MA 101 (Mathematics – I)
Integration : Lecture Notes

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1 Riemann integral

Integration Class 1

[**1.1**] DEFINITION A **partition** or **subdivision** P of an interval $[a, b]$ is a finite set $\{x_0, x_1, \dots, x_n\}$ such that $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. The points x_i are called the **nodes** of P. We will write P as $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$.

[**1.2**] EXAMPLE (i) Trivial partition: $P = \{a = x_0 < x_1 = b\}$.
(ii) $\mathbf{P}_n = \{a = x_0 < x_1 < \cdots < x_n = b\}$, where $n \in \mathbb{N}$ and $x_i = a + \frac{i}{n}(b - a)$. $\mathbf{P}_n$ divides $[a, b]$ into n subintervals of equal length.

For this section, f will always mean a function $f : [a, b] \to \mathbb{R}$ that is bounded.

[**1.3**] DEFINITION For a partition $P = \{a = x_0 < \cdots < x_n = b\}$ of $[a, b]$ define

$$m_k = \text{glb}\{f(x) : x \in [x_{k-1}, x_k]\}, M_k = \text{lub}\{f(x) : x \in [x_{k-1}, x_k]\},$$

lower sum of f w.r.t. P: $L(f, P) := \sum_{k=1}^{n} m_k(x_k - x_{k-1})$,
upper sum of f w.r.t. P: $U(f, P) := \sum_{k=1}^{n} M_k(x_k - x_{k-1})$.

[**1.4**] EXERCISE If $f(x) = x^4 - 4x^3 + 10$ for $x \in [1, 4]$ and $P = \{1 < 2 < 3 < 4\}$, calculate $U(f, P)$ and $L(f, P)$. [Fact: f is decreasing in $[1, 3]$ and increasing in $[3, 4]$.]

[**1.5**] RESULT *Let* $m = \text{glb}\{f(x) : x \in [a, b]\}$ *and* $M = \text{lub}\{f(x) : x \in [a, b]\}$. *Then*

$$m(b - a) \le L(f, P) \le U(f, P) \le M(b - a).$$

[**1.6**] DEFINITION

Lower integral of f: $L(f) = \underline{\int_a^b} f(x)dx := \text{lub}\{L(f, P) : P \text{ is a partition of } [a, b]\}$.

Upper integral of f: $U(f) = \overline{\int_a^b} f(x)dx := \text{glb}\{U(f, P) : P \text{ is a partition of } [a, b]\}$.

[**1.7**] RESULT $L(f) \le U(f)$. We will see soon why this is so.

[**1.8**] DEFINITION The function $f : [a, b] \to \mathbb{R}$ is said to be **(Riemann or Darboux) integrable** if $L(f) = U(f)$ on $[a, b]$. The common value is called the **integral** of f over $[a, b]$ and is denoted by $I(f)$ or $I_a^b(f)$ or $\int_a^b f$ or $\int_a^b f(x)dx$. By $\mathcal{R}[a, b]$ we denote the set of all integrable functions on $[a, b]$.

[1.9] EXAMPLE If $f : [a,b] \to \mathbb{R}$ is a constant function and $f(x) = c$, then f is integrable and $\int_a^b f = c(b-a)$.

[1.10] EXERCISE (1) Is the function $f(x) = 0$ for $0 \le x < 1$ and $f(1) = 1$, integrable?
(2) Is the Dirichlet function $f : [0,1]$ defined by $f(x) = 1$, if $x \in \mathbb{Q}$, and 0, otherwise, integrable?
(3) Is the function $f : [0,1]$ defined by $f(x) = x$, if $x \in \mathbb{Q}$, and 0, otherwise, integrable?
[Hint. Let $P = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ and $\frac{1}{2} \in [x_{i-1}, x_i]$. Then $U(f,P) \ge \frac{1}{2}(1 - x_{i-1}) \ge 1/4$. However, $L(f,P) = 0$.]

[1.11] DEFINITION For partitions P and Q of $[a,b]$, Q is called a **refinement** of P, if $P \subseteq Q$.

`Q`: When is $\mathbf{P}_m$ a refinement of $\mathbf{P}_n$?

[1.12] RESULT *If Q is a refinement of P, then $L(f,P) \le L(f,Q) \le U(f,Q) \le U(f,P)$.*

Proof. First, suppose $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ and Q has one more point s (say) than P, with $x_{i-1} < s < x_i <$. Then

$$m_i^{(1)} := \text{glb}\{f(x) : x \in [x_{i-1}, s]\} \ge m_i,$$

$$m_i^{(2)} := \text{glb}\{f(x) : x \in [s, x_i]\} \ge m_i.$$

Therefore, $L(f,Q) - L(f,P) = (m_i^{(1)} - m_i)(s - x_{i-1}) + (m_i^{(2)} - m_i)(x_{i-1} - s) \ge 0$, i.e., $L(f,P) \le L(f,Q)$. Now, it is clear that if Q is obtained by adding several (a finitely many) points to P, then $L(f,P) \le L(f,Q)$. Similarly, $U(f,Q) \le U(f,P)$. ■

[1.13] RESULT *If P and Q are partitions of $[a,b]$, then $L(f,P) \le U(f,Q)$. Therefore we have*

$$m(b-a) \le L(f) \le U(f) \le M(b-a).$$

Proof. $L(f,P) \le L(f, P \cup Q) \le U(f, P \cup Q) \le U(f,Q)$.

[1.14] RESULT *Suppose there is sequence (P_n) of partitions of $[a,b]$ such that $L(f,P_n) \to \alpha$ and $U(f,P_n) \to \alpha$. Then $f \in \mathcal{R}[a,b]$ and $\int_a^b f = \alpha$.*

Proof. $L(f) \ge \alpha$ and $U(f) \le \alpha$.

[1.15] EXERCISE

(1) For $f(x) = x$ on $[0,1]$ calculate $L(f, \mathbf{P}_n)$ and $U(f, \mathbf{P}_n)$, conclude $f \in \mathcal{R}[0,1]$, and find $\int_a^b f$.
(2) For $f(x) = x^2$ on $[0,1]$ calculate $L(f, \mathbf{P}_n)$ and $U(f, \mathbf{P}_n)$, conclude $f \in \mathcal{R}[0,1]$, and find $\int_a^b f$.
[Hint. $L(f, \mathbf{P}_n) = \sum_{i=1}^n \left(\frac{i-1}{n}\right)^2 \frac{1}{n} \to \frac{1}{3}, U(f, \mathbf{P}_n) = \sum_{i=1}^n \left(\frac{i}{n}\right)^2 \frac{1}{n} \to \frac{1}{3}$.]

[1.16] THEOREM **(Riemann condition for Integrability)** *A bounded function $f : [a,b] \to \mathbb{R}$ is integrable if and only if for each $\epsilon > 0$ there exists a partition P such that $U(f,P) - L(f,P) < \epsilon$.*

Proof. Exercise.

