MA 101 (Mathematics I)

Multivariable Calculus : Hints / Solutions of Tutorial Problem Set - 2

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1. Let $A = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < 1\}$ and $B = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$. Examine whether $A \cap B$ is (a) an open set (b) a closed set in $\mathbb{R}^3$.

   Solution: We have $(0, 0, 0) \in A \cap B$. If possible, let $(0, 0, 0) \in (A \cap B)^0$. Then there exists $r > 0$ such that $B_r\big((0, 0, 0)\big) \subseteq A \cap B$. Since $(0, 0, \frac{r}{2}) \in B_r\big((0, 0, 0)\big)$ but $(0, 0, \frac{r}{2}) \notin A \cap B$, we get a contradiction. Hence $(0, 0, 0) \notin (A \cap B)^0$. Therefore $A \cap B$ is not an open set in $\mathbb{R}^3$.
   Again, since $(1 - \frac{1}{n}, 0, 0) \in A \cap B$ for all $n \in \mathbb{N}$ and since $(1 - \frac{1}{n}, 0, 0) \to (1, 0, 0) \notin A \cap B$, $A \cap B$ is not a closed set in $\mathbb{R}^3$.

2. Show that $\{\mathbf{x} \in \mathbb{R}^m : 1 < \|\mathbf{x}\| \leq 2\}$ is neither an open set nor a closed set in $\mathbb{R}^m$.

   Solution: Let $S = \{\mathbf{x} \in \mathbb{R}^m : 1 < \|\mathbf{x}\| \leq 2\}$. Since $\|(2 + \frac{1}{n})\mathbf{e}_1\| = 2 + \frac{1}{n} > 2$ for all $n \in \mathbb{N}$, $(2 + \frac{1}{n})\mathbf{e}_1 \in \mathbb{R}^m \setminus S$ for all $n \in \mathbb{N}$. Also, $(2 + \frac{1}{n})\mathbf{e}_1 \to 2\mathbf{e}_1 \notin \mathbb{R}^m \setminus S$, since $\|2\mathbf{e}_1\| = 2$. Hence $\mathbb{R}^m \setminus S$ is not a closed set in $\mathbb{R}^m$ and consequently S is not an open set in $\mathbb{R}^m$.
   Again, since $\|(1 + \frac{1}{n})\mathbf{e}_1\| = 1 + \frac{1}{n} \in (1, 2]$ for all $n \in \mathbb{N}$, $(1 + \frac{1}{n})\mathbf{e}_1 \in S$ for all $n \in \mathbb{N}$. Also, $(1 + \frac{1}{n})\mathbf{e}_1 \to \mathbf{e}_1 \notin S$, since $\|\mathbf{e}_1\| = 1$. Hence S is not a closed set in $\mathbb{R}^m$.

3. State TRUE or FALSE with justification: If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous and if S is a bounded subset of $\mathbb{R}^2$, then $f(S)$ must be a bounded subset of $\mathbb{R}$.

   Solution: Since S is a bounded subset of $\mathbb{R}^2$, there exists $r > 0$ such that $S \subseteq B_r[\mathbf{0}]$. Now, since $B_r[\mathbf{0}]$ is a closed and bounded set in $\mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous, $f(B_r[\mathbf{0}])$ is a bounded set in $\mathbb{R}$. Hence there exists $M > 0$ such that $|f(\mathbf{x})| \leq M$ for all $\mathbf{x} \in B_r[\mathbf{0}]$. So, in particular, $|f(\mathbf{x})| \leq M$ for all $\mathbf{x} \in S$. Hence $f(S)$ is a bounded subset of $\mathbb{R}$. Therefore the given statement is TRUE.

4. Let S be a nonempty subset of $\mathbb{R}^m$ such that every continuous function $f : S \to \mathbb{R}$ is bounded. Show that S is a closed and bounded set in $\mathbb{R}^m$.

   Solution: If possible, let S be not closed in $\mathbb{R}^m$. Then there exists $\mathbf{x}_0 \in \mathbb{R}^m \setminus S$ and a sequence $(\mathbf{x}_n)$ in S such that $\mathbf{x}_n \to \mathbf{x}_0$. The function $f : S \to \mathbb{R}$, defined by $f(\mathbf{x}) = \frac{1}{\|\mathbf{x} - \mathbf{x}_0\|}$ for all $\mathbf{x} \in S$, is continuous but not bounded (since $\|\mathbf{x}_n - \mathbf{x}_0\| \to 0$ and so $f(\mathbf{x}_n) \to \infty$), which contradicts the hypothesis. Hence S must be a closed set in $\mathbb{R}^m$.
   Again, if possible, let S be not bounded in $\mathbb{R}^m$. Then the function $g : S \to \mathbb{R}$, defined by $g(\mathbf{x}) = \|\mathbf{x}\|$ for all $\mathbf{x} \in S$, is continuous but not bounded, which contradicts the hypothesis. Hence S must be bounded in $\mathbb{R}^m$.

