Mapeos Discretos

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1 Mapeos Discretos

1.1 1. Mapeo Logístico

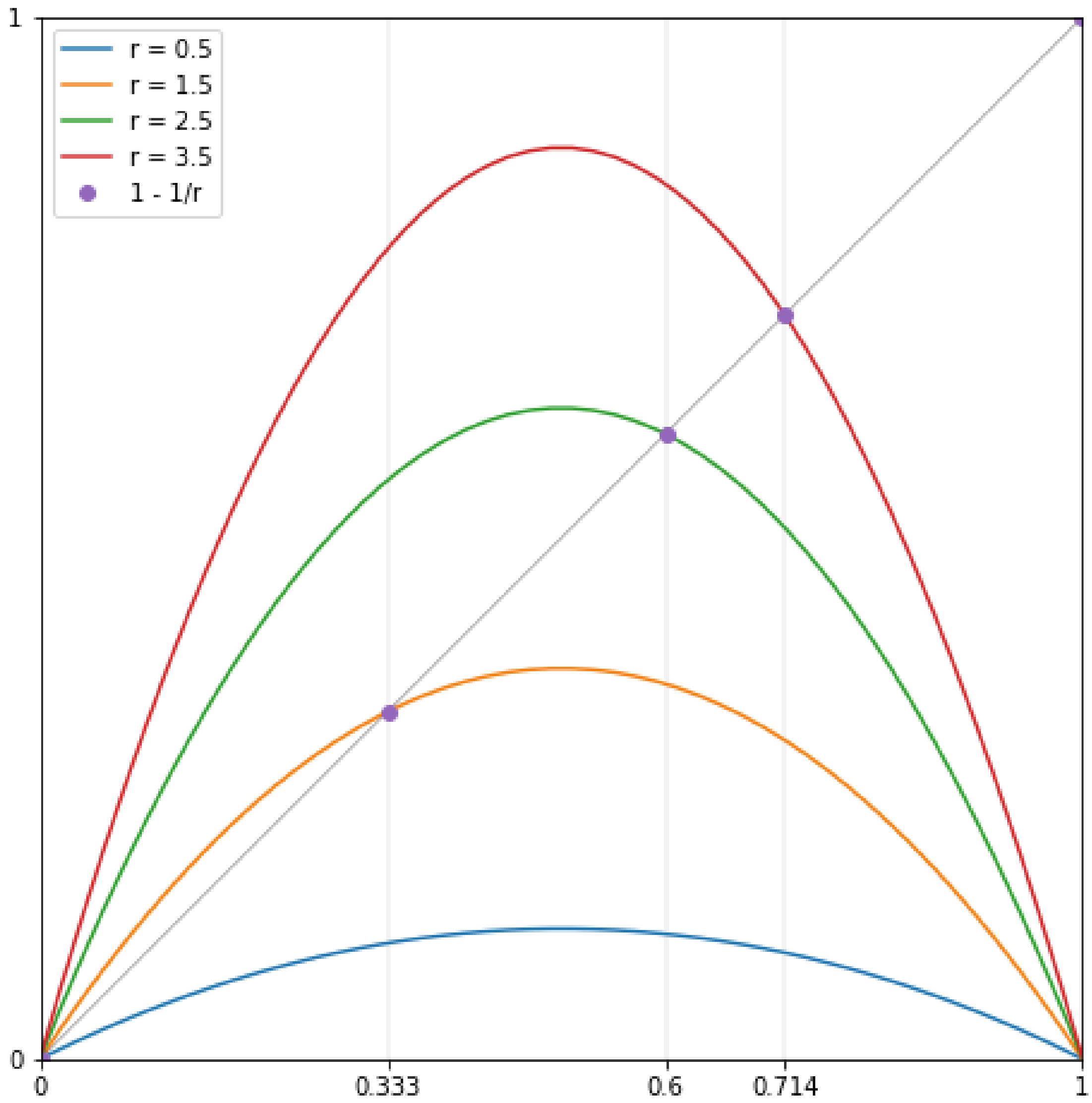
https://ipython-books.github.io/121-plotting-the-bifurcation-diagram-of-a-chaotic-dynamical-system/

[3]:
```
def logistic(r,x):
    return r*x*(1 - x)
```

[2]:
```
import numpy as np
import matplotlib.pyplot as plt
```

[81]:
```
x = np.linspace(0,1)
plt.figure(figsize=(8,8))
a = []
for i in range(4):
    plt.plot( x, logistic(i+0.5,x), label=f'r = {i+0.5:.1f}' )
    a.append(1 - 1/(i+0.5))
print(a)
a[0] = 0
a.append(1)
plt.plot(x,x,'k',lw=0.3)
plt.plot(a,a,'o',label='1 - 1/r')
plt.axis('scaled')
plt.xticks(a,[0,0.333,0.6,0.714,1])
plt.yticks([0,1])
plt.xlim(0,1)
plt.ylim(0,1)
plt.legend()
plt.grid(lw=0.3)
```

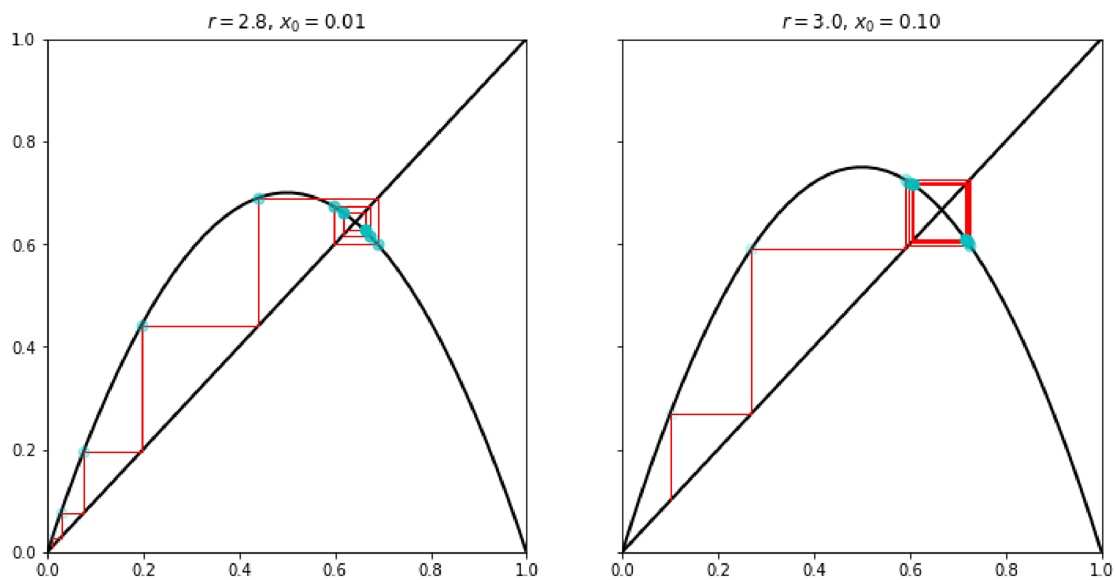
```
[-1.0, 0.33333333333333337, 0.6, 0.7142857142857143]
```



