MA 101 (Mathematics - I)

Multivariable Calculus : Examples from Lectures - 3

Example: If $f(x,y) = x^2 + xy + 2y$ for all $(x,y) \in \mathbb{R}^2$, $(x_0, y_0) \in \mathbb{R}^2$ and $\mathbf{u} = (u_1, u_2) \in \mathbb{R}^2$ with $\|\mathbf{u}\| = 1$, then $D_{\mathbf{u}} f(x_0, y_0) = (2x_0 + y_0)u_1 + (x_0 + 2)u_2$.

Proof: We have $D_{\mathbf{u}} f(x_0, y_0) = \lim\limits_{t\to 0} \frac{f((x_0,y_0)+t(u_1,u_2)) - f(x_0,y_0)}{t} = \lim\limits_{t\to 0} \frac{f(x_0+tu_1, y_0+tu_2) - f(x_0,y_0)}{t}$
$= \lim\limits_{t\to 0} \frac{(x_0+tu_1)^2 + (x_0+tu_1)(y_0+tu_2) + 2(y_0+tu_2) - (x_0^2 + x_0y_0 + 2y_0)}{t} = \lim\limits_{t\to 0} (2x_0u_1 + tu_1^2 + y_0u_1 + x_0u_2 + tu_1u_2 + 2u_2)$
$= (2x_0 + y_0)u_1 + (x_0 + 2)u_2$.

Example: If $f(x,y) = \sqrt{x^2+y^2}$ for all $(x,y) \in \mathbb{R}^2$, $(x_0,y_0) \in \mathbb{R}^2 \setminus \{(0,0)\}$ and $\mathbf{u} = (u_1,u_2) \in \mathbb{R}^2$ with $\|\mathbf{u}\| = 1$, then $D_{\mathbf{u}} f(x_0,y_0) = \frac{x_0u_1 + y_0u_2}{\sqrt{x_0^2+y_0^2}}$.
Also, $D_{\mathbf{u}} f(0,0)$ does not exist (in $\mathbb{R}$).

Proof: We have $D_{\mathbf{u}} f(x_0, y_0) = \lim\limits_{t\to 0} \frac{f((x_0,y_0)+t(u_1,u_2)) - f(x_0,y_0)}{t} = \lim\limits_{t\to 0} \frac{f(x_0+tu_1, y_0+tu_2) - f(x_0,y_0)}{t}$
$= \lim\limits_{t\to 0} \frac{\sqrt{(x_0+tu_1)^2+(y_0+tu_2)^2} - \sqrt{x_0^2+y_0^2}}{t} = \lim\limits_{t\to 0} \frac{(x_0+tu_1)^2+(y_0+tu_2)^2 - (x_0^2+y_0^2)}{t\left(\sqrt{(x_0+tu_1)^2+(y_0+tu_2)^2} + \sqrt{x_0^2+y_0^2}\right)}$
$= \lim\limits_{t\to 0} \frac{2x_0u_1 + tu_1^2 + 2y_0u_2 + tu_2^2}{\sqrt{(x_0+tu_1)^2+(y_0+tu_2)^2} + \sqrt{x_0^2+y_0^2}} = \frac{x_0u_1+y_0u_2}{\sqrt{x_0^2+y_0^2}}$.
Again, $D_{\mathbf{u}} f(0,0) = \lim\limits_{t\to 0} \frac{f((0,0)+t(u_1,u_2)) - f(0,0)}{t} = \lim\limits_{t\to 0} \frac{f(tu_1,tu_2)}{t} = \lim\limits_{t\to 0} \frac{\sqrt{t^2u_1^2+t^2u_2^2}}{t} = \lim\limits_{t\to 0} \frac{|t|}{t}$, which does not exist (in $\mathbb{R}$).

Example: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} 0 & \text{if } xy = 0, \\ 1 & \text{if } xy \neq 0. \end{cases}$

Then $f_x(0,0) = f_y(0,0) = 0$, but if $\mathbf{u} = (u_1,u_2) \in \mathbb{R}^2$ with $\|\mathbf{u}\| = 1$ and $u_1u_2 \neq 0$, then $D_{\mathbf{u}} f(0,0)$ does not exist.

Proof: We have $f_x(0,0) = \lim\limits_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = \lim\limits_{t\to 0} \frac{0-0}{t} = 0$
and $f_y(0,0) = \lim\limits_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = \lim\limits_{t\to 0} \frac{0-0}{t} = 0$.
Again, $D_{\mathbf{u}} f(0,0) = \lim\limits_{t\to 0} \frac{f((0,0)+t(u_1,u_2)) - f(0,0)}{t} = \lim\limits_{t\to 0} \frac{f(tu_1,tu_2)}{t} = \lim\limits_{t\to 0} \frac{1}{t}$, which does not exist (in $\mathbb{R}$).

