# Functional Programming
## An Introduction

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# Functional Programming
## Overview

What is functional programming?

- Functions are first class (objects).
  - That is, everything you can do with "data" can be done with functions themselves (such as passing a function to another function).
- Recursion is used as a primary control structure.
  - In some languages, no other "loop" construct exists.
- There is a focus on list processing.
  - Lists are often used with recursion on sub-lists as a substitute for loops.
- "Pure" functional languages eschew side-effects.
  - This excludes assignments to track the program state.
  - This discourages the use of statements in favor of expression evaluations.

Whys

- All these characteristics make for more rapidly developed, shorter, and less bug-prone code.
- A lot easier to prove formal properties of functional languages and programs than of imperative languages and programs.






# Functional Programming
## Overview

The basic idea is to model everything as a "mathematical function".

There are only two linguistic constructs:

- abstraction, used to define the function;
- application, used to call it.

No state concept

- this means no assignments are allowed
- variables are just names.

E.g., in $f(x) = x + 1$ the name $f$ is irrelevant,

- the function $g(x) = x + 1$ represents the same function;
- it can be referred as $x \mapsto x + 1$.






# Functional Programming
## λ-Calculus [Church and Kleene ~1930].

λ-expressions are made of constants, variables, λ, . and parenthesis

1. if $x$ is a variable or a constant then $x$ is a λ-expression;
2. if $x$ is a variable and $M$ is a λ-expression then $\lambda x.M$ is a λ-expression;
3. if $M, N$ are λ-expressions then $(MN)$ is a λ-expression.

Abstraction ∉ Application

λ-calculus provides only two basic operations: abstraction and application

- $\lambda x.x + 1$ is an example of abstraction that defines the successor;
- $(\lambda x.x + 1)7$ is an example of application that calculates the successor of 7;
  - application is left-associative, i.e., $MNP \equiv (MN)P$.

Binding, Free and Bound Variables

- in $\lambda x.xy$ $x$ is a bound variable whereas $y$ is unbound (free)
- in $\lambda x.\lambda y.xy$ (for short $\lambda xy.xy$) both variables are bound;
- in $(\lambda x.M)y$, all the occurrences of $x$ in $M$ are replaced by $y$ (denoted as $M[x/y]$) and brings to $M[x/y]$ as a result
  - e.g., $(\lambda x.x + 1)7 \to x + 1[x/7] \to 7 + 1 \to 8$.

# Functional Programming
## ML [Milner et al. ~1970]



ML is a general-purpose functional programming language developed by Robin Milner et al. in the 70ies.

- ML is the acronym for metalanguage, since it is an abstraction on polymorphic λ-calculus.

Features of ML include:

- a call-by-value evaluation strategy, first-class functions, parametric polymorphism,
- static typing, type inference, algebraic data types, pattern matching, and exception handling.

ML uses eager evaluation, which means that all sub-expressions are always evaluated.

- lazy evaluation can be achieved through the use of closures.

We will use OCaML (`http://caml.inria.fr`).



# Functional Programming
## ML/OCaML [Leroy et al. ~1980]



OCaML is an implementation of ML with extra functionality (object-orientation, modules, imperative statements, ...).

OCaML comes with

- an interpreter (`ocaml`) and
- a compiler (`ocamlc`).

```
let main() = print_string("Hello World in ML Style\n");;
main();;
```

```
[12:28]cazzola@surtur:~/lp/ml>ocamlc -o helloworld helloworld.ml
[12:28]cazzola@surtur:~/lp/ml>ls
helloworld*  helloworld.cmi  helloworld.cmo  helloworld.ml
[12:28]cazzola@surtur:~/lp/ml>helloworld
Hello World in ML Style.
[12:28]cazzola@surtur:~/lp/ml>rlwrap ocaml
        Objective Caml version 4.12.0

# let main() =  print_string("Hello World in ML Style.\n");;
val main : unit -> unit = <fun>
# main();;
Hello World in ML Style.
- : unit = ()
# ^D
[12:29]cazzola@surtur:~/lp/ml>
```



# Functional Programming
## ML Functions



ML derives directly from λ-calculus:

- functions are defined independently of their name

```
let succ = fun x -> x+1;;
let succ x = x+1;;
```

functions can be aliased

```
let succ' = succ;;
```

- calls are simply the application of the arguments to the function

```
succ 2;;
(fun x -> x+1) 2;;
```

```
[16:19]cazzola@surtur:~/lp/ml>ocaml
        Objective Caml version 4.12.0
# let succ = fun x -> x+1;;
val succ : int -> int = <fun>
# succ 7;;
- : int = 8
# succ -1;;
Error: This expression has type int -> int
       but an expression was expected of type int
```