[1.17] EXAMPLE Take $f(x) = x^3$ on $[0, 1]$. Let $\epsilon > 0$. Then

$$U(f, \mathbf{P}_n) - L(f, \mathbf{P}_n) = \frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{i}{n}\right)^3 - \left(\frac{i-1}{n}\right)^3\right] = \frac{1}{n}(f(1) - f(0)) = \frac{1}{n} < \epsilon$$

for large n. Thus, $f \in \mathcal{R}([0, 1])$.

`Q`: Suppose f is monotone on $[a, b]$. Is $f \in \mathcal{R}([a, b])$? Can we use the idea of above example?

[1.18] REMARK Let $f \in \mathcal{R}([a, b])$. Then, for each $n \in \mathbb{N}$, there is a partition P_n such that $U(f, P_n) - L(f, P_n) < \frac{1}{n}$. Since $L(f, P_n) \leq \int_a^b f \leq U(f, P_n)$, we then have

$$\lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n) = \int_a^b f.$$

Thus, if you can get hold of such a sequence of partitions, then you can (possibly) find out the integral taking a limit. However, it does not say how to find such a sequence.

[1.19] DEFINITION For a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a, b]$ the **mesh** of P is defined to be $\|P\| = \max\{x_i - x_{i-1} : 1 \leq i \leq n\}$, i.e., maximum length of the subintervals P produces.

[1.20] THEOREM **(Darboux condition)** *Let $f : [a, b] \to \mathbb{R}$ be bounded. Then $f \in \mathcal{R}([a, b])$ if and only if for every $\epsilon > 0$, there is $\delta > 0$ such that $U(f, P) - L(f, P) < \epsilon$ whenever $\|P\| < \delta$.*

Proof. Omitted.

[1.21] REMARK Suppose $f \in \mathcal{R}([a, b])$. Then, $\int_a^b f = \lim\limits_{n\to\infty} L(f, \mathbf{P}_n)$. Similarly, $\int_a^b f = \lim\limits_{n\to\infty} U(f, \mathbf{P}_n)$.

[1.22] EXERCISE Suppose you know that $\lim\limits_{n\to\infty} U(f, \mathbf{P}_n) = \ell$. Is it true that $f \in \mathcal{R}([a, b])$?
[Hint. Take the Dirichlet function on $[0, 1]$.]

Integration Class 2

[1.23] EXERCISE Suppose $f : [c, d] \to \mathbb{R}$ be bounded and $m = \text{glb}\{f(x) : x \in [c, d]\}$ and $M = \text{lub}\{f(x) : x \in [c, d]\}$. Show that $M - m = \text{lub}\{|f(x) - f(y)| : x, y \in [c, d]\}$.

[1.24] RESULT **(Algebra of integrals)** *Let $f, g \in \mathcal{R}([a, b])$, and $\alpha \in \mathbb{R}$. Then*

1. *$f + g \in \mathcal{R}([a, b])$ and $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.*
2. *$\alpha f \in \mathcal{R}([a, b])$ and $\int_a^b (\alpha f) = \alpha \int_a^b f$.*
3. *$|f| \in \mathcal{R}([a, b])$. (Converse?)*
4. *$f^2 \in \mathcal{R}([a, b])$.*

5. *$fg \in \mathcal{R}([a,b])$.*
6. *if $0 < m \le f \le M$, then $1/f \in \mathcal{R}([a,b])$.*
7. *$\max\{f,g\}, \min\{f,g\} \in \mathcal{R}([a,b])$.*
8. *If $a < c < b$, then $f \in \mathcal{R}([a,c]), f \in \mathcal{R}([c,b])$, and $\int_a^c f + \int_c^b f = \int_a^b f$.*

Proof.

1. Let $\epsilon > 0$. There are partitions P_1 and P_2 such that $U(f,P_1) - L(f,P_1) < \epsilon/2$ and $U(g,P_2) - L(g,P_2) < \epsilon/2$. Let $P = P_1 \cup P_2$. Then $U(f+g,P) - L(f+g,P) < \epsilon$, since
$$L(f,P) + L(g,P) \le L(f+g,P) \le U(f+g,P) \le U(f,P) + U(g,P).$$
Therefore $f+g \in \mathcal{R}([a,b])$. Now, note that $\int_a^b f + \int_a^b g$ and $\int_a^b (f+g)$ both lie in the interval $[L(f,P)+L(g,P), U(f,P)+U(g,P)]$ which is of length ϵ. Thus, $\left|(\int_a^b f + \int_a^b g) - \int_a^b (f+g)\right| < \epsilon$. Since ϵ is arbitrary, $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.
2. If $\alpha \ge 0$, then $\text{glb}\{\alpha f(x) : x \in [x_{i-1}, x_i]\} = \alpha m_i$, and so $L(\alpha f, P) = \alpha L(f,P)$, etc.
3. $U(|f|,P) - L(|f|,P) \le U(f,P) - L(f,P)$.
Converse is not true, e.g., $f(x) = 1$, if $x \in [0,1] \cap \mathbb{Q}$, and $f(x) = -1$, if $x \in [0,1] \cap \mathbb{Q}^c$.
4. There is $M > 0$ such that $|f(x)| \le M$ for $x \in [a,b]$. Then, for $x, y \in [a,b]$ we have $|f(x)^2 - f(y)^2| \le 2M|f(x) - f(y)|$. For a partition P,
$$\begin{aligned} U(f^2,P) - L(f^2,P) &= \sum_{i=1}^n (x_i - x_{i-1}) \,\text{lub}\{|f(x)^2 - f(y)^2| : x \in [x_{i-1}, x_i]\} \\ &\le 2M \sum_{i=1}^n (x_i - x_{i-1}) \,\text{lub}\{|f(x) - f(y)| : x \in [x_{i-1}, x_i]\} \\ &= 2M(U(f,P) - L(f,P)). \end{aligned}$$
5. Follows from $|(1/f)(x) - (1/f)(y)| \le \frac{1}{m^2}|f(x) - f(y)|$.
6. Follows from $fg = \frac{1}{2}\big((f+g)^2 - f^2 - g^2\big)$. Use the previous results.
7. $\max\{f,g\} = \frac{1}{2}\big(f + g + |f-g|\big)$, $\min\{f,g\} = \frac{1}{2}\big(f + g - |f-g|\big)$.
8. Let $\epsilon > 0$. Take Q such that $U(f,Q) - L(f,Q) < \epsilon$. Set $P = Q \cup \{c\}, P_1 = P \cap [a,c]$ and $P_2 = P \cap [c,b]$. Then $L(f,P) = L(f,P_1) + L(f,P_2)$ and $U(f,P_1) + U(f,P_2) = U(f,P)$. Thus,
$$\int_a^b f - \epsilon < L(f,P_1) + L(f,P_2) \le U(f,P_1) + U(f,P_2) < \int_a^b f + \epsilon.$$
Thus, $U(f,P_1) - L(f,P_1) < 2\epsilon$, yielding $f \in \mathcal{R}([a,c])$. Similarly, $f \in \mathcal{R}([c,b])$. Finally, observe that $\left|\big(\int_a^c f + \int_c^b f\big) - \int_a^b f\right| < \epsilon$.

