3.3 Functions

Definition 3.3.1 (Functions).

Let X, Y be sets, and let $P(x, y)$ be a property pertaining to an object $x \in X$ and an object $y \in Y$, such that for every $x \in X$, there is exactly one $y \in Y$ for which $P(x, y)$ is true (this is sometimes known as the vertical line test). Then we define the function $f : X \to Y$ defined by P on the domain X and range Y to be the object which, given any input $x \in X$, assigns an output $f(x) \in Y$, defined to be the unique object $f(x)$ for which $P(x, f(x))$ is true. Thus, for any $x \in X$ and $y \in Y$,

$$y = f(x) \iff P(x, y) \text{ is true.}$$

Definition 3.3.7 (Equality of functions).

Two functions $f : X \to Y$, $g : X \to Y$ with the same domain and range are said to be equal, $f = g$, if and only if $f(x) = g(x)$ for all $x \in X$. If $f(x)$ and $g(x)$ agree for some values of x, but not others, then we do not consider f and g to be equal. If two functions f, g have different domains, or different ranges, we also do not consider them to be equal.

Definition 3.3.11 (Composition).

Let $f : X \to Y$ and $g : Y \to Z$ be two functions, such that the range of f is the same set as the domain of g. We then define the composition $g \circ f : X \to Z$ of the two functions g and f to be the function defined explicitly by the formula

$$(g \circ f)(x) := g(f(x)).$$

If the range of f does not match the domain of g, we leave the composition $g \circ f$ undefined.

Lemma 3.3.13 (Composition is associative).

Let $f : Z \to W$, $g : Y \to Z$, and $h : X \to Y$ be functions. Then $f \circ (g \circ h) = (f \circ g) \circ h$.

Definition 3.3.15 (One-to-one functions).

A function f is one-to-one (or injective) if different elements map to different elements:

$$x \neq x' \implies f(x) \neq f(x').$$

Equivalently, a function is one-to-one if

$$f(x) = f(x') \implies x = x'.$$

Definition 3.3.18 (Onto functions).

A function f is onto (or surjective) if every element if Y comes from applying f to some element in X:

$$\text{For every } y \in Y, \text{ there exists } x \in X \text{ such that } f(x) = y.$$

Definition 3.3.21 (Bijective functions).

Functions $f : X \to Y$ which are both one-to-one and onto are also called bijective or invertible.

Exercises

Exercise 3.3.1

Show that the definition of equality in Definition 3.3.7 is reflexive, symmetric, and transitive. Also verify the substitution property: if $f, \tilde{f} : X \to Y$ and $g, \tilde{g} : Y \to Z$ are functions such that $f = \tilde{f}$ and $g = \tilde{g}$, then $g \circ f = \tilde{g} \circ \tilde{f}$.

Proof. Reflexivity: f and f have the same domain and range, and $f(x) = f(x)$ for all x in the domain of f. Therefore, f is equal to itself.

Symmetry: g and f have the same domain and range. For every x in the domain of g, we have $g(x) = f(x)$. Therefore, by Definition 3.3.7, $g(x)$ and $f(x)$ are equal.

Transitivity: Suppose f and g have the same domain and range, and for every x in the domain of f, $f(x) = g(x)$. And g and h have the same domain and range, and for every x in the domain of g, we have $g(x) = h(x)$. Then f and h have the same domain and range. $\forall x$ in the domain of f, we have $f(x) = g(x) = h(x)$. Therefore, f and h are equal.

Substitution property: Since $g \circ f, \tilde{g} \circ \tilde{f} : X \to Z$, they have the same domain and range. And for every $x \in X$, we have $f(x) = \tilde{f}(x)$, since $g = \tilde{g}$, we also have $g(f(x)) = \tilde{g}(f(x)) = \tilde{g}(\tilde{f}(x))$. Therefore, $g \circ f = \tilde{g} \circ \tilde{f}$. □

Exercise 3.3.2

Let $f : X \to Y$ and $g : Y \to Z$ be functions. Show that if f and g are both injective, then so is $g \circ f$; similarly, show that if f and g are both surjective, then so is $g \circ f$.

