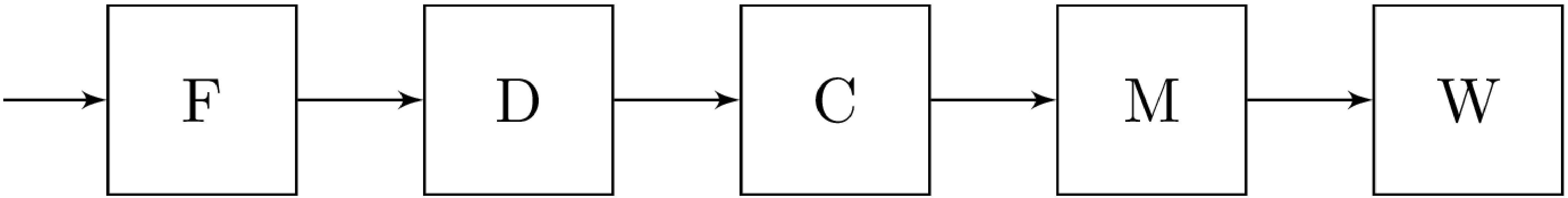


Figure 8.4: Pipeline for Question 2

Chapter 9

Glossary

```
Consider using a manual approach to style listlisting,
what I do for pandoc, a combination of prism.js with <pre> and <code> tags inbewteen.
```

Chapter 10

ELEC 460: Control Theory II

Z-transform

$\mathcal{Z}\{f_1(t) \pm f_2(t)\} = F_1(z) + F_2(z)$	Addition
$\mathcal{Z}\{af(t)\} = aF(z)$	Multiplication by a Constant
$\mathcal{Z}\{f(t - nT)\} = z^{-n}F(z)$	Shifting
$\mathcal{Z}\{f(t + kT)\} = z^k F(z) - z^k f(0) - \cdots - zf(kT - T)$	Shifting (cont'd)
$\mathcal{Z}\{e^{\mp at}f(t)\} = F(ze^{\pm at})$	Complex Translation
$\lim_{k\to\infty} f(kT) = \lim_{z\to 0} F(z)$	Initial Value Theorem
If $(1 - z^{-1})F(z)$ has all singularities inside unit disk $\lvert z\rvert = 1$, then $\lim_{k\to\infty} f(kT) = \lim_{z\to 1}(1 - z^{-1})F(z)$	Final Value Theorem
$\mathcal{Z}\left\{\frac{\partial}{\partial a} f(t,a)\right\} = \frac{\partial}{\partial a}F(z,a)$	Partial differentiation

$$G(z) = \mathcal{Z}\left\{\left(\frac{1 - e^{-s}}{s}\right)\left[\frac{1}{s+1}\right]\left[\frac{1}{s}\right]\right\} \to G_1(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{1}{s^2(s+1)}\right\}$$

All first-column elements of the Routh array are to be of the same sign. $a_0 s^n + a_1 s^{n-1} + \cdots + an - 1s + a_n = 0$, first row is even entries a_0, a_2, next row is a_1, a_2, b entries are the same are jury-marden table.

ZOH(zero-hold-system) $\quad f^*(t) = \sum_{k=-\infty}^{\infty} f(kT)\delta(t - kT) \quad G_h(s) = \frac{1-e^{-Ts}}{s}$

Stability Test for Digital Systems

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n \quad G(z) = \frac{A(z)}{P(z)}$$

Stability Condition: $P(z) \neq 0 \quad |z| \geq 1$ (Draw Unit Circle to test stability)

Routh-Stability in Digital Domain: $s = \dfrac{z+1}{z-1} \quad z = \dfrac{s+1}{s-1}$

Jury-Marden Table Uses function P of z

$$b_k = \det \begin{bmatrix} a_n & a_{n-1-k} \\ a_0 & a_{k+1} \end{bmatrix}$$

$$k = 0, 1, \cdots n-1$$

$$c_k = \det \begin{bmatrix} b_{n-1} & b_{n-2-k} \\ b_0 & b_{k+1} \end{bmatrix}$$

