

Technical note: Heston's characteristic function

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Let (W, W^\perp) be a two-dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, which satisfies the usual conditions. We consider the Heston model :

$$\begin{aligned} dS_t &= S_t \sqrt{V_t} dB_t, \quad S_0 > 0, \\ dV_t &= (a + bV_t) dt + c\sqrt{V_t} dW_t, \quad V_0 \geq 0. \end{aligned}$$

where $a \geq 0$, $b, c \in \mathbb{R}$, and $B = \rho W + \sqrt{1 - \rho^2} W^\perp$. In the sequel, we will denote by $\mathbb{E}[\cdot]$ the expectation under the measure \mathbb{P} .

A martingale argument

In this part, we will state a general result to rigorously justify a semimartingale is a true martingale. It will be the cornerstone to conclude the proof for the characteristic function, based on a martingale argument. By the way, the following Lemma proves in particular that the process $(S_t)_{t \geq 0}$ is a martingale.

Lemma. *Let $h \in L^\infty(\mathbb{R}^+, \mathbb{R})$ and define*

$$U_t = \int_0^t h(s) \sqrt{V_s} dB_s.$$

Then the stochastic exponential $\exp(U_t - \frac{1}{2}\langle U \rangle_t)$ is martingale. In particular, S is a martingale ($h \equiv 1$).

Proof. Let $M_t := \exp(U_t - \frac{1}{2}\langle U \rangle_t)$. Since M is a nonnegative local martingale, it is a supermartingale. Indeed, there exists by definition an increasing sequence of stopping times $(\sigma_n)_{n \geq 0}$ such that $\sigma_n \uparrow \infty$ and $(M_{t \wedge \sigma_n})_{t \geq 0}$ is a martingale. Because of $\sigma_n \uparrow \infty$ and $t \wedge \sigma_n \uparrow t$, it leads to

$$M_{t \wedge \sigma_n} \xrightarrow[n \rightarrow \infty]{\text{a.s.}} M_t,$$

The conditional version of the Fatou's lemma applies here, because $M \geq 0$, and yields

$$\mathbb{E}[M_t | \mathcal{F}_s] = \mathbb{E} \left[\liminf_{n \rightarrow \infty} M_{t \wedge \sigma_n} \middle| \mathcal{F}_s \right] \leq \liminf_{n \rightarrow \infty} \mathbb{E}[M_{t \wedge \sigma_n} | \mathcal{F}_s] = M_s.$$

In particular, it implies that $\mathbb{E}[M_T] \leq \mathbb{E}[M_0] = 1$ for all $T \in \mathbb{R}^+$. Therefore, it remains to show that $\mathbb{E}[M_T] \geq 1$ for all $T \in \mathbb{R}^+$. To this end, define stopping times $\tau_n = \inf\{t \geq 0, V_t > n\} \wedge T$. Then $(M_{t \wedge \tau_n})_{t \geq 0}$ is uniformly integrable martingale for each n by the Novikov's criterion:

$$\mathbb{E} \left[\exp \left(\frac{1}{2} \int_0^{\tau_n} h(t)^2 V_t dt \right) \right] \leq \exp \left(\frac{1}{2} \|h\|_\infty^2 T n \right) < \infty,$$

and may define probability measures \mathbb{Q}^n by

$$\frac{d\mathbb{Q}^n}{d\mathbb{P}} = M_{\tau_n} = \exp \left(\int_0^{\tau_n} h(s) \sqrt{V_s} dB_s - \frac{1}{2} \int_0^{\tau_n} h(s)^2 V_s ds \right).$$

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By Girsanov's theorem, $dW_t^n := dW_t - \mathbf{1}_{t \leq \tau_n} \rho h(t) \sqrt{V_t} dt$ is a Brownian motion under \mathbb{Q}^n , and it leads to:

$$dV_t = (a + (b + \mathbf{1}_{t \leq \tau_n} c \rho h(t)) V_t) dt + c \sqrt{V_t} dW_t^n$$

