① a) $\vec{V_1} = (1, 3, -2)$, $\vec{V_2} = (3, 7, -2)$ $\quad \vec{e_1}, \vec{e_2} - ?$

$$\vec{e_1} = \frac{\vec{V_1}}{\|\vec{V_1}\|} = \frac{(1, 3, -2)}{\sqrt{1+9+4}} = \frac{(1, 3, -2)}{\sqrt{14}}$$

$$\vec{w_2} = \vec{V_2} - \frac{(\vec{e_1}, \vec{V_2})}{(\vec{e_1}, \vec{e_1})} \vec{e_1} = (3, 7, -2) - \frac{(3+21+4)}{\sqrt{14}} \cdot \frac{(1, 3, -2)}{\sqrt{14}} =$$

$$= (3-2, 7-6, -2+4) = (1, 1, 2)$$

$$\vec{e_2} = \frac{\vec{W_2}}{\|\vec{W_2}\|} = \frac{(1, 1, 2)}{\sqrt{1+1+4}} = \frac{1}{\sqrt{16}} \cdot (1, 1, 2)$$

$$\vec{e_1} = \frac{(1, 3, -2)}{\sqrt{14}} \quad ; \quad \vec{e_2} = \frac{(1, 1, 2)}{\sqrt{16}}$$

b) $\vec{V}_1=(1,3,1)$; $\vec{V}_2=(5,1,3)$; $\vec{V}_3=(1,6,-8)$. $\vec{e}_1,\vec{e}_2,\vec{e}_3$ - ?

$\vec{e}_1=\frac{\vec{V}_1}{\|\vec{V}_1\|}=\frac{(1,3,1)}{\sqrt{1+9+1}}=\frac{1}{\sqrt{11}}(1,3,1)$

$\vec{e}_2=\frac{\vec{\omega}_2}{\|\vec{\omega}_2\|}$; $\vec{\omega}_2=\vec{V}_2-(\vec{e}_1,\vec{V}_2)\vec{e}_1=(5,1,3)-\frac{1}{\sqrt{11}}\overbrace{(5+3+3)}^{11}\frac{(1,3,1)}{\sqrt{11}}=$

$=(5-1,\ 1-3,\ 3-1)=(4,-2,2)\Rightarrow \vec{e}_2=\frac{1}{\sqrt{6}}(2,-1,1)$

$\vec{e}_3=\frac{\vec{\omega}_3}{\|\vec{\omega}_3\|}$; $\vec{\omega}_3=\vec{V}_3-(\vec{e}_1,\vec{V}_3)\vec{e}_1-(\vec{e}_2,\vec{V}_3)\vec{e}_2=(1,6,-8)-$

$-\frac{(1+18-8)}{11}(1,3,1)-\frac{(2-6-8)}{6}(2,-1,1)=(1,6,-8)-(1,3,1)+$

$+(4,-2,2)=(4,1,\underset{-7}{9})\Rightarrow \vec{e}_3=\frac{(4,1,9)}{\sqrt{16+1+\underset{49}{81}}}=\frac{1}{\sqrt{98}_{66}}(4,1,9)^{-7}$

$\vec{e}_1=\frac{1}{\sqrt{11}}(1,3,1)$; $\vec{e}_2=\frac{1}{\sqrt{6}}(2,-1,1)$; $\vec{e}_3=\frac{1}{\sqrt{66}}(4,1,-7)$

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c) $\vec{V}_1=(1,2,3)$; $\vec{V}_2=(2,1,1)$; $\vec{V}_3=(6,-7,-2)$

$\vec{e}_1=\frac{\vec{V}_1}{\|\vec{V}_1\|}=\frac{(1,2,3)}{\sqrt{1+4+9}}=\frac{1}{\sqrt{14}}(1,2,3)$

$\vec{e}_2=\frac{\vec{\omega}_2}{\|\vec{\omega}_2\|}$; $\vec{\omega}_2=\vec{V}_2-(\vec{e}_1,\vec{V}_2)\vec{e}_1=(2,1,1)-\frac{(2+2+3)}{14}(1,2,3)=\frac{1}{2}(3,0,-1)$

$\vec{e}_2=\frac{(\frac{3}{2},0,-\frac{1}{2})}{\sqrt{\frac{9}{4}+\frac{1}{4}}}=\frac{1}{\sqrt{10}}(3,0,-1)$

