

Chapter 5

The real numbers

5.1 Cauchy sequences

Definition 5.1.1 (Sequences).

Let m be an integer. A sequence $(a_n)_{n=m}^{\infty}$ of rational numbers is any function from the set $\{n \in \mathbf{Z} : n \geq m\}$ to \mathbf{Q} , i.e., a mapping which assigns to each integer n greater than or equal to m , a rational number a_n . More informally, a sequence $(a_n)_{n=m}^{\infty}$ of rational numbers is a collection of rationals $a_m, a_{m+1}, a_{m+2}, \dots$

Definition 5.1.3 (ε -steadiness).

Let $\varepsilon > 0$. A sequence $(a_n)_{n=0}^{\infty}$ is said to be ε -steady iff each pair a_j, a_k of sequence elements is ε -close for every natural number j, k . In other words, the sequence a_0, a_1, a_2, \dots is ε -steady iff $|a_j - a_k| \leq \varepsilon$ for all j, k .

Definition 5.1.6 (Eventual ε -steadiness).

Let $\varepsilon > 0$. A sequence $(a_n)_{n=0}^{\infty}$ is said to be eventually ε -steady iff the sequence $a_N, a_{N+1}, a_{N+2}, \dots$ is ε -steady for some natural number $N \geq 0$. In other words, the sequence a_0, a_1, a_2, \dots is eventually ε -steady iff there exists an $N \geq 0$ such that $|a_j - a_k| \leq \varepsilon$ for all $j, k \geq N$.

Definition 5.1.8 (Cauchy sequences).

A sequence $(a_n)_{n=0}^{\infty}$ of rational numbers is said to be a Cauchy sequence iff for every rational $\varepsilon > 0$, the sequence $(a_n)_{n=0}^{\infty}$ is eventually ε -steady. In other words, the sequence a_0, a_1, a_2, \dots is a Cauchy sequence iff for every $\varepsilon > 0$, there exists an $N \geq 0$ such that $d(a_j, a_k)$ for all $j, k \geq N$.

Proposition 5.1.11

The sequence a_1, a_2, a_3, \dots defined by $a_n := 1/n$ (i.e., the sequence $1, 1/2, 1/3, \dots$) is a Cauchy sequence.

Definition 5.1.12 (Bounded sequences).

Let $M \geq 0$ be rational. A finite sequence a_1, a_2, \dots, a_n is bounded by M iff $|a_i| \leq M$ for all $1 \leq i \leq n$. An infinite sequence $(a_n)_{n=1}^\infty$ is bounded by M iff $|a_i| \leq M$ for all $i \geq 1$. A sequence is said to be bounded iff it is bounded by M for some rational $M \geq 0$.

Lemma 5.1.14 (Finite sequences are bounded).

Every finite sequence a_1, a_2, \dots, a_n is bounded.

Lemma 5.1.15 (Cauchy sequences are bounded).

Every Cauchy sequence $(a_n)_{n=1}^\infty$ is bounded.

Exercise 5.1.1

Prove Lemma 5.1.15.

Proof. Suppose $(a_n)_{n=1}^\infty$ is a Cauchy sequence. So for $\varepsilon = 1$, there exists an $N \geq 1$ such that $d(a_j, a_k) \leq 1$ for all $j, k \geq N$. Then the sequence can be split into two parts: a_1, \dots, a_N and a_{N+1}, a_{N+2}, \dots . The former is a finite sequence, by Lemma 5.1.14, it is bounded. Suppose this finite sequence is bounded by M_1 . Consider a_{N+1}, a_{N+2}, \dots . Since it is 1-steady, for any $i > N + 1$, we have $|a_i - a_{N+1}| \leq 1$. Rearrange the inequalities, we have $a_{N+1} - 1 \leq a_i \leq a_{N+1} + 1$. Let $M_2 = a_{N+1} + 1$. Then $a_{N+1} < M_2$ and for every $i > N + 1$, we have $a_i \leq M_2$. So sequence a_{N+1}, a_{N+2}, \dots is bounded by M_2 . Let $M = \max\{M_1, M_2\}$. Then the Cauchy sequence $(a_n)_{n=1}^\infty$ is bounded by M . \square

5.2 Equivalent Cauchy sequences

Definition 5.2.1 (ε -close sequences).

Let $(a_n)_{n=0}^\infty$ and $(b_n)_{n=0}^\infty$ be two sequences, and let $\varepsilon > 0$. We say that the sequence $(a_n)_{n=0}^\infty$ is ε -close to $(b_n)_{n=0}^\infty$ iff a_n is ε -close to b_n for each $n \in \mathbf{N}$. In other words, the sequence a_0, a_1, a_2, \dots is ε -close to the sequence b_1, b_1, b_2, \dots iff $|a_n - b_n| \leq \varepsilon$ for all $n = 0, 1, 2, \dots$.

