

# Chapter 5

## The real numbers

### 5.1 Cauchy sequences

#### Definition 5.1.1 (Sequences).

Let  $m$  be an integer. A sequence  $(a_n)_{n=m}^{\infty}$  of rational numbers is any function from the set  $\{n \in \mathbf{Z} : n \geq m\}$  to  $\mathbf{Q}$ , i.e., a mapping which assigns to each integer  $n$  greater than or equal to  $m$ , a rational number  $a_n$ . More informally, a sequence  $(a_n)_{n=m}^{\infty}$  of rational numbers is a collection of rationals  $a_m, a_{m+1}, a_{m+2}, \dots$

#### Definition 5.1.3 ( $\varepsilon$ -steadiness).

Let  $\varepsilon > 0$ . A sequence  $(a_n)_{n=0}^{\infty}$  is said to be  $\varepsilon$ -steady iff each pair  $a_j, a_k$  of sequence elements is  $\varepsilon$ -close for every natural number  $j, k$ . In other words, the sequence  $a_0, a_1, a_2, \dots$  is  $\varepsilon$ -steady iff  $|a_j - a_k| \leq \varepsilon$  for all  $j, k$ .

#### Definition 5.1.6 (Eventual $\varepsilon$ -steadiness).

Let  $\varepsilon > 0$ . A sequence  $(a_n)_{n=0}^{\infty}$  is said to be eventually  $\varepsilon$ -steady iff the sequence  $a_N, a_{N+1}, a_{N+2}, \dots$  is  $\varepsilon$ -steady for some natural number  $N \geq 0$ . In other words, the sequence  $a_0, a_1, a_2, \dots$  is eventually  $\varepsilon$ -steady iff there exists an  $N \geq 0$  such that  $|a_j - a_k| \leq \varepsilon$  for all  $j, k \geq N$ .

#### Definition 5.1.8 (Cauchy sequences).

A sequence  $(a_n)_{n=0}^{\infty}$  of rational numbers is said to be a Cauchy sequence iff for every rational  $\varepsilon > 0$ , the sequence  $(a_n)_{n=0}^{\infty}$  is eventually  $\varepsilon$ -steady. In other words, the sequence  $a_0, a_1, a_2, \dots$  is a Cauchy sequence iff for every  $\varepsilon > 0$ , there exists an  $N \geq 0$  such that  $d(a_j, a_k)$  for all  $j, k \geq N$ .

**Proposition 5.1.11**

The sequence  $a_1, a_2, a_3, \dots$  defined by  $a_n := 1/n$  (i.e., the sequence  $1, 1/2, 1/3, \dots$ ) is a Cauchy sequence.

**Definition 5.1.12 (Bounded sequences).**

Let  $M \geq 0$  be rational. A finite sequence  $a_1, a_2, \dots, a_n$  is bounded by  $M$  iff  $|a_i| \leq M$  for all  $1 \leq i \leq n$ . An infinite sequence  $(a_n)_{n=1}^\infty$  is bounded by  $M$  iff  $|a_i| \leq M$  for all  $i \geq 1$ . A sequence is said to be bounded iff it is bounded by  $M$  for some rational  $M \geq 0$ .

**Lemma 5.1.14 (Finite sequences are bounded).**

Every finite sequence  $a_1, a_2, \dots, a_n$  is bounded.

**Lemma 5.1.15 (Cauchy sequences are bounded).**

Every Cauchy sequence  $(a_n)_{n=1}^\infty$  is bounded.

**Exercise 5.1.1**

Prove Lemma 5.1.15.

*Proof.* Suppose  $(a_n)_{n=1}^\infty$  is a Cauchy sequence. So for  $\varepsilon = 1$ , there exists an  $N \geq 1$  such that  $d(a_j, a_k) \leq 1$  for all  $j, k \geq N$ . Then the sequence can be split into two parts:  $a_1, \dots, a_N$  and  $a_{N+1}, a_{N+2}, \dots$ . The former is a finite sequence, by Lemma 5.1.14, it is bounded. Suppose this finite sequence is bounded by  $M_1$ . Consider  $a_{N+1}, a_{N+2}, \dots$ . Since it is 1-steady, for any  $i > N + 1$ , we have  $|a_i - a_{N+1}| \leq 1$ . Rearrange the inequalities, we have  $a_{N+1} - 1 \leq a_i \leq a_{N+1} + 1$ . Let  $M_2 = a_{N+1} + 1$ . Then  $a_{N+1} < M_2$  and for every  $i > N + 1$ , we have  $a_i \leq M_2$ . So sequence  $a_{N+1}, a_{N+2}, \dots$  is bounded by  $M_2$ . Let  $M = \max\{M_1, M_2\}$ . Then the Cauchy sequence  $(a_n)_{n=1}^\infty$  is bounded by  $M$ .  $\square$

## 5.2 Equivalent Cauchy sequences

### Definition 5.2.1 ( $\varepsilon$ -close sequences).

Let  $(a_n)_{n=0}^\infty$  and  $(b_n)_{n=0}^\infty$  be two sequences, and let  $\varepsilon > 0$ . We say that the sequence  $(a_n)_{n=0}^\infty$  is  $\varepsilon$ -close to  $(b_n)_{n=0}^\infty$  iff  $a_n$  is  $\varepsilon$ -close to  $b_n$  for each  $n \in \mathbf{N}$ . In other words, the sequence  $a_0, a_1, a_2, \dots$  is  $\varepsilon$ -close to the sequence  $b_1, b_1, b_2, \dots$  iff  $|a_n - b_n| \leq \varepsilon$  for all  $n = 0, 1, 2, \dots$ .

### Definition 5.2.3 (Eventually $\varepsilon$ -close sequences).

Let  $(a_n)_{n=0}^\infty$  and  $(b_n)_{n=0}^\infty$  be two sequences, and let  $\varepsilon > 0$ . We say that the sequence  $(a_n)_{n=0}^\infty$  is eventually  $\varepsilon$ -close to  $(b_n)_{n=0}^\infty$  iff there exists an  $N \geq 0$  such that the sequences  $(a_n)_{n=N}^\infty$  and  $(b_n)_{n=N}^\infty$  are  $\varepsilon$ -close. In other words,  $a_0, a_1, a_2, \dots$  is eventually  $\varepsilon$ -close to  $b_0, b_1, b_2, \dots$  iff there exists an  $N \geq 0$  such that  $|a_n - b_n| \leq \varepsilon$  for all  $n \geq N$ .

### Definition 5.2.6 (Equivalent sequences).

Two sequences  $(a_n)_{n=0}^\infty$  and  $(b_n)_{n=0}^\infty$  are equivalent iff for each rational  $\varepsilon > 0$ , the sequences  $(a_n)_{n=0}^\infty$  and  $(b_n)_{n=0}^\infty$  are eventually  $\varepsilon$ -close. In other words,  $a_0, a_1, a_2, \dots$  and  $b_0, b_1, b_2, \dots$  are equivalent iff for every for every rational  $\varepsilon > 0$ , there exists an  $N \geq 0$  such that  $|a_n - b_n| \leq \varepsilon$  for all  $n \geq N$ .

### Proposition 5.2.8

Let  $(a_n)_{n=1}^\infty$  and  $(b_n)_{n=1}^\infty$  be the sequences  $a_n = 1 + 10^{-n}$  and  $b_n = 1 - 10^{-n}$ . Then the sequences  $a_n, b_n$  are equivalent.

### Exercise 5.2.1

Show that if  $(a_n)_{n=1}^\infty$  and  $(b_n)_{n=1}^\infty$  are equivalent sequences of rationals, then  $(a_n)_{n=1}^\infty$  is a Cauchy sequence if and only if  $(b_n)_{n=1}^\infty$  is a Cauchy sequence.

*Proof.* Assume  $(a_n)_{n=1}^\infty$  and  $(b_n)_{n=1}^\infty$  are equivalent sequences of rationals, and  $(a_n)_{n=1}^\infty$  is a Cauchy sequence. We want to show that for any rational  $\varepsilon > 0$ , there exists  $N \geq 1$  such that  $b_N, b_{N+1}, \dots$  is  $\varepsilon$ -steady.