$$k = 0, 1, \cdots n-1$$

$$q_k = \det \begin{bmatrix} p_3 & p_{2-k} \\ p_0 & p_{k+1} \end{bmatrix}$$

$$k = 0, 1, 2$$

Row	z^0	z^1	z^2		z^{n-2}	z^{n-1}	z^n
1	a_n	a_{n-1}	a_{n-2}	$\cdots$	a_2	a_1	a_0
2	a_0	a_1	a_2	$\cdots$	a_{n-2}	a_{n-1}	a_n
3	b_{n-1}	b_{n-2}	b_{n-3}	$\cdots$	b_1	b_0	
4	b_0	b_1	b_2	$\cdots$	b_{n-2}	b_{n-1}	
5	c_{n-2}	c_{n-3}	c_{n-4}	$\cdots$	c_0		
6	c_0	c_1	c_2	$\cdots$	c_{n-2}		
2n-5	p_3	p_2	p_1	p_0			
2n-4	p_0	p_1	p_2	p_3			
2n-3	q_2	q_1	q_0				

Necessary and Sufficient Condition for Stability

1. $|a_n| < |a_0|$
2. $P(1) > 0$

3.

$$P(-1) > 0 \text{ for n even}$$
$$< 0 \text{ for n odd}$$

4. $b_{n-1} > |b_0|, |c_{n-2}| > |c_0|, \cdots |q_2| > |q_0|$

Special Case n =2
$P(z) = a_0 z^2 + a_1 z + a_2$

z^0	z^1	z^2
a_2	a_1	a_0

$P(z) \neq 0$ for $|z| \geq 1$ if and only if

1. $|a_2| < |a_0|$

2. $P(1) > 0$

3. $P(-1) > 0 \quad (n = 2)$

Root Locus presents the poles of the closed loop system when the gain K changes from zero to infinity.

Construction of the Root Locus

Open loop transfer function KH$(s)\,G\,(s) = K\dfrac{B(s)}{A(s)}$

m: the order of the **open-loop** numerator polynomial.

n: the order of the **open-loop** denominator polynomial. $q = n - m$

Rule 1: number of branches equals the number of poles of the open-loop transfer function

Rule 2: If the total number of poles and zeros of the open-loop system to the right of the s-point on the real axis is odd, then this point lies on the locus.

Rule 3: The locus starting point (K=0) are at the open-loop poles and the locus ending points (K=∞) are at the open loop zeros and n-m branches terminate at infinity.

Rule 4 and 5: Slope of asymptotes of root locus as ‘s’ approaches infinity.

Abscissa of the intersection between asymptotes of root locus and real-axis.

$$\sigma = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{q} \qquad \theta = \pm r\frac{180}{q} \quad \text{where r=1, 3, 5}$$

$$f(s) = A(s) + KB(s) = 0 \qquad and \qquad K = -\frac{A(s)}{B(s)}$$

$$\frac{\mathrm{dK}}{\mathrm{ds}} = -\frac{A'(s)B(s) - A(s)B'(s)}{B^2(s)} = 0$$

Rule 5:

Rule 6: Break-away and break-in points. From the characteristic equation

$$f(s) = A(s) + KB(s) = 0 \qquad and \qquad K = -\frac{A(s)}{B(s)}$$

The break-away and break-in points can be found from

$$\frac{\mathrm{dK}}{\mathrm{ds}} = -\frac{A'(s)B(s) - A(s)B'(s)}{B^2(s)} = 0$$

Rule 7: Angle of departure from complex poles or zeros. Subtract from 180^o the sum of all angles from all other zeros and poles of the open-loop system to the complex pole (or zero) with appropriate signs.

$$\text{Z-transform: Definition} \quad F(z) = Z[f(t)] - Z[f(kT)] = \sum_{k=0}^{\infty} f(kT)z^{-k}$$

$$e^*(\infty) = \lim_{z\to1}(1-z^{-1})E(z) \quad K_p = \lim_{z\to1} GH(z), \quad e^*(\infty) = \frac{1}{1+K_p}$$

$$e^*(\infty) = \frac{1}{K_v}, \quad K_v = \lim_{z\to1}\frac{(1-z^{-1})GH(z)}{T}$$

$$e^*(\infty) = \frac{1}{K_a}, \quad K_a = \lim_{z\to1}\frac{(1-z^{-1})^2GH(z)}{T^2}$$

Linear Factor Rule. For each factor of Q of the form $(ax+b)^m$, the partial fraction decomposition contains the following sum of m partial fractions:

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_m}{(ax+b)^m},$$

where the A_i are constants to be determined.

Quadratic Factor Rule. For each factor of Q of the form $(ax^2+bx+c)^m$, where ax^2+bx+c is an irreducible quadratic, the following sum of m partial fractions:

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_mx+B_m}{(ax^2+bx+c)^m},$$

where the A_i and B_i are constants to be determined.