# Functional Programming
## Name Scope



Scoping

- a new binding to a name hides the old bind;
- static binding is used in function definition (closure).
  - i.e., a triplet: args list, function body and environment (x, x+y, [5/y]).

```
[17:01]cazzola@surtur:~/lp/ml>ocaml
        OCaML version 4.12.0
# let f x = 5;;
val f : 'a -> int = <fun>
# let f x = 7;;
val f : 'a -> int = <fun>
# f 1;;
- : int = 7
# let y = 5;;
val y : int = 5
# let addy = fun x -> x+y;;
val addy : int -> int = <fun>
# addy 8;;
- : int = 13
# let y=10;;
val y : int = 10
# addy 8;;
- : int = 13
# (fun x -> x+y) 8;;
- : int = 18
[17:57]cazzola@surtur:~/lp/ml>
```

# Functional Programming
## High-Order Functions



In ML functions are first class citizens

- i.e., they can be used as values;
- when passed to a function this is an high-order function.

```
let compose f g x = f (g x);;
let compose' (f, g) x = f (g x);;
```

```
[15:30]cazzola@surtur:~/lp/ml>ocaml
# let compose f g x = f (g x);;
val compose : ('a -> 'b) -> ('c -> 'a) -> 'c -> 'b = <fun>
# let compose' (f,g) x = f (g x);;
val compose' : ('a -> 'b) * ('c -> 'a) -> 'c -> 'b = <fun>
# let succ = fun x -> x +1;;
val succ : int -> int = <fun>
# let plus1 = compose succ;;
val plus1 : ('_a -> int) -> '_a -> int = <fun>
# let plus1' = compose' succ;;
Error: This expression has type int -> int
       but an expression was expected of type ('a -> 'b) * ('c -> 'a)
# let plus2 = plus1 succ;;
val plus2 : int -> int = <fun>
# let plus2' = compose' (succ, succ);;
val plus2' : int -> int = <fun>
# plus2 7;;
- : int = 9
# plus2' 7;;
- : int = 9
```



# Functional Programming
## Functions & Pattern Matching



Functions can be defined by pattern matching.

```
match expression with
  | pattern when boolean expression -> expression
  | pattern when boolean expression -> expression
```

Patterns can contain

- constants, tuples, records, variant constructors and variable names;
- a catchall pattern denoted _ that matches any value; and
- sub-patterns containing alternatives, denoted $\mathbf{pat_1 \,|\, pat_2}$.

When a pattern matches

- the corresponding expression is returned.
- the (optional) when clause is a guard on the matching; it filters out undesired matchings.

```
let invert x =
  match x with
    | true -> false
    | false -> true ;;
let invert' = function
  true -> false | false -> true ;;
```





# Recursion
## Definition: Recursive Function



A function is called recursive when it is defined through itself.

Example: Factorial.

- $5! = 5 * 4 * 3 * 2 * 1$
- Note that: $5! = 5 * 4!$, $4! = 4 * 3!$ and so on.

Potentially a recursive computation.

From the mathematical definition:

$$n! = \begin{cases} 1 & \text{if } n=0, \\ n*(n-1)! & \text{otherwise.} \end{cases}$$

When n=0 is the base of the recursive computation (axiom) whereas the second step is the inductive step.





# Recursion
## What in ML?



Still, a function is recursive when its execution implies another invocation to itself.

- directly, i.e. in the function body there is an explicit call to itself;
- indirectly, i.e. the function calls another function that calls the function itself (mutual recursion).

```
let rec fact(n) = if n<=1 then 1 else n*fact(n-1);;
let main() =
  print_endline("fact(  5) : - "^string_of_int(fact(5)));
  print_endline("fact(  7) : - "^string_of_int(fact(7)));
  print_endline("fact( 15) : - "^string_of_int(fact(15)));
  print_endline("the largest admissible integer is ;- "^string_of_int(max_int));
  print_endline("fact( 25) : - "^string_of_int(fact(25)));;
main();;
```

```
[11:31]cazzola@surtur:~/lp/ml>ocamlc -o fact fact.ml
[11:31]cazzola@surtur:~/lp/ml>fact
fact(  5) : - 120
fact(  7) : - 5040
fact( 15) : - 1307674368000
the largest admissible integer is ;- 4611686018427387903
fact( 25) : - -2188836759280812032
[11:31]cazzola@surtur:~/lp/ml>
```