[1.25] RESULT *Suppose $f : [a,b] \to \mathbb{R}$, $a < c < b$, $f \in \mathcal{R}([a,c])$ and $f \in \mathcal{R}([c,b])$. Then, $f \in \mathcal{R}([a,b])$ and $\int_a^b f = \int_a^c f + \int_c^b f$.*

Proof. Exercise.

[**1.26**] EXAMPLE We have now many functions integrable on $[a, b]$

x, any polynomial, $\sin x$ (as monotone in subintervals), $x \sin x$, etc.

[**1.27**] RESULT *Let* $f : [a, b] \to \mathbb{R}$.

1. *If* $f \geq 0$ *and* $f \in \mathcal{R}([a, b])$, *then* $\int_a^b f \geq 0$.
2. *If* $f, g \in \mathcal{R}([a, b])$ *and* $f \leq g$, *then* $\int_a^b f \leq \int_a^b g$.
3. *If* $f \in \mathcal{R}([a, b])$, *then* $|\int_a^b f| \leq \int_a^b |f|$.

Proof. (1) Follows from the fact that $L(f, P) \geq 0$ for every partition p of $[a, b]$, since $f \geq 0$.

(2) $f \leq g$ implies $\int_a^b g - \int_a^b f = \int_a^b (g - f) \geq 0$, by (1).

(3) Note that $f \in \mathcal{R}([a, b])$ implies $|f| \in \mathcal{R}([a, b])$. [(3) of [**1.24**]]. Now, $-|f| \leq f \leq |f|$, and therefore by (2),

$$-\int_a^b |f| = \int_a^b -|f| \leq \int_a^b f \leq \int_a^b |f|, \quad \text{i.e. } |\int_a^b f| \leq \int_a^b |f|. \qquad \blacksquare$$

[**1.28**] DEFINITION Let $S \subseteq \mathbb{R}$. A function $f : S \to \mathbb{R}$ is **uniformly continuous** (on S), if given $\epsilon > 0$, there is $\delta > 0$ such that $x, y \in S, |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$.

[**1.29**] RESULT *(1) If* $f : S \to \mathbb{R}$ *is uniformly continuous, then* f *is continuous.*
(2) A continuous function f *on a closed interval is uniformly continuous.*

Proof. (1) Follows from the definition.
(2) Suppose f is continuous, but not uniformly continuous on $[a, b]$. Then, there is $\epsilon > 0$ such that for each $n \in \mathbb{N}$, there are $x_n, y_n \in [a, b]$ such that $|x_n - y_n| < 1/n$ and $|f(x_n) - f(y_n)| \geq \epsilon$. Since (x_n) is bounded, by BWT, (x_n) has convergent subsequence (x_{n_k}), converging to c, say. Then, $c \in [a, b]$. Further,

$$|y_{n_k} - c| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - c| < \frac{1}{n_k} + |x_{n_k} - c| \to 0,$$

that is, $y_n \to c$. Since f is continuous at c, we have $|f(x_{n_k}) - f(y_{n_k})| \to |c - c| = 0$. However, this cannot happen because $|f(x_{n_k}) - f(y_{n_k})| \geq \epsilon$ for every k. Hence, f must be uniformly continuous. $\blacksquare$

[**1.30**] THEOREM *If* f *is continuous on* $[a, b]$, *then* $f \in \mathcal{R}([a, b])$.

Proof. Let $\epsilon > 0$. Then there is $n \in \mathbb{N}$ such that

$$x, y \in [a, b], |x - y| < \frac{1}{n} \implies |f(x) - f(y)| < \frac{\epsilon}{b - a}.$$

Consider the partition $\mathbf{P}_n$ of $[a, b]$. For $1 \leq i \leq n$, we have

$$M_i - m_i = \text{lub}\{|f(x) - f(y)| : x, y \in [x_{i-1}, x_i]\} \leq \frac{\epsilon}{b - a}.$$

Consequently,

$$U(f, \mathbf{P}_n) - L(f, \mathbf{P}_n) = \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) \leq \left(\frac{\epsilon}{b - a}\right) \sum_{i=1}^{n} (x_i - x_{i-1}) = \epsilon. \qquad \blacksquare$$

[**1.31**] RESULT *If $f : [a,b] \to \mathbb{R}$ is bounded and continuous on $[a,b)$ (or on $(a,b]$), then $f \in \mathcal{R}([a,b])$.*

Proof. Assume $|f| < M$, and f is continuous on $[a,b)$. Let $\epsilon > 0$. Write $[a,b] = I_1 \cup I_2$, where

$$I_1 = \left[a, b - \frac{\epsilon}{4M}\right], \quad I_2 = \left[b - \frac{\epsilon}{4M}, b\right].$$

Since f is continuous on I_1, we have $f \in \mathcal{R}(I_1)$. So, there is a partition P_1 of I_1 such that $U(f,P_1) - L(f,P_1) < \epsilon/2$. Moreover, $\text{lub}\{|f(x) - f(y)| : x, y \in I_2\} \le 2M$. Now, $P = P_1 \cup \{b\}$ is a partition of $[a,b]$ and

$$U(f,P) - L(f,P) = (U(f,P_1) - L(f,P_1)) + 2M \cdot \frac{\epsilon}{4M} < \epsilon.$$

Therefore, $f \in \mathcal{R}([a,b])$. Similarly, when f is continuous on $(a,b]$. ∎

`Q:` Suppose $f : [a,b] \to \mathbb{R}$ is bounded and $C = \{x \in [a,b] : f \text{ is discontinuous at } x\}$.

1. Is $f \in \mathcal{R}([a,b])$, if $C = \{c\}$ where $c \in (a,b)$? `A:` Yes. Use [**1.31**] and [**1.25**].
2. Is $f \in \mathcal{R}([a,b])$, if C is finite? `A:` Yes. Use part (1) and [**1.25**].
3. Is $f \in \mathcal{R}([a,b])$, if $C = \{c_n : n \in \mathbb{N}\}$ where $c_n \to c \in [a,b]$? `A:` Yes. Use the idea of the proof of [**1.31**] and part (2).
4. Is $f \in \mathcal{R}([a,b])$, if C infinite? `A:` No. Take Dirichlet function.
5. Is $f \in \mathcal{R}([a,b])$, if C infinite having finitely many limit points? `A:` Yes. Use the idea of the proof of [**1.31**] and part (2).

[**1.32**] EXAMPLE

1. Let $f(x) = \sin\frac{1}{x}$ if $x \neq 0$, and $f(0) = 1$. Then, $f \in \mathcal{R}([0,1])$.
2. Let $f(x) = 0$ if $x \in (0,1]$, and $f(0) = c$. Then, $f \in \mathcal{R}([0,1])$. Further, $\int_0^1 f = \lim L(f, \mathbf{P}_n) = 0$.

[**1.33**] COROLLARY *Let $c_1, \ldots, c_k \in [a,b]$, and $f : [a,b] \to \mathbb{R}$ be such that $f(x) = 0$ for $x \notin \{c_1, \ldots, c_n\}$. Then $f \in \mathcal{R}([a,b])$ and $\int_a^b f = 0$.*

Proof. Exercise.

