MA 101 (Mathematics I)

Multivariable Calculus : Hints / Solutions of Practice Problem Set - 1

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1. If $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^m$, then show that $\big|\, \|\mathbf{x}\| - \|\mathbf{y}\| \,\big| \leq \|\mathbf{x} - \mathbf{y}\|$.

   Solution: We have $\|\mathbf{x}\| = \|\mathbf{x} - \mathbf{y} + \mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y}\|$ and so $\|\mathbf{x}\| - \|\mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\|$. Similarly $\|\mathbf{y}\| - \|\mathbf{x}\| \leq \|\mathbf{y} - \mathbf{x}\| = \|\mathbf{x} - \mathbf{y}\|$. Therefore $\big|\, \|\mathbf{x}\| - \|\mathbf{y}\| \,\big| \leq \|\mathbf{x} - \mathbf{y}\|$.

2. If $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^m$, then show that $\|\mathbf{x} + \mathbf{y}\|\, \|\mathbf{x} - \mathbf{y}\| \leq \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$.

   Solution: We have $\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\,\mathbf{x}\cdot\mathbf{y}$ and $\|\mathbf{x} - \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\,\mathbf{x}\cdot\mathbf{y}$.
   Hence $\|\mathbf{x} + \mathbf{y}\|^2\, \|\mathbf{x} - \mathbf{y}\|^2 = (\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)^2 - 4\,(\mathbf{x}\cdot\mathbf{y})^2 \leq (\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)^2$.
   Therefore $\|\mathbf{x} + \mathbf{y}\|\, \|\mathbf{x} - \mathbf{y}\| \leq \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$.

3. If $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^m$, then show that $\|\mathbf{x}\| \leq \max\{\|\mathbf{x} + \mathbf{y}\|,\, \|\mathbf{x} - \mathbf{y}\|\}$.

   Solution: If possible, let $\|\mathbf{x}\| > \max\{\|\mathbf{x}+\mathbf{y}\|,\, \|\mathbf{x}-\mathbf{y}\|\}$. Then $\|\mathbf{x}\| > \|\mathbf{x}+\mathbf{y}\|$ and $\|\mathbf{x}\| > \|\mathbf{x}-\mathbf{y}\|$ and so $2\|\mathbf{x}\| = \|(\mathbf{x} + \mathbf{y}) + (\mathbf{x} - \mathbf{y})\| \leq \|\mathbf{x} + \mathbf{y}\| + \|\mathbf{x} - \mathbf{y}\| < \|\mathbf{x}\| + \|\mathbf{x}\| = 2\|\mathbf{x}\|$, which is a contradiction. Hence $\|\mathbf{x}\| \leq \max\{\|\mathbf{x} + \mathbf{y}\|,\, \|\mathbf{x} - \mathbf{y}\|\}$.

4. If $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^m$, then show that $\|\mathbf{x} + \alpha\mathbf{y}\| \geq \|\mathbf{x}\|$ for all $\alpha \in \mathbb{R}$ iff $\mathbf{x} \cdot \mathbf{y} = 0$.

   Solution: We first assume that $\mathbf{x} \cdot \mathbf{y} = 0$. If $\alpha \in \mathbb{R}$, then we have
   $\|\mathbf{x} + \alpha\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2\,\mathbf{x}\cdot\alpha\mathbf{y} + \|\alpha\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2\alpha\,\mathbf{x}\cdot\mathbf{y} + |\alpha|^2\|\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + |\alpha|^2\|\mathbf{y}\|^2 \geq \|\mathbf{x}\|^2$ and hence $\|\mathbf{x} + \alpha\mathbf{y}\| \geq \|\mathbf{x}\|$.
   Conversely, let $\|\mathbf{x} + \alpha\mathbf{y}\| \geq \|\mathbf{x}\|$ for all $\alpha \in \mathbb{R}$. If possible, let $\mathbf{x} \cdot \mathbf{y} \neq 0$. Then $\mathbf{y} \neq \mathbf{0}$ and so $\|\mathbf{y}\| \neq 0$. If $\alpha = -\frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{y}\|^2}$, then $\alpha \in \mathbb{R}$ and we have
   $\|\mathbf{x} + \alpha\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2\alpha\,\mathbf{x}\cdot\mathbf{y} + |\alpha|^2\|\mathbf{y}\|^2 = \|\mathbf{x}\|^2 - \frac{(\mathbf{x}\cdot\mathbf{y})^2}{\|\mathbf{y}\|^2} < \|\mathbf{x}\|^2$. Thus $\|\mathbf{x} + \alpha\mathbf{y}\| < \|\mathbf{x}\|$, which is a contradiction. Therefore $\mathbf{x} \cdot \mathbf{y} = 0$.

5. Let $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^m$ and $\alpha > 0$. Show that $|\mathbf{x} \cdot \mathbf{y}| \leq \alpha\|\mathbf{x}\|^2 + \frac{1}{4\alpha}\|\mathbf{y}\|^2$.

   Solution: We have
   $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|\, \|\mathbf{y}\| = 2\sqrt{\alpha}\|\mathbf{x}\|\, \frac{1}{2\sqrt{\alpha}}\|\mathbf{y}\| \leq \left(\sqrt{\alpha}\|\mathbf{x}\|\right)^2 + \left(\frac{1}{2\sqrt{\alpha}}\|\mathbf{y}\|\right)^2 = \alpha\|\mathbf{x}\|^2 + \frac{1}{4\alpha}\|\mathbf{y}\|^2$.

6. Let $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^m$. Show that $\big|\, \|\mathbf{x}\| - \|\mathbf{y}\| \,\big| = \|\mathbf{x} - \mathbf{y}\|$ iff $\alpha\mathbf{x} = \beta\mathbf{y}$ for some $\alpha, \beta \geq 0$ with $(\alpha, \beta) \neq (0, 0)$.

