Kalman Filter
Linear Systems

- Observability
- Discrete-Time Kalman Filter (KF)

Kalman Filter
Linear Systems

- Consider a linear, (time-invariant,) deterministic, dynamical system, represented in discrete-time as:

$$\mathbf{x}_k = \mathbf{A}\mathbf{x}_{k-1} + \mathbf{B}\mathbf{u}_{k-1},$$
$$\mathbf{y}_k = \mathbf{H}\mathbf{x}_k,$$

where $\mathbf{x} \in \mathbb{R}^n$ is the n-dimensional state vector, $\mathbf{u} \in \mathbb{R}^p$ is the p-dimensional input vector, $\mathbf{y} \in \mathbb{R}^m$ is the m-dimensional output vector. Furthermore , $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times p}$ and $\mathbf{H} \in \mathbb{R}^{m\times n}$.

Kalman Filter

Linear Systens, Observability

Estimation problem for a state-space (hidden states) model vs the input/output model. Is it even possible ?

- Observability is a measure of how well internal states of a system can be inferred from knowledge of its external outputs. (the mathematical dual of the controllability concept)
- A dynamical system designed to estimate the state of a system from measurements of the outputs is called a state observer.
- If the original system is not observable, we cannot design a state observer that can fulfill its purpose.

Kalman Filter
Linear Kalman Filter

- If the system model is not deterministic, that is we can better model the real-life situation considering multivariate random noise, process noise $\mathbf{w} \in \mathbb{R}^n$ and measurement (sensor) noise $\mathbf{v} \in \mathbb{R}^m$, making the state and the output essentially random variables,

$$\mathbf{x}_k = \mathbf{A}\mathbf{x}_{k-1} + \mathbf{B}\mathbf{u}_{k-1} + \mathbf{G}\boldsymbol{w}_{k-1}$$
$$\mathbf{y}_k = \mathbf{H}\mathbf{x}_k + \boldsymbol{v}_k,$$

- then the Kalman filter framework is most suitable. Also, about noise (...), $E[\boldsymbol{w}_k] = 0_n$, $E[\boldsymbol{v}_k] = 0_m$ and covariances $E[\boldsymbol{w}_k \boldsymbol{w}_k^T] = \mathbf{Q} \in \mathbb{R}^{n\times n}$, and $E[\boldsymbol{v}_k \boldsymbol{v}_k^T] = \mathbf{R} \in \mathbb{R}^{m\times m}$.

Kalman Filter

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- Kalman Filter Predict Step:

$$\hat{\mathbf{x}}_{k|k-1} = \mathbf{A}\hat{\mathbf{x}}_{k-1} + \mathbf{B}\mathbf{u}_{k-1}$$
$$\mathbf{P}_{k|k-1} = \mathbf{A}\mathbf{P}_{k-1}\mathbf{A}^T + \mathbf{G}\mathbf{Q}\mathbf{G}^T$$

- Kalman Filter Update Step:

$$\hat{\mathbf{y}}_k = \mathbf{H}\hat{\mathbf{x}}_{k|k-1};\ \mathbf{P}_y = \mathbf{H}\mathbf{P}_{k|k-1}\mathbf{H}^T;\ \mathbf{P}_{xy} = \mathbf{P}_{k|k-1}\mathbf{H}^T$$
$$\mathbf{K}_k = \mathbf{P}_{xy}(\mathbf{P}_y + \mathbf{R})^{-1}$$
$$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k(\mathbf{y}_k - \hat{\mathbf{y}}_k)$$
$$\mathbf{P}_k = (\mathbf{I} - \mathbf{K}_k\mathbf{H})\mathbf{P}_{k|k-1}$$

- with some given initial conditions $\hat{\mathbf{x}}_0$, $\mathbf{P}_0$.

Kalman Filter

Linear Kalman Filter

- A covariance update equation that is more numerically stable, the Joseph Form.

$$\mathbf{P}_k = (\mathbf{I} - \mathbf{K}_k\mathbf{H})\mathbf{P}_{k|k-1}(\mathbf{I} - \mathbf{K}_k\mathbf{H})^T + \mathbf{K}_k\mathbf{R}\mathbf{K}_k^T$$