3.5 Cartesian products

Definition 3.5.1 (Ordered pair).

If x and y are any objects (possibly equal), we define the ordered pair (x, y) to be a new object, consisting of x as its first component and y as its second component. Two ordered pairs (x, y) and (x', y') are considered equal if and only if both their components match, i.e.

$$(x, y) = (x', y') \iff (x = x' \text{ and } y = y').$$

Definition 3.5.4 (Cartesian product).

If X and Y are sets, then we define the Cartesian product $X \times Y$ to be the collection of ordered pairs, whose first component lies in X and second component lies if Y, thus

$$X \times Y = \{(x, y) : x \in X, y \in Y\}$$

or equivalently

$$a \in (X \times Y) \iff (a = (x, y) \text{ for some } x \in X and y \in Y).$$

Definition 3.5.7 (Ordered n-tuple and n-fold Cartesian product).

Let n be a natural number. An ordered n-tuple $(x_i)_{1 \leq i \leq n}$ (also denoted $(x_1, \cdots, x_n)$) is a collection of objects x_i, one for every natural number i between 1 and n; we refer to x_i as the i^{th} component of the ordered n-tuple. Two ordered n-tuples $(x_i)_{1 \leq i \leq n}$ and $(y_i)_{1 \leq i \leq n}$ are said to be equal iff $x_i = y_i$ for all $1 \leq i \leq n$. If $(X_i)_{1 \leq i \leq n}$ is an ordered n-tuple of sets, we define their Cartesian product $\prod_{1 \leq i \leq n} X_i$ (also denoted $\prod_{i=1}^{n} X_i$ or $X_1 \times \cdots \times X_n$) by

$$\prod_{i \leq i \leq n} X_i := \{(x_i)_{1 \leq i \leq n} : x_i \in X_i \text{ for all } 1 \leq i \leq n\}.$$

Lemma 3.5.12 (Finite choice).

Let $n \geq 1$ be a natural number, and for each natural number $1 \leq i \leq n$, let X_i be a non-empty set. Then there exists an n-tuple $(x_i)_{1\leq i\leq n}$ such that $x_i \in X_i$ for all $1 \leq i \leq n$. In other words, if each X_i is non-empty, then the set $\prod_{1\leq i\leq n} X_i$ is also non-empty.

Exercises

Exercise 3.5.1

Suppose we define the ordered pair (x, y) for any objects x and y by the formula $(x, y) := \{\{x\}, \{x, y\}\}$ (thus using several applications of Axiom 3.4). Thus for instance $(1, 2)$ is the set $\{\{1\}, \{1, 2\}\}$, $(2, 1)$ is the set $\{\{2\}, \{2, 1\}\}$, and $(1, 1)$ is the set $\{1\}$. Show that such a definition indeed obeys the property (3.5), and also whenever X and Y are sets, the Cartesian product $X \times Y$ is also a set. Thus this definition can be validly used as a definition of an ordered pair. For an additional challenge, show that the alternate definition $(x, y) := \{x, \{x, y\}\}$ also verifies (3.5) and is thus also an acceptable definition of ordered pair.

1. Show that $(x, y) := \{\{x\}, \{x, y\}\}$ is a valid definition of an ordered pair.

   Proof. First, we need to show that

   $$(x, y) = (x', y') \iff (x = x' \text{ and } y = y').$$

   Suppose $(x, y) = (x', y')$. Then by definition, $\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\}$. If $x = y$, $\{\{x\}, \{x, y\}\} = \{\{x\}, \{x\}\} = \{\{x\}\}$. Then $\{\{x'\}, \{x', y'\}\}$ must also only contain one element. Thus, $\{x'\} = \{x', y'\}$. So $x' = y'$. Lastly, we have $\{\{x\}\} = \{\{x'\}\}$. So $x = x'$. Thus, $x = x' = y = y'$. It is the same thing if we assume $x' = y'$ at first. Now consider the case $x \neq y$. Then $\{x\}$ has one element and $\{x, y\}$ has two elements. And since $\{x'\}$ could only contain one element, we have the following relations:

$$\begin{cases} \{x\} = \{x'\} \implies x = x' \\ \{x, y\} = \{x', y'\} \text{ and } x = x' \implies y = y'. \end{cases}$$

It would be the same if we assume $x' \neq y'$. Thus, $(x, y) = (x', y') \implies (x = x' \text{ and } y = y')$

Suppose $x = x'$ and $y = y'$. Then we must have $\{x\} = \{x'\}$ and $\{x, y\} = \{x', y'\}$. Hence, $\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\}$. Thus, $(x = x' \text{ and } y = y') \implies (x, y) = (x', y')$.

