MA 101 (Mathematics – I)
Integration : Lecture Notes

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1 Riemann integral

Integration Class 1

[**1.1**] DEFINITION A **partition** or **subdivision** P of an interval $[a, b]$ is a finite set $\{x_0, x_1, \ldots, x_n\}$ such that $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. The points x_i are called the **nodes** of P. We will write P as $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$.

[**1.2**] EXAMPLE (i) Trivial partition: $P = \{a = x_0 < x_1 = b\}$.
(ii) $\mathbf{P}_n = \{a = x_0 < x_1 < \cdots < x_n = b\}$, where $n \in \mathbb{N}$ and $x_i = a + \frac{i}{n}(b - a)$. $\mathbf{P}_n$ divides $[a, b]$ into n subintervals of equal length.

For this section, f will always mean a function $f : [a, b] \to \mathbb{R}$ that is bounded.

[**1.3**] DEFINITION For a partition $P = \{a = x_0 < \cdots < x_n = b\}$ of $[a, b]$ define

$$m_k = \text{glb}\{f(x) : x \in [x_{k-1}, x_k]\}, M_k = \text{lub}\{f(x) : x \in [x_{k-1}, x_k]\},$$

lower sum of f w.r.t. P: $L(f, P) := \sum_{k=1}^{n} m_k(x_k - x_{k-1})$,
upper sum of f w.r.t. P: $U(f, P) := \sum_{k=1}^{n} M_k(x_k - x_{k-1})$.

[**1.4**] EXERCISE If $f(x) = x^4 - 4x^3 + 10$ for $x \in [1, 4]$ and $P = \{1 < 2 < 3 < 4\}$, calculate $U(f, P)$ and $L(f, P)$. [Fact: f is decreasing in $[1, 3]$ and increasing in $[3, 4]$.]

[**1.5**] RESULT *Let* $m = \text{glb}\{f(x) : x \in [a, b]\}$ *and* $M = \text{lub}\{f(x) : x \in [a, b]\}$. *Then*

$$m(b - a) \leq L(f, P) \leq U(f, P) \leq M(b - a).$$

[**1.6**] DEFINITION

Lower integral of f: $L(f) = \underline{\int_a^b} f(x)dx := \text{lub}\{L(f, P) : P \text{ is a partition of } [a, b]\}$.

Upper integral of f: $U(f) = \overline{\int_a^b} f(x)dx := \text{glb}\{U(f, P) : P \text{ is a partition of } [a, b]\}$.

[**1.7**] RESULT $L(f) \leq U(f)$. We will see soon why this is so.

[**1.8**] DEFINITION The function $f : [a, b] \to \mathbb{R}$ is said to be **(Riemann or Darboux) integrable** if $L(f) = U(f)$ on $[a, b]$. The common value is called the **integral** of f over $[a, b]$ and is denoted by $I(f)$ or $I_a^b(f)$ or $\int_a^b f$ or $\int_a^b f(x)dx$. By $\mathcal{R}[a, b]$ we denote the set of all integrable functions on $[a, b]$.

[1.9] EXAMPLE If $f : [a, b] \to \mathbb{R}$ is a constant function and $f(x) = c$, then f is integrable and $\int_a^b f = c(b - a)$.

[1.10] EXERCISE (1) Is the function $f(x) = 0$ for $0 \le x < 1$ and $f(1) = 1$, integrable?
(2) Is the Dirichlet function $f : [0, 1]$ defined by $f(x) = 1$, if $x \in \mathbb{Q}$, and 0, otherwise, integrable?
(3) Is the function $f : [0, 1]$ defined by $f(x) = x$, if $x \in \mathbb{Q}$, and 0, otherwise, integrable?
[Hint. Let $P = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ and $\frac{1}{2} \in [x_{i-1}, x_i]$. Then $U(f, P) \ge \frac{1}{2}(1 - x_{i-1}) \ge 1/4$. However, $L(f, P) = 0$.]

[1.11] DEFINITION For partitions P and Q of $[a, b]$, Q is called a **refinement** of P, if $P \subseteq Q$.

Q: When is $\mathbf{P}_m$ a refinement of $\mathbf{P}_n$?

[1.12] RESULT *If Q is a refinement of P, then $L(f, P) \le L(f, Q) \le U(f, Q) \le U(f, P)$.*

Proof. First, suppose $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ and Q has one more point s (say) than P, with $x_{i-1} < s < x_i <$. Then

$$m_i^{(1)} := \text{glb}\{f(x) : x \in [x_{i-1}, s]\} \ge m_i,$$

$$m_i^{(2)} := \text{glb}\{f(x) : x \in [s, x_i]\} \ge m_i.$$

Therefore, $L(f, Q) - L(f, P) = (m_i^{(1)} - m_i)(s - x_{i-1}) + (m_i^{(2)} - m_i)(x_{i-1} - s) \ge 0$, i.e., $L(f, P) \le L(f, Q)$. Now, it is clear that if Q is obtained by adding several (a finitely many) points to P, then $L(f, P) \le L(f, Q)$. Similarly, $U(f, Q) \le U(f, P)$. ∎

[1.13] RESULT *If P and Q are partitions of $[a, b]$, then $L(f, P) \le U(f, Q)$. Therefore we have*

$$m(b - a) \le L(f) \le U(f) \le M(b - a).$$

Proof. $L(f, P) \le L(f, P \cup Q) \le U(f, P \cup Q) \le U(f, Q)$.

[1.14] RESULT *Suppose there is sequence (P_n) of partitions of $[a, b]$ such that $L(f, P_n) \to \alpha$ and $U(f, P_n) \to \alpha$. Then $f \in \mathcal{R}[a, b]$ and $\int_a^b f = \alpha$.*

Proof. $L(f) \ge \alpha$ and $U(f) \le \alpha$.

[1.15] EXERCISE

(1) For $f(x) = x$ on $[0, 1]$ calculate $L(f, \mathbf{P}_n)$ and $U(f, \mathbf{P}_n)$, conclude $f \in \mathcal{R}[0, 1]$, and find $\int_a^b f$.
(2) For $f(x) = x^2$ on $[0, 1]$ calculate $L(f, \mathbf{P}_n)$ and $U(f, \mathbf{P}_n)$, conclude $f \in \mathcal{R}[0, 1]$, and find $\int_a^b f$.
[Hint. $L(f, \mathbf{P}_n) = \sum_{i=1}^n \left(\frac{i-1}{n}\right)^2 \frac{1}{n} \to \frac{1}{3}, U(f, \mathbf{P}_n) = \sum_{i=1}^n \left(\frac{i}{n}\right)^2 \frac{1}{n} \to \frac{1}{3}$.]

[1.16] THEOREM **(Riemann condition for Integrability)** *A bounded function $f : [a, b] \to \mathbb{R}$ is integrable if and only if for each $\epsilon > 0$ there exists a partition P such that $U(f, P) - L(f, P) < \epsilon$.*

Proof. Exercise.

[1.17] EXAMPLE Take $f(x) = x^3$ on $[0, 1]$. Let $\epsilon > 0$. Then

$$U(f, \mathbf{P}_n) - L(f, \mathbf{P}_n) = \frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{i}{n}\right)^3 - \left(\frac{i-1}{n}\right)^3\right] = \frac{1}{n}(f(1) - f(0)) = \frac{1}{n} < \epsilon$$

for large n. Thus, $f \in \mathcal{R}([0, 1])$.

`Q`: Suppose f is monotone on $[a, b]$. Is $f \in \mathcal{R}([a, b])$? Can we use the idea of above example?

[1.18] REMARK Let $f \in \mathcal{R}([a, b])$. Then, for each $n \in \mathbb{N}$, there is a partition P_n such that $U(f, P_n) - L(f, P_n) < \frac{1}{n}$. Since $L(f, P_n) \le \int_a^b f \le U(f, P_n)$, we then have

$$\lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n) = \int_a^b f.$$

Thus, if you can get hold of such a sequence of partitions, then you can (possibly) find out the integral taking a limit. However, it does not say how to find such a sequence.

[1.19] DEFINITION For a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a, b]$ the **mesh** of P is defined to be $\|P\| = \max\{x_i - x_{i-1} : 1 \le i \le n\}$, i.e., maximum length of the subintervals P produces.

[1.20] THEOREM **(Darboux condition)** *Let $f : [a, b] \to \mathbb{R}$ be bounded. Then $f \in \mathcal{R}([a, b])$ if and only if for every $\epsilon > 0$, there is $\delta > 0$ such that $U(f, P) - L(f, P) < \epsilon$ whenever $\|P\| < \delta$.*

Proof. Omitted.

[1.21] REMARK Suppose $f \in \mathcal{R}([a, b])$. Then, $\int_a^b f = \lim\limits_{n\to\infty} L(f, \mathbf{P}_n)$. Similarly, $\int_a^b f = \lim\limits_{n\to\infty} U(f, \mathbf{P}_n)$.

[1.22] EXERCISE Suppose you know that $\lim\limits_{n\to\infty} U(f, \mathbf{P}_n) = \ell$. Is it true that $f \in \mathcal{R}([a, b])$?
[Hint. Take the Dirichlet function on $[0, 1]$.]

Integration Class 2

[1.23] EXERCISE Suppose $f : [c, d] \to \mathbb{R}$ be bounded and $m = \text{glb}\{f(x) : x \in [c, d]\}$ and $M = \text{lub}\{f(x) : x \in [c, d]\}$. Show that $M - m = \text{lub}\{|f(x) - f(y)| : x, y \in [c, d]\}$.

[1.24] RESULT **(Algebra of integrals)** *Let $f, g \in \mathcal{R}([a, b])$, and $\alpha \in \mathbb{R}$. Then*

1. *$f + g \in \mathcal{R}([a, b])$ and $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.*
2. *$\alpha f \in \mathcal{R}([a, b])$ and $\int_a^b (\alpha f) = \alpha \int_a^b f$.*
3. *$|f| \in \mathcal{R}([a, b])$. (Converse?)*
4. *$f^2 \in \mathcal{R}([a, b])$.*

5. $fg \in \mathcal{R}([a,b])$.
6. *if* $0 < m \le f \le M$, *then* $1/f \in \mathcal{R}([a,b])$.
7. $\max\{f,g\}, \min\{f,g\} \in \mathcal{R}([a,b])$.
8. *If* $a < c < b$, *then* $f \in \mathcal{R}([a,c]), f \in \mathcal{R}([c,b])$, *and* $\int_a^c f + \int_c^b f = \int_a^b f$.

Proof.

