MA 101 (Mathematics - I)
Differentiability : Exercise set 1

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1. Discuss differentiability of $f : \mathbb{R} \to \mathbb{R}$, and continuity of f' wherever exists.

   (i) $f(x) = |x|$.

   (ii) $f(x) = |\sin x|$.

   (iii) $f(x) = \begin{cases} x^2, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

   (iv) $n \in \mathbb{N}$ and $f(x) = \begin{cases} x^n \sin \frac{1}{x}, & \text{if } x \neq 0. \\ 0, & \text{if } x = 0. \end{cases}$

2. Show that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable and is an even function, then f' is an odd function.

3. Let $f : (a, b) \to \mathbb{R}$ be differentiable at $c \in (a, b)$. Assume that $f'(c) \neq 0$. Show that there exists $\delta > 0$ such that for $x \in (c - \delta, c + \delta) \cap (a, b)$, we have $f(x) \neq f(c)$. Can you say something more, if $f'(x) > 0$? Similarly, if $f'(x) < 0$?

4. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $|f(x) - f(y)| \leq (x - y)^2$. Show that f is a constant function.

5. If $a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1} = 0$ for some real numbers a_i, then show that $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$ has a real root between 0 and 1.

6. Use the identity $1 + x + \cdots + x^n = \dfrac{1 - x^{n+1}}{1 - x}$ for $x \neq 1$ to arrive at a formula for the sum $1 + x + 2x^2 + \cdots + nx^n$.

7. Verify Chain Rule for f, g and $g \circ f$ at the point 0, where

$$f(x) = \begin{cases} x^2, & \text{if } x \in \mathbb{Q} \\ 0, & \text{otherwise,} \end{cases} \qquad g(x) = \begin{cases} \sin x, & \text{if } x \in \mathbb{Q} \\ x, & \text{otherwise.} \end{cases}$$

8. Find the number of real roots of the equation $x^4 + 2x^2 - 6x + 2 = 0$.

9. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $|f(x) - f(y)| \leq (x - y)^2$ for all $x, y \in \mathbb{R}$. Show that f is a constant function.

10. Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable at 0. If $f(\frac{1}{n}) = 0$ for all $n \in \mathbb{N}$, then find $f'(0)$ and $f''(0)$.

11. Let f be differentiable on $(0, \infty)$ and $\lim\limits_{x \to \infty} f'(x) = 0$. Put $g(x) = f(x + 1) - f(x)$. Show that $\lim\limits_{x \to \infty} g(x) = 0$.

12. If $f(x) = x^3 + x^2 - 5x + 3$ for $x \in \mathbb{R}$, then show that f is one-one on $[1, 5]$ but not one-one on $\mathbb{R}$.

13. Prove that for $x \geq -1$ and $\alpha > 1$, $(1 + x)^\alpha \geq 1 + \alpha x$.

14. (1) For $0 < x < y$, show that $\dfrac{y - x}{y} < \ln \dfrac{y}{x} < \dfrac{y - x}{x}$.
    (2) Deduce that if $e \leq a < b$, then $a^b > b^a$. (In particular $e^\pi > \pi^e$.)

15. Show that $0 < \frac{1}{x} \ln\left(\frac{e^x - 1}{x}\right) < 1$ for $x > 0$.

16. Find the points of local maximum and local minimum for $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = 1 - x^{2/3}$.