# Recursion
## Execution: What's Happen?

```
[11:45]cazzola@surtur:~/lp/ml>ocaml
        OCaML version 4.12.0
# let rec fact(n) =
    if n<=1
    then 1
    else n*fact(n-1);;
val fact : int -> int = <fun>
# fact 4;;
- : int = 24
[11:46]cazzola@surtur:~/lp/ml>
```

It runs `fact(4)`:
- a new frame with n = 4 is pushed on the stack;
- n is greater than 1;
- it calculates `4*fact(3)`, it returns 24

It runs `fact(3)`:
- a new frame with n = 3 is pushed on the stack;
- n is greater than 1;
- it calculates `3*fact(2)`, it returns 6

It runs `fact(2)`:
- a new frame with n = 2 is pushed on the stack;
- n is greater than 1;
- it calculates `2*fact(1)`, it returns 2

It runs `fact(1)`:
- a new frame with n = 1 is pushed on the stack;
- n is equal to 1;
- it returns 1





# Recursion
## Side Notes on the Execution

At any invocations the run-time environment creates an activation record or frame used to store the current values of:
- local variables, parameters and the location for the return value.

To have a frame for any invocation permits to:
- trace the execution flow;
- store the current state and restore it after the execution;
- avoid interferences on the local calculated values.

Warning:

Without any stopping rule, the inductive step will be applied "forever".
- Actually, the inductive step is applied until the memory reserved by the virtual machine is full.





# Recursion
## Case Study: Fibonacci Numbers

Leonardo Pisano, known as Fibonacci, in 1202 in his book "Liber Abaci" faced the (quite unrealistic) problem of determining:

> «how many pairs of rabbits can be produced from a single pair if each pair begets a new pair each month and every new pair becomes productive from the second month on, supposing that no pair dies»

To introduce a sequence whose i-th member is the sum of the 2 previous elements in the sequence. The sequence will be soon known as the Fibonacci numbers.





# Recursion
## Case Study: Fibonacci Numbers (Cont'd)

Fibonacci numbers are recursively defined:

$$f(n) = \begin{cases} 0 & \text{if } n=0, \\ 1 & \text{if } n=1 \text{ or } n=2, \\ f(n-1) + f(n-2) & \text{otherwise.} \end{cases}$$

The implementation comes forth from the definition:

```
open List;;
let rec fibo(n) = if n<=1 then n else fibo(n-1) + fibo(n-2);;
let main() =
  let in's = [5; 7; 15; 25; 30] in
    for i=0 to List.length in's -1 do
      print_endline(
        "fibo("^string_of_int(nth in's i)^") :- "^string_of_int(fibo(nth in's i)));
    done;;
main();;
```

```
[16:08]cazzola@surtur:~/lp/ml>ocamlc -o fibo fibo.ml
[16:14]cazzola@surtur:~/lp/ml>fibo
fibo(5) :- 5
fibo(7) :- 13
fibo(15) :- 610
fibo(25) :- 75025
fibo(30) :- 832040
[16:14]cazzola@surtur:~/lp/ml>
```



# Recursion
## Recursion Easier & More Elegant

The recursive solution is more intuitive:

```
let rec fibo(n) = if n<=1 then n else fibo(n-1) + fibo(n-2);;
```

The iterative solution is more cryptic:

```
let fibo(n) =
  let fib' = ref 0 and fib''= ref 1 and fib = ref 1 in
    if n<=1 then n
    else
      (for i=2 to n do
        fib := !fib' + !fib'';
        fib' := !fib'';
        fib'' := !fib;
      done;
      !fib);;
```

But ...





# Recursion
## Tail Recursion

The iterative implementation is more efficient:

```
[18:22]cazzola@surtur:~/lp/ml>time time_ifibo 50
fibo(50) :- 12586269025
0.000u 0.006s 0:00.00 0.0%      0+0k 0+0io 0pf+0w
[18:22]cazzola@surtur:~/ml/lp>time time_rfibo 50
fibo(50) :- 12586269025
1605.211u 1.688s 26:48.62 99.8% 0+0k 0+0io 0pf+0w
[18:49]cazzola@surtur:~/lp/ml>
```

The overhead is mainly due to the creation of the frame but this also affects the occupied memory.

