1 Introduction

In Part 1 is about Conditional Independence. Part 2 is about simulation and message-passing in a discrete version of a TrueSkill inspired factor graph. Please provide your solutions as PDF with the respective name. You can use for example Word or Tex to generate this PDF.

Part 1: Determine the Conditional Independence of pairs of nodes in a complex Bayes Network (3 Points)

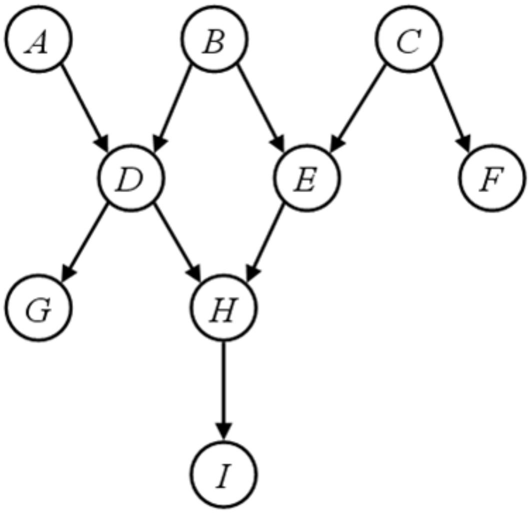

Figure 1: A directed graph with nodes A-I.

Consider the two graphs depicted in Figure 1. We look for pairs of variables that are conditionally independent!

a) Assume that node B and F are clamped/observed.

Are D and E conditionally independent?

Are A and E conditionally independent?

Are G and C conditionally independent?

b) Assume that node I and C are clamped/observed.

Are D and E conditionally independent?

Are A and E conditionally independent?

Are E and F conditionally independent?

c) Assume that node E and H are clamped/observed.

Are C and I conditionally independent?

Are A and I conditionally independent?

d) Assume that node D is clamped/observed.

Are A and H conditionally independent?

Are C and I conditionally independent?

Are G and F conditionally independent?

e) Now we look for a graph that satisfies certain relations. Consider four random variables (A, B, C, D). Find a graph that has the following relations.

1. A is a parent of D.
2. B is a child of C.
3. There is no edge between A and B.
4. D and C are conditionally independent when A is observed.
5. D and C are not conditionally independent when A is not observed.

Part 2: Factor Graphs and Message Passing in the Discrete TrueSkill Model (5 Points)

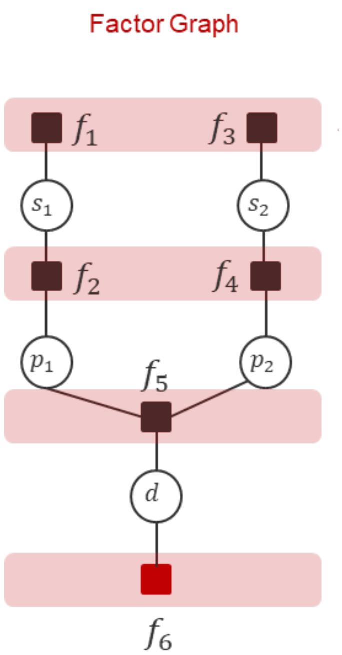

Figure 2: TrueSkill factor graph.

Consider the TrueSkill graph, see Figure 2, for given factors f_i, $i = 1, ..., 6$, with *discrete* values (use, e.g., the functions below for N different potential skill and performance levels):

$$f_1(s_1) := (2\pi\sigma_1^2)^{-1/2} \cdot \exp\left(-\frac{1}{2\sigma_1^2}(s_1 - \mu_1)^2\right), s_1 = 1, ..., N$$

$$f_2(s_1, p_1) = P(p_1|s_1) := (2\pi\beta^2)^{-1/2} \cdot \exp\left(-\frac{1}{2\beta^2}(p_1 - s_1)^2\right), s_1, p_1 = 1, ..., N$$

