- Read $\forall$ as ‘for each/for all/for every’ and ‘$\exists$’ as ‘there exists/there is’.
- Statement P: ‘$\forall x \in A$, x is a zyx’ means ‘each element in A is a zyx’.
  - P is considered true, if each element of A is a zyx.
  - P is considered false, if there is an element of A which is not a zyx.
  - Hence, the statement P is true for $A = \emptyset$.
- Take Q: ‘$\exists x \in A$ such that x is a zyx’. Means ‘there is an element in A which is a zyx’.
  - Q is considered true, if at least one element of A is a zyx.
  - Q is considered false, if no element of A is a zyx.
  - Thus, the statement Q is false for $A = \emptyset$.
- Notice the style of writing: ‘for each x, (comma) something’ and ‘there exists x such that (in place of comma) something’. Other variations exist.

[D]Real numbers A set F with two binary operations $+$, $*$ and a binary relation $\leq$ satisfying the following axioms. Here $a, b, c \in \mathbb{F}$. Write ab for $a * b$.

closure $\forall a, b$, we have $a + b, ab \in F$.

associative $a + (b + c) = (a + b) + c$, $a(bc) = (ab)c$ hold $\forall a, b, c$.

commutative $a + b = b + a$, $ab = ba$ hold for each a, b.

identity elements $\exists\, 0, 1 \in F$, $0 \neq 1$, s.t. $a + 0 = a$ and $a1 = a$ hold for each a.

additive inverse $\forall a$, $\exists z_a \in F$ s.t. $a + z_a = 0$.

multiplicative inverse $\forall a \neq 0$, $\exists y_a \in F$ such that $ay_a = 1$.

distributive $\forall a, b, c$, we have $a(b + c) = ab + ac$.

Such a set is unique. Notn: $\mathbb{R}$.

trichotomy $\forall a, b$, exactly one of $a < b$, $a = b$, $a > b$ holds true.

transitive $a \leq b, b \leq c \Rightarrow a \leq c$ holds $\forall a, b, c$.

positivity $\forall a, b, c$, we have $b < c \Rightarrow a + b < a + c$ and $a, b > 0 \Rightarrow ab > 0$.

special axiom If '$A, B \subseteq F$, nonempty and $a \leq b$ for each $a \in A$, $b \in B$', then '$\exists m \in F$ such that $a \leq m \leq b$ for each $a \in A$ and $b \in B$'.

[R] We can deduce many properties from this definition. (See notes for proofs.)

- The additive and multiplicative identities in $\mathbb{R}$ are unique.
- The additive inverse of an element a in $\mathbb{R}$ is unique. Denoted by $-a$.
- The multiplicative inverse of $a \neq 0$ in $\mathbb{R}$ is unique. Denoted by a^{-1} or $\frac{1}{a}$.
- $(a^{-1})^{-1} = a$ for each $a \neq 0$. Also $-0 = 0$ and $1^{-1} = 1$.
- $x0 = 0$ for each x. Also, if $ab = 0$ then either $a = 0$ or $b = 0$.
- $-a = (-1)a$ and $-(-a) = a$ for each a. So $(-1)(-1) = -(-1) = 1$.
- $a \geq 0$ implies $-a \leq 0$. Also $1 > 0$. (This means $1 \geq 0$ and $1 \neq 0$.)
- $a > 0$ implies $a^{-1} > 0$. Also $a \geq b$, $c \geq 0$ imply $ac \geq bc$.

[D] We inductively define the natural numbers $\mathbb{N}$ as $\{1, 1+1, 1+1+1, \ldots\}$. By definition it is a subset of $\mathbb{R}$. We use the symbols $1, 2, 3, \ldots$ for them.

[D] We define $\mathbb{Z} := \mathbb{N} \cup \{-n \mid n \in \mathbb{N}\} \cup \{0\}$ and $\mathbb{Q} := \{\frac{m}{n} \mid m \in \mathbb{Z}, n \in \mathbb{N}\}$.

- Give a real number which is $\geq$ each element of $S = \{1, 3, 5\}$. 6? 5? 4?

[D] Let $S \subseteq \mathbb{R}$, $a \in \mathbb{R}$. We call the number a an upper bound (ub) of S, if $a \geq s$ for each $s \in S$. The term lower bound (lb) is defined similarly.

