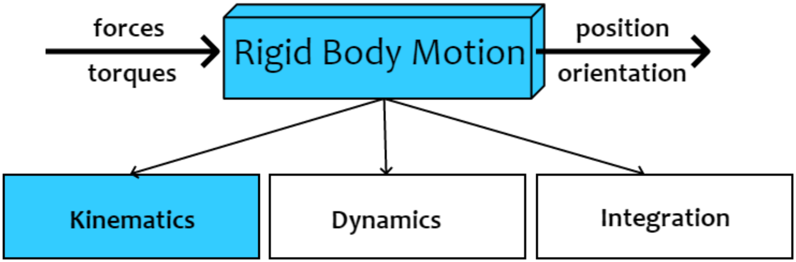
Quaternions and Euler Angles

Lecture 5



Agenda
Quaternions and Euler Angles

- Unit Quaternions (aka versors, or Euler parameters)
- Euler Angles

Motivation

Kinematics

- The rotation matrix consists of 9 elements, though only 3 are required to specify a rotation/orientation in 3D space. Can it be done better ?
- Unit Quaternions are a set of 4 numbers (a 4D vector). They are equivalent to rotation matrices, and are the most efficient and numerical stable tool for expressing rotations
- Euler angles are a set of 3 numbers, that are equivalent to rotation matrices but have the disadvantage that they have singularities in operation (divisions by zero in expressions) that need to be treated as special cases

Rotation Matrix

Kinematics

- Main relations:

$$\boldsymbol{r}^e = \mathbf{R}_b^e \boldsymbol{r}^b \tag{1}$$

$$\dot{\mathbf{R}}_b^e = [\omega^e]_\times \mathbf{R}_b^e = \mathbf{R}_b^e \left[\omega^b\right]_\times \tag{2}$$

Unit Quaternions

Kinematics

- Quaternions are a 4-dimensional vector $\boldsymbol{q} \in \mathbb{R}^4$, and can be seen as combination of a scalar and a 3-dimensional vector i.e $\boldsymbol{q} = \begin{bmatrix} s \\ \boldsymbol{v} \end{bmatrix} = \begin{bmatrix} s & v_1 & v_2 & v_3 \end{bmatrix}^T$.
- unit quaternion $\boldsymbol{q}$ has norm 1, that is $s^2 + v_1^2 + v_2^2 + v_3^2 = 1$
- A rotation matrix can be expressed as a function of a quaternion in the following way:

$$\mathbf{r}^e = \begin{bmatrix} -\boldsymbol{v} & s\mathbf{I}_3 + [\mathbf{v}]_\times \end{bmatrix} \begin{bmatrix} -\boldsymbol{v}^T \\ s\mathbf{I}_3 + [\mathbf{v}]_\times \end{bmatrix} \boldsymbol{r}^b = \mathbf{R}_b^e(\boldsymbol{q})\boldsymbol{r}^b \tag{3}$$

Unit Quaternions

Kinematics

$$\mathbf{R}_b^e(\mathbf{q}) = \begin{bmatrix} s^2+v_1^2-v_2^2-v_3^2 & 2v_1v_2-2sv_3 & 2v_1v_3+2sv_2 \\ 2v_1v_2+2v_3s & s^2-v_1^2+v_2^2-v_3^2 & -2sv_1+2v_2v_3 \\ 2v_1v_3-2sv_2 & 2v_2v_3+2v_1s & s^2-v_1^2-v_2^2+v_3^2 \end{bmatrix} \tag{4}$$

$$= 2\begin{bmatrix} s^2+v_1^2-0.5 & v_1v_2-sv_3 & v_1v_3+sv_2 \\ v_1v_2+v_3s & s^2+v_2^2-0.5 & -sv_1+v_2v_3 \\ v_1v_3-sv_2 & v_2v_3+v_1s & s^2+v_3^2-0.5 \end{bmatrix} \tag{5}$$

