Chapter 3

Set Theory

3.1 Fundamentals

Definition 3.1.1

(Informal) We define a *set* A to be any unordered collection of objects, e.g., $3, 8, 5, 2$ is a set. If x is an object, we say that x is an element of A or $x \in A$ if x lies in the collection; otherwise we say that $x \notin A$. For instance, $3 \in \{1, 2, 3, 4, 5\}$ but $7 \notin \{1, 2, 3, 4, 5\}$.

Axiom 3.1 (Sets are objects).

If A is a set, then A is also an object. In particular, given two sets A and B, it is meaningful to ask whether A is also an element of B.

Axiom 3.2 (Equality of sets).

Two sets A and B are equal, $A = B$, iff every element of A is an element of B and vice versa. To put it another way, $A = B$ if and only if every element x of A belongs also to B, and every element y of B belongs also to A.

Axiom 3.3 (Empty set).

There exists a set $\emptyset$, known as the empty set, which contains no elements, i.e., for every object x we have $x \notin \emptyset$.

Lemma 3.1.5 (Single choice).

Let A be a non-empty set. Then there exists an object x such that $x \in A$.

Axiom 3.4 (Singleton sets and pair sets).

If a is an object, then there exists a set $\{a\}$ whose only element is a, i.e., for every object y, we have $y \in \{a\}$ if and only if $y = a$; we refer to $\{a\}$ as the singleton set whose element is a. Furthermore, if a and b are objects, then there exists a set $\{a, b\}$ whose only elements are a and b; i.e., for every object y, we have $y \in \{a, b\}$ if and only if $y = a$ or $y = b$; we refer to this set as the pair set formed by a and b.

Axiom 3.5 (Pairwise union).

Given any two sets A, B, there exists a set $A \cup B$, called the union of A and B, which consists of all the elements which belong to A or B or both. In other words, for any object x,

$$x \in A \cup B \iff (x \in A \text{ or } x \in B).$$

Lemma 3.1.12

If a and b are objects, then $\{a, b\} = \{a\} \cup \{b\}$. If A, B, C are sets, then the union operation is commutative (i.e., $A \cup B = B \cup A$) and associative (i.e., $(A \cup B) \cup C = A \cup (B \cup C)$). Also, we have $A \cup A = A \cup \emptyset = \emptyset \cup A = A$.

Definition 3.1.14 (Subsets).

Let A, B be sets. We say that A is a subset of B, denoted $A \subseteq B$, iff every element of A is also an element of B, i.e.

$$\text{For any object } x,\ x \in A \iff x \in B,$$

We say that A is a proper subset of B, denoted $A \subsetneq B$, if $A \subseteq B$ and $A \neq B$.

Proposition 3.1.17 (Sets are partially ordered by set inclusion).

Let A, B, C be sets. If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$. If $A \subseteq B$ and $B \subseteq A$, then $A = B$. Finally, if $A \subsetneq B$ and $B \subsetneq C$ then $A \subsetneq C$.

Axiom 3.6 (Axiom of specification).

Let A be a set, and for each $x \in A$, let $P(x)$ be a property pertaining to x (i.e., $P(x)$ is either a true statement or a false statement). Then there exists a set, called $\{x \in A : P(x) \text{ is true}\}$ (or simply $\{x \in A : P(x)\}$ for short), whose elements are precisely the elements x in A for which $P(x)$ is true. In other words, for any object y,

$$y \in \{x \in A : P(x) \text{ is true}\} \iff (y \in A \text{ and } P(y) \text{ is true}).$$

Definition 3.1.22 (Intersections).

The intersection $S_1 \cap S_2$ of two sets is defined to be the set

$$S_1 \cap S_2 := \{x \in S_1 : x \in S_2\}.$$

In other words, $S_1 \cap S_2$ consists of all elements which belong to both S_1 and S_2. Thus, for all objects x,

$$x \in S_1 \cap S_2 \iff x \in S_1 \text{ and } x \in S_2.$$

Definition 3.1.26 (Difference sets).

