MA 101 (Mathematics I)

Multivariable Calculus : Practice Problem Set - 1

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1. If $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^m$, then show that
   (a) $\big|\, \|\mathbf{x}\| - \|\mathbf{y}\| \,\big| \le \|\mathbf{x} - \mathbf{y}\|$.
   (b) $\|\mathbf{x} + \mathbf{y}\|\, \|\mathbf{x} - \mathbf{y}\| \le \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$.
   (c) $\|\mathbf{x}\| \le \max\{\|\mathbf{x} + \mathbf{y}\|, \|\mathbf{x} - \mathbf{y}\|\}$.
   (d) $\|\mathbf{x} + \alpha\mathbf{y}\| \ge \|\mathbf{x}\|$ for all $\alpha \in \mathbb{R}$ iff $\mathbf{x} \cdot \mathbf{y} = 0$.

2. Let $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^m$ and $\alpha > 0$. Show that $|\mathbf{x} \cdot \mathbf{y}| \le \alpha\|\mathbf{x}\|^2 + \frac{1}{4\alpha}\|\mathbf{y}\|^2$.

3. Let $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^m$. Show that $\big|\, \|\mathbf{x}\| - \|\mathbf{y}\| \,\big| = \|\mathbf{x} - \mathbf{y}\|$ iff $\alpha\mathbf{x} = \beta\mathbf{y}$ for some $\alpha, \beta \ge 0$ with $(\alpha, \beta) \ne (0, 0)$.

4. Let $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^m$ and $r > 0$ such that $\mathbf{y} \cdot \mathbf{z} = 0$ for all $\mathbf{z} \in B_r(\mathbf{x})$. Show that $\mathbf{y} = \mathbf{0}$.

5. If $\mathbf{x}_0 \in \mathbb{R}^m$ and $r > 0$, then determine $\sup\{\|\mathbf{x} - \mathbf{y}\| : \mathbf{x}, \mathbf{y} \in B_r(\mathbf{x}_0)\}$ with justification.

6. Let $S \subseteq \mathbb{R}^m$ such that $S \subseteq B_r[\mathbf{x}_0]$ for some $\mathbf{x}_0 \in \mathbb{R}^m$ and for some $r > 0$. Show that S is a bounded set.

7. Let $\alpha \in (0, 1)$ and let $\mathbf{x}_n = \big(n^3\alpha^n, \frac{1}{n}[n\alpha]\big)$ for all $n \in \mathbb{N}$. (For each $x \in \mathbb{R}$, $[x]$ denotes the greatest integer not exceeding x.) Examine whether the sequence $(\mathbf{x}_n)$ converges in $\mathbb{R}^2$. Also, find $\lim\limits_{n\to\infty} \mathbf{x}_n$ if the sequence $(\mathbf{x}_n)$ converges in $\mathbb{R}^2$.

8. Let $(\mathbf{x}_n)$ be a sequence in $\mathbb{R}^m$ such that the series $\sum\limits_{n=1}^{\infty} n^2\|\mathbf{x}_n\|^2$ is convergent. Show that the series $\sum\limits_{n=1}^{\infty} \|\mathbf{x}_n\|$ is convergent.

9. Let $(\mathbf{x}_n)$ and $(\mathbf{y}_n)$ be sequences in $\mathbb{R}^m$ such that $\mathbf{x}_n \to \mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y}_n \to \mathbf{y} \in \mathbb{R}^m$. Show that $\mathbf{x}_n + \mathbf{y}_n \to \mathbf{x} + \mathbf{y}$ and $\mathbf{x}_n \cdot \mathbf{y}_n \to \mathbf{x} \cdot \mathbf{y}$.

10. Let $\mathbf{x} \in \mathbb{R}^m$ and let $(\mathbf{x}_n)$ be a sequence in $\mathbb{R}^m$ such that $\|\mathbf{x}_n\| \to \|\mathbf{x}\|$ and $\mathbf{x}_n \cdot \mathbf{x} \to \mathbf{x} \cdot \mathbf{x}$. Show that $(\mathbf{x}_n)$ is convergent.

11. State TRUE or FALSE with justification for each of the following statements.
    (a) If $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{x} \ne \mathbf{y}$ and $\|\mathbf{x}\| = 1 = \|\mathbf{y}\|$, then it is necessary that $\|\mathbf{x} + \mathbf{y}\| < 2$.
    (b) If $(\mathbf{x}_n)$ is a sequence in $\mathbb{R}^m$ such that for each $\mathbf{x} \in \mathbb{R}^m$, $\lim\limits_{n\to\infty} \mathbf{x}_n \cdot \mathbf{x}$ exists (in $\mathbb{R}$), then $\lim\limits_{n\to\infty} \|\mathbf{x}_n\|^2$ must exist (in $\mathbb{R}$).
    (c) There exists an unbounded sequence (x_n) of distinct real numbers such that the sequence $\big((x_n, \cos x_n)\big)$ in $\mathbb{R}^2$ has a convergent subsequence.