Observe that the expression $a + (b + \mathbf{1}_{t \leq \tau_n} c \rho h(t)) v$ satisfies a linear growth condition in v , uniformly in (t, ω) . Note that using Markov inequality

$$\mathbb{Q}^n(\tau_n < T) = \mathbb{Q}^n \left(\sup_{t \in [0, T]} V_t > n \right) \leq \frac{\mathbb{E}^{\mathbb{Q}^n} \left[\sup_{t \in [0, T]} V_t^p \right]}{n^p}, \quad \forall p \geq 1.$$

Moreover, it is clear that

$$\mathbb{E}^{\mathbb{Q}^n} \left[\sup_{t \in [0, T]} |V_t|^p \right] \leq C_p \mathbb{E}^{\mathbb{Q}^n} \left[|V_0|^p + \sup_{t \in [0, T]} \left| \int_0^t a + (b + \mathbf{1}_{s \leq \tau_n} c \rho h(s)) V_s ds \right|^p + \sup_{t \in [0, T]} \left| \int_0^t c \sqrt{V_s} dW_s^n \right|^p \right]$$

For all $t \leq T$,

$$\left| \int_0^t a + (b + \mathbf{1}_{s \leq \tau_n} c \rho h(s)) V_s ds \right| \leq |a|T + \bar{\beta} \int_0^T |V_s| ds, \quad \bar{\beta} := |b| + |c\rho|\|h\|_\infty,$$

therefore, using the fact that $(x + y)^p \leq 2^{p-1}(x^p + y^p)$ for $p \geq 1$, and the Hölder inequality, this gives

$$\sup_{t \in [0, T]} \left| \int_0^t a + (b + \mathbf{1}_{s \leq \tau_n} c \rho h(s)) V_s ds \right|^p \leq 2^{p-1} \left((|a|T)^p + \left(\int_0^T |V_s| ds \right)^p \right) \leq 2^{p-1} \left((|a|T)^p + T^{p-1} \int_0^T |V_s|^p ds \right)$$

On the other hand, BDG and Hölder inequalities leads to

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}^n} \left[\sup_{t \in [0, T]} \left| \int_0^t c \sqrt{V_s} dW_s^n \right|^p \right] &\leq C_p \mathbb{E}^{\mathbb{Q}^n} \left[\left(\int_0^T c^2 V_s ds \right)^{\frac{p}{2}} \right] \leq C_p T^{\frac{p}{2}-1} c^p \mathbb{E}^{\mathbb{Q}^n} \left[\int_0^T V_s^{\frac{p}{2}} ds \right] \\ &\leq C_p T^{\frac{p}{2}-1} c^p \left(T + \int_0^T \mathbb{E}^{\mathbb{Q}^n} [V_s^p] ds \right) \end{aligned}$$

where the last inequality comes from the fact that $x^{\frac{p}{2}} \leq 1 + x^p$ for all $x \geq 0$ and $p \geq 1$. Putting everything together, it yields to

$$\mathbb{E}^{\mathbb{Q}^n} \left[\sup_{t \in [0, T]} |V_t|^p \right] \leq C \left(1 + |V_0|^p + \int_0^T \mathbb{E}^{\mathbb{Q}^n} [V_s^p] ds \right),$$

with $C = C(p, T, a, b, c, \rho, \|h\|_\infty)$ which does not depend on n . Finally, we conclude that

$$\mathbb{E}^{\mathbb{Q}^n} \left[\sup_{t \in [0, T]} V_t^p \right] \leq C_0 e^{C_1 T}$$

using the fact that $\mathbb{E}[V_s^p] \leq \mathbb{E} \left[\sup_{u \in [0, s]} V_u^p \right]$ combined with the Grönwall inequality. In particular, this procedure implies $\mathbb{Q}^n(\tau_n < T) \xrightarrow[n \rightarrow \infty]{} 0$ (and in particular $\mathbb{Q}^n(\tau_n = T) \xrightarrow[n \rightarrow \infty]{} 1$). The fact that

$$\mathbb{E}[M_T] \geq \mathbb{E}[M_T \mathbf{1}_{\tau_n = T}] = \mathbb{Q}^n(\tau_n = T),$$

and $n \rightarrow \infty$ yields $\mathbb{E}[M_T] \geq 1$. This completes the proof. \square

Here, we consider a generic class of time-dependent Riccati equations that encompass the ones for the Heston model, in the form

$$\psi'(t) = a(t)\psi(t)^2 + b(t)\psi(t) + c(t), \quad \psi(0) = u_0, \quad t \in [0, T],$$

with $u_0 \in \mathbb{C}$, and $a, b, c : [0, T] \rightarrow \mathbb{C}$ measurable and bounded functions. If $\Re(e(u_0)) \leq 0$ and

$$\Im(a(t)) = 0, \quad a(t) > 0, \quad \Re(c(t)) + \frac{\Im(b(t))^2}{4a(t)} \leq 0, \quad t \in [0, T],$$

there exists a unique solution $\psi : [0, T] \rightarrow \mathbb{C}$ to the Riccati equations such that $\Re(\psi(t)) \leq 0$, for $t \in [0, T]$ and $\sup_{t \in [0, T]} |\psi(t)| < \infty$.

The characteristic function

The following theorem provides the joint conditional characteristic function of the log-price $\ln S$ and integrated variance $\bar{V} := \int_0^T V_s ds$ in the Heston model in terms of a solution to a system of time-dependent Riccati ODE.

Theorem. *Let $f, g : [0, T] \rightarrow \mathbb{C}$ be measurable and bounded functions such that*

$$\Re(g) + \frac{1}{2} ((\Re(f))^2 - \Re(f)) \leq 0.$$

Then, the joint conditional characteristic function of the log-price and the integrated variance is given by

$$\mathbb{E} \left[\exp \left(\int_t^T f(T-s) d \ln S_s + \int_t^T g(T-s) d \bar{V}_s \right) \middle| \mathcal{F}_t \right] = \exp (\phi(T-t) + \psi(T-t)V_t), \quad t \in [0, T],$$

where (ϕ, ψ) is the solution to the following system of time-dependent Riccati equations:

$$\begin{aligned} \phi'(t) &= a\psi(t), \quad \phi(0) = 0, \\ \psi'(t) &= \frac{c^2}{2}\psi(t)^2 + (\rho cf(t) + b)\psi(t) + g(t) + \frac{f(t)^2 - f(t)}{2} \end{aligned}$$

Proof. **Step 1.** We first argue the existence of a solution of the above system of Riccati equations. Let us rewrite the Riccati ODE as

$$\psi'(t) = a\psi(t)^2 + b(t)\psi(t) + c(t), \quad \psi(0) = 0,$$

with

$$a = \frac{c^2}{2}, \quad b(t) = \rho cf(t) + b, \quad c(t) = g(t) + \frac{f(t)^2 - f(t)}{2}$$

The two first conditions $\Im(a) = 0$ and $a > 0$ are trivially satisfied. Regarding the last one,

$$\begin{aligned} \Re(c(t)) + \frac{\Im(b(t))^2}{4a(t)} &= \Re(g(t)) + \frac{\Re(f(t)^2) - \Re(f(t))}{2} + \rho^2 \frac{\Im(f(t))^2}{2} \\ &= \Re(g(t)) + \frac{\Re(f(t))^2 - \Re(f(t))}{2} + (\rho^2 - 1) \frac{\Im(f(t))^2}{2} \leq 0. \end{aligned}$$

Consequently, there exists a unique $\psi : [0, T] \rightarrow \mathbb{C}$ solution to the above Riccati ODE such that $\Re(\psi(t)) \leq 0$, for all $t \in [0, T]$.