Since  $(a_n)_{n=1}^{\infty}$  is a Cauchy sequence and  $\frac{\varepsilon}{3} > 0$ , there exists  $N_1 \geq 1$  such that for all  $i, j \geq N_1$ ,  $|a_i - a_j| \leq \frac{\varepsilon}{3}$ . Since  $(a_n)_{n=1}^{\infty}$  and  $(b_n)_{n=1}^{\infty}$  are equivalent, and  $\frac{\varepsilon}{3} > 0$ , there exists  $N_2 \geq 1$  such that for all  $i \geq N_2$ ,  $|b_i - a_i| \leq \frac{\varepsilon}{3}$ . Let  $N = \max\{N_1, N_2\}$ . Consider arbitrary  $i, j \geq N$ . Since  $N \geq N_1$ , we have

$$|a_i - a_j| \leq \frac{\varepsilon}{3}.$$

Since  $N \geq N_2$ , we have

$$|b_i - a_i| \leq \frac{\varepsilon}{3}$$

and

$$|a_j - b_j| \leq \frac{\varepsilon}{3}.$$

Since

$$|a_i - a_j| \leq \frac{\varepsilon}{3}$$

and

$$|b_i - a_i| \leq \frac{\varepsilon}{3},$$

we have

$$|b_i - a_j| \leq |a_i - a_j| + |b_i - a_i| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}.$$

Since

$$|a_j - b_j| \leq \frac{\varepsilon}{3}$$

and

$$|b_i - a_j| \leq \frac{2\varepsilon}{3},$$

we have

$$|b_i - b_j| \leq |a_j - b_j| + |b_i - a_j| \leq \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon.$$

Thus, for any  $\varepsilon > 0$ , we can find  $N = \max\{N_1, N_2\}$  such that  $b_N, b_{N+1}, \dots$  is  $\varepsilon$ -steady. Therefore,  $(b_n)_{n=1}^{\infty}$  is a Cauchy sequence.

Similarly, we can show that if  $(a_n)_{n=1}^{\infty}$  and  $(b_n)_{n=1}^{\infty}$  are equivalent sequences of rationals and  $(b_n)_{n=1}^{\infty}$  is a Cauchy sequence, then  $(a_n)_{n=1}^{\infty}$  is a Cauchy sequence. Thus, if  $(a_n)_{n=1}^{\infty}$  and  $(b_n)_{n=1}^{\infty}$  are equivalent sequences of rationals, then  $(a_n)_{n=1}^{\infty}$  is a Cauchy

sequence if and only if  $(b_n)_{n=1}^\infty$  is a Cauchy sequence.  $\square$

### Exercise 5.2.2

Let  $\varepsilon > 0$ . Show that if  $(a_n)_{n=1}^\infty$  and  $(b_n)_{n=1}^\infty$  are eventually  $\varepsilon$ -close, then  $(a_n)_{n=1}^\infty$  is bounded if and only if  $(b_n)_{n=1}^\infty$  is bounded.

*Proof.* Suppose  $(a_n)_{n=1}^\infty$  and  $(b_n)_{n=1}^\infty$  are eventually  $\varepsilon$ -close and  $(a_n)_{n=1}^\infty$  is bounded. Since  $(a_n)_{n=1}^\infty$  and  $(b_n)_{n=1}^\infty$  are eventually  $\varepsilon$ -close, for any  $\varepsilon > 0$ , there exists  $N \geq 1$  such that for any  $i \geq N$ ,  $|a_i - b_i| \leq \varepsilon$ . Consider an arbitrary  $\varepsilon > 0$ . We can find  $N \geq 1$  such that for any  $i \geq N$ ,  $|a_i - b_i| \leq \varepsilon$ . Then

$$a_i - \varepsilon \leq b_i \leq a_i + \varepsilon.$$

Split  $(b_n)_{n=1}^\infty$  to  $b_1, \dots, b_N$  and  $b_{N+1}, b_{N+2}, \dots$ . The former is a finite sequence, so it is bounded by some rational number  $M_1$ . Since  $(a_n)_{n=1}^\infty$  is bounded, there exists  $M$  such that  $|a_i| \leq M$  for all  $i \geq 1$ . Then

$$-M \leq a_i \leq M.$$

So

$$-M - \varepsilon \leq b_i \leq M + \varepsilon.$$

Therefore,  $|b_i| \leq M + \varepsilon$  for all  $i \geq N + 1$ . Let  $M_0 = \max(M, M_1)$ . For any  $i \geq 1$ , we have  $|b_i| \leq M_0$ . Thus,  $(b_n)_{n=1}^\infty$  is bounded.

Similarly, we can show that if  $(a_n)_{n=1}^\infty$  and  $(b_n)_{n=1}^\infty$  are eventually  $\varepsilon$ -close, and  $(b_n)_{n=1}^\infty$  is bounded, then  $(a_n)_{n=1}^\infty$  is bounded.

Thus, if  $(a_n)_{n=1}^\infty$  and  $(b_n)_{n=1}^\infty$  are eventually  $\varepsilon$ -close, then  $(a_n)_{n=1}^\infty$  is bounded if and only if  $(b_n)_{n=1}^\infty$  is bounded.  $\square$

## 5.3 The construction of the real numbers

### Definition 5.3.1 (Real numbers).

A real number is defined to be an object of the form  $\text{LIM}_{n \rightarrow \infty} a_n$ , where  $\text{LIM}_{n \rightarrow \infty} a_n$  is a Cauchy sequence of rational numbers. Two real numbers  $\text{LIM}_{n \rightarrow \infty} a_n$  and  $\text{LIM}_{n \rightarrow \infty} b_n$  are said to be equal iff  $(a_n)_{n=1}^{\infty}$  and  $(b_n)_{n=1}^{\infty}$  are equivalent Cauchy sequences. The set of all real numbers is denoted  $\mathbf{R}$ .

### Proposition 5.3.3 (Formal limits are well-defined).

Let  $x = \text{LIM}_{n \rightarrow \infty} a_n$ ,  $y = \text{LIM}_{n \rightarrow \infty} b_n$ , and  $z = \text{LIM}_{n \rightarrow \infty} c_n$  be real numbers. Then, with the above definition of equality for real numbers, we have  $x = x$ . Also, if  $x = y$ , then  $y = x$ . Finally, if  $x = y$  and  $y = z$ , then  $x = z$ .

### Definition 5.3.4 (Addition of reals).

Let  $x = \text{LIM}_{n \rightarrow \infty} a_n$  and  $y = \text{LIM}_{n \rightarrow \infty} b_n$  be real numbers. Then we define the sum  $x + y$  to be  $x + y := \text{LIM}_{n \rightarrow \infty} (a_n + b_n)$ .

### Lemma 5.3.6 (Sum of Cauchy sequences is Cauchy).

Let  $x = \text{LIM}_{n \rightarrow \infty} a_n$  and  $y = \text{LIM}_{n \rightarrow \infty} b_n$  be real numbers. Then  $x + y$  is also a real number (i.e.,  $(a_n + b_n)_{n=1}^{\infty}$  is a Cauchy sequence of rationals).

### Lemma 5.3.7 (Sums of equivalent Cauchy sequences are equivalent).

Let  $x = \text{LIM}_{n \rightarrow \infty} a_n$ ,  $y = \text{LIM}_{n \rightarrow \infty} b_n$ , and  $x' = \text{LIM}_{n \rightarrow \infty} a'_n$  be real numbers. Suppose that  $x = x'$ . Then we have  $x + y = x' + y$ .

### Lemma 5.3.9 (Multiplication of reals).

Let  $x = \text{LIM}_{n \rightarrow \infty} a_n$  and  $y = \text{LIM}_{n \rightarrow \infty} b_n$  be real numbers. Then we define the product  $xy$  to be  $xy := \text{LIM}_{n \rightarrow \infty} a_n b_n$ .

**Proposition 5.3.10 (Multiplication is well defined).**

Let  $x = \text{LIM}_{n \rightarrow \infty} a_n$ ,  $y = \text{LIM}_{n \rightarrow \infty} b_n$ , and  $x' = \text{LIM}_{n \rightarrow \infty} a'_n$  be real numbers. Then  $xy$  is also a real number. Furthermore, if  $x = x'$ , then  $xy = x'y$ .

**Proposition 5.3.11**

All the laws of algebra from Proposition 4.1.6 hold not only for the integers, but for the reals as well.

**Definition 5.3.12 (Sequences bounded away from zero).**

A sequence  $(a_n)_{n=1}^{\infty}$  of rational numbers is said to be bounded away from zero iff there exists a rational number  $c > 0$  such that  $|a_n| > c$  for all  $n \geq 1$ .

**Lemma 5.3.14.**

Let  $x$  be a non-zero real number. Then  $x = \text{LIM}_{n \rightarrow \infty} a_n$  for some Cauchy sequence  $(a_n)_{n=1}^{\infty}$  which is bounded away from zero.

