# 2.2

## 2.2.1

For any natural numbers $a, b, c$, we have $(a + b) + c = a + (b + c)$.

*Proof.* Induct on $b$ by keeping $a$ and $c$ fixed. Consider the base case $b = 0$. In this case, LHS$=$ $(a + 0) + c = a + c$ and RHS$=$ $a + (0 + c) = a + c$. Now suppose that $(a+b)+c = a+(b+c)$. We need to show that $(a + (b++)) + c = a + ((b++) + c)$:

$$\begin{aligned} \text{LHS} &= (a + (b++)) + c = ((a + b)++) + c = (a + b + c)++, \\ \text{RHS} &= a + ((b++) + c) = a + ((b + c)++) = (a + b + c)++. \end{aligned}$$

Thus both sides are equal to each other, and we have closed the induction. □

## 2.2.2

Let $a$ be a positive number. Then there exists exactly one natural number $b$ such that $b++ = a$. (I'm assuming that it meant $a$ is a positive natural number.)

*Proof.* Induct on $a$. Since 0 is not positive, we consider the base case $a = 1$. We have $b++ = b+1 = 0+1 = 1$. Cancellation law tells us that $b = 0$, which is unique. Now suppose that there exists exactly one natural number $b_0$ such that $b_0++ = a$, we need to show that there exists exactly one natural number $b$ such that $b++ = a++$. By Cancellation law, we have $b = a = b_0++$. Since $b$ is the successor of $b_0$ and $b_0$ is unique, $b$ is also unique. Thus we have closed the induction. □

## 2.2.3

### (a)

$a \geq a$.

*Proof.* There exists a natural number 0 such that $a + 0 = a$. Thus, $a \geq a$. □

**(b)**

If $a \geq b$ and $b \geq c$, then $a \geq c$.

*Proof.* Since $a \geq b$, there exists a natural number $m$ such that $b + m = a$. Since $b \geq c$, there exists a natural number $n$ such that $c + n = b$. Then $c + (n + m) = (c + n) + m = b + m = a$. Therefore, $c \geq a$.

Thus, if $a \geq b$ and $b \geq c$, then $a \geq c$. □

**(c)**

If $a \geq b$ and $b \geq a$, then $a = b$.

*Proof.* Since $a \geq b$, there exists a natural number $m$ such that $b + m = a$. Since $b \geq a$, there exists a natural number $n$ such that $a + n = b$. Then we have $a + n = (b + m) + n = b + (m + n) = b$. By Cancellation law, we have $m + n = 0$ which leads to $m = 0$, $n = 0$. Thus, $a = a + 0 = b$.

Thus, if $a \geq b$ and $b \geq a$, then $a = b$. □

**(d)**

$a \geq b$ if and only if $a + c \geq b + c$.

*Proof.* First, we need to show that $a \geq b \Rightarrow a + c \geq b + c$. Since $a \geq b$, there exists a natural number $n$ such that $b + n = a$. Then we have $b + n + c = b + c + n = (b + c) + n = a + c$. Thus, $a + c \geq b + c$. Then, we need to show that $a + c \geq b + c \Rightarrow a \geq b$. Since $a + c \geq b + c$, there should be a natural number $n$ such that $b + c + n = b + n + c = (b + n) + c = a + c$. By Cancellation law, we have $b + n = a$. Thus, $a \geq b$.

Thus, if $a \geq b$ and $b \geq a$, then $a = b$. □

**(e)**

$a < b$ if and only if $a{+}{+} \leq b$.

*Proof.* First, we need to show that $a < b \Rightarrow a++ \leq b$. $a < b$ means there exists a natural number $n$ such that $a + n = b$, particularly, $a \neq b$. Then $n$ must not be zero. So $n$ is the predecessor of a natural number, denote it as $m$. Then we have $a + n = a + (m++) = (a + m)++ = (a++) + m = b$. Therefore, $a++ \leq b$. Then we need to show that $a++ \leq b \Rightarrow a < b$. There exists a natural number $n$ such that $(a++) + n = b$. $(a++) + n = (a + n)++ = a + (n++) = b$. Since $n++$ is the successor of $n$, $n++$ must not be equal to 0. If $a = b$, there will be $a + (n++) = a \Rightarrow n++ = 0$, contradiction. Therefore, $a \neq b$.

Thus, $a < b$ if and only if $a++ \leq b$. □

### (f)

$a < b$ if and only if $b = a + d$ for some positive number $d$.

*Proof.* First, we need to show that $a < b \Rightarrow b = a + d$ for some positive number $d$. There exists some natural number $d$ such that $a + d = b$, $a \neq b$. By Cancellation law, $d$ must not be zero. Therefore, $d$ is positive. Then, we need to show that $a + d = b$ for some positive $d \Rightarrow a < b$. We only need to prove $a \neq b$. If $a = b$, we have $a + d = a = a + 0$. By Cancellation law, $d = 0$ which contradicts to $d$ is positive. Therefore, $a \neq b$.

Thus, $a < b$ if and only if $b = a + d$ for some positive number $d$. □

## 2.2.4

Justify the three statements marked in the proof of Proposition 2.2.13.

### (a)

$0 \leq b$ for all $b$.

*Proof.* By definition of addition, we have $0 + b = b$. Thus, $0 \leq b$. □

### (b)

If $a > b$, then $a++ > b$.

*Proof.* By Proposition 2.2.12.e, we have $a > b \Rightarrow a \geq b++$. And by Proposition 2.2.12.d, $a + 1 \geq (b++) + 1$ that is equivalent to $a++ \geq b + 2$. Since 2 is positive, by Proposition 2.2.12.f, we have $a++ > b$. $\square$

**(c)**

If $a = b$, then $a++ > b$.