- What is the meaning of a is not an ub of S? $\exists s \in S$ such that $s > a$.

- Is 5 an ub of $\emptyset$? Yes. In fact, each real number is an ub of $\emptyset$.

[D] A set $S \subseteq \mathbb{R}$ is called bounded above if there is an ub of S in $\mathbb{R}$. The term bounded below is defined similarly.

- Let $S = \{1, 2, 4\}$. Is there a maximum of S? Is it in S? Is it an ub of S?

[D] Let $S \subseteq \mathbb{R}$ and a be an ub of S. If $a \in S$, then a is called a maximum of S. The term minimum is defined similarly.

[R] If a is a maximum of S, then it is unique. !!

So, we can use the notations $\max S$, $\min S$.

[Eg] Is $(0,1)$ bounded above? Is it bounded below?

[D] We say S is bounded if it is bounded above and below.

[Eg] A bounded set need not have a maximum or a minimum. $(0,1)$

[Eg] What is the set of all upper bounds of $S = \{0,1,2\}$? $[2,\infty)$.
Which one is the smallest ub? 2.

[R: lub property] Let $\emptyset \neq A \subseteq \mathbb{R}$ be bounded above. Then the set U of upper bounds of A always contains a minimum.

Po A is bounded above. So U is nonempty. So $a \leq u$, for each $a \in A$ and $u \in U$. By the special axiom, $\exists m \in \mathbb{R}$ such that $a \leq m \leq u$ for each $a \in A$ and $u \in U$. Is m an upper bound of A? Yes. So $m \in U$. Is m a lower bound of U? Yes. So $m = \min U$. $\square$

[D] Let $\emptyset \neq S \subseteq \mathbb{R}$ be bounded above. Then the least element of the upper bound set is called the least upper bound/supremum of S. ($\operatorname{lub} S$/$\sup S$.)

[D] The greatest lower bound/infimum of S is defined similarly. ($\operatorname{glb} S$/$\inf S$.)

[Eg] $\sup(0,2] = 2$. $\inf \mathbb{Q} \cap (0,2) = 0$. $\sup \emptyset$ does not exist in $\mathbb{R}$.

[Eg]. Let $\operatorname{lub} S = a$. Is $a - \epsilon$ an ub? (Here ϵ is some positive number.) No. So, $\exists s \in S$ such that $s > a - \epsilon$.

[R] $\operatorname{lub} S = a$ iff a is an ub of S and for each $\epsilon > 0$, $\exists s \in S$ such that $s > a - \epsilon$.

[Eg] Is $\mathbb{N}$ bounded above in $\mathbb{Q}$? No. Let $\frac{p}{q}$ be an upper bound of $\mathbb{N}$. Then $\frac{p}{q} \geq 1$. So $p \in \mathbb{N}$. So, $\frac{p}{q} \leq p < p + 1 \in \mathbb{N}$. So $\frac{p}{q}$ cannot be an ub of $\mathbb{N}$. A contradiction.

[Nonstandard notation] $-A := \{-a \mid a \in A\}$ and $\frac{1}{A} := \{\frac{1}{a} \mid a \in A, a \neq 0\}$.

[Eg] Is $A := (0,1)$ bounded above in $B := (-5,1)$? No. In $(-5,5)$? Yes.
So, A is not bounded above in B, but it is bounded above in a superset.

- Similarly, even though $\mathbb{N}$ is not bounded above in $\mathbb{Q}$, it may be bounded above in the superset $\mathbb{R}$. What is the truth?

[R] The set $\mathbb{N}$ is not bounded above in $\mathbb{R}$.

Po Suppose $\mathbb{N}$ is bounded above. Using lub property, let $x = \operatorname{lub} \mathbb{N}$. As $x - \frac{1}{2}$ is not an ub, $\exists n \in \mathbb{N}$ such that $x - \frac{1}{2} < n$. But then $x < n + \frac{1}{2} < n + 1 \in \mathbb{N}$, a contradiction. $\square$

[Eg] So $\sup \mathbb{N}$ ____________ in $\mathbb{R}$. does not exist

[R: glb property] Let $\emptyset \neq S \subseteq \mathbb{R}$ be bounded below. Then $\operatorname{glb} S$ exists in $\mathbb{R}$. ‼

[R: well ordering principle] Every nonempty subset of $\mathbb{N}$ contains a minimum. ‼

[R: Archimedean property] Let $\alpha > 0$. Then there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \alpha$.

