

2.2.1

For any natural numbers a, b, c , we have $(a + b) + c = a + (b + c)$.

Proof. Induct on b by keeping a and c fixed. Consider the base case $b = 0$. In this case, LHS = $(a + 0) + c = a + c$ and RHS = $a + (0 + c) = a + c$. Now suppose that $(a + b) + c = a + (b + c)$. We need to show that $(a + (b + +)) + c = a + ((b + +) + c)$:

$$\text{LHS} = (a + (b + +)) + c = ((a + b) + +) + c = (a + b + c) + +,$$

$$\text{RHS} = a + ((b + +) + c) = a + ((b + c) + +) = (a + b + c) + +.$$

Thus both sides are equal to each other, and we have closed the induction. □

2.2.2

Let a be a positive number. Then there exists exactly one natural number b such that $b + + = a$. (I'm assuming that it meant a is a positive natural number.)

Proof. Induct on a . Since 0 is not positive, we consider the base case $a = 1$. We have $b + + = b + 1 = 0 + 1 = 1$. Cancellation law tells us that $b = 0$, which is unique. Now suppose that there exists exactly one natural number b_0 such that $b_0 + + = a$, we need to show that there exists exactly one natural number b such that $b + + = a + +$. By Cancellation law, we have $b = a = b_0 + +$. Since b is the successor of b_0 and b_0 is unique, b is also unique. Thus we have closed the induction. □

2.2.3

(a)

$a \geq a$.

Proof. There exists a natural number 0 such that $a + 0 = a$. Thus, $a \geq a$. □

(b)

If $a \geq b$ and $b \geq c$, then $a \geq c$.

Proof. Since $a \geq b$, there exists a natural number m such that $b + m = a$. Since $b \geq c$, there exists a natural number n such that $c + n = b$. Then $c + (n + m) = (c + n) + m = b + m = a$. Therefore, $c \geq a$.

Thus, if $a \geq b$ and $b \geq c$, then $a \geq c$. \square

(c)

If $a \geq b$ and $b \geq a$, then $a = b$.

Proof. Since $a \geq b$, there exists a natural number m such that $b + m = a$. Since $b \geq a$, there exists a natural number n such that $a + n = b$. Then we have $a + n = (b + m) + n = b + (m + n) = b$. By Cancellation law, we have $m + n = 0$ which leads to $m = 0, n = 0$. Thus, $a = a + 0 = b$.

Thus, if $a \geq b$ and $b \geq a$, then $a = b$. \square

(d)

$a \geq b$ if and only if $a + c \geq b + c$.

Proof. First, we need to show that $a \geq b \Rightarrow a + c \geq b + c$. Since $a \geq b$, there exists a natural number n such that $b + n = a$. Then we have $b + n + c = b + c + n = (b + c) + n = a + c$. Thus, $a + c \geq b + c$. Then, we need to show that $a + c \geq b + c \Rightarrow a \geq b$. Since $a + c \geq b + c$, there should be a natural number n such that $b + c + n = b + n + c = (b + n) + c = a + c$. By Cancellation law, we have $b + n = a$. Thus, $a \geq b$.

Thus, if $a \geq b$ and $b \geq a$, then $a = b$. \square

(e)

$a < b$ if and only if $a + + \leq b$.

Proof. First, we need to show that $a < b \Rightarrow a + + \leq b$. $a < b$ means there exists a natural number n such that $a + n = b$, particularly, $a \neq b$. Then n must not be zero. So n is the predecessor of a natural number, denote it as m . Then we have $a + n = a + (m + +) = (a + m) + + = (a + +) + m = b$. Therefore, $a + + \leq b$. Then we need to show that $a + + \leq b \Rightarrow a < b$. There exists a natural number n such that $(a + +) + n = b$. $(a + +) + n = (a + n) + + = a + (n + +) = b$. Since $n + +$ is the successor of n , $n + +$ must not be equal to 0. If $a = b$, there will be $a + (n + +) = a \Rightarrow n + + = 0$, contradiction. Therefore, $a \neq b$.

Thus, $a < b$ if and only if $a + + \leq b$. □

(f)

$a < b$ if and only if $b = a + d$ for some positive number d .

