MA 101 (Mathematics - I)

Multivariable Calculus : Examples from Lectures - 1

Example: $S = \{(x, y, z) \in \mathbb{R}^3 : |x| + 2|y| + 3z^2 < 1\}$ is a bounded set in $\mathbb{R}^3$.
Proof: If $(x, y, z) \in S$, then $|x| + 2|y| + 3z^2 < 1$ and so $|x| < 1$, $|y| < \frac{1}{2}$ and $|z| < \frac{1}{\sqrt{3}}$. Hence $\|(x, y, z)\| = \sqrt{x^2 + y^2 + z^2} < \sqrt{1 + \frac{1}{4} + \frac{1}{3}} = \sqrt{\frac{19}{12}}$. Therefore S is a bounded set in $\mathbb{R}^3$.

Example: $S = \{(x, y) \in \mathbb{R}^2 : x + y \leq 1\}$ is an unbounded set in $\mathbb{R}^2$.
Proof: If possible, let S be a bounded set in $\mathbb{R}^2$. Then there exists $r > 0$ such that $\|(x, y)\| = \sqrt{x^2 + y^2} \leq r$ for all $(x, y) \in S$. Now, $(r + 1, -r) \in S$ and so we must have $\sqrt{(r + 1)^2 + r^2} \leq r$, which is not true. Therefore S is an unbounded set in $\mathbb{R}^2$.

Example: If $\mathbf{y}_0 \in \mathbb{R}^k$ and if $f(\mathbf{x}) = \mathbf{y}_0$ for all $\mathbf{x} \in \mathbb{R}^m$, then $f : \mathbb{R}^m \to \mathbb{R}^k$ is continuous.
Proof: Let $\mathbf{x}_0 \in \mathbb{R}^m$ and let $\varepsilon > 0$. Then for all $\mathbf{x} \in \mathbb{R}^m$, we have $\|f(\mathbf{x}) - f(\mathbf{x}_0)\| = \|\mathbf{y}_0 - \mathbf{y}_0\| = \|\mathbf{0}\| = 0$. Hence $\|f(\mathbf{x}) - f(\mathbf{x}_0)\| < \varepsilon$ for all $\mathbf{x} \in \mathbb{R}^m$ satisfying $\|\mathbf{x} - \mathbf{x}_0\| < 1$. Therefore f is continuous at $\mathbf{x}_0$. Since $\mathbf{x}_0 \in \mathbb{R}^m$ is arbitrary, f is continuous.

Example: If $f(\mathbf{x}) = x_j$ for all $\mathbf{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$, then $f : \mathbb{R}^m \to \mathbb{R}$ is continuous.
Proof: Let $\mathbf{x}_0 = \left(x_1^{(0)}, \ldots, x_m^{(0)}\right) \in \mathbb{R}^m$ and let $\varepsilon > 0$. Then for all $\mathbf{x} \in \mathbb{R}^m$, we have $|f(\mathbf{x}) - f(\mathbf{x}_0)| = |x_j - x_j^{(0)}| \leq \|\mathbf{x} - \mathbf{x}_0\|$. If $\delta = \varepsilon$, then $\delta > 0$ and $|f(\mathbf{x}) - f(\mathbf{x}_0)| < \varepsilon$ for all $\mathbf{x} \in \mathbb{R}^m$ satisfying $\|\mathbf{x} - \mathbf{x}_0\| < \delta$. Therefore f is continuous at $\mathbf{x}_0$. Since $\mathbf{x}_0 \in \mathbb{R}^m$ is arbitrary, f is continuous.

Example: The function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by $f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0), \end{cases}$
is continuous at $(0, 0)$.
Proof: Let $\varepsilon > 0$. Then for all $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, we have
$|f(x, y) - f(0, 0)| = \frac{|x|\,|y|}{\sqrt{x^2+y^2}} \leq \frac{\sqrt{x^2+y^2}\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = \sqrt{x^2 + y^2}$.
Also, $|f(x, y) - f(0, 0)| = 0$ if $(x, y) = (0, 0)$. Let $\delta = \varepsilon$. Then $\delta > 0$ and for all $(x, y) \in \mathbb{R}^2$ with $\|(x, y) - (0, 0)\| = \sqrt{x^2 + y^2} < \delta$, we have $|f(x, y) - f(0, 0)| < \varepsilon$. Therefore f is continuous at $(0, 0)$.