Given two sets A and B, we define the set $A - B$ or $A \backslash B$ to be the set A with any elements of B removed:

$$A \backslash B := \{x \in A : x \notin B\};$$

for instance, $\{1, 2, 3, 4\} \backslash \{2, 4, 6\} = \{1, 3\}$. In many cases B will be a subset of A, but not necessarily.

Proposition 3.1.27 (Sets form a boolean algebra).

Let A, B, C be sets, and let X be a set containing A, B, C as subsets.

1. (Minimal element) We have $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$.
2. (Maximal element) We have $A \cup X = X$ and $A \cap X = A$.
3. (Identity) We have $A \cap A = A$ and $A \cup A = A$.

4. (Commutativity) We have $A \cup B = B \cup A$ and $A \cap B = B \cap A$.

5. (Associativity) We have $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$.

6. (Distributivity) We have $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

7. (Partition) We have $A \cup (X \backslash A) = X$ and $A \cap (X \backslash A) = \emptyset$.

8. (De Morgan laws) We have $X \backslash (A \cup B) = (X \backslash A) \cap (X \backslash B)$ and $X \backslash (A \cap B) = (X \backslash A) \cup (X \backslash B)$.

Axiom 3.8 (Infinity).

There exists a set $\mathbb{N}$, whose elements are clled natural numbers, as well as an object 0 in $\mathbb{N}$, and an object $n++$ assigned to every natural number $n \in \mathbb{N}$, such that the Peano axioms hold.

Exercises

Exercise 3.1.1

Let a, b, c, d be objects such that $\{a, b\} = \{c, d\}$. Show that at least one of the two statements "$a = c$ and $b = d$" and "$a = d$ and $b = c$" hold.

Proof. Consider two cases: $a = b$ and $a \neq b$.
Case 1: $a = b$. Then $\{a, b\} = \{a\}$. By Axiom 3.2, if $\{a\}$ and $\{c, d\}$ are equal to each other, then every element belong to $\{c, d\}$ must also belong to $\{a\}$. Therefore, $c = a$, $d = a$. Since $a = b$, we have $a = b = c = d$. Thus, both statements hold.
Case 2: $a \neq b$. Similarly, by Axiom 3.2, every element belong to $\{a, b\}$ must also belong to $\{c, d\}$. So $\{c, d\}$, a set of two elements, contains two distinct elements a and b. Therefore, either $a = c, b = d$ or $a = d, b = c$ holds, exclusively.
Thus, we have shown that at least one of the two statements "$a = c$ and $b = d$" and "$a = d$ and $b = c$" hold. $\square$

Exercise 3.1.2

Using only Axiom 3.2, Axiom 3.1, Axiom 3.3, and Axiom 3.4, prove that the sets $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$, and $\{\emptyset, \{\emptyset\}\}$ are all distinct.

Proof. First, let's consider $\emptyset$. $\emptyset$ contains no element while other sets all have at least one element in it. Therefore, $\emptyset$ is distinct from $\{\emptyset\}$, $\{\{\emptyset\}\}$ and $\{\emptyset, \{\emptyset\}\}$. Then, let's consider $\{\emptyset\}$. Is it distinct from $\{\{\emptyset\}\}$ and $\{\emptyset, \{\emptyset\}\}$? We know that $\emptyset \in \{\emptyset\}$. But we have proved earlier $\emptyset$ and $\{\emptyset\}$ are not equal to each other, so $\emptyset \notin \{\{\emptyset\}\}$. So $\{\emptyset\}$ and $\{\{\emptyset\}\}$ are distinct. For the same reason, $\{\emptyset\} \notin \{\emptyset\}$. So $\{\emptyset\}$ and $\{\emptyset, \{\emptyset\}\}$ are also distinct. Last, consider $\{\{\emptyset\}\}$ and $\{\emptyset, \{\emptyset\}\}$. For the same reason ($\emptyset$ and $\{\emptyset\}$ are distinct), $\emptyset \notin \{\{\emptyset\}\}$. So $\{\{\emptyset\}\}$ and $\{\emptyset, \{\emptyset\}\}$ are distinct. Thus, we have proved the sets $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$, and $\{\emptyset, \{\emptyset\}\}$ are all distinct. $\square$

Exercise 3.1.3

Prove the remaining claims in Lemma 3.1.12.

