b) The series $1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{2^3} + \frac{1}{4^2} - \frac{1}{2^4} + \cdots$ converges absolutely.

c) Let $a_n, b_n > 0$. Suppose that $\sum a_n$ and $\sum b_n$ are convergent. Then the series $a_1 - b_1 + a_2 - b_2 + a_3 - b_3 + \cdots$ converges absolutely.

THEOREM 4.3.7 **(absolute convergence test)** *If $\sum a_n$ is absolutely convergent, then $\sum a_n$ is convergent.*!!

EXAMPLE 4.3.8 Consider $1 - \frac{1}{2} + \frac{1}{3} - \cdots$. Then

$$S_{2n} = 1 - \frac{1}{2} + \cdots + \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(2n-1)(2n)} \le \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)}.$$

Hence (S_{2n}) is convergent (why?). Let l be the limit. Then $S_{2n+1} = S_{2n} + \frac{1}{2n+1} \to l$. So $S_n \to l$. So the series is convergent. Does it converge absolutely?

4.4 Limit comparison test, condensation test

LEMMA 4.4.1 **(limit comparison test)** *Let $a_n, b_n > 0$, and $\lim \frac{a_n}{b_n} = c$.*

1. *If $c > 0$, then $\sum a_n$ and $\sum b_n$ converge or diverge together.*
2. *If $c = 0$ and $\sum b_n$ is convergent, then $\sum a_n$ is convergent.*

Proof.

1. Let $c > 0$. Then $\exists k \in \mathbb{N}$ such that $\frac{c}{2} \le \frac{a_n}{b_n} \le \frac{3c}{2}$, $\forall n \ge k$. Apply sandwich.
2. If $c = 0$, then $\exists k \in \mathbb{N}$ such that $\frac{a_n}{b_n} \le 1$, $\forall n \ge k$. So $0 < a_n \le b_n$, $\forall n \ge k$. Apply comparison test.■

EXAMPLE 4.4.2 1. $\sum \frac{1}{n!}$ is convergent as $\frac{1}{n!} < \frac{1}{2^{n-1}}$, and $\sum \frac{1}{2^{n-1}}$ converges. (Comparison test)

2. Take $a_n = \frac{2n}{n^2-n+1}$. Put $b_n = \frac{1}{n}$. Then $\frac{a_n}{b_n} \to 2$. By limit comparison test, $\sum a_n$ diverges.
3. Take $a_n = \frac{\ln n}{n^2}$, $n \ge 3$. Is $\sum a_n$ convergent? Yes. Put $b_n = \frac{1}{n^{3/2}}$. Then $\frac{a_n}{b_n} = \frac{\ln n}{\sqrt{n}} \to 0$. By limit comparison test $\sum a_n$ converges. (Try this for $\sum \frac{\ln n}{n^p}$, where $p > 1$ is fixed.)

LEMMA 4.4.3 **(Cauchy condensation test)** *Let $a_n \ge 0$ be decreasing. Then $\sum a_n$ is convergent iff $\sum 2^n a_{2^n}$ is convergent.*

Proof. Let $\sum a_n$ be convergent. Notice that

$$a_2 + (a_3 + a_4) + (a_5 + \cdots + a_8) + \cdots + (a_{2^{n-1}+1} + \cdots + a_{2^n}) \ge a_2 + 2a_4 + 4a_8 + \cdots + 2^{n-1}a_{2^n}.$$

Multiplying by 2 we get, $\sum 2^n a_{2^n}$ is convergent. On the other hand, if $\sum 2^n a_{2^n}$ is convergent, then

$$2a_2 + 4a_4 + \cdots + 2^n a_{2^n} \ge (a_2 + a_3) + (a_4 + \cdots + a_7) + \cdots + a_{2^{n+1}-1}.$$

Hence $\sum a_n$ converges. ■

EXAMPLE 4.4.4 The series $\sum\limits_{n\ge 2} \frac{1}{n(\ln n)^p}$, $p > 1$ is convergent, as $2^n a_{2^n} = \frac{2^n}{2^n(n\ln 2)^p} = \frac{1}{(\ln 2)^p}\frac{1}{n^p}$ and $\sum \frac{1}{n^p}$ converges for $p > 1$.

4.5 Ratio test and root test

THEOREM 4.5.1 **(D'Alembert's ratio test)**

1. *If* $\frac{|a_{n+1}|}{|a_n|} < \rho < 1$, *for each* $n \geq$ *some* k, *then* $\sum |a_n|$ *is convergent. (In such uses, it is assumed that* $a_n \neq 0$.*)*

2. *If* $\frac{|a_{n+1}|}{|a_n|} \geq 1$, *for each* $n \geq$ *some* k, *then* $\sum a_n$ *diverges.*

Proof.

1. We get $|a_{k+1}| \leq \rho |a_k|$, $|a_{k+2}| \leq \rho^2 |a_k|$ and so on. So
$$\sum |a_n| \leq (|a_1| + \cdots + |a_{k-1}|) + |a_k|(1 + r + r^2 + \ldots) \leq (|a_1| + \cdots + |a_{k-1}|) + \frac{1}{1-r}|a_k|.$$
Hence(?) it is convergent.

2. As the nth term does not go to 0. ∎

COROLLARY 4.5.2 **(ratio test-ii)** *Let* $\frac{|a_{n+1}|}{|a_n|} \to \rho$.

1. *If* $\rho < 0$, *then* $\sum |a_n|$ *converges.*

2. *If* $\rho > 1$, *then* $\sum a_n$ *diverges.*

3. *If* $\rho = 1$, *then* $\sum a_n$ *could be convergent or divergent.*

Proof. The first two are easy. For the last part, both $\sum n$ and $\sum \frac{1}{n^2}$, $\lim \frac{a_{n+1}}{a_n} = 1$. One of them is convergent and the other is divergent. ∎

EXAMPLE 4.5.3 1. $\sum \dfrac{2^n + 5}{3^n}$ converges as $\lim \frac{a_{n+1}}{a_n} \to \frac{2}{3}$.

2. $\sum \dfrac{x^n}{n!}$ for $x > 0$, converges as $\lim \frac{a_{n+1}}{a_n} \to 0$.

EXERCISE 4.5.4 *Test for convergence.*

1. $\sum \frac{(2n)!}{n!n!}$

2. $\sum \frac{n^5}{2^n}$

3. $\sum \frac{n!}{n^n}$

THEOREM 4.5.5 **(Cauchy's root test)**

1. *If* $\sqrt[n]{|a_n|} \leq \rho < 1$, *for each* $n \geq$ *some* k, *then* $\sum |a_n|$ *converges.*

2. *If* $\sqrt[n]{|a_n|} \geq 1$, *for each* $n \geq$ *some* k, *then* $\sum a_n$ *diverges.*

Proof. Exercise. ∎

THEOREM 4.5.6 **(root test-ii)** *Let* $\sqrt[n]{|a_n|} \to \rho$.

1. *If* $\rho < 1$, *then* $\sum |a_n|$ *converges.*

2. *If* $\rho > 1$, *then* $\sum a_n$ *diverges.*

3. *If* $\rho = 1$, *then the series may be convergent, may be divergent.*

Proof. Exercise. ■

EXAMPLE 4.5.7 1. Consider the series $\frac{1}{3}+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{2^4}+\frac{1}{3^5}+\frac{1}{2^6}+\cdots$. We cannot apply the ratio test. As $\sqrt[n]{a_n}\le\frac{1}{2}$, by root test the series converges.

2. Both the tests fail with $2+\frac{3}{2}+\frac{4}{3}+\frac{5}{4}+\cdots$.

EXERCISE 4.5.8 *Test for convergence.*

a) $\sum\left(\frac{n^5}{n^5+1}\right)^{n^6}$, *b)* $\sum\frac{(2n)^n}{2^{n^2}}$, *c)* $\sum\frac{5^n}{6^n+4^n}$.

EXERCISE 4.5.9 *Test for convergence* $\sum\frac{n!2^n}{n^{n^2}}$.

EXERCISE 4.5.10 *For which* $x\in\mathbb{R}$, *does* $1+2x+x^2+2x^3+x^4+\cdots$ *converge?*

EXERCISE 4.5.11 *Show that for a series of positive terms if ratio test concludes convergence, then root test also concludes convergence.*

4.6 Raabe's test

This is only an absolute convergent test.

THEOREM 4.6.1 **(Raabe's test-I)** *Let* $a_n\neq 0$ *and* $a>1$.

a) If $\dfrac{|a_{n+1}|}{|a_n|}\le 1-\dfrac{a}{n}$, *for all* $n\ge$ *some* n_0, *then* $\sum|a_n|$ *is convergent.*

b) If $\dfrac{|a_{n+1}|}{|a_n|}\ge 1-\dfrac{1}{n}$, *for all* $n\ge$ *some* $n_0>1$, *then* $\sum|a_n|$ *is divergent.*

Proof. a) Note that

$$n|a_{n+1}|\le(n-a)|a_n|\Rightarrow(a-1)|a_n|\le(n-1)|a_n|-n|a_{n+1}|.$$

Hence

$$(a-1)(|a_{n_0}|+\cdots+|a_k|)\le(n_0-1)|a_{n_0}|-k|a_{k+1}|\le(n_0-1)|a_{n_0}|.$$

We can now use monotone convergence theorem.**!!** We see that $\sum|a_n|$ is convergent.

b) From the hypothesis we see that

$$n|a_{n+1}|\ge(n-1)|a_n|\ge\cdots\ge(n_0-1)|a_{n_0}|=c>0\quad\text{(say)}.$$

Then $|a_{n+1}|\ge\dfrac{c}{n}$ for all $n\ge n_0$. By comparison test, we are done. ■

REMARK 4.6.2 Consider the item b) in the previous test.

1. We can replace the term $\frac{1}{n}$ with $\frac{1}{n+k}$, for some fixed positive integer k. The conclusion remains valid and the proof is similar.

2. The test does not talk about the convergence of $\sum a_n$. For example, we can take the series $\sum(-1)^n\frac{1}{n}$ and the test will tell us that it is not absolutely convergent. However, the series is convergent, as we know.

THEOREM 4.6.3 **(Raabe's test-II)** *Let* $a_n\neq 0$ *and* $n\left(1-\dfrac{|a_{n+1}|}{|a_n|}\right)\to a$.

a) If $a>1$ *then* $\sum|a_n|$ *is convergent.*

b) If $a<1$ *then* $\sum|a_n|$ *is divergent.**

c) If $a=1$ *then no conclusion.*

*It does not talk about $\sum a_n$. Take $a_n=(-1)^n\frac{1}{\sqrt{n}}$. Then $a=\frac{1}{2}$. We will see very soon that the series converges.

Proof. a) If $n\left(1-\frac{|a_{n+1}|}{|a_n|}\right) \to a > 1$, then for some $n \geq n_0$, we have $n\left(1-\frac{|a_{n+1}|}{|a_n|}\right) > b > 1$. That is

$$\frac{b}{n} < 1 - \frac{|a_{n+1}|}{|a_n|} \quad \text{or} \quad \frac{|a_{n+1}|}{|a_n|} < 1 - \frac{b}{n} \quad \text{for } n \geq n_0.$$

So it follows from the previous result.

b) If $n\left(1-\frac{|a_{n+1}|}{|a_n|}\right) \to a < 1$, then for some $n \geq n_0$, we have

$$n\left(1-\frac{|a_{n+1}|}{|a_n|}\right) < b < 1 \quad \text{or} \quad \frac{|a_{n+1}|}{|a_n|} > 1 - \frac{b}{n} > 1 - \frac{1}{n}.$$

So it follows from the previous result.

c) Consider $\sum \frac{1}{n}$ and $\sum \frac{1}{n(\ln n)^2)}$. ∎

Example 4.6.4 Take $a_n = \dfrac{3 \cdot 6 \cdot 9 \cdots (3n)}{7 \cdot 10 \cdot 13 \cdots (3n+4)} r^n$, $r > 0$. Then $\dfrac{a_{n+1}}{a_n} = \dfrac{3n+3}{3n+7} r \to r$.