Geometric Sum $\sum\limits_{k=-N}^{N} ar^{k-1} = a\frac{1-r^N}{1-r}$ $\sum_{i=0}^{\infty} a^i = \frac{1}{1-a}$

$x(k+2)-\frac{3}{2}x(k+1)+\frac{1}{2}x(k)=u(k), (x(0)=1, x(1)=5/2)$

$[z^2X(z)-z^2x(0)-zx(1)]-\frac{3}{2}(zX(z)-zx(0)]+\frac{1}{2}X(z)=\frac{z}{z-1}$

Effects of T on Transient Behaviour
$s=-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$,
ζ: damping ratio
ω_n: undamped natural frequency,
ω_d: damped natural frequency
$z=e^{Ts} \to z = \exp\left[T(-\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2})\right]$, and $|z|=e^{-T\zeta\omega_n}$, $\angle z = T\omega_n\sqrt{1-\zeta^2}=T\omega_d$. $\uparrow T$ makes system less stable (for the same gain K) than $\downarrow T$.

Matrix Inverses for 2x2 and 3x3

$$A^{-1}=\begin{bmatrix} a & b \\ c & d\end{bmatrix}^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a\end{bmatrix} \quad \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i\end{bmatrix}^{-1}$$

$$A^{-1}=\frac{1}{|A|}\begin{bmatrix} +\begin{vmatrix} e & f \\ h & i\end{vmatrix} & -\begin{vmatrix} b & c \\ h & i\end{vmatrix} & +\begin{vmatrix} b & c \\ e & f\end{vmatrix} \\ -\begin{vmatrix} d & f \\ g & i\end{vmatrix} & +\begin{vmatrix} a & c \\ g & i\end{vmatrix} & -\begin{vmatrix} a & c \\ d & f\end{vmatrix} \\ +\begin{vmatrix} d & e \\ g & h\end{vmatrix} & -\begin{vmatrix} a & b \\ g & h\end{vmatrix} & +\begin{vmatrix} a & b \\ d & e\end{vmatrix}\end{bmatrix}^{-1}$$

Bilinear Transform $\quad s=\frac{2(1-z^{-1})}{T(1+z^{-1})}$, $z=\dfrac{1+0.5Ts}{1-0.5Ts}$.

1. Stability $\Re[s]<0$

$$\Re\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)=\Re\left(\frac{2}{T}\frac{z-1}{z+1}\right)<0,\ z=\sigma+j\omega$$

$$\Re\frac{z-1}{z+1}=\Re\left[\frac{\sigma^2-1+\omega^2+2j\omega}{(\sigma+1)^2+\omega^2}\right]\to\sigma^2-1+\omega^2<0.$$

Solution of inhomogeneous state equations
scalar $\dot{x}=ax+bu \quad \dot{x}-ax=bu$

$$e^{-at}[\dot{x}(t)-ax(t)]=\underbrace{\frac{d}{dt}[e^{-at}x(t)]=e^{-at}bu(t)}_{\text{integrate } 0\to t}$$

$$e^{-at}x(t)-x(0)=\int_0^t e^{-a\tau}bu(\tau)d\tau$$

$$\to x(t)=e^{at}x(0)+e^{at}\int_0^t e^{-a\tau}bu(\tau)d\tau$$

matrix: $\dot{(x)}=Ax+bu$, but taking $\mathcal{L}^{-1}$ leads to $x(t)=e^{At}x(0)+\int_0^t e^{A(t-\tau)}bu(\tau)d\tau$

Controllable Canonical Form

$$G(z)=\frac{b_0+b_1z^{-1}+\cdots+b_nz^{-n}}{1+a_1z^{-1}+\cdots+a_n}=\frac{b_0z^n+b_1z^{n-1}+\cdots+b_n}{z^n+a_1z^{n-1}+\cdots+a_n}$$

$$G(z)=b_0+\frac{(b_1-a_1b_0)z^{-1}+(b_2-a_2b_0)z^{-2}+\cdots+(b_n-a_nb_0)z^{-n-1}}{1+a_1z^{-1}+a_2z^{-2}+\cdots+a_nz^{-n}}$$

$$\begin{bmatrix} x_1(k+1) \\ \vdots \\ x_n(k+1)\end{bmatrix}=\begin{bmatrix} 0 & 1 & 0 & \cdots \\ \vdots & \cdots & & \vdots \\ \vdots & & \cdots & 1 \\ -a_n & \cdots & \cdots & -a_1\end{bmatrix}\begin{bmatrix} x_1(k) \\ \vdots \\ x_n(k)\end{bmatrix}+\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1\end{bmatrix}u(k)$$

$$y(k)=\begin{bmatrix} b_n-a_nb_0, & b_{n-1}a_{n-1}b_0, & b_1-a_1b_0\end{bmatrix}\begin{bmatrix} x_1(k) \\ \vdots \\ x_n(k)\end{bmatrix}+b_0u(k)$$