Proof. First, prove the union operation is commutative (i.e., $A \cup B = B \cup A$). By definition, we know that $A \cup B$ consists of all the elements which belong to A or B, inclusively. And $B \cup A$ also consists of all the elements belong to A or B, inclusively. Therefore, $A \cup B$ and $B \cup A$ are containing exactly the same elements. Thus, $A \cup B = B \cup A$.

The second part is to prove $A \cup A = A \cup \emptyset = \emptyset \cup A = A$. First, let's consider $A \cup A$. By definition, $A \cup A$ consists of all the element x such that $x \in A$ or $x \in A$. So $A \cup A$ and A have exactly the same elements. Therefore $A \cup A = A$. Now let's consider $A \cup \emptyset$. $A \cup \emptyset$ consists of all the x such that $x \in A$ or $x \in \emptyset$. Since no element would belong to $\emptyset$. $A \cup \emptyset$ contains exactly the same elements as A. Therefore, $A \cup \emptyset = A$. And by commutative law, we have $A \cup \emptyset = \emptyset \cup A = A$.

Thus, we have proved $A \cup A = A \cup \emptyset = \emptyset \cup A = A$. $\square$

Exercise 3.1.4

Prove the remaining claims in Lemma 3.1.17.

Proof. Part I. If $A \subseteq B$ and $B \subseteq A$, then $A = B$. Translate the if statement into propositional logic: $(x \in A \Rightarrow x \in B) \wedge (x \in B \Rightarrow x \in A)$. Therefore, we have $x \in A \iff x \in B$. Thus, $A = B$.

Part II. If $A \subsetneq B$ and $B \subsetneq C$ then $A \subsetneq C$. Since $A \neq B$ and $B \neq C$, by transitivity, $A \neq C$. And by the first part of this proposition (if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$), we would have $A \subseteq C$. Since $A \subseteq C$ and $A \neq C$, $A \subsetneq C$. □

Exercise 3.1.5

Let A, B be sets. Show that the three statements $A \subseteq B$, $A \cup B = B$, $A \cap B = A$ are logically equivalent (any one of them implies the other two).

Proof. Rewrite the statements using propositional logic.

$A \subseteq B$: $x \in A \Rightarrow x \in B$.

$A \cup B = B$: $x \in A \vee x \in B \iff x \in B$.

$A \cap B = A$: $x \in A \wedge x \in B \iff x \in A$.

Assume $A \subseteq B$ is true. Then both $x \in A$ and $x \in B$ imply $x \in B$. So $x \in A \vee x \in B \Rightarrow x \in B$ stands. And $x \in B \Rightarrow x \in A \vee x \in B$ stands by rules of inference. Therefore, $A \subseteq B \Rightarrow A \cup B = B$. Assume $A \cup B = B$ is true. Then $x \in A \vee x \in B \Rightarrow x \in B$ stands. So $x \in A \Rightarrow x \in B$ is true which means $A \cup B = B \Rightarrow A \subseteq B$. Therefore, $A \subseteq B \iff A \cup B = B$.

Assume $A \subseteq B$ is true. We know that $x \in A \wedge x \in B \Rightarrow x \in A$ is true by rules of inference. And if $x \in A$, since $A \subseteq B$ is true, $x \in B$ is also true. So $x \in A \wedge x \in B$ is true. Therefore, $A \subseteq B \Rightarrow A \cap B = A$. Assume $A \cap B = A$ is true. Then if $x \in A$, $x \in A \wedge x \in B$ must be true. So $x \in B$ is true. $A \subseteq B$ is true. Therefore, $A \cap B \Rightarrow A \subseteq B$. Thus, $A \subseteq B \iff A \cap B = A$.

