3.6 Cardinality of sets

Definition 3.6.1 (Equal cardinality).

We say that two sets X and Y have equal cardinality iff there exists a bijection $f : X \to Y$ from X to Y.

Proposition 3.6.4

Let X, Y, Z be sets. Then X has equal cardinality with X. If X has equal cardinality with Y, then Y has equal cardinality with X. If X has equal cardinality with Y and Y has equal cardinality with Z, then X has equal cardinality with Z.

Definition 3.6.5

Let n be a natural number. A set X is said to have cardinality n, iff it has equal cardinality with $\{i \in \mathbf{N} : 1 \leq i \leq n\}$. We also say that X has n elements iff it has cardinality n.

Proposition 3.6.8 (Uniqueness of cardinality).

Let X be aset with some cardinality n. Then X cannot have any other cardinality, i.e., X cannot have cardinality m for any $m \neq n$.

Lemma 3.6.9

Suppose that $n \geq 1$, and X has cardinality n. Then X is non-empty, and if x is any element of X, then the set $X - \{x\}$ (i.e., X with the element x removed) has cardinality $n - 1$.

Definition 3.6.10 (Finite sets).

A set is finite iff it has cardinality n for some natural number n; otherwise, the set is called infinite. If X is a finite set, we use $\#(X)$ to denote the cardinality of X.

Theorem 3.6.12

The set of natural numbers $\mathbf{N}$ is infinite.

Proposition 3.6.14 (Cardinal arithmetic).

See Exercise 3.6.4.

Exercises

Exercise 3.6.1

Prove Proposition 3.6.4.

- X has equal cardinality with X.

  Proof. Define function $f : X \to X$ such that for each $x \in X$, $f(x) = x$. For $x_1 \neq x_2$, $f(x_1) = x_1$ and $f(x_2) = x_2$. So $f(x_1) \neq f(x_2)$. Therefore, f is injective. By definition, for every $x \in X$, $f(x) = x$. So f is surjective. Thus, f is bijective and X has equal cardinality with X. □

- If X has equal cardinality with Y, then Y has equal cardinality with X.

  Proof. Since X has equal cardinality with Y, there exists a bijective function $f : X \to Y$. Since f is bijective, there exists $f^{-1} : Y \to X$. For $y_1, y_2 \in Y$, if we have $f^{-1}(y_1) = f^{-1}(y_2)$, by the definition of function, $f(f^{-1}(y_1)) = f(f^{-1}(y_2))$, then $y_1 = y_2$. So f^{-1} is injective. For every $x \in X$, we have $f(x) \in Y$ such that $f^{-1}(f(x)) = x$. So f^{-1} is surjective. Thus, f^{-1} is bijective and Y has equal cardinality with X. □

- If X has equal cardinality with Y and Y has equal cardinality with Z, then X has equal cardinality with Z.

  Proof. Since X has equal cardinality with Y, there exists a bijective function $f : X \to Y$. Since Y has equal cardinality with Z, there exists a bijective function $g : Y \to Z$. By Exercise 3.3.7, $g \circ f : X \to Z$ is also bijective. Thus, X has equal cardinality with Z. □

Exercise 3.6.2

Show that a set X has cardinality 0 if and only if X is the empty set.

Proof. By definition 3.6.5, X has n elements iff it has cardinality n. So since X has cardinality 0, it has no element in it which means X is the empty set. On the other hand, if X is the empty set, it has 0 element and thus has cardinality 0. $\square$

Exercise 3.6.3

Let n be a natural number, and let $f : \{i \in \mathbf{N} : 1 \leq i \leq n\} \to \mathbf{N}$ be a function. Show that there exists a natural number M such that $f(i) \leq M$ for all $1 \leq i \leq n$. Thus finite subsets of the natural numbers are bounded.

Proof. For function $f : \{i \in \mathbf{N} : 1 \leq i \leq n\} \to \mathbf{N}$, we claim that $M = \max\{f(1), \ldots, f(n)\}$. Induct on n. Base case: $n = 1$. Let $M = \max\{f(1)\} = f(1)$. $M \leq f(i)$ for $1 \leq i \leq 1$. This proves the base case. Then suppose inductively that there exists $M' = \max\{f(1), \ldots, f(n)\}$ is the upper bound for $f : \{i \in \mathbf{N} : 1 \leq i \leq n\} \to \mathbf{N}$. Now consider $f : \{i \in \mathbf{N} : 1 \leq i \leq n+1\} \to \mathbf{N}$. Let $M = \max\{M', f(n+1)\}$. For all $1 \leq i \leq n$, we have $f(i) \leq M' \leq M$. Also we have $f(n+1) \leq M$. Thus, $f(i) \leq M$ for all $1 \leq i \leq n+1$. This closes the induction. $\square$

Exercise 3.6.4

Prove proposition 3.6.14.