$z\to s \quad zX(z)=\mathcal{Z}[x(k+1)] \quad sX(s)=\mathcal{L}[x(t)]$

Observable Canonical Form

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n\end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 & -a_n \\ 1 & \cdots & & -a_{n-1} \\ 0 & & \cdots & \vdots \\ 0 & 0 & 1 & -a_1\end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n\end{bmatrix}+\begin{bmatrix} b_n-a_nb_0 \\ \vdots \\ b_1-a_1b_0\end{bmatrix}u(k)$$

$$y(k)=\begin{bmatrix} 0, & \cdots & \cdots, & 0, & 1\end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n\end{bmatrix}+b_0u(k)$$

Part-Frac-Expansion Method, Dia Canonical

$$G(z) = b_0 + \frac{c_1}{z-p_1} + \cdots \cdots + \frac{c_n}{z-p_n}$$

$$\begin{bmatrix} x_1(k+1) \\ \vdots \\ \vdots \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} p_1 & 0 & \cdots & 0 \\ 0 & \vdots & & 0 \\ \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & p_n \end{bmatrix} \begin{bmatrix} x_1(k) \\ \vdots \\ \vdots \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + b_0 u(k)$$

Special Case

$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y + a_n = u \quad \dot{x} = Ax + bu$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 & \vdots & \vdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & & & & 1 \\ -a_n & -a_{n-1} & \cdots & \cdots & -a_1 \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \quad \begin{array}{l} c = [1\ 0\ ..\ 0] \quad y = cx + du \text{ and } d = 0 \\ Y(s) = [c(sI-A)^{-1}b + d]U(s) \\ \dfrac{Y(z)}{U(z)} = c(zI-A)^{-1}b + d \end{array}$$

Deadbeat Controller and Deadbeat Response

$x(k+1) = Gx(k) + Hu(k) \quad u(k) = -Kx(k)$

$x(k+1) = (G-HK)x(k) \quad x(k) = (G-HK)^k x(0)$

$x(k) = (G-HK)^k x(0) \quad x(k) = 0 \quad (for) k \geq \quad q(q \leq n)$

$\det(zI - G + HK) = z^n \quad N^n = 0$, N is nilpotent matrix.

Controllability A system is controllable, if and only if, it is possible to transfer the system state from any arbitrary initial state x(0) to the origin in finite time. initial state x(0) $\rightarrow$ desired state: x(n)=0. **Controllability condition for SI continuous systems:** $\det C = \det[b, \ Ab, \ \cdots, A^{n-1}b] \neq 0$.

Observability A system is observable if any initial state x(0) can be determined from a finite number of output observations. $\det O_c = \det \begin{bmatrix} c \\ cA \\ \vdots \\ cA^{n-1} \end{bmatrix} \neq 0.$

Continuous State Transition Matrix: $\phi(t)$, $\phi(t) = e^{At} = \mathcal{L}^{-1}[(sI-A)^{-1}]$, then $\dot{\phi}(t) = A\phi(t) \quad \phi(0) = I$, $\dot{(x)} = Ax$

Verification: $x(t) = \phi(0)x(0) = Ix(0)$ $\dot{x}(t) = \dot{\phi}(t)x(0) = A\phi(t)x(0) = Ax(t)$.

Properties of $\phi(t)$:

1) $\phi(0) = e^{A0} = I$
2) $\phi(t) = e^{At} = (e^{(-At)})^{-1} = [\phi(-t)]^{-1}$
3) $\phi(t_1 + t_2) = \phi(t_1)\phi(t_2) = \phi(t_2)\phi(t_1)$
4) $[\phi(t)]^n = \phi(nt)$
5) $\phi(t_0 - t_1)\phi(t_1 - t_2) = \phi(t_0 - t_2) = \phi(-t_1 + t_0)\phi(-t_2 + t_1)$

$e^{A(t_0-t_1)}e^{A(t_1-t_2)} = e^{A(t_0-t_2)} = e^{-A(t_1-t_0)}e^{-A(t_2-t_1)}$

BIBO Stability Output is bounded for any bounded input. CTS systems : Poles in left half plane, Discrete Systems: poles inside unit circle.

