Chapter 4

Integers and rationals

4.1 The integers

Definition 4.1.1 (Integers).

An integer is an expression of the form $a - -b$, where a and b are natural numbers. Two integers are considered to be equal, $a - -b = c - -d$, if and only if $a + d = c + b$. We let $\mathbf{Z}$ denote the set of all integers.

Definition 4.1.2

The sum of two integers, $(a - -b) + (c - -d)$, is defined by the formula

$$(a - -b) + (c - -d) := (a + c) - (b + d).$$

The product of two integers, $(a - -b) \times (c - -d)$, is defined by

$$(a - -b) \times (c - -d) := (ac + bd) - -(ad + bc).$$

Lemma 4.1.3 (Addition and multiplication are well-defined).

Let a, b, a', b', c, d be natural numbers. If $(a - -b) = (a' - -b')$, then $(a - -b) + (c - -d) = (a' - -b') + (c - -d)$ and $(a - -d) \times (c - -d) = (a' - -b') \times (c - -d)$, and also $(c - -d) + (a - -b) = (c - -d) + (a' - -b')$ and $(c - -d) \times (a - -b) = (c - -d) \times (a' - -b')$. Thus addition and multiplication are well-defined operations (equal inputs give equal outputs).

Definition 4.1.4 (Negation of integers).

If $(a - -b)$ is an integer, we define the negation $-(a - -b)$ to be the integer $(b - -a)$. In particular if $n = n - -0$ is a positive natural number, we can define its negation $-n = 0 - -n$.

Lemma 4.1.5 (Trichotomy of integers).

Let x be an integer. Then exactly one of the following three statements is true: (a) x is zero; (b) x is equal to a positive natural number n; or (c) x is the negation $-n$ of a positive natural number n.

Proposition 4.1.6 (Laws of algebra for integers).

Let x, y, z be integers. Then we have

$$x + y = y + x$$
$$(x + y) + z = x + (y + z)$$
$$x + 0 = 0 + x = x$$
$$x + (-x) = (-x) + x = 0$$
$$xy = yx$$
$$(xy)z = x(yz)$$
$$x1 = 1x = x$$
$$x(y + z) = xy + xz$$
$$(y + z)x = yx + zx$$

Proposition 4.1.8 (Integers have no zero divisors).

Let a and b be integers such that $ab = 0$. Then either $a = 0$ or $b = 0$ (or both).

Corollary 4.1.9 (Cancellation law for integers).

If a, b, c are integers such that $ac = bc$ and c is non-zero, then $a = b$.

Definition 4.1.10 (Ordering of the integers).

If n and m be integers. We say that n is greater than or equal to m, and write $n \geq m$ or $m \leq n$, iff we have $n = m + a$ for some natural number a. We say that n is strictly greater than m, and write $n > m$ or $m < n$, iff $n \geq m$ and $n \neq m$.

Lemma 4.1.11 (Properties of order).

Let a, b, c be integers.

(a) $a > b$ if and only if $a - b$ is a positive natural number.

(b) (Addition preserves order) If $a > b$, then $a + c > b + c$.

(c) (Positive multiplication preserves order) If $a > b$ and c is positive, then $ac > bc$.

(d) (Negation reverses order) If $a > b$ and $b > c$, then $a > c$.

(e) (Order trichotomy) Exactly one of the statements $a > b$, $a < b$, or $a = b$ is true.

Exercise 4.1.1

Verify that the definition of equality on the integers is both reflexive and symmetric.

Proof. Reflexivity: since summation is reflexive, we have $a + b = a + b$. Thus, by definition, $a - -b = a - -b$. Symmetry: assume $a - -b = c - -d$, then $a + d = c + b$. Since summation is symmetric, $c + b = a + d$. By definition, we have $c - -d = a - -b$. $\square$

Exercise 4.1.2

Show that the definition of negation on the integers is well-defined in the sense that $(a - -b) = (a' - -b')$, then $-(a - -b) = -(a' - -b')$ (so equal integers have equal negations).

Proof. Since $(a - -b) = (a' - -b')$, by definition, $a + b' = a' + b$. By the reflexivity and symmetry of summation, we have $b + a' = b' + a$. Thus, by definition, $b - -a = b' - -a'$. By definition of negation of integers, $-(a - -b) = -(a' - -b')$. $\square$

Exercise 4.1.3

Show that $(-1) \times a = -a$ for every integer a.

Proof. By definition, $-1 = (0 - -1)$ and $a = (a - -0)$. Then $(-1) \times a = (0 - -1) \times (a - -0) = (0 \times a + 1 \times 0) - -(0 \times 0 + 1 \times a) = 0 - -a = -a$. $\square$

Exercise 4.1.4

Prove the remaining identities in Proposition 4.1.6.

1. $x + y = y + x$.

   Proof. Suppose $x = a - -b$ and $y = c - -d$ for some natural numbers a, b, c, d. Then $x + y = (a - -b) + (c - -d) = (a + c) - -(b + d)$ and $y + x = (c - -d) + (a - -b) = (c + a) - -(d + b)$. By the symmetry property of summation, we have $(a + c) = (c + a)$ and $(b + d) = (d + b)$. Thus, $x + y = y + x$. □

2. $(x + y) + z = x + (y + z)$.

   Proof. Suppose $x = a - -b$, $y = c - -d$, and $z = e - -f$ for some natural numbers $a, b, c, d, e, f,$. Then

$$\begin{aligned}
(x + y) + z &= ((a - -b) + (c - -d)) + (e - -f) \\
&= ((a + c) - -(b + d)) + (e - -f) \\
&= ((a + c) + e) - -((b + d) + f) \\
&= (a + c + e) - -(b + d + f); \\
x + (y + z) &= (a - -b) + ((c - -d) + (e - -f)) \\
&= (a - -b) + ((c + e) - -(d + f)) \\
&= (a + (c + e)) - -(b + (d + f)) \\
&= (a + c + e) - -(b + d + f).
\end{aligned}$$

   Therefore, $(x + y) + z = x + (y + z)$. □

3. $x + 0 = 0 + x = x$.

   Proof. Since $x + y = y + x$, we have $x + 0 = 0 + x$. Let $x = a - -b$ for some natural numbers a, b, and write $0 = 0 - -0$. Then $x + 0 = (a - -b) + (0 - -0) = (a + 0) - -(b + 0) = a - -b = x$. Thus, $x + 0 = 0 + x = x$. □

4. $x + (-x) = (-x) + x = 0$.