If $r < 1$, convergent by ratio test. If $r > 1$, divergent by ratio test. If $r = 1$, ratio test cannot help. Apply Raabe's test:

$$n(1 - \frac{|a_{n+1}|}{|a_n|}) = n(1 - \frac{3n+3}{3n+7}) = \frac{4n}{3n+7} \to \frac{4}{3} > 0.$$

So by Raabe's test, the series is convergent.

x^n dominates any polynomial of degree $n-1$, for sufficiently large positive x.

Consider a polynomial $P(x) := x^n - a_1x^{n-1} - a_2x^{n-2} - \cdots - a_n$. Then for each $k \geq |a_1| + \cdots + |a_n| + 1$, we have

$$\begin{aligned}
|P(k)| &\geq (|a_1| + \cdots + |a_n| + 1)^n - |a_1|(|a_1| + \cdots + |a_n| + 1)^{n-1} \\
&\qquad - |a_2|(|a_1| + \cdots + |a_n| + 1)^{n-2} - \cdots - |a_n| \\
&\geq (|a_1| + \cdots + |a_n| + 1)^n - |a_1|(|a_1| + \cdots + |a_n| + 1)^{n-1} \\
&\qquad - |a_2|(|a_1| + \cdots + |a_n| + 1)^{n-1} - \cdots - |a_n|(|a_1| + \cdots + |a_n| + 1)^{n-1} \\
&= (|a_1| + \cdots + |a_n| + 1)^n - (|a_1| + |a_2| + \cdots + |a_n|)(|a_1| + \cdots + |a_n| + 1)^{n-1} \\
&\geq (|a_1| + \cdots + |a_n| + 1)^n - (|a_1| + |a_2| + \cdots + |a_n| + 1)(|a_1| + \cdots + |a_n| + 1)^{n-1} \\
&= 0.
\end{aligned}$$

Do not want to use Raabe's test?

We may still use, the ideas of geometric series and harmonic series to work it out. Let us take the previous example. Notice that,

$$(3n+3)^4(n+1)^5 < (3n+7)^4 n^5, \qquad \text{for all } n \geq \text{some } k, \tag{4.1}$$

as the coefficient of n^8 is higher in the rhs. This means, if $a_k < \frac{1}{k^{5/4}}$, then

$$a_{k+1} = a_k \frac{3k+3}{3k+7} \leq \frac{1}{k^{5/4}} \frac{3k+3}{3k+7} \leq \frac{1}{(k+1)^{5/4}} \quad \text{by the previous equation}$$

and $a_{k+2} \leq \frac{1}{(k+2)^{5/4}}$ and so on. But, then we can multiply the series by a scalar so that $a_k < \frac{1}{k^{5/4}}$. And so, we are done. The series is convergent.

Exercise 4.6.5 *Test for convergence.*

1. $\sum \left(\frac{1.3.5.\cdots(2n-1)}{2.4.6.\cdots(2n)}\right)^3$
2. $\sum \left(\frac{1.3.5.\cdots(2n-1)}{2.4.6.\cdots(2n)}\right)^2$

4.7 Alternating series

DEFINITION 4.7.1 An **alternating series** is a series with terms alternately positive and negative.

EXAMPLE 4.7.2 1. The series $1 - 1 + 1 - 1 + \cdots$ is an alternating series.

2. The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is called the **alternating harmonic series**.

THEOREM 4.7.3 **(Leibniz theorem)** *Let $u_n \geq 0$. The series $u_1 - u_2 + u_3 - u_4 + \cdots$ converges if $u_n \downarrow 0$.*

Proof. Let $u_n \downarrow 0$. Put $U_n = u_1 - u_2 + \cdots + (-1)^{n+1}u_n$. Then

$$U_{2n} = (u_1 - u_2) + (u_3 - u_4) + \cdots + (u_{2n-1} - u_{2n}) \leq U_{2n+2}$$

and

$$U_{2n} = u_1 + (-u_2 + u_3) + (-u_4 + u_5) + \cdots + (-u_{2n-2} + u_{2n-1}) - u_{2n} \leq u_1.$$

Hence $U_{2n} \uparrow$ and bounded above. By MCT, it converges to l (say). As $u_n \to 0$, we see that $U_{2n+1} \to l$. Hence, $U_n \to l$. ∎

EXERCISE 4.7.4 **(Dirichlet η function)** *For any fixed $p > 0$, show that the series $\eta(p) := \sum \frac{(-1)^{n-1}}{n^p}$ converges.*

EXERCISE 4.7.5 *Are there convergent alternating series $\sum(-1)^{n+1}u_n$, where u_n does not decrease to 0?*

4.8 Rearrangement of terms

Rearrangements

1. I have written 9 positive numbers: $a_1, \ldots, a_9$.
2. Without my knowledge my friend rearranges them: $b_1, \ldots, b_9$. Do you think both will have the same sum?
3. What if, I had $a_1, a_2, \ldots$ with $\sum a_n = a$?
4. Let $b_1, b_2, \ldots$ be a **rearrangement** (bijective image) of $a_1, a_2, \ldots$.
5. Will I have $\sum b_n$ convergent? If yes, is it equal to $\sum a_n$?

THEOREM 4.8.1 **(Rearrangement of terms of an absolutely convergent series)** *If $\sum a_n$ converges absolutely and (b_n) is a rearrangement of (a_n), then $\sum b_n$ converges absolutely and both converge to the same limit.*

Proof. First, we show it for $a_n \geq 0$. Let $\sum a_n = a$. Given n, there is a number n' such that the terms $b_1, \ldots, b_n$ can be found in $a_1, a_2, \ldots, a_{n'}$. So $B_n \leq A_{n'} \leq a$. But as $B_n \uparrow$, we see that $B_n \to b$ (say). So $b \leq a$.

Now that $\sum b_n = b$, and (a_n) is a rearrangement of (b_n), repeating the previous argument, we must have $a \leq b$. So $a = b$.

Next, if $\sum a_n$ converges absolutely, put $c_n = a_n + |a_n|$. Let (b_n) be a rearrangement of (a_n). Take the corresponding rearrangement of (c_n). Call it (d_n). As $c_n \geq 0$, we have $\sum c_n = \sum d_n$. As $\sum |a_n| = \sum |b_n|$, we get, $\sum a_n = \sum(c_n - |a_n|) = \sum(d_n - |b_n|) = \sum b_n$. ∎

Remark 4.8.2 So under absolute convergence, a series behaves as if it is a sum of finitely many numbers. You can rearrange it as you like and still have the same value. If we do not have an absolutely convergent series, can the value of the series change by rearrangement? The answer is yes, as observed by Riemann. (See the example and the result below.) This makes absolute convergence an important concept.

Example 4.8.3 Let $$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \cdots = S.$$

So $\frac{1}{2} - \frac{1}{4} + \frac{1}{6} + \cdots = \frac{S}{2}$.

Insert zeros: $$0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + 0 + \cdots = \frac{S}{2}.$$

Add them: $$1 + 0 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + 0 + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} \cdots = \frac{3S}{2}.$$

Now, drop the zeros. It is a rearrangement of the series on the top but the value is $\frac{3S}{2}$.

Riemann rearrangement theorem

Let $\sum a_n$ be conditionally convergent. Then it can be rearranged to converge to any fixed real number. It can also be rearranged to be divergent. We leave the proof as a guided exercise.

1. First argue that, without loss, we can assume that $0 < |a_n| < 1$.
2. Argue that $\sum\limits_{a_n > 0} a_n = \infty$ and $\sum\limits_{a_n < 0} a_n = -\infty$.
3. Make a list of the positive terms, as they appear in the series. Also make another, for the negative terms.
4. Suppose that we want to make to converge to a.
5. Select the first few positive terms so that the sum is more than a. (Why is this possible?)
6. Now add the next few negative terms, so that the sum is in $[a-1, a)$. (Why is this possible?)
7. Now add the next few positive terms so that the sum lies in $(a, a+1]$. Repeat. After a stage, we see that all $|a_n| < \frac{1}{2}$. (Why must such a stage exist?)
8. Now, proceed as before, to keep the sum in $[a - \frac{1}{2}, a)$ and $(a, a + \frac{1}{2}]$, alternately.
9. Show that if have reached a stage where the sums are in $[a - \frac{1}{k}, a)$ and $(a, a + \frac{1}{k}]$, alternately, then we can reach a stage where the sums are in $[a - \frac{1}{k+1}, a)$ and $(a, a + \frac{1}{k+1}]$, alternately.
10. This defines a rearrangement of the old series, inductively. Argue that, the series we obtain this way, converges to a.

Exercise 4.8.4 *Consider the alternating harmonic series with signs reversed:* $-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots$. *Give a rearrangement of this series which is divergent.*

4.9 Term-wise product

An important aspect in the study of series is to study the term-wise product of two series.

Abel's partial sum formula

Consider (a_n) and (b_n). Put $A_n = \sum_{i=1}^{n} a_i$ and $A_0 = 0$. Then

$$\begin{aligned}\sum_{k=1}^{n}(a_k b_k) &= \sum_{k=1}^{n}(A_k - A_{k-1})b_k = \sum_{k=1}^{n} A_k b_k - \sum_{k=1}^{n-1} A_k b_{k+1} \\ &= \sum_{k=1}^{n} A_k b_k - \sum_{k=1}^{n} A_k b_{k+1} + A_n b_{n+1} = A_n b_{n+1} + \sum_{k=1}^{n} A_k(b_k - b_{k+1}).\end{aligned}$$

■

LEMMA 4.9.1 **(Dirichlet's test)** *Let* $(A_n = \sum_{i=1}^{n} a_i)$ *be bounded and* $b_n \downarrow 0$. *Then* $\sum a_n b_n$ *converges.*

Proof. Assume that $|A_k| \leq M$ for all k. As $\lim A_n b_{n+1} = 0$, by Abel's partial sum formula, we only need to show that $\sum A_k(b_k - b_{k+1})$ is convergent. This series is absolutely convergent as the sequence of its partial sums (S_n) is an increasing sequence with

$$S_n = \sum_{k=1}^{n} |A_k(b_k - b_{k+1})| \leq M \sum_{k=1}^{n} (b_k - b_{k+1}) \leq M(b_1 - b_{n+1}) \leq M b_1 \text{ (MCT)}.$$ ■

LEMMA 4.9.2 **(Abel's test)** *Let* $\sum a_n$ *be convergent and* (b_n) *be monotone convergent. Then* $\sum a_n b_n$ *is convergent.*

Proof. Exercise. ■

EXAMPLE 4.9.3 Is $\sum \frac{\sin n}{n}$ convergent? Yes. To see that notice that

$$2 \sin 1 \sin k = \cos(k-1) - \cos(k+1).$$

So $\sum_{k=1}^{n} \sin k = \frac{\cos(0)+\cos(1)-\cos(n)-\cos(n+1)}{2 \sin 1}$ and

$$|\sum_{k=1}^{n} \sin k| \leq \frac{2}{\sin 1}.$$

As $\frac{1}{n} \downarrow 0$, by Dirichlet's test, we see that $\sum \frac{\sin n}{n}$ convergent.