Example: The function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by $f(x, y) = \begin{cases} 1 & \text{if } x^2 + y^2 \leq 1, \\ 2 & \text{if } x^2 + y^2 > 1, \end{cases}$
is continuous at $(x, y) \in \mathbb{R}^2$ iff $x^2 + y^2 \neq 1$.
Proof: Since f is a constant function on $S_1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ and also on $S_2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 > 1\}$, f is continuous on $S_1 \cup S_2$.
Now, let $(x_0, y_0) \in \mathbb{R}^2$ such that $x_0^2 + y_0^2 = 1$. Then $f(x_0, y_0) = 1$. We consider the following two possible cases.
Case 1: $x_0 \geq 0$.
In this case, $(x_0 + \frac{1}{n}, y_0) \to (x_0, y_0)$ and since $(x_0 + \frac{1}{n})^2 + y_0^2 > x_0^2 + y_0^2 = 1$ for all $n \in \mathbb{N}$, $f(x_0 + \frac{1}{n}, y_0) = 2 \to 2 \neq f(x_0, y_0)$.
Case 2: $x_0 < 0$.
In this case, $(x_0 - \frac{1}{n}, y_0) \to (x_0, y_0)$ and since $(x_0 - \frac{1}{n})^2 + y_0^2 > x_0^2 + y_0^2 = 1$ for all $n \in \mathbb{N}$, $f(x_0 - \frac{1}{n}, y_0) = 2 \to 2 \neq f(x_0, y_0)$.

Thus in either case f is not continuous at (x_0, y_0).
Therefore f is continuous at $(x, y) \in \mathbb{R}^2$ iff $x^2 + y^2 \neq 1$.

Example: The function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by $f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0), \end{cases}$
is not continuous at $(0, 0)$.
Proof: Since $(\frac{1}{n}, \frac{1}{n}) \to (0, 0)$ but $f(\frac{1}{n}, \frac{1}{n}) = \frac{\frac{1}{n^2}}{\frac{1}{n^2}+\frac{1}{n^2}} = \frac{1}{2} \to \frac{1}{2} \neq 0 = f(0, 0)$, f is not continuous at $(0, 0)$.

Example: The function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by $f(x, y) = \begin{cases} x^2 + y^2 & \text{if } x, y \in \mathbb{Q}, \\ 0 & \text{otherwise}, \end{cases}$
is continuous only at $(0, 0)$.
Proof: Let $\big((x_n, y_,)\big)$ be any sequence in $\mathbb{R}^2$ such that $(x_n, y_n) \to (0, 0)$. Then $x_n \to 0$ and $y_n \to 0$. Since $|f(x_n, y_n) - f(0, 0)| = |f(x_n, y_n)| \leq x_n^2 + y_n^2 \to 0$, $f(x_n, y_n) \to f(0, 0)$. Hence f is continuous at $(0, 0)$.
Let $(x_0, y_0) \in (\mathbb{Q} \times \mathbb{Q}) \setminus \{(0, 0)\}$. Then $f(x_0, y_0) = x_0^2 + y_0^2 \neq 0$. We know that there exists a sequence (x_n) in $\mathbb{R} \setminus \mathbb{Q}$ such that $x_n \to x_0$ and so $(x_n, y_0) \to (x_0, y_0)$. But $f(x_n, y_0) = 0$ for all $n \in \mathbb{N}$ and so $f(x_n, y_0) \to 0 \neq f(x_0, y_0)$. Hence f is not continuous at (x_0, y_0).
Again, let $(x_0, y_0) \in \mathbb{R}^2$ such that $x_0 \notin \mathbb{Q}$ or $y_0 \notin \mathbb{Q}$. Then $f(x_0, y_0) = 0$. We know that there exist sequences (x_n) and (y_n) in $\mathbb{Q}$ such that $x_n \to x_0$ and $y_n \to y_0$. Hence $(x_n, y_n) \to (x_0, y_0)$ but $f(x_n, y_n) = x_n^2 + y_n^2 \to x_0^2 + y_0^2 \neq f(x_0, y_0)$. Hence f is not continuous at (x_0, y_0).
Therefore f is continuous only at $(0, 0)$.

