Chapter 5

The real numbers

5.1 Cauchy sequences

Definition 5.1.1 (Sequences).

Let m be an integer. A sequence $(a_n)_{n=m}^{\infty}$ of rational numbers is any function from the set $\{n \in \mathbf{Z} : n \geq m\}$ to $\mathbf{Q}$, i.e., a mapping which assigns to each integer n greater than or equal to m, a rational number a_n. More informally, a sequence $(a_n)_{n=m}^{\infty}$ of rational numbers is a collection of rationals a_m, a_{m+1}, $a_{m+2}, \ldots$

Definition 5.1.3 (ε-steadiness).

Let $\varepsilon > 0$. A sequence $(a_n)_{n=0}^{\infty}$ is said to be ε-steady iff each pair a_j, a_k of sequence elements is ε-close for every natural number j, k. In other words, the sequence $a_0, a_1, a_2, \ldots$ is ε-steady iff $|a_j - a_k| \leq \varepsilon$ for all j, k.

Definition 5.1.6 (Eventual ε-steadiness).

Let $\varepsilon > 0$. A sequence $(a_n)_{n=0}^{\infty}$ is said to be eventually ε-steady iff the sequence $a_N, a_{N+1}, a_{N+2}, \ldots$ is ε-steady for some natural number $N \geq 0$. In other words, the sequence $a_0, a_1, a_2, \ldots$ is eventually ε-steady iff there exists an $N \geq 0$ such that $|a_j - a_k| \leq \varepsilon$ for all $j, k \geq N$.

Definition 5.1.8 (Cauchy sequences).

A sequence $(a_n)_{n=0}^{\infty}$ of rational numbers is said to be a Cauchy sequence iff for every rational $\varepsilon > 0$, the sequence $(a_n)_{n=0}^{\infty}$ is eventually ε-steady. In other words, the sequence $a_0, a_1, a_2, \ldots$ is a Cauchy sequence iff for every $\varepsilon > 0$, there exists an $N \geq 0$ such that $d(a_j, a_k)$ for all $j, k \geq N$.

Proposition 5.1.11

The sequence $a_1, a_2, a_3, \ldots$ defined by $a_n := 1/n$ (i.e., the sequence $1, 1/2, 1/3, \ldots$) is a Cauchy sequence.

Definition 5.1.12 (Bounded sequences).

Let $M \geq 0$ be rational. A finite sequence $a_1, a_2, \ldots, a_n$ is bounded by M iff $|a_i| \leq M$ for all $1 \leq i \leq n$. An infinite sequence $(a_n)_{n=1}^{\infty}$ is bounded by M iff $|a_i| \leq M$ for all $i \geq 1$. A sequence is said to be bounded iff it is bounded by M for some rational $M \geq 0$.

Lemma 5.1.14 (Finite sequences are bounded).

Every finite sequence $a_1, a_2, \ldots, a_n$ is bounded.

Lemma 5.1.15 (Cauchy sequences are bounded).

Every Cauchy sequence $(a_n)_{n=1}^{\infty}$ is bounded.

Exercise 5.1.1

Prove Lemma 5.1.15.

Proof. Suppose $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence. So for $\varepsilon = 1$, there exists an $N \geq 1$ such that $d(a_j, a_k) \leq 1$ for all $j, k \geq N$. Then the sequence can be splite into two parts: $a_1, \ldots, a_N$ and $a_{N+1}, a_{N+2}, \ldots$. The former is a finite sequence, by Lemma 5.1.14, it is bounded. Suppose this finite sequence is bounded by M_1. Consider $a_{N+1}, a_{N+2}, \ldots$. Since it is 1-steady, for any $i > N + 1$, we have $|a_i - a_{N+1}| \leq 1$. Rearrange the inequalities, we have $a_{N+1} - 1 \leq a_i \leq a_{N+1} + 1$. Let $M_2 = a_{N+1} + 1$. Then $a_{N+1} < M_2$ and for every $i > N + 1$, we have $a_i \leq M_2$. So sequence $a_{N+1}, a_{N+2}, \ldots$ is bounded by M_2. Let $M = \max\{M_1, M_2\}$. Then the Cauchy sequence $(a_n)_{n=1}^{\infty}$ is bounded by M. $\square$

5.2 Equivalent Cauchy sequences

Definition 5.2.1 (ε-close sequences).

Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be two sequences, and let $\varepsilon > 0$. We say that the sequence $(a_n)_{n=0}^{\infty}$ is ε-close to $(b_n)_{n=0}^{\infty}$ iff a_n is ε-close to b_n for each $n \in \mathbf{N}$. In other words, the sequence $a_0, a_1, a_2, \ldots$ is ε-close to the sequence $b_1, b_1, b_2, \ldots$ iff $|a_n - b_n| \leq \varepsilon$ for all $n = 0, 1, 2, \ldots$.

Definition 5.2.3 (Eventually ε-close sequences).

Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be two sequences, and let $\varepsilon > 0$. We say that the sequence $(a_n)_{n=0}^{\infty}$ is eventually ε-close to $(b_n)_{n=0}^{\infty}$ iff there exists an $N \geq 0$ such that the sequences $(a_n)_{n=N}^{\infty}$ and $(b_n)_{n=N}^{\infty}$ are ε-close. In other words, $a_0, a_1, a_2, \ldots$ is eventually ε-close to $b_0, b_1, b_2, \ldots$ iff there exists an $N \geq 0$ such that $|a_n - b_n| \leq \varepsilon$ for all $n \geq N$.

Definition 5.2.6 (Equivalent sequences).

Two sequences $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are equivalent iff for each rational $\varepsilon > 0$, the sequences $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are eventually ε-close. In other words, $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ are equivalent iff for every for every rational $\varepsilon > 0$, there exists an $N \geq 0$ such that $|a_n - b_n| \leq \varepsilon$ for all $n \geq N$.

