MA 101 (Mathematics I)

Multivariable Calculus : Hints / Solutions of Tutorial Problem Set - 1

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1. Let $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^m$. Show that $\|\mathbf{x}+\mathbf{y}\| = \|\mathbf{x}\| + \|\mathbf{y}\|$ iff $\mathbf{y} = \mathbf{0}$ or $\mathbf{x} = \alpha\mathbf{y}$ for some $\alpha \geq 0$.

   Solution: If $\mathbf{y} = \mathbf{0}$, then $\|\mathbf{x}+\mathbf{y}\| = \|\mathbf{x}\| = \|\mathbf{x}\| + \|\mathbf{y}\|$. Also, if $\mathbf{x} = \alpha\mathbf{y}$ for some $\alpha \geq 0$, then $\|\mathbf{x}+\mathbf{y}\| = \|(\alpha+1)\mathbf{y}\| = (\alpha+1)\|\mathbf{y}\|$ and $\|\mathbf{x}\| + \|\mathbf{y}\| = \alpha\|\mathbf{y}\| + \|\mathbf{y}\| = (\alpha+1)\|\mathbf{y}\|$, so that $\|\mathbf{x}+\mathbf{y}\| = \|\mathbf{x}\| + \|\mathbf{y}\|$.
   Conversely, let $\|\mathbf{x}+\mathbf{y}\| = \|\mathbf{x}\| + \|\mathbf{y}\|$ and let $\mathbf{y} \neq \mathbf{0}$. Then $\|\mathbf{x}+\mathbf{y}\|^2 = (\|\mathbf{x}\| + \|\mathbf{y}\|)^2$, which gives $\|\mathbf{x}\|^2 + 2\,\mathbf{x}\cdot\mathbf{y} + \|\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\,\|\mathbf{y}\| + \|\mathbf{y}\|^2$ and so $\mathbf{x}\cdot\mathbf{y} = \|\mathbf{x}\|\,\|\mathbf{y}\|$. Hence $|\mathbf{x}\cdot\mathbf{y}| = \|\mathbf{x}\|\,\|\mathbf{y}\|$ and by the equality condition in Cauchy-Schwarz inequality, we get $\mathbf{x} = \alpha\mathbf{y}$ for some $\alpha \in \mathbb{R}$. Since we have $\mathbf{x}\cdot\mathbf{y} = \|\mathbf{x}\|\,\|\mathbf{y}\|$, we obtain $\alpha\mathbf{y}\cdot\mathbf{y} = \|\alpha\mathbf{y}\|\,\|\mathbf{y}\|$, *i.e.* $\alpha\|\mathbf{y}\|^2 = |\alpha|\,\|\mathbf{y}\|^2$. Since $\|\mathbf{y}\| \neq 0$, we get $\alpha = |\alpha|$ and hence $\alpha \geq 0$.

2. Let $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^m$ and $r, s > 0$. Show that $B_r[\mathbf{x}] \cap B_s[\mathbf{y}] \neq \emptyset$ iff $\|\mathbf{x}-\mathbf{y}\| \leq r+s$.

   Solution: Let us first assume that $B_r[\mathbf{x}] \cap B_s[\mathbf{y}] \neq \emptyset$ Then there exists $\mathbf{z} \in B_r[\mathbf{x}] \cap B_s[\mathbf{y}]$ and so $\|\mathbf{z}-\mathbf{x}\| \leq r$, $\|\mathbf{z}-\mathbf{y}\| \leq s$. Hence $\|\mathbf{x}-\mathbf{y}\| = \|\mathbf{x}-\mathbf{z}+\mathbf{z}-\mathbf{y}\| \leq \|\mathbf{x}-\mathbf{z}\| + \|\mathbf{z}-\mathbf{y}\| \leq r+s$.
   Conversely, let $\|\mathbf{x}-\mathbf{y}\| \leq r+s$. If $\mathbf{z} = \frac{s}{r+s}\,\mathbf{x} + \frac{r}{r+s}\,\mathbf{y}$, then $\mathbf{z} \in \mathbb{R}^m$ and
   $\|\mathbf{z}-\mathbf{x}\| = \frac{1}{r+s}\|s\mathbf{x}+r\mathbf{y}-r\mathbf{x}-s\mathbf{x}\| = \frac{r}{r+s}\|\mathbf{x}-\mathbf{y}\| \leq r$, *i.e.* $\mathbf{z} \in B_r[\mathbf{x}]$. Similarly we get $\|\mathbf{z}-\mathbf{y}\| \leq s$ and so $\mathbf{z} \in B_s[\mathbf{y}]$. Hence $\mathbf{z} \in B_r[\mathbf{x}] \cap B_s[\mathbf{y}]$ and therefore $B_r[\mathbf{x}] \cap B_s[\mathbf{y}] \neq \emptyset$.

3. Let $(\mathbf{x}_n)$ be a sequence in $\mathbb{R}^m$. Show that $(\mathbf{x}_n)$ converges in $\mathbb{R}^m$ iff for each $\mathbf{x} \in \mathbb{R}^m$, the sequence $(\mathbf{x}_n\cdot\mathbf{x})$ converges in $\mathbb{R}$.

   Solution: Let us first assume that $(\mathbf{x}_n)$ converges in $\mathbb{R}^m$ and let $\mathbf{x}_0 \in \mathbb{R}^m$ such that $\mathbf{x}_n \to \mathbf{x}_0$. If $\mathbf{x} \in \mathbb{R}^m$, then for all $n \in \mathbb{N}$, $|\mathbf{x}_n\cdot\mathbf{x} - \mathbf{x}_0\cdot\mathbf{x}| = |(\mathbf{x}_n-\mathbf{x}_0)\cdot\mathbf{x}| \leq \|\mathbf{x}_n-\mathbf{x}_0\|\,\|\mathbf{x}\|$ (by Cauchy-Schwarz inequality). Since $\mathbf{x}_n \to \mathbf{x}_0$, we have $\|\mathbf{x}_n-\mathbf{x}_0\| \to 0$ and hence $|\mathbf{x}_n\cdot\mathbf{x} - \mathbf{x}_0\cdot\mathbf{x}| \to 0$. Therefore $\mathbf{x}_n\cdot\mathbf{x} \to \mathbf{x}_0\cdot\mathbf{x} \in \mathbb{R}$ and so the sequence $(\mathbf{x}_n\cdot\mathbf{x})$ converges in $\mathbb{R}$.
   Conversely, let the sequence $(\mathbf{x}_n\cdot\mathbf{x})$ converge in $\mathbb{R}$ for each $\mathbf{x} \in \mathbb{R}^m$. Let $\mathbf{x}_n = \left(x_1^{(n)}, \ldots, x_m^{(n)}\right)$ for all $n \in \mathbb{N}$. By the given condition, for each $j \in \{1, \ldots, m\}$, the sequence $\left(x_j^{(n)}\right) = (\mathbf{x}_n\cdot\mathbf{e}_j)$ converges in $\mathbb{R}$. Therefore the sequence $(\mathbf{x}_n)$ converges in $\mathbb{R}^m$.

4. (a) State TRUE or FALSE with justification: If $(\mathbf{x}_n)$ is a sequence in $\mathbb{R}^m$ having no convergent subsequence, then it is necessary that $\lim\limits_{n\to\infty} \|\mathbf{x}_n\| = \infty$.

   Solution: Let $r > 0$ and if possible, let $S = \{n \in \mathbb{N} : \|\mathbf{x}_n\| \leq r\}$ be an infinite set. Then there exists a strictly increasing sequence (n_k) in $\mathbb{N}$ such that $\|\mathbf{x}_{n_k}\| \leq r$ for all $k \in \mathbb{N}$. This implies that the subsequence $(\mathbf{x}_{n_k})$ of the sequence $(\mathbf{x}_n)$ is bounded in $\mathbb{R}^m$ and hence by the Bolzano-Weierstrass theorem in $\mathbb{R}^m$, $(\mathbf{x}_{n_k})$ has a convergent subsequence. This convergent subsequence is also a convergent subsequence of $(\mathbf{x}_n)$, which is a contradiction to the given condition. Therefore S is a finite set. Let $n_0 = 1$ if $S = \emptyset$ and $n_0 = \max S + 1$ if $S \neq \emptyset$. Then $\|\mathbf{x}_n\| > r$ for all

$n \geq n_0$ and hence $\lim\limits_{n\to\infty} \|\mathbf{x}_n\| = \infty$. Therefore the given statement is TRUE.

(b) State TRUE or FALSE with justification: If $\big((x_n, y_n)\big)$ is a bounded sequence in $\mathbb{R}^2$ such that every convergent subsequence of $\big((x_n, y_n)\big)$ converges to $(0,1)$, then $\big((x_n, y_n)\big)$ must converge to $(0,1)$.

