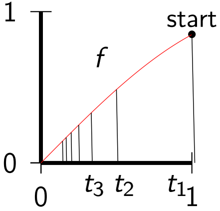
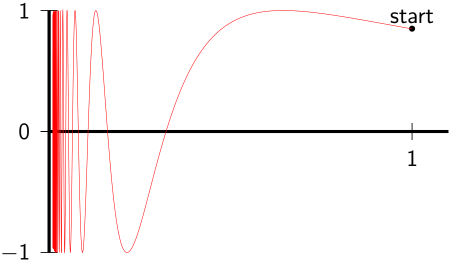
- Decided to collide against the wall (y-axis). Started at time $t=1$; collision at $t=0$. Path of f: $\sin(t)$. Path of g: $\sin(1/t)$.





Q Where should f collide?

Q Observe f at $t_n = 1, \frac{1}{2}, \frac{1}{3}, \cdots$. Does $(f(t_n))$ converge?

Q A friend observes f at $t_n = \frac{1}{n\sqrt{2}}$. Does $(f(t_n))$ converge?

Q If $t_n > 0$ and $t_n \to 0$, should $(f(t_n))$ have the same limit?

Q Take $t_n = \frac{2}{n\pi}$. Does $(g(t_n))$ converge?

D1[Sequential defn] Let $f : A \to \mathbb{R}$ and a be a cluster point of A. We say $\lim_{t \to a} f(t) = l$ if for each sequence $a_n \to a$, $a_n \neq a$, we have $f(a_n) \to l$.

D2[ϵ-δ-defn] We say $\lim_{t \to a} f(t) = l$ if for each $\epsilon > 0$, $\exists \delta > 0$ s.t.

$$\Big(0 < |t - a| < \delta, t \in A\Big) \Rightarrow |f(t) - l| < \epsilon.$$

- Here the value of δ depends on ϵ, the point a and on f.

- If f is a function, we define $f(B) := \{f(t) \mid t \in B \cap \operatorname{dom} f\}$.

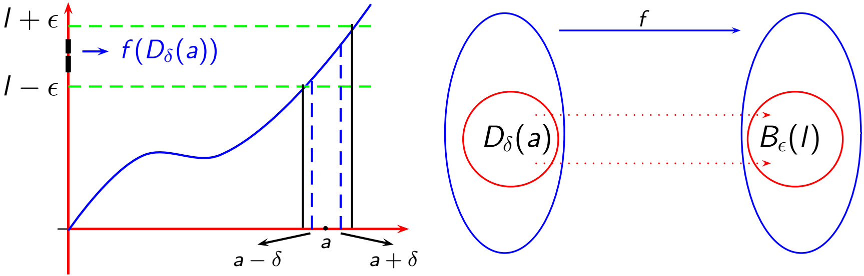
- Notice: $\{t \mid 0 < |t - a| < \delta, t \in A\} = D_\delta(a) \cap A$. And $\Big(0 < |t - a| < \delta, t \in A\Big) \Rightarrow |f(t) - l| < \epsilon$ means $f(D_\delta(a)) \subseteq B_\epsilon(l)$.

D2' We say $\lim_{t \to a} f(t) = l$ if each $B_\epsilon(l)$ contains some $f(D_\delta(a))$.

- Notice that to define limit at a, we need not have $a \in A$.

- It is useful to imagine dom f an interval or a disc, and a a point or boundary point of it.

  - By D2, $\lim_{x \to a} f(x) = l$ if each $B_\epsilon(l)$ contains some $f(D_\delta(a))$.



R Both the definitions are equivalent.

- Take $f(x) = x^2 - \frac{3}{\sqrt{x}}$ on $\mathbb{R}_+$. Show that $\lim\limits_{x\to 1} f(x) = -2$.

  A Let $a_n \to 1$, $a_n \neq 1$. So $a_n^2 - \frac{3}{\sqrt{a_n}} \to 1 - 3 = -2$, by limit theorems for sequences. By D1, $\lim\limits_{x\to 1} f(x) = -2$.

