3.4 Images and inverse images

Definition 3.4.1 (Images of sets).

If $f : X \to Y$ is a function from X to Y, and S is a set in X, we define $f(S)$ to be the set

$$f(S) := \{f(x) : x \in S\};$$

this set is a subset of Y, and is sometimes called the image of S under the map f. We sometimes call $f(S)$ the forward image of S to distinguish it from the concept of the inverse image $f^{-1}(S)$ of S, which is defined below.

Definition 3.4.5 (Inverse images).

If U is a subset of Y, we define the set $f^{-1}(U)$ to be the set

$$f^{-1}(U) := \{x \in X : f(x) \in U\}.$$

In other words, $f^{-1}(U)$ consists of all the elements of X which map into U:

$$f(x) \in U \iff x \in f^{-1}(U).$$

We feel $f^{-1}(U)$ the inverse image of U.

Axiom 3.11 (Power set axiom).

Let X and Y be sets. Then there exists a set, denoted Y^X, which consists of all the functions from X to Y, thus

$$f \in Y^X \iff (f \text{ is a function with domain } X \text{ and range } Y).$$

Lemma 3.4.10

Let X be a set. Then the set

$$\{Y : Y \text{ is a subset of } X\}$$

is a set.

Axiom 3.12 (Union).

Let A be a set, all of whose elements are themselves sets. Then there exists a set $\bigcup A$ whose elements are precisely those objects which are elements of the elements of A, thus for all objects x

$$x \in \bigcup A \iff (x \in S \text{ for some } S \in A).$$

Exercises

Exercise 3.4.1

Let $f : X \to Y$ be a bijective function, and let $f^{-1} : Y \to X$ be its inverse. Let V be any subset of Y. Prove that the forward image of V under f^{-1} is the same set as the inverse image of V under f; thus the fact that both sets are denoted by $f^{-1}(V)$ will not lead to any inconsistency.

Proof. Let U be the forward image of V under f^{-1},

$$U = \{f^{-1}(y) : y \in V\}.$$

And let W be the inverse image of V under f,

$$W = \{x \in X : f(x) \in V\}.$$

We need to show that $U = W$ which can be done by proving $x \in U \iff x \in W$.

First, consider an arbitrary $x \in U$. Since the range of f^{-1} is X, $x \in X$. And there exists exactly one $y \in V$ such that $x = f^{-1}(y)$. By definition of inverse, we have $f(x) = y \in V$. Therefore, $x \in W$.

Then, consider an arbitrary $x \in W$. Denote $y = f(x)$. Then we have $x \in X$ and $y = f(x) \in Y$. By definition, $x = f^{-1}(y)$. Therefore, $x \in U$.

Thus, $x \in V \iff x \in U$. The statement has been proved. $\square$

Exercise 3.4.2

Let $f : X \to Y$ be a function from one set X to another set Y, let S be a subset of X, and let U be a subset of Y. What, in general, can one say about $f^{-1}(f(S))$ and S? What about $f(f^{-1}(U))$ and U?

1. $S \subseteq f^{-1}(f(S))$.

   Proof. We need to show that $x \in S \implies x \in f^{-1}(f(S))$. Consider an arbitrary $x \in S$. Then $f(x) \in f(S)$. So $x = f^{-1}(f(x)) \in f^{-1}(f(S))$. $f^{-1}(f(S)) \subseteq S$ does not stand, see p.58 for a counterexample. Thus, in general, we have $S \subseteq f^{-1}(f(S))$. □

2. $f(f^{-1}(U)) \subseteq U$.

   Proof. We need to show that $y \in f(f^{-1}(U)) \implies y \in U$. Consider an arbitrary $y \in f(f^{-1}(U))$. Then there exists $x \in f^{-1}(U)$ such that $f(x) = y$. Since $x \in f^{-1}(U)$, by definition of inverse images, $f(x) = y \in U$. $U \subseteq f(f^{-1}(U))$ is not true, see p.58 for a counterexample. Thus, in general, we have $f(f^{-1}(U)) \subseteq U$. □

If f is bijective, we have $S = f^{-1}(f(S))$ and $f(f^{-1}(U)) = U$.

