MA 101 (Mathematics - I)
Exercise set 3

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1. (a) Can a power series $\sum_{n=0}^{\infty} a_n(x-3)^n$ converge at $x=0$ and diverge at $x=5$?

   (b) Suppose $\sum_{n=0}^{\infty} a_n x^n$ converges at -3 and diverges at 4. What can you say about the radius of convergence of the power series?

   (c) Prove or disprove: There is a power series about 0 which converges at π and diverges at $-\pi$.

2. Suppose $a_n > 0, a_n \to 0$ and $a_n^2 > \frac{1}{10^{50}} a_{n+1}$ for each n. Can you determine the domain of convergence of $\sum_{i=1}^{\infty} a_n x^n$?

3. Suppose (a_n) is a sequence converging to 0. One student found that the power series $a_1x+2a_2x^2+3a_3x^3+\cdots$ converges to f on the interval $(-5,5)$ and another student found that $1+a_1x+a_2x^2+a_3x^3+\cdots$ converges to g with radius of convergence as R. How are the answers of the two students related?

4. (a) Suppose that $\sum a_n x^n$ is a power series whose radius of convergence is R. Define $f(x) = \sum a_n x^n$ on $(-R, R)$. Argue that $\sum a_n x^n$ is the Taylor series of f about 0?

   (b) Use your argument and the Taylor series for $\sin x$ to find the Taylor series of $\sin x \cos 3x$ about 0.

5. Find radius of convergence and domain of convergence of the following power series:

   1. $\sum \frac{x^n}{n^2}$
   2. $\sum n(n+1)x^n$
   3. $\sum \frac{(-1)^n x^{2n}}{n^2}$
   4. $\sum \frac{3^n x^n}{2^n}$
   5. $\sum n^n x^n$
   6. $2x + \left(\frac{9}{4}x\right)^2 + \cdots + \left(\left(\frac{n+1}{n}\right)^n x\right)^n + \cdots$

6. Find the domain of converegnce of the power series $\frac{1}{a} - \left(\frac{1}{a}\right)^2 (x-a) + \left(\frac{1}{a}\right)^3 (x-a)^2 - \left(\frac{1}{a}\right)^4 (x-a)^3 + \cdots$. Can you give an explicit formula for the function represented by the power series?

7. (a) For $f(x) = x$ on $[0,1]$ calculate $L(f, \mathbf{P}_n)$ and $U(f, \mathbf{P}_n)$, conclude $f \in \mathcal{R}[0,1]$, and find $\int_a^b f$.

   (b) For $f(x) = x^2$ on $[0,1]$ calculate $L(f, \mathbf{P}_n)$ and $U(f, \mathbf{P}_n)$, conclude $f \in \mathcal{R}[0,1]$, and find $\int_a^b f$.

8. For the function $f : [-2,2] \to \mathbb{R}$ defined by $f(x) = x^5$, find a partition P such that $U(f,P) - L(f,P) < \epsilon$.

9. Give an example of a function on $[0,1]$ such that $L(f) = 1$ and $U(f) = 2$.

10. Show by definition that $f : [0,1] \to \mathbb{R}$ defined by $f(x) = x^n$ is integrable.

11. Suppose you know that $\lim_{n\to\infty} U(f, \mathbf{P}_n) = \ell$. Is it true that $f \in \mathcal{R}([a,b])$?

12. Show that $\frac{1}{2} \le \int_0^1 \frac{1+x-x^2}{1+x^4} dx \le \frac{5}{4}$.

13. Let f be continuous on $[a,b]$. If $\int_a^b f = 0$ then show that $f(c) = 0$ for at least one $c \in [a,b]$. Show that the result may not hold if f is not continuous.

14. If f is continuous on $[a,b]$ and $\int_a^b fg = 0$ for every $g \in \mathcal{R}[a,b]$, then show that $f = 0$.

15. Suppose $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{1}{2^n}, & \text{if } x = \frac{k}{2^n} \text{ for some } k, n \in \mathbb{N}, \text{ where } k \text{ is odd,} \\ 0, & \text{otherwise.} \end{cases}$$

Determine whether f is Riemann integrable, and if so, find $\int_0^1 f$.
