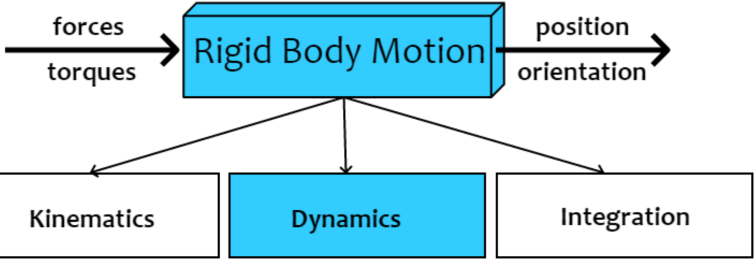
Dynamics Eqs of Rigid Body Motion

Lecture 6



Agenda
Dynamics

- Newton's laws of motion for point masses/particles
- Euler's laws of motion for rigid body
  - Translational equation of motion
  - Rotational equation of motion

Newton's Laws of Motion
Dynamics

Second Law

In an inertial reference frame, the sum of forces on a point mass/particle object is equal to mass of the object times the acceleration of the object,

$$\underbrace{\sum_k \boldsymbol{f}_k^e}_{\boldsymbol{f}_{total}} = m\boldsymbol{a}^e = m\ddot{\boldsymbol{p}}^e \Leftrightarrow \sum_k \begin{bmatrix} f_{k,x}^e \\ f_{k,y}^e \\ f_{k,z}^e \end{bmatrix} = m \begin{bmatrix} a_x^e \\ a_y^e \\ a_z^e \end{bmatrix} = \begin{bmatrix} \ddot{p}_x^e \\ \ddot{p}_y^e \\ \ddot{p}_z^e \end{bmatrix} \tag{1}$$

- For the purpose of studying quadrotor flight, the earth-fixed frame as used in the previous lectures is an inertial frame

Newton's Laws of Motion
Dynamics

Law of Action and Reaction

When a particle exerts a force on a second particle (upon some form of interaction, contact or at-a-distance), the second particle simultaneously exerts an equal in magnitude, opposite in direction, force onto the first particle.

- The direction of the two forces is along the straight line joining the point masses
- If i and j are two particles, and $\boldsymbol{f}_{ij}$ is the force with which particle i acts upon particle j, and $\boldsymbol{r}_i$ and $\boldsymbol{r}_j$ are position vectors then

$$\boldsymbol{f}_{ij} = -\boldsymbol{f}_{ji} \quad (2a) \qquad \boldsymbol{f}_{ij} = \pm \|\boldsymbol{f}_{ij}\| (\boldsymbol{r}_i - \boldsymbol{r}_j) \quad (2b)$$

Rigid Body
Dynamics

A rigid body object can be seen as system consisting of a very large (in the limit infinite) number of small (in the limit infinitesimal) point-mass particles, with the property that the relative positions of the particles wrt each other are constant (rigidity).

Translational Dynamic Equation

Dynamics

Let's take a system of N particles. There two types of forces acting on each particle i: external system force (from objects external to the system), and internal system forces (from the other particles):

$$\sum_k \boldsymbol{f}^e_{ik} + \sum_j \boldsymbol{f}^e_{ji} = m_i \boldsymbol{a}^e_i \tag{3}$$

$$\underbrace{\sum_k \sum_i \boldsymbol{f}^e_{ik}}_{=\boldsymbol{f}^e_{\text{ext, total}}} + \underbrace{\sum_i \sum_j \boldsymbol{f}^e_{ji}}_{=0,\text{ since } \boldsymbol{f}_{ji}=-\boldsymbol{f}_{ij}} = \sum_i m_i \boldsymbol{a}^e_i \tag{4}$$

Translation & Center of Mass

Dynamics

$$\sum_i m_i \boldsymbol{a}_i^e = \sum_i m_i \ddot{\boldsymbol{r}}_i^e = m \frac{d^2}{dt^2} \underbrace{\left(\frac{1}{m} \sum_i m_i \boldsymbol{r}_i^e \right)}_{\triangleq \boldsymbol{p}^e}$$

$$\boxed{\boldsymbol{p} = \frac{1}{m} \sum_i m_i \boldsymbol{r}_i} \tag{5}$$

$$\boxed{\boldsymbol{f}^e_{\text{ext, total}} = m \ddot{\boldsymbol{p}}^e = m \boldsymbol{a}_p^e} \tag{6}$$

where $\boldsymbol{r}$ is the center of mass position vector, and $m = \sum_i m_i$ is the total mass of the system. The sum of external forces equals the mass of the system multiplied with the acceleration of the center of mass.