$\vec{e}_3=\frac{\vec{\omega}_3}{\|\vec{\omega}_3\|}$; $\vec{\omega}_3=\vec{V}_3-(\vec{e}_1,\vec{V}_3)\vec{e}_1-(\vec{e}_2,\vec{V}_3)\vec{e}_2=(6,-7,-2)-\frac{(6-14-6)}{14}(1,2,3)-$

$-\frac{(18+2)}{10}(3,0,-1)=(6,-7,-2)+(1,2,3)-(6,0,-2)=(1,-5,3)$

$\vec{e}_3=\frac{(1,-5,3)}{\sqrt{1+25+9}}=\frac{1}{\sqrt{35}}(1,-5,3)$

$\vec{e}_1=\frac{1}{\sqrt{14}}(1,2,3)$; $\vec{e}_2=\frac{1}{\sqrt{10}}(3,0,-1)$; $\vec{e}_3=\frac{1}{\sqrt{35}}(1,-5,3)$

② $P_0(x)=1$; $P_1(x)=(x-1)$; $P_2(x)=x^2-x+1$ $\quad L^2[-0,5;0,5]$

$e_0(x)$; $e_1(x)$; $e_2(x)$ - ?

$$e_0(x)=\frac{P_0(x)}{\|P_0(x)\|}=\frac{1}{\int_{-1/2}^{1/2}dx}=\frac{1}{1}=1$$

$$e_1(x)=\frac{\omega_1(x)}{\|\omega_1(x)\|};\ \omega_1(x)=P_1(x)-\langle e_0,P_1\rangle e_0=x-1-\int_{-1/2}^{1/2}(x-1)dx=x-1+1=x$$

$$e_1(x)=x\cdot\left(\int_{-1/2}^{1/2}x^2dx\right)^{-1}=x\left(\frac{x^3}{3}\Big|_{-1/2}^{1/2}\right)^{-\frac{1}{2}}=\cancel{4\sqrt{3}x}=2\sqrt{3}x$$

$$e_2(x)=\frac{\omega_2(x)}{\|\omega_2(x)\|};\ \omega_2(x)=P_2(x)-\langle e_0,P_2\rangle e_0(x)-\langle e_1,P_2\rangle e_1(x)=$$

$$=x^2-x+1-\int_{-1/2}^{1/2}(x^2-x+1)dx-\left(\int_{-1/2}^{1/2}(x^2-x+1)2\sqrt{3}x\,dx\right)\cdot 2\sqrt{3}x=$$

$$=x^2-x+1-\frac{13}{12}+\frac{2\sqrt{3}}{2\sqrt{3}}x=x^2-\frac{1}{12}$$

$$e_2(x)=\left(x^2-\frac{1}{12}\right)\cdot\left(\int_{-1/2}^{1/2}\left(x^2-\frac{1}{12}\right)^2dx\right)^{-\frac{1}{2}}=6\sqrt{5}\left(x^2-1/12\right)$$

$$e_0(x)=1\qquad e_1(x)=2\sqrt{3}x\qquad e_2(x)=6\sqrt{5}\left(x^2-\frac{1}{12}\right)$$

③ $\exists f$: 1) $f\in C^k[a;b]\xrightarrow{WLOG}f\in C^k[-\pi;\pi]$

2) $f(x)=\sum_{n=-\infty}^{+\infty}\hat{f}_n e^{inx}\cdot\frac{1}{2\pi}$

3) $f^{(\eta)}(x)=f^{(\eta)}(2\pi+x)$, $\forall\eta=0,1,..,k$

$\overset{?}{\Longrightarrow}\ \hat{f}_n=o\left(\frac{1}{n^k}\right)$

$$\hat{f}_n=\int_{-\pi}^{\pi}f(x)e^{-inx}dx=\frac{f(x)e^{-inx}}{(-in)}\Big|_{-\pi}^{\pi}+\int_{-\pi}^{\pi}\frac{f'(x)e^{-inx}}{(in)}dx \ =$$

$$\frac{f(x)e^{-inx}}{(-in)}\Big|_{-\pi}^{\pi}=\frac{f(\pi)(e^{-i\pi})^n-f(-\pi)(e^{i\pi})^n}{(-in)}=\frac{f(\pi)}{(-in)}\left((-1)^n-(-1)^n\right)=0$$

