Question 1: The functions $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ are recursively defined as follows:

$$\begin{array}{lcll} f(0) & = & 1, & \\ f(n) & = & g(n, f(n-1)) & \text{if } n \geq 1, \\ g(m,0) & = & 0 & \text{if } m \geq 0, \\ g(m,n) & = & m + g(m, n-1) & \text{if } m \geq 0 \text{ and } n \geq 1. \end{array}$$

Solve these recurrences for f, i.e., express $f(n)$ in terms of n. Justify your answer.

Answer 1: The described recurrence states that both $f(0) = 0, g(m, 0) = 0$. With this, we show:

$$\begin{aligned} &f(0) = 0 \\ &f(1) = g(1,0) = 0 \\ &f(2) = g(2,1) = 2 + g(2,0) = 2 + 0 = 2 \\ &f(3) = g(3,2) = 3 + 3 + g(3,0) = 6 \\ &f(4) = g(4,3) = 4 + 4 + 4 + g(4,0) = 12 \\ &f(5) = g(5,4) = 5 + 5 + 5 + 5 + g(5,0) = 20 \\ &f(6) = g(6,5) = 6 + 6 + 6 + 6 + 6 + g(6,0) = 30 \end{aligned}$$

It follows that $f(n) = n(n-1)$ when $n \geq 2$

Question 2: The function $f : \mathbb{N} \to \mathbb{Z}$ is defined by

$$f(n) = 2n(n-6)$$

for each integer $n \geq 0$. Derive a recursive form of this function f. Show your work.

Answer 2:

$$\begin{aligned} f(1) &= 2(1)(1-6) = -10 \\ f(2) &= 2(2)(2-6) = -16 \\ f(3) &= 2(3)(3-6) = -18 \\ f(4) &= 2(4)(4-6) = -16 \\ f(5) &= 2(5)(5-6) = -10 \end{aligned}$$

Question 3: The function $f : \mathbb{N}^3 \to \mathbb{N}$ is defined as follows:

$$\begin{array}{lcll} f(k,n,0) & = & k + n & \text{if } k \geq 0 \text{ and } n \geq 0, \\ f(k,0,1) & = & 0 & \text{if } k \geq 0, \\ f(k,0,2) & = & 1 & \text{if } k \geq 0, \\ f(k,0,i) & = & k & \text{if } k \geq 0 \text{ and } i \geq 3, \\ f(k,n,i) & = & f(k, f(k, n-1, i), i-1) & \text{if } k \geq 0,\ i \geq 1, \text{ and } n \geq 1. \end{array}$$

Determine $f(2, 3, 2)$. Show your work.

Answer 3: Applying the above stated qualities to our data, $f(2, 3, 2)$, we see

$$f(k,n,0) = f(2,3,0) = 2 + 3 = 5 \tag{1}$$
$$f(k,0,1) = f(2,0,1) = 0 \tag{2}$$
$$f(k,0,2) = f(2,0,2) = 1 \tag{3}$$
$$f(k,0,i) = k \qquad \text{i} \geq 3 \tag{4}$$

$$
\begin{aligned}
f(k,n,i) =& f(2,3,2) \\
=& f(2,f(2,2,2),1) \\
=& f(2,f(2,f(2,1,2),1),1) \\
=& f(2,f(2,f(2,f(2,0,2),1),1),1) && (3) \\
=& f(2,f(2,f(2,1,1),0),0) \\
=& f(2,f(2,f(2,f(2,0,1),0),0),0) && (2) \\
=& f(2,f(2,f(2,0,0),0),0) && (4) \\
=& f(2,f(2,2,0),0) && (1) \\
=& f(2,4,0) && (1)
\end{aligned}
$$

Therefore, $f(2,3,2) = 6$.

Question 4: A *maximal run of ones* in a bitstring is a maximal consecutive substring of ones. For example, the bitstring 1100011110100111 has four maximal runs of ones: 11, 1111, 1, and 111. These maximal runs have lengths 2, 4, 1, and 3, respectively.

Let $n \geq 1$ be an integer and let B_n be the number of bitstrings of length n that do not contain any maximal run of ones of odd length; in other words, every maximal run of ones in these bitstrings has an even length.

- Determine B_1, B_2, and B_3.
- Determine the value of B_n, i.e., express B_n in terms of numbers that we have seen in class. Justify your answer. *Hint:* Derive a recurrence.

Answer 4:

Question 5: Let $n \geq 1$ be an integer and let S_n be the number of ways in which n can be written as a sum of 1s and 2s; the order in which the 1s and 2s occur in the sum matters. For example, $S_3 = 3$, because

$$3 = 1 + 1 + 1 = 1 + 2 = 2 + 1.$$

- Determine S_1, S_2, and S_4.
- Determine the value of S_n, i.e., express S_n in terms of numbers that we have seen in class. Justify your answer. *Hint:* Derive a recurrence.

Answer 5:

Question 6: Let n be a positive integer and consider a $5 \times n$ board B_n consisting of $5n$ cells, each one having sides of length one. The top part of the figure below shows B_{12}.

A *brick* is a horizontal or vertical board consisting of $2 \times 3 = 6$ cells; see the bottom part of the figure above. A *tiling* of the board B_n is a placement of bricks on the board such that

- the bricks exactly cover B_n and
- no two bricks overlap.

The figure below shows a tiling of B_{12}.

Let T_n be the number of different tilings of the board B_n.

- Let $n \geq 6$ be a multiple of 6. Determine the value of T_n. Justify your answer. *Hint:* Derive a recurrence.
- Let n be a positive integer that is not a multiple of 6. Prove that $T_n = 0$.

Answer 6:

Question 7: Consider the following recursive algorithm silly, which takes as input an integer $n \geq 1$ which is a power of 2:

For n a power of 2, let $F(n)$ be the number of times you fart when running algorithm silly(n).

- Determine the value of $F(n)$ and prove that your answer is correct. *Hint:* Derive a recurrence.

Answer 7:

Question 8:

Let $m \geq 1$ and $n \geq 1$ be integers. Consider m horizontal lines and n non-horizontal lines such that

- no two of the non-horizontal lines are parallel,
- no three of the $m + n$ lines intersect in one single point.

These lines divide the plane into regions (some of which are bounded and some of which are unbounded). Denote the number of these regions by $R_{m,n}$. From the figure below, you can see that $R_{4,3} = 23$.

- Derive a recurrence for the numbers $R_{m,n}$ and use it to prove that

$$R_{m,n} = 1 + m(n+1) + \binom{n+1}{2}.$$

Answer 8: