Chapter 4

Integers and rationals

4.1 The integers

Definition 4.1.1 (Integers).

An integer is an expression of the form $a--b$, where a and b are natural numbers. Two integers are considered to be equal, $a--b = c--d$, if and only if $a+d = c+b$. We let $\mathbf{Z}$ denote the set of all integers.

Definition 4.1.2

The sum of two integers, $(a--b)+(c--d)$, is defined by the formula

$$(a--b)+(c--d) := (a+c)-(b+d).$$

The product of two integers, $(a--b)\times(c--d)$, is defined by

$$(a--b)\times(c--d) := (ac+bd)--(ad+bc).$$

Lemma 4.1.3 (Addition and multiplication are well-defined).

Let a, b, a', b', c, d be natural numbers. If $(a--b) = (a'--b')$, then $(a--b)+(c--d) = (a'--b')+(c--d)$ and $(a--d)\times(c--d) = (a'--b')\times(c--d)$, and also $(c--d)+(a--b) = (c--d)+(a'--b')$ and $(c--d)\times(a--b) = (c--d)\times(a'--b')$. Thus addition and multiplication are well-defined operations (equal inputs give equal outputs).

Definition 4.1.4 (Negation of integers).

If $(a--b)$ is an integer, we define the negation $-(a--b)$ to be the integer $(b--a)$. In particular if $n = n--0$ is a positive natural number, we can define its negation $-n = 0--n$.

Lemma 4.1.5 (Trichotomy of integers).

Let x be an integer. Then exactly one of the following three statements is true: (a) x is zero; (b) x is equal to a positive natural number n; or (c) x is the negation $-n$ of a positive natural number n.

Proposition 4.1.6 (Laws of algebra for integers).

Let x, y, z be integers. Then we have

$$x + y = y + x$$
$$(x + y) + z = x + (y + z)$$
$$x + 0 = 0 + x = x$$
$$x + (-x) = (-x) + x = 0$$
$$xy = yx$$
$$(xy)z = x(yz)$$
$$x1 = 1x = x$$
$$x(y + z) = xy + xz$$
$$(y + z)x = yx + zx$$

Proposition 4.1.8 (Integers have no zero divisors).

Let a and b be integers such that $ab = 0$. Then either $a = 0$ or $b = 0$ (or both).

Corollary 4.1.9 (Cancellation law for integers).

If a, b, c are integers such that $ac = bc$ and c is non-zero, then $a = b$.

Definition 4.1.10 (Ordering of the integers).

If n and m be integers. We say that n is greater than or equal to m, and write $n \geq m$ or $m \leq n$, iff we have $n = m + a$ for some natural number a. We say that n is strictly greater than m, and write $n > m$ or $m < n$, iff $n \geq m$ and $n \neq m$.

Lemma 4.1.11 (Properties of order).

Let a, b, c be integers.

(a) $a > b$ if and only if $a - b$ is a positive natural number.

(b) (Addition preserves order) If $a > b$, then $a + c > b + c$.

(c) (Positive multiplication preserves order) If $a > b$ and c is positive, then $ac > bc$.

(d) (Negation reverses order) If $a > b$ and $b > c$, then $a > c$.

(e) (Order trichotomy) Exactly one of the statements $a > b$, $a < b$, or $a = b$ is true.

Exercise 4.1.1

Verify that the definition of equality on the integers is both reflexive and symmetric.

Proof. Reflexivity: since summation is reflexive, we have $a + b = a + b$. Thus, by definition, $a - -b = a - -b$. Symmetry: assume $a - -b = c - -d$, then $a + d = c + b$. Since summation is symmetric, $c + b = a + d$. By definition, we have $c - -d = a - -b$. $\square$

Exercise 4.1.2

Show that the definition of negation on the integers is well-defined in the sense that $(a - -b) = (a' - -b')$, then $-(a - -b) = -(a' - -b')$ (so equal integers have equal negations).

Proof. Since $(a - -b) = (a' - -b')$, by definition, $a + b' = a' + b$. By the reflexivity and symmetry of summation, we have $b + a' = b' + a$. Thus, by definition, $b - -a = b' - -a'$. By definition of negation of integers, $-(a - -b) = -(a' - -b')$. $\square$

Exercise 4.1.3

Show that $(-1) \times a = -a$ for every integer a.