Proposition 5.2.8

Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be the sequences $a_n = 1 + 10^{-n}$ and $b_n = 1 - 10^{-n}$. Then the sequences a_n, b_n are equivalent.

Exercise 5.2.1

Show that if $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent sequences of rationals, then $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence if and only if $(b_n)_{n=1}^{\infty}$ is a Cauchy sequence.

Proof. Assume $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent sequences of rationals, and $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence. We want to show that for any rational $\varepsilon > 0$, there exists $N \geq 1$ such that $b_N, b_{N+1}, \ldots$ is ε-steady.

Since $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence and $\frac{\varepsilon}{3} > 0$, there exists $N_1 \geq 1$ such that for all $i, j \geq N_1$, $|a_i - a_j| \leq \frac{\varepsilon}{3}$. Since $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent, and $\frac{\varepsilon}{3} > 0$, there exists $N_2 \geq 1$ such that for all $i \geq N_2$, $|b_i - a_i| \leq \frac{\varepsilon}{3}$. Let $N = \max\{N_1, N_2\}$. Consider arbitrary $i, j \geq N$. Since $N \geq N_1$, we have

$$|a_i - a_j| \leq \frac{\varepsilon}{3}.$$

Since $N \geq N_2$, we have

$$|b_i - a_i| \leq \frac{\varepsilon}{3}$$

and

$$|a_j - b_j| \leq \frac{\varepsilon}{3}.$$

Since

$$|a_i - a_j| \leq \frac{\varepsilon}{3}$$

and

$$|b_i - a_i| \leq \frac{\varepsilon}{3},$$

we have

$$|b_i - a_j| \leq |a_i - a_j| + |b_i - a_i| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}.$$

Since

$$|a_j - b_j| \leq \frac{\varepsilon}{3}$$

and

$$|b_i - a_j| \leq \frac{2\varepsilon}{3},$$

we have

$$|b_i - b_j| \leq |a_j - b_j| + |b_i - a_j| \leq \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon.$$

Thus, for any $\varepsilon > 0$, we can find $N = \max\{N_1, N_2\}$ such that $b_N, b_{N+1}, dotsc$ is ε-steady. Therefore, $(b_n)_{i=1}^{\infty}$ is a Cauchy sequence.

Similarly, we can show that if $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent sequences of rationals and $(b_n)_{i=1}^{\infty}$ is a Cauchy sequence, then $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence. Thus, if $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent sequences of rationals, then $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence if and only if $(b_n)_{n=1}^{\infty}$ is a Cauchy sequence. $\square$

Exercise 5.2.2

Let $\varepsilon > 0$. Show that if $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are eventually ε-close, then $(a_n)_{n=1}^{\infty}$ is bounded if and only if $(b_n)_{n=1}^{\infty}$ is bounded.

Proof. Suppose $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are eventually ε-close and $(a_n)_{n=1}^{\infty}$ is bounded. Since $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are eventually ε-close, for any $\varepsilon > 0$, there exists $N \geq 1$ such that for any $i \geq N$, $|a_i - b_i| \leq \varepsilon$. Consider an arbitrary $\varepsilon > 0$. We can find $N \geq 1$ such that for any $i \geq N$, $|a_i - b_i| \leq \varepsilon$. Then

$$a_i - \varepsilon \leq b_i \leq a_i + \varepsilon.$$

Split $(b_n)_{n=1}^{\infty}$ to $b_1, \ldots, b_N$ and $b_{N+1}, b_{N+2}, \ldots$. The former is a finite sequence, so it is bounded by some rational number M_1. Since $(a_n)_{n=1}^{\infty}$ is bounded, there exists M such that $|a_i| \leq M$ for all $i \geq 1$. Then

$$-M \leq a_i \leq M.$$

So

$$-M - \varepsilon \leq b_i \leq M + \varepsilon.$$

Therefore, $|b_i| \leq M + \varepsilon$ for all $i \geq N + 1$. Let $M_0 = \max(M, M_1)$. For any $i \geq 1$, we have $|b_i| \leq M_0$. Thus, $(b_n)_{n=1}^{\infty}$ is bounded.

Similarly, we can show that if $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are eventually ε-close, and $(b_n)_{n=1}^{\infty}$ is bounded, then $(a_n)_{n=1}^{\infty}$ is bounded.

Thus, if $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are eventually ε-close, then $(a_n)_{n=1}^{\infty}$ is bounded if and only if $(b_n)_{n=1}^{\infty}$ is bounded. $\square$

5.3 The construction of the real numbers

Definition 5.3.1 (Real numbers).

A real number is defined to be an object of the form $\mathrm{LIM}_{n\to\infty} a_n$, where $\mathrm{LIM}_{n\to\infty} a_n$ is a Cauchy sequence of rational numbers. Two real numbers $\mathrm{LIM}_{n\to\infty} a_n$ and $\mathrm{LIM}_{n\to\infty} b_n$are said to be equal iff $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent Cauchy sequences. The set of all real numbers is denoted $\mathbf{R}$.

Proposition 5.3.3 (Formal limits are well-defined).

Let $x = \mathrm{LIM}_{n\to\infty} a_n$, $y = \mathrm{LIM}_{n\to\infty} b_n$, and $z = \mathrm{LIM}_{n\to\infty} c_n$be real numbers. Then, with the above definition of equality for real numbers, we have $x = x$. Also, if $x = y$, then $y = x$. Finally, if $x = y$ and $y = z$, then $x = z$.

Definition 5.3.4 (Addition of reals).

