Numerical Mathematics I, 2015/2016, Lab session 7

Keywords: boundary value problem, finite difference approximation

Remarks

- Make a new folder called `NM1_LAB_7` for this lab session, save all your functions in this folder.
- Whenever a new `MATLAB` function is introduced, try figuring out yourself what this function does by typing `help <function>` in the command window.
- Make sure that you have done the preparation before starting the lab session. The answers should be worked out either by pen and paper (readable) or with any text processing software (LaTeX, Word, etc.).

1 Preparation

1.1 The Poisson problem

1. Study (Textbook, Section 8.2.1 & 8.2.5).
2. Consider the following *boundary value problem* (BVP)

$$\begin{array}{rcll} -\Delta u & = & f & \text{for } x \in \Omega, \\ u & = & g & \text{for } x \in \partial\Omega, \end{array} \tag{1}$$

   where $\Omega = (0,1)$ and $\partial\Omega = \{0,1\}$. Show that

$$\tilde{u}(x) = \sin(\pi x)$$

   is a solution to the Poisson problem (1) with $f(x) = \pi^2 \tilde{u}(x)$ and $g = 0$.

3. Let $h = 1/N$, and define a uniform grid as $x_i = ih$, for $i = 0, \dots, N$. Show that when using the second-order central finite difference approximation, we get

$$-f_i = \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + \mathcal{O}(h^2), \tag{2}$$

   for $i = 1, \dots, N-1$ (the interior nodes). The subscript i refers to evaluation at the point x_i.

4. Define the tridiagonal matrix $\mathbf{A} \in \mathbb{R}^{(N-1)\times(N-1)}$ as

$$\mathbf{A} := \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}.$$

   Show that the three-point scheme given by (2) leads to the following system of equations

$$\mathbf{A}\mathbf{u} = \mathbf{b},$$

   where $\mathbf{u} = (u_1, \dots, u_{N-1})^T$. Give $\mathbf{b}$ in terms of f_i and the boundary values g_0 and g_N.

1.2 The heat equation

1. Study (Textbook, Section 8.2.6).
2. Consider the *initial boundary value problem* (IBVP) given by

$$\begin{array}{rcll} \partial_t v & = & \mu\Delta v + p & \text{for } (x,t) \in \Omega \times [0,T], \\ v(x,t) & = & g(x) & \text{for } (x,t) \in \partial\Omega \times [0,T], \\ v(x,0) & = & v_0(x) & \end{array} \tag{3}$$

   where μ is the (positive) diffusion coefficient and the spatial domain is given by $\Omega = (0,1)$. Show that

$$\tilde{v}(x,t) = (1 + e^{-\gamma t})\tilde{u}(x)$$

   is a solution to (3) for

$$p(x,t) = \mu f(x) = \gamma \sin(\pi x), \quad g = 0, \tag{4}$$

   and $\gamma = \pi^2\mu$.
3. Show that discretising (3) in space gives the semi-discretisation given by

$$\frac{d\mathbf{v}}{dt}(t) = -\frac{\mu}{h^2}\mathbf{A}\mathbf{v}(t) + \mathbf{b}(t), \tag{5}$$

   where $\mathbf{v}(0) = \mathbf{v}^0$ is given. $\mathbf{b}(t)$ should be expressed in terms of $p_i(t)$, g_0 and g_N.
4. Show that applying the θ-method to (5) gives

$$\left(\mathbf{I} + \frac{\mu\Delta t}{h^2}\theta\mathbf{A}\right)\mathbf{v}^{n+1} = \left(\mathbf{I} - \frac{\mu\Delta t}{h^2}(1-\theta)\mathbf{A}\right)\mathbf{v}^n + \Delta t(\theta\mathbf{b}^{n+1} + (1-\theta)\mathbf{b}^n),$$

   where the superscript n means evaluation at time $t^n = n\Delta t$.
5. The eigenvalues of $\mathbf{A}$ are given by (see (Textbook, Exercise 8.2))

$$\lambda_k = 4\sin^2\left(\frac{k\pi}{2N}\right), \tag{6}$$

   for $k = 1, \ldots, N-1$. Show that the explicit Euler method ($\theta = 0$) is absolutely stable for

$$\frac{\mu\Delta t}{h^2} < \frac{1}{2}.$$

2 Lab experiments

2.1 The Poisson problem

Write a function `makeLaplace.m` that ouputs the $(N-1)\times(N-1)$ matrix $\mathbf{A}$ (which is the discrete analogue of the Laplace operator Δ), the header of your function should be of the following form

```
% INPUT
% N          number of subintervals
% OUTPUT
% L          discrete Laplace operator (3-point stencil)
function L = makeLaplace(N)
```

Write a MATLAB function called poissonSolveFD.m that solves the 1D Poisson problem on $\Omega = (0, 1)$ with Dirichlet boundary conditions. This function should use iterMethod.m from lab session 4 to solve the linear system of equations. The header of your function should be of the following form

```
% Solves the 1D Poisson problem
%       -laplace u = f on (0,1)
% with Dirichlet bdy conditions given by g
% INPUT
% f          right-hand side function
% g          Dirichlet boundary condition function
% N          number of subintervals
% precon     'none', 'jacobi' or 'gs'
% tol        tolerance for iterative solver
% OUTPUT
% sol        (N+1) x 1 solution array
% nodes      (N+1) x 1 array with location of spatial nodes
function [sol, nodes] = poissonSolveFD(f, g, N, precon, tol)
```

For iterMethod.m use the dynamic parameter which minimises the residual in the 2-norm (so dynamic = 2). Choose a preconditioner.

