TP 4 : Finite elements in 1D

We consider the interval $\mathrm{I} :=]0,1[$, a parameter $\mu \in \mathbb{R}$, as well as a function $f(x) = \cos(p\pi x)$ for a certain integer $p \in \mathbb{N}$, and we aim at finding $u : \mathrm{I} \to \mathbb{R}$ that solves the problem

$$\begin{cases} -\dfrac{d^2u}{dx^2} + \mu^2 u(x) = f(x), \; x \in \mathrm{I} \\ \dfrac{du}{dx}(0) = \dfrac{du}{dx}(1) = 0. \end{cases} \tag{1}$$

Question 1 After giving the explicit expression of $a(\;,\;)$ and $\ell(\;)$, show that Problem (1) can be put under the following variational form

$$\begin{cases} \text{Find } u \in \mathrm{H}^1(\mathrm{I}) \text{ such that} \\ a(u,v) = \ell(v) \quad \forall v \in \mathrm{H}^1(\mathrm{I}) \end{cases} \tag{2}$$

We wish to solve (2) numerically. Given a mesh width $h = 1/N$ for a certain N, we introduce a discretization grid $\mathrm{I} = \cup_{j=1}^{N} [x_{j-1}^h, x_j^h]$ with vertices $x_j^h = jh$. We introduce the space $\mathbb{P}_1$-Lagrange functions defined on this mesh by

$$\begin{aligned} \mathrm{V}_h(\mathrm{I}) := \{ v \in \mathscr{C}^0(\bar{\mathrm{I}}), \; \exists \alpha_j, \beta_j \in \mathbb{R} \text{ t.q.} \\ v(x) = \alpha_j x + \beta_j \text{ pour } x_{j-1}^h \leq x \leq x_j^h, \;\; \forall j = 1 \dots N \, \} \end{aligned}$$

The $\mathbb{P}_1$-Lagrange finite element method consists in solving a discrete version of (2) where $\mathrm{H}^1(\mathrm{I})$ is simply replaced by $\mathrm{V}_h(\mathrm{I})$,

$$\begin{cases} \text{Find } u_h \in \mathrm{V}_h(\mathrm{I}) \text{ such that} \\ a(u_h, v_h) = \ell(v_h) \quad \forall v_h \in \mathrm{V}_h(\mathrm{I}) \end{cases} \tag{3}$$

Question 2 We recall that $\mathrm{V}_h(\mathrm{I})$ is generated by the functions $\varphi_j^h(x) \in \mathrm{V}_h(\mathrm{I})$ defined by $\varphi_j^h(x_k^h) = 0$ if $j \neq k$, and $\varphi_j^h(x_j^h) = 1$, $j = 0, \dots N$. In particular $\dim \mathrm{V}_h(\mathrm{I}) = N+1$. Show that the Galerkin discretization method based on the shape functions $(\varphi_j^h)_{j=0\dots N}$ leads to equivalently reformulating Problem (3) as a linear system $\mathrm{A}_h \mathrm{U} = \mathrm{F}_h$ where you shall give the explicit expression of $\mathrm{A}_h \in \mathbb{R}^{(N+1)\times(N+1)}$ and $\mathrm{F}_h \in \mathbb{R}^{N+1}$ with respect to $a(\;,\;)$, $\ell(\;)$ and φ_j^h.

Question 3 Show that the matrix of Problem (3) admits the expression $\mathrm{A}_h = \mathrm{K}_h + \mu^2 \mathrm{M}_h$ where the mass matrix M_h and the stiffness matrix K_h are given by

$$\mathrm{K}_h = \frac{1}{h}\begin{bmatrix} 1 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \dots & 0 & -1 & 1 \end{bmatrix} \qquad \mathrm{M}_h = \frac{h}{6}\begin{bmatrix} 2 & 1 & 0 & \dots & 0 \\ 1 & 4 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & 4 & 1 \\ 0 & \dots & 0 & 1 & 2 \end{bmatrix}$$

Write a program where you assemble the matrices M_h, K_h given by the expression above, as well as the vector F_h.

Question 4 Using the matrices of the previsou question, solve Problem (3) numerically for 10 values of $N = 1/h$. You shall choose these values of N so that $\log_{10}(N)$ be uniformly distributed in the interval $[1, 3]$ (so that N ranges from 10 to 1000). You shall take $\mu = 1$ and $p = 3$ as concrete values of the parameters.

Question 5 Given an element $v_h \in \mathrm{V}_h(\mathrm{I})$ associated to the nodal values $\mathrm{V} = (v_h(x_j^h))_{j=0\dots N}$, prove that $\|v_h\|^2_{\mathrm{L}^2(\mathrm{I})} = \mathrm{V}^\top \mathrm{M}_h \mathrm{V}$ et $\|\nabla v_h\|^2_{\mathrm{L}^2(\mathrm{I})} = \mathrm{V}^\top \mathrm{K}_h \mathrm{V}$.

Find an explicit expression of the exact solution $u(x)$ to (1) with respect to p, μ. Let us denote $\Pi_h(u)(x) := \sum_{j=0}^{N} u(x_j^h)\varphi_j^h(x)$ the Lagrange interpolant of u over the grid. Using the work of the previous question, plot the error $\|\Pi_h(u) - u_h\|_{\mathrm{L}^2(\mathrm{I})}/\|u_h\|_{\mathrm{L}^2(\mathrm{I})}$ versus h. Choose a logarithmic scale for this plot both in x and y. What convergence rate do you observe ?

Question 6 Modify the assembly of the matrix A_h so as to compute the nodal values of the solution to the same problem as (1) but with homogeneous Dirichlet boundary conditions instead of Neumann conditions

$$\begin{cases} -\dfrac{d^2u}{dx^2} + \mu^2 u(x) = f(x), \;\; x \in \mathrm{I} \\ u(0) = u(1) = 0. \end{cases} \tag{4}$$