Question 1: See the top of the page.

Question 2: You flip a fair coin six times.

- What is the sample space? (Give the answer in plain English; do not list all elements of the sample space.)
- Define the events

$$A = \text{“the coin comes up heads at least four times”},$$

$$B = \text{“the number of heads is equal to the number of tails”},$$

and

$$C = \text{“there are at least four consecutive heads”}.$$

Determine $\Pr(A)$, $\Pr(B)$, $\Pr(C)$, $\Pr(A \mid B)$, and $\Pr(C \mid A)$. Show your work.

The format I will present, with a symbol H or T to the power of a certain integer will indicate the number of occurrences of that symbol. Therefore, H^0 would denote zero heads, and H^n would denote n consecutive of heads in the sequence...

Also, when $(\sum_{r=a}^{f} r) = 6$ is used, it's intended to mean the **sum of all the powers** must be 6.

$$A = \{H^a T^b H^c T^d H^e T^f : (a + c + e) \geq 4 \wedge \sum_{r=a}^{f} r \equiv 6\}$$

We can determine the size of A by splitting the problem into several *smaller* problems. Since the problem specifies that there must be **at least** 4, I can denote $A_{k \leq 6}$ to be the size of the set of possibilities with k number of heads.

$$|A| = \sum_{k=4}^{6} \binom{6}{k}$$
$$|A| = 22$$

As for B, where the number of heads must be equivalent, all we must do is choose three out of the sequence of six. This will implicitly ensure that three tails are also chosen.

The sum of variables a, c, and e represent the number of times heads would occur, and ensuring exactly one of them has a run of 4 or more **(because the sum of all variables must be equal 6)** would fit the event as described ...

Determining the size of C can be a bit tricky, but can be expressed in a comprehensive way through use of summations. No matter how many heads occur, the chance of actually having 4 heads in a row is the same as flipping a coin 4 times, and getting heads every time. Therefore, we have $\frac{1}{2}^4$...

But now we have to count this multiple times over, because the *bitstring* will be different when it contains 4, 5, and 6 heads. Let n be the length of the bitstring, and k be the number of consecutive heads...

$$\sum_{i=1}^{n-k+1} \frac{1}{2^k} = \sum_{i=1}^{3} \frac{1}{2^4} = \frac{1}{2^4} \cdot \sum_{i=1}^{3} 1 = \frac{3}{16} = \frac{12}{64} = Pr(C)$$

Now to determine the probabilities, the first thing we must do is establish what our sample space, S, is ...

Sample space, S, would count all possible combinations of coin flips, this is the same as the number of combinations that can form from a bit string of length n, because the options presented for each character can be either $HEADS$ or $TAILS$. Therefore,

$$|S| = 2^6 = 64$$

$$\begin{aligned} Pr(A) &= \frac{|A|}{|S|} \\ &= \frac{22}{64} \\ &= 0.34375 \end{aligned}$$

$$\begin{aligned} Pr(B) &= \frac{|B|}{|S|} \\ &= \frac{20}{64} \\ &= 0.3125 \end{aligned}$$

$$\begin{aligned} Pr(C) &= \frac{|C|}{|S|} \\ &= \frac{12}{64} \\ &= 0.18 \end{aligned}$$

For B to occur, there must be exactly 3 heads, for A to occur there must be at least 4 heads. Rendering the two sets **completely** disjoint. Meaning that the probability of their intersection will equal 0...

$$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)} = \frac{0}{Pr(B)} = 0$$

$Pr(C|A)$ is the probability intersection of C and A, over the probability of A. C is clearly a subset of A, as A contains all permutations of 4 or more heads, while C contains only the permutations that contain a run of 4 or more heads. Because of this, the probability of their intersection will yield only the probability of C ...

$$Pr(C|A) = \frac{Pr(C \cap A)}{Pr(A)} = \frac{\frac{12}{64}}{\frac{22}{64}} = \frac{12}{22} = 0.54$$

Question 3: You are given three events A, B, and C in a sample space S. Is the following true or false?

$$\Pr\left(A \cap \overline{B} \cap \overline{C}\right) = \Pr(A \cup B \cup C) - \Pr(B) - \Pr(C) + \Pr\left(B \cap C\right).$$

Justify your answer.