4.10 Exercises

EXERCISE 4.10.1 **(Practice)** *Find their values.*

1. $\sum \frac{1}{n(n+1)(n+2)}$
2. $\sum (\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}})$

EXERCISE 4.10.2 **(Practice)** *True or false?*

1. $\sum \frac{5^n}{6^n+4^n}$ *is convergent.*
2. $\sum (-1)^{n+1} \frac{1}{n^2}$ *is absolutely convergent.*
3. $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{\ln n}$ *is conditionally convergent.*
4. $\sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{\ln n^2}$ *is convergent.*
5. $\sum \frac{\cos(n\pi)}{n\sqrt{n}}$ *is absolutely convergent.*

6. $\sum(-1)^n\left(\sqrt{n+1}-\sqrt{n}\right)$ *is absolutely convergent.*

7. $\sum \frac{e^{n\pi}}{\pi^{ne}}$ *is convergent.*

8. $\sum(-1)^n \sin(\frac{1}{n})$ *is conditionally convergent.*

9. $\sum(-1)^n n \sin(\frac{1}{n})$ *is divergent.*

10. $\sum \frac{n}{(\ln n)^n}$ *is convergent.*

11. $\sum \ln(\frac{n}{n+1})$ *is convergent.*

12. $\sum \ln(\frac{1}{n})$ *is convergent.*

13. $\sum(1-\frac{1}{n})^n$ *is convergent.*

14. $\sum \frac{1+\cos n}{n^2}$ *is absolutely convergent.*

15. $\sum\limits_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$ *is divergent.*

16. $\sum\limits_{n=2}^{\infty} \frac{1}{(\ln n)^2}$ *is convergent.*

17. $\sum \frac{n^2+2^n}{n^2 2^n}$ *is convergent*

18. $\sum \frac{\tan^{-1}(n)}{n^{1.1}}$ *is convergent.*

19. $\sum \frac{1}{n.n^{1/n}}$ *is convergent.*

20. $\sum \frac{\ln n}{n^3}$ *is convergent.*

21. $\sum \frac{n!\ln n}{n^n}$ *is convergent.*

22. $\sum a_n$ *is convergent, where* $a_1 = 2$, $a_{n+1} = \frac{1+\sin n}{n} a_n$.

23. $\sum \frac{n}{(\ln n)^n}$ *is convergent.*

24. $\sum\limits_{n=5}^{\infty} \frac{n^3}{e^{\sqrt{n}}}$ *is convergent.*

EXERCISE 4.10.3 **(Gauss test: useful in some rare but tough cases)**

1. *Let* $a_n > 0$ *with* $\frac{a_n}{a_{n+1}} = \alpha + \frac{\beta}{n} + \frac{\gamma_n}{n^p}$, *where* $\alpha > 0$, $p > 1$ *and* (γ_n) *is bounded. Prove the following.*

   a) $\sum a_n$ *converges if* $\alpha > 1$ *and diverges if* $\alpha < 1$.

   *b)** *If* $\alpha = 1$, *then* $\sum a_n$ *converges if* $\beta > 1$ *and diverges if* $\beta \le 1$.

2. *Consider the series* $\dfrac{2^2}{3^2} + \dfrac{2^2\cdot 4^2}{3^2\cdot 5^2} + \dfrac{2^2\cdot 4^2\cdot 6^2}{3^2\cdot 5^2\cdot 7^2} + \cdots$. *Is it convergent?*

EXERCISE* 4.10.4 **(Information: do it if you have time)** *Let* p_i *be the ith prime. Then show that* $\sum \frac{1}{p_i} = \infty$.

EXERCISE* 4.10.5 **(Test your innovative self)** *Let* S *be the set of natural numbers whose only prime divisors are* $2, 3, 5, 7$. *Evaluate* $1 + \sum\limits_{n\in S} \frac{1}{n}$.

EXERCISE* 4.10.6 **(Work of an adamant person)** *Let* $a_n = \dfrac{1.3.5.\cdots(2n-1)}{2.4.6.\cdots(2n)}$ *and consider the series* $\sum a_n^3$. *Show, without using Raabe's test (or Gauss' test), that the series is convergent.*

EXERCISE 4.10.7 **(Extra practice)**

1. *Suppose that* $a_n \geq 0$, $p > 1$ *and* $\sum a_n$ *converges. Is* $\sum a_n^p$ *necessarily convergent?*
2. *Suppose that* $a_n > 0$ *and* $\sum a_n$ *converges. Is* $\sum \sqrt{a_n a_{n+1}}$ *necessarily convergent?*
3. *Let* $p > 1$ *and* a_n *be such that* $\lim n^p a_n$ *exists. Is* $\sum a_n$ *absolutely convergent?*
4. *Let* $a_n > 0$ *be decreasing and* $\sum a_n$ *be convergent. Is it necessary that* $\lim na_n = 0$*?*
5. *Let* $a_n > 0$ *be decreasing and* $\sum a_n$ *be divergent. Can* $\lim na_n$ *be* 0*?*
6. *If* $a_n > 0$ *is any sequence and* $\sum a_n$ *is convergent, then what can we say about* $\lim na_n$*?*

EXERCISE 4.10.8 **(More practice)** *If possible find examples with nonzero terms (wherever necessary).*

1. *Let* l *be a fixed real number. Give two series with distinct terms converging to* l.
2. *If* $\sum a_n$ *and* $\sum b_n$ *are divergent, is* $\sum(a_n + b_n)$ *necessarily divergent?*
3. *What happens to* $\sum(a_n + b_n)$ *when* $\sum a_n = a$ *and* $\sum b_n$ *diverges?*
4. *If* $\sum a_n$ *and* $\sum b_n$ *are convergent is* $\sum(a_n b_n)$ *necessarily convergent?*
5. *Let* $a_n, b_n \geq 0$. *Suppose that* $\sum a_n = l$ *and* $\sum b_n = t$. *Is* $\sum a_n b_n$ *convergent?*
6. *Let* $a_n \geq 0$. *If* $\sum a_n^2$ *converges, then is* $\sum a_n$ *convergent?*
7. *Can* $\sum(a_n b_n)$ *be convergent, given* $\sum a_n, \sum b_n$ *are divergent?*
8. *If* $\sum a_n$ *is convergent and* $\{b_n\}$ *is bounded, is* $\sum(a_n b_n)$ *necessarily convergent?*
9. *Let* $a_n \geq 0$ *and* $\sum a_n$ *be convergent. Is* $\sum \frac{\sqrt{a_n}}{n}$ *necessarily convergent?*
10. *If* $\sum a_n$ *converges and* $a_n \geq 0$ *then is* $\sum \frac{a_n}{n}$ *necessarily convergent?*
11. *Suppose that* $a_n > 0$ *and* $\lim\limits_{n\to\infty} a_n = 0$. *Is* $\sum\limits_{n=1}^{\infty} (-1)^{n+1} a_n$ *necessarily convergent?*

EXERCISE 4.10.9 **(Just an old classical exercise)** *Imagine you are in a period when information about p-series is not known. You still could argue for* $p = 1.5$. *Follow the instructions to see that. Show that for* $n \geq 1$, $\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \geq \frac{1}{2(n+1)\sqrt{n}}$. *Use this to show that* $\sum \frac{1}{n^{1.5}}$ *converges.*

EXERCISE* 4.10.10 **(Imagination)** *Let* S *be the set of natural numbers whose decimal representation does not involve a* 0. *Show that* $\sum\limits_{n\in S} \frac{1}{n}$ *is convergent.*

EXERCISE 4.10.11 **(Some old exam question)** *Fix* $a > 0$. *Find values of* p *for which the series* $\frac{1}{(1+a)^p} - \frac{1}{(2+a)^p} + \frac{1}{(3+a)^p} - \frac{1}{(4+a)^p} + \cdots$ *converges absolutely (conditionally)?*

EXERCISE 4.10.12 **(Analytical: some bits may be old exam (subjective type) questions)** *Let* $a_n > 0$, $\sum a_n \uparrow \infty$, *and* $S_n = a_1 + \cdots + a_n$. *Check for convergence of the following series.*

(a) $\sum \frac{a_n}{1+a_n}$ *(b)* $\sum \frac{a_n}{S_n}$ *(c)* $\sum \frac{a_n}{(S_n)^2}$ *(d)* $\sum \frac{a_n}{1+na_n}$ *(e)* $\sum \frac{a_n}{1+n^2 a_n}$

EXERCISE* 4.10.13 **(Remainder of convergent series)** *Suppose* $a_n > 0$ *and* $\sum a_n$ *converges. Let* $r_n = a_n + a_{n+1} + a_{n+2} + \cdots$. *Show that* $\sum \frac{a_n}{r_n}$ *diverges whereas* $\sum \frac{a_n}{\sqrt{r_n}}$ *converges.*

Chapter 5

Limits of functions and continuity

After this chapter I should be able to answer the following.

1. Show that there are two diametrically opposite points on the surface of the Earth at which the temperatures are the same.

2. Imagine holding a nonuniform elastic rubber band, say $[-1, 1]$ at two ends. Fixing your right hand, move the left hand to -5. Now fixing your left hand move the right hand to 9. Do you think as a result, there is a point on the band which has not moved?

3. Consider $f(x) = x^9 - x^8 + 5x^5 - 7x^3 + 37$. Does there exist a value of $\delta > 0$ such that $f(x) \in (396, 400)$ whenever $x \in (2-\delta, 2+\delta)$? Does there exist a value of $\delta > 0$ such that $f(x) \in (398, 400)$ whenever $x \in (2, 2+\delta)$?

4. A machines job is to mark a time $t \in [1, 2]$, where a real function $f(t)$ is the maximum among all functional values over $[1, 2]$. (This may be needed for many purposes, for example given a set of waves, I may be required to mark the peaks.) Suppose that a function $f[1, 2] \to [5, 6]$ is given. Argue by giving an example that, the machine may not even mark a single point.

5. Do we know a class of functions, for which the machine will return a point?

6. Some fixed function $f : [1, 2] \to [-9, 9]$, with $f(L = 1) = -9$ and $f(R = 2) = 9$ is given. There is a very talented salesman who wants to sale me a zero finding machine.

   The machine finds the mid point $\frac{L+R}{2}$. If $f(\frac{L+R}{2}) < 0$, it sets (redefines) $L = \frac{L+R}{2}$ and if $f(\frac{L+R}{2}) > 0$, it sets $L = \frac{L+R}{2}$. If $f(\frac{L+R}{2}) = 0$, it returns $\frac{L+r}{2}$ and stops. Otherwise, it continues and returns us a sequence of mid points.

   The salesman claims that

   (a) either the machine returns a point where f is zero

   (b) or it will return a convergent sequence of mid points (M_n) with M_n converging to some $l \in [1, 2]$ and that $f(l) = 0$.

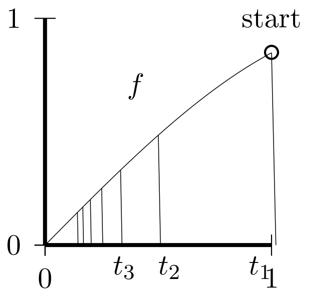
   Argue by giving an example, that such a claim is a lie.

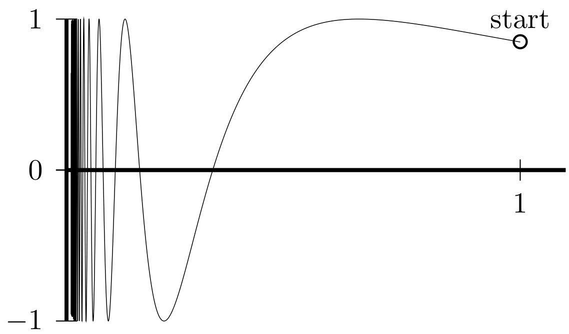
7. Do we know a class of functions, for which the salesman's claim will be true?

