Chapter 5

The real numbers

5.1 Cauchy sequences

Definition 5.1.1 (Sequences).

Let m be an integer. A sequence $(a_n)_{n=m}^{\infty}$ of rational numbers is any function from the set $\{n \in \mathbf{Z} : n \geq m\}$ to $\mathbf{Q}$, i.e., a mapping which assigns to each integer n greater than or equal to m, a rational number a_n. More informally, a sequence $(a_n)_{n=m}^{\infty}$ of rational numbers is a collection of rationals a_m, a_{m+1}, $a_{m+2}, \ldots$

Definition 5.1.3 (ε-steadiness).

Let $\varepsilon > 0$. A sequence $(a_n)_{n=0}^{\infty}$ is said to be ε-steady iff each pair a_j, a_k of sequence elements is ε-close for every natural number j, k. In other words, the sequence $a_0, a_1, a_2, \ldots$ is ε-steady iff $|a_j - a_k| \leq \varepsilon$ for all j, k.

Definition 5.1.6 (Eventual ε-steadiness).

Let $\varepsilon > 0$. A sequence $(a_n)_{n=0}^{\infty}$ is said to be eventually ε-steady iff the sequence $a_N, a_{N+1}, a_{N+2}, \ldots$ is ε-steady for some natural number $N \geq 0$. In other words, the sequence $a_0, a_1, a_2, \ldots$ is eventually ε-steady iff there exists an $N \geq 0$ such that $|a_j - a_k| \leq \varepsilon$ for all $j, k \geq N$.

Definition 5.1.8 (Cauchy sequences).

A sequence $(a_n)_{n=0}^{\infty}$ of rational numbers is said to be a Cauchy sequence iff for every rational $\varepsilon > 0$, the sequence $(a_n)_{n=0}^{\infty}$ is eventually ε-steady. In other words, the sequence $a_0, a_1, a_2, \ldots$ is a Cauchy sequence iff for every $\varepsilon > 0$, there exists an $N \geq 0$ such that $d(a_j, a_k)$ for all $j, k \geq N$.

Proposition 5.1.11

The sequence $a_1, a_2, a_3, \dots$ defined by $a_n := 1/n$ (i.e., the sequence $1, 1/2, 1/3, \dots$) is a Cauchy sequence.

Definition 5.1.12 (Bounded sequences).

Let $M \geq 0$ be rational. A finite sequence $a_1, a_2, \dots, a_n$ is bounded by M iff $|a_i| \leq M$ for all $1 \leq i \leq n$. An infinite sequence $(a_n)_{n=1}^{\infty}$ is bounded by M iff $|a_i| \leq M$ for all $i \geq 1$. A sequence is said to be bounded iff it is bounded by M for some rational $M \geq 0$.

Lemma 5.1.14 (Finite sequences are bounded).

Every finite sequence $a_1, a_2, \dots, a_n$ is bounded.

Lemma 5.1.15 (Cauchy sequences are bounded).

Every Cauchy sequence $(a_n)_{n=1}^{\infty}$ is bounded.

Exercise 5.1.1

Prove Lemma 5.1.15.

Proof. Suppose $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence. So for $\varepsilon = 1$, there exists an $N \geq 1$ such that $d(a_j, a_k) \leq 1$ for all $j, k \geq N$. Then the sequence can be splite into two parts: $a_1, \dots, a_N$ and $a_{N+1}, a_{N+2}, \dots$. The former is a finite sequence, by Lemma 5.1.14, it is bounded. Suppose this finite sequence is bounded by M_1. Consider $a_{N+1}, a_{N+2}, \dots$. Since it is 1-steady, for any $i > N+1$, we have $|a_i - a_{N+1}| \leq 1$. Rearrange the inequalities, we have $a_{N+1} - 1 \leq a_i \leq a_{N+1} + 1$. Let $M_2 = a_{N+1} + 1$. Then $a_{N+1} < M_2$ and for every $i > N+1$, we have $a_i \leq M_2$. So sequence $a_{N+1}, a_{N+2}, \dots$ is bounded by M_2. Let $M = \max\{M_1, M_2\}$. Then the Cauchy sequence $(a_n)_{n=1}^{\infty}$ is bounded by M. $\square$

5.2 Equivalent Cauchy sequences

Definition 5.2.1 (ε-close sequences).

Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be two sequences, and let $\varepsilon > 0$. We say that the sequence $(a_n)_{n=0}^{\infty}$ is ε-close to $(b_n)_{n=0}^{\infty}$ iff a_n is ε-close to b_n for each $n \in \mathbf{N}$. In other words, the sequence $a_0, a_1, a_2, \ldots$ is ε-close to the sequence $b_1, b_1, b_2, \ldots$ iff $|a_n - b_n| \leq \varepsilon$ for all $n = 0, 1, 2, \ldots$.

Definition 5.2.3 (Eventually ε-close sequences).

Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be two sequences, and let $\varepsilon > 0$. We say that the sequence $(a_n)_{n=0}^{\infty}$ is eventually ε-close to $(b_n)_{n=0}^{\infty}$ iff there exists an $N \geq 0$ such that the sequences $(a_n)_{n=N}^{\infty}$ and $(b_n)_{n=N}^{\infty}$ are ε-close. In other words, $a_0, a_1, a_2, \ldots$ is eventually ε-close to $b_0, b_1, b_2, \ldots$ iff there exists an $N \geq 0$ such that $|a_n - b_n| \leq \varepsilon$ for all $n \geq N$.

Definition 5.2.6 (Equivalent sequences).

Two sequences $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are equivalent iff for each rational $\varepsilon > 0$, the sequences $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are eventually ε-close. In other words, $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ are equivalent iff for every for every rational $\varepsilon > 0$, there exists an $N \geq 0$ such that $|a_n - b_n| \leq \varepsilon$ for all $n \geq N$.

Proposition 5.2.8

Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be the sequences $a_n = 1 + 10^{-n}$ and $b_n = 1 - 10^{-n}$. Then the sequences a_n, b_n are equivalent.

Exercise 5.2.1

Show that if $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent sequences of rationals, then $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence if and only if $(b_n)_{n=1}^{\infty}$ is a Cauchy sequence.

Proof. Assume $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent sequences of rationals, and $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence. We want to show that for any rational $\varepsilon > 0$, there exists $N \geq 1$ such that $b_N, b_{N+1}, \ldots$ is ε-steady.

Since $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence and $\frac{\varepsilon}{3} > 0$, there exists $N_1 \geq 1$ such that for all $i, j \geq N_1$, $|a_i - a_j| \leq \frac{\varepsilon}{3}$. Since $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent, and $\frac{\varepsilon}{3} > 0$, there exists $N_2 \geq 1$ such that for all $i \geq N_2$, $|b_i - a_i| \leq \frac{\varepsilon}{3}$. Let $N = \max\{N_1, N_2\}$. Consider arbitrary $i, j \geq N$. Since $N \geq N_1$, we have

$$|a_i - a_j| \leq \frac{\varepsilon}{3}.$$

Since $N \geq N_2$, we have

$$|b_i - a_i| \leq \frac{\varepsilon}{3}$$

and

$$|a_j - b_j| \leq \frac{\varepsilon}{3}.$$

Since

$$|a_i - a_j| \leq \frac{\varepsilon}{3}$$

and

$$|b_i - a_i| \leq \frac{\varepsilon}{3},$$

we have

$$|b_i - a_j| \leq |a_i - a_j| + |b_i - a_i| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}.$$

Since

$$|a_j - b_j| \leq \frac{\varepsilon}{3}$$

and

$$|b_i - a_j| \leq \frac{2\varepsilon}{3},$$

we have

$$|b_i - b_j| \leq |a_j - b_j| + |b_i - a_j| \leq \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon.$$

Thus, for any $\varepsilon > 0$, we can find $N = \max\{N_1, N_2\}$ such that $b_N, b_{N+1}, dotsc$ is ε-steady. Therefore, $(b_n)_{i=1}^{\infty}$ is a Cauchy sequence.

Similarly, we can show that if $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent sequences of rationals and $(b_n)_{i=1}^{\infty}$ is a Cauchy sequence, then $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence. Thus, if $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent sequences of rationals, then $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence if and only if $(b_n)_{n=1}^{\infty}$ is a Cauchy sequence. □

Exercise 5.2.2

Let $\varepsilon > 0$. Show that if $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are eventually ε-close, then $(a_n)_{n=1}^{\infty}$ is bounded if and only if $(b_n)_{n=1}^{\infty}$ is bounded.