Let $x = \mathrm{LIM}_{n\to\infty} a_n$ and $y = \mathrm{LIM}_{n\to\infty} b_n$be real numbers. Then we define the sum $x + y$ to be $x + y := \mathrm{LIM}_{n\to\infty}(a_n + b_n)$.

Lemma 5.3.6 (Sum of Cauchy sequences is Cauchy).

Let $x = \mathrm{LIM}_{n\to\infty} a_n$ and $y = \mathrm{LIM}_{n\to\infty} b_n$be real numbers. Then $x + y$ is also a real number (i.e., $(a_n + b_n)_{n=1}^{\infty}$ is a Cauchy sequence of rationals).

Lemma 5.3.7 (Sums of equivalent Cauchy sequences are equivalent).

Let $x = \mathrm{LIM}_{n\to\infty} a_n$, $y = \mathrm{LIM}_{n\to\infty} b_n$, and $x' = \mathrm{LIM}_{n\to a'_n}$ be real numbers. Suppose that $x = x'$. Then we have $x + y = x' + y$.

Lemma 5.3.9 (Multiplication of reals).

Let $x = \mathrm{LIM}_{n\to\infty} a_n$ and $y = \mathrm{LIM}_{n\to\infty} b_n$be real numbers. Then we define the product xy to be $xy := \mathrm{LIM}_{n\to\infty} a_n b_n$.

Proposition 5.3.10 (Multiplication is well defined).

Let $x = \mathrm{LIM}_{n\to\infty} a_n$, $y = \mathrm{LIM}_{n\to\infty} b_n$, and $x' = \mathrm{LIM}_{n\to\infty} a'_n$ be real numbers. Then xy is also a real number. Furthermore, if $x = x'$, then $xy = x'y$.

Proposition 5.3.11

All the laws of algebra from Proposition 4.1.6 hold not only for the integers, but for the reals as well.

Definition 5.3.12 (Sequences bounded away from zero).

A sequence $(a_n)_{n=1}^{\infty}$ of rational numbers is said to be bounded away from zero iff there exists a raional number $c > 0$ such that $|a_n| > c$ for all $n \geq 1$.

Lemma 5.3.14.

Let x be a non-zero real number. Then $x = \mathrm{LIM}_{n\to\infty} a_n$ for some Cauchy sequence $(a_n)_{n=1}^{\infty}$ which is bounded away from zero.

Lemma 5.3.15.

Suppose that $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence which is bounded away from zero. Then the sequence $(a_n^{-1})_{n=1}^{\infty}$ is also a Cauchy sequence.

Definition 5.3.16 (Reciprocals of real numbers).

Let x be a non-zero real number. Let $(a_n)_{n=1}^{\infty}$ be a Cauchy sequence bounded away from zero such that $x = \mathrm{LIM}_{n\to\infty} a_n$. Then we define the reciprocal x^{-1} by the formula $x^{-1} := \mathrm{LIM}_{n\to\infty} a_n^{-1}$.

Lemma 5.3.17 (Reciprocation is well defined).

Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be two Cauchy sequences bounded away from zero such that $\mathrm{LIM}_{n\to\infty} a_n = \mathrm{LIM}_{n\to\infty} b_n$(i.e., the two sequences are equivalent). Then $\mathrm{LIM}_{n\to\infty} a_n^{-1} = \mathrm{LIM}_{n\to\infty} b_n^{-1}$.

Exercise 5.3.1

Prove Proposition 5.3.3.

Proof. Reflexivity. Since $x = \text{LIM}_{n\to\infty} a_n$, $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence. Obviously, $(a_n)_{n=1}^{\infty}$ and $(a_n)_{n=1}^{\infty}$ are equivalent. Therefore, $\text{LIM}_{n\to\infty} a_n = \text{LIM}_{n\to\infty} a_n$ $(x = x)$.

Symmetry. Assume $x = y$, then Cauchy sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent. So for every $\varepsilon > 0$, there exists $N \geq 1$ such that for every $i \geq N$, $|a_i - b_i| = |b_i - a_i| \leq \varepsilon$. Therefore, $(b_n)_{n=1}^{\infty}$ and $(a_n)_{n=1}^{\infty}$ are equivalent. Thus, $\text{LIM}_{n\to\infty} b_n = \text{LIM}_{n\to\infty} a_n$ $(y = x)$.

Transitivity. Assume $x = y$ and $y = z$. We want to show that the Cauchy sequences $(a_n)_{n=1}^{\infty}$ and $(c_n)_{n=1}^{\infty}$ are equivalent, that is, for any $\varepsilon > 0$, there exists $N \geq 1$ such that for all $i \geq N$, we have $|a_i - c_i| \leq \varepsilon$. Suppose ε is an arbitrary positive rational number. Since $x = y$, the Cauchy sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent. Then there exists N_1 such that for every $i \geq N_1$, we have $|a_i - b_i| \leq \frac{\varepsilon}{2}$. Since $y = z$, the Cauchy sequences $(b_n)_{n=1}^{\infty}$ and $(c_n)_{n=1}^{\infty}$ are equivalent. Then there exists N_2 such that for every $i \geq N_2$, we have $|b_i - c_i| \leq \frac{\varepsilon}{2}$. Let $N = \max(N_1, N_2)$, then for all $i \geq N$, $|a_i - c_i| \leq |a_i - b_i| + |b_i - c_i| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \epsilon$. Thus, $(a_n)_{n=1}^{\infty}$ and $(c_n)_{n=1}^{\infty}$ are equivalent. So $x = z$. □

Exercise 5.3.2

Prove Proposition 5.3.10.

