Chapter 4

Integers and rationals

4.1 The integers

Definition 4.1.1 (Integers).

An integer is an expression of the form $a - -b$, where a and b are natural numbers. Two integers are considered to be equal, $a - -b = c - -d$, if and only if $a + d = c + b$. We let $\mathbf{Z}$ denote the set of all integers.

Definition 4.1.2

The sum of two integers, $(a - -b) + (c - -d)$, is defined by the formula

$$(a - -b) + (c - -d) := (a + c) - (b + d).$$

The product of two integers, $(a - -b) \times (c - -d)$, is defined by

$$(a - -b) \times (c - -d) := (ac + bd) - -(ad + bc).$$

Lemma 4.1.3 (Addition and multiplication are well-defined).

Let a, b, a', b', c, d be natural numbers. If $(a - -b) = (a' - -b')$, then $(a - -b) + (c - -d) = (a' - -b') + (c - -d)$ and $(a - -d) \times (c - -d) = (a' - -b') \times (c - -d)$, and also $(c - -d) + (a - -b) = (c - -d) + (a' - -b')$ and $(c - -d) \times (a - -b) = (c - -d) \times (a' - -b')$. Thus addition and multiplication are well-defined operations (equal inputs give equal outputs).

Definition 4.1.4 (Negation of integers).

If $(a - -b)$ is an integer, we define the negation $-(a - -b)$ to be the integer $(b - -a)$. In particular if $n = n - -0$ is a positive natural number, we can define its negation $-n = 0 - -n$.

Lemma 4.1.5 (Trichotomy of integers).

Let x be an integer. Then exactly one of the following three statements is true: (a) x is zero; (b) x is equal to a positive natural number n; or (c) x is the negation $-n$ of a positive natural number n.

Proposition 4.1.6 (Laws of algebra for integers).

Let x, y, z be integers. Then we have

$$\begin{aligned}
x + y &= y + x \\
(x + y) + z &= x + (y + z) \\
x + 0 = 0 + x &= x \\
x + (-x) = (-x) + x &= 0 \\
xy &= yx \\
(xy)z &= x(yz) \\
x1 = 1x &= x \\
x(y + z) &= xy + xz \\
(y + z)x &= yx + zx
\end{aligned}$$

Proposition 4.1.8 (Integers have no zero divisors).

Let a and b be integers such that $ab = 0$. Then either $a = 0$ or $b = 0$ (or both).

Corollary 4.1.9 (Cancellation law for integers).

If a, b, c are integers such that $ac = bc$ and c is non-zero, then $a = b$.

Definition 4.1.10 (Ordering of the integers).

If n and m be integers. We say that n is greater than or equal to m, and write $n \geq m$ or $m \leq n$, iff we have $n = m + a$ for some natural number a. We say that n is strictly greater than m, and write $n > m$ or $m < n$, iff $n \geq m$ and $n \neq m$.

Lemma 4.1.11 (Properties of order).

Let a, b, c be integers.

(a) $a > b$ if and only if $a - b$ is a positive natural number.

(b) (Addition preserves order) If $a > b$, then $a + c > b + c$.

(c) (Positive multiplication preserves order) If $a > b$ and c is positive, then $ac > bc$.

(d) (Negation reverses order) If $a > b$ and $b > c$, then $a > c$.

(e) (Order trichotomy) Exactly one of the statements $a > b$, $a < b$, or $a = b$ is true.

Exercise 4.1.1

Verify that the definition of equality on the integers is both reflexive and symmetric.

Proof. Reflexivity: since summation is reflexive, we have $a + b = a + b$. Thus, by definition, $a - -b = a - -b$. Symmetry: assume $a - -b = c - -d$, then $a + d = c + b$. Since summation is symmetric, $c + b = a + d$. By definition, we have $c - -d = a - -b$. □

Exercise 4.1.2

Show that the definition of negation on the integers is well-defined in the sense that $(a - -b) = (a' - -b')$, then $-(a - -b) = -(a' - -b')$ (so equal integers have equal negations).

