Department of Mathematics **IIT Guwahati**

Quiz1	**MA101**	**12-12-2020**	**Total marks: 24**	**Time: 65 min**

- The first question is writing your roll number. It is compulsory.
- Each other question carries 2 marks.
- For single-correct-option question, you get 2 for correct answer, -1 for wrong answer and 0 for not attempting.
- For multiple-correct-option question, you get 2 for correct answer and 0 for wrong answer or not attempting.
- You get extra 5 minutes for submission. REMEMBER to press the SUBMIT button by 11:04:59. The form will not accept responses after that.
- The form permits ONLY ONE submission. It does not allow REVISION.

1. Write your roll number.

2. (Multiple correct options) Let A and B be nonempty subsets of $\mathbb{R}$ such that for each $a \in A$ and $b \in B$, we have $a^2 \leq b$. Then which of the following statements are correct?

   A) $\mathsf{lub}\, A$ must exist in $\mathbb{R}$

   B) $\mathsf{lub}\, B$ must exist in $\mathbb{R}$

   C) $\mathsf{glb}\, A$ must exist in $\mathbb{R}$

   D) $\mathsf{glb}\, B$ must exist in $\mathbb{R}$

   Sol. A), C), D)

   Since both the sets are nonempty, let $a_0 \in A$, $b_0 \in B$. So $b_0 \geq a_0^2 \geq 0$ is a nonnegative number. Hence, $-\sqrt{b_0} \leq a \leq \sqrt{b_0}$ for each $a \in A$. Hence A is nonempty and bounded. Hence $\mathsf{lub}\, A$ and $\mathsf{glb}\, A$ will exist in $\mathbb{R}$.

   As $a_0^2 \leq b$ for each $b \in B$, we see that 0 is a lower bound of B. As B is nonempty, and bounded below, $\mathsf{glb}\, B$ will exist in $\mathbb{R}$.

   B) need not be true, for example, take $A = (-1, 0)$ and $B = (2, \infty)$.

3. For a natural number n define $val(n) = n(1 - \dfrac{n \ln n}{(n+1)\ln(n+1)})$. Let

   $$(p_n) = (3, 2, 7, 5, 13, 11, 19, 17, \ldots)$$

   be the sequence of prime numbers. We want to find $\lim\limits_{n\to\infty} val(p_n)$. Then which of the following options is correct?

   A) Limit exists and it is less than half.

   B) Limit exists and it is half.

   C) Limit exists and it is more than half.

   D) Limit does not exist.

   Sol. C), as the limit is 1.

   The main sequence $(val(n))$ has limit 1. Hence $(val(q_n))$ has limit 1, where (q_n) is the subsequence of primes. Hence the further subsequences $(val(q_{2n}))$ and $(val(q_{2n-1}))$ also converge to 1. Hence

   $$(val(q_2),\ val(q_1),\ val(q_4),\ val(q_3),\ \cdots)$$

also converge to 1. To show that the main sequence converges to 1 we see that

$$\begin{aligned}\lim n\left(1-\frac{n\ln n}{(n+1)\ln(n+1)}\right) &= \lim n\left(\frac{(n+1)\ln(n+1)-n\ln n}{(n+1)\ln(n+1)}\right)\\ &= \lim\left(\frac{(n+1)\ln(n+1)-n\ln n}{\ln(n+1)}\right) \quad \text{(if exists).}\end{aligned}$$

Now

$$\lim\left(\frac{(n+1)\ln(n+1)-n\ln n}{\ln(n+1)}\right) = \lim\left(\frac{\ln(n+1)+n\ln(n+1)-n\ln n}{\ln(n+1)}\right) = 1+\lim\frac{n\ln\frac{n+1}{n}}{\ln(n+1)} = 1,$$

as $\ln(1+\frac{1}{n}) \leq \frac{1}{n}$. Students may use LHospital, but they should not.

4. Consider the following two statements.

   Statement 1: Take $a_n = (1+\frac{1}{\sqrt{n}})^n$. Then the sequence (a_n) is convergent.