Therefore, $(x, y) = (x', y') \iff (x = x' \text{ and } y = y')$. This definition verifies (3.5).

Then, we need to show that whenever X and Y are sets, the Cartesian product $X \times Y$ is a set. Use the definition above: $(x, y) = \{\{x\}, \{x, y\}\}$. The powerset of $X \cup Y$ is $\{0, \{x\}, \{y\}, \{x, y\}\}$ which contains the elements in (x, y). Then the powerset of the powerset of $X \cup Y$ contains (x, y). The elements in $\mathcal{P}(\mathcal{P}(X \cup Y))$ is a set, thus, the Cartesian product is a set. More specifically,

$$X \times Y = \{z \in \mathcal{P}(\mathcal{P}(X \cup Y)) : z \text{ contains exactly one singleton set and one pair set}\}.$$

$\square$

2. Show that $(x, y) = \{x, \{x, y\}\}$ is also a valid definition of an ordered pair.

Proof. We need to show that

$$\{x, \{x, y\}\} = \{x', \{x', y'\}\} \iff x = x' \text{ and } y = y'.$$

Suppose $\{x, \{x, y\}\} = \{x', \{x', y'\}\}$. Denote $A = \{x, y\}$, $B = \{x', y'\}$. Then $\{x, A\} = \{x', B\}$. Since sets are objects, $\{x, A\}$ and $\{x', B\}$ are both pair sets. Since $x \in \{x, \{x, y\}\}$ and $x \in \{x, \{x, y\}\} \implies x \in \{x', \{x', y'\}\}$. So either $x = x'$ or $x = \{x', y'\}$. Assume $x = \{x', y'\}$. Then the only option left is $x' = \{x, y\}$. As x and x' are both sets, having $x \in x'$ and $x' \in x$ at the same time violates the statements in Exercise 3.2.2. Therefore, $x = x'$ and $\{x, y\} = \{x', y'\} = \{x, y'\}$. Thus, $y = y'$.

Suppose $x = x'$ and $y = y'$. Then clearly we have $\{x, y\} = \{x', y'\}$. So $\{x, \{x, y\}\} = \{x', \{x', y'\}\}$.

Thus, $(x, y) := \{x, \{x, y\}\}$ verifies (3.5). $\square$

Exercise 3.5.2

Suppose we define an ordered n-tuple to be a surjective function $x : \{i \in \mathbb{N} : 1 \leq i \leq n = n\} \to X$ whose range is some arbitrary set X (so different ordered n-tuples are allowed to have different ranges); we then write x_i for $x(i)$, and also write x as $(x_i)_{1\leq i\leq n}$. Using this definition, verify that we have $(x_i)_{1\leq i\leq n} = (y_i)_{1\leq i\leq n}$ if and only if $x_i = y_i$ for all $1 \leq i \leq n$. Also, show that if $(X_i)_{1\leq i\leq n}$ is an ordered n-tuple of sets, then the Cartesian product, as defined in Definition 3.5.7, is indeed a set.

1. $(x_i)_{1\leq i\leq n} = (y_i)_{1\leq i\leq n} \iff x_i = y_i$ for all $1 \leq i \leq n$.

   Proof. Apparently, x and y have the same domain $\{i \in \mathbb{N} : 1 \leq i \leq n\}$. Suppose $y : \{i \in \mathbb{N} : 1 \leq i \leq n\} \to Y$.

   Suppose $(x_i)_{1\leq i\leq n} = (y_i)_{1\leq i\leq n}$. Since x and y are two functions, we must have $X{=}Y$ so that they have the same range. And by Definition 3.3.7, we have $x(i) = y(i)$ for all $1 \leq i \leq n$. Therefore, $(x_i)_{1\leq i\leq n} = (y_i)_{1\leq i\leq n} \implies x_i = y_i$ for all $1 \leq i \leq n$..