**Lemma 5.3.15.**

Suppose that  $(a_n)_{n=1}^{\infty}$  is a Cauchy sequence which is bounded away from zero. Then the sequence  $(a_n^{-1})_{n=1}^{\infty}$  is also a Cauchy sequence.

**Definition 5.3.16 (Reciprocals of real numbers).**

Let  $x$  be a non-zero real number. Let  $(a_n)_{n=1}^{\infty}$  be a Cauchy sequence bounded away from zero such that  $x = \text{LIM}_{n \rightarrow \infty} a_n$ . Then we define the reciprocal  $x^{-1}$  by the formula  $x^{-1} := \text{LIM}_{n \rightarrow \infty} a_n^{-1}$ .

**Lemma 5.3.17 (Reciprocation is well defined).**

Let  $(a_n)_{n=1}^{\infty}$  and  $(b_n)_{n=1}^{\infty}$  be two Cauchy sequences bounded away from zero such that  $\text{LIM}_{n \rightarrow \infty} a_n = \text{LIM}_{n \rightarrow \infty} b_n$  (i.e., the two sequences are equivalent). Then  $\text{LIM}_{n \rightarrow \infty} a_n^{-1} = \text{LIM}_{n \rightarrow \infty} b_n^{-1}$ .

### Exercise 5.3.1

Prove Proposition 5.3.3.

*Proof.* Reflexivity. Since  $x = \text{LIM}_{n \rightarrow \infty} a_n$ ,  $(a_n)_{n=1}^{\infty}$  is a Cauchy sequence. Obviously,  $(a_n)_{n=1}^{\infty}$  and  $(a_n)_{n=1}^{\infty}$  are equivalent. Therefore,  $\text{LIM}_{n \rightarrow \infty} a_n = \text{LIM}_{n \rightarrow \infty} a_n$  ( $x = x$ ).

Symmetry. Assume  $x = y$ , then Cauchy sequences  $(a_n)_{n=1}^{\infty}$  and  $(b_n)_{n=1}^{\infty}$  are equivalent. So for every  $\varepsilon > 0$ , there exists  $N \geq 1$  such that for every  $i \geq N$ ,  $|a_i - b_i| = |b_i - a_i| \leq \varepsilon$ . Therefore,  $(b_n)_{n=1}^{\infty}$  and  $(a_n)_{n=1}^{\infty}$  are equivalent. Thus,  $\text{LIM}_{n \rightarrow \infty} b_n = \text{LIM}_{n \rightarrow \infty} a_n$  ( $y = x$ ).

Transitivity. Assume  $x = y$  and  $y = z$ . We want to show that the Cauchy sequences  $(a_n)_{n=1}^{\infty}$  and  $(c_n)_{n=1}^{\infty}$  are equivalent, that is, for any  $\varepsilon > 0$ , there exists  $N \geq 1$  such that for all  $i \geq N$ , we have  $|a_i - c_i| \leq \varepsilon$ . Suppose  $\varepsilon$  is an arbitrary positive rational number. Since  $x = y$ , the Cauchy sequences  $(a_n)_{n=1}^{\infty}$  and  $(b_n)_{n=1}^{\infty}$  are equivalent. Then there exists  $N_1$  such that for every  $i \geq N_1$ , we have  $|a_i - b_i| \leq \frac{\varepsilon}{2}$ . Since  $y = z$ , the Cauchy sequences  $(b_n)_{n=1}^{\infty}$  and  $(c_n)_{n=1}^{\infty}$  are equivalent. Then there exists  $N_2$  such that for every  $i \geq N_2$ , we have  $|b_i - c_i| \leq \frac{\varepsilon}{2}$ . Let  $N = \max(N_1, N_2)$ , then for all  $i \geq N$ ,  $|a_i - c_i| \leq |a_i - b_i| + |b_i - c_i| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ . Thus,  $(a_n)_{n=1}^{\infty}$  and  $(c_n)_{n=1}^{\infty}$  are equivalent. So  $x = z$ .  $\square$

### Exercise 5.3.2

Prove Proposition 5.3.10.

*Proof.*  $xy$  is a real number. We want to show that for any  $\varepsilon > 0$ , there exists  $N \geq 1$  such that  $|a_i b_i - a_j b_j| \leq \varepsilon$  for any  $i, j \geq N$ . Consider an arbitrary  $\varepsilon > 0$ . Since  $(a_n)_{n=1}^{\infty}$  and  $(b_n)_{n=1}^{\infty}$  are Cauchy sequences, by Lemma 5.1.15, they are both bounded. Assume  $(a_n)_{n=1}^{\infty}$  is bounded by  $M_1$  and  $(b_n)_{n=1}^{\infty}$  is bounded by  $M_2$ . Since  $a_n$  is a Cauchy sequence, there exists  $N_1 \geq 1$  such that for all  $i, j \geq N_1$ , we have

$$|a_i - a_j| \leq \frac{\varepsilon}{2M_1}.$$

Similarly, since  $(b_n)_{n=1}^{\infty}$  is a Cauchy sequence, there exists  $N_2 \geq 1$  such that for all



$i, j \geq N_2$ , we have

$$|b_i - b_j| \leq \frac{\varepsilon}{2M_2}.$$

Let  $N = \max(N_1, N_2)$ , consider an arbitrary pair of  $i, j \geq N$ . Then we have

$$\begin{aligned} |a_j b_j - a_i b_i| &= |a_j b_j - a_j b_i + a_j b_i - a_i b_i| \\ &\leq |a_j b_j - a_j b_i| + |a_j b_i - a_i b_i| \\ &= |a_j| \cdot |b_j - b_i| + |b_i| \cdot |a_j - a_i| \\ &\leq M_1 \cdot \frac{\varepsilon}{2M_1} + M_2 \cdot \frac{\varepsilon}{2M_2} \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Therefore, for every  $\varepsilon > 0$ , we can find an  $N = \max(N_1, N_2)$  such that  $a_N b_N, a_{N+1} b_{N+1}, \dots$  is  $\varepsilon$ -close. Thus,  $(a_n b_n)_{n=1}^\infty$  is a Cauchy sequence and  $xy$  is a real number.

Since  $(b_n)_{n=1}^\infty$  is a Cauchy sequence, it must be bounded by some rational number  $M$ . Since  $(a_n)_{n=1}^\infty$  and  $(a'_n)_{n=1}^\infty$  are equivalent, for every  $\varepsilon > 0$ , there exists  $N \geq 1$  such that for all  $i \geq N$ , we have

$$|a_i - a'_i| \leq \frac{\varepsilon}{M}.$$

Therefore, for all  $i \geq N$ ,

$$\begin{aligned} |a_i b_i - a'_i b_i| &= |b_i| \cdot |a_i - a'_i| \\ &\leq M \cdot \frac{\varepsilon}{M} \\ &= \varepsilon. \end{aligned}$$

Thus,  $(a_n b_n)_{n=1}^\infty$  and  $(a'_n b_n)_{n=1}^\infty$  are equivalent and that  $xy = x'y$ . □

### Exercise 5.3.3

Let  $a, b$  be rational numbers. Show that  $a = b$  if and only if  $\text{LIM}_{n \rightarrow \infty} a_n = \text{LIM}_{n \rightarrow \infty} b_n$  (i.e., the Cauchy sequences  $a, a, a, a, \dots$  and  $b, b, b, b, \dots$  are equivalent if and only if  $a = b$ ).

This allows us to embed the rational numbers inside the real numbers in a well-defined manner.

*Proof.* Suppose the Cauchy sequences  $a, a, a, a, \dots$  and  $b, b, b, b, \dots$  are equivalent. Assume  $a \neq b$ , then  $|a - b| > 0$ . Since the two sequences are equivalent, for every  $\varepsilon > 0$ , there exists  $N \geq 1$  such that  $|a_i - b_i| \leq \varepsilon$ . Let  $\varepsilon = \frac{|a-b|}{2} > 0$ . Then no matter what value  $i$  is, we have

$$|a_i - b_i| = |a - b| > \frac{|a - b|}{2} = \frac{\varepsilon}{2}$$

which contradicts the definition of equivalent sequences. Therefore,  $a = b$ .

Suppose  $a = b$ . Then for any  $\varepsilon > 0$ , let  $N = 1$ , we have

$$|a_i - b_i| = |a - b| = 0 < \varepsilon$$

for all  $i \geq N$ . Therefore,  $(a_n)_{n=1}^{\infty}$  and  $(b_n)_{n=1}^{\infty}$  are equivalent.