Po Otherwise $\mathbb{N}$ becomes bounded above in $\mathbb{R}$. $\square$

[Eg] Does $A = \{x > 0 \mid x^6 < 3\}$ have a maximum? No. If yes, let it be a. So $\frac{3-a^6}{(1+a)^6} > 0$. By Archimedean principle $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{3-a^6}{(1+a)^6}$. So

$$\begin{aligned}
\left(a + \frac{1}{n}\right)^6 &= a^6 + \binom{6}{1} a^5 \frac{1}{n} + \binom{6}{2} a^4 \frac{1}{n^2} + \cdots + \binom{6}{6} \frac{1}{n^6} \\
&\leq a^6 + \binom{6}{1} a^5 \frac{1}{n} + \binom{6}{2} a^4 \frac{1}{n} + \cdots + \binom{6}{6} \frac{1}{n} \\
&< a^6 + \frac{1}{n}\left[a^6 + \binom{6}{1} a^5 + \binom{6}{2} a^4 + \cdots + \binom{6}{6}\right] \\
&= a^6 + \frac{1}{n}(1+a)^6 < 3.
\end{aligned}$$

Thus $a + \frac{1}{n} \in A$, a contradiction.

[R: kth root] Fix $a > 0$ and $k \in \mathbb{N}$. Then there is a unique $b > 0$ s.t. $b^k = a$. ‼ We call this number b the kth root of a. Notn. $\sqrt[k]{a}$, $a^{\frac{1}{k}}$

[Eg] In the above, if we take $a = 2 = k$, we get a real number b such that $b^2 = 2$. We denote this b by $\sqrt{2}$. It is easy to show that $\sqrt{2} \notin \mathbb{Q}$. So it is an irrational number. In fact, $\sqrt{n}$ is an irrational number, when n is not a perfect square. ‼

[R: greatest integer function] Let $\alpha \in \mathbb{R}$. Then there exists a unique $z \in \mathbb{Z}$ such that $z \leq \alpha < z + 1$. ‼ We denote it by $[\alpha]$.

[R.] Each interval (a, b) contains a rational and an irrational number.

Po Take $n \in \mathbb{N}$, large s.t. length of $(na, nb) > 3$. It has at least two integers, $m = [na] + 1$ and $m + 1$. So $m, m + \frac{1}{\sqrt{2}} \in (na, nb)$. So $\frac{m}{n}, \frac{m+\frac{1}{\sqrt{2}}}{n} \in (a, b)$. □

[Ex.] Each (a, b) contains infinitely many rationals and irrationals.

[R: nested interval theorem] Let $[a_1, b_1] \supseteq [a_2, b_2] \supseteq \cdots$ and $I = \bigcap_{n=1}^{\infty} [a_n, b_n]$. Then $I \neq \emptyset$.

$$\big[a_1 \ \big[a_2 \ \big[a_3 \qquad \big[a_n \qquad b_n\big] \qquad b_3\big] \ b_2\big] \ b_1\big]$$

Po Let $L = \{a_1, a_2, \cdots\}$ (left endpoints), $R = \{b_1, b_2, \cdots\}$ (right endpoints). Is $a_n \leq a_{n+m} \leq b_{n+m} \leq b_m$, for each $n, m \in \mathbb{N}$? Yes. Apply the special property: $\exists z$ such that $a_i \leq z \leq b_j$ for each $i, j \in \mathbb{N}$. In particular, $z \in [a_n, b_n]$ for each $n \in \mathbb{N}$. Hence z is in the intersection I. $\square$

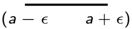
- One can show that the intersection I is actually a closed interval.