Proof. Since $(a - -b) = (a' - -b')$, by definition, $a + b' = a' + b$. By the reflexivity and symmetry of summation, we have $b + a' = b' + a$. Thus, by definition, $b - -a = b' - -a'$. By definition of negation of integers, $-(a - -b) = -(a' - -b')$. □

Exercise 4.1.3

Show that $(-1) \times a = -a$ for every integer a.

Proof. By definition, $-1 = (0 - -1)$ and $a = (a - -0)$. Then $(-1) \times a = (0 - -1) \times (a - -0) = (0 \times a + 1 \times 0) - -(0 \times 0 + 1 \times a) = 0 - -a = -a$. □

Exercise 4.1.4

Prove the remaining identities in Proposition 4.1.6.

1. $x + y = y + x$.

   Proof. Suppose $x = a - -b$ and $y = c - -d$ for some natural numbers a, b, c, d. Then $x + y = (a - -b) + (c - -d) = (a + c) - -(b + d)$ and $y + x = (c - -d) + (a - -b) = (c + a) - -(d + b)$. By the symmetry property of summation, we have $(a + c) = (c + a)$ and $(b + d) = (d + b)$. Thus, $x + y = y + x$. $\square$

2. $(x + y) + z = x + (y + z)$.

   Proof. Suppose $x = a - -b$, $y = c - -d$, and $z = e - -f$ for some natural numbers $a, b, c, d, e, f,$. Then

$$\begin{aligned}
(x + y) + z &= ((a - -b) + (c - -d)) + (e - -f) \\
&= ((a + c) - -(b + d)) + (e - -f) \\
&= ((a + c) + e) - -((b + d) + f) \\
&= (a + c + e) - -(b + d + f); \\
x + (y + z) &= (a - -b) + ((c - -d) + (e - -f)) \\
&= (a - -b) + ((c + e) - -(d + f)) \\
&= (a + (c + e)) - -(b + (d + f)) \\
&= (a + c + e) - -(b + d + f).
\end{aligned}$$

   Therefore, $(x + y) + z = x + (y + z)$. $\square$

3. $x + 0 = 0 + x = x$.

   Proof. Since $x + y = y + x$, we have $x + 0 = 0 + x$. Let $x = a - -b$ for some natural numbers a, b, and write $0 = 0 - -0$. Then $x + 0 = (a - -b) + (0 - -0) = (a + 0) - -(b + 0) = a - -b = x$. Thus, $x + 0 = 0 + x = x$. $\square$

4. $x + (-x) = (-x) + x = 0$.

   Proof. Since $x + y = y + x$, we have $x + (-x) = (-x) + x$. Let $x = a - -b$ for some natural numbers a, b, then $-x = b - -a$. Write 0 as $0 - -0$. Then $x + (-x) = (a - -b) + (b - -a) = (a + b) - -(b + a)$. Since $(a + b) + 0 = (b + a) + 0 = a + b$, we have that $(a + b) - -(b + a) = 0 - -0$. So $x + (-x) = 0$. Thus, $x + (-x) = (-x) + x = 0$. □

5. $xy = yx$.

   Proof. Let $x = a - -b$ and $y = c - -d$ for some natural numbers a, b, c, d. Then

$$\begin{aligned} xy &= (a - -b) \times (c - -d) \\ &= (ac + bd) - -(ad + bc); \\ yx &= (c - -d) \times (a - -b) \\ &= (ca + db) - -(cb + da) \\ &= (ac + bd) - -(ad + bc). \end{aligned}$$

   Therefore, $xy = yx$. □

6. $(xy)z = x(yz)$.
   Has been proved on page 79.

7. $x1 = 1x = x$.

   Proof. Since $xy = yx$, we have $x1 = 1x$. Let $x = a - -b$ for some natural numbers a, b. $1x = (1 - -0)(a - -b) = 1a - -1b = a - -b = x$. Thus, $x1 = 1x = x$. □

8. $x(y + z) = xy + xz$.