   Solution: We first assume that $\big|\, \|\mathbf{x}\| - \|\mathbf{y}\| \,\big| = \|\mathbf{x} - \mathbf{y}\|$. Then $\big|\, \|\mathbf{x}\| - \|\mathbf{y}\| \,\big|^2 = \|\mathbf{x} - \mathbf{y}\|^2$, which gives $\|\mathbf{x}\|\, \|\mathbf{y}\| = \mathbf{x} \cdot \mathbf{y}$. So $\|\mathbf{x}\|\, \|\mathbf{y}\| = |\mathbf{x} \cdot \mathbf{y}|$ and hence by the equality condition in Cauchy-Schwarz inequality, we get $\mathbf{y} = \mathbf{0}$ or $\mathbf{x} = t\mathbf{y}$ for some $t \in \mathbb{R}$. If $\mathbf{y} = \mathbf{0}$, then by taking $\alpha = 0$, $\beta = 1$, we find that $\alpha\mathbf{x} = \beta\mathbf{y}$ and α, $\beta \geq 0$ with $(\alpha, \beta) \neq (0, 0)$. Again, if $\mathbf{y} \neq \mathbf{0}$ and $\mathbf{x} = t\mathbf{y}$, then since we have $\|\mathbf{x}\|\, \|\mathbf{y}\| = \mathbf{x} \cdot \mathbf{y}$, we obtain $\|t\mathbf{y}\|\, \|\mathbf{y}\| = t\mathbf{y} \cdot \mathbf{y}$, *i.e.* $|t|\, \|\mathbf{y}\|^2 = t\|\mathbf{y}\|^2$. Since $\|\mathbf{y}\| \neq 0$, we get $|t| = t$ and hence $t \geq 0$. Taking $\alpha = 1$, $\beta = t$, we find that $\alpha\mathbf{x} = \beta\mathbf{y}$ and

$\alpha,\,\beta \geq 0$ with $(\alpha,\beta) \neq (0,0)$.

Conversely, let $\alpha\mathbf{x} = \beta\mathbf{y}$ for some $\alpha,\,\beta \geq 0$ with $(\alpha,\beta) \neq (0,0)$. Then $\alpha \neq 0$ or $\beta \neq 0$. We first assume that $\alpha \neq 0$. Then $\mathbf{x} = t\mathbf{y}$, where $t = \frac{\beta}{\alpha} \geq 0$. Now,
$\big|\, \|\mathbf{x}\| - \|\mathbf{y}\| \,\big| = \big|\, \|t\mathbf{y}\| - \|\mathbf{y}\| \,\big| = |t-1|\,\|\mathbf{y}\|$ and $\|\mathbf{x}-\mathbf{y}\| = \|t\mathbf{y}-\mathbf{y}\| = |t-1|\,\|\mathbf{y}\|$. Therefore $\big|\, \|\mathbf{x}\|-\|\mathbf{y}\| \,\big| = \|\mathbf{x}-\mathbf{y}\|$. Similarly we obtain $\big|\, \|\mathbf{x}\|-\|\mathbf{y}\| \,\big| = \|\mathbf{x}-\mathbf{y}\|$ if we assume that $\beta \neq 0$.

7. Let $\mathbf{x},\,\mathbf{y} \in \mathbb{R}^m$ and $r > 0$ such that $\mathbf{y}\cdot\mathbf{z} = 0$ for all $\mathbf{z} \in B_r(\mathbf{x})$. Show that $\mathbf{y} = \mathbf{0}$.

   Solution: If possible, let $\mathbf{y} \neq \mathbf{0}$. Then $\|\mathbf{y}\| \neq 0$. If $\mathbf{z} = \mathbf{x} + \frac{r}{2\|\mathbf{y}\|}\mathbf{y}$, then $\mathbf{z} \in \mathbb{R}^m$ and since $\|\mathbf{z}-\mathbf{x}\| = \frac{r}{2} < r$, $\mathbf{z} \in B_r(\mathbf{x})$. Hence $\mathbf{y}\cdot\mathbf{z} = 0$ and so $\mathbf{y}\cdot\mathbf{x} + \frac{r}{2\|\mathbf{y}\|}\|\mathbf{y}\|^2 = 0$. Since $\mathbf{x} \in B_r(\mathbf{x})$, $\mathbf{y}\cdot\mathbf{x} = 0$ and so from above, we get $\|\mathbf{y}\| = 0$, which is a contradiction. Therefore $\mathbf{y} = \mathbf{0}$.

8. If $\mathbf{x}_0 \in \mathbb{R}^m$ and $r > 0$, then determine $\sup\{\|\mathbf{x}-\mathbf{y}\| : \mathbf{x},\,\mathbf{y} \in B_r(\mathbf{x}_0)\}$ with justification.

   Solution: For all $\mathbf{x},\,\mathbf{y} \in B_r(\mathbf{x}_0)$, $\|\mathbf{x}-\mathbf{y}\| \leq \|\mathbf{x}-\mathbf{x}_0\| + \|\mathbf{x}_0-\mathbf{y}\| < r + r = 2r$ and so $2r$ is an upper bound of $\{\|\mathbf{x}-\mathbf{y}\| : \mathbf{x},\mathbf{y} \in B_r(\mathbf{x}_0)\}$. Let $\varepsilon > 0$ such that $\varepsilon < r$. Then $\mathbf{x}_0 + (r-\frac{\varepsilon}{3})\mathbf{e}_1,\ \mathbf{x}_0 - (r-\frac{\varepsilon}{3})\mathbf{e}_1 \in \mathbb{R}^m$ and since $\|\mathbf{x}_0 + (r-\frac{\varepsilon}{3})\mathbf{e}_1 - \mathbf{x}_0\| = r - \frac{\varepsilon}{3} < r$, $\|\mathbf{x}_0 - (r-\frac{\varepsilon}{3})\mathbf{e}_1 - \mathbf{x}_0\| = r - \frac{\varepsilon}{3} < r$, we have $\mathbf{x}_0 + (r-\frac{\varepsilon}{3})\mathbf{e}_1,\ \mathbf{x}_0 - (r-\frac{\varepsilon}{3})\mathbf{e}_1 \in B_r(\mathbf{x}_0)$. Also, $\big\|\big(\mathbf{x}_0 + (r-\frac{\varepsilon}{3})\mathbf{e}_1\big) - \big(\mathbf{x}_0 - (r-\frac{\varepsilon}{3})\mathbf{e}_1\big)\big\| = 2r - \frac{2\varepsilon}{3} > 2r - \varepsilon$ and hence $2r-\varepsilon$ is not an upper bound of $\{\|\mathbf{x}-\mathbf{y}\| : \mathbf{x},\,\mathbf{y} \in B_r(\mathbf{x}_0)\}$. Therefore $\sup\{\|\mathbf{x}-\mathbf{y}\| : \mathbf{x},\,\mathbf{y} \in B_r(\mathbf{x}_0)\} = 2r$.

