Exercise 1

The french election system uses a form of single choice runoff voting. Each seat in the assembly is individually elected, with voters casting a vote for a single party in up to two rounds. If a party wins the majority in the first round, it wins the seat. If it does not, the top two parties and all parties under 12.5% move on to the second round. After re-casting of votes, the most popular party wins the seat.

Cases in which three parties remain in the running for the second round are called 'triangulaires'. Usually, less than ten or so out of 577 seat-elections result in triangulaires, with a previous record of 105. The recent election, however, resulted in 306 triangulaires, owing to the recent rise of the National Rally party - a third highly popular party/alliance next to the Ensemble and the Popular Front.

In most of these triangulaires, the politically more aligned Ensemble and Popular Front were then able to unite, dropping their candidates whenever in third place, resulting in massively increased votership for their allies' candidate, thereby winning more seats.

Interesting computational problems may be
(a) Manipulation by Cooperative Candidate Removal: Given a set of elections where candidates a and b together are not the most popular, can either a or b drop out of select elections such that a and b together win the most seats?
(b) Manipulation by Shifting Position: Given an election where candidate a does not win, assuming political positions are on some n-dimensional vector, with voters voting for the closest candidate, can candidate a win by changing positions by a vector up to k in length?
(c) French Election Scheme Robustness: Given some voting rule, consider many random elections. In what percentage of elections is (a) possible?

Exercise 2

Programatically generated answers, via brute-forcing. One run takes roughly 20s.
Source code available in my github[1]. Enjoy!

Candidates	AV	CC	PAV	JR	EJR	Core Stability
$\{c_6, c_7, c_8, c_9\}$	False	True	False	True	True	True
$\{c_5, c_7, c_8, c_9\}$	False	True	False	True	True	True
$\{c_5, c_6, c_7, c_9\}$	False	True	False	True	True	True
$\{c_4, c_7, c_8, c_9\}$	False	True	False	True	True	True
$\{c_4, c_6, c_8, c_9\}$	False	True	False	True	True	True
$\{c_4, c_6, c_7, c_9\}$	False	True	False	True	True	True
$\{c_4, c_5, c_8, c_9\}$	False	True	False	True	True	True
$\{c_4, c_5, c_6, c_9\}$	False	True	False	True	True	True
$\{c_3, c_7, c_8, c_9\}$	False	True	False	True	True	True
$\{c_3, c_6, c_7, c_9\}$	False	True	False	True	True	True
$\{c_3, c_5, c_8, c_9\}$	False	True	False	True	True	True
$\{c_3, c_5, c_6, c_9\}$	False	True	False	True	True	True
$\{c_3, c_4, c_8, c_9\}$	False	True	False	True	True	True
$\{c_3, c_4, c_6, c_9\}$	False	True	False	True	True	True
$\{c_2, c_7, c_8, c_9\}$	False	True	False	True	True	True
$\{c_2, c_6, c_7, c_9\}$	False	True	False	True	True	True
$\{c_2, c_5, c_8, c_9\}$	False	True	False	True	True	True
$\{c_2, c_5, c_7, c_9\}$	False	True	False	True	True	True
$\{c_2, c_5, c_6, c_9\}$	False	True	False	True	True	True
$\{c_2, c_4, c_8, c_9\}$	False	True	False	True	True	True
$\{c_2, c_4, c_7, c_9\}$	False	True	False	True	True	True
$\{c_2, c_4, c_6, c_9\}$	False	True	False	True	True	True
$\{c_2, c_3, c_7, c_9\}$	False	True	False	True	True	True
$\{c_2, c_3, c_5, c_9\}$	False	True	False	True	False. $S = \{v_3, v_4, v_5\}$, $l = 2$	False. $S = \{v_3, v_4, v_5\}$ prefer $U = \{c_6, c_8$
$\{c_2, c_3, c_4, c_9\}$	False	True	False	True	True	True
$\{c_1, c_7, c_8, c_9\}$	False	True	False	True	True	True
$\{c_1, c_6, c_8, c_9\}$	False	True	False	True	True	True
$\{c_1, c_6, c_7, c_9\}$	False	True	False	True	True	True
$\{c_1, c_6, c_7, c_8\}$	True	True	True	True	True	True
$\{c_1, c_5, c_8, c_9\}$	False	True	False	True	True	True
$\{c_1, c_5, c_7, c_9\}$	False	True	False	True	True	True
$\{c_1, c_5, c_7, c_8\}$	False	True	False	True	True	True
$\{c_1, c_5, c_6, c_9\}$	False	True	False	True	True	True
$\{c_1, c_5, c_6, c_8\}$	False	True	False	True	True	True
$\{c_1, c_5, c_6, c_7\}$	False	True	False	True	True	True
$\{c_1, c_4, c_8, c_9\}$	False	True	False	True	True	True
$\{c_1, c_4, c_7, c_9\}$	False	True	False	True	True	True
$\{c_1, c_4, c_7, c_8\}$	True	True	True	True	True	True
$\{c_1, c_4, c_6, c_9\}$	False	True	False	True	True	True
$\{c_1, c_4, c_6, c_8\}$	False	True	False	True	True	True
$\{c_1, c_4, c_6, c_7\}$	True	True	True	True	True	True
$\{c_1, c_4, c_5, c_8\}$	False	True	False	True	True	True
$\{c_1, c_4, c_5, c_7\}$	False	True	False	True	True	True
$\{c_1, c_4, c_5, c_6\}$	False	True	False	True	True	True
$\{c_1, c_3, c_8, c_9\}$	False	True	False	True	True	True
$\{c_1, c_3, c_7, c_9\}$	False	True	False	True	True	True
$\{c_1, c_3, c_7, c_8\}$	False	True	False	True	True	True