INTERNAL (Also asymptotic stability) Def: Equilibrium state: Continuous systems: Assume u(t) = 0; $\dot{x}_e = 0 = Ax_e + bu \rightarrow x_e = 0$

Discrete systems: Assume u(k) = 0; $x_e(k+1) = 0 = x_e(k) + Gx_e(k) \rightarrow x_e = 0$

Def: A system is asymptotically stable if any initial condition x(0) converges to the equilibrium state $x_e = 0$. (It is assumed $u(t) = 0, t \leq 0$ or $u(k) = 0, k \geq 0$)

Condition for asymptotic stability:

CTS $\Re[\lambda_i\{A\}] < 0$

Discrete $|\lambda_i\{G\}|$, all eigenvalues in unit circle

BIBO Stability $\rightarrow$ Asymptotic Stability (AS)

BIBO Stability & no pole zero cancellation $\rightarrow$ AS

Eigenvalues of A are the solutions of $\det(I\lambda - A) = 0$,

Poles of G(z) are the zeros of denominator poly. $G(z) = d + c(zI-A)^{-1}b$ where $(zI-A)^{-1} = \frac{\text{adj}(zI-A)}{\det(zI-A)}$

C: nonsingular if system controllable. If the system is controllable, any closed-loop poles can be obtained,

Feed-forward observers State Observer: $\tilde{X}(k+1) = G\tilde{x}(k) + Hu(k)$ $\tilde{y}(k) = c\tilde{x}(k)$, Observed state: $\tilde{x}(k)$, Observation error: $e(k) = x(k) - \tilde{x}(k)$, $e(k+1) = Ge(k)$, Dynamics of error depend on G

Prediction (full order) observer where the estimate $\bar{x}(k+1)$ is obtained based on measurements of up to y(k).

$$\bar{x}(k+1) = G\bar{x}(k) + Hu(k) + k_e[y(k) - \bar{y}(k)]$$
$$\bar{x}(k+1) = [G - k_e c]\tilde{x}(k) + Hu(k) + k_e cx(k)$$

k_e for this observer can be obtained using $k_e = O^{-1}\bar{A}^{-1}(\alpha - a)^T$, where

$$\bar{A} = \begin{bmatrix} 1 & 0 \cdots & \cdots & 0 \\ a_1 & 1 & & \\ \vdots & a_1 & . & \\ \vdots & . & . & \\ a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \end{bmatrix},$$

lower triangular Toeplitz matrix, A square matrix that is not singular, i.e., one that has a matrix inverse.

Current observer where the estimate is obtained based on measurements up to $y(k+1)$.

$$\tilde{x}(k+1) = G\tilde{x}(k) + Hu(k) + K_e[y(k+1) - c\tilde{x}(k+1)]$$
$$\bar{z}(k+1) = c\bar{x}(k+1)$$

ASYMPTOTIC OBSERVERS 4 CTS SYS

$$\dot{x}(t) = Ax(t) + bu(t) \quad x(0-) = x_0$$
$$y(t) = cx(t) \quad t > 0-$$
$$Ox(0-) = [y(0-), \ \cdots \ y^{n-1}(0-)]$$

Open-loop Observer Use $(\{A, B, c\}, \{u(t), t > 0\}$, and $x_0) \rightarrow \{x(t), t > 0-\}$,

Effects of disturbance ϵ: $\tilde{x}_0 = x_0 - \epsilon$, $|\epsilon| \ll |x_0|$, $\dot{\tilde{x}}(t) = A\tilde{x}(t) + bu(t)$, $\tilde{x}(0-) = \tilde{x}_0 = x_0 - \epsilon$, $\dot{e}(t) = Ae(t)$, $e(0-) = \epsilon$, A is unstable $e(t) \rightarrow \infty$

Closed-loop observer: Output Error: $y(t) - \tilde{y}(t) = y(t) - c\tilde{x}(t) = c[x(t) - \tilde{x}(t = ce(t)$, Observer $\tilde{x}(t) =$