Test the consistency of the finite difference approximation by confirming (Textbook, Proposition 8.1). For $N = 2^i$, where $i = 2, \ldots 13$ compute the finite difference approximation to the Poisson problem with homogeneous Dirichlet boundary conditions (hence $g = 0$) and let the right-hand side function be given by

$$f(x) = \pi^2 \sin(\pi x).$$

Summarise your results in one figure where you plot the maximum error against h together with the line $y = h^2$.

2.2 The heat equation

Test problem

Write a MATLAB function heatSolveTheta.m that solves the IBVP (3) using the semi-discretisation given by (5). The time integration should be done using odeSolveTheta.m from the previous lab session.

The header of your function should look like

```
% Solves the 1D heat eqn
%       du/dt = mu laplace u + p(x,t) on (0,1) x (0, tEnd)
% and initial condition u0 (the Dirichlet boundary
% conditions are imposed by u0)
% INPUT
% p          right-hand side forcing term (function of x and t)
% u0Func     initial condition function (function of x)
% mu         diffusion coefficient
% theta      parameter for time integration
% tEnd       end time
% N          number of subintervals
% dt         step-size
% OUTPUT
```

θ	Δt	$\frac{\mu \Delta t}{h^2}$	$e_{\max}(T)$
0	0.625		
0	1.25		
0	1.35		
1/2	100		
1/2	200		
1	100		
1	200		

Table 1: Summary of results for the heat equation.

```
% tArray     array containing the time points
% solArray   array containing the solution at each time level
%            (the ith row equals the solution at time tArray(i))
%            (nrTimeSteps + 1) x (N+1) array
% nodes      (N+1) x 1 array with location of spatial nodes
function [tArray, solArray, nodes] = heatSolveTheta(p,...
    u0Func , mu, theta, tEnd, N, dt)
```

Test your implementation on the IBVP where p is given by (4), $g = 0$, $\mu = 10^{-3}$, $T = 1000$ and let $v_i^0 = \tilde{v}(x_i, 0)$. For the spatial discretisation choose $N = 20$. Let $e_{\max}(T)$ be the maximum error at $t = T$. Test the combinations of $\theta, \Delta t$ as shown Table 1, and summarise your results in such a table.

Application

Consider heating a one-dimensional iron bar with a length of one metre. Let the heat source be located at the centre and be of length 20 centimetres. Suppose that the heat source produces heat with a rate of $2 \cdot 10^7 W \cdot m^{-3}$. The melting temperature of iron is $T^* = 1811K$, the density is given by $\rho = 7.874 \cdot 10^3 kg \cdot m^{-3}$, the thermal conductivity is given by $k = 8.04 \cdot 10^1 W \cdot m^{-1} \cdot K^{-1}$ and finally the specific heat capacity is given by $c = 4.5 \cdot 10^2 J \cdot kg^{-1} \cdot K^{-1}$. Let the initial temperature be given by $293K$, and assume that the bar is kept at constant temperature of $293K$ at both ends.

How long must heat be applied in order for the centre of the bar to reach the melting temperature T^*? Choose θ, h and Δt. Approximate the heat source by a “step function”.

3 Discussion

3.1 The Poisson problem

1. Do your experiments show that the three-point stencil yields a second-order accurate approximations?

2. Explain how from (6) it follows that

$$K(\mathbf{A}) = \frac{\lambda_{\max}(\mathbf{A})}{\lambda_{\min}(\mathbf{A})} = \frac{4}{(\pi h)^2} + \mathcal{O}(1).$$

   How is this relevant?

3.2 The heat equation

1. What happens to the error for $t = T$ when halving Δt (for $\theta = 0, 1/2, 1$)?

2. Explain why for this particular problem, using $\theta \geq 1/2$, the step-size may be chosen as large as 200 while still getting an accurate solution.

3. Consider using the explicit Euler method for finding the stationary solution of (5), where $\mathbf{b}$ is constant in time. Explain why in this case integrating in time is equivalent to directly applying Richardson iteration by setting $\frac{d\mathbf{v}}{dt}(t) = 0$. Give the iteration matrix $\mathbf{B}_\alpha$ and show that in this context, the parameter α is given by

$$\alpha = \Delta t.$$

   How does the *optimal* parameter α_{opt} relate to chosing the step-size Δt? Hint: Recall that $\alpha_{\text{opt}}(\mathbf{M}) = \frac{2}{\lambda_{\min}(\mathbf{M}) + \lambda_{\max}(\mathbf{M})}$, use (6) to analytically find an approximation to α_{opt}.

4. How long does it take untill the centre of the iron bar reaches its melting temperature? Can you estimate the discretisation error in this case?