Proof. xy is a real number. We want to show that for any $\varepsilon > 0$, there exists $N \geq 1$ such that $|a_i b_i - a_j b_j| \leq \varepsilon$ for any $i, j \geq N$. Consider an arbitrary $\varepsilon > 0$. Since $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are Cauchy sequences, by Lemma 5.1.15, they are both bounded. Assume $(a_n)_{n=1}^{\infty}$ is bounded by M_1 and $(b_n)_{n=1}^{\infty}$ is bounded by M_2. Since a_n is a Cauchy sequence, there exists $N_1 \geq 1$ such that for all $i, j \geq N_1$, we have

$$|a_i - a_j| \leq \frac{\varepsilon}{2M_1}.$$

Similarly, since $(b_n)_{n=1}^{\infty}$ is a Cauchy sequence, there exists $N_2 \geq 1$ such that for all

$i, j \geq N_2$, we have

$$|b_i - b_j| \leq \frac{\varepsilon}{2M_2}.$$

Let $N = \max(N_1, N_2)$, consider an arbitrary pair of $i, j \geq N$. Then we have

$$\begin{aligned}
|a_j b_j - a_i b_i| &= |a_j b_j - a_j b_i + a_j b_i - a_i b_i| \\
&\leq |a_j b_j - a_j b_i| + |a_j b_i - a_i b_i| \\
&= |a_j| \cdot |b_j - b_i| + |b_i| \cdot |a_j - a_i| \\
&\leq M_1 \cdot \frac{\varepsilon}{2M_1} + M_2 \cdot \frac{\varepsilon}{2M_2} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Therefore, for every $\varepsilon > 0$, we can find an $N = \max(N_1, N_2)$ such that $a_N b_N, a_{N+1} b_{N+1}, \ldots$ is ε-close. Thus, $(a_n b_n)_{n=1}^{\infty}$ is a Cauchy sequence and xy is a real number.

Since $(b_n)_{n=1}^{\infty}$ is a Cauchy sequence, it must be bounded by some rational number M. Since $(a_n)_{n=1}^{\infty}$ and $(a'_n)_{n=1}^{\infty}$ are equivalent, for every $\varepsilon > 0$, there exists $N \geq 1$ such that for all $i \geq N$, we have

$$|a_i - a'_i| \leq \frac{\varepsilon}{M}.$$

Therfore, for all $i \geq N$,

$$\begin{aligned}
|a_i b_i - a'_i b_i| &= |b_i| \cdot |a_i - a'_i| \\
&\leq M \cdot \frac{\varepsilon}{M} \\
&= \varepsilon.
\end{aligned}$$

Thus, $(a_n b_n)_{n=1}^{\infty}$ and $(a'_n b_n)_{n=1}^{\infty}$ are equivalent and that $xy = x'y$. □

Exercise 5.3.3

Let a, b be rational numbers. Show that $a = b$ if and only if $\mathrm{LIM}_{n\to\infty} a_n = \mathrm{LIM}_{n\to\infty} b_n$(i.e., the Cauchy sequences $a, a, a, a, \ldots$ and $b, b, b, b, \ldots$ are equivalent if and only if $a = b$). This allows us to embed the rational numbers inside the real numbers in a well-defined manner.

Proof. Suppose the Cauchy sequences $a, a, a, a, \ldots$ and $b, b, b, b, \ldots$ are equivalent. Assume $a \neq b$, then $|a - b| > 0$. Since the two sequences are equivalent, for every $\varepsilon > 0$, there exists $N \geq 1$ such that $|a_i - b_i| \leq \varepsilon$. Let $\varepsilon = \frac{|a-b|}{2} > 0$. Then no matter what value i is, we have

$$|a_i - b_i| = |a - b| > \frac{|a - b|}{2} = \frac{\varepsilon}{2}$$

which contradicts the definition of equivalent sequences. Therefore, $a = b$.

Suppose $a = b$. Then for any $\varepsilon > 0$, let $N = 1$, we have

$$|a_i - b_i| = |a - b| = 0 < \varepsilon$$

for all $i \geq N$. Therefore, $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent.

Thus, $a = b$ if and only if $\mathrm{LIM}_{n\to\infty} a_n = \mathrm{LIM}_{n\to\infty} b_n$. □

Exercise 5.3.4

Let $(a_n)_{n=0}^{\infty}$ be a sequence of rational numbers which is bounded. Let $(b_n)_{n=0}^{\infty}$ be another sequence of rational numbers which is equivalent to $(a_n)_{n=0}^{\infty}$. Show that $(b_n)_{n=0}^{\infty}$ is also bounded.

Proof. Suppose $(a_n)_{n=0}^{\infty}$ is bounded by M. Similar to Exercise 5.2.2, we can split $(b_n)_{n=1}^{\infty}$ to $b_0, \ldots, b_N$ and $b_{N+1}, b_{N+2}, \ldots$ such that the former is bounded by some rational number M_1 and the latter is bounded by $M + \varepsilon$ for any $\varepsilon > 0$. Let $M_0 = \max(M, M_1)$, then $(b_n)_{n=0}^{\infty}$ is bounded by M_0. □

Exercise 5.3.5

Show that $\mathrm{LIM}_{n\to\infty} 1/n = 0$.

Proof. We want to show that $a_n = 1/n$ and $0, 0, 0, \ldots$ are equivalent. Consider an arbitrary $\varepsilon > 0$. Let $N = \lceil \frac{1}{\varepsilon} \rceil$, we have

$$|a_i - 0| = a_i \leq a_N = \frac{1}{N} \leq \varepsilon$$

for all $i \geq N$. Thus, $\text{LIM}_{n\to\infty} 1/n = 0$. $\square$

5.4 Ordering the reals

Definition 5.4.1.