9. Let $S \subseteq \mathbb{R}^m$ such that $S \subseteq B_r[\mathbf{x}_0]$ for some $\mathbf{x}_0 \in \mathbb{R}^m$ and for some $r > 0$. Show that S is a bounded set.

   Solution: If $\mathbf{x} \in S$, then $\mathbf{x} \in B_r[\mathbf{x}_0]$ and hence $\|\mathbf{x}\| = \|\mathbf{x}-\mathbf{x}_0+\mathbf{x}_0\| \leq \|\mathbf{x}-\mathbf{x}_0\|+\|\mathbf{x}_0\| \leq r+\|\mathbf{x}_0\|$. Therefore S is a bounded set in $\mathbb{R}^m$.

10. Let $\alpha \in (0,1)$ and let $\mathbf{x}_n = \big(n^3\alpha^n,\, \frac{1}{n}[n\alpha]\big)$ for all $n \in \mathbb{N}$. (For each $x \in \mathbb{R}$, $[x]$ denotes the greatest integer not exceeding x.) Examine whether the sequence $(\mathbf{x}_n)$ converges in $\mathbb{R}^2$. Also, find $\lim\limits_{n\to\infty} \mathbf{x}_n$ if the sequence $(\mathbf{x}_n)$ converges in $\mathbb{R}^2$.

    Solution: Let $x_n = n^3\alpha^n$ and $y_n = \frac{1}{n}[n\alpha]$ for all $n \in \mathbb{N}$.
    Since $\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim\limits_{n\to\infty} (1+\frac{1}{n})^3\alpha = \alpha < 1$, the sequence (x_n) converges in $\mathbb{R}$ to 0. Again, since $[n\alpha] \leq n\alpha < [n\alpha]+1$ for all $n \in \mathbb{N}$, we have $n\alpha - 1 < [n\alpha] \leq n\alpha$ for all $n \in \mathbb{N}$ and so it follows that $\alpha - \frac{1}{n} < y_n \leq \alpha$ for all $n \in \mathbb{N}$. Hence by sandwich theorem, the sequence (y_n) converges in $\mathbb{R}$ to α. Therefore the sequence $(\mathbf{x}_n)$ converges in $\mathbb{R}^2$ and $\lim\limits_{n\to\infty} \mathbf{x}_n = (0,\alpha)$.

11. Let $(\mathbf{x}_n)$ be a sequence in $\mathbb{R}^m$ such that the series $\sum\limits_{n=1}^{\infty} n^2\|\mathbf{x}_n\|^2$ is convergent. Show that the series $\sum\limits_{n=1}^{\infty} \|\mathbf{x}_n\|$ is convergent.

    Solution: For all $n \in \mathbb{N}$, using Cauchy-Schwarz inequality, we have
    $\sum\limits_{k=1}^{n} \|\mathbf{x}_k\| = \sum\limits_{k=1}^{n} k\,\|\mathbf{x}_k\|\,\frac{1}{k} \leq \Big(\sum\limits_{k=1}^{n} k^2\|\mathbf{x}_k\|^2\Big)^{\frac{1}{2}} \Big(\sum\limits_{k=1}^{n} \frac{1}{k^2}\Big)^{\frac{1}{2}} \leq \Big(\sum\limits_{k=1}^{\infty} k^2\|\mathbf{x}_k\|^2\Big)^{\frac{1}{2}} \Big(\sum\limits_{k=1}^{\infty} \frac{1}{k^2}\Big)^{\frac{1}{2}} < \infty$. This shows that the sequence $\Big(\sum\limits_{k=1}^{n} \|\mathbf{x}_k\|\Big)$ of partial sums of the series $\sum\limits_{n=1}^{\infty} \|\mathbf{x}_n\|$ of non-negative real

numbers is bounded above and hence the sequence $\left(\sum_{k=1}^{n} \|\mathbf{x}_k\|\right)$ converges in $\mathbb{R}$. Consequently the series $\sum_{n=1}^{\infty} \|\mathbf{x}_n\|$ is convergent in $\mathbb{R}$.

12. Let $(\mathbf{x}_n)$ and $(\mathbf{y}_n)$ be sequences in $\mathbb{R}^m$ such that $\mathbf{x}_n \to \mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y}_n \to \mathbf{y} \in \mathbb{R}^m$. Show that $\mathbf{x}_n + \mathbf{y}_n \to \mathbf{x} + \mathbf{y}$ and $\mathbf{x}_n \cdot \mathbf{y}_n \to \mathbf{x} \cdot \mathbf{y}$.