$A\tilde{x}(t) + bu(t) + l[y(t) - c\tilde{x}(t)$, $\tilde{x}(t_o) = \tilde{x}_o$ $\tilde{x}_o$ an estimated initial state vector l: feedback gain vector.
Observer design: $l = O^{-1}\tilde{A}^{-1}(\alpha - a)^T$

1. Pole Placement CTS

$$\dot{x}(t) = Ax(t) + bu(t) \quad y(t) = cx(t)$$

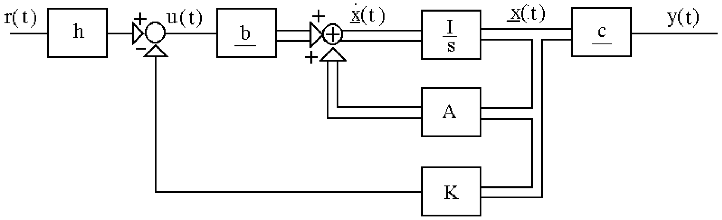
$$a(s) = \det(sI - A) = s^n + a - 1s^{n-1} + \cdots + a_n$$

Find a feedback gain vector K so that the characteristic polynomial of the resulting closed-loop system is given by the polynomial:

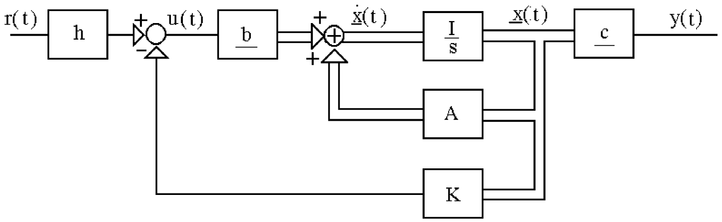
$$\alpha(s) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n \quad u(t) = hr(t) - Kx(t)$$

$$\dot{x}(t) = (A - bK)x(t) + bhr(t) \quad y = cx(t)$$

$$\alpha - a = KC\tilde{A}^T \quad K = (\alpha - a)\tilde{A}^{-T}C^{-1}$$



2. Tracking a Reference Signal



Tracking: y(t) should follow r(t) at steady state(ss) i.e, y(k) follows r(k) at ss. Find h in $u(k) = hr(k) - Kx(k)$ for tracking. ss $x(k+1) = x(k)$

$$\dot{x}(t) = Ax + bu = x(t) \downarrow$$

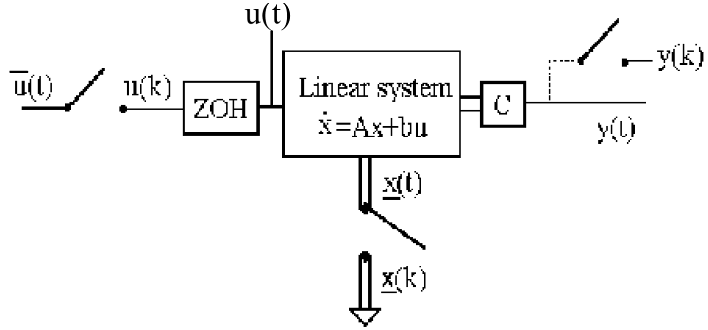
$$x(t) = (A - bK)x(t) + bhr(t)$$

$$y(t) = c(I - G + HK)^{-1}Hr(t) \quad y(t) = r(t)$$

$$h = \frac{-1}{c(A - bK)^{-1}b}$$

Estimation of unmeasurable state variables is commonly called observation. $G(s) = C(sI - A)^{-1}B$, $\Delta(\lambda) = (\lambda^2 + 2\zeta\omega_n + \omega_n^2)(\lambda + \zeta\omega_n)$
Overdamped $\zeta > 1$, Critically Damped $\zeta = 1$, Underdamped(oscillations) $0 < \zeta < 1$ $\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$
$t_s = \frac{4}{\sigma} = \frac{4}{\zeta\omega_n}$ (2% *band*), $t_s = \frac{3}{\sigma} = \frac{3}{\zeta\omega_n}$ (5% *band*),

Discretization of CTS-Time State Equations:



$\dot{x} = Ax + bu \quad G(T) = e^{AT} = \phi(T)$ $H(T) = (\int_0^T e^{AT}$ d

3. Integral Error Feedback Discrete $x(k+1) = Gx(k) + Hu(k) + w(k) \quad y(k) = c(k)x(k)$ w(k) is unknown but constant disturbance.

