MA 101 (Mathematics - I)
Integration : Exercise set 2

Fundamental Theorems of Calculus

1. Let $f : [a, b] \to \mathbb{R}$ be continuous. Assume that there exist distinct constants α and β such that for all $c \in [a, b]$, we have $\alpha \displaystyle\int_a^c f(x)dx + \beta \int_c^b f(x)dx = 0$. Show that $f = 0$.

2. Let $g : \mathbb{R} \to \mathbb{R}$ be differentiable. Let $F(x) = \int_0^{g(x)} t^2 dt$. Prove that $F'(x) = g^2(x)g'(x)$ for all $x \in \mathbb{R}$. If $G(x) = \int_{h(x)}^{g(x)} t^2 dt$, then what is $G'(x)$?

3. Let $f : [a, b] \to \mathbb{R}$ be continuous and $g : [c, d] \to [a.b]$ be differentiable. Define $\phi(x) = \int_a^{g(x)} f(t)dt$. Prove that ϕ is differentiable and compute its derivative.

4. If f'' is continuous on $[a, b]$, show that $\int_a^b xf''(x)dx = [bf'(b) - f(b)] - [af'(a) - f(a)]$.

5. Let $f > 0$ be continuous on $[1, \infty)$ and suppose that for $x > 0$, $\displaystyle\int_1^x f(t)dt \leq (f(x))^2$. Prove that $f(x) \geq \frac{1}{2}(x - 1)$.

6. If $f : [a, b] \to \mathbb{R}$ is continuously differentiable, show that $\displaystyle\lim_{n\to\infty} \int_a^b f(x) \cos nx \, dx = 0$.

7. If $f : [0; 1] \to \mathbb{R}$ is continuous, then show that $\int_0^x \big(\int_0^u f(t)dt \big) du = \int_0^x (x - u)f(u)du$.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and let $g(x) = \int_0^x (x - t)f(t)dt$ for $x \in \mathbb{R}$. Prove that $g''(x) = f(x)$ for all $x \in \mathbb{R}$.

Improper Integrals

9. Compute the following improper integrals or prove their divergence.

   (a) $\displaystyle\int_1^\infty \frac{dx}{x^4}$ (b) $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}}$ (c) $\displaystyle\int_0^\infty e^{-ax}dx$ (d) $\displaystyle\int_0^1 \frac{dx}{\sqrt{1 - x^2}}$

   (e) $\displaystyle\int_0^1 \frac{dx}{x \ln x}$ (f) $\displaystyle\int_0^1 \frac{dx}{x \ln(\ln x)}$ (g) $\displaystyle\int_0^1 \frac{dx}{x(\ln x)^{3/2}}$ (h) $\displaystyle\int_0^1 \frac{\sqrt{x}\,dx}{(1 + x)^2}$

10. Test the following improper integrals for convergence.

    (a) $\displaystyle\int_0^1 \frac{\sqrt{x}}{\sqrt{1 - x^4}} dx$ (b) $\displaystyle\int_0^{\pi/2} \frac{\ln \sin x}{\sqrt{x}} dx$

11. Determine the values of k for which the following integrals converge.

    (a) $\displaystyle\int_a^b \frac{dx}{(b - x)^k}$, $(b < a)$ (b) $\displaystyle\int_0^\pi \frac{dx}{\sin^k x}$

12. For what values of k and t the integral $\displaystyle\int_0^\infty \frac{x^k}{1 + x^t} dx$ is convergent?

13. Examine whether the following integrals are convergent

    (a) $\displaystyle\int_0^\infty \sin(x^2)dx$, (b) $\displaystyle\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$.

14. The integral $\int_a^\infty f(x)dx$ is said converge **absolutely** if $\int_a^\infty |f(x)|dx$ is convergent. If the integral $\int_a^\infty f(x)dx$ converges, but not absolutely, then it is said to converge **conditionally**. Show that $\displaystyle\int_1^\infty \frac{\sin x}{x^p}dx$ converges absolutely if $p > 1$ and conditionally if $0 < p \leq 1$.

Applications of integrals

15. Find the area of the region enclosed by the curve $y = \sqrt{|x+1|}$ and the line $5y = x + 7$.

16. Find the area enclosed by the curve $r = a\cos 3\theta,\ -\pi/6 \leq \theta \leq \pi/6$.

17. Sketch the curve $x = a\sin 2t,\ y = a\sin t$ and find the area of one of its loops.

18. Find the length of the curve $y = \int_0^x \sqrt{\cos 2t}\,dt,\ 0 \leq x \leq \frac{\pi}{4}$.

19. A curve is given by the equation

$$x = a(\cos\theta + \theta\sin\theta),\ y = a(\sin\theta - \theta\cos\theta).$$

    Find the length of the arc from $\theta = 0$ to $\theta = \alpha$.

20. Consider the funnel formed by revolving the curve $y = \frac{1}{x}$ about the x-axis, between $x = 1$ and $x = a$, where $a > 1$. If V_a and S_a denote respectively the volume and the surface area of the funnel, then show that $\lim\limits_{a\to\infty} V_a = \pi$ and $\lim\limits_{a\to\infty} S_a = \infty$.

21. Find the volume and area of the curved surface of a paraboloid of revolution formed by revolving the parabola $y^2 = 4ax$ about the x-axis, and bounded by the section $x = x_1$.

22. Show that the area of the surface obtained by revolving the cardioid $r = 1 + \cos\theta$ about the x-axis is $\frac{32}{5}\pi a^2$.