D A sequence of elements in S is a function $f : \mathbb{N} \to S$. We view it as an ordered list $(a_1, a_2, \ldots)$, where $a_i = f(i)$.

Eg $(1, 2, 3, 4, 5 \cdots)$. Better to write as (n). ‼

- $(\frac{1}{n})$ means $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots)$.
- $(a) = (a, a, a, \ldots)$ is called a constant sequence.
- We can define a sequence by a rule:

$$a_n = n\text{th decimal digit, in the decimal representation of } \sqrt{2}.$$

We may not know the exact value of a_n, but it is well-defined.

- We can define (a_n) recursively: $a_1 = a_2 = 1$, $a_{n+2} = a_{n+1} + a_n$, $\forall n$.

D Let us call the part $a_k, a_{k+1}, a_{k+2}, \cdots$ of (a_n), a tail of (a_n).

Q Suppose $a_n = \frac{1}{n}$ is the purchase capacity of 1 rupee on nth day.

- What will be the purchase capacity of a rupee ultimately? 0.

- It means, given any positive number ϵ, there comes a day m such that, then onwards the purchase capacity of the rupee is less than ϵ.

- That is, $\forall \epsilon > 0$, $\exists m$ such that $a_n < \epsilon$, for each $n \geq m$.

- That is, the set $B_\epsilon(0)$ contains a tail of (a_n), namely, $a_m, a_{m+1}, \ldots$.

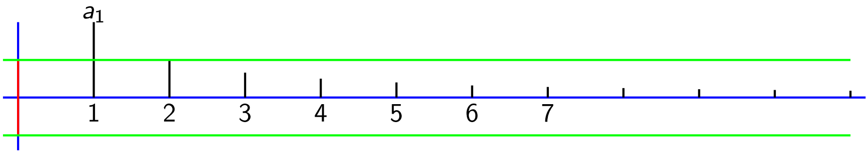
- In short, each $B_\epsilon(0)$ contains a tail of (a_n).

D We say $a_n \to a$ if each $B_\epsilon(a)$ contains a tail of (a_n). That is, if

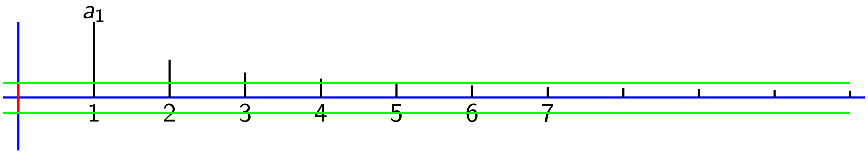
$$\forall \epsilon > 0, \quad \exists m \in \mathbb{N} \quad \text{s.t.} \quad a_n \in B_\epsilon(a), \text{ for each } n \geq m.$$

In this case, we say a is the limit of (a_n) and (a_n) converges to a.

Eg Take $a_n = \frac{1}{n}$. View them as the height some lines.

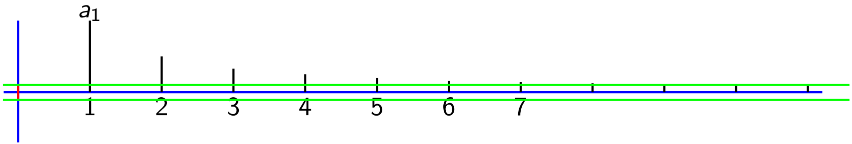


- Does $B_{\frac{1}{2}}(0)$ contain a tail? Yes. $a_3, a_4, \ldots$.



- Does $B_{\frac{1}{5}}(0)$ contain a tail? Yes. $a_6, a_7, \ldots$.

- Take $a_n = \frac{1}{n}$.



- Does $B_\epsilon(0)$ contain a tail? Yes. For $n \geq [\frac{1}{\epsilon}] + 1$. !! So $\frac{1}{n} \to 0$.

D We also write $\lim\limits_{n\to\infty} a_n = a$ to mean that $a_n \to a$. Divergent means not convergent. Note that $a_n \in B_\epsilon(a)$ means $|a_n - a| < \epsilon$.