As expressed in one of Prof. Smid's lectures, Venn Diagrams remain a competent way to express the relationship between sets.

Through some manipulation it becomes easy to see that the right hand side is equivilent to the left.

$$
\begin{aligned}
&= Pr(A \cup B \cup C) - Pr(B) - Pr(C) + Pr(B \cap C) \\
&= Pr(A \cup B \cup C) - (Pr(B) + Pr(C) - Pr(B \cap C)) \\
&= Pr(A \cup B \cup C) - Pr(B \cup C)
\end{aligned}
$$

This statement becomes easier to express in a venn diagram.

Therefore, the expression is $TRUE$, as seen from the illustration of the relationships between the three different sets.

Question 4: Consider two events A and B in a sample space S.

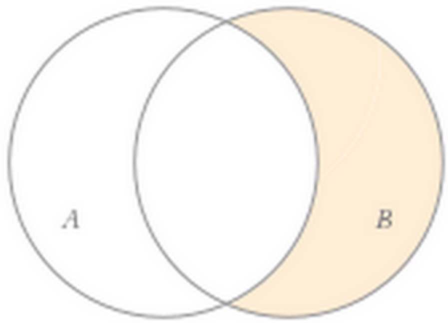
- Assume that $\Pr(A) = 1/2$ and $\Pr\left(B \mid \overline{A}\right) = 3/5$. Determine $\Pr(A \cup B)$.
- Assume that $\Pr(A \cup B) = 5/6$ and $\Pr\left(\overline{A} \mid \overline{B}\right) = 1/3$. Determine $\Pr(B)$.

Show your work.

1.)

$$
\begin{aligned}
\frac{3}{5} &= Pr(B|\overline{A}) \\
\frac{3}{5} &= \frac{Pr(B \cap \overline{A})}{1 - Pr(A)} \\
\frac{3}{5} &= \frac{Pr(B \cap \overline{A})}{1/2} \\
\frac{3}{10} &= Pr(B \cap \overline{A})
\end{aligned}
$$

Below is an illustration of $B \cap \overline{A} \ldots$



Through further manipulation, it can be shown that $B \cap \overline{A} \equiv Pr(B) - Pr(A \cap B) \ldots$

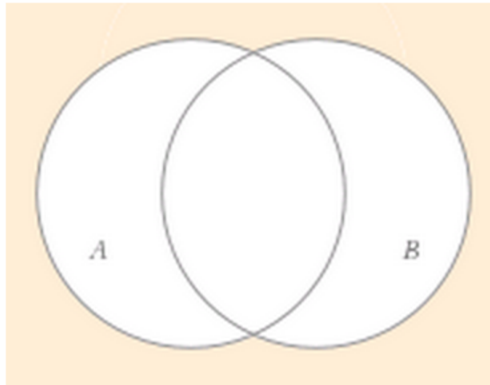
$$\begin{aligned} Pr(B \cap \overline{A}) &= Pr(B) \cdot Pr(\overline{A}) \\ &= Pr(B) \cdot (1 - Pr(A)) \\ &= Pr(B) - Pr(B) \cdot Pr(A) \\ &= Pr(B) - Pr(A \cap B) \end{aligned}$$

Because it can be expressed like such, we can make the claim that adding A would give us the probability of the union of A and B.

$$\begin{aligned} Pr(A \cup B) &= Pr(A) + Pr(B) - Pr(A \cap B) \\ &= Pr(A) + Pr(\overline{A} \cap B) \\ &= \frac{1}{2} + \frac{3}{10} \\ &= \frac{8}{10} \end{aligned}$$

2.)

