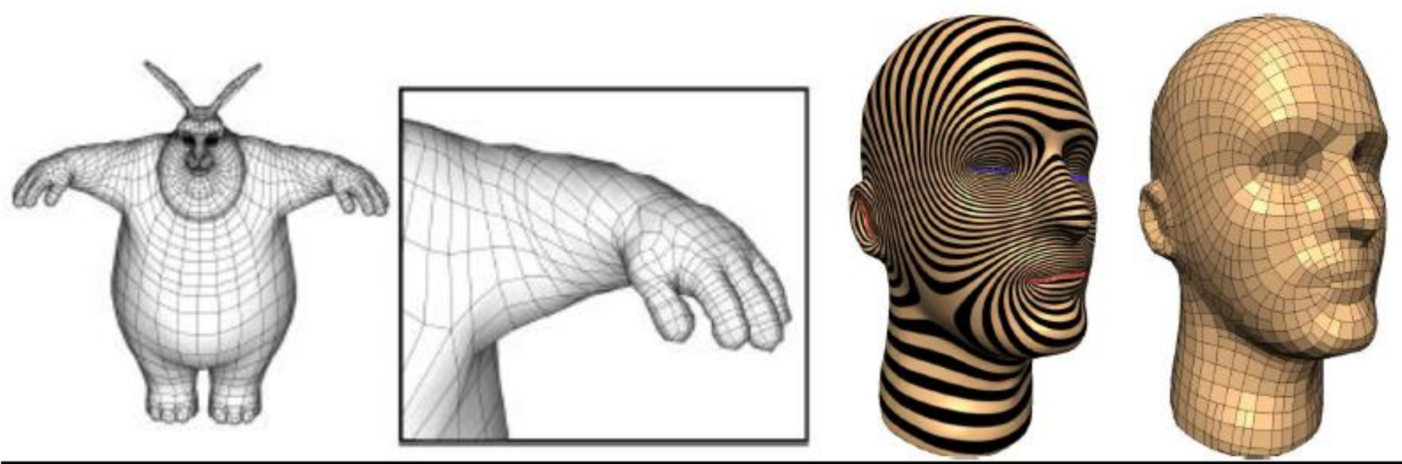
Curves

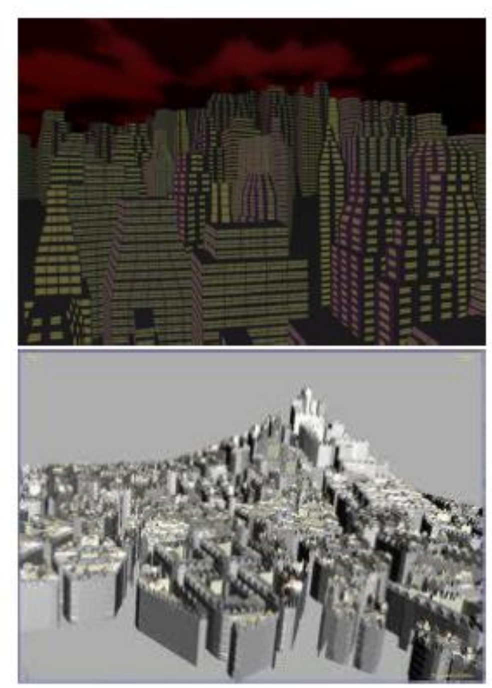
How to create a virtual world?

