MA 101 (Mathematics - I)
Tutorial 6: Differentiation 5b, Integration 1, 2

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1. (a) Find an explicit formula for the function represented by the power series $\sum_{n=1}^{\infty} nx^n$ and indicate its domain of convergence.

   (b) Find an explicit formula for the function represented by the power series $\sum_{n=1}^{\infty} n^2x^n$ in its interval of convergence. Use it to find the sum of $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ and $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$.

2. Find the Taylor series of the function $f(x) = (1+x)e^{-x} - (1-x)e^{x}$ about 0. Using this, find the sum of the series
$$\frac{1}{3!} + \frac{2}{5!} + \cdots + \frac{n}{(2n+1)!} + \cdots .$$

3. Let $f : [a,b] \to R$ be a bounded function. If there is a partition P of $[a,b]$ such that $L(f,P) = U(f,P)$, then prove that f is a constant function.

4. Define $f : [-1,1] \to \mathbb{R}$ by
$$f(x) = \begin{cases} 1, & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Show that $f \in \mathcal{R}([a,b])$ and that $\int_{-1}^{1} f = 0$.

   (b) Define $F(x) = \int_{-1}^{x} f$ on $[-1,1]$. Show that f is differentiable. In particular, $F'(0) = f(0)$ although f is not continuous at 0.

5. Show that (a) $\displaystyle\int_0^1 \frac{x^4}{\sqrt{1+4x^{90}}} \geq \frac{1}{5\sqrt{5}}$. (b) $\displaystyle\int_0^3 \frac{x^3(x-4)}{1+x^{10}} \sin(2020x)dx \leq 81$.

6. (1) If $f \in \mathcal{R}[a,b]$, $f \geq 0$ and $\int_a^b f = 0$, then show that $f = 0$ at each point of continuity of f.
   (2) If f is continuous, $f \geq 0$ and $\int_a^b f = 0$, then conclude that $f = 0$ on $[a,b]$.
   (3) Show that the results need not hold if $f \geq 0$ is not assumed.

7. Let $f > 0$ and continuous on $[a,b]$. Let $M = \max f$ on $[a,b]$. Show that
$$\lim_{n\to\infty} \left(\int_a^b (f(x))^n dx \right)^{1/n} = M.$$
