Master de sciences et technologie 1 | MU4MA029
Mention : mathématiques et applications | 2021-2022

TP 5 : Mass and stiffness

In accordance with previous tutorial sheets, a mesh shall be represented by two tables : `vtx` that models the vertices, and `elt` that models the elements (triangles in 2D, edges in 1D).

Exercise 1 : mass matrix

Question 1.1 Write a function `Mloc` taking as input argument a table of vertices `vtx`, as well as a table `e` of 3 integers representing a triangle. Denoting $|\tau|$ the surface area of the triangle triangle represented by `e`, the function `Mloc` must return as output the elementary 2D mass matrix defined by :

$$\mathrm{M}^{\mathrm{loc}} = \frac{|\tau|}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

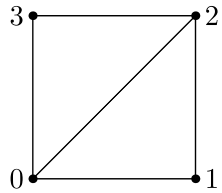
Question 1.2 Modify the function `Mloc` so that it can take as input argument a table of vertices `vtx` as well as a table `e` of 2 integers representing an edge. In this case, if $|\gamma|$ refers to the lengthof the edge represented by `e`, the function `Mloc` must return as output an elementary 1D mass matrix defined by

$$\mathrm{M}^{\mathrm{loc}} = \frac{|\gamma|}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Question 1.3 Write a function `Mass` taking as input two tables (`vtx`, `elt`) and returning as output a variable of type `coo_matrix` modeling the mass matrix associated to the mesh, see the module `scipy.sparse`. This function shall work properly for a 2D triangular mesh as well as for a 1D edge based contour mesh.

Question 1.4 We consider the mesh below. Compute the mass matrix of this triangulation by means of your function `Mass`. Compute by hand the explicit expression of this very same matrix. Check that both results coincide. Do the same calculation for the boundary mesh.

```
vtx =
[[0., 0.]
 [1., 0.]
 [1., 1.]
 [0., 1.]]
```



Question 1.5 Test the assembly of your mass matrix by computing the area of the mesh `maillage6.msh` by means of the mass matrix. Do the same verification for the boundary, computing the length of the boundary of this mesh.

Exercise 2 : stiffness matrix

Question 2.1 Go back to questions 1.1 and 1.2 and, this time, write a function `Kloc` with the same input arguments avec as `Mloc` but returning as output argument an elementary stiffness matrix $\mathrm{K}^{\mathrm{loc}} = (\mathrm{K}^{\mathrm{loc}}_{j,k})_{j,k=1,\dots,d+1}$. This routine should be functional for 2D triangular meshes. We recall that, on a triangle τ, this matrix is defined by

$$\begin{aligned}\mathrm{K}^{\mathrm{loc}}_{j,k} &= \int_{\tau} (\nabla\varphi_j)^\top (\nabla\varphi_k)\, d\boldsymbol{x} \\ &= (\nabla\varphi_j)^\top (\nabla\varphi_k)\, |\tau|\end{aligned}$$

where the φ_j are shape functions whose gradient is constant vector field (since the φ_j are affine functions).

Question 2.2 Write a function `Rig` taking as input the two tables (`vtx`, `elt`) and returning as output a variable of type `coo_matrix` modeling the stiffness matrix associated to the corresponding mesh. This function should be functional in 2D.

Question 2.3 We come back to the mesh with two elements from Question 1.4. Compute the stiffness matrix over this mesh by means of the function `Rig`. Make the same computation by hand. Check that both results coincide.

Question 2.4 We consider the mesh file `maillage6.msh`. We denote Ω the computational domain and K the associated stiffness matrix assembled by means of the function `Rig`. We consider two functions $u_k(\boldsymbol{x}) = \boldsymbol{\alpha}_k^\top \boldsymbol{x} + \beta_k, k = 1, 2$ where $\boldsymbol{\alpha}_k \in \mathbb{R}^2, \beta_k \in \mathbb{R}$ are randomly generated. We denote $\mathrm{U}_k = (u_k(\boldsymbol{x}_j))_{j=1,\dots,\mathrm{N}}$ the nodal value vectors (where $\mathrm{N} =$ the number of vertices in the mesh). Prove theoretically that we have following the identity

$$\mathrm{U}_1^\top \cdot \mathrm{K} \cdot \mathrm{U}_2 = \boldsymbol{\alpha}_1^\top \boldsymbol{\alpha}_2\, |\Omega|$$

and check that this relation is indeed satisfied with your code for each random draw of the coefficients.