MA 101 (Mathematics - I)

Quiz - III

Maximum Marks : 25

Date: February 20, 2021 **Time:** 10 am - 11 am

Instructions:

- The answers of this Quiz question paper are to be filled in the Quiz - III response form. You get exactly one hour time (from 10 am to 11 am) for doing this.
- You should submit the response form at 11 am (or before). Although you get extra 5 minutes for submission only (the portal will close at 11:05 am), it is advised not to take any risk of submitting after 11 am. No request will be entertained if you fail to submit your responses through the portal due to any reason.
- The response form permits only one submission. It does not allow revision.

Type of Questions and Marking scheme:

- The first question is writing your Roll number. It is compulsory. It has no marks.
- Q.2 to Q.5 are of single correct option type questions, where exactly one option is correct. Each of these questions carries 2 marks for correct answer, -1 mark for incorrect answer, and 0 mark for no answer. No answer will be considered if you do not mark any option or if you mark option (E) given in the response form.
- Q.6 to Q.12 are of multiple correct option type questions, where one or more of the options is (are) correct. In each of these questions, you get 2 marks if you choose all the correct options and choose no wrong option. In all other cases, you get 0 mark.
- Q.13 is also of multiple correct option type questions, where one or more of the options is (are) correct. For this question, you get 3 marks if you choose all the correct options and choose no wrong option. In all other cases, you get 0 mark.

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1. Write your Roll number.

2. Let C be the curve of intersection of the surfaces $3x^2y + y^2z + 2 = 0$ and $2xz - x^2y = 3$ in $\mathbb{R}^3$. Then the tangent line to the curve C at the point $(1, -1, 1)$ passes through the point
(A) $\left(\frac{3}{2}, \frac{4}{3}, \frac{7}{3}\right)$ (B) $\left(2, \frac{13}{3}, \frac{5}{3}\right)$ (C) $\left(0, -\frac{17}{3}, \frac{2}{3}\right)$ (D) $\left(-\frac{1}{2}, -\frac{11}{3}, -1\right)$

   Answer : (B)

   Explanation: Let $f(x, y, z) = 3x^2y+y^2z+2$ and $g(x, y, z) = 2xz-x^2y-3$ for all $(x, y, z) \in \mathbb{R}^3$. Then $\nabla f(1, -1, 1) = (-6, 1, 1)$ and $\nabla g(1, -1, 1) = (4, -1, 2)$. If a, b, c are the direction ratios of the tangent line to the curve C at $(1, -1, 1)$, then $(a, b, c) \cdot \nabla f(1, -1, 1) = 0$ and $(a, b, c) \cdot \nabla g(1, -1, 1) = 0$. So $-6a + b + c = 0$ and $4a - b + 2c = 0$ and hence $c = \frac{2}{3}a$, $b = \frac{16}{3}a$. Therefore the tangent line to C at $(1, -1, 1)$ is given by $\ell(t) = (1, -1, 1)+t(1, \frac{16}{3}, \frac{2}{3})$, $t \in \mathbb{R}$. Taking $t = 1$, we get option (B). Other options can be rejected by taking $t = \frac{1}{2}$, $t = -1$ and $t = -\frac{3}{2}$.

3. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function such that $f(x+xy^3, x, y) = 5y - 3x$ for all $x, y \in \mathbb{R}$. If $\mathbf{u} = \big(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\big)$ and $D_{\mathbf{u}}f(4,2,1) = 4\sqrt{3}$, then $\|\nabla f(4,2,1)\|^2$ is equal to
(A) 48 (B) 96 (C) 100 (D) 102

Answer : (D)

Explanation: Let $u(x,y)x + xy^3$, $v(x,y) = x$ and $w(x,y) = y$ for all $(x,y) \in \mathbb{R}^2$. Then by chain rule, we get $f_u(4,2,1)u_x(2,1) + f_v(4,2,1)v_x(2,1) + f_w(4,2,1)w_x(2,1) = -3$ and $f_u(4,2,1)u_y(2,1) + f_v(4,2,1)v_y(2,1) + f_w(4,2,1)w_y(2,1) = 5$. Writing $a = f_u(4,2,1)$, $b = f_v(4,2,1)$ and $c = f_w(4,2,1)$, we get $2a + b = -3$ and $6a + c = 5$. Again, $D_{\mathbf{u}}f(4,2,1) = \nabla f(4,2,1) \cdot \mathbf{u} = 4\sqrt{3}$ and from this, we get $a + b + c = -12$. Thus $a = 2$, $b = c = -7$ and therefore $\|\nabla f(4,2,1)\|^2 = a^2 + b^2 + c^2 = 102$.