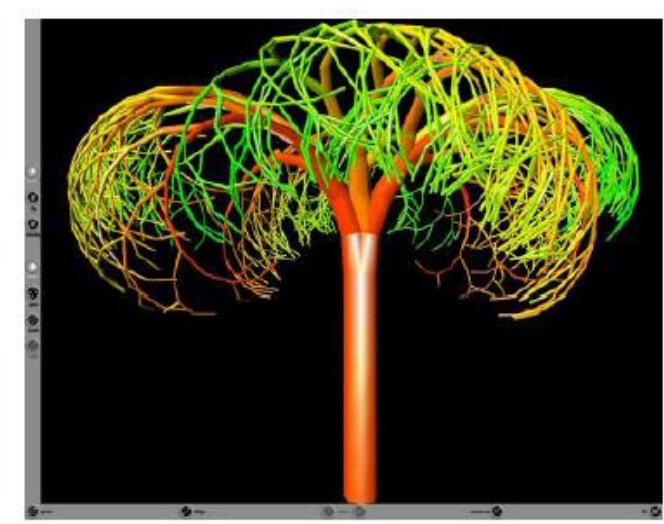
To compose scenes

We need to define objects

- Characters
- Terrains
- Objects (trees, furniture, buildings etc)

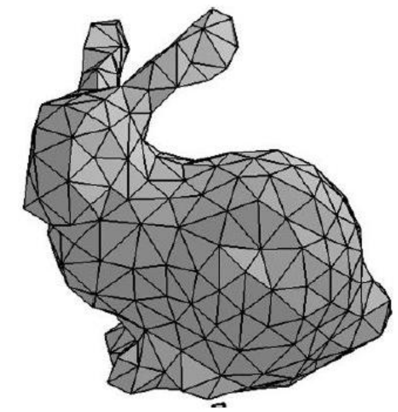


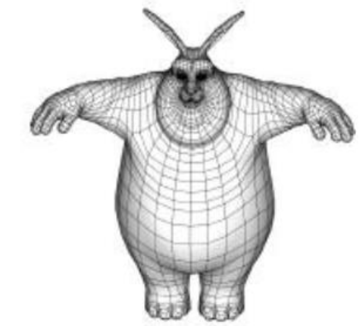




Geometric representations

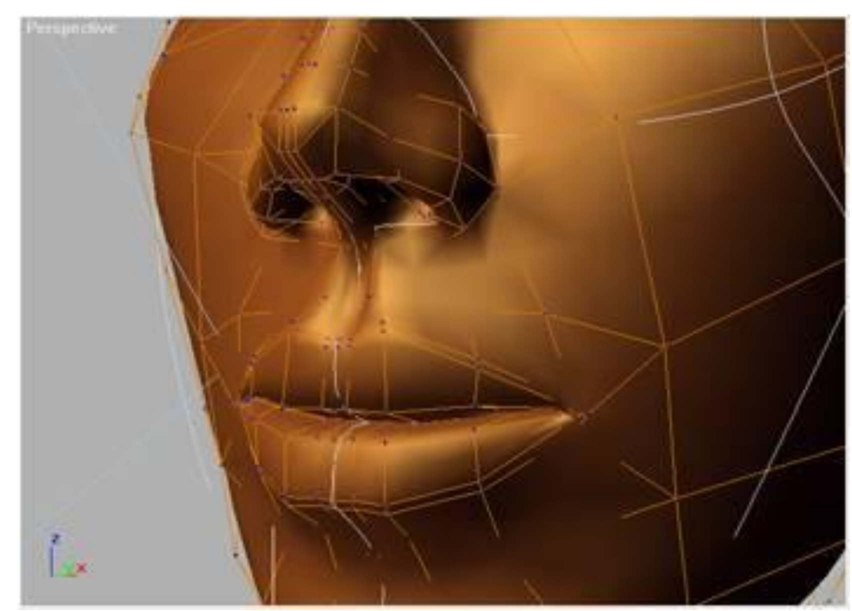
- Meshes
  - Triangle, quadrilateral, polygon





- Implicit surfaces
  - Blobs, metaballs

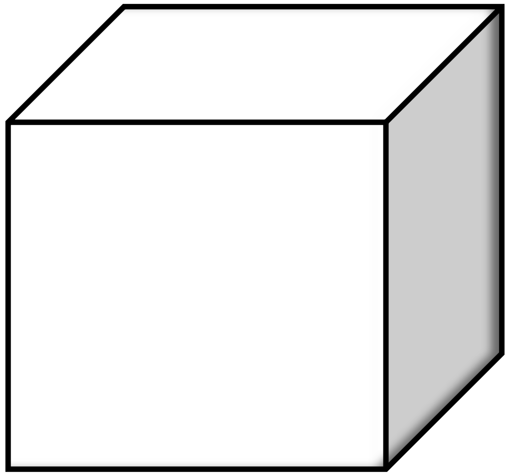
- Parametric surfaces / curves
  - Polynomials
  - Bezier curves, B-splines