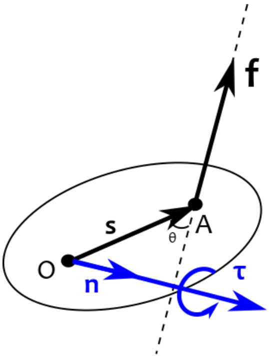
Torque
Dynamics

Because of the rigidity constraint, forces do not only push or pull an object through space (translate it), but also tend to rotate it. This effect is expressed by the torque.

“Torque is the tendency of a force to turn or twist. If a force is used to begin to spin an object or to stop an object from spinning, a torque is made” (wikipedia)

Torque

Dynamics

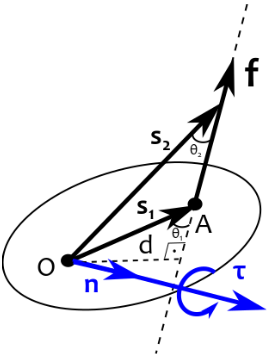


Let $\boldsymbol{f}$ be an external force acting on a body, and point O not on the line of operation of the force. And let vector $\boldsymbol{s}$ be defined by point O and any point on the line of action of the force $\boldsymbol{f}$. Then the torque about point O will be

$$\tau_O = [\boldsymbol{s}]_\times \boldsymbol{f} = \|\boldsymbol{s}\| \, \|\boldsymbol{f}\| \sin(\theta)\boldsymbol{n} \tag{7}$$

Torque

Dynamics



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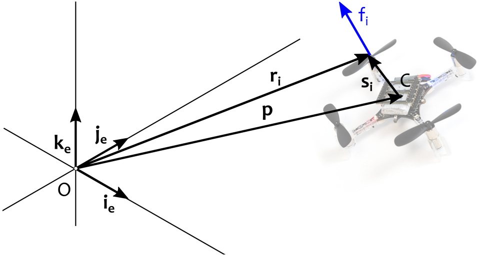
$$\tau_O = [\boldsymbol{s}]_\times \boldsymbol{f} = \|f\| \cdot d \cdot \boldsymbol{n} \tag{8}$$

Fixed, Sliding and Free vectors

Dynamics

- Position vector - is a fixed vector, its initial point is determined by the coordinate system of reference frame
- Force in the context of a rigid body study - is a sliding vector, meaning its initial point can be anywhere on the line of operation
- Velocity, acceleration - are free vectors, only the magnitude and direction that are relevant

Rotational Equation of Motion
Dynamics



First relation, where $\boldsymbol{f}_i$ external force

$$\boldsymbol{r}_i = \boldsymbol{p} + \boldsymbol{s}_i$$
$$[\boldsymbol{r}_i]_\times = [\boldsymbol{p}]_\times + [\boldsymbol{s}_i]_\times$$
$$[\boldsymbol{r}_i]_\times \boldsymbol{f}_i = [\boldsymbol{p}]_\times \boldsymbol{f}_i + [\boldsymbol{s}_i]_\times \boldsymbol{f}_i$$
$$\sum_i \tau_{O,i} = [\boldsymbol{p}]_\times \sum_i \boldsymbol{f}_i + \sum_i \tau_{C,i}$$
$$\boxed{\tau_{O,ext} = [\boldsymbol{p}]_\times \boldsymbol{f}_{total,ext} + \tau_{C,ext}} \tag{9}$$