    Solution: Since $\mathbf{x}_n \to \mathbf{x}$ and $\mathbf{y}_n \to \mathbf{y}$, $\|\mathbf{x}_n - \mathbf{x}\| \to 0$ and $\|\mathbf{y}_n - \mathbf{y}\| \to 0$. Hence $\|(\mathbf{x}_n + \mathbf{y}_n) - (\mathbf{x} + \mathbf{y})\| \le \|\mathbf{x}_n - \mathbf{x}\| + \|\mathbf{y}_n - \mathbf{y}\| \to 0$. Therefore $\|(\mathbf{x}_n + \mathbf{y}_n) - (\mathbf{x} + \mathbf{y})\| \to 0$ and so $\mathbf{x}_n + \mathbf{y}_n \to \mathbf{x} + \mathbf{y}$.
    Again, $|\mathbf{x}_n \cdot \mathbf{y}_n - \mathbf{x} \cdot \mathbf{y}| = |\mathbf{x}_n \cdot \mathbf{y}_n - \mathbf{x}_n \cdot \mathbf{y} + \mathbf{x}_n \cdot \mathbf{y} - \mathbf{x} \cdot \mathbf{y}| = |\mathbf{x}_n \cdot (\mathbf{y}_n - \mathbf{y}) + (\mathbf{x}_n - \mathbf{x}) \cdot \mathbf{y}|$ $\le |\mathbf{x}_n \cdot (\mathbf{y}_n - \mathbf{y})| + |(\mathbf{x}_n - \mathbf{x}) \cdot \mathbf{y}| \le \|\mathbf{x}_n\| \, \|\mathbf{y}_n - \mathbf{y}\| + \|\mathbf{x}_n - \mathbf{x}\| \, \|\mathbf{y}\|$ for all $n \in \mathbb{N}$. Since $(\mathbf{x}_n)$ is a convergent sequence in $\mathbb{R}^m$, $(\mathbf{x}_n)$ is bounded in $\mathbb{R}^m$. Hence there exists $r > 0$ such that $\|\mathbf{x}_n\| \le r$ for all $n \in \mathbb{N}$. Therefore $|\mathbf{x}_n \cdot \mathbf{y}_n - \mathbf{x} \cdot \mathbf{y}| \le \|\mathbf{x}_n\| \, \|\mathbf{y}_n - \mathbf{y}\| + \|\mathbf{x}_n - \mathbf{x}\| \, \|\mathbf{y}\| \to 0$ and so $|\mathbf{x}_n \cdot \mathbf{y}_n - \mathbf{x} \cdot \mathbf{y}| \to 0$. Hence $\mathbf{x}_n \cdot \mathbf{y}_n \to \mathbf{x} \cdot \mathbf{y}$.

13. Let $\mathbf{x} \in \mathbb{R}^m$ and let $(\mathbf{x}_n)$ be a sequence in $\mathbb{R}^m$ such that $\|\mathbf{x}_n\| \to \|\mathbf{x}\|$ and $\mathbf{x}_n \cdot \mathbf{x} \to \mathbf{x} \cdot \mathbf{x}$. Show that $(\mathbf{x}_n)$ is convergent.

    Solution: Since $\|\mathbf{x}_n - \mathbf{x}\|^2 = \|\mathbf{x}_n\|^2 - 2\,\mathbf{x}_n \cdot \mathbf{x} + \|\mathbf{x}\|^2 \to \|\mathbf{x}\|^2 - 2\,\mathbf{x} \cdot \mathbf{x} + \|\mathbf{x}\|^2 = 2\|\mathbf{x}\|^2 - 2\|\mathbf{x}\|^2 = 0$, we have that $\|\mathbf{x}_n - \mathbf{x}\| \to 0$ and hence $\mathbf{x}_n \to \mathbf{x}$. Therefore $(\mathbf{x}_n)$ is convergent in $\mathbb{R}^m$.

14. State TRUE or FALSE with justification: If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{x} \ne \mathbf{y}$ and $\|\mathbf{x}\| = 1 = \|\mathbf{y}\|$, then it is necessary that $\|\mathbf{x} + \mathbf{y}\| < 2$.

    Solution: We have $\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\,\mathbf{x} \cdot \mathbf{y} = 2 + 2\,\mathbf{x} \cdot \mathbf{y}$ and $\|\mathbf{x} - \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\,\mathbf{x} \cdot \mathbf{y} = 2 - 2\,\mathbf{x} \cdot \mathbf{y}$. Hence $\|\mathbf{x} + \mathbf{y}\|^2 = 2 + 2 - \|\mathbf{x} - \mathbf{y}\|^2 < 4$, since $\|\mathbf{x} - \mathbf{y}\| > 0$. So $\|\mathbf{x} + \mathbf{y}\| < 2$. Therefore the given statement is TRUE.

15. State TRUE or FALSE with justification: If $(\mathbf{x}_n)$ is a sequence in $\mathbb{R}^m$ such that for each $\mathbf{x} \in \mathbb{R}^m$, $\lim_{n\to\infty} \mathbf{x}_n \cdot \mathbf{x}$ exists (in $\mathbb{R}$), then $\lim_{n\to\infty} \|\mathbf{x}_n\|^2$ must exist (in $\mathbb{R}$).

    Solution: For each $n \in \mathbb{N}$, let $\mathbf{x}_n = \left(x_1^{(n)}, ..., x_m^{(n)}\right)$.
    By the given condition, $\lim_{n\to\infty} x_j^{(n)} = \lim_{n\to\infty} \mathbf{x}_n \cdot \mathbf{e}_j$ exists (in $\mathbb{R}$) for $j = 1, \ldots, m$. Consequently $\lim_{n\to\infty} \|\mathbf{x}_n\|^2 = \lim_{n\to\infty} \left((x_1^{(n)})^2 + \cdots + (x_m^{(n)})^2\right)$ exists (in $\mathbb{R}$). Therefore the given statement is TRUE.

16. State TRUE or FALSE with justification: There exists an unbounded sequence (x_n) of distinct real numbers such that the sequence $\big((x_n, \cos x_n)\big)$ in $\mathbb{R}^2$ has a convergent subsequence.

    Solution: The sequence $(x_n) = \left(1, \frac{1}{2}, 2, \frac{1}{3}, 3, \frac{1}{4}, \ldots\right)$ in $\mathbb{R}$ is unbounded and its subsequence $(x_{2n}) = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right)$ converges in $\mathbb{R}$. By continuity of the cosine function, the sequence $(\cos x_{2n})$ also converges in $\mathbb{R}$. Hence the subsequence $\big((x_{2n}, \cos x_{2n})\big)$ of the sequence $\big((x_n, \cos x_n)\big)$ converges in $\mathbb{R}^2$. Therefore the given statement is TRUE.

17. Let $S = \{(x,y) \in \mathbb{R}^2 : x \neq y\}$ and let $f : S \to \mathbb{R}$ be defined by $f(x,y) = \frac{x+y}{x-y}$ for all $(x,y) \in S$. Show by using the definition of continuity that f is continuous at $(1,2)$.