[**1.34**] RESULT *Let $f \in \mathcal{R}([a,b])$ and $g : [a,b] \to \mathbb{R}$ be such $g(x) \neq f(x)$ for only finitely many points $x \in [a,b]$. Then $g \in \mathcal{R}([a,b])$ and $\int_a^b g = \int_a^b f$.*

Proof. Apply [**1.33**] to $f - g$.

`Q:` Can you improve the above result?

[1.35] EXAMPLE The Thomae's function is integrable: $f : [0,1] \to \mathbb{R}$ where

$$f(x) = \begin{cases} 1/q, & \text{if } x = p/q \in \mathbb{Q}, \gcd(p,q) = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Let $\epsilon > 0$. Since for any partition P, we have $L(f, P) = 0$, it is enough to find a partition P such that $U(f, P) < \epsilon$. Now, there are only finitely many points $0, c_1, \ldots, c_k, 1$ (in increasing order) in $[0,1]$ where f takes value $> \epsilon/2$. Choose $\delta < \frac{\epsilon}{4(k+1)}$ so that we get a partition

$$P = \{0 < \delta < c_1 - \delta < c_1 + \delta < \cdots < 1 - \delta < 1\}$$

of $[a,b]$. Then the contribution of $[0,\delta]$ and $[1-\delta, 1]$ to $U(f,P)$ is $\delta + \delta = 2\delta$. The total contribution of the intervals $[c_i - \delta, c_i + \delta]$ is $\leq k \cdot 2\delta$. The contribution of the rest of the intervals is less than $\epsilon/2$, since the total length of these intervals is less than 1 ans $f(x) \leq \epsilon/2$ for x in these intervals. Hence,

$$U(f,P) < 2\delta + 2k\delta + \epsilon/2 = 2(k+1)\delta + \epsilon/2 < \epsilon.$$

This shows, $f \in \mathcal{R}([0,1]$. Further, $\int_a^b f = \lim L(f, \mathbf{P}_n) = 0$.

[1.36] EXAMPLE Composition of integrable functions need not be integrable. Take f as Thomae's function on $[0,1]$ and $g : [0,1] \to \mathbb{R}$ defined by $g(0) = 0$ and $g(x) = 1$, elsewhere. Then $g \circ f$ is the Dirichlet function!

Integration Class 3

[1.37] THEOREM **(Mean Value Theorem for Integrals)** *Suppose $f \in \mathcal{R}([a,b])$, and $m = \operatorname{glb} f$, $M = \operatorname{lub} f$ on $[a,b]$. Then there exists $\alpha \in [m, M]$ such that $\int_a^b f = \alpha(b-a)$. If f is continuous, then there is $c \in [a,b]$ such that $\int_a^b f = f(c)(b-a)$.*

Proof. Follows from $m(b-a) \leq \int_a^b \leq M(b-a)$, and IVT, if f is continuous. ∎

[1.38] THEOREM **(First Fundamental Theorem of Calculus)** *Let $f \in \mathcal{R}([a,b])$ and $F(x) = \int_a^x f$ for $x \in [a,b]$. Then, F is continuous on $[a,b]$. Further, if f is continuous at $c \in [a,b]$, then F is differentiable at c and $F'(c) = f(c)$.*

Proof. Choose M such that $|f| \leq M$. Then for $a \leq x < y \leq b$ we have

$$|F(y) - f(x)| = \left| \int_x^y f \right| \leq \int_x^y |f| \leq M(y-x).$$

Thus, F is (Lipschitz) continuous.

Next, suppose f is continuous at $c \in [a,b]$. Let $\epsilon > 0$. There exists $\delta > 0$ such that

$$|f(x) - f(c)| < \epsilon \text{ for } x \in (c-\delta, c+\delta) \cap [a,b].$$

Now, for $x \in (c, c+\delta) \cap [a,b]$, we have

$$\begin{aligned} \left| \frac{F(x) - F(c)}{x-c} - f(c) \right| &= \left| \frac{F(x) - F(c)}{x-c} - \frac{f(c)(x-c)}{x-c} \right| = \left| \frac{\int_c^x f - f(c) \int_c^x 1}{x-c} \right| \\ &\leq \frac{\int_c^x |f - f(c)|}{x-c} \leq \frac{\int_c^x \epsilon}{x-c} = \epsilon. \end{aligned}$$

This shows, $F'_+(c) = f(c)$, if $c < b$. Similarly, $F'_-(c) = f(c)$. ∎

[1.39] DEFINITION If $f \in \mathcal{R}([a,b])$, then we define $\int_b^a f$ to be $-\int_a^b f$. This also means $\int_c^c f = 0$.

[1.40] REMARK If f is continuous at x, then $\lim\limits_{h\to 0} \dfrac{1}{h}\displaystyle\int_x^{x+h} f(t)dt = f(x)$.

[1.41] RESULT **(Leibniz Rule)** *Let f be continuous on $[a,b]$, and $u, v : [c,d] \to \mathbb{R}$ be differentiable. Then*

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt = f(v(x))v'(x) - f(u(x))u'(x).$$

Proof. Put $F(x) = \int_a^x f$. Then, $\int_{u(x)}^{v(x)} f(t)dt = F(v(x)) - F(u(x))$. Now, differentiate both sides and use chain rule. ■

[1.42] THEOREM **(Second Fundamental Theorem of Calculus)** *Suppose $f \in \mathcal{R}([a,b])$ and $F : [a,b] \to \mathbb{R}$ be such that $F'(x) = f(x)$ for $x \in [a,b]$. Then $\int_a^b f = F(b) - F(a) = F\big|_a^b$.*

Proof. Consider the partition $\mathbf{P}_n = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a,b]$. Then, by MVT, $F(x_i) - F(x_{i-1}) = f(t_i)(x_i - x_{i-1})$ for some $t_i \in [x_{i-1}, x_i]$, $1 \le i \le n$. Thus,

$$F(b) - F(a) = \sum_{i=1}^{n}(F(x_i) - F(x_{i-1}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}),$$

that is, $L(f,\mathbf{P}_n) \le F(b) - F(a) \le U(f,\mathbf{P}_n)$. Since $L(f,\mathbf{P}_n) \to \int_a^b f$, $U(f,\mathbf{P}_n) \to \int_a^b f$, we must have $\int_a^b f = F(b) - F(a)$. ■

[1.43] COROLLARY **(Integration by parts)** *Let F and G be differentiable with their derivatives f and g are integrable on $[a,b]$. Then, $\int_a^b Fg = (FG)\big|_a^b - \int_a^b fG$.*

Proof. Since $(FG)' = fG + Fg \in \mathcal{R}([a,b])$, FTC-2 gives $\int_a^b fG + \int_a^b Fg = (FG)\big|_a^b$. ■

[1.44] THEOREM **(Substitution rule)** *Let $f : [m,M] \to \mathbb{R}$ be continuous, and $\phi : [a,b] \to [m,M]$, and ϕ' is continuous. Then $\displaystyle\int_a^b (f\circ\phi)\phi' = \int_{\phi(a)}^{\phi(b)} f$.*