Before we begin, it should be made clear that $Pr(\overline{A} \cap \overline{B})$ is the inverse of $Pr(\overline{\overline{A} \cap \overline{B}}) \equiv Pr(A \cup B)$ *DeMorgan's law*. As shown in the diagram below which illustrates $Pr(\overline{A} \cap \overline{B}) \ldots$



Therefore, if we know that $Pr(A \cup B) = 5/6$, and $Pr(\overline{A} \cap \overline{B})$ is the inverse. Then $Pr(S) - Pr(A \cup B) \equiv Pr(\overline{A} \cap \overline{B})$. It is known that the $Pr(S) = 1 \ldots$

$$Pr(\overline{A} \cap \overline{B}) = 1 - \frac{5}{6} = \frac{1}{6}$$

$$\begin{aligned}
\frac{1}{3} &= \frac{Pr(\overline{A} \cap \overline{B})}{Pr(\overline{B})} \\
\frac{1}{3} &= \frac{\frac{1}{6}}{1 - Pr(B)} \\
\frac{3}{6} &= 1 - Pr(B) \\
1 - \frac{3}{6} &= Pr(B) \\
\frac{1}{2} &= Pr(B)
\end{aligned}$$

Question 5: Let S be a set consisting of 6 positive integers and 8 negative integers. Choose a 4-element subset of S uniformly at random, and multiply the elements in this subset. Denote the product by x. Determine the probability that $x > 0$. Show your work.

To be able to accumulate all the positive integers, we need an even number of negative numbers. Out of the four numbers we need to choose, there are only three cases of an even number of negative numbers. There being no negatives, two negatives, and four negatives. Therefore, to calculate the total number of ways this can occur, we must take the sum of all the cases.

Let $2k$, be an integer that represents the number of negative numbers.

$$A = \begin{cases} \binom{6}{4} & \text{if } 2k \equiv 0 \\ \binom{6}{4-2k}\binom{8}{2k} & \text{if } 4 - 2k > 0 \\ \binom{8}{2k} & \text{if } 4 - 2k \equiv 0 \end{cases}$$

The sum of all these would present us with the total number of combinations of choosing 4 that would yield positive. We must next determine our sample space. This would simply be all the ways that you can choose 4 from 14.

$$S = \binom{14}{4}$$

Then to calculate the probability...

$$\begin{aligned} Pr(A) &= \frac{\binom{6}{4} + \binom{6}{2}\binom{8}{2} + \binom{8}{4}}{\binom{14}{4}} \\ &= \frac{505}{1001} \\ &= 0.504 \end{aligned}$$

Question 6: Give an example of a sample space S and six events A, B, C, D, E, and F such that

- $\Pr(A \mid B) = \Pr(A)$,
- $\Pr(C \mid D) < \Pr(C)$,
- $\Pr(E \mid F) > \Pr(E)$.

Justify your answer.

Hint: The sequence of six events may contain duplicates. Try to make the sample space S as small as you can.

Let our sample space be the different results of a dice roll.

$$\begin{aligned} S &= \{1, 2, 3, 4, 5, 6\} \\ |S| &= 6 \\ Pr(S) &= 1 \end{aligned}$$

I'll define the events as follows ...

$$\begin{aligned}
A &= \text{Result of roll is even} \\
B &= \text{Result of roll is less than three} \\
C &= \text{Result of roll mod 2 is equal to zero} \\
D &= \text{Result or roll is odd} \\
E &= \text{Result of roll is 1} \\
F &= \text{Result of roll is less than 2}
\end{aligned}$$

$$Pr(A|B) \equiv Pr(A)$$

This will occur if A and B are independent events. This can be shown to be true by seeing if the equality holds. I will denote which elements of my sample space match the event by showing it as a subscript of ...

$$\{1_B, 2_{A,B}, 3, 4_A, 5, 6_A\}$$

$$\begin{aligned}
A &= \{2, 4, 6\} \\
|A| &= 3 \\
Pr(A) &= \frac{1}{2} \\
B &= \{1, 2\} \\
|B| &= 2 \\
Pr(B) &= \frac{1}{3} \\
A \cap B &= \{2\} \\
Pr(A \cap B) &= \frac{1}{6} \\
&= Pr(A) \cdot Pr(B) \\
&= \frac{1}{2} \cdot \frac{1}{3}
\end{aligned}$$

Since A and B are independent of each other, but not disjoint, the claim can be made that the probability of A given B is equal to the probability of A.

$$\begin{aligned}
Pr(A|B) &= \frac{Pr(A) \cdot Pr(B)}{Pr(B)} \\
&= Pr(A)
\end{aligned}$$

$$Pr(C|D) < Pr(C)$$

One way for this to occur is to have C and D be completely **disjoint**. C contains only even numbers, while D odd. Therefore, $Pr(C|D) = 0$, which is less than $Pr(C) \equiv \frac{1}{2}$.

$$Pr(E|F) > Pr(E)$$

An efficient solution to this is have event E equivalent to event F.

$$E = \{1\}$$
$$F = \{1\}$$

Since these events are identical, the following arithmetic will hold ...

$$\begin{aligned} Pr(E|F) &= \frac{Pr(E \cap F)}{Pr(F)} \\ &= \frac{Pr(F)}{Pr(F)} \\ &= 1 \end{aligned}$$

The probability of E given F will then be 1, and is greater than the probability of event E occurring, $\frac{1}{6}$.

Question 7: Let $n \geq 3$ be an integer, consider a uniformly random permutation of the set $\{1, 2, \ldots, n\}$, and define the events

$$A = \text{"in this permutation, 2 is to the left of 3"}$$

and

$$B = \text{"in this permutation, 1 is to the left of 2 and 1 is to the left of 3"}.$$

Are these two events independent? Justify your answer.

Hint: Use the Product Rule to count the number of permutations that define A and B.

Two events can be shown to be independent if the following equality holds:

$$Pr(A \cap B) = Pr(A) \cdot Pr(B)$$

For this to be proven, we must compare the probabilities of both sides.

The event A will occur with a probability of $\frac{1}{2}$. This is simply because 2 can only be to the left or right of 3. The arrangement of all other integers is irrelevant, because it can be factored out from both sides.

$$Pr(A) = \frac{1}{2}$$

As for B, the same methodology can be applied, except three cases would need to be specified. Of the possible arrangements, we only care about the ones where 1 is to left of both 2 and 3. Since the extra numbers are irrelevant, we can show all permutations of 1,2, and 3 to see how many combinations fit this event.

$$\text{S} = \begin{cases} \{1,2,3\}, \\ \{1,3,2\}, \\ \{2,1,3\}, \\ \{2,3,1\}, \\ \{3,1,2\}, \\ \{3,2,1\} \end{cases}$$

There are only two cases that match the event B, (1,2,3, and 1,3,2), henceforth $Pr(B) = \frac{1}{3}$.

As for $Pr(A \cap B)$, there is only 1 element in the defined set S where **1 is to the left of 2 and 3**, and **2 is to the left of 3** ...

Therefore $Pr(A \cap B) = \frac{|(A \cap B)|}{|S|} = \frac{1}{6}$

Finally, can come back to the equality ...

$$Pr(A \cap B) = Pr(A) \cdot Pr(B)$$
$$\frac{1}{6} = \frac{1}{2} \cdot \frac{1}{3}$$
$$\frac{1}{6} = \frac{1}{6}$$

Therefore, A and B are independent.

Question 8: Consider two players P_1 and P_2:

- P_1 has one fair coin.
- P_2 has two coins. One of them is fair, whereas the other one is 2-headed (Her Majesty is on both sides of this coin).

The two players P_1 and P_2 play a game in which they alternate making turns: P_1 starts, after which it is P_2's turn, after which it is P_1's turn, after which it is P_2's turn, etc.

- When it is P_1's turn, she flips her coin once.
- When it is P_2's turn, he does the following:
  - P_2 chooses one of his two coins uniformly at random. Then he flips the chosen coin once.
  - If the first flip did not results in heads, then P_2 repeats this process one more time: P_2 again chooses one of his two coins uniformly at random and flips the chosen coin once.

The player who flips heads first is the winner of the game.

- Determine the probability that P_2 wins this game, assuming that all random choices and coin flips made are mutually independent. Justify your answer.

