Technical note: Stein-Stein's characteristic function

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Let $(W, W^\perp)$ be a two-dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geqslant 0}, \mathbb{P})$, which satisfies the usual conditions. We consider the Stein-Stein model :

$$dS_t = S_t \sigma_t \, dB_t, \quad S_0 > 0,$$
$$d\sigma_t = (a + b\sigma_t) \, dt + c \, dW_t, \quad \sigma_0 \in \mathbb{R},$$
$$V_t = \sigma_t^2.$$

where $a, b, c \in \mathbb{R}$, and $B = \rho W + \sqrt{1 - \rho^2} W^\perp$. In the sequel, we will denote by $\mathbb{E}[\cdot]$ the expectation under the measure $\mathbb{P}$. Note that V_t is a semimartingale satisfying

$$dV_t = (2a\sigma_t + 2bV_t + c^2) \, dt + 2c\sigma_t \, dW_t, \quad V_0 = \sigma_0^2,$$

where $\sqrt{V_t} = |\sigma_t|$, i.e. $\sigma_t = \text{sign}(\sigma_t) \sqrt{V_t}$, with the convention $\text{sign}(0) = 0$. By a direct application of the Levy's theorem,

$$\widetilde{W}_t := \int_0^t \text{sign}(\sigma_s) \, dW_s$$

is a $\mathscr{F}$-Brownian motion, . In particular, it leads to the following expression

$$dV_t = (2a\sigma_t + 2bV_t + c^2) \, dt + 2c\sqrt{V_t} \, d\widetilde{W}_t.$$

When $a = 0$, we retrieve that V is a CIR-type diffusion and yields to an the Heston-type diffusion:

$$d\widetilde{S}_t = \widetilde{S}_t \sqrt{\widetilde{V}_t} \, d\widetilde{B}_t, \quad S_0 > 0,$$
$$d\widetilde{V}_t = (\widetilde{a} + \widetilde{b}\widetilde{V}_t) \, dt + \widetilde{c}\sqrt{\widetilde{V}_t} \, d\widetilde{W}_t, \quad \widetilde{V}_0 \geqslant 0,$$

with $\widetilde{a} := c^2$, $\widetilde{b} = 2b$, $\widetilde{c} = 2c$ and $\widetilde{B} := \rho\widetilde{W} + \sqrt{1 - \rho^2}\widetilde{W}^\perp$, with

$$\widetilde{W}_t^\perp := \int_0^t \text{sign}(\sigma_s) \, dW_s^\perp.$$

In fact, the process $\boldsymbol{W} := (\widetilde{W}, \widetilde{W}^\perp)$ is a two-dimensional Brownian motion on the same filtered probability space, which is another application of the Levy's theorem:

$$\left\langle \widetilde{W}, \widetilde{W}^\perp \right\rangle_t = \int_0^t \text{sign}(\sigma_s)^2 \, d\left\langle W, W^\perp \right\rangle_s = 0 \quad \implies \quad \langle \boldsymbol{W} \rangle_t = t I_2.$$

Time-dependent Riccati ODEs: existence and uniqueness

Here, we consider a generic class of time-dependent Riccati equations that encompass the ones for the Heston model, in the form

$$\psi'(t) = a(t)\psi(t)^2 + b(t)\psi(t) + c(t), \quad \psi(0) = u_0, \quad t \in [0, T],$$

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with $u_0 \in \mathbb{C}$, and $a, b, c, : [0, T] \to \mathbb{C}$ measurable and bounded functions. If $\mathfrak{Re}(u_0) \leqslant 0$ and

$$\mathfrak{Im}(a(t)) = 0, \quad a(t) > 0, \quad \mathfrak{Re}(c(t)) + \frac{\mathfrak{Im}(b(t))^2}{4a(t)} \leqslant 0, \quad t \in [0, T],$$

there exists a unique solution $\psi : [0, T] \to \mathbb{C}$ to the Riccati equations such that $\mathfrak{Re}(\psi(t)) \leqslant 0$, for $t \in [0, T]$ and $\sup_{t \in [0,T]} |\psi(t)| < \infty$.