By transitivity, we have $A \cup B = B \iff A \cap B = A$. Thus, these three statements are logically equivalent. □

Exercise 3.1.6

Prove Proposition 3.1.27.

1. $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$.

Proof. $A \cup \emptyset = A$ has been proved in 3.1.3. $A \cap \emptyset$ consists of all x such that $x \in A \land x \in \emptyset$. Since $x \in \emptyset$ is always false, $A \cap \emptyset$ has no element in it. Thus, $A \cap \emptyset = \emptyset$. $\square$

2. $A \cup X = X$ and $A \cap X = A$.

   Proof. We have proved in 3.1.5 that $A \subseteq X$, $A \cup X = X$ and $A \cap X = A$ are logically equivalent. $\square$

3. $A \cap A = A$ and $A \cup A = A$.

   Proof. $x \in A \iff x \in A$, so $A \subseteq A$. Therefore, by 3.1.5 we have $A \cap A = A$ and $A \cup A = A$. $\square$

4. $A \cup B = B \cup A$ and $A \cap B = B \cap A$.

   Proof. $A \cup B = B \cup A$ has been proved in 3.1.3. On the other hand,

   $x \in A \land x \in B \iff x \in B \land x \in A$. Therefore, $A \cap B = B \cap A$. $\square$

5. $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$.

   Proof. $(A \cup B) \cup C = A \cup (B \cup C)$ has been proved in Lemma 3.1.12.

   $x \in A \land (x \in B \land x \in C) \iff (x \in A \land x \in B) \land x \in C$. Hence, $(A \cap B) \cap C = A \cap (B \cap C)$. $\square$

6. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

   Proof. This can be proved using distribution law in propositional logic.

   Since $x \in A \land (x \in B \lor x \in C) \iff (x \in A \land x \in B) \lor (x \in A \lor x \in C)$ and

   $x \in A \lor (x \in B \land x \in C) \iff (x \in A \lor x \in B) \land (x \in A \lor x \in C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ are also true. $\square$

7. $A \cup (X \backslash A) = X$ and $A \cap (X \backslash A) = \emptyset$.

Proof. $A \cup (X \backslash A) \iff (x \in A \vee (x \in X \wedge x \notin A)) \iff (x \in A \vee x \in X) \iff A \cup X$. From 3.1.5, we have $A \cup X = X$. Thus, $A \cup (X \backslash A) = X$.

$A \cap (X \backslash A) \iff x \in A \wedge (x \in X \wedge x \notin A) \iff (x \in A \wedge x \notin A) \wedge x \in X$. $(x \in A \wedge x \notin A) \wedge x \in X$ is always false. Therefore, $A \cap (X \backslash A) = \emptyset$. $\square$

8. $X \backslash (A \cup B) = (X \backslash A) \cap (X \backslash B)$ and $X \backslash (A \cap B) = (X \backslash A) \cup (X \backslash B)$.

Proof. $X \backslash (A \cup B) \iff x \in X \wedge \sim (x \in A \vee x \in B) \iff (x \in X \wedge x \notin A) \wedge (x \in X \wedge x \notin B) \iff (X \backslash A) \cap (X \backslash B)$.

$X \backslash (A \cap B) \iff x \in X \wedge \sim (x \in A \wedge x \in B) \iff x \in X \wedge (x \notin A \vee x \notin B) \iff (x \in X \wedge x \notin A) \vee (x \in X \wedge x \in B) \iff (X \backslash A) \cup (X \backslash B)$. $\square$

Exercise 3.1.7

Let A, B, C be sets. Show that $A \cap B \subseteq A$ and $A \cap B \subseteq B$. Furthermore, show that $C \subseteq A$ and $C \subseteq B$ if and only if $C \subseteq A \cap B$. In a similar spirit, show that $A \subseteq A \cup B$ and $B \subseteq A \cup B$, and furthermore that $A \subseteq C$ and $B \subseteq C$ if and only if $A \cup B \subseteq C$.