Proof. Since $x + y = y + x$, we have $x + (-x) = (-x) + x$. Let $x = a - -b$ for some natural numbers a, b, then $-x = b - -a$. Write 0 as $0 - -0$. Then $x + (-x) = (a - -b) + (b - -a) = (a + b) - -(b + a)$. Since $(a + b) + 0 = (b + a) + 0 = a + b$, we have that $(a + b) - -(b + a) = 0 - -0$. So $x + (-x) = 0$. Thus, $x + (-x) = (-x) + x = 0$. $\square$

5. $xy = yx$.

Proof. Let $x = a - -b$ and $y = c - -d$ for some natural numbers a, b, c, d. Then

$$\begin{aligned} xy &= (a - -b) \times (c - -d) \\ &= (ac + bd) - -(ad + bc); \\ yx &= (c - -d) \times (a - -b) \\ &= (ca + db) - -(cb + da) \\ &= (ac + bd) - -(ad + bc). \end{aligned}$$

Therefore, $xy = yx$. $\square$

6. $(xy)z = x(yz)$.

Has been proved on page 79.

7. $x1 = 1x = x$.

Proof. Since $xy = yx$, we have $x1 = 1x$. Let $x = a - -b$ for some natural numbers a, b. $1x = (1 - -0)(a - -b) = 1a - -1b = a - -b = x$. Thus, $x1 = 1x = x$. $\square$

8. $x(y + z) = xy + xz$.

Proof. Let $x = a - -b$, $y = c - -d$, and $z = e - -f$ for some natural numbers a, b, c, d, e, f. Then

$$\begin{aligned}
x(y+z) &= (a - -b)((c - -d) + (e - -f)) \\
&= (a - -b)((c+e) - -(d+f)) \\
&= (a(c+e) + b(d+f)) - -(a(e+f) + b(c+d)) \\
&= (ac + ae + bd + bf) - -(ae + af + bc + bd); \\
xy + xz &= (a - -b)(c - -d) + (a - -b)(e - -f) \\
&= ((ac+bd) - -(ad+bc)) + ((ae+bf) - -(af+be)) \\
&= ((ac+bd) + (ae+bf)) - -((ad+bc) + (af+be)) \\
&= (ac + ae + bd + bf) - -(ae + af + bc + bd).
\end{aligned}$$

Therefore, $x(y+z) = xy + xz$. □

9. $(y+z)x = yx + zx$.

Proof. Since $xy = yx$, we have $(y+z)x = x(y+z)$. By using identities, we get $xy + xz = yx + zx$. And because $x(y+z) = xy + xz$, $(y+z)x = yx + zx$. □

Exercise 4.1.5

Prove Proposition 4.1.8.

Proof. From now on we could just use $-$ instead of $--$. Let $a = c - d$ and $b = e - f$ for some natural numbers c, d, e, f. So $ab = 0 \implies (c-d)(e-f) = 0$. Assume $c - d \geq 0$ and $e - f \geq 0$, by Lemma 2.3.3, at least one of $a = (c-d)$ and $b = e - f$ is equal to 0. If at least one of $(c-d)$ and $(e-f)$ is negative, without loss of generality, assume $c - d < 0$. Then $-(c-d) = d - c > 0$ and we have $(d-c)(e-f) = -1 \times 0 = 0$. By Lemma 2.3.3, at least one of $-a = d - c$ and $b = e - f$ is equal to zero, and this statement is equivalent to either $a = 0$ or $b = 0$ (or both). □

Exercise 4.1.6

Prove Corollary 4.1.9.

Proof. $ac = bc \implies ac - bc = ac + (-b)c = (a + (-b))c = (a - b)c = 0$ by Proposition 4.1.6. By Proposition 4.1.8, at least one of $(a - b)$ and c is equal to 0. Since $c \neq 0$, $a - b = 0$. Thus, $a = b$. $\square$

Exercise 4.1.7

Prove Lemma 4.1.11.

(a) *Proof.* We need to show that $a > b \iff a - b$ is a positive natural number. Suppose $a > b$. By definition, there exists a positive natural number n such that $a = b + n$. So $a - b = n > 0$ as required. Suppose $a - b$ is a positive natural number. Then $a - b = n \iff a = b + n$ for some positive integer n. Thus, $a > b$. $\square$

(b) *Proof.* Since $a > b$, there exists a positive natural number n such that $a - b = n$. Then $(a + c) - b = c + n \iff (a + c) = (b + c) + n$. Therefore, $a + c > b + c$. $\square$

(c) *Proof.* Since $a > b$, there exists a positive natural number n such that $a - b = n$. Since $(a - b)$ and n are both natural numbers, we have $c(a - b) = cn$ for any positive integer c. Then $ac = bc + cn$ where cn is a positive natural number. Therefore, $ac > bc$. $\square$

(d) *Proof.* Since $a > b$, there exists a positive natural number n such that $a - b = n$. Then $-b = -a + n$. Since $n > 0$, $-b > -a$. $\square$

(e) *Proof.* Since $a > b$, there exists a positive natural number n such that $a - b = n$. Since $b > c$, there exists a positive natural number m such that $b - c = m$. Then $(a - b) + (b - c) = a - c = n + m$ where $(n + m)$ is a positive natural number. Therefore, $a > c$. $\square$

(f) *Proof.* Since $a - b$ is an integer, by Lemma 4.1.5, exactly one of the following three statement is true:

(a) $a - b$ is zero. Then $a = b$.

(b) $a - b$ is equal to a positive natural number n. $a - b = n$, so $a > b$.

(c) $-(a - b) = b - a$ is equal to a positive natural number n. $b - a = n$, so $b > a$ which is equivalent to $a < b$.

□

Exercise 4.1.8

Show that the principle of induction does not apply directly to the integers. More precisely, give an example of a property $P(n)$ pertaining to an integer n such that $P(0)$ is true, and that $P(n)$ implies $P(n++)$ for all integers n, but that $P(n)$ is not true for all integers n. Thus induction is not as useful a tool for dealing with the integers as it is with the natural numbers.

Proof. A counterexample of $P(n)$ could be $f(n) = n^2$ is a monotonically increasing function. □

4.2 The rationals

Definition 4.2.1

A rational number is an expression of the form $a//b$, where a and b are integers and b is non-zero; $a//0$ is not considered to be a rational number. Two rational numbers are considered to be equal, $a//b = c//d$, if and only if $ad = cb$. The set of all rational numbers is denoted **Q**.

Definition 4.2.2

If $a//b$ and $c//d$ are rational numbers, we define their sum

$$(a//b) + (c//d) := (ad + bc)//(bd)$$

their product

$$(a//b) * (c//d) := (ac)//(bd)$$

and the negation

$$-(a//b) := (-a)//b.$$

Lemma 4.2.3

The sum, product, and negation operations on rational numbers are well-defined, in the sense that if one repalce $a//b$ with another rational number $a'//b'$ which is equal to $a//b$, then the output of the above operations remains unchanged, and similarly for $c//d$.

Proposition 4.2.4 (Laws of algebra for rationals).

Let x, y, z be rationals. Then the following laws of algebra hold:

$$x + y = y + x$$
$$(x + y) + z = x + (y + z)$$
$$x + 0 = 0 + x = x$$
$$x + (-x) = (-x) + x = 0$$
$$xy = yx$$
$$(xy)z = x(yz)$$
$$x1 = 1x = x$$
$$x(y + z) = xy + xz$$
$$(y + z)x = yx + zx.$$

If x is non-zero, we also have

$$xx^{-1} = x^{-1}x = 1.$$

Definition 4.2.6

A rational number x is said to be positive iff we have $x = a/b$ for some positive integers a and b. It is said to be negative iff we have $x = -y$ for some positive rational y (i.e., $x = (-a)/b$ for some positive integers a and b).

Lemma 4.2.7 (Trichotomy of rationals).

Let x be a rational number. Then exactly one of the following three statements is true: (a) x is equal to 0. (b) x is a positive rational number. (c) x is a negative rational number.

Definition 4.2.8 (Ordering of the rationals).

Let x and y be rational numbers. We say that $x > y$ iff $x - y$ is a positive rational number, and $x < y$ iff $x - g$ is a negative rational number. We write $x \geq y$ iff either $x > y$ or $x = y$, and similarly define $x \leq y$.

Proposition 4.2.9 (Basic properties of order on the rationals).

Let x, y, z be rational numbers. Then the following properties hold.

(a) (Order trichotomy) Exactly one of the three statements $x = y$, $x < y$, or $x > y$ is true.

(b) (Order is anti-symmetric) One has $x < y$ if and only if $y > x$.

(c) (Order is transitive) If $x < y$ and $y < z$, then $x < z$.

(d) (Addition preserves order) If $x < y$, then $x + z < y + z$.

(e) (Positive multiplication preserves order) If $x < y$ and z is positive, then $xz < yz$.

Exercise 4.2.1

Show that the definition of equality for the rational numbers is reflexive, symmetric, and transitive.

Proof. Reflexivity: suppose a, b are some natural numbers. Since $ab = ab$, by definition, we have $a//b = a//b$. Symmetry: suppose we have $a//b = c//d$ for some natural numbers a, b, c, d. Since $ad = cb$ implies $cb = ad$, by definition, we have $c//d = a//b$. Transitivity: suppose we have $a//b = c//d$ and $c//d = e//f$ for some natural numbers a, b, c, d, e, f. Then we have $ad = cb$ and $cf = ed$. So $adf = cbf$, then $adf = (af)d = b(cf) = b(ed) = (eb)d$. Since $d \neq 0$, by Corollary 4.1.9, $af = eb$. By definition, $a//b = e//f$. $\square$

Exercise 4.2.2

Prove the remaining components of Lemma 4.2.3.

Proof. Multiplication: suppose $a//b = a'//b'$ where a, b, a', b' are some natural numbers. We want to show that $(a//b) * (c//d) = (a'//b') * (c//d)$. Since $a//b = a'//b'$, we have $ab' = a'b$. Then $(ab')(cd) = (a'b)(cd)$, by identities, we have $(ac)(b'd) = (bd)(a'c)$. By definition of equality for the rationals, we have $(ac)//(bd) = (a'c)//(b'd)$. Thus, $(a//b) * (c//d) = (a'//b') * (c//d)$.

Negation: suppose $a//b = a'//b'$ where a, b, a', b' are some natural numbers. Since $ab' = a'b$, we have $(-a)b' = -ab' = -a'b = (-a')b$. Then by definition, we have $-(a//b) = (-a)//b = (-a')//b' = -(a'//b')$ as required. $\square$

Exercise 4.2.3

Prove the remaining components of Proposition 4.2.4.

1. $x + y = y + x$.

   Proof. Let $x = a//b$ and $y = c//d$ for some integers a, b, c, d and $b, d \neq 0$. Then

$$\begin{aligned} x + y &= a//b + c//d \\ &= (ad + bc)//bd \\ y + x &= c//d + a//b \\ &= (cb + da)//db \\ &= (ad + bc)//bd. \end{aligned}$$

   Therefore, $x + y = y + x$. $\square$

2. $(x + y) + z = x + (y + z)$.

   Proof. The proof is on page 84. $\square$

3. $x + 0 = 0 + x = x$.

Proof. Since $x + y = y + x$ for rational numbers x, y, we have $x + 0 = 0 + x$. Let $x = a//b$ for some integers a, b and $b \neq 0$. Write 0 as $0//1$. Then

$$\begin{aligned} x + 0 &= a//b + 0//1 \\ &= (a + 0)//b \\ &= a//b \\ &= x. \end{aligned}$$

Therefore, $x + 0 = 0 + x = 0$ □

4. $x + (-x) = (-x) + x = 0$.

Proof. Since $x+y = y+x$, $x+(-x) = (-x)+x$. Let $x = a//b$ for some integers a, b and $b \neq 0$. Then $-x = -(a//b)$.

$$\begin{aligned} x + (-x) &= a//b - (a//b) \\ &= a//b + (-a)//b \\ &= (ab + (-a)b)//b^2 \\ &= 0. \end{aligned}$$

Therefore, $x + (-x) = (-x) + x = 0$. □

5. $xy = yx$.

Proof. Let $x = a//b$ and $y = c//d$ for some integers a, b, c, d and $b, d \neq 0$. Then

$$\begin{aligned} xy &= (ac)//(bd) \\ yx &= (ca)//(db) \\ &= (ac)//(bd). \end{aligned}$$

Therefore, $xy = yx$. □

6. $(xy)z = x(yz)$.