Proof. For $x \in [m,M]$, let $F(x) = \int_{\phi(a)}^x f$. Then $F' = f$, by FTC-1. So, by chain rule, $(F\circ\phi)' = (f\circ\phi)\phi' \in \mathcal{R}([a,b])$, as f and ϕ' are continuous. Hence, by FTC-2,

$$\int_a^b (f\circ\phi)\phi' = (F\circ\phi)(b) - (F\circ\phi)(a) = (F\circ\phi)(b) = \int_{\phi(a)}^{\phi(b)} f. \quad ■$$

[1.45] THEOREM **(Weighted Mean Value Theorem for Integrals)** *Let f, g be continuous on $[a,b]$. Assume that g does not change sign on $[a,b]$. Then for some $c \in [a,b]$ we have $\int_a^b fg = f(c)\int_a^b g$.*

Proof. Assume $g \ge 0$. Let $m = \operatorname{glb} f$, $M = \operatorname{lub} f$ on $[a,b]$. Then $mg(x) \le f(x)g(x) \le Mg(x)$ for $x \in [a,b]$, and therefore

$$m\int_a^b g \le \int_a^b fg \le M\int_a^b g.$$

Since f is continuous, there is $c \in [a,b]$ such that $\int_a^b fg = f(c)\int_a^b g$. ■

[**1.46**] THEOREM **(Second Mean Value Theorem for Integrals)** *Let g be continuous on $[a,b]$ and f be continuously differentiable on $[a,b]$. Suppose that f' does not change sign on $[a,b]$. Then there exists $c \in [a,b]$ such that*

$$\int_a^b fg = f(a)\int_a^c g + f(b)\int_c^b g.$$

Proof. Put $G(x) = \int_a^x g$. Since g is continuous, $G'(x) = g(x)$. Integration by parts gives

$$\int_a^b fg = \int_a^b fG' = fG\Big|_a^b - \int_a^b f'G = f(b)G(b) - \int_a^b f'G.$$

Since f', G are continuous, and f' does not change sign on $[a,b]$, by the Weighted Mean Value Theorem, for some $c \in [a,b]$

$$\int_a^b f'G = G(c)\int_a^b f' = G(c)\big[f(b) - f(a)\big].$$

Thus,

$$\begin{aligned}\int_a^b fg &= f(b)G(b) - \int_a^b f'G = f(b)G(b) - G(c)\big[f(b) - f(a)\big]\\ &= f(a)G(c) + f(b)\big[G(b) - G(c)\big] = f(a)\int_a^c g + f(b)\int_c^b g.\end{aligned}$$ ■

[**1.47**] THEOREM **(Term by term integration)** *Suppose $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ on $(-R,R)$. Then, for $x \in (-R,R)$, $\displaystyle\int_0^x f = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$.*

Proof. For $x \in (-R,R)$, $\sum a_n x^n$ is absolutely convergent, and therefore $\sum \frac{a_n}{n+1} x^{n+1}$ is absolutely convergent, by comparison. Let $g(x) = \sum \frac{a_n}{n+1} x^{n+1}$ on $(-R,R)$. Then, g is differentiable and $g'(x) = \sum a_n x^n = f(x)$ (term by term differentiation). By FTC-2, we have

$$\int_0^x f = g(x) - g(0) = \sum \frac{a_n}{n+1} x^{n+1}.$$ ■

[**1.48**] EXAMPLE **(Old friend revisiting)** As $\dfrac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$ on $(-1,1)$, we get

$$\ln(1+x) = \int_0^x \frac{1}{1+x} = \sum \int_0^x (-1)^{n-1} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad -1 < x < 1.$$

Integration Class 4

2 Improper integrals

We now define and discuss integrals when either the function or the interval is unbounded. These integrals are called **Improper integrals**.

[2.1] DEFINITION **(Integral of unbounded functions over bounded intervals)**

- Let $f : (a, b] \to \mathbb{R}$ and $f(a+) = \pm\infty$. Suppose $f \in \mathcal{R}([t, b])$ for every $t \in (a, b)$. Then, we define $\int_a^b f = \lim\limits_{t \to a+} \int_t^b f$, if it exists.
- If $f : [a, b) \to \mathbb{R}$ and $f(b-) = \pm\infty$, and $f \in \mathcal{R}([a, t])$ for every $t \in (a, b)$, then, we define $\int_a^b f = \lim\limits_{t \to b-} \int_a^t f$, if it exists.
- If $f : [a, c) \cup (c, b]$ and $\lim\limits_{t \to c} = \pm\infty$, then we define $\int_a^b f := \int_a^c f + \int_c^b f$, if the later two integrals exist.

[2.2] EXAMPLE Consider $f(x) = 1/x^2$ on $(0, 1]$. Then $\lim\limits_{t \to 0+} \displaystyle\int_t^1 f = \lim\limits_{t \to 0+} (-1/x)\Big|_t^1 = \infty$.

[2.3] EXAMPLE Consider $f(x) = 1/\sqrt{x}$ on $(0, 1]$. Then $\lim\limits_{t \to 0+} \displaystyle\int_t^1 f = \lim\limits_{t \to 0+} (2\sqrt{x})\Big|_t^1 = 2$.

[2.4] DEFINITION **(Integral of unbounded intervals)**

- Suppose that for every $t \in (a, \infty)$, $\int_a^t f$ exists (that is, either $f \in \mathcal{R}([a, t])$ or the improper integral exists). We define $\int_a^\infty f = \lim\limits_{t \to \infty} \int_a^t f$, if the limit exists.
- If $\int_t^a f$ exists for every $t \in (-\infty, a)$, then we define $\int_{-\infty}^a f = \lim\limits_{t \to -\infty} \int_t^a f$, if the limit exists.
- We define $\int_{-\infty}^\infty f = \int_{-\infty}^0 f + \int_0^\infty f$, if the integrals on the right exist and finite. (Here, 0 can be replaced by any real number a.)

[2.5] EXAMPLE Consider $f(x) = 1/x^2$ on $[1, \infty)$. Then $\lim\limits_{t \to \infty} \displaystyle\int_1^t f = \lim\limits_{t \to \infty} (-1/x)\Big|_1^t = 1$.

[2.6] EXAMPLE Consider $f(x) = 1/\sqrt{x}$ on $[1, \infty)$. Then $\lim\limits_{t \to \infty} \displaystyle\int_t^1 f = \lim\limits_{t \to \infty} (2\sqrt{x})\Big|_1^t = \infty$.

[2.7] DEFINITION If the improper integral $\int_a^b f$ exists and finite, then it is said to **converge** to the (finite) value. In other cases, it is said to **diverge**.

[2.8] EXAMPLE Since $\int_t^1 \frac{1}{x} dx = \ln t$ and $\lim\limits_{t \to 0+} \ln t$ is not finite, $\int_0^1 \frac{1}{x}$ is not convergent. Similarly, $\int_1^\infty \frac{1}{x} dx$ is divergent.

[**2.9**] EXAMPLE The improper integral $\int_0^1 \frac{1}{\sqrt{x}}dx$ converges to 2.

[**2.10**] EXERCISE Suppose $p > 0$. Show that $\int_0^1 \frac{1}{x^p}dx$ converges if and only if $p < 1$.