[1] https://github.com/Fritz-D/AlgCDM

$\{c_1, c_3, c_6, c_9\}$	False	True	False	True	True	True
$\{c_1, c_3, c_6, c_8\}$	False	True	False	True	True	True
$\{c_1, c_3, c_6, c_7\}$	False	True	False	True	True	True
$\{c_1, c_3, c_5, c_9\}$	False	True	False	True	False. $S = \{v_3, v_4, v_5\}$, $l = 2$	False. $S = \{v_3, v_4, v_5\}$ prefer $U = \{c_6, c_8\}$
$\{c_1, c_3, c_5, c_8\}$	False	True	False	True	True	True
$\{c_1, c_3, c_5, c_7\}$	False	True	False	True	True	True
$\{c_1, c_3, c_5, c_6\}$	False	True	False	True	True	True
$\{c_1, c_3, c_4, c_9\}$	False	True	False	True	True	True
$\{c_1, c_3, c_4, c_8\}$	False	True	False	True	True	True
$\{c_1, c_3, c_4, c_7\}$	False	True	False	True	True	True
$\{c_1, c_3, c_4, c_6\}$	False	True	False	True	True	True
$\{c_1, c_3, c_4, c_5\}$	False	True	False	True	False. $S = \{v_3, v_4, v_5\}$, $l = 2$	False. $S = \{v_3, v_4, v_5\}$ prefer $U = \{c_6, c_8\}$
$\{c_1, c_2, c_8, c_9\}$	False	True	False	True	True	True
$\{c_1, c_2, c_7, c_9\}$	False	True	False	True	True	True
$\{c_1, c_2, c_7, c_8\}$	False	True	False	True	True	True
$\{c_1, c_2, c_6, c_9\}$	False	True	False	True	True	True
$\{c_1, c_2, c_6, c_8\}$	False	True	False	True	True	True
$\{c_1, c_2, c_6, c_7\}$	False	True	False	True	True	True
$\{c_1, c_2, c_5, c_8\}$	False	True	False	True	True	True
$\{c_1, c_2, c_5, c_7\}$	False	True	False	True	True	True
$\{c_1, c_2, c_5, c_6\}$	False	True	False	True	True	True
$\{c_1, c_2, c_4, c_8\}$	False	True	False	True	True	True
$\{c_1, c_2, c_4, c_7\}$	False	True	False	True	True	True
$\{c_1, c_2, c_4, c_6\}$	False	True	False	True	True	True
$\{c_1, c_2, c_3, c_9\}$	False	True	False	True	True	True
$\{c_1, c_2, c_3, c_8\}$	False	True	False	True	True	True
$\{c_1, c_2, c_3, c_7\}$	False	True	False	True	True	True
$\{c_1, c_2, c_3, c_6\}$	False	True	False	True	True	True
$\{c_1, c_2, c_3, c_4\}$	False	True	False	True	True	True

Exercise 3

a)
Yes.
In short, we can sum up approval counts for all candidates. Iff a candidate c has an approval count $s > \frac{n}{k}$ then there is a group $S \subset V$ with $|S| \geq \frac{n}{k}$ and $c \in \cap_{i \in S} A_i$ and vice versa.

Then, for each such candidate we iterate over voters approving of that candidate. For each voter, we check if they approve of no candidate in W. Iff we find $\frac{n}{k}$ or more such disgruntled voters, we have a 1-cohesive group such that JR is not satisfied. If not, we don't.

This is roughly in $\mathcal{O}(|C|^2|V|)$.

b)
Let $W \in \binom{C}{k}$ be a committee maximizing the CC-score. For contradiction, let $S \subset V$ be a subset of voters such that $|S| \geq \frac{n}{k}$ and $| \cap_{i \in S} A_i | \geq 1$ but $u_i(W) < 1$ for all $i \in S$. Let $c^* \in \cap_{i \in S} A_i$ with $c^* \notin W$.

Intuitively, this implies there is a set of at least $\frac{n}{k}$ completely unrepresented in W. That could be represented were c^* to be added. Hence, the score of $W \cup \{c^*\}$ is at least $\frac{n}{k}$ larger than the CC-score of W.

We know therefore that at most $\frac{(k-1)n}{k}$ voters are represented in W. As there are k candidates in W, there is at least one candidate $c' \in W$ that can be removed without decreasing the CC-score by more than $\frac{n}{k}$.

