MA 101 (Mathematics I)

Multivariable Calculus : Hints / Solutions of Practice Problem Set - 3

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1. If $f : \mathbb{R}^m \to \mathbb{R}$ satisfies $|f(\mathbf{x})| \le \|\mathbf{x}\|^2$ for all $\mathbf{x} \in \mathbb{R}^m$, then examine whether f is differentiable at $\mathbf{0}$.

   Solution: Since $|f(\mathbf{0})| \le \|\mathbf{0}\|^2 = 0$, we have $f(\mathbf{0}) = 0$. If $\alpha = \mathbf{0}$, then $\alpha \in \mathbb{R}^m$ and for all $\mathbf{h} \in \mathbb{R}^m \setminus \{\mathbf{0}\}$, we have $\frac{|f(\mathbf{h})-f(\mathbf{0})-\alpha\cdot\mathbf{h}|}{\|\mathbf{h}\|} \le \|\mathbf{h}\|$. Hence it follows that $\lim\limits_{\mathbf{h}\to\mathbf{0}} \frac{|f(\mathbf{h})-f(\mathbf{0})-\alpha\cdot\mathbf{h}|}{\|\mathbf{h}\|} = 0$. Therefore f is differentiable at $\mathbf{0}$.

2. Let $f(\mathbf{x}) = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$. Examine whether $f : \mathbb{R}^m \to \mathbb{R}$ is differentiable at $\mathbf{0}$.

   Solution: Since $\lim\limits_{t\to 0} \frac{f(\mathbf{0}+t\mathbf{e}_1)-f(\mathbf{0})}{t} = \lim\limits_{t\to 0} \frac{\|t\mathbf{e}_1\|}{t} = \lim\limits_{t\to 0} \frac{|t|}{t}$ does not exist (in $\mathbb{R}$), $\frac{\partial f}{\partial x_1}(\mathbf{0})$ does not exist (in $\mathbb{R}$). Consequently f is not differentiable at $\mathbf{0}$.

3. If $f(x,y) = \sqrt{|xy|}$ for all $(x,y) \in \mathbb{R}^2$, then examine whether $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(0,0)$.

   Hint: Here $f_x(0,0) = f_y(0,0) = 0$.
   Since $\lim\limits_{(h,k)\to(0,0)} \frac{f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)}{\sqrt{h^2+k^2}} = \lim\limits_{(h,k)\to(0,0)} \frac{\sqrt{|hk|}}{\sqrt{h^2+k^2}} \ne 0$, f is not differentiable at $(0,0)$.

4. If $f(x,y) = \big|\,|x|-|y|\,\big| - |x| - |y|$ for all $(x,y) \in \mathbb{R}^2$, then examine whether $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(0,0)$.

   Solution: We have $f_x(0,0) = \lim\limits_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0$ and $f_y(0,0) = \lim\limits_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = 0$. Now $\lim\limits_{(h,k)\to(0,0)} \frac{|f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)|}{\sqrt{h^2+k^2}} = \lim\limits_{(h,k)\to(0,0)} \frac{|f(h,k)|}{\sqrt{h^2+k^2}} \ne 0$, since $(\frac{2}{n},\frac{1}{n}) \to (0,0)$ but $\lim\limits_{n\to\infty} \frac{|f(\frac{2}{n},\frac{1}{n})|}{\sqrt{\frac{4}{n^2}+\frac{1}{n^2}}} = \frac{2}{\sqrt{5}} \ne 0$. Hence f is not differentiable at $(0,0)$.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} 1 & \text{if } y < x^2 < 2y, \\ 0 & \text{otherwise.} \end{cases}$
   Examine whether f is differentiable at $(0,0)$.

   Solution: We have $(\frac{1}{\sqrt{n+1}}, \frac{1}{n+2}) \to (0,0)$ but $f(\frac{1}{\sqrt{n+1}}, \frac{1}{n+2}) = 1 \to 1 \ne 0 = f(0,0)$. Hence f is not continuous at $(0,0)$ and consequently f is not differentiable at $(0,0)$.