Solution: Let $\varepsilon > 0$. For all $(x,y) \in S$, we have $|f(x,y) - f(1,2)| = \left|\frac{x+y}{x-y} + 3\right| = 2\left|\frac{2x-y}{x-y}\right|$. If $(x,y) \in S$ and $\|(x,y)-(1,2)\| = \sqrt{(x-1)^2+(y-2)^2} < \frac{1}{4}$, then $|x-1| < \frac{1}{4}$ and $|y-2| < \frac{1}{4}$, and so $|x-y| = |1-((2-y)+(x-1))| \geq 1-|(2-y)+(x-1)| \geq 1-(|2-y|+|x-1|) \geq 1-(\frac{1}{4}+\frac{1}{4}) = \frac{1}{2}$. Again, if $r > 0$ and $(x,y) \in S$ such that $\|(x,y)-(1,2)\| = \sqrt{(x-1)^2+(y-2)^2} < r$, then $|x-1| < r$ and $|y-2| < r$, and so $|2x-y| = |2(x-1)+2-y| \leq 2|x-1|+|y-2| < 3r$. Hence if we choose $\delta = \min\{\frac{1}{4}, \frac{\varepsilon}{12}\}$, then $\delta > 0$ and for all $(x,y) \in S$ satisfying $\|(x,y)-(1,2)\| < \delta$, we have $|f(x,y) - f(1,2)| < 12\delta \leq \varepsilon$. Therefore f is continuous at $(1,2)$.

18. If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous and $f(x,y) = x^2 + y^2$ for all $x \in \mathbb{Q}$ and for all $y \in \mathbb{R} \setminus \mathbb{Q}$, then determine $f(\sqrt{2}, 2)$.

Solution: We know that there exist sequences (x_n) in $\mathbb{Q}$ and (y_n) in $\mathbb{R} \setminus \mathbb{Q}$ such that $x_n \to \sqrt{2}$ and $y_n \to 2$. Hence $(x_n, y_n) \to (\sqrt{2}, 2)$. Since f is continuous at $(\sqrt{2}, 2)$, we have
$f(\sqrt{2}, 2) = \lim\limits_{n\to\infty} f(x_n, y_n) = \lim\limits_{n\to\infty} (x_n^2 + y_n^2) = \lim\limits_{n\to\infty} x_n^2 + \lim\limits_{n\to\infty} y_n^2 = 2 + 4 = 6.$

19. Examine the continuity of $f : \mathbb{R}^2 \to \mathbb{R}$ at $(0,0)$, where for all $(x,y) \in \mathbb{R}^2$,
$$f(x,y) = \begin{cases} xy & \text{if } xy \geq 0, \\ -xy & \text{if } xy < 0. \end{cases}$$
Solution: Let $\big((x_n, y_n)\big)$ be any sequence in $\mathbb{R}^2$ such that $(x_n, y_n) \to (0,0)$. Then $x_n \to 0$ and $y_n \to 0$. We have $|f(x_n, y_n)| = |x_n y_n| \to 0$ and hence $f(x_n, y_n) \to 0 = f(0,0)$. Therefore f is continuous at $(0,0)$.

20. Examine the continuity of $f : \mathbb{R}^2 \to \mathbb{R}$ at $(0,0)$, where for all $(x,y) \in \mathbb{R}^2$,
$$f(x,y) = \begin{cases} \frac{xy^3}{x^2+y^4} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Solution: Let $\varepsilon > 0$. Then for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$, we have
$|f(x,y) - f(0,0)| = \frac{|x|y^2}{x^2+y^4}|y| \leq \frac{1}{2}|y| \leq \frac{1}{2}\sqrt{x^2+y^2}$.
Since $f(0,0) = 0$, we get $|f(x,y) - f(0,0)| \leq \frac{1}{2}\sqrt{x^2+y^2}$ for all $(x,y) \in \mathbb{R}^2$. Let $\delta = 2\varepsilon$. Then $\delta > 0$ and for all $(x,y) \in \mathbb{R}^2$ with $\|(x,y)-(0,0)\| = \sqrt{x^2+y^2} < \delta$, we have $|f(x,y)-f(0,0)| < \varepsilon$. Therefore f is continuous at $(0,0)$.

21. Examine the continuity of $f : \mathbb{R}^2 \to \mathbb{R}$ at $(0,0)$, where for all $(x,y) \in \mathbb{R}^2$,
$$f(x,y) = \begin{cases} 1 & \text{if } x > 0 \text{ and } 0 < y < x^2, \\ 0 & \text{otherwise.} \end{cases}$$

Solution: Since $\left(\frac{1}{n}, \frac{1}{2n^2}\right) \to (0,0)$ but $f\left(\frac{1}{n}, \frac{1}{2n^2}\right) = 1 \to 1 \neq 0 = f(0,0)$, f is not continuous at $(0,0)$.

22. Determine all the points of $\mathbb{R}^2$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous, if for all $(x,y) \in \mathbb{R}^2$,
$$f(x,y) = \begin{cases} \frac{xy}{x-y} & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Solution: If $\varphi(x,y) = xy$ and $\psi(x,y) = x - y$ for all $(x,y) \in \mathbb{R}^2$, then as polynomial functions,

$\varphi,\,\psi:\mathbb{R}^2\to\mathbb{R}$ are continuous and $\psi(x,y)\neq 0$ for all $(x,y)\in\mathbb{R}^2$ with $x\neq y$. Hence f is continuous at each $(x,y)\in\mathbb{R}^2$ with $x\neq y$.

Let $x\in\mathbb{R}\setminus\{0\}$. Then $(x+\frac{1}{n},x)\to(x,x)$ but $f(x+\frac{1}{n},x)=nx^2+x\not\to 0=f(x,x)$. So f is not continuous at (x,x).

Again, $(\frac{1}{n}+\frac{1}{n^2},\frac{1}{n})\to(0,0)$ but $f(\frac{1}{n}+\frac{1}{n^2},\frac{1}{n})=1+\frac{1}{n}\to 1\neq 0=f(0,0)$. So f is not continuous at $(0,0)$.

Therefore the set of points of continuity of f is $\{(x,y)\in\mathbb{R}^2: x\neq y\}$.