Let $(a_n)_{n=1}^{\infty}$ be a sequence of rationals. We say that this sequence is positively bounded away from zero iff we have a positive rational $c > 0$ such that $a_n \geq c$ for all $n \geq 1$ (in particular, the sequence is entirely positive). The sequence is negatively bounded away from zero iff we have a negative rational $-c < 0$ such that $a_n \leq -c$ for all $n \geq 1$ (in particular, the sequence is entirely negative).

Definition 5.4.3.

A real number x is said to be positive iff it can be written as $x = \text{LIM}_{n\to\infty} a_n$ for some Cauchy sequence $(a_n)_{n=1}^{\infty}$ which is positively bounded away from zero. x is said to be negative iff it can be written as $x = \text{LIM}_{n\to\infty} a_n$ for some sequence $(a_n)_{n=1}^{\infty}$ which is negatively bounded away from zero.

Proposition 5.4.4 (Basic properties of positive reals).

For every real number x, exactly one of the following three statements is true: (a) x is zero; (b) x is positive; (c) x is negative. A real number x is negative if and only if $-x$ is positive. If x and y are positive, then so are $x + y$ and xy.

Definition 5.4.5 (Absolute value).

Let x be a real number. We define the absolute value $|x|$ of x to equal x if x is positive, $-x$ when x is negative, and 0 when x is zero.

Definition 5.4.6 (Ordering of the real numbers).

Let x and y be real numbers. We say that x is greater than y, and write $x > y$, iff $x - y$ is a positive real number, and $x < y$ iff $x - y$ is a negative real number. We define $x \geq y$ iff $x > y$ or $x = y$, and similarly define $x \leq y$.

Proposition 5.4.7.

All the claims in Proposition 4.2.9 which held for rationals, continue to hold for real numbers.

Proposition 5.4.8.

let x be a positive real number. Then x^{-1} is also positive. Also, if y is another positive number and $x > y$, then $x^{-1} < y^{-1}$.

Proposition 5.4.9 (The non-negative reals are closed).

Let $a_1, a_2, a_3, \ldots$ be a Cauchy sequence of non-negative rational numbers. Then $\mathrm{LIM}_{n\to\infty} a_n$ is a non-negative real number.

Corollary 5.4.10.

Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be Cauchy sequences of rationals such that $a_n \geq b_n$ for all $n \geq 1$. Then $\mathrm{LIM}_{n\to\infty} a_n \geq \mathrm{LIM}_{n\to\infty} b_n$.

Proposition 5.4.12 (Bounding of reals by rationals).

Let x be a positive real number. Then there exists a positive rational number q such that $q \leq x$, and there exists a positive integer N such that $x \leq N$.

Corollary 5.4.13 (Archimedean property).

Let x and ε be any positive real numbers. Then there exists a positive integer M such that $M\varepsilon > x$.

Proposition 5.4.14.

Given any two real numbers $x < y$, we can find a rational number q such that $x < q < y$.

Exercise 5.4.1.

Prove Proposition 5.4.4.

Proof. Assume $x = \mathrm{LIM}_{n\to\infty} a_n$.
At least one of the three statements is true. If $(a_n)_{n=1}^{\infty}$ is equivalent to $(0)_{n=1}^{\infty}$, x is 0. If the Cauchy sequence $(a_n)_{n=1}^{\infty}$ is not equivalent to $(0)_{n=1}^{\infty}$, by Lemma 5.3.14, $(a_n)_{n=1}^{\infty}$ is bounded away from zero. Then there exists a rational number $c > 0$ such that $|a_i| \geq c$ for all $i \geq 1$. Since $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence, let $\varepsilon = c/2$, then there exists an there exists $N \geq 1$ such that $|a_i - a_j| \leq \varepsilon = c/2$ for all $i, j \geq N$. Let $j = N$, we have $|a_i - a_N| \leq c/2$ for all $i \geq N$. Since $(a_n)_{n=1}^{\infty}$ is bounded away from 0 by c, a_N cannot be 0. If $a_N > 0$, we have $a_N \geq c$ and $a_N - c/2 \leq a_i \leq a_N + c/2$. So $a_i \geq a_N - c/2 \geq c/2 > 0$, and $(a_n)_{n=1}^{\infty}$ is eventually positively bounded away from zero. In particular, $a_i \geq c$ for all $i \geq N$. Let $b_i = c$ when $i < N$ and $b_i = a_i$ when $i \geq N$. Then $(b_n)_{n=1}^{\infty}$ is equivalent to $(a_n)_{n=1}^{\infty}$ and $x = \mathrm{LIM}_{n\to\infty} a_n = \mathrm{LIM}_{n\to\infty} b_n$ is a positive real number. Similarly, we can show that if $a_N < 0$, x would be negative. Thus, at least one of the three statements is true.

At most one of the three statements is true. Suppose x is zero. For any $c > 0$, there exists $N \geq 1$ such that $|a_i - 0| = |a_i| \leq \frac{c}{2} < c$. Therefore, $(a_n)_{n=1}^{\infty}$ is not bounded away from zero. Thus, x is not positive nor negative. Suppose x is positive. Then there exists $c > 0$ such that $a_i > c > 0$ for all $i \geq 1$. So for any $c' > 0$, $a_i > 0 > -c'$. Therefore, x cannot be negative. Similarly, if x is negative, it cannot be positive. Thus, at most one of the three statements is true.

x is negative $\iff$ $-x$ is positive. We know that $-x = \mathrm{LIM}(-a_n)_{n=1}^{\infty}$. Suppose x is negative. Then there exists $c > 0$ such that $-a_i < -c$ for all $i \geq 1$. So $a_i > c$ for all $i \geq 1$. Thus, $(a_n)_{n=1}^{\infty}$ is positively bounded away from zero, and $x = \mathrm{LIM}_{n\to\infty} a_n$ is positive. Similarly, we can show that if $-x$ is positive, then x is negative.