Los puntos fijos de $f(x) = rx(1-x)$ se encuentran en 0 y $1 - \frac{1}{r}$ y son estables para $1 < r < 3$, así que hay bifurcaciones en $r = 1$ y $r = 3$

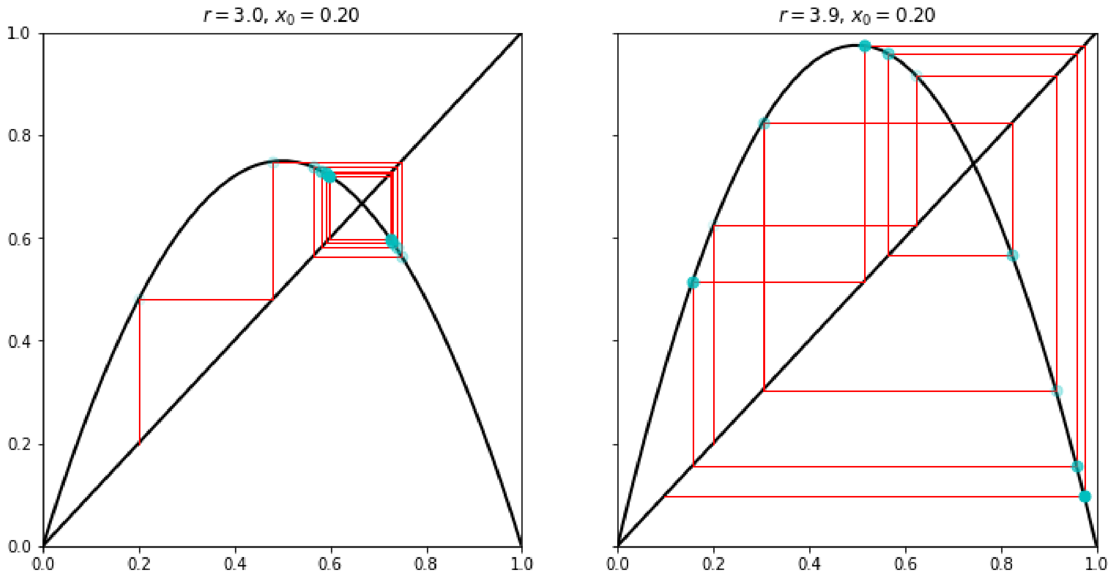
[95]:
```
def plot_system(r, x0, n, ax=None):
    # Plot the function and the y=x diagonal line.
    t = np.linspace(0,1)
    ax.plot(t, logistic(r, t), 'k', lw=2)
    ax.plot([0, 1], [0, 1], 'k', lw=2)

    # Recursively apply y=f(x) and plot two lines:
    # (x, x) -> (x, y)
    # (x, y) -> (y, y)
```

```python
    x = x0
    for i in range(n):
        y = logistic(r, x)
        # Plot the two lines.
        ax.plot([x, x], [x, y], 'r', lw=1)
        ax.plot([x, y], [y, y], 'r', lw=1)
        # Plot the positions with increasing opacity.
        ax.plot([x], [y], 'oc', ms=7,
                alpha=(i+1)/n)
        x = y

    ax.set_xlim(0, 1)
    ax.set_ylim(0, 1)
    ax.set_title(f"$r={r:.1f}, \, x_0={x0:.2f}$")


fig, (ax1, ax2) = plt.subplots(1, 2, figsize=(12, 6), sharey=True)
plot_system(2.8, .01, 10, ax=ax1)
plot_system(3, .1, 10, ax=ax2)
```



Cambiando el valor inicial a $x_0 = 0.2$

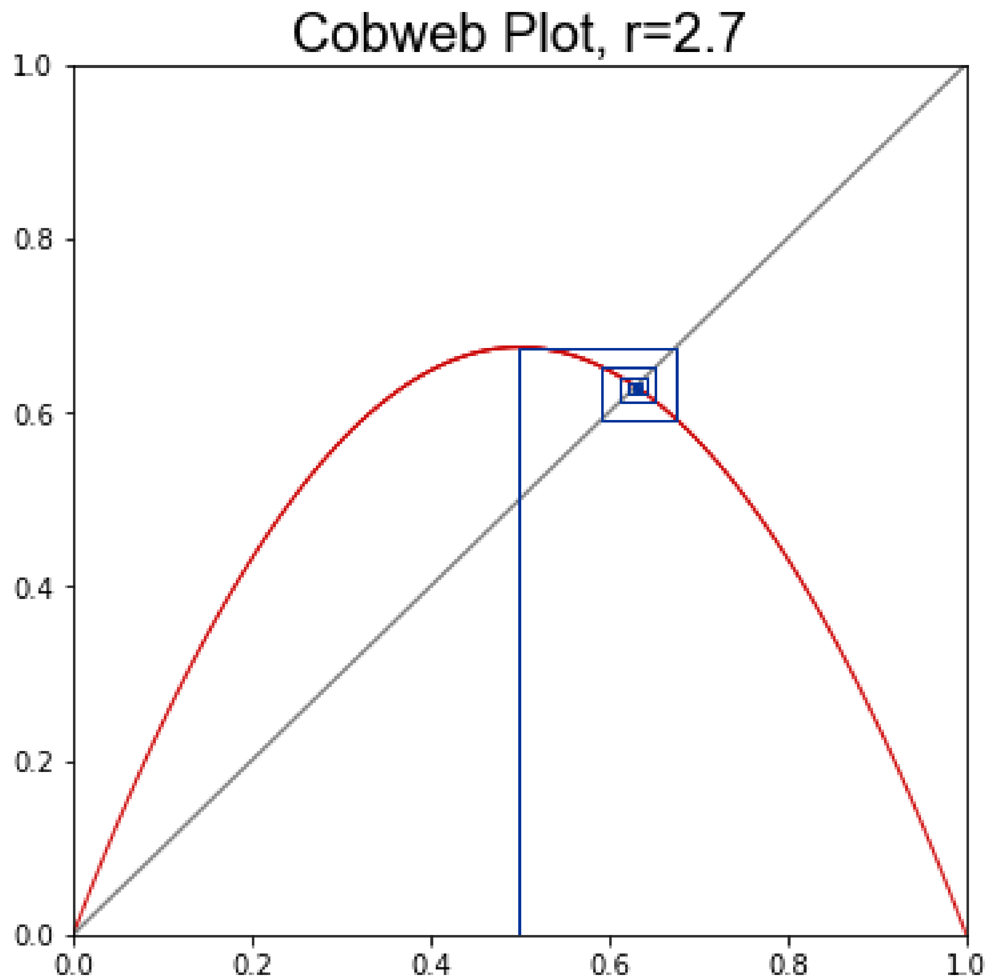
[106]:
```python
fig, (ax1, ax2) = plt.subplots(1, 2, figsize=(12, 6), sharey=True)
plot_system(3, .2, 10, ax=ax1)
plot_system(3.9, .2, 10, ax=ax2)
```



https://notebook.community/gboeing/pynamical/examples/pynamical-demo-cobweb-plots

