Tutorial 4: continuity

1. T/F? Let (x_n) be a fixed sequence of points in $[a,b]$ such that $x_n \to c$. Let $f : [a,b] \to \mathbb{R}$ be a function such that $f(x_n) \to f(c)$. Then f is continuous at c.

   Sol. F. (Reamrk: Not just one sequence, it should be true for each sequence (x_n) converging to c in order to be continuous at c.) Take the Dirichlet type function, $f = 1$ on $\mathbb{Q} \cap [-1,1]$ and $f = 0$ on the rest. One has infinitely many rational sequences (x_n) converging to 0 and for each of them $f(x_n) \to f(0)$. But f is not continuous at 0.

2. T/F? If $f+g$ is continuous at a, then f may be discontinuous at a.

   Sol. T. Take f to be the Dirichlet's function and $g = -f$.

3. T/F? If $f+g$ is continuous at a, then f may be discontinuous at a, even if g is continuous at a.

   Sol. F. Follows from the algebra of continuous functions.

4. T/F? Let f be continuous at a. Then $|f|$ is also continuous at a.

   Sol. T.

5. T/F? Let f and g be continuous on $\mathbb{R}$. Then $h(x) := \max\{g(x), f(x)\}$ is continuous.

   Sol. T. As $\max\{f(x), g(x)\} = \frac{f(x)+g(x)}{2} + \frac{|f(x)-g(x)|}{2}$.

6. T/F? Let f be continuous. Then the **positive part** $f_+(x)$ which is defined as $\max\{0, f(x)\}$ is continuous.

   Sol. T. Follows from the algebra of continuous functions.

7. T/F? Let f be continuous on $\mathbb{R}$ with $f(x+y) = f(x) + f(y)$ for each x, y. Then there exists a c such $f(x) = cx$ for all x.

   Sol. T. Put $c = f(1)$. So $f(m) = cm$ for each $m \in \mathbb{N}$. Note that $f(0) = f(0) + f(0)$, so $f(0) = 0$. So $f(m) = cm$ for each $m \in \mathbb{Z}$. So $f(\frac{p}{q}) = c\frac{p}{q}$ for each $p \in \mathbb{Z}, q \in \mathbb{N}$. Let a be an irrational number. Then $f(a) = \lim f(\frac{[10^n a]}{10^n}) = \lim c\frac{[10^n a]}{10^n} = ca$.

8. Let f, g be continuous at 0. Then $f \circ g$ must be continuous at 0.

   Sol. F. Consider $g(x) = x + .5$, and $f(x) = [x + .5]$. Both are continuous at 0. But $f(g(x)) = [x + 1]$, is not continuous at 0.

9. There is a bijection $f : [0, 1] \to \mathbb{R}$. I claim that such a bijection is discontinuous, how?

   Sol. If f is continuous, then $f([0, 1])$ would be bounded.

10. There is no continuous function f from $(0, 1)$ onto $(0, 1) \cup (2, 3)$. Why?

    Sol. If there is one such f, then there exists $a \in (0, 1)$ such that $f(a) = 2.5$ and there exists $b \in (0, 1)$ such that $f(b) = 0.5$. By IVT, there exists a c between a and b such that $f(c) = 1.5$. This is a contradiction.

11. Let $f : [0, 2] \to \mathbb{R}$ be continuous such that $f(0) = 5, f(1) = 4, f(2) = 9$. Then there must be a point $c \in [0, 1]$ such that $f(2c) = 2f(c)$. Why?

    Sol. Take $g(x) = f(2x) - 2f(x)$. It is continuous with $g(0) = -10$ and $g(1) = 1$. Apply IVT.

12. Let $f : \mathbb{R} \to \mathbb{R}$ be a monotone increasing function. Fix any $a \in \mathbb{R}$. Then $\lim\limits_{x \to a-} f(x)$ must exist. Why?

    Sol. Let $A = (-\infty, a)$. So $f(A)$ is nonempty and bounded above (by $f(a)$). Put $l = \text{lub}\, f(A)$. We show that $f(a+) = l$. For that consider $l - \epsilon$. This is not an upper bound of $f(A)$. So there exists $b \in A$ such that $f(b) > l - \epsilon$. In that case $f((b, a)) \subseteq (l - \epsilon, l)$. Choose $\delta = |a - b|$. Thus for each $x \in (a - \delta, a)$ we have $|f(x) - l| < \epsilon$.

13. T/F? Let $A = (0, 1) \cup (3, 4)$ and $f : A \to \mathbb{R}$ be defined as $f(x) = 1$ if $x \in A$ and $f(x) = 2$, otherwise. Then f is continuous on A.

    Sol. True. By definition. If you are not convinced, give me a point of discontinuity.