5. Let $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$ and let $f : S \to \mathbb{R}$ be continuous. Show that there exist $\alpha,\, \beta \in \mathbb{R}$ with $\alpha \leq \beta$ such that $f(S) = [\alpha,\, \beta]$.

   Solution: We know that $S = B_1[\mathbf{0}]$ is a closed and bounded set in $\mathbb{R}^3$. Since $f : S \to \mathbb{R}$

is continuous, there exist $\mathbf{x}_0,\ \mathbf{y}_0 \in S$ such that $f(\mathbf{x}_0) \le f(\mathbf{x}) \le f(\mathbf{y}_0)$ for all $\mathbf{x} \in S$. Taking $\alpha = f(\mathbf{x}_0)$ and $\beta = f(\mathbf{y}_0)$, we find that $\alpha,\ \beta \in \mathbb{R}$, $\alpha \le \beta$ and $f(S) \subseteq [\alpha, \beta]$. Again, if $t \in [0,1]$, then $(1-t)\mathbf{x}_0 + t\mathbf{y}_0 \in \mathbb{R}^3$ and since $\|(1-t)\mathbf{x}_0 + t\mathbf{y}_0\| \le (1-t)\|\mathbf{x}_0\| + t\|\mathbf{y}_0\| \le 1 - t + t = 1$, $(1-t)\mathbf{x}_0 + t\mathbf{y}_0 \in S$. Let $F(t) = (1-t)\mathbf{x}_0 + t\mathbf{y}_0$ and $\varphi(t) = f(F(t))$ for all $t \in [0,1]$. Since the functions $F : [0,1] \to S$ and $f : S \to \mathbb{R}$ are continuous, $\varphi = f \circ F : [0,1] \to \mathbb{R}$ is continuous. Assuming $\alpha < \beta$, let $\gamma \in (\alpha, \beta) = (\varphi(0), \varphi(1))$. Then by the intermediate value property of the continuous function φ, there exists $t_0 \in (0,1)$ such that $\gamma = \varphi(t_0) = f(F(t_0)) \in f(S)$, since $F(t_0) \in S$. Therefore $f(S) = [\alpha, \beta]$.

6. (a) Examine whether $\lim\limits_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{(x^2+y^2)^2}$ exists (in $\mathbb{R}$) and find its value if it exists (in $\mathbb{R}$).

   Solution: Let $\big((x_n, y_n)\big)$ be any sequence in $\mathbb{R}^2 \setminus \{(0,0)\}$ such that $(x_n, y_n) \to (0,0)$. Then $x_n^2 + y_n^2 \neq 0$ for all $n \in \mathbb{N}$ and $x_n^2 + y_n^2 \to 0$ in $\mathbb{R}$. Since $\lim\limits_{t\to0} \frac{1-\cos t}{t^2} = \lim\limits_{t\to0} \frac{\sin t}{2t} = \frac{1}{2}$, we have $\lim\limits_{n\to\infty} \frac{1-\cos(x_n^2+y_n^2)}{(x_n^2+y_n^2)^2} = \frac{1}{2}$. It follows that $\lim\limits_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{(x^2+y^2)^2}$ exists and its value is $\frac{1}{2}$.

   (b) Examine whether $\lim\limits_{(x,y)\to(0,0)} \frac{y}{x^2+y^2} \sin \frac{1}{x^2+y^2}$ exists (in $\mathbb{R}$) and find its value if it exists (in $\mathbb{R}$).

   Solution: Let $f(x,y) = \frac{y}{x^2+y^2} \sin \frac{1}{x^2+y^2}$ for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$.
   Since $\left(0, \frac{\sqrt{2}}{\sqrt{(4n+1)\pi}}\right) \to (0,0)$ and $f\left(0, \frac{\sqrt{2}}{\sqrt{(4n+1)\pi}}\right) = \sqrt{2n\pi + \frac{\pi}{2}} \to \infty$, $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist (in $\mathbb{R}$).

7. Let S be a nonempty open set in $\mathbb{R}$ and let $F : S \to \mathbb{R}^m$ be a differentiable function such that $\|\, F(t) \,\|$ is constant for all $t \in S$. Show that $F(t) \cdot F'(t) = 0$ for all $t \in S$.

   Solution: Let $c \in \mathbb{R}$ such that $\|\, F(t) \,\| = c$ for all $t \in S$. Then $F(t) \cdot F(t) = \|\, F(t) \,\|^2 = c^2$ for all $t \in S$. Hence $\frac{d}{dt}\big(F(t) \cdot F(t)\big) = 0$ for all $t \in S$. This gives $F'(t) \cdot F(t) + F(t) \cdot F'(t) = 0$ for all $t \in S$. So $2\,F(t) \cdot F'(t) = 0$ for all $t \in S$. Therefore $F(t) \cdot F'(t) = 0$ for all $t \in S$.