[108]:
```
from pynamical import cobweb_plot
import pandas as pd, numpy as np, matplotlib.pyplot as plt, IPython.display as␣
 ↪IPdisplay, glob
from PIL import Image
```

[118]:
```
cobweb_plot(r=2.7,cobweb_x=0.5)
```



1.1.1 Animación

[140]:
```
# configure the animation: 70 rates between 0 and 3.999 plus 15 paused frames␣
↪each at start and end
save_folder = 'images/cobweb-animate'
gif_filepath = 'images/animated-logistic-cobweb.gif'
start_r = 0
end_r = 3.99
steps = 140
rates = np.linspace(start_r, end_r, steps)
```

[141]:
```
# create a plot for each growth rate
for n, r in enumerate(rates):
    filename = '{:03d}'.format(n)
    title = 'Logistic Map Cobweb Plot, r={:.2f}'.format(r)
```

```
    cobweb_plot(r=r, filename=filename, folder=save_folder, dpi=90,␣
↪title=title, show=False)
    plt.close()
```

[142]:
```
# create a tuple of display durations, one for each frame
first_last = 100 #show the first and last frames for 100 ms
standard_duration = 10 #show all other frames for 10 ms
durations = tuple([first_last] + [standard_duration] * (len(rates) - 2) +␣
↪[first_last])
```

[143]:
```
# load all the static images into a list then save as an animated gif
images = [Image.open(image) for image in glob.glob('{}/*.png'.
↪format(save_folder))]
gif = images[0]
gif.info['duration'] = durations #ms per frame
gif.info['loop'] = 0 #how many times to loop (0=infinite)
gif.save(fp=gif_filepath, format='gif', save_all=True, append_images=images[1:])
```

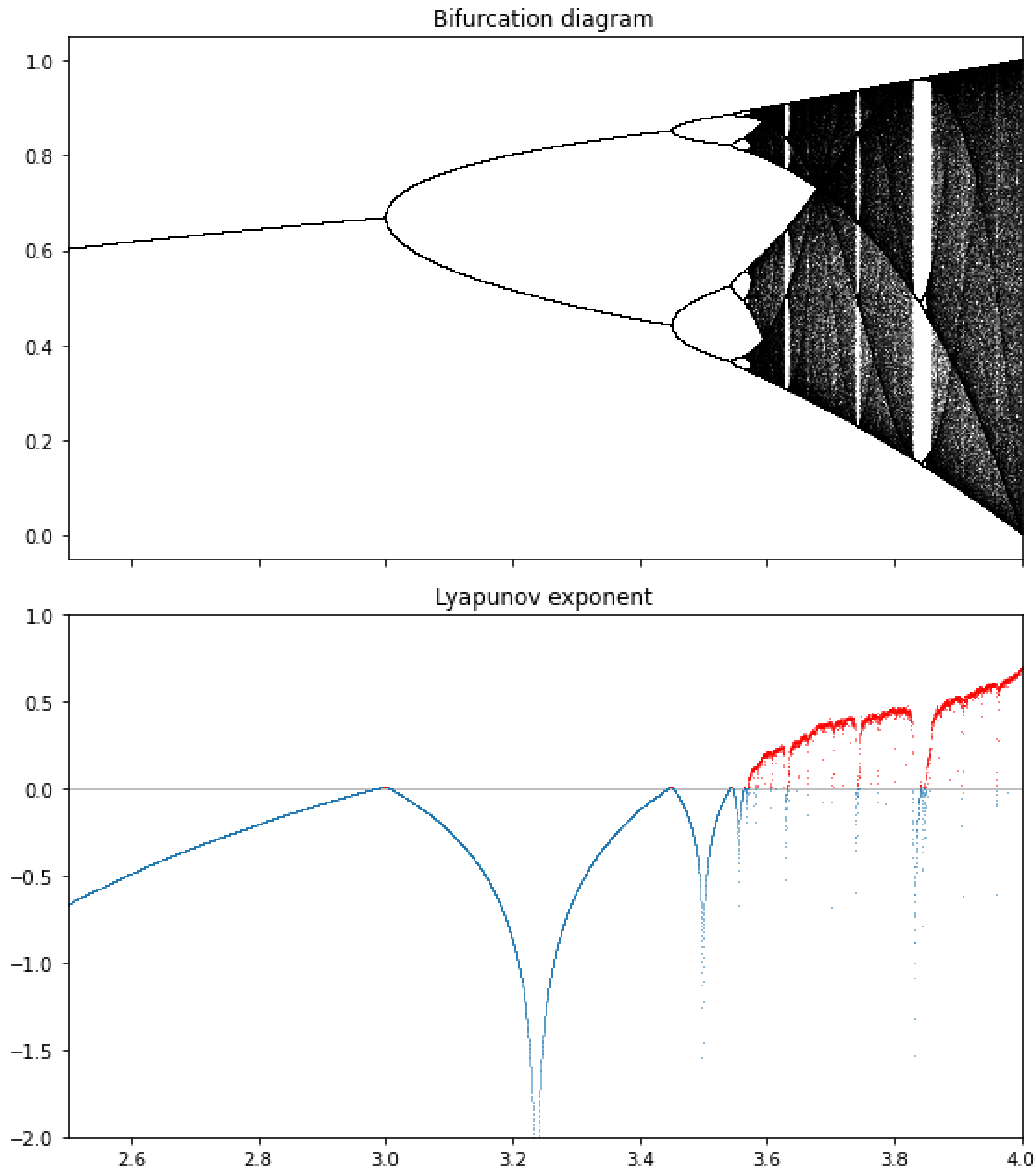
[144]:
```
IPdisplay.Image(url=gif_filepath)
```

[144]: <IPython.core.display.Image object>

1.1.2 Diagrama de bifurcación

[146]:
```
n = 10000
r = np.linspace(2.5, 4.0, n)

# We use 1000 iterations of the logistic map and keep the last 100 iterations␣
↪to display the bifurcation diagram:

iterations = 1000
last = 100

# We initialize our system with the same initial condition x0=0.00001:

x = 1e-5 * np.ones(n)

# We also compute an approximation of the Lyapunov exponent for every value of␣
↪r. The Lyapunov exponent is defined by:
# (r)=limn→∞1n i=0n−1log  dfrdx(x(r)i)

# We first initialize the lyapunov vector:

lyapunov = np.zeros(n)
```