To determine the probability that event $P2$ wins will occur, we must first determine the probability of their loss. For player two to lose, a sequence like such must occur . . .

$$\text{"P1 Wins"} = \begin{cases} H_{p1}, \\ T_{p1}T_{p2}T_{p2}H_{p1}, \\ ... \end{cases}$$

It becomes clear that player 1 will win after n sequences of 3 consecutive tails. Next we must find the probability of such event occurring.

For the first tail, this has the odds of $\frac{1}{2}$ of happening, as $P1$'s coin is fair. But since $P2$ will gets two chances, for both chances, the odds are $\frac{1}{4}$ that he will get a tail (*because of the trick coin*), we cannot just do $\frac{1}{2}^n$, as you would playing the game with a fair coin.

Thus, the probability of $P1$ winning is as follows:

$$Pr(T^{3n}H) = \{\frac{1}{2} \cdot \frac{1}{4}^2\}^n \cdot \frac{1}{2} = \frac{1}{32}^n \cdot \frac{1}{2}$$

But this is still in terms of n, where n is the number of times this sequence occurs. We must find where this infinite series converges . . .

$$\begin{aligned} Pr(P1Wins) &= \sum_{n=0}^{\infty} \frac{1}{2}(\frac{1}{32})^n \\ &= \frac{1}{2} \cdot \sum_{n=0}^{\infty}(\frac{1}{32})^n \\ &= \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{32}} \\ &= \frac{1}{\frac{62}{32}} \\ &= \frac{32}{62} \\ &= 0.516 \end{aligned}$$

Now to get $P2$'s probability, we get the difference of the total probability, and the probability of $P1$ winning.

$$Pr(P2Wins) = 1 - Pr(P1Wins) = 1 - 0.516 = 0.484$$

The probability of $P2$ winning is 0.484.

Question 9: You would like to generate a *biased* random bit: With probability 2/3, this bit is 0, and with probability 1/3, it is 1. You find a *fair* coin in your pocket: This coin comes up heads (H) with probability 1/2 and tails (T) with probability 1/2. In this question, you will show that this coin can be used to generate a biased random bit.

Consider the following recursive algorithm , which does not take any input:

Algorithm

// all coin flips made are mutually independent
flip the coin;
if the result is H

then return 0
else $b =$
return $1 - b$
endif

- The sample space S is the set of all sequences of coin flips that can occur when running algorithm . Determine this sample space S.
- Prove that algorithm returns 0 with probability 2/3.

This problem is essentially the same as two players, $P1$ and $P2$ flipping a fair coin to see whom will land heads first. In that situation, $P1$ has the edge, as going first gives them a probability of 0.66 of winning before $P2$.
This recursive function can be expressed in a similar manner. First I will define two events ...

$$\text{generateBiasedBit} = \begin{cases} 0 : \text{P1 Wins} \\ 1 : \text{P2 Wins} \end{cases}$$

$$Pr(generateBiasedBit = 0) = \frac{2}{3}$$

This can be shown to true for any combination structured as such:

$$\{T^{2n}H\} : n \in Z$$

$$0 = \begin{cases} H_{p1}, \\ T_{p1}T_{p2}H_{p1}, \\ T_{p1}T_{p2}T_{p1}T_{p2}H_{p1} \\ ... \end{cases}$$

Every second iteration will subtract 1 from 1, resulting in zero. We can show this to be true by showing that $Pr(generateBiasedBit = 0) = Pr(\{T^{2n}H\})$

$$
\begin{aligned}
Pr(\{T^{2n}H\}) &= \frac{1}{2}^{2n} \cdot \frac{1}{2} \\
&= \sum_{n=0}^{\infty} \frac{1}{2}^{2n} \cdot \frac{1}{2} \\
&= \frac{1}{2} \sum_{n=0} \infty \frac{1}{4}^{n} \\
&= \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{4}} \\
&= \frac{1}{2} \cdot \frac{1}{\frac{3}{4}} \\
&= \frac{1}{\frac{6}{4}} \\
&= \frac{4}{6} & = \frac{2}{3}
\end{aligned}
$$

Through use of the convergence of infinite series' can variable n be eliminated from the probability, and upon further manipulation it becomes evident that the probability of $generateBiasedBit = 0$ is indeed $\frac{2}{3}$.