$$f_3(s_2) := (2\pi\sigma_2^2)^{-1/2} \cdot \exp\left(-\frac{1}{2\sigma_2^2}(s_2-\mu_2)^2\right), s_2 = 1, ..., N$$

$$f_4(s_2, p_2) = P(p_2|s_2) := (2\pi\beta^2)^{-1/2} \cdot \exp\left(-\frac{1}{2\beta^2}(p_2-s_2)^2\right), s_2, p_2 = 1, ..., N$$

$$f_5(p_1, p_2, d) = P(d|p_1, p_2) := 1_{\{d=p_1-p_2\}},\ p_1, p_2 = 1, ..., S,\ d = -N, ..., N$$

$$f_6(d) := 1_{\{d>0\}} \propto P(d|y=1), d = -N, ..., N$$

(a) Compute the *joint* probability $p(s_1, s_2, p_1, p_2, d)$, which (unnormalized) can be written as:

$$p(s_1, s_2, p_1, p_2, d) \propto f_1(s_1) \cdot f_2(s_1, p_1) \cdot f_3(s_2) \cdot f_4(s_2, p_2) \cdot f_5(p_1, p_2, d) \cdot f_6(d)$$

Then, obtain all variables' *marginals* using the (unnormalized) expressions

$$p(s_1) \propto \sum_{s_2}\sum_{p_1}\sum_{p_2}\sum_{d} f_1(s_1) \cdot f_2(s_1, p_1) \cdot f_3(s_2) \cdot f_4(s_2, p_2) \cdot f_5(p_1, p_2, d) \cdot f_6(d)$$

$$p(s_2) \propto \sum_{s_1}\sum_{p_1}\sum_{p_2}\sum_{d} f_1(s_1) \cdot f_2(s_1, p_1) \cdot f_3(s_2) \cdot f_4(s_2, p_2) \cdot f_5(p_1, p_2, d) \cdot f_6(d)$$

$$p(p_1) \propto \sum_{s_1}\sum_{s_2}\sum_{p_2}\sum_{d} f_1(s_1) \cdot f_2(s_1, p_1) \cdot f_3(s_2) \cdot f_4(s_2, p_2) \cdot f_5(p_1, p_2, d) \cdot f_6(d)$$

$$p(p_2) \propto \sum_{s_1}\sum_{s_2}\sum_{p_1}\sum_{d} f_1(s_1) \cdot f_2(s_1, p_1) \cdot f_3(s_2) \cdot f_4(s_2, p_2) \cdot f_5(p_1, p_2, d) \cdot f_6(d)$$

$$p(d) \propto \sum_{s_1}\sum_{s_2}\sum_{p_1}\sum_{p_2} f_1(s_1) \cdot f_2(s_1, p_1) \cdot f_3(s_2) \cdot f_4(s_2, p_2) \cdot f_5(p_1, p_2, d) \cdot f_6(d)$$

(b) Now, compute these marginals more efficiently via *message passing*. You may compute messages and marginals in the following order:

$$m_{f_1 \to S_1}(s_1) = f_1(s_1)$$

$$m_{f_3 \to S_2}(s_2) = f_3(s_2)$$

$$p(s_1) \propto m_{f_1 \to S_1}(s_1) \cdot \underbrace{m_{f_2 \to S_1}(s_1)}_{uniform}$$

$$p(s_2) \propto m_{f_3 \to S_2}(s_2) \cdot \underbrace{m_{f_4 \to S_2}(s_2)}_{uniform}$$

$$m_{f_2 \to P_1}(p_1) \propto \sum_{s_1} f_2(s_1, p_1) \cdot \underbrace{m_{S_1 \to f_2}(s_1)}_{p(s_1)/\underbrace{m_{f_2 \to S_1}(s_1)}_{uniform}} = \sum_{s_1} f_2(s_1, p_1) \cdot p(s_1)$$

