

# Technical note: Heston's characteristic function

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February 18, 2026

Let  $(W, W^\perp)$  be a two-dimensional Brownian motion defined on a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ , which satisfies the usual conditions. We consider the Heston model :

$$\begin{aligned} dS_t &= S_t \sqrt{V_t} dB_t, \quad S_0 > 0, \\ dV_t &= (a + bV_t) dt + c\sqrt{V_t} dW_t, \quad V_0 \geq 0. \end{aligned}$$

where  $a \geq 0, b, c \in \mathbb{R}$ , and  $B = \rho W + \sqrt{1 - \rho^2} W^\perp$ . In the sequel, we will denote by  $\mathbb{E}[\cdot]$  the expectation under the measure  $\mathbb{P}$ .

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## A martingale argument

In this part, we will state a general result to rigorously justify a semimartingale is a true martingale. It will be the cornerstone to conclude the proof for the characteristic function, based on a martingale argument. By the way, the following Lemma proves in particular that the process  $(S_t)_{t \geq 0}$  is a martingale.

**Lemma.** *Let  $h \in L^\infty(\mathbb{R}^+, \mathbb{R})$  and define*

$$U_t = \int_0^t h(s) \sqrt{V_s} dB_s.$$

*Then the stochastic exponential  $\exp(U_t - \frac{1}{2}\langle U \rangle_t)$  is martingale. In particular,  $S$  is a martingale ( $h \equiv 1$ ).*

*Proof.* Let  $M_t := \exp(U_t - \frac{1}{2}\langle U \rangle_t)$ . Since  $M$  is a nonnegative local martingale, it is a supermartingale. Indeed, there exists by definition an increasing sequence of stopping times  $(\sigma_n)_{n \geq 0}$  such that  $\sigma_n \uparrow \infty$  and  $(M_{t \wedge \sigma_n})_{t \geq 0}$  is a martingale. Because of  $\sigma_n \uparrow \infty$  and  $t \wedge \sigma_n \uparrow t$ , it leads to

$$M_{t \wedge \sigma_n} \xrightarrow[n \rightarrow \infty]{\text{a.s.}} M_t,$$

The conditional version of the Fatou's lemma applies here, because  $M \geq 0$ , and yields

$$\mathbb{E}[M_t | \mathcal{F}_s] = \mathbb{E} \left[ \liminf_{n \rightarrow \infty} M_{t \wedge \sigma_n} \middle| \mathcal{F}_s \right] \leq \liminf_{n \rightarrow \infty} \mathbb{E}[M_{t \wedge \sigma_n} | \mathcal{F}_s] = M_s.$$

In particular, it implies that  $\mathbb{E}[M_T] \leq \mathbb{E}[M_0] = 1$  for all  $T \in \mathbb{R}^+$ . Therefore, it remains to show that  $\mathbb{E}[M_T] \geq 1$  for all  $T \in \mathbb{R}^+$ . To this end, define stopping times  $\tau_n = \inf\{t \geq 0, V_t > n\} \wedge T$ . Then  $(M_{t \wedge \tau_n})_{t \geq 0}$  is uniformly integrable martingale for each  $n$  by the Novikov's criterion:

$$\mathbb{E} \left[ \exp \left( \frac{1}{2} \int_0^{\tau_n} h(t)^2 V_t dt \right) \right] \leq \exp \left( \frac{1}{2} \|h\|_\infty^2 T n \right) < \infty,$$

and may define probability measures  $\mathbb{Q}^n$  by

$$\frac{d\mathbb{Q}^n}{d\mathbb{P}} = M_{\tau_n} = \exp \left( \int_0^{\tau_n} h(s) \sqrt{V_s} dB_s - \frac{1}{2} \int_0^{\tau_n} h(s)^2 V_s ds \right).$$

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By Girsanov's theorem,  $dW_t^n := dW_t - \mathbf{1}_{t \leq \tau_n} \rho h(t) \sqrt{V_t} dt$  is a Brownian motion under  $\mathbb{Q}^n$ , and it leads to:

$$dV_t = (a + (b + \mathbf{1}_{t \leq \tau_n} c \rho h(t)) V_t) dt + c \sqrt{V_t} dW_t^n$$

Observe that the expression  $a + (b + \mathbf{1}_{t \leq \tau_n(\omega)} \rho h(t)) v$  satisfies a linear growth condition in  $v$ , uniformly in  $(t, \omega)$ . Note that using Markov inequality

$$\mathbb{Q}^n(\tau_n < T) = \mathbb{Q}^n \left( \sup_{t \in [0, T]} V_t > n \right) \leq \frac{\mathbb{E}^{\mathbb{Q}^n} \left[ \sup_{t \in [0, T]} V_t^p \right]}{n^p}, \quad \forall p \geq 1.$$

Moreover, it is clear that

$$\mathbb{E}^{\mathbb{Q}^n} \left[ \sup_{t \in [0, T]} |V_t|^p \right] \leq C_p \mathbb{E}^{\mathbb{Q}^n} \left[ |V_0|^p + \sup_{t \in [0, T]} \left| \int_0^t a + (b + \mathbf{1}_{s \leq \tau_n} c \rho h(s)) V_s ds \right|^p + \sup_{t \in [0, T]} \left| \int_0^t c \sqrt{V_s} dW_s^n \right|^p \right]$$

For all  $t \leq T$ ,

$$\left| \int_0^t a + (b + \mathbf{1}_{s \leq \tau_n} c \rho h(s)) V_s ds \right| \leq |a|T + \bar{\beta} \int_0^T |V_s| ds, \quad \bar{\beta} := |b| + |c\rho| \|h\|_\infty,$$

therefore, using the fact that  $(x + y)^p \leq 2^{p-1}(x^p + y^p)$  for  $p \geq 1$ , and the Hölder inequality, this gives

$$\sup_{t \in [0, T]} \left| \int_0^t a + (b + \mathbf{1}_{s \leq \tau_n} c \rho h(s)) V_s ds \right|^p \leq 2^{p-1} \left( (|a|T)^p + \left( \int_0^T |V_s| ds \right)^p \right) \leq 2^{p-1} \left( (|a|T)^p + T^{p-1} \int_0^T |V_s|^p ds \right)$$

On the other hand, BDG and Hölder inequalities leads to

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}^n} \left[ \sup_{t \in [0, T]} \left| \int_0^t c \sqrt{V_s} dW_s^n \right|^p \right] &\leq C_p \mathbb{E}^{\mathbb{Q}^n} \left[ \left( \int_0^T c^2 V_s ds \right)^{\frac{p}{2}} \right] \leq C_p T^{\frac{p}{2}-1} c^p \mathbb{E}^{\mathbb{Q}^n} \left[ \int_0^T V_s^{\frac{p}{2}} ds \right] \\ &\leq C_p T^{\frac{p}{2}-1} c^p \left( T + \int_0^T \mathbb{E}^{\mathbb{Q}^n} [V_s^p] ds \right) \end{aligned}$$

where the last inequality comes from the fact that  $x^{\frac{p}{2}} \leq 1 + x^p$  for all  $x \geq 0$  and  $p \geq 1$ . Putting everything together, it yields to

$$\mathbb{E}^{\mathbb{Q}^n} \left[ \sup_{t \in [0, T]} |V_t|^p \right] \leq C \left( 1 + |V_0|^p + \int_0^T \mathbb{E}^{\mathbb{Q}^n} [V_s^p] ds \right),$$

with  $C = C(p, T, a, b, c, \rho, \|h\|_\infty)$  which does not depend on  $n$ . Finally, we conclude that

$$\mathbb{E}^{\mathbb{Q}^n} \left[ \sup_{t \in [0, T]} V_t^p \right] \leq C_0 e^{C_1 T}$$

using the fact that  $\mathbb{E}[V_s^p] \leq \mathbb{E} \left[ \sup_{u \in [0, s]} V_u^p \right]$  combined with the Grönwall inequality. In particular, this procedure implies  $\mathbb{Q}^n(\tau_n < T) \xrightarrow{n \rightarrow \infty} 0$  (and in particular  $\mathbb{Q}^n(\tau_n = T) \xrightarrow{n \rightarrow \infty} 1$ ). The fact that

$$\mathbb{E}[M_T] \geq \mathbb{E}[M_T \mathbf{1}_{\tau_n = T}] = \mathbb{Q}^n(\tau_n = T),$$

and  $n \rightarrow \infty$  yields  $\mathbb{E}[M_T] \geq 1$ . This completes the proof.  $\square$