Example: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

If $\mathbf{u} = (u_1,u_2) \in \mathbb{R}^2$ with $\|\mathbf{u}\| = 1$, then $D_{\mathbf{u}} f(0,0) = \begin{cases} \frac{u_1^2}{u_2} & \text{if } u_2 \neq 0, \\ 0 & \text{if } u_2 = 0. \end{cases}$

Proof: We have $D_{\mathbf{u}} f(0,0) = \lim\limits_{t\to 0} \frac{f((0,0)+t(u_1,u_2)) - f(0,0)}{t} = \lim\limits_{t\to 0} \frac{f(tu_1,tu_2)}{t}$
$= \lim\limits_{t\to 0} \frac{u_1^2u_2}{t^2u_1^4+u_2^2} = \begin{cases} \frac{u_1^2}{u_2} & \text{if } u_2 \neq 0, \\ 0 & \text{if } u_2 = 0. \end{cases}$

Example: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

Then $f_{xy}(0,0) = -1 \neq 1 = f_{yx}(0,0)$.

Proof: We have $f_{xy}(0,0) = \lim\limits_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k}$ and $f_{yx}(0,0) = \lim\limits_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$.
Now, $f_x(0,0) = \lim\limits_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim\limits_{h\to 0} \frac{0-0}{h} = 0$ and $f_y(0,0) = \lim\limits_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim\limits_{k\to 0} \frac{0-0}{k} = 0$.
Also, if $h \in \mathbb{R} \setminus \{0\}$, then $f_y(h,0) = \lim\limits_{k\to 0} \frac{f(h,k)-f(h,0)}{k} = \lim\limits_{k\to 0} \frac{h(h^2-k^2)}{h^2+k^2} = h$ and if $k \in \mathbb{R} \setminus \{0\}$, then $f_x(0,k) = \lim\limits_{h\to 0} \frac{f(h,k)-f(0,k)}{h} = \lim\limits_{h\to 0} \frac{k(h^2-k^2)}{h^2+k^2} = -k$. Hence $f_{xy}(0,0) = \lim\limits_{k\to 0} \frac{-k-0}{k} = -1$ and

$f_{yx}(0,0) = \lim\limits_{h\to 0} \frac{h-0}{h} = 1.$

Example: If $f(x,y) = 2x^2 + y^3$ for all $(x,y) \in \mathbb{R}^2$, then $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(1,1)$ and $f'(1,1) = [4 \quad 3]$.

Proof: For all $(h,k) \in \mathbb{R}^2$, we have $f((1,1)+(h,k)) - f(1,1) = f(1+h, 1+k) - f(1,1) = 2(1+h)^2 + (1+k)^3 - 3 = 4h + 2h^2 + 3k + 3k^2 + k^3$. Let $\alpha = (4,3)$. Then $\alpha \in \mathbb{R}^2$ and $\lim\limits_{(h,k)\to(0,0)} \frac{|f((1,1)+(h,k)) - f(1,1) - \alpha \cdot (h,k)|}{\|(h,k)\|} = \lim\limits_{(h,k)\to(0,0)} \frac{|2h^2+3k^2+k^3|}{\sqrt{h^2+k^2}} = 0$, since for all $(h,k) \in \mathbb{R}^2 \setminus \{(0,0)\}$, we have $\frac{|2h^2+3k^2+k^3|}{\sqrt{h^2+k^2}} \le 2\frac{|h|}{\sqrt{h^2+k^2}}|h| + 3\frac{|k|}{\sqrt{h^2+k^2}}|k| + \frac{|k|}{\sqrt{h^2+k^2}}k^2 \le 2|h| + 3|k| + k^2$ and since $2|h| + 3|k| + k^2 \to 0$ as $(h,k) \to (0,0)$.

Therefore f is differentiable at $(1,1)$ and $f'(1,1) = [4 \quad 3]$.

Example: The function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by $f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0), \end{cases}$

is differentiable at $(0,0)$ and $f'(0,0) = [0 \quad 0]$.