1. Let $\epsilon > 0$. There are partitions P_1 and P_2 such that $U(f,P_1) - L(f,P_1) < \epsilon/2$ and $U(g,P_2) - L(g,P_2) < \epsilon/2$. Let $P = P_1 \cup P_2$. Then $U(f+g,P) - L(f+g,P) < \epsilon$, since
$$L(f,P) + L(g,P) \le L(f+g,P) \le U(f+g,P) \le U(f,P) + U(g,P).$$
Therefore $f+g \in \mathcal{R}([a,b])$. Now, note that $\int_a^b f + \int_a^b g$ and $\int_a^b (f+g)$ both lie in the interval $[L(f,P)+L(g,P), U(f,P)+U(g,P)]$ which is of length ϵ. Thus, $\left|(\int_a^b f + \int_a^b g) - \int_a^b (f+g)\right| < \epsilon$. Since ϵ is arbitrary, $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.
2. If $\alpha \ge 0$, then $\operatorname{glb}\{\alpha f(x) : x \in [x_{i-1}, x_i]\} = \alpha m_i$, and so $L(\alpha f, P) = \alpha L(f,P)$, etc.
3. $U(|f|,P) - L(|f|,P) \le U(f,P) - L(f,P)$.
Converse is not true, e.g., $f(x) = 1$, if $x \in [0,1] \cap \mathbb{Q}$, and $f(x) = -1$, if $x \in [0,1] \cap \mathbb{Q}^c$.
4. There is $M > 0$ such that $|f(x)| \le M$ for $x \in [a,b]$. Then, for $x, y \in [a,b]$ we have $|f(x)^2 - f(y)^2| \le 2M|f(x) - f(y)|$. For a partition P,
$$\begin{aligned} U(f^2,P) - L(f^2,P) &= \sum_{i=1}^n (x_i - x_{i-1}) \operatorname{lub}\{|f(x)^2 - f(y)^2| : x \in [x_{i-1}, x_i]\} \\ &\le 2M \sum_{i=1}^n (x_i - x_{i-1}) \operatorname{lub}\{|f(x) - f(y)| : x \in [x_{i-1}, x_i]\} \\ &= 2M(U(f,P) - L(f,P)). \end{aligned}$$
5. Follows from $|(1/f)(x) - (1/f)(y)| \le \frac{1}{m^2}|f(x) - f(y)|$.
6. Follows from $fg = \frac{1}{2}\big((f+g)^2 - f^2 - g^2\big)$. Use the previous results.
7. $\max\{f,g\} = \frac{1}{2}\big(f + g + |f-g|\big)$, $\min\{f,g\} = \frac{1}{2}\big(f + g - |f-g|\big)$.
8. Let $\epsilon > 0$. Take Q such that $U(f,Q) - L(f,Q) < \epsilon$. Set $P = Q \cup \{c\}, P_1 = P \cap [a,c]$ and $P_2 = P \cap [c,b]$. Then $L(f,P) = L(f,P_1) + L(f,P_2)$ and $U(f,P_1) + U(f,P_2) = U(f,P)$. Thus,
$$\int_a^b f - \epsilon < L(f,P_1) + L(f,P_2) \le U(f,P_1) + U(f,P_2) < \int_a^b f + \epsilon.$$
Thus, $U(f,P_1) - L(f,P_1) < 2\epsilon$, yielding $f \in \mathcal{R}([a,c])$. Similarly, $f \in \mathcal{R}([c,b])$. Finally, observe that $\left|\big(\int_a^c f + \int_c^b f\big) - \int_a^b f\right| < \epsilon$.

[1.25] RESULT *Suppose* $f : [a,b] \to \mathbb{R}$, $a < c < b$, $f \in \mathcal{R}([a,c])$ *and* $f \in \mathcal{R}([c,b])$. *Then,* $f \in \mathcal{R}([a,b])$ *and* $\int_a^b f = \int_a^c f + \int_c^b f$.

Proof. Exercise.

[**1.26**] EXAMPLE We have now many functions integrable on $[a, b]$

x, any polynomial, $\sin x$ (as monotone in subintervals), $x \sin x$, etc.

[**1.27**] RESULT *Let $f : [a, b] \to \mathbb{R}$.*

1. *If $f \geq 0$ and $f \in \mathcal{R}([a, b])$, then $\int_a^b f \geq 0$.*
2. *If $f, g \in \mathcal{R}([a, b])$ and $f \leq g$, then $\int_a^b f \leq \int_a^b g$.*
3. *If $f \in \mathcal{R}([a, b])$, then $|\int_a^b f| \leq \int_a^b |f|$.*

Proof. (1) Follows from the fact that $L(f, P) \geq 0$ for every partition p of $[a, b]$, since $f \geq 0$.

(2) $f \leq g$ implies $\int_a^b g - \int_a^b f = \int_a^b (g - f) \geq 0$, by (1).