12. Let $S = \{(x,y) \in \mathbb{R}^2 : x \neq y\}$ and let $f : S \to \mathbb{R}$ be defined by $f(x,y) = \frac{x+y}{x-y}$ for all $(x,y) \in S$. Show by using the definition of continuity that f is continuous at $(1,2)$.

13. If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous and $f(x,y) = x^2 + y^2$ for all $x \in \mathbb{Q}$ and for all $y \in \mathbb{R} \setminus \mathbb{Q}$, then determine $f(\sqrt{2}, 2)$.

14. Examine the continuity of $f : \mathbb{R}^2 \to \mathbb{R}$ at $(0,0)$, where for all $(x,y) \in \mathbb{R}^2$,
(a) $f(x,y) = \begin{cases} xy & \text{if } xy \geq 0, \\ -xy & \text{if } xy < 0. \end{cases}$
(b) $f(x,y) = \begin{cases} \frac{xy^3}{x^2+y^4} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$
(c) $f(x,y) = \begin{cases} 1 & \text{if } x > 0 \text{ and } 0 < y < x^2, \\ 0 & \text{otherwise.} \end{cases}$

15. Determine all the points of $\mathbb{R}^2$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous, if for all $(x,y) \in \mathbb{R}^2$,
(a) $f(x,y) = \begin{cases} \frac{xy}{x-y} & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$
(b) $f(x,y) = \begin{cases} xy & \text{if } xy \in \mathbb{Q}, \\ -xy & \text{if } xy \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

16. Let α, β be positive real numbers and let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by
$f(x,y) = \begin{cases} \frac{|x|^\alpha |y|^\beta}{x^2+xy+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$
Show that f is continuous iff $\alpha + \beta > 2$.

17. Let S be a nonempty subset of $\mathbb{R}^m$ and let $f_j : S \to \mathbb{R}$ for each $j \in \{1, \ldots, k\}$. If $f(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_k(\mathbf{x}))$ for all $\mathbf{x} \in S$, then show that $f : S \to \mathbb{R}^k$ is continuous at $\mathbf{x}_0 \in S$ iff f_j is continuous at $\mathbf{x}_0$ for each $j \in \{1, \ldots, k\}$.

18. Examine the continuity of $f : \mathbb{R}^2 \to \mathbb{R}^2$ at $(0,0)$, where for all $(x,y) \in \mathbb{R}^2$,
$f(x,y) = \begin{cases} \left(\frac{x^3}{x^2+y^2}, \sin(x^2+y^2)\right) & \text{if } (x,y) \neq (0,0), \\ (0,0) & \text{if } (x,y) = (0,0). \end{cases}$

19. If $f, g : S \subseteq \mathbb{R}^m \to \mathbb{R}^k$ are continuous at $\mathbf{x}_0 \in S$ and if $\varphi(\mathbf{x}) = f(\mathbf{x}) \cdot g(\mathbf{x})$ for all $\mathbf{x} \in S$, then show that $\varphi : S \to \mathbb{R}$ is continuous at $\mathbf{x}_0$.

20. Let $f : S \subseteq \mathbb{R}^m \to \mathbb{R}^k$ be continuous at $\mathbf{x}_0 \in S^0$ and let $f(\mathbf{x}_0) \neq \mathbf{0}$. Show that there exists $r > 0$ such that $f(\mathbf{x}) \neq \mathbf{0}$ for all $\mathbf{x} \in B_r(\mathbf{x}_0)$.

21. Let S be an open subset of $\mathbb{R}^m$ and let $f : S \to \mathbb{R}^k$ and $g : S \to \mathbb{R}^k$ be continuous at $\mathbf{x}_0 \in S$. If for each $\varepsilon > 0$, there exist $\mathbf{x}, \mathbf{y} \in B_\varepsilon(\mathbf{x}_0)$ such that $f(\mathbf{x}) = g(\mathbf{y})$, then show that $f(\mathbf{x}_0) = g(\mathbf{x}_0)$.

22. If $S = \{(x,y) \in \mathbb{R}^2 : x + y \geq 2\}$, then determine (with justification) S^0.

23. If $S = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : x_m = 1\}$, then determine (with justification) S^0.

24. If $\mathbf{x} \in \mathbb{R}^m$ and $r > 0$, then determine (with justification) all the interior points of $B_r[\mathbf{x}]$.

25. Examine whether $\{(x, y) \in \mathbb{R}^2 : 0 < x < y\}$ is an open set in $\mathbb{R}^2$.