Regular Languages

CS F331 Programming Languages
CSCE A331 Programming Language Concepts
Lecture Slides
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Review
Introduction to Syntax & Semantics

"**Dynamic**" means *at runtime*.

"**Static**" means *before runtime*.

Syntax is the correct *structure* of code.

Semantics is the *meaning* of code.

In programming languages like C++ and Java, which have static type checking, type errors lie in a gray area between syntax and semantics. We classify these under **static semantics**. What code does when executed involves **dynamic semantics**.

Coming Up

- How the syntax of a programming language is specified.
- How such specifications are used.
- Write a **lexer** and a **parser**; the latter checks syntactic correctness.
- Later, a brief study of semantics.

Formal Languages & Grammars — Formal Languages [1/2]

A (**formal**) **language** is a *set of strings*.

Not the same as a programming language!

For now, we write strings without quotes (for example, *abc*). We represent the empty string with a lower-case Greek epsilon (ε).

Example of a language over {0, 1}:

{ε, 01, 0101, 010101, 01010101, …}

Important examples of formal languages:

- The set of all lexemes in some category, for some programming language (e.g., the set of all legal C++ identifiers).
- The set of all syntactically correct programs, in some programming language (e.g., the set of all syntactically correct Lua programs).

Formal Languages & Grammars — Formal Languages [2/2]

Two ways to describe a formal language.

- With a **generator**: something that can produce the strings in a formal language—all of them, and nothing else.
- With a **recognizer**: a way of determining whether a given string lies in the formal language.

Generally:

- Generators are easier to construct.
- Recognizers are more useful.

It is common to begin with a generator and then construct a recognizer based on it.

Formal Languages & Grammars — Grammars [1/4]

A (**phrase-structure**) **grammar** is a kind of language generator.

Needed

- A collection of **terminal symbols**. This is our alphabet.
- A collection of **nonterminal symbols**. These are like variables that eventually turn into something else. One nonterminal symbol is the **start symbol**.

Conventions, for now:

- Lower-case letters are terminal symbols (*a b x*)
- Upper-case letters are nonterminal symbols (*C Q S*)
- *S* is the start symbol.

Formal Languages & Grammars — Grammars [2/4]

A **grammar** is a list of one or more **productions**. A *production* is a rule for altering strings by substituting one substring for another. The strings are made of terminal and nonterminal symbols.

Here is a grammar with three productions.

$S \rightarrow yS$

$S \rightarrow x$

$S \rightarrow \varepsilon$

An important application of grammars is specifying programming-language syntax. Since the late 1970s, nearly all programming languages have used a grammar for their a syntax specification.

Formal Languages & Grammars — Grammars [3/4]

Grammar

1. $S \rightarrow yS$
2. $S \rightarrow x$
3. $S \rightarrow \varepsilon$

The numbers and underlining are annotations that I find helpful. Strictly speaking, they are not part of the derivation.

Derivation of *yyy*

$\underline{S}$
1
$y\underline{S}$
1
$yy\underline{S}$
1
$yyy\underline{S}$
3
yyy

No "ε" appears here.

Using a grammar:

- Begin with the start symbol.
- Repeat:
  - Apply a production, replacing the left-hand side of the production (which must be a contiguous collection of symbols in the current string) with the right-hand side.
- We can stop only when there are no more nonterminals.

The result is a **derivation** of the final string.

Formal Languages & Grammars — Grammars [4/4]

Grammar

$S \rightarrow yS$

$S \rightarrow x$

$S \rightarrow \varepsilon$

The language **generated** by a grammar consists of all strings for which there is a derivation.

- So *yyy* lies in the language generated by the above grammar.

Avoid saying, "any number of …".