1. Let X be a finite set, and let x be an object which is not an element of X. Then $X \cup \{x\}$ is finite and $\#(X \cup \{x\}) = \#(X) + 1$.

   Proof. Use n to denote the cardinality of X. By Lemma 3.6.9, $\#(X) = \#((X \cup \{x\}) - \{x\}) = \#(X \cup \{x\}) - 1$. So $\#(X \cup \{x\}) = \#(X) + 1 = n + 1$ which is also a natural number. Thus, $X \cup \{x\}$ is finite and $\#(X \cup \{x\}) = \#(X) + 1$. $\square$

2. Let X and Y be finite sets. Then $X \cup Y$ is finite and $\#(X \cup Y) \leq \#(X) + \#(Y)$. If in addition X and Y are disjoint, then $\#(X \cup Y) = \#(X) + \#(Y)$.

Proof. Use m to denote the cardinality of $X = \{x_1, \dots, x_m\}$ and n to denote the cardinality of $Y = \{y_1 \dots, y_n\}$. Induct on n. The base case is when $\#(Y) = n = 0$. $\#(X \cup Y) = \#(X) \leq \#(X) + \#(Y) = \#(X)$. Now suppose inductively $\#(X \cup Y) \leq \#(X) + \#(Y)$ when $\#(Y) = n$ ($Y = \{y_1, \dots, y_n\}$). Consider when $Y = \{y_1, \dots, y_n, y_{n+1}\}$ and $\#(Y) = n+1$. If $y_{n+1} \in \{x_1, \dots, x_m\} \cup \{y_1, \dots, y_n\}$, then

$$\begin{aligned} \#(\{x_1, \dots, x_m\} \cup \{y_1, \dots, y_{n+1}\}) &= \#(\{x_1, \dots, x_m\} \cup \{y_1, \dots, y_n\}) \\ &\leq m + n < m + (n+1) = \#(X) + \#(Y). \end{aligned}$$

So in this case, $\#(X \cup Y) < \#(X) + \#(Y)$. If $y_{n+1} \notin \{x_1, \dots, x_m\} \cup \{y_1, \dots, y_n\}$, then

$$\#(\{x_1, \dots, x_m\} \cup \{y_1, \dots, y_{n+1}\}) = \#(\{x_1, \dots, x_m\} \cup \{y_1, \dots, y_n\}) + 1,$$

since by induction hypothesis we have,

$$\#(\{x_1, \dots, x_m\} \cup \{y_1, \dots, y_n\}) \leq m + n,$$

then

$$\#(\{x_1, \dots, x_m\} \cup \{y_1, \dots, y_{n+1}\}) \leq m + (n+1) = \#(X) + \#(Y).$$

So in this case, $\#(X \cup Y) \leq \#(X) + \#(Y)$. Thus, in both cases, we have $\#(X \cup Y) \leq \#(X) + \#(Y)$. This closes the induction. Hence, since both X and Y are finite, $X \cup Y$ is also finite.

If X and Y are disjoint, by Lemma 3.6.9, we have

$$\begin{gathered} \#(X \cup Y - \{y_1\}) = \#(X \cup Y) - 1, \\ \#((X \cup Y - \{y_1\})) = \#(X \cup Y - \{y_1\}) - 1, \\ \vdots \\ \#((X \cup Y - \{y_1\} - \cdots - \{y_{n-1}\}) - \{y_n\}) = \#(X \cup Y - \cdots - \{y_{n-1}\}) - 1. \end{gathered}$$

Sum these n equations up, we have

$$\#(X) = \#((X \cup Y - \{y_1\} - \cdots - \{y_{n-1}\}) - \{y_n\}) = \#(X \cup Y) - \#(Y).$$

Thus, $\#(X \cup Y) = \#(X) + \#(Y)$. □

3. Let X be a finite set, and let Y be a subset of X. Then Y is finite, and $\#(Y) \leq \#(X)$. If in addition $Y \neq X$, then we have $\#(Y) < \#(X)$.

