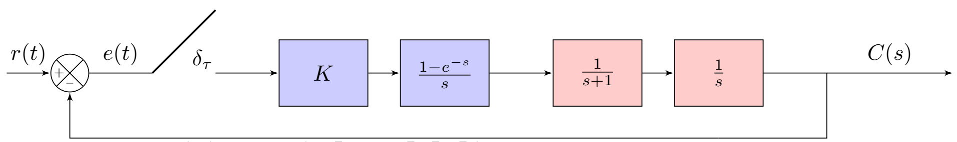
Z-transform

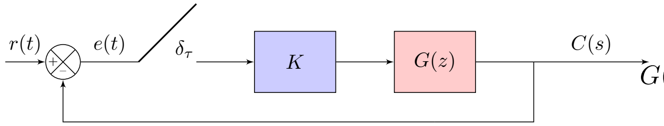
$\mathcal{Z}\{f_1(t) \pm f_2(t)\} = F_1(z) + F_2(z)$	Addition
$\mathcal{Z}\{af(t)\} = aF(z)$	Multiplication by a Constant
$\mathcal{Z}\{f(t-nT)\} = z^{-n}F(z)$	Shifting
$\mathcal{Z}\{f(t+kT)\} = z^kF(z) - z^kf(0) - \cdots - zf(kT-T)$	Shifting (cont'd)
$\mathcal{Z}\{e^{\mp at}f(t)\} = F(ze^{\pm at})$	Complex Translation
$\lim_{k\to\infty} f(kT) = \lim_{z\to 0} F(z)$	Initial Value Theorem
If $(1-z^{-1})F(z)$ has all singularities inside unit disk $\lvert z\rvert = 1$, then $\lim_{k\to\infty} f(kT) = \lim_{z\to 1}(1-z^{-1})F(z)$	Final Value Theorem
$\mathcal{Z}\left\{\frac{\partial}{\partial a}f(t,a)\right\} = \frac{\partial}{\partial a}F(z,a)$	Partial differentiation

Sampling Theory

$\mathcal{Z}\{G_1(s)G_2(s)\} = G_1G_2(z) = G_2G_1(z)$ In General $G_1(z)G_2(z) \neq G_1G_2(z)$



$$G(z) = \mathcal{Z}\left\{\left(\frac{1-e^{-s}}{s}\right)\left[\frac{1}{s+1}\right]\left[\frac{1}{s}\right]\right\} \to G_1(z) = (1-z^{-1})\mathcal{Z}\{G_{rest}(s)\}$$



$$G(z) = \frac{ze^{-1} + (1-ze^{-1})}{(z-1)(z-e^{-1})}$$

$$\text{ZOH(zero-hold-system)} \quad f^*(t) = \sum_{k=-\infty}^{\infty} f(kT)\delta(t-kT) \quad G_h(s) = \frac{1-e^{-Ts}}{s}$$

Stability Test for Digital Systems

$$P(z) = a_0z^n + a_1z^{n-1} + \cdots + a_{n-1}z + a_n \quad G(z) = \frac{A(z)}{P(z)}$$

Stability Condition: $P(z) \neq 0 \quad |z| \geq 1$ (Draw Unit Circle to test stability)

Routh-Stability in Digital Domain: $s = \dfrac{z+1}{z-1} \quad z = \dfrac{s+1}{s-1}$