Proof. By definition, $-1 = (0 - -1)$ and $a = (a - -0)$. Then $(-1) \times a = (0 - -1) \times (a - -0) = (0 \times a + 1 \times 0) - -(0 \times 0 + 1 \times a) = 0 - -a = -a$. $\square$

Exercise 4.1.4

Prove the remaining identities in Proposition 4.1.6.

1. $x + y = y + x$.

   Proof. Suppose $x = a - -b$ and $y = c - -d$ for some natural numbers a, b, c, d. Then $x + y = (a - -b) + (c - -d) = (a + c) - -(b + d)$ and $y + x = (c - -d) + (a - -b) = (c + a) - -(d + b)$. By the symmetry property of summation, we have $(a + c) = (c + a)$ and $(b + d) = (d + b)$. Thus, $x + y = y + x$. □

2. $(x + y) + z = x + (y + z)$.

   Proof. Suppose $x = a - -b$, $y = c - -d$, and $z = e - -f$ for some natural numbers $a, b, c, d, e, f,$. Then

$$
\begin{aligned}
(x + y) + z &= ((a - -b) + (c - -d)) + (e - -f) \\
&= ((a + c) - -(b + d)) + (e - -f) \\
&= ((a + c) + e) - -((b + d) + f) \\
&= (a + c + e) - -(b + d + f); \\
x + (y + z) &= (a - -b) + ((c - -d) + (e - -f)) \\
&= (a - -b) + ((c + e) - -(d + f)) \\
&= (a + (c + e)) - -(b + (d + f)) \\
&= (a + c + e) - -(b + d + f).
\end{aligned}
$$

   Therefore, $(x + y) + z = x + (y + z)$. □

3. $x + 0 = 0 + x = x$.

   Proof. Since $x + y = y + x$, we have $x + 0 = 0 + x$. Let $x = a - -b$ for some natural numbers a, b, and write $0 = 0 - -0$. Then $x + 0 = (a - -b) + (0 - -0) = (a + 0) - -(b + 0) = a - -b = x$. Thus, $x + 0 = 0 + x = x$. □

4. $x + (-x) = (-x) + x = 0$.

Proof. Since $x + y = y + x$, we have $x + (-x) = (-x) + x$. Let $x = a - -b$ for some natural numbers a, b, then $-x = b - -a$. Write 0 as $0 - -0$. Then $x + (-x) = (a - -b) + (b - -a) = (a + b) - -(b + a)$. Since $(a + b) + 0 = (b + a) + 0 = a + b$, we have that $(a + b) - -(b + a) = 0 - -0$. So $x + (-x) = 0$. Thus, $x + (-x) = (-x) + x = 0$. □

5. $xy = yx$.

Proof. Let $x = a - -b$ and $y = c - -d$ for some natural numbers a, b, c, d. Then

$$\begin{aligned} xy &= (a - -b) \times (c - -d) \\ &= (ac + bd) - -(ad + bc); \\ yx &= (c - -d) \times (a - -b) \\ &= (ca + db) - -(cb + da) \\ &= (ac + bd) - -(ad + bc). \end{aligned}$$

Therefore, $xy = yx$. □

6. $(xy)z = x(yz)$.
Has been proved on page 79.

7. $x1 = 1x = x$.

Proof. Since $xy = yx$, we have $x1 = 1x$. Let $x = a - -b$ for some natural numbers a, b. $1x = (1 - -0)(a - -b) = 1a - -1b = a - -b = x$. Thus, $x1 = 1x = x$. □

8. $x(y + z) = xy + xz$.

Proof. Let $x = a - -b$, $y = c - -d$, and $z = e - -f$ for some natural numbers a, b, c, d, e, f. Then

$$\begin{aligned}
x(y + z) &= (a - -b)((c - -d) + (e - -f)) \\
&= (a - -b)((c + e) - -(d + f)) \\
&= (a(c + e) + b(d + f)) - -(a(e + f) + b(c + d)) \\
&= (ac + ae + bd + bf) - -(ae + af + bc + bd); \\
xy + xz &= (a - -b)(c - -d) + (a - -b)(e - -f) \\
&= ((ac + bd) - -(ad + bc)) + ((ae + bf) - -(af + be)) \\
&= ((ac + bd) + (ae + bf)) - -((ad + bc) + (af + be)) \\
&= (ac + ae + bd + bf) - -(ae + af + bc + bd).
\end{aligned}$$

Therefore, $x(y + z) = xy + xz$. □

9. $(y + z)x = yx + zx$.