$f^{(\eta)}(x)\to$ Similarly, $\eta=0,1,..,k$

$$\textcircled{=} \int_{-\pi}^{\pi} \frac{f'(x)}{(in)} e^{-inx}\,dx = \int_{-\pi}^{\pi} \frac{f^{(k)}(x)\, e^{-inx}}{(in)^k}\,dx = \frac{1}{n^k}\cdot\underbrace{\frac{1}{i^k}\int_{-\pi}^{\pi} f^{(k)}(x)\, e^{-inx}\,dx}_{I_n}$$

$$4\ I_n \Rightarrow \frac{\widehat{f^{(k)}_n}}{i^k} = I_n\ ;\ f^{(k)}(x) \in C[-\pi;\pi] \Rightarrow \widehat{f^{(k)}_n} \xrightarrow[n\to\infty]{} 0 \left(\begin{array}{l}\text{Riemann-}\\ \text{-Lebesgue}\\ \text{Lem.}\end{array}\right)$$

$$\Rightarrow I_n \xrightarrow[n\to\infty]{} 0 \Rightarrow \hat{f}_n = \frac{I_n}{n^k}\cdot\frac{1}{i^k}\ ;\ \hat{f}_n \xrightarrow[n\to\infty]{} 0,\ \hat{f}_n = o\left(\frac{1}{n^k}\right)$$

$$\frac{I_n}{n^k} \xrightarrow[n\to\infty]{} 0$$

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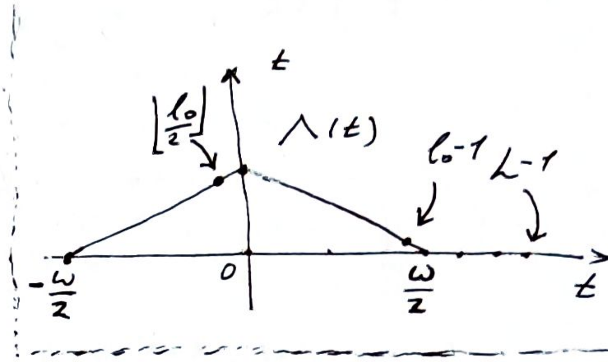
(4) $\Lambda(t) = \begin{cases} 1-\frac{2}{\omega}|x|, & |x|<\frac{\omega}{2} \\ 0, & |x|\geq \frac{\omega}{2}\end{cases}$; $f_s = \frac{1}{T}$; y_k - ?



$$\Lambda_\eta = 1-\frac{2}{\omega}\left|-\frac{\omega}{2}+\eta T\right| = 1-\frac{2}{\omega}\left|\frac{\eta}{f_s}-\frac{\omega}{2}\right|$$

$\eta = 0,\dots,l_0-1$; $l_0 - 1 = \left\lfloor \frac{\omega}{T}\right\rfloor = \lfloor \omega f_s\rfloor$

$$\left\lfloor \frac{l_0}{2}\right\rfloor = \left\lfloor\frac{\omega f_s+1}{2}\right\rfloor$$

$$y_k = \sum_{\eta=0}^{L-1}\Lambda_\eta e^{\frac{-2\pi i k}{L}\cdot\eta} = \sum_{\eta=0}^{l_0-1}\left(1-\frac{2}{\omega}\left|\frac{\eta}{f_s}-\frac{\omega}{2}\right|\right)e^{-\frac{2\pi i k}{L}\cdot\eta} = \sum_{\eta=0}^{l_0-1} e^{\frac{-2\pi i k}{L}\cdot\eta} -$$

$$- \frac{2}{\omega}\sum_{\eta=0}^{\lfloor \frac{l_0}{2}\rfloor}\left(\frac{\omega}{2}-\frac{\eta}{f_s}\right)e^{-\frac{2\pi i k}{L}\eta} - \frac{2}{\omega}\sum_{\eta=\lfloor \frac{l_0}{2}\rfloor+1}^{l_0-1}\left(-\frac{\omega}{2}+\frac{\eta}{f_s}\right)e^{-\frac{2\pi i k}{L}\cdot\eta} = e^{\frac{-i\pi k}{L}(l_0-1)}\frac{\sin(\pi l_0 k/L)}{\sin(\pi k/L)} -$$

