MA 101 (Mathematics - I)
Tutorial Problems 4: (Differentiability 1, 2)

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1. Give an example of a continuous function on $\mathbb{R}$ which is not differentiable exactly at (i) 1, (ii) $1, 2, 3$, (iii) every integer.

2. Let $r > 0$ be a rational number, and $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^r sin\frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$. Determine those values of r for which f is differentiable.

3. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $c \in \mathbb{R}$, show that $f'(c) = \lim \Big(n\big(f(c + \frac{1}{n}) - f(c)\big)\Big)$. Show by an example that the existence of the limit of this sequence does not imply the existence of $f'(c)$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Let $n \in \mathbb{N}$, $a \in \mathbb{R}$. Find the limit $\lim\limits_{x \to a} \dfrac{a^n f(x) - x^n f(a)}{x - a}$.

5. Let $f : I \to \mathbb{R}$ be differentiable at $c \in (a, b)$, and $x_n < c < y_n$ in I such that $y_n - x_n \to 0$. Find $\lim\limits_{n \to \infty} \dfrac{f(y_n) - f(x_n)}{y_n - x_n}$, if it exists.

6. Suppose $f : [a, b] \to \mathbb{R}$ is differentiable on (a, b) and $\lim\limits_{x \to a+} f'(x) = \ell$. Show that f is differentiable at a and $f'(a) = \ell$ if and only if f is continuous at a.

7. Let $f : [a, b] \to [a, b]$ be differentiable. Assume that $f'(x) \neq 1$ for $x \in [a, b]$. Prove that f has a unique fixed point in $[a, b]$.

8. Let $f : [a, b] \to \mathbb{R}$ be differentiable. Assume that there exists no $x \in [a, b]$ such that $f(x) = 0 = f'(x)$. Prove that the number of zeroes of f in $[a, b]$ is finite.

9. Show that $\frac{\sin x}{x}$ is strictly increasing on $(0, \pi/2)$.

10. Consider the function $h : \mathbb{R} \to \mathbb{R}$ given by $h(x) = x^3 + 2x + 1$. Show that h is a bijection, and therefore has an inverse h^{-1} on $\mathbb{R}$. Find $(h^{-1})'(y)$ at the points y corresponding to $x = 0, 1, -1$.

11. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable such that $f(0) = f(1) = 0$ and $f'(0) > 0, f'(1) > 0$. Show that there are distinct $c_1, c_2 \in (0, 1)$ such that $f'(c_1) = f'(c_2) = 0$.
