MA 101 (Mathematics I)

Multivariable Calculus : Hints / Solutions of Tutorial Problem Set - 3

---

1. Let S be a nonempty open subset of $\mathbb{R}^2$ and let $f : S \to \mathbb{R}$ be such that the partial derivatives f_x and f_y exist at each point of S. If $f_x : S \to \mathbb{R}$ and $f_y : S \to \mathbb{R}$ are bounded, then show that f is continuous.

   Solution: Since f_x and f_y are bounded, there exist $M_1, M_2 > 0$ such that $|f_x(x, y)| \le M_1$ and $|f_y(x, y)| \le M_2$ for all $(x, y) \in S$. Let $(x_0, y_0) \in S$. Since S is open in $\mathbb{R}^2$, there exists $r > 0$ such that $B_r\big((x_0, y_0)\big) \subseteq S$. For all $h, k \in \mathbb{R}$ with $|h| < \frac{r}{2}$, $|k| < \frac{r}{2}$, we have $|f(x_0 + h, y_0 + k) - f(x_0, y_0)| = |f(x_0 + h, y_0 + k) - f(x_0, y_0 + k) + f(x_0, y_0 + k) - f(x_0, y_0)|$
   $\le |f(x_0 + h, y_0 + k) - f(x_0, y_0 + k)| + |f(x_0, y_0 + k) - f(x_0, y_0)|$
   $= |h||f_x(x_0 + \theta_1 h, y_0 + k)| + |k||f_y(x_0, y_0 + \theta_2 k)|$ for some $\theta_1, \theta_2 \in (0, 1)$ (using Lagrange's mean value theorem of single real variable). Hence if $\varepsilon > 0$, then choosing $\delta = \min\{\frac{r}{2}, \frac{\varepsilon}{M_1+M_2}\} > 0$, we find that $|f(x_0 + h, y_0 + k) - f(x_0, y_0)| \le M_1|h| + M_2|k| < \varepsilon$ for all $(h, k) \in \mathbb{R}^2$ with $\|(h, k)\| = \sqrt{h^2 + k^2} < \delta$. Therefore f is continuous at (x_0, y_0). Since $(x_0, y_0) \in S$ is arbitrary, f is continuous.

2. Find all $\mathbf{u} \in \mathbb{R}^2$ with $\|\mathbf{u}\| = 1$ for which the directional derivative $D_{\mathbf{u}}f(0, 0)$ exists (in $\mathbb{R}$), if for all $(x, y) \in \mathbb{R}^2$, $f(x, y) = \begin{cases} 1 & \text{if } y < x^2 < 2y, \\ 0 & \text{otherwise.} \end{cases}$

   Solution: Let $\mathbf{u} = (u_1, u_2) \in \mathbb{R}^2$ with $\|\mathbf{u}\| = 1$. We have
   $\lim\limits_{t\to 0} \frac{f((0,0)+t\mathbf{u})-f(0,0)}{t} = \lim\limits_{t\to 0} \frac{f(tu_1,tu_2)}{t} = \lim\limits_{t\to 0} \frac{0}{t} = 0$. (The inequalities $tu_2 < t^2u_1^2 < 2tu_2$ are equivalent to the inequalities (i) $u_2 < tu_1^2 < 2u_2$ if $t > 0$ and (ii) $u_2 > tu_1^2 > 2u_2$ if $t < 0$. We can make $|tu_1^2|$ arbitrarily small for sufficiently small $|t| > 0$ and hence for such t, at least one inequality in each of (i) and (ii) cannot be satisfied. Thus we get $f(tu_1, tu_2) = 0$ for sufficiently small $|t| > 0$.)
   Therefore $D_{\mathbf{u}}f(0, 0)$ exists (and equals 0) for each $\mathbf{u} \in \mathbb{R}^2$ with $\|\mathbf{u}\| = 1$.