- Take $f(x) = \sqrt{x}$ on $\mathbb{R}_+$. Show that $\lim\limits_{x\to 1} f(x) \neq 2$.

  A Take $a_n = 1 - \frac{1}{n}$. Then $a_n \to 1$, $a_n \neq 1$. But $f(a_n) = \sqrt{a_n} \to 1 \neq 2$. By D1, $\lim\limits_{x\to 1} f(x) \neq 2$.

- Take $f(x) = \begin{cases} 1, & x < 0 \\ 0, & x \geq 0. \end{cases}$ Show that $\lim\limits_{x\to 0} f(x)$ does not exist.

  A Take $a_n = (-.1)^n$. Then $a_n \to 0$, $a_n \neq 0$. But $(f(a_n)) = (1, 0, 1, 0, \cdots)$ diverges. By D1, $\lim\limits_{x\to 0} f(x)$ does not exist.

- Did you notice? We used particular examples of (a_n), to show $\lim f \neq l$. We started with an arbitrary (a_n), to argue $\lim f = l$.

- Take $f(x) = \begin{cases} 0 & x = 0 \\ \sin(\frac{1}{x}) & x \neq 0. \end{cases}$ Then $\lim_{x \to 0} f(x)$ does not exist.

A Take $a_n = \frac{2}{n\pi}$. Then $a_n \to 0$, $a_n \neq a$. But $(f(a_n)) = (1, 0, -1, 0, 1, \cdots)$ diverges. By D1, $\lim_{x \to 0} f(x)$ does not exist.

- Take $f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q}. \end{cases}$ Fix any a. Then $\lim_{x \to a} f(x)$ does not exist.

A Let (r_n) be a sequence of rationals s.t. $r_n \neq a$, $r_n \to a$. Then $f(r_n) \to 0$. Let (i_n) be a sequence of irrationals s.t. $i_n \neq a$, $i_n \to a$. Then $f(i_n) \to 1$. Hence $\lim_{x \to a} f(x)$ does not exist, by D1.

Note For the limit to be l, we should have $f(a_n) \to l$, for each sequence $a_n \to a$, $a_n \neq a$.

- Take $f(x) = x^2$. Show that $\lim_{x\to 2} f(x) = 4$.

A Let $\epsilon > 0$. We are looking for a $0 < \delta < 1$ s.t. $f(D_\delta(2)) \subseteq B_\epsilon(4)$.

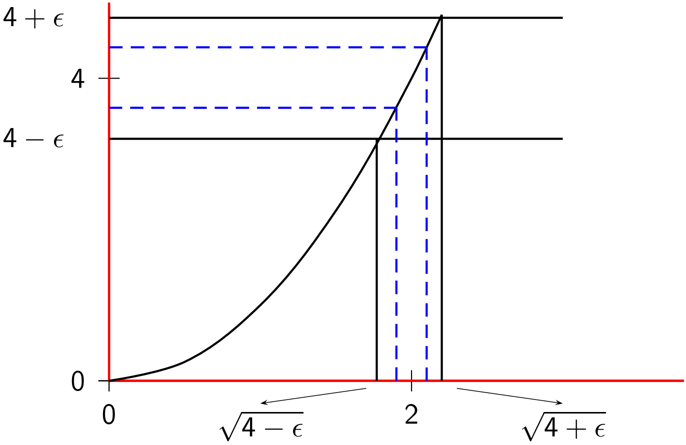
As f is increasing, we have

$$
\begin{aligned}
f(D_\delta(2)) \subseteq B_\epsilon(4) \quad & \Leftarrow \quad 4-\epsilon \le (2-\delta)^2 < (2+\delta)^2 \le 4+\epsilon \\
& \Leftarrow \quad \sqrt{4-\epsilon} \le 2-\delta < 2+\delta \le \sqrt{4+\epsilon} \\
& \Leftarrow \quad \sqrt{4-\epsilon} - 2 \le -\delta < \delta \le \sqrt{4+\epsilon} - 2. \\
& \Leftarrow \quad \delta = \min\{|\sqrt{4-\epsilon} - 2|, |\sqrt{4+\epsilon} - 2|\}.
\end{aligned}
$$

Then $\delta > 0$ and we are done.