Jury-Marden Table Uses function P of z

$$b_k = \det\begin{bmatrix} a_n & a_{n-1-k} \\ a_0 & a_{k+1}\end{bmatrix}$$

$k = 0, 1, \cdots n-1$

$$c_k = \det\begin{bmatrix} b_{n-1} & b_{n-2-k} \\ b_0 & b_{k+1}\end{bmatrix}$$

$k = 0, 1, \cdots n-1$

$$q_k = \det\begin{bmatrix} p_3 & p_{2-k} \\ p_0 & p_{k+1}\end{bmatrix}$$

$k = 0, 1, 2$

Row	z^0	z^1	z^2		z^{n-2}	z^{n-1}	z^n
1	a_n	a_{n-1}	a_{n-2}	$\cdots$	a_2	a_1	a_0
2	a_0	a_1	a_2	$\cdots$	a_{n-2}	a_{n-1}	a_n
3	b_{n-1}	b_{n-2}	b_{n-3}	$\cdots$	b_1	b_0	
4	b_0	b_1	b_2	$\cdots$	b_{n-2}	b_{n-1}	
5	c_{n-2}	c_{n-3}	c_{n-4}	$\cdots$	c_0		
6	c_0	c_1	c_2	$\cdots$	c_{n-2}		
2n-5	p_3	p_2	p_1	p_0			
2n-4	p_0	p_1	p_2	p_3			
2n-3	q_2	q_1	q_0				

Necessary and Sufficient Condition for Stability

1. $|a_n| < |a_0|$
2. $P(1) > 0$
3. $P(-1) > 0$ for n even; < 0 for n odd
4. $b_{n-1} > |b_0|, |c_{n-2}| > |c_0|, \cdots |q_2| > |q_0|$

Special Case n =2

$P(z) = a_0z^2 + a_1z + a_2$

z^0	z^1	z^2
a_2	a_1	a_0

$P(z) \neq 0$ for $|z| \geq 1$ if and only if

1. $|a_2| < |a_0|$
2. $P(1) > 0$
3. $P(-1) > 0 \quad (n = 2)$

Root Locus presents the poles of the closed loop system when the gain K changes from zero to infinity.

Construction of the Root Locus

Open loop transfer function KH $(s)\,G\,(s) = K\dfrac{B(s)}{A(s)}$

m: the order of the **open-loop** numerator polynomial.

n: the order of the **open-loop** denominator polynomial. $q = n - m$

Rule 1: number of branches equals the number of poles of the open-loop transfer function

Rule 2: If the total number of poles and zeros of the open-loop system to the right of the s-point on the real axis is odd, then this point lies on the locus.

Rule 3: The locus starting point (K=0) are at the open-loop poles and the locus ending points (K=∞) are at the open loop zeros and n-m branches terminate at infinity.

Rule 4 and 5: Slope of asymptotes of root locus as 's' approaches infinity. Abscissa of the intersection between asymptotes of root locus and real-axis.

$$\sigma = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{q} \quad \theta = \pm r\frac{180}{q} \quad \text{where r=1, 3, 5}$$

Rule 6: Break-away and break-in points. From the characteristic equation

$$f\,(s) = A\,(s) + KB\,(s) = 0 \quad \textit{and} \quad K = -\frac{A\,(s)}{B\,(s)}$$

The break-away and break-in points can be found from

$$\frac{\mathrm{dK}}{\mathrm{ds}} = -\frac{A'\,(s)\,B\,(s) - A\,(s)\,B'\,(s)}{B^2\,(s)} = 0$$

Rule 7: Angle of departure from complex poles or zeros. Subtract from 180^o the sum of all angles from all other zeros and poles of the open-loop system to the complex pole (or zero) with appropriate signs.

$$\text{Z-transform: Definition} \quad F(z) = Z[f(t)] - Z[f(kT)] = \sum_{k=0}^{\infty} f(kT)z^{-k}$$

$$e^*(\infty) = \lim_{z\to 1}(1-z^{-1})E(z)$$

$$K_p = \lim_{z\to 1} G(z), \quad e^*(\infty) = \frac{1}{1+K_p}$$

$$e^*(\infty) = \frac{1}{K_v}, \quad K_v = \frac{1}{T}\lim_{z\to 1}(z-1)G(z)$$

$$e^*(\infty) = \frac{1}{K_a}, \quad K_a = \frac{1}{T^2}\lim_{z\to 1}(z-1)^2 G(z)$$

Linear Factor Rule. For each factor of Q of the form $(ax+b)^m$, the partial fraction decomposition contains the following sum of m partial fractions:

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_m}{(ax+b)^m},$$

where the A_i are constants to be determined.