Definition 5.2.3 (Eventually ε -close sequences).

Let $(a_n)_{n=0}^\infty$ and $(b_n)_{n=0}^\infty$ be two sequences, and let $\varepsilon > 0$. We say that the sequence $(a_n)_{n=0}^\infty$ is eventually ε -close to $(b_n)_{n=0}^\infty$ iff there exists an $N \geq 0$ such that the sequences $(a_n)_{n=N}^\infty$ and $(b_n)_{n=N}^\infty$ are ε -close. In other words, a_0, a_1, a_2, \dots is eventually ε -close to b_0, b_1, b_2, \dots iff there exists an $N \geq 0$ such that $|a_n - b_n| \leq \varepsilon$ for all $n \geq N$.

Definition 5.2.6 (Equivalent sequences).

Two sequences $(a_n)_{n=0}^\infty$ and $(b_n)_{n=0}^\infty$ are equivalent iff for each rational $\varepsilon > 0$, the sequences $(a_n)_{n=0}^\infty$ and $(b_n)_{n=0}^\infty$ are eventually ε -close. In other words, a_0, a_1, a_2, \dots and b_0, b_1, b_2, \dots are equivalent iff for every for every rational $\varepsilon > 0$, there exists an $N \geq 0$ such that $|a_n - b_n| \leq \varepsilon$ for all $n \geq N$.

Proposition 5.2.8

Let $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ be the sequences $a_n = 1 + 10^{-n}$ and $b_n = 1 - 10^{-n}$. Then the sequences a_n, b_n are equivalent.

Exercise 5.2.1

Show that if $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ are equivalent sequences of rationals, then $(a_n)_{n=1}^\infty$ is a Cauchy sequence if and only if $(b_n)_{n=1}^\infty$ is a Cauchy sequence.

Proof. Assume $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ are equivalent sequences of rationals, and $(a_n)_{n=1}^\infty$ is a Cauchy sequence. We want to show that for any rational $\varepsilon > 0$, there exists $N \geq 1$ such that b_N, b_{N+1}, \dots is ε -steady.

Since $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence and $\frac{\varepsilon}{3} > 0$, there exists $N_1 \geq 1$ such that for all $i, j \geq N_1$, $|a_i - a_j| \leq \frac{\varepsilon}{3}$. Since $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent, and $\frac{\varepsilon}{3} > 0$, there exists $N_2 \geq 1$ such that for all $i \geq N_2$, $|b_i - a_i| \leq \frac{\varepsilon}{3}$. Let $N = \max\{N_1, N_2\}$. Consider arbitrary $i, j \geq N$. Since $N \geq N_1$, we have

$$|a_i - a_j| \leq \frac{\varepsilon}{3}.$$

Since $N \geq N_2$, we have

$$|b_i - a_i| \leq \frac{\varepsilon}{3}$$

and

$$|a_j - b_j| \leq \frac{\varepsilon}{3}.$$

Since

$$|a_i - a_j| \leq \frac{\varepsilon}{3}$$

and

$$|b_i - a_i| \leq \frac{\varepsilon}{3},$$

we have

$$|b_i - a_j| \leq |a_i - a_j| + |b_i - a_i| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}.$$

Since

$$|a_j - b_j| \leq \frac{\varepsilon}{3}$$

and

$$|b_i - a_j| \leq \frac{2\varepsilon}{3},$$

we have

$$|b_i - b_j| \leq |a_j - b_j| + |b_i - a_j| \leq \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon.$$

Thus, for any $\varepsilon > 0$, we can find $N = \max\{N_1, N_2\}$ such that b_N, b_{N+1}, \dots is ε -steady. Therefore, $(b_n)_{n=1}^{\infty}$ is a Cauchy sequence.

Similarly, we can show that if $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent sequences of rationals and $(b_n)_{n=1}^{\infty}$ is a Cauchy sequence, then $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence. Thus, if $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent sequences of rationals, then $(a_n)_{n=1}^{\infty}$ is a Cauchy

sequence if and only if $(b_n)_{n=1}^\infty$ is a Cauchy sequence. \square

Exercise 5.2.2

Let $\varepsilon > 0$. Show that if $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ are eventually ε -close, then $(a_n)_{n=1}^\infty$ is bounded if and only if $(b_n)_{n=1}^\infty$ is bounded.