4. Consider the following two statements **P** and **Q**.
P : There exists a continuous function from $\{(x,y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 2\}$ onto $\{x \in \mathbb{R} : |x| > 1\}$.
Q : There exists a one-one continuous function from $\{(x,y) \in \mathbb{R}^2 : y^2 = 2x\}$ onto $\{x \in \mathbb{R} : x > 2\}$. Then
(A) both **P** and **Q** are true (B) **P** is true but **Q** is false
(C) **Q** is true but **P** is false (D) both **P** and **Q** are false

Answer : (C)

Explanation: (The explanation of **P** is exactly similar to the last part of the solution for Ex.5 of Tutorial Problem Set - 2.) If possible, let there exist a continuous function f from $S_1 = \{(x,y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 2\}$ onto $S_2 = \{(x \in \mathbb{R} : |x| > 1\}$. Then there exist $\mathbf{x}_1, \mathbf{x}_2 \in S_1$ such that $f(\mathbf{x}_1) = -2$ and $f(\mathbf{x}_2) = 2$. There exists a curve (*i.e.* a continuous function) $\gamma : [0,1] \to S_1$ such that $\gamma(0) = \mathbf{x}_1$ and $\gamma(1) = \mathbf{x}_2$. (If $\|\mathbf{x}_1\| = \|\mathbf{x}_2\|$, then γ can be taken as the arc of a circle and if $\|\mathbf{x}_1\| \neq \|\mathbf{x}_2\|$, then γ can be taken as the arc of a circle followed by a straight line segment.) If $\varphi(t) = f(\gamma(t))$ for all $t \in [0,1]$, then $\varphi : [0,1] \to S_2$ is continuous and $\varphi(0) = -2 < 0$, $\varphi(1) = 2 > 0$. Hence there exists $t_0 \in (0,1)$ such that $\varphi(t_0) = 0$, *i.e.* $f(\gamma(t_0)) = 0$. Since $\gamma(t_0) \in S_1$ and $0 \notin S_2$, we get a contradiction. Therefore **P** is false. Again, if $f(x,y) = y$ for all $(x,y) \in S = \{(x,y) \in \mathbb{R}^2 : y^2 = 2x\}$, then $f : S \to \mathbb{R}$ is one-one, onto and continuous. Also, if $g(x) = e^x + 2$ for all $x \in \mathbb{R}$, then $g : \mathbb{R} \to (2, \infty)$ is one-one, onto and continuous. Hence $g \circ f : S \to (2, \infty)$ is one-one, onto and continuous. Therefore **Q** is true.

5. For $c \in \mathbb{R}$ with $c > 1$, let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = (cx^2 + y^2)e^{-x^2-y^2}$ for all $(x,y) \in \mathbb{R}^2$. Consider the following two statements **P** and **Q**.
P : There exists $c \in \mathbb{R}$ with $c > 1$ such that f has at least six critical points.
Q : There exists $c \in \mathbb{R}$ with $c > 1$ such that f has at least three saddle points.
Then
(A) both **P** and **Q** are true (B) **P** is true but **Q** is false

(C) **Q** is true but **P** is false (D) both **P** and **Q** are false

Answer : (D)

