Chapter 3

Set Theory

Definition 3.1.1

(Informal) We define a *set* A to be any unordered collection of objects, e.g., $3, 8, 5, 2$ is a set. If x is an object, we say that x is an element of A or $x \in A$ if x lies in the collection; otherwise we say that $x \notin A$. For instance, $3 \in \{1, 2, 3, 4, 5\}$ but $7 \notin \{1, 2, 3, 4, 5\}$.

Axiom 3.1 (Sets are objects).

If A is a set, then A is also an object. In particular, given two sets A and B, it is meaningful to ask whether A is also an element of B.

Axiom 3.2 (Equality of sets).

Two sets A and B are equal, $A = B$, iff every element of A is an element of B and vice versa. To put it another way, $A = B$ if and only if every element x of A belongs also to B, and every element y of B belongs also to A.

Axiom 3.3 (Empty set).

There exists a set $\emptyset$, known as the empty set, which contains no elements, i.e., for every object x we have $x \notin \emptyset$.

Lemma 3.1.5 (Single choice).

Let A be a non-empty set. Then there exists an object x such that $x \in A$.

Axiom 3.4 (Singleton sets and pair sets).

If a is an object, then there exists a set $\{a\}$ whose only element is a, i.e., for every object y, we have $y \in \{a\}$ if and only if $y = a$; we refer to $\{a\}$ as the singleton set whose element is a. Furthermore, if a and b are objects, then there exists a set $\{a, b\}$

whose only elements are a and b; i.e., for every object y, we have $y \in \{a, b\}$ if and only if $y = a$ or $y = b$; we refer to this set as the pair set formed by a and b.

Exercises

Exercise 3.1.1

Let a, b, c, d be objects such that $\{a, b\} = \{c, d\}$. Show that at least one of the two statements "$a = c$ and $b = d$" and "$a = d$ and $b = c$" hold.

Proof. Consider two cases: $a = b$ and $a \neq b$.
Case 1: $a = b$. Then $\{a, b\} = \{a\}$. By Axiom 3.2, if $\{a\}$ and $\{c, d\}$ are equal to each other, then every element belong to $\{c, d\}$ must also belong to $\{a\}$. Therefore, $c = a$, $d = a$. Since $a = b$, we have $a = b = c = d$. Thus, both statements hold.
Case 2: $a \neq b$. Similarly, by Axiom 3.2, every element belong to $\{a, b\}$ must also belong to $\{c, d\}$. So $\{c, d\}$, a set of two elements, contains two distinct elements a and b. Therefore, either $a = c, b = d$ or $a = d, b = c$ holds, exclusively.
Thus, we have shown that at least one of the two statements "$a = c$ and $b = d$" and "$a = d$ and $b = c$" hold. $\square$

Exercise 3.1.2

Using only Axiom 3.2, Axiom 3.1, Axiom 3.3, and Axiom 3.4, prove that the sets $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$, and $\{\emptyset, \{\emptyset\}\}$ are all distinct.

Proof. First, let's consider $\emptyset$. $\emptyset$ contains no element while other sets all have at least one element in it. Therefore, $\emptyset$ is distinct from $\{\emptyset\}$, $\{\{\emptyset\}\}$ and $\{\emptyset, \{\emptyset\}\}$. Then, let's consider $\{\emptyset\}$. Is it distinct from $\{\{\emptyset\}\}$ and $\{\emptyset, \{\emptyset\}\}$? We know that $\emptyset \in \{\emptyset\}$. But we have proved earlier $\emptyset$ and $\{\emptyset\}$ are not equal to each other, so $\emptyset \notin \{\{\emptyset\}\}$. So $\{\emptyset\}$ and $\{\{\emptyset\}\}$ are distinct. For the same reason, $\{\emptyset\} \notin \{\emptyset\}$. So $\{\emptyset\}$ and $\{\emptyset, \{\emptyset\}\}$ are also distinct. Last, consider $\{\{\emptyset\}\}$ and $\{\emptyset, \{\emptyset\}\}$. For the same reason ($\emptyset$ and $\{\emptyset\}$ are distinct), $\emptyset \notin \{\{\emptyset\}\}$. So $\{\{\emptyset\}\}$ and $\{\emptyset, \{\emptyset\}\}$ are distinct. Thus, we have proved the sets $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$, and $\{\emptyset, \{\emptyset\}\}$ are all distinct. $\square$