Exercise 3.4.3

Let A, B be two subsets of a set X, and let $f : X \to Y$ be a function. Show that $f(A \cap B) \subseteq f(A) \cap f(B)$, that $f(A) \backslash f(B) \subseteq f(A \backslash B)$, $f(A \cup B) = f(A) \cup f(B)$. For the first two statements, is it true that the $\subseteq$ relation can be improved to $=$?

1. $f(A \cap B) \subseteq f(A) \cap f(B)$.

   Proof. We need to show that $y \in f(A \cap B) \implies y \in f(A) \cap f(B)$. Assume $y \in f(A \cap B)$, then there exists $x \in A \cap B$ such that $y = f(x)$. $x \in A \cap B \iff (x \in A) \wedge (x \in B)$. $x \in A \implies y = f(x) \in f(A)$, $x \in B \implies y = f(x) \in f(B)$. So $(y \in f(A)) \wedge (y \in f(B))$. Therefore, $y \in f(A) \cap f(B)$.

The $\subseteq$ relation cannot be improved to $=$. A counterexample: $A : \{0, 1\}$, $B : \{1, 2\}$, $f(0) = 2$, $f(1) = 1$, $f(2) = 2$. $\square$

2. $f(A)\backslash f(B) \subseteq f(A\backslash B)$.

   Proof. We need to show that $y \in f(A)\backslash f(B) \implies y \in f(A\backslash B)$. Assume $y \in f(A)\backslash f(B)$ which means $y \in f(A) \wedge y \notin f(B)$. Since $y \in f(A)$, there exists $x \in A$ such that $f(x) = y$. On the other hand, $y \notin f(B)$ so $x \notin B$ (otherwise we will have $y = f(x) \in B$). So there exists $(x \in A) \wedge (x \notin B) \iff x \in (A\backslash B)$ such that $y = f(x)$. Thus, $y \in A\backslash B$.

   The $\subseteq$ relation cannot be improved to $=$. A counterexample: $A : \{1, 2\}$, $B : \{2\}$, $f(1) = 1$, $f(2) = 1$. $\square$

3. $f(A \cup B) = f(A) \cup f(B)$.

   Proof. We need to show that $y \in f(A \cup B) \iff y \in f(A) \cup f(B)$.

   First, suppose $y \in f(A \cup B)$. Then there exists $x \in A \cup B$ such that $y = f(x)$. $x \in A \cup B \implies (x \in A) \vee (x \in B)$. If $x \in A$, since $y = f(x)$, $y \in f(A)$. If $x \in B$, since $y = f(x)$, $y \in f(B)$. So $y \in f(A)$ or $y \in f(B)$. Thus, $y \in f(A) \cup f(B)$.

   Then, suppose $y \in f(A) \cup f(B)$. If $y \in f(A)$, $\exists x \in A$ such that $y = f(x)$. $x \in A \implies x \in A \cup B$. So $y \in f(A \cup B)$. Similarly, if $y \in f(B)$, we also conclude that $y \in f(A \cup B)$. Therefore, in both cases, we have $y \in f(A \cup B)$. Thus, $f(A \cup B) = f(A) \cup f(B)$. $\square$

Exercise 3.4.4

Let $f : X \to Y$ be a function from one set X to another set Y, and let U, V be subsets of Y. Show that $f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$, that $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$, and that $f^{-1}(U\backslash V) = f^{-1}(U)\backslash f^{-1}(V)$.

1. $f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$.

Proof. We need to show that $x \in f^{-1}(U \cup V) \iff x \in f^{-1}(U) \cup f^{-1}(V)$.

First, suppose $x \in f^{-1}(U \cup V)$. Then there exists $y \in U \cup V$ such that $f(x) = y$. If $y \in U$, $x \in f^{-1}(U)$. If $y \in V$, $x \in f^{-1}(V)$. So $x \in f^{-1}(U)$ or $x \in f^{-1}(V)$. Thus, $x \in f^{-1}(U) \cup f^{-1}(V)$.

Then, suppose $x \in f^{-1}(U) \cup f^{-1}(V)$ which means $x \in f^{-1}(U)$ or $x \in f^{-1}(V)$. If $x \in f^{-1}(U)$, then $\exists y \in U$ such that $y = f(x)$. If $x \in f^{-1}(V)$, then $\exists y \in V$ such that $y = f(x)$. So $y = f(x) \in U$ or $y = f(x) \in V$. So $y = f(x) \in U \cup V$. Thus, $x \in f^{-1}(U \cup V)$.