5.1 The concept of limit

Two mad flies f and g

1. Decided to collide against the wall (y-axis).
2. They started at time $t = 1$ and the time of collision is $t = 0$.
3. The path followed by f is $\sin(t)$ and the path followed by g is $\sin(1/t)$.





4. Where should f collide?
5. Observe f at $t_n = 1, \frac{1}{2}, \frac{1}{3}, \cdots$. Does $(f(t_n))$ converge?
6. A friend observes f at $t_n = \frac{1}{n\sqrt{2}}$. Does $(f(t_n))$ converge?
7. If $t_n > 0$ and $t_n \to 0$, should $(f(t_n))$ have the same limit?
8. Take $t_n = \frac{2}{n\pi}$. Does $(g(t_n))$ converge?

DEFINITION 5.1.1 **(D1: sequential definition)** Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ be a function. Suppose that a is a cluster point of A. We say $\boldsymbol{\lim\limits_{t\to a} f(t) = l}$ if for each sequence $a_n \to a$, $a_n \neq a$, we have $f(a_n) \to l$.

Notice that to define limit at a, we need not have $a \in A$.

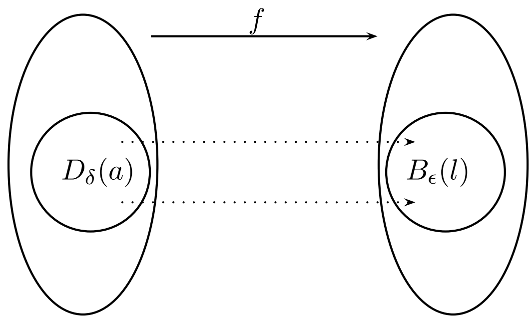
DEFINITION 5.1.2 **(D2: ϵ-δ-definition or the geometric definition)** Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ be a function. Suppose that a is a cluster point of A. We say $\lim\limits_{\mathbf{t}\to\mathbf{a}} \mathbf{f}(\mathbf{t}) = \mathbf{l}$ if for each $\epsilon > 0$, $\exists \delta > 0$ such that

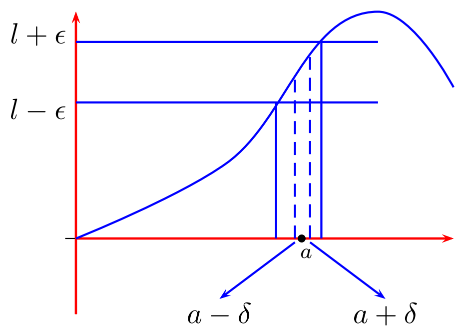
$$\Big(0 < |t-a| < \delta, t \in A\Big) \Rightarrow |f(t) - l| < \epsilon.$$

In the geometric definition, the value of δ depends on ϵ, the point a and on f. Recall that, $f(B) := \{f(t) \mid t \in B \cap \mathsf{dom}\, f\}$. Also notice that $\{t : 0 < |t-a| < \delta, t \in A\} = D_\delta(a) \cap A$ and $\Big(0 < |t-a| < \delta, t \in A\Big) \Rightarrow |f(t) - l| < \epsilon$ means $f(D_\delta(a)) \subseteq B_\epsilon(l)$. Hence the geometric definition may also be given in a more convenient manner.

DEFINITION 5.1.3 **(D2': ϵ-δ-definition or the geometric definition)** Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ be a function. Suppose that a is a cluster point of A. We say $\lim\limits_{\mathbf{t}\to\mathbf{a}} \mathbf{f}(\mathbf{t}) = \mathbf{l}$ if <u>each</u> $B_\epsilon(l)$ contains <u>some</u> $f(D_\delta(a))$.

Below we supply two pictures explaining the geometric definition of limits (D2). It is useful to imagine $\mathsf{dom}\, f$ an interval or a disc and a a point or boundary point of it.





THEOREM 5.1.4 *Both the definitions of the limit of a function at a point, are equivalent. That is, if* $\lim\limits_{x\to a} f(x) = l$ *by D1, then* $\lim\limits_{x\to a} f(x) = l$ *by D2, and vice versa.*

Proof. (Optional) Suppose that $\lim\limits_{x\to a} f(x) = l$ by D1 but $\lim\limits_{x\to a} f(x) = l$ by D2 is not true. So some $B_\epsilon(l)$ does not contain any $f(D_\delta(a))$. In particular $f(D_1(a))$ is not contained in $B_\epsilon(l)$. So $\exists a_1 \in D_1(a)$ for which $f(a_1)$ is not in $B_\epsilon(l)$. Similarly, $\exists a_n \in D_{1/n}(a)$ for which $f(a_n)$ is not in $B_\epsilon(l)$. That is all these $f(a_n)$ are always outside $B_\epsilon(l)$. Hence $f(a_n) \not\to l$. But $a_n \to a$ and $a_n \neq a$. As D1 holds, we must have $f(a_n) \to l$. A contradiction.

Conversely, suppose that $\lim\limits_{x\to a} f(x) = l$ by D2. We want to show that $\lim\limits_{x\to a} f(x) = l$ by D1. For that let $a_n \to a$, $a_n \neq a$. We want to show that $f(a_n) \to l$. For that let $\epsilon > 0$. We want to show that $B_\epsilon(l)$ contains a tail of $(f(a_n))$. Now as $\lim\limits_{x\to a} f(x) = l$ by D2, $B_\epsilon(l)$ contains some $f(D_\delta(a))$. As $a_n \to a$, $a_n \neq a$, this $D_\delta(a)$ contains a tail of (a_n), say $(a_m, a_{m+1}, \ldots)$. As $f(D_\delta(a)) \subseteq B_\epsilon(l)$, we see that the points $f(a_m), f(a_{m+1}), \ldots$ are contained in $B_\epsilon(l)$. Which we wanted to show. ∎

5.2 Proving limits by D1 (sequential method)

EXAMPLE 5.2.1 Take $f(x) = x^2 - \frac{3}{\sqrt{x}}$ on $\mathbb{R}_+$. Show that $\lim\limits_{x\to 1} f(x) = -2$.

Answer. Let $a_n \to 1$, $a_n \neq 1$. So $a_n^2 - \frac{3}{\sqrt{a_n}} \to 1 - 3 = -2$, by the limit theorems for sequences. By D1, $\lim\limits_{x\to 1} f(x) = -2$.

EXAMPLE 5.2.2 Take $f(x) = \sqrt{x}$ on $\mathbb{R}_+$. Show that $\lim\limits_{x\to 1} f(x) \neq 2$.

Answer. Take $a_n = 1 - \frac{1}{n}$. Then $a_n \to 1$, $a_n \neq 1$. But $f(a_n) = \sqrt{a_n} \to 1 \neq 2$. By D1, $\lim\limits_{x\to 1} f(x) \neq 2$.

EXAMPLE 5.2.3 Take $f(x) = \begin{cases} 1, & x < 0 \\ 0, & x \geq 0. \end{cases}$ Show that $\lim\limits_{x\to 0} f(x)$ does not exist.

Answer. Take $a_n = (-.1)^n$. Then $a_n \to 0$, $a_n \neq 0$. But $(f(a_n)) = (1, 0, 1, 0, \cdots)$ diverges. By D1, $\lim\limits_{x\to 0} f(x)$ does not exist.

EXAMPLE 5.2.4 Take $f(x) = \begin{cases} 0 & x = 0 \\ \sin(\frac{1}{x}) & x \neq 0. \end{cases}$ Then $\lim\limits_{x\to 0} f(x)$ does not exist.

Answer. Take $a_n = \frac{2}{n\pi}$. Then $a_n \to 0$, $a_n \neq 0$. But $(f(a_n)) = (1, 0, -1, 0, 1, \cdots)$ diverges. By D1, $\lim\limits_{x\to 0} f(x)$ does not exist.

EXAMPLE 5.2.5 Take $f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q}. \end{cases}$ Fix any a. Then $\lim\limits_{x\to a} f(x)$ does not exist.

Answer. Let (r_n) be a sequence of rationals such that $r_n \neq a$, $r_n \to a$. Then $f(r_n) \to 0$. Let (i_n) be a sequence of irrationals such that $i_n \neq a$, $i_n \to a$. Then $f(i_n) \to 1$. Hence $\lim\limits_{x\to a} f(x)$ does not exist, by D1. (For the limit to be l, we should have $f(a_n) \to l$, for each sequence $a_n \to a$, $a_n \neq a$.)

Did you notice?

We used particular examples of (a_n), to show $\lim f \neq l$. We started with an arbitrary (a_n), to argue $\lim f = l$.

Below 'LTf' strands for 'Limit theorems of functions'.

Exercise 5.2.6 **(LTf)** *Let $S \cap T = \emptyset$, and a be a cluster point of both sets. Suppose that $f : S \to \mathbb{R}$, $g : T \to \mathbb{R}$ are functions. Define*

$$h(x) = \begin{cases} f(x) & x \in S \\ g(x) & x \in T. \end{cases}$$

Then $\lim\limits_{x\to a} h(x) = l$ iff $\lim\limits_{x\to a} f(x) = l = \lim\limits_{x\to a} g(x)$.

In particular, if $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = 1$, then $\lim\limits_{x\to a} h(x)$ does not exist. This was the case of a previous example.

5.3 Proving limits by D2 (ϵ-δ method)

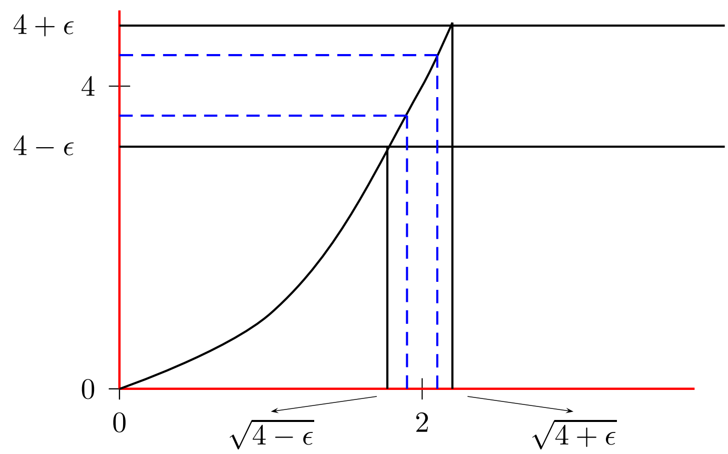
Example 5.3.1 Take $f(x) = x^2$. Show that $\lim\limits_{x\to 2} f(x) = 4$.

Answer. Let $\epsilon > 0$. We are looking for a $0 < \delta < 1$ such that $f(D_\delta(2)) \subseteq B_\epsilon(4)$. As f is increasing, we have

$$\begin{aligned} f(D_\delta(2)) \subseteq B_\epsilon(4) \quad &\Leftarrow \quad 4-\epsilon \leq (2-\delta)^2 < (2+\delta)^2 \leq 4+\epsilon \\ &\Leftarrow \quad \sqrt{4-\epsilon} \leq 2-\delta < 2+\delta \leq \sqrt{4+\epsilon} \\ &\Leftarrow \quad \sqrt{4-\epsilon}-2 \leq -\delta < \delta \leq \sqrt{4+\epsilon}-2 \\ &\Leftarrow \quad \delta = \min\{|\sqrt{4-\epsilon}-2|, |\sqrt{4+\epsilon}-2|\}. \end{aligned}$$

Then $\delta > 0$ and we are done.