Unit Quaternions

Kinematics

- The identical rotation is represented by unit quaternion $\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$
- The time derivative of the quaternion is the following

$$\dot{\boldsymbol{q}} = \frac{1}{2} \begin{bmatrix} -\boldsymbol{v}^T \\ s\mathbf{I}_3 + [\mathbf{v}]_\times \end{bmatrix} \boldsymbol{\omega}^b = \frac{1}{2} \begin{bmatrix} -v_1\omega_x^b - v_2\omega_y^b - v_3\omega_z^b \\ s\omega_x^b - v_3\omega_y^b + v_2\omega_z^b \\ v_3\omega_x^b + s\omega_y^b - v_1\omega_z^b \\ -v_2\omega_x^b + v_1\omega_y^b + s\omega_z^b \end{bmatrix} \tag{6}$$

Euler Angles
Kinematics

The two interpretations of the rotation matrix $\mathbf{R}_b^e$:

- The passive interpretation, multiplication with a vector expressed in the b-frame results in the same vector now expressed in the e-frame, $\boldsymbol{r}^e = \mathbf{R}_b^e \boldsymbol{r}^b$.
- The active interpretation, is that it takes the e-frame basis vectors and rotates them over to the b-frame basis vector, in effect rotating the e-frame over to the b-frame

$$\text{Since } \mathbf{R}_b^e = \begin{bmatrix} i_{b,x}^e & j_{b,x}^e & k_{b,x}^e \\ i_{b,y}^e & j_{b,y}^e & k_{b,y}^e \\ i_{b,z}^e & j_{b,z}^e & k_{b,z}^e \end{bmatrix}, \ \boldsymbol{i}_b^e = \mathbf{R}_b^e \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \boldsymbol{j}_b^e = \mathbf{R}_b^e \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \ \boldsymbol{k}_b^e = \mathbf{R}_b^e \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Euler Angles

Kinematics

- The rotation $\mathbf{R}_b^e$ of the e-frame $\boldsymbol{i}_e, \boldsymbol{j}_e, \boldsymbol{k}_e$ such that it becomes identical/overlaps with the b-frame, $\boldsymbol{i}_b, \boldsymbol{j}_b, \boldsymbol{k}_b$, can be split up in a series of three individual rotations around the coordinate system axes. This allows to write the rotation matrix $\mathbf{R}_b^e$ as a product of three 2D rotations.
- For example,

$$\mathbf{R}_b^e = \mathbf{R}_z(\phi)\mathbf{R}_y(\theta)\mathbf{R}_x(\psi), \text{ where}$$

$$\mathbf{R}_z(\phi) = \begin{bmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{R}_y(\theta) = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix}, \mathbf{R}_x(\psi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\psi & -s\psi \\ 0 & s\psi & c\psi \end{bmatrix}$$

Euler Angles

Kinematics

$$\mathbf{R}_b^e = \begin{bmatrix} c\theta c\phi & s\psi s\theta c\phi - c\psi s\phi & c\psi s\theta c\phi + s\psi s\phi \\ c\theta s\phi & s\psi s\theta s\phi + c\psi c\phi & c\psi s\theta s\phi - s\psi c\phi \\ -s\theta & s\psi c\theta & c\psi c\theta \end{bmatrix} \quad (7)$$

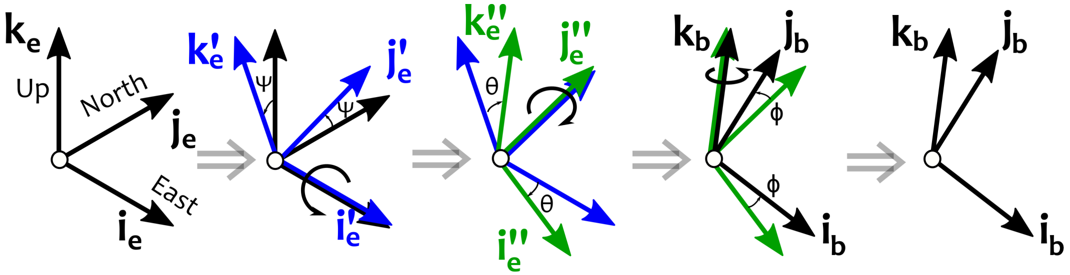
- here is ψ is roll (rotation around the x-axis) , θ is pitch (rotation around the new y-axis), ϕ is yaw (rotation around the new z-axis)
- Notice the order of rotation:

$$\boldsymbol{i}_b^e = \mathbf{R}_z\left(\mathbf{R}_y\left(\mathbf{R}_x \boldsymbol{i}_e^e\right)\right) \quad (8)$$

- Rotations (matrix products in general) are not commutative:

$$\mathbf{R}_1\mathbf{R}_2 \neq \mathbf{R}_2\mathbf{R}_1$$

Euler Angles
Kinematics



An x-y-z intrinsic rotation taking the e-frame to the b-frame

Euler Angles

Kinematics

- We can write other combinations for the 2D axes rotation, there exist in all 12 intrinsic combinations and 12 extrinsic combinations. We are going to look only at the one described above - the x,y,z intrinsic sequence (called also Tait-Bryan angles or Cardan angles)
- We are going to use the Euler angles to make plots of the rotation, we are interested in transforming the rotation matrix to Euler angles.

Euler Angles
Kinematics

$$\mathbf{R}_b^e = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} = \begin{bmatrix} c\theta c\phi & s\psi s\theta c\phi - c\psi s\phi & c\psi s\theta c\phi + s\psi s\phi \\ c\theta s\phi & s\psi s\theta s\phi + c\psi c\phi & c\psi s\theta s\phi - s\psi c\phi \\ -s\theta & s\psi c\theta & c\psi c\theta \end{bmatrix}$$

The solutions are, see note by Gregory G. Slabaugh in the lecture resources, if $R_{31} \neq \pm 1$

$$R_{31} = -\sin(\theta) \rightarrow \theta_1 = \arcsin(R_{31}), \theta_2 = \pi - \arcsin(R_{31}) \tag{9}$$

$$\frac{R_{32}}{R_{33}} = \tan(\psi) \rightarrow \psi_1 = \operatorname{atan2}\left(\frac{R_{32}}{\cos\theta_1}, \frac{R_{33}}{\cos\theta_1}\right), \psi_2 = \operatorname{atan2}\left(\frac{R_{32}}{\cos\theta_2}, \frac{R_{33}}{\cos\theta_2}\right) \tag{10}$$

$$\frac{R_{21}}{R_{11}} = \tan(\phi) \rightarrow \phi_1 = \operatorname{atan2}\left(\frac{R_{21}}{\cos\theta_1}, \frac{R_{11}}{\cos\theta_1}\right), \phi_2 = \operatorname{atan2}\left(\frac{R_{21}}{\cos\theta_2}, \frac{R_{11}}{\cos\theta_2}\right) \tag{11}$$

and if $R_{31} = \pm 1$,

$$\theta = \pm\frac{\pi}{2}, \psi = \pm\phi + \operatorname{atan2}\left(\pm R_{12}, \pm R_1 3\right), \phi \in \mathbb{R} \tag{12}$$

Derivates of Euler Angles

Kinematics

Looking at the relation $\dot{\mathbf{R}}_b^e = \mathbf{R}_b^e \left[\omega^b\right]_\times$, we can derive the expressions of the Euler angles derivatives. First, looking at term R_{31}

$$-\frac{ds\theta}{dt} = s\psi c\theta\omega_z - c\psi c\theta\omega_y \Rightarrow \boxed{\dot{\theta} = -s\psi\omega_z + c\psi\omega_y} \tag{13}$$

Then looking at the term R_{32},

$$\frac{ds\psi c\theta}{dt} = c\theta\omega_z + c\psi c\theta\omega_x \Rightarrow \boxed{\dot{\psi} = \omega_x + s\psi\tan\theta\omega_y + c\psi\tan\theta\omega_z} \tag{14}$$

And finally, looking at R_{21},

$$\frac{ds\theta s\phi}{dt} = (s\psi s\theta s\phi + c\psi c\phi)\omega_z - (c\psi s\theta s\phi - s\psi c\psi)\omega_y \Rightarrow \boxed{\dot{\phi} = \frac{c\psi}{c\theta}\omega_z + \frac{s\psi}{c\theta}\omega_y} \tag{15}$$

Derivates of Euler Angles

Kinematics

This can also be written in a matrix form:

$$\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} 1 & s\psi \tan\theta & c\psi \tan\theta \\ 0 & c\psi & -s\psi \\ 0 & \dfrac{s\psi}{c\theta} & \dfrac{c\psi}{c\theta} \end{bmatrix} \omega^b \tag{16}$$