Proof. Let $x = a - -b$, $y = c - -d$, and $z = e - -f$ for some natural numbers a, b, c, d, e, f. Then

$$\begin{aligned}
x(y+z) &= (a - -b)((c - -d) + (e - -f)) \\
&= (a - -b)((c+e) - -(d+f)) \\
&= (a(c+e) + b(d+f)) - -(a(e+f) + b(c+d)) \\
&= (ac + ae + bd + bf) - -(ae + af + bc + bd); \\
xy + xz &= (a - -b)(c - -d) + (a - -b)(e - -f) \\
&= ((ac+bd) - -(ad+bc)) + ((ae+bf) - -(af+be)) \\
&= ((ac+bd) + (ae+bf)) - -((ad+bc) + (af+be)) \\
&= (ac + ae + bd + bf) - -(ae + af + bc + bd).
\end{aligned}$$

Therefore, $x(y+z) = xy + xz$. □

9. $(y+z)x = yx + zx$.

   Proof. Since $xy = yx$, we have $(y+z)x = x(y+z)$. By using identities, we get $xy + xz = yx + zx$. And because $x(y+z) = xy + xz$, $(y+z)x = yx + zx$. □

Exercise 4.1.5

Prove Proposition 4.1.8.

Proof. From now on we could just use $-$ instead of $--$. Let $a = c - d$ and $b = e - f$ for some natural numbers c, d, e, f. So $ab = 0 \implies (c-d)(e-f) = 0$. Assume $c - d \geq 0$ and $e - f \geq 0$, by Lemma 2.3.3, at least one of $a = (c-d)$ and $b = e - f$ is equal to 0. If at least one of $(c-d)$ and $(e-f)$ is negative, without loss of generality, assume $c - d < 0$. Then $-(c-d) = d - c > 0$ and we have $(d-c)(e-f) = -1 \times 0 = 0$. By Lemma 2.3.3, at least one of $-a = d - c$ and $b = e - f$ is equal to zero, and this statement is equivalent to either $a = 0$ or $b = 0$ (or both). □

Exercise 4.1.6

Prove Corollary 4.1.9.

Proof. $ac = bc \implies ac - bc = ac + (-b)c = (a + (-b))c = (a - b)c = 0$ by Proposition 4.1.6. By Proposition 4.1.8, at least one of $(a - b)$ and c is equal to 0. Since $c \neq 0$, $a - b = 0$. Thus, $a = b$. $\square$

Exercise 4.1.7

Prove Lemma 4.1.11.

(a) *Proof.* We need to show that $a > b \iff a - b$ is a positive natural number. Suppose $a > b$. By definition, there exists a positive natural number n such that $a = b + n$. So $a - b = n > 0$ as required. Suppose $a - b$ is a positive natural number. Then $a - b = n \iff a = b + n$ for some positive integer n. Thus, $a > b$. $\square$

(b) *Proof.* Since $a > b$, there exists a positive natural number n such that $a - b = n$. Then $(a + c) - b = c + n \iff (a + c) = (b + c) + n$. Therefore, $a + c > b + c$. $\square$

(c) *Proof.* Since $a > b$, there exists a positive natural number n such that $a - b = n$. Since $(a - b)$ and n are both natural numbers, we have $c(a - b) = cn$ for any positive integer c. Then $ac = bc + cn$ where cn is a positive natural number. Therefore, $ac > bc$. $\square$

(d) *Proof.* Since $a > b$, there exists a positive natural number n such that $a - b = n$. Then $-b = -a + n$. Since $n > 0$, $-b > -a$. $\square$

(e) *Proof.* Since $a > b$, there exists a positive natural number n such that $a - b = n$. Since $b > c$, there exists a positive natural number m such that $b - c = m$. Then $(a - b) + (b - c) = a - c = n + m$ where $(n + m)$ is a positive natural number. Therefore, $a > c$. $\square$

(f) *Proof.* Since $a - b$ is an integer, by Lemma 4.1.5, exactly one of the following three statement is true:

(a) $a - b$ is zero. Then $a = b$.

(b) $a - b$ is equal to a positive natural number n. $a - b = n$, so $a > b$.

(c) $-(a - b) = b - a$ is equal to a positive natural number n. $b - a = n$, so $b > a$ which is equivalent to $a < b$.

□

Exercise 4.1.8

Show that the principle of induction does not apply directly to the integers. More precisely, give an example of a property $P(n)$ pertaining to an integer n such that $P(0)$ is true, and that $P(n)$ implies $P(n++)$ for all integers n, but that $P(n)$ is not true for all integers n. Thus induction is not as useful a tool for dealing with the integers as it is with the natural numbers.