Q To show $\lim_{x\to 2} x^3 = 8$, we take $\delta = \min\{|\sqrt[3]{8-\epsilon} - 2|, |\sqrt[3]{8+\epsilon} - 2|\}$.

- Pictures can help to guess δ. Take $f(x) = x^2$. Then $\lim_{x \to 2} f(x) = 4$.



- Intervals at 2 suggest: $\delta \leq \min\{2 - \sqrt{4-\epsilon}, \sqrt{4+\epsilon} - 2\}$. Which one?

- Take $f(x) = x^2$ on $\mathbb{R}$. Then $\lim\limits_{x \to 2} f(x) \neq 3.99$.

A Each $D_\delta(2)$ contains a number more than 2. So each $f(D_\delta(2))$ contains a number more than 4. Put $\epsilon = .01$. Then $B_\epsilon(3.99) = (3.98, 4)$. So $\exists$ no $\delta > 0$ such that $f(D_\delta(2)) \subseteq B_\epsilon(3.99)$. Thus by D2, $\lim\limits_{x \to 2} f(x) \neq 3.99$.

- Take $f(x) = \begin{cases} 1, & x < 0 \\ 0, & x \geq 0. \end{cases}$ Then $\lim\limits_{x \to 0} f(x)$ does not exist.

A If it exists, let it be l. Each $D_\delta(0)$ contains $+$ve and $-$ve numbers. So each $f(D_\delta(0))$ contains 1 and 0. Put $\epsilon = .1$. As $B_\epsilon(l)$ has length $.2$, it cannot contain two integers. So $\exists$ no $\delta > 0$ such that $f(D_\delta(0)) \subseteq B_\epsilon(l)$. So $\lim\limits_{x \to 0} f(x) \neq l$, a contradiction.

- We don't want to find the limits, every time from the definitions. So we require some tools to find limit for nontrivial functions.

- We say a function f is bounded on A if the set $f(A)$ is bounded in $\mathbb{R}$.

R Let $\lim\limits_{x\to c} f(x) = l$. Then f is bounded on some $D_\delta(c)$. Follows from D2.

R[Sandwich] Let $f \le h \le g$ on A and $\lim\limits_{x\to c} f = l = \lim\limits_{x\to c} g$. Then $\lim\limits_{x\to c} h = l$. Follows from D1. Here c is a cluster point of A.

R We have $\lim\limits_{x\to c} f(x) = 0$ iff $\lim\limits_{x\to c} |f(x)| = 0$. Follows from D1.

R. Let $\lim\limits_{x\to c} f(x) = l$ and $\lim\limits_{x\to c} g(x) = m$. Then

a) $\lim\limits_{x\to c}(f(x) + g(x)) = l + m$.

b) $\lim\limits_{x\to c} f(x)g(x) = lm$.

c) $\lim\limits_{x\to c}(\alpha f)(x) = \alpha l$.

d) If $f \ge 0$ on $\operatorname{dom} f$, then $l \ge 0$.

e) If $l > 0$, then $f > 0$ on a $D_\delta(c)$ and $\lim\limits_{x\to c} \frac{1}{f(x)} = \frac{1}{l}$.

f) If $f \ge 0$ and $k \in \mathbb{N}$, then $\lim\limits_{x\to c} \sqrt[k]{f(x)} = \sqrt[k]{l}$.

Use D2 for the first part of e) and D1 for the rest.