Therefore, the CC-score for $W \cup \{c^*\} \backslash \{c'\}$ is greater than the CC-score for W, a contradiction to W maximizing the CC-score. q.e.d.

Exercise 4

Given some Thiele Rule with an associate weight vector $\mathbf{w} = (w_1, w_2, w_3, \dots)$ with $w_1 = 1$ and $\exists w_i \in \mathbf{w} : w_i \neq \frac{1}{i}$. Let $j = \min\{i \mid w_i \in \mathbf{w} : w_i \neq \frac{1}{i}\}$, i.e. w_j is the first weight discordant to the harmonic series. Based on whether w_j is smaller or larger than $\frac{1}{j}$ we can construct elections to show this rule does not satisfy EJR.

Case 1: $w_j > \frac{1}{j} \implies w_j = \frac{1}{j} \cdot x, x > 1$
Construct an election with a committee size k and a large number of voters V with $|V| = n$ and Candidates C such that it has

(a) exactly $j\frac{n}{k} - 1$ voters voting for exactly candidates c_1 to c_j, i.e. a j-cohesive subset of voters $S_1 \subset V$ with $|S_1| = j\frac{n}{k} - 1$ and a set of candidates $C_1 \subset C$ with $|C_1| = j$ and $\forall i \in S_1 : u_i(C_1) = j$.
(b) exactly $\frac{n}{k}$ voters voting for exactly one candidate $c \notin C_1$, i.e. a 1-cohesive subset of voters $S_2 \subset V$ with $|S_2| = \frac{n}{k}$ and a candidate $c \in C \backslash C_1$ with $\forall i \in S_2 : u_i(c) = 1$.
(c) some arbitrarily large number of extra voters voting for nobody (or nobody relevant, at least)

Let W be a committee maximizing the Score according to the Thiele Rule. Assume our Thiele Rule is EJR compliant. Then, $c \in W$ and therefore $Score(W\backslash\{c\}) = Score(W) - \frac{n}{k}$, i.e. we can remove candidate c while losing $\frac{n}{k}$ in score.
At the same time, we know $\exists c_i \in C_1 : C_1 \notin W$ such that $C_1 \cap W = C_1\backslash\{c_i\}$ as well as $Score(W \cup \{c_i\} = Score(W) + (j\frac{n}{k} - 1)w_j$, i.e. we can add a candidate our voters in (a) want and gain $(j\frac{n}{k} - 1)w_j = (j\frac{n}{k} - 1)(\frac{1}{j}x) = x(\frac{n}{k} - \frac{1}{j})$.

We know that $x(\frac{n}{k} - \frac{1}{j}) > \frac{n}{k} \equiv x > \frac{jn}{jn-k}$, which holds for arbitrarily large n. Then, $Score(\{c_i\} \cup W\backslash\{c\}) = Score(W) - \frac{n}{k} + x(\frac{n}{k} - \frac{1}{j}) > Score(W)$, contradicting our assumption of this Thiele Rule fulfilling EJR.

Case 2: $w_j < \frac{1}{j} \implies w_j = \frac{1}{j} \cdot x, 0 < x < 1$
Construct an election with a committee size k and a large number of voters V with $|V| = n$ and Candidates C such that it has

(a) exactly $j\frac{n}{k}$ voters voting for exactly candidates c_1 to c_j, i.e. a j-cohesive subset of voters $S_1 \subset V$ with $|S_1| = j\frac{n}{k}$ and a set of candidates $C_1 \subset C$ with $|C_1| = j$ and $\forall i \in S_1 : u_i(C_1) = j$.
(b) exactly $\frac{n}{k} - 1$ voters voting for exactly one candidate $c \notin C_1$, i.e. a 1-cohesive subset of voters $S_2 \subset V$ with $|S_2| = \frac{n}{k} - 1$ and a candidate $c \in C\backslash C_1$ with $\forall i \in S_2 : u_i(c) = 1$.
(c) some arbitrarily large number of extra voters voting for nobody (or nobody relevant, at least)

Let W be a committee maximizing the Score according to the Thiele Rule. Assume our Thiele Rule is EJR compliant. Then, $C_1 \subseteq W$ and therefore $\exists c \in C_1 : Score(W\backslash\{c\}) = Score(W) - j\frac{n}{k}w_j$, i.e. a candidate we can remove from the winning committee while only losing $j\frac{n}{k}w_j = j\frac{n}{k}(\frac{1}{j}x) = \frac{n}{k}x < \frac{n}{k}$ in score.
At the same time, we know $Score(W \cup \{c\}) = Score(W) + \frac{n}{k} - 1$ - we can add the candidate our voters from (b) want to gain $\frac{n}{k} - 1$ in score.

We know that $\frac{n}{k} - 1 > \frac{xn}{k} \equiv \frac{n-k}{n} > x$, which holds for our arbitrarily large n. Then, $Score(\{c\} \cup W\backslash\{c_1\} = Score(W) + \frac{xn}{k} + \frac{n}{k} - 1 > Score(W)$, contradicting our assumption of this Thiele Rule fulfilling EJR.