6. Let $f(x,y) = \begin{cases} (x^2+y^2)\cos\left(\frac{1}{x^2+y^2}\right) & \text{if } (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}, \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$
   Examine whether $f : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable.

   Solution: For all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$, we have $f_x(x,y) = 2x\cos\left(\frac{1}{x^2+y^2}\right) + \frac{2x}{x^2+y^2}\sin\left(\frac{1}{x^2+y^2}\right)$. Now $\left(\frac{\sqrt{2}}{\sqrt{(4n+1)\pi}}, 0\right) \to (0,0)$ but $f_x\left(\frac{\sqrt{2}}{\sqrt{(4n+1)\pi}}, 0\right) = \sqrt{2(4n+1)\pi} \to \infty$. Hence $\lim\limits_{(x,y)\to(0,0)} f_x(x,y)$ does not exist (in $\mathbb{R}$) and consequently f_x is not continuous at $(0,0)$. Therefore f is not continuously differentiable.

7. Let $\alpha \in \mathbb{R}$ and $\alpha > 0$. If $f(x,y) = |xy|^\alpha$ for all $(x,y) \in \mathbb{R}^2$, then determine all values of α for which $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(0,0)$.

Solution: We have $f_x(0,0) = \lim\limits_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = \lim\limits_{t\to 0} \frac{0-0}{t} = 0$ and $f_y(0,0) = \lim\limits_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = \lim\limits_{t\to 0} \frac{0-0}{t} = 0$. For all $(h,k) \in \mathbb{R}^2 \setminus \{(0,0)\}$, let $\varphi(h,k) = \frac{|f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)|}{\sqrt{h^2+k^2}} = \frac{|hk|^\alpha}{\sqrt{h^2+k^2}}$. If $\alpha > \frac{1}{2}$, then $\varphi(h,k) \le \frac{(h^2+k^2)^{\alpha/2}(h^2+k^2)^{\alpha/2}}{\sqrt{h^2+k^2}} = (h^2+k^2)^{\alpha-\frac{1}{2}}$ and so $\lim\limits_{(h,k)\to(0,0)} \varphi(h,k) = 0$. Consequently f is differentiable at $(0,0)$.
Again, if $\alpha \le \frac{1}{2}$, then $(\frac{1}{n},\frac{1}{n}) \to (0,0)$ but $\varphi(\frac{1}{n},\frac{1}{n}) = \frac{1}{\sqrt{2}} n^{1-2\alpha} \not\to 0$ (for $\alpha = \frac{1}{2}$, $\varphi(\frac{1}{n},\frac{1}{n}) \to \frac{1}{\sqrt{2}}$ and for $\alpha < \frac{1}{2}$, the sequence $\big(\varphi(\frac{1}{n},\frac{1}{n})\big)$ is unbounded). Hence $\lim\limits_{(h,k)\to(0,0)} \varphi(h,k) \ne 0$ and so f is not differentiable at $(0,0)$.

8. Let $f(x,y) = |xy|$ for all $(x,y) \in \mathbb{R}^2$. Determine all the points of $\mathbb{R}^2$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable.