Thus,  $a = b$  if and only if  $\text{LIM}_{n \rightarrow \infty} a_n = \text{LIM}_{n \rightarrow \infty} b_n$ . □

#### Exercise 5.3.4

Let  $(a_n)_{n=0}^{\infty}$  be a sequence of rational numbers which is bounded. Let  $(b_n)_{n=0}^{\infty}$  be another sequence of rational numbers which is equivalent to  $(a_n)_{n=0}^{\infty}$ . Show that  $(b_n)_{n=0}^{\infty}$  is also bounded.

*Proof.* Suppose  $(a_n)_{n=0}^{\infty}$  is bounded by  $M$ . Similar to Exercise 5.2.2, we can split  $(b_n)_{n=1}^{\infty}$  to  $b_0, \dots, b_N$  and  $b_{N+1}, b_{N+2}, \dots$  such that the former is bounded by some rational number  $M_1$  and the latter is bounded by  $M + \varepsilon$  for any  $\varepsilon > 0$ . Let  $M_0 = \max(M, M_1)$ , then  $(b_n)_{n=0}^{\infty}$  is bounded by  $M_0$ . □

#### Exercise 5.3.5

Show that  $\text{LIM}_{n \rightarrow \infty} 1/n = 0$ .

*Proof.* We want to show that  $a_n = 1/n$  and  $0, 0, 0, \dots$  are equivalent. Consider an arbitrary  $\varepsilon > 0$ . Let  $N = \lceil \frac{1}{\varepsilon} \rceil$ , we have

$$|a_i - 0| = a_i \leq a_N = \frac{1}{N} \leq \varepsilon$$

for all  $i \geq N$ . Thus,  $\text{LIM}_{n \rightarrow \infty} 1/n = 0$ . □

## 5.4 Ordering the reals

### Definition 5.4.1.

Let  $(a_n)_{n=1}^{\infty}$  be a sequence of rationals. We say that this sequence is positively bounded away from zero iff we have a positive rational  $c > 0$  such that  $a_n \geq c$  for all  $n \geq 1$  (in particular, the sequence is entirely positive). The sequence is negatively bounded away from zero iff we have a negative rational  $-c < 0$  such that  $a_n \leq -c$  for all  $n \geq 1$  (in particular, the sequence is entirely negative).

### Definition 5.4.3.

A real number  $x$  is said to be positive iff it can be written as  $x = \text{LIM}_{n \rightarrow \infty} a_n$  for some Cauchy sequence  $(a_n)_{n=1}^{\infty}$  which is positively bounded away from zero.  $x$  is said to be negative iff it can be written as  $x = \text{LIM}_{n \rightarrow \infty} a_n$  for some sequence  $(a_n)_{n=1}^{\infty}$  which is negatively bounded away from zero.

### Proposition 5.4.4 (Basic properties of positive reals).

For every real number  $x$ , exactly one of the following three statements is true: (a)  $x$  is zero; (b)  $x$  is positive; (c)  $x$  is negative. A real number  $x$  is negative if and only if  $-x$  is positive. If  $x$  and  $y$  are positive, then so are  $x + y$  and  $xy$ .

### Definition 5.4.5 (Absolute value).

Let  $x$  be a real number. We define the absolute value  $|x|$  of  $x$  to equal  $x$  if  $x$  is positive,  $-x$  when  $x$  is negative, and 0 when  $x$  is zero.

### Definition 5.4.6 (Ordering of the real numbers).

Let  $x$  and  $y$  be real numbers. We say that  $x$  is greater than  $y$ , and write  $x > y$ , iff  $x - y$  is a positive real number, and  $x < y$  iff  $x - y$  is a negative real number. We define  $x \geq y$  iff  $x > y$  or  $x = y$ , and similarly define  $x \leq y$ .

**Proposition 5.4.7.**

All the claims in Proposition 4.2.9 which held for rationals, continue to hold for real numbers.

**Proposition 5.4.8.**

let  $x$  be a positive real number. Then  $x^{-1}$  is also positive. Also, if  $y$  is another positive number and  $x > y$ , then  $x^{-1} < y^{-1}$ .

**Proposition 5.4.9 (The non-negative reals are closed).**

Let  $a_1, a_2, a_3, \dots$  be a Cauchy sequence of non-negative rational numbers. Then  $\text{LIM}_{n \rightarrow \infty} a_n$  is a non-negative real number.

**Corollary 5.4.10.**

Let  $(a_n)_{n=1}^{\infty}$  and  $(b_n)_{n=1}^{\infty}$  be Cauchy sequences of rationals such that  $a_n \geq b_n$  for all  $n \geq 1$ . Then  $\text{LIM}_{n \rightarrow \infty} a_n \geq \text{LIM}_{n \rightarrow \infty} b_n$ .

**Proposition 5.4.12 (Bounding of reals by rationals).**

Let  $x$  be a positive real number. Then there exists a positive rational number  $q$  such that  $q \leq x$ , and there exists a positive integer  $N$  such that  $x \leq N$ .

**Corollary 5.4.13 (Archimedean property).**

Let  $x$  and  $\varepsilon$  be any positive real numbers. Then there exists a positive integer  $M$  such that  $M\varepsilon > x$ .

**Proposition 5.4.14.**

Given any two real numbers  $x < y$ , we can find a rational number  $q$  such that  $x < q < y$ .

**Exercise 5.4.1.**

Prove Proposition 5.4.4.

*Proof.* Assume  $x = \text{LIM}_{n \rightarrow \infty} a_n$ .

At least one of the three statements is true. If  $(a_n)_{n=1}^\infty$  is equivalent to  $(0)_{n=1}^\infty$ ,  $x$  is 0. If the Cauchy sequence  $(a_n)_{n=1}^\infty$  is not equivalent to  $(0)_{n=1}^\infty$ , by Lemma 5.3.14,  $(a_n)_{n=1}^\infty$  is bounded away from zero. Then there exists a rational number  $c > 0$  such that  $|a_i| \geq c$  for all  $i \geq 1$ . Since  $(a_n)_{n=1}^\infty$  is a Cauchy sequence, let  $\varepsilon = c/2$ , then there exists an  $N \geq 1$  such that  $|a_i - a_j| \leq \varepsilon = c/2$  for all  $i, j \geq N$ . Let  $j = N$ , we have  $|a_i - a_N| \leq c/2$  for all  $i \geq N$ . Since  $(a_n)_{n=1}^\infty$  is bounded away from 0 by  $c$ ,  $a_N$  cannot be 0. If  $a_N > 0$ , we have  $a_N \geq c$  and  $a_N - c/2 \leq a_i \leq a_N + c/2$ . So  $a_i \geq a_N - c/2 \geq c/2 > 0$ , and  $(a_n)_{n=1}^\infty$  is eventually positively bounded away from zero. In particular,  $a_i \geq c$  for all  $i \geq N$ . Let  $b_i = c$  when  $i < N$  and  $b_i = a_i$  when  $i \geq N$ . Then  $(b_n)_{n=1}^\infty$  is equivalent to  $(a_n)_{n=1}^\infty$  and  $x = \text{LIM}_{n \rightarrow \infty} a_n = \text{LIM}_{n \rightarrow \infty} b_n$  is a positive real number. Similarly, we can show that if  $a_N < 0$ ,  $x$  would be negative. Thus, at least one of the three statements is true.

At most one of the three statements is true. Suppose  $x$  is zero. For any  $c > 0$ , there exists  $N \geq 1$  such that  $|a_i - 0| = |a_i| \leq \frac{c}{2} < c$ . Therefore,  $(a_n)_{n=1}^\infty$  is not bounded away from zero. Thus,  $x$  is not positive nor negative. Suppose  $x$  is positive. Then there exists  $c > 0$  such that  $a_i > c > 0$  for all  $i \geq 1$ . So for any  $c' > 0$ ,  $a_i > 0 > -c'$ . Therefore,  $x$  cannot be negative. Similarly, if  $x$  is negative, it cannot be positive. Thus, at most one of the three statements is true.

$x$  is negative  $\iff -x$  is positive. We know that  $-x = \text{LIM}_{n \rightarrow \infty} (-a_n)$ . Suppose  $x$  is negative. Then there exists  $c > 0$  such that  $-a_i < -c$  for all  $i \geq 1$ . So  $a_i > c$  for all  $i \geq 1$ . Thus,  $(a_n)_{n=1}^\infty$  is positively bounded away from zero, and  $x = \text{LIM}_{n \rightarrow \infty} a_n$  is positive. Similarly, we can show that if  $-x$  is positive, then  $x$  is negative.