1. If if f and g are both injective, then so is $g \circ f$.

   Proof. f is injective:

$$x \in X, x' \in X, x \neq x' \implies f(x) \neq f(x').$$

   g is injective:

$$f(x) \in Y, f(x') \in Y, f(x) \neq f(x') \implies g(f(x)) \neq g(f(x')).$$

   Therefore, $x \neq x' \implies (g \circ f)(x) \neq (g \circ f)(x')$. Thus, $g \circ f$ is injective. □

2. If f and g are both surjective, then so is $g \circ f$.

   Proof. f is surjective:

$$\text{For every } y \in Y, \text{ there exists } x \in X \text{ such that } f(x) = y.$$

   g is surjective:

$$\text{For every } z \in Z, \text{ there exists } y \in Y \text{ such that } g(y) = z.$$

   Therefore, for every $z \in Z$, there exists $x \in X$ such that $(g \circ f)(x) = g(f(x)) = g(y) = z$. Thus, $g \circ f$ is surjective. □

Exercise 3.3.3

When is the empty function injective? surjective? bijective?

The empty function is of the form $f : \emptyset \to X$. It is always injective no matter what X is. It is surjective if X is $\emptyset$. It is bijective if X is $\emptyset$.

Exercise 3.3.4

In this section we give some cancellation laws for composition. Let $f : X \to Y$, $\tilde{f} : X \to Y$, $g : Y \to Z$, and $\tilde{g} : Y \to Z$ be functions. Show that if $g \circ f = g \circ \tilde{f}$ and g is injective, then $f = \tilde{f}$. Is the same statement true if g is not injective? Show that if $g \circ f = \tilde{g} \circ f$ and f is surjective, then $g = \tilde{g}$. Is the same statement true if f is not surjective?

1. *Proof.* Suppose x is an arbitrary object in X, $y = f(x) \in Y$, $y' = \tilde{f}(x) \in Y$. Since $g \circ f = g \circ \tilde{f}$, $g(f(x)) = g(\tilde{f}(x))$. Because g is injective, $g(f(x)) = g(\tilde{f}(x)) \implies f(x) = \tilde{f}(x)$. And $f, \tilde{f}$ have the same domain and range. Thus, $f = \tilde{f}$.

   This won't be true if g is not injective. Counterexample: $g(1) = 3, g(2) = 3, f(1) = 1, \tilde{f}(1) = 2$. In this case, $g(f(1)) = g(\tilde{f}(1)) = 3$, but $f \neq \tilde{f}$. □

2. *Proof.* Since f is surjective, for every $y \in Y$, there exists $x \in X$ such that $y = f(x)$. Also for every $x \in X$, we have $g(f(x)) = \tilde{g}(f(x))$ as $g \circ f = g \circ \tilde{f}$. Therefore, for every $y \in Y$, there exists $x \in X$ such that $g(y) = g(f(x)) = \tilde{g}(f(x)) = \tilde{g}(y)$. As $g, \tilde{g}$ have the same domain and range, $g = \tilde{g}$.

   This won't be true if f is not surjective. Counterexample: $f : \{0, 1\} \to \{1, 2, 3\}$, $g : \{1, 2, 3\} \to \{4, 5, 6\}$, $\tilde{g} : \{1, 2, 3\} \to \{4, 5, 7\}$. □

Exercise 3.3.5

Let $f : X \to Y$ and $g : Y \to Z$ be functions. Show that if $g \circ f$ is injective, then f must be injective. Is it true that g must also be injective? Show that if $g \circ f$ is surjective, then g must be surjective. Is it true that f must also be surjective?

1. *Proof.* $g \circ f$ is injective $\implies (x \neq x' \implies g(f(x)) \neq g(f(x')))$. Suppose f is not injective, that is, $\exists x, x' \in X$ such that $x \neq x'$ and $f(x) = f(x')$. Then by definition, $g(f(x)) = g(f(x'))$ so $g \circ f$ is not injective. (contradiction) Therefore, f must be injective. However, g does not have to be injective. Counterexample: $f : \{0, 1\} \to \{1, 2, 3\}$, $g : \{1, 2, 3\} \to \{4, 5, 5\}$. ($f$ does not have to be surjective.) $\square$

2. *Proof.* $g \circ f$ is surjective $\implies \forall z \in Z, \exists x \in X$ such that $g(f(x)) = z$. Assume g is not surjective. Then $\exists z \in Z$ such that $\forall y \in Y, g(y) \neq z$. So there does not exist $x \in X$ such that $g(y) = g(f(x)) = z$. It implies $g \circ f$ is not surjective. (contradiction) Thus, g must be surjective. f does not have to be surjective. Counterexample: $f : \{1\} \to \{2, 3\}, f(1) := 2, g : \{2, 3\} \to \{4\}, g(2) := 4, g(3) := 4$. ($g$ does not have to be injective.) $\square$

Exercise 3.3.6

Let $f : X \to Y$ be a bijective function, and let $f^{-1} : Y \to X$ be its inverse. Verify the cancellation laws $f^{-1}(f(x)) = x$ for all $x \in X$ and $f(f^{-1}(y)) = y$ for all $y \in Y$. Conclude that f^{-1} is also invertible, and has f as its inverse (thus $(f^{-1})^{-1} = f$).