Solution: Let $S_1 = \{(x,y) \in \mathbb{R}^2 : xy > 0\}$ and $S_2 = \{(x,y) \in \mathbb{R}^2 : xy < 0\}$. Then $f(x,y) = xy$ for all $(x,y) \in S_1$ and $f(x,y) = -xy$ for all $(x,y) \in S_2$. Since $f_x(x,y) = y$ and $f_y(x,y) = x$ for all $(x,y) \in S_1$, we find that both $f_x : S_1 \to \mathbb{R}$ and $f_y : S_1 \to \mathbb{R}$ are continuous. Hence f is differentiable at every point of S_1. By a similar argument, we can show that f is differentiable at every point of S_2. If $\alpha(\ne 0) \in \mathbb{R}$, then $f_y(\alpha,0) = \lim\limits_{t\to 0} \frac{f(\alpha,t)-f(\alpha,0)}{t} = \lim\limits_{t\to 0} \frac{|\alpha||t|}{t}$ does not exist (in $\mathbb{R}$) and similarly $f_x(0,\alpha)$ does not exist (in $\mathbb{R}$). Hence f is not differentiable at any point $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ for which $xy = 0$. Again, $f_x(0,0) = \lim\limits_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = 0$, $f_y(0,0) = \lim\limits_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = 0$ and $\lim\limits_{(h,k)\to(0,0)} \frac{|f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)|}{\sqrt{h^2+k^2}} = \lim\limits_{(h,k)\to(0,0)} \frac{|h||k|}{\sqrt{h^2+k^2}} = 0$ (since $|h||k| \le h^2+k^2$ for all $(h,k) \in \mathbb{R}^2$). Hence f is differentiable at $(0,0)$. Therefore the set of all points of $\mathbb{R}^2$ at which f is differentiable is $\{(x,y) \in \mathbb{R}^2 : xy \ne 0\} \cup \{(0,0)\}$.

9. Let $f(x,y) = (xy)^{\frac{2}{3}}$ for all $(x,y) \in \mathbb{R}^2$. Determine all the points of $\mathbb{R}^2$ at which $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable.

Solution: Let $S = \{(x,y) \in \mathbb{R}^2 : xy \ne 0\}$. Since $f_x(x,y) = \frac{2}{3}x^{-\frac{1}{3}}y^{\frac{2}{3}}$ and $f_y(x,y) = \frac{2}{3}x^{\frac{2}{3}}y^{-\frac{1}{3}}$ for all $(x,y) \in S$, we find that both $f_x : S \to \mathbb{R}$ and $f_y : S \to \mathbb{R}$ are continuous. Hence f is differentiable at every point of S. If $\alpha(\ne 0) \in \mathbb{R}$, then $f_y(\alpha,0) = \lim\limits_{t\to 0} \frac{f(\alpha,t)-f(\alpha,0)}{t} = \lim\limits_{t\to 0} \frac{\alpha^{\frac{2}{3}}}{t^{\frac{1}{3}}}$ does not exist (in $\mathbb{R}$) and similarly $f_x(0,\alpha)$ does not exist (in $\mathbb{R}$). Hence f is not differentiable at any point $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ for which $xy = 0$. Again, $f_x(0,0) = \lim\limits_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = 0$, $f_y(0,0) = \lim\limits_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = 0$ and $\lim\limits_{(h,k)\to(0,0)} \frac{|f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)|}{\sqrt{h^2+k^2}} = \lim\limits_{(h,k)\to(0,0)} \frac{|h|^{\frac{2}{3}}|k|^{\frac{2}{3}}}{\sqrt{h^2+k^2}} = 0$ (since $|h|^{\frac{2}{3}}|k|^{\frac{2}{3}} \le (h^2+k^2)^{\frac{2}{3}}$ for all $(h,k) \in \mathbb{R}^2$). Hence f is differentiable at $(0,0)$. Therefore the set of all points of $\mathbb{R}^2$ at which f is differentiable is $\{(x,y) \in \mathbb{R}^2 : xy \ne 0\} \cup \{(0,0)\}$.

10. Determine all the points of $\mathbb{R}^2$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable, if for all $(x,y) \in \mathbb{R}^2$,
$$f(x,y) = \begin{cases} x^2+y^2 & \text{if both } x,y \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$$
Solution: Since $|f(x,y)| \le x^2+y^2 = \|(x,y)\|^2$ for all $(x,y) \in \mathbb{R}^2$, by Ex.12(a) of Practice

Problem Set - 3, f is differentiable at $(0,0)$.