This can be avoided with a tail recursive solution:

```
let rec trfiboaux n m fib_m' fib_m =
  if (n=m) then fib_m
  else (trfiboaux n (m+1) fib_m (fib_m'+fib_m));;
let fibo n = if n<=1 then 1 else trfiboaux n 1 0 1;;
```

```
[16:59]cazzola@surtur:~/lp/ml>time trfibo 50
fibo(50) :- 12586269025
0.000u 0.005s 0:00.00 0.0%      0+0k 0+0io 0pf+0w
[16:59]cazzola@surtur:~/lp/ml>
```

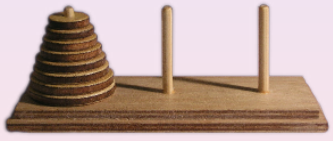




# The Towers of Hanoi
## Definition (É douard Lucas, 1883)

### Problem Description

There are 3 available pegs and several holed disks that should be stacked on the pegs. The diameter of the disks differs from disk to disk each disk can be stacked only on a larger disk.



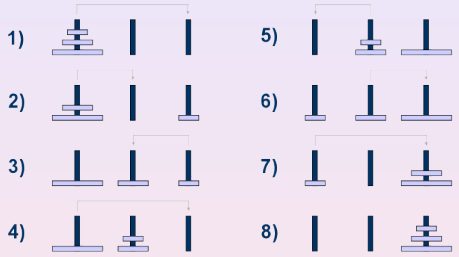
The goal of the game is to move all the disks, one by one, from the first peg to the last one without ever violate the rules.





# The Towers of Hanoi
## The Recursive Algorithm

### 3-Disks Algorithm




### n-Disks Algorithm

Base: n=1, move the disk from the source (S) to the target (T);

Step: move n-1 disks from S to the first free peg (F), move the last disk to the target peg (T), finally move the n-1 disks from F to T.



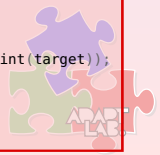
# The Towers of Hanoi
## ML/OCaML Implementation

```
type peg = string*string*string ;;
type pegs = {mutable src: peg; mutable trg: peg; mutable aux:peg} ;;

let nth(x,y,z) n = match n with 1 -> x | 2 -> y | 3 -> z ;;
let set_nth(x,y,z) w n = match n with 1 -> (w,y,z) | 2 -> (x,w,z) | 3 -> (x,y,w) ;;

let set_nth_peg ps p n =
  match n with 1 -> ps.src <- p| 2 -> ps.trg <- p | 3 -> ps.aux <- p ;;
let nth_peg ps n = match n with 1 -> ps.src | 2 -> ps.trg | 3 -> ps.aux ;;

let top(x,y,z) =
  match x,y,z with "0","0","0" -> 3 | "0","0", _ -> 2 | "0", _, _ -> 1 | _, _, _ -> 0 ;;
let p:pegs={src=("1","2","3"); trg=("0","0","0"); aux=("0","0","0")} in
  let rec display ps n =
    if n <4 then (
      print_endline("  "^nth ps.src n^"    "^nth ps.trg n^"    "^nth ps.aux n);
      display ps (n+1);)
  and move ps source target =
    let s=(top (nth_peg ps source))+1 and t= top (nth_peg ps target) in (
      set_nth_peg ps (set_nth (nth_peg ps target) (nth (nth_peg ps source) s) t) target;
      set_nth_peg ps (set_nth (nth_peg ps source) "0" s) source;
      display ps 1;)
  and move_disks ps disks source target aux =
    if disks <=1 then (
      print_endline("moving from "^string_of_int(source)^" to "^string_of_int(target));
      move ps source target;)
    else (
      move_disks ps (disks-1) source aux target;
      print_endline("moving from "^string_of_int(source)^" to "^string_of_int(target));
      move ps source target;
      move_disks ps (disks-1) aux target source;
    );
  in (print_endline("Start!!!");display p 1; move_disks p 3 1 3 2;) ;;
```







# The Towers of Hanoi
## 3-Disks Run

```
[16:21]cazzola@surtur:~/lp/ml>ocamlc -o hanoi2 hanoi2.ml
[16:21]cazzola@surtur:~/lp/ml>hanoi2
Start!!!            moving from 1 to 2   moving from 1 to 3   moving from 2 to 3
  1    0    0          0    0    0          0    0    0          0    0    0
  2    0    0          0    0    0          0    1    0          0    0    2
  3    0    0          3    2    1          0    2    3          1    0    3
moving from 1 to 3   moving from 3 to 2   moving from 2 to 1   moving from 1 to 3
  0    0    0          0    0    0          0    0    0          0    0    1
  2    0    0          0    1    0          0    0    0          0    0    2
  3    0    1          3    2    0          1    2    3          0    0    3
[16:21]cazzola@surtur:~/lp/ml>
```





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