   Suppose $x_i = y_i$ for all $1 \leq i \leq n$. Since x and y are both surjective and $\{x_1, \cdots, x_n\} = \{y_1, \cdots, y_n\}$, $X = Y = \{x_1, \cdots, x_n\} = \{y_1, \cdots, y_n\}$. Thus, x and y have the same range. And because they also have the same domain and $x_i = y_i$ for all $1 \leq i \leq n$, $x = y$. Therefore, we have proved $x_i = y_i$ for all $1 \leq i \leq n \implies x = y$.

   Thus, $(x_i)_{1\leq i\leq n} = (y_i)_{1\leq i\leq n} \iff x_i = y_i$ for all $1 \leq i \leq n$. $\square$

2. If $(X_i)_{1\leq i\leq n}$ is an ordered n-tuple of sets, then the Cartesian product is a set.

Proof. Denote $A = \{X_1, X_2, \cdots, X_n\}$, so every element of A is a set itself and by the union axiom we have $\bigcup A$ being the set consists of all the elements of the elements of A. Denote $I = \{i \in \mathbb{N} : 1 \leq i \leq n\}$. Then, the mapping function x would be partial functions with domain I which is also a subset of I and range being a subset of $\bigcup A$. Denote it as X. Thus, by Exercise 3.4.7, the collection of all these partial functions is a set. By Definition 3.5.7, the Cartesian product would be a subset of the set of all these partial functions. Let $P(x)$ be $x_i \in X_i$ for all $1 \leq i \leq n$. By Axiom of specification, there exists a set $\{x \in X : P(x) \text{ is true}\}$ which is the same as the Cartesian product. Therefore, the Cartesian product is indeed a set. □

Exercise 3.5.3

Show that the definitions of equality for ordered pair and ordered n-tuple obey the reflexivity, symmetry, and transitivity axioms.

- reflexivity

  Proof. For the ordered pair (x, y), since $x = x$ and $y = y$, we have $(x, y) = (x, y)$. For the ordered n-tuple $(x_i)_{1 \leq i \leq n}$, since $x_i = x_i$ for $1 \leq i \leq n$, by definition, we have $(x_i)_{1 \leq i \leq n}$. □

- symmetry

  Proof. We want to show $(x, y) = (x', y') \iff (x', y') = (x, y)$. Assume $(x, y) = (x', y')$. Then $x = x'$ and $y = y'$. By symmetry property of equality, we have $x' = x$ and $y' = y$. By definition, we have $(x', y') = (x, y)$. Similarly, we can show that $(x', y') = (x, y) \implies (x, y) = (x', y')$. Thus, $(x, y) = (x', y') \iff (x', y') = (x, y)$.

  For ordered n-tuple, we want to show that $(x_i)_{1 \leq i \leq n} = (y_i)_{1 \leq i \leq n} \iff (y_i)_{1 \leq i \leq n} = (x_i)_{1 \leq i \leq n}$. Assume $(x_i)_{1 \leq i \leq n} = (y_i)_{1 \leq i \leq n}$. Then $x_i = y_i$ for every $1 \leq i \leq n$. By the symmetry property of equality, we have $y_i = x_i$ for every $1 \leq i \leq n$. Therefore, by definition of ordered n-tuple, we have $(y_i)_{1 \leq i \leq n} = (x_i)_{1 \leq i \leq n}$. The approach is the same for the other way around. Thus, $(x_i)_{1\leq i\leq n} = (y_i)_{1\leq i\leq n} \iff (y_i)_{1\leq i\leq n} = (x_i)_{1\leq i\leq n}$. $\square$