Proof. Suppose $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ are eventually ε -close and $(a_n)_{n=1}^\infty$ is bounded. Since $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ are eventually ε -close, for any $\varepsilon > 0$, there exists $N \geq 1$ such that for any $i \geq N$, $|a_i - b_i| \leq \varepsilon$. Consider an arbitrary $\varepsilon > 0$. We can find $N \geq 1$ such that for any $i \geq N$, $|a_i - b_i| \leq \varepsilon$. Then

$$a_i - \varepsilon \leq b_i \leq a_i + \varepsilon.$$

Split $(b_n)_{n=1}^\infty$ to b_1, \dots, b_N and b_{N+1}, b_{N+2}, \dots . The former is a finite sequence, so it is bounded by some rational number M_1 . Since $(a_n)_{n=1}^\infty$ is bounded, there exists M such that $|a_i| \leq M$ for all $i \geq 1$. Then

$$-M \leq a_i \leq M.$$

So

$$-M - \varepsilon \leq b_i \leq M + \varepsilon.$$

Therefore, $|b_i| \leq M + \varepsilon$ for all $i \geq N + 1$. Let $M_0 = \max(M, M_1)$. For any $i \geq 1$, we have $|b_i| \leq M_0$. Thus, $(b_n)_{n=1}^\infty$ is bounded.

Similarly, we can show that if $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ are eventually ε -close, and $(b_n)_{n=1}^\infty$ is bounded, then $(a_n)_{n=1}^\infty$ is bounded.

Thus, if $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ are eventually ε -close, then $(a_n)_{n=1}^\infty$ is bounded if and only if $(b_n)_{n=1}^\infty$ is bounded. \square

5.3 The construction of the real numbers

Definition 5.3.1 (Real numbers).

A real number is defined to be an object of the form $\text{LIM}_{n \rightarrow \infty} a_n$, where $\text{LIM}_{n \rightarrow \infty} a_n$ is a Cauchy sequence of rational numbers. Two real numbers $\text{LIM}_{n \rightarrow \infty} a_n$ and $\text{LIM}_{n \rightarrow \infty} b_n$ are said to be equal iff $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent Cauchy sequences. The set of all real numbers is denoted \mathbf{R} .

Proposition 5.3.3 (Formal limits are well-defined).

Let $x = \text{LIM}_{n \rightarrow \infty} a_n$, $y = \text{LIM}_{n \rightarrow \infty} b_n$, and $z = \text{LIM}_{n \rightarrow \infty} c_n$ be real numbers. Then, with the above definition of equality for real numbers, we have $x = x$. Also, if $x = y$, then $y = x$. Finally, if $x = y$ and $y = z$, then $x = z$.

Definition 5.3.4 (Addition of reals).

Let $x = \text{LIM}_{n \rightarrow \infty} a_n$ and $y = \text{LIM}_{n \rightarrow \infty} b_n$ be real numbers. Then we define the sum $x + y$ to be $x + y := \text{LIM}_{n \rightarrow \infty} (a_n + b_n)$.

Lemma 5.3.6 (Sum of Cauchy sequences is Cauchy).

Let $x = \text{LIM}_{n \rightarrow \infty} a_n$ and $y = \text{LIM}_{n \rightarrow \infty} b_n$ be real numbers. Then $x + y$ is also a real number (i.e., $(a_n + b_n)_{n=1}^{\infty}$ is a Cauchy sequence of rationals).

Lemma 5.3.7 (Sums of equivalent Cauchy sequences are equivalent).

Let $x = \text{LIM}_{n \rightarrow \infty} a_n$, $y = \text{LIM}_{n \rightarrow \infty} b_n$, and $x' = \text{LIM}_{n \rightarrow \infty} a'_n$ be real numbers. Suppose that $x = x'$. Then we have $x + y = x' + y$.

Lemma 5.3.9 (Multiplication of reals).

Let $x = \text{LIM}_{n \rightarrow \infty} a_n$ and $y = \text{LIM}_{n \rightarrow \infty} b_n$ be real numbers. Then we define the product xy to be $xy := \text{LIM}_{n \rightarrow \infty} a_n b_n$.

Proposition 5.3.10 (Multiplication is well defined).

Let $x = \text{LIM}_{n \rightarrow \infty} a_n$, $y = \text{LIM}_{n \rightarrow \infty} b_n$, and $x' = \text{LIM}_{n \rightarrow \infty} a'_n$ be real numbers. Then xy is also a real number. Furthermore, if $x = x'$, then $xy = x'y$.

Proposition 5.3.11

All the laws of algebra from Proposition 4.1.6 hold not only for the integers, but for the reals as well.

Definition 5.3.12 (Sequences bounded away from zero).

A sequence $(a_n)_{n=1}^{\infty}$ of rational numbers is said to be bounded away from zero iff there exists a rational number $c > 0$ such that $|a_n| > c$ for all $n \geq 1$.

Lemma 5.3.14.

Let x be a non-zero real number. Then $x = \text{LIM}_{n \rightarrow \infty} a_n$ for some Cauchy sequence $(a_n)_{n=1}^{\infty}$ which is bounded away from zero.

Lemma 5.3.15.