Po Is L is bounded above? Yes, each b_i is an ub. So $\operatorname{lub} L$ exists in $\mathbb{R}$. Let it be a. Similarly, put $b = \operatorname{glb} R$. If $z \in [a, b]$, then $a_n \leq a \leq z \leq b \leq b_n$. So $[a, b] \subseteq [a_n, b_n]$. So it is contained in I. If $z < a = \operatorname{lub} L$, then $\exists a_{n_0} \in L$ such that $a_{n_0} > z$. So $z \notin [a_{n_0}, b_{n_0}]$. So $z \notin I$. Similarly, if $z > b$, then $z \notin I$. Hence, $I = [a, b]$. $\square$

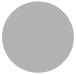
[D] In $\mathbb{R}^n$, 0 means $(0, \ldots, 0)$. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Then $|x - y| := \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$.

- Let $a \in \mathbb{R}^n$. We define $B_\alpha(a) := \{x \mid |a - x| < \alpha\}$ and $D_\alpha(a) := \{x \mid 0 < |a - x| < \alpha\}$. Here $\alpha > 0$.

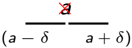
[Eg] In $\mathbb{R}$, the set $B_\epsilon(a) = (a - \epsilon, a + \epsilon)$.



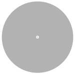
- In $\mathbb{R}^2$, $B_1(0)$ means the open unit disc centered at 0.



- In $\mathbb{R}$, the set $D_\delta(a) = (a - \delta, a) \cup (a, a + \delta)$



- In $\mathbb{R}^2$, $D_1(0)$ means the open unit disc without center 0.



- $B_\alpha(a)$ is called the open ball of radius α centered at a. Also called a neighbourhood of a.

- $D_\alpha(a)$ is called the punctured open ball of radius α centered at a. Also called a deleted neighbourhood of a.

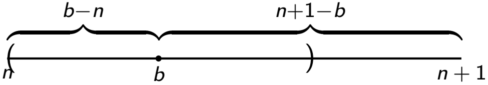
- Each $B_\epsilon(0)$ means $B_\epsilon(0)$ for each $\epsilon > 0$.

- In $\mathbb{R}$, does $D_1(0)$ contain a rational number? What about $D_{\frac{1}{2}}(0)$? $D_\epsilon(0)$?

[D] Let $A \subseteq \mathbb{R}$. Then $a \in \mathbb{R}$ is called a cluster point of A, if $\underline{\text{each } D_\epsilon(a)}$ contains at least one point of A. Also called a limit point of A. The definition is similar for subsets of $\mathbb{R}^n$.

[Eg] Each real number is a cluster point of $\mathbb{Q}$. Also of $\mathbb{Q}^c$.

- Does there exists a limit point of $\mathbb{N}$ in $\mathbb{R}$? In other words, does there exist a real number b such that each $D_\epsilon(b)$ contains a natural number? No. If $n < b < n+1$, then take $\epsilon = \min\{b-n, n+1-b\}$. Then $D_\epsilon(b) \cap \mathbb{N} = \emptyset$.



If $b = n$, take $\epsilon = 1$. Then $D_\epsilon(b) \cap \mathbb{N} = \emptyset$. If $b < 1$, argue similarly.

[Eg] Take $A = (0, 1)$ and $a = 0.9$. Can you find a $B_\delta(a) \subseteq A$? Draw picture and tell $\delta =$ .1 or less.

- What if I take $a = .99$? Will you be able to give a $B_\delta(a)$?
- Take $A = B_1(0)$ in $\mathbb{R}^2$ and $a = (x, y)$. Can we give a $B_\delta(a) \subseteq A$? Yes, $\delta \leq 1 - \sqrt{x^2 + y^2}$.

[D] A set $A \subseteq \mathbb{R}^n$ is open if A contains a neighbourhood around <u>each</u> point a of A. A set A is closed if A^c is open.

[Eg] $(0, 1)$ is open in $\mathbb{R}$. Any $B_\alpha(a)$ is open in $\mathbb{R}^n$.

- Union of open sets is open.
- $[0, 1)$ is not open in $\mathbb{R}$. Here 0 creates a problem.
- $[0, 1)$ is not closed in $\mathbb{R}$. As its complement $(-\infty, 0) \cup [1, \infty)$ is not open.
- Consider $B_1(0) \cup \{(1, 0)\}$ in $\mathbb{R}^2$. It is neither open nor closed.