MA 101 (Mathematics - I)
Exercise set 3: Solutions and Hints

---

1. (a) Can a power series $\sum_{n=0}^{\infty} a_n(x-3)^n$ converge at $x=0$ and diverge at $x=5$?

   (b) Suppose $\sum_{n=0}^{\infty} a_n x^n$ converges at -3 and diverges at 4. What can you say about the radius of convergence of the power series?

   (c) Prove or disprove: There is a power series about 0 which converges at π and diverges at $-\pi$.

> **Solution:**
> (a) No. If the power series about 3 converge a 3, then it must converge in $(0,6)$.
> (b) $R \in [3,4]$.
> (c) We know the domain of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{n}$ is $[-1,1)$.
> Check: $\sum_{n=1}^{\infty} \frac{x^n}{n(-\pi)^n}$ has domain of convergence $(-\pi,\pi]$.

2. Suppose $a_n > 0, a_n \to 0$ and $a_n^2 > \frac{1}{10^{50}} a_{n+1}$ for each n. Can you determine the domain of convergence of $\sum_{i=1}^{\infty} a_n x^n$?

> **Solution:** $\left|\frac{a_n}{a_{n+1}}\right| = \frac{a_n}{a_{n+1}} > \frac{1}{10^{50} a_n} \to \infty$, since $a_n > 0$ and $a_n \to 0$. Hence $R = \infty$.
> The domain of convergence is $\mathbb{R}$.

3. Suppose (a_n) is a sequence converging to 0. One student found that the power series $a_1 x + 2a_2x^2 + 3a_3x^3 + \cdots$ converges to f on the interval $(-5,5)$ and another student found that $1 + a_1x + a_2x^2 + a_3x^3 + \cdots$ converges to g with radius of convergence as R. How are the answers of the two students related?

> **Solution:** $R \geq 5$.

4. (a) Suppose that $\sum a_n x^n$ is a power series whose radius of convergence is R. Define $f(x) = \sum a_n x^n$ on $(-R,R)$. Argue that $\sum a_n x^n$ is the Taylor series of f about 0?

   (b) Use your argument and the Taylor series for $\sin x$ to find the Taylor series of $\sin x \cos 3x$ about 0.

Solution: (a) Yes, because $a_n = \frac{f^{(n)}(0)}{n!}$.
(b) Use $\sin x \cos x = \frac{1}{2}(\sin 4x - \sin 2x)$ and the Taylor series for $\sin x$.

5. Find radius of convergence and domain of convergence of the following power series:

1. $\sum \dfrac{x^n}{n^2}$
2. $\sum n(n+1)x^n$
3. $\sum \dfrac{(-1)^n x^{2n}}{n^2}$
4. $\sum \dfrac{3^n x^n}{2^n}$
5. $\sum n^n x^n$
6. $2x + \left(\dfrac{9}{4}x\right)^2 + \cdots + \left(\left(\dfrac{n+1}{n}\right)^n x\right)^n + \cdots$

Solution: Let $R =$ radius of convergence, and $D =$ domain of convergence.

1. $R = 1$; $D = [-1, 1]$
2. $R = 1$; $D = (-1, 1)$
3. $R = 1$; $D = [-1, 1]$
4. $R = 2/3$; $D = (-2/3, 2/3)$
5. $R = 0$; $D = \{0\}$
6. $|a_n|^{1/n} = \left(\frac{n+1}{n}\right)^n \to e$. Therefore, $R = 1/e$. At $x = \pm(1/e)$, $a_n x^n$ does not converge to 0. Hence, $D = (-\frac{1}{e}, \frac{1}{e})$.

6. Find the domain of converegnce of the power series $\dfrac{1}{a} - \left(\dfrac{1}{a}\right)^2 (x-a) + \left(\dfrac{1}{a}\right)^3 (x-a)^2 - \left(\dfrac{1}{a}\right)^4 (x-a)^3 + \cdots$. Can you give an explicit formula for the function represented by the power series?

Solution: $(0, 2a)$; $f(x) = 1/x$.