Suppose that $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence which is bounded away from zero. Then the sequence $(a_n^{-1})_{n=1}^{\infty}$ is also a Cauchy sequence.

Definition 5.3.16 (Reciprocals of real numbers).

Let x be a non-zero real number. Let $(a_n)_{n=1}^{\infty}$ be a Cauchy sequence bounded away from zero such that $x = \text{LIM}_{n \rightarrow \infty} a_n$. Then we define the reciprocal x^{-1} by the formula $x^{-1} := \text{LIM}_{n \rightarrow \infty} a_n^{-1}$.

Lemma 5.3.17 (Reciprocation is well defined).

Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be two Cauchy sequences bounded away from zero such that $\text{LIM}_{n \rightarrow \infty} a_n = \text{LIM}_{n \rightarrow \infty} b_n$ (i.e., the two sequences are equivalent). Then $\text{LIM}_{n \rightarrow \infty} a_n^{-1} = \text{LIM}_{n \rightarrow \infty} b_n^{-1}$.

Exercise 5.3.1

Prove Proposition 5.3.3.

Proof. Reflexivity. Since $x = \text{LIM}_{n \rightarrow \infty} a_n$, $(a_n)_{n=1}^\infty$ is a Cauchy sequence. Obviously, $(a_n)_{n=1}^\infty$ and $(a_n)_{n=1}^\infty$ are equivalent. Therefore, $\text{LIM}_{n \rightarrow \infty} a_n = \text{LIM}_{n \rightarrow \infty} a_n$ ($x = x$).

Symmetry. Assume $x = y$, then Cauchy sequences $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ are equivalent. So for every $\varepsilon > 0$, there exists $N \geq 1$ such that for every $i \geq N$, $|a_i - b_i| = |b_i - a_i| \leq \varepsilon$. Therefore, $(b_n)_{n=1}^\infty$ and $(a_n)_{n=1}^\infty$ are equivalent. Thus, $\text{LIM}_{n \rightarrow \infty} b_n = \text{LIM}_{n \rightarrow \infty} a_n$ ($y = x$).

Transitivity. Assume $x = y$ and $y = z$. We want to show that the Cauchy sequences $(a_n)_{n=1}^\infty$ and $(c_n)_{n=1}^\infty$ are equivalent, that is, for any $\varepsilon > 0$, there exists $N \geq 1$ such that for all $i \geq N$, we have $|a_i - c_i| \leq \varepsilon$. Suppose ε is an arbitrary positive rational number. Since $x = y$, the Cauchy sequences $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ are equivalent. Then there exists N_1 such that for every $i \geq N_1$, we have $|a_i - b_i| \leq \frac{\varepsilon}{2}$. Since $y = z$, the Cauchy sequences $(b_n)_{n=1}^\infty$ and $(c_n)_{n=1}^\infty$ are equivalent. Then there exists N_2 such that for every $i \geq N_2$, we have $|b_i - c_i| \leq \frac{\varepsilon}{2}$. Let $N = \max(N_1, N_2)$, then for all $i \geq N$, $|a_i - c_i| \leq |a_i - b_i| + |b_i - c_i| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Thus, $(a_n)_{n=1}^\infty$ and $(c_n)_{n=1}^\infty$ are equivalent. So $x = z$. \square

Exercise 5.3.2

Prove Proposition 5.3.10.

Proof. xy is a real number. We want to show that for any $\varepsilon > 0$, there exists $N \geq 1$ such that $|a_i b_i - a_j b_j| \leq \varepsilon$ for any $i, j \geq N$. Consider an arbitrary $\varepsilon > 0$. Since $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ are Cauchy sequences, by Lemma 5.1.15, they are both bounded. Assume $(a_n)_{n=1}^\infty$ is bounded by M_1 and $(b_n)_{n=1}^\infty$ is bounded by M_2 . Since a_n is a Cauchy sequence, there exists $N_1 \geq 1$ such that for all $i, j \geq N_1$, we have

$$|a_i - a_j| \leq \frac{\varepsilon}{2M_1}.$$

Similarly, since $(b_n)_{n=1}^\infty$ is a Cauchy sequence, there exists $N_2 \geq 1$ such that for all

$i, j \geq N_2$, we have

$$|b_i - b_j| \leq \frac{\varepsilon}{2M_2}.$$

Let $N = \max(N_1, N_2)$, consider an arbitrary pair of $i, j \geq N$. Then we have

$$\begin{aligned} |a_j b_j - a_i b_i| &= |a_j b_j - a_j b_i + a_j b_i - a_i b_i| \\ &\leq |a_j b_j - a_j b_i| + |a_j b_i - a_i b_i| \\ &= |a_j| \cdot |b_j - b_i| + |b_i| \cdot |a_j - a_i| \\ &\leq M_1 \cdot \frac{\varepsilon}{2M_1} + M_2 \cdot \frac{\varepsilon}{2M_2} \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Therefore, for every $\varepsilon > 0$, we can find an $N = \max(N_1, N_2)$ such that $a_N b_N, a_{N+1} b_{N+1}, \dots$ is ε -close. Thus, $(a_n b_n)_{n=1}^\infty$ is a Cauchy sequence and xy is a real number.

