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COMP 2804 — Solutions Assignment 1

Question 1: On the first page of your assignment, write your name and student number.

Solution:

- Name: James Bond
- Student number: 007

Question 2: Let m and n be integers with $m \geq n \geq 1$. How many ways are there to place m books on n shelves, if there must be at least one book on each shelf? As in Section 1.3 of the textbook, the order on each shelf matters. Justify your answer.

Solution: We are going to do the following:

- Start with all shelves being empty.
- Choose n books out of a total of m books (these books will be placed in the last step; for the moment, we put them aside). There are $\binom{m}{n}$ ways to do this.
- Place the remaining $m - n$ books on the n shelves as we did in class. We have seen in class that there are
$$\frac{(n + (m-n) - 1)!}{(n-1)!} = \frac{(m-1)!}{(n-1)!}$$
ways to do this.
- Place the n books that have not been placed yet; each of them is placed at the far-left of one shelf, and exactly one book will be placed on each of the n shelves. This corresponds to a permutation of these n books; we know that there are $n!$ permutations. Thus, there are $n!$ ways to do this.

By the Product Rule, the total number of ways to place the books is equal to
$$\binom{m}{n} \frac{(m-1)!}{(n-1)!} n!.$$

If you want, you can write this as
$$\frac{m!}{(m-n)!} \cdot \frac{(m-1)!}{(n-1)!}.$$

Question 3: In how many ways can you paint 200 distinct chairs, if 33 of them must be painted red, 66 of them must be painted blue, and 101 of them must be painted green? Justify your answer.

Solution: We are going to do the following:

- Choose 33 chairs out of 200 and paint them red. There are $\binom{200}{33}$ ways to do this.
- Out of the remaining 167 chairs, pick 66 and paint them blue. There are $\binom{167}{66}$ ways to do this.
- Out of the remaining 101 chairs, pick 101 and paint them green. There is $\binom{101}{101} = 1$ way to do this.

By the Product Rule, the total number of ways to paint the chairs is equal to

$$\binom{200}{33} \cdot \binom{167}{66} \cdot 1.$$

If you want, you can write this as

$$\frac{200!}{33!66!101!}.$$

Question 4: There are $n \geq 4$ students in Carleton's Computer Science program. The Carleton Computer Science Society has a Board of Directors, consisting of one president and three vice-presidents. The entire board consists of four distinct students. Prove that

$$n\binom{n-1}{3} = (n-3)\binom{n}{3},$$

by counting, in two different ways, the number of ways to choose a Board of Directors.

Solution:
First way:

- First task: Choose a president; there are n ways to do this.
- Second task: Choose 3 vice-presidents. Since the president has already been chosen, there are are $\binom{n-1}{3}$ way to do this.

By the Product Rule, the number of way to choose a Board of Directors is equal to

$$n\binom{n-1}{3}. \tag{1}$$

Second way:

- First task: Choose 3 vice-president; there are $\binom{n}{3}$ ways to do this.
- Second task: Choose a president. Since the vice-presidents have already been chosen, there are are $\binom{n-3}{1} = n - 3$ ways to do this.

By the Product Rule, the number of way to choose a Board of Directors is equal to

$$\binom{n}{3}(n-3). \tag{2}$$

The values of (1) and (2) must be equal, because both of them count the number of way to choose a Board of Directors.

Question 5: How many bitstrings of length 77 are there that start with 010 (i.e., have 010 at positions 1, 2, and 3) or have 101 at positions 2, 3, and 4, or have 010 at positions 3, 4, and 5. Justify your answer.

Solution: Let

- A be the set of all bitstrings of length 77 that start with 010,
- B be the set of all bitstrings of length 77 that have 101 at positions 2, 3, and 4,
- C be the set of all bitstrings of length 77 that have 010 at positions 3, 4, and 5.

We have to determine the size of the union of A, B, and C.

