MA 101 (Mathematics - I)

Multivariable Calculus : Examples from Lectures - 2

Example: $S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$ is a closed set but not an open set in $\mathbb{R}^2$.
Proof: Let $\big((x_n, y_n)\big)$ be any sequence in S such that $(x_n, y_n) \to (x,y) \in \mathbb{R}^2$. Then $x_n \to x$ and $y_n \to y$. Hence $x_n^2 + y_n^2 \to x^2 + y^2$. Also, $x_n^2 + y_n^2 \le 1$ for all $n \in \mathbb{N}$ and so $x^2 + y^2 \le 1$. Thus $(x,y) \in S$ and therefore S is a closed set in $\mathbb{R}^2$.
Again, since $(1 + \frac{1}{n}, 0) \in \mathbb{R}^2 \setminus S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 > 1\}$ for all $n \in \mathbb{N}$ and $(1 + \frac{1}{n}, 0) \to (1,0) \notin \mathbb{R}^2 \setminus S$, $\mathbb{R}^2 \setminus S$ is not a closed set in $\mathbb{R}^2$. Consequently S is not an open set in $\mathbb{R}^2$.

Example: If $\mathbf{x}_0 \in \mathbb{R}^m$ and $r > 0$, then $B_r[\mathbf{x}_0]$ is a closed set but not an open set in $\mathbb{R}^m$.
Proof: Let $(\mathbf{x}_n)$ be any sequence in $B_r[\mathbf{x}_0]$ such that $\mathbf{x}_n \to \mathbf{x} \in \mathbb{R}^m$. Since $\big|\, \|\mathbf{x}_n - \mathbf{x}_0\| - \|\mathbf{x} - \mathbf{x}_0\| \,\big| \le \|(\mathbf{x}_n - \mathbf{x}_0) - (\mathbf{x} - \mathbf{x}_0)\| = \|\mathbf{x}_n - \mathbf{x}\| \to 0$ (see Ex.1(a) of Practice Problem Set - 1), we find that $\|\mathbf{x}_n - \mathbf{x}_0\| \to \|\mathbf{x} - \mathbf{x}_0\|$. Again, since $\|\mathbf{x}_n - \mathbf{x}_0\| \le r$ for all $n \in \mathbb{N}$, it follows that $\|\mathbf{x} - \mathbf{x}_0\| \le r$. Thus $\mathbf{x} \in B_r[\mathbf{x}_0]$ and therefore $B_r[\mathbf{x}_0]$ is a closed set in $\mathbb{R}^m$.
Again, $\mathbf{x}_0 + (1 + \frac{1}{n}) r \mathbf{e}_1 \in \mathbb{R}^m$ and $\|\mathbf{x}_0 + (1 + \frac{1}{n}) r \mathbf{e}_1 - \mathbf{x}_0\| = (1 + \frac{1}{n}) r > r$ for all $n \in \mathbb{N}$. Hence $\mathbf{x}_0 + (1 + \frac{1}{n}) r \mathbf{e}_1 \in \mathbb{R}^m \setminus B_r[\mathbf{x}_0] = \{\mathbf{x} \in \mathbb{R}^m : \|\mathbf{x} - \mathbf{x}_0\| > r\}$ for all $n \in \mathbb{N}$. Also, $\mathbf{x}_0 + r\mathbf{e}_1 \in \mathbb{R}^m$ and $\|\mathbf{x}_0 + (1 + \frac{1}{n}) r \mathbf{e}_1 - (\mathbf{x}_0 + r\mathbf{e}_1)\| = \frac{r}{n} \to 0$ and so $\mathbf{x}_0 + (1 + \frac{1}{n}) r \mathbf{e}_1 \to \mathbf{x}_0 + r\mathbf{e}_1$. Since $\|\mathbf{x}_0 + r\mathbf{e}_1 - \mathbf{x}_0\| = r$, $\mathbf{x}_0 + r\mathbf{e}_1 \notin \mathbb{R}^m \setminus B_r[\mathbf{x}_0]$ and therefore $\mathbb{R}^m \setminus B_r[\mathbf{x}_0]$ is not a closed set in $\mathbb{R}^m$. Consequently $B_r[\mathbf{x}_0]$ is not an open set in $\mathbb{R}^m$.

Example: $S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ is an open set but not a closed set in $\mathbb{R}^2$.
Proof: Let $\big((x_n, y_n)\big)$ be any sequence in $\mathbb{R}^2 \setminus S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \ge 1\}$ such that $(x_n, y_n) \to (x,y) \in \mathbb{R}^2$. Then $x_n \to x$ and $y_n \to y$. Hence $x_n^2 + y_n^2 \to x^2 + y^2$. Also, $x_n^2 + y_n^2 \ge 1$ for all $n \in \mathbb{N}$ and so $x^2 + y^2 \ge 1$. Thus $(x,y) \in \mathbb{R}^2 \setminus S$ and therefore $\mathbb{R}^2 \setminus S$ is a closed set in $\mathbb{R}^2$. Consequently S is an open set in $\mathbb{R}^2$.
Again, since $(1 - \frac{1}{n}, 0) \in S$ for all $n \in \mathbb{N}$ and $(1 - \frac{1}{n}, 0) \to (1,0) \notin S$, S is not a closed set in $\mathbb{R}^2$.