3. State TRUE or FALSE with justification: If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous such that all the directional derivatives of f at $(0, 0)$ exist (in $\mathbb{R}$), then f must be differentiable at $(0, 0)$.

   Solution: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \begin{cases} \frac{x^2y\sqrt{x^2+y^2}}{x^4+y^2} & \text{if } (x, y) \ne (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$
   We know that f is continuous at each point of $\mathbb{R}^2 \setminus \{(0, 0)\}$. Let $\varepsilon > 0$. We have
   $|f(x, y) - f(0, 0)| = \big|\frac{x^2y}{x^4+y^2}\big|\sqrt{x^2 + y^2} \le \frac{1}{2}\sqrt{x^2 + y^2}$ for all $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ and
   $|f(x, y) - f(0, 0)| = 0$ if $(x, y) = (0, 0)$. Hence choosing $\delta = 2\varepsilon > 0$, we find that
   $|f(x, y) - f(0, 0)| < \varepsilon$ for all $(x, y) \in \mathbb{R}^2$ satisfying $\|(x, y) - (0, 0)\| = \sqrt{x^2 + y^2} < \delta$. This shows that f is continuous at $(0, 0)$ and therefore f is continuous.
   If $\mathbf{u} = (u_1, u_2) \in \mathbb{R}^2$ with $\|\mathbf{u}\| = 1$, then $\lim\limits_{t\to 0} \frac{f((0,0)+t\mathbf{u})-f(0,0)}{t} = \lim\limits_{t\to 0} \frac{u_1^2u_2|t|\sqrt{u_1^2+u_2^2}}{t^2u_1^4+u_2^2} = 0$, *i.e.* $D_{\mathbf{u}}f(0, 0)$ exists. Hence all the directional derivatives of f at $(0, 0)$ exist.
   Again, $\lim\limits_{(h,k)\to(0,0)} \frac{|f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)|}{\sqrt{h^2+k^2}} = \lim\limits_{(h,k)\to(0,0)} \frac{h^2k}{h^4+k^2} \ne 0$, since $(\frac{1}{n}, \frac{1}{n^2}) \to (0, 0)$ but

$\frac{\frac{1}{n^2}\cdot\frac{1}{n^2}}{\frac{1}{n^4}+\frac{1}{n^4}} = \frac{1}{2} \not\to 0$. Hence f is not differentiable at $(0,0)$.
Therefore the given statement is FALSE.

4. Determine all the points of $\mathbb{R}^2$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable, if for all $(x,y) \in \mathbb{R}^2$,
$f(x,y) = \begin{cases} x^{4/3}\sin\left(\frac{y}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$

   Solution: Let $E = \{(x,y) \in \mathbb{R}^2 : x \neq 0\}$. Since $f_x(x,y) = \frac{4}{3}x^{1/3}\sin\left(\frac{y}{x}\right) - \frac{y}{x^{2/3}}\cos\left(\frac{y}{x}\right)$ and $f_y(x,y) = x^{1/3}\cos\left(\frac{y}{x}\right)$ for all $(x,y) \in E$, $f_x : E \to \mathbb{R}$ and $f_y : E \to \mathbb{R}$ are continuous. Hence f is differentiable at all $(x,y) \in E$. Let $y_0 \in \mathbb{R}$ and let $\varepsilon > 0$. Then
$f_x(0,y_0) = \lim\limits_{h\to 0} \frac{f(h,y_0)-f(0,y_0)}{h} = \lim\limits_{h\to 0} h^{1/3}\sin\left(\frac{y_0}{h}\right) = 0$ (since $|h^{1/3}\sin\left(\frac{y_0}{h}\right)| \leq |h|^{1/3}$ for all $h \in \mathbb{R}\setminus\{0\}$) and $f_y(0,y_0) = \lim\limits_{k\to 0}\frac{f(0,y_0+k)-f(0,y_0)}{k} = 0$. Also, for all $(x,y) \in E$, we have $f_y(x,y) = x^{1/3}\cos\left(\frac{y}{x}\right)$, and so $|f_y(x,y) - f_y(0,y_0)| \leq |x|^{1/3} < \varepsilon$ for all $(x,y) \in B_\delta\big((0,y_0)\big)$, where $\delta = \varepsilon^3 > 0$. Thus $f_x(0,y_0)$ exists (in $\mathbb{R}$), $f_y(x,y)$ exists (in $\mathbb{R}$) for all $(x,y) \in \mathbb{R}^2$ and $f_y : \mathbb{R}^2 \to \mathbb{R}$ is continuous at $(0,y_0)$. Hence by Ex.21 of Practice Problem Set - 3, f is differentiable at $(0,y_0)$. Therefore f is differentiable at all points of $\mathbb{R}^2$.

