Question 1: See the top of the page.

Question 2: Consider a fair coin that has 0 on one side and 1 on the other side. We flip this coin, independently, twice. Define the following random variables:

$$\begin{aligned} X &= \text{the result of the first coin flip,} \\ Y &= \text{the sum of the results of the two coin flips,} \\ Z &= X \cdot Y. \end{aligned}$$

- Determine the distribution functions of X, Y, and Z.
- Are X and Y independent random variables?
- Are X and Z independent random variables?
- Are Y and Z independent random variables?

A **distribution function** maps any real number x to the real number $Pr(X = x)$

Variable X describes *the result of the first coin flip*, before analysing the probabilities, the sample space, S, must first be established ...

$$S = \{0, 1\}$$

For every uniformly random $x \in X$ there is a $\frac{1}{2}$ chance of it being picked.

$$\text{X} = \begin{cases} Pr(X = 0) & = \frac{1}{2} \\ Pr(X = 1) & = \frac{1}{2} \end{cases}$$

Event Y describes variables which are the *sum of the first two coin flips*. To illustrate the probabilities, a new sample space must be defined. This sample space only containing the sum of of the flips.

$$S = \{0, 1, 2\}$$

$$\text{Y} = \begin{cases} Pr(Y = 0) \begin{cases} 0 + 0 \end{cases} & = \frac{1}{4} \\ \\ Pr(Y = 1) \begin{cases} 1 + 0 \\ 0 + 1 \end{cases} & = \frac{2}{4} \\ \\ Pr(Y = 2) \begin{cases} 1 + 1 \end{cases} & = \frac{1}{4} \end{cases}$$

$$Z = \begin{cases} Pr(Z=0) \begin{cases} 0 \times 0 \\ 0 \times 1 \end{cases} & = \frac{3}{5} \\ \\ Pr(Z=1) \begin{cases} 1 \times 1 \end{cases} & = \frac{1}{5} \\ \\ Pr(Z=2) \begin{cases} 1 \times 2 \end{cases} & = \frac{1}{5} \end{cases}$$

Are X and Y independent random variables?
We can determine if this is true by calculating the probabilities of their intersection vs. the product of each of their probabilities. . .

$$Pr(X = x \wedge Y = y) = Pr(X = x) \cdot Pr(Y = y)$$

If an inequivilence can be shown for any $X = x$ and $Y = y$, it proves that the two variables are **dependent**.

Let $X = 0, Y = 0$...
Our sample space will be that of Y's, as it will show all the cases of both $X = x$ and $Y = y$, for any x or y.

$$S = \{(0,0),(0,1),(1,0),(1,1)\}$$

Now we must determine the probability of variable $X = 0$ and $Y = 0$. This can only **happen one way**, The first flip has to be zero, and the second flip has to be 1. Therefore our probability becomes. . .

$$Pr(X = 0 \wedge Y = 0) = \frac{1}{4}$$

The probability of $X = 0$ is just $\frac{1}{2}$, and the probability of $Y = 0$ is $\frac{1}{4}$.

$$\frac{1}{4} \neq \frac{1}{2} \cdot \frac{1}{4}$$

Therefore X and Y are not independent.

Are X and Z independent?

The formula must again be applied. And to prove that it is not independent, I will determine a case where the equality **doesn't** hold...
Let $X = 0$, and $Z = 0$. First we must determine the sample space. This is just the sample space of Z. The probability where $X = 0$ is again $\frac{1}{2}$, and $Z = 0$ will only occur when $X = 0$. $Pr(Z = 0) = \frac{1}{2}$. And since Z will only equal 0, when $X = 0$, then it becomes clear that they're **not actually independent**.

$$Pr(X = 0 \wedge Z = 0) = \frac{1}{2} \neq \frac{1}{2} \times \frac{1}{2}$$

Are Y and Z independent variables?

I will show that it is not when $Y = 2$, and $Z = 2$. Since **Z will only equal 2 when Y is two**, it is easy to see that their relationship is dependent.

$$Pr(Y = 2) = \frac{1}{4}$$

$$Pr(Z = 2) = \frac{|\{1, 2\}|}{4} = \frac{1}{4}$$

$$Pr(Y = 2 \land Z = 2) = \frac{1}{4} \neq \frac{1}{4}^2$$

Question 3: Consider two random variables X and Y. If X and Y are independent random variables, then it can be shown that

$$E(XY) = E(X) \cdot E(Y).$$

In this exercise, you will show that the converse of this statement is, in general, not true.