Solution: If possible, let $(x_n, y_n) \not\to (0,1)$. Then there exists $\varepsilon > 0$ such that $(x_n, y_n) \notin B_\varepsilon\big((0,1)\big)$ for infinitely many $n \in \mathbb{N}$ and hence we can find a strictly increasing sequence (n_k) in $\mathbb{N}$ such that $(x_{n_k}, y_{n_k}) \notin B_\varepsilon\big((0,1)\big)$ for all $k \in \mathbb{N}$. Since $\big((x_n, y_n)\big)$ is bounded, its subsequence $\big((x_{n_k}, y_{n_k})\big)$ is also bounded and hence by the Bolzano-Weierstrass theorem in $\mathbb{R}^2$, $\big((x_{n_k}, y_{n_k})\big)$ has a convergent subsequence $\big((x_{n_{k_l}}, y_{n_{k_l}})\big)$. Now, $\big((x_{n_{k_l}}, y_{n_{k_l}})\big)$ is also a subsequence of $\big((x_n, y_n)\big)$ and hence by the given condition $\big(x_{n_{k_l}}, y_{n_{k_l}}\big) \to (0,1)$. But this contradicts the fact that $\big(x_{n_{k_l}}, y_{n_{k_l}}\big) \notin B_\varepsilon\big((0,1)\big)$ for all $l \in \mathbb{N}$. Hence $(x_n, y_n) \to (0,1)$. Therefore the given statement is TRUE.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} \frac{xy}{x^2-y^2} & \text{if } x^2 \neq y^2, \\ 0 & \text{if } x^2 = y^2. \end{cases}$
Determine all the points of $\mathbb{R}^2$ where f is continuous.

Solution: If $\varphi(x,y) = xy$ and $\psi(x,y) = x^2 - y^2$ for all $(x,y) \in \mathbb{R}^2$, then as polynomial functions, $\varphi,\ \psi : \mathbb{R}^2 \to \mathbb{R}$ are continuous and $\psi(x,y) \neq 0$ for all $(x,y) \in \mathbb{R}^2$ with $x^2 \neq y^2$. Hence f is continuous at each $(x,y) \in \mathbb{R}^2$ with $x^2 \neq y^2$.
Let $(x,y) \in \mathbb{R}^2$ such that $x^2 = y^2 \neq 0$. Then $(x + \frac{x}{n}, y) \to (x,y)$ but $|f(x + \frac{x}{n}, y)| = \frac{n+1}{2+\frac{1}{n}} \to \infty$ and so $f(x + \frac{x}{n}, y) \not\to 0 = f(x,y)$. Hence f is not continuous at (x,y).
Again, $(\frac{2}{n}, \frac{1}{n}) \to (0,0)$ but $f(\frac{2}{n}, \frac{1}{n}) = \frac{2}{3}$ for all $n \in \mathbb{N}$, so that $f(\frac{2}{n}, \frac{1}{n}) \not\to 0 = f(0,0)$. Hence f is not continuous at $(0,0)$.
Therefore the set of points of continuity of f is $\{(x,y) \in \mathbb{R}^2 : x^2 \neq y^2\}$.

6. Let α, β be positive real numbers and let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by
$f(x,y) = \begin{cases} \frac{|x|^\alpha |y|^\beta}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$
Show that f is continuous iff $\alpha + \beta > 1$.

Solution: Let $\alpha + \beta > 1$ and let $\big((x_n, y_n)\big)$ be any sequence in $\mathbb{R}^2$ such that $(x_n, y_n) \to (0,0)$. Then $x_n \to 0$ and $y_n \to 0$. For all $n \in \mathbb{N}$ for which $(x_n, y_n) \neq (0,0)$, we have
$0 \leq f(x_n, y_n) \leq \frac{(x_n^2+y_n^2)^{\frac{\alpha}{2}} (x_n^2+y_n^2)^{\frac{\beta}{2}}}{\sqrt{x_n^2+y_n^2}} = (x_n^2 + y_n^2)^{\frac{1}{2}(\alpha+\beta-1)}$ and since $f(0,0) = 0$,
we have $0 \leq f(x_n, y_n) \leq (x_n^2 + y_n^2)^{\frac{1}{2}(\alpha+\beta-1)}$ for all $n \in \mathbb{N}$. Since $(x_n^2 + y_n^2)^{\frac{1}{2}(\alpha+\beta-1)} \to 0$, we get $f(x_n, y_n) \to 0 = f(0,0)$. This shows that f is continuous at $(0,0)$. Also, it is clear (by similar arguments given in other examples) that f is continuous at each $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Therefore f is continuous.
Conversely, let f be continuous and if possible, let $\alpha + \beta \leq 1$. We have $(\frac{1}{n}, \frac{1}{n}) \to (0,0)$ but $f(\frac{1}{n}, \frac{1}{n}) = \frac{1}{\sqrt{2}} n^{1-(\alpha+\beta)} \not\to 0 = f(0,0)$ (because for $\alpha+\beta = 1$, $f(\frac{1}{n}, \frac{1}{n}) \to \frac{1}{\sqrt{2}}$ and for $\alpha+\beta < 1$, the sequence $\big(f(\frac{1}{n}, \frac{1}{n})\big)$ is unbounded). Hence f is not continuous at $(0,0)$, which is a contradiction.