Motivation

Smoothness

Many applications require smooth surfaces



[scene360.com]

Can produce smooth surfaces with less parameters

- Easier to design
- Can efficiently preserve complex structures

Original Spline

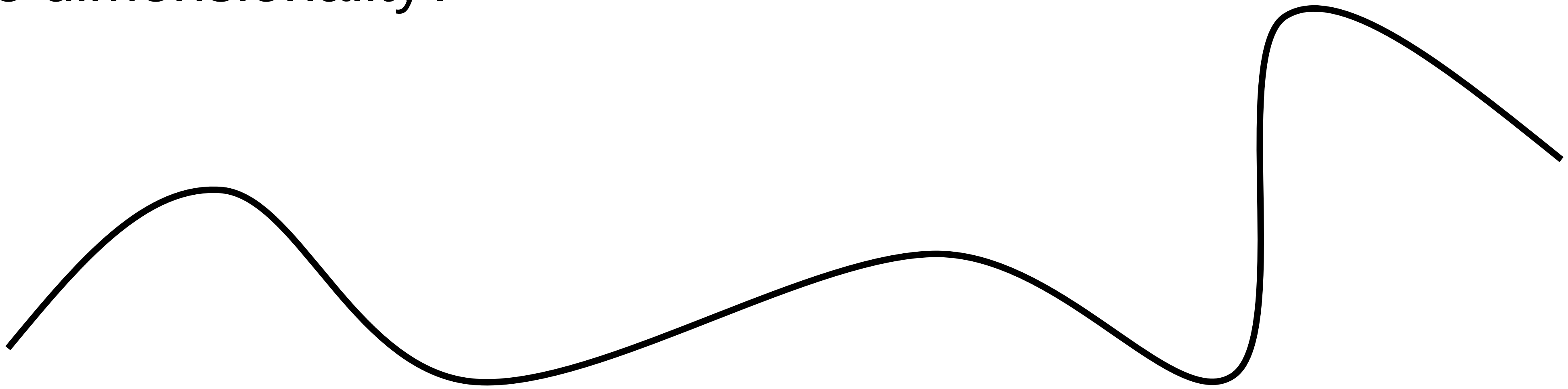
From draftsmanship to CG

- Control
  - user specified control points
  - analogy: ducks
- Smoothness
  - smooth functions
  - usually low order polynomials
  - analogy: physical constraints, optimization

What is a curve?

A set of points that the pen traces over an interval of time

What is the dimensionality?



Implicit form: $\mathrm{f}(\mathrm{x}, \mathrm{y}) = x^2 + y^2 - 1 = 0$

- Find the points that satisfy the equation

Parametric form: $(\mathrm{x}, \mathrm{y}) = \mathrm{f}(\mathrm{t}) = (cost, sint), \quad t \in [0,2\pi)$