Assume  $y = \text{LIM}_{n \rightarrow \infty} b_n$ . Suppose  $x$  and  $y$  are positive. Then there exist  $c_1, c_2 > 0$  such that  $a_i \geq c_1$  and  $b_i \geq c_2$  for all  $i \geq 1$ . Let  $c = c_1 + c_2$ , we have  $(a_i + b_i) \geq c = c_1 + c_2$  for all  $i \geq 1$ . Therefore,  $x + y$  is positive. Let  $c' = c_1 c_2$ , we have  $a_i b_i \geq c' = c_1 c_2$  for all  $i \geq 1$ . Therefore,  $xy$  is positive.  $\square$

**Exercise 5.4.2.**

Prove the remaining claims in Proposition 5.4.7.

- (a) *Proof.* Since  $x - y$  is a real number, by Proposition 5.4.4, exactly one of the three statements  $x - y = 0$ ,  $x - y > 0$ , or  $x - y < 0$  is true. Thus, exactly one of  $x = y$ ,  $x > y$ , or  $x < y$  is true.  $\square$
- (b) *Proof.* Since  $x - y$  is a real number, by Proposition 5.4.4,  $x - y$  is negative iff  $-(x - y) = y - x$  is positive. Thus,  $x < y$  iff  $y > x$ .  $\square$
- (c) *Proof.* Since  $x < y$ , we have  $y > x$ , hence,  $y - x$  is positive. Similarly, since  $y < z$ ,  $z - y$  is positive. By Proposition 5.4.4,  $z - x = (y - x) + (z - y)$  is positive. Therefore,  $x < z$ .  $\square$
- (d) *Proof.* Since  $x < y$ , we have  $x - y = x - y + 0 = x - y + z - z = (x + z) - (y + z) < 0$ . Therefore,  $x + z < y + z$ .  $\square$

**Exercise 5.4.3.**

Show that for every real number  $x$  there is exactly one integer  $N$  such that  $N \leq x < N + 1$ . (This integer  $N$  is called the integer part of  $x$ , and is sometimes denoted  $N = \lfloor x \rfloor$ .)

*Proof.* Denote  $x = (a_n)_{n=1}^{\infty}$  where  $(a_n)_{n=1}^{\infty}$  is a Cauchy sequence. Let  $\varepsilon = \frac{1}{2}$ . Then there exists an  $N \geq 1$  such that  $|x_i - x_N| \leq \varepsilon = \frac{1}{2}$  for all  $i \geq N$ . Therefore,  $|x - x_N| \leq \frac{1}{2}$ . So

$$-\frac{1}{2} \leq x - x_N \leq \frac{1}{2} \implies x_N - \frac{1}{2} \leq x \leq x_N + \frac{1}{2}.$$

Since  $x_N$  is rational,  $x_N - \frac{1}{2}$  and  $x_N + \frac{1}{2}$  are also rational. Then there exists exactly one integer  $n$  such that

$$n \leq x_N - \frac{1}{2} < n + 1$$

and

$$n + 1 \leq x_N + \frac{1}{2} < n + 2.$$

Then there are two cases. If  $x < n + 1$ , there exists exactly one integer  $N$  such that

$$N = n \leq x_N - \frac{1}{2} \leq x < n + 1 = N + 1.$$

If  $x \geq n + 1$ , there exists exactly one integer  $N$  such that

$$N = n + 1 \leq x \leq x_N + \frac{1}{2} < n + 2 = N + 1.$$

Therefore, for every real number  $x$  there is exactly one integer  $N$  such that  $N \leq x < N + 1$ .  $\square$

**Exercise 5.4.4.**

Show that for any positive real number  $x > 0$  there exists a positive integer  $N$  such that  $x > 1/N > 0$ .

*Proof.* Let  $x = \text{LIM}_{n \rightarrow \infty} a_n$ . Since  $x$  is a positive real number, it is positively bounded away from zero. Then there exists a positive rational number  $c > 0$ , such that  $a_n \geq c > 0$  for  $n \geq 1$ . By Corollary 5.4.10, we have  $x = \text{LIM}_{n \rightarrow \infty} a_n \geq c > 0$ . Since  $c > c/2 > 0$ , we have  $x \geq c > c/2 > 0$ . Let  $N = \frac{2}{c} + 1$ , then  $0 < \frac{1}{N} < \frac{c}{2}$ . Then we have  $x > \frac{1}{N} > 0$ .  $\square$

**Exercise 5.4.5.**

Prove Proposition 5.4.14.

*Proof.*  $y - x > 0 \implies y - x > 0$ . By Exercise 5.4.4, there exists a positive integer  $N$  such that  $y - x > 1/N > 0$ . Then we have  $Ny > Nx + 1 > Nx$ . And by Exercise 5.4.3, there exists exactly one integer  $n$  such that  $n \leq Nx < n + 1$ . Then we also have  $n + 1 \leq Nx + 1 < n + 2$ . Therefore,  $n \leq Nx < n + 1 \leq Nx + 1 < Ny$ . Therefore, there exists an integer between  $Nx$  and  $Ny$ . Divide the inequalities by  $N$ , we have  $x < \frac{n+1}{N} < y$  where  $\frac{n+1}{N}$  is a rational number. Therefore, if  $x < y$ , we can find a rational number  $q$  such that  $x < q < y$ .  $\square$

**Exercise 5.4.6.**

Let  $x, y$  be real numbers and let  $\varepsilon > 0$  be a positive real. Show that  $|x - y| < \varepsilon$  if and only if  $y - \varepsilon < x < y + \varepsilon$ , and that  $|x - y| \leq \varepsilon$  if and only if  $y - \varepsilon \leq x \leq y + \varepsilon$ .

- $|x - y| < \varepsilon \iff y - \varepsilon < x < y + \varepsilon$ .

*Proof.* Suppose  $|x - y| < \varepsilon$ . By definition, if  $x - y > 0$ , we have  $x - y < \varepsilon \implies x < y + \varepsilon$ , and if  $x - y < 0$ , we have  $y - x < \varepsilon \implies y - \varepsilon < x$ . Combining the two inequalities, we have  $y - \varepsilon < x < y + \varepsilon$ .

Suppose  $y - \varepsilon < x < y + \varepsilon$ . Then  $-\varepsilon < x - y < \varepsilon$ . So if  $x - y$  is positive,  $|x - y| = x - y < \varepsilon$ . Otherwise,  $|x - y| = y - x < \varepsilon$ . Therefore,  $|x - y| < \varepsilon$ .

Thus,  $|x - y| < \varepsilon \iff y - \varepsilon < x < y + \varepsilon$ .  $\square$

- $|x - y| \leq \varepsilon \iff y - \varepsilon \leq x \leq y + \varepsilon$ .

*Proof.* The proof is almost identical to the previous one.  $\square$

**Exercise 5.4.7.**

Let  $x$  and  $y$  be real numbers. Show that  $x \leq y + \varepsilon$  for all real numbers  $\varepsilon > 0$  if and only if  $x \leq y$ . Show that  $|x - y| \leq \varepsilon$  for all real numbers  $\varepsilon > 0$  if and only if  $x = y$ .

- $x \leq y + \varepsilon$  for all  $\varepsilon > 0 \iff x \leq y$ .

*Proof.* Suppose  $x \leq y + \varepsilon$  for all  $\varepsilon > 0$ . Assume  $x > y$ . Then we have  $x - y > \frac{x - y}{2} > 0$ . Let  $\varepsilon = \frac{x - y}{2}$ . We have  $x \leq y + \varepsilon = y + \frac{x - y}{2} \implies x \leq y$ . (contradiction) Therefore, we must have  $x \leq y$ .

Suppose  $x \leq y$ . For all  $\varepsilon > 0$ , we have  $x \leq y < y + \varepsilon$ .

Thus,  $x \leq y + \varepsilon$  for all  $\varepsilon > 0 \iff x \leq y$ .  $\square$

- $|x - y| \leq \varepsilon$  for all real numbers  $\varepsilon > 0 \iff x = y$ .



*Proof.* Suppose  $|x - y| \leq \varepsilon$  for all real numbers. By Exercise 5.4.6, we have  $-\varepsilon < x - y < \varepsilon$ . Since  $x < y + \varepsilon$ , we have  $x \leq y$ . Since  $y < x + \varepsilon$ , we have  $y \leq x$ . And since  $y \leq x$  and  $x \leq y$ , we have  $x = y$ .

