Technical note: Heston's characteristic function

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February 23, 2026

Let $(W, W^\perp)$ be a two-dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geqslant 0}, \mathbb{P})$, which satisfies the usual conditions. We consider the Heston model :

$$
\begin{aligned}
dS_t &= S_t \sqrt{V_t}\, dB_t, \quad S_0 > 0, \\
dV_t &= (a + bV_t)\, dt + c\sqrt{V_t}\, dW_t, \quad V_0 \geqslant 0.
\end{aligned}
$$

where $a \geqslant 0$, $b, c \in \mathbb{R}$, and $B = \rho W + \sqrt{1 - \rho^2} W^\perp$. In the sequel, we will denote by $\mathbb{E}[\cdot]$ the expectation under the measure $\mathbb{P}$.

A martingale argument

In this part, we will state a general result to rigorously justify a semimartingale is a true martingale. It will be the cornerstone to conclude the proof for the characteric function, based on a martingale argument. By the way, the following Lemma proves in particular that the process $(S_t)_{t \geqslant 0}$ is a martingale.

Lemma. *Let $h \in L^\infty(\mathbb{R}^+, \mathbb{R})$ and define*

$$
U_t = \int_0^t h(s) \sqrt{V_s}\, dB_s.
$$

Then the stochastic exponential $\exp\left(U_t - \frac{1}{2}\langle U \rangle_t\right)$ is martingale. In particular, S is a martingale $(h \equiv 1)$.

Proof. Let $M_t := \exp\left(U_t - \frac{1}{2}\langle U \rangle_t\right)$. Since M is a nonnegative local martingale, it is a supermartingale. Indeed, there exists by definition an increasing sequence of stopping times $(\sigma_n)_{n \geqslant 0}$ such that $\sigma_n \uparrow \infty$ and $(M_{t \wedge \sigma_n})_{t \geqslant 0}$ is a martingale. Because of $\sigma_n \uparrow \infty$ and $t \wedge \sigma_n \uparrow t$, it leads to

$$
M_{t \wedge \sigma_n} \xrightarrow[n \to \infty]{\text{a.s.}} M_t,
$$

The conditional version of the Fatou's lemma applies here, because $M \geqslant 0$, and yields

$$
\mathbb{E}[M_t | \mathscr{F}_s] = \mathbb{E}\left[\liminf_{n \to \infty} M_{t \wedge \sigma_n} \,\Big|\, \mathscr{F}_s \right] \leqslant \liminf_{n \to \infty} \mathbb{E}\left[M_{t \wedge \sigma_n} | \mathscr{F}_s \right] = M_s.
$$

In particular, it implies that $\mathbb{E}[M_T] \leqslant \mathbb{E}[M_0] = 1$ for all $T \in \mathbb{R}^+$. Therefore, it remains to show that $\mathbb{E}[M_T] \geqslant 1$ for all $T \in \mathbb{R}^+$. To this end, define stopping times $\tau_n = \inf\{t \geqslant 0,\, V_t > n\} \wedge T$. Then $(M_{t \wedge \tau_n})_{t \geqslant 0}$ is uniformly integrable martingale for each n by the Novikov's criterion:

$$
\mathbb{E}\left[\exp\left(\frac{1}{2} \int_0^{\tau_n} h(t)^2 V_t\, dt \right) \right] \leqslant \exp\left(\frac{1}{2} \|h\|_\infty^2 T n \right) < \infty,
$$

and may define probability measures $\mathbb{Q}^n$ by

$$
\frac{d\mathbb{Q}^n}{d\mathbb{P}} = M_{\tau_n} = \exp\left(\int_0^{\tau_n} h(s) \sqrt{V_s}\, dB_s - \frac{1}{2} \int_0^{\tau_n} h(s)^2 V_s\, ds \right).
$$

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By Girsanov's theorem, $dW_t^n := dW_t - \mathbb{1}_{t \leqslant \tau_n} \rho h(t) \sqrt{V_t} \, dt$ is a Brownian motion under $\mathbb{Q}^n$, and it leads to:

$$dV_t = \left(a + \left(b + \mathbb{1}_{t \leqslant \tau_n} c\rho h(t)\right) V_t\right) dt + c\sqrt{V_t} \, dW_t^n$$

