Control Theory Homework 4

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Consider classical benchmark system in control theory - inverted pendulum on a cart (Figure 1). It is nonlinear under-actuated system that has the following dynamics

$$(M+m)\ddot{x} - ml\cos(\theta)\ddot{\theta} + ml\sin(\theta)\dot{\theta}^2 = F \tag{1}$$

$$-\cos(\theta)\ddot{x} + l\ddot{\theta} - g\sin(\theta) = 0 \tag{2}$$

where $g = 9.81$ is gravitational acceleration.

(d) $M = 11.6, m = 2.7, l = 0.57$

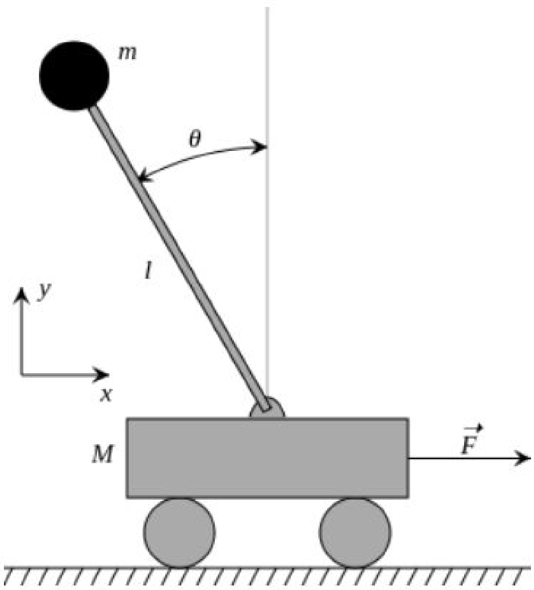

Figure 1: A schematic drawing of the inverted pendulum on a cart. The rod is considered massless. The mass of the cart and the point mass at the end of the rod are denoted by M and m. The rod has a length l.

A)

(A) write equations of motion of the system in manipulator form

$$M(q)\ddot{q} + n(q, \dot{q}) = Bu$$

where $u = F$, $q = \left[\begin{array}{cc} x & \theta \end{array}\right]^T$ is vector of generalized coordinates;

Our given equations are

$$(M + m)\ddot{x} - ml\cos(\theta)\ddot{\theta} + ml\sin(\theta)\dot{\theta}^2 = F \tag{1}$$

$$-\cos(\theta)\ddot{x} + l\ddot{\theta} - g\sin(\theta) = 0 \tag{2}$$

For me

$$M = 11.6, m = 2.7, g = 9.81, l = 0.57 \tag{3}$$

Manipulator form is

$$\begin{bmatrix} M + m & -ml\sin(\theta) \\ -\cos(\theta) & l \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} ml\sin(\theta)\dot{\theta}^2 \\ -g\sin(\theta) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} F \tag{4}$$

$$\begin{bmatrix} 14.3 & -1.539 * \sin(\theta) \\ -\cos(\theta) & 0.57 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 1.539 * \sin(\theta)\dot{\theta}^2 \\ -9.81 * \sin(\theta) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} F \tag{5}$$

B)

(B) write dynamics of the system in control affine nonlinear form

$$\dot{z} = f(z) + g(z)u$$

where $z = \begin{bmatrix} x & \theta & \dot{x} & \dot{\theta} \end{bmatrix}^T$ is vector of states of the system;

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} z_3 \\ z_4 \\ \frac{mg \sin z_2 \cos z_2 - ml \sin z_2 z_4^2}{M + m \sin^2 z_2} \\ \frac{(M+m) g \tan z_2 - ml \sin z_2 z_4^2}{\frac{(M+m)l}{\cos z_2} - ml \cos z_2} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M + m \sin^2 z_2} \\ \frac{1}{\frac{(M+m)l}{\cos z_2} - ml \cos z_2} \end{bmatrix} F \tag{6}$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} z_3 \\ z_4 \\ \frac{26.487 * \sin z_2 \cos z_2 - 1.539 * \sin z_2 z_4^2}{11.6 + 2.7 * \sin^2 z_2} \\ \frac{140.283 * \tan z_2 - 1.539 * \sin z_2 z_4^2}{\frac{8.151}{\cos z_2} - 1.539 * \cos z_2} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{11.6 + 2.7 * \sin^2 z_2} \\ \frac{1}{\frac{8.151}{\cos z_2} - 1.539 * \cos z_2} \end{bmatrix} F \tag{7}$$

$$z_1 = x, z_2 = \theta, z_3 = \dot{x}, z_4 = \dot{\theta} \tag{8}$$

C)