Quadratic Factor Rule. For each factor of Q of the form $(ax^2+bx+c)^m$, where ax^2+bx+c is an irreducible quadratic, the partial fraction decomposition contains the following sum of m partial fractions:

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_mx+B_m}{(ax^2+bx+c)^m},$$

where the A_i and B_i are constants to be determined.

$$x(k+2) - \frac{3}{2}x(k+1) + \frac{1}{2}x(k) = u(k), (x(0)=1, x(1) =$$

$$[z^2X(z) - z^2x(0) - zx(1)] - \frac{3}{2}(zX(z) - zx(0)] + \frac{1}{2}X(z)$$

$$[z^2 - 1.5z + 0.5z]X(z) = \frac{z}{z-1} + z^2 + (2.5-1.5)z$$

$$X(z) = \frac{z[1+(z+1)(z-1]}{(z-1)(z-1)(z-0.5)} = \frac{z^3}{(z-1)^2(z-0.5)}$$

$$\frac{X(z)}{z} = \frac{z^2}{(z-1)^2(z-0.5)} = \frac{A_{11}}{(z-1)^2} + \frac{A_{12}}{z-1} + \frac{A_{13}}{z-0.5}$$

Geometric Sum $\sum_{k=-N}^{N} ar^{k-1} = a\frac{1-r^N}{1-r}$ $\sum_{i=0}^{\infty} a^i = \frac{1}{1-a}$

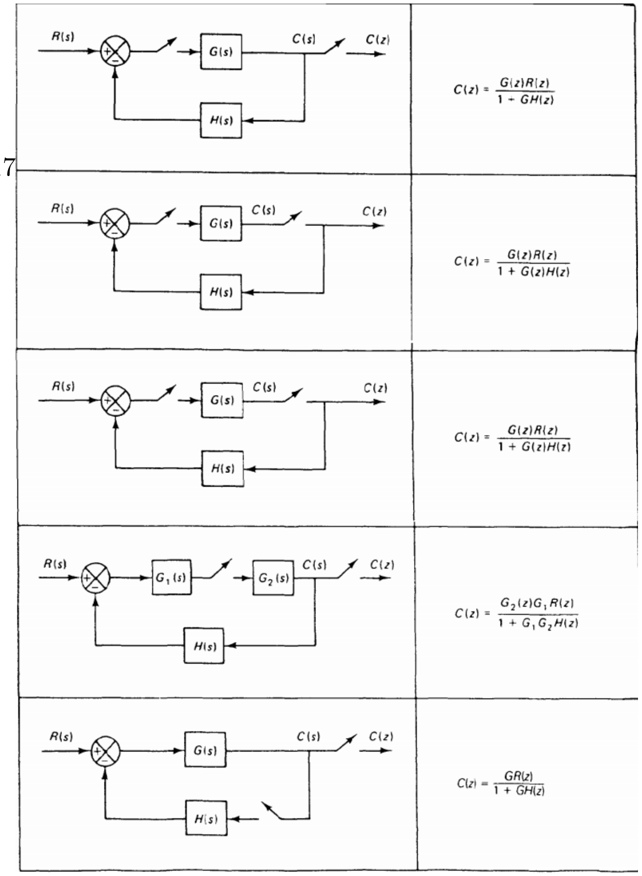
Example Partial Fractions

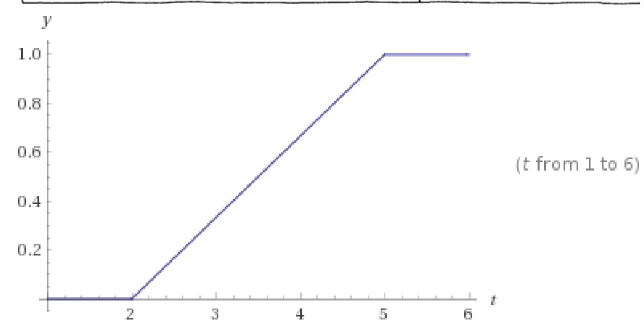
$$\frac{20}{(s+3)\,(s^2+6\,s+25)} \to \frac{5}{4\,(s+3)} - \frac{\frac{5\,s}{4}+\frac{15}{4}}{s^2+6\,s+25}$$

