Chapter 3

Set Theory

Definition 3.1.1

(Informal) We define a *set* A to be any unordered collection of objects, e.g., $3, 8, 5, 2$ is a set. If x is an object, we say that x is an element of A or $x \in A$ if x lies in the collection; otherwise we say that $x \notin A$. For instance, $3 \in \{1, 2, 3, 4, 5\}$ but $7 \notin \{1, 2, 3, 4, 5\}$.

Axiom 3.1 (Sets are objects).

If A is a set, then A is also an object. In particular, given two sets A and B, it is meaningful to ask whether A is also an element of B.

Axiom 3.2 (Equality of sets).

Two sets A and B are equal, $A = B$, iff every element of A is an element of B and vice versa. To put it another way, $A = B$ if and only if every element x of A belongs also to B, and every element y of B belongs also to A.

Axiom 3.3 (Empty set).

There exists a set $\emptyset$, known as the empty set, which contains no elements, i.e., for every object x we have $x \notin \emptyset$.

Lemma 3.1.5 (Single choice).

Let A be a non-empty set. Then there exists an object x such that $x \in A$.

Axiom 3.4 (Singleton sets and pair sets).

If a is an object, then there exists a set $\{a\}$ whose only element is a, i.e., for every object y, we have $y \in \{a\}$ if and only if $y = a$; we refer to $\{a\}$ as the singleton set whose element is a. Furthermore, if a and b are objects, then there exists a set $\{a, b\}$

whose only elements are a and b; i.e., for every object y, we have $y \in \{a, b\}$ if and only if $y = a$ or $y = b$; we refer to this set as the pair set formed by a and b.

Axiom 3.5 (Pairwise union).

Given any two sets A, B, there exists a set $A \cup B$, called the union of A and B, which consists of all the elements which belong to A or B or both. In other words, for any object x,

$$x \in A \cup B \iff (x \in A \text{ or } x \in B).$$

Lemma 3.1.12

If a and b are objects, then $\{a, b\} = \{a\} \cup \{b\}$. If A, B, C are sets, then the union operation is commutative (i.e., $A \cup B = B \cup A$) and associative (i.e., $(A \cup B) \cup C = A \cup (B \cup C)$). Also, we have $A \cup A = A \cup \emptyset = \emptyset \cup A = A$.

Definition 3.1.14 (Subsets).

Let A, B be sets. We say that A is a subset of B, denoted $A \subseteq B$, iff every element of A is also an element of B, i.e.

$$\text{For any object } x,\ x \in A \iff x \in B,$$

We say that A is a proper subset of B, denoted $A \subsetneq B$, if $A \subseteq B$ and $A \neq B$.

Proposition 3.1.17 (Sets are partially ordered by set inclusion).

Let A, B, C be sets. If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$. If $A \subseteq B$ and $B \subseteq A$, then $A = B$. Finally, if $A \subsetneq B$ and $B \subsetneq C$ then $A \subsetneq C$.

Exercises

Exercise 3.1.1

Let a, b, c, d be objects such that $\{a, b\} = \{c, d\}$. Show that at least one of the two statements "$a = c$ and $b = d$" and "$a = d$ and $b = c$" hold.

Proof. Consider two cases: $a = b$ and $a \neq b$.
Case 1: $a = b$. Then $\{a, b\} = \{a\}$. By Axiom 3.2, if $\{a\}$ and $\{c, d\}$ are equal to each other, then every element belong to $\{c, d\}$ must also belong to $\{a\}$. Therefore, $c = a$, $d = a$. Since $a = b$, we have $a = b = c = d$. Thus, both statements hold.
Case 2: $a \neq b$. Similarly, by Axiom 3.2, every element belong to $\{a, b\}$ must also belong to $\{c, d\}$. So $\{c, d\}$, a set of two elements, contains two distinct elements a and b. Therefore, either $a = c, b = d$ or $a = d, b = c$ holds, exclusively.
Thus, we have shown that at least one of the two statements "$a = c$ and $b = d$" and "$a = d$ and $b = c$" hold. □

Exercise 3.1.2

Using only Axiom 3.2, Axiom 3.1, Axiom 3.3, and Axiom 3.4, prove that the sets $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$, and $\{\emptyset, \{\emptyset\}\}$ are all distinct.

Proof. First, let's consider $\emptyset$. $\emptyset$ contains no element while other sets all have at least one element in it. Therefore, $\emptyset$ is distinct from $\{\emptyset\}$, $\{\{\emptyset\}\}$ and $\{\emptyset, \{\emptyset\}\}$. Then, let's consider $\{\emptyset\}$. Is it distinct from $\{\{\emptyset\}\}$ and $\{\emptyset, \{\emptyset\}\}$? We know that $\emptyset \in \{\emptyset\}$. But we have proved earlier $\emptyset$ and $\{\emptyset\}$ are not equal to each other, so $\emptyset \notin \{\{\emptyset\}\}$. So $\{\emptyset\}$ and $\{\{\emptyset\}\}$ are distinct. For the same reason, $\{\emptyset\} \notin \{\emptyset\}$. So $\{\emptyset\}$ and $\{\emptyset, \{\emptyset\}\}$ are also distinct. Last, consider $\{\{\emptyset\}\}$ and $\{\emptyset, \{\emptyset\}\}$. For the same reason ($\emptyset$ and $\{\emptyset\}$ are distinct), $\emptyset \notin \{\{\emptyset\}\}$. So $\{\{\emptyset\}\}$ and $\{\emptyset, \{\emptyset\}\}$ are distinct. Thus, we have proved the sets $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$, and $\{\emptyset, \{\emptyset\}\}$ are all distinct. □

Exercise 3.1.3

Prove the remaining claims in Lemma 3.1.12.

Proof. First, prove the union operation is commutative (i.e., $A \cup B = B \cup A$). By definition, we know that $A \cup B$ consists of all the elements which belong to A or B, inclusively. And $B \cup A$ also consists of all the elements belong to A or B, inclusively. Therefore, $A \cup B$ and $B \cup A$ are containing exactly the same elements. Thus, $A \cup B = B \cup A$.

The second part is to prove $A \cup A = A \cup \emptyset = \emptyset \cup A = A$. First, let's consider $A \cup A$. By definition, $A \cup A$ consists of all the element x such that $x \in A$ or $x \in A$. So $A \cup A$ and A have exactly the same elements. Therefore $A \cup A = A$. Now let's consider $A \cup \emptyset$. $A \cup \emptyset$ consists of all the x such that $x \in A$ or $x \in \emptyset$. Since no element would belong to $\emptyset$. $A \cup \emptyset$ contains exactly the same elements as A. Therefore, $A \cup \emptyset = A$. And by commutative law, we have $A \cup \emptyset = \emptyset \cup A = A$.

Thus, we have proved $A \cup A = A \cup \emptyset = \emptyset \cup A = A$. $\square$

Exercise 3.1.4

Prove the remaining claims in Lemma 3.1.17.

Proof. Part I. If $A \subseteq B$ and $B \subseteq A$, then $A = B$. Translate the if statement into propositional logic: $(x \in A \Rightarrow x \in B) \wedge (x \in B \Rightarrow x \in A)$. Therefore, we have $x \in A \iff x \in B$. Thus, $A = B$.

Part II. If $A \subsetneq B$ and $B \subsetneq C$ then $A \subsetneq C$. Since $A \neq B$ and $B \neq C$, by transitivity, $A \neq C$. And by the first part of this proposition (if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$), we would have $A \subseteq C$. Since $A \subseteq C$ and $A \neq C$, $A \subsetneq C$. $\square$