Proof. Suppose $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are eventually ε-close and $(a_n)_{n=1}^{\infty}$ is bounded. Since $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are eventually ε-close, for any $\varepsilon > 0$, there exists $N \geq 1$ such that for any $i \geq N$, $|a_i - b_i| \leq \varepsilon$. Consider an arbitrary $\varepsilon > 0$. We can find $N \geq 1$ such that for any $i \geq N$, $|a_i - b_i| \leq \varepsilon$. Then

$$a_i - \varepsilon \leq b_i \leq a_i + \varepsilon.$$

Split $(b_n)_{n=1}^{\infty}$ to $b_1, \ldots, b_N$ and $b_{N+1}, b_{N+2}, \ldots$. The former is a finite sequence, so it is bounded by some rational number M_1. Since $(a_n)_{n=1}^{\infty}$ is bounded, there exists M such that $|a_i| \leq M$ for all $i \geq 1$. Then

$$-M \leq a_i \leq M.$$

So

$$-M - \varepsilon \leq b_i \leq M + \varepsilon.$$

Therefore, $|b_i| \leq M + \varepsilon$ for all $i \geq N + 1$. Let $M_0 = \max(M, M_1)$. For any $i \geq 1$, we have $|b_i| \leq M_0$. Thus, $(b_n)_{n=1}^{\infty}$ is bounded.

Similarly, we can show that if $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are eventually ε-close, and $(b_n)_{n=1}^{\infty}$ is bounded, then $(a_n)_{n=1}^{\infty}$ is bounded.

Thus, if $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are eventually ε-close, then $(a_n)_{n=1}^{\infty}$ is bounded if and only if $(b_n)_{n=1}^{\infty}$ is bounded. □

5.3 The construction of the real numbers

Definition 5.3.1 (Real numbers).

A real number is defined to be an object of the form $\mathrm{LIM}_{n\to\infty}a_n$, where $\mathrm{LIM}_{n\to\infty}a_n$ is a Cauchy sequence of rational numbers. Two real numbers $\mathrm{LIM}_{n\to\infty}a_n$ and $\mathrm{LIM}_{n\to\infty}b_n$ are said to be equal iff $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent Cauchy sequences. The set of all real numbers is denoted $\mathbf{R}$.

Proposition 5.3.3 (Formal limits are well-defined).

Let $x = \mathrm{LIM}_{n\to\infty}a_n$, $y = \mathrm{LIM}_{n\to\infty}b_n$, and $z = \mathrm{LIM}_{n\to\infty}c_n$ be real numbers. Then, with the above definition of equality for real numbers, we have $x = x$. Also, if $x = y$, then $y = x$. Finally, if $x = y$ and $y = z$, then $x = z$.

Definition 5.3.4 (Addition of reals).

Let $x = \mathrm{LIM}_{n\to\infty}a_n$ and $y = \mathrm{LIM}_{n\to\infty}b_n$ be real numbers. Then we define the sum $x + y$ to be $x + y := \mathrm{LIM}_{n\to\infty}(a_n + b_n)$.

Lemma 5.3.6 (Sum of Cauchy sequences is Cauchy).

Let $x = \mathrm{LIM}_{n\to\infty}a_n$ and $y = \mathrm{LIM}_{n\to\infty}b_n$ be real numbers. Then $x + y$ is also a real number (i.e., $(a_n + b_n)_{n=1}^{\infty}$ is a Cauchy sequence of rationals).

Lemma 5.3.7 (Sums of equivalent Cauchy sequences are equivalent).

Let $x = \mathrm{LIM}_{n\to\infty}a_n$, $y = \mathrm{LIM}_{n\to\infty}b_n$, and $x' = \mathrm{LIM}_{n\to a'_n}$ be real numbers. Suppose that $x = x'$. Then we have $x + y = x' + y$.

Lemma 5.3.9 (Multiplication of reals).

Let $x = \mathrm{LIM}_{n\to\infty}a_n$ and $y = \mathrm{LIM}_{n\to\infty}b_n$ be real numbers. Then we define the product xy to be $xy := \mathrm{LIM}_{n\to\infty}a_nb_n$.

Proposition 5.3.10 (Multiplication is well defined).

Let $x = \mathrm{LIM}_{n\to\infty} a_n$, $y = \mathrm{LIM}_{n\to\infty} b_n$, and $x' = \mathrm{LIM}_{n\to\infty} a'_n$ be real numbers. Then xy is also a real number. Furthermore, if $x = x'$, then $xy = x'y$.

Proposition 5.3.11

All the laws of algebra from Proposition 4.1.6 hold not only for the integers, but for the reals as well.

Definition 5.3.12 (Sequences bounded away from zero).

A sequence $(a_n)_{n=1}^{\infty}$ of rational numbers is said to be bounded away from zero iff there exists a raional number $c > 0$ such that $|a_n| > c$ for all $n \geq 1$.

Lemma 5.3.14.

Let x be a non-zero real number. Then $x = \mathrm{LIM}_{n\to\infty} a_n$ for some Cauchy sequence $(a_n)_{n=1}^{\infty}$ which is bounded away from zero.

Lemma 5.3.15.

Suppose that $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence which is bounded away from zero. Then the sequence $(a_n^{-1})_{n=1}^{\infty}$ is also a Cauchy sequence.

Definition 5.3.16 (Reciprocals of real numbers).

Let x be a non-zero real number. Let $(a_n)_{n=1}^{\infty}$ be a Cauchy sequence bounded away from zero such that $x = \mathrm{LIM}_{n\to\infty} a_n$. Then we define the reciprocal x^{-1} by the formula $x^{-1} := \mathrm{LIM}_{n\to\infty} a_n^{-1}$.

Lemma 5.3.17 (Reciprocation is well defined).

Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be two Cauchy sequences bounded away from zero such that $\mathrm{LIM}_{n\to\infty} a_n = \mathrm{LIM}_{n\to\infty} b_n$ (i.e., the two sequences are equivalent). Then $\mathrm{LIM}_{n\to\infty} a_n^{-1} = \mathrm{LIM}_{n\to\infty} b_n^{-1}$.

Exercise 5.3.1.

Prove Proposition 5.3.3.

Proof. Reflexivity. Since $x = \text{LIM}_{n\to\infty}a_n$, $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence. Obviously, $(a_n)_{n=1}^{\infty}$ and $(a_n)_{n=1}^{\infty}$ are equivalent. Therefore, $\text{LIM}_{n\to\infty}a_n = \text{LIM}_{n\to\infty}a_n$ $(x = x)$.

Symmetry. Assume $x = y$, then Cauchy sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent. So for every $\varepsilon > 0$, there exists $N \geq 1$ such that for every $i \geq N$, $|a_i - b_i| = |b_i - a_i| \leq \varepsilon$. Therefore, $(b_n)_{n=1}^{\infty}$ and $(a_n)_{n=1}^{\infty}$ are equivalent. Thus, $\text{LIM}_{n\to\infty}b_n = \text{LIM}_{n\to\infty}a_n$ $(y = x)$.

Transitivity. Assume $x = y$ and $y = z$. We want to show that the Cauchy sequences $(a_n)_{n=1}^{\infty}$ and $(c_n)_{n=1}^{\infty}$ are equivalent, that is, for any $\varepsilon > 0$, there exists $N \geq 1$ such that for all $i \geq N$, we have $|a_i - c_i| \leq \varepsilon$. Suppose ε is an arbitrary positive rational number. Since $x = y$, the Cauchy sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent. Then there exists N_1 such that for every $i \geq N_1$, we have $|a_i - b_i| \leq \frac{\varepsilon}{2}$. Since $y = z$, the Cauchy sequences $(b_n)_{n=1}^{\infty}$ and $(c_n)_{n=1}^{\infty}$ are equivalent. Then there exists N_2 such that for every $i \geq N_2$, we have $|b_i - c_i| \leq \frac{\varepsilon}{2}$. Let $N = \max(N_1, N_2)$, then for all $i \geq N$, $|a_i - c_i| \leq |a_i - b_i| + |b_i - c_i| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \epsilon$. Thus, $(a_n)_{n=1}^{\infty}$ and $(c_n)_{n=1}^{\infty}$ are equivalent. So $x = z$. □

Exercise 5.3.2

Prove Proposition 5.3.10.

Proof. xy is a real number. We want to show that for any $\varepsilon > 0$, there exists $N \geq 1$ such that $|a_ib_i - a_jb_j| \leq \varepsilon$ for any $i, j \geq N$. Consider an arbitrary $\varepsilon > 0$. Since $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are Cauchy sequences, by Lemma 5.1.15, they are both bounded. Assume $(a_n)_{n=1}^{\infty}$ is bounded by M_1 and $(b_n)_{n=1}^{\infty}$ is bounded by M_2. Since a_n is a Cauchy sequence, there exists $N_1 \geq 1$ such that for all $i, j \geq N_1$, we have

$$|a_i - a_j| \leq \frac{\varepsilon}{2M_1}.$$

Similarly, since $(b_n)_{n=1}^{\infty}$ is a Cauchy sequence, there exists $N_2 \geq 1$ such that for all

$i, j \geq N_2$, we have

$$|b_i - b_j| \leq \frac{\varepsilon}{2M_2}.$$

Let $N = \max(N_1, N_2)$, consider an arbitrary pair of $i, j \geq N$. Then we have

$$\begin{aligned}
|a_j b_j - a_i b_i| &= |a_j b_j - a_j b_i + a_j b_i - a_i b_i| \\
&\leq |a_j b_j - a_j b_i| + |a_j b_i - a_i b_i| \\
&= |a_j| \cdot |b_j - b_i| + |b_i| \cdot |a_j - a_i| \\
&\leq M_1 \cdot \frac{\varepsilon}{2M_1} + M_2 \cdot \frac{\varepsilon}{2M_2} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Therefore, for every $\varepsilon > 0$, we can find an $N = \max(N_1, N_2)$ such that $a_N b_N, a_{N+1} b_{N+1}, \ldots$ is ε-close. Thus, $(a_n b_n)_{n=1}^{\infty}$ is a Cauchy sequence and xy is a real number.

Since $(b_n)_{n=1}^{\infty}$ is a Cauchy sequence, it must be bounded by some rational number M. Since $(a_n)_{n=1}^{\infty}$ and $(a'_n)_{n=1}^{\infty}$ are equivalent, for every $\varepsilon > 0$, there exists $N \geq 1$ such that for all $i \geq N$, we have

$$|a_i - a'_i| \leq \frac{\varepsilon}{M}.$$

Therfore, for all $i \geq N$,

$$\begin{aligned}
|a_i b_i - a'_i b_i| &= |b_i| \cdot |a_i - a'_i| \\
&\leq M \cdot \frac{\varepsilon}{M} \\
&= \varepsilon.
\end{aligned}$$

Thus, $(a_n b_n)_{n=1}^{\infty}$ and $(a'_n b_n)_{n=1}^{\infty}$ are equivalent and that $xy = x'y$. $\square$

Exercise 5.3.3

Let a, b be rational numbers. Show that $a = b$ if and only if $\mathrm{LIM}_{n\to\infty} a_n = \mathrm{LIM}_{n\to\infty} b_n$ (i.e., the Cauchy sequences $a, a, a, a, \ldots$ and $b, b, b, b, \ldots$ are equivalent if and only if $a = b$). This allows us to embed the rational numbers inside the real numbers in a well-defined manner.

Proof. Suppose the Cauchy sequences $a, a, a, a, \ldots$ and $b, b, b, b, \ldots$ are equivalent. Assume $a \neq b$, then $|a - b| > 0$. Since the two sequences are equivalent, for every $\varepsilon > 0$, there exists $N \geq 1$ such that $|a_i - b_i| \leq \varepsilon$. Let $\varepsilon = \frac{|a-b|}{2} > 0$. Then no matter what value i is, we have

$$|a_i - b_i| = |a - b| > \frac{|a - b|}{2} = \frac{\varepsilon}{2}$$

which contradicts the definition of equivalent sequences. Therefore, $a = b$.

Suppose $a = b$. Then for any $\varepsilon > 0$, let $N = 1$, we have

$$|a_i - b_i| = |a - b| = 0 < \varepsilon$$

for all $i \geq N$. Therefore, $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent.

Thus, $a = b$ if and only if $\text{LIM}_{n\to\infty} a_n = \text{LIM}_{n\to\infty} b_n$. □

Exercise 5.3.4

Let $(a_n)_{n=0}^{\infty}$ be a sequence of rational numbers which is bounded. Let $(b_n)_{n=0}^{\infty}$ be another sequence of rational numbers which is equivalent to $(a_n)_{n=0}^{\infty}$. Show that $(b_n)_{n=0}^{\infty}$ is also bounded.

Proof. Suppose $(a_n)_{n=0}^{\infty}$ is bounded by M. Similar to Exercise 5.2.2, we can split $(b_n)_{n=1}^{\infty}$ to $b_0, \ldots, b_N$ and $b_{N+1}, b_{N+2}, \ldots$ such that the former is bounded by some rational number M_1 and the latter is bounded by $M + \varepsilon$ for any $\varepsilon > 0$. Let $M_0 = \max(M, M_1)$, then $(b_n)_{n=0}^{\infty}$ is bounded by M_0. □

Exercise 5.3.5

Show that $\text{LIM}_{n\to\infty} 1/n = 0$.

Proof. We want to show that $a_n = 1/n$ and $0, 0, 0, \ldots$ are equivalent. Consider an arbitrary $\varepsilon > 0$. Let $N = \lceil \frac{1}{\varepsilon} \rceil$, we have

$$|a_i - 0| = a_i \leq a_N = \frac{1}{N} \leq \varepsilon$$

for all $i \geq N$. Thus, $\mathrm{LIM}_{n\to\infty} 1/n = 0$. $\square$