Let $(x_0,y_0)\in\mathbb{R}^2\setminus\{(0,0)\}$. If $(x_0,y_0)\in\mathbb{Q}\times\mathbb{Q}$, then $(x_0+\frac{\sqrt{2}}{n},y_0)\to(x_0,y_0)$ but $f(x_0+\frac{\sqrt{2}}{n},y_0)=0\to 0\neq x_0^2+y_0^2=f(x_0,y_0)$. Again if $(x_0,y_0)\notin\mathbb{Q}\times\mathbb{Q}$, then we choose rational sequences (x_n) and (y_n) such that $x_n\to x_0$ and $y_n\to y_0$. Then $(x_n,y_n)\to(x_0,y_0)$ but $f(x_n,y_n)=x_n^2+y_n^2\to x_0^2+y_0^2\neq 0=f(x_0,y_0)$. Hence f is not continuous at (x_0,y_0) and consequently f is not differentiable at (x_0,y_0).

11. State TRUE or FALSE with justification: If $S=\{(x,y)\in\mathbb{R}^2: x^2+y^2<1\}$ and if $f(x,y)=|xy|$ for all $(x,y)\in S$, then $f:S\to\mathbb{R}$ is differentiable.

Solution: Clearly $(\frac{1}{2},0)\in S$. Since $\lim\limits_{t\to 0}\frac{f(\frac{1}{2},t)-f(\frac{1}{2},0)}{t}=\lim\limits_{t\to 0}\frac{|t|}{2t}$ does not exist (in $\mathbb{R}$), $f_y(\frac{1}{2},0)$ does not exist (in $\mathbb{R}$). Hence f is not differentiable at $(\frac{1}{2},0)$ and so f is not differentiable. Therefore the given statement is FALSE.

12. Let $f:\mathbb{R}^2\to\mathbb{R}$ be such that f_x exists (in $\mathbb{R}$) at all points of $B_\delta\big((x_0,y_0)\big)$ for some $(x_0,y_0)\in\mathbb{R}^2$ and $\delta>0$, f_x is continuous at (x_0,y_0) and $f_y(x_0,y_0)$ exists (in $\mathbb{R}$). Show that f is differentiable at (x_0,y_0).

Solution: For all $(h,k)\in B_\delta\big((0,0)\big)$, we have $f(x_0+h,y_0+k)-f(x_0,y_0)=f(x_0+h,y_0+k)-f(x_0,y_0+k)+f(x_0,y_0+k)-f(x_0,y_0)$. Now, by the mean value theorem for single real variable, we get $f(x_0+h,y_0+k)-f(x_0,y_0+k)=hf(x_0+\theta h,y_0+k)$ for some $\theta\in(0,1)$. Again, if $\varepsilon(k)=\frac{f(x_0,y_0+k)-f(x_0,y_0)}{k}-f_y(x_0,y_0)$ for all $k\in\mathbb{R}\setminus\{0\}$ with $|k|<\delta$ and $\varepsilon(0)=0$, then $f(x_0,y_0+k)-f(x_0,y_0)=kf_y(x_0,y_0)+k\varepsilon(k)$ for all $k\in\mathbb{R}$ with $|k|<\delta$ and $\varepsilon(k)\to 0$ as $k\to 0$.
Now, $\lim\limits_{(h,k)\to(0,0)}\frac{|f(x_0+h,y_0+k)-f(x_0,y_0)-hf_x(x_0,y_0)-kf_y(x_0,y_0)|}{\sqrt{h^2+k^2}}$
$\leq \lim\limits_{(h,k)\to(0,0)}\left(\frac{|h|}{\sqrt{h^2+k^2}}|f_x(x_0+\theta h,y_0+k)-f_x(x_0,y_0)|+\frac{|k|}{\sqrt{h^2+k^2}}|\varepsilon(k)|\right)$
$\leq \lim\limits_{(h,k)\to(0,0)}(|f_x(x_0+\theta h,y_0+k)-f_x(x_0,y_0)|+|\varepsilon(k)|)=0$ (since f_x is continuous at (x_0,y_0)).
Therefore f is differentiable at (x_0,y_0).