3.3 Functions

Definition 3.3.1 (Functions).

Let X, Y be sets, and let $P(x, y)$ be a property pertaining to an object $x \in X$ and an object $y \in Y$, such that for every $x \in X$, there is exactly one $y \in Y$ for which $P(x, y)$ is true (this is sometimes known as the vertical line test). Then we define the function $f : X \to Y$ defined by P on the domain X and range Y to be the object which, given any input $x \in X$, assigns an output $f(x) \in Y$, defined to be the unique object $f(x)$ for which $P(x, f(x))$ is true. Thus, for any $x \in X$ and $y \in Y$,

$$y = f(x) \iff P(x, y) \text{ is true.}$$

Definition 3.3.7 (Equality of functions).

Two functions $f : X \to Y$, $g : X \to Y$ with the same domain and range are said to be equal, $f = g$, if and only if $f(x) = g(x)$ for all $x \in X$. If $f(x)$ and $g(x)$ agree for some values of x, but not others, then we do not consider f and g to be equal. If two functions f, g have different domains, or different ranges, we also do not consider them to be equal.

Definition 3.3.11 (Composition).

Let $f : X \to Y$ and $g : Y \to Z$ be two functions, such that the range of f is the same set as the domain of g. We then define the composition $g \circ f : X \to Z$ of the two functions g and f to be the function defined explicitly by the formula

$$(g \circ f)(x) := g(f(x)).$$

If the range of f does not match the domain of g, we leave the composition $g \circ f$ undefined.

Lemma 3.3.13 (Composition is associative).

Let $f : Z \to W$, $g : Y \to Z$, and $h : X \to Y$ be functions. Then $f \circ (g \circ h) = (f \circ g) \circ h$.

Definition 3.3.15 (One-to-one functions).

A function f is one-to-one (or injective) if different elements map to different elements:

$$x \neq x' \implies f(x) \neq f(x').$$

Equivalently, a function is one-to-one if

$$f(x) = f(x') \implies x = x'.$$

Definition 3.3.18 (Onto functions).

A function f is onto (or surjective) if every element if Y comes from applying f to some element in X:

$$\text{For every } y \in Y, \text{ there exists } x \in X \text{ such that } f(x) = y.$$

Definition 3.3.21 (Bijective functions).

Functions $f : X \to Y$ which are both one-to-one and onto are also called bijective or invertible.

Exercises

Exercise 3.3.1

Show that the definition of equality in Definition 3.3.7 is reflexive, symmetric, and transitive. Also verify the substitution property: if $f, \tilde{f} : X \to Y$ and $g, \tilde{g} : Y \to Z$ are functions such that $f = \tilde{f}$ and $g = \tilde{g}$, then $g \circ f = \tilde{g} \circ \tilde{f}$.

Proof. Reflexivity: f and f have the same domain and range, and $f(x) = f(x)$ for all x in the domain of f. Therefore, f is equal to itself.

Symmetry: g and f have the same domain and range. For every x in the domain of g, we have $g(x) = f(x)$. Therefore, by Definition 3.3.7, $g(x)$ and $f(x)$ are equal.

Transitivity: Suppose f and g have the same domain and range, and for every x in the domain of f, $f(x) = g(x)$. And g and h have the same domain and range, and for every x in the domain of g, we have $g(x) = h(x)$. Then f and h have the same domain and range. $\forall x$ in the domain of f, we have $f(x) = g(x) = h(x)$. Therefore, f and h are equal.

Substitution property: Since $g \circ f, \tilde{g} \circ \tilde{f} : X \to Z$, they have the same domain and range. And for every $x \in X$, we have $f(x) = \tilde{f}(x)$, since $g = \tilde{g}$, we also have $g(f(x)) = \tilde{g}(f(x)) = \tilde{g}(\tilde{f}(x))$. Therefore, $g \circ f = \tilde{g} \circ \tilde{f}$. □

Exercise 3.3.2

Let $f : X \to Y$ and $g : Y \to Z$ be functions. Show that if f and g are both injective, then so is $g \circ f$; similarly, show that if f and g are both surjective, then so is $g \circ f$.

1. If if f and g are both injective, then so is $g \circ f$.

   Proof. f is injective:

   $$x \in X, x' \in X, x \neq x' \implies f(x) \neq f(x').$$

   g is injective:

   $$f(x) \in Y, f(x') \in Y, f(x) \neq f(x') \implies g(f(x)) \neq g(f(x')).$$

   Therefore, $x \neq x' \implies (g \circ f)(x) \neq (g \circ f)(x')$. Thus, $g \circ f$ is injective. □

2. If f and g are both surjective, then so is $g \circ f$.

   Proof. f is surjective:

   $$\text{For every } y \in Y, \text{ there exists } x \in X \text{ such that } f(x) = y.$$

   g is surjective:

   $$\text{For every } z \in Z, \text{ there exists } y \in Y \text{ such that } g(y) = z.$$

   Therefore, for every $z \in Z$, there exists $x \in X$ such that $(g \circ f)(x) = g(f(x)) = g(y) = z$. Thus, $g \circ f$ is surjective. □

Exercise 3.3.3

When is the empty function injective? surjective? bijective?

The empty function is of the form $f : \emptyset \to X$. It is always injective no matter what X is. It is surjective if X is $\emptyset$. It is bijective if X is $\emptyset$.