Department of Mathematics
IIT Guwahati

Quiz 2	**MA101**	**02-01-2021**	**Total points: 26**	**Time: 1 hour (10:00-11:00 am)**

- Keep the "**Quiz 2 Response Form**" ready so that you can keep filling with your answers.
- Note the Exam duration is 10:00-11:00 am. Start submitting your Response Form at 11:00 am.
- You may take screenshots of your responses before submitting. They may help in case your submission fails. Screenshots should clearly depict the time. Also tick the box "Send me an email receipt of my responses" at the bottom of the Form before submitting. The email you receive will also be used for viewing the marks, so do not delete it.
- The submission portal will be active till 11:07:00 am, but there will be heavy penalty for submission after 11:03:00 am, 20% of your mark will be deducted for every minute and fraction thereof.
- The response form permits ONLY ONE submission. It does not allow REVISION.

- The first question is writing your **Roll Number**. It is compulsory.
- There are **Ten** Quiz questions each for $2/3$ points. The points are indicated in the questions.
- There will not be any negative marks for incorrect answers.

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1. **Write your Roll Number (correctly).**

2. (3 points)
(Multiple correct options) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 1 - (1-x)^{\frac{2}{3}}$. Which of the following statements are true?

A) f is a differentiable function.
B) There is a unique point of local maximum for f in $\mathbb{R}$.
C) There are more than one point of local minimum for f in $\mathbb{R}$.
D) f has a Taylor series about the point $x = 0$.

Solution: (B), (D)
We have $f'(x) = \frac{2}{3}(1-x)^{-1/3}$ at $\mathbb{R} \setminus \{1\}$. At $x = 1$, using L'Hopital rules, we have

$$\lim_{x\to 1-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1-} \frac{1-(1-x)^{\frac{2}{3}}}{x-1} = \infty, \quad \lim_{x\to 1+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1+} \frac{1-(1-x)^{\frac{2}{3}}}{x-1} = -\infty.$$

A) is false, f is not differentiable at 1.
B) True, f is continuous, $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$. Hence, f has local maxima at $x = 1$.
C) False. There is not any point of local minimum.
D) True. f is infinitely differentiable in $(-1, 1)$.

3. (2 points)

Statement 1: Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^5 + 2x + 3$. Then $(f^{-1})'(0)$ equals $\frac{1}{2}$.

Statement 2: $\lim\limits_{n\to\infty} \lim\limits_{x\to 0+} \dfrac{e^{-\frac{1}{x}}}{x^n} = 1.$

Which of the following options is correct?

A) Both Statement 1 and Statement 2 are True.
B) Statement 1 is True, but Statement 2 is False.
C) Statement 2 is True, but Statement 1 is False.
D) Both Statement 1 and Statement 2 are False.

Solution: (D)
(1) False. $(f^{-1})'(0) = (f^{-1})'(f(-1)) = 1/f'(-1) = 1/7$.
(2) False. $\lim\limits_{x\to 0+} \dfrac{e^{-\frac{1}{x}}}{x^n} = 0$ for every $n \in \mathbb{N}$.

4. (3 points)

Statement 1: The radius of convergence of the power series $(1+1) + (3+5)x + (3^2+5^2)x^2 + \cdots$ is $1/5$.

Statement 2: For $x \in [-1, 1]$, if one approximates $\cos x$ by $1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!}$, then the result will be correct up to at least 3 decimals.

Which of the following options is correct?

A) Both Statement 1 and Statement 2 are True.
B) Statement 1 is True, but Statement 2 is False.
C) Statement 2 is True, but Statement 1 is False.
D) Both Statement 1 and Statement 2 are False.

Solution: (A)
(1) True. $|a_n|^{1/n} = (3^n + 5^n)^{1/n} = 5(1 + (3/5)^n)^{1/n} \to 5$, since $1 < (1 + (3/5)^n)^{1/n} < 2^{1/n} \to 1$.
(2) True. $R_7 = \dfrac{f^{(8)}(t)x^8}{8!} = \dfrac{\cos t\, x^8}{8!}$, for some t between x and 0. So, $|R_7| \leq \dfrac{1}{8!} = \dfrac{1}{42,000} < 10^{-4} \times 0.23$.

5. (3 points)

Suppose it is given that $f : (0, 1) \to \mathbb{R}$ is differentiable and that f' is bounded. The following statements contain an assertion and arguments for a proof of the assertion, but the statements are not in correct order.