Example: If $\mathbf{x}_0 \in \mathbb{R}^m$ and $r > 0$, then $B_r(\mathbf{x}_0)$ is an open set but not a closed set in $\mathbb{R}^m$.
Proof: Let $(\mathbf{x}_n)$ be any sequence in $\mathbb{R}^m \setminus B_r(\mathbf{x}_0) = \{\mathbf{x} \in \mathbb{R}^m : \|\mathbf{x} - \mathbf{x}_0\| \ge r\}$ such that $\mathbf{x}_n \to \mathbf{x} \in \mathbb{R}^m$. Since $\big|\, \|\mathbf{x}_n - \mathbf{x}_0\| - \|\mathbf{x} - \mathbf{x}_0\| \,\big| \le \|(\mathbf{x}_n - \mathbf{x}_0) - (\mathbf{x} - \mathbf{x}_0)\| = \|\mathbf{x}_n - \mathbf{x}\| \to 0$, we find that $\|\mathbf{x}_n - \mathbf{x}_0\| \to \|\mathbf{x} - \mathbf{x}_0\|$. Again, since $\|\mathbf{x}_n - \mathbf{x}_0\| \ge r$ for all $n \in \mathbb{N}$, it follows that $\|\mathbf{x} - \mathbf{x}_0\| \ge r$. Thus $\mathbf{x} \in \mathbb{R}^m \setminus B_r(\mathbf{x}_0)$ and therefore $\mathbb{R}^m \setminus B_r(\mathbf{x}_0)$ is a closed set in $\mathbb{R}^m$. Consequently $B_r(\mathbf{x}_0)$ is an open set in $\mathbb{R}^m$.
Again, $\mathbf{x}_0 + (1 - \frac{1}{n}) r \mathbf{e}_1 \in \mathbb{R}^m$ and $\|\mathbf{x}_0 + (1 - \frac{1}{n}) r \mathbf{e}_1 - \mathbf{x}_0\| = (1 - \frac{1}{n}) r < r$ for all $n \in \mathbb{N}$. Hence $\mathbf{x}_0 + (1 - \frac{1}{n}) r \mathbf{e}_1 \in B_r(\mathbf{x}_0)$ for all $n \in \mathbb{N}$. Also, $\mathbf{x}_0 + r\mathbf{e}_1 \in \mathbb{R}^m$ and $\|\mathbf{x}_0 + (1 - \frac{1}{n}) r \mathbf{e}_1 - (\mathbf{x}_0 + r\mathbf{e}_1)\| = \frac{r}{n} \to 0$ and so $\mathbf{x}_0 + (1 - \frac{1}{n}) r \mathbf{e}_1 \to \mathbf{x}_0 + r\mathbf{e}_1$. Since $\|\mathbf{x}_0 + r\mathbf{e}_1 - \mathbf{x}_0\| = r$, $\mathbf{x}_0 + r\mathbf{e}_1 \notin B_r(\mathbf{x}_0)$ and therefore $B_r(\mathbf{x}_0)$ is not a closed set in $\mathbb{R}^m$.

Example: $\mathbb{R}^m$ is both an open set and a closed set in $\mathbb{R}^m$.
Proof: If $\mathbf{x}_0 \in \mathbb{R}^m$, then $B_1(\mathbf{x}_0) \subseteq \mathbb{R}^m$ and so $\mathbf{x}_0$ is an interior point of $\mathbb{R}^m$. Since $\mathbf{x}_0 \in \mathbb{R}^m$ is arbitrary, it follows that $\mathbb{R}^m$ is an open set in $\mathbb{R}^m$.
Again, if $(\mathbf{x}_n)$ is any sequence in $\mathbb{R}^m$ and $(\mathbf{x}_n)$ is convergent in $\mathbb{R}^m$, then $\lim\limits_{n\to\infty} \mathbf{x}_n \in \mathbb{R}^m$. Therefore

$\mathbb{R}^m$ is a closed set in $\mathbb{R}^m$.

Example: $S = \{(x, y) \in \mathbb{R}^2 : 1 < x \leq 2\}$ is neither an open set nor a closed set in $\mathbb{R}^2$.
Proof: Since $(1 + \frac{1}{n}, 0) \in S$ for all $n \in \mathbb{N}$ and $(1 + \frac{1}{n}, 0) \to (1, 0) \notin S$, S is not a closed set in $\mathbb{R}^2$. Again, since $(2 + \frac{1}{n}, 0) \in \mathbb{R}^2 \setminus S$ for all $n \in \mathbb{N}$ and $(2 + \frac{1}{n}, 0) \to (2, 0) \notin \mathbb{R}^2 \setminus S$, $\mathbb{R}^2 \setminus S$ is not a closed set in $\mathbb{R}^2$. Consequently S is not an open set in $\mathbb{R}^2$.