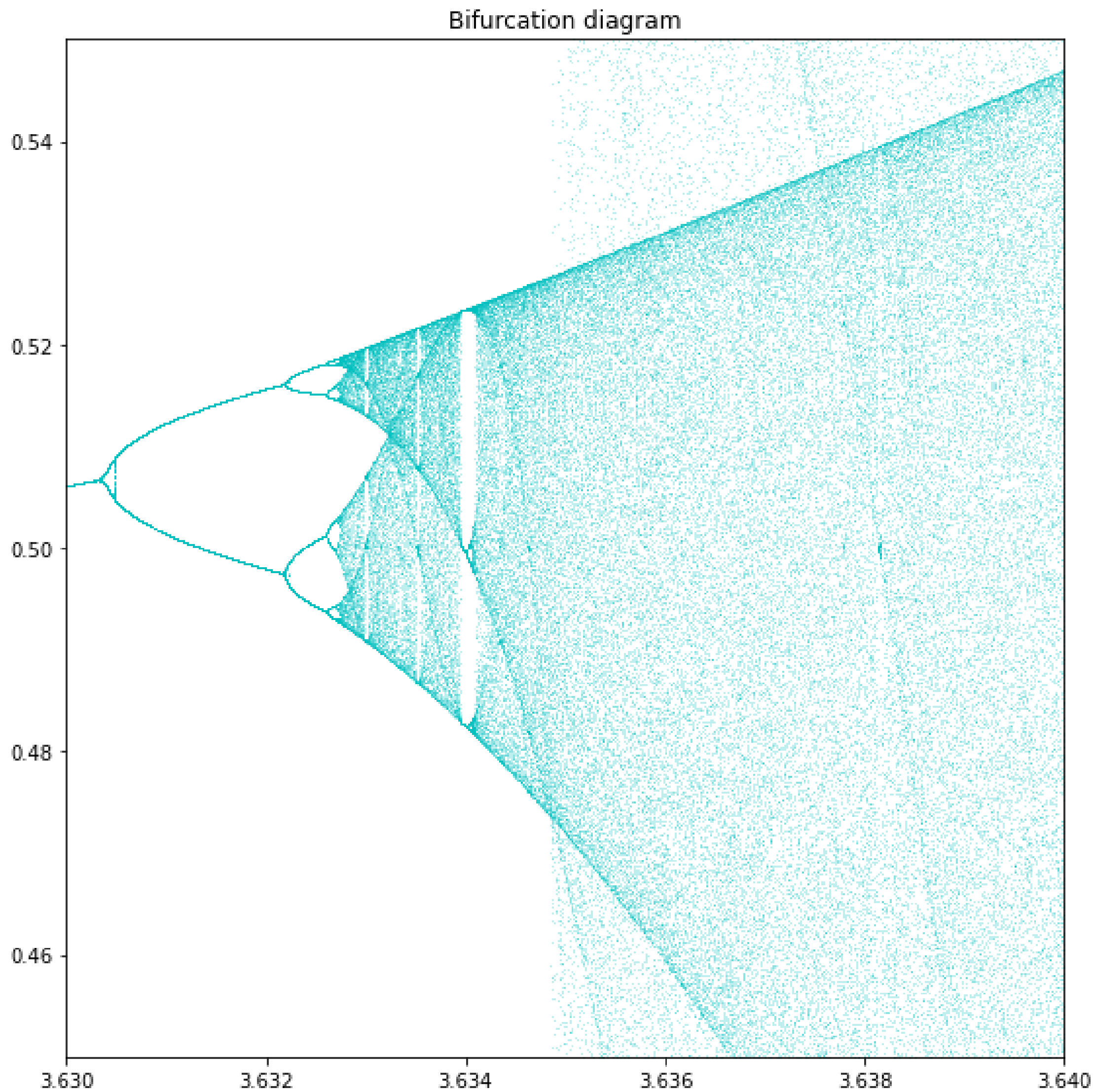
```python
# Now, we simulate the system and plot the bifurcation diagram. The simulation␣
↪only involves the iterative evaluation of the logistic() function on our␣
↪vector x. Then, to display the bifurcation diagram, we draw one pixel per␣
↪point x(r)n during the last 100 iterations:

fig, (ax1, ax2) = plt.subplots(2, 1, figsize=(8, 9),
                               sharex=True)
for i in range(iterations):
    x = logistic(r, x)
    # We compute the partial sum of the
    # Lyapunov exponent.
    lyapunov += np.log(abs(r - 2 * r * x))
    # We display the bifurcation diagram.
    if i >= (iterations - last):
        ax1.plot(r, x, ',k', alpha=.25)
ax1.set_xlim(2.5, 4)
ax1.set_title("Bifurcation diagram")

# We display the Lyapunov exponent.
# Horizontal line.
ax2.axhline(0, color='k', lw=.5, alpha=.5)
# Negative Lyapunov exponent.
ax2.plot(r[lyapunov < 0],
         lyapunov[lyapunov < 0] / iterations,
         ',', alpha=.5, ms=.5)
# Positive Lyapunov exponent.
ax2.plot(r[lyapunov >= 0],
         lyapunov[lyapunov >= 0] / iterations,
         ',r', alpha=.5, ms=.5)
ax2.set_xlim(2.5, 4)
ax2.set_ylim(-2, 1)
ax2.set_title("Lyapunov exponent")
plt.tight_layout()
```



Si hacemos un zoom entre $x = 3.63$ y 3.64 y $f(x)$ entre 0.45 y 0.55, observamos que efectivamente el diagrama de bifurcación tiene una estructura fractal.

[155]:
```
n = 10000
x0 = 3.63
xf = 3.64
r = np.linspace(x0,xf,n)
iterations = 1000
last = 100
```