The characteristic function

Theorem. *Let $f, g : [0, T] \to \mathbb{C}$ be measurable and bounded functions such that $\mathfrak{Re}(f) \in [0, 1]$ and $\mathfrak{Re}(g) \leqslant 0$. In particular it implied that*

$$\mathfrak{Re}(g) + \tfrac{1}{2}\left((\mathfrak{Re}(f))^2 - \mathfrak{Re}(f) \right) \leqslant 0.$$

Then, the joint conditional characteristic function of the log-price and the integrated variance is given by

$$\mathbb{E}\left[\exp\left(\int_t^T f(T-s)\, d\ln S_s + \int_t^T g(T-s)\, d\overline{V}_s \right) \Bigg| \mathscr{F}_t \right] = \exp\left(\phi(T-t) + \chi(T-t)\sigma_t + \psi(T-t)V_t \right), t \in [0, T],$$

where (ϕ, χ, ψ) is the solution to the following system of time-dependent Riccati equations:

$$
\begin{aligned}
\phi'(t) &= a\chi(t) + \tfrac{c^2}{2}\chi(t)^2 + c^2\psi(t), \quad \phi(0) = 0, \\
\chi'(t) &= \left(2c^2\psi(t) + c\rho f(t) + b \right)\chi(t) + 2a\psi(t), \quad \chi(0) = 0, \\
\psi'(t) &= 2c^2\psi(t)^2 + (2\rho c f(t) + 2b)\,\psi(t) + g(t) + \tfrac{f(t)^2 - f(t)}{2}, \quad \psi(0) = 0.
\end{aligned}
$$

Proof. **Step 1.** We first argue the existence of a solution of the above system of Riccati equations. Let us rewrite the Riccati ODE as

$$\psi'(t) = a\psi(t)^2 + b(t)\psi(t) + c(t), \quad \psi(0) = 0,$$

with

$$a = 2c^2, \quad b(t) = 2(\rho c f(t) + b), \quad c(t) = g(t) + \tfrac{f(t)^2 - f(t)}{2}$$

The two first conditions $\mathfrak{Im}(a) = 0$ and $a > 0$ are trivially satisfied. Regarding the last one,

$$
\begin{aligned}
\mathfrak{Re}(c(t)) + \tfrac{\mathfrak{Im}(b(t))^2}{4a(t)} &= \mathfrak{Re}(g(t)) + \tfrac{\mathfrak{Re}(f(t)^2) - \mathfrak{Re}(f(t))}{2} + \tfrac{4\rho^2 c^2\, \mathfrak{Im}(f(t))^2}{8c^2} \\
&= \mathfrak{Re}(g(t)) + \tfrac{\mathfrak{Re}(f(t))^2 - \mathfrak{Re}(f(t))}{2} + (\rho^2 - 1)\tfrac{\mathfrak{Im}(f(t))^2}{2} \leqslant 0.
\end{aligned}
$$

Consequently, there exists a unique $\psi : [0, T] \to \mathbb{C}$ solution to the above Riccati ODE such that $\mathfrak{Re}(\psi(t)) \leqslant 0$, for all $t \in [0, T]$.

Step 2. We are now able to prove the expression for the characteristic function. Define the processes

$$U_t := \phi(T-t) + \chi(T-t)\sigma_t + \psi(T-t)V_t + \int_0^t f(T-s)\, d\ln S_s + \int_0^t g(T-s)\, d\overline{V}_s, \quad M_t := \exp(U_t).$$

This amounts to showing that M is a martingale. We first show that M is a local martingale using the Itô's formula. It is clear that

$$dM_t = M_t\left(dU_t + \tfrac{1}{2}d\langle U\rangle_t \right),$$

with

$$
\begin{aligned}
dU_t &= \Big(-\phi'(T-t) - \chi'(T-t)\sigma_t - \psi'(T-t)V_t + \chi(T-t)(a + b\sigma_t) + \psi(T-t)(2a\sigma_t + 2bV_t + c^2) - f(T-t)\tfrac{V_t}{2} \\
&\quad + g(T-t)V_t \Big)dt + \left((2c\psi(T-t) + \rho f(T-t))\sigma_t + c\chi(T-t) \right)dW_t + \sqrt{1-\rho^2}f(T-t)\sigma_t\, dW_t^\perp \\
d\langle U\rangle_t &= \Big(\left((2c\psi(T-t) + \rho f(T-t))^2 + (1-\rho^2)f(T-t)^2 \right)V_t + c^2\chi(T-t)^2 \\
&\quad + 2c\chi(T-t)(2c\psi(T-t) + \rho f(T-t))\sigma_t \Big)dt
\end{aligned}
$$

Therefore the drift of dM_t/M_t is given by

$$
\begin{aligned}
0 = {} & -\phi'(T-t) - \chi'(T-t)\sigma_t - \psi'(T-t)V_t + \chi(T-t)(a+b\sigma_t) + \psi(T-t)(2a\sigma_t + 2bV_t + c^2) - f(T-t)\tfrac{V_t}{2} \\
& + g(T-t)V_t + \tfrac{V_t}{2}\left((2c\psi(T-t) + \rho f(T-t))^2 + (1-\rho^2)f(T-t)^2\right) + \tfrac{c^2}{2}\chi(T-t)^2 \\
& + c\chi(T-t)(2c\psi(T-t) + \rho f(T-t))\sigma_t \\
= {} & -\phi'(T-t) + a\chi(T-t) + c^2\psi(T-t) + \tfrac{c^2}{2}\chi(T-t)^2 \\
& + \left(-\chi'(T-t) + b\chi(T-t) + 2a\psi(T-t) + 2c^2\chi(T-t)\psi(T-t) + c\rho\chi(T-t)f(T-t)\right)\sigma_t \\
& + \left(-\psi'(T-t) + 2b\psi(T-t) + \tfrac{f(T-t)^2 - f(T-t)}{2} + g(T-t) + 2c^2\psi(T-t)^2 + 2\rho c\psi(T-t)f(T-t)\right)V_t
\end{aligned}
$$

which leads to the above Riccati equations.