Proof. A counterexample of $P(n)$ could be $f(n) = n^2$ is a monotonically increasing function. □

4.2 The rationals

Definition 4.2.1

A rational number is an expression of the form $a//b$, where a and b are integers and b is non-zero; $a//0$ is not considered to be a rational number. Two rational numbers are considered to be equal, $a//b = c//d$, if and only if $ad = cb$. The set of all rational numbers is denoted **Q**.

Definition 4.2.2

If $a//b$ and $c//d$ are rational numbers, we define their sum

$$(a//b) + (c//d) := (ad + bc)//(bd)$$

their product

$$(a//b) * (c//d) := (ac)//(bd)$$

and the negation

$$-(a//b) := (-a)//b.$$

Lemma 4.2.3

The sum, product, and negation operations on rational numbers are well-defined, in the sense that if one repalce $a//b$ with another rational number $a'//b'$ which is equal to $a//b$, then the output of the above operations remains unchanged, and similarly for $c//d$.

Proposition 4.2.4 (Laws of algebra for rationals).

Let x, y, z be rationals. Then the following laws of algebra hold:

$$x + y = y + x$$
$$(x + y) + z = x + (y + z)$$
$$x + 0 = 0 + x = x$$
$$x + (-x) = (-x) + x = 0$$
$$xy = yx$$
$$(xy)z = x(yz)$$
$$x1 = 1x = x$$
$$x(y + z) = xy + xz$$
$$(y + z)x = yx + zx.$$

If x is non-zero, we also have

$$xx^{-1} = x^{-1}x = 1.$$

Definition 4.2.6

A rational number x is said to be positive iff we have $x = a/b$ for some positive integers a and b. It is said to be negative iff we have $x = -y$ for some positive rational y (i.e., $x = (-a)/b$ for some positive integers a and b).

Lemma 4.2.7 (Trichotomy of rationals).

Let x be a rational number. Then exactly one of the following three statements is true: (a) x is equal to 0. (b) x is a positive rational number. (c) x is a negative rational number.

Definition 4.2.8 (Ordering of the rationals).

Let x and y be rational numbers. We say that $x > y$ iff $x - y$ is a positive rational number, and $x < y$ iff $x - g$ is a negative rational number. We write $x \geq y$ iff either $x > y$ or $x = y$, and similarly define $x \leq y$.

Proposition 4.2.9 (Basic properties of order on the rationals).

Let x, y, z be rational numbers. Then the following properties hold.

(a) (Order trichotomy) Exactly one of the three statements $x = y$, $x < y$, or $x > y$ is true.

(b) (Order is anti-symmetric) One has $x < y$ if and only if $y > x$.

(c) (Order is transitive) If $x < y$ and $y < z$, then $x < z$.

(d) (Addition preserves order) If $x < y$, then $x + z < y + z$.

(e) (Positive multiplication preserves order) If $x < y$ and z is positive, then $xz < yz$.

Exercise 4.2.1

Show that the definition of equality for the rational numbers is reflexive, symmetric, and transitive.

Proof. Reflexivity: suppose a, b are some natural numbers. Since $ab = ab$, by definition, we have $a//b = a//b$. Symmetry: suppose we have $a//b = c//d$ for some natural numbers a, b, c, d. Since $ad = cb$ implies $cb = ad$, by definition, we have $c//d = a//b$. Transitivity: suppose we have $a//b = c//d$ and $c//d = e//f$ for some natural numbers a, b, c, d, e, f. Then we have $ad = cb$ and $cf = ed$. So $adf = cbf$, then $adf = (af)d = b(cf) = b(ed) = (eb)d$. Since $d \neq 0$, by Corollary 4.1.9, $af = eb$. By definition, $a//b = e//f$. $\square$

Exercise 4.2.2

Prove the remaining components of Lemma 4.2.3.