$$\frac{5\,z}{4\,(z-\mathrm{e}^{-3})} + \frac{5\,z\,\mathrm{e}^3\,(\cos(4) - z\,\mathrm{e}^3)}{4\,(\mathrm{e}^6\,z^2 - 2\,\cos(4)\,\mathrm{e}^3\,z+1)} \text{ Z-table} = 17$$

	$f(t)$	$F(s)$	$F(z)$	$f(kT)$
1.	$u(t)$	$\frac{1}{s}$	$\frac{z}{z-1}$	$u(kT)$
2.	t	$\frac{1}{s^2}$	$\frac{Tz}{(z-1)^2}$	kT
3.	t^n	$\frac{n!}{s^{n+1}}$	$\lim_{a\to 0}(-1)^n\frac{d^n}{da^n}\left[\frac{z}{z-e^{-aT}}\right]$	$(kT)^n$
4.	e^{-at}	$\frac{1}{s+a}$	$\frac{z}{z-e^{-aT}}$	e^{-akT}
5.	t^ne^{-at}	$\frac{n!}{(s+a)^{n+1}}$	$(-1)^n\frac{d^n}{da^n}\left[\frac{z}{z-e^{-aT}}\right]$	$(kT)^ne^{-akT}$
6.	$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$	$\frac{z\sin\omega T}{z^2-2z\cos\omega T+1}$	$\sin\omega kT$
7.	$\cos\omega t$	$\frac{s}{s^2+\omega^2}$	$\frac{z(z-\cos\omega T)}{z^2-2z\cos\omega T+1}$	$\cos\omega kT$
8.	$e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$	$\frac{ze^{-aT}\sin\omega T}{z^2-2ze^{-aT}\cos\omega T+e^{-2aT}}$	$e^{-akT}\sin\omega kT$
9.	$e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$	$\frac{z^2-ze^{-aT}\cos\omega T}{z^2-2ze^{-aT}\cos\omega T+e^{-2aT}}$	$e^{-akT}\cos\omega kT$

Ch.eqn $= \Delta P(z) = z^2 + (K-4)z + 0.8 = 0$
Inputting $z=1$ and $z=-1$, $K = -0.8-1+4 = 2.2$, $K = (1)^2 + 4 + 0.8 = 5.8$, for stability $-1 < K < 1$.





$$\frac{1}{3}((t-2)u(t-2) - (t-5)u(t-5)) \to \frac{z^2+z+1}{3(z-1)z^4}$$

$$= \frac{1(z^{-3}+z^{-4}+z^{-5})}{3(1-z^{-1})} \quad \frac{1}{3}z^{-3} + \frac{2}{3}z^{-4} + z^{-5} + z^{-6} + \cdots$$