- $|A| = 2^{74}$
- $|B| = 2^{74}$
- $|C| = 2^{74}$
- $|A \cap B| = 2^{73}$
- $|A \cap C| = 2^{72}$
- $|B \cap C| = 2^{73}$
- $|A \cap B \cap C| = 2^{72}$

Using the Inclusion-Exclusion formula, we get

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ &= 2^{74} + 2^{74} + 2^{74} - 2^{73} - 2^{72} - 2^{73} + 2^{72} \\ &= 3 \cdot 2^{74} - 2 \cdot 2^{73} \\ &= 3 \cdot 2^{74} - 2^{74} \\ &= 2 \cdot 2^{74} \\ &= 2^{75}. \end{aligned}$$

Question 6: Let $n \geq 1$ be an integer. Use Newton's Binomial Theorem to argue that

$$\sum_{k=1}^{n} \binom{n}{k} 10^k \cdot 26^{n-k} = 36^n - 26^n.$$

Solution: Recall Newton's Binomial Theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Take $x = 26$ and $y = 10$. Then we get

$$36^n = \sum_{k=0}^{n} \binom{n}{k} 10^k \cdot 26^{n-k}.$$

The term for $k = 0$ in the summation is equal to

$$\binom{n}{0} 10^0 \cdot 26^{n-0} = 26^n.$$

Thus,

$$\begin{aligned} 36^n &= \sum_{k=0}^{n} \binom{n}{k} 10^k \cdot 26^{n-k} \\ &= 26^n + \sum_{k=1}^{n} \binom{n}{k} 10^k \cdot 26^{n-k}. \end{aligned}$$

Question 7: Let $n \geq 1$ be an integer. We consider passwords consisting of n characters, each character being a digit or a lowercase letter. A password must contain at least one digit.

- Use the method we used in class (see Section 3.3 in the textbook) to show that the number of passwords is equal to $36^n - 26^n$.

  Solution: Let A be the set of all passwords. We have to determine the size of A.

  Let U be the set of all strings consisting of n characters, each character being a digit or a lowercase letter. The set U has size 36^n. The complement of A, i.e., the set $U \setminus A$, is the set of all strings consisting of n characters, each character being a lowercase letter. The set $U \setminus A$ has size 26^n.

  By the Complement Rule, we have

  $$|A| = |U| - |U \setminus A| = 36^n - 26^n.$$

- Let k be an integer with $1 \leq k \leq n$. Prove that the number of passwords with exactly k digits is equal to $\binom{n}{k} 10^k \cdot 26^{n-k}$.

  Solution: Let A_k be the set of all passwords with exactly k digits. We obtain all passwords of A_k in the following way:

- Choose k positions out of n positions. There are $\binom{n}{k}$ ways to do this.
- In each of the k chosen positions, write a digit. There are 10^k ways to do this.
- In each of the $n-k$ remaining positions, write a lowercase letter. There are 26^{n-k} ways to do this.

By the Product Rule, the size of A_k is equal to

$$|A_k| = \binom{n}{k} 10^k \cdot 26^{n-k}.$$

- Explain why the above two parts imply the expression in Question 6.

  Solution: Since A is the union of the pairwise disjoint sets A_k, $1 \leq k \leq n$, the Sum Rule implies that

$$|A| = \sum_{i=1}^{k} |A_k|.$$

  The left-hand side is equal to $36^n - 26^n$, whereas the right-hand side is equal to

$$\sum_{i=1}^{k} \binom{n}{k} 10^k \cdot 26^{n-k}.$$

Question 8: Let n and k be integers with $n \geq k+2$ and $k \geq 2$. Prove the following identity using a combinatorial proof (as in the proof of Theorem 3.7.2 in the textbook):

$$\binom{n}{k} = \binom{n-2}{k} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2}.$$

Solution: Let S be a set of size n. We know that there are $\binom{n}{k}$ subsets of S. We are now going to count these subsets in a different way:

Take two elements x and y in S. Each subset of S having size k is of exactly one of the following four types:

1. The subset does not contain x and does not contain y.

   There are $\binom{n-2}{k}$ such subsets.

2. The subset contains x and does not contain y.

   There are $\binom{n-2}{k-1}$ such subsets.

3. The subset contains y and does not contain x.

   There are $\binom{n-2}{k-1}$ such subsets.

4. The subset contains both x and y.

   There are $\binom{n-2}{k-2}$ such subsets.

By taking the sum of these four binomial coefficients, we see that the number of subsets of S having size k is equal to

$$\binom{n-2}{k} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2}.$$

Question 9: Consider five points in a square with sides of length one. Use the Pigeonhole Principle to prove that there are two of these points having distance at most $1/\sqrt{2}$.

Solution: Divide the square into four subsquares, each having sides of length 1/2. Since there are five points, there must be a subsquare with at least two points. Take two points p and q in the same subsquare. The distance between p and q is at most the diagonal of a subsquare, which, by Pythagoras, is equal to

$$\frac{1}{2} \cdot \sqrt{2} = 1/\sqrt{2}.$$