Rotational Equation of Motion

Dynamics

Second relation, starting from Netwons Second law for a system of particles

$$\boldsymbol{f}_i^e + \sum_j \boldsymbol{f}_{ji}^e = m_i \boldsymbol{a}_i^e$$

$$[\boldsymbol{r}_i^e]_\times \boldsymbol{f}_i^e + \sum_j [\boldsymbol{r}_i^e]_\times \boldsymbol{f}_{ji}^e = m_i [\boldsymbol{r}_i^e]_\times \boldsymbol{a}_i^e$$

$$\sum_i [\boldsymbol{r}_i^e]_\times \boldsymbol{f}_i^e + \sum_i \sum_j [\boldsymbol{r}_i^e]_\times \boldsymbol{f}_{ji}^e = \sum_i m_i [\boldsymbol{r}_i^e]_\times \boldsymbol{a}_i^e$$

$$\tau_{O,ext}^e + (\dots [\boldsymbol{r}_i^e]_\times \boldsymbol{f}_{ji}^e - \left[\boldsymbol{r}_j^e\right]_\times \boldsymbol{f}_{ji}^e \dots) = \sum_i m_i [\boldsymbol{r}_i^e]_\times \boldsymbol{a}_i^e$$

Rotational Equation of Motion

Dynamics

$$[\boldsymbol{r}_i^e]_\times \boldsymbol{f}_{ji}^e - \left[\boldsymbol{r}_j^e\right]_\times \boldsymbol{f}_{ji}^e = \left[\boldsymbol{r}_i^e - \boldsymbol{r}_j^e\right]_\times (\pm) \left\|\boldsymbol{f}_{ji}^e\right\| (\boldsymbol{r}_i^e - \boldsymbol{r}_j^e) = 0 = \tau_{O,int}$$

So,

$$\boxed{\tau_O^e = \sum_i m_i \, [\boldsymbol{r}_i^e]_\times \, \boldsymbol{a}_i^e} \tag{10}$$

Rotational Equation of Motion

Dynamics

We next look at the right-hand side term, and begin to expand it

$$\sum_i m_i \left[\boldsymbol{r}_i^e\right]_\times \boldsymbol{a}_i^e = \sum_i m_i \left[\boldsymbol{p}^e + \boldsymbol{s}_i^e\right]_\times \boldsymbol{a}_i^e$$

from the kinematics lecture we know that

$$\boldsymbol{a}_i^e = \boldsymbol{a}_p^e + \left[\alpha^e\right]_\times \boldsymbol{s}_i^e + \left[\omega^e\right]_\times \left[\omega^e\right]_\times \boldsymbol{s}_i^e$$

thus

$$\begin{aligned} \tau_O^e &= \sum_i m_i \left[\boldsymbol{p}^e + \boldsymbol{s}_i^e\right]_\times \left(\boldsymbol{a}_p^e + \left[\alpha^e\right]_\times \boldsymbol{s}_i^e + \left[\omega^e\right]_\times \left[\omega^e\right]_\times \boldsymbol{s}_i^e\right) \\ &= \boldsymbol{T}_1 + \boldsymbol{T}_2 + \boldsymbol{T}_3 + \boldsymbol{T}_4 + \boldsymbol{T}_5 + \boldsymbol{T}_6 \end{aligned}$$

Rotational Equation of Motion

Dynamics

$$\boldsymbol{T}_1 = \sum_i m_i \left[\boldsymbol{p}^e\right]_\times a_p^e = m \left[\boldsymbol{p}^e\right]_\times a_p^e = \left[\boldsymbol{p}^e\right] \boldsymbol{f}^e_{total,ext}$$

$$\boldsymbol{T}_2 = \sum_i m_i \left[\boldsymbol{p}^e\right]_\times \left[\alpha^e\right]_x \boldsymbol{s}_i^e = \left[\boldsymbol{p}^e\right]_\times \left[\alpha^e\right]_x \sum_i m_i \boldsymbol{s}_i^e = 0$$