Q. What language does this grammar generate?

A. The set of all strings that consist of zero or more *y*'s, followed by an optional *x*.

$\{\varepsilon, y, yy, yyy, \ldots, x, yx, yyx, yyyx, \ldots\}$

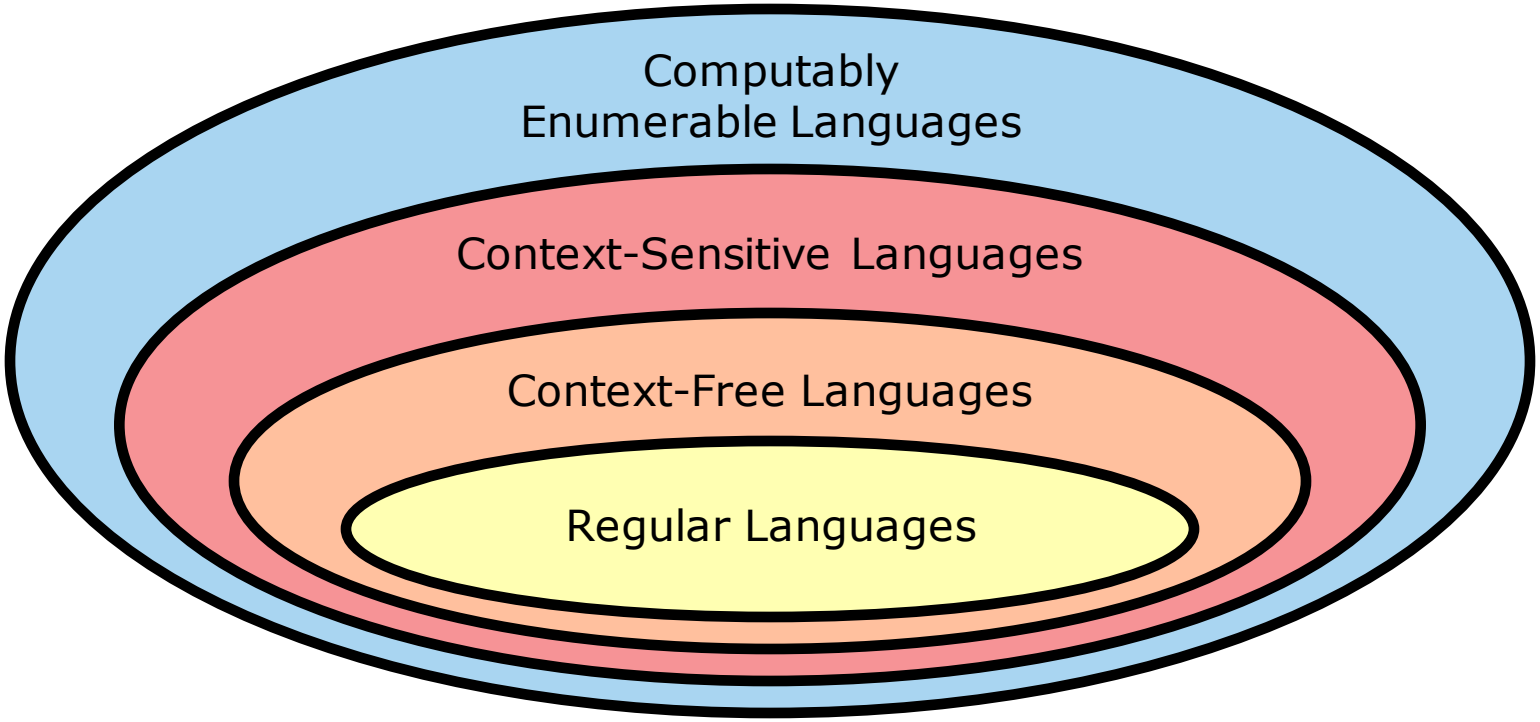
Formal Languages & Grammars — The Chomsky Hierarchy [1/2]

The **Chomsky Hierarchy** classifies languages according to the kinds of grammars that can generate them.

Language Category		**Generator**	**Recognizer**
Number	**Name**		
Type 3	Regular	Grammar in which each production has one of the following forms. • $A \rightarrow \varepsilon$ • $A \rightarrow b$ • $A \rightarrow bC$ Another kind of generator: **regular expressions** (covered later).	Deterministic Finite Automaton Think: Program that uses a small, fixed amount of memory.
Type 2	Context-Free	Grammar in which the left-hand side of each production consists of a single nonterminal. • $A \rightarrow$ [anything]	Nondeterministic Push-Down Automaton Think: Finite Automaton + Stack (roughly).
Type 1	Context-Sensitive	*Don't worry about it.*	*Don't worry about it.*
Type 0	Computably Enumerable	Grammar (no restrictions).	Turing Machine Think: Computer Program

Formal Languages & Grammars — The Chomsky Hierarchy [2/2]

Each category of languages in the Chomsky Hierarchy is contained in the next. So every regular language is context-free, etc.