Proof. Since $xy = yx$, we have $(y + z)x = x(y + z)$. By using identities, we get $xy + xz = yx + zx$. And because $x(y + z) = xy + xz$, $(y + z)x = yx + zx$. □

Exercise 4.1.5

Prove Proposition 4.1.8.

Proof. From now on we could just use $-$ instead of $--$. Let $a = c - d$ and $b = e - f$ for some natural numbers c, d, e, f. So $ab = 0 \implies (c - d)(e - f) = 0$. Assume $c - d \geq 0$ and $e - f \geq 0$, by Lemma 2.3.3, at least one of $a = (c - d)$ and $b = e - f$ is equal to 0. If at least one of $(c - d)$ and $(e - f)$ is negative, without loss of generality, assume $c - d < 0$. Then $-(c - d) = d - c > 0$ and we have $(d - c)(e - f) = -1 \times 0 = 0$. By Lemma 2.3.3, at least one of $-a = d - c$ and $b = e - f$ is equal to zero, and this statement is equivalent to either $a = 0$ or $b = 0$ (or both). □

Exercise 4.1.6

Prove Corollary 4.1.9.

Proof. $ac = bc \implies ac - bc = ac + (-b)c = (a + (-b))c = (a - b)c = 0$ by Proposition 4.1.6. By Proposition 4.1.8, at least one of $(a - b)$ and c is equal to 0. Since $c \neq 0$, $a - b = 0$. Thus, $a = b$. $\square$

Exercise 4.1.7

Prove Lemma 4.1.11.

(a) *Proof.* We need to show that $a > b \iff a - b$ is a positive natural number. Suppose $a > b$. By definition, there exists a positive natural number n such that $a = b + n$. So $a - b = n > 0$ as required. Suppose $a - b$ is a positive natural number. Then $a - b = n \iff a = b + n$ for some positive integer n. Thus, $a > b$. $\square$

(b) *Proof.* Since $a > b$, there exists a positive natural number n such that $a - b = n$. Then $(a + c) - b = c + n \iff (a + c) = (b + c) + n$. Therefore, $a + c > b + c$. $\square$

(c) *Proof.* Since $a > b$, there exists a positive natural number n such that $a - b = n$. Since $(a - b)$ and n are both natural numbers, we have $c(a - b) = cn$ for any positive integer c. Then $ac = bc + cn$ where cn is a positive natural number. Therefore, $ac > bc$. $\square$

(d) *Proof.* Since $a > b$, there exists a positive natural number n such that $a - b = n$. Then $-b = -a + n$. Since $n > 0$, $-b > -a$. $\square$

(e) *Proof.* Since $a > b$, there exists a positive natural number n such that $a - b = n$. Since $b > c$, there exists a positive natural number m such that $b - c = m$. Then $(a - b) + (b - c) = a - c = n + m$ where $(n + m)$ is a positive natural number. Therefore, $a > c$. $\square$

(f) *Proof.* Since $a - b$ is an integer, by Lemma 4.1.5, exactly one of the following three statement is true:

(a) $a - b$ is zero. Then $a = b$.

(b) $a - b$ is equal to a positive natural number n. $a - b = n$, so $a > b$.

(c) $-(a - b) = b - a$ is equal to a positive natural number n. $b - a = n$, so $b > a$ which is equivalent to $a < b$.

□

Exercise 4.1.8

Show that the principle of induction does not apply directly to the integers. More precisely, give an example of a property $P(n)$ pertaining to an integer n such that $P(0)$ is true, and that $P(n)$ implies $P(n++)$ for all integers n, but that $P(n)$ is not true for all integers n. Thus induction is not as useful a tool for dealing with the integers as it is with the natural numbers.

Proof. A counterexample of $P(n)$ could be $f(n) = n^2$ is a monotonically increasing function. □