

# Chapter 3

## Set Theory

### 3.1 Fundamentals

#### Definition 3.1.1

(Informal) We define a *set*  $A$  to be any unordered collection of objects, e.g.,  $3, 8, 5, 2$  is a set. If  $x$  is an object, we say that  $x$  is an element of  $A$  or  $x \in A$  if  $x$  lies in the collection; otherwise we say that  $x \notin A$ . For instance,  $3 \in \{1, 2, 3, 4, 5\}$  but  $7 \notin \{1, 2, 3, 4, 5\}$ .

#### Axiom 3.1 (Sets are objects).

If  $A$  is a set, then  $A$  is also an object. In particular, given two sets  $A$  and  $B$ , it is meaningful to ask whether  $A$  is also an element of  $B$ .

#### Axiom 3.2 (Equality of sets).

Two sets  $A$  and  $B$  are equal,  $A = B$ , iff every element of  $A$  is an element of  $B$  and vice versa. To put it another way,  $A = B$  if and only if every element  $x$  of  $A$  belongs also to  $B$ , and every element  $y$  of  $B$  belongs also to  $A$ .

#### Axiom 3.3 (Empty set).

There exists a set  $\emptyset$ , known as the empty set, which contains no elements, i.e., for every object  $x$  we have  $x \notin \emptyset$ .

#### Lemma 3.1.5 (Single choice).

Let  $A$  be a non-empty set. Then there exists an object  $x$  such that  $x \in A$ .

**Axiom 3.4 (Singleton sets and pair sets).**

If  $a$  is an object, then there exists a set  $\{a\}$  whose only element is  $a$ , i.e., for every object  $y$ , we have  $y \in \{a\}$  if and only if  $y = a$ ; we refer to  $\{a\}$  as the singleton set whose element is  $a$ . Furthermore, if  $a$  and  $b$  are objects, then there exists a set  $\{a, b\}$  whose only elements are  $a$  and  $b$ ; i.e., for every object  $y$ , we have  $y \in \{a, b\}$  if and only if  $y = a$  or  $y = b$ ; we refer to this set as the pair set formed by  $a$  and  $b$ .

**Axiom 3.5 (Pairwise union).**

Given any two sets  $A, B$ , there exists a set  $A \cup B$ , called the union of  $A$  and  $B$ , which consists of all the elements which belong to  $A$  or  $B$  or both. In other words, for any object  $x$ ,

$$x \in A \cup B \iff (x \in A \text{ or } x \in B).$$

**Lemma 3.1.12**

If  $a$  and  $b$  are objects, then  $\{a, b\} = \{a\} \cup \{b\}$ . If  $A, B, C$  are sets, then the union operation is commutative (i.e.,  $A \cup B = B \cup A$ ) and associative (i.e.,  $(A \cup B) \cup C = A \cup (B \cup C)$ ). Also, we have  $A \cup A = A \cup \emptyset = \emptyset \cup A = A$ .

**Definition 3.1.14 (Subsets).**

Let  $A, B$  be sets. We say that  $A$  is a subset of  $B$ , denoted  $A \subseteq B$ , iff every element of  $A$  is also an element of  $B$ , i.e.

$$\text{For any object } x, x \in A \iff x \in B,$$

We say that  $A$  is a proper subset of  $B$ , denoted  $A \subsetneq B$ , if  $A \subseteq B$  and  $A \neq B$ .

**Proposition 3.1.17 (Sets are partially ordered by set inclusion).**

Let  $A, B, C$  be sets. If  $A \subseteq B$  and  $B \subseteq C$  then  $A \subseteq C$ . If  $A \subseteq B$  and  $B \subseteq A$ , then  $A = B$ . Finally, if  $A \subsetneq B$  and  $B \subsetneq C$  then  $A \subsetneq C$ .

**Axiom 3.6 (Axiom of specification).**

Let  $A$  be a set, and for each  $x \in A$ , let  $P(x)$  be a property pertaining to  $x$  (i.e.,  $P(x)$  is either a true statement or a false statement). Then there exists a set, called  $\{x \in A : P(x) \text{ is true}\}$  (or simply  $\{x \in A : P(x)\}$  for short), whose elements are precisely the elements  $x$  in  $A$  for which  $P(x)$  is true. In other words, for any object  $y$ ,

$$y \in \{x \in A : P(x) \text{ is true}\} \iff (y \in A \text{ and } P(y) \text{ is true}).$$