23. Determine all the points of $\mathbb{R}^2$ where $f:\mathbb{R}^2\to\mathbb{R}$ is continuous, if for all $(x,y)\in\mathbb{R}^2$,
$$f(x,y)=\begin{cases} xy & \text{if } xy\in\mathbb{Q},\\ -xy & \text{if } xy\in\mathbb{R}\setminus\mathbb{Q}.\end{cases}$$

Solution: Let $(x,y)\in\mathbb{R}^2$ such that $xy=0$ and let $\big((x_n,y_n)\big)$ be any sequence in $\mathbb{R}^2$ such that $(x_n,y_n)\to(x,y)$. Then $x_n\to x$ and $y_n\to y$. We have $|f(x_n,y_n)|=|x_ny_n|\to|xy|=0$ and so $f(x_n,y_n)\to 0=f(x,y)$. Hence f is continuous at (x,y).

Again, let $(x,y)\in\mathbb{R}^2$ such that $xy\neq 0$. We consider the following two possible cases.

Case (i): $xy\in\mathbb{R}\setminus\mathbb{Q}$.

We can find two sequences (x_n) and (y_n) in $\mathbb{Q}$ such that $x_n\to x$ and $y_n\to y$. Then $\big((x_n,y_n)\big)$ is a sequence in $\mathbb{R}^2$ such that $(x_n,y_n)\to(x,y)$ but $f(x_n,y_n)=x_ny_n\to xy\neq -xy=f(x,y)$. Hence f is not continuous at (x,y).

Case (ii): $xy\in\mathbb{Q}$.

Since $x\neq 0$, we can find a sequence (x_n) in $\mathbb{Q}\setminus\{0\}$ and a sequence (y_n) in $\mathbb{R}\setminus\mathbb{Q}$ such that $x_n\to x$ and $y_n\to y$. Then $\big((x_n,y_n)\big)$ is a sequence in $\mathbb{R}^2$ such that $(x_n,y_n)\to(x,y)$ but $f(x_n,y_n)=-x_ny_n\to -xy\neq xy=f(x,y)$. Hence f is not continuous at (x,y).

Therefore the set of points of continuity of f is $\{(x,y)\in\mathbb{R}^2: xy=0\}$.

24. Let α, β be positive real numbers and let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by
$$f(x,y)=\begin{cases} \frac{|x|^\alpha|y|^\beta}{x^2+xy+y^2} & \text{if } (x,y)\neq(0,0),\\ 0 & \text{if } (x,y)=(0,0).\end{cases}$$
Show that f is continuous iff $\alpha+\beta>2$.

Solution: Let $\alpha+\beta>2$ and let $\big((x_n,y_n)\big)$ be any sequence in $\mathbb{R}^2$ such that $(x_n,y_n)\to(0,0)$. Then $x_n\to 0$ and $y_n\to 0$. For all $n\in\mathbb{N}$ for which $(x_n,y_n)\neq(0,0)$, we have

$0\le f(x_n,y_n)\le\frac{(x_n^2+y_n^2)^{\frac{\alpha}{2}}(x_n^2+y_n^2)^{\frac{\beta}{2}}}{\frac{1}{2}(x_n^2+y_n^2)}=2(x_n^2+y_n^2)^{\frac{1}{2}(\alpha+\beta-2)}$ and since $f(0,0)=0$,

we have $0\le f(x_n,y_n)\le 2(x_n^2+y_n^2)^{\frac{1}{2}(\alpha+\beta-2)}$ for all $n\in\mathbb{N}$. Since $2(x_n^2+y_n^2)^{\frac{1}{2}(\alpha+\beta-2)}\to 0$, we get $f(x_n,y_n)\to 0=f(0,0)$. This shows that f is continuous at $(0,0)$. Also, it is clear (by similar arguments given in other examples) that f is continuous at each $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$. Therefore f is continuous.

Conversely, let f be continuous and if possible, let $\alpha+\beta\le 2$. We have $(\frac{1}{n},\frac{1}{n})\to(0,0)$ but $f(\frac{1}{n},\frac{1}{n})=\frac{1}{3}n^{2-(\alpha+\beta)}\not\to 0=f(0,0)$ (because for $\alpha+\beta=2$, $f(\frac{1}{n},\frac{1}{n})\to\frac{1}{3}$ and for $\alpha+\beta<2$, the sequence $\big(f(\frac{1}{n},\frac{1}{n})\big)$ is unbounded). Hence f is not continuous at $(0,0)$, which is a contradiction. Therefore $\alpha+\beta>2$.

25. Let S be a nonempty subset of $\mathbb{R}^m$ and let $f_j : S \to \mathbb{R}$ for each $j \in \{1, \ldots, k\}$. If $f(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_k(\mathbf{x}))$ for all $\mathbf{x} \in S$, then show that $f : S \to \mathbb{R}^k$ is continuous at $\mathbf{x}_0 \in S$ iff f_j is continuous at $\mathbf{x}_0$ for each $j \in \{1, \ldots, k\}$.

Solution: We first assume that f is continuous at $\mathbf{x}_0$ and let $(\mathbf{x}_n)$ be any sequence in S such that $\mathbf{x}_n \to \mathbf{x}_0$. Then $\big(f_1(\mathbf{x}_n), \ldots, f_k(\mathbf{x}_n)\big) = f(\mathbf{x}_n) \to f(\mathbf{x}_0) = \big(f_1(\mathbf{x}_0), \ldots, f_k(\mathbf{x}_0)\big)$ and hence $f_j(\mathbf{x}_n) \to f_j(\mathbf{x}_0)$ for each $j \in \{1, \ldots, k\}$. Consequently f_j is continuous at $\mathbf{x}_0$ for each $j \in \{1, \ldots, k\}$.
Conversely, let f_j be continuous at $\mathbf{x}_0$ for each $j \in \{1, \ldots, k\}$ and let $(\mathbf{x}_n)$ be any sequence in S such that $\mathbf{x}_n \to \mathbf{x}_0$. Then $f_j(\mathbf{x}_n) \to f_j(\mathbf{x}_0)$ for each $j \in \{1, \ldots, k\}$ and hence
$f(\mathbf{x}_n) = \big(f_1(\mathbf{x}_n), \ldots, f_k(\mathbf{x}_n)\big) \to \big(f_1(\mathbf{x}_0), \ldots, f_k(\mathbf{x}_0)\big) = f(\mathbf{x}_0)$. Therefore f is continuous at $\mathbf{x}_0$.