Assume $y = \mathrm{LIM}_{n\to\infty} b_n$. Suppose x and y are positive. Then there exist $c_1, c_2 > 0$ such that $a_i \geq c_1$ and $b_i \geq c_2$ for all $i \geq 1$. Let $c = c_1 + c_2$, we have $(a_i + b_i) \geq c = c_1 + c_2$ for all $i \geq 1$. Therefore, $x + y$ is positive. Let $c' = c_1 c_2$, we have $a_i b_i \geq c' = c_1 c_2$ for all $i \geq 1$. Therefore, xy is positive. $\square$

Exercise 5.4.2.

Prove the remaining claims in Proposition 5.4.7.

(a) *Proof.* Since $x - y$ is a real number, by Proposition 5.4.4, exactly one of the three statements $x - y = 0$, $x - y > 0$, or $x - y < 0$ is true. Thus, exactly one of $x = y$, $x > y$, or $x < y$ is true. □

(b) *Proof.* Since $x - y$ is a real number, by Proposition 5.4.4, $x - y$ is negative iff $-(x - y) = y - x$ is positive. Thus, $x < y$ iff $y > x$. □

(c) *Proof.* Since $x < y$, we have $y > x$, hence, $y - x$ is positive. Similarly, since $y < z$, $z - y$ is positive. By Proposition 5.4.4, $z - x = (y - x) + (z - y)$ is positive. Therefore, $x < z$. □

(d) *Proof.* Since $x < y$, we have $x - y = x - y + 0 = x - y + z - z = (x + z) - (y + z) < 0$. Therefore, $x + z < y + z$. □

Exercise 5.4.3.

Show that for every real number x there is exactly one integer N such that $N \leq x < N + 1$. (This integer N is called the integer part of x, and is sometimes denoted $N = \lfloor x \rfloor$.)

Proof. Denote $x = (a_n)_{n=1}^{\infty}$ where $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence. Let $\varepsilon = \frac{1}{2}$. Then there exists an $N \geq 1$ such that $|x_i - x_N| \leq \varepsilon = \frac{1}{2}$ for all $i \geq N$. Therefore, $|x - x_N| \leq \frac{1}{2}$. So

$$-\frac{1}{2} \leq x - x_N \leq \frac{1}{2} \implies x_N - \frac{1}{2} \leq x \leq x_N + \frac{1}{2}.$$

Since x_N is rational, $x_N - \frac{1}{2}$ and $x_N + \frac{1}{2}$ are also rational. Then there exists exactly one integer n such that

$$n \leq x_N - \frac{1}{2} < n + 1$$

and

$$n + 1 \leq x_N + \frac{1}{2} < n + 2.$$

Then there are two cases. If $x < n+1$, there exists exactly one integer N such that

$$N = n \leq x_N - \frac{1}{2} \leq x < n+1 = N+1.$$

If $x \geq n+1$, there exists exists exactly one integer N such that

$$N = n+1 \leq x \leq x_N + \frac{1}{2} < n+2 = N+1.$$

Therefore, for every real number x there is exactly one integer N such that $N \leq x < N+1$. □

Exercise 5.4.4.

Show that for any positive real number $x > 0$ there exists a positive integer N such that $x > 1/N > 0$.

Proof. Let $x = \mathrm{LIM}_{n\to\infty} a_n$. Since x is a positive real number, it is positively bounded away from zero. Then there exists a positive rational number $c > 0$, such that $a_n \geq c > 0$ for $n \geq 1$. By Corollary 5.4.10, we have $x = \mathrm{LIM}_{n\to\infty} a_n \geq c > 0$. Since $c > c/2 > 0$, we have $x \geq c > c/2 > 0$. Let $N = \frac{2}{c} + 1$, then $0 < \frac{1}{N} < \frac{c}{2}$. Then we have $x > \frac{1}{N} > 0$. □

Exercise 5.4.5.

Prove Proposition 5.4.14.

Proof. $y - x > 0 \implies y - x > 0$. By Exercise 5.4.4, there exists a positive integer N such that $y - x > 1/N > 0$. Then we have $Ny > Nx + 1 > Nx$. And by Exercise 5.4.3, there exists exactly one integer n such that $n \leq Nx < n+1$. Then we also have $n+1 \leq Nx+1 < n+2$. Therefore, $n \leq Nx < n+1 \leq Nx+1 < Ny$. Therefore, there exists an integer between Nx and Ny. Divide the inequalities by N, we have $x < \frac{n+1}{N} < y$ where $\frac{n+1}{N}$ is a rational number. Therefore, if $x < y$, we can find a rational number q such that $x < q < y$. □

Exercise 5.4.6.

Let x, y be real numbers and let $\varepsilon > 0$ be a positive real. Show that $|x - y| < \varepsilon$ if and only if $y - \varepsilon < x < y + \varepsilon$, and that $|x - y| \leq \varepsilon$ if and only if $y - \varepsilon \leq x \leq y + \varepsilon$.

- $|x - y| < \varepsilon \iff y - \varepsilon < x < y + \varepsilon$.

  Proof. Suppose $|x - y| < \varepsilon$. By definition, if $x - y > 0$, we have $x - y < \varepsilon \implies x < y + \varepsilon$, and if $x - y < 0$, we have $y - x < \varepsilon \implies y - \varepsilon < x$. Combining the two inequalities, we have $y - \varepsilon < x < y + \varepsilon$.

  Suppose $y - \varepsilon < x < y + \varepsilon$. Then $-\varepsilon < x - y < \varepsilon$. So if $x - y$ is positive, $|x - y| = x - y < \varepsilon$. Otherwise, $|x - y| = y - x < \varepsilon$. Therefore, $|x - y| < \varepsilon$.

