

## 3.3 Functions

### Definition 3.3.1 (Functions).

Let  $X, Y$  be sets, and let  $P(x, y)$  be a property pertaining to an object  $x \in X$  and an object  $y \in Y$ , such that for every  $x \in X$ , there is exactly one  $y \in Y$  for which  $P(x, y)$  is true (this is sometimes known as the vertical line test). Then we define the function  $f : X \rightarrow Y$  defined by  $P$  on the domain  $X$  and range  $Y$  to be the object which, given any input  $x \in X$ , assigns an output  $f(x) \in Y$ , defined to be the unique object  $f(x)$  for which  $P(x, f(x))$  is true. Thus, for any  $x \in X$  and  $y \in Y$ ,

$$y = f(x) \iff P(x, y) \text{ is true.}$$

### Definition 3.3.7 (Equality of functions).

Two functions  $f : X \rightarrow Y, g : X \rightarrow Y$  with the same domain and range are said to be equal,  $f = g$ , if and only if  $f(x) = g(x)$  for all  $x \in X$ . If  $f(x)$  and  $g(x)$  agree for some values of  $x$ , but not others, then we do not consider  $f$  and  $g$  to be equal. If two functions  $f, g$  have different domains, or different ranges, we also do not consider them to be equal.

### Definition 3.3.11 (Composition).

Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be two functions, such that the range of  $f$  is the same set as the domain of  $g$ . We then define the composition  $g \circ f : X \rightarrow Z$  of the two functions  $g$  and  $f$  to be the function defined explicitly by the formula

$$(g \circ f)(x) := g(f(x)).$$

If the range of  $f$  does not match the domain of  $g$ , we leave the composition  $g \circ f$  undefined.

### Lemma 3.3.13 (Composition is associative).

Let  $f : Z \rightarrow W, g : Y \rightarrow Z$ , and  $h : X \rightarrow Y$  be functions. Then  $f \circ (g \circ h) = (f \circ g) \circ h$ .

**Definition 3.3.15 (One-to-one functions).**

A function  $f$  is one-to-one (or injective) if different elements map to different elements:

$$x \neq x' \implies f(x) \neq f(x').$$

Equivalently, a function is one-to-one if

$$f(x) = f(x') \implies x = x'.$$

**Definition 3.3.18 (Onto functions).**

A function  $f$  is onto (or surjective) if every element of  $Y$  comes from applying  $f$  to some element in  $X$ :

$$\text{For every } y \in Y, \text{ there exists } x \in X \text{ such that } f(x) = y.$$

**Definition 3.3.21 (Bijective functions).**

Functions  $f : X \rightarrow Y$  which are both one-to-one and onto are also called bijective or invertible.

**Exercises****Exercise 3.3.1**

Show that the definition of equality in Definition 3.3.7 is reflexive, symmetric, and transitive. Also verify the substitution property: if  $f, \tilde{f} : X \rightarrow Y$  and  $g, \tilde{g} : Y \rightarrow Z$  are functions such that  $f = \tilde{f}$  and  $g = \tilde{g}$ , then  $g \circ f = \tilde{g} \circ \tilde{f}$ .

*Proof.* Reflexivity:  $f$  and  $f$  have the same domain and range, and  $f(x) = f(x)$  for all  $x$  in the domain of  $f$ . Therefore,  $f$  is equal to itself.

Symmetry:  $g$  and  $f$  have the same domain and range. For every  $x$  in the domain of  $g$ , we have  $g(x) = f(x)$ . Therefore, by Definition 3.3.7,  $g(x)$  and  $f(x)$  are equal.

Transitivity: Suppose  $f$  and  $g$  have the same domain and range, and for every  $x$  in the domain of  $f$ ,  $f(x) = g(x)$ . And  $g$  and  $h$  have the same domain and range, and

for every  $x$  in the domain of  $g$ , we have  $g(x) = h(x)$ . Then  $f$  and  $h$  have the same domain and range.  $\forall x$  in the domain of  $f$ , we have  $f(x) = g(x) = h(x)$ . Therefore,  $f$  and  $h$  are equal.

