MA 101 (Mathematics - I)
Differentiation : Exercise set 2: Hints and Solutions

CMVT/L'Hôpital's Rules

1. Use CMVT to derive the following: Suppose f, g are differentiable on $[a, b]$ and $|f'(x)| \geq |g'(x)| > 0$ for all x. Show that for $a \leq x < y \leq b$, $|f(y) - f(x)| \geq |g(y) - g(x)|$.

> **Solution:** (Hint.) There is $c \in (x, y)$ such that $\dfrac{f(y) - f(x)}{g(y) - g(x)} = \dfrac{f'(c)}{g'(c)}$. Now use the given condition.

2. Find the following by using L'Hôpital's Rules, whenever needed. Do not forget to check the conditions needed for using L'Hôpital's Rules.
(i) $\displaystyle\lim_{x\to 0+} \frac{\sqrt{1+x} - 1}{\sqrt{x}}$ (ii) $\displaystyle\lim_{x\to \frac{\pi}{2}} \frac{1 - \sin x}{1 + \cos 2x}$ (iii) $\displaystyle\lim_{x\to 0} \frac{x^2 \sin \frac{1}{x}}{\sin x}$ (iv) $\displaystyle\lim_{x\to 0+} \left(\frac{\sin x}{x}\right)^{1/x}$
(v) $\displaystyle\lim_{x\to 0+} \frac{e^{-1/x^2}}{x}$ (vi) $\displaystyle\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$ (vii) $\displaystyle\lim_{x\to \infty} \frac{x - \sin x}{2x + \sin x}$ (viii) $\displaystyle\lim_{x\to \pi/2-} (\sec x - \tan x)$.

> **Solution:**
>
> (i) 0, (ii) 1/4
>
> (iii) 0. Since $\displaystyle\lim_{x\to 0} \frac{x}{\sin x} = 1$, and $\displaystyle\lim_{x\to 0} x \sin \frac{1}{x} = 0$, we have $\displaystyle\lim_{x\to 0} \frac{x^2 \sin \frac{1}{x}}{\sin x} = 0$.
>
> (iv) Let $h(x) = \left(\dfrac{\sin x}{x}\right)^{1/x}$ for $x > 0$.
>
> Then, $\ln h(x) = \dfrac{\ln \sin x - \ln x}{x} = \dfrac{f_1(x)}{g_1(x)}$ is in $\left(\frac{0}{0}\right)$ form as $x \to 0+$.
>
> Now, $\dfrac{f_1'(x)}{g_1'(x)} = \dfrac{x \cos x - \sin x}{x \sin x} = \dfrac{f_2(x)}{g_2(x)}$ is in $\left(\frac{0}{0}\right)$ form as $x \to 0+$.
>
> Again, $\dfrac{f_2'(x)}{g_2'(x)} = \dfrac{x \sin x}{\sin x + x \cos x} = \dfrac{f_3(x)}{g_3(x)}$ is in $\left(\frac{0}{0}\right)$ form as $x \to 0+$.
>
> Now, $\dfrac{f_3'(x)}{g_3'(x)} = \dfrac{\sin x + x \cos x}{2 \cos x - x \sin x} \to 0$ as $x \to 0+$. Therefore, by LH1, $\displaystyle\lim_{x\to 0+} \ln h(x) = 0$.
>
> Since Exp is continuous, $\displaystyle\lim_{x\to 0+} h(x) = e^0 = 1$.
>
> (v) Put $x = 1/t$ for $x > 0$. Then, $0 < \dfrac{e^{-1/x^2}}{x} = \dfrac{t}{e^{t^2}} < \dfrac{t}{t^2} = x$, since $e^{t^2} > t^2$. Now, use sandwich theorem.
>
> (vi) 0,

(vii) For $x > 0$, put $f(x) = \dfrac{x - \sin x}{2x + \sin x} = \dfrac{1 - \frac{\sin x}{x}}{2 + \frac{\sin x}{x}}$. Now, given $\epsilon > 0$, we have $\left|\dfrac{\sin x}{x}\right| \leq \dfrac{1}{x} < \epsilon$, if $x > \frac{1}{\epsilon}$. Thus, $\lim\limits_{x\to\infty} \dfrac{\sin x}{x} = 0$, and hence $\lim\limits_{x\to\infty} \dfrac{x - \sin x}{2x + \sin x} = 1/2$.