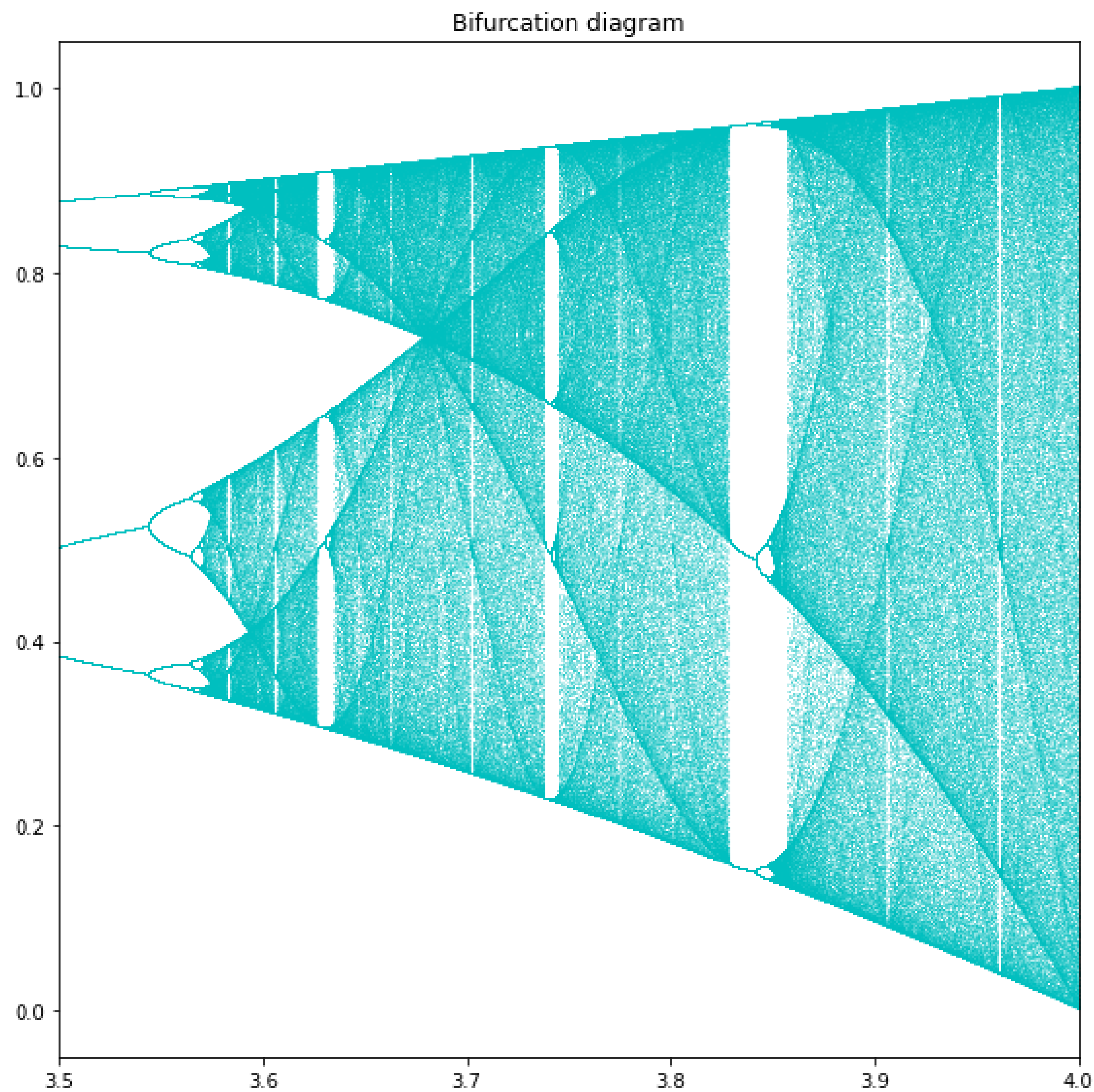
```
x = 1e-5 * np.ones(n)
lyapunov = np.zeros(n)

fig,ax1 = plt.subplots(1, 1, figsize=(8,8),
                               sharex=True)
for i in range(iterations):
    x = logistic(r, x)
    # We compute the partial sum of the Lyapunov exponent.
    lyapunov += np.log(abs(r - 2*r*x))
    # We display the bifurcation diagram.
    if i >= (iterations - last):
        ax1.plot(r, x, ',c', alpha=.25)
ax1.set_xlim(x0,xf)
ax1.set_ylim(0.45,0.55)
ax1.set_title("Bifurcation diagram")
plt.tight_layout()
```



Veamos otros detalles

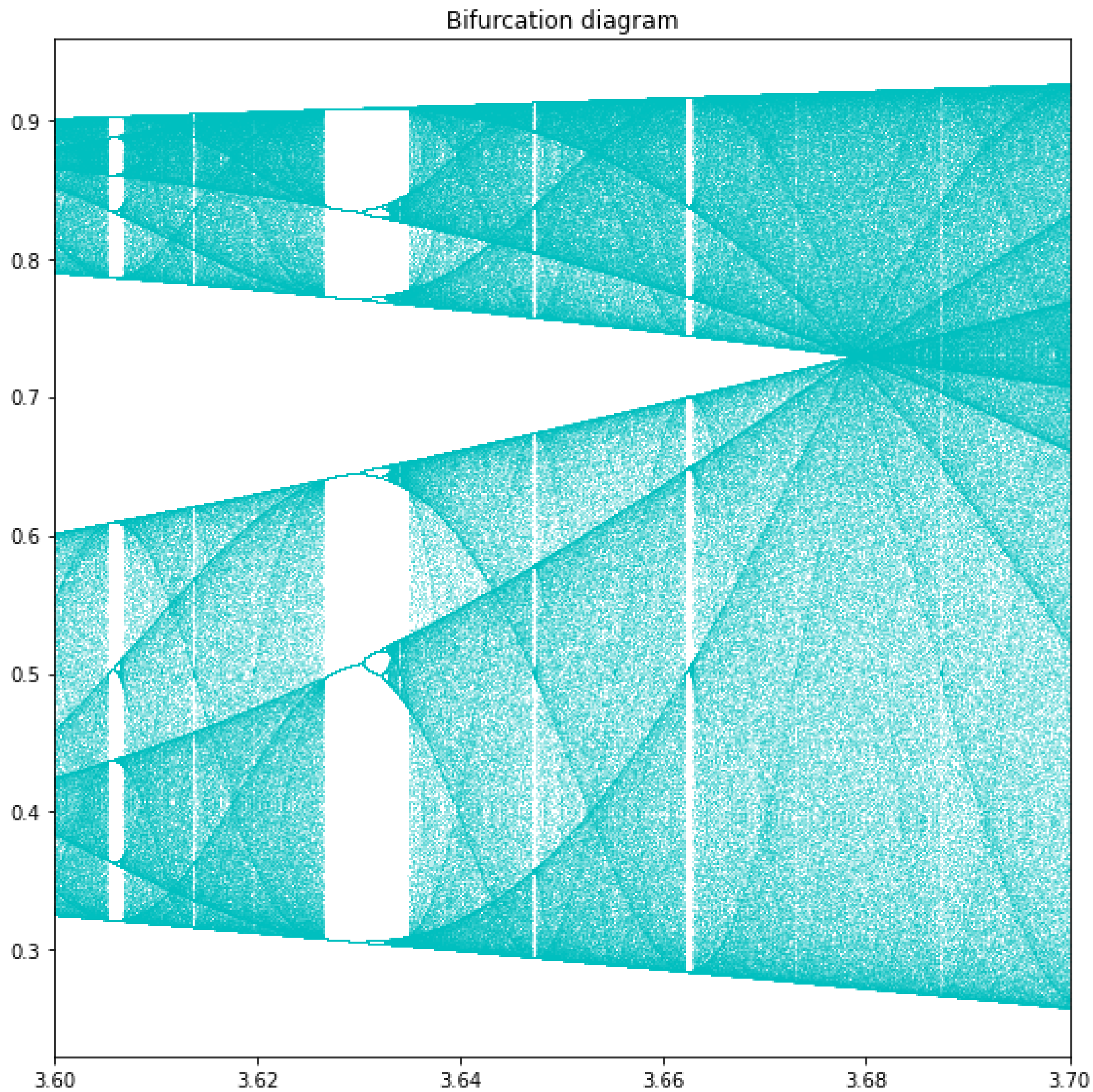
[148]:
```
n = 10000
x0 = 3.5
xf = 4
r = np.linspace(x0,xf,n)
iterations = 1000
last = 100
x = 1e-5 * np.ones(n)
lyapunov = np.zeros(n)

fig,ax1 = plt.subplots(1, 1, figsize=(8,8),
                               sharex=True)
```

```python
for i in range(iterations):
    x = logistic(r, x)
    # We compute the partial sum of the Lyapunov exponent.
    lyapunov += np.log(abs(r - 2*r*x))
    # We display the bifurcation diagram.
    if i >= (iterations - last):
        ax1.plot(r, x, ',c', alpha=.25)
ax1.set_xlim(x0,xf)
ax1.set_title("Bifurcation diagram")
plt.tight_layout()
```