Since $(b_n)_{n=1}^\infty$ is a Cauchy sequence, it must be bounded by some rational number M . Since $(a_n)_{n=1}^\infty$ and $(a'_n)_{n=1}^\infty$ are equivalent, for every $\varepsilon > 0$, there exists $N \geq 1$ such that for all $i \geq N$, we have

$$|a_i - a'_i| \leq \frac{\varepsilon}{M}.$$

Therefore, for all $i \geq N$,

$$\begin{aligned} |a_i b_i - a'_i b_i| &= |b_i| \cdot |a_i - a'_i| \\ &\leq M \cdot \frac{\varepsilon}{M} \\ &= \varepsilon. \end{aligned}$$

Thus, $(a_n b_n)_{n=1}^\infty$ and $(a'_n b_n)_{n=1}^\infty$ are equivalent and that $xy = x'y$. □

Exercise 5.3.3

Let a, b be rational numbers. Show that $a = b$ if and only if $\text{LIM}_{n \rightarrow \infty} a_n = \text{LIM}_{n \rightarrow \infty} b_n$ (i.e., the Cauchy sequences a, a, a, a, \dots and b, b, b, b, \dots are equivalent if and only if $a = b$).

This allows us to embed the rational numbers inside the real numbers in a well-defined manner.

Proof. Suppose the Cauchy sequences a, a, a, a, \dots and b, b, b, b, \dots are equivalent. Assume $a \neq b$, then $|a - b| > 0$. Since the two sequences are equivalent, for every $\varepsilon > 0$, there exists $N \geq 1$ such that $|a_i - b_i| \leq \varepsilon$. Let $\varepsilon = \frac{|a-b|}{2} > 0$. Then no matter what value i is, we have

$$|a_i - b_i| = |a - b| > \frac{|a - b|}{2} = \frac{\varepsilon}{2}$$

which contradicts the definition of equivalent sequences. Therefore, $a = b$.

Suppose $a = b$. Then for any $\varepsilon > 0$, let $N = 1$, we have

$$|a_i - b_i| = |a - b| = 0 < \varepsilon$$

for all $i \geq N$. Therefore, $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent.

Thus, $a = b$ if and only if $\text{LIM}_{n \rightarrow \infty} a_n = \text{LIM}_{n \rightarrow \infty} b_n$. □

Exercise 5.3.4

Let $(a_n)_{n=0}^{\infty}$ be a sequence of rational numbers which is bounded. Let $(b_n)_{n=0}^{\infty}$ be another sequence of rational numbers which is equivalent to $(a_n)_{n=0}^{\infty}$. Show that $(b_n)_{n=0}^{\infty}$ is also bounded.

Proof. Suppose $(a_n)_{n=0}^{\infty}$ is bounded by M . Similar to Exercise 5.2.2, we can split $(b_n)_{n=1}^{\infty}$ to b_0, \dots, b_N and b_{N+1}, b_{N+2}, \dots such that the former is bounded by some rational number M_1 and the latter is bounded by $M + \varepsilon$ for any $\varepsilon > 0$. Let $M_0 = \max(M, M_1)$, then $(b_n)_{n=0}^{\infty}$ is bounded by M_0 . □

Exercise 5.3.5

Show that $\text{LIM}_{n \rightarrow \infty} 1/n = 0$.

Proof. We want to show that $a_n = 1/n$ and $0, 0, 0, \dots$ are equivalent. Consider an arbitrary $\varepsilon > 0$. Let $N = \lceil \frac{1}{\varepsilon} \rceil$, we have

$$|a_i - 0| = a_i \leq a_N = \frac{1}{N} \leq \varepsilon$$

for all $i \geq N$. Thus, $\text{LIM}_{n \rightarrow \infty} 1/n = 0$. □

5.4 Ordering the reals

Definition 5.4.1.

Let $(a_n)_{n=1}^{\infty}$ be a sequence of rationals. We say that this sequence is positively bounded away from zero iff we have a positive rational $c > 0$ such that $a_n \geq c$ for all $n \geq 1$ (in particular, the sequence is entirely positive). The sequence is negatively bounded away from zero iff we have a negative rational $-c < 0$ such that $a_n \leq -c$ for all $n \geq 1$ (in particular, the sequence is entirely negative).

Definition 5.4.3.