- Easier to draw

Spline segments

Linear Segment

A line segment connecting point $\boldsymbol{p_o}$ to $\boldsymbol{p_1}$

Such that $f(0) = \boldsymbol{p_0}$ and $f(1) = \boldsymbol{p_1}$

$$f_x(t) = (1-t)\boldsymbol{x_o} + t\boldsymbol{x_1}$$

$$f_y(t) = (1-t)\boldsymbol{y_o} + t\boldsymbol{y_1}$$

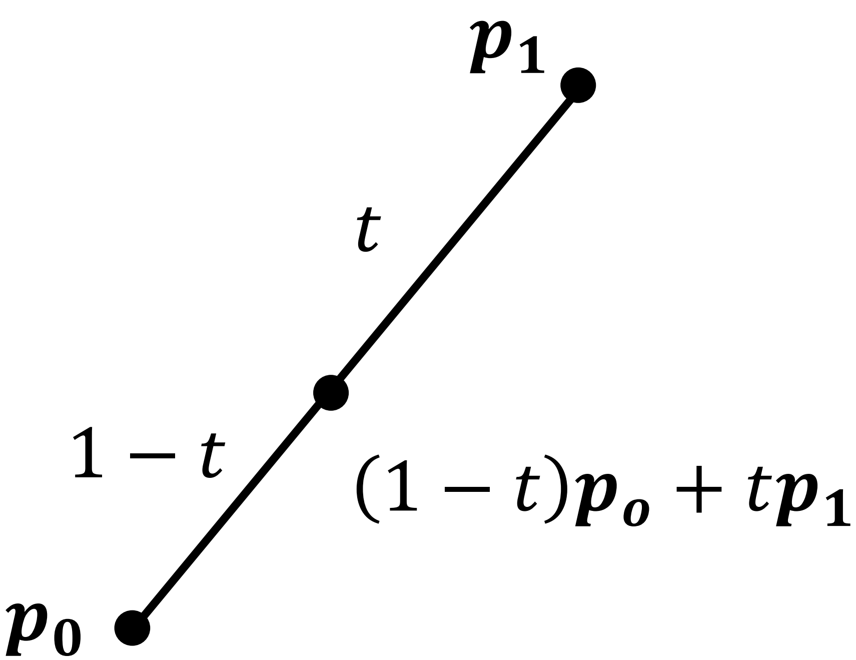
Vector formulation

$$f(t) = (1-t)\boldsymbol{p_o} + t\boldsymbol{p_1}$$

Matrix formulation

$$f(t) = (t \quad 1)\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix}\begin{pmatrix} \boldsymbol{p_0} \\ \boldsymbol{p_1} \end{pmatrix}$$

$$f(t) = (t \quad 1)\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{p_0} \\ \boldsymbol{p_1} \end{pmatrix}$$



Matrix form of spline

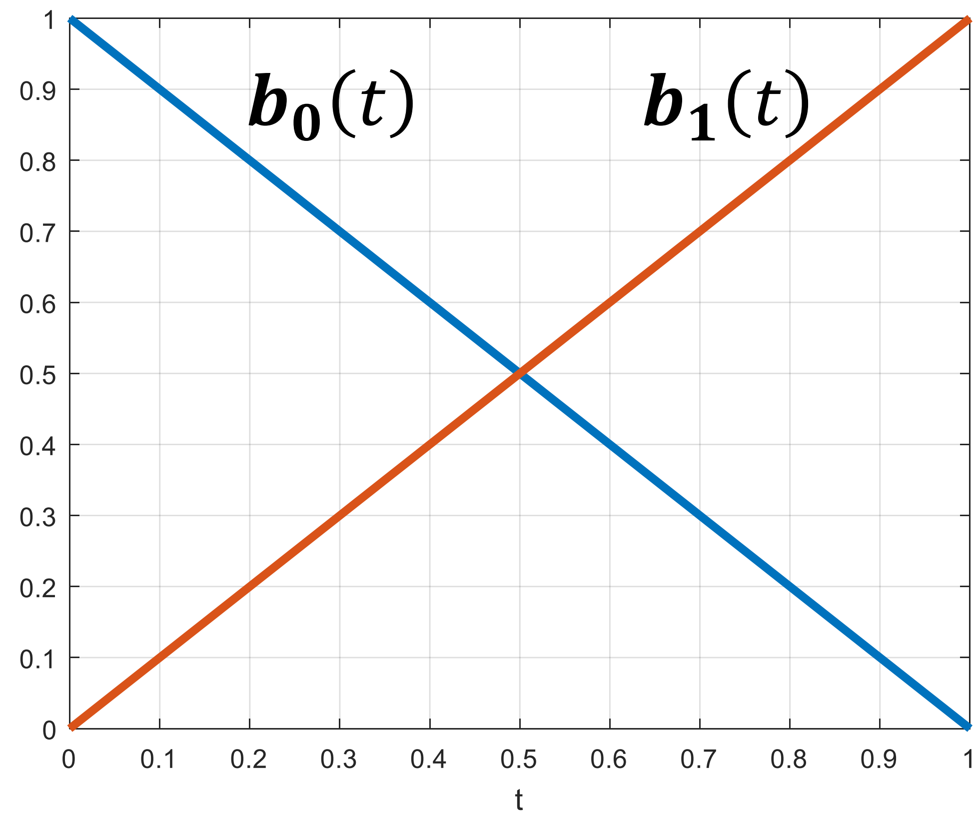
$$f(t) = \begin{pmatrix} t & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{p_0} \\ \boldsymbol{p_1} \end{pmatrix}$$