Thus, we have shown that $f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$. □

2. $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$.

Proof. We need to show that $x \in f^{-1}(U \cap V) \iff x \in f^{-1}(U) \cap f^{-1}(V)$.

First, suppose $x \in f^{-1}(U \cup V)$. Then $\exists y \in U \cap V$ such that $y = f(x)$. Since $y \in U$, $x \in f^{-1}(U)$. Since $y \in V$, $x \in f^{-1}(V)$. And because $x \in f^{-1}(U)$ and $x \in f^{-1}(V)$, $x \in f^{-1}(U) \cap f^{-1}(V)$.

Then, suppose $x \in f^{-1}(U) \cap f^{-1}(V)$. Then there exists $y = f(x)$ such that $y = f(x)$, $y \in U$ and $y \in V$. So $y = f(x) \in U \cap V$. Thus, $x \in f^{-1}(U \cap V)$.

Thus, $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$. □

3. $f^{-1}(U \backslash V) = f^{-1}(U) \backslash f^{-1}(V)$.

Proof. We need to show that $x \in f^{-1}(U \backslash V) \iff x \in f^{-1}(U) \backslash f^{-1}(V)$. First, suppose $x \in f^{-1}(U \backslash V)$. Then there exists $(y \in U) \wedge y \notin V$ such that $f(x) = y$. $y \in U \implies x \in f^{-1}(U)$. On the other hand, $x \notin f^{-1}(V)$ (otherwise $y = f(x) \in V$). So $(x \in f^{-1}(U)) \wedge (x \notin f^{-1}(V))$. Hence, $x \in f^{-1}(U) \backslash f^{-1}(V)$.

Then, suppose $x \in f^{-1}(U) \backslash f^{-1}(V)$ which means $x \in f^{-1}(U) \wedge x \notin f^{-1}(V)$. Since $x \in f^{-1}(U)$, there exists $y \in U$ such that $y = f(x)$. And since $x \notin f^{-1}(V)$, we must have $y \notin V$. So there exists $y \in U \wedge y \notin V \iff y \in U \backslash V$ such that $f(x) = y$. Hence, $x \in f^{-1}(U \backslash V)$.

Thus, $f^{-1}(U \backslash V) = f^{-1}(U) \backslash f^{-1}(V)$. □

Exercise 3.4.5

Let $f : X \to Y$ be a function from one set X to another set Y. Show that $f(f^{-1}(S)) = S$ for every $S \subseteq Y$ if and only if f is surjective. Show that $f^{-1}(f(S)) = S$ for every $S \subseteq X$ if and only if f is injective.

1. $f(f^{-1}(S)) = S$ for every $S \subseteq Y$ if and only if f is surjective.

   Proof. We need to show that $y \in f(f^{-1}(S)) = S \iff f$ is surjective. And for the LHS, we have proved in 3.4.2 that $f(f^{-1}(S)) \subseteq S$ not matter what kind of function f is. So it would be sufficient to show that $S \subseteq f(f^{-1}(S))$.

   First, suppose f is surjective. We want to show that $y \in S \implies y \in f(f^{-1}(S))$. Since f is surjective and $S \in Y$, there must exist $x \in X$ such that $f(x) = y$. Because $y = f(x)$ and $y \in S$, $x \in f^{-1}(S)$. Since $x \in f^{-1}(S)$ and $y = f(x)$, $y \in f(f^{-1}(S))$.

   Then, suppose $y \in S \implies y \in f(f^{-1}(S))$. We want to show that f is surjective. Assume f is not surjective. Then there exists y and $S \subseteq Y$, such that $y \in S$ and $\forall x \in X,\ f(x) \neq y$. Since $f^{-1}(S)$ is a subset of X, for all objects $x \in f^{-1}(S)$, $f(x) \neq y$. Thus, $y \notin f(f^{-1}(S))$, contradiction. Thus, f is surjective.

   Thus, $f(f^{-1}(S)) = S$ for every $S \subseteq Y$ if and only if f is surjective. $\square$

2. $f^{-1}(f(S)) = S$ for every $S \subseteq X$ if and only if f is injective.

   Proof. We need to show that $f^{-1}(f(S)) = S \iff f$ is injective. For the LHS, it is not necessary to show that $S \subseteq f^{-1}(f(S))$ since we have proved in 3.4.2 that it stands generally. So we only need to show $f^{-1}(f(S)) \subseteq S$ for every $S \subseteq X \iff f$ is injective.

   First, suppose f is injective. Assume $x \in f^{-1}(f(S))$. Then there exists $y \in f(S)$ such that $y = f(x)$. Since $y \in f(S)$, there exists $x' \in S$ such that $y = f(x')$. And because f is injective, $x = x'$. Therefore, $x \in S$.

   Next, suppose $x \in f^{-1}(f(S)) \implies x \in S$. Assume f is not injective. Then $\exists x, x' \in X, x \neq x'$ and $f(x) = f(x') = y$. Let S be $\{x'\}$. In this case, $y \in f(S)$ and $x \in f^{-1}(f(S))$. But $x \notin S$, contradiction. Hence, f is injective.

Thus, $f^{-1}(f(S)) = S$ for every $S \subseteq X$ if and only if f is injective. $\square$

Exercise 3.4.6

Prove Lemma 3.4.10. (Hint: start with the set $\{0,1\}^X$ and apply the replacement axiom, replacing each function f with the object $f^{-1}(\{1\})$.)

Proof. Consider the set $\{0,1\}^X$ which is set of all functions that map from X to $\{0,1\}$. Denote $\{0,1\}^X$ as $\mathcal{P}(X)$. Let statement $P(f,Y)$ be $Y = f^{-1}(\{1\})$ is a subset of X. For any f, there exists at most Y for which $P(f,Y)$ is true. Then, by axiom of replacement, there exists a set $\{Y : P(Y) \text{ is true for some } f \in \mathcal{P}(X)\}$, such that for any object z,

$$z \in \{Y : P(Y) \text{ is true for some } f \in \mathcal{P}(X)\} \iff P(x,z) \text{ is true for some } f \in \mathcal{P}(X).$$

So such a set Y exists and this is exactly the set of all the subsets of X. Thus, all the subsets of X is a set. $\square$

Exercise 3.4.7

Let X, Y be sets. Define a partial function from X to Y to be any function $f : X' \to Y'$ whose domain X' is a subset of X, and whose range Y' is a subset of Y. Show that the collection of all partial functions from X to Y is itself a set.

Proof. $\{0,1\}^X$ is the set of all subsets of X, $X' \in \{0,1\}^X$. Similarly, $Y' \in \{0,1\}^Y$. $Y'^{X'}$ for some $X' \in \{0,1\}^X$, $Y' \in \{0,1\}^Y$ is the set of all partial functions from X' to Y'. Using the union axiom to iterate over all $X' \in \{0,1\}^X$ and $Y' \in \{0,1\}^Y$ and obtain the set all partial functions.

Consider an arbitrary $Y_0 \in \{0,1\}^Y$. Let ${Y_0}^{X'}$ be the set consists of all the set in the form of ${Y_0}^{X'}$ where $X' \in \{0,1\}^X$. (For example, if $X = \{0,1\}$, ${Y_0}^{X'}$ would be $\{{Y_0}^{\{0\}}\}, {Y_0}^{\{1\}}, {Y_0}^{\{0,1\}}, \emptyset$.) So every element of ${Y_0}^{X'}$ is a set itself. Apply the union axiom, there exists $\bigcup {Y_0}^{X'}$ whose elements are the elements of the elements of ${Y_0}^{X'}$,

$$f \in \bigcup {Y_0}^{X'} \iff (f \in S \text{ for some } S \in {Y_0}^{X'}).$$

Every f in $\bigcup {Y_0}^{X'}$ is a partial function from Y_0 to some X' where $X' \in \{0,1\}^X$.

Then, generalize Y_0. Let ${Y'}^{X'}$ be the set of all the sets in the form of $\bigcup {Y'}^{X'}$ where $Y' \in \{0,1\}^Y$. Every element of ${Y'}^{X'}$ is a set. By the union axiom, there exists $\bigcup {Y'}^{X'}$ whose elements are the elements of the elements of ${Y'}^{X'}$,

$$f \in \bigcup {Y'}^{X'} \iff (f \in S \text{ for some } S \in {Y'}^{X'}).$$

Therefore, every f is a partial function from some $Y' \in \{0,1\}^Y$ and $X' \in \{0,1\}^X$. Thus, $\bigcup {Y'}^{X'}$ is a set, and it is the collection of all partial functions from X to Y. $\square$