MA 101 (Mathematics - I)
Tutorial Problems 5: (Differentiation 3,4,5a)

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1. Find the following by using L'Hôpital's Rules, whenever needed. Do not forget to check the conditions needed for using L'Hôpital's Rules.

   (i) $\lim\limits_{x\to 0+}\left(\dfrac{1}{x}-\dfrac{1}{\operatorname{Arctan} x}\right)$ (ii) $\lim\limits_{x\to\infty}\dfrac{x+\ln x}{x\ln x}$ (iii) $\lim\limits_{x\to 0+}(1+3/x)^x$ (iv) $\lim\limits_{x\to\infty} x^{1/x}$

2. Let $f:\mathbb{R}\to\mathbb{R}$ have second derivative at $c\in\mathbb{R}$. Prove that $\lim\limits_{h\to 0}\dfrac{f(c+h)-2f(c)+f(c-h)}{h^2}=f''(c)$.
   Give example of a function $f:\mathbb{R}\to\mathbb{R}$ and a point $c\in\mathbb{R}$ for which the above limit exists, but $f''(c)$ does not exist.

3. For $x>0$ show that $|(1+x)^{1/3}-(1+\frac{1}{3}x-\frac{1}{9}x^2)|<(5/81)x^3$. Use this inequality to approximate $\sqrt[3]{1.2}$ and $\sqrt[3]{2}$, and find the bounds for errors in the estimations.

4. Find the Taylor series of $\sin x\cos 3x$ about 0. What is the domain of convergence?

5. (a) Show that for $n\geq 0$, $\lim\limits_{x\to 0}\dfrac{e^{-1/x^2}}{x^n}=0$.

   (b) Use induction to prove Leibniz's rule for the n-th derivative of a product:
   $$(fg)^{(n)}(x)=\sum_{k=0}^{n}f^{(n-k)}(x)g^{(k)}(x).$$

   (c) Define $f:\mathbb{R}\to\mathbb{R}$ by
   $$f(x)=\begin{cases} e^{-1/x^2}, & \text{if } x\neq 0,\\ 0, & \text{if } x=0.\end{cases}$$
   Show that f is infinitely differentiable on $\mathbb{R}$ and find $f^{(n)}(0)$ for $n\in\mathbb{N}$.

   (d) What is the Taylor series of f about 0, and on what interval is it defined?

   (e) Does the remainder term for f at 0 in Taylor's theorem converge to zero as $n\to\infty$?

   (f) On what set does the Taylor series converge to f?

6. Suppose $a_n\to a$, $b_n\to b$, and $a<b$. Suppose (c_n) is given by $c_{2n-1}=a_n, c_{2n}=b_n$ for $n\in\mathbb{N}$. What can you say about $\limsup c_n$ and $\liminf c_n$? Justify your claim. (Note that b can be ∞ and a can be $-\infty$.)