$$- \frac{\left(1-e^{\frac{-2\pi i k}{L}\left(\lfloor \frac{l_0}{2}\rfloor+1\right)}\right)}{\left(1-e^{-\frac{2\pi i k}{L}}\right)} + e^{-\frac{2\pi i k}{L}\left(\lfloor \frac{l_0}{2}\rfloor+1\right)}\frac{\left(1-e^{\frac{-2\pi i k}{L}\left(l_0-\lfloor \frac{l_0}{2}\rfloor-1\right)}\right)}{1-e^{-\frac{2\pi i k}{L}}} +$$

$$+ \frac{2}{\omega f_s}\underbrace{\sum_{\eta=0}^{\lfloor \frac{l_0}{2}\rfloor}\eta\left(e^{\frac{-2\pi i k}{L}}\right)^\eta}_{\Sigma_1} - \frac{2}{\omega f_s}\underbrace{\sum_{\eta=\lfloor \frac{l_0}{2}\rfloor+1}^{l_0-1}\eta\left(e^{\frac{-2\pi i k}{L}}\right)^\eta}_{\Sigma_2} =$$

We know: $\sum_{\eta=\alpha}^{\beta} \rho^\eta = \frac{\rho^\alpha(1-\rho^{\beta-\alpha+1})}{1-\rho}$; $\sum_{\eta=\alpha}^\beta \eta \rho^\eta = \frac{d}{d\rho}\left(\sum_{\eta=\alpha}^{\beta} \rho^\eta\right) \Rightarrow$

$$\Rightarrow \sum_{\eta=\alpha}^\beta \eta \rho^\eta = \frac{\rho^\beta(\beta \rho^2+(-\beta-1)\rho)+\rho^\alpha((1-\alpha)\rho+\alpha)}{\rho(\rho-1)^2}$$

∡ $\Sigma_1 \Rightarrow \alpha=0,\ \beta=\left\lfloor \frac{l_0}{2}\right\rfloor,\ \rho = e^{-\frac{2\pi i k}{L}}$; $]\ \rho = e^{\frac{-2\pi i k}{L}}$

$$\Sigma_1 = \frac{\rho^{\lfloor \frac{l_0}{2}\rfloor}\left(\lfloor \frac{l_0}{2}\rfloor\rho^2-\left(\lfloor \frac{l_0}{2}\rfloor+1\right)\rho\right)+\rho}{\rho(\rho-1)^2}$$

∡ $\Sigma_2 \Rightarrow \alpha = \left\lfloor \frac{l_0}{2}\right\rfloor+1,\ \beta = l_0-1,\ \rho = e^{-\frac{2\pi i k}{L}}$

$$\Sigma_2 = \frac{\rho^{l_0-1}\left((l_0-1)\rho^2-(l_0)\rho\right)+\rho^{\lfloor \frac{l_0}{2}\rfloor+1}\left(\left(-\lfloor \frac{l_0}{2}\rfloor\right)\rho+\lfloor \frac{l_0}{2}\rfloor+1\right)}{\rho(\rho-1)^2}$$

$\exists\; g = e^{-\frac{2\pi i k}{L}}$

$$\Leftrightarrow \frac{(1-g^{l_0})}{1-g} - \frac{(1-g^{\left(\lfloor\frac{l_0}{2}\rfloor+1\right)})}{1-g} + g^{\left(\lfloor\frac{l_0}{2}\rfloor+1\right)}\frac{(1-g^{\left(l_0-\lfloor\frac{l_0}{2}\rfloor-1\right)})}{1-g} +$$

$$+\frac{2}{\omega f_s}\left[\frac{g^{\lfloor\frac{l_0}{2}\rfloor}\left(\lfloor\frac{l_0}{2}\rfloor g^2-\left(\lfloor\frac{l_0}{2}\rfloor+1\right)g\right)+g+g^{l_0-1}\left((l_0-1)g^2-l_0 g\right)+g^{\lfloor\frac{l_0}{2}\rfloor+1}\left(\lfloor\frac{l_0}{2}\rfloor+1-\lfloor\frac{l_0}{2}\rfloor g\right)}{g(g-1)^2}\right] =$$

$$= \frac{1-g^{l_0}-1+g^{\left(\lfloor\frac{l_0}{2}\rfloor+1\right)}+g^{\left(\lfloor\frac{l_0}{2}\rfloor+1\right)}-g^{l_0}}{1-g} + \frac{2}{\omega f_s(g-1)^2}\left[g^{\lfloor\frac{l_0}{2}\rfloor+1}\cdot\lfloor\frac{l_0}{2}\rfloor - g^{\lfloor\frac{l_0}{2}\rfloor}\left(\lfloor\frac{l_0}{2}\rfloor+1\right)+1+\right.$$