**Definition 3.1.22 (Intersections).**

The intersection  $S_1 \cap S_2$  of two sets is defined to be the set

$$S_1 \cap S_2 := \{x \in S_1 : x \in S_2\}.$$

In other words,  $S_1 \cap S_2$  consists of all elements which belong to both  $S_1$  and  $S_2$ . Thus, for all objects  $x$ ,

$$x \in S_1 \cap S_2 \iff x \in S_1 \text{ and } x \in S_2.$$

**Definition 3.1.26 (Difference sets).**

Given two sets  $A$  and  $B$ , we define the set  $A - B$  or  $A \setminus B$  to be the set  $A$  with any elements of  $B$  removed:

$$A \setminus B := \{x \in A : x \notin B\};$$

for instance,  $\{1, 2, 3, 4\} \setminus \{2, 4, 6\} = \{1, 3\}$ . In many cases  $B$  will be a subset of  $A$ , but not necessarily.

**Proposition 3.1.27 (Sets form a boolean algebra).**

Let  $A, B, C$  be sets, and let  $X$  be a set containing  $A, B, C$  as subsets.

1. (Minimal element) We have  $A \cup \emptyset = A$  and  $A \cap \emptyset = \emptyset$ .
2. (Maximal element) We have  $A \cup X = X$  and  $A \cap X = A$ .
3. (Identity) We have  $A \cap A = A$  and  $A \cup A = A$ .

4. (Commutativity) We have  $A \cup B = B \cup A$  and  $A \cap B = B \cap A$ .
5. (Associativity) We have  $(A \cup B) \cup C = A \cup (B \cup C)$  and  $(A \cap B) \cap C = A \cap (B \cap C)$ .
6. (Distributivity) We have  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$  and  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ .
7. (Partition) We have  $A \cup (X \setminus A) = X$  and  $A \cap (X \setminus A) = \emptyset$ .
8. (De Morgan laws) We have  $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$  and  $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$ .

### Axiom 3.8 (Infinity).

There exists a set  $\mathbb{N}$ , whose elements are called natural numbers, as well as an object 0 in  $\mathbb{N}$ , and an object  $n++$  assigned to every natural number  $n \in \mathbb{N}$ , such that the Peano axioms hold.

## Exercises

### Exercise 3.1.1

Let  $a, b, c, d$  be objects such that  $\{a, b\} = \{c, d\}$ . Show that at least one of the two statements " $a = c$  and  $b = d$ " and " $a = d$  and  $b = c$ " hold.

*Proof.* Consider two cases:  $a = b$  and  $a \neq b$ .

Case 1:  $a = b$ . Then  $\{a, b\} = \{a\}$ . By Axiom 3.2, if  $\{a\}$  and  $\{c, d\}$  are equal to each other, then every element belong to  $\{c, d\}$  must also belong to  $\{a\}$ . Therefore,  $c = a$ ,  $d = a$ . Since  $a = b$ , we have  $a = b = c = d$ . Thus, both statements hold.

Case 2:  $a \neq b$ . Similarly, by Axiom 3.2, every element belong to  $\{a, b\}$  must also belong to  $\{c, d\}$ . So  $\{c, d\}$ , a set of two elements, contains two distinct elements  $a$  and  $b$ . Therefore, either  $a = c, b = d$  or  $a = d, b = c$  holds, exclusively.

Thus, we have shown that at least one of the two statements " $a = c$  and  $b = d$ " and " $a = d$  and  $b = c$ " hold.  $\square$

**Exercise 3.1.2**

Using only Axiom 3.2, Axiom 3.1, Axiom 3.3, and Axiom 3.4, prove that the sets  $\emptyset$ ,  $\{\emptyset\}$ ,  $\{\{\emptyset\}\}$ , and  $\{\emptyset, \{\emptyset\}\}$  are all distinct.