A real number x is said to be positive iff it can be written as $x = \text{LIM}_{n \rightarrow \infty} a_n$ for some Cauchy sequence $(a_n)_{n=1}^{\infty}$ which is positively bounded away from zero. x is said to be negative iff it can be written as $x = \text{LIM}_{n \rightarrow \infty} a_n$ for some sequence $(a_n)_{n=1}^{\infty}$ which is negatively bounded away from zero.

Proposition 5.4.4 (Basic properties of positive reals).

For every real number x , exactly one of the following three statements is true: (a) x is zero; (b) x is positive; (c) x is negative. A real number x is negative if and only if $-x$ is positive. If x and y are positive, then so are $x + y$ and xy .

Definition 5.4.5 (Absolute value).

Let x be a real number. We define the absolute value $|x|$ of x to equal x if x is positive, $-x$ when x is negative, and 0 when x is zero.

Definition 5.4.6 (Ordering of the real numbers).

Let x and y be real numbers. We say that x is greater than y , and write $x > y$, iff $x - y$ is a positive real number, and $x < y$ iff $x - y$ is a negative real number. We define $x \geq y$ iff $x > y$ or $x = y$, and similarly define $x \leq y$.

Proposition 5.4.7.

All the claims in Proposition 4.2.9 which held for rationals, continue to hold for real numbers.

Proposition 5.4.8.

let x be a positive real number. Then x^{-1} is also positive. Also, if y is another positive number and $x > y$, then $x^{-1} < y^{-1}$.

Proposition 5.4.9 (The non-negative reals are closed).

Let a_1, a_2, a_3, \dots be a Cauchy sequence of non-negative rational numbers. Then $\text{LIM}_{n \rightarrow \infty} a_n$ is a non-negative real number.

Corollary 5.4.10.

Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be Cauchy sequences of rationals such that $a_n \geq b_n$ for all $n \geq 1$. Then $\text{LIM}_{n \rightarrow \infty} a_n \geq \text{LIM}_{n \rightarrow \infty} b_n$.

Proposition 5.4.12 (Bounding of reals by rationals).

Let x be a positive real number. Then there exists a positive rational number q such that $q \leq x$, and there exists a positive integer N such that $x \leq N$.

Corollary 5.4.13 (Archimedean property).

Let x and ε be any positive real numbers. Then there exists a positive integer M such that $M\varepsilon > x$.

Proposition 5.4.14.

Given any two real numbers $x < y$, we can find a rational number q such that $x < q < y$.

Exercise 5.4.1.

Prove Proposition 5.4.4.

Proof. Assume $x = \text{LIM}_{n \rightarrow \infty} a_n$.

At least one of the three statements is true. If $(a_n)_{n=1}^\infty$ is equivalent to $(0)_{n=1}^\infty$, x is 0. If the Cauchy sequence $(a_n)_{n=1}^\infty$ is not equivalent to $(0)_{n=1}^\infty$, by Lemma 5.3.14, $(a_n)_{n=1}^\infty$ is bounded away from zero. Then there exists a rational number $c > 0$ such that $|a_i| \geq c$ for all $i \geq 1$. Since $(a_n)_{n=1}^\infty$ is a Cauchy sequence, let $\varepsilon = c/2$, then there exists an $N \geq 1$ such that $|a_i - a_j| \leq \varepsilon = c/2$ for all $i, j \geq N$. Let $j = N$, we have $|a_i - a_N| \leq c/2$ for all $i \geq N$. Since $(a_n)_{n=1}^\infty$ is bounded away from 0 by c , a_N cannot be 0. If $a_N > 0$, we have $a_N \geq c$ and $a_N - c/2 \leq a_i \leq a_N + c/2$. So $a_i \geq a_N - c/2 \geq c/2 > 0$, and $(a_n)_{n=1}^\infty$ is eventually positively bounded away from zero. In particular, $a_i \geq c$ for all $i \geq N$. Let $b_i = c$ when $i < N$ and $b_i = a_i$ when $i \geq N$. Then $(b_n)_{n=1}^\infty$ is equivalent to $(a_n)_{n=1}^\infty$ and $x = \text{LIM}_{n \rightarrow \infty} a_n = \text{LIM}_{n \rightarrow \infty} b_n$ is a positive real number. Similarly, we can show that if $a_N < 0$, x would be negative. Thus, at least one of the three statements is true.

At most one of the three statements is true. Suppose x is zero. For any $c > 0$, there exists $N \geq 1$ such that $|a_i - 0| = |a_i| \leq \frac{c}{2} < c$. Therefore, $(a_n)_{n=1}^\infty$ is not bounded away from zero. Thus, x is not positive nor negative. Suppose x is positive. Then there exists $c > 0$ such that $a_i > c > 0$ for all $i \geq 1$. So for any $c' > 0$, $a_i > 0 > -c'$. Therefore, x cannot be negative. Similarly, if x is negative, it cannot be positive. Thus, at most one of the three statements is true.

x is negative $\iff -x$ is positive. We know that $-x = \text{LIM}_{n \rightarrow \infty} (-a_n)$. Suppose x is negative. Then there exists $c > 0$ such that $-a_i < -c$ for all $i \geq 1$. So $a_i > c$ for all $i \geq 1$. Thus, $(a_n)_{n=1}^\infty$ is positively bounded away from zero, and $x = \text{LIM}_{n \rightarrow \infty} a_n$ is positive. Similarly, we can show that if $-x$ is positive, then x is negative.

Assume $y = \text{LIM}_{n \rightarrow \infty} b_n$. Suppose x and y are positive. Then there exist $c_1, c_2 > 0$ such that $a_i \geq c_1$ and $b_i \geq c_2$ for all $i \geq 1$. Let $c = c_1 + c_2$, we have $(a_i + b_i) \geq c = c_1 + c_2$ for all $i \geq 1$. Therefore, $x + y$ is positive. Let $c' = c_1 c_2$, we have $a_i b_i \geq c' = c_1 c_2$ for all $i \geq 1$. Therefore, xy is positive. \square

Exercise 5.4.2.

Prove the remaining claims in Proposition 5.4.7.

- (a) *Proof.* Since $x - y$ is a real number, by Proposition 5.4.4, exactly one of the three statements $x - y = 0$, $x - y > 0$, or $x - y < 0$ is true. Thus, exactly one of $x = y$, $x > y$, or $x < y$ is true. \square
- (b) *Proof.* Since $x - y$ is a real number, by Proposition 5.4.4, $x - y$ is negative iff $-(x - y) = y - x$ is positive. Thus, $x < y$ iff $y > x$. \square
- (c) *Proof.* Since $x < y$, we have $y > x$, hence, $y - x$ is positive. Similarly, since $y < z$, $z - y$ is positive. By Proposition 5.4.4, $z - x = (y - x) + (z - y)$ is positive. Therefore, $x < z$. \square
- (d) *Proof.* Since $x < y$, we have $x - y = x - y + 0 = x - y + z - z = (x + z) - (y + z) < 0$. Therefore, $x + z < y + z$. \square

Exercise 5.4.3.

Show that for every real number x there is exactly one integer N such that $N \leq x < N + 1$. (This integer N is called the integer part of x , and is sometimes denoted $N = \lfloor x \rfloor$.)

Proof. Denote $x = (a_n)_{n=1}^{\infty}$ where $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence. Let $\varepsilon = \frac{1}{2}$. Then there exists an $N \geq 1$ such that $|x_i - x_N| \leq \varepsilon = \frac{1}{2}$ for all $i \geq N$. Therefore, $|x - x_N| \leq \frac{1}{2}$. So

$$-\frac{1}{2} \leq x - x_N \leq \frac{1}{2} \implies x_N - \frac{1}{2} \leq x \leq x_N + \frac{1}{2}.$$

Since x_N is rational, $x_N - \frac{1}{2}$ and $x_N + \frac{1}{2}$ are also rational. Then there exists exactly one integer n such that

$$n \leq x_N - \frac{1}{2} < n + 1$$

and

$$n + 1 \leq x_N + \frac{1}{2} < n + 2.$$

Then there are two cases. If $x < n + 1$, there exists exactly one integer N such that

$$N = n \leq x_N - \frac{1}{2} \leq x < n + 1 = N + 1.$$

If $x \geq n + 1$, there exists exactly one integer N such that

$$N = n + 1 \leq x \leq x_N + \frac{1}{2} < n + 2 = N + 1.$$

Therefore, for every real number x there is exactly one integer N such that $N \leq x < N + 1$. \square

Exercise 5.4.4.

Show that for any positive real number $x > 0$ there exists a positive integer N such that $x > 1/N > 0$.

Proof. Let $x = \text{LIM}_{n \rightarrow \infty} a_n$. Since x is a positive real number, it is positively bounded away from zero. Then there exists a positive rational number $c > 0$, such that $a_n \geq c > 0$ for $n \geq 1$. By Corollary 5.4.10, we have $x = \text{LIM}_{n \rightarrow \infty} a_n \geq c > 0$. Since $c > c/2 > 0$, we have $x \geq c > c/2 > 0$. Let $N = \frac{2}{c} + 1$, then $0 < \frac{1}{N} < \frac{c}{2}$. Then we have $x > \frac{1}{N} > 0$. \square

Exercise 5.4.5.

Prove Proposition 5.4.14.

Proof. $y - x > 0 \implies y - x > 0$. By Exercise 5.4.4, there exists a positive integer N such that $y - x > 1/N > 0$. Then we have $Ny > Nx + 1 > Nx$. And by Exercise 5.4.3, there exists exactly one integer n such that $n \leq Nx < n + 1$. Then we also have $n + 1 \leq Nx + 1 < n + 2$. Therefore, $n \leq Nx < n + 1 \leq Nx + 1 < Ny$. Therefore, there exists an integer between Nx and Ny . Divide the inequalities by N , we have $x < \frac{n+1}{N} < y$ where $\frac{n+1}{N}$ is a rational number. Therefore, if $x < y$, we can find a rational number q such that $x < q < y$. \square

Exercise 5.4.6.

Let x, y be real numbers and let $\varepsilon > 0$ be a positive real. Show that $|x - y| < \varepsilon$ if and only if $y - \varepsilon < x < y + \varepsilon$, and that $|x - y| \leq \varepsilon$ if and only if $y - \varepsilon \leq x \leq y + \varepsilon$.

- $|x - y| < \varepsilon \iff y - \varepsilon < x < y + \varepsilon$.

Proof. Suppose $|x - y| < \varepsilon$. By definition, if $x - y > 0$, we have $x - y < \varepsilon \implies x < y + \varepsilon$, and if $x - y < 0$, we have $y - x < \varepsilon \implies y - \varepsilon < x$. Combining the two inequalities, we have $y - \varepsilon < x < y + \varepsilon$.

Suppose $y - \varepsilon < x < y + \varepsilon$. Then $-\varepsilon < x - y < \varepsilon$. So if $x - y$ is positive, $|x - y| = x - y < \varepsilon$. Otherwise, $|x - y| = y - x < \varepsilon$. Therefore, $|x - y| < \varepsilon$.

Thus, $|x - y| < \varepsilon \iff y - \varepsilon < x < y + \varepsilon$. \square

- $|x - y| \leq \varepsilon \iff y - \varepsilon \leq x \leq y + \varepsilon$.

Proof. The proof is almost identical to the previous one. \square

Exercise 5.4.7.

Let x and y be real numbers. Show that $x \leq y + \varepsilon$ for all real numbers $\varepsilon > 0$ if and only if $x \leq y$. Show that $|x - y| \leq \varepsilon$ for all real numbers $\varepsilon > 0$ if and only if $x = y$.

- $x \leq y + \varepsilon$ for all $\varepsilon > 0 \iff x \leq y$.

Proof. Suppose $x \leq y + \varepsilon$ for all $\varepsilon > 0$. Assume $x > y$. Then we have $x - y > \frac{x - y}{2} > 0$. Let $\varepsilon = \frac{x - y}{2}$. We have $x \leq y + \varepsilon = y + \frac{x - y}{2} \implies x \leq y$. (contradiction) Therefore, we must have $x \leq y$.

Suppose $x \leq y$. For all $\varepsilon > 0$, we have $x \leq y < y + \varepsilon$.

Thus, $x \leq y + \varepsilon$ for all $\varepsilon > 0 \iff x \leq y$. \square

- $|x - y| \leq \varepsilon$ for all real numbers $\varepsilon > 0 \iff x = y$.

Proof. Suppose $|x - y| \leq \varepsilon$ for all real numbers. By Exercise 5.4.6, we have $-\varepsilon < x - y < \varepsilon$. Since $x < y + \varepsilon$, we have $x \leq y$. Since $y < x + \varepsilon$, we have $y \leq x$. And since $y \leq x$ and $x \leq y$, we have $x = y$.

Suppose $x = y$. Then $|x - y| = 0 \leq \varepsilon$ for all $\varepsilon > 0$.

Thus, $|x - y| \leq \varepsilon$ for all real numbers $\varepsilon > 0 \iff x = y$. □

Exercise 5.4.8.

Let $(a_n)_{n=1}^{\infty}$ be a Cauchy sequence of rationals, and let x be a real number. Show that if $a_n \leq x$ for all $n \geq 1$, then $\text{LIM}_{n \rightarrow \infty} a_n \leq x$. Similarly, show that if $a_n \geq x$ for all $n \geq 1$, then $\text{LIM}_{n \rightarrow \infty} a_n \geq x$.

Proof. Suppose $a_n \leq x$ for all $n \geq 1$. Assume $\text{LIM}_{n \rightarrow \infty} a_n > x$. Let $y = \text{LIM}_{n \rightarrow \infty} a_n$. If $y = \text{LIM}_{n \rightarrow \infty} a_n > x$, by Proposition 5.4.14, there exists a rational number q such that $y = \text{LIM}_{n \rightarrow \infty} a_n > q > x$. On the other hand, we have $a_n \leq x < q$ for all $n \geq 1$. By Corollary 5.4.10, we have $y = \text{LIM}_{n \rightarrow \infty} a_n \leq q$ which contradicts $y > q$. Therefore, $\text{LIM}_{n \rightarrow \infty} a_n \leq x$. The second statement can be proved in a similar way. □