Here, we consider a generic class of time-dependent Riccati equations that encompass the ones for the Heston model, in the form

$$\psi'(t) = a(t)\psi(t)^2 + b(t)\psi(t) + c(t), \quad \psi(0) = u_0, \quad t \in [0, T],$$

with  $u_0 \in \mathbb{C}$ , and  $a, b, c, : [0, T] \rightarrow \mathbb{C}$  measurable and bounded functions. If  $\Re(u_0) \leq 0$  and

$$\Im(a(t)) = 0, \quad a(t) > 0, \quad \Re(c(t)) + \frac{\Im(b(t))^2}{4a(t)} \leq 0, \quad t \in [0, T],$$

there exists a unique solution  $\psi : [0, T] \rightarrow \mathbb{C}$  to the Riccati equations such that  $\Re(\psi(t)) \leq 0$ , for  $t \in [0, T]$  and  $\sup_{t \in [0, T]} |\psi(t)| < \infty$ .

### The characteristic function

The following theorem provides the joint conditional characteristic function of the log-price  $\ln S$  and integrated variance  $\bar{V} := \int_0^T V_s ds$  in the Heston model in terms of a solution to a system of time-dependent Riccati ODE.

**Theorem.** Let  $f, g : [0, T] \rightarrow \mathbb{C}$  be measurable and bounded functions such that

$$\Re(g) + \frac{1}{2} ((\Re(f))^2 - \Re(f)) \leq 0.$$

Then, the joint conditional characteristic function of the log-price and the integrated variance is given by

$$\mathbb{E} \left[ \exp \left( \int_t^T f(T-s) d\ln S_s + \int_t^T g(T-s) d\bar{V}_s \right) \middle| \mathcal{F}_t \right] = \exp(\phi(T-t) + \psi(T-t)V_t), \quad t \in [0, T],$$

where  $(\phi, \psi)$  is the solution to the following system of time-dependent Riccati equations:

$$\begin{aligned} \phi'(t) &= a\psi(t), \quad \phi(0) = 0, \\ \psi'(t) &= \frac{c^2}{2}\psi(t)^2 + (\rho cf(t) + b)\psi(t) + g(t) + \frac{f(t)^2 - f(t)}{2} \end{aligned}$$

*Proof. Step 1.* We first argue the existence of a solution of the above system of Riccati equations. Let us rewrite the Riccati ODE as

$$\psi'(t) = a\psi(t)^2 + b(t)\psi(t) + c(t), \quad \psi(0) = 0,$$

with

$$a = \frac{c^2}{2}, \quad b(t) = \rho cf(t) + b, \quad c(t) = g(t) + \frac{f(t)^2 - f(t)}{2}$$

The two first conditions  $\Im(a) = 0$  and  $a > 0$  are trivially satisfied. Regarding the last one,

$$\begin{aligned} \Re(c(t)) + \frac{\Im(b(t))^2}{4a(t)} &= \Re(g(t)) + \frac{\Re(f(t)^2) - \Re(f(t))}{2} + \rho^2 \frac{\Im(f(t))^2}{2} \\ &= \Re(g(t)) + \frac{\Re(f(t))^2 - \Re(f(t))}{2} + (\rho^2 - 1) \frac{\Im(f(t))^2}{2} \leq 0. \end{aligned}$$

Consequently, there exists a unique  $\psi : [0, T] \rightarrow \mathbb{C}$  solution to the above Riccati ODE such that  $\Re(\psi(t)) \leq 0$ , for all  $t \in [0, T]$ .