Eg a) Fix $a > 0$. Then $a_n = \frac{a}{n} \to 0$. How? Let $\epsilon > 0$. Then $a_n \in B_\epsilon(0)$, for all $n > m = [\frac{a}{\epsilon}]$. Will $m = [\frac{a}{\epsilon}] + 4$ work?

b) Take $a_n = a$. Then $a_n \to a$. How? Let $\epsilon > 0$. Then $a_n \in B_\epsilon(a)$, for all $n \geq m = 1$.

c) Take $a_n = n$. Can we find $a \in \mathbb{R}$ s.t. each $B_\alpha(a)$ contains a tail? No. Suppose there is such an a. Then each $B_\alpha(a)$ contains a tail. In particular, $B_1(a)$ contains a tail. So it contains infinitely many integers. But that cannot happen as $B_1(a)$ has length 2. So (a_n) is divergent.

d) Take $a_n = (-1)^n$. Can we find $a \in \mathbb{R}$ s.t. each $B_\alpha(a)$ contains a tail? No. $B_{0.1}(a)$ has length 0.2. It cannot contain a tail, as a tail contains two distinct integers.

e) Let $a_n = \frac{2n+5}{3n+1}$. Then $a_n \to \frac{2}{3}$. This is so, as

$$\left|\frac{2}{3} - \frac{2n+5}{3n+1}\right| = \frac{13}{9n+3} < \frac{13}{9n} < \frac{2}{n} < \epsilon$$

for $n > [\frac{2}{\epsilon}]$.

R The sequence $a_n \to 0$ iff the sequence $|a_n| \to 0$.

Po This is so, because, $a_n \in B_\epsilon(0) \Leftrightarrow |a_n| \in B_\epsilon(0)$. □

R Fix $0 < a < 1$. Take $a_n = a^n$. Then $a_n \to 0$.

Po Let $\epsilon > 0$. Write $\frac{1}{a} = 1 + b$, where $b > 0$. So

$$|a^n - 0| = a^n = \frac{1}{(1+b)^n} \overset{\text{binomial}}{\leq} \frac{1}{nb} \leq \epsilon$$

for $n > [\frac{1}{b\epsilon}]$. □

R: sandwich lemma Let $a_n \leq b_n \leq c_n$. If $a_n \to l$ and $c_n \to l$, then $b_n \to l$.

Po Let $\epsilon > 0$. As $a_n \to l$, $\exists m$ s.t. $a_n \in B_\epsilon(l)$, for each $n \geq m$. As $c_n \to l$, $\exists m'$ s.t. $c_n \in B_\epsilon(l)$ for each $n \geq m'$. So $a_n, c_n \in B_\epsilon(l)$ for each $n \geq N := \max\{m, m'\}$. Thus $b_n \in B_\epsilon(l)$ for each $n \geq N$. □

Eg Note that $0 \leq |\frac{\sin n}{n}| \leq \frac{1}{n}$. So $|\frac{\sin n}{n}| \to 0$. So $\frac{\sin n}{n} \to 0$.

R Let $a_n \to a$ and fix $c \in \mathbb{R}$. Then $ca_n \to ca$. ‼

R (a_n) converges to a iff a tail of (a_n) converges to a.

D $a_n \not\to l$ means $a_n \to l$ is not true.

- So, some $B_\epsilon(l)$ does not contain any tail.
- That is, $\exists \epsilon > 0$ s.t. $B_\epsilon(l)$ misses infinitely many terms of (a_n).

R: uniqueness of limit Let $a_n \to l$ and $a_n \to k$. Then $l = k$.

Po Assume that $l \neq k$. Put $\epsilon = |l - k|/2$. As $a_n \to l$, $\exists m$ s.t. $a_m, a_{m+1}, \ldots \in B_\epsilon(l)$. So $a_m, a_{m+1}, \ldots \notin B_\epsilon(k)$, as $B_\epsilon(l) \cap B_\epsilon(k) = \emptyset$.
So, $B_\epsilon(k)$ misses infinitely many terms of (a_n). So $a_n \not\to k$. $\Rightarrow\Leftarrow$

D We say (a_n) is bounded above if $\exists M \in \mathbb{R}$ s.t. $a_n \leq M$ for all n. Bounded below is defined similarly. A sequence is bounded if it is bounded below and above.

Eg (2^n) is <u>bounded below</u>, <u>not bounded</u>. $((-1)^n)$ is <u>bounded</u>.

R Let (a_n) be convergent. Then (a_n) is bounded.

Po Let $a_n \to a$. So $\exists m$ s.t. $a_m, a_{m+1}, \ldots \in B_1(a)$. Put $M := \max\{|a_1|, \ldots, |a_{m-1}|, |a| + 1\}$. Then $|a_n| \leq M$ for all n. $\square$

- Converse is not true. $((-1)^n)$ is bounded but not convergent.