Blending functions $\boldsymbol{b}(t)$ specify how to blend the values of the control point vector

$$\boldsymbol{f}(t) = \boldsymbol{b_0}(t)\boldsymbol{p_0} + \boldsymbol{b_1}(t)\boldsymbol{p_1}$$

$$\boldsymbol{b_0}(t) = 1 - t$$

$$\boldsymbol{b_1}(t) = t$$



Matrix form of spline

Blending functions

$$\boldsymbol{f}(t) = (t^2 \;\; t \;\; 1)\begin{pmatrix} 2 & -4 & 2 \\ -3 & 4 & -1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{p_0} \\ \boldsymbol{p_1} \\ \boldsymbol{p_2} \end{pmatrix}$$

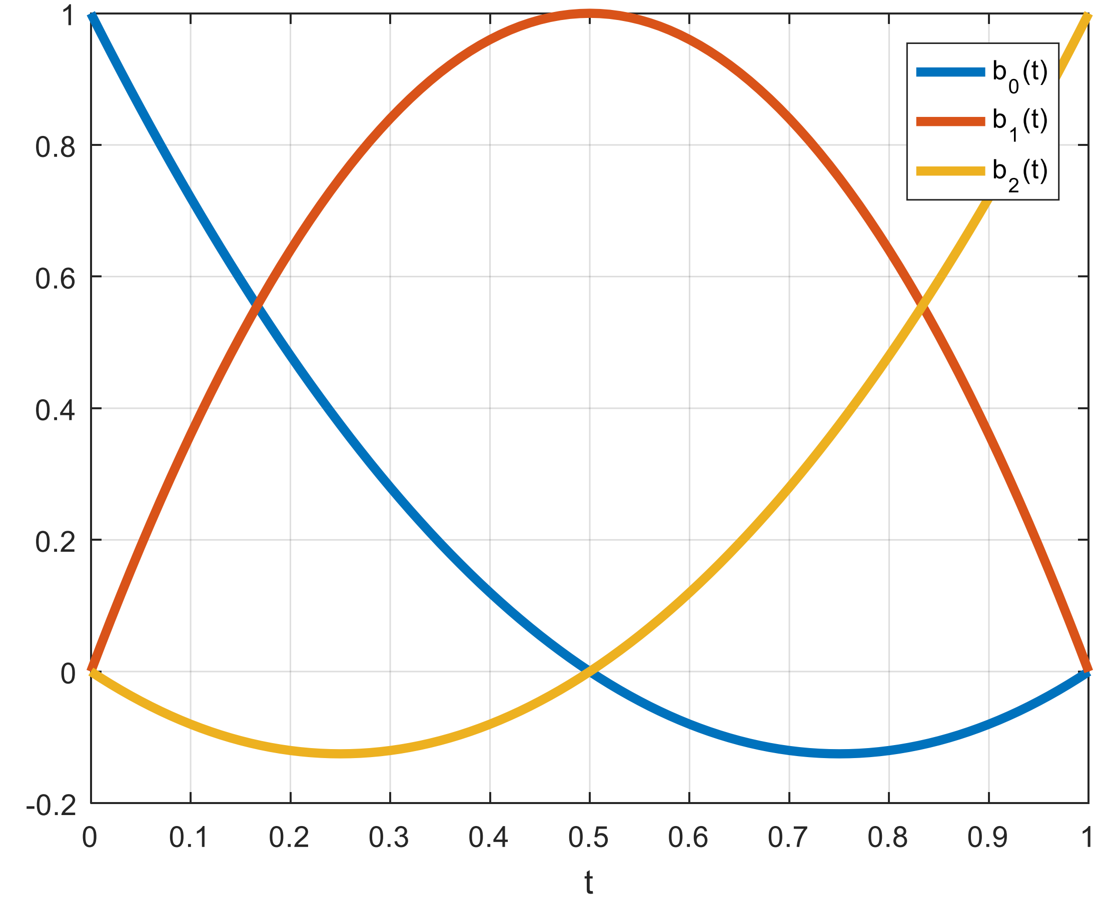
Blending functions $\boldsymbol{b}(t)$ specify how to blend the values of the control point vector