Observe that the expression $a + \left(b + \mathbb{1}_{t \leqslant \tau_n(\omega)} \rho h(t)\right) v$ satisfies a linear growth condition in v, uniformly in (t, ω). Note that using Markov inequaliy

$$\mathbb{Q}^n(\tau_n < T) = \mathbb{Q}^n\left(\sup_{t \in [0,T]} V_t > n\right) \leqslant \frac{\mathbb{E}^{\mathbb{Q}^n}\left[\sup_{t \in [0,T]} V_t^p\right]}{n^p}, \quad \forall p \geqslant 1.$$

Moreover, it is clear that

$$\mathbb{E}^{\mathbb{Q}^n}\left[\sup_{t \in [0,T]} |V_t|^p\right] \leqslant C_p \mathbb{E}^{\mathbb{Q}^n}\left[|V_0|^p + \sup_{t \in [0,T]} \left|\int_0^t a + \left(b + \mathbb{1}_{s \leqslant \tau_n} c\rho h(s)\right) V_s \, ds\right|^p + \sup_{t \in [0,T]} \left|\int_0^t c\sqrt{V_s} \, dW_s^n\right|^p\right]$$

For all $t \leqslant T$,

$$\left|\int_0^t a + \left(b + \mathbb{1}_{s \leqslant \tau_n} c\rho h(s)\right) V_s \, ds\right| \leqslant |a|T + \overline{\beta} \int_0^T |V_s| \, ds, \quad \overline{\beta} := |b| + |c\rho| \|h\|_\infty,$$

therefore, using the fact that $(x + y)^p \leqslant 2^{p-1}(x^p + y^p)$ for $p \geqslant 1$, and the Hölder inequality, this gives

$$\sup_{t \in [0,T]} \left|\int_0^t a + \left(b + \mathbb{1}_{s \leqslant \tau_n} c\rho h(s)\right) V_s \, ds\right|^p \leqslant 2^{p-1}\left((|a|T)^p + \left(\int_0^T |V_s| \, ds\right)^p\right) \leqslant 2^{p-1}\left((|a|T)^p + T^{p-1}\int_0^T |V_s|^p \, ds\right)$$

On the other hand, BDG and Hölder inequalities leads to

$$\mathbb{E}^{\mathbb{Q}^n}\left[\sup_{t \in [0,T]} \left|\int_0^t c\sqrt{V_s} \, dW_s^n\right|^p\right] \leqslant C_p \mathbb{E}^{\mathbb{Q}^n}\left[\left(\int_0^T c^2 V_s \, ds\right)^{\frac{p}{2}}\right] \leqslant C_p T^{\frac{p}{2}-1} c^p \mathbb{E}^{\mathbb{Q}^n}\left[\int_0^T V_s^{\frac{p}{2}} \, ds\right]$$

$$\leqslant C_p T^{\frac{p}{2}-1} c^p \left(T + \int_0^T \mathbb{E}^{\mathbb{Q}^n}\left[V_s^p\right] ds\right)$$

where the last inequality comes from the fact that $x^{\frac{p}{2}} \leqslant 1 + x^p$ for all $x \geqslant 0$ and $p \geqslant 1$. Putting everything together, it yields to

$$\mathbb{E}^{\mathbb{Q}^n}\left[\sup_{t \in [0,T]} |V_t|^p\right] \leqslant C\left(1 + |V_0|^p + \int_0^T \mathbb{E}^{\mathbb{Q}^n}[V_s^p] \, ds\right),$$

with $C = C(p, T, a, b, c, \rho, \|h\|_\infty)$ which does not depend on n. Finally, we conclude that

$$\mathbb{E}^{\mathbb{Q}^n}\left[\sup_{t \in [0,T]} V_t^p\right] \leqslant C_0 e^{C_1 T}$$

using the fact that $\mathbb{E}[V_s^p] \leqslant \mathbb{E}\left[\sup_{u \in [0,s]} V_u^p\right]$ combined with the Grönwall inequality. In particular, this procedure implies $\mathbb{Q}^n(\tau_n < T) \xrightarrow[n \to \infty]{} 0$ (and in particular $\mathbb{Q}^n(\tau_n = T) \xrightarrow[n \to \infty]{} 1$). The fact that

$$\mathbb{E}[M_T] \geqslant \mathbb{E}[M_T \mathbb{1}_{\tau_n = T}] = \mathbb{Q}^n(\tau_n = T),$$

and $n \to \infty$ yields $\mathbb{E}[M_T] \geqslant 1$. This completes the proof. $\qquad \square$

