Exercise 1

a)

Round 1 Votes

11	5	6	3	6	1	4	1
a	e	b	d	c	d	e	b
b	d	a	c	a	b	d	c
c	b	d	b	d	a	c	a
d	c	e	e	b	c	a	e
e	a	c	a	e	e	b	d

Plurality Votes

a	b	c	d	e
11	7	6	4	9

Eliminating 'd'.

Round 2 Votes

11	5	6	3	6	1	4	1
a	e	b		c		e	b
b		a	c	a	b		c
c	b		b		a	c	a
	c	e	e	b	c	a	e
e	a	c	a	e	e	b	

Plurality Votes

a	b	c	d	e
11	8	9	/	9

Eliminating 'b'.

Round 3 Votes

11	5	6	3	6	1	4	1
a	e			c		e	
		a	c	a			c
c					a	c	a
	c	e	e		c	a	e
e	a	c	a	e	e		

Plurality Votes

a	b	c	d	e
18	/	10	/	9

Eliminating 'e'.

Round 4 Votes

11	5	6	3	6	1	4	1
a				c			
		a	c	a			c
c					a	c	a
	c				c	a	
	a	c	a				

Plurality Votes

a	b	c	d	e
18	/	19	/	/

Eliminating 'a'.

Round 5 Votes

11	5	6	3	6	1	4	1
				c			
			c				c
c						c	
	c				c		
		c					

Plurality Votes

a	b	c	d	e
/	/	19	/	/

Candidate 'c' wins.

b)
Consider a profile $\mathcal{P}$:

10	6	7	2
a	b	c	b
b	c	a	a
c	a	b	c

In this profile, Plurality votes begin with a,b and c having 10,8 and 7 votes - c is then eliminated, bringing the votes to 17 and 8 for a and b, leading to a win for a.

Now consider profile $\mathcal{P}'$:

10	6	7	2
a	b	c	a
b	c	a	b
c	a	b	c

The only difference to $\mathcal{P}$ is that the group of two voters moved a up by one position. Now, the first candidate to be eliminated is b. This brings the votes to 12 and 13 for a and c respectively, leading to a win for c.

Therefore, STV is not monotonic.

Exercise 2

Plurality:
Majority Winner: Yes. A majority winner trivially wins.

Majority Loser: No. Consider profile 1, in which a is ranked last by 52% of voters but wins.

Condorcet Winner: No. Consider profile 2, in which b is a condorcet winner but loses.

Condorcet Loser: No. Consider profile 3, in which d is a condorcet loser but wins.

IIA: No. Consider profile 4, in which c is a winner and a is a loser, and profile 4' where a is a winner and c is a loser, even though only the relative ranking of b and c was changed.

48	26	26
a	b	c
b	c	b
c	a	a

(a) Profile 1

5	5	5
a	b	c
d	d	d
b	c	a
c	a	b

(b) Profile 2

5	5	5	6
a	b	c	d
b	c	a	a
c	a	b	b
d	d	d	c

(c) Profile 3

8	4	5
a	c	c
b	a	b
c	b	a

(d) Profile 4

8	4	5
a	c	b
b	a	c
c	b	a

(e) Profile 4'

Borda:
Majority Winner: No. Consider Profile 5, in which a receives 10 points, b receives 13 points by Borda, although a is majority winner and b is not.

Majority Loser: Yes. Consider the majority loser with the most possible points, i.e. a majority loser x that is ranked last $50 + \epsilon\%$ and first $50 - \epsilon\%$ of the time, with ϵ being an arbitrarily small number, resulting in a score $s_x < |V| \cdot \frac{|C|}{2}$. As there are in total $|V| \cdot \frac{|C|^2+|C|}{2}$ points, this leaves $|V| \cdot \frac{|C|^2}{2}$ points to be distributed amongst $|C| - 1$ remaining candidates. Assuming that the maximum score of other candidates is minimized, i.e. equally distributed, other candidates would receive a score $|V| \cdot \frac{|C|^2}{2\cdot(|C|-1)} > |V| \cdot \frac{|C|^2}{2\cdot|C|} = |V| \cdot \frac{|C|}{2} > s_x$. The majority loser is therefore a loser in the most favourable situation, and therefore the loser in any case. Note that for easy of calculation we used a Borda version wherein the score of all positions is increased by 1 over the common version - but this is logically equivalent to normal Borda.

Condorcet Winner: No. A majority winner is necessarily a condorcet winner - but does not necessarily win.