$$\boldsymbol{f}(t) = \boldsymbol{b_0}(t)\boldsymbol{p_0} + \boldsymbol{b_1}(t)\boldsymbol{p_1} + \boldsymbol{b_2}(t)\boldsymbol{p_2}$$

$$\boldsymbol{b_0}(t) = 2t^2 - 3t + 1$$

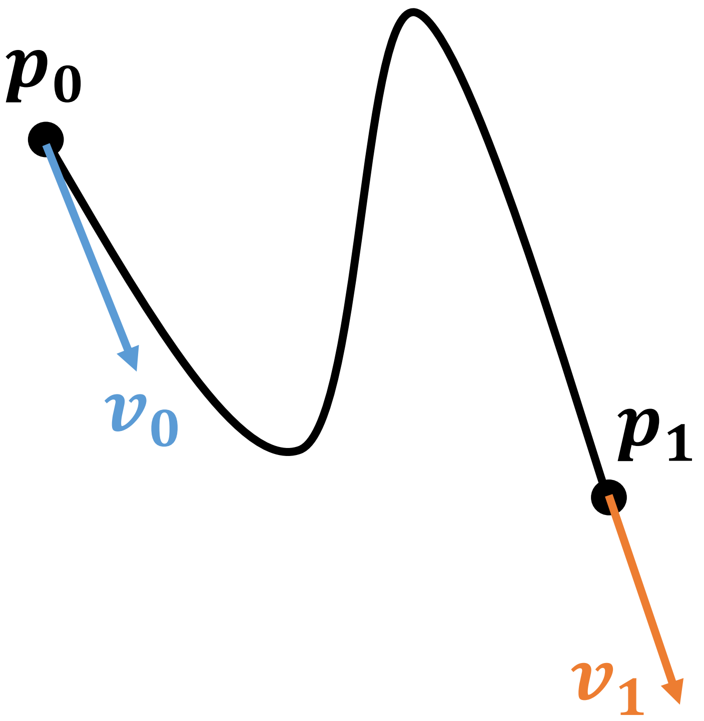
$$\boldsymbol{b_1}(t) = -4t^2 - 4t$$

$$\boldsymbol{b_2}(t) = 2t^2 - 1$$



Hermite spline

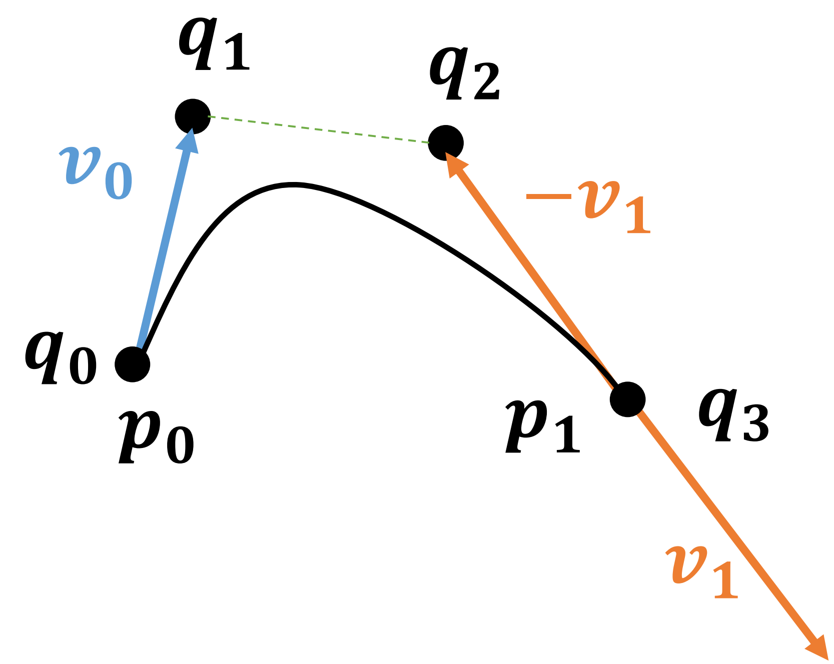
- Piecewise cubic ($\boldsymbol{f}(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$)
- Additional constraint on tangents (derivatives)
  - $\boldsymbol{f}(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$
  - $\boldsymbol{f}'^{(t)} = a_1 + 2a_2 t + 3a_3 t^2$
  - $\boldsymbol{p_0} = f(0) \quad = a_0$
  - $\boldsymbol{p_1} = f(1) \quad = a_0 + a_1 + a_2 + a_3$
  - $\boldsymbol{v}_1 = f'(0) \quad = a_1$
  - $\boldsymbol{v}_2 = f'(1) \quad = a_1 + 2a_2 + \ 3a_3$



- Simpler matrix form

$$\boldsymbol{f}(t) = (t^3 \; t^2 \;\; t \;\; 1)\begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} \boldsymbol{p_0} \\ \boldsymbol{p_1} \\ \boldsymbol{v_1} \\ \boldsymbol{v_2} \end{pmatrix}$$