(C) linearize nonlinear dynamics of the systems around equilibrium point $\bar{z} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$

$$\delta \dot{z} = A \delta z + B \delta u$$

Firstly, I need to find when f(z) equals to zero to get values of z_1, z_2, z_3, z_4.

$$\begin{cases} z_3 = 0 \\ z_4 = 0 \\ \frac{26.487*\sin z_2 \cos z_2 - 1.539*\sin z_2 z_4^2}{11.6+2.7*\sin^2 z_2} = 0 \\ \frac{140.283*\tan z_2 - 1.539*\sin z_2 z_4^2}{\frac{8.151}{\cos z_2} - 1.539*\cos z_2} = 0 \end{cases}$$

After simplification:

$$\begin{cases} z_3 = 0 \\ z_4 = 0 \\ \sin z_2 \cos z_2 = 0 \\ \tan z_2 = 0 \end{cases}$$

$$Thus, z_1 \text{ is any number }, z_2 = n\pi \ (\cos z_2 \neq 0), z_3 = 0, z_4 = 0 \tag{9}$$

Secondly, I need to find partial derivatives of all entries of f(z) with respect to z_1, z_2, z_3 and z_4 and construct matrix A out of them. As a result I got Matrix A:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2.28 & 0 & 0 \\ 0 & 21.21 & 0 & 0 \end{bmatrix} \tag{10}$$

Lastly, I need take derivatives of g(z) with respect to F and put them in matrix B. So.

$$B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{11.6} \\ \frac{1}{8.151} \end{bmatrix} \tag{11}$$

D)

(D) check stability of the linearized system using any method you like;

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eigenvalues of A: [ 0.          0.          4.60543158 -4.60543158]
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Thus, system is unstable.

E)

(E) check if linearized system is controllable; if not - try another variant or change values of your variant and find controllable.

Controllability Matrix of system:

$$\begin{bmatrix} 0 & 0.0862 & 0 & 0.2797 \\ 0 & 0.1227 & 0 & 2.6021 \\ 0.0862 & 0 & 0.2797 & 0 \\ 0.1227 & 0 & 2.6021 & 0 \end{bmatrix} \tag{12}$$

The system is controllable if the controllability matrix has full row rank, rank of this matrix is 4, so it is controllable system.