   Alternative solution: As shown above f is differentiable at all $(x,y) \in \mathbb{R}^2$ for which $x \neq 0$. Let $y_0 \in \mathbb{R}$. Then as shown above $f_x(0,y_0) = f_y(0,y_0) = 0$. For all $(h,k) \in \mathbb{R}^2$ with $h \neq 0$, we have
$\varepsilon(h,k) = \frac{|f(h,y_0+k)-f(0,y_0)-hf_x(0,y_0)-kf_y(0,y_0)|}{\sqrt{h^2+k^2}} = \frac{h^{4/3}|\sin(\frac{y_0+k}{h})|}{\sqrt{h^2+k^2}} = |h|^{1/3}\frac{|h|}{\sqrt{h^2+k^2}}|\sin(\frac{y_0+k}{h})| \leq |h|^{1/3}$.
Also, $\varepsilon(0,k) = 0$ for all $k \in \mathbb{R}\setminus\{0\}$. Hence it follows that $\lim\limits_{(h,k)\to(0,0)} \varepsilon(h,k) = 0$. Consequently f is differentiable at $(0,y_0)$. Therefore f is differentiable at all points of $\mathbb{R}^2$.

5. Let $f : S \subseteq \mathbb{R}^m \to \mathbb{R}$ be differentiable at $\mathbf{x}_0 \in S^0$ and let $f(\mathbf{x}_0) = 0$. If $g : S \to \mathbb{R}$ is continuous at $\mathbf{x}_0$, then show that $fg : S \to \mathbb{R}$, defined by $(fg)(\mathbf{x}) = f(\mathbf{x})g(\mathbf{x})$ for all $\mathbf{x} \in S$, is differentiable at $\mathbf{x}_0$.