Suppose  $x = y$ . Then  $|x - y| = 0 \leq \varepsilon$  for all  $\varepsilon > 0$ .

Thus,  $|x - y| \leq \varepsilon$  for all real numbers  $\varepsilon > 0 \iff x = y$ .  $\square$

### Exercise 5.4.8.

Let  $(a_n)_{n=1}^{\infty}$  be a Cauchy sequence of rationals, and let  $x$  be a real number. Show that if  $a_n \leq x$  for all  $n \geq 1$ , then  $\text{LIM}_{n \rightarrow \infty} a_n \leq x$ . Similarly, show that if  $a_n \geq x$  for all  $n \geq 1$ , then  $\text{LIM}_{n \rightarrow \infty} a_n \geq x$ .

*Proof.* Suppose  $a_n \leq x$  for all  $n \geq 1$ . Assume  $\text{LIM}_{n \rightarrow \infty} a_n > x$ . Let  $y = \text{LIM}_{n \rightarrow \infty} a_n$ . If  $y = \text{LIM}_{n \rightarrow \infty} a_n > x$ , by Proposition 5.4.14, there exists a rational number  $q$  such that  $y = \text{LIM}_{n \rightarrow \infty} a_n > q > x$ . On the other hand, we have  $a_n \leq x < q$  for all  $n \geq 1$ . By Corollary 5.4.10, we have  $y = \text{LIM}_{n \rightarrow \infty} a_n \leq q$  which contradicts  $y > q$ . Therefore,  $\text{LIM}_{n \rightarrow \infty} a_n \leq x$ . The second statement can be proved in a similar way.  $\square$

## 5.5 The least upper bound property

### Definition 5.5.1 (Upper bound).

Let  $E$  be a subset of  $\mathbf{R}$ , and let  $M$  be a real number. We say that  $M$  is an upper bound for  $E$ , iff we have  $x \leq M$  for every element  $x$  in  $E$ .

### Definition 5.5.5 (Least upper bound).

Let  $E$  be a subset of  $\mathbf{R}$ , and  $M$  be a real number. We say that  $M$  is a least upper bound for  $E$  iff (a)  $M$  is an upper bound for  $E$ , and also (b) any other upper bound  $M'$  for  $E$  must be larger than or equal to  $M$ .

### Proposition 5.5.8 (Uniqueness of least upper bound).

Let  $E$  be a subset of  $\mathbf{R}$ . Then  $E$  can have at most one least upper bound.

**Theorem 5.5.9 (Existence of least upper bound).**

Let  $E$  be a non-empty subset of  $\mathbf{R}$ . If  $E$  has an upper bound, (i.e.,  $E$  has some upper bound  $M$ ), then it must have exactly one least upper bound.

**Definition 5.5.10 (Supremum).**

Let  $E$  be a subset of the real numbers. If  $E$  is non-empty and has some upper bound, we define  $\sup(E)$  to be the least upper bound of  $E$  (this is well-defined by Theorem 5.5.9). We introduce two additional symbols,  $+\infty$  and  $-\infty$ . If  $E$  is non-empty and has no upper bound, we set  $\sup(E) := +\infty$ ; if  $E$  is empty, we set  $\sup(E) := -\infty$ . We refer to  $\sup(E)$  as the supremum of  $E$ , and also denote it by  $\sup E$ .

**Exercise 5.5.1.**

Let  $E$  be a subset of the real numbers  $\mathbf{R}$ , and suppose that  $E$  has a least upper bound  $M$  which is a real number, i.e.,  $M = \sup(E)$ . Let  $-E$  be the set

$$-E := \{-x : x \in E\}.$$

Show that  $-M$  is the greatest lower bound of  $-E$ , i.e.,  $-M = \inf(-E)$ .

*Proof.* For all  $-x \in -E$ , since  $M \geq x$ , we have  $-M \leq -x$ . Therefore,  $-M$  is a lower bound for  $-E$ . Assume there exists some  $\varepsilon > 0$  such that  $-M + \varepsilon$  is also a lower bound for  $-E$ . Then for all  $-x \in -E$ , we have  $-M + \varepsilon \leq -x \implies M - \varepsilon \geq x$ , hence  $M - \varepsilon$  is an upper bound for  $E$  which contradicts the fact that  $M$  is the least upper bound for  $E$ . Therefore,  $-M$  is the least lower bound for  $-E$ .  $\square$

**Exercise 5.5.2.**

Let  $E$  be a non-empty subset of  $\mathbf{R}$ , let  $n \geq 1$  be an integer, and let  $L < K$  be integers. Suppose that  $K/n$  is an upper bound for  $E$ , but that  $L/n$  is not an upper bound for  $E$ . Without using Theorem 5.5.9, show that there exists an integer  $L < m \leq K$  such that  $m/n$  is an upper bound for  $E$ , but that  $(m-1)/n$  is not an upper bound for  $E$ .

*Proof.* Since  $L < K$  and  $K/n$  is an upper bound for  $E$ , there exists an integer  $m$  such that  $L < m \leq K$  and  $m/n$  is an upper bound for  $E$  (for example, we can let  $m = K$ ). **Suppose for all such  $m$ ,  $(m - 1)/n$  is also an upper bound for  $E$ .** Since such  $(m - 1)/n$  is an upper bound for  $E$ , we must have  $(m - 1) > L$ . (Then since  $(m - 1)/n$  is an upper bound for  $E$  and  $L < (m - 1) \leq K$ ,  $(m - 2)/n$  is also an upper bound for  $E$  and for similar reason,  $L < (m - 2) \leq K$ .)

Let  $P(i)$  be  $(m - i)/n$  is an upper bound for  $E$  (and  $L < m - i \leq K$ ). We want to show that  $P(i)$  is true for all natural number  $i$ . Then the base case is true since there exists an  $m$  that satisfies the conditions. Assume inductively  $P(i)$  is true. Since  $(m - i)/n$  is an upper bound for  $E$  and  $L < (m - i) \leq K$ , by our assumption,  $(m - i - 1)/n = (m - (i + 1))/n$  is also an upper bound for  $E$  and it is less than  $(m - i)/n$ , hence  $K \geq (m - (i + 1)) > L$ . Therefore,  $P(i + 1)$  is true which closes the induction.

Thus,  $L < (m - i) \leq K$  and  $(m - i)/n$  is an upper bound for  $E$  for all natural number  $i$ . Let  $i = \lceil m - L \rceil \geq m - L$ . Then  $(m - i)/n \leq L/n$  is an upper bound for  $E$  which contradicts the fact that  $L/n$  is not an upper bound for  $E$ .

Thus, there exists  $L < m \leq K$  such that  $m/n$  is an upper bound for  $E$ , but that  $(m - 1)/n$  is not an upper bound for  $E$ .  $\square$

### Exercise 5.5.3.

Let  $E$  be a non-empty subset of  $\mathbf{R}$ , let  $n \geq 1$  be an integer, and let  $m, m'$  be integers with the properties that  $m/n$  and  $(m' - 1)/n$  are not upper bounds for  $E$ . Show that  $m = m'$ .

*Proof.* Assume  $m \neq m'$ . Without loss of generality, suppose  $m' > m$ , since  $m, m'$  are integers, we have  $m' > m' - 1 \geq m \implies \frac{m'}{n} > \frac{m' - 1}{n} \geq \frac{m}{n}$ . By Theorem 5.5.9,  $E$  has the least upper bound, denote it by  $M$ . Then we have

$$\frac{m - 1}{n} < M \leq \frac{m}{n} \leq \frac{m' - 1}{n} < \frac{m'}{n}.$$

But since  $(m' - 1)/n$  is not an upper bound for  $E$ ,  $\frac{m' - 1}{n} < M$ , a contradiction. Therefore,  $m = m'$ .  $\square$

**Exercise 5.5.4.**

Let  $q_1, q_2, q_3, \dots$  be a sequence of rational numbers with the property that  $|q_n - q'_n| \leq \frac{1}{M}$  whenever  $M \geq 1$  is an integer and  $n, n' \geq M$ . Show that  $q_1, q_2, q_3, \dots$  is a Cauchy sequence. Furthermore, if  $S := \text{LIM}_{n \rightarrow \infty} q_n$ , show that  $|q_M - S| \leq \frac{1}{M}$  for every  $M \geq 1$ .

