Personal Project 4

T-117-STR, Discrete Mathematics I, 2023-3

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1 Problem 1.

1.1 Given Matrix:

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

1.2 Rule:

$$A^n = \begin{bmatrix} n+1 & n \\ -n & 1-n \end{bmatrix} for\ n \geq 1.$$

1.3 Base Step:

$$Lets\ prove\ that\ this\ rule\ holds\ true\ for\ the\ integer\ K = 1$$

$$A^1 = \begin{bmatrix} 1+1 & 1 \\ -1 & 1-1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

1.4 Solution:

Now let's prove that the rule also applies for $K = 2$

$$A^2 = \begin{bmatrix} 2 \times 2 + 1 \times (-1) & 2 \times 1 + 1 \times 0 \\ (-1) \times 2 + 0 \times (-1) & (-1) \times 1 + 0 \times 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 4 - 1 & 2 + 0 \\ -2 + 0 & -1 + 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}$$

Now that we can see that this rule applies to 1 *and* 2 *we can imagine that we pick any integer* k *so we get*

$$A^k = \begin{bmatrix} k+1 & k \\ -k & 1-k \end{bmatrix}.$$

Then for the rule to be true, $k + 1$ *should also apply.*

$$A^{k+1} = \begin{bmatrix} (k+1)+1 & k+1 \\ -(k+1) & 1-(k+1) \end{bmatrix} = \begin{bmatrix} k+2 & k+1 \\ -k-1 & -k \end{bmatrix}$$

Which should be equal to $A^1 \times A^k$

$$A^1 \times A^k = \begin{bmatrix} 2(k+1) - k & 2k + 1 - k \\ -1(k+1) + 0 \times (-k) & -1(k) + 0 \times (1-k) \end{bmatrix}$$

$$A^1 \times A^k = \begin{bmatrix} 2k + 2 - k & k+1 \\ -k - 1 + 0 & -k + 0 \end{bmatrix}$$

$$A^1 \times A^k = \begin{bmatrix} k+2 & k+1 \\ -k-1 & -k \end{bmatrix} = A^{k+1}$$

Now we have shown what we wanted to show.

2 Problem 2.

$$Let\ us\ solve\ the\ following\ using\ a\ inductive\ hypothesis:$$

$$The\ sum\ of\ the\ first\ n\ powers\ of\ two\ is\ 2^n - 1:$$
$$P(n): 2^0 + 2^1 + 2^2 + 2^3 + \cdots + 2^{n-1} = 2^n - 1$$

2.1 Base Step:

$$Let's\ first\ see\ wether\ or\ not\ this\ applies\ to\ P(1):$$
$$P(1) = 2^0 = 1 = 2^1 - 1$$

$$This\ holds\ true\ for\ P(1)$$

2.2 Solution:

$$Now\ let's\ imagine\ ourselves\ at\ some\ step\ k:$$
$$P(k) = 2^0 + 2^1 + 2^2 + 2^3 \cdots + 2^{k-1} = 2^k - 1$$

$$For\ the\ rule\ to\ apply\ we\ must\ prove\ that\ P(k+1)\ also\ follows\ this\ rule:$$

$$P(k+1) = 2^0 + 2^1 + 2^2 + 2^3 \cdots + 2^{(k+1)-1} = 2^{k+1} - 1$$

$$The\ Sequence\ leading\ the\ +2^k\ is\ infact\ equal\ to\ P(k)\ cause\ what\ comes\ before\ 2^k\ is\ 2^{k-1}:$$

$$P(k+1) = P(k) + 2^k = 2^{k+1} - 1$$

$$Now\ we\ simplify:$$

$$P(k+1) = 2^k - 1 + 2^k = 2^{k+1} - 1$$
$$P(k+1) = 2 \times 2^k - 1 = 2^{k+1} - 1$$
$$P(k+1) = 2^{k+1} - 1 = 2^{k+1} - 1$$

$$We\ have\ now\ proved\ the\ rule\ true.$$

3 Problem 3.

$$Let's\ find\ the\ recursive\ definition\ of\ this\ function\ for\ all\ n \geq 1:$$

$$f(n) = 5n + 2$$

$$Lets\ first\ try\ this\ function\ with\ small\ n's:$$

$$f(1) = 5 \times 1 + 2 = 7$$
$$f(2) = 5 \times 2 + 2 = 12$$
$$f(3) = 5 \times 3 + 2 = 17$$
$$f(4) = 5 \times 4 + 2 = 22$$

$$If\ we\ look\ closely\ we\ see\ that\ f(n+1) - f(n) = 5.$$

$$5(n+1) + 2 - (5n + 2)$$
$$5n + 5 + 2 - 5n - 2$$
$$5n - 5n + 5 - 2 + 2 = 5$$

$$This\ means\ that\ the\ recursive\ function\ is:$$
$$f(n) = f(n-1) + 5$$

4 Problem 4.

Let's find how many swedish vehicle registration numbers exist based on certain criteria

$[C] = Character\ [D] = Digit$

4.1 A):

No Criteria :

$$[C][C][C][C][D][D][C/D]$$
$$[32][32][32][32][10][10][42]$$
$$32^4 \times 10^2 \times 42 = 4404019200$$

4.2 B):

Start with X and end with 1

$$X[C][C][C][D][D]1$$
$$[1][32][32][32][10][10][1]$$
$$1 \times 32^3 \times 10^2 \times 1 = 3276800$$

4.3 C):

Has the string 42 *in included, exclude the letter A and the digit* 7.

$$[C][C][C][C]42[C/D]$$
$$[31][31][31][31][1][1][40]$$
$$31^4 \times 1^2 \times 40 = 36940840$$
$$[C][C][C][C][9]42$$
$$[31][31][31][31][9][1][1]$$
$$31^4 \times 9 = 8311689$$
$$36940840 + 8311689 = 45252529$$

5 Problem 5.

$Passwords\ consist\ of\ 7\ symbols\ that\ can\ be\ either\ uppercase\ letters,\ special\ characters\ or\ digits.$

$Assuming\ there\ are\ 32\ letters,\ 20\ special\ characters\ and\ 10\ digits,$

$how\ many\ passwords\ exist\ with\ the\ following\ criteria.$

5.1 A):

$Only\ one\ type\ of\ character:$

$$[C][C][C][C][C][C][C]$$
$$[32][32][32][32][32][32][32]$$
$$32^7 = 34359738368$$
$$[S][S][S][S][S][S][S]$$
$$[20][20][20][20][20][20][20]$$
$$20^7 = 1280000000$$
$$[D][D][D][D][D][D][D]$$
$$10^7 = 10000000$$
$$34359738368 + 1280000000 + 10000000 = 35649738368\ passwords$$

5.2 B):

$Passwords\ with\ symbol\ in\ last\ position\ or\ second\ to\ last\ position:$

$$So\ first\ we\ have\ to\ work\ out\ how\ many\ passwords\ there\ are$$

$$where\ the\ manditory\ symbol\ is\ in\ the\ second-to-last\ position.$$

$$[32+20+10][32+20+10][32+20+10][32+20+10][32+20+10][20][32+20+10]$$

$$[62][62][62][62][62][20][62]$$

$$62^6 \times 20$$

$$Now\ in\ the\ last\ position.$$

$$[62][62][62][62][62][62][20]$$

$$62^6 \times 20$$

$$(62^6 \times 20)^2 - 62^4$$

$$64^4\ came\ from\ the\ letters\ that\ the\ two\ sets\ had\ in\ common$$

$$with\ each\ other\ or\ |LA \cap LB|$$

$$The\ formula\ for\ this\ being$$

$$|LA \cup LB| = |LA| + |LB| - |LA \cap LB|$$