   Statement 2: If (c_n) and (d_n) are two Cauchy sequences, then $(c_n d_n)$ must be a Cauchy sequence.

   Which of the following options is correct?

   A) Statement 1 is correct but Statement 2 is wrong.

   B) Statement 2 is correct but Statement 1 is wrong.

   C) Both Statement 1 and Statement 2 are wrong.

   D) Both Statement 1 and statement 2 are correct.

   Sol. B).

   For all $n > 1$, we have $(1+\frac{1}{n})^n > 2$. Hence, for n^2, we have $a_{n^2} = (1+\frac{1}{\sqrt{n^2}})^{n^2} > 2^n$. That is, the sequence is not bounded above (writing 'unbounded above', though conveys the message, is hardly used). So it is divergent. So Statement 1 is wrong.

   Statement 2 is correct as Cauchy sequences are convergent sequences.

5. Consider the following two statements.

   Statement 1: $\sum(-1)^n \ln\left(1+\frac{1}{n^2}\right)$ is absolutely convergent.

   Statement 2: Let $a_n \geq 0$ and $\sum a_n$ be convergent. Then the sequence (na_n) must be convergent.

   Which of the following options is correct?

   A) Statement 1 is correct but Statement 2 is wrong.

   B) Statement 2 is correct but Statement 1 is wrong.

   C) Both Statement 1 and Statement 2 are wrong.

   D) Both Statement 1 and statement 2 are correct.

   Sol. A)

   Recall that $\ln\left(1+\frac{1}{n^2}\right) < \frac{1}{n^2}$. So by comparison test, the first one is absolutely convergent.

   Next, consider $0+1+0+\frac{1}{2}+0+0+0+\frac{1}{4}+\cdots$. Then $\lim\limits_{n\to\infty}(na_n)$ does not exist, as $3^k a_{3^k} \to 0$ and $2^k a_{2^k} \to 2$.

6. Consider the following two statements.

Statement 1: There exists an increasing sequence (a_n) such that $\sum a_n = 2020$.

Statement 2: $\sum\limits_{n\geq 1} \frac{1}{n^{1+\frac{1}{n}}}$ is divergent.

Which of the following options is correct?

A) Statement 1 is correct but Statement 2 is wrong.

B) Statement 2 is correct but Statement 1 is wrong.

C) Both Statement 1 and Statement 2 are wrong.

D) Both Statement 1 and statement 2 are correct.

Sol. B).

The first one is false. If $\sum a_n$ is convergent, then it means $a_n \to 0$. But as (a_n) is increasing, terms can only be nonpositive (means ≤ 0). Hence their sum cannot be 2020, a positive number.

The second one is true. By limit comparison test with $\sum \frac{1}{n}$.

7. Consider the following two statements.

Statement 1: Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Then $Z = \{x \mid f(x) = 5\}$ contains all its cluster points.

Statement 2: Let f be continuous with $\lim\limits_{h\to 0} \frac{f(h)}{h^2} = 2$ and $\lim\limits_{h\to 0} \frac{f(h)}{h} = l$. Then $\lim\limits_{h\to 0} \frac{l}{h} = 2$.

Which of the following options is correct?

A) Statement 1 is correct but Statement 2 is wrong.

B) Statement 2 is correct but Statement 1 is wrong.

C) Both Statement 1 and Statement 2 are wrong.

D) Both Statement 1 and statement 2 are correct.

Sol. A)

Let a be a cluster point of Z. Then there is a sequence of point $a_n \in Z$ such that $a_n \to a$, $a_n \neq a$. As f is continuous at a we have $f(a_n) \to f(a)$. As $(f(a_n)) = (5, 5, \ldots)$, we have $f(a) = 5$. That is $a \in Z$.

For the second one, take $f(h) = 2h^2$. Then $l = 0$.

(General answer: Note that if $\lim\limits_{h\to 0} \frac{f(h)}{h^2} = k$, then $\lim\limits_{h\to 0} \frac{f(h)}{h} = \lim\limits_{h\to 0} h\frac{f(h)}{h^2} = \lim\limits_{h\to 0} h \lim\limits_{h\to 0} \frac{f(h)}{h^2} = 0k = 0$. Thus $l = 0$. Thus $\lim\limits_{h\to 0} \frac{l}{h} = 0$. And this limit will be equal to k if and only if k is zero.)

8. (Multiple correct options) I have a polynomial $p(x)$ of degree 2021. Define $f : \mathbb{R} \to \mathbb{R}$ as

$$f(x) = \begin{cases} |p(x)| & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise.} \end{cases}$$

Then which of the following options are correct?

A) The function is discontinuous at each point.

B) The set $\{a \mid f(x) \text{ is continuous at } a\}$ is nonempty and finite.

C) The function is continuous at every point a where $\lim\limits_{x\to a} f(x)$ exist.

D) The limit $\lim\limits_{x\to a} f(x)$ exist at infinitely many points $a \in \mathbb{R}$.

Sol. We know that $p(x)$ has a real zero, say α. Write $p(x) = (x-\alpha)q(x)$. Then $0 \le f(x) \le |x-\alpha||q(x)|$. Note that $\lim_{x\to\alpha} |x-\alpha||q(x)| = 0 \times |q(\alpha)| = 0$. By sandwich lemma, we have $\lim_{x\to\alpha} f(x) = 0 = p(\alpha)$. So $f(x)$ is continuous at α. In fact, in a similar way, f is continuous at each real zero of $p(x)$.

Now suppose $p(\beta) \neq 0$. We show that $\lim_{x\to\beta} f(x)$ does not exist. Suppose that the limit exists and let it be l. Take a sequence of rationals (r_n) converging to β and a sequence of irrationals (i_n) converging to β. Then the sequence $f(r_n) = |p(r_n)| \to |p(\beta)|$. Hence l must be $|p(\beta)| \neq 0$. The sequence $f(i_n) = 0 \to 0$. Hence l must be 0. This is a contradiction.

Hence B) and C) are correct.

9. Consider the following two statements.

   Statement 1: Let $f : \mathbb{R} \to \mathbb{R}$ with $\lim_{h\to 0}\Big(f(x+h) - f(x-h)\Big) = 0$ for all x. Then f is continuous on $\mathbb{R}$.

   Statement 2: If $f : \mathbb{R} \to \mathbb{R}$ is continuous and one-one, then it must be strictly monotone (means strictly increasing or strictly decreasing).

   Which of the following options is correct?

   A) Statement 1 is correct but Statement 2 is wrong.

   B) Statement 2 is correct but Statement 1 is wrong.

   C) Both Statement 1 and Statement 2 are wrong.

   D) Both Statement 1 and statement 2 are correct.

   Sol. B)

   The first one is false. Define $f(0) = 1$ and $f(x) = 0$ else. Then f satisfies the property, but not continuous.

   The second one is true. Suppose that the conclusion is not true. Then $\exists a < b < c$ such that $f(a) < f(b) > f(c)$ or $f(a) > f(b) < f(c)$. Assume that $\exists a < b < c$ such that $f(a) < f(b) > f(c)$. Take $m = \max\{f(a), f(c)\}$. By IVT, there exists points $p \in (a,b), q \in (b,c)$ such that $f(p) = f(q) = \frac{f(b)+m}{2}$. A contradiction.

10. Consider the following two statements.

    Statement 1: Let $f, g : \mathbb{R} \to \mathbb{R}$ with $\lim_{x\to 5} f(x) = l$ and $\lim_{x\to 5} g(x) = k$. Then

    $$\lim_{x\to 5} \max\{f(x), g(x)\} = \max\{l, k\}.$$

    Statement 2: Let $f : \mathbb{R} \to \mathbb{R}$ be continuous such at two positive points a, b, we have $f(a) = 5a$ and $f(b) = 7b$. There there must exist a point c for which $f(c) = 6c$.

    Which of the following options is correct?

    A) Statement 1 is correct but Statement 2 is wrong.

    B) Statement 2 is correct but Statement 1 is wrong.

    C) Both Statement 1 and Statement 2 are wrong.

    D) Both Statement 1 and statement 2 are correct.

    Sol. D)

    Since $\max\{f(x), g(x)\} = (f(x)+g(x))/2 + |f(x)-g(x)|/2$, and limits for $f(x)$ and $g(x)$ exist, applying limit theorems for functions we see that the statement is correct.

    For the second one, Consider $g(x) = f(x) - 6x$. Then $g(a) = -a$ and $g(b) = b$. Notice that g is continuous. So by IVT, there is a point c where $g(c) = 0$, that is $f(c) = 6c$. So the statement is correct.

11. Consider the following two statements.

Statement 1: The number of continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that the image $f(\mathbb{R}) \subseteq \{0, \pi, e, \sqrt{2}\}$ is at least 2.

Statement 2: It is given that $\lim_{x\to 5}\Big((f(x))^3 - (f(x))^2 + f(x) - 1\Big)$ exists and it is 0. Then $\lim_{x\to 5} f(x)$ must exist.

Which of the following options is correct?

A) Statement 1 is correct but Statement 2 is wrong.

B) Statement 2 is correct but Statement 1 is wrong.

C) Both Statement 1 and Statement 2 are wrong.

D) Both Statement 1 and statement 2 are correct.

Sol. D)

Such functions have to be constant functions, by IVT. In that case there are 4 such functions. So the statement is true.

For the second one note that as $\lim_{x\to 5}\Big((f(x))^3 - (f(x))^2 + f(x) - 1\Big) = 0$, we have

$$\lim_{x\to 5}\Big|\Big((f(x))^3 - (f(x))^2 + f(x) - 1\Big)\Big| = 0.$$

Also,

$$\Big|\Big((f(x))^3 - (f(x))^2 + f(x) - 1\Big)\Big| = \Big|(f(x)-1)\Big((f(x))^2+1\Big)\Big| \geq |f(x)-1| \geq 0.$$

Applying sandwich, we see that $\lim_{x\to 5} f(x) = 1$.

12. (Multiple correct options) Let $0 < a < b$. Which of these irrational numbers are necessarily in the interval (a, b)? Here $[x]$ means the greatest integer function.

A) $[a] + \dfrac{1}{500\sqrt{2}}$

B) $\dfrac{[na] + \frac{1}{500\sqrt{2}}}{n}$, where $n = [\frac{3}{b-a}]$

C) $\dfrac{[na] + 1 + \frac{1}{500\sqrt{2}}}{n}$, where $n = [\frac{3}{b-a}]$

D) $\dfrac{[na] + 1 + \frac{1}{500\sqrt{2}}}{n}$, where $n = [\frac{2}{b-a}]$

Sol. Question dropped as I forgot to add $+1$ in $n = [\frac{2}{b-a}]$ and $n = [\frac{3}{b-a}]$.

13. Let $p(x) = x^4 + 5x^3 - 3x^2$ and A be an (arbitrary) infinite bounded set of positive real numbers. Define $B = \{x^4 + 5y^3 - 3z^2 \mid x, y, z \in A\}$. Then which of the following statements is correct?

A) We must have $\operatorname{lub} B = p(\operatorname{lub} A)$.

B) We must have $\operatorname{lub} B < p(\operatorname{lub} A)$.

C) We must have $\operatorname{lub} B > p(\operatorname{lub} A)$.

D) No comparisons can be made between $\operatorname{lub} B$ and $p(\operatorname{lub} A)$ in general.

Sol. C).

The set being infinite is nonempty. It is given to be bounded. So $\operatorname{glb} A$ and $\operatorname{lub} A$ exist. Note that $\operatorname{glb} A < \operatorname{lub} A$, as A has at least two numbers. So

$$\begin{aligned}\operatorname{lub} B &= (\operatorname{lub} A)^4 + 5(\operatorname{lub} A)^3 - 3(\operatorname{glb} A)^2 \\ &> (\operatorname{lub} A)^4 + 5(\operatorname{lub} A)^3 - 3(\operatorname{lub} A)^2 = p(\operatorname{lub} A).\end{aligned}$$