*Proof.* We need to show that for every rational  $\varepsilon > 0$ , there exists  $N \geq 1$  such that for  $|x_i - x_j| \leq \varepsilon$  for all  $i, j \geq N$ . When  $\varepsilon \geq 1$ , let  $M = N = 1$ , we have  $|q_i - q_j| \leq 1 \leq \varepsilon$  for all  $i, j \geq N$ . If  $0 < \varepsilon < 1$ , Let  $M = N = \lceil \frac{1}{\varepsilon} \rceil \geq \frac{1}{\varepsilon}$ . Then

$$|q_i - q_j| \leq \frac{1}{\lceil \frac{1}{\varepsilon} \rceil} \leq \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$$

for all  $i, j \geq N$ . Therefore, for every rational  $\varepsilon > 0$ , there exists  $N \geq 1$  such that for  $|x_i - x_j| \leq \varepsilon$  for all  $i, j \geq N$ . Hence,  $q_1, q_2, q_3, \dots$  is a Cauchy sequence.

Let  $N = M$ , we have  $|q_M - q_n| \leq \frac{1}{M}$  for all  $n \geq N$ . Let  $a_n = q_n$  for  $n \geq M$  and  $a_n = q_M$  for  $n < M$ . Then we have  $q_M - \frac{1}{M} \leq a_n \leq q_M + \frac{1}{M}$  for all  $n \geq 1$ . By Exercise 5.4.8,  $q_M - \frac{1}{M} \leq S = \text{LIM}_{n \rightarrow \infty} a_n = \text{LIM}_{n \rightarrow \infty} q_n \leq q_M + \frac{1}{M}$ . Therefore,  $|q_M - S| \leq \frac{1}{M}$  for every  $M \geq 1$ .  $\square$

**Exercise 5.5.5.**

Establish an analogue of Proposition 5.4.14, in which “rational” is replaced by “irrational”.

*Proof.* Assume for any  $x < y$ , there does not exist an irrational number  $z$  such that  $x < z < y$ . Suppose  $x < z < y$  where  $z$  is a rational number. Then  $x + \sqrt{2} < z + \sqrt{2} < y + \sqrt{2}$ . By assumption,  $z + \sqrt{2}$  is a rational number. So  $z + \sqrt{2} = \frac{m}{n}$  for some integers  $m, n$  ( $n \neq 0$ ). Suppose  $z = \frac{a}{b}$  for integers  $a, b$  ( $b \neq 0$ ). Then  $\sqrt{2} = \frac{m}{n} - \frac{a}{b} = \frac{bn - ma}{bn}$  where  $bn - ma$  and  $bn$  are integers ( $bn \neq 0$ ). By definition,  $\sqrt{2}$  is a rational number (a contradiction). Therefore, for any  $x < y$ , there exists an irrational number  $z$  such that  $x < z < y$ .  $\square$

## 5.6 Real exponentiation, part I

### Definition 5.6.1 (Exponentiating a real by a natural number).

Let  $x$  be a real number. To raise  $x$  to the power 0, we define  $x^0 := 1$ . Now suppose recursively that  $x^n$  has been defined for some natural number  $n$ , then we define  $x^{n+1} := x^n \times x$ .

### Definition 5.6.2 (Exponentiating a real by an integer).

Let  $x$  be a non-zero real number. Then for any negative integer  $-n$ , we define  $x^{-n} := 1/x^n$ .

### Proposition 5.6.3.

All the properties in Propositions 4.3.10 and 4.3.12 remain valid if  $x$  and  $y$  are assumed to be real numbers instead of rational numbers.

### Definition 5.6.4.

Let  $x \geq 0$  be a non-negative real, and let  $n \geq 1$  be a positive integer. We define  $x^{1/n}$ , also known as the  $n^{\text{th}}$  root of  $x$ , by the formula

$$x^{1/n} := \sup\{y \in \mathbf{R} : y \geq 0 \text{ and } y^n \leq x\}.$$

### Lemma 5.6.5 (Existence of $n^{\text{th}}$ roots).

Let  $x \geq 0$  be a non-negative real, and let  $n \geq 1$  be a positive integer. Then the set  $E := \{y \in \mathbf{R} : y \geq 0 \text{ and } y^n \leq x\}$  is non-empty and is also bounded above. In particular,  $x^{1/n}$  is a real number.

### Lemma 5.6.6.

Let  $x, y \geq 0$  be non-negative reals, and let  $n, m \geq 1$  be positive integers.

- (a) If  $y = x^{1/n}$ , then  $y^n = x$ .
- (b) Conversely, if  $y^n = x$ , then  $y = x^{1/n}$ .

- (c)  $x^{1/n}$  is a non-negative real number, and is positive iff  $x$  is positive.
- (d) We have  $x > y$  if and only if  $x^{1/n} > y^{1/n}$ .
- (e) If  $x > 1$ , then  $x^{1/k}$  is a decreasing function of  $k$ . If  $x < 1$ , then  $x^{1/k}$  is an increasing function of  $k$ . If  $x = 1$ , then  $x^{1/k} = 1$  for all  $k$ .
- (f) We have  $(xy)^{1/n} = x^{1/n}y^{1/n}$ .
- (g) We have  $(x^{1/n})^{1/m} = x^{1/nm}$ .

**Definition 5.6.7.**

Let  $x > 0$  be a positive real number, and let  $q$  be a rational number. To define  $x^q$ , we write  $q = a/b$  for some integer  $a$  and positive integer  $b$ , and define

$$x^q := (x^{1/b})^a.$$

**Lemma 5.6.8.**

Let  $a, a'$  be integers and  $b, b'$  be positive integers such that  $a/b = a'/b'$ , and let  $x$  be a positive real number. Then we have  $(x^{1/b'})^{a'} = (x^{1/b})^a$ .

**Lemma 5.6.9.**

Let  $x, y > 0$  be positive reals, and let  $q, r$  be rationals.

- (a)  $x^q$  is a positive real.
- (b)  $x^{q+r} = x^q x^r$  and  $(x^q)^r = x^{qr}$ .
- (c)  $x^{-q} = 1/x^q$ .
- (d) If  $q > 0$ , then  $x > y$  if and only if  $x^q > y^q$ .
- (e) If  $x > 1$ , then  $x^q > x^r$  if and only if  $q > r$ . If  $x < 1$ , then  $x^q > x^r$  if and only if  $q < r$ .

**Exercise 5.6.1.**

Prove Lemma 5.6.6.

- (a) *Proof.* We need to show that both  $y^n > x$  and  $y^n < x$  lead to contradictions.

Suppose  $y^n > x$ . Similar to the proof of Proposition 5.5.12, we want to show that there exists a real number  $r$  and  $\varepsilon > 0$  such that  $y^n > (y - \varepsilon)^n \geq y^n - r\varepsilon \geq x$ . Hence  $y$  is not the least upper bound for  $E$ , a contradiction. Use induction to show that there exists a real number  $r \geq 0$  such that  $(y - \varepsilon)^n \geq y^n - r\varepsilon$ . When  $n = 0$ , let  $r = 0$ , we have  $1 \geq 1$  which proves the base case. Assume inductively that the statement is true for  $n$ . Then  $(y - \varepsilon)^{n+1} \geq (y^n - r\varepsilon)(y - \varepsilon) \geq y^{n+1} - (y^n + ry)\varepsilon$  where  $y^n + ry$  is a non-negative real number. Thus, the statement is true for all natural number  $n$ . Let  $\varepsilon \leq \frac{y^n - x}{r}$ , we have  $y^n > (y - \varepsilon)^n \geq y^n - r\varepsilon \geq x$ . We thus have  $y$  not being the least upper bound for  $E$ , a contradiction.

Suppose  $y^n < x$ . We want to show that there exists positive numbers  $\varepsilon, r$  such that  $y^n < (y + \varepsilon)^n \leq y^n + r\varepsilon \leq x$  so that  $y$  is not an upper bound for  $E$ . Since we have  $(y + \varepsilon)^n \leq y^n + ny^{n-1}\varepsilon$ , for any  $\varepsilon > 0$ , we can let  $r = ny^{n-1}$ . For any fixed  $r$ , we let  $0 < \varepsilon \leq \frac{x - y^n}{r}$ , so  $y^n + r\varepsilon \leq x$ . Therefore,  $y^n$  is not an upper bound for  $E$ , a contradiction.