We can see a picture for this. Intervals at 2 suggest that $\delta \leq \min\{2-\sqrt{4-\epsilon}, \sqrt{4+\epsilon}-2\} = \sqrt{4+\epsilon}-2$.



Exercise 5.3.2 *To show $\lim\limits_{x\to 2} x^3 = 8$, we take $\delta =$ _____.*

EXAMPLE 5.3.3 Take $f(x) = x^2$ on $\mathbb{R}$. Then $\lim\limits_{x\to 2} f(x) \neq 3.99$.

Answer. Each $D_\delta(2)$ contains a number more than 2. So each $f(D_\delta(2))$ contains a number more than 4. Put $\epsilon = .01$. Then $B_\epsilon(3.99) = (3.98, 4)$.

So $\exists$ no $\delta > 0$ such that $f(D_\delta(2)) \subseteq B_\epsilon(3.99)$. Thus by D2, $\lim\limits_{x\to 2} f(x) \neq 3.99$.

EXAMPLE 5.3.4 Take $f(x) = 1$ if $x < 0$ and $f(x) = 0$ if $x \geq 0$. Then $\lim\limits_{x\to 0} f(x)$ does not exist.

Answer. If it exists, let it be l. Each $D_\delta(0)$ contains +ve and −ve numbers. So each $f(D_\delta(0))$ contains 1 and 0. Take $\epsilon = 0.1$. As $B_\epsilon(l)$ has length 0.2, it cannot contain two integers. So $\exists$ no $\delta > 0$ such that $f(D_\delta(0)) \subseteq B_\epsilon(l)$. So $\lim\limits_{x\to 0} f(x) \neq l$, a contradiction.

EXERCISE 5.3.5 *Do them by both the methods.*

1. *Take* $f(x) = x$ *on* $\mathbb{R}$. *Then* $\lim\limits_{x\to 1} f(x) \neq 1.001$.

2. *Define* $f(x) = \begin{cases} 1 & x < 0 \\ 0 & x \geq 0. \end{cases}$ *Then* $\lim\limits_{x\to 0} f(x)$ *does not exist.*

3. *Let* $f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1/x & \text{if } x > 0. \end{cases}$ *Show that* $\lim\limits_{x\to 0} f(x)$ *does not exist.*

4. *Let* $f(x) = \begin{cases} 0 & \text{if } x \in D \\ 1 & \text{if } x \notin \mathbb{D}. \end{cases}$ *Here* D *means the set of* **dyadic rational** *numbers, that is, the rational numbers in the lowest terms with denominator a power of* 2. *It is a dense subset of* $\mathbb{R}$. *Show that* $\lim\limits_{x\to 0} f(x)$ *does not exist.*

5. *Take* $f(x) = \sin(x)$. *Then* $\lim\limits_{x\to 0} \sin(x) = 0$.

EXERCISE 5.3.6 *True or false? Let* $x_n \to a$ *and* $\lim\limits_{n\to\infty} f(x_n) \neq l$. *Then* $\lim\limits_{x\to a} f(x) \neq l$.

EXERCISE* 5.3.7 *Consider* $f(x) = 3x^3 - 5x^2 + 27x + 9$ *on* $\mathbb{R}$. *Then* $\lim\limits_{x\to 2} f(x) = 67$. *Can we show this from the the sequential definition? Yes. Easy. Can we show this by the* ϵ-δ-*definition? Yes. Tricky.*

5.4 Limit theorems for functions

We do not want to talk about the limits every time from the definition. Note that our arguments get tougher with nontrivial functions. So we require some tools to find limit for nontrivial functions.

DEFINITION 5.4.1 We say a function f is **bounded** on A if $f(A)$ is bounded.

THEOREM 5.4.2 **(LTf)** *Let* $\lim\limits_{x\to c} f(x) = l$. *Then* f *is bounded on some* $D_\delta(c)$.

Proof. Follows easily from the ϵ-δ-definition. ∎

LEMMA 5.4.3 **(LTf)(sandwich)** *Let* $f \leq h \leq g$ *on* A *and* c *be a cluster point of* A. *If* $\lim\limits_{x\to c} f = l = \lim\limits_{x\to c} g$, *then* $\lim\limits_{x\to c} h$ *exists and is* l.

Proof. Follows easily from any of the definitions. Try both. You need to use the property that $B_\epsilon(l)$ is an interval. That is, if $x, y \in B_\epsilon(l)$, and z is in between x and y, then $z \in B_\epsilon(l)$. ∎

EXERCISE 5.4.4 **(LTf)** *Show that* $\lim\limits_{x\to c} f(x) = l \Rightarrow \lim\limits_{x\to c} |f(x)| = |l|$. *Also* $\lim\limits_{x\to c} f(x) = 0$ *iff* $\lim\limits_{x\to c} |f(x)| = 0$.

THEOREM 5.4.5 **(LTf)** *Let* $\lim\limits_{x\to c} f(x) = l$ *and* $\lim\limits_{x\to c} g(x) = m$. *Fix* $\alpha \in \mathbb{R}$. *Then*

a) $\lim\limits_{x\to c}(f(x)+g(x)) = l+m$.

b) $\lim\limits_{x\to c} f(x)g(x) = lm$.

c) $\lim\limits_{x\to c}(\alpha f)(x) = \alpha l$.

d) *If* $f \geq 0$ *on* $\mathsf{dom}\, f$, *then* $l \geq 0$.

e) *If* $l > 0$, *then* $f > 0$ *on a* $D_\delta(c)$ *and* $\lim\limits_{x\to c} \frac{1}{f(x)} = \frac{1}{l}$.

f) *If* $f \geq 0$ *and* $k \in \mathbb{N}$, *then* $\lim\limits_{x\to c} \sqrt[k]{f(x)} = \sqrt[k]{l}$.

Proof. Use D2 for the first half of e), and D1 for the rest. ∎

REMARK 5.4.6 **(LTf)(Very important)**

1. Let $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$. As we know that $\lim\limits_{x\to a} x = a$, by the limit theorems of functions, we have
$$\lim_{x\to c} P(x) = P(c).$$

2. **(Rational polynomials)** As $\lim\limits_{x\to a} x = a$, we have
$$\lim_{x\to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)},$$
when $Q(a) \neq 0$. Compare this with Exercise 5.3.7.

3. As $-|\theta| \leq \sin\theta \leq |\theta|$, we have $\lim\limits_{\theta\to 0} \sin\theta = 0$. Hence, $\lim\limits_{\theta\to 0}\cos\theta = \lim\limits_{\theta\to 0}(1-2\sin^2\frac{\theta}{2}) = 1$. Hence,
$$\lim_{x\to a}\sin(x) = \lim_{x\to a}\Big(\sin(x-a)\cos a + \cos(x-a)\sin a\Big) = \sin(a).$$

4. We have similar limit results for trigonometric polynomials and rational functions.

5. Recall from (3.1), that for $0 < \theta < \frac{\pi}{2}$, we have $\sin\theta \leq 2\sin(\theta/2) \leq \theta \leq \sin\theta + (1-\cos\theta)$. So
$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1,$$
as $0 \leq \frac{1-\cos\theta}{\theta} = \left|\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}\sin\frac{\theta}{2}\right| \leq |\sin\frac{\theta}{2}|$.

6. For $|x| < 1$, argue that $1 + x \leq e^x \leq 1 + x + x^2$.**!!** † So by sandwich lemma, $\lim\limits_{x\to 0} e^x = 1$. Thus
$$\lim_{x\to a} e^x = \lim_{y\to 0} e^{a+y} = e^a \lim_{y\to 0} e^y = e^a.$$

7. For $x \neq 0$, we have $-|x| \leq x\sin(\frac{1}{x}) \leq |x|$. So
$$\lim_{x\to 0} x\sin(\frac{1}{x}) = 0.$$

† We are using e^x for $\exp x$.

EXAMPLE 5.4.7 1. $\lim\limits_{x\to 1}\dfrac{x^2+x-2}{x^2-x}=\lim\limits_{x\to 1}\dfrac{(x+2)(x-1)}{x(x-1)}=^{\ddagger}\lim\limits_{x\to 1}\dfrac{x+2}{x}=3.$

2. $\lim\limits_{h\to 0}\dfrac{\sqrt{5+h}-\sqrt{5}}{h}=\lim\limits_{h\to 0}\dfrac{\sqrt{5+h}-\sqrt{5}}{(\sqrt{5+h}-\sqrt{5})(\sqrt{5+h}+\sqrt{5})}=\lim\limits_{h\to 0}\dfrac{1}{\sqrt{5+h}+\sqrt{5}}=\dfrac{1}{2\sqrt{5}}.$

EXERCISE 5.4.8 *Determine* $\lim\limits_{x\to 0}\frac{f(a+x)-f(a)}{x}$ *for* $f(x)=\sin(x), e^x, x^n$.

EXERCISE 5.4.9 *Define* $\lim\limits_{x\to c} f(x)=\infty$ *in both ways. Compare with texts.*

EXERCISE 5.4.10 *Define* $\lim\limits_{x\to\infty} f(x)=l$ *in both ways. Compare with the texts. Did you find it similar to that of* $\lim\limits_{n\to\infty} a_n=l$, *where* $a_n=f(n)$?

5.5 One sided limits

DEFINITION 5.5.1 Let $f: A\to\mathbb{R}$ and c be a cluster point of $(c,\infty)\cap A$.[§] We say $\boldsymbol{\lim\limits_{x\to c^+} f(x)=l}$, if each $B_\epsilon(l)$ contains some $f(c,c+\delta)$. That is,

$$\forall\epsilon>0,\ \exists\,\delta>0 \text{ such that } (c<x<c+\delta, x\in A)\Rightarrow |f(x)-l|<\epsilon.$$

It is also called the **right hand limit $\boldsymbol{f(c+)}$**. The **left hand limit $\boldsymbol{f(c-)}$** is defined similarly.

EXAMPLE 5.5.2 *Take* $f(x)=[x]$. *Then* $f(2-)=1$ *and* $f(2+)=2$.

EXERCISE 5.5.3 *Write a sequential definition of left/right hand limit.*

LEMMA 5.5.4 *Let* $D_\epsilon(a)\subseteq \mathsf{dom}(f)$ *for some* $\epsilon>0$. *Then* $\lim\limits_{x\to a} f(x)=l$ *iff* $f(a+), f(a-)$ *exist and* $f(a+)=f(a-)=l$.

Proof. Exercise. ■

REMARK 5.5.5 1. We never talked about the limit of $\ln(x)$. This is because it will be discussed later when we discuss the continuity of the inverse of a strictly increasing function.

2. Remember to write 'by the sequential definition of limits' in place of 'by D1' and 'by the ϵ-δ-definition of limits' in place of 'by D2'.

Mixing functions and must know examples

1. Define $f:\mathbb{R}\to\mathbb{R}$ as $f(x)=\begin{cases} x^3 & \text{if } x\in\mathbb{Q}\\ 0 & \text{else.}\end{cases}$ Show that $\lim\limits_{x\to 0} f(x)=0$ and limit at any other points does not exist.

   Answer. For $a=0$. Let $x_n\to 0$, $x_n\neq 0$. Then $0\le |f(x_n)|\le x_n^3$. Using sandwich lemma, we see that $|f(x_n)|\to 0$. Hence $f(x_n)\to 0$. Thus by sequential definition $\lim\limits_{x\to 0} f(x)=0$.