Time-dependent Riccati ODEs: existence and uniqueness

Here, we consider a generic class of time-dependent Riccati equations that encompass the ones for the Heston model, in the form

$$\psi'(t) = a(t)\psi(t)^2 + b(t)\psi(t) + c(t), \quad \psi(0) = u_0, \quad t \in [0, T],$$

with $u_0 \in \mathbb{C}$, and $a, b, c, : [0, T] \to \mathbb{C}$ measurable and bounded functions. If $\mathfrak{Re}(u_0) \leqslant 0$ and

$$\mathfrak{Im}(a(t)) = 0, \quad a(t) > 0, \quad \mathfrak{Re}(c(t)) + \frac{\mathfrak{Im}(b(t))^2}{4a(t)} \leqslant 0, \quad t \in [0, T],$$

there exists a unique solution $\psi : [0, T] \to \mathbb{C}$ to the Riccati equations such that $\mathfrak{Re}(\psi(t)) \leqslant 0$, for $t \in [0, T]$ and $\sup_{t \in [0,T]} |\psi(t)| < \infty$.

The characteristic function

The following theorem provides the joint conditional characteristic function of the log-price $\ln S$ and integrated variance $\overline{V} := \int_0^{\cdot} V_s \, ds$ in the Heston model in terms of a solution to a system of time-dependent Riccati ODE.

Theorem. *Let $f, g : [0, T] \to \mathbb{C}$ be measurable and bounded functions such that*

$$\mathfrak{Re}(g) + \tfrac{1}{2}\left((\mathfrak{Re}(f))^2 - \mathfrak{Re}(f)\right) \leqslant 0.$$

Then, the joint conditional characteristic function of the log-price and the integrated variance is given by

$$\mathbb{E}\left[\exp\left(\int_t^T f(T-s)\, d\ln S_s + \int_t^T g(T-s)\, d\overline{V}_s\right) \Bigg| \mathscr{F}_t\right] = \exp\left(\phi(T-t) + \psi(T-t)V_t\right), \quad t \in [0, T],$$

where (ϕ, ψ) is the solution to the following system of time-dependent Riccati equations:

$$\phi'(t) = a\psi(t), \quad \phi(0) = 0,$$
$$\psi'(t) = \tfrac{c^2}{2}\psi(t)^2 + (\rho c f(t) + b)\psi(t) + g(t) + \tfrac{f(t)^2 - f(t)}{2}$$

Proof. **Step 1.** We first argue the existence of a solution of the above system of Riccati equations. Let us rewrite the Riccati ODE as

$$\psi'(t) = a\psi(t)^2 + b(t)\psi(t) + c(t), \quad \psi(0) = 0,$$

with

$$a = \tfrac{c^2}{2}, \quad b(t) = \rho c f(t) + b, \quad c(t) = g(t) + \tfrac{f(t)^2 - f(t)}{2}$$

The two first conditions $\mathfrak{Im}(a) = 0$ and $a > 0$ are trivially satisfied. Regarding the last one,

$$\mathfrak{Re}(c(t)) + \tfrac{\mathfrak{Im}(b(t))^2}{4a(t)} = \mathfrak{Re}(g(t)) + \tfrac{\mathfrak{Re}(f(t)^2) - \mathfrak{Re}(f(t))}{2} + \rho^2 \tfrac{\mathfrak{Im}(f(t))^2}{2}$$
$$= \mathfrak{Re}(g(t)) + \tfrac{\mathfrak{Re}(f(t))^2 - \mathfrak{Re}(f(t))}{2} + (\rho^2 - 1)\tfrac{\mathfrak{Im}(f(t))^2}{2} \leqslant 0.$$

Consequently, there exists a unique $\psi : [0, T] \to \mathbb{C}$ solution to the above Riccati ODE such that $\mathfrak{Re}(\psi(t)) \leqslant 0$, for all $t \in [0, T]$.