Our next topics concern the smallest two categories in the Chomsky Hierarchy.

Regular Languages
Introduction

We now look closer at the smallest of the four categories of languages in the Chomsky Hierarchy: the regular languages.

Regular languages have two important applications.

- They are heavily used in text search/replace.
- In most programming languages, the set of all lexemes (words, roughly) of a particular kind forms a regular language. Thus we make use of regular languages in the early stages of compilation, when we break up a program into lexemes—a process called **lexical analysis**, or **lexing**.

Regular Grammars & Languages — Definitions

A **regular grammar** is a grammar, each of whose productions looks like one of the following.

$$A \to \varepsilon \qquad A \to b \qquad A \to bC$$

That is, the left-hand side of each production is a single nonterminal, while the right-hand side is one of:

- the empty string
- a single terminal, or
- a single terminal followed by a single nonterminal—which may be the same as the left-hand side.

A **regular language** is a language that is generated by some regular grammar.

Regular Grammars & Languages — Examples [1/3]

Here is an example of a regular grammar.

$S \rightarrow \varepsilon$
$S \rightarrow t$
$S \rightarrow xB$
$B \rightarrow yS$

Q. What language does this grammar generate?

A. The set of all strings that consist of zero or more concatenated copies of xy, followed by an optional t.

$\{\varepsilon, xy, xyxy, xyxyxy, \ldots, t, xyt, xyxyt, xyxyxyt, \ldots\}$

So this language is a regular language.

Regular Grammars & Languages — Examples [2/3]

Here is another grammar. This is *not* a regular grammar.

$S \rightarrow A$

$S \rightarrow At$

$A \rightarrow Axy$

$A \rightarrow \varepsilon$

Q. What language does this grammar generate?

A. The set of all strings that consist of zero or more concatenated copies of xy, followed by an optional t.

$\{\varepsilon, xy, xyxy, xyxyxy, \ldots, t, xyt, xyxyt, xyxyxyt, \ldots\}$

Q. Is this a regular language?

A. Yes! Because it is generated by a regular grammar: the one on the previous slide.

Regular Grammars & Languages — Examples [3/3]

There are languages that are not regular.

For example, this grammar …

$S \rightarrow aSa$

$S \rightarrow b$

… generates the following language.

$$\{b, aba, aabaa, aaabaaa, \ldots\} = \{ a^k b a^k \mid k \geq 0 \}$$

There is *no* regular grammar that generates this language. It is not a regular language.

I am not saying this is obvious; but it is true. (Proving that a language is not regular is beyond the scope of this class.)

Finite Automata — Basics [1/4]

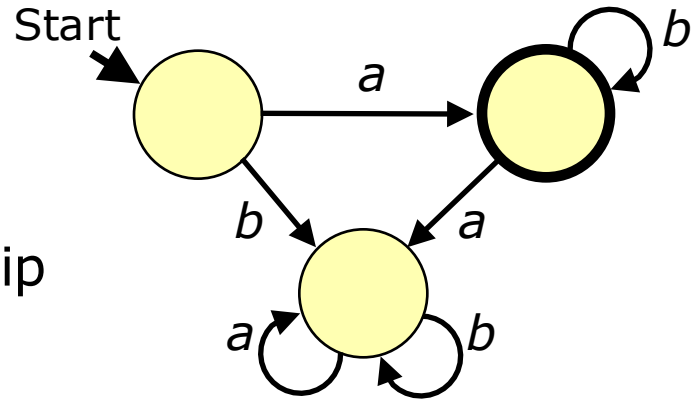
A **deterministic finite automaton** (Latin plural "**automata**"), or **DFA**, is a kind of recognizer for regular languages.

A DFA consists of a finite collection of **states**, with **transitions** between these states.

- One state is the **start state**.
- Some states may be **accepting states**.
- Each transition begins at a state, ends at a state, and is associated with a character in the alphabet—that is, some terminal symbol.
- For each character, each state has *exactly one* transition leaving it that is associated with that character.

Here is a diagram of a DFA with three states, one of which is accepting.