- transitivity

  Proof. The proof for ordered pair is omitted since it is only a special case of ordered $n-$tuple. We need to show that $(x_i)_{1\leq i\leq n} = (y_i)_{1\leq i\leq n}$ and $(y_i)_{1\leq i\leq n} = (z_i)_{1\leq i\leq n} \implies (x_i)_{1\leq i\leq n} = (z_i)_{1\leq i\leq n}$. Since $(x_i)_{1\leq i\leq n} = (y_i)_{1\leq i\leq n}$, we have $x_i = y_i$ for $1 \leq i \leq n$. Since $(y_i)_{1\leq i\leq n} = (z_i)_{1\leq i\leq n}$, we have $y_i = z_i$ for $1 \leq i \leq n$. By transitivity property of equality, we have $x_i = z_i$. By definition of ordered n-tuple, $(x_i)_{1\leq i\leq n} = (z_i)_{1\leq i\leq n}$. $\square$

Exercise 3.5.4

Let A, B, C be sets. Show that $A\times(B\cup C) = (A\times B)\cup(A\times C)$, that $A\times(B\cap C) = (A\times B)\cap(A\times C)$, and that $A\times(B\backslash C) = (A\times B)\backslash(A\times C)$.

1. $A\times(B\cup C) = (A\times B)\cup(A\times C)$.

   Proof. We need to show that $(x, y) \in A\times(B\cup C) \iff (x, y) \in (A\times B)\cup(A\times C)$.

   Suppose $(x, y) \in A\times(B\cup C)$. By definition, we have $x \in A$ and $y \in (B\cup C)$. $y \in B\cup C \iff y \in B$ or $y \in C$. Therefore, $(x, y) \in (A\times B)$ or $(x, y) \in (A\times C)$. Hence, $(x, y) \in (A\times B)\cup(A\times C)$.

   Suppose $(x, y) \in (A\times B)\cup(A\times C)$. Then we have either $(x, y) \in A\times B$ or $(x, y) \in A\times C$. $(x, y) \in A\times B \implies x \in A$ and $y \in B$. $(x, y) \in A\times C \implies x \in A$ and $y \in C$. Therefore, we have $x \in A$ and $y \in B\cup C$. Hence, $(x, y) \in A\times(B\cup C)$.

   Thus, $A\times(B\cup C) = (A\times B)\cup(A\times C)$. $\square$

2. $A\times(B\cap C) = (A\times B)\cap(A\times C)$.

Proof. We need to show that $(x, y) \in A \times (B \cap C) \iff (x, y) \in (A \times B) \cap (A \times C)$.

Suppose $(x, y) \in A \times (B \cap C)$. Then $x \in A$ and $y \in B \cap C$. $y \in B \cap C \iff y \in B$ and $y \in C$. Then we have $(x, y) \in A \times B$ and $(x, y) \in A \times C$. Hence, $(x, y) \in (A \times B) \cap (A \times C)$.

Suppose $(x, y) \in (A \times B) \cap (A \times C)$. Then $(x, y) \in A \times B$ and $(x, y) \in A \times C$. $(x, y) \in A \times B \implies x \in A$ and $y \in B$. $(x, y) \in A \times C \implies x \in A$ and $y \in C$. Overall, we have $x \in A$ and $y \in B \cap C$. Hence, $(x, y) \in A \times (B \cap C)$.

Thus, $A \times (B \cap C) = (A \times B) \cap (A \times C)$. □

3. $A \times (B \backslash C) = (A \times B) \backslash (A \times C)$.

Proof. We need to show that $(x, y) \in A \times (B \backslash C) \iff (x, y) \in (A \times B) \backslash (A \times C)$.

Suppose $(x, y) \in A \times (B \backslash C)$. Then $x \in A$ and $y \in B$ and $y \notin C$. $x \in A$ and $y \in B \implies (x, y) \in A \times B$. $y \notin C \implies (x, y) \notin A \times C$. Therefore, $(x, y) \in (A \times B) \backslash (A \times C)$.

Suppose $(x, y) \in (A \times B) \backslash (A \times C)$. Then $(x, y) \in A \times B$ and $(x, y) \notin A \times C$. $(x, y) \in A \times B \implies x \in A$ and $y \in B$. $(x, y) \notin A \times C$ and $x \in A \implies y \notin C$. Therefore, we have $x \in A$ and $y \in B$ and $y \notin C$. Hence, $(x, y) \in A \times (B \backslash C)$.

Thus, $A \times (B \backslash C) = (A \times B) \backslash (A \times C)$. □

Exercise 3.5.5

Let A, B, C, D be sets. Show that $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$. Is it true that $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$? Is it true that $(A \times B) \backslash (C \times D) = (A \backslash C) \times (B \backslash D)$?