Proof. Part I. $A \cap B \subseteq A$ and $A \cap B \subseteq B$. $x \in A \wedge x \in B \Rightarrow x \in A$, so $A \cap B \subseteq A$. Similarly, we can show that $A \cap B \subseteq B$.

Part II. $C \subseteq A$ and $C \subseteq B \iff C \subseteq A \cap B$. $(x \in C \Rightarrow x \in A) \wedge (x \in C \Rightarrow x \in B) \iff (x \in C \Rightarrow (x \in A \wedge x \in B))$. Therefore, $C \subseteq A$ and $C \subseteq B \iff C \subseteq A \cap B$.

Part III. $A \subseteq A \cup B$ and $B \subseteq A \cup B$. According to propositional logic, $x \in A \Rightarrow x \in A \vee x \in B$. Therefore, $A \subseteq A \cup B$. Similarly, we can show that $B \subseteq A \cup B$.

Part IV. $A \subseteq C$ and $B \subseteq C \iff A \cup B \subseteq C$. $x \in A \Rightarrow x \in C$ and $x \in B \Rightarrow x \in C$ is equivalent to $(x \in A \vee x \in B) \Rightarrow x \in C$. Thus, $A \subseteq C$ and $B \subseteq C \iff A \cup B \subseteq C$. $\square$

Exercise 3.1.8

Let A, B be sets. Prove the absorption laws $A \cap (A \cup B) = A$ and $A \cup (A \cap B) = A$.

Proof. $x \in A \wedge (x \in A \vee x \in B) \iff x \in A$. Therefore, $A \cap (A \cup B) = A$.

$x \in A \vee (x \in A \wedge x \in B) \iff x \in A$. Therefore, $A \cup (A \cap B) = A$. $\square$

Exercise 3.1.9

Let A, B, X be sets such that $A \cup B = X$ and $A \wedge B = \emptyset$. Show that $A = X \backslash B$ and $B = X \backslash A$.

Proof. Since $A \cap B = \emptyset$, A and B have no element in common. So if $x \in A$, then $x \notin B$. And $x \in A \vee x \in B \Rightarrow x \in X$, so $x \in A \Rightarrow x \in X$. Therefore, $x \in A \Rightarrow x \in X \wedge x \notin B$. On the other hand, if $x \in X$, then $x \in A \vee x \in B$. But $x \notin B$, so $x \in A$. Therefore, $A = X \backslash B$. The proof of $B = X \backslash A$ is somewhat identical. $\square$

Exercise 3.1.10

Let A and B be sets. Show that the three sets $A \backslash B$, $A \cap B$, and $B \backslash A$ are disjoint, and that their union is $A \cup B$.

Proof. $A \cap B$ consists of all elements x such that $x \in A \wedge x \in B$. $A \backslash B \iff x \in A \wedge x \notin B$. So the elements in $A \cap B$ must not in $A \backslash B$. $B \backslash A \iff x \in B \wedge x \notin A$. So the elements of $B \backslash A$ are not belong to A, the elements in $A \cap B$ must not in $B \backslash A$. Therefore, $A \cap B$ and $A \backslash B$ are disjoint, $A \cap B$ and $B \backslash A$ are disjoint. Since elements in $A \backslash B$ are belong to A but the elements in $B \backslash A$ are not belong to A. So they are disjoint. So these three sets are disjoint. $(x \in A \wedge x \notin B) \vee (x \in A \wedge x \in B) \vee (x \in B \wedge x \notin A) \iff x \in A \vee x \in B$. Therefore, their union is $A \cup B$. $\square$

Exercise 3.1.11

Show that the axiom of replacement implies the axiom of specification.

Proof. $\square$