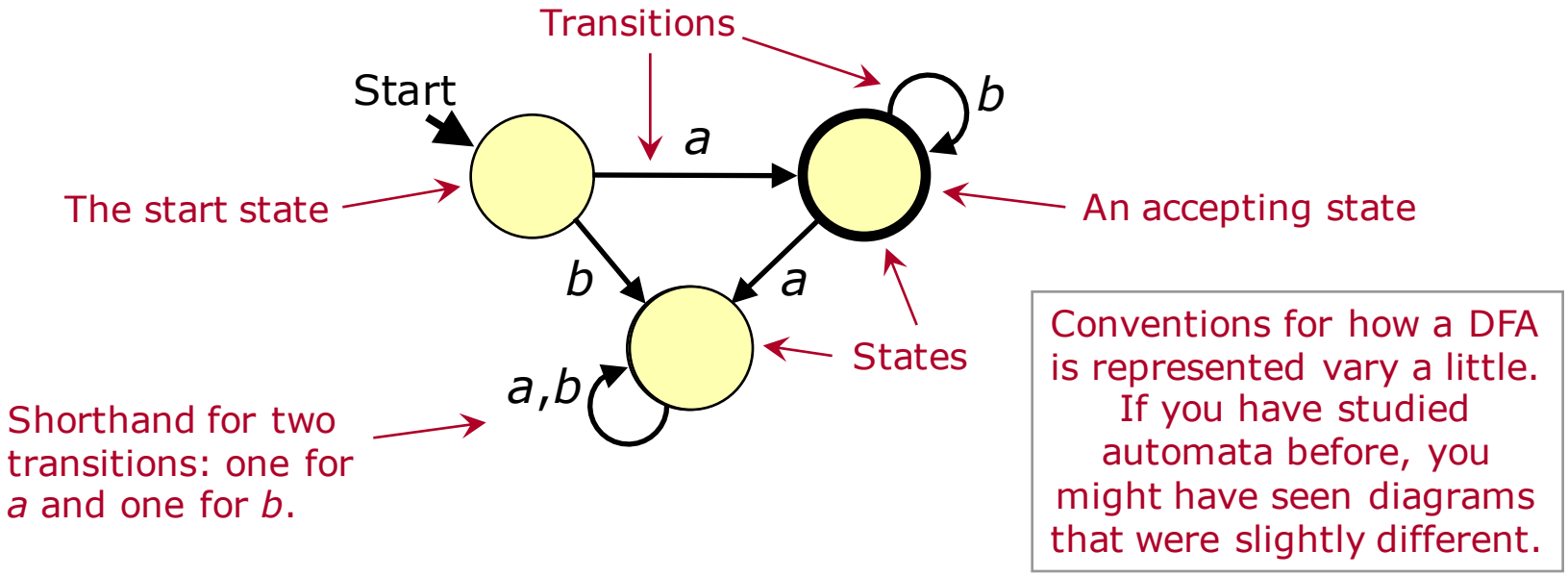
On the next slide we make the relationship between a DFA and its diagram more precise.



Finite Automata — Basics [2/4]

To make a diagram of a DFA, we draw a circle (or other enclosed shape) for each state. For each transition, we draw an arrow from the state it begins at to the state it ends at, labeling the arrow with the transition's character.

We make accepting states bold, and we draw an arrow labeled "Start" to the start state.



Finite Automata — Basics [3/4]

We use a DFA as a recognizer as follows.

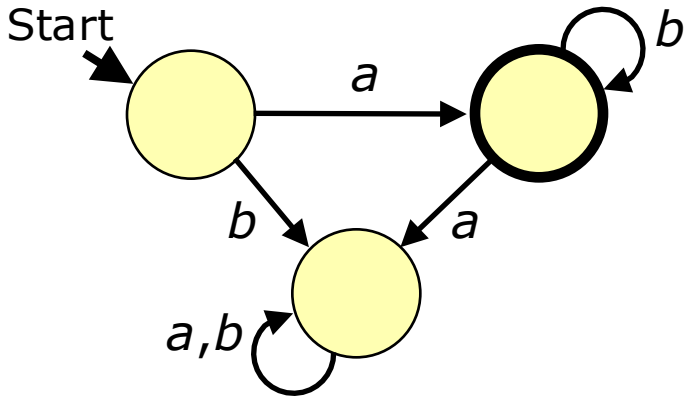
- We are always in one of the states, beginning with the start state.
- We proceed in steps. At each, we read a character from the input and follow the transition beginning at the current state and labeled with the character that was read. Where this ends is our new state.
- If, when we reach the end of the input, we are in an accepting state, then we **accept** the input.

The set of all inputs that are accepted is the language **recognized** by the DFA.

Q. What language does this DFA recognize?

A. The set of all strings that consist of an *a* followed by zero or more *b*'s.

{*a*, *ab*, *abb*, *abbb*, *abbbb*, *abbbbb*, …}



Finite Automata — Basics [4/4]

Fact. The languages that are recognized by DFAs are precisely the regular languages.

That is:

- For each DFA, the language it recognizes is a regular language.
- For each regular language, there is a DFA that recognizes it.

A DFA is a kind of **state machine**: it has a state, and it transitions to a new state based, in part, on its current state.

We will use the state-machine idea again later in the semester when we write code to do lexical analysis. We will keep track of a state and change that state based on characters read.

Finite Automata — DFAs & Regular Grammars [1/4]

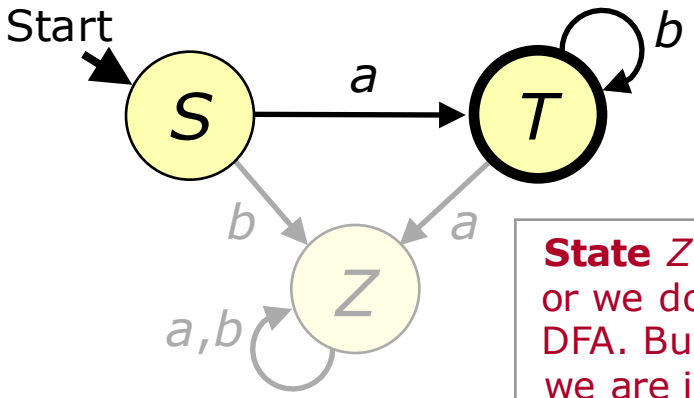
Given a DFA, we can easily transform it into a regular grammar. We can do this in such a way that that the grammar generates the same language that the automaton recognizes.

Procedure

- Make one nonterminal symbol for each state. The nonterminal corresponding to the start state is the start symbol.
- For each transition, make a production. A transition from state A to state B labeled with character x gives the production $A \rightarrow xB$.
- For each accepting state, make a production whose right-hand side is the empty string. If state A is accepting, then write $A \rightarrow \varepsilon$.

- For each state, 1 nonterminal. Start state gives start symbol.
- For each transition, 1 production. *A* to *B* by *x* gives $A \rightarrow xB$.
- For each accepting state, 1 production. *A* accepting gives $A \rightarrow \varepsilon$.

DFA with states labeled



State *Z* must exist, or we do not have a DFA. But note: once we are in *Z*, we can never accept.

Regular Grammar

$S \rightarrow aT$

$S \rightarrow bZ$

$T \rightarrow aZ$

$T \rightarrow bT$

$Z \rightarrow aZ$

$Z \rightarrow bZ$

$T \rightarrow \varepsilon$

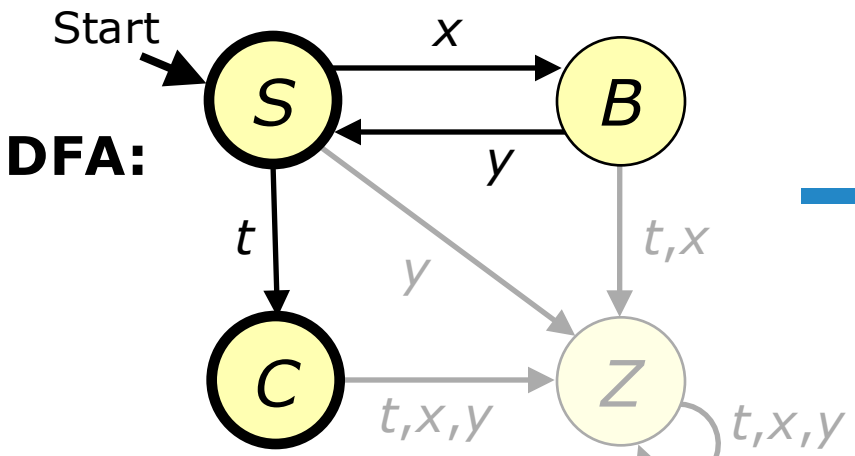
Productions involving *Z* can never be used in a derivation, so we may ignore them.