Proof. First, we need to show that $a < b \Rightarrow b = a + d$ for some positive number d . There exists some natural number d such that $a + d = b$, $a \neq b$. By Cancellation law, d must not be zero. Therefore, d is positive. Then, we need to show that $a + d = b$ for some positive $d \Rightarrow a < b$. We only need to prove $a \neq b$. If $a = b$, we have $a + d = a = a + 0$. By Cancellation law, $d = 0$ which contradicts to d is positive. Therefore, $a \neq b$.

Thus, $a < b$ if and only if $b = a + d$ for some positive number d . □

2.2.4

Justify the three statements marked in the proof of Proposition 2.2.13.

(a)

$0 \leq b$ for all b .

Proof. By definition of addition, we have $0 + b = b$. Thus, $0 \leq b$. □

(b)

If $a > b$, then $a++ > b$.

Proof. By Proposition 2.2.12.e, we have $a > b \Rightarrow a \geq b++$. And by Proposition 2.2.12.d, $a+1 \geq (b++) + 1$ that is equivalent to $a++ \geq b+2$. Since 2 is positive, by Proposition 2.2.12.f, we have $a++ > b$. \square

(c)

If $a = b$, then $a++ > b$.

Proof. We know from Proposition 2.2.12.a that $a \geq a$, so $a \geq a = b$. And again by Proposition 2.2.12.d, we have $a++ = a+1 \geq b+1$. Since 1 is positive, by Proposition 2.2.12.f, $a++ > b$. \square

2.2.5

Proposition 2.2.14 (Strong principle of induction). Let m_0 be a natural number, and let $P(m)$ be a property pertaining to an arbitrary natural number m . Suppose that for each $m \geq m_0$, we have the following implication: if $P(m')$ is true for all natural numbers $m_0 \leq m' < m$, then $P(m)$ is also true. Then we can conclude that $P(m)$ is true for all natural numbers $m \geq m_0$.

Proof. Let $Q(n)$ be the property that $P(m)$ is true for all $m_0 \leq m < n$. Induct on n . Consider the base case $n = 0$. This is vacuously true. In fact, $Q(n)$ is vacuously true for all $n \leq m_0$. So we can assume $n > m_0$ to see if the implication stands. Suppose $Q(n)$ is true, that is, $P(m)$ is true for all $m_0 \leq m < n$. We want to show that $Q(n+1)$ is also true. As stated in Proposition 2.2.14, if $Q(n)$ is true, then $P(n)$ is also true. So $P(m)$ is true for all $m_0 \leq m \leq n$. Hence, $P(m)$ is true for all $m_0 \leq m < n+1$. ($m \leq n \Leftrightarrow m < n+1$ can be shown using prop 2.2.12.) Thus, $Q(n+1)$ is true. This closes the induction. \square

2.2.6

Let n be a natural number, and let $P(m)$ be a property pertaining to the natural numbers such that whenever $P(m++)$ is true, then $P(m)$ is true. Suppose that $P(n)$ is also true. Prove that $P(m)$ is true for all natural numbers $m \leq n$. (Principle of backwards induction.)

Proof. Apply induction to n . For the base case $n = 0$, suppose $P(0)$ is true. In this case, m can only be 0 ($m + k = 0 \Rightarrow m = 0, k = 0$). Since $P(0)$ is true, the base case is proved. Next, suppose if $P(n)$ is true then $P(m)$ is true for all natural numbers $m \leq n$. We want to show that if $P(n++)$ is true then $P(m)$ is true for all natural numbers $m \leq n++$. $m \leq n$ means there exists a natural number a such that $m + a = n++$. a is either 0 or a positive number. If a is 0, $m = n++$. If a is positive, $m < n++$ (by prop 2.2.12.f), this is equivalent to $m \leq n$ (can be shown using prop 2.2.12.). For $m = n++$, $P(m)$ is true because of the assumption. For each $m \leq n$, $P(m)$ is also true by induction hypothesis. Therefore, $P(m)$ is true for all natural numbers $m \leq n++$. And we have closed the induction. \square

In the above proofs, $n++$ and $n+1$ got mixed up because $n++ = n+1$ has been illustrated on Page 26 (and the $+1$ version is a little easier). But $++$ is a more desirable expression since it stands for the successor in a general way.