   Proof. Assume $Y \neq X$. Denote $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_m\}$. Induct on n. When $n \leq m$, the statement is vacuously true. Suppose inductively that $\#(Y) < \#(X)$ is true. Consider when $X = \{x_1, \ldots, x_{n+1}\}$, $\#(\{x_1, \ldots, x_{n+1}\}) = \#(\{x_1, \ldots, x_n\})+1$. So $\#(\{y_1, \ldots, y_m\}) < \#(\{x_1, \ldots, x_n\}) < \#(\{x_1, \ldots, x_n\})+1 = \#(\{x_1, \ldots, x_{n+1}\})$. This closes the induction. For the case $Y = X$, $\#(Y) = \#(X)$, so $\#(Y) \leq \#(X)$. Since $\#(Y) \leq \#(X)$ and X is finite, Y is also finite. □

4. If X is a finite set, and $f : X \to Y$ is a function, then $f(X)$ is a finite set with $\#(f(X)) \leq \#(X)$. If in addition f is one-to-one, then $\#(f(X)) = \#(X)$.

   Proof. Denote $X = \{x_1, \ldots, x_n\}$. Induct on n. When $n = 0$, $\#f(X) = \#(X) = 0$. The base case is proved. Suppose inductively $\#(f(X)) \leq \#(X)$ is true for $n \in \mathbf{N}$. Now consider $X = \{x_1, \ldots, x_n, x_{n+1}\}$. By Lemma 3.6.9, $\#(\{x_1, \ldots, x_{n+1}\}) = \#(\{x_1, \ldots, x_n\}) + 1$. By Proposition 3.6.14-(b), $f(\{x_1, \ldots, x_n, x_{n+1}\}) = f(\{x_1, \ldots, x_n\}) \cup f(x_{n+1}) \leq \#f(\{x_1, \ldots, x_n\}) + 1 = \#(\{x_1, \ldots, x_{n+1}\})$. This closes the induction.

   If f is one-to-one, the proof is similar and we only need to modify a bit from the previous one. The proof of the base case stays the same. Suppose inductively $\#(f(X)) = \#(X)$. Now $f(\{x_1, \ldots, x_{n+1}\}) = f(\{x_1, \ldots, x_n\}) \cup f(x_{n+1})$, since f is injective, these two sets are disjoint. By Proposition 3.6.14.(b), $\#(f\{x_1, \ldots, x_{n+1}\}) = \#(X) + 1 = \#(\{x_1, \ldots, x_{n+1}\})$. This closes the induction. □

5. Let X and Y be finite sets. Then Cartesian product $X \times Y$ is finite and $\#(X \times Y) = \#(X) \times \#(Y)$.

   Proof. Let X has equal cardinality with $\{i \in \mathbf{N} : 1 \leq i \leq n\}$ and y has equal cardinality with $\{i \in \mathbf{N} : 1 \leq i \leq m\}$, use f to denote this function. The statement we need to prove is $\#(X \times Y)$ has equal cardinality with $\{i \in \mathbf{N} : 1 \leq i \leq nm\}$. Induct on n. When $n = 0$, $\#(X \times Y) = \#(X) \times \#(Y) = 0$. Suppose the statement is true for $n \in \mathbf{N}$. Consider when X has the same cardinality with $\{i \in \mathbf{N} : 1 \leq i \leq n+1\}$. By induction hypothesis, there exists a bijective function from $\#(X \times Y)$ to $\{i \in \mathbf{N} : 1 \leq i \leq nm\}$. Define the map from $X \times Y$ (partially) to $\{i \in \mathbf{N} : 1 \leq i \leq n+1\}$ as: for $x = x_{n+1} \in X$, $y \in Y$, $g(x, y) = nm + f(y)$. We need to verify that g is also bijective.

   For any $j \in \{i \in \mathbf{N} : 1 \leq i \leq nm\}$, by induction hypothesis, there exists a bijective function h from $X \times Y$ to $\{i \in \mathbf{N} : 1 \leq i \leq nm\}$, so there exists some $x \in X, y \in Y$ such that $h(x, y) = j$. Let $g(x, y) = h(x, y) = j$ for $x \in X - \{x_{n+1}\}$ and $y \in Y$. For any $j \in \{i \in \mathbf{N} : nm + 1 \leq i \leq (n+1)m\}$, we have $g(n+1, j - nm)$. Thus, function g is surjective. Suppose $x_1, x_2 \in X - \{x_{n+1}\}$, $y_1, y_2 \in Y$, $(x_1, y_1) \neq (x_2, y_2)$, by induction hypothesis, $g(x_1, y_1) \neq g(x_2, y_2)$. For $x_1 \in X - \{x_{n+1}\}$, $x_2 = x_{n+1}$, $y_1, y_2 \in Y$, we have $g(x_1, y_1) \leq nm$ and $g(x_2, y_2) > nm$. So $g(x_1, y_1) \neq g(x_2, y_2)$. For $x_1 = x_2 = x_{n+1}$ and $y_1, y_2 \in Y, y_1 \neq y_2$, by definition, $g(x_1, y_1) \neq g(x_2, y_2)$. Thus, g is injective. So g is bijective and thus $\#(X \times Y)$ has equal cardinality with $\{i \in \mathbf{N} : 1 \leq i \leq (n+1)m\}$. □

