Chapter 5

The real numbers

5.1 Cauchy sequences

Definition 5.1.1 (Sequences).

Let m be an integer. A sequence $(a_n)_{n=m}^{\infty}$ of rational numbers is any function from the set $\{n \in \mathbf{Z} : n \geq m\}$ to $\mathbf{Q}$, i.e., a mapping which assigns to each integer n greater than or equal to m, a rational number a_n. More informally, a sequence $(a_n)_{n=m}^{\infty}$ of rational numbers is a collection of rationals a_m, a_{m+1}, $a_{m+2}, \ldots$

Definition 5.1.3 (ε-steadiness).

Let $\varepsilon > 0$. A sequence $(a_n)_{n=0}^{\infty}$ is said to be ε-steady iff each pair a_j, a_k of sequence elements is ε-close for every natural number j, k. In other words, the sequence $a_0, a_1, a_2, \ldots$ is ε-steady iff $|a_j - a_k| \leq \varepsilon$ for all j, k.

Definition 5.1.6 (Eventual ε-steadiness).

Let $\varepsilon > 0$. A sequence $(a_n)_{n=0}^{\infty}$ is said to be eventually ε-steady iff the sequence $a_N, a_{N+1}, a_{N+2}, \ldots$ is ε-steady for some natural number $N \geq 0$. In other words, the sequence $a_0, a_1, a_2, \ldots$ is eventually ε-steady iff there exists an $N \geq 0$ such that $|a_j - a_k| \leq \varepsilon$ for all $j, k \geq N$.

Definition 5.1.8 (Cauchy sequences).

A sequence $(a_n)_{n=0}^{\infty}$ of rational numbers is said to be a Cauchy sequence iff for every rational $\varepsilon > 0$, the sequence $(a_n)_{n=0}^{\infty}$ is eventually ε-steady. In other words, the sequence $a_0, a_1, a_2, \ldots$ is a Cauchy sequence iff for every $\varepsilon > 0$, there exists an $N \geq 0$ such that $d(a_j, a_k)$ for all $j, k \geq N$.

Proposition 5.1.11

The sequence $a_1, a_2, a_3, \ldots$ defined by $a_n := 1/n$ (i.e., the sequence $1, 1/2, 1/3, \ldots$) is a Cauchy sequence.

Definition 5.1.12 (Bounded sequences).

Let $M \geq 0$ be rational. A finite sequence $a_1, a_2, \ldots, a_n$ is bounded by M iff $|a_i| \leq M$ for all $1 \leq i \leq n$. An infinite sequence $(a_n)_{n=1}^{\infty}$ is bounded by M iff $|a_i| \leq M$ for all $i \geq 1$. A sequence is said to be bounded iff it is bounded by M for some rational $M \geq 0$.

Lemma 5.1.14 (Finite sequences are bounded).

Every finite sequence $a_1, a_2, \ldots, a_n$ is bounded.

Lemma 5.1.15 (Cauchy sequences are bounded).

Every Cauchy sequence $(a_n)_{n=1}^{\infty}$ is bounded.

Exercise 5.1.1

Prove Lemma 5.1.15.

Proof. Suppose $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence. So for $\varepsilon = 1$, there exists an $N \geq 1$ such that $d(a_j, a_k) \leq 1$ for all $j, k \geq N$. Then the sequence can be splite into two parts: $a_1, \ldots, a_N$ and $a_{N+1}, a_{N+2}, \ldots$. The former is a finite sequence, by Lemma 5.1.14, it is bounded. Suppose this finite sequence is bounded by M_1. Consider $a_{N+1}, a_{N+2}, \ldots$. Since it is 1-steady, for any $i > N+1$, we have $|a_i - a_{N+1}| \leq 1$. Rearrange the inequalities, we have $a_{N+1} - 1 \leq a_i \leq a_{N+1} + 1$. Let $M_2 = a_{N+1} + 1$. Then $a_{N+1} < M_2$ and for every $i > N+1$, we have $a_i \leq M_2$. So sequence $a_{N+1}, a_{N+2}, \ldots$ is bounded by M_2. Let $M = \max\{M_1, M_2\}$. Then the Cauchy sequence $(a_n)_{n=1}^{\infty}$ is bounded by M. $\square$

5.2 Equivalent Cauchy sequences

Definition 5.2.1 (ε-close sequences).

Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be two sequences, and let $\varepsilon > 0$. We say that the sequence $(a_n)_{n=0}^{\infty}$ is ε-close to $(b_n)_{n=0}^{\infty}$ iff a_n is ε-close to b_n for each $n \in \mathbf{N}$. In other words, the sequence $a_0, a_1, a_2, \ldots$ is ε-close to the sequence $b_1, b_1, b_2, \ldots$ iff $|a_n - b_n| \leq \varepsilon$ for all $n = 0, 1, 2, \ldots$.

Definition 5.2.3 (Eventually ε-close sequences).

Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be two sequences, and let $\varepsilon > 0$. We say that the sequence $(a_n)_{n=0}^{\infty}$ is eventually ε-close to $(b_n)_{n=0}^{\infty}$ iff there exists an $N \geq 0$ such that the sequences $(a_n)_{n=N}^{\infty}$ and $(b_n)_{n=N}^{\infty}$ are ε-close. In other words, $a_0, a_1, a_2, \ldots$ is eventually ε-close to $b_0, b_1, b_2, \ldots$ iff there exists an $N \geq 0$ such that $|a_n - b_n| \leq \varepsilon$ for all $n \geq N$.

Definition 5.2.6 (Equivalent sequences).

Two sequences $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are equivalent iff for each rational $\varepsilon > 0$, the sequences $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are eventually ε-close. In other words, $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ are equivalent iff for every for every rational $\varepsilon > 0$, there exists an $N \geq 0$ such that $|a_n - b_n| \leq \varepsilon$ for all $n \geq N$.

Proposition 5.2.8

Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be the sequences $a_n = 1 + 10^{-n}$ and $b_n = 1 - 10^{-n}$. Then the sequences a_n, b_n are equivalent.

Exercise 5.2.1

Show that if $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent sequences of rationals, then $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence if and only if $(b_n)_{n=1}^{\infty}$ is a Cauchy sequence.

Proof. Assume $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent sequences of rationals, and $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence. We want to show that for any rational $\varepsilon > 0$, there exists $N \geq 1$ such that $b_N, b_{N+1}, \ldots$ is ε-steady.

Since $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence and $\frac{\varepsilon}{3} > 0$, there exists $N_1 \geq 1$ such that for all $i, j \geq N_1$, $|a_i - a_j| \leq \frac{\varepsilon}{3}$. Since $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent, and $\frac{\varepsilon}{3} > 0$, there exists $N_2 \geq 1$ such that for all $i \geq N_2$, $|b_i - a_i| \leq \frac{\varepsilon}{3}$. Let $N = \max\{N_1, N_2\}$. Suppose $i \geq N$ is an arbitrary natural number. Since $N \geq N_1$, we have

$$|a_i - a_{i+1}| \leq \frac{\varepsilon}{3}.$$

Since $N \geq N_2$, we have

$$|b_i - a_i| \leq \frac{\varepsilon}{3}$$

and

$$|a_{i+1} - b_{i+1}| \leq \frac{\varepsilon}{3}.$$

Since

$$|a_i - a_{i+1}| \leq \frac{\varepsilon}{3}$$

and

$$|b_i - a_i| \leq \frac{\varepsilon}{3},$$

we have

$$|b_i - a_{i+1}| \leq |a_i - a_{i+1}| + |b_i - a_i| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}.$$

Since

$$|a_{i+1} - b_{i+1}| \leq \frac{\varepsilon}{3}$$

and

$$|b_i - a_{i+1}| \leq \frac{2\varepsilon}{3},$$

we have

$$|b_i - b_{i+1}| \leq |a_{i+1} - b_{i+1}| + |b_i - a_{i+1}| \leq \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon.$$

Thus, for any $\varepsilon > 0$, we can find $N = \max\{N_1, N_2\}$ such that $b_N, b_{N+1}, dotsc$ is ε-steady. Therefore, $(b_n)_{i=1}^{\infty}$ is a Cauchy sequence.

Similarly, we can show that if $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent sequences of rationals and $(b_n)_{i=1}^{\infty}$ is a Cauchy sequence, then $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence. Thus, if $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent sequences of rationals, then $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence if and only if $(b_n)_{n=1}^\infty$ is a Cauchy sequence. $\square$

Exercise 5.2.2

Let $\varepsilon > 0$. Show that if $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ are eventually ε-close, then $(a_n)_{n=1}^\infty$ is bounded if and only if $(b_n)_{n=1}^\infty$ is bounded.

Proof. Suppose $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ are eventually ε-close and $(a_n)_{n=1}^\infty$ is bounded. Since $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ are eventually ε-close, for any $\varepsilon > 0$, there exists $N \geq 1$ such that for any $i \geq N$, $|a_i - b_i| \leq \varepsilon$. Consider an arbitrary $\varepsilon > 0$. We can find $N \geq 1$ such that for any $i \geq N$, $|a_i - b_i| \leq \varepsilon$. Then

$$a_i - \varepsilon \leq b_i \leq a_i + \varepsilon.$$

Since $(a_n)_{n=1}^\infty$ is bounded, there exists M such that $|a_i| \leq M$ for all $i \geq 1$. Then

$$-M \leq a_i \leq M.$$

So

$$-M - \varepsilon \leq b_i \leq M + \varepsilon.$$

Therefore, $|b_i| \leq M + \varepsilon$ for all $i \geq 1$. Thus, $(b_n)_{n=1}^\infty$ is bounded.

Similarly, we can show that if $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ are eventually ε-close, and $(b_n)_{n=1}^\infty$ is bounded, then $(a_n)_{n=1}^\infty$ is bounded.

Thus, if $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ are eventually ε-close, then $(a_n)_{n=1}^\infty$ is bounded if and only if $(b_n)_{n=1}^\infty$ is bounded. $\square$