Proof. Let $x = a//b$, $y = c//d$, and $z = e//f$ for some integers a, b, c, d, e, f and $b, d, e \neq 0$. Then

$$\begin{aligned}
(xy)z &= ((a//b) * (c//d)) * (e//f) \\
&= (ac//bd) * (e//f) \\
&= ((ac)e)//((bd)f) \\
&= (ace)//(bdf) \\
x(yz) &= (a//b) * ((c//d) * (e//f)) \\
&= (a//b) * ((ce)//(df)) \\
&= (a(ce))//(b(df)) \\
&= (ace)//bdf.
\end{aligned}$$

Therefore, $(xy)z = x(yz)$. □

7. $x1 = 1x = x$.

Proof. Since $xy = yx$, we have $x1 = 1x$. Let $x = a//b$ for some integers a, b and $b \neq 0$. Then

$$\begin{aligned}
x1 &= (a//b) * (1//1) \\
&= (a1)//(b1) \\
&= a//b \\
&= x.
\end{aligned}$$

Thus, $x1 = 1x = x$. □

8. $x(y + z) = xy + xz$.

Proof. Let $x = a//b$, $y = c//d$, and $z = e//f$ for some integers a, b, c, d, e, f

and $b, d, e \neq 0$. Then

$$\begin{aligned}
x(y+z) &= (a//b) * ((cf+de)//(df)) \\
&= (a(ef+de))//(bdf) \\
&= (acf+ade)//(bdf) \\
xy+xz &= (a//b)*(c//d)+(a//b)*(e//f) \\
&= (ac)//(bd)+(ae)//(bf) \\
&= (acbf+bdae)//(b^2df) \\
&= (acf+ade)//(bdf).
\end{aligned}$$

Thus, $x(y+z) = xy + xz$. □

9. $(y+z)x = yx + zx$.

   Proof. Since $xy = yx$, we have $x(y+z) = (y+z)x$. By using identities, we have $xy + xz = yx + zx$. Therefore, since $x(y+z) = xy + xz$, we have $(y+z)x = yx + zx$. □

10. If x is non-zero, then $xx^{-1} = x^{-1}x = 1$.

    Proof. Let $x = a//b$ where a, b are non-zero integers. Then $x^{-1} = b//a$. Since $xy = yx$, we have $xx^{-1} = x^{-1}x$. Then $xx^{-1} = (ab)//(ba)$. Since $(ab)1 = 1(ba)$, we have $xx^{-1} = (ab)//(ba) = 1//1 = 1$. Thus, $xx^{-1} = x^{-1}x = 1$. □

Exercise 4.2.4

Prove Lemma 4.2.7.

Proof. Let $x = a//b$ where a, b are integers and $b \neq 0$. Consider all the possible combinations of a and b:

- $a = 0$. $x = 0//b = 0$.
- $a > 0$, $b > 0$. By definition, $x = a//b$ is positive.

- $a > 0$, $b < 0$. Then $-b > 0$. So $x = -(a//(-b))$ where $a//(-b)$ is a positive rational number. Therefore, x is negative.

- $a < 0$, $b > 0$. Similarly, we can show that $x = a//b$ is negative.

- $a < 0$, $b < 0$. Then $-a > 0$ and $-b > 0$. Since $x = a//b = (-a)//(-b)$, by definition, x is positive.

So we have proved that at least one of the statements is true. Then we need to check that at most one of them is true. Assume $a = 0$. Then by Trichotomy of integers, a cannot be positive nor negative. So by definition, $x = a//b$ cannot be positive nor negative. Thus, if a rational number is 0, it cannot be positive nor negative. Then we need to show that a rational number cannot be positive and negative at the same time. Assume $x = a//b$ is positive, then by definition, $a > 0$, $b > 0$. Assume x is also negative, so there exists a positive $-y = -c//d = x$. Then $ad = -bc$ which leads to an integer being negative and positive at the same time (contradiction). Thus, a rational number cannot be positive and negative at the same time. Therefore, at most of the three statements is true. Hence, exactly one of the three statements is true. $\square$

Exercise 4.2.5

Prove Proposition 4.2.9.

(a) *Proof.* $x - y$ is a rational number, by Lemma 4.2.7, exactly one of $x - y = 0$, $x - y > 0$, or $x - y < 0$ is true. Thus, exactly one of the three statements $x = y$, $x < y$, or $x > y$ is true. $\square$

(b) *Proof.* Assume $x < y$. So $x - y = r$ is a negative rational number. Then $-r$ is positive and $y - x = -r$. Therefore, $y > x$. Assume $y > x$. So $y - x = r$ is a positive rational number. Then $-r$ is negative and $x - y = -r$. Therefore, $x < y$. $\square$

(c) *Proof.* $x < y \implies y - x = r$ where r is a positive rational number. $y < z \implies z - y = s$ where s is a positive rational number. Then $z - x = z - (y + r) = s + r$ which is also a positive rational number. Thus, $x < z$. $\square$

(d) *Proof.* $x < y \implies y - x = r$ where r is a positive rational number. Then $(y + z) - (x + z) = r > 0$. Therefore, $x + z < y + z$. □

(e) *Proof.* $x < y \implies y - x = r$ where r is a positive rational number. We have $(y - x)z = yz - xz = zr > 0$. Therefore, $xz < yz$. □

Exercise 4.2.6

Show that if x, y, z are rational numbers such that $x < y$ and z is negative, then $xz > yz$.

Proof. $x < y \implies y - x = r$ where r is a positive rational number. Since z is negative, $-z$ is positive. Then $(y - x)(-z) = -(y - x)z = (x - y)z = xz - yz = (-z)r > 0$. Therefore, $xz > yz$. □

4.3 Absolute value and exponentiation

Definition 4.3.1 (Absolute value).

If x is a rational number, the absolute value $|x|$ of x is defined as follows. If x is positive, then $|x| := x$. If x is negative, then $|x| := -x$. If x is zero, then $|x| := 0$.

Definition 4.3.2 (Distance).

Let x and y be rational numbers. The quantity $|x - y|$ is called the distance between x and y and is sometimes denoted $d(x, y)$, thus $d(x, y) := |x - y|$. For instance, $d(3, 5) = 2$.

Proposition 4.3.3 (Basic properties of absolute value and distance).

Let x, y, z be rational numbers.

(a) (Non-degeneracy of absolute value) We have $|x| \geq 0$. Also, $|x| = 0$ if and only if x is 0.

(b) (Triangle inequality for absolute value) We have $|x + y| \leq |x| + |y|$.

(c) We have the inequalities $-y \leq x \leq y$ if and only if $y \geq |x|$. In particular, we have $-|x| \leq x \leq |x|$.

(d) (Multiplicativity of absolute value) We have $|xy| = |x||y|$. In particualr, $|-x| = |x|$.

(e) (Non-degeneracy of distance) We have $d(x, y) \geq 0$. Also, $d(x, y) = 0$ if and only if $x = y$.

(f) (Symmetry of distance) $d(x, y) = d(y, x)$.

(g) (Triangle inequality for distance) $d(x, z) \leq d(x, y) + d(y, z)$.

Definition 4.3.4 (ε-closeness).

Let $\varepsilon > 0$ be a rational number, and let x, y be raional numbers. We say that y is ε-close to x iff we have $d(y, x) < \varepsilon$.

Proposition 4.3.7

Let x, y, z, w be rational numbers.

(a) If $x = y$, then x is ε-close to y for every $\varepsilon > 0$. Conversely, if x is ε-close to y for every $\varepsilon > 0$, then we have $x = y$.

(b) Let $\varepsilon > 0$. If x is ε-close to y, then y is ε-close to x.

(c) Let ε , $\delta > 0$. If x is ε-close to y, and y is δ-close to z, then x and z are $(\varepsilon + delta)$-close.

(d) Let ε , $\delta > 0$. If x and y are ε-close , and z and w are δ-close , then $x + z$ and $y + w$ are $(\varepsilon + \delta)$-close, and $x - z$ and $y - w$ are also $(\varepsilon + \delta)$-close.

(e) Let $\varepsilon > 0$. If x and y are ε-close , they are also ε'-close for every $\varepsilon' > \varepsilon$.

(f) Let $\varepsilon > 0$. If x and y are ε-close to x, and w is between y and z, then w is also ε-close to x.

(g) Let $\varepsilon > 0$. If x nad y are ε-close , and z is non-zero, then xz and yz are $\varepsilon|z|$-close.

(h) Let ε , $\delta > 0$. If x and y are ε-close , and z and w are δ-close , then xz and yw are $(\varepsilon|z| + \delta|x| + \varepsilon\delta)$-close.

Definition 4.3.9 (Exponentiation to a natural number).

Let x be a rational number. To raise x to the power 0, we define $x^0 := 1$; in particular we define $0^0 := 1$. Now suppose inductively that x^n has been defined for some natural number n, then we define $x^{n+1} := x^n \times x$.

Proposition 4.3.10 (Properties of exponentiation, I)

Let x, y be rational numbers, and let n, m be natural numbers.

(a) We have $x^n x^m = x^{n+m}$, $(x^n)^m = x^{nm}$, and $(xy)^n = x^n y^n$.

(b) Suppose $n > 0$. Then we have $x^n = 0$ if and only if $x = 0$.

(c) If $x \geq y \geq 0$, then $x^n \geq y^n \geq 0$. If $x > y \geq 0$ and $n > 0$, then $x^n > y^n \geq 0$.

(d) We have $|x^n| = |x|^n$.

Definition 4.3.11 (Exponentiation to a negative number).

Let x be a non-zero rational number. Then for any negative integer $-n$, we define $x^{-n} := 1/x^n$.

Proposition 4.3.12 (Properties of exponentiation, II).

(a) We have $x^n x^m = x^{n+m}$, $(x^n)^m = x^{nm}$, and $(xy)^n = x^n y^n$.

(b) If $x \geq y \geq 0$, then $x^n \geq y^n > 0$ if n is positive, and $0 < x^n \leq y^n$ if n is negative.

(c) If $x, y > 0$, $n \neq 0$, and $x^n = y^n$, then $x = y$.

(d) We have $|x^n| = |x|^n$.

Exercise 4.3.1

(a) *Proof.* If x is positive, $|x| = x > 0$. If $x = 0$, $|x| = 0$. If x is negative, $|x| = -x > 0$. Therefore, $|x| \geq 0$. Suppose $|x| = 0$. Since if x is positive or negative, $|x|$ would be positive, x can only be 0. And $|x| = 0$, so $x = 0$. If $x = 0$, by definition, $|x| = 0$. Thus, $|x| = 0$ if and only if x is 0. □

(b) *Proof.* If $x = 0$ or $y = 0$ (or both), we have $|x + y| = |x| + |y|$.

If $x > 0$, $y > 0$, we have $|x + y| = x + y = |x| + |y|$.

If $x < 0$, $y < 0$, we have $|x + y| = -(x + y) = (-x) + (-y) = |x| + |y|$.

If $x > 0\,(y > 0)$, $y < 0\,(x < 0)$, and $x + y > 0$, we have $|x| + |y| = x - y > x + y = |x + y|$.

If $x > 0\,(y > 0)$, $y < 0\,(x < 0)$, and $x + y < 0$, we have $|x| + |y| = x - y > -x - y = -(x + y) = |x + y|$.

If $x > 0\,(y > 0)$, $y < 0\,(x < 0)$, and $x + y = 0$, we have $|x| + |y| \geq |x + y| = 0$ by (a).

Thus, $|x + y| \leq |x| + |y|$. □

(c) *Proof.* We need to show that $-y \leq x \leq y \iff y \geq |x|$.

Suppose $-y \leq x \leq y$. Since the inequalities stand, y cannot be negative. If $x = 0$, we have $|x| = 0$ and $y \geq 0 = |x|$. If $x > 0$, we have $y \geq x = |x|$. If $x < 0$, we have $-y \leq x \implies y \geq -x = |x|$. Thus, $-y \leq x \leq y \implies y \geq |x|$.

Suppose $y \geq |x|$. When $x = 0$, we have $y \geq 0$ and $y \leq 0$. So $-y \leq x \leq y$. When $x > 0$, we have $y \geq x$. Since $y \geq 0 \implies -y \leq 0$, $x \geq -y$. So $-y \leq x \leq y$. When $x < 0$, we have $y \geq |x| = -x \implies x \geq -y$. Since $y \geq |x| \geq 0$, we have $y \geq x$. So $-y \leq x \leq y$. Therefore, in all cases we have $-y \leq x \leq y$. Thus, $y \geq |x| \implies -y \leq x \leq y$.

Thus, $-y \leq x \leq y \iff y \geq |x|$.

And since $|x| \geq |x|$, we have $-|x| \leq x \leq |x|$. □

(d) *Proof.* If $x = 0$ or $y = 0$ (or both), we have $|xy| = |x||y| = 0$.