26. Examine the continuity of $f : \mathbb{R}^2 \to \mathbb{R}^2$ at $(0,0)$, where for all $(x,y) \in \mathbb{R}^2$,
$$f(x,y) = \begin{cases} \left(\frac{x^3}{x^2+y^2}, \sin(x^2+y^2)\right) & \text{if } (x,y) \neq (0,0), \\ (0,0) & \text{if } (x,y) = (0,0). \end{cases}$$

Solution: For all $(x,y) \in \mathbb{R}^2$, let $\varphi(x,y) = \sin(x^2+y^2)$ and $\psi(x,y) = \begin{cases} \frac{x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$
Since $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is a composition of a polynomial function and the sine function, both of which are continuous, φ is continuous at $(0,0)$.
Again, let $\varepsilon > 0$. Then for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$, we have
$|\psi(x,y) - \psi(0,0)| = \frac{x^2}{x^2+y^4}|x| \leq |x| \leq \sqrt{x^2+y^2}$.
Since $\psi(0,0) = 0$, we get $|\psi(x,y) - \psi(0,0)| \leq \sqrt{x^2+y^2}$ for all $(x,y) \in \mathbb{R}^2$. Let $\delta = \varepsilon$. Then $\delta > 0$ and for all $(x,y) \in \mathbb{R}^2$ with $\|(x,y) - (0,0)\| = \sqrt{x^2+y^2} < \delta$, we have $|\psi(x,y) - \psi(0,0)| < \varepsilon$. Therefore ψ is continuous at $(0,0)$.
Consequently (by Ex.17 of Practice Problem Set - 1) f is continuous at $(0,0)$.

27. If $f, g : S \subseteq \mathbb{R}^m \to \mathbb{R}^k$ are continuous at $\mathbf{x}_0 \in S$ and if $\varphi(\mathbf{x}) = f(\mathbf{x}) \cdot g(\mathbf{x})$ for all $\mathbf{x} \in S$, then show that $\varphi : S \to \mathbb{R}$ is continuous at $\mathbf{x}_0$.

Solution: Let $(\mathbf{x}_n)$ be any sequence in S such that $\mathbf{x}_n \to \mathbf{x}_0$. Since f and g are continuous at $\mathbf{x}_0$, $f(\mathbf{x}_n) \to f(\mathbf{x}_0)$ and $g(\mathbf{x}_n) \to g(\mathbf{x}_0)$. Hence (by Ex.9 of Practice Problem Set - 1) $\varphi(\mathbf{x}_n) = f(\mathbf{x}_n) \cdot g(\mathbf{x}_n) \to f(\mathbf{x}_0) \cdot g(\mathbf{x}_0) = \varphi(\mathbf{x}_0)$. Therefore φ is continuous at $\mathbf{x}_0$.

28. Let $f : S \subseteq \mathbb{R}^m \to \mathbb{R}^k$ be continuous at $\mathbf{x}_0 \in S^0$ and let $f(\mathbf{x}_0) \neq \mathbf{0}$. Show that there exists $r > 0$ such that $f(\mathbf{x}) \neq \mathbf{0}$ for all $\mathbf{x} \in B_r(\mathbf{x}_0)$.

Solution: Since $\mathbf{x}_0 \in S^0$, there exists $s > 0$ such that $B_s(\mathbf{x}_0) \subseteq S$. Again, since $f(\mathbf{x}_0) \neq \mathbf{0}$, $\frac{1}{2}\|f(\mathbf{x}_0)\| > 0$. By the continuity of f at $\mathbf{x}_0$, there exists $\delta > 0$ such that $\|f(\mathbf{x}) - f(\mathbf{x}_0)\| < \frac{1}{2}\|f(\mathbf{x}_0)\|$ for all $\mathbf{x} \in S$ satisfying $\|\mathbf{x} - \mathbf{x}_0\| < \delta$. Taking $r = \min\{s, \delta\} > 0$, we find that $\|f(\mathbf{x}) - f(\mathbf{x}_0)\| < \frac{1}{2}\|f(\mathbf{x}_0)\|$ for all $\mathbf{x} \in B_r(\mathbf{x}_0)$. If possible, let $f(\mathbf{x}) = \mathbf{0}$ for some $\mathbf{x} \in B_r(\mathbf{x}_0)$. Then from above, we get $\|f(\mathbf{x}_0)\| < \frac{1}{2}\|f(\mathbf{x}_0)\|$, which is not true. Therefore $f(\mathbf{x}) \neq \mathbf{0}$ for all $\mathbf{x} \in B_r(\mathbf{x}_0)$.

29. Let S be an open subset of $\mathbb{R}^m$ and let $f : S \to \mathbb{R}^k$ and $g : S \to \mathbb{R}^k$ be continuous at $\mathbf{x}_0 \in S$. If for each $\varepsilon > 0$, there exist $\mathbf{x}, \mathbf{y} \in B_\varepsilon(\mathbf{x}_0)$ such that $f(\mathbf{x}) = g(\mathbf{y})$, then show that $f(\mathbf{x}_0) = g(\mathbf{x}_0)$.

Solution: By the given condition, for each $n \in \mathbb{N}$, there exist $\mathbf{x}_n,\ \mathbf{y}_n \in B_{\frac{1}{n}}(\mathbf{x}_0)$ such that $f(\mathbf{x}_n) = g(\mathbf{y}_n)$. So $\|\mathbf{x}_n - \mathbf{x}_0\| < \frac{1}{n} \to 0$ and $\|\mathbf{y}_n - \mathbf{x}_0\| < \frac{1}{n} \to 0$. Hence $\mathbf{x}_n \to \mathbf{x}_0$ and $\mathbf{y}_n \to \mathbf{x}_0$. Since f and g are continuous at $\mathbf{x}_0$, $f(\mathbf{x}_n) \to f(\mathbf{x}_0)$ and $g(\mathbf{y}_n) \to g(\mathbf{x}_0)$. Therefore $f(\mathbf{x}_0) = g(\mathbf{x}_0)$.