Example: Let $p : \mathbb{R}^m \to \mathbb{R}$ be a polynomial function,
i.e. $p(x_1, \ldots, x_m) = \sum_{j_1=0}^{k_1} \cdots \sum_{j_m=0}^{k_m} a_{j_1,\ldots,j_m} x_1^{j_1} \cdots x_m^{j_m}$ for all $(x_1, \ldots, x_m) \in \mathbb{R}^m$, where $a_{j_1,\ldots,j_m} \in \mathbb{R}$ for all $j_1, \ldots, j_m$, and $k_1, \ldots, k_m$ are non-negative integers. Then p is continuous.
Proof: We know that every constant function from $\mathbb{R}^m$ to $\mathbb{R}$ is continuous. Also, we know that for each $j \in \{1, \ldots, m\}$, the function $f_j : \mathbb{R}^m \to \mathbb{R}$, defined by $f_j(x_1, \ldots, x_m) = x_j$ for all $(x_1, \ldots, x_m) \in \mathbb{R}^m$, is continuous. Hence repeated applications of the product rule of continuous functions give that each of the the functions $g_{j_1,\ldots,j_m} : \mathbb{R}^m \to \mathbb{R}$, defined by
$g_{j_1,\ldots,j_m}(x_1, \ldots, x_m) = a_{j_1,\ldots,j_m} x_1^{j_1} \cdots x_m^{j_m}$ for all $(x_1, \ldots, x_m) \in \mathbb{R}^m$, is continuous,
where $j_i \in \{0, 1, \ldots, k_i\}$ for $i = 1, \ldots, m$. Now, by the sum rule of continuous functions, $p = \sum_{j_1=0}^{k_1} \cdots \sum_{j_m=0}^{k_m} g_{j_1,\ldots,j_m}$ is continuous.

Example: The function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by $f(x, y) = \begin{cases} \frac{x^2+y^2}{x+y} & \text{if } x + y \neq 0, \\ 0 & \text{if } x + y = 0, \end{cases}$
is continuous at $(x, y) \in \mathbb{R}^2$ iff $x + y \neq 0$.
Proof: Let $f_1(x, y) = x^2 + y^2$ and $f_2(x, y) = x + y$ for all $(x, y) \in \mathbb{R}^2$. As polynomial functions, $f_1 : \mathbb{R}^2 \to \mathbb{R}$ and $f_2 : \mathbb{R}^2 \to \mathbb{R}$ are continuous. If $S = \{(x, y) \in \mathbb{R}^2 : x + y \neq 0\}$, then $f_2(x, y) \neq 0$ for all $(x, y) \in S$. Hence $\frac{f_1}{f_2} : S \to \mathbb{R}$ is continuous and so it follows that f is continuous at each point of S.
Now, let $x \in \mathbb{R} \setminus \{0\}$. Then $(x + \frac{1}{n}, -x) \to (x, -x)$ but
$f(x + \frac{1}{n}, -x) = n[(x + \frac{1}{n})^2 + x^2] = 2nx^2 + 2x + \frac{1}{n} \to \infty \neq 0 = f(x, -x)$. Hence f is not continuous at $(x, -x)$.
Again, $(\frac{1}{n} + \frac{1}{n^2}, -\frac{1}{n}) \to (0, 0)$ but $f(\frac{1}{n} + \frac{1}{n^2}, -\frac{1}{n}) = n^2[(\frac{1}{n} + \frac{1}{n^2})^2 + \frac{1}{n^2}] = (1 + \frac{1}{n})^2 + 1 \to 2 \neq 0 = f(0, 0)$.