(3) Note that $f \in \mathcal{R}([a, b])$ implies $|f| \in \mathcal{R}([a, b])$. [(3) of [**1.24**]]. Now, $-|f| \leq f \leq |f|$, and therefore by (2),

$$-\int_a^b |f| = \int_a^b -|f| \leq \int_a^b f \leq \int_a^b |f|, \text{ i.e. } |\int_a^b f| \leq \int_a^b |f|. \qquad \blacksquare$$

[**1.28**] DEFINITION Let $S \subseteq \mathbb{R}$. A function $f : S \to \mathbb{R}$ is **uniformly continuous** (on S), if given $\epsilon > 0$, there is $\delta > 0$ such that $x, y \in S, |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$.

[**1.29**] RESULT *(1) If $f : S \to \mathbb{R}$ is uniformly continuous, then f is continuous.*
(2) A continuous function f on a closed interval is uniformly continuous.

Proof. (1) Follows from the definition.
(2) Suppose f is continuous, but not uniformly continuous on $[a, b]$. Then, there is $\epsilon > 0$ such that for each $n \in \mathbb{N}$, there are $x_n, y_n \in [a, b]$ such that $|x_n - y_n| < 1/n$ and $|f(x_n) - f(y_n)| \geq \epsilon$. Since (x_n) is bounded, by BWT, (x_n) has convergent subsequence (x_{n_k}), converging to c, say. Then, $c \in [a, b]$. Further,

$$|y_{n_k} - c| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - c| < \frac{1}{n_k} + |x_{n_k} - c| \to 0,$$

that is, $y_n \to c$. Since f is continuous at c, we have $|f(x_{n_k}) - f(y_{n_k})| \to |c - c| = 0$. However, this cannot happen because $|f(x_{n_k}) - f(y_{n_k})| \geq \epsilon$ for every k. Hence, f must be uniformly continuous. $\blacksquare$

[**1.30**] THEOREM *If f is continuous on $[a, b]$, then $f \in \mathcal{R}([a, b])$.*

Proof. Let $\epsilon > 0$. Then there is $n \in \mathbb{N}$ such that

$$x, y \in [a, b], |x - y| < \frac{1}{n} \implies |f(x) - f(y)| < \frac{\epsilon}{b - a}.$$

Thus, for $\mathbf{P}_n$, $M_i - m_i = \text{lub}\{|f(x) - f(y)| : x, y \in [x_{i-1}, x_i]\} \leq \frac{\epsilon}{b-a}$. Consequently,

$$U(f, \mathbf{P}_n) - L(f, \mathbf{P}_n) = \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) \leq \frac{\epsilon}{b - a} = \epsilon. \qquad \blacksquare$$

[1.31] RESULT *If $f : [a, b] \to \mathbb{R}$ is bounded and continuous on $[a, b)$ (or on $(a, b]$), then $f \in \mathcal{R}([a, b])$.*

Proof. Assume $|f| < M$ f is continuous on $[a, b)$. Let $\epsilon > 0$. Write $[a, b] = I_1 \cup I_2$, where

$$I_1 = \left[a, b - \frac{\epsilon}{4M}\right], \quad I_2 = \left[b - \frac{\epsilon}{4M}, b\right].$$

Since f is continuous on I_1, we have $f \in \mathcal{R}(I_1)$. So, there is a partition P_1 of I_1 such that $U(f, P_1) - L(f, P_1) < \epsilon/2$. Moreover, $\text{lub}\{|f(x) - f(y)| : x, y \in I_2\} \leq 2M$. Now, $P = P_1 \cup \{b\}$ is a partition of $[a, b]$ and

$$U(f, P) - L(f, P) = (U(f, P_1) - L(f, P_1)) + 2M \cdot \frac{\epsilon}{4M} < \epsilon.$$

Therefore, $f \in \mathcal{R}([a, b])$. Similarly, when f is continuous on $(a, b]$. ∎

`Q`: Suppose $f : [a, b] \to \mathbb{R}$ is bounded and $C = \{x \in [a, b] : f \text{ is discontinuous at } x\}$.