   For $a\neq 0$. Let $x_n\to a$ be a sequence of rationals with $x_n\neq a$. Then $f(x_n)\to a^3$. Let $y_n\to a$ be a sequence of irrationals with $x_n\neq a$. Then $f(y_n)\to 0$. Hence for the sequence $(z_n)=(x_1,y_1,x_2,y_2,\ldots)$, we see that $(f(z_n))$ does not converge. Hence the limit $\lim\limits_{x\to a} f(x)$ does not exist.

[‡] Why is this equality justified?

[§] Are there sets A for which this intersection need not have c as a cluster point.

2. Observe how we have mixed two different functions to create one.

3. **(Must know examples)** Try to think, if you can give another (of a different type).

   (a) Give $f : \mathbb{R} \to \mathbb{R}$ for which limit does not exist at 0.

   (b) Give $f : \mathbb{R} \to \mathbb{R}$ for which limit exists only at 0.

   (c) Give $f : \mathbb{R} \to \mathbb{R}$ for which limit does not exist at integers.

   (d) Give $f : \mathbb{R} \to \mathbb{R}$ for which limit exists only at integers.

   (e) Give $f : \mathbb{R} \to \mathbb{R}$ for which limit does not exist at any point.

   (f) Give $f : \mathbb{R} \to \mathbb{R}$ for which limit exists at all points.

■

5.6 Exercises

EXERCISE* 5.6.1 **(Understanding)** *Take $f(x) = x^5+7x^4-10x^3+5$. We want to show that $\lim\limits_{x\to 1} f(x) = 3$. For that we start with 'Let $\epsilon > 0$'. Then*

$$\delta = \min\{\sqrt[5]{1+\frac{\epsilon}{3}}-1,\ \sqrt[4]{1+\frac{\epsilon}{21}}-1,\ \sqrt[3]{1+\frac{\epsilon}{30}}-1\}$$

is an appropriate value.

EXERCISE 5.6.2 **(Understanding)** *Suppose that $\lim\limits_{x\to a} f(x) = l$. Is it true that $\lim\limits_{x\to a}[f(x)] = [l]$, where $[\cdot]$ is the bracket function? What if I have an additional condition $f(x) \geq l$? What if I use the additional condition $f(x) \leq l$ instead?*

EXERCISE 5.6.3 **(Understanding)** *At which points $a > 0$, the limit $\lim\limits_{x\to a} \frac{[x^2]}{x^2}$ exist?*

EXERCISE 5.6.4 **(Thomae function: important example, many times used in future)** *Let $f : \mathbb{R} \to \mathbb{R}$ is defined as*

$$f(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q} \in \mathbb{Q},\ \gcd(p,q) = 1 \\ f(x) = 0 & x \notin \mathbb{Q}. \end{cases}$$

1. *Fix $a \in \mathbb{R}$. Show that there are at most $\binom{n+1}{2}$ rationals in $D_{\frac{1}{2}}(a)$ with denominator from $J_n := \{1, 2, 3, \dots, n\}$.*

2. *Argue that $\exists 0 < \delta < \frac{1}{2}$, such that $D_\delta(a)$ does not contain a rational with denominator from J_n.*

3. *Argue that $f(D_\delta(a)) \subseteq B_{\frac{1}{n}}(0)$. Hence conclude that $\lim\limits_{x\to a} f(x) = 0$.*

4. *Conclude that $\lim\limits_{x\to a} f(x) = f(a)$ only on $\mathbb{Q}^c$.*

EXERCISE* 5.6.5 **(May leave it, if you have no time)** *Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions such that $\lim\limits_{x\to a} f(x) = l$ and $\lim\limits_{x\to l} g(x) = m$. Must $\lim\limits_{x\to a}(g \circ f)(x)$ exist? Careful here.*

EXERCISE 5.6.6 **(Regular practice)** *Evaluate. Argue it in two ways for better knowledge.*

1. $\lim\limits_{x\to 0+} \frac{1}{x}$

2. $\lim\limits_{x\to 2} f(x)$ *where* $f(x) = \begin{cases} x^2 & x \in \mathbb{N} \\ \frac{1}{x} & \textit{otherwise.} \end{cases}$

3. $\lim_{x\to\pi} f(x)$ *where* $f(x) = \sin(1/x)$

4. $\lim_{x\to 0} f(x)$ *where* $f(x) = [\sin(1/x)]$

5. $\lim_{x\to 0} f(x)$ *where* $f(x) = [\frac{\sin(1/x)}{2}]$

EXERCISE 5.6.7 **(Remember something similar for sequences?)** *Let f be an odd function and $f(0+) = 0$. Do we then know $\lim_{x\to 0} f(x)$? What happens if we were given that $f(0+) = 4$?*

EXERCISE 5.6.8 **(Answer in one minute)** *Let $f : D_\epsilon(a) \to \mathbb{R}$ with $f(a-) = 5$. If $\lim_{x\to a} f(x)$ exists, then it must be _____.*

EXERCISE 5.6.9 **(Practice and future use)** *Find $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$.*

(a) $f(x) = x^3$
(b) $f(x)$ *is a polynomial*
(c) $f(x) = 1/x,\ x \neq 0$
(d) $f(x) = \sin(x)$
(e) $f(x) = \cos(x)$
(f) $f(x) = e^x$
(g) $f(x) = x^2 \sin(\frac{1}{x})$ *if* $x =\neq 0$, *and* $f(0) = 0$
(h) $f(x) = k^x,\ k > 0$. *Use that there exists α such that $e^\alpha = k$.*

EXERCISE 5.6.10 **(Understanding, avoid if pressed for time)** *Find a $\delta > 0$, such that $f(D_\delta(a)) \subseteq B_\alpha(l)$.*

1. $f(x) = \cos(x),\ a = 0,\ l = 1,\ \alpha = 0.1$. *Is this possible for each $\alpha > 0$?*

2. $f(x) = \sin(x),\ a = 0,\ l = 1,\ \alpha = 2$. *Is this possible for each $\alpha > 0$?*

3. $f(x) = [x],\ a = 0,\ l = 0,\ \alpha = 2$. *Is this possible for each $\alpha > 0$?*

EXERCISE 5.6.11 **(Repeat type)** *Take $f(x) = x^2$. As it is a polynomial, $\lim_{x\to a} f(x) = f(a) = a^2$. Find the maximum value of $\delta > 0$, such that $f(D_\delta(a)) \subseteq B_1(a^2)$.*

(a) $a = 0$, *(b)* $a = 1$, *(c)* $a = 2$, *(d)* $a = 10$.
Do you think it depends on point a?

EXERCISE 5.6.12 **(What is wrong with this definition?)** *My friend gives a new definition of limit and say that it is equivalent to our old definition. He says '$\lim_{x\to c} f(x) = l$, if for each $\epsilon > 0$, there exists a value of x such that $|f(x) - l| \leq \epsilon$'. Is he correct?*

EXERCISE 5.6.13 **(Frequently asked in interviews)** *Define f on $\mathbb{R}$ as*

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{1}{n},\ n \in \mathbb{N} \\ 0 & \text{else.} \end{cases}$$

Find the points a at which $\lim_{x\to a} f(x)$ exists.

EXERCISE* 5.6.14 **(For future, may leave now)** *Take some function $f : \mathbb{R}^2 \to \mathbb{R}$. Select any line L passing through 0 and select any sequence of points (x_n), $x_n \neq 0$ of L converging to 0. It is given that $f(x_n) \to l$ for each such line and each such selected sequence. Does that mean $\lim_{x\to 0} f(x) = l$?*

EXERCISE 5.6.15 **(Frequently asked in interviews)** *Does $\lim_{x\to 0^+} x\cos(\cot x)$ exist? Explain.*

EXERCISE 5.6.16 **(Limits of an increasing function)** *Let $f : \mathbb{R} \to \mathbb{R}$ be a monotone increasing function. Fix any $a \in \mathbb{R}$. Then $\lim_{x\to a+} f(x)$ must exist.*

5.7 Continuity

DEFINITION 5.7.1 Let $f : A \to \mathbb{R}$ and $a \in A$.¶ We say f is **continuous** at a, one of the following conditions is satisfied.

D1 The sequence $f(a_n) \to f(a)$ for each sequence $a_n \to a$, $\underline{a_n \in A}$.

D2 Each $B_\epsilon(f(a))$ contains a $f(B_\delta(a))$. That is,

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \Big(x \in A, |x - a| < \delta\Big) \Rightarrow |f(x) - f(a)| < \epsilon.$$

REMARK 5.7.2 1. The above two condition are equivalent. This can be easily shown in a way similar to the definition of limit.

2. If $a \in A$ is a cluster point of A, then 'f is continuous at a' means $\lim\limits_{x \to a} f(x) = f(a)$.**!!**

3. If $a \in A$, but it is NOT a cluster point of A, then 'f is continuous at a' by definition.**!!** *

4. We say f is **discontinuous** at a to mean that f is not continuous at a.

5. We say f is continuous <u>on</u> D, if it is continuous at each point $a \in D$.

REMARK 5.7.3 **(Property of a continuous function)** Notice that by D1, if f continuous at a and $a_n \to a$, then $f(a_n) \to f(a)$. In words it said 'a continuous function takes a convergent sequence to a convergent sequence'.

EXERCISE 5.7.4 *Define f on $[1, 2) \cup \{3\}$ as $f = 1$ on $[1, 2)$ and $f(3) = 2$. Apply D1,D2.*

THEOREM 5.7.5 *From the limit theorems of functions, we already know that rational functions involving $\sqrt[k]{x}$, $\sin(x)$, e^x are <u>continuous</u> wherever defined.***!!**

EXAMPLE 5.7.6 1. The function $f(x) = [x]$ is discontinuous at integers and continuous at every other point. This type of discontinuity where both side limits exist but are not equal, is sometimes called **jump discontinuity**.

2. Let $f(0) = 1$, $f(x) = 2$, if $x \in \mathbb{R} \setminus \{0\}$. Then f is discontinuous at 0. If we redefine $f(0) = 2$, then it is continuous. Such a type of discontinuity is called **removable discontinuity**.

3. Let $f(x) = \begin{cases} \sin(\frac{1}{x}) & x > 0, \\ 0 & x \le 0. \end{cases}$ Then f is continuous on $\mathbb{R} \setminus \{0\}$.

   If you want to argue that, let $a \neq 0$. Let $a_n \to a$. By the limit theorems for sequences, a_n may be assumed nonzero and so $\frac{1}{a_n} \to \frac{1}{a}$. Since $\sin x$ is continuous, we get that $\sin(1/a_n) \to \sin(1/a)$. So f is continuous at a.

   Note that $f(0+)$ does not exist, hence $\lim\limits_{x \to 0} f(x)$ does not exist. This type of discontinuity is known as a **discontinuity of the second kind** whereas a jump or a removable discontinuity is called a **discontinuity of the first kind**.

4. Consider the **Dirichlet's function** $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{Q}^c. \end{cases}$ We already know that $\lim\limits_{x \to a} f(x)$ exists at no point a. So f has a discontinuity of the second kind at each $a \in \mathbb{R}$.

¶We definite continuity at points inside A.

*To use sequential definition, it is important to note if (a_n) is a sequence of points of A which converges to a, then $a_n = a$ for each $n \ge$ some k. To use the other definition, note that there exists a $\delta > 0$ such that $B_\delta(a) \cap A = \{a\}$.

5. Let $f(x)=\begin{cases} x\sin(\frac{1}{x}) & x>0,\\ 0 & x\le 0.\end{cases}$ Then f is continuous on $\mathbb{R}$. The argument is similar. At $a=0$, we now have $\lim\limits_{x\to 0} f(x)=0=f(0)$. So f is continuous at 0.