Proof. $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D) \iff ((x, y) \in (A \times B) \cap (C \times D) \iff (x, y) \in (A \cap C) \times (B \cap D))$. Suppose $x \in (A \times B) \cap (C \times D)$. $(x, y) \in A \times B \implies x \in A$ and $y \in B$. $(x, y) \in C \times D \implies x \in C$ and $y \in D$. Therefore, we have $x \in A \cap C$ and $y \in B \times D$. Thus, $(x, y) \in (A \cap C) \times (B \cap D)$. Suppose $(x, y) \in (A \cap C) \times (B \cap D)$. Then $x \in A \cap C$ and $y \in B \cap D$. $x \in A \cap C \implies x \in A$

and $x \in C$. $y \in B \cap D \implies y \in B$ and $y \in D$. $x \in A$ and $y \in B \implies (x, y) \in A \times B$. $x \in C$ and $y \in D \implies (x, y) \in C \times D$. Hence, $(x, y) \in (A \times B) \cap (C \times D)$. Thus, $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

$(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$ is not true. Assume $(x, y) \in (A \cup C) \times (B \cup D)$. And suppose $x \in A$ and $y \in D$. Then $(x, y) \in A \times D$ and $(x, y) \notin A \times B$ and $(x, y) \notin C \times D$. Hence, $x \notin (A \times B) \cup (C \times D)$. Thus, $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$ is not true.

$(A \times B) \backslash (C \times D) = (A \backslash C) \times (B \backslash D)$ is not true. A counterexample: $x \in A \cap C$ and $y \in B \backslash D$. Then $(x, y) \in (A \times B) \backslash (C \times D)$ but $(x, y) \notin (A \backslash C) \times (B \backslash D)$. Thus, $(A \times B) \backslash (C \times D) = (A \backslash C) \times (B \backslash D)$ is not true. □

Exercise 3.5.6

Let A, B, C, D be non-empty sets. Show that $A \times B \subseteq C \times D$ if and only if $A \subseteq C$ and $B \subseteq D$, and that $A \times B = C \times D$ if and only if $A = C$ and $B = D$. What happens if the hypotheses that the A, B, C, D are all non-empty are removed?

1. $A \times B \subseteq C \times D \iff A \subseteq C$ and $B \subseteq D$.

   Proof. Suppose $A \times B \subseteq C \times D$, that is $(x, y) \in A \times B \implies (x, y) \in C \times D$. Since A, B, C, D are non-empty, we have two conditions: $x \in A \implies x \in C$ and $y \in B \implies y \in D$. Thus, $A \subseteq C$ and $B \subseteq D$.

   Suppose $A \subseteq C$ and $B \subseteq D$. Then we have $x \in A \implies x \in C$ and $y \in B \implies y \in D$. Combining these two conditions, $(x, y) \in A \times B \implies (x, y) \in C \times D$. Hnece, $A \times B \subseteq C \times D$.

   Thus, $A \times B \subseteq C \times D \iff A \subseteq C$ and $B \subseteq D$. □

2. $A \times B = C \times D \iff A = C$ and $B = D$.

   Proof. Suppose $A \times B = C \times D$. Then $(x, y) \in A \times B \iff (x, y) \in C \times D$. Since A, B, C, D are non-empty, we have $x \in A \iff x \in C$ and $y \in B \iff y \in D$. Therefore, $A = C$ and $B = D$.

Suppose $A = C$ and $B = D$. Then $x \in A \iff x \in C$ and $y \in B \iff y \in D$. Therefore, $(x, y) \in A \times B \iff (x, y) \in C \times D$. Hence, $A \times B = C \times D$.

Thus, $A \times B = C \times D \iff A = C$ and $B = D$. $\square$

If the hypothese that the A, B, C, D are all non-empty are removed, the equalities will not hold any more. A counterexample would be A is non-empty, $B = \emptyset$, $C = \emptyset$, and D is non-empty. $A \times B \subseteq C \times D$ but A is not a subset of C. $A \times B = C \times D$ but $A \neq C$.