1. Is $f \in \mathcal{R}([a, b])$, if $C = \{c\}$ where $c \in (a, b)$? `A`: Yes. Use **[1.31]** and **[1.25]**.
2. Is $f \in \mathcal{R}([a, b])$, if C is finite? `A`: Yes. Use part (1) and **[1.25]**.
3. Is $f \in \mathcal{R}([a, b])$, if $C = \{c_n : n \in \mathbb{N}\}$ where $c_n \to c \in [a, b]$? `A`: Yes. Use the idea of the proof of **[1.31]** and part (2).
4. Is $f \in \mathcal{R}([a, b])$, if C infinite? `A`: No. Take Dirichlet function.
5. Is $f \in \mathcal{R}([a, b])$, if C infinite having finitely many limit points? `A`: Yes. Use the idea of the proof of **[1.31]** and part (2).

[1.32] EXAMPLE

1. Let $f(x) = \sin \frac{1}{x}$ if $x \neq 0$, and $f(0) = 1$. Then, $f \in \mathcal{R}([0, 1])$.
2. Let $f(x) = 0$ if $x \in (0, 1]$, and $f(0) = c$. Then, $f \in \mathcal{R}([0, 1])$. Further, $\int_0^1 f = \lim L(f, \mathbf{P}_n) = 0$.

[1.33] COROLLARY *Let $c_1, \ldots, c_k \in [a, b]$, and $f : [a, b] \to \mathbb{R}$ be such that $f(x) = 0$ for $x \notin \{c_1, \ldots, c_n\}$. Then $f \in \mathcal{R}([a, b])$ and $\int_a^b f = 0$.*

Proof. Exercise.

[1.34] RESULT *Let $f \in \mathcal{R}([a, b])$ and $g : [a, b] \to \mathbb{R}$ be such $g(x) \neq f(x)$ for only finitely many points $x \in [a, b]$. Then $g \in \mathcal{R}([a, b])$ and $\int_a^b g = \int_a^b f$.*

Proof. Apply **[1.33]** to $f - g$.

`Q`: Can you improve the above result?

[1.35] EXAMPLE The Thomae's function is integrable: $f : [0, 1] \to \mathbb{R}$ where

$$f(x) = \begin{cases} 1/q, & \text{if } x = p/q \in \mathbb{Q}, \gcd(p, q) = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Let $\epsilon > 0$. Since for any partition P, we have $L(f, P) = 0$, it is enough to find a partition P such that $U(f, P) < \epsilon$. Now, there are only finitely many points $0, c_1, \ldots, c_k, 1$ (in increasing order) in $[0, 1]$ where f takes value $> \epsilon/2$. Choose $\delta < \frac{\epsilon}{4(k+1)}$ so that we get a partition

$$P = \{0 < \delta < c_1 - \delta < c_1 + \delta < \cdots < 1 - \delta < 1\}$$

of $[a, b]$. Then the contribution of $[0, \delta]$ and $[1 - \delta, 1]$ to $U(f, P)$ is $\delta + \delta = 2\delta$. The total contribution of the intervals $[c_i - \delta, c_i + \delta]$ is $\leq k \cdot 2\delta$. The contribution of the rest of the intervals is less than $\epsilon/2$, since the total length of these intervals is less than 1 ans $f(x) \leq \epsilon/2$ for x in these intervals. Hence,

$$U(f, P) < 2\delta + 2k\delta + \epsilon/2 = 2(k + 1)\delta + \epsilon/2 < \epsilon.$$

This shows, $f \in \mathcal{R}([0, 1]$. Further, $\int_a^b f = \lim L(f, \mathbf{P}_n) = 0$.

[1.36] EXAMPLE Composition of integrable functions need not be integrable. Take f as Thomae's function on $[0, 1]$ and $g : [0, 1] \to \mathbb{R}$ defined by $g(0) = 0$ and $g(x) = 1$, elsewhere. Then $g \circ f$ is the Dirichlet function!

<u>**To be continued.**</u>