(a) $(f(\frac{1}{n}))$ is a convergent sequence.
(b) There exists $\alpha > 0$ such that $|f'(x)| \leq \alpha$ for $x \in (0, 1)$.
(c) $(f(\frac{1}{n}))$ is a Cauchy sequence.

(d) For $m > n \geq 1$, there exists $x_{mn} \in [\frac{1}{m}, \frac{1}{n}]$ such that $|f(\frac{1}{n}) - f(\frac{1}{m})| = |f'(x_{mn})|\,|\frac{1}{n} - \frac{1}{m}| \leq \alpha\,|\frac{1}{n} - \frac{1}{m}|$.

The correct order of the statements (arguments ended by the conclusion) is

A) (b), (d), (a), (c)
B) (c), (b), (d), (a)
C) (a), (b), (d), (c)
D) (b), (d), (c), (a)

Solution: (D)

There exists $\alpha > 0$ such that $|f'(x)| \leq \alpha$ for $x \in (0,1)$. By MVT

$$|f(1/n) - f(1/m)| = |f'(x_{mn})|\,|1/n - 1/m| \leq \alpha\,|1/n - 1/m|$$

for all $m \geq 2, n \geq 2$, where x_{mn} lies between $1/n$ and $1/m$. Hence it follows that $(f(1/n))$ is a Cauchy sequence. Since a Cauchy sequence is convergent, $\lim_{n\to\infty} f(1/n)$ exists.

6. (2 points)
Let $f, g : (0,1)$ be given by $f(x) = x^2 \sin\frac{1}{x}, g(x) = \sin x$. The following argument has a flaw, because the conclusion is wrong.

A) $\lim\limits_{x\to 0+} \dfrac{f(x)}{g(x)} = 0.$

B) Therefore, by L'Hópital rule, $\lim\limits_{x\to 0+} \dfrac{f'(x)}{g'(x)} = 0.$

C) Therefore, $\lim\limits_{x\to 0+} (2x\sin\frac{1}{x} - \cos\frac{1}{x}) = 0$, since $\lim\limits_{x\to 0+} \cos x = 1 \neq 0$.

D) Hence, $\lim\limits_{x\to 0+} \cos\frac{1}{x} = \lim\limits_{x\to 0+} 2x\sin\frac{1}{x} = 0$.

From steps (A) to step (D), which is the first one which does not hold?

Solution: (B) It uses the converse of the assertion in LH1, which is not true.

7. (2 points)
(Multiple correct options) Select the correct statements.

A) There is a Riemann integrable function on $[0,1]$ which is discontinuous at infinitely many points.

B) There is a continuous function $f : [0,1] \to \mathbb{R}$ such that $f \geq 0$, $f \neq 0$ and $\int_0^1 f = 0$.

C) If $0 \leq f(x) \leq x^2$ on $[0,1]$, then f must be Riemann integrable.

D) If $f : \mathbb{R} \to [0,\infty)$ is such that $\int_1^\infty f = 10$, then $\lim\limits_{n\to\infty} \int_n^{n+1} f = 0$.

Solution: (A), (D)
A) True: For example, take Thomae function.

B) False: with the given conditions $\int_0^1 f = 0$.
C) False: Take $f : [0,1] \to \mathbb{R}$ defined by $f(x) = x^2$ if $x \in \mathbb{Q}$ and o, otherwise.
D) True: Let $F(t) = \int_0^t f$. Then, $\lim_{n\to\infty} F(n) = 10$. Therefore, $\lim_{n\to\infty}(F(n+1) - F(n)) = 0$.

8. (2 points)
Statement 1: Let $f : [1,3] \to \mathbb{R}$ be defined by $f(x) = 1$, if x is rational, and $f(x) = 2$, if x is irrational. Then, there is a partition P of $[1,3]$ such that $U(P,f) - L(P,f) < 2$.
Statement 2: If $f : [0,1] \to \mathbb{R}$ is continuous and $\int_0^x f(t)dt = \int_x^1 f(t)dt$ for all $x \in [0,1]$, then $f(x) = 0$ for all $x \in [0,1]$.

Which of the following options is correct?

A) Both Statement 1 and Statement 2 are True.
B) Statement 1 is True, but Statement 2 is False.
C) Statement 2 is True, but Statement 1 is False.
D) Both Statement 1 and Statement 2 are False.

Solution: (C)
(1) False. For any partition P of $[1,3]$, $U(P,f) - L(P,f)$ equals 2.
(2) True. Let $F(x) = \int_0^x f(t)dt$. Since f is continuous, F is differentiable, and $F'(x) = f(x)$ for $x \in [0,1]$. By the given condition, $F(x) = F(1) - F(x)$, that is, $2F(x) = F(1)$. This gives $2F'(x) = 0$, that is, $f(x) = 0$ for $x \in [0,1]$.

9. (3 points)
Statement 1: There is a function $f : \mathbb{R} \to \mathbb{R}$ which is Riemann integrable on every closed interval $[a,b]$, but the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = \int_x^{x+1} f(t)dt$ is NOT differentiable.
Statement 2: The function $f : [0,\pi/2] \to \mathbb{R}$ defined by $f(x) = \int_{\sin x}^{\cos x} \sqrt{1-t^2}\,dt$ has a local extremum in $(0,\pi/2)$.

Which of the following options is correct?

A) Both Statement 1 and Statement 2 are True.
B) Statement 1 is True, but Statement 2 is False.
C) Statement 2 is True, but Statement 1 is False.
D) Both Statement 1 and Statement 2 are False.

Solution: (B)
(1) True. Take $f(x) = 0$ for $x < 0$ and 1 for $x \geq 0$. Then

$$g(x) = \begin{cases} 0, & x \leq -1, \\ x+1, & -1 < x < 0, \\ 1, & x \geq 1. \end{cases}$$

(2) False. $f'(x) = \sqrt{1-\cos^2 x}(-\sin x) - \sqrt{1-\sin^2 x}(\cos x) = -1$. So, f is does not have any extremum in the interior.

10. (3 points)

Statement 1: The improper integral $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}dx$ diverges.

Statement 2: The integral $\displaystyle\int_0^{1/e} \frac{dx}{x(\ln x)^2}$ converges to 1.

Which of the following options is correct?

A) Both Statement 1 and Statement 2 are True.
B) Statement 1 is True, but Statement 2 is False.
C) Statement 2 is True, but Statement 1 is False.
D) Both Statement 1 and Statement 2 are False.

Solution: (C)

(1) False. Since $0 \le \dfrac{\sin^2 x}{x^2} \le \dfrac{1}{x^2}$ and $\int_1^\infty \frac{1}{x^2}dx$ converges. Therefore, by comparison test, $\int_1^\infty \frac{\sin^2 x}{x^2}dx$ converges. Also note that $\frac{\sin^2 x}{x^2}$ is continuous on $(0,1]$ and $\to 1$ as $x \to 0$. Thus, $\int_0^1 \frac{\sin^2 x}{x^2}dx$ exists in the sense of Riemann, (because the integrand can be redefined as a continuous function). Thus the improper integral converges.

(2) True. $\displaystyle\int_t^{1/e} \frac{dx}{x(\ln x)^2} = -\frac{1}{\ln x}\Big|_t^{1/e} = -\Big[\frac{1}{\ln(1/e)} - \frac{1}{\ln t}\Big] = 1 + \frac{1}{\ln t} \to 1$, as $t \to 0+$.

11. (3 points)

Statement 1: The length of the curve $r = 3(1+\cos\theta)$, $-\pi < \theta \le \pi$, is 24.

Statement 2: The volume of the solid produced by revolving the region enclosed by the curve $y^2 = x$ and the line $y = x$ about the x-axis is $\pi/6$.

Which of the following options is correct?

A) Both Statement 1 and Statement 2 are True.
B) Statement 1 is True, but Statement 2 is False.
C) Statement 2 is True, but Statement 1 is False.
D) Both Statement 1 and Statement 2 are False.

Solution: (A)

(1) Length $= 2\int_0^\pi \sqrt{r^2 + r'^2}\,d\theta = 6\int_0^\pi \sqrt{1+\cos^2\theta + 2\cos\theta + \sin^2\theta}\,d\theta = 6\int_0^\pi \sqrt{2+2\cos\theta}\,d\theta$

$$= 6\int_0^\pi \sqrt{4\cos^2(\theta/2)}\,d\theta = 12\int_0^\pi \cos(\theta/2)\,d\theta = 24\int_0^{\pi/2} \cos(t)\,dt = 24.$$

(2) Volume of the solid formed $= \pi\int_0^1 (x - x^2)dx = \pi(\frac{1}{2} - \frac{1}{3}) = \pi/6$.