Step 2. We are now able to prove the expression for the characteristic function. Define the processes

$$U_t := \phi(T-t) + \psi(T-t)V_t + \int_0^t f(T-s)\, d\ln S_s + \int_0^t g(T-s)\, d\overline{V}_s, \quad M_t := \exp(U_t).$$

This amounts to showing that M is a martingale. We first show that M is a local martingale using the Itô's formula. It is clear that

$$dM_t = M_t\left(dU_t + \tfrac{1}{2}d\langle U\rangle_t\right),$$

with

$$dU_t = \left(-\phi'(T-t) - \psi'(T-t)V_t + \psi(T-t)(a + bV_t) - f(T-t)\tfrac{V_t}{2} + g(T-t)V_t\right)dt$$
$$\qquad + (c\psi(T-t) + \rho f(T-t))\sqrt{V_t}\, dW_t + \sqrt{1-\rho^2}f(T-t)\sqrt{V_t}\, dW_t^\perp$$
$$d\langle U\rangle_t = \left((c\psi(T-t) + \rho f(T-t))^2 + (1-\rho^2)f(T-t)^2\right)V_t\, dt$$

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Therefore the drift of dM_t/M_t is given by

$$0 = -\phi'(T-t) - \psi'(T-t)V_t + \psi(T-t)(a+bV_t) - f(T-t)\tfrac{V_t}{2} + g(T-t)V_t$$
$$+ \tfrac{V_t}{2}\left((c\psi(T-t) + \rho f(T-t))^2 + (1-\rho^2)f(T-t)^2\right)$$
$$\iff 0 = -\phi'(T-t) + a\psi(T-t) + V_t\left(-\psi'(T-t) + (c\rho f(T-t)+b)\psi(T-t) + g(T-t) + \tfrac{c^2}{2}\psi(T-t)^2 + \right.$$
$$\left. + \tfrac{1}{2}\left(f(T-t)^2 - f(T-t)\right)\right)$$

which leads to the above Riccati equations. Now, it remains to show that M is true martingale.

$$\mathfrak{Re}(U_t) \leqslant \int_0^t \mathfrak{Re}(f(T-t))\, d\ln S_s + \int_0^t \mathfrak{Re}(g(T-s))\, d\overline{V}_s$$
$$= \int_0^t \mathfrak{Re}(g(T-s)) - \frac{1}{2}\mathfrak{Re}(f(T-s))\, d\overline{V}_s + \int_0^t \mathfrak{Re}(f(T-s))\sqrt{V_s}\, dB_s$$
$$\leqslant -\frac{1}{2}\int_0^t \mathfrak{Re}(f(T-s))^2 V_s\, ds + \int_0^t \mathfrak{Re}(f(T-s))\sqrt{V_s}\, dB_s := \widetilde{U}_t$$

It follows that $|M_t| = \exp\left(\mathfrak{Re}(U_t)\right) \leqslant \exp(\widetilde{U}_t)$, where $\exp(\widetilde{U})$ is a true martingale. This a simple application of the Lemma stated before with $h = \mathfrak{Re}(f)$ which is bounded by assumption. This shows that M is local martingale which satisfies for all $t \in [0,T]$, $\mathbb{E}[|M_t|] < \infty$. Said differently, it implies that M is a true martingale. $\qquad\square$

A useful particular case. Let $u, w \in \mathbb{C}$ such that $\mathfrak{Re}(w) + \tfrac{1}{2}\left(\mathfrak{Re}(u)^2 - \mathfrak{Re}(u)\right) \leqslant 0$. The functions $f(t) = u$ and $g(t) = w$ are *a fortiori* bounded. The system of Riccati ODE could be rewrite as follows

$$\phi'(t) = a\psi(t), \quad \phi(0) = 0,$$
$$\psi'(t) = \tfrac{c^2}{2}\psi(t)^2 + (\rho cu + b)\psi(t) + w + \tfrac{u^2-u}{2}, \quad \psi(0) = 0.$$

In this case, the solution is explicit :

$$\psi(t) = \frac{\beta(u) - D(u,w)}{c^2}\frac{1 - e^{-D(u,w)t}}{1 - G(u,w)e^{-D(u,w)t}},$$
$$\phi(t) = \frac{a}{c^2}\left((\beta(u) - D(u,w))t - 2\ln\left(\frac{G(u,w)e^{-D(u,w)t}-1}{G(u,w)-1}\right)\right),$$
$$\beta(u) = -b - u\rho c, \quad D(u,w) = \sqrt{\beta(u)^2 + c^2(-2w + u - u^2)}, \quad G(u,w) = \frac{\beta(u) - D(u,w)}{\beta(u) + D(u,w)}.$$

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