Eg $(n^2 - n)$ is not convergent, because it is not bounded.

Q I have a sequence $a_n \to a$. Zyx replaced the first 500 terms by some numbers I do not know. Should the new sequence be convergent?

R Let $a_n \to a$ and $b_n \to b$. Then $(a_n + b_n) \to (a + b)$. ‼

R Let $a_n \to a$ and $b_n \to b$. Then $(a_n b_n) \to ab$.

Po Recall: convergent implies bounded. Take $M > |a_n|, |b_n|, |a|, |b|$.

As $a_n \to a$, $\exists n_0$ s.t. $|a_n - a| < \frac{\epsilon}{2M}$ for each $n \geq n_0$.

As $b_n \to b$, $\exists n_1$ s.t. $|b_n - b| < \frac{\epsilon}{2M}$ for each $n \geq n_1$. Then

$$|a_n b_n - ab| \leq |a_n - a||b_n| + |a||b_n - b| \leq \frac{\epsilon}{2M} M + M \frac{\epsilon}{2M} = \epsilon,$$

for each $n \geq \max\{n_0, n_1\}$.

R Let $a_n \to a$. If $a > 0$, then (a_n) has a positive tail. Furthermore, if a_n are nonzero, then $\frac{1}{a_n} \to \frac{1}{a}$. ‼

R Let $a_n \to a$, $a_n \geq 0$ and $k \in \mathbb{N}$. Then $a \geq 0$ and $\sqrt[k]{a_n} \to \sqrt[k]{a}$. ‼

Ex Let $a_n \to a$. Then $|a_n| \to |a|$. The converse is not true.

Ex Show that $a_n, b_n \to a$ iff $(a_1, b_1, a_2, b_2, \ldots) \to a$.

R: ratio test Let $a_n \neq 0$ for each n. Suppose that $\lim |\frac{a_{n+1}}{a_n}| = a$. Then

$a < 1 \Rightarrow a_n \to 0$ and $a > 1 \Rightarrow (a_n)$ is divergent.

Po Let $a < 1$. As $|\frac{a_{n+1}}{a_n}| \to a$, $\exists m$ s.t. $|\frac{a_{n+1}}{a_n}| < \frac{1+a}{2} = r$ (say), for each $n \geq m$. So $0 < |a_{m+k}| < |a_m| r^k$. So $|a_{m+k}| \overset{\text{sandwich}}{\to} 0$. So $a_{m+k} \to 0$. So $a_n \to 0$. Other one follows as $a_n \not\to 0$. $\square$

Eg Take $a_n = \dfrac{5^n}{n!}$. Then $a_n \to 0$ as $\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{5}{n+1} \to 0$. Alternately

$$n > 5 \Rightarrow \frac{5^n}{n!} = \frac{5^5}{5!} \; \frac{5}{6} \; \frac{5}{7} \; \cdots \; \frac{5}{n} \leq \frac{5^5}{5!} \left(\frac{5}{6}\right)^{n-5} \to 0.$$

D We say (a_n) is increasing if $a_n \leq a_{n+1}$, $\forall n$. We say it is strictly increasing if $a_n < a_{n+1}$, $\forall n$. Decreasing and strictly decreasing sequences are defined similarly. These are called monotone sequences.

Eg (n) and $(\frac{1}{n})$ are monotone. $((-1)^n)$ is not monotone.

Q Take $a_n = -\frac{1}{n}$. Then $\lim_{n\to\infty} a_n = 0 = \text{lub}\{-1, -\frac{1}{2}, \ldots\}$. In general?

Monotone convergence theorem(MCT) Let $a_n \uparrow$ (increasing), bounded above. Put $A = \{a_1, a_2, \ldots\}$. Then $A \neq \emptyset$, bounded above and $a_n \to \text{lub}\, A$.

Po It is easy to show that A is nonempty and bounded above. Let $a = \text{lub}\, A$. Let $\epsilon > 0$. So $a - \epsilon$ is not an upper bound of A. So, $\exists m$ s.t. $a - \epsilon < a_m$. So $a - \epsilon \leq a_m \leq a_{m+1} \leq \cdots \leq a$. That is, $a_{n_0}, a_{n_0+1}, \ldots \in B_\epsilon(a)$. So $a_n \to a$.

$\square$

C Let $a_n \downarrow$ (decreasing), bounded below. Then $a_n \to \text{glb}\{a_1, a_2, \dots\}$.

- So a monotone sequence is convergent iff it is bounded.

Eg Take $a_1 = 1$ and $a_{n+1} = \frac{3a_n+7}{5}$, for $n \in \mathbb{N}$. Convergent?

Technique check if (a_n) is monotone and bounded.

Notice: $a_1 < a_2$. Also $a_n \le a_{n+1} \Rightarrow \frac{3a_n+7}{5} \le \frac{3a_{n+1}+7}{5}$.

So $a_n \uparrow$, by induction. Also $a_n \le 4$, by induction.

Hence by MCT, (a_n) converges. Let $a_n \to l$. So $l = \frac{3l+7}{5}$. So $l = \frac{7}{2}$.

- Wrong to use the last step, directly. Try $a_1 = 1$, $a_{n+1} = 3a_n + 7$.

Eg Take $a_n = (1 + \frac{1}{n})^n$. We show that (a_n) is convergent. Note that

$$\frac{\binom{n}{k}}{n^k} = \frac{n(n-1)\cdots(n-k+1)}{n^k k!} = \frac{1(1-\frac{1}{n})\cdots(1-\frac{k-1}{n})}{k!} \le \frac{\binom{n+1}{k}}{(n+1)^k}, \frac{1}{k!}.$$

Also, we have

$$1+\frac{\binom{n}{1}}{n}+\frac{\binom{n}{2}}{n^2}+\frac{\binom{n}{3}}{n^3}+\cdots+\frac{\binom{n}{n}}{n^n}\le 1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}<3.$$

By MCT, (a_n) is convergent. The limit is called e. Argue $e\in(2,3)$.

Eg Fix $x>0$. Put $m=[x]+1$. So $\rho:=\frac{x}{m}<1$. Consider $A_n=\sum_{i=0}^{n}\frac{x^k}{k!}$.

Then
$$A_{m+k}-A_m=\frac{x^{m+1}}{(m+1)!}+\cdots+\frac{x^{m+k}}{(m+k)!}$$
$$\le\frac{x^m}{m!}\left(\rho+\cdots+\rho^k\right)\le\frac{x^m}{m!}\left(\frac{1}{1-\rho}\right),$$
a fixed number, for all n.

As $A_n\uparrow$ and bounded above, By MCT, (A_n) is convergent. The limit is called $\exp(x)$.

Ex Do the exercise about $\exp(x)$ from the notes.

D We define $\ln x$ as the inverse function of $\exp x$. That is, for $x > 0$, define $\ln x = a$ where $\exp a = x$. We define $x^b = \exp(b \ln x)$, for $x > 0, b \in \mathbb{R}$.

D Let $n_1 < n_2 < \cdots$ be some natural numbers. Then we call (a_{n_k}) a subsequence of (a_n). It's terms are $\quad a_{n_1}, \;\; a_{n_2}, \;\; a_{n_3}, \;\; \cdots.$

Eg $(2n)$ is a subsequence of (n). So is $2, 3, 5, \cdots$ (prime numbers).

- $1, 1, 2, 3, 4, 5, \cdots$ is not a subsequence, as the original sequence does not have two 1's.

- $1, 2, 4, 3, 5, 6, \cdots$ is not a subsequence, as the order of 3 and 4 in the original sequence is not preserved.

R Let $a_n \to l$ and (a_{n_k}) be a subsequence. View (a_{n_k}) as a new sequence (b_k). Then $\lim\limits_{k\to\infty} b_k = l$.

Po If $B_\epsilon(l)$ contains a tail of (a_n), then it will also contain a tail of the subsequence (b_k). $\square$

Note. One of the following is sufficient to prove that (a_n) is divergent.

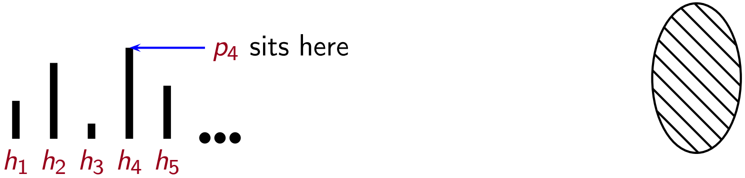
a) (a_n) has a divergent subsequence.

b) (a_n) has two subsequences converging to two different limits.

c) (a_n) is not bounded.