Hermite to Bézier



Specify tangents as points

- $p_0 = q_0, p_1 = q_3, v_0 = 3(q_1 - q_0), v_1 = 3(q_3 - q_2)$
- $\begin{pmatrix} \boldsymbol{p_0} \\ \boldsymbol{p_1} \\ \boldsymbol{v_1} \\ \boldsymbol{v_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 1 & 0 \\ 0 & 0 & -3 & 3 \end{pmatrix} \begin{pmatrix} \boldsymbol{q_0} \\ \boldsymbol{q_1} \\ \boldsymbol{q_2} \\ \boldsymbol{q_3} \end{pmatrix}$
- Update Hermite eq. (from previous slide)

$$\boldsymbol{f}(t) = (t^3\ t^2\ \ t\ \ 1) \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 1 & 0 \\ 0 & 0 & -3 & 3 \end{pmatrix} \begin{pmatrix} \boldsymbol{q_0} \\ \boldsymbol{q_1} \\ \boldsymbol{q_2} \\ \boldsymbol{q_3} \end{pmatrix}$$

Bézier matrix

$$\boldsymbol{f}(t) = (t^3\ t^2\ \ t\ \ 1)\begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 2 & 0 & -6 & 6 \end{pmatrix}\begin{pmatrix} \boldsymbol{q_0} \\ \boldsymbol{q_1} \\ \boldsymbol{q_2} \\ \boldsymbol{q_3} \end{pmatrix}$$

