Homework 1.

- The file name of your homework (in PDF) should be in the format: “學號-ds-作業編號.pdf”. For example: **00757999-ds-hw1.pdf**
- Please submit your homework to Tronclass **before 23:59, October 19, 2025**

Questions:

1. (10%) Order the following function by growth rate in increasing order:
   n^2, $n\ log(log\ \text{n})$, $n\ log(n^2)$, n^n

2. (10%) Ackerman's function $A(m, n)$ is defined as:

$$A(m,n) = \begin{cases} n+1, & \text{if } m = 0 \\ A(m-1,1), & \text{if } n = 0 \\ A(m-1, A(m, n-1)), & \text{otherwise} \end{cases}$$

   What is the value of $A(3,4)$?

3. (12%) Show that the following statements are correct:
   (a) $5n^2 - 6n = \Theta(n^2)$
   (b) $n! = O(n^n)$

4. (12%) Show that the following statements are incorrect:
   (a) $10n^2 + 9 = O(n)$
   (b) $3^n = O(2^n)$

5. (16%) (from NTHU 106) Which ones of the following conjectures are correct? Please justify your answers:
   (a) $3n^2 - 100n + 6 = O(n)$
   (b) $log(n!) = \Theta(n\ log\ n)$
   (c) $n! = O(n^n)$
   (d) $(n+a)^b = \Theta(n^b)$ where a and b are real constants
   (e) $(n+a)^b = \Omega(n^{b+1})$ where a and b are real constants

6. (20%) Fill the step count table (yellow regions) for the following program:

Statement	s/e	Frequency	Total Steps
void *mult*(**int** *a*[][*MAX_SIZE*],**int** *b*[][*MAX_SIZE*], **int** *c*[][*MAX_SIZE*])	0	0	0
{	0	0	0
int *i*, *j*, *k*;	0	0	0
for (*i* = 0; *i* < *MAX_SIZE*; *i*++)	1		
for (*j* = 0; *j* < *MAX_SIZE*; *j*++) {	1		
c [*i*][*j*] = 0;	1		
for (*k* = 0; *k* < *MAX_SIZE*; *k*++)	1		
c [*i*][*j*] += *a* [*i*][*k*] * *b* [*k*][*j*];	1		
}	0	0	0
}	0	0	0
Total			

7. (10%) Given an four-dimensional array A[100][200][300][400], if the address of A[3][20][32][45] is α + X, what is the value of X ?

   Note: (1) α is the address of A[0][0][0][0]

   (2) Only consider row major here.

8. (10%) (from NTCU 107) Consider the following function. Please derive the time complexity of function foo using recurrence relation. Please show your derivation step by step.

```
int i[n]; /* Assume that the values of all integers in i[n] have been initialized */
int foo(int a, int b, int c) {
    int d;
    if (a>b) return -1;
    d=(a+b)/2;
    if (c==i[d])
        return d;
    else if (c>i[d])
        return foo(d+1,b,c);
    else
        return foo(a, d-1,c);
}
```

Answer:

1. n^n = Biggest

$$\frac{n\log(n^2)}{n^2} = \frac{2\cancel{n}\log(n)}{n^{\cancel{2}}} \to 0 \text{ as } n \to \infty$$