Condorcet Loser: Yes. Consider a voting profile that leads to the election of a condorcet loser under Borda. The lowest possible score of a winner occurs when scores are equally distributed. This implies the condorcet loser has a score greater equal $|V| \cdot \frac{|C|-1}{2}$, and is therefore ranked in front of other candidates at least $x \geq |V| \cdot \frac{|C|-1}{2}$ times. To qualify as a condorcet loser, it must however be ranked below other candidates more than $y > \frac{|V|}{2} \cdot (|C| - 1)$ times. Therefore, it must be ranked in relation to other candidates more than $x + y > |V| \cdot (|C| - 1)$ times, which is impossible. Therefore, a condorcet loser cannot win under Borda.

IIA: No. Consider profiles 6 and 6'. In profile 6, candidates a b and c respectively achieve scores of 8, 3 and 10, with c winning and the rest losing. In profile 6', they respectively achieve scores of 8, 6 and 7, with a winning and the rest losing. As only the ordering of b and c was changed, but a went from being a loser to a winner, these profiles violate IIA under Borda.

5	4
a	b
b	c
c	a

(a) Profile 5

4	3
a	c
c	b
b	a

(b) Profile 6

4	3
a	b
c	c
b	a

(c) Profile 6'

Copeland: Majority Winner: Yes. A majority winner is necessarily a condorcet winner (see below).

Majority Loser: Yes. A majority loser is necessarily a condorcet loser (see below).

Condorcet Winner: Yes. A condorcet winner by definition wins all pairwise comparisons and therefore achieves the maximum score $|C| - 1$, winning.

Condorcet Loser: Yes. A condorcet loser by definition loses all pairwise comparisons and therefore achieves the minimum score 0, losing.

IIA: No. Consider a preference profile with candidates a, b, c, and pairwise rankings of $a > b$, $a == c$, $b == c$. The only winner by Copeland is a, with b and c being losers. Now modify the profile such that $a > b$, $a == c$, $b > c$. Now, the winners are a and b, with only c losing. As only the ordering between b and c, but not between a and b changed, this violates IIA.

Exercise 4

a)
The borda score s_a for a candidate a is exactly the number of candidates voters have ranked below a. Therefore, dividing the borda score s_a by the number of candidates $|V|$ yields the average rank of candidate a. As such, the candidate with the maximal borda score has the maximal average ranking.

b)
For a set of alternatives C, the positional scoring rule induced by the vector $(s_1, s_2, \dots, s_{|C|})$ is monotonic if and only if $s_1 \geq s_2 \geq \dots \geq s_{|C|}$.

$\rightarrow$
...$(s_1, s_2, \dots, s_{|C|})$ is monotonic if $s_1 \geq s_2 \geq \dots \geq s_{|C|}$.
Consider a preference profile $\mathcal{P}$ over a set of alternatives C with a positional scoring rule induced by $(s_1, s_2, \dots, s_{|C|})$ with $s_1 \geq s_2 \geq \dots \geq s_{|C|}$, which has some winner $a \in f(\mathcal{P})$ that is not in the highest position for all voters - other profiles are uninteresting to us.
Now construct a preference profile $\mathcal{P}'$ by moving any candidate $a \in f(\mathcal{P})$ up one position in some vote, and therefore moving some candidate b down one position. Say that a was moved from the xth position to the $x-1$th position. Then, the score of a in $\mathcal{P}$ being s_a and the score of a in $\mathcal{P}'$ being s'_a, we know that $s'_a = s_a - s_x + s_{x-1}$. Since $s_{x-1} > s_x$, therefore $s'_a > s_a$. Similarly, $s'_b < s_b$. All other scores remain unchanged.
Therefore, $a \in f(\mathcal{P}')$ and the positional scoring rule is monotonic.

$\leftarrow$
...$(s_1, s_2, \dots, s_{|C|})$ is not monotonic if not $s_1 \geq s_2 \geq \dots \geq s_{|C|}$. Consider a preference profile $\mathcal{P}$ over a set of alternatives C with a positional scoring rule induced by $v = (s_1, s_2, \dots, s_{|C|})$ with $\exists s_x, s_{x-1} \in v : s_x > s_{x-1}$, with a winner $a \in f(\mathcal{P})$ where a is in the xth position in at least one vote within $\mathcal{P}$. Then, moving a up by one place in such a vote results in a score decrease of $s_x - s_{x-1}$.
We can now construct an example in which monotonicity is broken.
Consider the alternative b with the second highest total score in $\mathcal{P}$. Construct a preference profile $\mathcal{P}'$ from $mathcalP$ by swapping a and b. In the profile $\mathcal{P}'' = \mathcal{P} + \mathcal{P}'$, both a and b now achieve the same score and are winners. Now, modify $\mathcal{P}''$ by moving a from the xth position to the $x-1$th position in one vote. This decreases the score of a, leaving b as the only winner.
Therefore, the positional scoring rule under consideration is not monotonic.