6. Take $f(x)=\begin{cases} x & x\in\mathbb{Q}\\ 0, & x\notin\mathbb{Q}\end{cases}$. It is discontinuous at each point except 0.

   Answer. Let $a\neq 0$. Let (r_n) be a sequence of rationals converging to a. Similarly, let (i_n) be a sequence of irrationals converging to a. Then $f(r_n)\to a$ and $f(i_n)\to 0$. Hence $\lim\limits_{x\to a} f(x)$ does not exist as $a\neq 0$.

   Now, let $a=0$. Note that $-|x|\le f(x)\le |x|$ and $\lim\limits_{x\to 0}(\pm|x|)=0$. By sandwich lemma, $\lim\limits_{x\to 0} f(x)0=f(0)$. So f is continuous at $a=0$.

Definition 5.7.7 We say f is **left continuous** at c, if $f(c-)=f(c)$. Similarly, we define the **right continuity**.

Example 5.7.8 1. The function $[x]$ is right continuous at every point in $\mathbb{R}$ and left continuous at non-integer points.

2. Let $B_\delta(c)\in \mathsf{dom}(f)$. Then f is continuous at c iff it is left continuous at c and right continuous at c.

5.8 Algebra of continuous functions

Lemma 5.8.1 **(algebra)** *Let $f,g:D\to\mathbb{R}$ be continuous at c.*

a) Then $f\pm g$, fg, are continuous at c.

b) If $f(c)>0$, then $f>0$ in some $B_\delta(c)$ and $\frac{1}{f}$ is continuous at c.

c) If $f\ge 0$ on D and $n\in\mathbb{N}$, then $f^{\frac{1}{n}}$ is continuous at c.

Proof. For b) first part, use ϵ-δ definition. For others use sequential definition. ∎

Lemma 5.8.2 *If f is continuous at a, then so is $|f|$. The converse is not true.*

Proof. Use the sequential argument. For the next part, take $f(x)=-1$ for $x\le 0$, and $f(x)=1$ for $x>0$. ∎

Lemma 5.8.3 *If f and g are continuous at a, then so is $h=\min\{f,g\}$.*

Proof. As $h(x)=(f+g)/2-|f-g|/2$. ∎

Lemma 5.8.4 **(Composition)** *If f is continuous at a and g is continuous at $f(a)$, then $g\circ f$ is continuous at a.*

Proof. Let $a_n\to a$. As f is continuous at a, we get $f(a_n)\to f(a)$. As g is continuous at $f(a)$, we get $g(f(a_n))\to g(f(a))$.[†] Try the ϵ-δ proof too. ∎

Exercise 5.8.5 *Can $g\circ f$ be continuous at a, even if g is not continuous at $f(a)$?*

[†]If I mimic this for limits, where will I have a problem?

5.9 Maximum, minimum

DEFINITION 5.9.1 Let $f : A \to \mathbb{R}$ and $a \in A$. We say f has an **absolute maximum** at a, if $f(a) \geq f(x)$ for each $x \in A$. The term **absolute minimum** is defined similarly.

THEOREM 5.9.2 **(Property of a continuous function)** *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is bounded.*

Proof. Suppose that it is not bounded. So, $\exists x_n \in [a, b]$ such that $|f(x_n)| \to \infty$. Is the sequence (x_n) bounded? Yes. Then by BWT, we must have a convergent subsequence, say, (x_{n_k}) with the limit, say, l. As $a \leq x_{n_k} \leq b$, we get $a \leq l \leq b$. As $x_{n_k} \to l$, and f continuous at l, we have $f(x_{n_k}) \to f(l)$. So $|f(x_{n_k})| \to |f(l)|$. But by limit theorems for sequences, we see that $f(x_{n_k})| \to \infty$. A contradiction. ∎

THEOREM 5.9.3 **(Property of a continuous function)** *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f has an absolute maximum (minimum) in $[a, b]$.*

Proof. We know that f is bounded on the interval. Let $p = \sup f([a, b])$. Is $p - \frac{1}{n}$ an upper bound of $f([a, b])$? No. So, $\exists y_n \in [a, b]$ such that $f(y_n) \geq p - \frac{1}{n}$. Is (y_n) a bounded sequence? Yes. So, by BWT, $\exists$ a convergent subsequence, say, $y_{n_k} \to t$. As $a \leq y_{n_k} \leq b$, we get $a \leq t \leq b$. As $y_{n_k} \to t$, and f continuous at t, we have $f(y_{n_k}) \to f(t)$. As $p - \frac{1}{n_k} \leq f(y_{n_k}) \leq p$, we get $f(y_{n_k}) \to p$. So $f(t) = p$. ∎

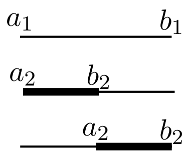
EXERCISE 5.9.4 *I have a function $f : [0, 1]$ <u>onto</u> $\mathbb{N}$. Is it cts?*

EXERCISE 5.9.5 *Does $f(x) = x$ have an absolute maximum on $(0, 1)$? Does it contradict max-min thm?*

zyyx

5.10 Intermediate value theorem

THEOREM 5.10.1 **(property of a continuous function)(bisection method)** *Let $f : [a, b] \to \mathbb{R}$ be continuous such that $f(a) < 0$ and $f(b) > 0$. Then there exists a $c \in (a, b)$ such that $f(c) = 0$.*



Proof. Call $a_1 = a$, $b_1 = b$, and put $I_1 = [a_1, b_1]$.

If $f(\frac{a+b}{2}) > 0$, then put $a_2 = a_1$, $b_2 = \frac{a+b}{2}$, and $I_2 = [a_2, b_2]$.

If $f(\frac{a+b}{2}) < 0$, then put $a_2 = \frac{a+b}{2}$, $b_2 = b_1$, and $I_2 = [a_2, b_2]$.

Assume that we never get $f(\frac{a_n+b_n}{2}) = 0$. Notice that, we always have $f(a_n) < 0$, $f(b_n) > 0$, and length $I_{n+1} = \frac{1}{2}$ length I_n with $I_{n+1} \subseteq I_n$. By nested interval theorem, $\bigcap\limits_{n=1}^{\infty} [a_n, b_n] = \{c\}$. As $a_n \to c$ and $f(a_n) < 0$, we get $f(c) \leq 0$. As $b_n \to c$ and $f(b_n) > 0$, we get $f(c) \geq 0$. Hence $f(c) = 0$. ∎

EXERCISE 5.10.2 *Where did we use continuity in the proof of Theorem 5.10.1?*

COROLLARY 5.10.3 **(Intermediate value theorem (IVT))** *Let $f : [a, b] \to \mathbb{R}$ be continuous. Let m (resp. M) be the absolute minimum (resp. maximum) value of f on $[a, b]$ and suppose that $m < v < M$. Then $\exists c \in (a, b)$ such that $f(c) = v$.*

Proof. Use the previous result with $f(x) - v$. ∎

COROLLARY 5.10.4 *Let $f : [a, b]$ be continuous. Let m = absolute minimum value of f on $[a, b]$ and M = absolute maximum value of f on $[a, b]$. Then $f([a, b]) = [m, M]$.*

Proof. Follows from the definition of an interval and using IVT. ∎

Definition 5.10.5 A point x for which $f(x) = x$, is called a **fixed point** of f.

Corollary 5.10.6 **(Fixed point)** *Let $f : [0,1] \to [0,1]$ be continuous. Then $\exists c \in [0,1]$ such that $f(c) = c$.*

Proof. If $f(0) = 0$ or $f(1) = 1$, we are done. Otherwise, we must have $f(0) > 0$ and $f(1) < 1$. Consider the function $g(x) = f(x) - x$. Apply IVT. ∎

Example 5.10.7 **(Application)** The equation $p(x) = x^3 - 5x^2 + 17x + 18$ has at least one real zero.
Answer. Notice that

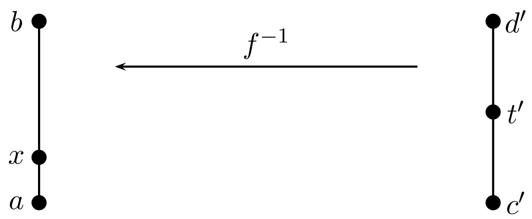
$$x > 5 + 17 + 18 \Rightarrow x^3 > (5 + 17 + 18)x^2 \geq 5x^2 + 17x + 18 \geq \pm 5x^2 \pm 17x \pm 18.$$

That is, $x > 5 + 17 + 18 \Rightarrow p(x) > 0$ and $x < -(5 + 17 + 18) \Rightarrow p(x) < 0$. Apply IVT.

Exercise 5.10.8 *The equation $x^2 - |\sin x| = 0$ has at least 3 real roots.*

Corollary 5.10.9 **(Application: inverse continuity)** *Let $f : [a,b] \to [c', d']$ be strictly increasing, onto and continuous. Then f^{-1} is strictly increasing and continuous.*

Proof. To show continuity of f^{-1}, let $c' < t' < d'$. Put $x = f^{-1}(t')$. Is $a < x < b$? Yes.



Now, take a small ϵ such that $[x - \epsilon, x + \epsilon] \subseteq (a, b)$. As f is strictly increasing, and continuous, using IVT, we have $f(x - \epsilon, x + \epsilon) = (f(x - \epsilon), f(x + \epsilon))$ and it contains t'. So $\exists \delta$ such that $(t' - \delta, t' + \delta) \subseteq f(x - \epsilon, x + \epsilon)$. This means, $f^{-1}(t' - \delta, t' + \delta) \subseteq (x - \epsilon, x + \epsilon)$. So f^{-1} is continuous at t'.
Similarly, f is continuous at c' and d'. ∎

Exercise 5.10.10 *Can you give a proof of the previous theorem by using sequential argument?*

Corollary 5.10.11 *Thus $\ln x$ (being the inverse of $\exp x$) is continuous on $(0, \infty)$.*

5.11 Exercises

Exercise 5.11.1 *If $f : \mathbb{R} \to \mathbb{R}$ is continuous and one-one, then it is strictly monotone.*

Exercise 5.11.2 *Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(f(x)) = -x$. Then f is not continuous.*

Exercise 5.11.3 *Let $f : [0,3] \to \mathbb{R}$ be continuous and $f(0) = f(3)$. Then $\exists c \in [0,3]$ such that $f(c+1) = f(c)$.*

Exercise 5.11.4 **(positive part)** *Let f be continuous. Then the* **positive part** $\boldsymbol{f_+(x)}$:$= \max\{0, f(x)\}$ *of f is continuous.*

Exercise 5.11.5 *Give a function $f : \mathbb{R} \to \mathbb{R}$, which is continuous only at integers.*

EXERCISE 5.11.6 *Recall the Thomae function,* $f(x) = \begin{cases} \frac{1}{q}, & x = \frac{p}{q} \in \mathbb{Q}, \gcd(p,q) = 1 \\ 0, & x \notin \mathbb{Q}. \end{cases}$
It is cts only at irrationals.

EXERCISE 5.11.7 *Let f be continuous on $\mathbb{R}$ with $f(x+y) = f(x) + f(y)$. Then $f(x) = cx$, for some c.*

EXERCISE 5.11.8 **(Lipschitz/contraction/fixed point)** *A function $f : A \to \mathbb{R}$ is* **Lipschitz** *if $\exists M > 0$ such that*

$$x, y \in A \qquad \Rightarrow \qquad |f(x) - f(y)| \le M|x - y|.$$

A **contraction** *is a Lipschitz function with $M < 1$.*

a) A Lipschitz function is continuous. Is the converse true?

b) Give a continuous function f on $\mathbb{R}$ such that $f(x) \neq x$ for each x.

c) Let f be a contraction on $\mathbb{R}$ and $a \in \mathbb{R}$. Then the sequence $a, f(a), f(f(a)), \cdots$ is convergent. Conclude that f has a fixed point.