Hence f is not continuous at $(0,0)$.

Therefore f is continuous at $(x,y) \in \mathbb{R}^2$ iff $x+y \neq 0$.

Example: If $f(x,y) = e^{\sin(x^2+y^2)}$ for all $(x,y) \in \mathbb{R}^2$, then $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous.

Proof: Let $f_1(x,y) = x^2 + y^2$ for all $(x,y) \in \mathbb{R}^2$, $f_2(t) = \sin t$ for all $t \in \mathbb{R}$ and $f_3(t) = e^t$ for all $t \in \mathbb{R}$. Since $(f_3 \circ (f_2 \circ f_1))(x,y) = f(x,y)$ for all $(x,y) \in \mathbb{R}^2$, $f_3 \circ (f_2 \circ f_1) = f$. Now, we know that $f_2 : \mathbb{R} \to \mathbb{R}$ and $f_3 : \mathbb{R} \to \mathbb{R}$ are continuous. Also, as a polynomial function, $f_1 : \mathbb{R}^2 \to \mathbb{R}$ is continuous. Hence $f_2 \circ f_1 : \mathbb{R}^2 \to \mathbb{R}$ is continuous and therefore $f = f_3 \circ (f_2 \circ f_1)$ is continuous.

Example: If $S = \{(x,y) \in \mathbb{R}^2 : x + y \leq 0\}$, then $(-1,0) \in S^0$.

Proof: Let $r = \frac{1}{\sqrt{2}}$ and let $(x,y) \in B_r\big((-1,0)\big)$. Then $\|(x,y) - (-1,0)\| = \sqrt{(x+1)^2 + y^2} < r$. By Cauchy-Schwarz inequality, we have $x + 1 + y \leq \sqrt{(x+1)^2 + y^2}\,\sqrt{1^2 + 1^2} < \sqrt{2}r = 1$. Hence $x + y < 0$ and so $(x,y) \in S$. Thus $B_r\big((-1,0)\big) \subseteq S$ and therefore $(-1,0) \in S^0$.

Example: If $S = \{(x,y) \in \mathbb{R}^2 : x + y \leq 0\}$, then $(0,0) \notin S^0$.

Proof: If possible, let $(0,0) \in S^0$. Then there exists $r > 0$ such that $B_r\big((0,0)\big) \subseteq S$. Now, $\|(\frac{r}{2},0) - (0,0)\| = \|(\frac{r}{2},0)\| = \frac{r}{2} < r$ and so $(\frac{r}{2},0) \in B_r\big((0,0)\big)$. However, $(\frac{r}{2},0) \notin S$, which is a contradiction. Therefore $(0,0) \notin S^0$.

Example: $S = \{(x,y) \in \mathbb{R}^2 : x + y < 0\}$ is an open set in $\mathbb{R}^2$.

Proof: Let $(x_0,y_0) \in S$ so that $x_0 + y_0 < 0$. Let $r = \frac{-x_0-y_0}{\sqrt{2}} > 0$ and let $(x,y) \in B_r\big((x_0,y_0)\big)$. Then $\|(x,y) - (x_0,y_0)\| = \sqrt{(x-x_0)^2 + (y-y_0)^2} < r$. By Cauchy-Schwarz inequality, we have $x - x_0 + y - y_0 \leq \sqrt{(x-x_0)^2 + (y-y_0)^2}\,\sqrt{1^2+1^2} < \sqrt{2}r = -x_0 - y_0$. Hence $x + y < 0$ and so $(x,y) \in S$. Thus $B_r\big((x_0,y_0)\big) \subseteq S$ and therefore (x_0,y_0) is an interior point of S. Since $(x_0,y_0) \in S$ is arbitrary, it follows that S is an open set in $\mathbb{R}^2$.