*Proof.* We know from Proposition 2.2.12.a that $a \geq a$, so $a \geq a = b$. And again by Proposition 2.2.12.d, we have $a{+}{+} = a{+}1 \geq b{+}1$. Since 1 is positive, by Proposition 2.2.12.f, $a++ > b$.

$\square$

## 2.2.5

Proposition 2.2.14 (Strong principle of induction). Let $m_0$ be a natural number, and let $P(m)$ be a property pertaining to an arbitrary natural number $m$. Suppose that for each $m \geq m_0$, we have the following implication: if $P(m')$ is true for all natural numbers $m_0 \leq m' < m$, then $P(m)$ is also true. Then we can conclude that $P(m)$ is true for all natural numbers $m \geq m_0$.

*Proof.* Let $Q(n)$ be the property that $P(m)$ is true for all $m_0 \leq m < n$. Induct on $n$. Consider the base case $n = 0$. This is vacuously true. In fact, $Q(n)$ is vacuously true for all $n \leq m_0$. So we can assume $n > m_0$ to see if the implication stands. Suppose $Q(n)$ is true, that is, $P(m)$ is true for all $m_0 \leq m < n$. We want to show that $Q(n{+}1)$ is also true. As stated in Proposition 2.2.14, if $Q(n)$ is true, then $P(n)$ is also true. So $P(m)$ is true for all $m_0 \leq m \leq n$. Hence, $P(m)$ is true for all $m_0 \leq m < n + 1$. ($m \leq n \Leftrightarrow m < n + 1$ can be shown using prop 2.2.12.) Thus, $Q(n + 1)$ is true. This closes the induction. $\square$

## 2.2.6

Let $n$ be a natural number, and let $P(m)$ be a property pertaining to the natural numbers such that whenever $P(m{+}{+})$ is true, then $P(m)$ is true. Suppose that $P(n)$

is also true. Prove that $P(m)$ is true for all natural numbers $m \leq n$. (Principle of backwards induction.)

*Proof.* Apply induction to $n$. For the base case $n = 0$, suppose $P(0)$ is true. In this case, $m$ can only be 0 ($m + k = 0 \Rightarrow m = 0, k = 0$). Since $P(0)$ is true, the base case is proved. Next, suppose if $P(n)$ is true then $P(m)$ is true for all natural numbers $m \leq n$. We want to show that if $P(n++)$ is true then $P(m)$ is true for all natural numbers $m \leq n++$. $m \leq n$ means there exists a natural number $a$ such that $m + a = n++$. $a$ is either 0 or a positive number. If $a$ is 0, $m = n++$. If $a$ is positive, $m < n++$ (by prop 2.2.12.f), this is equivalent to $m \leq n$ (can be shown using prop 2.2.12.). For $m = n++$, $P(m)$ is true because of the assumption. For each $m \leq n$, $P(m)$ is also true by induction hypothesis. Therefore, $P(m)$ is true for all natural numbers $m \leq n++$. And we have closed the induction. $\square$

In the above proofs, $n++$ and $n + 1$ got mixed up because $n++ = n + 1$ has been illustrated on Page 26 (and the +1 version is a little easier). But ++ is a more desirable expression since it stands for the successor in a general way.

## 2.3

### Definition 2.3.1 (Multiplication of natural numbers).

Let $m$ be a natural number. To multiply zero to $m$, we define $0 \times m := 0$. Now suppose inductively that we have defined how to multiply $n$ to $m$. Then we can multiply $n++$ to $m$ by defining $(n++) \times m := (n \times m) + m$.

### 2.3.1

**Lemma 2.3.2** (Multiplication is commutatitive). Let $n, m$ be natural numbers. Then $n \times m = m \times n$.

*Proof.* First, we want to show that $m \times 0 = 0$. Induct on $m$. When $m = 0$, by definition $0 \times m = 0$ for every $m$, so $0 \times 0 = 0$. Suppose $m \times 0 = 0$, we want to show that $(m++) \times 0 = 0$. By definition, we got $(m++) \times 0 = (m \times 0) + 0$ which is equal to $0 + 0 = 0$. This closes the induction.

Then, we want to show that $n \times (m++) = n \times m + n$. Induct on $n$ by keeping $m$ fixed. Consider the base case $n = 0$. The LHS is equal to $0 \times (m++) = 0$ by definition. The RHS is equal to $0 \times m + 0$ which is also 0. Now suppose inductively $n \times (m++) = n \times m + n$. We need to show that $(n++) \times (m++) = (n++) \times m + (n++)$.

$$\begin{aligned}
\text{LHS} &= (n++) \times (m++) = n \times (m++) + (m++) \\
&= n \times m + n + (m++) = n \times m + (n+m)++, \\
\text{RHS} &= (n++) \times m + (n++) = n \times m + m + (n++) = n \times m + (n+m)++.
\end{aligned}$$

Thus, both sides are equal to each other. This closes the induction.

Now we can use the things above to show Lemma 2.3.2. We induct on $n$ by keeping $m$ fixed. Consider the base case $n = 0$. $0 \times m = m \times 0 = 0$ by definition and the lemma we have shown above. Assume inductively $n \times m = m \times n$. We want to show that $(n++) \times m = m \times (n++)$. By definition, we have the LFS is equal to $(n++) \times m = (n \times m) + m$. By the lemma we proved above, we have the RHS is equal to $m \times (n++) = m \times n + m = n \times m + m$. So both sides are equal to each other. We have closed the induction. $\square$