Example: $(0, 0)$ and $(1, 0)$ are limit points of $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ but $(1, 1)$ is not a limit point of S.
Proof: Let $r > 0$.
Then $(\frac{r}{2}, 0) \in \big(B_r\big((0, 0)\big) \setminus \{(0, 0)\}\big) \cap S$ if $r < 2$ and $(\frac{1}{2}, 0) \in \big(B_r\big((0, 0)\big) \setminus \{(0, 0)\}\big) \cap S$ if $r \geq 2$. Thus in either case $\big(B_r\big((0, 0)\big) \setminus \{(0, 0)\}\big) \cap S \neq \emptyset$ and therefore $(0, 0)$ is a limit point of S.
Again, $(1 - \frac{r}{2}, 0) \in \big(B_r\big((1, 0)\big) \setminus \{(1, 0)\}\big) \cap S$ if $r < 4$ and $(0, 0) \in \big(B_r\big((1, 0)\big) \setminus \{(1, 0)\}\big) \cap S$ if $r \geq 4$. Thus in either case $\big(B_r\big((1, 0)\big) \setminus \{(1, 0)\}\big) \cap S \neq \emptyset$ and therefore $(1, 0)$ is a limit point of S.
Now, let $s = \sqrt{2} - 1 > 0$ and let $(x, y) \in B_s\big((1, 1)\big)$. Then we have (using Ex.1(a) of Practice Problem Set - 1) $s > \|(x, y) - (1, 1)\| \geq \|(1, 1)\| - \|(x, y)\| = \sqrt{2} - \|(x, y)\|$ and hence $\|(x, y)\| > 1$. Thus $(x, y) \notin S$ and this shows that $\big(B_s\big((1, 1)\big) \setminus \{(1, 1)\}\big) \cap S = \emptyset$. Therefore $(1, 1)$ is not a limit point of S.

Example: $\lim\limits_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = 0$.
Proof: First method (by directly using definition):
Let $\varepsilon > 0$. For all $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, we have $\left|\frac{x^3}{x^2+y^2}\right| = \frac{x^2}{x^2+y^2}|x| \leq |x| \leq \sqrt{x^2 + y^2}$. If $\delta = \varepsilon$, then $\delta > 0$ and $\left|\frac{x^3}{x^2+y^2} - 0\right| < \varepsilon$ for all $(x, y) \in \mathbb{R}^2$ satisfying $0 < \|(x, y) - (0, 0)\| = \sqrt{x^2 + y^2} < \delta$. Therefore $\lim\limits_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = 0$.

Second method (by using sequential criterion):
Let $\big((x_n, y_n)\big)$ be any sequence in $\mathbb{R}^2 \setminus \{(0, 0)\}$ such that $(x_n, y_n) \to (0, 0)$. Then $x_n \to 0$ and $y_n \to 0$. Since $\left|\frac{x_n^3}{x_n^2+y_n^2}\right| = \frac{x_n^2}{x_n^2+y_n^2}|x_n| \leq |x_n| \to 0$, it follows that $\frac{x_n^3}{x_n^2+y_n^2} \to 0$. Therefore $\lim\limits_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = 0$.

Example: $\lim\limits_{(x,y)\to(0,0)} \frac{x^2 y}{x^4+y^2}$ does not exist (in $\mathbb{R}$).
Proof: Let $f(x, y) = \frac{x^2 y}{x^4+y^2}$ for all $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. We have that $(\frac{1}{n}, 0) \to (0, 0)$ and $(\frac{1}{n}, \frac{1}{n^2}) \to (0, 0)$. Since $f(\frac{1}{n}, 0) = 0 \to 0$ and $f(\frac{1}{n}, \frac{1}{n^2}) = \frac{1}{2} \to \frac{1}{2}$, it follows that $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist (in $\mathbb{R}$)

Example: $\lim\limits_{(x,y)\to(0,0)} \frac{1}{x^2+y^2} = \infty$.
Proof: Let $r > 0$. If $\delta = \frac{1}{\sqrt{r}}$, then $\delta > 0$ and for all $(x, y) \in \mathbb{R}^2$ with $0 < \|(x, y) - (0, 0)\| = \sqrt{x^2 + y^2} < \delta$, we have $\frac{1}{x^2+y^2} > \frac{1}{\delta^2} = r$. Hence $\lim\limits_{(x,y)\to(0,0)} \frac{1}{x^2+y^2} = \infty$.

Example: $\lim\limits_{(x,y)\to(0,0)} \frac{1}{x+y} \neq \infty$.
Proof: Since $(-\frac{1}{n}, 0) \to (0, 0)$ but $\frac{1}{-\frac{1}{n}+0} = -n \not\to \infty$, $\lim\limits_{(x,y)\to(0,0)} \frac{1}{x+y} \neq \infty$.