Let X be the random variable that takes each of the values -1, 0, and 1 with probability 1/3. Let Y be the random variable with value $Y = X^2$.

- Prove that X and Y are not independent.
- Prove that $E(XY) = E(X) \cdot E(Y)$.

It is easy to see that X and Y are dependent because multiple variables $x \in X$ will map to the same value in Y. Since $Y = X^2$, both -1 and 1 will produce the same result in Y.

$$Pr(X = -1 \land Y = 1) = Pr(X = -1) \cdot Pr(Y = 1)$$

$Y = 1$ when X is either -1 or 1. Which means that there are $\frac{2}{3}$ options that map to $Y = 1$. When we restrict it to only $X = -1$, this becomes $\frac{1}{3}\ldots$

$$Pr(X = -1) = \frac{1}{3}$$

$$Pr(Y = 1) = \frac{2}{3}$$

$$Pr(X = -1 \land Y = 1) = \frac{1}{3}$$

$$\frac{1}{3} \neq \frac{1}{3} \cdot \frac{2}{3}$$

$$\frac{1}{3} \neq \frac{2}{9}$$

Therefore it is evident that these variables are **not independent**, with the inequality above as evidence.

Now to show that $E(XY) = E(X) \cdot E(Y)$. $X \times Y$ is actually synonymous to X^3, because variable $Y = X^2$. And X^3 for the values of -1, 0, 1 will be equivalent to just X.

$$E(X^3) = E(X) + E(Y)$$
$$\sum_{w \in X^3} X^3(w) \cdot Pr(w) = (\sum_{w \in X} X(w) \cdot Pr(w)) \cdot E(Y)$$
$$-\frac{1}{3} + \frac{1}{3} = (-\frac{1}{3} + \frac{1}{3}) \cdot E(Y)$$
$$0 = 0$$

Therefore, because the two sets were independent, yet the expected value of XY is equivalent to the sum of the expected value of X and Y that the **converse is generally not true**...

Question 4: Lindsay and Simon want to play a game in which the expected amount of money that each of them wins is equal to zero. After having chosen a number x, the game is played as follows: Lindsay rolls a fair die, independently, three times.

- If none of the three rolls results in 6, then Lindsay pays one dollar to Simon.
- If exactly one of the rolls results in 6, then Simon pays one dollar to Lindsay.
- If exactly two rolls result in 6, then Simon pays two dollars to Lindsay.
- If all three rolls result in 6, then Simon pays x dollars to Lindsay.

Determine the value of x.

The first thing we must do is determine the distribution function. The values would be the amount gained or lost. I'll do it from the perspective of Lindsay *(in terms of gaining or losing money)*.

$$\text{X} : \begin{cases} 1\$: \frac{5}{6}^3 \\ -1\$: 3(\frac{1}{6} \times \frac{5}{6}^2) \\ -2\$: 3(\frac{1}{6^2} \times \frac{5}{6}) \\ -x\$: \frac{1}{6^3} \end{cases}$$

The case where 1 and two dollars goes from Simon to Lindsay, the odds are multiplied by 3 to account for the different dice result permutations that could occur. The expected value is already known, so now we must solve for x...

$$E(x) = \sum_{w \in S} X(w) \cdot Pr(w)$$
$$0 = -(\frac{5}{6})^3 - 3\frac{1}{6} \times (\frac{5}{6})^2 - 6((\frac{1}{6})^2 \times \frac{5}{6}) - x(\frac{1}{6})^3$$
$$x = 20$$

Therefore the value of x will be 20, through algebraic deduction...

Question 5: Let $n \geq 1$ be an integer and consider a uniformly random permutation $a_1, a_2, \ldots, a_n$ of the set $\{1, 2, \ldots, n\}$. Define the random variable X to be the number of indices i for which $1 \leq i < n$ and $a_i < a_{i+1}$.

Determine the expected value $E(X)$ of X. (*Hint:* Use indicator random variables.)

If the set containing the permutations, $a_1 ... a_n$, is truly random, then the probability that the next item is **less than the current is just as good as the probability that a fair coin will turn heads**. The probability for each is just $\frac{1}{2}$. To determine the expected value, each of the probabilities for every index i must be summated, less the last (because there is no next element after the n^{th}).

$$Pr(X_i = 1) = \frac{1}{2}$$

$$X_i = \begin{cases} 1 : a_i < a_{i+1} \\ 0 : \text{Otherwise} \end{cases}$$

It follows from the *Linearity of Expectation* that...

$$\begin{aligned} E(X) &= \sum_{i=1}^{n-1} E(X_i) \\ &= \sum_{i=1}^{n-1} Pr(X_i = 1) \\ &= \frac{n-1}{2} \end{aligned}$$

Therefore, through use of *linearity of expecation, th.6.5.2* that the expected value of $X = \frac{n-1}{2}$.