Therefore $\alpha + \beta > 1$.

7. Let $f : S \subseteq \mathbb{R}^2 \to \mathbb{R}$ and let $(x_0, y_0) \in S$. Let $A = \{x \in \mathbb{R} : (x, y_0) \in S\}$ and $B = \{y \in \mathbb{R} : (x_0, y) \in S\}$. Define $\varphi(x) = f(x, y_0)$ for all $x \in A$ and $\psi(y) = f(x_0, y)$ for all $y \in B$. If f is continuous at (x_0, y_0), then show that $\varphi : A \to \mathbb{R}$ is continuous at x_0 and $\psi : B \to \mathbb{R}$ is continuous at y_0. Is the converse true? Justify.

   Solution: Let (x_n) be a sequence in A such that $x_n \to x_0$ and let (y_n) be a sequence in B such that $y_n \to y_0$. Then $(x_n, y_0), (x_0, y_n) \in S$ for all $n \in \mathbb{N}$ and $(x_n, y_0) \to (x_0, y_0)$, $(x_0, y_n) \to (x_0, y_0)$. Since f is continuous at (x_0, y_0), $\varphi(x_n) = f(x_n, y_0) \to f(x_0, y_0) = \varphi(x_0)$ and $\psi(y_n) = f(x_0, y_n) \to f(x_0, y_0) = \psi(y_0)$. Therefore φ is continuous at x_0 and ψ is continuous at y_0.

   The converse is not true, in general. For example, let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by
   $$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$
   Then f is not continuous at $(0, 0)$, since $(\frac{1}{n}, \frac{1}{n}) \to (0, 0)$ but $f(\frac{1}{n}, \frac{1}{n}) = \frac{1}{2} \to \frac{1}{2} \neq 0 = f(0, 0)$. However, $\varphi(x) = f(x, 0) = 0$ for all $x \in \mathbb{R}$ and $\psi(y) = f(0, y) = 0$ for all $y \in \mathbb{R}$. So $\varphi : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$ are continuous at 0.

8. If $S = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 3\}$, then determine (with justification) S^0.

   Solution: Let $(x_0, y_0) \in S$ and let $0 < x_0 < 3$. If $r = \min\{x_0, 3 - x_0\}$, then $r > 0$. Let $(x, y) \in B_r\big((x_0, y_0)\big)$. Then $|x - x_0| \leq \sqrt{(x - x_0)^2 + (y - y_0)^2} < r$. Hence $x - x_0 < r \leq 3 - x_0$, which gives $x < 3$, and $x_0 - x < r \leq x_0$, which gives $x > 0$. Therefore $(x, y) \in S$ and so $B_r\big((x_0, y_0)\big) \subseteq S$. Hence $(x_0, y_0) \in S^0$.

   Now, let $y \in \mathbb{R}$.

   If possible, let $(0, y) \in S^0$. Then there exists $r > 0$ such that $B_r\big((0, y)\big) \subseteq S$. Since $\|(-\frac{r}{2}, y) - (0, y)\| = \frac{r}{2} < r$, $(-\frac{r}{2}, y) \in B_r\big((0, y)\big)$ and since $-\frac{r}{2} < 0$, $(-\frac{r}{2}, y) \notin S$. Thus we get a contradiction. Hence $(0, y) \notin S^0$.

   Again, if possible, let $(3, y) \in S^0$. Then there exists $r > 0$ such that $B_r\big((3, y)\big) \subseteq S$. Since $\|(3 + \frac{r}{2}, y) - (3, y)\| = \frac{r}{2} < r$, $(3 + \frac{r}{2}, y) \in B_r\big((3, y)\big)$ and since $3 + \frac{r}{2} > 3$, $(3 + \frac{r}{2}, y) \notin S$. Thus we get a contradiction. Hence $(3, y) \notin S^0$.

   Therefore $S^0 = \{(x, y) \in \mathbb{R}^2 : 0 < x < 3\}$.