- rational polynomials As $\lim_{x \to a} x = a$, we have $\lim_{x \to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}$, if $Q(a) \neq 0$.
- As $-|\theta| \leq \sin\theta \leq |\theta|$, we have $\lim_{\theta \to 0} \sin\theta = 0$. Hence, $\lim_{\theta \to 0} \cos\theta = 1$. Hence, $\lim_{x \to a} \sin(x) = \lim_{x \to a} \Big(\sin(x-a)\cos a + \cos(x-a)\sin a \Big) = \sin(a)$.
- Similar results for trigonometric polynomials and rational functions.
- For $0 < \theta < \frac{\pi}{2}$, we have $\sin\theta \leq \theta \leq \sin\theta + (1 - \cos\theta)$. So $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$.
- For $|x| < 1$, we have $1 + x \leq e^x \leq 1 + x + x^2$!! . So $\lim_{x \to 0} e^x = 1$. Thus $\lim_{x \to a} e^x = \lim_{y \to 0} e^{a+y} = e^a \lim_{y \to 0} e^y = e^a$.
- For $x \neq 0$, we have $-|x| \leq f(x) = x\sin(\frac{1}{x}) \leq |x|$. So $\lim_{x \to 0} f(x) = 0$.
- $\lim_{x \to 1} \frac{x^2+x-2}{x^2-x} = \lim_{x \to 1} \frac{(x+2)(x-1)}{x(x-1)} = (?) \lim_{x \to 1} \frac{x+2}{x} = 3$.

Ex Define $\lim_{x \to c} f(x) = \infty$ in both ways. Compare with the texts.

Ex Define $\lim_{x \to \infty} f(x) = l$ in both ways. Similar to $\lim_{n \to \infty} a_n = l$, where $a_n = f(n)$.

D Let $f : A \to \mathbb{R}$ and c be a cluster point of $(c, \infty) \cap A$.(?)

We say $\lim_{x \to c^+} f(x) = l$, if each $B_\epsilon(l)$ contains some $f(c, c + \delta)$.

That is, $\forall \epsilon > 0$, $\exists\, \delta > 0$ s.t. $(c < x < c + \delta, x \in A) \Rightarrow |f(x) - l| < \epsilon$.

It is the right hand limit $f(c+)$. Define left hand limit $f(c-)$ similarly.

Eg Take $f(x) = [x]$. Then $f(2-) = 1$ and $f(2+) = 2$.

Ex Write a sequential definition of left/right hand limit.

R Let $D_\epsilon(a) \subseteq \mathrm{dom}(f)$ for an ϵ. Then $\lim_{x \to a} f(x) = l$ iff $f(a+) = f(a-) = l$. !!

- Let $f : A \to \mathbb{R}$ and $\underline{a \in A}$. We say f is continuous at a,

D1 if $f(a_n) \to f(a)$ for each sequence $a_n \to a$, $\underline{a_n \in A}$. (property)

D2 if each $B_\epsilon(f(a))$ contains a $f(B_\delta(a))$. That is,

$\forall\epsilon > 0$, $\exists\delta > 0$ such that $\Big(x \in A, |x - a| < \delta\Big) \Rightarrow |f(x) - f(a)| < \epsilon$.

- If $a \in A$ is a cluster point of A, then 'f is cts at a' means $\lim\limits_{x\to a} f(x) = f(a)$.
- If $a \in A$ is NOT a cluster point of A, then 'f is cts at a, by definition.
- 'f is discontinuous at a' means 'f is not continuous at a'.
- We say f is continuous <u>on</u> D, if it is continuous at each $a \in D$.

Ex Define f on $[1, 2) \cup \{3\}$ as $f = 1$ on $[1, 2)$ and $f(3) = 2$. Apply D1,D2.

R Rational functions involving $\sqrt[k]{x}$, $\sin(x)$, e^x are continuous <u>wherever defined</u>.

Eg(Dirichlet's function) $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ is discontinuous at each point. In fact, limit does not exist at any point.

Eg Take $f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$ Then f is continuous at each point.

A Let $a \neq 0$. Let $a_n \to a$. By LT(s), a_n may be assumed nonzero and so $\frac{1}{a_n} \to \frac{1}{a}$. As $\sin x$ is cts at $\frac{1}{a}$, we get $\sin \frac{1}{a_n} \to \sin \frac{1}{a}$. So $a_n \sin \frac{1}{a_n} \to a \sin \frac{1}{a}$. So f is continuous at a.

Let $a = 0$. Note that $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} x \sin \frac{1}{x} = 0 = f(0)$. So f is cts at 0.