[**2.11**] EXERCISE Examine for convergence: $\displaystyle\int_0^1 \frac{dx}{\sqrt{x-x^2}}$.

[**2.12**] EXAMPLE Consider $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{1}{\sqrt{|x-1|}}, & \text{if } x \neq 1, \\ 0, & \text{if } x = 1. \end{cases}$$

Then,

$$\int_0^5 f = \int_0^1 f + \int_1^5 f = \lim_{t\to 1-} \int_0^t \frac{dx}{\sqrt{1-x}} + \lim_{t\to 1+} \int_t^5 \frac{dx}{\sqrt{x-1}} = 2 + 4 = 6,$$

that is, $\int_0^5 f$ is convergent.
What about $\int_{-\infty}^{\infty} f$? No, since $\lim\limits_{t\to\infty} \int_5^t f = 2 \lim\limits_{t\to\infty} (\sqrt{t-1} - 2) = \infty$.

[**2.13**] EXAMPLE For $p > 0$, $\displaystyle\int_1^\infty \frac{1}{x^p}dx$ is convergent if and only if $p > 1$.
For $p \neq 1$, we have

$$\int_1^t \frac{1}{x^p}dx = \frac{1}{1-p}\left[t^{1-p} - 1\right] = \begin{cases} \frac{1}{p-1}, & \text{if } p > 1, \\ \infty, & \text{if } p < 1. \end{cases}$$

Moreover, $\lim\limits_{t\to\infty} \int_1^t \frac{1}{x}dx = \lim\limits_{t\to\infty} \ln t = \infty$.

[**2.14**] REMARK The improper integral $\int_{-\infty}^{\infty} f$ may diverge even if $\lim\limits_{t\to\infty} \int_{-t}^t f$ exists. For example, $\int_{-t}^t xdx = 0$, yielding $\lim\limits_{t\to\infty} \int_{-t}^t f = 0$, but $\int_0^\infty xdx = \infty$, and therefore $\int_{-\infty}^\infty xdx$ diverges.

[**2.15**] THEOREM **(Cauchy criterion)** *The improper integral $\int_a^\infty f$ converges if and only if for each $\epsilon > 0$, there exists $\alpha > a$ such that $\int_a^\alpha f$ is convergent and $|\int_s^t f| < \epsilon$ for all $t > s \geq \alpha$.*

[**2.16**] THEOREM **(Integral test)** *Suppose $f : [1,\infty) \to \mathbb{R}$ is positive and decreasing. Then $\int_1^\infty f$ converges if and only if $\sum\limits_{n=1}^{\infty} f(n)$ converges.*

Proof. Note that $P = \{n < n+1 < \cdots < m-1 < m\}$ is a partition of $[n, m]$. Therefore

$$\sum_{k=n}^{m-1} f(k+1) = L(f,P) \leq \int_n^m f \leq U(f,P) = \sum_{k=n}^{m-1} f(k).$$

Now use Cauchy criterion for convergence of series and [**2.15**]. ∎

[**2.17**] THEOREM **(Comparison test)** *Let $0 \leq f \leq g$ on $[a,\infty)$ and $\int_a^t f$ exists for each $t > a$. If $\int_a^\infty g$ is convergent, then $\int_a^\infty f$ is convergent.*

[2.18] EXAMPLE For $t > 1$, we have $0 < t^2 < e^{t^2}$, and so, $0 < e^{-t^2} < \frac{1}{t^2}$. Since e^{-x^2} is continuous on $[1, \infty)$ and $\int_1^\infty \frac{dx}{x^2}$ is convergent, $\int_1^\infty e^{-x^2} dx$ is convergent.

[2.19] EXERCISE Test convergence of (a) $\int_{-\infty}^\infty e^{-t}$, (b) $\int_0^\infty \frac{dx}{1+x^2}$, (c) $\int_{-\infty}^\infty te^{-t^2}$,

[2.20] THEOREM **(Limit comparison test)** *Let $f \geq 0, g > 0$ on $[a, \infty)$, and $\int_a^x f, \int_a^x g$ exist for $x > a$. Suppose $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = \ell \in \mathbb{R}$.*

1. *If $\ell > 0$, then $\int_a^\infty f$ and $\int_a^\infty g$ converge or diverge together.*
2. *If $\ell = 0$ and $\int_a^\infty g$ converges, then $\int_a^\infty f$ converges.*

[2.21] EXERCISE Examine for convergence.

(a) $\displaystyle\int_1^\infty \frac{\sin^2 t}{t^2} dt$ (b) $\displaystyle\int_1^\infty \frac{dt}{t\sqrt{1+t^2}} dt$ (c) $\displaystyle\int_1^\infty e^{-t} t^p dt.$

[2.22] RESULT **(Absolute convergence)** *If $\int_a^\infty |f|$ converges, and $\int_a^x f$ exists for $x > a$, then $\int_a^\infty f$ converges.*

Proof. Use Cauchy criterion.

[2.23] EXAMPLE By **[2.22]**, $\displaystyle\int_1^\infty \frac{\cos t}{1+t^2} dt$ converges.

[2.24] THEOREM **(Dirichlet test[I])** *Suppose g is continuous and f is monotonic and continuously differentiable on $[a, \infty)$, and $\lim\limits_{t\to\infty} f(t) = 0$. Suppose there is $M \in \mathbb{R}$ such that $\left|\int_a^x g\right| \leq M$ for $x \in (a, \infty)$. Then $\int_a^\infty fg$ is convergent.*

Proof. Let $a \leq s < t$. Then there is $c \in [s, t]$ such that

$$\left|\int_s^t fg\right| = \left|f(s)\int_s^c g + f(t)\int_c^t g\right| \leq f(s)\left|\int_s^c g\right| + f(t)\left|\int_c^t g\right| \leq 2M(f(s) + f(t)),$$

since $\left|\int_s^c g\right| = \left|\int_a^c g - \int_a^s g\right| \leq \left|\int_a^c g\right| + \left|\int_a^s g\right| = 2M$. As $\lim\limits_{t\to\infty} f(t) = 0$, there is $\alpha \geq a$ such that $f(x) < \epsilon/(4M)$ for every $x \geq \alpha$. Then for $\alpha \leq s < t$, we have $\left|\int_s^t fg\right| < \epsilon$. Now use Cauchy criterion. ∎

[2.25] EXERCISE Examine for convergence.

(a) $\displaystyle\int_1^\infty \frac{\sin t}{t} dt$ (b) $\displaystyle\int_1^\infty \frac{dt}{t\sqrt{1+t^2}} dt$ (c) $\displaystyle\int_1^\infty \frac{\sin(x^2)}{\sqrt{x}} dx.$

[2.26] EXERCISE Determine all real numbers p for which the integral $\displaystyle\int_0^\infty \frac{e^{-x} - 1}{x^p} dx$ is convergent.