Step 2. We are now able to prove the expression for the characteristic function. Define the processes

$$U_t := \phi(T-t) + \psi(T-t)V_t + \int_0^t f(T-s) d \ln S_s + \int_0^t g(T-s) d \bar{V}_s, \quad M_t := \exp(U_t).$$

This amounts to showing that M is a martingale. We first show that M is a local martingale using the Itô's formula. It is clear that

$$dM_t = M_t (dU_t + \frac{1}{2} d\langle U \rangle_t),$$

with

$$dU_t = (-\phi'(T-t) - \psi'(T-t)V_t + \psi(T-t)(a + bV_t) - f(T-t)\frac{V_t}{2} + g(T-t)V_t) dt$$

$$+ (c\psi(T-t) + \rho f(T-t))\sqrt{V_t} dW_t + \sqrt{1 - \rho^2} f(T-t)\sqrt{V_t} dW_t^\perp$$

$$d\langle U \rangle_t = ((c\psi(T-t) + \rho f(T-t))^2 + (1 - \rho^2)f(T-t)^2) V_t dt$$

Therefore the drift of dM_t/M_t is given by

$$\begin{aligned} 0 &= -\phi'(T-t) - \psi'(T-t)V_t + \psi(T-t)(a+bV_t) - f(T-t)\frac{V_t}{2} + g(T-t)V_t \\ &\quad + \frac{V_t}{2}((c\psi(T-t)+\rho f(T-t))^2 + (1-\rho^2)f(T-t)^2) \\ \iff 0 &= -\phi'(T-t) + a\psi(T-t) + V_t \left(-\psi'(T-t) + (c\rho f(T-t) + b)\psi(T-t) + g(T-t) + \frac{c^2}{2}\psi(T-t)^2 + \right. \\ &\quad \left. + \frac{1}{2}(f(T-t)^2 - f(T-t)) \right) \end{aligned}$$

which leads to the above Riccati equations. Now, it remains to show that M is true martingale.

$$\begin{aligned} \Re(U_t) &\leq \int_0^t \Re(f(T-s)) d\ln S_s + \int_0^t \Re(g(T-s)) d\bar{V}_s \\ &= \int_0^t \Re(g(T-s)) - \frac{1}{2} \Re(f(T-s)) d\bar{V}_s + \int_0^t \Re(f(T-s)) \sqrt{V_s} dB_s \\ &\leq -\frac{1}{2} \int_0^t \Re(f(T-s))^2 V_s ds + \int_0^t \Re(f(T-s)) \sqrt{V_s} dB_s := \tilde{U}_t \end{aligned}$$

It follows that $|M_t| = \exp(\Re(U_t)) \leq \exp(\tilde{U}_t)$, where $\exp(\tilde{U})$ is a true martingale. This a simple application of the Lemma stated before with $h = \Re(f)$ which is bounded by assumption. This shows that M is local martingale which satisfies for all $t \in [0, T]$, $\mathbb{E}[|M_t|] < \infty$. Said differently, it implies that M is a true martingale. \square

A useful particular case. Let $u, w \in \mathbb{C}$ such that $\Re(w) + \frac{1}{2}(\Re(u)^2 - \Re(u)) \leq 0$. The functions $f(t) = u$ and $g(t) = w$ are *a fortiori* bounded. The system of Riccati ODE could be rewrite as follows

$$\begin{aligned} \phi'(t) &= a\psi(t), \quad \phi(0) = 0, \\ \psi'(t) &= \frac{c^2}{2}\psi(t)^2 + (\rho cu + b)\psi(t) + w + \frac{u^2+u}{2}, \quad \psi(0) = 0. \end{aligned}$$

In this case, the solution is explicit :

$$\begin{aligned} \psi(t) &= \frac{\beta(u)-D(u,w)}{c^2} \frac{1-e^{-D(u,w)t}}{1-G(u,w)e^{-D(u,w)t}}, \\ \phi(t) &= \frac{a}{c^2} \left((\beta(u) - D(u,w))t - 2 \ln \left(\frac{G(u,w)e^{-D(u,w)t}-1}{G(u,w)-1} \right) \right), \\ \beta(u) &= -b - u\rho c, \quad D(u,w) = \sqrt{\beta(u)^2 + c^2(-2w + u - u^2)}, \quad G(u,w) = \frac{\beta(u)-D(u,w)}{\beta(u)+D(u,w)}. \end{aligned}$$