   Solution: Since f is differentiable at $\mathbf{x}_0$, there exists $\alpha \in \mathbb{R}^m$ such that
$\lim\limits_{\mathbf{h}\to\mathbf{0}} \frac{|f(\mathbf{x}_0+\mathbf{h})-f(\mathbf{x}_0)-\alpha\cdot\mathbf{h}|}{\|\mathbf{h}\|} = 0$. For all $\mathbf{h} \in \mathbb{R}^m$ for which $\mathbf{x}_0 + \mathbf{h} \in S$, we have
$(fg)(\mathbf{x}_0+\mathbf{h}) - (fg)(\mathbf{x}_0) - g(\mathbf{x}_0)\alpha\cdot\mathbf{h} = (f(\mathbf{x}_0+\mathbf{h}) - f(\mathbf{x}_0) - \alpha\cdot\mathbf{h})g(\mathbf{x}_0+\mathbf{h}) + (g(\mathbf{x}_0+\mathbf{h}) - g(\mathbf{x}_0))\alpha\cdot\mathbf{h}$.
Hence for all $\mathbf{h} \in \mathbb{R}^m\setminus\{\mathbf{0}\}$ for which $\mathbf{x}_0 + \mathbf{h} \in S$, we have
$\frac{|(fg)(\mathbf{x}_0+\mathbf{h})-(fg)(\mathbf{x}_0)-g(\mathbf{x}_0)\alpha\cdot\mathbf{h}|}{\|\mathbf{h}\|} \leq \frac{|(f(\mathbf{x}_0+\mathbf{h})-f(\mathbf{x}_0)-\alpha\cdot\mathbf{h})|}{\|\mathbf{h}\|}|g(\mathbf{x}_0+\mathbf{h})| + |g(\mathbf{x}_0+\mathbf{h}) - g(\mathbf{x}_0)|\frac{|\alpha\cdot\mathbf{h}|}{\|\mathbf{h}\|}$. Since g is continuous at $\mathbf{x}_0$, $\lim\limits_{\mathbf{h}\to\mathbf{0}} g(\mathbf{x}_0+\mathbf{h}) = g(\mathbf{x}_0)$ and since $|\alpha\cdot\mathbf{h}| \leq \|\alpha\|\,\|\mathbf{h}\|$, it follows that $\lim\limits_{\mathbf{h}\to\mathbf{0}} \frac{|(fg)(\mathbf{x}_0+\mathbf{h})-(fg)(\mathbf{x}_0)-g(\mathbf{x}_0)\alpha\cdot\mathbf{h}|}{\|\mathbf{h}\|} = 0$. Since $g(\mathbf{x}_0)\alpha \in \mathbb{R}^m$, we conclude that fg is differentiable at $\mathbf{x}_0$.

6. Show that $f : S \subseteq \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(x_0,y_0) \in S^0$ iff there exist functions $\varphi, \psi : S \to \mathbb{R}$ such that φ, ψ are continuous at (x_0,y_0) and $f(x,y) - f(x_0,y_0) = (x-x_0)\varphi(x,y) + (y-y_0)\psi(x,y)$ for all $(x,y) \in S$.

   Solution: We first assume that f is differentiable at (x_0,y_0). Then $\alpha = f_x(x_0,y_0)$ and $\beta = f_y(x_0,y_0)$ exist (in $\mathbb{R}$). For each $(x,y) \in S$, let
$g(x,y) = f(x,y) - f(x_0,y_0) - \alpha(x-x_0) - \beta(y-y_0)$,

$\varphi(x,y) = \begin{cases} \alpha + \frac{(x-x_0)g(x,y)}{(x-x_0)^2+(y-y_0)^2} & \text{if } (x,y) \neq (x_0,y_0), \\ \alpha & \text{if } (x,y) = (x_0,y_0), \end{cases}$

and $\psi(x,y) = \begin{cases} \beta + \frac{(y-y_0)g(x,y)}{(x-x_0)^2+(y-y_0)^2} & \text{if } (x,y) \neq (x_0,y_0), \\ \beta & \text{if } (x,y) = (x_0,y_0). \end{cases}$

If $(x,y) \in S \setminus \{(x_0,y_0)\}$, then

$(x-x_0)\varphi(x,y) + (y-y_0)\psi(x,y) = \alpha(x-x_0) + \beta(y-y_0) + g(x,y) = f(x,y) - f(x_0,y_0)$. Also, if $(x,y) = (x_0,y_0)$, then $(x-x_0)\varphi(x,y) + (y-y_0)\psi(x,y) = 0 = f(x,y) - f(x_0,y_0)$. Hence $f(x,y) - f(x_0,y_0) = (x-x_0)\varphi(x,y) + (y-y_0)\psi(x,y)$ for all $(x,y) \in S$.

Again, for all $(x,y) \in S \setminus \{(x_0,y_0)\}$, we have

$|\varphi(x,y) - \varphi(x_0,y_0)| = \frac{|x-x_0|\,|g(x,y)|}{(x-x_0)^2+(y-y_0)^2} \leq \frac{|g(x,y)|}{\sqrt{(x-x_0)^2+(y-y_0)^2}}$. Since f is differentiable at (x_0,y_0), $\lim\limits_{(x,y)\to(x_0,y_0)} \frac{|g(x,y)|}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$ and hence it follows that $\lim\limits_{(x,y)\to(x_0,y_0)} \varphi(x,y) = \varphi(x_0,y_0)$. Therefore φ is continuous at (x_0,y_0). Similarly we can show that ψ is continuous at (x_0,y_0).