Proof: We have $f_x(0,0) = \lim\limits_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim\limits_{h\to 0} \frac{0-0}{h} = 0$

and $f_y(0,0) = \lim\limits_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim\limits_{k\to 0} \frac{0-0}{k} = 0.$

Now, $\lim\limits_{(h,k)\to(0,0)} \frac{|f((0,0)+(h,k)) - f(0,0) - hf_x(0,0) - kf_y(0,0)|}{\sqrt{h^2+k^2}} = \lim\limits_{(h,k)\to(0,0)} \frac{|h|\,|k|\,|h^2-k^2|}{(h^2+k^2)\sqrt{h^2+k^2}} = 0$,

since for all $(h,k) \in \mathbb{R}^2 \setminus \{(0,0)\}$, $|h^2 - k^2| \le h^2 + k^2$ and $|hk| \le \frac{1}{2}(h^2+k^2)$,

and hence $\frac{|hk|\,|h^2-k^2|}{(h^2+k^2)\sqrt{h^2+k^2}} \le \frac{1}{2}\sqrt{h^2+k^2} \to 0$ as $(h,k) \to (0,0)$.

Therefore f is differentiable at $(0,0)$. Also, $f'(0,0) = [f_x(0,0) \quad f_y(0,0)] = [0 \quad 0]$.

Example: The function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by $f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0), \end{cases}$

is not differentiable at $(0,0)$.

Proof: We have $f_x(0,0) = \lim\limits_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim\limits_{h\to 0} \frac{0-0}{h} = 0$

and $f_y(0,0) = \lim\limits_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim\limits_{k\to 0} \frac{0-0}{k} = 0.$

Now, $\lim\limits_{(h,k)\to(0,0)} \frac{|f((0,0)+(h,k)) - f(0,0) - hf_x(0,0) - kf_y(0,0)|}{\sqrt{h^2+k^2}} = \lim\limits_{(h,k)\to(0,0)} \frac{|hk|}{h^2+k^2} \neq 0$, since $(\frac{1}{n}, \frac{1}{n}) \to (0,0)$ but $\frac{\frac{1}{n}\cdot\frac{1}{n}}{\frac{1}{n^2}+\frac{1}{n^2}} = \frac{1}{2} \to \frac{1}{2} \neq 0$.

Therefore f is not differentiable at $(0,0)$.

Example: The function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right) & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

is differentiable at $(0,0)$ although neither $f_x : \mathbb{R}^2 \to \mathbb{R}$ nor $f_y : \mathbb{R}^2 \to \mathbb{R}$ is continuous at $(0,0)$.

Proof: We have $f_x(0,0) = \lim\limits_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim\limits_{h\to 0} h \sin\frac{1}{h^2} = 0$

and $f_y(0,0) = \lim\limits_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim\limits_{k\to 0} k \sin\frac{1}{k^2} = 0.$

Now, $\lim\limits_{(h,k)\to(0,0)} \frac{|f((0,0)+(h,k)) - f(0,0) - hf_x(0,0) - kf_y(0,0)|}{\sqrt{h^2+k^2}} = \lim\limits_{(h,k)\to(0,0)} \sqrt{h^2+k^2}\sin\left(\frac{1}{h^2+k^2}\right) = 0$,

since for all $(h,k) \in \mathbb{R}^2 \setminus \{(0,0)\}$, $\left|\sqrt{h^2+k^2}\sin\left(\frac{1}{h^2+k^2}\right)\right| \le \sqrt{h^2+k^2}$ and $\lim\limits_{(h,k)\to(0,0)} \sqrt{h^2+k^2} = 0$.

Therefore f is differentiable at $(0,0)$.

Again, for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$, we have $f_x(x,y) = 2x\sin\left(\frac{1}{x^2+y^2}\right) - \frac{2x}{x^2+y^2}\cos\left(\frac{1}{x^2+y^2}\right)$ and $f_y(x,y) = 2y\sin\left(\frac{1}{x^2+y^2}\right) - \frac{2y}{x^2+y^2}\cos\left(\frac{1}{x^2+y^2}\right)$.

Since $\left(\frac{1}{\sqrt{2n\pi}}, 0\right) \to (0,0)$ but $f_x\left(\frac{1}{\sqrt{2n\pi}}, 0\right) = -\sqrt{2n\pi} \not\to 0 = f_x(0,0)$, f_x is not continuous at $(0,0)$.

Also, since $\left(0, \frac{1}{\sqrt{2n\pi}}\right) \to (0,0)$ but $f_y\left(0, \frac{1}{\sqrt{2n\pi}}\right) = -\sqrt{2n\pi} \not\to 0 = f_y(0,0)$, f_y is not continuous at $(0,0)$.