Proof. Since f is surjective, for every $y \in Y$, there exists $x \in X$ such that $f(x) = y$. Suppose $\exists x, x' \in X$ such that $f^{-1}(f(x)) = x' \neq x$. Then there exists $y \in Y$, such that $f(x) = y$ and $f^{-1}(y) = x'(f(x') = y)$. So f is not injective. (contradiction) Thus, $f^{-1}(f(x)) = x$ for all $x \in X$.

Since f is surjective, for every $y \in Y$, there exists $x \in X$ such that $f^{-1}(y) = x$. Suppose $f(x) = y$ and $\exists y' \in Y, y' \neq y, f(f^{-1}(y)) = y'$. Then we have $f(f^{-1}(y)) = f(x) = y' \neq y$ (contradiction). Thus, $f(f^{-1}(y)) = y$.

Since $f^{-1}(f(x)) = x$ for all $x \in X$, f^{-1} is surjective. Suppose there exists $y, y' \in Y$, $x \in X$, $y \neq y'$, $f^{-1}(y) = f^{-1}(y') = x$. By cancellation law, we have $f(x) = y$ and $f(x) = y'$ (contradiction). Therefore, f^{-1} is injective. Thus, f^{-1} is bijective.

$(f^{-1})^{-1}$ has X as its domain and Y as its range, the inverse of f^{-1}. Then we need to show that for every $x \in X$, we have $f(x) = (f^{-1})^{-1}(x)$. Let x be an arbitrary object in X, $y = f(x)$. By definition of inverse, $f^{-1}(y) = x$. Again by definition of inverse, $(f^{-1})^{-1}(x) = y$. Therefore, $f(x) = (f^{-1})^{-1}(x)$. Thus, $(f^{-1})^{-1} = f$. $\square$

Exercise 3.3.7

Let $f : X \to Y$ and $g : Y \to Z$ be functions. Show that if f and g are bijective, then so is $g \circ f$, and we have $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Proof. We have shown in 3.3.2, if f and g are both injective/surjective, $f \circ g$ is also injective/surjective. By symmetry, $g \circ f$ is also injective/surjective. As bijective $\iff$ injective and surjective, f and g are bijective $\iff$ $g \circ f$ is bijective.

$g \circ f : X \to Z$, so $(g \circ f)^{-1} : Z \to X$. Since $f : X \to Y$ and $g : Y \to Z$, $g^{-1} : Z \to Y$, $f : Y \to X$, we have $f^{-1} \circ g^{-1} : Z \to X$. Therefore, $(g \circ f)^{-1}$ and $f^{-1} \circ g^{-1}$ have the same domain and range. Then consider an arbitrary obejct $z \in Z$. Since g and f are both bijective, there exist exactly one x and one y such that $g(y) = z$ and $f(x) = y$. So $(g \circ f)(x) = z$. And by definition of inverse, we have $(g \circ f)^{-1}(z) = x$. Again by definition, we have $g^{-1}(z) = y$, $f^{-1}(y) = x$. So $(f^{-1} \circ g^{-1})(z) = x$. Therefore, for every $z \in Z$, $(g \circ f)^{-1}(z) = (f^{-1} \circ g^{-1})(z)$. Thus, $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. □

Exercise 3.3.8

If X is a subset of Y, let $\iota_{X \to Y}$ be the inclusion map from X to Y, defined by mapping $x \mapsto x$ for all $x \in X$, i.e., $\iota_{X \to Y}(x) := x$ for all $x \in X$. The map $\iota_{X \to X}$ is in particular called the identity map on X.