*Proof.* First, let's consider  $\emptyset$ .  $\emptyset$  contains no element while other sets all have at least one element in it. Therefore,  $\emptyset$  is distinct from  $\{\emptyset\}$ ,  $\{\{\emptyset\}\}$  and  $\{\emptyset, \{\emptyset\}\}$ . Then, let's consider  $\{\emptyset\}$ . Is it distinct from  $\{\{\emptyset\}\}$  and  $\{\emptyset, \{\emptyset\}\}$ ? We know that  $\emptyset \in \{\emptyset\}$ . But we have proved earlier  $\emptyset$  and  $\{\emptyset\}$  are not equal to each other, so  $\emptyset \notin \{\{\emptyset\}\}$ . So  $\{\emptyset\}$  and  $\{\{\emptyset\}\}$  are distinct. For the same reason,  $\{\emptyset\} \notin \{\emptyset, \{\emptyset\}\}$ . So  $\{\emptyset\}$  and  $\{\emptyset, \{\emptyset\}\}$  are also distinct. Last, consider  $\{\{\emptyset\}\}$  and  $\{\emptyset, \{\emptyset\}\}$ . For the same reason ( $\emptyset$  and  $\{\emptyset\}$  are distinct),  $\emptyset \notin \{\{\emptyset\}\}$ . So  $\{\{\emptyset\}\}$  and  $\{\emptyset, \{\emptyset\}\}$  are distinct. Thus, we have proved the sets  $\emptyset$ ,  $\{\emptyset\}$ ,  $\{\{\emptyset\}\}$ , and  $\{\emptyset, \{\emptyset\}\}$  are all distinct.  $\square$

**Exercise 3.1.3**

Prove the remaining claims in Lemma 3.1.12.

*Proof.* First, prove the union operation is commutative (i.e.,  $A \cup B = B \cup A$ ). By definition, we know that  $A \cup B$  consists of all the elements which belong to  $A$  or  $B$ , inclusively. And  $B \cup A$  also consists of all the elements belong to  $A$  or  $B$ , inclusively. Therefore,  $A \cup B$  and  $B \cup A$  are containing exactly the same elements. Thus,  $A \cup B = B \cup A$ .

The second part is to prove  $A \cup A = A \cup \emptyset = \emptyset \cup A = A$ . First, let's consider  $A \cup A$ . By definition,  $A \cup A$  consists of all the element  $x$  such that  $x \in A$  or  $x \in A$ . So  $A \cup A$  and  $A$  have exactly the same elements. Therefore  $A \cup A = A$ . Now let's consider  $A \cup \emptyset$ .  $A \cup \emptyset$  consists of all the  $x$  such that  $x \in A$  or  $x \in \emptyset$ . Since no element would belong to  $\emptyset$ .  $A \cup \emptyset$  contains exactly the same elements as  $A$ . Therefore,  $A \cup \emptyset = A$ . And by commutative law, we have  $A \cup \emptyset = \emptyset \cup A = A$ .

Thus, we have proved  $A \cup A = A \cup \emptyset = \emptyset \cup A = A$ .  $\square$

**Exercise 3.1.4**

Prove the remaining claims in Lemma 3.1.17.

*Proof.* Part I. If  $A \subseteq B$  and  $B \subseteq A$ , then  $A = B$ . Translate the if statement into propositional logic:  $(x \in A \Rightarrow x \in B) \wedge (x \in B \Rightarrow x \in A)$ . Therefore, we have  $x \in A \iff x \in B$ . Thus,  $A = B$ .

Part II. If  $A \subsetneq B$  and  $B \subsetneq C$  then  $A \subsetneq C$ . Since  $A \neq B$  and  $B \neq C$ , by transitivity,  $A \neq C$ . And by the first part of this proposition (if  $A \subseteq B$  and  $B \subseteq C$  then  $A \subseteq C$ ), we would have  $A \subseteq C$ . Since  $A \subseteq C$  and  $A \neq C$ ,  $A \subsetneq C$ .  $\square$

### Exercise 3.1.5

Let  $A, B$  be sets. Show that the three statements  $A \subseteq B$ ,  $A \cup B = B$ ,  $A \cap B = A$  are logically equivalent (any one of them implies the other two).

*Proof.* Rewrite the statements using propositional logic.

$$A \subseteq B: x \in A \Rightarrow x \in B.$$

$$A \cup B = B: x \in A \vee x \in B \iff x \in B.$$

$$A \cap B = A: x \in A \wedge x \in B \iff x \in A.$$

Assume  $A \subseteq B$  is true. Then both  $x \in A$  and  $x \in B$  imply  $x \in B$ . So  $x \in A \vee x \in B \Rightarrow x \in B$  stands. And  $x \in B \Rightarrow x \in A \vee x \in B$  stands by rules of inference. Therefore,  $A \subseteq B \Rightarrow A \cup B = B$ . Assume  $A \cup B = B$  is true. Then  $x \in A \vee x \in B \Rightarrow x \in B$  stands. So  $x \in A \Rightarrow x \in B$  is true which means  $A \cup B = B \Rightarrow A \subseteq B$ . Therefore,  $A \subseteq B \iff A \cup B = B$ .