7. (a) For $f(x) = x$ on $[0,1]$ calculate $L(f, \mathbf{P}_n)$ and $U(f, \mathbf{P}_n)$, conclude $f \in \mathcal{R}[0,1]$, and find $\int_a^b f$.

   (b) For $f(x) = x^2$ on $[0,1]$ calculate $L(f, \mathbf{P}_n)$ and $U(f, \mathbf{P}_n)$, conclude $f \in \mathcal{R}[0,1]$, and find $\int_a^b f$.

Solution: (b) $L(f, \mathbf{P}_n) = \sum_{i=1}^{n} \left(\frac{i-1}{n}\right)^2 \frac{1}{n} \to \frac{1}{3}, \quad U(f, \mathbf{P}_n) = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \frac{1}{n} \to \frac{1}{3}.$

8. For the function $f : [-2, 2] \to \mathbb{R}$ defined by $f(x) = x^5$, find a partition P such that $U(f, P) - L(f, P) < \epsilon$.

Solution: For $n \in \mathbb{N}$,

$$U(f, \mathbf{P}_n) - L(f, \mathbf{P}_n) = \sum_{i=1}^{n} \left[\left(\frac{i}{n}\right)^5 - \left(\frac{i-1}{n}\right)^5\right] \frac{1}{n} = \frac{1}{n}.$$

So, for $n > 1/\epsilon$, we have $U(f, \mathbf{P}_n) - L(f, \mathbf{P}_n) < \epsilon$.

9. Give an example of a function on $[0, 1]$ such that $L(f) = 1$ and $U(f) = 2$.

Solution: Define $f : [0, 1] \to \mathbb{R}$ by $f(x) = \begin{cases} 2, & \text{if } x \in \mathbb{Q}, \\ 1, & \text{otherwise.} \end{cases}$

10. Show by definition that $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = x^n$ is integrable.

Solution: Use the argument similar to that of Q.8.

11. Suppose you know that $\lim_{n\to\infty} U(f, \mathbf{P}_n) = \ell$. Is it true that $f \in \mathcal{R}([a, b])$?

Solution: No. Take the Dirichlet function on $[0, 1]$.

12. Show that $\frac{1}{2} \le \int_0^1 \frac{1+x-x^2}{1+x^4} dx \le \frac{5}{4}$.

Solution: $\dfrac{1}{2} \le \dfrac{1+x-x^2}{1+x^4} \le 1 + x - x^2$ on $[0, 1]$. Therefore,

$$\frac{1}{2} = \int_0^1 \frac{1}{2}\, dx \le \int_0^1 \frac{1+x-x^2}{1+x^4}\, dx \le \int_0^1 (1 + x - x^2) dx = \frac{7}{6} < \frac{5}{4}.$$

13. Let f be continuous on $[a, b]$. If $\int_a^b f = 0$ then show that $f(c) = 0$ for at least one $c \in [a, b]$. Show that the result may not hold if f is not continuous.

Solution: If false, then using the extreme value theorem, either $f(x) \leq M < 0$ or $f(x) > m > 0$ for $x \in [a, b]$.
Use the monotonicity of the integral.

(Alternately, use mean value theorem of integrals.)

For the function $f : [-1, 1] \to \mathbb{R}$ defined by $f(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ -1, & \text{if } x < 0, \end{cases}$
$\int_{-1}^{1} f(x)dx = 0$. However, $f(x) \neq 0$ for all $x \in [-1, 1]$.

14. If f is continuous on $[a, b]$ and $\int_a^b fg = 0$ for every $g \in \mathcal{R}[a, b]$, then show that $f = 0$.

Solution: Hint: Take $g = f$.

15. Suppose $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{1}{2^n}, & \text{if } x = \frac{k}{2^n} \text{ for some } k, n \in \mathbb{N}, \text{ where } k \text{ is odd}, \\ 0, & \text{otherwise.} \end{cases}$$

Determine whether f is Riemann integrable, and if so, find $\int_0^1 f$.

Solution: Riemann Integrable. The function is similar to the Thomae's function. You can deal it similarly.