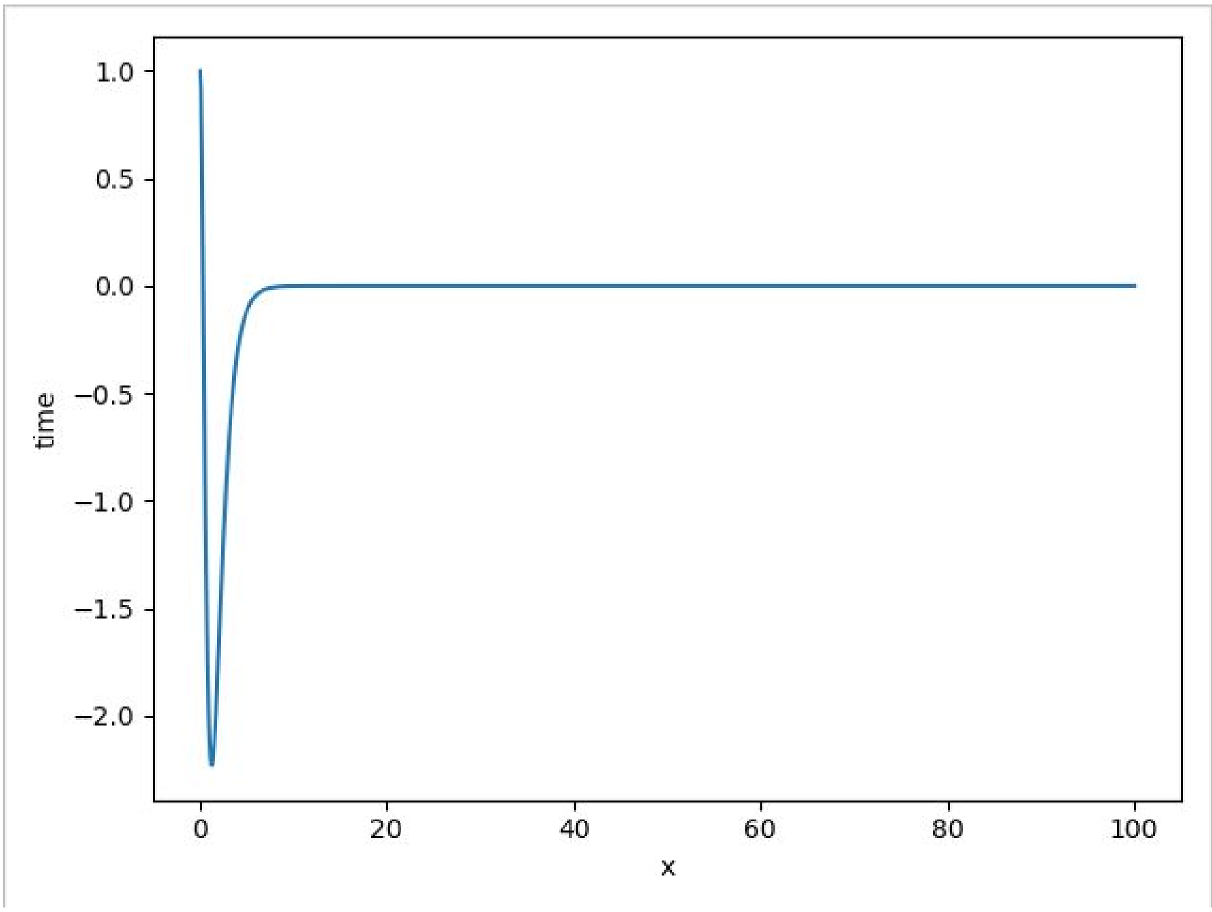
F)

(F) (for the controllable system) design state feedback controller for linearized system using pole placement method. Assess the performance of the controller for variety of initial conditions. Justify the choice of initial conditions. Solve the task by two ways: using root-locus and with python. Compare them;

