3.6 Cardinality of sets

Definition 3.6.1 (Equal cardinality).

We say that two sets X and Y have equal cardinality iff there exists a bijection $f : X \to Y$ from X to Y.

Proposition 3.6.4

Let X, Y, Z be sets. Then X has equal cardinality with X. If X has equal cardinality with Y, then Y has equal cardinality with X. If X has equal cardinality with Y and Y has equal cardinality with Z, then X has equal cardinality with Z.

Definition 3.6.5

Let n be a natural number. A set X is said to have cardinality n, iff it has equal cardinality with $\{i \in \mathbf{N} : 1 \leq i \leq n\}$. We also say that X has n elements iff it has cardinality n.

Proposition 3.6.8 (Uniqueness of cardinality).

Let X be aset with some cardinality n. Then X cannot have any other cardinality, i.e., X cannot have cardinality m for any $m \neq n$.

Lemma 3.6.9

Suppose that $n \geq 1$, and X has cardinality n. Then X is non-empty, and if x is any element of X, then the set $X - \{x\}$ (i.e., X with the element x removed) has cardinality $n - 1$.

Definition 3.6.10 (Finite sets).

A set is finite iff it has cardinality n for some natural number n; otherwise, the set is called infinite. If X is a finite set, we use $\#(X)$ to denote the cardinality of X.

Theorem 3.6.12

The set of natural numbers $\mathbf{N}$ is infinite.

Proposition 3.6.14 (Cardinal arithmetic).

See Exercise 3.6.4.

Exercises

Exercise 3.6.1

Prove Proposition 3.6.4.

- X has equal cardinality with X.

  Proof. Define function $f : X \to X$ such that for each $x \in X$, $f(x) = x$. For $x_1 \neq x_2$, $f(x_1) = x_1$ and $f(x_2) = x_2$. So $f(x_1) \neq f(x_2)$. Therefore, f is injective. By definition, for every $x \in X$, $f(x) = x$. So f is surjective. Thus, f is bijective and X has equal cardinality with X. □

- If X has equal cardinality with Y, then Y has equal cardinality with X.

  Proof. Since X has equal cardinality with Y, there exists a bijective function $f : X \to Y$. Since f is bijective, there exists $f^{-1} : Y \to X$. For $y_1, y_2 \in Y$, if we have $f^{-1}(y_1) = f^{-1}(y_2)$, by the definition of function, $f(f^{-1}(y_1)) = f(f^{-1}(y_2))$, then $y_1 = y_2$. So f^{-1} is injective. For every $x \in X$, we have $f(x) \in Y$ such that $f^{-1}(f(x)) = x$. So f^{-1} is surjective. Thus, f^{-1} is bijective and Y has equal cardinality with X. □

- If X has equal cardinality with Y and Y has equal cardinality with Z, then X has equal cardinality with Z.

  Proof. Since X has equal cardinality with Y, there exists a bijective function $f : X \to Y$. Since Y has equal cardinality with Z, there exists a bijective function $g : Y \to Z$. By Exercise 3.3.7, $g \circ f : X \to Z$ is also bijective. Thus, X has equal cardinality with Z. □

Exercise 3.6.2

Show that a set X has cardinality 0 if and only if X is the empty set.

Proof. By definition 3.6.5, X has n elements iff it has cardinality n. So since X has cardinality 0, it has no element in it which means X is the empty set. On the other hand, if X is the empty set, it has 0 element and thus has cardinality 0. $\square$

Exercise 3.6.3

Let n be a natural number, and let $f : \{i \in \mathbf{N} : 1 \leq i \leq n\} \to \mathbf{N}$ be a function. Show that there exists a natural number M such that $f(i) \leq M$ for all $1 \leq i \leq n$. Thus finite subsets of the natural numbers are bounded.

Proof. For function $f : \{i \in \mathbf{N} : 1 \leq i \leq n\} \to \mathbf{N}$, we claim that $M = \max\{f(1), \ldots, f(n)\}$. Induct on n. Base case: $n = 1$. Let $M = \max\{f(1)\} = f(1)$. $M \leq f(i)$ for $1 \leq i \leq 1$. This proves the base case. Then suppose inductively that there exists $M' = \max\{f(1), \ldots, f(n)\}$ is the upper bound for $f : \{i \in \mathbf{N} : 1 \leq i \leq n\} \to \mathbf{N}$. Now consider $f : \{i \in \mathbf{N} : 1 \leq i \leq n+1\} \to \mathbf{N}$. Let $M = \max\{M', f(n+1)\}$. For all $1 \leq i \leq n$, we have $f(i) \leq M' \leq M$. Also we have $f(n+1) \leq M$. Thus, $f(i) \leq M$ for all $1 \leq i \leq n+1$. This closes the induction. $\square$

Exercise 3.6.4

Prove proposition 3.6.14.

1. Let X be a finite set, and let x be an object which is not an element of X. Then $X \cup \{x\}$ is finite and $\#(X \cup \{x\}) = \#(X) + 1$.

   Proof. Use n to denote the cardinality of X. By Lemma 3.6.9, $\#(X) = \#((X \cup \{x\}) - \{x\}) = \#(X \cup \{x\}) - 1$. So $\#(X \cup \{x\}) = \#(X) + 1 = n + 1$ which is also a natural number. Thus, $X \cup \{x\}$ is finite and $\#(X \cup \{x\}) = \#(X) + 1$. $\square$

2. Let X and Y be finite sets. Then $X \cup Y$ is finite and $\#(X \cup Y) \leq \#(X) + \#(Y)$. If in addition X and Y are disjoint, then $\#(X \cup Y) = \#(X) + \#(Y)$.

Proof. Use m to denote the cardinality of $X = \{x_1, \ldots, x_m\}$ and n to denote the cardinality of $Y = \{y_1 \ldots, y_n\}$. Induct on n. The base case is when $\#(Y) = n = 0$. $\#(X \cup Y) = \#(X) \le \#(X) + \#(Y) = \#(X)$. Now suppose inductively $\#(X \cup Y) \le \#(X) + \#(Y)$ when $\#(Y) = n$ ($Y = \{y_1, \ldots, y_n\}$). Consider when $Y = \{y_1, \ldots, y_n, y_{n+1}\}$ and $\#(Y) = n+1$. If $y_{n+1} \in \{x_1, \ldots, x_m\} \cup \{y_1, \ldots, y_n\}$, then

$$\begin{aligned}\#(\{x_1, \ldots, x_m\} \cup \{y_1, \ldots, y_{n+1}\}) &= \#(\{x_1, \ldots, x_m\} \cup \{y_1, \ldots, y_n\}) \\ &\le m + n < m + (n+1) = \#(X) + \#(Y).\end{aligned}$$

So in this case, $\#(X \cup Y) < \#(X) + \#(Y)$. If $y_{n+1} \notin \{x_1, \ldots, x_m\} \cup \{y_1, \ldots, y_n\}$, then

$$\#(\{x_1, \ldots, x_m\} \cup \{y_1, \ldots, y_{n+1}\}) = \#(\{x_1, \ldots, x_m\} \cup \{y_1, \ldots, y_n\}) + 1,$$

since by induction hypothesis we have,

$$\#(\{x_1, \ldots, x_m\} \cup \{y_1, \ldots, y_n\}) \le m + n,$$

then

$$\#(\{x_1, \ldots, x_m\} \cup \{y_1, \ldots, y_{n+1}\}) \le m + (n+1) = \#(X) + \#(Y).$$

So in this case, $\#(X \cup Y) \le \#(X) + \#(Y)$. Thus, in both cases, we have $\#(X \cup Y) \le \#(X) + \#(Y)$. This closes the induction. Hence, since both X and Y are finite, $X \cup Y$ is also finite.

If X and Y are disjoint, by Lemma 3.6.9, we have

$$\begin{gathered}\#(X \cup Y - \{y_1\}) = \#(X \cup Y) - 1, \\ \#((X \cup Y - \{y_1\})) = \#(X \cup Y - \{y_1\}) - 1, \\ \vdots \\ \#((X \cup Y - \{y_1\} - \cdots - \{y_{n-1}\}) - \{y_n\}) = \#(X \cup Y - \cdots - \{y_{n-1}\}) - 1.\end{gathered}$$

Sum these n equations up, we have

$$\#(X) = \#((X \cup Y - \{y_1\} - \cdots - \{y_{n-1}\}) - \{y_n\}) = \#(X \cup Y) - \#(Y).$$

Thus, $\#(X \cup Y) = \#(X) + \#(Y)$. $\square$

3. Let X be a finite set, and let Y be a subset of X. Then Y is finite, and $\#(Y) \leq \#(X)$. If in addition $Y \neq X$, then we have $\#(Y) < \#(X)$.

   Proof. Assume $Y \neq X$. Denote $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_m\}$. Induct on n. When $n \leq m$, the statement is vacuously true. Suppose inductively that $\#(Y) < \#(X)$ is true. Consider when $X = \{x_1, \ldots, x_{n+1}\}$, $\#(\{x_1, \ldots, x_{n+1}\}) = \#(\{x_1, \ldots, x_n\})+1$. So $\#(\{y_1, \ldots, y_m\}) < \#(\{x_1, \ldots, x_n\}) < \#(\{x_1, \ldots, x_n\})+1 = \#(\{x_1, \ldots, x_{n+1}\})$. This closes the induction. For the case $Y = X$, $\#(Y) = \#(X)$, so $\#(Y) \leq \#(X)$. Since $\#(Y) \leq \#(X)$ and X is finite, Y is also finite. $\square$

4. If X is a finite set, and $f : X \to Y$ is a function, then $f(X)$ is a finite set with $\#(f(X)) \leq \#(X)$. If in addition f is one-to-one, then $\#(f(X)) = \#(X)$.

