Tutorial problems: MA101-Calculus **IIT Guwahati, 2020**

Tutorial 1: Realnumbers1,2,3, Sequence1

1. Let S and T be nonempty and bounded above. Define $S+T = \{s+t \mid s \in S, t \in T\}$. Then show that $\sup(S+T) = \sup S + \sup T$.

   Sol. Let $a = \sup S$ and $b = \sup T$. In particular, a is an upper bound of S and b is an upper bound of B. That is, $\forall\, s \in S$, $a \geq s$, and $\forall\, t \in T$, $b \geq t$. Thus $\forall\, s \in S, t \in T$, we have $a+b \geq s+t$. That is, $a+b$ is an upper bound of $S+T$.

   Since $S+T \neq \emptyset$ and bounded above, $\sup(S+T)$ exists in $\mathbb{R}$. Let $d = \sup(S+T)$. As $a+b$ is already an upper bound, we will have $d \leq a+b$.

   Now we show $d = a+b$. Suppose it is not true. Then $d < a+b$. Write $d = a+b-\epsilon$, where $\epsilon > 0$. As $a = \sup S$, $\exists s \in S$ s.t. $s > a - \epsilon/2$. As $b = \sup T$, $\exists t \in T$ s.t. $t > b - \epsilon/2$. So, we have $s+t \in S+T$ and $s+t > a+b-\epsilon = d$. Hence d cannot be an upper bound of $S+T$. Therefore it cannot be the least upper bound of $S+T$. This contradicts the fact that $d = \sup(S+T)$. So $d = a+b$.

2. Give a finite set, a countable set and an uncountable set $S \subseteq \mathbb{R}$ such that $\operatorname{lub} S \in S$. Give a finite set, a countable set and an uncountable set $S \subseteq \mathbb{R}$ such that $\operatorname{lub} S \notin S$.

   Sol. First: $\{1\}$, $-\mathbb{N}$, $(0,1]$.

   Second: For each nonempty finite set S, $\operatorname{lub} S \in S$. For $S = \emptyset$, $\operatorname{lub} S$ does not exist. So $S = \emptyset$, the condition $\operatorname{lub} S \in S$ cannot hold. $-\{\frac{1}{n} : n \in \mathbb{N}\}$, $(0,1)$.

3. Let A and B be nonempty and bounded sets such that $A \cap B \neq \emptyset$. Order lub's of $A \cup B$, A and $A \cap B$.

   Sol. $\operatorname{lub} A \cap B \leq \operatorname{lub} A \leq \operatorname{lub} A \cup B$.

4. Determine the sets $\bigcap\limits_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n})$ and $\bigcap\limits_{n=1}^{\infty}(0, \frac{1}{n}]$.

   Sol. (a) Ans: $\{0\}$. Note that $0 \in$ all these sets. For any $x > 0$, by Archimedean principle, there is $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < x$. Thus $x \notin (-\frac{1}{n_0}, \frac{1}{n_0})$. So x cannot be in the intersection. Similarly any $x < 0$ cannot be in the intersection.

(b) Ans: $\emptyset$. Argue!

5. Let $S \subseteq [1,2]$ be an infinite set. Show that it has a limit point.

   Sol. Divide the interval into two halves and select the one which contains an infinite subset of S. Call it $I_1 = [a_1, b_1]$. Now consider that interval and further divide that and continue. We will have closed intervals $[a_1, b_1] \supseteq [a_2, b_2] \supseteq \ldots$.

   Using nested interval theorem, let $a \in \cap[a_n, b_n]$. We now show that a is a limit point of S. Consider a $D_\epsilon(a)$. Select n such that $\frac{1}{2^n} < \epsilon$. Since the length of $I_n = [a_n, b_n]$ is $\frac{1}{2^n}$, and $a \in I_n$, we see that $(a-\epsilon, a+\epsilon)$ completely contains I_n. But, recall that I_n contains infinitely many points of S. Hence, $D_\epsilon(a) = (a-\epsilon, a) \cup (a, a+\epsilon)$ contains infinitely many points of S.

6. Let $a < b$. Supply 3 rationals and 3 irrationals inside (a, b).

   Sol. Put $n = [\frac{3}{b-a}] + 1$. Then the numbers are

$$\frac{[na]+1+\frac{1}{2}}{n}, \frac{[na]+1+\frac{1}{3}}{n}, \frac{[na]+1+\frac{1}{4}}{n}$$

   and

$$\frac{[na]+1+\frac{1}{2\sqrt{2}}}{n}, \frac{[na]+1+\frac{1}{3\sqrt{2}}}{n}, \frac{[na]+1+\frac{1}{4\sqrt{2}}}{n}.$$

7. Consider the sequence $(a_n = \frac{1}{n})$.

   a) Let $a \neq 0$. Then $a_n \not\to a$ as $\exists \epsilon > 0$ such that $B_\epsilon(a)$ misses infinitely many terms of (a_n). Give a value for ϵ.

   Sol. $|a|/2$

   b) $a_n \to 0$ as each $B_\epsilon(a)$ contains a tail (which may depend on ϵ) of (a_n). Which tail?

   Sol. $a_{[1/\epsilon]+1}, a_{[1/\epsilon]+2}, \ldots$. If someone gives an existential argument using Archimedean property, then it is fine. The intention of the exercise was to familiarize the students with the notations.

8. Let $s > 0$. Is $\frac{[10^n s]}{10^n} \to s$?

   Sol. Yes. Recall that for any real number a, we have $[a] \le a < [a]+1$.

   Hence, $\quad [10^n s] \le 10^n s < [10^n s] + 1.$

   So $\quad 0 \le 10^n s - [10^n s] \le 1.$

That is, $0 \leq s - \frac{[10^n s]}{10^n} < \frac{1}{10^n}$.

Sandwich lemma implies $s - \frac{[10^n s]}{10^n} \to 0$. Then by definition, $\frac{[10^n s]}{10^n} \to s$.