Assume  $A \subseteq B$  is true. We know that  $x \in A \wedge x \in B \Rightarrow x \in A$  is true by rules of inference. And if  $x \in A$ , since  $A \subseteq B$  is true,  $x \in B$  is also true. So  $x \in A \wedge x \in B$  is true. Therefore,  $A \subseteq B \Rightarrow A \cap B = A$ . Assume  $A \cap B = A$  is true. Then if  $x \in A$ ,  $x \in A \wedge x \in B$  must be true. So  $x \in B$  is true.  $A \subseteq B$  is true. Therefore,  $A \cap B \Rightarrow A \subseteq B$ . Thus,  $A \subseteq B \iff A \cap B = A$ .

By transitivity, we have  $A \cup B = B \iff A \cap B = A$ . Thus, these three statements are logically equivalent.  $\square$

### Exercise 3.1.6

Prove Proposition 3.1.27.

1.  $A \cup \emptyset = A$  and  $A \cap \emptyset = \emptyset$ .

*Proof.*  $A \cup \emptyset = A$  has been proved in 3.1.3.  $A \cap \emptyset$  consists of all  $x$  such that  $x \in A \wedge x \in \emptyset$ . Since  $x \in \emptyset$  is always false,  $A \cap \emptyset$  has no element in it. Thus,  $A \cap \emptyset = \emptyset$ .  $\square$

2.  $A \cup X = X$  and  $A \cap X = A$ .

*Proof.* We have proved in 3.1.5 that  $A \subseteq X$ ,  $A \cup X = X$  and  $A \cap X = A$  are logically equivalent.  $\square$

3.  $A \cap A = A$  and  $A \cup A = A$ .

*Proof.*  $x \in A \iff x \in A$ , so  $A \subseteq A$ . Therefore, by 3.1.5 we have  $A \cap A = A$  and  $A \cup A = A$ .  $\square$

4.  $A \cup B = B \cup A$  and  $A \cap B = B \cap A$ .

*Proof.*  $A \cup B = B \cup A$  has been proved in 3.1.3. On the other hand,  $x \in A \wedge x \in B \iff x \in B \wedge x \in A$ . Therefore,  $A \cap B = B \cap A$ .  $\square$

5.  $(A \cup B) \cup C = A \cup (B \cup C)$  and  $(A \cap B) \cap C = A \cap (B \cap C)$ .

*Proof.*  $(A \cup B) \cup C = A \cup (B \cup C)$  has been proved in Lemma 3.1.12.  $x \in A \wedge (x \in B \wedge x \in C) \iff (x \in A \wedge x \in B) \wedge x \in C$ . Hence,  $(A \cap B) \cap C = A \cap (B \cap C)$ .  $\square$

6.  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$  and  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ .

*Proof.* This can be proved using distribution law in propositional logic.

Since  $x \in A \wedge (x \in B \vee x \in C) \iff (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)$  and  $x \in A \vee (x \in B \wedge x \in C) \iff (x \in A \vee x \in B) \wedge (x \in A \vee x \in C)$ ,  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$  and  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$  are also true.  $\square$

7.  $A \cup (X \setminus A) = X$  and  $A \cap (X \setminus A) = \emptyset$ .

*Proof.*  $A \cup (X \setminus A) \iff (x \in A \vee (x \in X \wedge x \notin A)) \iff (x \in A \vee x \in X) \iff A \cup X$ . From 3.1.5, we have  $A \cup X = X$ . Thus,  $A \cup (X \setminus A) = X$ .

$A \cap (X \setminus A) \iff x \in A \wedge (x \in X \wedge x \notin A) \iff (x \in A \wedge x \notin A) \wedge x \in X$ .  $(x \in A \wedge x \notin A) \wedge x \in X$  is always false. Therefore,  $A \cap (X \setminus A) = \emptyset$ .  $\square$

8.  $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$  and  $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$ .

*Proof.*  $X \setminus (A \cup B) \iff x \in X \wedge \sim (x \in A \vee x \in B) \iff (x \in X \wedge x \notin A) \wedge (x \in X \wedge x \notin B) \iff (X \setminus A) \cap (X \setminus B)$ .

$X \setminus (A \cap B) \iff x \in X \wedge \sim (x \in A \wedge x \in B) \iff x \in X \wedge (x \notin A \vee x \notin B) \iff (x \in X \wedge x \notin A) \vee (x \in X \wedge x \notin B) \iff (X \setminus A) \cup (X \setminus B)$ .  $\square$