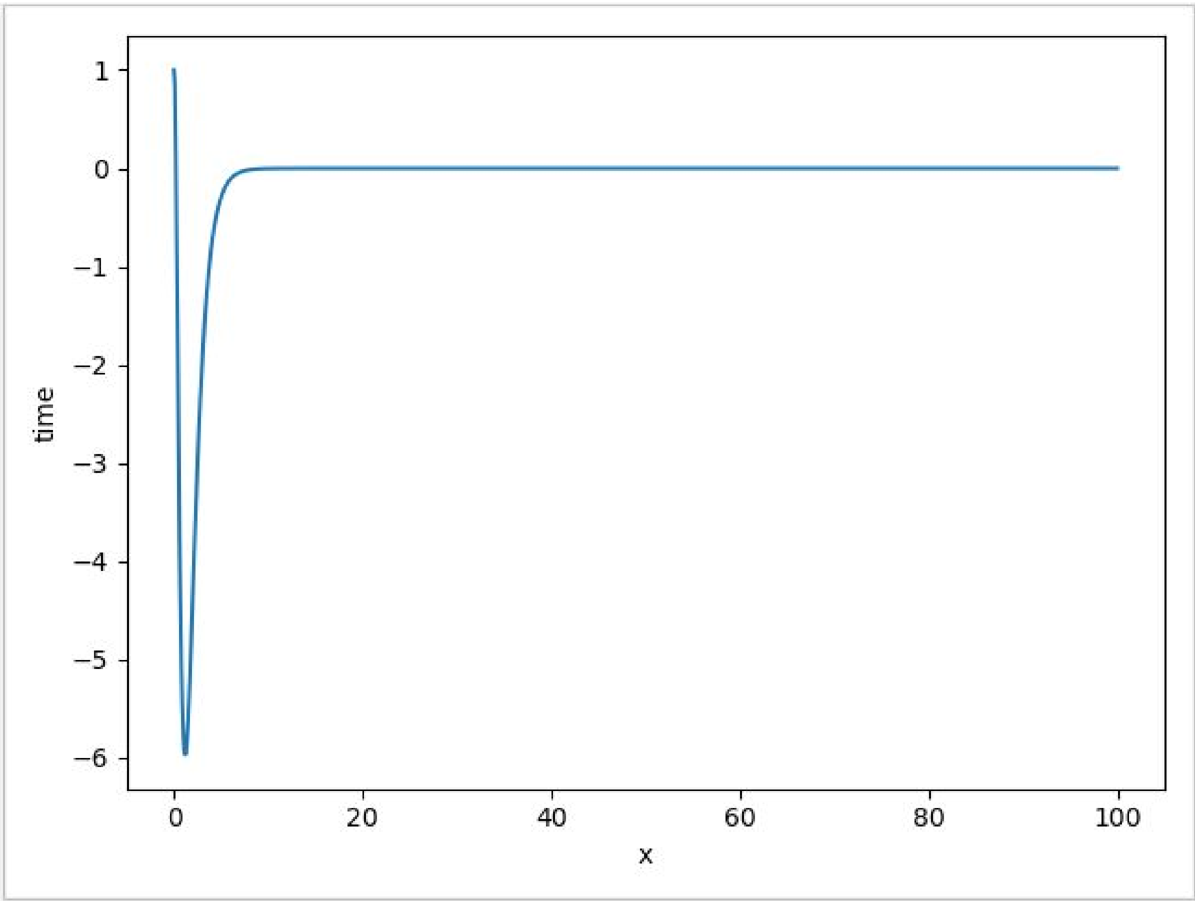
1)Root-locus: ¯_(ツ)_/¯

2) Python: For pole values I chose different negative numbers, plots has big overshoot in the beginning because I chose it badly. Also I chose this initial conditions in order to check all possible starting positions.

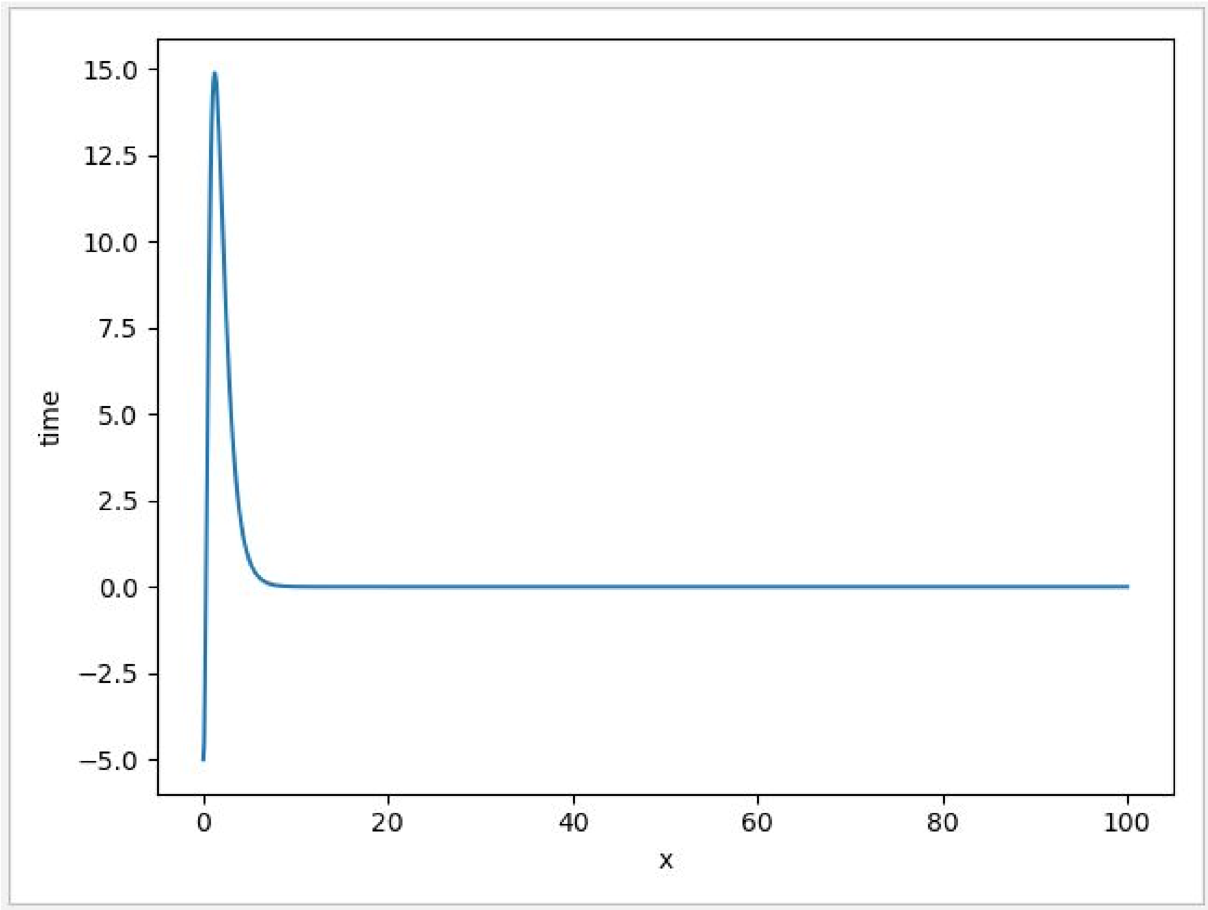
Plot with initial conditions [1, 1, 1, 1]:



Plot with initial conditions [1, 2, 3, 4]:



Plot with initial conditions [-5, -6, -7, -8]:

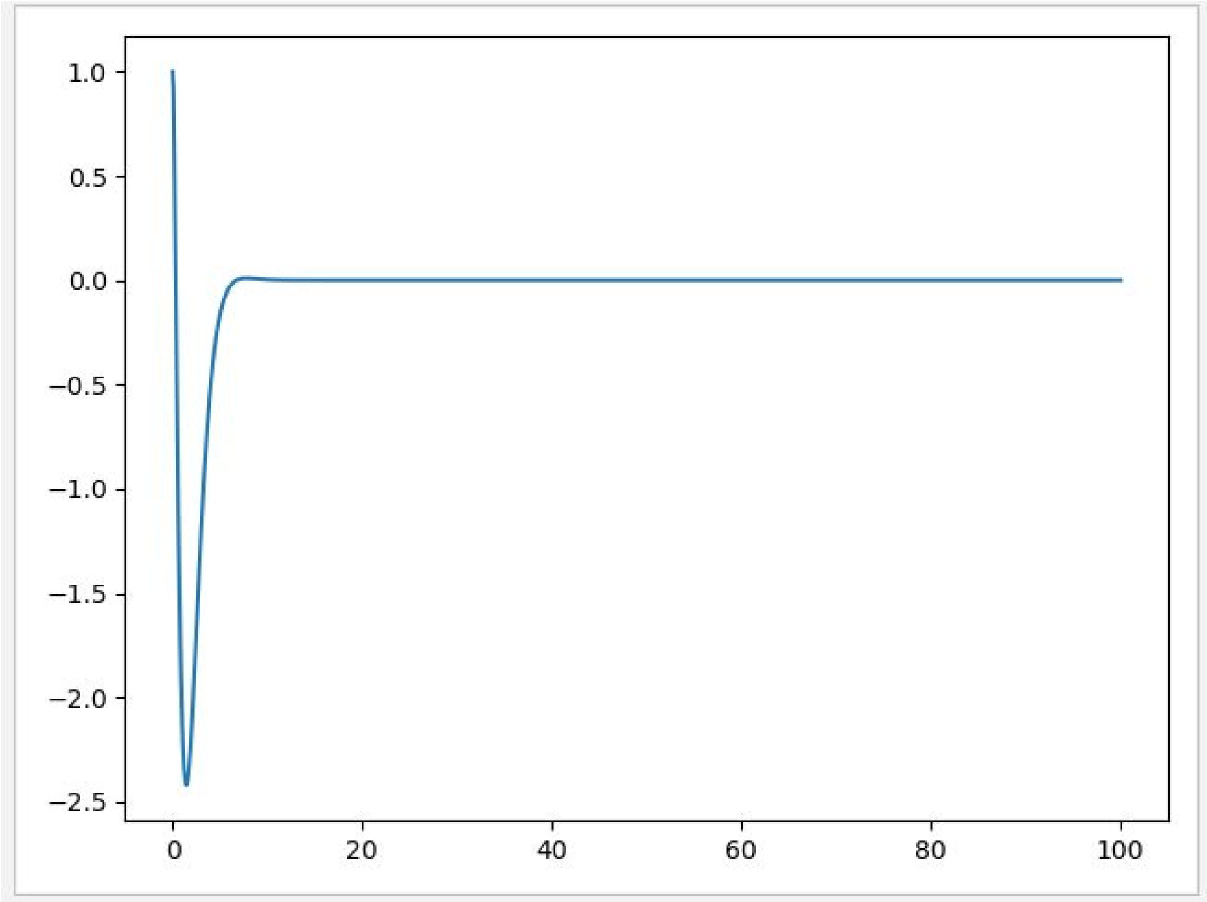


G)