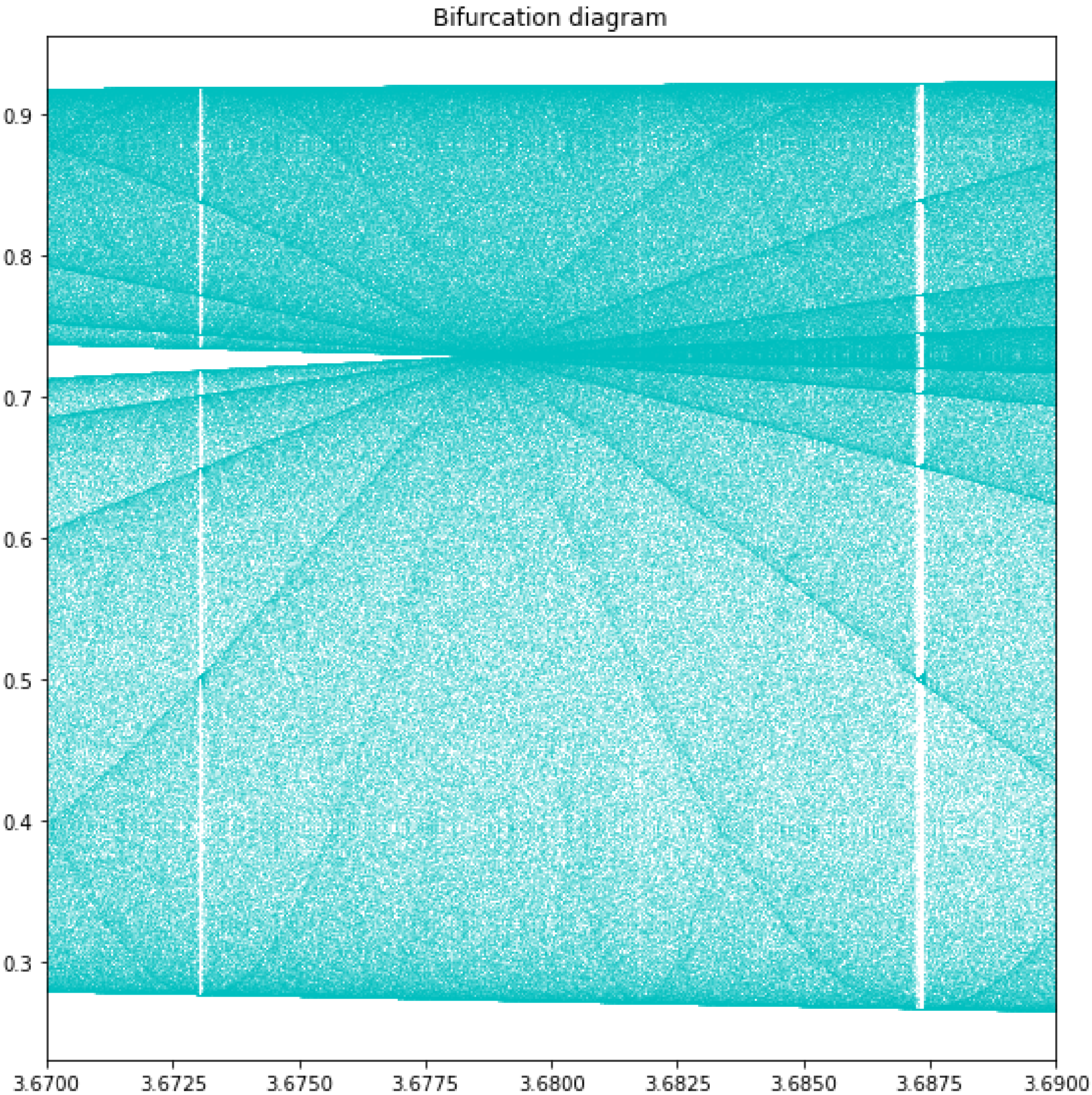
[149]:

```
n = 10000
x0 = 3.6
xf = 3.7
r = np.linspace(x0,xf,n)
iterations = 1000
last = 100
x = 1e-5 * np.ones(n)
lyapunov = np.zeros(n)

fig,ax1 = plt.subplots(1, 1, figsize=(8,8),
                               sharex=True)
for i in range(iterations):
    x = logistic(r, x)
    # We compute the partial sum of the Lyapunov exponent.
    lyapunov += np.log(abs(r - 2*r*x))
    # We display the bifurcation diagram.
    if i >= (iterations - last):
        ax1.plot(r, x, ',c', alpha=.25)
ax1.set_xlim(x0,xf)
ax1.set_title("Bifurcation diagram")
plt.tight_layout()
```



[153]:
```
n = 10000
x0 = 3.67
xf = 3.69
r = np.linspace(x0,xf,n)
iterations = 1000
last = 100
x = 1e-5 * np.ones(n)
lyapunov = np.zeros(n)

fig,ax1 = plt.subplots(1, 1, figsize=(8,8),
                               sharex=True)
for i in range(iterations):
    x = logistic(r, x)
```

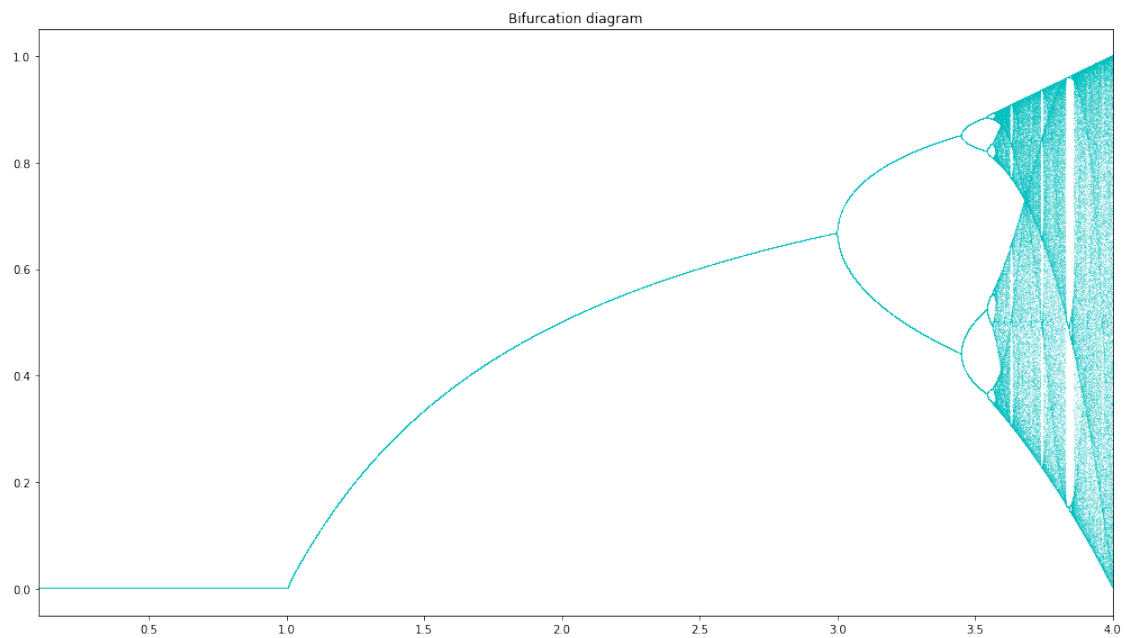
```
    # We compute the partial sum of the Lyapunov exponent.
    lyapunov += np.log(abs(r - 2*r*x))
    # We display the bifurcation diagram.
    if i >= (iterations - last):
        ax1.plot(r, x, ',c', alpha=.25)
ax1.set_xlim(x0,xf)
ax1.set_title("Bifurcation diagram")
plt.tight_layout()
```



Veamos el mapeo completo

[37]:
```
n = 10000
x0 = 0.1
```

```python
xf = 4
r = np.linspace(x0,xf,n)
iterations = 1000
last = 100
x = 1e-5 * np.ones(n)
lyapunov = np.zeros(n)

fig,ax1 = plt.subplots(1, 1, figsize=(14,8),
                               sharex=True)
for i in range(iterations):
    x = logistic(r, x)
    # We compute the partial sum of the
    # Lyapunov exponent.
    lyapunov += np.log(abs(r - 2*r*x))
    # We display the bifurcation diagram.
    if i >= (iterations - last):
        ax1.plot(r, x, ',c', alpha=.25)
ax1.set_xlim(x0,xf)
ax1.set_title("Bifurcation diagram")
plt.tight_layout()
```



$ $

1.2 2. Conjunto de Mandelbrot

https://levelup.gitconnected.com/mandelbrot-set-with-python-983e9fc47f56

El conjunto de Mandelbrot es el conjunto de valores de c en el plano complejo para los cuales la órbita del punto crítico $z = 0$ bajo la iteración del mapeo

$$z_{n+1} = z_n^2 + c$$

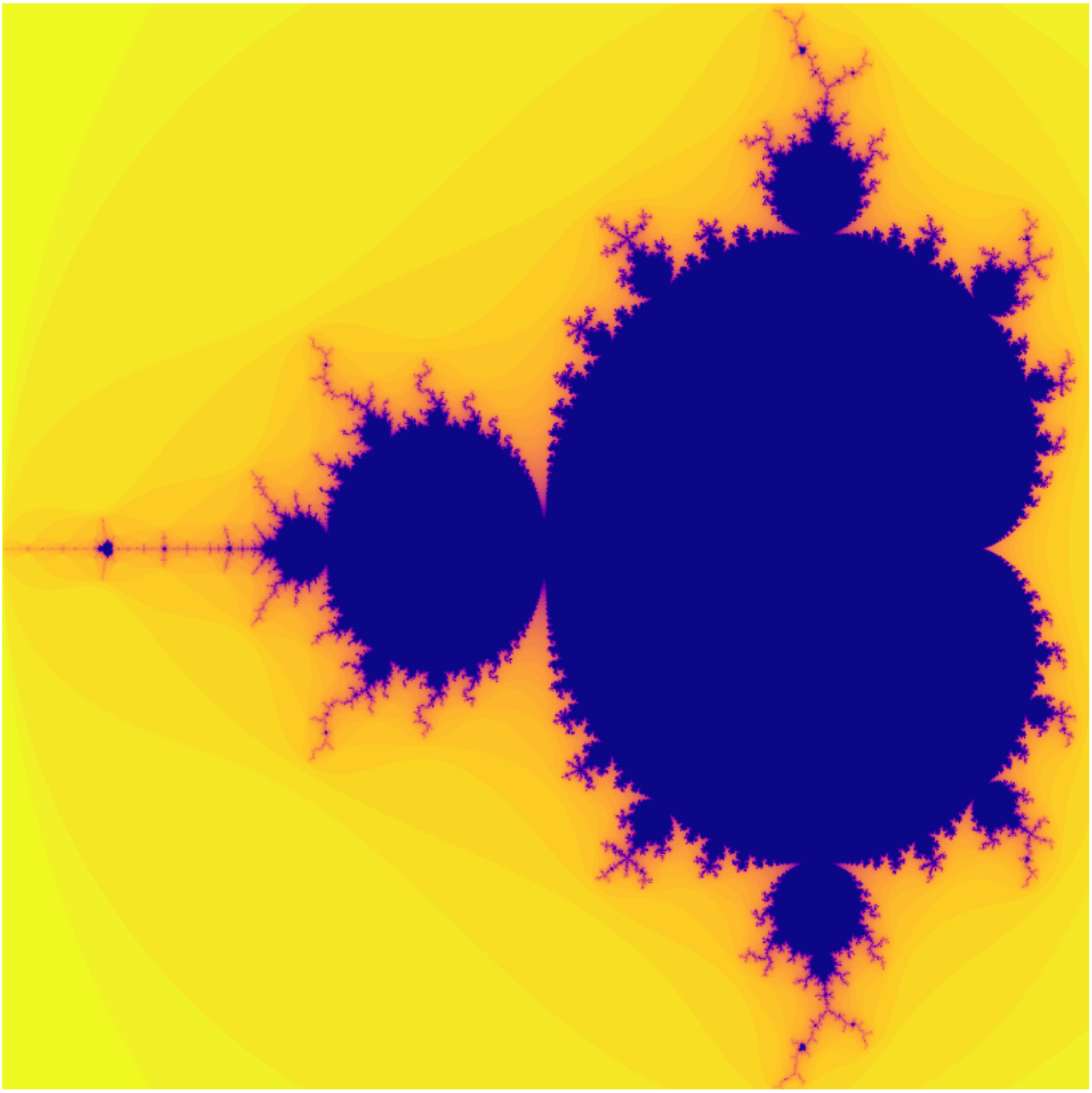
permanece acotada. Por lo tanto, un número complejo c pertenece al conjunto de Mandelbrot si, empezando con $z_0 = 0$ y aplicando la iteración, el valor absoluto de z_n permanece acotado para toda $n > 0$, donde esa cota es 2.