  Thus, $|x - y| < \varepsilon \iff y - \varepsilon < x < y + \varepsilon$. $\square$

- $|x - y| \leq \varepsilon \iff y - \varepsilon \leq x \leq y + \varepsilon$.

  Proof. The proof is almost identical to the previous one. $\square$

Exercise 5.4.7.

Let x and y be real numbers. Show that $x \leq y + \varepsilon$ for all real numbers $\varepsilon > 0$ if and only if $x \leq y$. Show that $|x - y| \leq \varepsilon$ for all real numbers $\varepsilon > 0$ if and only if $x = y$.

- $x \leq y + \varepsilon$ for all $\varepsilon > 0 \iff x \leq y$.

  Proof. Suppose $x \leq y + \varepsilon$ for all $\varepsilon > 0$. Assume $x > y$. Then we have $x - y > \frac{x-y}{2} > 0$. Let $\varepsilon = \frac{x-y}{2}$. We have Then we have $x \leq y + \varepsilon = y + \frac{x-y}{2} \implies x \leq y$. (contradiction) Therefore, we must have $x \leq y$.

  Suppose $x \leq y$. For all $\varepsilon > 0$, we have $x \leq y < y + \varepsilon$.

  Thus, $x \leq y + \varepsilon$ for all $\varepsilon > 0 \iff x \leq y$. $\square$

- $|x - y| \leq \varepsilon$ for all real numbers $\varepsilon > 0 \iff x = y$.

Proof. Suppose $|x - y| \leq \varepsilon$ for all real numbers. By Exercise 5.4.6, we have $-\varepsilon < x - y < \varepsilon$. Since $x < y + \varepsilon$, we have $x \leq y$. Since $y < x + \varepsilon$, we have $y \leq x$. And since $y \leq x$ and $x \leq y$, we have $x = y$.

Suppose $x = y$. Then $|x - y| = 0 \leq \varepsilon$ for all $\varepsilon > 0$.

Thus, $|x - y| \leq \varepsilon$ for all real numbers $\varepsilon > 0 \iff x = y$. □

Exercise 5.4.8.

Let $(a_n)_{n=1}^{\infty}$ be a Cauchy sequence of rationals, and let x be a real number. Show that if $a_n \leq x$ for all $n \geq 1$, then $\mathrm{LIM}_{n\to\infty} a_n \leq x$. Similarly, show that if $a_n \geq x$ for all $n \geq 1$, then $\mathrm{LIM}_{n\to\infty} a_n \geq x$.

Proof. Suppose $a_n \leq x$ for all $n \geq 1$. Assume $\mathrm{LIM}_{n\to\infty} a_n > x$. Let $y = \mathrm{LIM}_{n\to\infty} a_n$. If $y = \mathrm{LIM}_{n\to\infty} a_n > x$, by Proposition 5.4.14, ther exists a rational number q such that $y = \mathrm{LIM}_{n\to\infty} a_n > q > x$. On the other hand, we have $a_n \leq x < q$ for all $n \geq 1$. By Corollary 5.4.10, we have $y = \mathrm{LIM}_{n\to\infty} a_n \leq q$ which contradicts $y > q$. Therefore, $\mathrm{LIM}_{n\to\infty} a_n \leq x$. The second statement can be proved in a similar way. □

5.5 The least upper bound property

Definition 5.5.1 (Upper bound).

Let E be a subset of $\mathbf{R}$, and let M be a real number. We say that M is an upper bound for E, iff we have $x \leq M$ for every element x in E.

Definition 5.5.5 (Least upper bound).

Let E be a subset of $\mathbf{R}$, and M be a real number. We say that M is a least upper bound for E iff (a) M is an upper bound for E, and also (b) any other upper bound M' for E must be larger than or equal to M.

Proposition 5.5.8 (Uniqueness of least upper bound).

Let E be a subset of $\mathbf{R}$. Then E can have at most one least upper bound.

Theorem 5.5.9 (Existence of least upper bound).

Let E be a non-empty subset of $\mathbf{R}$. If E has an upper bound, (i.e., E has some upper bound M), then it must have exactly one least upper bound.

Definition 5.5.10 (Supremum).

Let E be a subset of the real numbers. If E is non-empty and has some upper bound, we define $\sup(E)$ to be the least upper bound of E (this is well-defined by Theorem 5.5.9). We introduce two additional symbols, $+\infty$ and $-\infty$. If E is non-empty and has no upper bound, we set $\sup(E) := +\infty$; if E is empty, we set $\sup(E) := -\infty$. We refer to $\sup(E)$ as the supremum of E, and also denote it by $\sup E$.

Exercise 5.5.1.

Let E be a subset of the real numbers $\mathbf{R}$, and suppose that E has a least upper bound M which is a real number, i.e., $M = \sup(E)$. Let $-E$ be the set

$$-E := \{-x : x \in E\}.$$

Show that $-M$ is the greaest lower bound of $-E$, i.e., $-M = \inf(-E)$.

Proof. For all $-x \in -E$, since $M \geq x$, we have $-M \leq -x$. Therefore, $-M$ is a lower bound for $-E$. Assume there exists some $\varepsilon > 0$ such that $-M + \varepsilon$ is also a lower bound for $-E$. Then for all $-x \in -E$, we have $-M + \varepsilon \leq -x \implies M - \varepsilon \geq x$, hence $M - \varepsilon$ is an upper bound for E which contradicts the fact that M is the least upper bound for E. Therefore, $-M$ is the least lower bound for $-E$. $\square$

Exercise 5.5.2.