EXERCISE* 5.11.9 *A function f defined on $[a,b]$ is said to have* **intermediate value property** *if for each $r, s \in [a,b]$, the interval joining $f(r)$ and $f(r)$ is found in the range $f([a,b])$. Thus if f is continuous function, then it has the intermediate value property. Show that a function having intermediate value property need not be continuous.*

EXERCISE* 5.11.10 *A* **separation** *for a set S is a pair (A,B) of disjoint opens sets such that $S \cap A \neq \emptyset$, $S \cap B \neq \emptyset$, and $S \subseteq A \cup B$. A set for which no separation exists, is called* **connected**.

1. *Let $S \subseteq \mathbb{R}$ be nonempty and connected. Then S is an interval.*

2. *Argue that intervals are connected subsets of $\mathbb{R}$.*

3. *Let f be a continuous function defined on an interval I. Show that $f(I)$ is connected.*

EXERCISE 5.11.11 **(attaining bounds)** *Let $S \subseteq \mathbb{R}$ be closed and bound. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Then f attains its bounds in S. That is, there is a point in S where f is minimum and a point where it is maximum.*

EXERCISE 5.11.12 **(continuity/monotonic function)** *Let $f : \mathbb{R} \to \mathbb{R}$ be a monotonic increasing function, $a \in \mathbb{R}$.*

(a) Then show that both $\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$ exist and satisfy

$$\lim_{x \to a^-} f(x) = \sup\{f(x) : x \in (a-1, a)\} \le \lim_{x \to a^+} f(x) = \inf\{f(x) : x \in (a, a+1)\}.$$

(b) Conclude that f has no discontinuity of second kind and no removable discontinuity.

(c) Conclude that f can be discontinuous at most at countably many points.

EXERCISE 5.11.13 **(uniform continuity (required later))** *We say f is* **uniformly cts on S** *if for each $\alpha > 0$, $\exists \delta > 0$ (global) such that*

$$(y, z \in S, \ |y - z| \le \delta) \quad \Rightarrow \quad |f(y) - f(z)| \le \alpha.$$

Roughly it means on each interval of length δ the function can vary at most α.

Contrast

continuity	*uniform continuity*
essentially defined at a point only	*defined directly on a domain*
$\delta > 0$ *at one point may not work for another point*	$\delta > 0$ *works throughout the domain*

1. *The function $f(x) = x$ is uniformly continuous on $\mathbb{R}$. What about $f(x) = |x|$? Yes.*

2. *Take $f(x) = x^2$ on $\mathbb{R}$. Can we find a $\delta > 0$ such that $|y - z| \le \delta \Rightarrow |f(y) - f(z)| \le 1$? Conclude that it is not uniformly continuous on $\mathbb{R}$.*

3. *The function $f(x) = 1/x$, is not uniformly continuous on $(0, 1)$.*

4. *If f is uniformly continuous on S then it is continuous at each point of S.*

5. *Let f be cts on $[a, b]$. Then it is uniformly continuous on $[a, b]$.*

6. **(criterion for uniform continuity)** *Let $f : S \to \mathbb{R}$. Show that f is uniformly continuous iff for every pair of sequences $\{x_n\}, \{y_n\}$ with $\lim |x_n - y_n| = 0$, we have $\lim |f(x_n) - f(y_n)| = 0$.*

7. *Show that $f(x) = \sin(1/x)$ is not uniformly continuous on $(0, 1)$.*

8. **(image of a bounded set under an uniformly continuous function)** *Let S be bounded and f be uniformly continuous on S. Show that $f(S)$ is bounded. Is this necessarily true for a continuous function?*

EXERCISE 5.11.14 **(convex and concave functions)** *A function f defined on (a, b) is said to be* **convex** *if for each $x, y \in (a, b)$, $\lambda \in (0, 1)$, the function satisfies $f(\lambda x + (1 - \lambda)y) \le \lambda f(x) + (1 - \lambda)f(y)$. A function f is* **concave** *if $-f$ is convex.*

(a) Let $0 < a < 1$ and $p, q \in \mathbb{N}$, $p \le q$. Show that $\dfrac{1 + a + a^2 + \cdots + a^{p-1}}{1 + a + a^2 + \cdots + a^{q-1}} \ge \dfrac{p}{q}$.

(b) Let $0 < a < 1$ and $p, q \in \mathbb{N}, p \le q$. Show that $\frac{1-a^{\frac{p}{q}}}{1-a} \ge \frac{p}{q}$.

(c) Let $0 < a < 1$. Recall that for $0 < x$, we define $x^r = e^{r \ln(x)}$. Show that $f(x) = x^r$ is continuous on $(0, \infty)$. Argue that $g(x) = \frac{1-a^x}{1-a} \ge x$ for all $x \in (0, 1)$. Hence, argue that if $0 < a < b, 0 < p < 1$, then $a^p b^{1-p} \le p\,a + (1 - p)b$. Hence argue that $h(x) = e^x$ is convex.

(d) If f, g are convex on the same domain, then $f + g$ is convex, cf is convex for positive c.

(e) If f is convex on (a, b) then $g(x) = f(a + x)$ is convex on $(0, b - a)$.

(f) Show that every convex function on (a, b) is continuous.

(g) Suppose f is convex and g is increasing, convex. Show that $g(f)$ is convex.

EXERCISE 5.11.15 *Show using only calculus that every polynomial in x of odd degree has at least one real root.*

EXERCISE 5.11.16 *Write the domain and the range (as subsets of $\mathbb{R}$) of the function* $\frac{x^2+2x-27}{x^3+1}$.

EXERCISE 5.11.17 *Explain why $\cos(x) = x$, has at least one solution. Exactly how many solutions it has in $\mathbb{R}$?*

EXERCISE* 5.11.18 *Suppose that $f : \mathbb{R} \to \mathbb{R}$. Show that the following are equivalent.*

1. *f is continuous.*

2. *For all open sets $O \subseteq \mathbb{R}$, $f^{-1}(O) = \{x \in \mathbb{R} \mid f(x) \in O\}$ is open.*

3. *For all closed sets $F \subseteq \mathbb{R}$, $f^{-1}(F) = \{x \in \mathbb{R} \mid f(x) \in F\}$ is closed.*

Did you relate this to an earlier exercise where you showed that $\{x \mid p(x) < 0\}$ was open, where $p(x)$ was a polynomial?

EXERCISE 5.11.19 *Let f be continuous on $\mathbb{R}$ satisfying $f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$. Show that f is convex.*

EXERCISE 5.11.20 *Test for uniform continuity.*

a) $f(x) = x^r$, $r > 0$ *fixed, on* $(5, 7)$.

b) $f(x) = \frac{1}{1+x^2}$ *on* $\mathbb{R}$.

c) $f(x) = x\sin(1/x)$ *on* $(0, 1)$.

d) $f(x) = \sin(\sin(x))$ *on* $\mathbb{R}$.

e) $f(x) = |x|^r$, $0 < r \leq 1$ *on* $\mathbb{R}$.

EXERCISE 5.11.21 **(magic)** *Give a bijection from $\mathbb{R}$ to $\mathbb{R}$ which is not monotonic. I guarantee that its not continuous. How?*

EXERCISE 5.11.22 **(existence?)** *Does there exist a continuous $f : [0, 1] \to [0, 1]$ taking each image exactly twice?*

EXERCISE 5.11.23 **(Existence? Nope!)** *Let f be cts on $\mathbb{R}$ and a a local max. Does there exist an interval $(a - \epsilon, a)$ on which f increases?*

EXERCISE 5.11.24 **(what are you saying!)** *Let f be periodic and continuous on $\mathbb{R}$ with period 1. Show that $\exists a$ such that $f(a + e\pi) = f(a)$.*

EXERCISE 5.11.25 *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Show that $Z = \{x \mid f(x) = 0\}$ is closed.*

EXERCISE 5.11.26 **(imagination)** *Suppose you stretch a nonuniform rubber band by moving one end to the right and one end to the left. Is it true that at least one point will remain at its original position?*

EXERCISE* 5.11.27 **(IVP + 'what' would imply continuity?)** *Let f on $\mathbb{R}$ satisfy IVP. Assume that $f^{-1}(x)$ is closed $\forall x \in \mathbb{Q}$. Show that f is continuous.*

EXERCISE 5.11.28 *Let $f : [a, b] \to [a, b]$ be continuous. Suppose that for each $x \in [a, b]$, $\exists y \in [a, b]$ such that $|f(y)| \leq \frac{|f(x)|}{2}$. Is $f(p) = 0$ for some $p \in [a, b]$?*

EXERCISE 5.11.29 *Let f be continuous with $f(x) \to 0$ as $x \to \pm\infty$. Should f have an absolute maximum or minimum?*

EXERCISE 5.11.30 *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $A \subseteq \mathbb{R}$. Show that $f(\overline{A}) \subseteq \overline{f(A)}$. Can the inclusion be proper? Did you relate it to some earlier exercise?*

EXERCISE 5.11.31 *Write yes/no and give examples, wherever necessary.*

a) *Let $x_n \in (0, 1)$ such that $\{x_n\}$ is a converges in $\mathbb{R}$. Let f is continuous on $(0, 1)$. Does that mean $\{f(x_n)\}$ is a convergent sequence?*

b) *Let $f(h)$ be a continuous function of h. Assume that $\lim\limits_{h\to 0} \frac{f(h)}{h^2} = k$ and $\lim\limits_{h\to 0} \frac{f(h)}{h} = l$ both exist. Does that mean $\lim\limits_{h\to 0} \frac{l}{h} = k$?*

c) If f, g are continuous at 0, is it necessary that $f \circ g$ is continuous at 0?

d) Suppose that a continuous function f is never zero on $[0,1]$. Does it mean that f does not change sign on $[0,1]$?

e) Let $f : [a,b] \to [a,b]$ be continuous. Is it always true that $\exists$, $c \in [a,b]$ such that $f(c) = c$?

f) Let $f : [a,b] \to [a,b]$ be continuous such that for each $x \in [a,b]$, there exists $y \in [a,b]$ such that $f(y) \leq \frac{f(x)}{2}$. Is it necessary that there is a point $p \in [a,b]$ such that $f(p) = 0$.

g) Let $f, g : A \to \mathbb{R}$ be uniformly continuous. Then

1. *$f + g$ is uniformly continuous.*
2. *fg is uniformly continuous.*
3. *$\frac{1}{f}$ is uniformly continuous if $f \neq 0$.*
4. *$g \circ f$ is uniformly continuous if range$(f) \subseteq$ domain(g).*
5. *fg is uniformly continuous if f, g are bounded.*
6. *fg is uniformly continuous if f is bounded.*

h) Suppose f is continuous on (a,b). Is it necessary that f is uniformly continuous?

i) Suppose f is continuous on $\mathbb{R}$. Is it necessary that f is uniformly continuous on (a,b)?

j) Suppose f is continuous on $\mathbb{R}$ and is bounded. Is it necessary that f is uniformly continuous?

k) Suppose f is continuous on $(0,1)$ and is bounded. Is it necessary that f is uniformly continuous?

l) Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies $\lim\limits_{h \to 0} \Big(f(x+h) - f(x-h) \Big) = 0$, for some x. Does that mean f is continuous at x?

m) Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies $\lim\limits_{h \to 0} \Big(f(x+h) - f(x-h) \Big) = 0$, for all x. Does that mean f is continuous?

n) Is it necessary that a convex function f on $[a,b]$ is continuous?