[I]Dirichlet test works with much weaker conditions: *Let f be bounded and monotonic in $[a, \infty)$ and $\lim\limits_{t\to\infty} f(t) = 0$. Suppose there is $M \in \mathbb{R}$ such that $\left|\int_a^x g\right| \leq M$ for $x \in (a, \infty)$. Then $\int_a^\infty fg$ is convergent.* This follows from the following version of the **Second mean value theorem for integration (Dixon, 1929)**: *If g is integrable on $[a, b]$ and f monotonic on $[a, b]$, then there is $c \in [a, b]$ such that $\int_a^b fg = f(a)\int_a^c g + f(b)\int_c^b g$.*

Integration Class 5

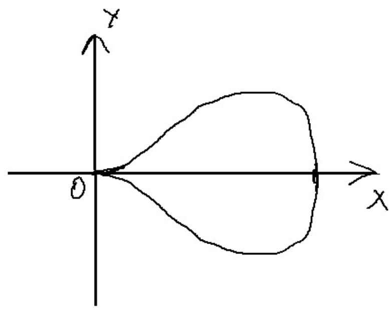
3 Applications

3.1 Area bounded by curves

[**3.1**] DEFINITION

1. For continuous functions $f, g : [a, b] \to \mathbb{R}$, the **area between** $y = f(x)$ and $y = g(x)$ from a to b is defined to be $\int_a^b |f - g|$.
2. (Polar coordinates) For a nonnegative continuous function $f : [\alpha, \beta] \to \mathbb{R}$, we define the area bounded by $r = f(\theta)$ and the lines $\theta = \alpha$ and $\theta = \beta$ to be $\frac{1}{2}\int_\alpha^\beta (f(\theta))^2 d\theta$.[I]

[**3.2**] EXAMPLE Find the ratio between the area of the region bounded by the curve $a^4y^2 = x^5(2a - x)$ and the area inside the circle whose radius is a.
The first curve has the graph as give below. It meets the x-axis at $x = 0$ and $x = 2a$.

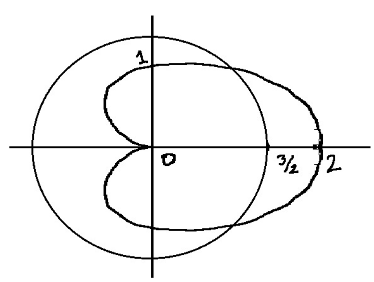


The area of the region bounded by the first curve is (putting $x = 2a\sin^2\theta$)

$$\begin{aligned} A &= \frac{2}{a^2}\int_0^{2a} x^{5/2}\sqrt{2a - x}dx = \frac{2}{a^2}\int_0^{\pi/2} (2a)^{5/2}\sin^5\theta \cdot \sqrt{2a}\cos\theta \cdot 4a\sin\theta d\theta \\ &= 64a^2\int_0^{\pi/2} \sin^6\theta\cos^2\theta d\theta = 64a^2\,\frac{5\cdot 3\cdot 1\cdot 1}{8\cdot 6\cdot 4\cdot 2}\,\frac{\pi}{2} = \frac{5a^2\pi}{4}. \end{aligned}$$

The required ratio is 5/4.

[**3.3**] EXERCISE Find the area of the region inside the cardioid $r = 1 + \cos\theta$ and also inside the circle $r = \frac{3}{2}$.
Note that the curves meet at $\theta = \pm\pi/3$.



[I]The rational behind this is the following: Suppose $\{\theta_0 < \theta_1 < \cdots < \theta_n\}$ is a partition of $[\alpha, \beta]$. The sectoral area bounded by $\theta = \theta_{i-1}$, $\theta = \theta_i$ and $r = f(\theta)$ is approximately $\frac{1}{2}(\theta_i - \theta_{i-1})(f(\theta_i))^2$ (half of the area of a rectangle with length $f(\theta_i)$ and height $f(\theta_i)(\theta_i - \theta_{i-1})$.) Therefore, area is given by $\frac{1}{2}\int_\alpha^\beta (f(\theta))^2 d\theta$.

The area bounded by the cardioid $A_1 = \int_0^{\pi} (1+\cos\theta)^2 d\theta =$ ______

The area between the two curves $A_2 = \int_0^{\pi/3} \left((1+\cos\theta)^2 - (3/2)^2\right) d\theta =$ ______

So, the required area $= A_1 - A_2 =$ ______.

3.2 Length of smooth curves

[3.4] DEFINITION

1. Let $y = f(x)$, where $f : [a,b] \to \mathbb{R}$ is such that f' is continuous. Then, the **length of the curve** is defined to be $L = \int_a^b \sqrt{1+(f'(x))^2}dx$.
2. **(Parametric form)** Let $x = \phi(t)$, $y = \psi(t)$, where $\phi, \psi : [a,b] \to \mathbb{R}$ are such that ϕ' and ψ' are continuous. Then, the **length of the curve** is defined to be
$$L = \int_a^b \sqrt{(\phi'(t))^2 + (\psi'(t))^2}dt.$$
3. **(Polar form)** Let $r = f(\theta)$, where $f : [\alpha, \beta] \to \mathbb{R}$ is such that f' is continuous. Then, the **length of the curve** is defined to be $L = \int_\alpha^\beta \sqrt{r^2 + (f'(\theta))^2}d\theta$.

[3.5] REMARK Suppose $\{a_0 < a_1 < \cdots < a_n\}$ is a partition of $[a,b]$. The length of the chord joining $(a_{i-1}, f(a_{i-1})$ and $(a_i, f(a_i))$ is

$$\sqrt{(a_i - a_{i-1})^2 + (f(a_i) - f(a_{i-1})^2} = (a_i - a_{i-1})\sqrt{1 + \left(\frac{f(a_i) - f(a_{i-1})}{a_i - a_{i-1}}\right)^2} = (a_i - a_{i-1})\sqrt{1+(f'(t_i))^2}$$

for some $t_i \in (x_{i-1}, x_i)$ (by MVT). Thus, the sum S of the lengths of the chords satisfies

$$L(g,P) \le S \le U(g,P)$$

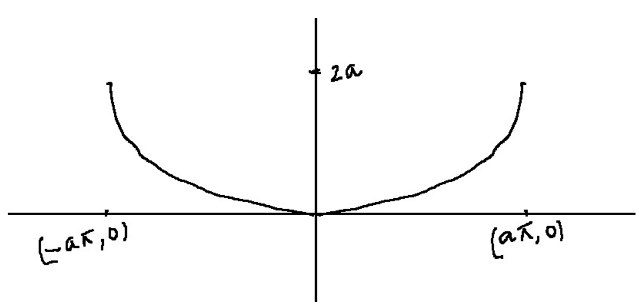
where $g(x) = \sqrt{1+(f'(x))^2}$. This motivated to define the length of the curve as $L = \int_a^b \sqrt{1+(f'(x))^2}dx$.

Similarly, the definitions in parametric and polar forms are motivated.

[3.6] EXAMPLE

1. The length of $y = x^2$ on $[0,2]$ is given by $L = \int_0^2 \sqrt{1+4x^2}dx =$ ______.
2. The length of the arc of the parabola $y^2 = 4ax$ cut off by its latus rectum is
$$L = 2\int_0^a \sqrt{1+(f'(x))^2}dx = 2\int_0^a \left(\sqrt{1+\frac{a}{x}}\right)dx = \cdots = 2a(\sqrt{2} + \ln(1+\sqrt{2})).$$

[3.7] EXAMPLE Find the length of the following curve given by $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta), -\pi \le \theta \le \pi$.