If $x > 0$, $y > 0$, then $|xy| = xy = |x||y|$.

If $x > 0$ $(y > 0)$, $y < 0$ $(x < 0)$, then $|xy| = -xy = x(-y) = |x||y|$.

If $x < 0$, $y < 0$, then $|xy| = xy = (-x)(-y) = |x||y|$.

Thus, $|xy| = |x||y|$.

Let $y = -1$, we have $|-x| = x$. $\square$

(e) *Proof.* Since $d(x, y)$ is an absolute value, by (a) we have $d(x, y) \geq 0$. By (a), we also have $d(x, y) = |x - y| = 0$ if and only if $x - y = 0 \iff x = y$. $\square$

(f) *Proof.* By (d), we have $d(x, y) = |x - y| = |y - x| = d(y, x)$. $\square$

(g) *Proof.* By (b), we have $d(x, z) = |x-z| = |(x-y)+(y-z)| \leq |x-y|+|y-z| = d(x, y) + d(y, z)$. $\square$

Exercise 4.3.2

(a) *Proof.* Suppose $x = y$, then $|x-y| = 0 \leq \varepsilon$ for every $\varepsilon > 0$. Suppose x is ε-close to y for every $\varepsilon > 0$. If $x \neq y$, then $|x - y| = a > 0$. Let $\varepsilon = a/2$, we have $|x - y| = 2\varepsilon > \varepsilon$ (contradiction). Therefore, $x = y$. $\square$

(b) *Proof.* Since x is ε-close to y, $|x-y| \leq \varepsilon$. As $|y-x| = |x-y|$, we have $|y-x| \leq \varepsilon$. Therefore, y is ε-close to x. $\square$

(c) *Proof.* Since x is ε-close to y, $|x - y| \leq \varepsilon$. Since y is δ-close to z, $|y - z| \leq \delta$. By Proposition 4.3.3, we have $|x - z| \leq |x - y| + |y - z| \leq (\varepsilon + \delta)$. Thus, x and z are $(\varepsilon + \delta)$-close. $\square$

(d) *Proof.* Since x is ε-close to y, $|x-y| \leq \varepsilon$. Since z is δ-close to w, $|z-w| \leq \delta$. By Proposition 4.3.3, $|(x+z)-(y+w)| = |(x-y)+(z-w)| \leq |x-y|+|z-w| \leq \varepsilon+\delta$. Therefore, $x+z$ and $y+w$ are $(\varepsilon+\delta)$-close. Since $|z-w| \leq \delta$ implies $|w-z| \leq \delta$, by Proposition 4.3.3, we have $|(x - z) - (y - w)| = |(x - y) + (w - z)| \leq |x - y| + |w - z| \leq \varepsilon + \delta$. Therefore, $x - z$ and $y - w$ are also $(\varepsilon + \delta)$-close. $\square$

(e) *Proof.* Since x and y are ε-close , we have $|x-y| \le \varepsilon$. Since $\varepsilon < \varepsilon'$, $|x-y| \le \varepsilon < \varepsilon'$ for every $\varepsilon' > \varepsilon$ which also implies $|x-y| \le \varepsilon'$ for every $\varepsilon' > \varepsilon$. Thus, x and y are ε'-close for every $\varepsilon' > \varepsilon$. $\square$

(f) *Proof.* Without loss of generality, assume $y \le w \le z$.

$x \ge z$. Since y is ε-close to x, $|x-y| = x-y \le \varepsilon$. Then $|w-x| = x-w \ge x-y = |x-y| \le \varepsilon$. So w is ε-close to x.

$x \le y$. Since z is ε-close to x, $|z-x| = z-x \le \varepsilon$. Then $|w-x| = w-x \le z-x = |z-x| \le \varepsilon$. So w is ε-close to x.

$y \le x \le z$. We have $|w-x| \le \max(|z-x|, |x-y|) \le \varepsilon$. So w is ε-close to x. $\square$

(g) *Proof.* Since x and y are ε-close, $|x-y| \le \varepsilon$. Since $|z| \ge 0$, we have $|x-y||z| \le \varepsilon|z|$. By Proposition 4.3.3, $|xz - yz| = |(x-y)z| = |x-y||z| \le \varepsilon|z|$. Thus, xz and yz are $\varepsilon|z|$-close. $\square$

Exercise 4.3.3

Prove Proposition 4.3.10.

(a) *Proof.* $x^n x^m = x^{n+m}$. Induct on n. When $n = 0$, we have $x^0 x^m = x^{0+m} = x^m$. The base case is proved. Assume inductively $x^n x^m = x^{n+m}$. Then $x^{n+1} x^m = x \cdot x^n \cdot x^m = x \cdot x^{n+m} = x^{(n+1)+m}$. This closes the induction.

$(x^n)^m = x^{nm}$. Induct on m. When $m = 0$, we have $(x^n)^0 = x^{n \cdot 0} = 1$. Then assume inductively $(x^n)^m = x^{nm}$. Then we have $(x^n)^{m+1} = (x^n)^m \cdot x^n = x^{mn} \cdot x^n = x^{mn+n} = x^{n(m+1)}$. This closes the induction.

$(xy)^n = x^n y^n$. Induct on n. When $n = 0$, we have $(xy)^0 = x^0 y^0 = 1$. Assume inductively $(xy)^n = x^n y^n$. Then $(xy)^{n+1} = (xy)^n (xy) = x^n y^n xy = x^{n+1} y^{n+1}$. This closes the induction. $\square$

(b) *Proof.* $x^n = 0 \implies x = 0$. Induct on n. When $n = 1$, $x^n = x^1 = 0 \implies x = 0$. The base case is proved. Assume inductively $x^n = 0 \implies x = 0$. Then if we have $x^{n+1} = 0$, by definition, $x^n \cdot x = 0$. Then either $x^n = 0$ or $x = 0$. If $x^n = 0$, by induction hypothesis, $x = 0$. Thus, in both cases, we have $x = 0$. Then $x^{n+1} = 0 \implies x = 0$. This closes the induction.

$x = 0 \implies x^n = 0$. If $x = 0$, we have $x^n = x \cdot x^{n-1} = 0$. Thus, $x = 0 \implies x^n = 0$.

Therefore, $x^n = 0$ if and only if $x = 0$. □

(c) *Proof.* $x \geq y \geq 0 \implies x^n \geq y^n \geq 0$. Induct on n. When $n = 0$, if $x \geq y \geq 0$, we have $x^0 \geq y^0 \geq 0$. Now assume inductively $x \geq y \geq 0 \implies x^n \geq y^n \geq 0$. Then $x^{n+1} = x \cdot x^n \geq x \cdot y^n \geq y \cdot y^n = y^{n+1}$. And $y^{n+1} = y \cdot y^n \geq y^n = 0$. Therefore, $x^{n+1} \geq y^{n+1} \geq 0$. This closes the induction.

The latter part can be shown in a similar way. □

(d) *Proof.* $|x^n| = |x|^n$. Induct on n. When $n = 0$, $|x^0| = |x|^0 = 1$. Assume inductively $|x^n| = |x|^n$. Then $|x^{n+1}| = |x^n \cdot x| = |x^n||x| = |x|^n \cdot |x| = |x|^{n+1}$. This closes the induction. □

Exercise 4.3.4

Prove Proposition 4.3.12.

(a) $x^n x^m = x^{n+m}$.

- $n \geq 0$, $m \geq 0$. Has been proved in Exercise 4.3.3.
- $n < 0$, $m < 0$. Since $-n > 0$ and $-m > 0$, we have $x^n x^m = \frac{1}{x^{-n}} \cdot \frac{1}{x^{-m}} = \frac{1}{x^{-(n+m)}} = x^{n+m}$.
- $n \geq 0$ ($m \geq 0$), $m < 0$ ($n < 0$), $n + m \geq 0$. Since $-m > 0$, we have $x^n x^m = (x^{n+m} x^{-m}) x^m = x^{n+m}(x^{-m} x^m) = x^{n+m}$.
- $n \geq 0$ ($m \geq 0$), $m < 0$ ($n < 0$), $n + m < 0$. Then $-n - m > 0$. $x^{n+m} = \frac{1}{x^{-n-m}} \implies \frac{1}{x^n} x^{n+m} = \frac{1}{x^{-n-m}} \cdot \frac{1}{x^n} = \frac{1}{x^{-m}} = x^m$. Therefore, $x^n x^m = x^{n+m}$.

$(x^n)^m = x^{nm}$.
First, we need to show that $\frac{1}{x^n} = (\frac{1}{x})^n$ for natural number n. Induct on n.

When $n = 0$, $\frac{1}{x^0} = (\frac{1}{x})^0 = 1$. Assume inductively $\frac{1}{x^n} = (\frac{1}{x})^n$. Then $\frac{1}{x^{n+1}} = \frac{1}{x^n} \cdot \frac{1}{x} = (\frac{1}{x})^n \cdot \frac{1}{x} = (\frac{1}{x})^{n+1}$. This closes the induction.

- $n \geq 0, m \geq 0$. Has been proved in Exercise 4.3.3.
- $n < 0, m < 0$. $(x^n)^m = \frac{1}{(x^n)^{-m}} = \frac{1}{(\frac{1}{n})^{-m}} = \frac{1}{\frac{1}{(x^{-n})^{-m}}} = \frac{1}{\frac{1}{x^{(-n)(-m)}}} = \frac{1}{\frac{1}{x^{nm}}} = x^{nm}$.
- $n \geq 0$ $(m \geq 0)$, $m < 0$ $(n < 0)$. Then $(x^n)^m = \frac{1}{(x^n)^{-m}} = \frac{1}{x^{-nm}} = x^{nm}$.

$(xy)^n = x^n y^n$. We have proved the case when $n \geq 0$. If $n < 0$, we have $(xy)^n = \frac{1}{(xy)^{-n}} = \frac{1}{x^{-n}y^{-n}} = \frac{1}{x^{-n}}\frac{1}{y^{-n}} = x^n y^n$.

(b) $x \geq y > 0 \implies x^n \geq y^n > 0$ when $n > 0$. Consider the base case when $n = 1$. If $x \geq y > 0$, we have $x^1 \geq y^1 > 0$. Assume inductively $x \geq y > 0 \implies x^n \geq y^n > 0$. Then $x^{n+1} = x^n \cdot x \geq y^n \cdot x \geq y^n \cdot y = y^{n+1}$, $y^{n+1} = y^n \cdot y > 0 \cdot y = 0$. So $x^{n+1} \geq y^{n+1} > 0$. This closes the induction.

When n is negative, use the conclusion above. Since $-n > 0$, we have $x^{-n} \geq y^{-n} > 0$. Since $x^{-n} \geq y^{-n}$, by multiplying both sides by $x^n y^n$ (which is positive), we have $y^n \geq x^n$. Since $x^{-n} = \frac{1}{x^n} > 0$, we have $x^n > 0$. Therefore, $0 < x^n \leq y^n$.

(c) When $n > 0$, suppose $x^n = y^n$ and $x \neq y$. If $x > y$, $x^n > y^n$. If $y > x$, $y^n > x^n$. In either case, $x^n \neq y^n$ (contradiction). Therefore, $x = y$. When $n < 0$, since $-n > 0$, we have $x^{-n} = y^{-n} \implies x = y$. By multiplyting both sides of $x^{-n} = y^{-n}$, we have $x^n = y^n \iff x^{-n} = y^{-n}$. Thus, $x^n = y^n \iff x^{-n} = y^{-n} \implies x = y$. Therefore, if $x, y > 0, n \neq 0$, and $x^n = y^n$, then $x = y$.

(d) The case $n \geq 0$ has been proved in Exercise 4.3.3. When $n < 0$, we have $-n > 0$, then $|x^{-n}| = |x|^{-n}$. And $|x^{-n}| = |(\frac{1}{x})^n| = |\frac{1}{x^n}| = \frac{1}{|x^n|}$, $|x|^{-n} = \frac{1}{|x|^n}$. Since $\frac{1}{|x^n|} = \frac{1}{|x|^n}$, we have $|x^n| = |x|^n$.

Exercise 4.3.5

Prove that $2^N \geq N$ for all positive integers N.

Proof. When $N = 1$, $2^1 \geq 1$ as desired. Assume inductively $2^N \geq N$. Then $2^{N+1} = 2^N \cdot 2 \geq 2N = N + N \geq N + 1$. Therefore, $2^N \geq N$ is true for all positive integers N. $\square$

4.4 Gaps in the rational numbers

Proposition 4.4.1 (Interspersing of integers by rationals).

Let x be a rational number. then there exists an integer n such that $n \leq x < n + 1$. In fact, this integer is unique. In particualr, there exists a natural number N such that $N > x$.

Proposition 4.4.3 (Interspersing of rationals by rationals).

If x and y are two rationals such that $x < y$, then there exists a third rational z such that $x < z < y$.

Proposition 4.4.4

There does not exist any rational number x for which $x^2 = 2$.

Proposition 4.4.5

For every rational number $\varepsilon > 0$, there exists a non-negative rational number x such that $x^2 < 2 < (x + \varepsilon)^2$.

Exercise 4.4.1

Prove Proposition 4.4.1.

Proof. Consider $x \geq 0$. Then $x = \frac{a}{b}$ where a, b are natural numbers and $b \neq 0$. Since a is a natural number and $b > 0$, by Proposition 2.3.9, $a = bn + r$ where $0 \leq r < n$. Therefore, $bn \leq a = bn + r < b(n + 1)$. Thus, $n \leq x = \frac{a}{b} < n + 1$.

Now consider $x < 0$. We have $-x = \frac{a}{b}$. By Proposition 2.3.9, we have $a = bm + r$ where $0 \leq r < b$. If $r > 0$, $bm < a < b(m + 1)$ so $m < -x < m + 1$. Then $-(m+1) < x < -m$ where $-(m+1)$ and $-m$ are integers. Since $-(m+1) < x < -m$

we can also say $-(m+1) \leq x < -m$. If $r = 0$, we can write $a = b(m+1)+r = b(m+1)$ since $-x > 0$. Then we have $m < -x \leq (m+1)$ and therefore $-(m+1) \leq x < m$. Thus, in both cases, we can find an integer n such that $n \leq x < n+1$. Therefore, for any rational number x, there exists an integer n such that $n \leq x < n+1$.

Assume we have $n \leq x < n+1$ and $m \leq x < m+1$ where n, m are integers and $n \neq m$. Without loss of generality, suppose $m > n$. Then $m \geq n+1$. So we have $x \geq m \geq n+1$ and $x < n+1$ at the same time (contradiction). Therefore, $m > n$ does not hold. Similarly, we can show that $m < n$ does not hold either. Thus, $m = n$ and the integer is unique.

Since there exists an integer n such that $n \leq x < n+1$, let $N = n+1$, we have $N > x$. □

Exercise 4.4.2

A definition: a sequence $a_0, a_1, a_2, \ldots$ of numbers (natural numbers, integers, rationals, or reals) is said to be infinite descent if we have $a_n > a_{n+1}$ for all natural numbers n.

(a) Prove the principle of infinite descent: that it is not possible to have a sequence of natural numbers which is in infinite descent.

Proof. Assume that one can find a sequence of natrual numbers which is in infinite descent. Show that $a_n \geq k$ for all $k \in \mathbf{N}$ and all $n \in \mathbf{N}$. Induct on k. When $k = 0$, since a_n is a natural number for all $n \in \mathbf{N}$. Therefore, $a_n \geq 0$ for all $n \in \mathbf{N}$. Assume inductively $a_n \geq k$ for all $n \in \mathbf{N}$. We want to show that $a_n \geq k+1$ for all $n \in \mathbf{N}$. Consider an arbitrary $n \in \mathbf{N}$. Since $a_n > a_{n+1}$ and they are natural numbers, we have $a_n \geq a_{n+1}+1$. And by induction hypothesis, we have $a_{n+1} \geq k$. Therefore, we have $a_n \geq g_{n+1}+1 \geq k+1$. Thus, $a_n \geq k+1$ for all $n \in \mathbf{N}$. This closes the induction.

Then, since a_0 is a natural number, we have $a_n \geq a0$ for all $n \in \mathbf{N}$. So $a_1 \geq a_0$. But by the definition of infinite descent, we have $a_0 > a_1$. (Contradiction.) Therefore, it is not possible to have a sequence of natural numbers which is in infinite descent. □

(b) Does the principle of infinite descent work if the sequence $a_1, a_2, a_3, \ldots$ is allowed to take integer values instead of natural number values? What about if it is allowed to take positive rational values instead of natural numbers? Explain.

Proof. The principle of infinite descent does not work if we are allowed to take integer values. Since for every $a_n \in \mathbf{Z}$, we can take $a_{n+1} = a_n - 1$ such that $a_{n+1} \in \mathbf{Z}$ and $a_{n+1} < a_n$. It also does not work for positive rational values. Since natural numbers do not have a upper bound, for every $a_n = \frac{p}{q}$ where p, q are positive integers, we can find $a_{n+1} = \frac{p}{q+1}$ such that $p, (q+1)$ are positive integers. Therefore, for every $a_n \in \mathbf{R}$ $(n \in \mathbf{Z}^+)$, we can find $a_{n+1} < a_n$. $\square$

Exercise 4.4.3

Fill in the gaps marked in the proof of Proposition 4.4.4.

Proof. Every natrual number is either even or odd, but not both. Assume p is both even and odd. Then $p = 2m = 2n + 1$ for some natural numbers m, n. Then $2(m - n) = 1$ which means $1 = 2 \times 0 + 1$ is even. (Contradiction.) Thus, natural numbers cannot be both even and odd. Then we need to show that every natural number n is either even or odd. Induct on n. $n = 0$ is even. Assume inductively n is either even or odd. If n is odd, there exists $m \in \mathbf{N}$ such that $n = 2m$. Then $n + 1 = 2m + 1$ which is odd. If n is even, then there exists $m \in \mathbf{N}$ such that $n = 2m + 1$. Then $n + 1 = 2m + 1 + 1 = 2(m + 1)$ which is even. Therefore, $n + 1$ is either odd or even. Thus, every natrual number is either even or odd, but not both.

If p is odd, then p^2 is also odd. Since p is odd, there exists a natural number $m \in \mathbf{N}$ such that $p = 2m+1$. Then $p^2 = (2m+1)(2m+1) = 4m^2+4m+1 = 2(2m^2+2m)+1$ where $(2m^2 + 2m)$ is a natural number. Thus, p^2 is odd.

$p^2 = 2q^2 \implies q < p$. For positive natural numbers p, q, assume $p = q$. Then $p^2 = q^2$ and $p^2 < p^2 + p^2 = q^2 + q^2 = 2q^2$. (contradiction) Assume $p < q$. Then $p^2 = p \times p < q \times p < q \times q < 2 \times q \times q = 2q^2$. (contradiction) Therefore, there must be $q < p$. $\square$