[37]:
```
import matplotlib.pyplot as plt
import numpy as np

def get_iter(c:complex, thresh:int =4, max_steps:int=25) -> int:
    # Z_(n) = (Z_(n-1))^2 + c
    # Z_(0) = c
    z=c
    i=1
    while i<max_steps and (z*z.conjugate()).real<thresh:
        z = z*z + c
        i+=1
    return i
```

[38]:
```
def plotter(n, thresh, max_steps=25):
    mx = 2.5 / (n-1)
    my = 2.25 / (n-1)
    mapper = lambda x,y: (mx*x - 2, my*y - 1.13)
    img=np.full((n,n), 255)
    for x in range(n):
        for y in range(n):
            it = get_iter(complex(*mapper(x,y)), thresh=thresh,␣
↪max_steps=max_steps)
            img[y][x] = 255 - it
    return img
```

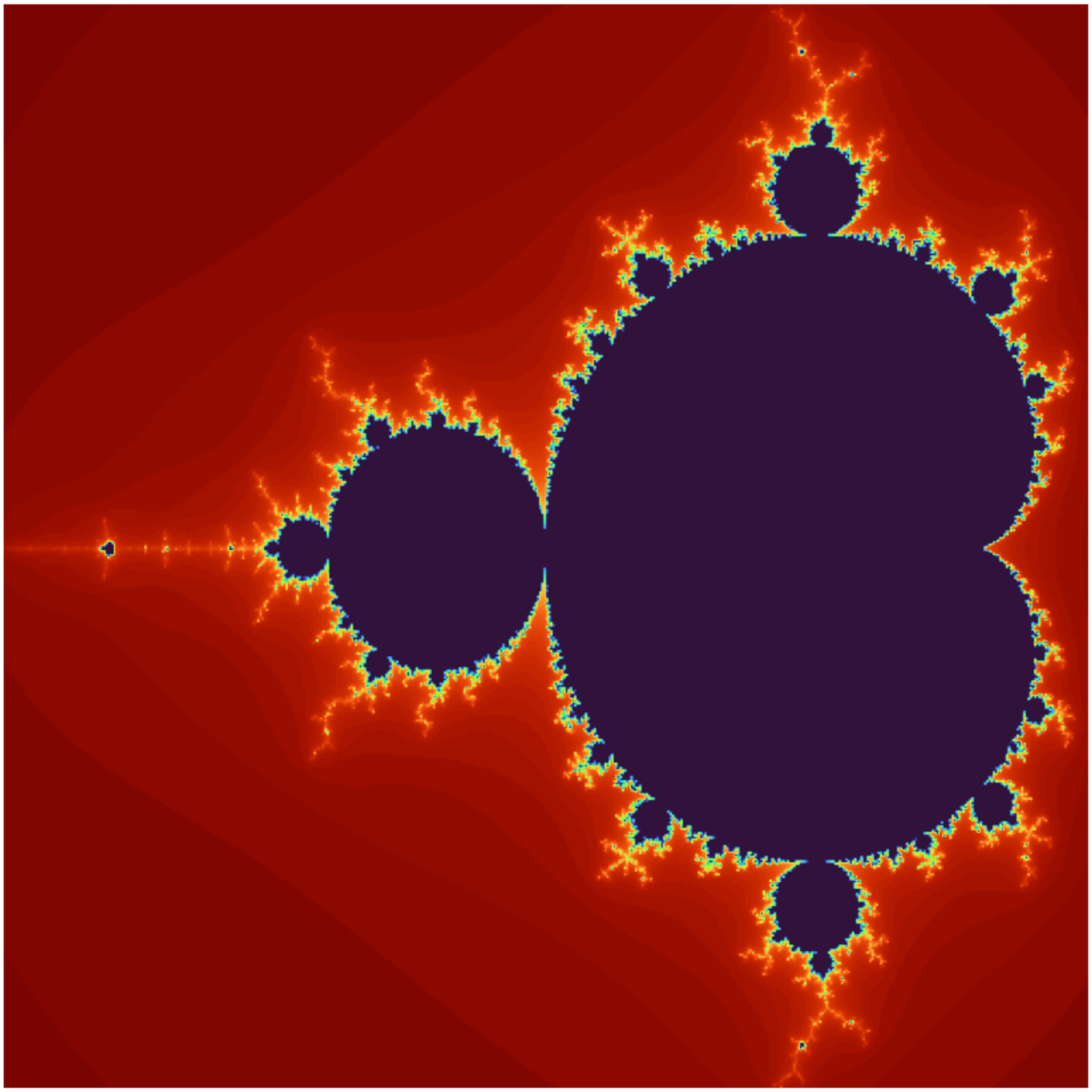
Código de colores `plasma`

[39]:
```
n=1000
img = plotter(n, thresh=4, max_steps=50)
plt.figure(figsize=(15,15))
plt.imshow(img, cmap="plasma")
plt.axis("off")
plt.show()
```



Código de colores `turbo`

[6]:
```
n=500
img = plotter(n, thresh=8, max_steps=100)
plt.figure(figsize=(15,15))
plt.imshow(img, cmap="turbo")
plt.axis("off")
plt.show()
```



1.3 3

La relación entre el mapeo logístico y el conjunto de Mandelbrot M está dada por la intersección de M con el eje real, la cual es el intervalo $[-2, 1/4]$. Los parámetros a lo largo de este intervalo se pueden poner en correspondencia uno a uno con aquellos de la familia logística real

$$x_{n+1} = rx_n(1 - x_n), \quad r \in [1, 4]$$

La correspondencia está dada por

$$z = r\left(\frac{1}{2} - x\right), \quad c = \frac{r}{2}\left(1 - \frac{r}{2}\right)$$

De hecho, esto da una correspondencia entre el espacio de parámetros completo de la familia logística y la del conjunto de Mandelbrot.

1.4 4 La constante de Feigenbaum

La **constante de Feigenbaum** es la proporción entre cada intervalo de bifurcation y el siguiente entre cada dupliación de periodo, de un mapeo de 1 parámetro

$$x_{i+1} = f(x_i)$$

donde $f(x)$ es una función parametrizada por el parámetro de bifurcación a.

Está dada por

$$\delta = \lim_{n\to\infty} \frac{a_{n-1} - a_{n-2}}{a_n - a_{n-1}} = 4.669\,201\,609\,\ldots$$

donde a_n son valores discretos de a cuando se duplica el n-ésimo periodo.

En el caso del **mapeo logístico**:

n	Periodo	Parámetro de bifurcación a_n	Razón $\frac{a_{n-1} - a_{n-2}}{a_n - a_{n-1}}$
1	2	3	—
2	4	3.4494897	—
3	8	3.5440903	4.7514
4	16	3.5644073	4.6562
5	32	3.5687594	4.6683
6	64	3.5696916	4.6686
7	128	3.5698913	4.6692
8	256	3.5699340	4.6694

En el caso del **conjunto de Mandelbrot**, la constante de Feigenbaum es la razón entre los diámetros de círculos sucesivos en el eje real del plano complejo (como se observa en la siguiente animación).

n	Periodo	Parámetro de bifurcación c_n	Razón $\dfrac{c_{n-1} - c_{n-2}}{c_n - c_{n-1}}$
1	2	-0.75	—
2	4	-1.25	—
3	8	-1.3680989	4.2337
4	16	-1.3940462	4.5515
5	32	-1.3996312	4.6458
6	64	-1.4008287	4.6639
7	128	-1.4010853	4.6682
8	256	-1.4011402	4.6689
9	512	-1.401151982029	$\vdots$
10	1024	-1.401154502237	$\vdots$
∞		$-1.4011551890\ldots$	$\vdots$

El parámetro de bifurcación c_n converge al punto de Feigenbaum $c = -1.401155\ldots$ y la razón converge a la constante de Feigenbaum.

Otros mapeos también reproducen esta razón, así que la constante de Feigenbaum en teoría de bifurcaciones es análoga a en geometría y a e en cálculo.