Question 6: Lindsay Bangs[1] and Simon Pratt[2] visit their favorite pub that has 10 different beers. Both Lindsay and Simon order, independently of each other, a uniformly random subset of 5 beers.

- One of the beers available is *Leo's Early Breakfast IPA*. Determine the probability that this is one of the beers that Lindsay orders.
- Let X be the random variable whose value is the number of beers that are ordered by both Lindsay and Simon. Determine the expected value $E(X)$ of X. (*Hint:* Use indicator random variables.)

The probability that one of the beers available is **Leo's Early Breakfast IPA** is the same probability that any one beer is chosen. In a scenario where we would choose this beer, the first thing we must do is choose it. This leaves 9 different beers, and 4 different choices. This can occur $\binom{9}{4}$ ways, as all permutations need to be considered. Then, we determine the total number of permutations, which is the number of ways you can choose 5 beers from 10.

Thus the probability that this beer, or any specific beer in general becomes ...

$$\frac{\binom{9}{4}}{\binom{10}{5}} = \frac{1}{2}$$

[1]President of the Carleton Computer Science Society (2014–2015)
[2]President of the Carleton Computer Science Society (2013–2014)

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To determine if both drinkers have chosen a beer, it must be true that Lindsay and Simon have both had this beer. Yielding the following probability that a given beer has been drunk or not yet. Since the two subsets are **independent of each other**, their product becomes the probability of both.

$$Pr(X_i = 1) = \frac{\binom{9}{4}}{\binom{10}{5}}^2 = \frac{1}{4}$$

Using random indicator variables, we can count the choices we want (a beer to be chosen by both parties), and ignore those that have been chosen by either Lindsay or Simon.

$$X_i : \begin{cases} 1 : \text{"Beer has been chosen by both Lindsay and Simon"} \\ 0 : \text{"Otherwise"} \end{cases}$$

To determine the expected value, the expected value for each specific beer being chosen or not by either parties must be accounted for. Thus, using *linearity of expectation* the expected value of the number of beers that they both drink can be determined as such...

$$\begin{aligned} E(X) &= \sum_{i=1}^{10} E(X_i) \\ &= \sum_{i=1}^{10} Pr(X_i = 1) \\ &= 10 \cdot \frac{1}{4} \end{aligned}$$

Question 7: Lindsay and Simon have discovered a new pub that has n different beers $B_1, B_2, \ldots, B_n$, where $n \geq 1$ is an integer. They want to try all different beers in this pub and agree on the following approach: During a period of n days, they visit the pub every day. On each day, they drink one of the beers. Lindsay drinks the beers in order, i.e., on the i-th day, she drinks beer B_i. Simon takes a uniformly random permutation $a_1, a_2, \ldots, a_n$ of the set $\{1, 2, \ldots, n\}$ and drinks beer B_{a_i} on the i-th day.

Let X be the random variable whose value is the number of days during which Lindsay and Simon drink the same beer. Determine the expected value $E(X)$ of X. (*Hint:* Use indicator random variables.)

$$X = \text{"number of days when Lindsay and Simon drink the same beer"}$$

$$X_i = \begin{cases} 1 : \text{"Simon drinks beer 'i' on the i'th day"} \\ 0 : \text{"otherwise"} \end{cases}$$

The probability of this event can be expressed as $\frac{1}{n}$, as he will drink one of n beers for every day i of all n days.

Using linearity of expectation, the expected value can be determined as the sum of the instances where Simon drinks the i^{th} beer, on that day (along with Lindsay).

$$\begin{aligned} E(X) &= \sum_{i=1}^{n} E(X_i) \\ &= \sum_{i=1}^{n} Pr(X_i = 1) \\ &= n \cdot \frac{1}{n} \\ &= 1 \end{aligned}$$

It is seen that the expected value for the number of days where they will drink the same beer is just 1, as deduced by the application of the linearity of expectation with use of indicator random variables.

Question 8: Let $G = (V, E)$ be a graph with vertex set V and edge set E. A subset I of V is called an *independent set* if for any two distinct vertices u and v in I, (u, v) is not an edge in E. For example, in the following graph, $I = \{a, e, i\}$ is an independent set.

Let n and m denote the number of vertices and edges in G, respectively, and assume that $m \geq n/2$. This exercise will lead you through a proof of the fact that G contains an independent set of size at least $n^2/(4m)$.

Consider the following algorithm, in which all random choices made are mutually independent:

Algorithm (G)

> **Step 1:** Set $H = G$.
>
> **Step 2:** Let $d = 2m/n$. For each vertex v of H, with probability $1 - 1/d$, delete the vertex v, together with its incident edges, from H.
>
> **Step 3:** As long as the graph H contains edges, do the following: Pick an arbitrary edge (u, v) in H, and remove the vertex u, together with its incident edges, from H.
>
> **Step 4:** Let I be the vertex set of the graph H. Return I.

- Argue that the set I that is returned by this algorithm is an independent set in G.
- Let X and Y be the random variables whose values are the number of vertices and edges in the graph H after Step 2, respectively. Prove that

$$E(X) = n^2/(2m)$$

and

$$E(Y) = n^2/(4m).$$

- Let Z be the random variable whose value is the size of the independent set I that is returned by the algorithm. Argue that

$$Z \geq X - Y.$$

- Prove that

$$E(Z) \geq n^2/(4m).$$

- Argue that this implies that the graph G contains an independent set of size at least $n^2/(4m)$.

The I that will be returned is the vertex set of the graph H processed from our graph G after the removals have occurred. It's guaranteed that this algorithm will return a set of independent vertices's, as it will first remove **all vertices's along with all of it's incident edges, with a given probability**. After this process, it will process the **rest of the edges** and eliminate one of the two edges, and pop the edge, and keep evaluating the graph while *there are still edges...*

This guarantees that there will be no connected vertexes left, as if there is, one of the two will be removed, and will mark one to be a part of the result I.

Variable X is described to be the **number of vertices after Step 2)**. To reiterate, Step 2) is where the algorithm would *pop* off a vertex, along with its adjacent edges. The randomness of this process allows for a randomly outputted **independent set**. It can be shown that ...

$$E(X) = \frac{n^2}{2m}$$

The vertices that *survive* Step 2) are not guaranteed to be independent. Since there exists the restriction that $m \geq \frac{n}{2}$ it can be deduced that **the largest independent set will occur the less edges there are**. This can therefore be identified as our **upper bound** on the number of vertices left after Step 2 as been processed. All vertices can be removed with a probability of $1 - \frac{1}{d}$, thus the probability that each vertex hasn't been removed is just $\frac{1}{d}$. This random variable will be described as $E(X_i)$, denoting the aforementioned.

$$\begin{aligned} E(X) &= \sum_{i=1}^{n} E(x_i) \\ &= n \cdot \frac{1}{d} \\ &= n \cdot \frac{n}{2m} \\ &= \frac{n^2}{2m} \end{aligned}$$

The expected value of Y describes the **number of edges that remain after Step 2**. Since an edge will only remain if both of it's vertices, (u, v), are not removed, this random variable can be expressed in terms of the remaining vertices. Each edge has **two vertices**, and both must not be removed. Therefore the expected number of edges is just $\frac{1}{2d}$ for all n vertices.

$$\begin{aligned} E(Y) &= \sum_{i=1}^{n} E(y_i) \\ &= n \cdot \frac{1}{2d} \\ &= n \cdot \frac{n}{4m} \\ &= \frac{n^2}{4m} \end{aligned}$$

Since we only account for one edge per two vertices, the expected value of Y can be seen as a **lower bound**.
It can be shown that the **size of Z** is greater or equal to $X - Y$ simply because of the definition of Z. X is the number of vertices left after step 2, and Y the number of edges. During step 3, **at least one edge** is removed for every vertex removed. There are Y edges left, this means that the iterations that occur in step 3 will happen **at most Y times**. Meaning that at worst case, $X - Y \equiv Z$, but in any case where more than one edge is removed per vertex, then $X - Y > Z$.

$$\begin{aligned} E(Z) &\geq E(X) - E(Y) \\ &\geq \frac{n^2}{2m} - \frac{1}{2}(\frac{n^2}{2m}) \\ &\geq \frac{1}{2}(\frac{n^2}{2m}) \\ &\geq \frac{n^2}{4m} \end{aligned}$$

The result of this algorithm must be an independent set size of at least $\frac{n^2}{4m}$ as it is the expected value, or the average. If the size of the set was to be smaller, it would contradict the determined expected value. Therefore it must be true that graph G must contain an independent set of at least size $\frac{n^2}{4m}$.