$n\log(n^2) < n^2 < n^n$

$\log n < n^2$

$n\log(\log n) < n\log(n^2) < n^2 < n^n$

2. $A(0,n) = n+1$

$A(1,n) = A(0, A(1,n-1)) = A(1,n-1)+1$

$A(1,0) = A(0,1) = 2$

$A(1,1) = A(0, A(1,0)) = 2+1 = 3$

$A(1,2) = A(0, A(1,1)) = 3+1 = 4$

$A(1,n) = n+2$

$A(2,n) = A(1, A(2,n-1)) = A(2,n-1)+2$

$A(2,0) = A(1,1) = 3$

$A(2,1) = A(1, A(2,0)) = 5$

$A(2,2) = A(1, A(2,1)) = 7$

$A(2,n) = 2n+3$

$A(3,n)$

$A(3,0) = A(2,1) = 2\cdot 1+3 = 5 \xrightarrow{+8} = 8-3$

$A(3,1) = A(2, A(3,0)) = 2(5)+3 = 13 \xrightarrow{+16} = 16-3$

$A(3,2) = A(2, A(3,1)) = 2(13)+3 = 29 = 32-3$

$A(3,n) = 2^{3+n} - 3$

$A(3,4) = 2^7 - 3 = 128-3 = 125$

3. a. $5n^2 - 6n = \Theta(n^2)$

$c_1 g(n) \le f(n) \le c_2 g(n)$

$c_1 = 4 \quad c_2 = 5$ ✓ $\quad n_0 = 6$

$4n^2 \le 5n^2 - 6n \le 5n^2$ for all $n \ge 6$

Correct

b. $n! = O(n^n)$

$n! \le c n^n \quad c = 1$

$n! \le 1 \cdot n^n$ for all $n \ge 1$

Correct

4. a. $10n^2 + 9 \le c\cancel{n}$

$10n + \frac{9}{n} \le c$

as $n \to \infty$

$\infty \le c$ is wrong

b. $3^n \le c \cdot 2^n$

$\frac{3^n}{2^n} \le c$

$\left(\frac{3}{2}\right)^n \le c$

as $n \to \infty$

$\infty \le c$ is wrong

5. a. False

$3n - 100 + \frac{6}{n} \leq C$

as $n \to \infty$

$\infty \leq C$ is wrong

b. True

$C_1 \cdot n \cdot \log n \leq \log(n!) \leq C_2 \cdot n \cdot \log n$

Upper bound $C_2 = 1$

$\log(n!) = \log(1 \cdot 2 \cdots n)$

$= \log(1) + \log(2) + \cdots + \log(n)$

$= \sum_{k=1}^{n} \log(k) \leq n \log(n)$

lower bound $C_1 = \frac{1}{2}$

$\log(n!) \geq \sum_{k=n/2}^{n} \log(k)$

$\sum_{k=n/2}^{n} \log(k) \geq n/2 \cdot \log(n/2)$

$\log(n!) \geq n/2 \cdot \log(n/2)$

$= \frac{n}{2} (\log(n) - \log(2))$

drop the constants

$\log(n!) \geq n \log(n)$

Upper and lower bound

Proved

5.C. True

$n! \leq c \cdot n^n$

$c = 1$

$n! \leq n^n$ for all $n \geq 1$

D. True

$C_1 n^b \leq (n+a)^b \leq C_2 n^b$

Substitute

$C_1 \leq \left(\frac{\cancel{n}+a}{\cancel{n}}\right)^b \leq C_2$

$C_1 \leq (1 + \frac{a}{n})^b \leq C_2$ exists for all $n \geq 1$

as $n \to \infty$

$C_1 \leq 1 \leq C_2$

5.e. False

$(n+a)^b \geq C \cdot n^{b+1}$

$\left(\frac{\cancel{n}+a}{\cancel{n}}\right)^b \geq C \cdot n$

$(1 + \frac{a}{n})^b \geq C \cdot n$

as $n \to \infty$

$1 \geq \infty$ is wrong

6. (20%) Fill the step count table (yellow regions) for the following program: let $n = MAX\_SIZE$

Statement	s/e	Frequency	Total Steps
void *mult*(**int** *a*[][*MAX_SIZE*],**int** *b*[][*MAX_SIZE*], **int** *c*[][*MAX_SIZE*])	0	0	0
{	0	0	0
int *i*, *j*, *k*;	0	0	0
for (*i* = 0; *i* < *MAX_SIZE*; *i*++)	1	$n+1$	$n+1$
for (*j* = 0; *j* < *MAX_SIZE*; *j*++) {	1	n^2+n	n^2+n
c [*i*][*j*] = 0;	1	n^2	n^2
for (*k* = 0; *k* < *MAX_SIZE*; *k*++)	1	n^3+n^2	n^3+n^2
c [*i*][*j*] += *a* [*i*][*k*] * *b* [*k*][*j*];	1	n^3	n^3
}	0	0	0
}	0	0	0
Total		$O(n^3)$	$2n^3+3n^2+2n+1$

7. $\alpha + x$

$$x = 3 \times 200 \times 300 \times 400 + 20 \times 300 \times 400 + 32 \times 400 + 45$$

$$= \underline{74,412,845}$$

8. $$T(a, b, c) = \begin{cases} -1 & \text{if } a > b \\ d & \text{if } c == i[(a+b)/2] \\ T((a+b)/2 + 1, b, c) & \text{if } c > i[(a+b)/2] \\ T(a, (a+b)/2 - 1, c) & \text{if } c < i[(a+b)/2] \end{cases}$$

The function does constant-time work like comparison and assignments, then the function calls itself recursively on half of the array so each recursions halves the size.

$$T(n) = T(n/2) + c$$

$$T(n/2) = T(n/4) + 2c$$

$$T(n/4) = T(n/8) + 3c$$

So, $T(n) = T(n/2^k) + kc$

The recursion stops when base case is reached which is when $T(n/2^k)$ takes $T(1)$.

So, $n/2^k = 1$

$n = 2^k$

$k = \log_2 n$

So,

$$T(n) = T(1) + c \log n$$

$$T(n) = O(\log n)$$