Explanation: Let $c \in \mathbb{R}$ such that $c > 1$. We have $f_x(x,y) = 2xe^{-x^2-y^2}(c-cx^2-y^2)$, $f_y(x,y) = 2ye^{-x^2-y^2}(1-cx^2-y^2)$, $f_{xx}(x,y) = 2e^{-x^2-y^2}(c-5cx^2-y^2+2cx^4+2x^2y^2)$, $f_{yy}(x,y) = 2e^{-x^2-y^2}(1-cx^2-5y^2+2cx^2y^2+2y^4)$ and $f_{xy}(x,y) = 4xye^{-x^2-y^2}(cx^2+y^2-c-1)$ for all $(x,y) \in \mathbb{R}^2$. Solving the system of equations $f_x(x,y) = 0$, $f_y(x,y) = 0$, we get $(0,0)$, $(0,1)$, (0,-1), $(1,0)$ and $(-1,0)$ as all the critical points of f. Hence **P** is false. Let $D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - f_{xy}(x,y)^2$ for all $(x,y) \in \mathbb{R}^2$. Then $D(0,0) = 4c > 0$, $D(0,1) = D(0,-1) = \frac{8}{e^2}(1-c) < 0$ and $D(1,0) = D(-1,0) = \frac{8c}{e^2}(c-1) > 0$. Hence $(0,1)$ and $(0,-1)$ are all the saddle points of f. Therefore **Q** is false.

6. Let $S = [0,1] \times [0,1] \subseteq \mathbb{R}^2$ and let $f : S \to \mathbb{R}$ be defined by
$$f(x,y) = \begin{cases} x\left(\cos\left(\frac{1}{x+y}\right) - 1\right) - y & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$
Then
(A) f is continuous at each point of S
(B) there exists exactly one $\mathbf{x}_0 \in S$ such that $f(\mathbf{x}_0) = \sup\{f(\mathbf{x}) : \mathbf{x} \in S\}$
(C) there exist more than one $\mathbf{x}_0 \in S$ such that $f(\mathbf{x}_0) = \sup\{f(\mathbf{x}) : \mathbf{x} \in S\}$
(D) there exist no $\mathbf{y}_0 \in S$ such that $f(\mathbf{y}_0) = \inf\{f(\mathbf{x}) : \mathbf{x} \in S\}$

Answer : (A), (C)

Explanation: Note that $|f(x,y) - f(0,0)| \leq 2|x| + |y|$ for all $(x,y) \in S$ and hence it follows that f is continuous at $(0,0)$. Therefore f is continuous at each point of S. Hence option (A) is correct. Since S is a closed and bounded set in $\mathbb{R}^2$, there exist $\mathbf{x}_0$, $\mathbf{y}_0 \in S$ such that $f(\mathbf{x}_0) = \sup\{f(\mathbf{x}) : \mathbf{x} \in S\}$ and $f(\mathbf{y}_0) = \inf\{f(\mathbf{x}) : \mathbf{x} \in S\}$. So option (D) is wrong. Again, note that $f(x,y) \leq 0 = f(0,0) = f(\frac{1}{2\pi}, 0)$ for all $(x,y) \in S$ and so $\sup\{f(\mathbf{x}) : \mathbf{x} \in S\} = 0$ and it is attained by f at $(0,0)$ and $(\frac{1}{2\pi}, 0)$. Therefore option (B) is wrong and option (C) is correct.

7. For $r, s, t \in (0, \infty)$, let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} \frac{\sin(|x|^r|y|^s)}{(x^2+y^2)^t} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$
Then
(A) f is differentiable at $(0,0)$ if $r + s > 2t + 1$
(B) f is differentiable at $(0,0)$ only if $r + s > 2t + 1$
(C) f is differentiable at $(0,0)$ if $r > t + 1$ and $s > t + 1$
(D) f is differentiable at $(0,0)$ only if $r + s > 2t + 2$

Answer : (A), (B), (C)

Explanation: We have $f_x(0,0) = \lim\limits_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = \lim\limits_{t\to 0} \frac{0-0}{t} = 0$ and $f_y(0,0) = \lim\limits_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = \lim\limits_{t\to 0} \frac{0-0}{t} = 0$. Hence for all $(h,k) \in \mathbb{R}^2 \setminus \{(0,0)\}$, $\varepsilon(h,k) = \frac{|f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)|}{\sqrt{h^2+k^2}} = \frac{|\sin(|h|^r|k|^s)|}{(h^2+k^2)^t\sqrt{h^2+k^2}}$. If $r + s > 2t + 1$, then $\varepsilon(h,k) \leq \frac{|h|^r|k|^s}{(\sqrt{h^2+k^2})^{2t+1}} \leq (\sqrt{h^2+k^2})^{r+s-2t-1}$ and hence $\lim\limits_{(h,k)\to(0,0)} \varepsilon(h,k) = 0$. Therefore f is differentiable at $(0,0)$. Thus options (A), (C) are correct and option (D) is not correct. Again, let $r + s \leq 2t + 1$. Then $(\frac{1}{n}, \frac{1}{n}) \to (0,0)$ but

$\varepsilon(\frac{1}{n},\frac{1}{n}) = \frac{1}{(\sqrt{2})^{2t+1}} n^{2t+1}\sin\left(\frac{1}{n^{r+s}}\right) = \frac{1}{(\sqrt{2})^{2t+1}} \frac{\sin\left(\frac{1}{n^{r+s}}\right)}{\frac{1}{n^{r+s}}} n^{2t+1-r-s} \not\to 0$. Hence $\lim\limits_{(h,k)\to(0,0)} \varepsilon(h,k) \neq 0$ and consequently f is not differentiable at $(0,0)$. Therefore option (B) is correct.

8. Let $f:\mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} x^2+y^2 & \text{if } x,y \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$
Then
(A) $f_x(x_0,y_0)$ exists (in $\mathbb{R}$) for exactly one point $(x_0,y_0)\in\mathbb{R}^2$
(B) $f_x(x_0,y_0)$ exist (in $\mathbb{R}$) for all points $(x_0,y_0)\in\mathbb{R}^2$ with $x_0,\ y_0\in\mathbb{Q}$
(C) $f_x(x_0,y_0)$ exist (in $\mathbb{R}$) for all points $(x_0,y_0)\in\mathbb{R}^2$ with $x_0\in\mathbb{R}\setminus\mathbb{Q}$, $y_0\in\mathbb{Q}$
(D) $f_y(x_0,y_0)$ exist (in $\mathbb{R}$) for infinitely many points $(x_0,y_0)\in\mathbb{R}^2$

Answer : (D)

Explanation: If $(x_0,y_0)\in\mathbb{R}^2$ such that $y_0\in\mathbb{R}\setminus\mathbb{Q}$, then $f_x(x_0,y_0) = \lim\limits_{t\to 0}\frac{f(x_0+t,y_0)-f(x_0,y_0)}{t} = \lim\limits_{t\to 0}\frac{0-0}{t} = 0$. Hence option (A) is not correct. If $(x_0,y_0)\in\mathbb{R}^2$ such that $x_0\in\mathbb{R}\setminus\mathbb{Q}$, then $f_y(x_0,y_0) = \lim\limits_{t\to 0}\frac{f(x_0,y_0+t)-f(x_0,y_0)}{t} = \lim\limits_{t\to 0}\frac{0-0}{t} = 0$. Hence option (D) is correct. Let $(x_0,y_0)\in\mathbb{Q}^2\setminus\{(0,0)\}$. Then for all $t\in\mathbb{R}\setminus\mathbb{Q}$, we have $\frac{f(x_0+t,y_0)-f(x_0,y_0)}{t} = -\frac{x_0^2+y_0^2}{t}$ and since $x_0^2+y_0^2\neq 0$, it follows that $f_x(x_0,y_0) = \lim\limits_{t\to 0}\frac{f(x_0+t,y_0)-f(x_0,y_0)}{t}$ does not exist (in $\mathbb{R}$). Hence option (B) is not correct. Let $(x_0,y_0)\in\mathbb{R}^2$ such that $x_0\in\mathbb{R}\setminus\mathbb{Q}$. Then there exists a sequence (r_n) in $\mathbb{Q}$ such that $r_n\to x_0$. Since for all $t\in\mathbb{R}\setminus\{0\}$, $\frac{f(x_0+t,y_0)-f(x_0,y_0)}{t} = \frac{f(x_0+t,y_0)}{t}$ and since $\left|\frac{f(r_n,y_0)}{r_n-x_0}\right| = \frac{r_n^2+y_0^2}{|r_n-x_0|}\to\infty$, it follows that $f_x(x_0,y_0) = \lim\limits_{t\to 0}\frac{f(x_0+t,y_0)-f(x_0,y_0)}{t}$ does not exist (in $\mathbb{R}$). Hence option (C) is not correct.

9. If $S = \{(a_1,a_2,a_3,a_4,a_5)\in\mathbb{R}^5 : \text{there exists } x\in\mathbb{R} \text{ such that } a_1x^4+a_2x^3+a_3x^2+a_4x+a_5 = 0\}$, then
(A) $(4,-7,2,-3,-1)\in S^0$
(B) $(3,1,-5,-2,1)\in S^0$
(C) $(-2,0,4,-6,3)\in S^0$
(D) $(-6,-2,0,-6,-1)\in S^0$

Answer : (A), (B), (C), (D)

Explanation: For each $\mathbf{a} = (a_1,\ldots,a_5)\in\mathbb{R}^5$, let $p_{\mathbf{a}}(x) = a_1x^4+a_2x^3+a_3x^2+a_4x+a_5$ for all $x\in\mathbb{R}$. If $\mathbf{a} = (4,-7,2,-3,-1)$, then $p_{\mathbf{a}}(0) = -1 < 0$ and $p_{\mathbf{a}}(-1) = 15 > 0$. Let $r = \frac{1}{10}$ and $\mathbf{b} = (b_1,\ldots,b_5)\in B_r(\mathbf{a})$. Then $|b_j - a_j| \le \|\mathbf{b}-\mathbf{a}\| < \frac{1}{10}$ for all $j\in\{1,\ldots,5\}$ and hence $p_{\mathbf{b}}(0) < -1+\frac{1}{10} < 0$ and $p_{\mathbf{b}}(-1) > 15 - \frac{5}{10} > 0$. Therefore there exists $x\in\mathbb{R}$ such that $p_{\mathbf{b}}(x) = 0$ and so $\mathbf{b}\in S$. Thus $B_r(\mathbf{a})\subseteq S$ and hence $\mathbf{a}\in S^0$. Therefore option (A) is correct. If $\mathbf{a} = (3,1,-5,-2,1)$, then $p_{\mathbf{a}}(0) = 1 > 0$ and $p_{\mathbf{a}}(1) = -2 < 0$. If $\mathbf{a} = (-2,0,4,-6,3)$, then $p_{\mathbf{a}}(0) = 3 > 0$ and $p_{\mathbf{a}}(1) = -1 < 0$. If $\mathbf{a} = (-6,-2,0,-6,-1)$, then $p_{\mathbf{a}}(0) = -1 < 0$ and $p_{\mathbf{a}}(-1) = 1 > 0$. Therefore proceeding exactly as in case of option (A), we conclude that options (B), (C) and (D) are also correct.

10. Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by $f(x,y) = \begin{cases} \frac{xy^2+x^3+xy^4}{x^2+y^4} & \text{if } (x,y)\neq(0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$
Then which of the following statements is (are) FALSE ?

(A) Starting at $(0,0)$, among all $\mathbf{u} \in \mathbb{R}^2$ with $\|\mathbf{u}\| = 1$, f increases most rapidly in the direction of $\mathbf{u} = (1,0)$

(B) Starting at $(0,0)$, among all $\mathbf{u} \in \mathbb{R}^2$ with $\|\mathbf{u}\| = 1$, f increases most rapidly in the direction of $\mathbf{u} = (0,1)$

(C) Starting at $(0,0)$, among all $\mathbf{u} \in \mathbb{R}^2$ with $\|\mathbf{u}\| = 1$, f increases most rapidly in the direction of $\mathbf{u} = (-1,0)$

(D) Starting at $(0,0)$, among all $\mathbf{u} \in \mathbb{R}^2$ with $\|\mathbf{u}\| = 1$, f decreases most rapidly in the direction of $\mathbf{u} = (0,-1)$

Answer : (A), (B), (C), (D)

Explanation: If $\mathbf{u} = (u_1, u_2) \in \mathbb{R}^2$ with $\|\mathbf{u}\| = 1$, then
$D_{\mathbf{u}}f(0,0) = \lim\limits_{t \to 0} \frac{f(tu_1, tu_2)}{t} = \begin{cases} \frac{1}{u_1} & \text{if } u_1 \neq 0, \\ 0 & \text{if } u_1 = 0. \end{cases}$
Thus $D_{\mathbf{u}}f(0,0) \to \infty$ if $u_1 \to 0+$ and so the statements in (A), (B), (C) are false. Again, since $D_{\mathbf{u}}f(0,0) \to -\infty$ as $u_1 \to 0-$, the statement in (D) is false.

11. Which of the following is (are) closed set(s) in $\mathbb{R}^2$?

(A) $\left\{\left(x, \frac{1}{x}\right) : x \in \mathbb{R},\, x > 0\right\}$

(B) $\left\{\left(x, \frac{1}{x}\right) : x \in \mathbb{R},\, x > 0\right\} \bigcup \left\{(x,y) \in \mathbb{R}^2 : x \geq 0,\, y \geq 0,\, xy = 0\right\}$

(C) $\left\{\left(\frac{1}{4^m}, \frac{1}{5^n}\right) : m, n \in \mathbb{N}\right\}$

(D) $\left\{\left(\frac{1}{4^m}, \frac{1}{5^n}\right) : m, n \in \mathbb{N}\right\} \bigcup \left\{(0,0)\right\}$

Answer : (A), (B)

Explanation: Let $\big((x_n, y_n)\big)$ be any sequence in $S_1 = \left\{\left(x, \frac{1}{x}\right) : x \in \mathbb{R},\, x > 0\right\}$ and let $(x_n, y_n) \to (x,y) \in \mathbb{R}^2$. Then $x_n \to x$, $y_n \to y$ and so $x_n y_n \to xy$. Since $x_n > 0$ for all $n \in \mathbb{N}$, we get $x \geq 0$. Also, since $x_n y_n = 1$ for all $n \in \mathbb{N}$, we get $xy = 1$. Hence $x > 0$ and $y = \frac{1}{x}$. Thus $(x,y) \in S_1$ and so S_1 is a closed set in $\mathbb{R}^2$. Therefore option (A) is correct. Let $\big((x_n, y_n)\big)$ be any sequence in $S_1 \cup S_2$ such that $(x_n, y_n) \to (x,y) \in \mathbb{R}^2$, where $S_2 = \{(x,y) \in \mathbb{R}^2 : x \geq 0,\, y \geq 0,\, xy = 0\}$. If $(x_n, y_n) \in S_1$ for infinitely many $n \in \mathbb{N}$, then we get a subsequence $\big((x_{n_k}, y_{n_k})\big)$ of $\big((x_n, y_n)\big)$ such that $(x_{n_k}, y_{n_k}) \in S_1$ for all $k \in \mathbb{N}$ and $(x_{n_k}, y_{n_k}) \to (x,y)$. Following the argument given for option (A), we get $(x,y) \in S_1 \subseteq S_1 \cup S_2$. On the other hand, if $(x_n, y_n) \in S_1$ only for finitely many $n \in \mathbb{N}$, then $(x_n, y_n) \in S_2$ for infinitely many $n \in \mathbb{N}$. So we get a subsequence $\big((x_{n_k}, y_{n_k})\big)$ of $\big((x_n, y_n)\big)$ such that $(x_{n_k}, y_{n_k}) \in S_2$ for all $k \in \mathbb{N}$ and $(x_{n_k}, y_{n_k}) \to (x,y)$. Now, $x_{n_k} \to x$, $y_{n_k} \to y$ and so $x_{n_k} y_{n_k} \to xy$. Since $x_{n_k} \geq 0$ and $y_{n_k} \geq 0$ for all $k \in \mathbb{N}$, we get $x \geq 0$ and $y \geq 0$. Also, since $x_{n_k} y_{n_k} = 0$ for all $k \in \mathbb{N}$, we get $xy = 0$. Hence $(x,y) \in S_2 \subseteq S_1 \cup S_2$. Therefore $S_1 \cup S_2$ is a closed set in $\mathbb{R}^2$. So option (B) is correct. Let S_3 denote either of the sets in (C) and (D). Then $\left(\frac{1}{4}, \frac{1}{5^n}\right) \in S_3$ for all $n \in \mathbb{N}$ and $\left(\frac{1}{4}, \frac{1}{5^n}\right) \to \left(\frac{1}{4}, 0\right) \notin S_3$. Hence S_3 is not closed in $\mathbb{R}^2$ and so options (C) and (D) are not correct.

12. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that $f_x(x,y)$ and $f_y(x,y)$ exist (in $\mathbb{R}$) for all $(x,y) \in \mathbb{R}^2$. Which of the following statements is (are) always true?

(A) If both $f_x : \mathbb{R}^2 \to \mathbb{R}$ and $f_y : \mathbb{R}^2 \to \mathbb{R}$ are bounded, then f is continuous

(B) If f is discontinuous, then at least one of $f_x : \mathbb{R}^2 \to \mathbb{R}$ and $f_y : \mathbb{R}^2 \to \mathbb{R}$ is unbounded

(C) If f is discontinuous, then both $f_x : \mathbb{R}^2 \to \mathbb{R}$ and $f_y : \mathbb{R}^2 \to \mathbb{R}$ are unbounded
(D) If both $f_x : \mathbb{R}^2 \to \mathbb{R}$ and $f_y : \mathbb{R}^2 \to \mathbb{R}$ are unbounded, then f is unbounded

Answer : (A), (B), (C)

Explanation: By Ex.1 of Tutorial Problem Set - 3, option (A) is correct and by considering the contrapositive statement in (A), option (B) is also correct. By using the idea of the solution of Ex.21 of Practice Problem Set - 3 in the solution of Ex.1 of Tutorial Problem Set - 3, we can show that even if any one (instead of both) of f_x and f_y is bounded, then also f is continuous. Considering the contrapositive statement, we see that option (C) is correct. By considering $f(x,y) = \sin(x^2) + \sin(y^2)$ for all $(x,y) \in \mathbb{R}^2$, we can see that option (D) is not correct.

13. For $m, n, k, \ell \in \mathbb{N}$ with k, ℓ even, let $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ be defined by $f(x,y) = \frac{x^m y^n}{x^k + y^\ell}$ for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Then $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists (in $\mathbb{R}$) if
(A) $m = 5,\ n = 2,\ k = \ell = 6$ (B) $m = 2,\ n = 3,\ k = 6,\ \ell = 2$
(C) $m = 2,\ n = 3,\ k = 6,\ \ell = 4$ (D) $m = n = 3,\ k = 8,\ \ell = 4$

Answer : (A), (B), (C), (D)

Explanation: Let $m = 5$, $n = 2$, $k = \ell = 6$. If $0 < |x| \le |y|$, then $|f(x,y)| \le \frac{|x|^5 y^2}{y^6} \le \frac{|y|^5 y^2}{y^6} = |y| \le \sqrt{x^2+y^2}$ and if $|y| < |x|$, then $|f(x,y)| \le \frac{|x|^5 y^2}{x^6} \le \frac{|x|^5 x^2}{x^6} = |x| \le \sqrt{x^2+y^2}$. Hence it follows that $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$ and therefore option (A) is correct. If $m = 2$, $n = 3$, $k = 6$, $\ell = 2$, then $|f(x,y)| = \frac{y^2}{x^6+y^2} x^2 |y| \le x^2|y| \le (x^2+y^2)^{3/2}$ for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ and hence it follows that $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$. Therefore option (B) is correct. Let $m = 2$, $n = 3$, $k = 6$, $\ell = 4$. If $0 < |x| \le |y|^{2/3}$, then $|f(x,y)| \le \frac{x^2|y|^3}{y^4} \le \frac{|y|^{4/3}|y|^3}{y^4} = |y|^{1/3} \le (x^2+y^2)^{1/6}$ and if $|y|^{2/3} < |x|$, then $|f(x,y)| \le \frac{x^2|y|^3}{x^6} \le \frac{|x|^2|x|^{9/2}}{x^6} = |x|^{1/2} \le (x^2+y^2)^{1/4}$. Hence it follows that $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$ and therefore option (C) is correct. Let $m = n = 3$, $k = 8$, $\ell = 4$. If $0 < |x| \le |y|^{1/2}$, then $|f(x,y)| \le \frac{|x|^3|y|^3}{y^4} \le \frac{|y|^{3/2}|y|^3}{y^4} = |y|^{1/2} \le (x^2+y^2)^{1/4}$ and if $|y|^{1/2} < |x|$, then $|f(x,y)| \le \frac{|x|^3|y|^3}{x^8} \le \frac{|x|^3 x^6}{x^8} = |x| \le \sqrt{x^2+y^2}$. Hence it follows that $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$ and therefore option (D) is correct.

——— **END** ———