Conversely, let there exist functions $\varphi, \psi : S \to \mathbb{R}$ such that φ, ψ are continuous at (x_0,y_0) and $f(x,y) - f(x_0,y_0) = (x-x_0)\varphi(x,y) + (y-y_0)\psi(x,y)$ for all $(x,y) \in S$. Then for all $(x,y) \in S \setminus \{(x_0,y_0)\}$, we have $\frac{|f(x,y)-f(x_0,y_0)-(x-x_0)\varphi(x_0,y_0)-(y-y_0)\psi(x_0,y_0)|}{\sqrt{(x-x_0)^2+(y-y_0)^2}}$

$\leq \frac{|x-x_0|}{\sqrt{(x-x_0)^2+(y-y_0)^2}}|\varphi(x,y) - \varphi(x_0,y_0)| + \frac{|y-y_0|}{\sqrt{(x-x_0)^2+(y-y_0)^2}}|\psi(x,y) - \psi(x_0,y_0)|$

$\leq |\varphi(x,y) - \varphi(x_0,y_0)| + |\psi(x,y) - \psi(x_0,y_0)|$. Since φ and ψ are continuous at (x_0,y_0), $\lim\limits_{(x,y)\to(x_0,y_0)} |\varphi(x,y) - \varphi(x_0,y_0)| = 0$ and $\lim\limits_{(x,y)\to(x_0,y_0)} |\psi(x,y) - \psi(x_0,y_0)| = 0$.

Hence $\lim\limits_{(x,y)\to(x_0,y_0)} \frac{|f(x,y)-f(x_0,y_0)-(x-x_0)\varphi(x_0,y_0)-(y-y_0)\psi(x_0,y_0)|}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$ and therefore f is differentiable at (x_0,y_0).

7. Let the temperature $T(x,y)$ at any point $(x,y) \in \mathbb{R}^2$ be given by $T(x,y) = 2x^2 + xy + y^2$. An insect is at the point $(1,1)$.
   (a) What is the best direction for the insect to move to feel cooler?
   (b) In which direction should the insect move to feel no change in temperature?

Solution: Since $T_x(x,y) = 4x + y$ and $T_y(x,y) = x + 2y$ for all $(x,y) \in \mathbb{R}^2$, $T_x : \mathbb{R}^2 \to \mathbb{R}$ and $T_y : \mathbb{R}^2 \to \mathbb{R}$ are continuous and hence $T : \mathbb{R}^2 \to \mathbb{R}$ is differentiable.

Since $\nabla T(1,1) = \big(T_x(1,1), T_y(1,1)\big) = (5,3)$, the temperature will decrease fastest in the direction of $-\frac{1}{\|\nabla T(1,1)\|}\nabla T(1,1) = \big(-\frac{5}{\sqrt{34}}, -\frac{3}{\sqrt{34}}\big)$ and so this is the best direction for the insect to start moving to feel cooler.

Again, if $\mathbf{u} = (u_1,u_2) \in \mathbb{R}^2$ with $\|\mathbf{u}\| = 1$, is the direction for the insect to feel no change in temperature, then we must have $D_{\mathbf{u}}T(1,1) = \nabla T(1,1) \cdot \mathbf{u} = 0$. This gives $5u_1 + 3u_2 = 0$. Since we also have $u_1^2 + u_2^2 = 1$, we get $\mathbf{u} = \big(\frac{3}{\sqrt{34}}, -\frac{5}{\sqrt{34}}\big)$ or $\big(-\frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}}\big)$.