$$\left.+g^{l_0}(l_0-1)-l_0 g^{l_0-1}+g^{\lfloor\frac{l_0}{2}\rfloor}\cdot\lfloor\frac{l_0}{2}\rfloor+g^{\lfloor\frac{l_0}{2}\rfloor}-\lfloor\frac{l_0}{2}\rfloor g^{\lfloor\frac{l_0}{2}\rfloor+1}\right] = \frac{2\left(g^{\lfloor\frac{l_0}{2}\rfloor+1}-g^{l_0}\right)}{1-g}+$$

$$+\frac{2}{\omega f_s(g-1)^2}\left(1-l_0 g^{l_0-1}+g^{l_0}(l_0-1)\right) = \frac{2}{1-\exp\left(\frac{-2\pi i k}{L}\right)}\cdot\left(e^{-\frac{2\pi i k}{L}\left(\lfloor\frac{l_0}{2}\rfloor+1\right)}-\right.$$

$$\left.\cancel{+e^{-\frac{2\pi i k}{L}\lfloor\frac{l_0}{2}\rfloor}} - e^{-\frac{2\pi i k l_0}{L}}\right) + \frac{2}{\omega f_s\left(1-\exp\left(-\frac{2\pi i k}{L}\right)\right)^2}\left(1-l_0\exp\left(\frac{-2\pi i k}{L}(l_0-1)\right)+\right.$$

$$\left.+\exp\left(\frac{-2\pi i k}{L}\cdot l_0\right)(l_0-1)\right) \Leftrightarrow \left\{\begin{array}{l} l_0 = \lfloor\omega f_s\rfloor+1;\quad \lfloor\frac{l_0}{2}\rfloor+1=\lfloor\frac{l_0+2}{2}\rfloor = \\ \lfloor\frac{l_0}{2}\rfloor = \lfloor\frac{\lfloor\omega f_s\rfloor+1}{2}\rfloor;\quad = \lfloor\frac{\lfloor\omega f_s\rfloor+3}{2}\rfloor \end{array}\right\}$$

$$\Leftrightarrow 2\exp\left(-\frac{i\pi k}{L}\left(\frac{\lfloor\frac{l_0}{2}\rfloor+1+\overset{+1}{l_0}}{2}\right)\right)\frac{\sin\left(\frac{\pi k}{L}\left(\frac{\lfloor\frac{l_0}{2}\rfloor+1-l_0}{2}\right)\right)}{\sin\left(\frac{\pi k}{L}\right)} +$$

$$+\frac{2}{\omega f_s}\cdot\frac{1}{\left(1-\exp\left(-i\frac{2\pi k}{L}\right)\right)^2}\cdot\left(1-l_0\exp\left(-i\frac{2\pi k}{L}(l_0-1)\right)+(l_0-1)e^{-i\frac{2\pi k l_0}{L}}\right) =$$

$$= 2\exp\left(\frac{-i\pi k}{2L}\left(\left\lfloor\frac{\lfloor \omega f_s\rfloor+1}{2}\right\rfloor+\lfloor \omega f_s\rfloor+3\right)\right)\frac{\sin\left(\frac{\pi k}{2L}\left(\left\lfloor\frac{\lfloor \omega f_s\rfloor+1}{2}\right\rfloor-\lfloor \omega f_s\rfloor\right)\right)}{\sin\left(\frac{\pi k}{L}\right)}+$$

$$+\frac{2}{\omega f_s}\cdot\frac{1}{\left(1-\exp\left(-i\frac{2\pi k}{L}\right)\right)^2}\cdot\left(1-\left(\lfloor \omega f_s\rfloor+1\right)\cdot\exp\left(-i\frac{2\pi k}{L}\left(\lfloor \omega f_s\rfloor\right)\right)+\cancel{\lfloor \omega f_s\rfloor e^{-i\frac{2\pi k}{L}(\lfloor \omega f_s\rfloor)}}\right.$$

$$\left.+\lfloor \omega f_s\rfloor\exp\left(-i\frac{2\pi k}{L}\left(\lfloor \omega f_s\rfloor+1\right)\right)\right)=y_k$$