Substitution property: Since  $g \circ f, \tilde{g} \circ \tilde{f} : X \rightarrow Z$ , they have the same domain and range. And for every  $x \in X$ , we have  $f(x) = \tilde{f}(x)$ , since  $g = \tilde{g}$ , we also have  $g(f(x)) = \tilde{g}(f(x)) = \tilde{g}(\tilde{f}(x))$ . Therefore,  $g \circ f = \tilde{g} \circ \tilde{f}$ .  $\square$

### Exercise 3.3.2

Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be functions. Show that if  $f$  and  $g$  are both injective, then so is  $g \circ f$ ; similarly, show that if  $f$  and  $g$  are both surjective, then so is  $g \circ f$ .

1. If  $f$  and  $g$  are both injective, then so is  $g \circ f$ .

*Proof.*  $f$  is injective:

$$x \in X, x' \in X, x \neq x' \implies f(x) \neq f(x').$$

$g$  is injective:

$$f(x) \in Y, f(x') \in Y, f(x) \neq f(x') \implies g(f(x)) \neq g(f(x')).$$

Therefore,  $x \neq x' \implies (g \circ f)(x) \neq (g \circ f)(x')$ . Thus,  $g \circ f$  is injective.  $\square$

2. If  $f$  and  $g$  are both surjective, then so is  $g \circ f$ .

*Proof.*  $f$  is surjective:

For every  $y \in Y$ , there exists  $x \in X$  such that  $f(x) = y$ .

$g$  is surjective:

For every  $z \in Z$ , there exists  $y \in Y$  such that  $g(y) = z$ .

Therefore, for every  $z \in Z$ , there exists  $x \in X$  such that  $(g \circ f)(x) = g(f(x)) = g(y) = z$ . Thus,  $g \circ f$  is surjective.  $\square$

### Exercise 3.3.3

When is the empty function injective? surjective? bijective?

The empty function is of the form  $f : \emptyset \rightarrow X$ . It is always injective no matter what  $X$  is. It is surjective if  $X$  is  $\emptyset$ . It is bijective if  $X$  is  $\emptyset$ .

### Exercise 3.3.4

In this section we give some cancellation laws for composition. Let  $f : X \rightarrow Y$ ,  $\tilde{f} : X \rightarrow Y$ ,  $g : Y \rightarrow Z$ , and  $\tilde{g} : Y \rightarrow Z$  be functions. Show that if  $g \circ f = g \circ \tilde{f}$  and  $g$  is injective, then  $f = \tilde{f}$ . Is the same statement true if  $g$  is not injective? Show that if  $g \circ f = \tilde{g} \circ f$  and  $f$  is surjective, then  $g = \tilde{g}$ . Is the same statement true if  $f$  is not surjective?

1. *Proof.* Suppose  $x$  is an arbitrary object in  $X$ ,  $y = f(x) \in Y$ ,  $y' = \tilde{f}(x) \in Y$ . Since  $g \circ f = g \circ \tilde{f}$ ,  $g(f(x)) = g(\tilde{f}(x))$ . Because  $g$  is injective,  $g(f(x)) = g(\tilde{f}(x)) \implies f(x) = \tilde{f}(x)$ . And  $f, \tilde{f}$  have the same domain and range. Thus,  $f = \tilde{f}$ .

This won't be true if  $g$  is not injective. Counterexample:  $g(1) = 3, g(2) = 3, f(1) = 1, \tilde{f}(1) = 2$ . In this case,  $g(f(1)) = g(\tilde{f}(1)) = 3$ , but  $f \neq \tilde{f}$ .  $\square$

2. *Proof.* Since  $f$  is surjective, for every  $y \in Y$ , there exists  $x \in X$  such that  $y = f(x)$ . Also for every  $x \in X$ , we have  $g(f(x)) = \tilde{g}(f(x))$  as  $g \circ f = \tilde{g} \circ f$ . Therefore, for every  $y \in Y$ , there exists  $x \in X$  such that  $g(y) = g(f(x)) = \tilde{g}(f(x)) = \tilde{g}(y)$ . As  $g, \tilde{g}$  have the same domain and range,  $g = \tilde{g}$ .

This won't be true if  $f$  is not surjective. Counterexample:  $f : \{0, 1\} \rightarrow \{1, 2, 3\}$ ,  $g : \{1, 2, 3\} \rightarrow \{4, 5, 6\}$ ,  $\tilde{g} : \{1, 2, 3\} \rightarrow \{4, 5, 7\}$ .  $\square$

### Exercise 3.3.5

Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be functions. Show that if  $g \circ f$  is injective, then  $f$  must be injective. Is it true that  $g$  must also be injective? Show that if  $g \circ f$  is surjective, then  $g$  must be surjective. Is it true that  $f$  must also be surjective?