Eg $f(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ is continuous at each point except 0.

A For $a \neq 0$, similar to the previous argument. For $a = 0$, recall that $\lim\limits_{x \to 0} \sin \frac{1}{x}$ does not exist. So f is not continuous at 0.

Eg Take $f(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q}. \end{cases}$ It is discontinuous at each point except 0.

A Let $a \neq 0$. Recall that $a_n = \dfrac{[10^n a]}{10^n} \to a$ and $b_n = \dfrac{[10^n a]}{10^n} + \dfrac{\sqrt{2}}{n} \to a$. But $f(a_n) \to a$ and $f(b_n) \to 0 \neq a$. Hence f is not continuous at a.

Let $a = 0$. Note that $-|x| \leq f(x) \leq |x|$ and $\lim_{x \to 0}(\pm|x|) = 0$. By sandwich lemma, $\lim_{x \to 0} f(x) = 0 = f(0)$. So f is continuous at $a = 0$.

R(Combination of cts functions) If $f, g : D \to \mathbb{R}$ are cts at a and $\beta \in \mathbb{R}$, then

a) $f + g$, βf, fg are cts at a.

b) If $f(a) > 0$, then $f > 0$ in some $B_\delta(a)$ and $1/f$ is cts at a.

c) If $f \geq 0$ on D and $k \in \mathbb{N}$, then $\sqrt[k]{f}$ is cts at a.

Po For b) first part, use ϵ-δ definition. For others use sequential definition.

R If f is cts at a, then so is $|f|$. Use the sequential argument.

- Converse true? No. Take $f(x) = -1$ for $x \leq 0$ and $f(x) = 1$ for $x > 0$.

R If f and g are cts at a, then so is $h = \min\{f, g\}$. As $h(x) = \frac{f+g}{2} - \frac{|f-g|}{2}$.

R(Composition) If f is cts at a and g is cts at $f(a)$, then $g \circ f$ is cts at a.

Po Let $a_n \to a$. As f is cts at a, we get $f(a_n) \to f(a)$. As g is cts at $f(a)$, we get $g(f(a_n)) \to g(f(a))$. If I mimic this for limits, where will I have a problem? Try the ϵ-δ proof too.

Eg Can $g \circ f$ be cts at a, even if g is not cts at $f(a)$? Yes. Take $f = 0$ and $g(x) = [x]$.

D Let $f : A \to \mathbb{R}$ and $a \in A$. We say f has an absolute maximum at a, if $f(a) \geq f(x)$ for each $x \in A$. Absolute minimum is defined similarly.

R(maximum-minimum theorem) Let $f : [a, b] \to \mathbb{R}$ be cts. Then f is bounded on $[a, b]$. Also f has an absolute maximum (minimum) in $[a, b]$.

Po Suppose it is not bounded. So, $\exists x_n \in [a, b]$ s.t. $|f(x_n)| \to \infty$.

Is (x_n) bounded? By BWT, we have a conv subsequence, say, $x_{n_k} \to l$.

As $a \leq x_{n_k} \leq b$, we get $a \leq l \leq b$.

Is $x_{n_k} \to l$? Is f cts at l? So $f(x_{n_k}) \to f(l)$. So $|f(x_{n_k})| \to |f(l)|$. $\Rightarrow\Leftarrow$

(cont.) Let $p = \sup f([a, b])$.

Is $p - \frac{1}{n}$ an upper bound of $f([a, b])$? So, $\exists\, y_n \in [a, b]$ s.t. $f(y_n) \geq p - \frac{1}{n}$.

Is (y_n) bounded? So, by BWT, $\exists$ a conv subsequence, say, $y_{n_k} \to t$.

Is $a \leq y_{n_k} \leq b$? So $a \leq t \leq b$. Is $y_{n_k} \to t$? Is f cts at t? So $f(y_{n_k}) \to f(t)$.

As $p - \frac{1}{n_k} \leq f(y_{n_k}) \leq p$, we get $f(y_{n_k}) \to p$. So $f(t) = p$.