This is because $\boldsymbol{p} = \frac{1}{m} \sum_i m_i \boldsymbol{r}_i = \frac{1}{m} \sum_i \left(m_i \boldsymbol{p} + m_i \boldsymbol{s}_i\right) = \boldsymbol{p} + \frac{1}{m} \sum_i m_i \boldsymbol{s}_i \Rightarrow$

$$\sum_i m_i \boldsymbol{s}_i = 0 \tag{11}$$

Rotational Equation of Motion

Dynamics

$$\boldsymbol{T}_3 = \sum_i m_i \left[\boldsymbol{p}^e\right]_\times \left[\omega^e\right]_\times \left[\omega^e\right]_\times \boldsymbol{s}_i^e = \left[\boldsymbol{p}^e\right]_\times \left[\omega^e\right]_\times \left[\omega^e\right]_\times \sum_i m_i \boldsymbol{s}_i^e = 0$$

$$\boldsymbol{T}_4 = \sum_i m_i \left[\boldsymbol{s}_i^e\right]_\times \boldsymbol{a}_p^e = \left[\sum_i m_i \boldsymbol{s}_i^e\right]_\times \boldsymbol{a}_p^e = 0$$

$$\boldsymbol{T}_5 = \sum_i m_i \left[\boldsymbol{s}_i^e\right]_\times \left[\alpha^e\right]_x \boldsymbol{s}_i^e = -\underbrace{\sum_i m_i \left[\boldsymbol{s}_i^e\right]_\times \left[\boldsymbol{s}_i^e\right]_x}_{\mathbf{J}^e} \alpha^e = \mathbf{J}^e \alpha^e$$

where we used the fact that $\left[\boldsymbol{a}\right]_\times \boldsymbol{b} = -\left[\boldsymbol{b}\right]_\times \boldsymbol{a}$.

Rotational Equation of Motion

Dynamics

$$
\begin{aligned}
\boldsymbol{T}_6 &= \sum_i m_i \left[\boldsymbol{s}_i^e\right]_\times \left[\omega^e\right]_\times \overbrace{\left[\omega^e\right]_\times \boldsymbol{s}_i^e} = -\sum_i m_i \underbrace{\left[\boldsymbol{s}_i^e\right]_\times \left[\omega^e\right]_\times} \left[\boldsymbol{s}_i^e\right]_\times \omega^e = \\
&= -\sum_i m_i \left(\left[\left[\boldsymbol{s}_i^e\right]_\times \omega^e\right]_\times + \left[\omega^e\right]_\times \left[\boldsymbol{s}_i^e\right]_\times \right) \left[\boldsymbol{s}_i^e\right]_\times \omega^e = \\
&= -\sum_i m_i \left[\omega^e\right]_\times \left[\boldsymbol{s}_i^e\right]_\times \left[\boldsymbol{s}_i^e\right]_\times \omega^e = \left[\omega^e\right]_\times \underbrace{\left(-\sum_i m_i \left[\boldsymbol{s}_i^e\right]_\times \left[\boldsymbol{s}_i^e\right]_\times \right)}_{\mathbf{J}^e} \omega^e
\end{aligned}
$$

We used the following relations $\left[[a]_\times \boldsymbol{b}\right]_\times = [a]_\times [b]_\times - [b]_\times [a]_\times$, such that $[a]_\times [b]_\times = \left[[a]_\times \boldsymbol{b}\right]_\times + [b]_\times [a]_\times$, and $[\boldsymbol{a}]_\times \boldsymbol{a} = 0$

Rotational Equation of Motion

Dynamics

$$\tau_O^e = \boldsymbol{T}_1 + \boldsymbol{T}_5 + \boldsymbol{T}_6$$
$$\tau_O^e - [\boldsymbol{p}^e]\, \boldsymbol{f}_{total,ext}^e = \mathbf{J}^e \alpha^e + [\omega^e]_\times \mathbf{J}^e \omega^e$$

If we use eq. (**??**) to obtain the rotational eq of motion as:

$$\boxed{\tau_C^e = \mathbf{J}^e \alpha^e + [\omega^e]_\times \mathbf{J}^e \omega^e} \tag{12}$$

where

$$\boxed{\mathbf{J}^e = -\sum_i m_i\, [\boldsymbol{s}_i^e]_\times\, [\boldsymbol{s}_i^e]_\times} \tag{13}$$