Language Recognized/Generated

{*a*, *ab*, *abb*, *abbb*, *abbbb*, *abbbbb*, …}

Another Example

DFA:



As before, state Z must exist, but we do not really use it.

Regular Grammar:

$S \rightarrow tC$
$S \rightarrow xB$
$S \rightarrow yZ$
$B \rightarrow tZ$
$B \rightarrow xZ$
$B \rightarrow yS$
$C \rightarrow tZ$
$C \rightarrow xZ$
$C \rightarrow yZ$
$Z \rightarrow tZ$
$Z \rightarrow xZ$
$Z \rightarrow yZ$
$S \rightarrow \varepsilon$
$C \rightarrow \varepsilon$

Once again, productions involving *Z* can never be used in a derivation, so we may ignore them.

Language Recognized/Generated:

{ε, *xy*, *xyxy*, *xyxyxy*, …,
t, *xyt*, *xyxyt*, *xyxyxyt*, …}

Finite Automata — DFAs & Regular Grammars [4/4]

We can also do the transformation the other way: begin with a regular grammar, and produce a DFA that recognizes the same language the grammar generates. However, the process is more complex, and we will not cover it.

You may stop reading this slide here.

For those with some knowledge of automata:

- The complexity in the Regular Grammar → DFA process is due to the rule that, for each character, each state in a DFA has *exactly one* transition leaving it that is associated with that character.
- Transforming a regular grammar into an automaton in the obvious way can violate that rule, giving an NFA, not a DFA ("N" for "nondeterministic").
- An NFA can be transformed into a DFA, but the result can have a huge number of states. An n-state NFA gives a DFA with 2^n states.
- Typically, there are unnecessary states, which can be eliminated.

Regular Expressions — Introduction

We wish to define a kind of generator called a *regular expression*. We will cover both their syntax and their semantics.

Before we do this, let us consider a kind of expression that all of us are familiar with: the *arithmetic expression*.

As practice, we will describe the syntax and semantics of arithmetic expressions, using informal methods. Afterward, we will describe regular expressions in a similar way.

Regular Expressions — Arithmetic Expressions [1/4]

An **arithmetic expression** is an expression involving numbers, numeric variables, and arithmetic operators (+ – * /) as usual.

We are *not* describing regular expressions here!

Here is an example of an arithmetic expression.

$$34*(3-n)+(5.6/g+3)$$

We describe the syntax and semantics of arithmetic expressions.

- **Syntax** refers to correct structure. Knowing the syntax of arithmetic expressions allows us to say whether some given string is a correctly written arithmetic expression, and, if it is, how it is put together.
- **Semantics** refers to meaning. For arithmetic expressions, this would specify an expression's numerical value.

Regular Languages
Regular Expressions — Arithmetic Expressions [2/4]

We can specify the syntax of arithmetic expressions by showing how to build them up from small pieces.

We are *not* describing regular expressions here!

First we list the pieces.

- A *numeric literal* is an arithmetic expression: 26.5.
- A *variable* is an arithmetic expression: *x*.

Next we list the ways to build new arithmetic expressions out of existing ones. If *A* and *B* are arithmetic expressions, then so are each of the following.

- *–A*
- *A***B*
- *A*/*B*
- *A*+*B*
- *A*–*B*

Regular Expressions — Arithmetic Expressions [3/4]

The list, again:

- $-A$
- $A*B$
- A/B
- $A+B$
- $A-B$

We are *not* describing regular expressions here!

The above goes from highest precedence (unary "–") to lowest (binary "–"). Unary minus is right-associative, while all four binary operators are left-associative.

- **Left-associative** means, for example, that 1–2–3 is the same as (1–2)–3, not 1–(2–3).

If we want to override these precedence & associativity rules, then we can use parentheses for grouping. If particular, if A is an arithmetic expression, then so is the following.

- (A)

Regular Expressions — Arithmetic Expressions [4/4]

We have defined the syntax of arithmetic expressions. Using the rules covered, we can look at some text and determine whether the text is actually an arithmetic expression. We can also figure out the structure of the expression: how it is put together.

We are *not* describing regular expressions here!

However, we do *not* have enough information to find the value of an arithmetic expression. The rules covered so far do not tell us what such an expression means: its semantics.

