Fall 2021 Math340-101. Midterm

Friday, October 15^{th}, 2021, IN CLASS

Time $\leq$ 50min

Last name ....

First name ....

Student number ....

Grade: out of 50.

Student's signature: ....

There are total 4 problems.

This is a closed-book examination. ONLY pen/pencil/eraser are allowed.

1. $\boxed{\text{11 points}}$ Definitions: when you write a definition, be sure to define each variable you are using.

   (a) $\boxed{\text{2 points}}$ Give the definition of a convex set $C \subset \mathbb{R}^n$.

   (b) $\boxed{\text{3 points}}$ Give the definition of a half-space in $\mathbb{R}^n$. What is its dimension?

   (c) $\boxed{\text{3 points}}$ Give the definition of a hyperplane in $\mathbb{R}^n$. What is its dimension?

   (d) $\boxed{\text{3 points}}$ In $\mathbb{R}^7$, how many (linearly independent) hyperplanes do you need, for their intersection to be a 2-dimensional plane? Justify your answer by giving the general rule for the dimension of the intersection of hyperplanes.

2. $\boxed{\text{13 points}}$ Alice owns a restaurant and wants to buy flax seeds and pumpkin seeds, but her provider only offers multi-grain packages. The first package contains 25g of sunflower seeds, 20g of pumpkin seeds, 20g of flax seeds, and costs \$3. The second package contains 40g of pumpkin seeds, 25g of chia seeds, 30g of flax seeds, and 20g of sunflower seeds, and costs \$6. The third package contains 40g of flax seeds, 30g of pumpkin seeds, and 20g of chia seeds, and costs \$5. Alice needs 300g of pumpkin seeds and 500g of flax seeds, and wants to know how many of each package she should buy in order to be cost-efficient.

   (a) $\boxed{\text{5 points}}$ Model this problem as an LP problem

   (b) $\boxed{\text{3 points}}$ Write it in standard inequality form.

   (c) $\boxed{\text{5 points}}$ Alice's friend Bob claims that the optimal solution is to get 14 of the third package. Is it possible that his claim is correct? You don't need to solve the LP problem to answer.

3. $\boxed{\text{10 points}}$ Consider the following LP problem, for $\alpha, \beta \in \mathbb{R}$:

$$\begin{aligned} \max \quad & \alpha x_1 + \beta x_2 \\ \text{s.t.} \quad & x_1 - 2x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Find (necessary and sufficient) conditions on α, β for this problem to be unbounded.

Note: This means that giving one example of α, β that yields an unbounded problem is not enough. You must find all values of α, β that result in an unbounded LP problem.

4. $\boxed{\text{16 points}}$ Consider the following LP problem:

$$\begin{aligned} \max \quad & 3x_1 + x_2 \\ \text{s.t.} \quad & x_1 - x_2 \leq -1 \\ & -x_1 - x_2 \leq -3 \\ & 2x_1 - x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

(a) $\boxed{\text{4 points}}$ Draw the feasible region of this problem, as well as the gradient of the objective function.

(b) $\boxed{\text{12 points}}$ Solve this LP problem algebraically using the 2-phase simplex algorithm.
Hint: this problem requires a *total* of $\{\max 2y + z \mid y + z \leq 3,\ y \leq 1,\ y, z \geq 0\}$ pivots.

(i) $\boxed{\text{6 points}}$ Solve the first phase. **Don't forget to use Anstee's rule**.

(ii) $\boxed{\text{6 points}}$ Solve the second phase. **Don't forget to use Anstee's rule**.