(viii) 0.

3. Let f be a differentiable on $(0, \infty)$ and suppose that $\lim\limits_{x\to\infty} \big(f(x) + f'(x)\big) = L$. Show that $\lim\limits_{x\to\infty} f(x) = L$ and $\lim\limits_{x\to\infty} f'(x) = 0$.

Solution: (Hint.) Note that $f(x) = \frac{e^x f(x)}{e^x}$. Now use L'Hôpital's Rule II, which holds also in the case when $\lim\limits_{x\to\infty} e^x f(x) = 0$ is not given. See L'Hôpital's Rule given in Bartle and Sherbert (6.3.5).

4. Try to use L'Hôpital's Rule to find the limit of $\dfrac{\tan x}{\sec x}$ as $x \to (\pi/2)-$. Also, evaluate it directly by changing to sines and cosines.

Solution: For $f(x) = \tan x, g(x) = \sec x$, $\dfrac{f'(x)}{g'(x)} = \dfrac{g(x)}{f(x)}$. So, the existence of the limit $\lim\limits_{x\to(\pi/2)-} \dfrac{f(x)}{g(x)}$ cannot be concluded by L'Hôpital's Rules.

However, for $0 < x < \pi/2$, $\frac{f(x)}{g(x)} = \sin x \to 1$ as $x \to (\pi/2)-$.

Taylor's Theorem

5. Let x_0 be a fixed in $\mathbb{R}$. Find the n-th Taylor polynomial and the remainder for the following functions f about x_0, and check for $x \in \mathbb{R}$ whether the remainder term converges to zero as $n \to \infty$.

(i) $f(x) := e^x$ on $\mathbb{R}$, (ii) $f(x) := \sin x$ on $\mathbb{R}$,

Solution: (i) $T_n(f, x_0)(x) = e^{x_0} + \dfrac{e^{x_0}}{1!}(x - x_0) + \cdots + \dfrac{e^{x_0}}{n!}(x - x_0)^n$,

$R_n = \frac{e^t}{(n+1)!}(x - x_0)^{n+1}$ for some t between x and x_0. For any $x \in \mathbb{R}$, $R_n \to 0$ as $n \to \infty$.

(ii) $T_n(f, x_0)(x) = \sin x_0 + \dfrac{\cos x_0}{1!}(x - x_0) - \dfrac{\sin x_0}{2!}(x - x_0)^2 - \dfrac{\cos x_0}{3!}(x - x_0)^3 + \cdots + \dfrac{f^{(n)}(x_0)}{n!}(x - x_0)^n$,

$R_n = \frac{f^{(n+1)}(t)}{(n+1)!}(x - x_0)^{n+1}$ for some t between x and x_0. For any $x \in \mathbb{R}$, $|R_n| \leq \frac{|x - x_0|^{n+1}}{(n+1)!} \to 0$ as $n \to \infty$.

6. Show that for any $k \in \mathbb{N}$ and for all $x > 0$

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots - \frac{x^{2k}}{2k} < \ln(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{x^{2k+1}}{2k+1}.$$

Solution: Applying Taylor's theorem to the function $\ln(1+x)$ gives for $x > 0$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots - \frac{x^{2k}}{2k} + \frac{c^{2k+1}}{2k+1}$$

for some $c \in (0, x)$. Since $0 < \dfrac{c^{2k+1}}{2k+1} < \dfrac{x^{2k+1}}{2k+1}$, we get the inequalities.