TABLE 2–1 TABLE OF z TRANSFORMS

	$X(s)$	$x(t)$	$x(kT)$ or $x(k)$	$X(z)$
1.	—	—	Kronecker delta $\delta_0(k)$ $1, \quad k = 0$ $0, \quad k \neq 0$	1
2.	—	—	$\delta_0(n-k)$ $1, \quad n = k$ $0, \quad n \neq k$	z^{-k}
3.	$\frac{1}{s}$	$1(t)$	$1(k)$	$\frac{1}{1-z^{-1}}$
4.	$\frac{1}{s+a}$	e^{-at}	e^{-akT}	$\frac{1}{1-e^{-aT}z^{-1}}$
5.	$\frac{1}{s^2}$	t	kT	$\frac{Tz^{-1}}{(1-z^{-1})^2}$
6.	$\frac{2}{s^3}$	t^2	$(kT)^2$	$\frac{T^2z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$
7.	$\frac{6}{s^4}$	t^3	$(kT)^3$	$\frac{T^3z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4}$
8.	$\frac{a}{s(s+a)}$	$1-e^{-at}$	$1-e^{-akT}$	$\frac{(1-e^{-aT})z^{-1}}{(1-z^{-1})(1-e^{-aT}z^{-1})}$
9.	$\frac{b-a}{(s+a)(s+b)}$	$e^{-at}-e^{-bt}$	$e^{-akT}-e^{-bkT}$	$\frac{(e^{-aT}-e^{-bT})z^{-1}}{(1-e^{-aT}z^{-1})(1-e^{-bT}z^{-1})}$
10.	$\frac{1}{(s+a)^2}$	te^{-at}	kTe^{-akT}	$\frac{Te^{-aT}z^{-1}}{(1-e^{-aT}z^{-1})^2}$
11.	$\frac{s}{(s+a)^2}$	$(1-at)e^{-at}$	$(1-akT)e^{-akT}$	$\frac{1-(1+aT)e^{-aT}z^{-1}}{(1-e^{-aT}z^{-1})^2}$
12.	$\frac{2}{(s+a)^3}$	t^2e^{-at}	$(kT)^2e^{-akT}$	$\frac{T^2e^{-aT}(1+e^{-aT}z^{-1})z^{-1}}{(1-e^{-aT}z^{-1})^3}$
13.	$\frac{a^2}{s^2(s+a)}$	$at-1+e^{-at}$	$akT-1+e^{-akT}$	$\frac{[(aT-1+e^{-aT})+(1-e^{-aT}-aTe^{-aT})z^{-1}]z^{-1}}{(1-z^{-1})^2(1-e^{-aT}z^{-1})}$
14.	$\frac{\omega}{s^2+\omega^2}$	$\sin \omega t$	$\sin \omega kT$	$\frac{z^{-1}\sin \omega T}{1-2z^{-1}\cos \omega T+z^{-2}}$
15.	$\frac{s}{s^2+\omega^2}$	$\cos \omega t$	$\cos \omega kT$	$\frac{1-z^{-1}\cos \omega T}{1-2z^{-1}\cos \omega T+z^{-2}}$
16.	$\frac{\omega}{(s+a)^2+\omega^2}$	$e^{-at}\sin \omega t$	$e^{-akT}\sin \omega kT$	$\frac{e^{-aT}z^{-1}\sin \omega T}{1-2e^{-aT}z^{-1}\cos \omega T+e^{-2aT}z^{-2}}$
17.	$\frac{s+a}{(s+a)^2+\omega^2}$	$e^{-at}\cos \omega t$	$e^{-akT}\cos \omega kT$	$\frac{1-e^{-aT}z^{-1}\cos \omega T}{1-2e^{-aT}z^{-1}\cos \omega T+e^{-2aT}z^{-2}}$
18.			a^k	$\frac{1}{1-az^{-1}}$
19.			a^{k-1} $k = 1, 2, 3, \ldots$	$\frac{z^{-1}}{1-az^{-1}}$
20.			ka^{k-1}	$\frac{z^{-1}}{(1-az^{-1})^2}$
21.			k^2a^{k-1}	$\frac{z^{-1}(1+az^{-1})}{(1-az^{-1})^3}$
22.			k^3a^{k-1}	$\frac{z^{-1}(1+4az^{-1}+a^2z^{-2})}{(1-az^{-1})^4}$
23.			k^4a^{k-1}	$\frac{z^{-1}(1+11az^{-1}+11a^2z^{-2}+a^3z^{-3})}{(1-az^{-1})^5}$
24.			$a^k \cos k\pi$	$\frac{1}{1+az^{-1}}$ $\frac{z}{z+a}$
25.			$\frac{k(k-1)}{2!}$	$\frac{z^{-2}}{(1-z^{-1})^3}$