Proof. Multiplication: suppose $a//b = a'//b'$ where a, b, a', b' are some natural numbers. We want to show that $(a//b)*(c//d) = (a'//b')*(c//d)$. Since $a//b = a'//b'$, we have $ab' = a'b$. Then $(ab')(cd) = (a'b)(cd)$, by identities, we have $(ac)(b'd) = (bd)(a'c)$. By definition of equality for the rationals, we have $(ac)//(bd) = (a'c)//(b'd)$. Thus, $(a//b) * (c//d) = (a'//b') * (c//d)$.

Negation: suppose $a//b = a'//b'$ where a, b, a', b' are some natural numbers. Since $ab' = a'b$, we have $(-a)b' = -ab' = -a'b = (-a')b$. Then by definition, we have $-(a//b) = (-a)//b = (-a')//b' = -(a'//b')$ as required. $\square$

Exercise 4.2.3

Prove the remaining components of Proposition 4.2.4.

1. $x + y = y + x$.

   Proof. Let $x = a//b$ and $y = c//d$ for some integers a, b, c, d and $b, d \neq 0$. Then

$$\begin{aligned} x + y &= a//b + c//d \\ &= (ad + bc)//bd \\ y + x &= c//d + a//b \\ &= (cb + da)//db \\ &= (ad + bc)//bd. \end{aligned}$$

   Therefore, $x + y = y + x$. $\square$

2. $(x + y) + z = x + (y + z)$.

   Proof. The proof is on page 84. $\square$

3. $x + 0 = 0 + x = x$.

Proof. Since $x + y = y + x$ for rational numbers x, y, we have $x + 0 = 0 + x$. Let $x = a//b$ for some integers a, b and $b \neq 0$. Write 0 as $0//1$. Then

$$\begin{aligned} x + 0 &= a//b + 0//1 \\ &= (a + 0)//b \\ &= a//b \\ &= x. \end{aligned}$$

Therefore, $x + 0 = 0 + x = 0$ $\square$

4. $x + (-x) = (-x) + x = 0$.

Proof. Since $x+y = y+x$, $x+(-x) = (-x)+x$. Let $x = a//b$ for some integers a, b and $b \neq 0$. Then $-x = -(a//b)$.

$$\begin{aligned} x + (-x) &= a//b - (a//b) \\ &= a//b + (-a)//b \\ &= (ab + (-a)b)//b^2 \\ &= 0. \end{aligned}$$

Therefore, $x + (-x) = (-x) + x = 0$. $\square$

5. $xy = yx$.

Proof. Let $x = a//b$ and $y = c//d$ for some integers a, b, c, d and $b, d \neq 0$. Then

$$\begin{aligned} xy &= (ac)//(bd) \\ yx &= (ca)//(db) \\ &= (ac)//(bd). \end{aligned}$$

Therefore, $xy = yx$. $\square$

6. $(xy)z = x(yz)$.

Proof. Let $x = a//b$, $y = c//d$, and $z = e//f$ for some integers a, b, c, d, e, f and $b, d, e \neq 0$. Then

$$\begin{aligned}
(xy)z &= ((a//b) * (c//d)) * (e//f) \\
&= (ac//bd) * (e//f) \\
&= ((ac)e)//((bd)f) \\
&= (ace)//(bdf) \\
x(yz) &= (a//b) * ((c//d) * (e//f)) \\
&= (a//b) * ((ce)//(df)) \\
&= (a(ce))//(b(df)) \\
&= (ace)//bdf.
\end{aligned}$$

Therefore, $(xy)z = x(yz)$. □

7. $x1 = 1x = x$.

Proof. Since $xy = yx$, we have $x1 = 1x$. Let $x = a//b$ for some integers a, b and $b \neq 0$. Then

$$\begin{aligned}
x1 &= (a//b) * (1//1) \\
&= (a1)//(b1) \\
&= a//b \\
&= x.
\end{aligned}$$

Thus, $x1 = 1x = x$. □

8. $x(y + z) = xy + xz$.

Proof. Let $x = a//b$, $y = c//d$, and $z = e//f$ for some integers a, b, c, d, e, f

and $b, d, e \neq 0$. Then

$$\begin{aligned}
x(y+z) &= (a//b) * ((cf+de)//(df)) \\
&= (a(ef+de))//(bdf) \\
&= (acf+ade)//(bdf) \\
xy+xz &= (a//b) * (c//d) + (a//b) * (e//f) \\
&= (ac)//(bd) + (ae)//(bf) \\
&= (acbf+bdae)//(b^2df) \\
&= (acf+ade)//(bdf).
\end{aligned}$$

Thus, $x(y+z) = xy + xz$. □

9. $(y+z)x = yx + zx$.