We can specify the semantics of arithmetic expressions based on our description of the syntax.

- The value of a numeric literal is its numeric value.
- The value of a variable is its numeric value.
- The value of –*A* is –1 times the value of *A*.
- The value of *A***B* is the product of the value of *A* and the value of *B*.
- Etc.

Regular Expressions — Syntax [1/2]

Now we specify the syntax of **regular expressions**. As we did with arithmetic expressions, we do this by showing how to build them up from small pieces.

First we list the pieces.

- A *single character* is a regular expression: *a*.
- The *empty string* is a regular expression: ε.

Next we list the ways to build new regular expressions out of existing ones. If *A* and *B* are regular expressions, then so are each of the following.

- *A**
- *AB*
- *A*|*B*

Regular Expressions — Syntax [2/2]

The list, again:

- *A**
- *AB*
- *A*|*B*

The above goes from high to low precedence. All are left-associative. Parentheses can be used for grouping, to override precedence & associativity. In particular, if *A* is a regular expression, then so is the following.

- (*A*)

So, for example, here is a regular expression: (*a*|*x*)**cb*

We can now determine whether a given string is a regular expression, and find its structure, if so. Next we discuss what regular expressions mean: their semantics.

Regular Expressions — Semantics [1/2]

Regular expressions are a kind of language generator. A regular expression is said to **match** certain strings. The language generated by the regular expression consists of all strings that it matches.

Once again, we can describe the semantics based on our description of the syntax.

Here are the rules for matching the pieces.

- A single character matches itself, and nothing else.
- The empty string matches itself, and nothing else.

Regular Expressions — Semantics [2/2]

Now suppose that *A* and *B* are regular expressions.

- *A** matches the concatenation of zero of more strings, each of which is matched by *A*.
  - Note that *A** matches the empty string, no matter what *A* is.
- *AB* matches the concatenation of any string matched by *A* and any string matched by *B*.
- *A*|*B* matches all strings matched by *A* and also all strings matched by *B*.
- (*A*) matches the same strings that are matched by *A*.

The asterisk (*), used as above, is called the **Kleene Star**, after Stephen Kleene, a 20th century mathematician who worked in mathematical logic. "Kleene" is, somewhat mysteriously, pronounced KLAY-nee.

Regular Expressions — Language Generated [1/4]

Again, the language generated by a regular expression consists of all strings that it matches.

Fact. The languages that are generated by regular expressions are precisely the regular languages.

That is:

- For each regular expression, the language it generates is a regular language.
- For each regular language, there is regular expression that generates it.

Regular Expressions — Language Generated [2/4]

Consider the regular expression mentioned previously:

$$(a|x)^*cb$$

What language does this regular expression generate?

Each of the expressions "*a*" and "*x*" matches itself.

The expression "*a*|*x*" matches two strings: "*a*" and "*x*".

So the expression "(*a*|*x*)*" matches any string consisting of nothing but *a*'s and *x*'s. For example, it matches "*aaaxaxaaaxxx*". It also matches the empty string.

We conclude that the expression "(*a*|*x*)**cb*" matches zero or more *a*'s and/or *x*'s, followed by *c*, followed by *b*. For example, it matches *cb*, *acb*, *xcb*, *aacb*, *axcb*, *xacb*, *xxcb*, *aaacb*, *aaxcb*, etc.

Regular Expressions — Language Generated [3/4]

Watch out for precedence! In particular, the Kleene star is a high-precedence operator.

For example, as we have said, this regular expression

$$(a|x)^*$$

matches any string consisting of nothing but *a*'s and *x*'s.

On the other hand, the following two regular expressions

$$a|x^*$$
$$a|(x^*)$$

(which are essentially the same) match the string "*a*", along with any string consisting of zero or more *x*'s: ε, *x*, *xx*, *xxx*, etc.

Regular Expressions — Language Generated [4/4]

Recall the regular language for which we covered several grammars and a DFA:

$$\{\varepsilon, xy, xyxy, xyxyxy, \ldots, t, xyt, xyxyt, xyxyxyt, \ldots\}$$

Q. What regular expression generates the above language?

A. $(xy)^*(\varepsilon|t)$

Regular Languages
TO BE CONTINUED …

Regular Languages will be continued next time.