Step 3. Now, it remains to show that M is a true martingale, where

$$
\frac{dM_t}{M_t} = \left((2c\psi(T-t) + \rho f(T-t))\sigma_t + c\chi(T-t)\right) dW_t + \sqrt{1-\rho^2} f(T-t)\sigma_t\, dW_t^\perp.
$$

For the sake of brevity, this step is ommited here. It is presented in details in E. Abi Jaber. *The characteristic function of Gaussian stochastic volatility models: an analytic expression*. 2021. $\qquad\square$

The link with the Heston's characteristic function

When $a = 0$, the equation $\chi'(t) = \left(2c^2\psi(t) + c\rho f(t) + b\right)\chi(t) + 2a\psi(t)$, $\chi(0) = 0$ collapses to $\chi'(t) = \left(2c^2\psi(t) + c\rho f(t) + b\right)\chi(t)$, $\chi(0) = 0$ and yields $\chi(t) = 0$. In this case, the system could be rewritten as

$$
\begin{aligned}
\phi'(t) &= c^2\psi(t), \quad \phi(0) = 0, \\
\psi'(t) &= 2c^2\psi(t)^2 + (2\rho cf(t) + 2b)\psi(t) + g(t) + \tfrac{f(t)^2 - f(t)}{2}, \quad \psi(0) = 0.
\end{aligned}
$$

Note that this system is the one arising from the characteristic function of the Heston model where the spot variance is given by $d\widetilde{V}_t = (\widetilde{a} + \widetilde{b}\widetilde{V}_t)\,dt + \widetilde{c}\sqrt{\widetilde{V}_t}\,d\widetilde{W}_t$. To make clear the notations, we will introduce:

$$
\varphi_{\text{SteinStein}}(t,T,f,g,S_t,\sigma_t,a,b,c,\rho) = \mathbb{E}\left[\exp\left(\int_t^T f(T-s)\,d\ln S_s + \int_t^T g(T-s)\,d\overline{V}_s\right)\bigg|\mathscr{F}_t\right],
$$

$$
\varphi_{\text{Heston}}(t,T,f,g,\widetilde{S}_t,\widetilde{V}_t,\widetilde{a},\widetilde{b},\widetilde{c},\rho) = \mathbb{E}\left[\exp\left(\int_t^T f(T-s)\,d\ln \widetilde{S}_s + \int_t^T g(T-s)\,d\overline{\widetilde{V}}_s\right)\bigg|\mathscr{F}_t\right],
$$

and the last observation translates that

$$
\varphi_{\text{SteinStein}}(t,T,f,g,S_t,\sigma_t,0,b,c,\rho) = \varphi_{\text{Heston}}(t,T,f,g,S_t,\sigma_t^2,c^2,2b,2c,\rho).
$$

For $a \neq 0$,

$$
\varphi_{\text{SteinStein}}(t,T,f,g,S_t,\sigma_t,a,b,c,\rho) = \varphi_{\text{Heston}}(t,T,f,g,S_t,\sigma_t^2,c^2,2b,2c,\rho)\exp\left(\widetilde{\phi}(T-t) + \chi(T-t)\sigma_t\right),
$$

with $\widetilde{\phi}'(t) = a\chi(t) + \tfrac{c^2}{2}\chi(t)^2$, $\widetilde{\phi}(0) = 0$. More explicitely, when $f(t) = u$ and $g(t) = w$,

$$
\widetilde{\phi}(t) = a^2\frac{\beta - D}{2D^2c^2}\left(\beta(Dt - 4) + D(Dt - 2) + \frac{4e^{-\frac{1}{2}Dt}\left(\frac{D^2 - 2\beta^2}{\beta + D}e^{-\frac{1}{2}Dt} + 2\beta\right)}{1 - Ge^{-Dt}}\right),
$$

$$
\chi(t) = a\frac{\beta - D}{Dc^2}\frac{\left(1 - e^{-\frac{1}{2}Dt}\right)^2}{1 - Ge^{-Dt}}, \quad \beta := -2(\rho cu + b), \quad D := \sqrt{\beta^2 - 8c^2\left(w + \frac{u^2 - u}{2}\right)}, \quad G := \frac{\beta - D}{\beta + D}.
$$