TP 1 : Reminders on Python

Python/Numpy shall be used to answer the questions. Computations by hand are not required.

Exercice 1 : vectors, matrices

Question 1.1 Consider the vectors $u = (1, 2, 3, 4)^T$, $v = (-1, 0, 1, 2)^T$ et $w = (2, -2, 1, 0)^T$. Compute uv^T, $v^T w$, $\|u\|_2$, $\|v\|_1$, $\|u - v\|_\infty$.

Question 1.2 For a given integer n, define the following matrices of $\mathbb{R}^{n\times n}$: the matrix full of zeros, the identity matrix, the matrix full of 1, the diagonal matrix where the diagonal corresponds to a random vector of $\mathbb{R}^n$.

Question 1.3 Set

$$A := \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \quad B := \begin{pmatrix} 0 & 1 & 2 & 3 \\ -1 & 0 & 1 & 2 \\ -2 & -1 & 0 & 1 \\ -3 & -2 & -1 & 0 \end{pmatrix}.$$

1. Compute the determinant of each of thoses two matrices, then compute its spectrum.
2. Compute the determinant of the following block-wise defined matrix :

$$C = \begin{pmatrix} A & B^T \\ B & A \end{pmatrix}.$$

Question 1.4

1. Write a function returning a matrix X of dimension $n \times n$ with coefficients $X_{i,j} = 2^{i-j}$.
2. Write a function returning a matrix X of dimension $n \times n$ with coefficients $X_{i,j} = 1/(i + j - 1)$.

Question 1.5 Consider the matrix A of Question 1.3 above. For a given vector u_0 of $\mathbb{R}^4$, we build (as far as possible!) the sequence of vectors $(u_n)_n$ and the sequence of real numbers $(\alpha_n)_n$ through the formula

$$\alpha_n = \|Au_n\|_2, \quad u_{n+1} = \frac{1}{\alpha_n} Au_n.$$

Plot the first 20 values of α_n. Comment. The initial guess shall be taken as $u_0 = (1, 1, 1, 1)^T$, and then $u_0 = (0, 1, 2, 3)^T$ and finally a u_0 generated randomly.

Exercice 2 : graphics

1. Write a function that, for a given time $t \geq 0$, plots the function $f_t : [0, 2\pi] \to \mathbb{R}$ defined by $f_t(x) := \sin(x - t)$.
2. Plots severalof those functions f_t for $t \in [0, 4]$ on the same figure.

Exercice 3 : eigenvalues and eigenvectors

For $n > 0$, set $h = 1/(n + 1)$ and define $A_h \in \mathbb{R}^{n\times n}$ by

$$A_h := \frac{1}{h^2} \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{bmatrix}$$

1. Write a function taking an integer n as input, and returning the matrix A_h as output.
2. It can be proved that the eigenvalues of the matrices A_h are given by $\lambda_{h,k} = (2/h^2)(1 - \cos(k\pi h)), k = 1 \dots n$. Verify this formula by computing the eigenvalues of the matrices A_h using Numpy.
3. Plot in log-log scale the quadratic condition number of the matrix A_h versus n. Comment.

Exercice 4 : linear systems

1. We keep the notations of the previous exercise, and we consider the vector $b = (1, \dots, 1)^T$. Solve the linear system $A_h u = b$. Do this computation for the values $n \in \{5, 10, 15, 20\}$. Plot the solution taking the following points $x_{k,h} = kh, k = 1 \dots n$ as abscissa.
2. Same question taking for the right hand side $b = (f(x_{1,h}), \dots, f(x_{n,h}))^T$ with $f(x) := \sin(p\pi x)$ for a certain integer p chosen by yourself.

Exercice 5 : finite differences

Let f be a function (at least continuous) defined by $\mathbb{R}$. For a given $h > 0$, we define the functions $d_h^+ f$, $d_h^- f$, $d_h^0 f$:

$$d_h^+ f(x) = \frac{f(x+h) - f(x)}{h}, d_h^- f(x) = \frac{f(x) - f(x-h)}{h}, d_h^0 f(x) = \frac{f(x+h) - f(x-h)}{2h}.$$

These are finite difference approcimations (respectively "forward", "backward" and "centered") of the function f' (if it exists !). We shall take $f(x) = \sin(2\pi x)$ for example.

1. On the same figure, plot the function f' and the functions $d_h^+ f$, $d_h^- f$ and $d_h^0 f$, on the interval $I = [-1, 1]$. We shall take $h = 0.1$, and then $h = 0.05$, ...
2. Plot the error $\max_{x \in I} |f'(x) - df_h^+(x)|$ versus h. The "max" shall be computed over 1000 points on the interval I and the parameter h shall span the interval $[0.001, 0.1]$. Determine (numerically) how the error decreases with respect to h, *i.e.* assuming that this error behaves like h^p as h tends to 0 and determine p.
3. Same question with $d_h^0 f$.