Eg Take $a_n = \sin n$. Convergent? Assume that it converges to l. Use $\sin(2n+1) = \sin 2n \cos 1 + (1-2\sin^2 n)\sin 1$ to get $l \neq 0$. Use $\sin(n+1)+\sin(n-1) = 2\sin n \cos 1$ to get $l = 0$. $\Rightarrow\Leftarrow$

- Persons $p_1, p_2, p_3, \ldots$ want to watch a movie. Their seats have heights h_i. All seats are in one line facing the screen.



- Can p_2 watch the movie? No, as p_4 has a higher seat.

- Let W be the set of persons who can watch the movie. W is finite or not.

- If $W = \{p_{k_1}, \ldots, p_{k_n}\}$, take $n_0 > k_1, \cdots, k_n$. If $W = \emptyset$, take $n_0 = 1$.

- Is $p_{n_0} \in W$? No. So, $\exists$ a higher seat in front. That is, $\exists n_1 > n_0$ s.t. $h_{n_1} > h_{n_0}$. Is $p_{n_1} \in W$? So, $\exists n_2 > n_1$ s.t. $h_{n_2} > h_{n_1}$. Continue.

- We get $h_{n_0} < h_{n_1} < h_{n_2} < \cdots$, an ________ subsequence of (h_n).

- Assume that $W = \{p_{n_1}, p_{n_2}, \ldots\}$ is infinite, where $n_1 < n_2 < \cdots$.
  - Then $h_{n_1} \geq h_{n_2} \geq \cdots$, as they all can watch the movie.
  - In either case, the sequence (h_n) has a monotone subsequence.

Monotone subsequence theorem (MST)

a) If $a_n \geq 0$, then (a_n) has a monotone subsequence.

b) If $a_n < 0$, then (a_n) has a monotone subsequence.

c) Every sequence of real numbers has a monotone subsequence.

Bolzano-Weierstrass theorem Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Application $(\sin n)$ has a convergent subsequence.

- What is the distance between 1 and 3? Between x and y?

D A sequence (a_n) is called Cauchy if 'for each $\epsilon > 0$, there is a tail in which terms have a distance $< \epsilon$ among them'. That is,

$$\forall \epsilon > 0,\ \exists n_0 \text{ s.t. } |a_n - a_m| < \epsilon \text{ for each } m, n \geq n_0.$$

Eg Is $(-1)^n$ Cauchy? No, as there is no tail in which, terms have distance less than 1 among them.

- Is $(\frac{1}{n})$ Cauchy? Yes. Let $\epsilon > 0$. Choose $k \in \mathbb{N}$ s.t. $\frac{1}{k} < \epsilon$. From kth term onwards, the distance among the terms is always less than ϵ.

R If (a_n) is Cauchy, then (a_n) is bounded. !!

R: Cauchy criterion. (a_n) is convergent iff (a_n) is Cauchy. !!

Eg Take $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Is it convergent? No. How? Because, it is not Cauchy. How? Ok. Take $\epsilon = \frac{1}{3}$. Can we find a n_0 s.t. $|a_n - a_m| < \epsilon$ for each $m, n \geq n_0$? Suppose, we can. Then we have $|a_{2n_0} - a_{n_0}| < \frac{1}{3}$. But, $|a_{2n_0} - a_{n_0}| = \frac{1}{n_0+1} + \cdots + \frac{1}{2n_0} > \frac{1}{2}$. $\Rightarrow\Leftarrow$

D (a_n) is contractive if $\exists c \in (0,1)$ s.t. $|a_{n+2} - a_{n+1}| \leq c|a_{n+1} - a_n|$, $\forall n$.

R A contractive sequence is Cauchy. Hence it is convergent. ‼

Eg Take $a_1 = 2$, $a_{n+1} = 2 + \frac{1}{a_n}$. Convergent? Note that

$$|a_{n+2} - a_{n+1}| = \left|\frac{1}{a_{n+1}} - \frac{1}{a_n}\right| = \left|\frac{|a_{n+1} - a_n|}{a_{n+1}a_n}\right| \leq \frac{1}{4}|a_{n+1} - a_n|.$$

So, it is convergent. The limit l must satisfy $l^2 - 2l - 1 = 0$. So $l = 1 + \sqrt{2}$ or $l = 1 - \sqrt{2}$. As $a_n \geq 2$, we get $l \geq 2$. So $l = 1 + \sqrt{2}$.