Rotational Equation of Motion

Dynamics

We are also interested in expressing the rotational equation of motion in body-frame coordinates. We'll can do some algebra to obtain that

$$\boxed{\boldsymbol{\tau}_c^b = \mathbf{J}^b \boldsymbol{\alpha}^b + \left[\boldsymbol{\omega}^b\right]_\times \mathbf{J}^b \boldsymbol{\omega}^b}, \tag{14}$$

where $\boxed{\mathbf{J}^b = \mathbf{R}_e^b \mathbf{J}^e \mathbf{R}_b^e}$, and we used the fact that $[\mathbf{A}\boldsymbol{a}]_\times = \mathbf{A}\,[\boldsymbol{a}]_\times\,\mathbf{A}^T$.
And finally, we can also express the relation as:

$$\boxed{\boldsymbol{\alpha}^b = \dot{\boldsymbol{\omega}}^b = \left(\mathbf{J}^b\right)^{-1}\left(-\left[\omega^b\right]_\times \mathbf{J}^b \boldsymbol{\omega}^b + \boldsymbol{\tau}_c^b\right)} \tag{15}$$

Inertia Matrix

Dynamics

- The global inertia matrix $\mathbf{J}^e$ was expressed in terms of the body inertia matrix $\mathbf{J}^b$ by the following

$$
\begin{aligned}
\mathbf{J}^e &= -\sum_i m_i \left[\boldsymbol{s}_i^e\right]_\times \left[\boldsymbol{s}_i^e\right]_\times = -\sum_i m_i \left[\mathbf{R}_b^e \boldsymbol{s}_i^b\right]_\times \left[\mathbf{R}_b^e \boldsymbol{s}_i^b\right]_\times \\
&= -\sum_i m_i \mathbf{R}_b^e \left[\boldsymbol{s}_i^b\right]_\times \underbrace{\mathbf{R}_e^b \mathbf{R}_b^e}_{\mathbf{I}_3} \left[\boldsymbol{s}_i^b\right]_\times \mathbf{R}_e^b = \mathbf{R}_b^e \underbrace{\left(-\sum_i m_i \left[\boldsymbol{s}_i^b\right]_\times \left[\boldsymbol{s}_i^b\right]_\times\right)}_{\mathbf{J}^b} \mathbf{R}_e^b
\end{aligned}
$$

- While $\boldsymbol{s}_e^b$ is variable in time, vector $\boldsymbol{s}_i^b$ is constant, meaning $\mathbf{J}^e$ is time dependent, while $\mathbf{J}^b$ is constant

Inertia Matrix
Dynamics

$$\mathbf{J}^b = -\sum_i m_i \left[\boldsymbol{s}_i^b\right]_\times \left[\boldsymbol{s}_i^b\right]_\times = -\sum_i m_i \begin{bmatrix} 0 & -z_i & y_i \\ z_i & 0 & -x_i \\ -y_i & x_i & 0 \end{bmatrix} \begin{bmatrix} 0 & -z_i & y_i \\ z_i & 0 & -x_i \\ -y_i & x_i & 0 \end{bmatrix}$$

$$\mathbf{J}^b = \sum_i m_i \begin{bmatrix} y_i^2 + z_i^2 & -x_i y_i & -x_i z_i \\ -x_i y_i & x_i^2 + z_i^2 & -y_i z_i \\ -x_i z_i & -y_i z_i & x_i^2 + y_i^2 \end{bmatrix}, \text{ where } \boldsymbol{s}_i^b = \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} \tag{16}$$

$$\mathbf{J}^b = \begin{bmatrix} \int_V (y_i^2 + z_i^2) dm & -\int_V x_i y_i dm & -\int_V x_i z_i dm \\ -\int_V x_i y_i dm & \int_V (x_i^2 + z_i^2) dm & -\int_V y_i z_i dm \\ -\int_V x_i z_i dm & -\int_V y_i z_i dm & \int_V (x_i^2 + y_i^2) dm \end{bmatrix} \tag{17}$$