1. *Proof.*  $g \circ f$  is injective  $\implies (x \neq x' \implies g(f(x)) \neq g(f(x')))$ . Suppose  $f$  is not injective, that is,  $\exists x, x' \in X$  such that  $x \neq x'$  and  $f(x) = f(x')$ . Then by definition,  $g(f(x)) = g(f(x'))$  so  $g \circ f$  is not injective. (contradiction) Therefore,  $f$  must be injective. However,  $g$  does not have to be injective. Counterexample:  $f : \{0, 1\} \rightarrow \{1, 2, 3\}$ ,  $g : \{1, 2, 3\} \rightarrow \{4, 5, 5\}$ . ( $f$  does not have to be surjective.)  $\square$
2. *Proof.*  $g \circ f$  is surjective  $\implies \forall z \in Z, \exists x \in X$  such that  $g(f(x)) = z$ . Assume  $g$  is not surjective. Then  $\exists z \in Z$  such that  $\forall y \in Y, g(y) \neq z$ . So there does not exist  $x \in X$  such that  $g(y) = g(f(x)) = z$ . It implies  $g \circ f$  is not surjective. (contradiction) Thus,  $g$  must be surjective.  $f$  does not have to be surjective. Counterexample:  $f : \{1\} \rightarrow \{2, 3\}$ ,  $f(1) := 2$ ,  $g : \{2, 3\} \rightarrow \{4\}$ ,  $g(2) := 4$ ,  $g(3) := 4$ . ( $g$  does not have to be injective.)  $\square$

### Exercise 3.3.6

Let  $f : X \rightarrow Y$  be a bijective function, and let  $f^{-1} : Y \rightarrow X$  be its inverse. Verify the cancellation laws  $f^{-1}(f(x)) = x$  for all  $x \in X$  and  $f(f^{-1}(y)) = y$  for all  $y \in Y$ . Conclude that  $f^{-1}$  is also invertible, and has  $f$  as its inverse (thus  $(f^{-1})^{-1} = f$ ).

*Proof.* Since  $f$  is surjective, for every  $y \in Y$ , there exists  $x \in X$  such that  $f(x) = y$ . Suppose  $\exists x, x' \in X$  such that  $f^{-1}(f(x)) = x' \neq x$ . Then there exists  $y \in Y$ , such that  $f(x) = y$  and  $f^{-1}(y) = x'(f(x') = y)$ . So  $f$  is not injective. (contradiction) Thus,  $f^{-1}(f(x)) = x$  for all  $x \in X$ .

Since  $f$  is surjective, for every  $y \in Y$ , there exists  $x \in X$  such that  $f^{-1}(y) = x$ . Suppose  $f(x) = y$  and  $\exists y' \in Y, y' \neq y, f(f^{-1}(y)) = y'$ . Then we have  $f(f^{-1}(y)) = f(x) = y' \neq y$  (contradiction). Thus,  $f(f^{-1}(y)) = y$ .

Since  $f^{-1}(f(x)) = x$  for all  $x \in X$ ,  $f^{-1}$  is surjective. Suppose there exists  $y, y' \in Y$ ,  $x \in X$ ,  $y \neq y'$ ,  $f^{-1}(y) = f^{-1}(y') = x$ . By cancellation law, we have  $f(x) = y$  and  $f(x) = y'$  (contradiction). Therefore,  $f^{-1}$  is injective. Thus,  $f^{-1}$  is bijective.

$(f^{-1})^{-1}$  has  $X$  as its domain and  $Y$  as its range, the inverse of  $f^{-1}$ . Then we need to show that for every  $x \in X$ , we have  $f(x) = (f^{-1})^{-1}(x)$ . Let  $x$  be an arbitrary object in  $X$ ,  $y = f(x)$ . By definition of inverse,  $f^{-1}(y) = x$ . Again by definition of inverse,  $(f^{-1})^{-1}(x) = y$ . Therefore,  $f(x) = (f^{-1})^{-1}(x)$ . Thus,  $(f^{-1})^{-1} = f$ .  $\square$