Let E be a non-empty subset of $\mathbf{R}$, let $n \geq 1$ be an integer, and let $L < K$ be integers. Suppose that K/n is an upper bound for E, but that L/n is not an upper bound for E. Without using Theorem 5.5.9, show that there eixsts an integer $L < m \leq K$ such that m/n is an upper bound for E, but that $(m-1)/n$ is not an upper bound for E.

Proof. Since $L < K$ and K/n is an upper bound for E, there exists an integer m such that $L < m \leq K$ and m/n is an upper bound for E (for example, we can let $m = K$). **Suppose for all such** m, $(m-1)/n$ **is also an upper bound for** E. Since such $(m-1)/n$ is an upper bound for E, we must have $(m-1) > L$. (Then since $(m-1)/n$ is an upper bound for E and $L < (m-1) \leq K$, $(m-2)/n$ is also an upper bound for E and for similar reason, $L < (m-2) \leq K$.)
Let $P(i)$ be $(m-i)/n$ is an upper bound for E (and $L < m - i \leq K$). We want to show that $P(i)$ is true for all natural number i. Then the base case is true since there exists an m that satisfies the conditions. Assume inductively $P(i)$ is true. Since $(m-i)/n$ is an upper bound for E and $L < (m-i) \leq K$, by our assumption, $(m-i-1)/n = (m-(i+1))/n$ is also an upper bound for E and it is less than $(m-i)/n$, hence $K \geq (m-(i+1)) > L$. Therefore, $P(i+1)$ is true which closes the induction.
Thus, $L < (m-i) \leq K$ and $(m-i)/n$ is an upper bound for E for all natural number i. Let $i = \lceil m - L \rceil \geq m - L$. Then $(m-i)/n \leq L/n$ is an upper bound for E which contradicts the fact that L/n is not an upper bound for E.

Thus, there exists $L < m \leq K$ such that m/n is an upper bound for E, but that $(m-1)/n$ is not an upper bound for E. □

Exercise 5.5.3.

Let E be a non-empty subset of $\mathbf{R}$, let $n \geq 1$ be an integer, and let m, m' be integers with the properties that m/n and $(m'-1)/n$ are not upper bounds for E. Show that $m = m'$.

Proof. Assume $m \neq m'$. Without loss of generality, suppose $m' > m$, since m, m' are integers, we have $m' > m' - 1 \geq m \implies \frac{m'}{n} > \frac{m'-1}{n} \geq \frac{m}{n}$. By Theorem 5.5.9, E has the least upper bound, denote it by M. Then we have

$$\frac{m-1}{n} < M \leq \frac{m}{n} \leq \frac{m'-1}{n} < \frac{m'}{n}.$$

But since $(m'-1)/n$ is not an upper bound for E, $\frac{m'-1}{n} < M$, a contradiction. Therefore, $m = m'$. □

Exercise 5.5.4.

Let $q_1, q_2, q_3, \dots$ be a sequence of rational numbers with the property that $|q_n - q'_n| \le \frac{1}{M}$ whenever $M \ge 1$ is an integer and $n, n' \ge M$. Show that $q_1, q_2, q_3, \dots$ is a Cauchy sequence. Furthermore, if $S := \mathrm{LIM}_{n\to\infty} q_n$, show that $|q_M - S| \le \frac{1}{M}$ for every $M \ge 1$.

Proof. We need to show that for every rational $\varepsilon > 0$, there exists $N \ge 1$ such that for $|x_i - x_j| \le \varepsilon$ for all $i, j \ge N$. When $\varepsilon \ge 1$, let $M = N = 1$, we have $|q_i - q_j| \le 1 \le \varepsilon$ for all $i, j \ge N$. If $0 < \varepsilon < 1$, Let $M = N = \lceil \frac{1}{\varepsilon} \rceil \ge \frac{1}{\varepsilon}$. Then

$$|q_i - q_j| \le \frac{1}{\lceil \frac{1}{\varepsilon} \rceil} \le \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$$

for all $i, j \ge N$. Therefore, for every rational $\varepsilon > 0$, there exists $N \ge 1$ such that for $|x_i - x_j| \le \varepsilon$ for all $i, j \ge N$. Hence, $q_1, q_2, q_3, \dots$ is a Cauchy sequence.

Let $N = M$, we have $|q_M - q_n| \le \frac{1}{M}$ for all $n \ge N$. Let $a_n = q_n$ for $n \ge M$ and $a_n = q_M$ fpr $n < M$. Then we have $q_M - \frac{1}{M} \le a_n \le q_M + \frac{1}{M}$. for all $n \ge 1$. By Exercise 5.4.8, $q_M - \frac{1}{M} \le S = \mathrm{LIM}_{n\to\infty} a_n = \mathrm{LIM}_{n\to\infty} q_n \le q_M + \frac{1}{M}$. Therefore, $|q_M - S| \le \frac{1}{M}$ for every $M \ge 1$. $\square$

Exercise 5.5.5.

Establish an analogue of Proposition 5.4.14, in which "rational" is replaced by "irrational".

Proof. Assume for any $x < y$, there does not exist an irrational number z such that $x < z < y$. Suppose $x < z < y$ where z is a rational number. Then $x + \sqrt{2} < z + \sqrt{2} < y + \sqrt{2}$. By assumption, $z + \sqrt{2}$ is a rational number. So $z + \sqrt{2} = \frac{m}{n}$ for some integers m, n ($n \ne 0$). Suppose $z = \frac{a}{b}$ for integers a, b ($b \ne 0$). Then $\sqrt{2} = \frac{m}{n} - \frac{a}{b} = \frac{bn - ma}{bn}$ where $bn - ma$ and bn are integers ($bn \ne 0$). By definition, $\sqrt{2}$ is a rational number (a contradiction). Therefore, for any $x < y$, there exists an irrational number z such that $x < z < y$. $\square$