Thus,  $y^n = x$ . □

- (b) *Proof.* Since  $y^n = x$ , for every  $y' \in E$ , we have  $x = y^n \geq y' \implies y$  is an upper bound for  $E$ . And  $\forall \varepsilon > 0$ ,  $(y - \varepsilon)^n < y^n = x$ . Therefore, for all  $\varepsilon > 0$ ,  $(y - \varepsilon)$  is not an upper bound for  $E$ . Hence  $y$  is the least upper bound for  $E$ , by definition, we have  $y = x^{1/n}$ . □

- (c) *Proof.* For all  $y \in E$ , we have  $x \geq y \geq 0$ . Therefore,  $x^{1/n}$  is non-negative. Suppose  $x^{1/n} > 0$ . We have  $y = x^{1/n} > 0$ . By (a), we have  $x = y^n > 0$ . Suppose  $x > 0$ . We then have  $y^n = x > 0$ . So  $y = x^{1/n} > 0$ . Thus,  $x^{1/n} > 0 \iff x > 0$ . □

- (d) *Proof.* Denote  $x = a^n$  and  $y = b^n$ . If  $x = a^n > b^n = y$ , we have  $x^{1/n} = a > b = y^{1/n}$  (otherwise there will be contradictions). On the other hand, if  $a > b$ , we have  $x = a^n > b^n = y$ . Thus,  $x > y \iff x^{1/n} > y^{1/n}$ . □

- (e) *Proof.*  $x > 1$ . Suppose  $k_1 > k_2$ , then  $k_1 \geq k_2 + 1$ .  $x^{1/k} = y_1 \iff y_1^{k_1} = x$ ,  $x^{1/k} = y_1 \iff y_1^{k_1} = x$ , the remaining part is to show that  $y_1 < y_2$ . Since  $x > 1$ , we have  $y_1, y_2 > 1$ . Then

$$y_2^{k_2} = y_1^{k_1} \geq y_1^{k_2+1} = y_1^{k_2} \cdot y_1, \\ \left(\frac{y_2}{y_1}\right)^{k_2} \geq y_1 > 1.$$

Therefore,

$$\frac{y_2}{y_1} > 1 \implies y_1 < y_2.$$

Thus,  $x^{1/k}$  is a decreasing function of  $k$  when  $x > 1$ .

$x < 1$ . In this case, we have  $0 \leq y_1, y_2 < 1$ , and we need to show that  $y_1 > y_2$ . Then

$$y_2^{k_2} = y_1^{k_1} \leq y_1^{k_2+1} = y_1^{k_2} \cdot y_1, \\ \left(\frac{y_2}{y_1}\right)^{k_2} \leq y_1 < 1.$$

Therefore,

$$\frac{y_2}{y_1} < 1 \implies y_1 > y_2.$$

Thus,  $x^{1/k}$  is an increasing function of  $k$  when  $x > 1$ .

If  $x = 1$ . we have  $y^n = 1$  so  $y = 1$ . Therefore,  $x^{1/n} = y = 1$ .  $\square$

- (f) *Proof.* Denote  $(xy)^{1/n}$  by  $a$ . Then  $a = (xy)^{1/n} \iff a^n = xy$ . Suppose  $b^n = x$  and  $c^n = y$ . We need to show that  $a = bc$ . By substitution, we have  $a^n = b^n c^n = (bc)^n$ , hence  $a = bc$ . Therefore,  $(xy)^{1/n} = x^{1/n} y^{1/n}$ .  $\square$
- (g) *Proof.* Let  $a = x^{1/nm}$ , then  $a^{nm} = x$ . Let  $b = (x^{1/n})^{1/m}$ , then  $b^m = x^{1/n}$ . We want to show that  $a = b$ . Since  $(b^m)^n = b^{mn} = b^{nm} = a^{nm}$ , we have  $a = b$ . Thus,  $(x^{1/n})^{1/m} = x^{1/nm}$ .  $\square$

### Exercise 5.6.2.

Prove Lemma 5.6.9.

- (a) *Proof.* Write  $q = a/b$  for some integer  $a$  and some positive integers  $b$ . Then



$x^q = x^{a/b} = (x^{1/b})^a$ . By Lemma 5.6.6, since  $x > 0$ ,  $x^{1/b} > 0$ . Therefore,  $x^q = (x^{1/b})^a > 0$ . Thus,  $x^q$  is a positive real.  $\square$

(b) *Proof.* Write  $q = a/b$  and  $r = c/d$  for some integers  $a, c$  and some positive integers  $b, d$ . Then

$$\begin{aligned}
 x^{q+r} &= (x^{\frac{a}{b} + \frac{c}{d}}) \\
 &= x^{\frac{ad+bc}{bd}} \\
 &= (x^{\frac{1}{bd}})^{ad+bc} \\
 &= x^{\frac{ad}{bd}} \cdot x^{\frac{bc}{bd}} \\
 &= x^{\frac{a}{b}} \cdot x^{\frac{c}{d}} \\
 &= x^q \cdot x^r. \\
 ((x^q)^r)^{bd} &= ((x^{\frac{a}{b}})^{\frac{c}{d}})^{bd} \\
 &= (((x^{\frac{1}{b}})^a)^{\frac{1}{d}})^{c}^{bd} \\
 &= (((x^{\frac{1}{b}})^a)^{\frac{1}{d}})^{d}^{bc} \\
 &= ((x^{\frac{1}{b}})^a)^{bc} \\
 &= ((x^{\frac{1}{b}})^b)^{ac} \\
 &= x^{ac} \\
 &= (x^{\frac{ac}{bd}})^{bd} \\
 &= (x^{qr})^{bd},
 \end{aligned}$$

hence  $(x^q)^r = x^{qr}$ .  $\square$

(c) *Proof.* Write  $q = a/b$  for some integer  $a$  and some positive integer  $b$ . Then

$$\begin{aligned}
 x^{-q} &= x^{-\frac{a}{b}} \\
 &= (x^{\frac{1}{b}})^{-a} \\
 &= \frac{1}{(x^{\frac{1}{b}})^a} \\
 &= \frac{1}{x^q}.
 \end{aligned}$$

$\square$

(d) *Proof.* Write  $q = a/b$  for some positive integers  $a, b$ .

Suppose  $x > y$ . By Lemma 5.6.6, we have  $x^{\frac{1}{b}} > y^{\frac{1}{b}}$ . By Proposition 4.3.10, we have  $(x^{\frac{1}{b}})^a > (y^{\frac{1}{b}})^a$ . Therefore,  $x^q = (x^{\frac{1}{b}})^a > (y^{\frac{1}{b}})^a = x^r$ .

Suppose  $x^q = (x^{\frac{1}{b}})^a > (y^{\frac{1}{b}})^a = x^r$ . By Lemma 5.6.6,  $x^{\frac{1}{b}} = ((x^{\frac{1}{b}})^a)^{1/a} > ((y^{\frac{1}{b}})^a)^{1/a} = y^{1/b}$ . Apply Lemma 5.6.6 once again, we have  $x > y$ .  $\square$

(e) *Proof.* Write  $q = a/b$  and  $r = c/d$  for some integers  $a, c$  and some positive integers  $b, d$ .  $q - r = \frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$ .

$x > 1$ .  $x^q = x^{\frac{a}{b}} > x^{\frac{c}{d}} = x^r \iff (x^{\frac{a}{b}})^{\frac{1}{ac}} = x^{\frac{1}{bc}} > x^{\frac{1}{ad}} = (x^{\frac{c}{d}})^{\frac{1}{ac}} \iff bc < ad$  (since  $x > 1$ ). Therefore,  $x^q \iff q > r$ .

$x < 1$ .  $x^q = x^{\frac{a}{b}} > x^{\frac{c}{d}} = x^r \iff (x^{\frac{a}{b}})^{\frac{1}{ac}} = x^{\frac{1}{bc}} > x^{\frac{1}{ad}} = (x^{\frac{c}{d}})^{\frac{1}{ac}} \iff bc > ad$  (since  $x < 1$ ). Therefore,  $x^q \iff q < r$ .  $\square$

### Exercise 5.6.3.

If  $x$  is a real number, show that  $|x| = (x^2)^{1/2}$ .

*Proof.* If  $x > 0$ ,  $(x^2)^{1/2} = x = |x|$ . If  $x < 0$ ,  $(x^2)^{1/2} = ((-x)^2)^{1/2} = -x = |x|$ . If  $x = 0$ ,  $(x^2)^{1/2} = 0 = |0|$ . Thus,  $|x| = (x^2)^{1/2}$ .  $\square$