   Proof. Since $xy = yx$, we have $x(y+z) = (y+z)x$. By using identities, we have $xy + xz = yx + zx$. Therefore, since $x(y+z) = xy + xz$, we have $(y+z)x = yx + zx$. □

10. If x is non-zero, then $xx^{-1} = x^{-1}x = 1$.

    Proof. Let $x = a//b$ where a, b are non-zero integers. Then $x^{-1} = b//a$. Since $xy = yx$, we have $xx^{-1} = x^{-1}x$. Then $xx^{-1} = (ab)//(ba)$. Since $(ab)1 = 1(ba)$, we have $xx^{-1} = (ab)//(ba) = 1//1 = 1$. Thus, $xx^{-1} = x^{-1}x = 1$. □

Exercise 4.2.4

Prove Lemma 4.2.7.

Proof. Let $x = a//b$ where a, b are integers and $b \neq 0$. Consider all the possible combinations of a and b:

- $a = 0$. $x = 0//b = 0$.
- $a > 0$, $b > 0$. By definition, $x = a//b$ is positive.

- $a > 0$, $b < 0$. Then $-b > 0$. So $x = -(a//(-b))$ where $a//(-b)$ is a positive rational number. Therefore, x is negative.
- $a < 0$, $b > 0$. Similarly, we can show that $x = a//b$ is negative.
- $a < 0$, $b < 0$. Then $-a > 0$ and $-b > 0$. Since $x = a//b = (-a)//(-b)$, by definition, x is positive.

So we have proved that at least one of the statements is true. Then we need to check that at most one of them is true. Assume $a = 0$. Then by Trichotomy of integers, a cannot be positive nor negative. So by definition, $x = a//b$ cannot be positive nor negative. Thus, if a rational number is 0, it cannot be positive nor negative. Then we need to show that a rational number cannot be positive and negative at the same time. Assume $x = a//b$ is positive, then by definition, $a > 0$, $b > 0$. Assume x is also negative, so there exists a positive $-y = -c//d = x$. Then $ad = -bc$ which leads to an integer being negative and positive at the same time (contradiction). Thus, a rational number cannot be positive and negative at the same time. Therefore, at most of the three statements is true. Hence, exactly one of the three statements is true. $\square$

Exercise 4.2.5

Prove Proposition 4.2.9.

(a) *Proof.* $x - y$ is a rational number, by Lemma 4.2.7, exactly one of $x - y = 0$, $x - y > 0$, or $x - y < 0$ is true. Thus, exactly one of the three statements $x = y$, $x < y$, or $x > y$ is true. $\square$

(b) *Proof.* Assume $x < y$. So $x - y = r$ is a negative rational number. Then $-r$ is positive and $y - x = -r$. Therefore, $y > x$. Assume $y > x$. So $y - x = r$ is a positive rational number. Then $-r$ is negative and $x - y = -r$. Therefore, $x < y$. $\square$

(c) *Proof.* $x < y \implies y - x = r$ where r is a positive rational number. $y < z \implies z - y = s$ where s is a positive rational number. Then $z - x = z - (y + r) = s + r$ which is also a positive rational number. Thus, $x < z$. $\square$

(d) *Proof.* $x < y \implies y - x = r$ where r is a positive rational number. Then $(y + z) - (x + z) = r > 0$. Therefore, $x + z < y + z$. □

(e) *Proof.* $x < y \implies y - x = r$ where r is a positive rational number. We have $(y - x)z = yz - xz = zr > 0$. Therefore, $xz < yz$. □

Exercise 4.2.6

Show that if x, y, z are rational numbers such that $x < y$ and z is negative, then $xz > yz$.

Proof. $x < y \implies y-x = r$ where r is a positive rational number. Since z is negative, $-z$ is positive. Then $(y - x)(-z) = -(y - x)z = (x - y)z = xz - yz = (-z)r > 0$. Therefore, $xz > yz$. □