Chapter 3

Set Theory

Definition 3.1.1

(Informal) We define a set A to be any unordered collection of objects, e.g., 3, 8, 5, 2 is a set. If x is an object, we say that x is an element of A or $x \in A$ if x lies in the collection; otherwise we say that $x \notin A$. For instance, $3 \in \{1, 2, 3, 4, 5\}$ but $7 \notin \{1, 2, 3, 4, 5\}$.

Axiom 3.1 (Sets are objects).

If A is a set, then A is also an object. In particular, given two sets A and B, it is meaningful to ask whether A is also an element of B.

Axiom 3.2 (Equality of sets).

Two sets A and B are equal, A = B, iff every element of A is an element of B and vice versa. To put it another way, A = B if and only if every element x of A belongs also to B, and every element y of B belongs also to A.

Axiom 3.3 (Empty set).

There exists a set \emptyset , known as the empty set, which contains no elements, i.e., for every object x we have $x \notin \emptyset$.

Lemma 3.1.5 (Single choice).

Let A be a non-empty set. Then there exists an object x such that $x \in A$.

Axiom 3.4 (Singleton sets and pair sets).

If a is an object, then there exists a set $\{a\}$ whose only element is a, i.e., for every object y, we have $y \in \{a\}$ if and only if y = a; we refer to $\{a\}$ as the singleton set whose element is a. Furthermore, if a and b are objects, then there exists a set $\{a,b\}$

whose only elements are a and b; i.e., for every object y, we have $y \in \{a, b\}$ if and only if y = a or y = b; we refer to this set as the pair set formed by a and b.

Exercises

Exercise 3.1.1

Let a, b, c, d be objects such that $\{a, b\} = \{c, d\}$. Show that at least one of the two statements "a = c and b = d" and "a = d and b = c" hold.

Proof. Consider two cases: a = b and $a \neq b$.

Case 1: a = b. Then $\{a, b\} = \{a\}$. By Axiom 3.2, if $\{a\}$ and $\{c, d\}$ are equal to each other, then every element belong to $\{c, d\}$ must also belong to $\{a\}$. Therefore, c = a, d = a. Since a = b, we have a = b = c = d. Thus, both statements hold.

Case 2: $a \neq b$. Similarly, by Axiom 3.2, every element belong to $\{a, b\}$ must also belong to $\{c, d\}$. So $\{c, d\}$, a set of two elements, contains two distinct elements a and b. Therefore, either a = c, b = d or a = d, b = c holds, exclusively.

Thus, we have shown that at least one of the two statements "a=c and b=d" and "a=d and b=c" hold.