**Step 2.** We are now able to prove the expression for the characteristic function. Define the processes

$$U_t := \phi(T-t) + \psi(T-t)V_t + \int_0^t f(T-s) d\ln S_s + \int_0^t g(T-s) d\bar{V}_s, \quad M_t := \exp(U_t).$$

This amounts to showing that  $M$  is a martingale. We first show that  $M$  is a local martingale using the Itô's formula. It is clear that

$$dM_t = M_t (dU_t + \frac{1}{2} d\langle U \rangle_t),$$

with

$$\begin{aligned} dU_t &= (-\phi'(T-t) - \psi'(T-t)V_t + \psi(T-t)(a + bV_t) - f(T-t)\frac{V_t}{2} + g(T-t)V_t) dt \\ &\quad + (c\psi(T-t) + \rho f(T-t))\sqrt{V_t} dW_t + \sqrt{1 - \rho^2} f(T-t)\sqrt{V_t} dW_t^\perp \\ d\langle U \rangle_t &= ((c\psi(T-t) + \rho f(T-t))^2 + (1 - \rho^2)f(T-t)^2) V_t dt \end{aligned}$$

Therefore the drift of  $dM_t/M_t$  is given by

$$\begin{aligned}
0 &= -\phi'(T-t) - \psi'(T-t)V_t + \psi(T-t)(a+bV_t) - f(T-t)\frac{V_t}{2} + g(T-t)V_t \\
&\quad + \frac{V_t}{2} \left( (c\psi(T-t) + \rho f(T-t))^2 + (1-\rho^2)f(T-t)^2 \right) \\
\iff 0 &= -\phi'(T-t) + a\psi(T-t) + V_t \left( -\psi'(T-t) + (c\rho f(T-t) + b)\psi(T-t) + g(T-t) + \frac{c^2}{2}\psi(T-t)^2 + \right. \\
&\quad \left. + \frac{1}{2}(f(T-t)^2 - f(T-t)) \right)
\end{aligned}$$

which leads to the above Riccati equations. Now, it remains to show that  $M$  is true martingale.

$$\begin{aligned}
\Re(U_t) &\leq \int_0^t \Re(f(T-s)) d\ln S_s + \int_0^t \Re(g(T-s)) d\bar{V}_s \\
&= \int_0^t \Re(g(T-s)) - \frac{1}{2} \Re(f(T-s)) d\bar{V}_s + \int_0^t \Re(f(T-s)) \sqrt{V_s} dB_s \\
&\leq -\frac{1}{2} \int_0^t \Re(f(T-s))^2 V_s ds + \int_0^t \Re(f(T-s)) \sqrt{V_s} dB_s := \tilde{U}_t
\end{aligned}$$

It follows that  $|M_t| = \exp(\Re(U_t)) \leq \exp(\tilde{U}_t)$ , where  $\exp(\tilde{U})$  is a true martingale. This is a simple application of the Lemma stated before with  $h = \Re(f)$  which is bounded by assumption. This shows that  $M$  is local martingale which satisfies for all  $t \in [0, T]$ ,  $\mathbb{E}[|M_t|] < \infty$ . Said differently, it implies that  $M$  is a true martingale.  $\square$

**A useful particular case.** Let  $u, w \in \mathbb{C}$  such that  $\Re(w) + \frac{1}{2}(\Re(u)^2 - \Re(u)) \leq 0$ . The functions  $f(t) = u$  and  $g(t) = w$  are *a fortiori* bounded. The system of Riccati ODE could be rewrite as follows

$$\begin{aligned}
\phi'(t) &= a\psi(t), \quad \phi(0) = 0, \\
\psi'(t) &= \frac{c^2}{2}\psi(t)^2 + (\rho cu + b)\psi(t) + w + \frac{u^2+u}{2}, \quad \psi(0) = 0.
\end{aligned}$$

In this case, the solution is explicit :

$$\begin{aligned}
\psi(t) &= \frac{\beta(u) - D(u, w)}{c^2} \frac{1 - e^{-D(u, w)t}}{1 - G(u, w)e^{-D(u, w)t}}, \\
\phi(t) &= \frac{a}{c^2} \left( (\beta(u) - D(u, w))t - 2 \ln \left( \frac{G(u, w)e^{-D(u, w)t} - 1}{G(u, w) - 1} \right) \right), \\
\beta(u) &= -b - u\rho c, \quad D(u, w) = \sqrt{\beta(u)^2 + c^2(-2w + u - u^2)}, \quad G(u, w) = \frac{\beta(u) - D(u, w)}{\beta(u) + D(u, w)}.
\end{aligned}$$