Bisection method. Let $f : [a, b] \to \mathbb{R}$ be continuous with $f(a) < 0$ and $f(b) > 0$.

Call $a_1 = a$, $b_1 = b$ and $I_1 = [a_1, b_1]$.

If $f(\frac{a+b}{2}) > 0$, then put $a_2 = a_1$, $b_2 = \frac{a+b}{2}$, $I_2 = [a_2, b_2]$.

If $f(\frac{a+b}{2}) < 0$, then put $a_2 = \frac{a+b}{2}$, $b_2 = b_1$, $I_2 = [a_2, b_2]$.

Assume that we never get $f(\frac{a_n+b_n}{2}) = 0$. Notice that, we always have

$$f(a_n) < 0, \quad f(b_n) > 0, \quad \text{length } I_{n+1} = \tfrac{1}{2} \text{ length } I_n, \text{ and } \quad I_{n+1} \subseteq I_n.$$

By nested interval theorem, $\bigcap\limits_{n=1}^{\infty} [a_n, b_n] = \{c\}$. As $a_n \to c$ and $f(a_n) < 0$, we get $f(c) \leq 0$. As $b_n \to c$ and $f(b_n) > 0$, we get $f(c) \geq 0$. So $f(c) = 0$.

R Let $f : [a, b] \to \mathbb{R}$ be cts with $f(a)f(b) < 0$. Then $\exists c \in (a, b)$ s.t. $f(c) = 0$.

R(IVT) Let $f : [a, b] \to \mathbb{R}$ be cts. Let $m = \min f$ and $M = \max f$ on $[a, b]$. Take an intermediate value k in (m, M). Then $\exists c \in (a, b)$ s.t. $f(c) = k$.

Po Use previous result with $f(x) - k$.

R Let $f : [a, b]$ be cts. Let $m = \min f$ and $M = \max f$ on $[a, b]$. Then $f([a, b]) = [m, M]$.

R(fixed point) Let $f : [0, 1] \to [0, 1]$ be cts. Then $\exists c \in [0, 1]$ s.t. $f(c) = c$.

Po If $f(0) = 0$ or $f(1) = 1$, we are done. Otherwise, we have $f(0) > 0$ <u>and</u> $f(1) < 1$. Consider $g(x) = f(x) - x$. Apply IVT.

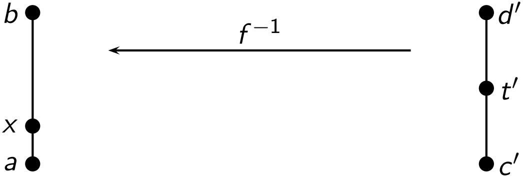
Eg The equation $p(x) = x^3 - 5x^2 + 17x + 18$ has at least one real zero.

A $x > 40 \Rightarrow x^3 > \underbrace{(5 + 17 + 18)}_{40} x^2 \geq 5x^2 + 17x + 18 \geq \pm 5x^2 \pm 17x \pm 18$.

That is, $x > 40 \Rightarrow p(x) > 0$ and $x < -40 \Rightarrow p(x) < 0$. Apply IVT.

R(Inverse continuity) Let $f : [a, b] \to [c', d']$ be strictly increasing, onto and continuous. Then f^{-1} is strictly increasing and continuous.

Po Continuity of f^{-1}: let $c' < t' < d'$. Put $x = f^{-1}(t')$. Is $a < x < b$? Yes.



Now, take some $[x - \epsilon, x + \epsilon] \subseteq (a, b)$.

Then $f(x - \epsilon, x + \epsilon) = (f(x - \epsilon), f(x + \epsilon))$, as f is strictly increasing and cts. Also it contains t'. So some $(t' - \delta, t' + \delta) \subseteq f(x - \epsilon, x + \epsilon)$. That is, $f^{-1}(t' - \delta, t' + \delta) \subseteq (x - \epsilon, x + \epsilon)$. So f^{-1} is cts at t'.

Similarly, f is cts at c' and d'.

Cor Thus $\ln x$ is cts on $(0, \infty)$.