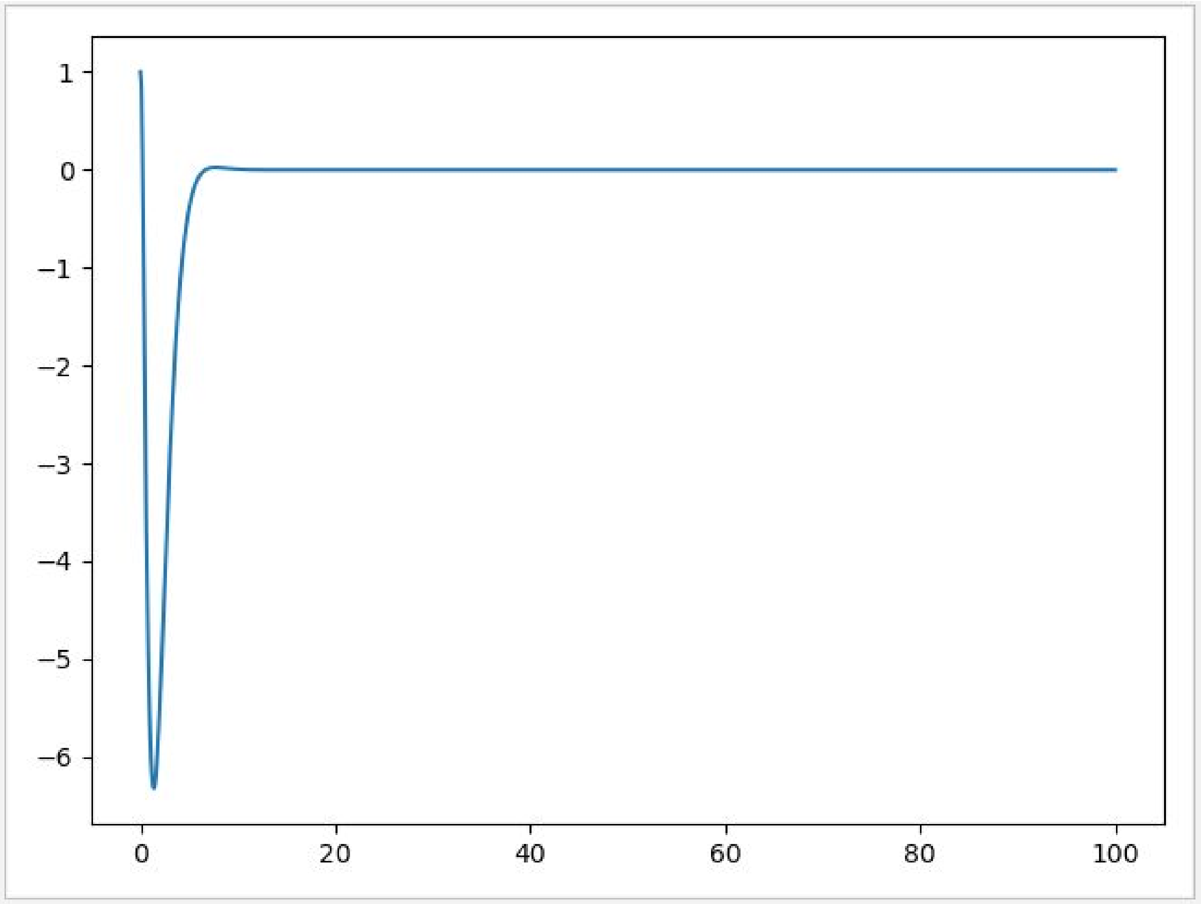
(G) (for the controllable system) design linear quadratic regulator for linearized system. Assess the performance of the controller for variety of initial conditions. Justify the choice of initial conditions;

I chose the same initial conditions.

Plot with initial conditions [1, 1, 1, 1]:



Plot with initial conditions [1, 2, 3, 4]:



Plot with initial conditions [-5, -6, -7, -8]: