## Tutorial 2: Sequence2,3, Series 1

1. Let $a_1 = 1$, $a_2 = 3$, $a_3 = 7$, and define $a_{n+3} = \dfrac{a_n + a_{n+1} + a_{n+2}}{3}$, for $n \geq 1$. Is $(a_n)$ convergent?

*Sol.* Let $m_1 = \min\{a_1, a_2, a_3\}$ and $M_1 = \max\{a_1, a_2, a_3\}$. So $m_1 \leq a_1, a_2, a_3 \leq M_1$ and at least one of them equals $M_1$. So

$$\frac{2m_1 + M_1}{3} \leq \frac{a_1 + a_2 + a_3}{3} = a_4 \quad \text{and} \quad a_4 = \frac{a_1 + a_2 + a_3}{3} \leq M_1.$$

Similarly,

$$\frac{8m_1 + M_1}{9} = \frac{m_1 + m_1 + \frac{2m_1+M_1}{3}}{3} \leq \frac{a_2 + a_3 + a_4}{3} = a_5 \leq M_1.$$

and

$$\frac{26m_1 + M_1}{27} = \frac{m_1 + m_1 + \frac{8m_1+M_1}{9}}{3} \leq \frac{a_3 + a_4 + a_5}{3} = a_6 \leq M_1.$$

As $\frac{26m_1+M_1}{27} \leq \frac{8m_1+M_1}{9} \leq \frac{2m_1+M_1}{3}$, we see that

$$\frac{26m_1 + M_1}{27} \leq a_4, a_5, a_6 \leq M_1. \tag{1}$$

Put length $l_1 = M_1 - m_1$. Put $m_2 = \min\{a_4, a_5, a_6\}$ and $M_2 = \max\{a_4, a_5, a_6\}$. Observe that

$$a_4, a_5, a_6, a_7, \cdots \in [m_2, M_2] \subseteq [m_1, M_1], \quad \text{and length} \quad l_2 = M_2 - m_2 \leq \frac{26}{27} l_1.$$

Similarly, put $m_3 = \min\{a_7, a_8, a_9\}$ and $M_3 = \max\{a_7, a_8, a_9\}$. Then

$$a_7, a_8, a_9, a_{10}, \cdots \in [m_3, M_3] \subseteq [m_2, M_2], \quad \text{and length} \quad l_3 = M_3 - m_3 \leq \frac{26}{27} l_2 \leq \left(\frac{26}{27}\right)^2 l_1.$$

As $(\frac{26}{27})^n \to 0$, we see that $(a_n)$ is Cauchy. Hence convergent. The limit is $\frac{a_1+2a_2+3a_3}{6}$. To see this, define $b_1 = \frac{a_1+2a_2+3a_3}{6}$, $b_2 = \frac{a_2+2a_3+3a_4}{6}$, and so on. Notice that

$$b_n = \frac{a_n + 2a_{n+1} + 3a_{n+2}}{6} = \frac{a_n + 2a_{n+1} + a_{n-1} + a_n + a_{n+1}}{6} = \frac{a_{n-1} + 2a_n + 3a_{n+1}}{6} = b_{n-1},$$

So it is a constant sequence. Hence

$$\lim b_n = b_1. \tag{2}$$

On the other hand, by LTs (abbreviation for limit theorems for sequences),

$$\lim b_n = \lim \frac{a_n + 2a_{n+1} + 3a_{n+2}}{6} = \frac{6l}{6} = l. \tag{3}$$

Hence $l = \frac{a_1+2a_2+3a_3}{6}$.

2. Let $S \neq \emptyset$ and $(a_n)$ be a decreasing sequence of upper bounds of $S$. Let $a_n \to a$. Show that $a$ is an upper bound.

   *Sol.* Suppose that $a$ is not an upper bound of $S$. Then $\exists s \in S$ such that $s > a$. Put $\epsilon = s - a$. Taking this $\epsilon$, as $a_n \to a$, there exists $n_0$ such that

   $$a_{n_0}, a_{n_0+1}, a_{n_0+2}, \ldots \in B_\epsilon(a) = (a - \epsilon, a + \epsilon).$$

   In particular, $a_{n_0} < a + \epsilon = s$. That is, $a_{n_0}$ is (strictly) less than an element $s \in S$. Then $a_{n_0}$ cannot be an upper bound of $S$. This is a contradiction, as each term of the sequence was an upper bound of $S$.

3. Test for convergence: $(a_n)$ where $a_1 = 2$, $a_{n+1} = \sqrt{2a_n - 1}$ for $n \in \mathbb{N}$.

   *Sol.* Note that $a_1 > 1$ and assuming that $a_n > 1$, we have $a_{n+1} = \sqrt{2a_n - 1} > 1$. Thus by PMI (principle of mathematical induction), $a_n > 1$ for all $n$. So the sequence is bounded below by $1$.

   (Notice that, if we started with $a_1 > 0$, we would have faced difficulty in showing that $a_n > 0$. So a clever start helped us.)

   Note that $a_{n+1}^2 - a_n^2 = (2a_n - 1) - a_n^2 = -(a_n - 1)^2 < 0$. Hence, the sequence is strictly decreasing.

   Hence, by MCT (monotone convergence theorem), it is convergent. Let the limit be $l$. As $a_{n+1} = \sqrt{2a_n - 1}$, by LTs, we have $l = \sqrt{2l - 1}$. Solving this we get $l = 1$.

4. There are two particles $A$ and $B$, placed at $0$ and $1$, on day $1$, respectively. On day $n+1$, particle $A$ moves right by one tenth of the distance between the particles on the $n$th day and particle $B$ moves left by two tenth of the distance between the particles on $n$th day. Do you think they will meet eventually? If so, where?

   Let us randomize it a little bit. On $n + 1$th day a coin is tossed.

(a) If it is 'head', then $A$ moves right by one tenth of the distance between the particles on the $n$th day and particle $B$ moves left by two tenth of the distance between the particles on $n$th day.

(b) If it is 'tail', then $A$ moves right by two tenth of the distance between the particles on the $n$th day and particle $B$ moves left by four tenth of the distance between the particles on $n$th day.

Now, what is your answer and how do you argue?

*Sol.* First part. Let $d_n$ be the distance between them on $n$th day. So $0 \le d_{n+1} = \frac{7}{10} d_n$, $d_1 = 1$. So $d_{n+1} = (\frac{7}{10})^n \to 0$. So, they will meet eventually.

To find the position, let $a_n$ be the position of $A$ on $n$th day. Then

$$(a_n) = (0,\ \frac{1}{10},\ \frac{1}{10} + \frac{7}{10^2},\ \frac{1}{10} + \frac{7}{10^2} + \frac{7^2}{10^3},\ \ldots).$$

So $a_n \to \frac{1}{10} \frac{1}{1 - \frac{7}{10}} = \frac{1}{3}$. So they will meet at $\frac{1}{3}$.

Alternately, you can find the limit of

$$(b_n) = (1,\ 1 - \frac{2}{10},\ 1 - \frac{2}{10} - \frac{2 \times 7}{10^2},\ 1 - \frac{2}{10} - \frac{2 \times 7}{10^2} - \frac{2 \times 7^2}{10^3},\ \ldots)$$

and that limit is $1 - \frac{2}{10} \frac{1}{1 - \frac{7}{10}} = \frac{1}{3}$.

Alternately, once you show that they will meet, note that for each movement of $A$, the particle $B$ moves twice as much. Hence, the limit has to be $\frac{1}{3}$. (Persons using such 'relatively more English' type of arguments must have the ability to convert this into a more rigorous proof. Otherwise, using such an argument will mean bluffing.)

Second part. Let $d_n$ be the distance between them on $n$th day. So $0 \le d_{n+1} \le \frac{7}{10} d_n$, $d_1 = 1$. So $d_n \to 0$. Hence, they will meet.

Note that for each movement of $A$, the particle $B$ moves twice as much. Hence, the limit has to be $\frac{1}{3}$. (Here is an example of that vague, relatively more English type of statement. Hence we supply a rigorous proof below. As we move to higher level texts in mathematics, such 'relatively more English' type of proofs will be common, with the assumption that the reader already knows how to convert them to more rigorous proofs.)

Let $a_n$ be the position of $A$ on $n$th day. Then $b_n = 1 - 2a_n$. Notice that $a_n \uparrow$ and bounded above by $1$. By MCT, $a_n \to a$ (say). Hence by LTs, $b_n \to 1 - 2a$ (say). Since we also know that $d_n = (b_n - a_n) \to 0$, both these sequences must have the same limit. That is, $a = 1 - 2a$. Hence, $a = \frac{1}{3}$.

5. Is $\sum \frac{n! \ln n}{n^n}$ convergent?

   *Sol.* Yes. When $n > 24$, we have

   $$\ln n! = (\ln 2 + \ln 3 + \ln 4) + \ln 5 + \cdots \ln n \leq (n-3) \ln n,$$

   that is, we have $n! < n^{n-3}$. So for $n > 24$, we have

   $$\frac{n!\ \ln n}{n^n} \leq \frac{n^{n-2}}{n^n} = \frac{1}{n^2}.$$

   Hence the given series is convergent, by comparison test.

6. Is $\sum \frac{n!^{2n}}{n^{n^2}}$ convergent?

   *Sol.* No. The series is divergent as the $n$th term does not go to $0$. To see this, put

   $$b_n = a_n^{\frac{1}{n}} = \frac{n!^2}{n^n}.$$

   We will show that $b_n \geq 1$. (Hence we get that $a_n \not\to 0$.) To see that $b_n \geq 1$, observe that for $k = 0, 1, 2, \ldots, n-1$, we have $(k+1)(n-k) \geq n$. Hence

   $$n!^2 = (1.2.3. \cdots .(n-1).n)(1.2.3. \cdots .(n-1).n) = (1.n)(2.(n-1))(3.(n-2)) \cdots (n.1) \geq n^n.$$

   Alternately, to show that $b_n \geq 1$, notice that

   $$\frac{b_{n+1}}{b_n} = \left(\frac{n}{n+1}\right)^n (n+1) \geq \frac{1}{3}(n+1) > 1 \quad \text{for} \quad n \geq 3,$$

   that is $b_n$ increases starting from $b_3$ and $b_3 > 1$. The first two terms are $1$.

7. Is $\sum \frac{e^{n\pi}}{\pi^{ne}}$ convergent?

   *Sol.* No. As $e^\pi > \pi^e$, the $n$th term does not go to $0$. To see an argument, let $e \leq x < y$. Put $x = e^r$, $y = e^s$. So $1 \leq r < s$. We have

   $$\frac{s}{r} = 1 + \frac{s-r}{r} \leq 1 + (s-r) \leq e^{s-r} = \frac{e^s}{e^r}.$$

   So $sx \leq ry$, that is, $x \ln y \leq y \ln x$. Thus $\exp(x \ln y) \leq \exp(y \ln x)$, that is, $y^x < x^y$.

8. Is $\sum\limits_{n=2}^{\infty} \frac{1}{(\ln n)^2}$ convergent?

*Sol.* No. We know that for $x > 3$, we have

$$\exp x > 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} > 1 + x + \frac{x^2}{2} + \frac{x^2}{2} > x^2.$$

Hence,

$$(\ln x)^2 < x \quad \text{for} \quad x > 27.$$

So $\frac{1}{(\ln n)^2} > \frac{1}{n}$ for $n \geq 27$. So the given series is divergent by comparison test.

9. Is $\sum\limits_{n=2}^{\infty} (-1)^n \frac{\ln n}{\ln n^2}$ convergent?

*Sol.* Not convergent $a_n = \frac{1}{2} \not\to 0$.

10. Is $\sum \frac{\cos(n\pi)}{n\sqrt{n}}$ convergent?

*Sol.* Yes. It is absolutely convergent. Note that $|a_n| \leq \frac{1}{n^{1.5}}$ and $\sum \frac{1}{n^{1.5}}$ is convergent. Hence by comparison test, the given series is absolutely convergent.

11. Fix $x \in \mathbb{R}$. We already know that $a_n = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$ converges and the limit is defined as $\exp x$. We also know that $a_n = (1 + \frac{1}{n})^n$ converges and the limit is $e$.

(a) Let $(n_k)$ be a sequence of natural numbers diverging to $\infty$. Then $\lim\limits_{k\to\infty} (1 + \frac{1}{n_k})^{n_k} = e$.

*Sol.* Follows from the definition. Let $\epsilon > 0$. Since $\lim\limits_{k\to\infty} (1 + \frac{1}{n})^n = e$, $\exists m$ such that

$$a_m, a_{m+1}, \ldots, \in (e - \epsilon, e + \epsilon).$$

Since $n_k \to \infty$, with $m$ given, $\exists p$ such that $n_p, n_{p+1}, \ldots \geq m$. Hence

$$a_{n_p}, a_{n_{p+1}}, \ldots \in (e - \epsilon, e + \epsilon).$$

So, by definition, $\lim\limits_{k\to\infty} a_{n_k} = e$.

(b) Let $a_k > 0$ be a sequence of rationals diverging to $\infty$. Show that $\lim(1 + \frac{1}{a_k})^{a_k} \to e$.

*Sol.* We already know that

if $n_k \leq a_k \leq n_k + 1$, then $(1 + \frac{1}{n_k+1})^{n_k} \leq (1 + \frac{1}{a_k})^{a_k} \leq (1 + \frac{1}{n_k})^{n_k+1}$.

These are three sequences. Use sandwich.

(c) Show that, for any rational number $x = \frac{p}{q} > 0$, $e^x = \lim\left(1 + \frac{x}{n}\right)^n$.

*Sol.* Notice that $a_k = \frac{k}{x} \to \infty$. By the previous item

$$\lim(1 + \frac{1}{a_k})^{a_k} = \lim(1 + \frac{x}{k})^{k/x} \to e.$$

Hence $\lim(1 + \frac{x}{k})^k \to e^x$.

(d) Let $x > 0$ be a rational. Show that $\exp x = e^x$.

(Put $s_n = \left(1 + \frac{x}{n}\right)^n$, $t_n = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$ and $r_k = \binom{n}{0} + \binom{n}{1}\frac{x}{n} + \cdots + \binom{n}{k}\frac{x^k}{n^k}$. Let $\alpha > 0$. Show that $\exists k$ such that $\forall\, n > k$ we have $r_{k-1} \leq s_n \leq r_{k-1} + \alpha$.)

*Sol.* As $\frac{x^k}{k!} \to 0$, $\exists k \in \mathbb{N}$ such that $\frac{x^k}{k!} < \frac{\alpha}{2}$ and $\frac{x}{k} < \frac{1}{2}$. Now for each $n > k$, we have

$$\binom{n}{k}\frac{x^k}{n^k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{n^k}\frac{x^k}{k!} \leq \frac{x^k}{k!}. \tag{4}$$

Thus

$$\begin{aligned}\binom{n}{k}\frac{x^k}{n^k} + \binom{n}{k+1}\frac{x^{k+1}}{n^{k+1}} + \cdots + \binom{n}{n}\frac{x^n}{n^n} &\leq \frac{x^k}{k!} + \frac{x^{k+1}}{(k+1)!} + \cdots + \frac{x^n}{n!} \\ &\leq \frac{\alpha}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-k}}\right) \leq \alpha.\end{aligned}$$

Hence we have

$$r_{k-1} \leq s_n \leq r_{k-1} + \alpha, \qquad \forall\, n > k. \tag{5}$$

In the above equation, making $n \to \infty$ we see that (from the middle term of (4), we see that $r_{k-1} \to t_{k-1}$ as $n \to \infty$)

$$t_{k-1} \leq e^x \leq t_{k-1} + \alpha. \tag{6}$$

In the above equation, making $k \to \infty$ we have

$$\exp x \leq e^x \leq \exp x + \alpha. \tag{7}$$

As this is true for each $\alpha > 0$, we are done.

(e) Conclude that $\lim\left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right) = e$.

*Sol.* By the previous item $\exp x = e^x$ for each positive rational. In particular, the lhs $\exp(1)$ and the rhs is $e$ are equal. So this is the first time we are having a new expression for $e$.

(f) [Irrationality of $e$] Conclude that $e$ is irrational.

(Assume that $e = \frac{p}{q}$, $\gcd(p, q) = 1$. Then $q!e - q!(1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{q!})$ must be a positive integer. Can it be?)

*Sol.* Let $a_n = 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}$. Assume that $e = \frac{p}{q}$, $\gcd(p, q) = 1$. Then $q!e = q!\frac{p}{q}$ is a positive integer. As $q!(1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{q!})$ is a positive integer less than $q!e$ (which is a positive integer by the previous line), we see that $q!e - q!(1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{q!})$ is a positive integer.

But for $n > q + 2$, we have

$$\begin{aligned}
& q!a_n - q!(1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{q!}) \\
= \; & q!(\frac{1}{(q+1)!} + \frac{1}{(q+2)!} + \frac{1}{(q+3)!} + \cdots + \frac{1}{n!}) \\
\leq \; & \frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \frac{1}{(q+1)(q+2)^2} + \cdots + \frac{1}{n!} \\
\leq \; & \frac{q+2}{(q+1)^2}.
\end{aligned}$$

Taking limit as $n \to \infty$, we get

$$q!e - q!(1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{q!}) \leq \frac{q+2}{(q+1)^2} < 1.$$

This is, a contradiction as this difference is supposed to be a positive integer.