1. Show that if $X \subseteq Y \subseteq Z$ then $\iota_{Y \to Z} \circ \iota_{X \to Y} = \iota_{X \to Z}$.

   Proof. Both $\iota_{Y \to Z} \circ \iota_{X \to Y}$ and $\iota_{X \to Z}$ have X as domain and Z as range. Consider an arbitrary object $x \in X$. $\iota_{X \to Z}(x) = x$. $\iota_{X \to Y}(x) = x$, $(\iota_{Y \to Z} \circ \iota_{X \to Y})(x) = \iota_{Y \to Z}(\iota_{X \to Y}(x)) = \iota_{Y \to Z}(x) = x$. Therefore, for every $x \in X$, $(\iota_{Y \to Z} \circ \iota_{X \to Y})(x) = \iota_{X \to Z}(x)$. Thus, $\iota_{Y \to Z} \circ \iota_{X \to Y} = \iota_{X \to Z}$. □

2. Show that if $f : A \to B$ is any function, then $f = f \circ \iota_{A \to A} = \iota_{B \to B} \circ f$.

   Proof. Obviously, f, $f \circ \iota_{A \to A}$, and $\iota_{B \to B} \circ f$ all have the same domain and range. Consider an arbitrary $x \in A$. $(f \circ \iota_{A \to A})(x) = f(\iota_{A \to A}(x)) = f(x)$.

$(\iota_{B\to B} \circ f)(x) = \iota_{B\to B}(f(x)) = f(x)$. Therefore, for every $x \in A$, $f(x) = (f \circ \iota_{A\to A})(x) = (\iota_{B\to B} \circ f)(x)$. Thus, $f = f \circ \iota_{A\to A} = \iota_{B\to B} \circ f$. □

3. Show that, if $f : A \to B$ is a bijective function, then $f \circ f^{-1} = \iota_{B\to B}$ and $f^{-1} \circ f = \iota_{A\to A}$.

   (a) $f \circ f^{-1} = \iota_{B\to B}$.

   Proof. $f^{-1} : B \to A$, $f \circ f^{-1} : B \to B$. So $f \circ f^{-1}$ and $\iota_{B\to B}$ have the same domain and range. Consider an arbitrary $y \in B$. $\iota_{B\to B}(y) = y$. Since f is bijective, there exists exactly one $x \in A$ such that $f(x) = y$. Then $(f \circ f^{-1})(y) = f(f^{-1}(y)) = f(x) = y = \iota_{B\to B}(y)$. Thus, $f \circ f^{-1} = \iota_{B\to B}$. □

   (b) $f^{-1} \circ f = \iota_{A\to A}$.

   Proof. $f : A \to B$, $f^{-1} \circ f : B \to A$. So $f^{-1} \circ f$ and $\iota_{A\to A}$ have the same domain and range. Consider an arbitrary $x \in A$. $\iota_{A\to A}(x) = x$. Let $f(x) = y$. By definition of inverse, $f^{-1}(y) = x$. So $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x = \iota_{A\to A}(x)$. Thus, $f^{-1} \circ f = \iota_{A\to A}$. □

4. Show that if X and Y are disjoint sets, and $f : X \to Z$ and $g : Y \to Z$ are functions, then there is a unique function $h : X \cup Y \to Z$ such that $h \circ \iota_{X\to X\cup Y} = f$ and $h \circ \iota_{Y\to X\cup Y} = g$.

   Proof. The existence of h: $\forall x \in X \cup Y$, if $x \in X$, $h(x) := f(x)$, if $x \in Y$, $h(x) := g(x)$. Then we can know that $h \circ \iota_{X\to X\cup Y}$ and f both have domain X and range Z. For an arbitrary $x \in X$, $(h \circ \iota_{X\to X\cup Y})(x) = h(\iota_{X\to X\cup Y}(x)) = f(x)$. Thus, $h \circ \iota_{X\to X\cup Y} = f$. Similarly, we can show that $h \circ \iota_{Y\to X\cup Y} = g$. To check the uniqueness of h, we need to show that if there exists another function h' with the same domain, range and properties as h, then $h' = h$. Consider an arbitrary $x \in X$. $(h' \circ \iota_{X\to X\cup Y})(x) = h'(\iota_{X\to X\cup Y}(x)) = h'(x)$. Since $h' \circ \iota_{X\to X\cup Y} = f$, we must have $(h' \circ \iota_{X\to X\cup Y})(x) = f(x) = (h \circ \iota_{X\to X\cup Y})(x)$. Similarly, we can show that for every $y \in Y$, we have $(h' \circ \iota_{Y\to X\cup Y})(y) = g(y) = (h \circ \iota_{Y\to X\cup Y})(y)$. Since

h' and h have the same domain and range, we can conclude that $h = h'$. Thus, h is unique. $\square$