3.4 Images and inverse images

Definition 3.4.1 (Images of sets).

If $f : X \to Y$ is a function from X to Y, and S is a set in X, we define $f(S)$ to be the set

$$f(S) := \{f(x) : x \in S\};$$

this set is a subset of Y, and is sometimes called the image of S under the map f. We sometimes call $f(S)$ the forward image of S to distinguish it from the concept of the inverse image $f^{-1}(S)$ of S, which is defined below.

Definition 3.4.5 (Inverse images).

If U is a subset of Y, we define the set $f^{-1}(U)$ to be the set

$$f^{-1}(U) := \{x \in X : f(x) \in U\}.$$

In other words, $f^{-1}(U)$ consists of all the elements of X which map into U:

$$f(x) \in U \iff x \in f^{-1}(U).$$

We feel $f^{-1}(U)$ the inverse image of U.

Axiom 3.11 (Power set axiom).

Let X and Y be sets. Then there exists a set, denoted Y^X, which consists of all the functions from X to Y, thus

$$f \in Y^X \iff (f \text{ is a function with domain } X \text{ and range } Y).$$

Lemma 3.4.10

Let X be a set. Then the set

$$\{Y : Y \text{ is a subset of } X\}$$

is a set.

Axiom 3.12 (Union).

Let A be a set, all of whose elements are themselves sets. Then there exists a set $\bigcup A$ whose elements are precisely those objects which are elements of the elements of A, thus for all objects x

$$x \in \bigcup A \iff (x \in S \text{ for some } S \in A).$$

Exercises

Exercise 3.4.1

Let $f : X \to Y$ be a bijective function, and let $f^{-1} : Y \to X$ be its inverse. Let V be any subset of Y. Prove that the forward image of V under f^{-1} is the same set as the inverse image of V under f; thus the fact that both sets are denoted by $f^{-1}(V)$ will not lead to any inconsistency.

Proof. Let U be the forward image of V under f^{-1},

$$U = \{f^{-1}(y) : y \in V\}.$$

And let W be the inverse image of V under f,

$$W = \{x \in X : f(x) \in V\}.$$

We need to show that $U = W$ which can be done by proving $x \in U \iff x \in W$.

First, consider an arbitrary $x \in U$. Since the range of f^{-1} is X, $x \in X$. And there exists exactly one $y \in V$ such that $x = f^{-1}(y)$. By definition of inverse, we have $f(x) = y \in V$. Therefore, $x \in W$.

Then, consider an arbitrary $x \in W$. Denote $y = f(x)$. Then we have $x \in X$ and $y = f(x) \in Y$. By definition, $x = f^{-1}(y)$. Therefore, $x \in U$.

Thus, $x \in V \iff x \in U$. The statement has been proved. $\square$

Exercise 3.4.2

Let $f : X \to Y$ be a function from one set X to another set Y, let S be a subset of X, and let U be a subset of Y. What, in general, can one say about $f^{-1}(f(S))$ and S? What about $f(f^{-1}(U))$ and U?

1. $S \subseteq f^{-1}(f(S))$.

   Proof. We need to show that $x \in S \implies x \in f^{-1}(f(S))$. Consider an arbitrary $x \in S$. Then $f(x) \in f(S)$. So $x = f^{-1}(f(x)) \in f^{-1}(f(S))$. $f^{-1}(f(S)) \subseteq S$ does not stand, see p.58 for a counterexample. Thus, in general, we have $S \subseteq f^{-1}(f(S))$. □

2. $f(f^{-1}(U)) \subseteq U$.

   Proof. We need to show that $y \in f(f^{-1}(U)) \implies y \in U$. Consider an arbitrary $y \in f(f^{-1}(U))$. Then there exists $x \in f^{-1}(U)$ such that $f(x) = y$. Since $x \in f^{-1}(U)$, by definition of inverse images, $f(x) = y \in U$. $U \subseteq f(f^{-1}(U))$ is not true, see p.58 for a counterexample. Thus, in general, we have $f(f^{-1}(U)) \subseteq U$. □

If f is bijective, we have $S = f^{-1}(f(S))$ and $f(f^{-1}(U)) = U$.