2.2.1

For any natural numbers a, b, c, we have $(a+b)+c = a+(b+c)$.

Proof. Induct on b by keeping a and c fixed. Consider the base case $b = 0$. In this case, LHS$= (a+0)+c = a+c$ and RHS$= a+(0+c) = a+c$. Now suppose that $(a+b)+c = a+(b+c)$. We need to show that $(a+(b++))+c = a+((b++)+c)$:

$$\begin{aligned} \text{LHS} &= (a+(b++))+c = ((a+b)++)+c = (a+b+c)++, \\ \text{RHS} &= a+((b++)+c) = a+((b+c)++) = (a+b+c)++. \end{aligned}$$

Thus both sides are equal to each other, and we have closed the induction. □

2.2.2

Let a be a positive number. Then there exists exactly one natural number b such that $b++ = a$. (I'm assuming that it meant a is a positive natural number.)

Proof. Induct on a. Since 0 is not positive, we consider the base case $a = 1$. We have $b++ = b+1 = 0+1 = 1$. Cancellation law tells us that $b = 0$, which is unique. Now suppose that there exists exactly one natural number b_0 such that $b_0++ = a$, we need to show that there exists exactly one natural number b such that $b++ = a++$. By Cancellation law, we have $b = a = b_0++$. Since b is the successor of b_0 and b_0 is unique, b is also unique. Thus we have closed the induction. □

2.2.3

(a)

$a \geq a$.

Proof. There exists a natural number 0 such that $a+0 = a$. Thus, $a \geq a$. □

(b)

If $a \geq b$ and $b \geq c$, then $a \geq c$.

Proof. Since $a \geq b$, there exists a natural number m such that $b + m = a$. Since $b \geq c$, there exists a natural number n such that $c + n = b$. Then $c + (n + m) = (c + n) + m = b + m = a$. Therefore, $c \geq a$.

Thus, if $a \geq b$ and $b \geq c$, then $a \geq c$. □

(c)

If $a \geq b$ and $b \geq a$, then $a = b$.

Proof. Since $a \geq b$, there exists a natural number m such that $b + m = a$. Since $b \geq a$, there exists a natural number n such that $a + n = b$. Then we have $a + n = (b + m) + n = b + (m + n) = b$. By Cancellation law, we have $m + n = 0$ which leads to $m = 0$, $n = 0$. Thus, $a = a + 0 = b$.

Thus, if $a \geq b$ and $b \geq a$, then $a = b$. □

(d)

$a \geq b$ if and only if $a + c \geq b + c$.

Proof. First, we need to show that $a \geq b \Rightarrow a + c \geq b + c$. Since $a \geq b$, there exists a natural number n such that $b + n = a$. Then we have $b + n + c = b + c + n = (b + c) + n = a + c$. Thus, $a + c \geq b + c$. Then, we need to show that $a + c \geq b + c \Rightarrow a \geq b$. Since $a + c \geq b + c$, there should be a natural number n such that $b + c + n = b + n + c = (b + n) + c = a + c$. By Cancellation law, we have $b + n = a$. Thus, $a \geq b$.

Thus, if $a \geq b$ and $b \geq a$, then $a = b$. □

(e)

$a < b$ if and only if $a{+}{+} \leq b$.

Proof. First, we need to show that $a < b \Rightarrow a++ \leq b$. $a < b$ means there exists a natural number n such that $a + n = b$, particularly, $a \neq b$. Then n must not be zero. So n is the predecessor of a natural number, denote it as m. Then we have $a + n = a + (m++) = (a + m)++ = (a++) + m = b$. Therefore, $a++ \leq b$. Then we need to show that $a++ \leq b \Rightarrow a < b$. There exists a natural number n such that $(a++) + n = b$. $(a++) + n = (a + n)++ = a + (n++) = b$. Since $n++$ is the successor of n, $n++$ must not be equal to 0. If $a = b$, there will be $a + (n++) = a \Rightarrow n++ = 0$, contradiction. Therefore, $a \neq b$.

Thus, $a < b$ if and only if $a++ \leq b$. □

(f)

$a < b$ if and only if $b = a + d$ for some positive number d.

Proof. First, we need to show that $a < b \Rightarrow b = a + d$ for some positive number d. There exists some natural number d such that $a + d = b$, $a \neq b$. By Cancellation law, d must not be zero. Therefore, d is positive. Then, we need to show that $a + d = b$ for some positive $d \Rightarrow a < b$. We only need to prove $a \neq b$. If $a = b$, we have $a + d = a = a + 0$. By Cancellation law, $d = 0$ which contradicts to d is positive. Therefore, $a \neq b$.

Thus, $a < b$ if and only if $b = a + d$ for some positive number d. □

2.2.4

Justify the three statements marked in the proof of Proposition 2.2.13.

Proof. □

2.2.5