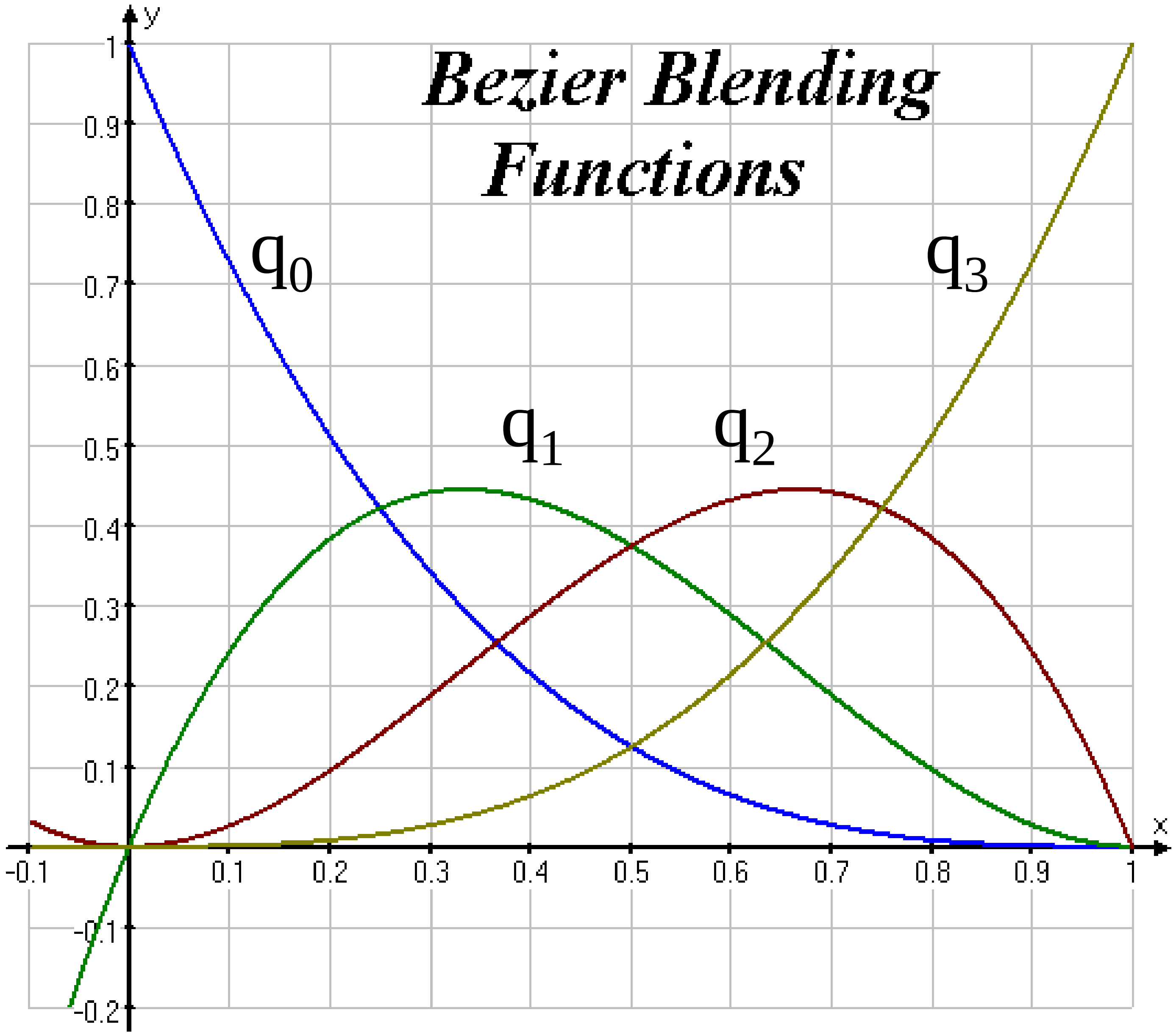
- $\boldsymbol{f}(t) = \sum_{n=0}^{d} \boldsymbol{b_{n,3}} \boldsymbol{q_n}$
- Blending functions $\boldsymbol{b}(t)$ has a special name in this case:
- Bernstein polynomials

$$b_{n,k} = \binom{n}{k} t^k (1-t)^{n-k}$$

and that defines Bézier curves for any degree

Bézier blending functions

- The functions sum to 1 at any point along the curve.
- Endpoints have full weight



Another view to Bézier segments

de Casteljau algorithm

Blend each linear spline with α and $\beta = 1 - \alpha$