We have

$$(x'(\theta))^2 + (y'(\theta))^2 = a^2\big((1+\cos\theta)^2 + \sin^2\theta\big) = 2a^2(1+\cos\theta) = 4a^2\cos^2(\theta/2).$$

Therefore

$$L = 2\int_0^\pi \sqrt{(x'(\theta))^2 + (y'(\theta))^2}d\theta = 4a\int_0^\pi \cos(\theta/2) = 8a.$$

[**3.8**] EXAMPLE Find the length of the cardioid $r = a(1+\cos\theta)$.
We have

$$r^2 + (f'(\theta))^2 = a^2\big((1+\cos\theta)^2 + \sin^2\theta\big) = 2a^2\cos^2(\theta/2).$$

$$L = 2\int_0^\pi \sqrt{r^2 + (f'(\theta))^2}d\theta = 4a\int_0^\pi \cos(\theta/2) = 8a.$$

[**3.9**] EXERCISE Find the lengths of the following curves.
(i) $y = \int_0^x \sqrt{\cos 2t}\, dt,\ 0 \le x \le \pi/4$.
(ii) $x = e^t\cos t, y = e^t \sin t$, where $0 \le t \le \pi/2$.
(iii) the cardioid $r = 1 - \cos\theta$.

3.3 Volumes given by integrals

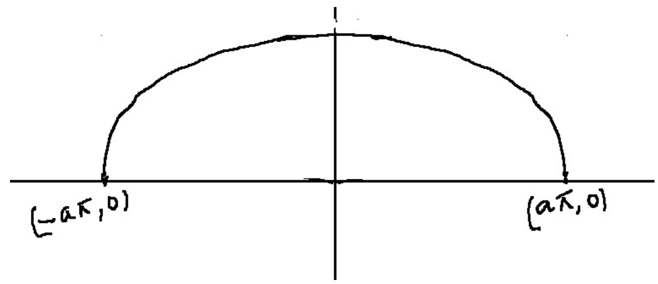
[**3.10**] DEFINITION Suppose a solid lies between planes perpendicular to the x-axis at $x = a$ and $x = b$. Suppose the cross sectional area perpendicular to the x-axis for $a \le x \le b$ be $A(x)$ and $A : [a, b] \to \mathbb{R}$ is continuous. Then the **volume of the solid by slicing** is defined to be $V = \int_a^b A(x)dx$.

[**3.11**] DEFINITION (1) Suppose $f : [a, b] \to \mathbb{R}$ be continuous. Then the **volume of the solid of revolution** of the curve $y = f(x)$ on $[a, b]$ is defined to be $V = \pi\int_a^b (f(x))^2dx$. (Note that $A(x) = \pi(f(x))^2$.)
(2) The **volume of washer** given by revolution of $f, g : [a, b] \to \mathbb{R}$ $(0 \le f \le g)$ is defined to be $V = \pi\int_a^b \Big((f(x))^2 - (g(x))^2\Big)dx$.

[**3.12**] EXAMPLE Find the volume of the solid obtained by revolving the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ about the x-axis.
The required volume is

$$V = 2\pi\int_0^a y^2dx = \frac{2\pi b^2}{a^2}\int_0^a (a^2 - x^2)dx = \frac{4\pi ab^2}{3}.$$

[**3.13**] EXAMPLE Find the volume of the solid obtained by revolving the cycloid $x = a(\theta + \sin\theta),\ y = a(1+\cos\theta), -\pi \le \theta \le \pi$.



The volume is given by

$$V = 2\pi \int_0^{a\pi} y^2 dx = 2\pi \int_0^{\pi} a^2(1+\cos\theta)^2 a(1+\cos\theta)d\theta = 2\pi a^3 \int_0^{\pi} 8\cos^6(\theta/2)d\theta = 5\pi^2 a^3.$$

[**3.14**] EXAMPLE Consider the solid of revolution of $y^2 = 4x$ about the x-axis. The volume of the solid bounded by $x = 0$ and $x = 5$ is $\pi \int_0^5 4x dx$.

[**3.15**] EXERCISE Find the volume of a sphere of radius r.

[**3.16**] EXERCISE A round hole of radius $\sqrt{3}$ is bored through the centre of a solid sphere of radius 2. Find the volume of the portion bored out.
[The volume of the portion is the sum of volume of a cylinder (with length 2 and radius $\sqrt{3}$ and $2\pi \int_1^2 (4-x^2)dx$.]

3.4 Area of surface of revolution

[**3.17**] DEFINITION Suppose $f : [a,b] \to \mathbb{R}$ be continuous. Then the **area of the surface of revolution** of the curve $y = f(x)$ on $[a,b]$ is defined to be $S = 2\pi \int_a^b f(x)\sqrt{1+(f'(x))^2}dx$.

[**3.18**] REMARK Suppose $\{a_0 < a_1 < \cdots < a_n\}$ is a partition of $[a,b]$. The length of the chord joining $(a_{i-1}, f(a_{i-1}))$ and $(a_i, f(a_i))$ is $\ell_i = (a_i - a_{i-1})\sqrt{1+(f'(t_i))^2}$ for some $t_i \in (x_{i-1}, x_i)$ (see [**3.5**]). Now, the surface of revolution of the portion of the curve for $a_{i-1} \le x \le a_i$ will be approximately $s_i = \ell_i \cdot 2\pi f(t_i) = \left(2\pi f(t_i)\sqrt{1+(f'(t_i))^2}\right)(a_i - a_{i-1})$. Thus, the definition [**3.17**] makes sense.

[**3.19**] EXAMPLE Consider the surface of revolution of $y^2 = 4x$ about the x-axis. The area bounded by $x = 0$ and $x = 5$ is $4\pi \int_0^5 \sqrt{1+x}\, dx$.

[**3.20**] RESULT *If the curve is given in polar coordinates by $r = g(\theta), s \le \theta \le t$, then the surface of revolution of the curve about x-axis will be*

$$S = 2\pi \int_s^t (r\sin\theta)\sqrt{\left(r^2 + (dr/d\theta)^2\right)}\, d\theta, \quad \text{where} \quad r = g(\theta).$$

[**3.21**] EXERCISE Show that the area of the surface obtained by revolving the cardioid $r = 1+\cos\theta$ about the x-axis is $\frac{32}{5}\pi a^2$.

[3.22] EXERCISE Consider the funnel formed by revolving the curve $y = \frac{1}{x}$ about the x-axis, between $x = 1$ and $x = a$, $(a > 1)$. Let V_a and S_a denote respectively the volume and the surface of the funnel. Show that $\lim_{a\to\infty} V_a = \pi$ and $\lim_{a\to\infty} S_a = \infty$.

[3.23] EXERCISE Find the volume and area of the curved surface of a paraboloid of revolution formed by revolving the parabola $y^2 = 4ax$ about the x-axis, and bounded by the section $x = x_1$.

End of single variable calculus