6. Let X and Y be finite sets. Then the set Y^X is finite and $\#(Y^X) = \#(Y)^{\#(X)}$.

   Proof. Denote $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$. The statement we need to prove is Y^X has equal cardinality with $\{i \in \mathbf{N} : 1 \leq i \leq m^n\}$. Induct on n. When $n = 0$, the number of functions from X to the empty set is 1 which is equal to m^0. This proved the base case. Suppose inductively Y^X has equal cardinality with $\{i \in \mathbf{N} : 1 \leq i \leq m^n\}$ when $X = \{x_1, \ldots, x_n\}$. Now consider when $X = \{x_1, \ldots, x_{n+1}\}$. We want to show that it has equal cardinality with $M = \{i \in \mathbf{N} : 1 \leq i \leq m^{n+1}\}$. Define function g that maps function f such that $f(x_{n+1}) = y_i \in Y$ to $(m^n + i) \in M$. The proof of the bijectivity of g is similar to (e). Once we have proved g is bijective, the induction is closed. □

Exercise 3.6.5

Let A and B be sets. Show that $A \times B$ and $B \times A$ have equal cardinality by constructing an explicit bijection between the two sets. Then use Proposition 3.6.14 to conclude an alternate proof of Lemma 2.3.2.

Proof. Firstly, we need to define a bijective function $f : A \times B \to B \times A$ such that for $a \in A$ and $b \in B$, $f(a,b) = (b,a)$. Suppose $(a_1, b_1), (a_2, b_2) \in A \times B$, $f(a_1, b_1) = f(a_2, b_2)$. Then $(b_1, a_1) = (b_2, a_2)$. By definition of ordered pair, $a_1 = a_2$ and $b_1 = b_2$. Therefore, $(a_1, b_1) = (a_2, b_2)$. Thus, f is injective. For any $(b, a) \in B \times A$, there exists $(a, b) \in A \times B$ such that $f(a, b) = (b, a)$. So f is surjective. Hence, f is bijective, and $A \times B$ and $B \times A$ have equal cardinality. Suppose $\#(A) = n$ and $\#(B) = m$. By Proposition 3.6.14,

$$\#(A \times B) = \#(A) \times \#(B) = n \times m,$$
$$\#(B \times A) = \#(B) \times \#(A) = m \times n,$$

and since

$$\#(A \times B) = \#(B \times A),$$

we have

$$n \times m = m \times n.$$

□

Exercise 3.6.6

Let A, B, C be sets. Show that the sets $(A^B)^C$ and $A^{B \times C}$ have equal cardinality by constructing an explicit bijection between the two sets. Conclude that $(a^b)^c = a^{bc}$ for any natural numbers a, b, c. Use similar argument to also conclude $a^b \times a^c = a^{b+c}$.

Proof. Suppose we have $f : B \to A$, $g : C \to A^B$ and $l : B \times C \to A$, let $h : (A^B)^C \to A^{B \times C}$ be a function such that for $g \in (A^B)^C$, for all $b \in B, c \in C$, $(h(g))(b, c) =$

$(g(c))(b)$. Suppose $g_1, g_2 \in (A^B)^C$ and $h(g_1) = h(g_2)$. Then by definition, for any $b \in B$, $c \in C$, $(g_1(c))(b) = (g_2(c))(b)$. Let $g_1(c) = f_1$ and $g_2(c) = f_2$. Since for all $b \in B$, $f_1(b) = f_2(b)$, we have $f_1 = f_2$. Then for all $c \in C$, $f_1 = g_1(c) = g_2(c) = f_2$, so $g_1 = g_2$. Thus, h is injective. For any function l, for any $b \in B$, $c \in C$, let $f(b) = l(b, c)$ and $g(c) = f$, by definition, we have $(h(g))(b, c) = l$. Thus, h is surjective. Since h is both injective and surjective, h is bijective. Thus, $(A^B)^C$ and $A^{B\times C}$ have equal cardinality.

Suppose $\#(A) = a$, $\#(B) = b$ and $\#(C) = c$. By Proposition 3.6.14,

$$\#(A^B)^C = \#(A^B)^{\#(C)} = (\#(A)^{\#(B)})^{\#(C)} = (a^b)^c,$$
$$\#(A^{B\times C}) = \#(A)^{\#(B\times C)} = \#(A)^{\#(B)\times\#(C)} = a^{b\times c},$$

since $\#((A^B)^C) = \#(A^{B\times C})$, we have

$$(a^b)^c = a^{b\times c}.$$

Suppose $f : B \to A$, $g : C \to A$, $l : B \cup C \to A$ and B and C are disjoint. Define $h : A^B \times A^C \to A^{B\cup C}$ as for $b \in B$, $c \in C$, $(h(f, g))(b) = f(b)$ and $(h(f, g))(c) = g(c)$. We can show that h is bijective in a similar way. Use this argument and Proposition 3.6.14 denote $\#(A) = a, \#(B) = b, \#(C) = c$, we can show that $a^b \times a^c = a^{b+c}$. □

Exercise 3.6.7

Let A and B be sets. Let us say that A has lesser or equal cardinality to B if there exists an injection $f : A \to B$ from A to B. Show that if A and B are finite sets, then A has lesser or equal cardinality to B iff $\#(A) \leq \#(B)$.

Proof. We need to show that $\exists$ injection $f : A \to B \iff \#(A) \leq \#(B)$.

Suppose $\exists$ injection $f : A \to B$. By Proposition 3.6.14, $\#(f(A)) = \#(A)$. By definition of function, if $a \in A$, $f(a) \in B$. So $f(A) \subseteq B$. Again by Proposition 3.6.14, $\#f(A) \leq \#f(B)$. Hence, $\#(A) = \#f(A) \leq \#(B)$.

Suppose $\#(A) \leq \#(B)$. Let $\#(A) = m$ and $\#(B) = n$. By definition, there exists a bijection $f : A \to \{i \in \mathbf{N} : 1 \leq i \leq m\}$ and a bijection $g : B \to \{i \in \mathbf{N} : 1 \leq i \leq n\}$. As we have shown in Exercise 3.6.1, g^{-1} is also bijective. Consider $h = g^{-1} \circ f$.

$\{i \in \mathbf{N} : 1 \leq i \leq m\} \in \{i \in \mathbf{N} : 1 \leq i \leq n\}$, so the range of f is within the domain of g^{-1}. Then h is a valid function from A to B. We need to show that h is injective. Suppose we have $a_1, a_2 \in A$ such that $g^{-1}(f(a_1)) = g^{-1}(f(a_2))$. Since g^{-1} is bijective, there must be $f(a_1) = f(a_2)$. Since f is bijective, we have $a_1 = a_2$. Thus, h is injective. By definition, A has lesser or equal cardinality to B. □

Exercise 3.6.8

Let A and B be sets such that there exists an injection $f : A \to B$ from A to B. Assume also that A is non-empty. Show that there exists a surjection $g : B \to A$ from B to A.

Proof. Define $g : B \to A$ as below

$$g(b) = \begin{cases} a \text{ such that } f(a) = b, & b \in f(A) \\ 0, & \text{otherwise} \end{cases}.$$

Then for any $a \in A$, we have $f(a) \in B$ such that $g(f(a)) = a$. Hence, g is surjective. □

Exercise 3.6.9

Let A and B be finite sets. Show that $A \cup B$ and $A \cap B$ are also finite sets, and that $\#(A) + \#(B) = \#(A \cup B) + \#(A \cap B)$.

Proof. By Exercise 3.1.10, $A \cup B = (A \backslash B) \cup (B \backslash A) \cup (A \cap B)$ and $A \backslash B$, $B \backslash A$, and $A \cap B$ are disjoint. $A = A \backslash B + A \cap B \implies \#(A) = \#(A \backslash B) + \#(A \cap B)$. $B = B \backslash A + A \cap B \implies \#(B) = \#(B \backslash A) + \#(A \cap B)$. So $\#(A) + \#(B) = (\#(A \backslash B) + \#(B \backslash A) + \#(A \cap B)) + \#(A \cap B) = \#(A \cap B) + \#(A \cap B)$. □

Exercise 3.6.10

Let $A_1, \ldots, A_n$ be finite sets such that $\#(\bigcup_{i \in \{1,\ldots,n\}} A_i) > n$. Show that there exists $i \in \{1, \ldots, n\}$ such that $\#(A_i) \geq 2$. (This is known as the pigeonhole principle.)

Proof. Suppose for all $i \in \{1, \dots, n\}$, $\#(A_i) \leq 1$. Then $\#(\bigcup_{i \in \{1,\dots,n\}} A_i) = \#(A_1 \cup \dots \cup A_n) \leq \#(A_1) + \cdots + \#(A_n) \leq n$. (Contradiction.) Therefore, there exists $i \in \{1, \dots, n\}$ such that $\#(A) \geq 2$. $\square$