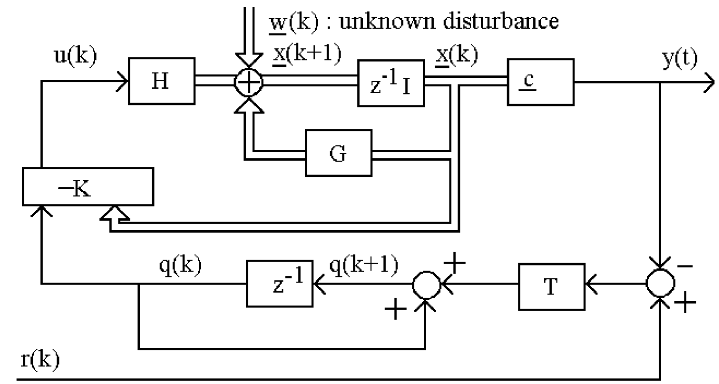
Problem: Design a state-feedback controller so that 1) The CL eigenvalues are at prescribed locations.

2) The output y(k) follows the reference r(k) for any w(k) (constant, but unknown) at steady state.

$$\begin{bmatrix} x(k+1) \\ q(k+1) \end{bmatrix} = \begin{bmatrix} G & 0 \\ -T_c & 1 \end{bmatrix} \begin{bmatrix} x(k) \\ q(k) \end{bmatrix} + \begin{bmatrix} H \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ T \end{bmatrix} r(k) + \begin{bmatrix} w(k) \\ 0 \end{bmatrix} \quad q(k+1) = q(k) + T(r(k) - y(k))$$

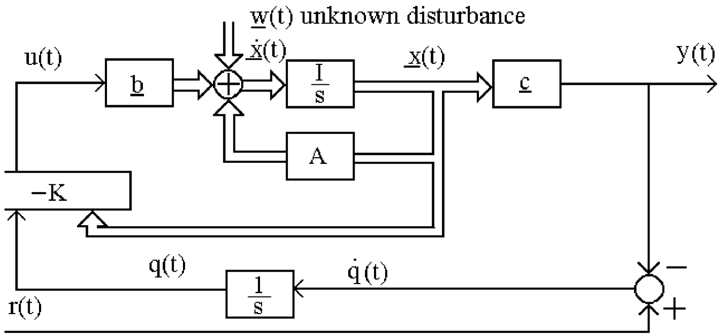
Find $K = [K_x, K_q]$,$K = (\alpha - a)\tilde{A}^{-T}C^{-1}$,

$$\begin{bmatrix} x(k+1) \\ q(k+1) \end{bmatrix} = \begin{bmatrix} G - HK_x & -HK_q \\ -T_c & 1 \end{bmatrix} \begin{bmatrix} x(k) \\ q(k) \end{bmatrix} + \begin{bmatrix} 0 \\ T \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ T \end{bmatrix} r(k) + \begin{bmatrix} w(k) \\ 0 \end{bmatrix} \quad u(k) = -[K_x, K_q] \begin{bmatrix} x(k) \\ q(k) \end{bmatrix}$$



3. Integral Error Feedback CTS

$$\begin{bmatrix} x(k+1) \\ q(k+1) \end{bmatrix} = \begin{bmatrix} A & 0 \\ -c & 1 \end{bmatrix} \begin{bmatrix} x(k) \\ q(k) \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(k) + \begin{bmatrix} w(k) \\ 0 \end{bmatrix} \quad q(k) = (r(k) - y(k))$$