   Proof. Denote $X = \{x_1, \ldots, x_n\}$. Induct on n. When $n = 0$, $\#f(X) = \#(X) = 0$. The base case is proved. Suppose inductively $\#(f(X)) \leq \#(X)$ is true for $n \in \mathbf{N}$. Now consider $X = \{x_1, \ldots, x_n, x_{n+1}\}$. By Lemma 3.6.9, $\#(\{x_1, \ldots, x_{n+1}\}) = \#(\{x_1, \ldots, x_n\}) + 1$. By Proposition 3.6.14-(b), $f(\{x_1, \ldots, x_n, x_{n+1}\}) = f(\{x_1, \ldots, x_n\}) \cup f(x_{n+1}) \leq \#f(\{x_1, \ldots, x_n\}) + 1 = \#(\{x_1, \ldots, x_{n+1}\})$. This closes the induction.

   If f is one-to-one, the proof is similar and we only need to modify a bit from the previous one. The proof of the base case stays the same. Suppose inductively $\#(f(X)) = \#(X)$. Now $f(\{x_1, \ldots, x_{n+1}\}) = f(\{x_1, \ldots, x_n\}) \cup f(x_{n+1})$, since f is injective, these two sets are disjoint. By Proposition 3.6.14.(b), $\#(f\{x_1, \ldots, x_{n+1}\}) = \#(X) + 1 = \#(\{x_1, \ldots, x_{n+1}\})$. This closes the induction. $\square$

5. Let X and Y be finite sets. Then Cartesian product $X \times Y$ is finite and $\#(X \times Y) = \#(X) \times \#(Y)$.

   Proof. Let X has equal cardinality with $\{i \in \mathbf{N} : 1 \leq i \leq n\}$ and y has equal cardinality with $\{i \in \mathbf{N} : 1 \leq i \leq m\}$, use f to denote this function. The statement we need to prove is $\#(X \times Y)$ has equal cardinality with $\{i \in \mathbf{N} : 1 \leq i \leq nm\}$. Induct on n. When $n = 0$, $\#(X \times Y) = \#(X) \times \#(Y) = 0$. Suppose the statement is true for $n \in \mathbf{N}$. Consider when X has the same cardinality with $\{i \in \mathbf{N} : 1 \leq i \leq n+1\}$. By induction hypothesis, there exists a bijective function from $\#(X \times Y)$ to $\{i \in \mathbf{N} : 1 \leq i \leq nm\}$. Define the map from $X \times Y$ (partially) to $\{i \in \mathbf{N} : 1 \leq i \leq n+1\}$ as: for $x = x_{n+1} \in X$, $y \in Y$, $g(x, y) = nm + f(y)$. We need to verify that g is also bijective.

   For any $j \in \{i \in \mathbf{N} : 1 \leq i \leq nm\}$, by induction hypothesis, there exists a bijective function h from $X \times Y$ to $\{i \in \mathbf{N} : 1 \leq i \leq nm\}$, so there exists some $x \in X, y \in Y$ such that $h(x, y) = j$. Let $g(x, y) = h(x, y) = j$ for $x \in X - \{x_{n+1}\}$ and $y \in Y$. For any $j \in \{i \in \mathbf{N} : nm + 1 \leq i \leq (n+1)m\}$, we have $g(n+1, j - nm)$. Thus, function g is surjective. Suppose $x_1, x_2 \in X - \{x_{n+1}\}$, $y_1, y_2 \in Y$, $(x_1, y_1) \neq (x_2, y_2)$, by induction hypothesis, $g(x_1, y_1) \neq g(x_2, y_2)$. For $x_1 \in X - \{x_{n+1}\}$, $x_2 = x_{n+1}$, $y_1, y_2 \in Y$, we have $g(x_1, y_1) \leq nm$ and $g(x_2, y_2) > nm$. So $g(x_1, y_1) \neq g(x_2, y_2)$. For $x_1 = x_2 = x_{n+1}$ and $y_1, y_2 \in Y, y_1 \neq y_2$, by definition, $g(x_1, y_1) \neq g(x_2, y_2)$. Thus, g is injective. So g is bijective and thus $\#(X \times Y)$ has equal cardinality with $\{i \in \mathbf{N} : 1 \leq i \leq (n+1)m\}$. □

6. Let X and Y be finite sets. Then the set Y^X is finite and $\#(Y^X) = \#(Y)^{\#(X)}$.

   Proof. Denote $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$. The statement we need to prove is Y^X has equal cardinality with $\{i \in \mathbf{N} : 1 \leq i \leq m^n\}$. Induct on n. When $n = 0$, the number of functions from X to the empty set is 1 which is equal to m^0. This proved the base case. Suppose inductively Y^X has equal cardinality with $\{i \in \mathbf{N} : 1 \leq i \leq m^n\}$ when $X = \{x_1, \ldots, x_n\}$. Now consider when $X = \{x_1, \ldots, x_{n+1}\}$. We want to show that it has equal cardinality with $M = \{i \in \mathbf{N} : 1 \leq i \leq m^{n+1}\}$. Define function g that maps function f such that $f(x_{n+1}) = y_i \in Y$ to $(m^n + i) \in M$. The proof of the bijectivity of g is similar to (e). Once we have proved g is bijective, the induction is closed. $\square$