TP 6 : Neumann/Dirichlet boundary value problem

In accordance with previous tutorial sheets, a mesh shall be modelled by two tables : `vtx` that models the vertices, and `elt` that models the mesh cells (triangles in 2D, edges in 1D).

Exercice 1 : computing the normal vector field

Here we consider meshes that have been generated by means of the function `GenerateMesh` written in exercise 2 of tutorial sheet no.2.

Question 1.1 Write a function taking as input a pair `(vtx,belt)` that represents the boundary of a mesh generated by means of `GenerateMesh`, and that returns as output a table `nrm` of `nb_belt` arrays of $\mathbb{R}^2$ such that `nrm[j]` is the outgoing vector normal to element no.`j` on the boundary.

Question 1.2 Plot the outgoing normal vector field through the boundary of the mesh obtained by a call to `GenerateMesh('rectangle.msh'` $,2\pi,\pi,10,10)$.

Exercice 2 : Neumann problem

In this exercise we consider the computational domain $\Omega =]0,2\pi[\times]0,\pi[$ and the corresponding mesh obtained by means of the routine `GenerateMesh`. We shall denote $\boldsymbol{n}$ the outgoing vector field normal to the boundary of Ω.

Question 2.1 We consider given $\boldsymbol{d}\in\mathbb{R}^2$, $|\boldsymbol{d}|=1$, a parameter $\mu>0$, and a function $u_{\mathrm{ex}}^{\mathrm{N}}(\boldsymbol{x}) := \sinh(\mu\,\boldsymbol{d}\cdot(\boldsymbol{x}-\boldsymbol{x}_c))$ with $\boldsymbol{x}_c=(\pi,\pi/2)$. Plot $u_{\mathrm{ex}}^{\mathrm{N}}$ over the domain Ω taking $\boldsymbol{d}=(1/\sqrt{2},1/\sqrt{2})$ and $\mu=2$. What is the result of $-\Delta u_{\mathrm{ex}}^{\mathrm{N}}+\mu^2 u_{\mathrm{ex}}^{\mathrm{N}}$? What is the solution to the following boundary value problem ?

$$\begin{cases} \text{Find } u\in \mathrm{H}^1(\Omega) \text{ such that}\\ \Delta u-\mu^2 u = 0 \quad \text{in } \Omega,\\ \partial_{\boldsymbol{n}} u = \partial_{\boldsymbol{n}} u_{\mathrm{ex}}^{\mathrm{N}} \quad \text{on } \partial\Omega. \end{cases} \tag{1}$$

Question 2.2 Compute the nodal values of the numerical solution u_h^{N} obtained by means of a $\mathbb{P}_1$-Lagrange finite element method applied to Problem (1).

Question 2.3 Let $\Pi_h : \mathscr{C}^0(\overline{\Omega}) \to \mathrm{V}_h(\Omega)$ refer to the $\mathbb{P}_1$-Lagrange interpolation operator. Produce a graphical plot of $u_h^{\mathrm{N}} - \Pi_h(u_{\mathrm{ex}}^{\mathrm{N}})$. This function should be small compared to 1 over the whole domain Ω for a sufficiently fine mesh.

Question 2.4 Plot the function $h \mapsto \|u_h^{\mathrm{N}} - \Pi_h(u_{\mathrm{ex}}^{\mathrm{N}})\|_{\mathrm{L}^2(\Omega)}/\|\Pi_h(u_{\mathrm{ex}}^{\mathrm{N}})\|_{\mathrm{L}^2(\Omega)}$, where h refers to the meshwidth. What is the convergence rate ? On the same figure, plot the function $h \mapsto \|u_h^{\mathrm{N}} - \Pi_h(u_{\mathrm{ex}}^{\mathrm{N}})\|_{\mathrm{H}^1(\Omega)}/\|\Pi_h(u_{\mathrm{ex}}^{\mathrm{N}})\|_{\mathrm{H}^1(\Omega)}$. What is the rate of convergence with this second plot ?

Exercice 3 : Dirichlet problem

In this exercise we consider the same computational domain $\Omega =]0, 2\pi[\times]0, \pi[$ as in the previous exercise, and $\mu > 0$ refers again to the same fixed parameter.

Question 2.1 Again we consider the function $f(\boldsymbol{x}) = \sin(x_1)\sin(x_2)$. Compute Δf. What is the solution $u_{\mathrm{ex}}^{\mathrm{D}}(\boldsymbol{x})$ of the following boundary value problem ?

$$\begin{cases} \text{Trouver } u \in \mathrm{H}^1(\Omega) \text{ tel que} \\ -\Delta u + \mu^2 u = f \quad \text{dans } \Omega, \\ u = 0 \quad \text{sur } \partial\Omega. \end{cases} \tag{2}$$

Question 2.2 Solve Problem (2) by the $\mathbb{P}_1$-Lagrange finite element method. We shall denote u_h^{D} the obtained discrete solution.

Question 2.3 Plot $u_h^{\mathrm{D}} - \Pi_h(u_{\mathrm{ex}}^{\mathrm{D}})$ graphically taking $\mu = 2$.

Question 2.4 Compute $\|f\|_{\mathrm{L}^2(\Omega)}^2$ by hands. Plot the function $h \mapsto \|u_h^{\mathrm{D}} - \Pi_h(u_{\mathrm{ex}}^{\mathrm{D}})\|_{\mathrm{L}^2(\Omega)}/\|f\|_{\mathrm{L}^2(\Omega)}$, where h refers to the meshwidth, with the value $\mu = 2$.