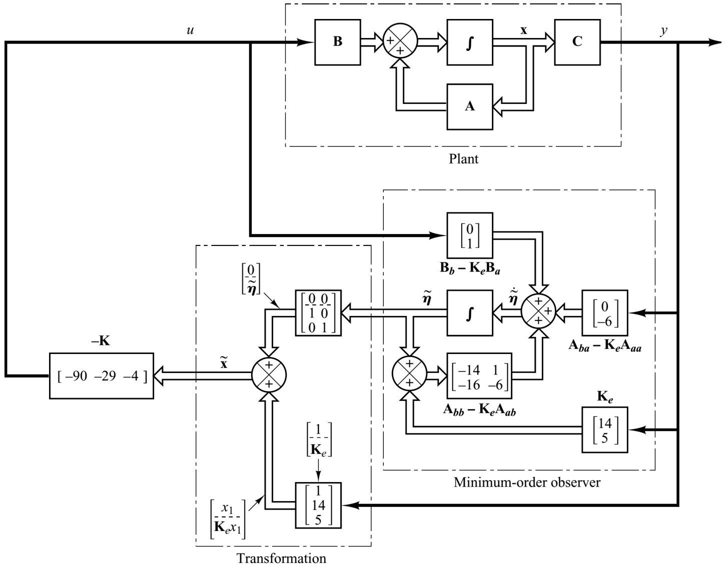
Combined Observer-Controller

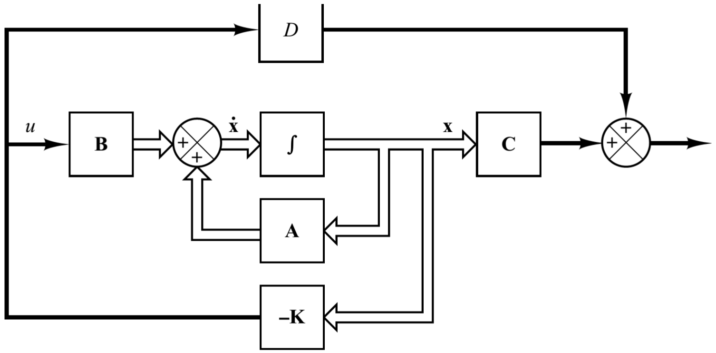
Observer feedback: $l(y(t) - c\tilde{x}(t))$,

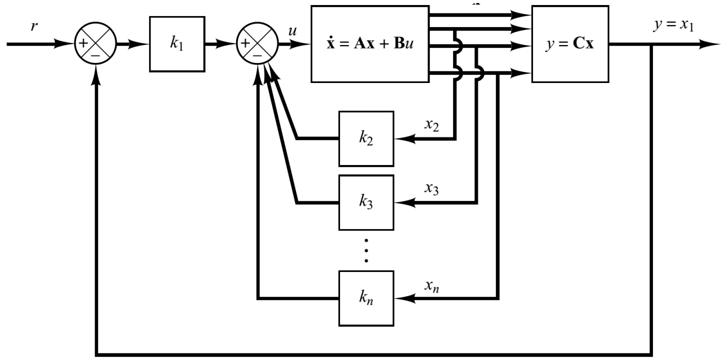
Feedback control signal $u(t) = -K\tilde{x}(t) + v(t)$
Observation error: $\dot{e}(t) = (A - lc)e(t)$
$\det \begin{bmatrix} sI - A & bK \\ -lc & sI - A + lc + bK \end{bmatrix}$
$= \det(sI - A + bk)\det(sI - A + lc)$ $\begin{bmatrix} \dot{x}(t) \\ \dot{\tilde{x}}(t) \end{bmatrix} =$

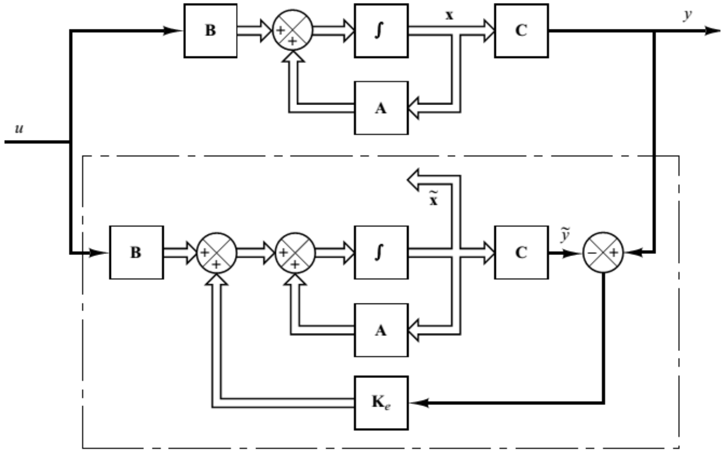
$$\begin{bmatrix} A & -bK \\ lc & A - lc - bK \end{bmatrix} \begin{bmatrix} x(t) \\ \tilde{x}(t) \end{bmatrix} + \begin{bmatrix} b \\ b \end{bmatrix} v(t) \quad \begin{bmatrix} x(t_0) \\ \tilde{x}(t_0) \end{bmatrix} = \begin{bmatrix} x_0 \\ \tilde{x}_0 \end{bmatrix}$$

Quad Form: $ax^2 + bx + c = 0 \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$,
Steady-state error is defined as the difference between the input (command) and the output of a system in the limit as time goes to infinity. $x_1(1) = x_2(0)$.









Full-order state observer