Exercise 5

a)
Consider that the score s_a of a candidate a given a preference profile $\mathcal{P}$ according to Borda's rule corresponds exactly to the sum over all voters of said candidates rank - i.e. the number of candidates the candidate in question is ranked above, in total:

$$s_a = \sum_{i \in V} |\{b \in C : b \prec_i a\}| = \sum_{b \in C} |\{i \in V : b \prec_i a\}|$$

Furthermore, we know that according to the Monotony of Multiplication and Monotony of Addition, multiplying all scores by two and then subtracting the total number of Candidates(-1) does not change the ordering of scores. i.e. such a voting rule would deliver the exact same winners and losers as Borda. Therefore, we may calculate borda scores as

$$s_a \propto s'_a = \sum_{i \in V} |\{b \in C : b \prec_i a\}| \cdot 2 - |C| - 1$$

Furthermore, we know that $\forall i \in V, a \in C : |\{b \in C : b \prec_i a\}| + |\{b \in C : a \prec_i b\}| = |C| - 1$, i.e. each voter must rank each candidate in comparison to other candidates exactly $|C| - 1$ times, and therefore

$$s'_a = \sum_{i \in V} |\{b \in C : b \prec_i a\}| - |\{b \in C : a \prec_i b\}| = \sum_{b \in C} |\{i \in V : b \prec_i a\}| - |\{i \in V : a \prec_i b\}|$$

With the groundwork out of the way, we can calculate the Borda's rule ranking from a weighted pairwise majority tournament graph by summing the weights of all outgoing edges for a given node corresponding to candidate a, yielding a score s''_a as follows:

$$s''_a = \sum_{b \in C} \mathrm{Net}_{\mathcal{P}}(a, b) = \sum_{b \in C} |\{i \in V : b \prec_i a\}| - |\{i \in V : a \prec_i b\}| = s'_a \propto s_a$$

Ranking according to these scores then yields exactly the Borda's rule ranking, as elaborated above.

b)
Fundamentally, Borda-equivalent scoring rules, such as a Borda variant with scores multiplied by 2 are monotonic, not Borda and yet are in C2. Similarly, trivial rules that assign the same score to all positions (violating Pareto) are monotonic, not Borda and in C2.

Let's however consider all other monotonic positional scoring rules:
Consider any given such scoring rule made up by a vector $(s_1, s_2, \dots, s_{|C|})$ with $s_1 \geq s_2 \geq \dots \geq s_{|C|}$ alongside some candidates C.

Assign all candidates a number. Consider a preference profile $\mathcal{P}$ made up of exactly two voters v_1 with preferences $c_1 \succ c_2 \succ \dots \succ c_{|C|}$ and a voter v_2 with preferences $c_{|C|} \succ c_{|C|-1} \succ \dots \succ c_1$, i.e. preferences exactly inverted from v_1. Therefore, $\forall a, b \in C : \mathrm{Net}_{\mathcal{P}}(a, b) = 0$.

Now we can consider two cases. Either (a) all candidates achieve the exact same score under the given scoring rule or (b) they do not.

In case (a) we then know that $s_1 + s_{|C|} = s_2 + s_{|C|-1}$ etc, and therefore $s_1 - s_2 = s_{|C|-1} - s_{|C|}$ etc. This implies that the increments between scores in the scoring vector are all some constant c, and the scoring rule is therefore either trivial ($c = 0$) or Borda-equivalent.

As such, we discard case (a) and consider case (b). In case (b) there is then some candidate c with a highest score.

We can now consider any preference profile $\mathcal{P}'$ where the set of winners does not include c. Assume our scoring rule is in $C2$, and a function ϕ exists that computes the correct winners for $\mathcal{P}'$ from the weighted pairwise majority tournament, i.e. $\phi(\mathcal{P}') = f(\mathcal{P}')$. We can now add copies of $\mathcal{P}$ to $\mathcal{P}'$. Each copy added increases the score of c more, relative to other candidates, without changing the weighted pairwise majority tournament graph.

Therefore, there is some number $k \in \mathbb{N}$ such that for $\mathcal{P}'' = k \cdot \mathcal{P} + \mathcal{P}'$ with $c \notin f(\mathcal{P}')$ yet $c \in f(\mathcal{P}'')$ and therefore $f(\mathcal{P}') \neq f(\mathcal{P}'')$ but $\phi(\mathcal{P}') = \phi(\mathcal{P}'')$. In our constructed preference profile, the function ϕ therefore cannot compute the correct result, and the positional scoring rule is not in C2.