$$m_{f_4 \to P_2}(p_2) \propto \sum_{s_2} f_4(s_2, p_2) \cdot \underbrace{m_{S_2 \to f_4}(s_2)}_{p(s_2)/\underbrace{m_{f_4 \to S_2}(s_2)}_{uniform}} = \sum_{s_2} f_4(s_2, p_2) \cdot p(s_2)$$

$$p(p_1) \propto m_{f_2 \to P_1}(p_1) \cdot \underbrace{m_{f_5 \to P_1}(p_1)}_{uniform} = m_{f_2 \to P_1}(p_1)$$

$$p(p_2) \propto m_{f_4 \to P_2}(p_2) \cdot \underbrace{m_{f_5 \to P_2}(p_2)}_{uniform} = m_{f_4 \to P_2}(p_2)$$

$$m_{P_1 \to f_5}(p_1) \propto p(p_1) / \underbrace{m_{f_5 \to P_1}(p_1)}_{uniform} = p(p_1)$$

$$m_{P_2 \to f_5}(p_2) \propto p(p_2) / \underbrace{m_{f_5 \to P_2}(p_2)}_{uniform} = p(p_2)$$

$$m_{f_5 \to d}(d) \propto \sum_{p_1} \sum_{p_2} f_5(p_1, p_2, d) \cdot m_{P_1 \to f_5}(p_1) \cdot m_{P_2 \to f_5}(p_2)$$

$$p(d) \propto m_{f_5 \to d}(d) \cdot \underbrace{m_{f_6 \to d}(d)}_{uniform} = m_{f_5 \to d}(d)$$

Now, we observe the game outcome, i.e. player 1 wins ($y = 1$). We compute:

$$m_{f_6 \to d}(d) \propto f_6(d)$$

$$p(d) \propto m_{f_5 \to d}(d) \cdot m_{f_6 \to d}(d) = P(d|y = 1) = P(d|d > 0)$$

$$m_{d \to f_5}(d) \propto p(d)/m_{f_5 \to d}(d) \cdot 1_{\{m_{f_5 \to d}(d) > 0\}}$$

$$m_{f_5 \to P_1}(p_1) \propto \sum_{d} \sum_{p_2} \underbrace{1_{\{d = p_1 - p_2\}}}_{f_5(p_1, p_2, d)} \cdot m_{d \to f_5}(d) \cdot m_{P_2 \to f_5}(p_2)$$

$$m_{f_5 \to P_2}(p_2) \propto \sum_{d} \sum_{p_1} \underbrace{1_{\{d = p_1 - p_2\}}}_{f_5(p_1, p_2, d)} \cdot m_{d \to f_5}(d) \cdot m_{P_1 \to f_5}(p_1)$$

$$p(p_1) \propto m_{f_2 \to P_1}(p_1) \cdot m_{f_5 \to P_1}(p_1)$$

$$p(p_2) \propto m_{f_4 \to P_2}(p_2) \cdot m_{f_5 \to P_2}(p_2)$$

$$m_{f_2 \to S_1}(s_1) \propto \sum_{p_1} f_2(s_1, p_1) \cdot \underbrace{m_{P_1 \to f_2}(p_1)}_{p(p_1)/m_{f_2 \to P_1}(p_1)}$$

$$m_{f_4 \to S_2}(s_2) \propto \sum_{p_2} f_4(s_2, p_2) \cdot \underbrace{m_{P_2 \to f_4}(p_2)}_{p(p_2)/m_{f_4 \to P_2}(p_2)}$$

$$p(s_1) \propto m_{f_1 \to S_1}(s_1) \cdot m_{f_2 \to S_1}(s_1)$$

$$p(s_2) \propto m_{f_3 \to S_2}(s_2) \cdot m_{f_4 \to S_2}(s_2)$$

(c) Solve the model for the exemplary parameters $N = 20$, $\mu_1 = 8$, $\mu_2 = 12$, $\sigma_1^2 = 2$, $\sigma_2^2 = 2$, $\beta^2 = 3$. Compare the runtime of the solution for (a) and (b).