30. If $S = \{(x, y) \in \mathbb{R}^2 : x + y \geq 2\}$, then determine (with justification) S^0.

Solution: Let $(x_0, y_0) \in S$ with $x_0 + y_0 > 2$. Let $r = \frac{x_0+y_0-2}{\sqrt{2}} > 0$ and let $(x, y) \in B_r\big((x_0, y_0)\big)$. Then $\|(x, y) - (x_0, y_0)\| = \sqrt{(x - x_0)^2 + (y - y_0)^2} < r$. By Cauchy-Schwarz inequality, we have $x_0 - x + y_0 - y \leq \sqrt{(x_0 - x)^2 + (y_0 - y)^2}\,\sqrt{1^2 + 1^2} < \sqrt{2}r = x_0 + y_0 - 2$. Hence $x + y > 2$ and so $(x, y) \in S$. Thus $B_r\big((x_0, y_0)\big) \subseteq S$ and therefore $(x_0, y_0) \in S^0$.
Now, let $(x_0, y_0) \in S$ such that $x_0 + y_0 = 2$ and if possible, let $(x_0, y_0) \in S^0$. Then there exists $r > 0$ such that $B_r\big((x_0, y_0)\big) \subseteq S$. Since $\|(x_0 - \frac{r}{2}, y_0) - (x_0, y_0)\| = \|(-\frac{r}{2}, 0)\| = \frac{r}{2} < r$, $(x_0 - \frac{r}{2}, y_0) \in B_r\big((x_0, y_0)\big)$. However, $(x_0 - \frac{r}{2}, y_0) \notin S$, since $x_0 - \frac{r}{2} + y_0 = 2 - \frac{r}{2} < 2$. Thus we get a contradiction. Hence $(x_0, y_0) \notin S^0$.
Therefore $S^0 = \{(x, y) \in \mathbb{R}^2 : x + y > 2\}$.

31. If $S = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : x_m = 1\}$, then determine (with justification) S^0.

Solution: If possible, let $S^0 \neq \emptyset$. Then there exists $\mathbf{x} = (x_1, \ldots, x_m) \in S^0$ and hence there exists $r > 0$ such that $B_r(\mathbf{x}) \subseteq S$. If $\mathbf{y} = (x_1, \ldots, x_{m-1}, x_m + \frac{r}{2})$, then $\|\mathbf{y} - \mathbf{x}\| = \frac{r}{2} < r$ and so $\mathbf{y} \in B_r(\mathbf{x})$. But $\mathbf{y} \notin S$, because $x_m + \frac{r}{2} = 1 + \frac{r}{2} \neq 1$. Thus we get a contradiction. Therefore $S^0 = \emptyset$.

32. If $\mathbf{x} \in \mathbb{R}^m$ and $r > 0$, then determine (with justification) all the interior points of $B_r[\mathbf{x}]$.

Solution: Let $\mathbf{y} \in B_r(\mathbf{x})$. Then $\|\mathbf{y} - \mathbf{x}\| < r$. If $s = r - \|\mathbf{y} - \mathbf{x}\|$, then $s > 0$. Let $\mathbf{z} \in B_s(\mathbf{y})$. Then $\|\mathbf{z} - \mathbf{y}\| < s$ and so $\|\mathbf{z} - \mathbf{x}\| = \|\mathbf{z} - \mathbf{y} + \mathbf{y} - \mathbf{x}\| \leq \|\mathbf{z} - \mathbf{y}\| + \|\mathbf{y} - \mathbf{x}\| < s + \|\mathbf{y} - \mathbf{x}\| = r$. Hence $\mathbf{z} \in B_r[\mathbf{x}]$ and so $B_s(\mathbf{y}) \subseteq B_r[\mathbf{x}]$. Therefore $\mathbf{y} \in \big(B_r[\mathbf{x}]\big)^0$.
Again, let $\mathbf{y} \in B_r[\mathbf{x}]$ such that $\|\mathbf{y} - \mathbf{x}\| = r$. If possible, let $\mathbf{y} \in \big(B_r[\mathbf{x}]\big)^0$. Then there exists $s > 0$ such that $B_s(\mathbf{y}) \subseteq B_r[\mathbf{x}]$. Now, $\mathbf{y} + \frac{s}{2r}(\mathbf{y} - \mathbf{x}) \in B_s(\mathbf{y})$, since
$\|\mathbf{y} + \frac{s}{2r}(\mathbf{y} - \mathbf{x}) - \mathbf{y}\| = \frac{s}{2r}\|\mathbf{y} - \mathbf{x}\| = \frac{s}{2} < s$, but $\mathbf{y} + \frac{s}{2r}(\mathbf{y} - \mathbf{x}) \notin B_r[\mathbf{x}]$, because
$\|\mathbf{y} + \frac{s}{2r}(\mathbf{y} - \mathbf{x}) - \mathbf{x}\| = (1 + \frac{s}{2r})\|\mathbf{y} - \mathbf{x}\| = r + \frac{s}{2} > r$. Thus we get a contradiction. Hence $\mathbf{y} \notin \big(B_r[\mathbf{x}]\big)^0$.
Therefore $\big(B_r[\mathbf{x}]\big)^0 = B_r(\mathbf{x})$.

33. Examine whether $\{(x, y) \in \mathbb{R}^2 : 0 < x < y\}$ is an open set in $\mathbb{R}^2$.

Solution: Let $S = \{(x, y) \in \mathbb{R}^2 : 0 < x < y\}$ and let $(x_0, y_0) \in S$. If $r = \min\big\{x_0, \frac{y_0 - x_0}{\sqrt{2}}\big\}$, then $r > 0$. Let $(x, y) \in B_r\big((x_0, y_0)\big)$. Then $\|(x, y) - (x_0, y_0)\| = \sqrt{(x - x_0)^2 + (y - y_0)^2} < r$. Hence $x_0 - x \leq |x - x_0| < r \leq x_0$ and so $x > 0$. Also, using Cauchy-Schwarz inequality, we have $x - x_0 + y_0 - y \leq \sqrt{(x - x_0)^2 + (y_0 - y)^2}\,\sqrt{1^2 + 1^2} < \sqrt{2}r \leq y_0 - x_0$ and hence $x - y < 0$, *i.e.*

$x < y$. Thus $(x, y) \in S$ and so $(x_0, y_0) \in S^0$. Since $(x_0, y_0) \in S$ is arbitrary, it follows that S is an open set in $\mathbb{R}^2$.