7. For a differentiable function $f : [a, b] \to \mathbb{R}$, a point $c \in (a, b)$ is called a **point of inflection** of f if $f(x) - f(c) - f'(c)(x - c)$ changes sign as x increases through c in an interval containing c.
Suppose $n \in \mathbb{N}$ is odd, $f^{(n)}$ is continuous, $f'(c) = \cdots = f^{(n-1)}(c) = 0$ and $f^{(n)}(c) \neq 0$. Show that c is a point of inflection for f.

Solution: (Note: Continuity of $f^{(n)}$, which is needed, was missing in the original question.)
Under the given condition, using Taylor's theorem to f about c, we have

$$f(x) - f(c) - f'(c)(x - c) = \frac{f^{(n)}(d)}{n!}(x - c)^n,$$

for some d between x and c. Since $f^{(n)}$ is continuous and $f^{(n)}(c) \neq 0$, $f^{(n)}(x)$ assumes same sign as $f^{(n)}(c)$ in some interval $(c - \delta, c + \delta)$. Thus, $f(x) - f(c) - f'(c)(x - c)$ assumes sign as that of $f^{(n)}(c)$ when $x \in (c, c + \delta)$ and assumes sign as that of $-f^{(n)}(c)$ when $x \in (c - \delta, c)$. Thus, the result follows.

8. What is the Taylor series for a polynomial?

Solution: The polynomial itself.

9. Consider the function

$$f(t) = \begin{cases} e^{-1/t}, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0. \end{cases}$$

Show that
(1) f is infinitely differentiable on $\mathbb{R}$.
(2) f has a Taylor series about the point 0.
(3) the Taylor series converges to a function different from f.

Solution: The problem is similar to the problem 5 of Tutorial 5.

10. Determine whether $x = 0$ is a point of local maximum/minimum of the following functions defined on $\mathbb{R}$:
(i) $f(x) := x^4 - x^3 + 2$, (ii) $g(x) := x - \sin x$, (iii) $h(x) = \sin x + \frac{1}{6}x^3$, (iv) $k(x) := \cos x - 1 + \frac{1}{2}x^2$.

Solution: (i) Local minimum, (ii) Neither, (iii) Neither, (iv) Local minimum.

Limit superior/inferior

11. Find limit superior and limit inferior of the following sequences.

(1) $a_n = \frac{n}{n+1}$, if n is odd, and $a_n = \frac{1}{n}$, if n is even.
(2) $a_n = (-1)^n(1 - \frac{1}{n})$.
(3) $a_n = (-1)^n(n + \frac{1}{2^n})$
(4) $(1, -1, \frac{1}{2}, -2, \frac{1}{3}, -3, \ldots)$
(5) $(-1)^n(1 - \frac{1}{n})n^{1/n}$

Solution: (1) For $n \in \mathbb{N}$, $\text{lub}\{a_n, a_{n+1}, \ldots\} = 1$, and $\text{glb}\{a_n, a_{n+1}, \ldots\} = 0$.
Therefore, $\limsup a_n = 1$, and $\liminf a_n = 0$.

(2) $(a_n) = (0, \frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \ldots)$. For $n \in \mathbb{N}$, $\text{lub}\{a_n, a_{n+1}, \ldots\} = 1$, and $\text{glb}\{a_n, a_{n+1}, \ldots\} = -1$.
Therefore, $\limsup a_n = 1$, and $\liminf a_n = -1$.

(3) (a_n) is not bounded above, and not bounded below. Therefore, $\limsup a_n = \infty$, and $\liminf a_n = -\infty$.

(4) For $n = 2k - 1$ and $n = 2k$, $k \in \mathbb{N}$, $\text{lub}\{a_n, a_{n+1}, \ldots\} = \frac{1}{k} \to 0$ as $k \to \infty$. Therefore, $\limsup a_n = 0$. As (a_n) is not bounded below, $\liminf a_n = -\infty$.

(5) The given sequence is $b_n = a_n n^{1/n}$, where $a_n = (-1)^n(1 - \frac{1}{n})$. Since $n^{1/n} \to 1$, we have $\limsup b_n = \limsup a_n = 1$, and $\liminf b_n = \liminf a_n = -1$ (see [6.5] of Differential Notes).