MA 101 (Mathematics - I)

Differentiability : Exercise set 1: Hints ans solutions

1. Discuss differentiability of $f : \mathbb{R} \to \mathbb{R}$, and continuity of f' wherever exists.

   (i) $f(x) = |x|$.

   (ii) $f(x) = |\sin x|$.

   (iii) $f(x) = \begin{cases} x^2, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

   (iv) $n \in \mathbb{N}$ and $f(x) = \begin{cases} x^n \sin \frac{1}{x}, & \text{if } x \neq 0. \\ 0, & \text{if } x = 0. \end{cases}$

Solution: (Hints.)

(i) Not differentiable at $x = 0$.

(ii) Not differentiable at $x = n\pi,\ n \in \mathbb{Z}$. Draw the graph.

(iii) Differentiable at 0. Not continuous at $x \neq 0$, so differentiable exactly at 0.

(iv) If and only if $n > 1$.

2. Show that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable and is an even function, then f' is an odd function.

Solution: For $c \in \mathbb{R}$

$$\frac{f(x) - f(-c)}{x - (-c)} = \frac{f(-x) - f(c)}{x + c} = -\frac{f(-x) - f(c)}{(-x) - c} \to -f'(c), \text{ as } -x \to c, \text{ i.e., as } x \to -c.$$

Therefore, for any $x \in \mathbb{R}$, $f'(-x) = -f'(x)$.

3. Let $f : (a, b) \to \mathbb{R}$ be differentiable at $c \in (a, b)$. Assume that $f'(c) \neq 0$. Show that there exists $\delta > 0$ such that for $x \in (c - \delta, c + \delta) \cap (a, b)$, we have $f(x) \neq f(c)$. Can you say something more, if $f'(x) > 0$? Similarly, if $f'(x) < 0$?

Solution: Choose $\epsilon > 0$ such that $0 \notin \big(f'(c) - \epsilon, f'(c) + \epsilon\big)$, e.g. $\epsilon = |f'(c)|/2$. Since $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c} = f'(c)$, there exists $\delta > 0$ such that for $x \in (c - \delta, c + \delta) \cap (a, b)$, we have

$$f'(c) - \epsilon < \frac{f(x) - f(c)}{x - c} < f'(c) + \epsilon.$$

Therefore for $x \in (c - \delta, c + \delta) \cap (a, b)$, $f(x) - f(c) \neq 0$. For the second part, look at the sign of $\dfrac{f(x) - f(c)}{x - c}$ (see [2.11] of Differentiation Notes.)

4. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $|f(x) - f(y)| \leq (x - y)^2$. Show that f is a constant function.

Solution: (Hint.) Show that $f'(x) = 0$ for every $x \in \mathbb{R}$.

5. If $a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1} = 0$ for some real numbers a_i, then show that $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$ has a real root between 0 and 1.

> **Solution:** (Hint.) Define $f : [0,1] \to \mathbb{R}$ by $f(x) = a_0 x + \frac{a_1 x^2}{2} + \frac{a_2 x^3}{3} + \cdots + \frac{a_n x^{n+1}}{n+1}$ and use Rolle's theorem on $[0,1]$.

6. Use the identity $1+x+\cdots+x^n = \dfrac{1-x^{n+1}}{1-x}$ for $x \neq 1$ to arrive at a formula for the sum $1+x+2x^2+\cdots+nx^n$.

> **Solution:** (Hint.) Differentiate both sides, multiply by x and add 1, and get the sum as
> $(n+1)\dfrac{x^{n+1}}{x-1} + \dfrac{x(1-x^{n+1})}{(1-x)^2} + 1.$

7. Verify Chain Rule for f, g and $g \circ f$ at the point 0, where

$$f(x) = \begin{cases} x^2, & \text{if } x \in \mathbb{Q} \\ 0, & \text{otherwise,} \end{cases} \qquad g(x) = \begin{cases} \sin x, & \text{if } x \in \mathbb{Q} \\ x, & \text{otherwise.} \end{cases}$$

> **Solution:** $g \circ f$ is given by $(g \circ f)(x) = \sin x^2$, if $x \in \mathbb{Q}$ and $(g \circ f)(x) = 0$, if $x \notin \mathbb{Q}$.
> Now, for $x \neq 0$
>
> $$\frac{f(x) - f(0)}{x - 0} = \begin{cases} x, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{otherwise,} \end{cases} \quad \to 0 \;\text{ as } x \to 0.$$
>
> Hence, $f'(0) = 0$. Again, for $x \neq 0$
>
> $$\frac{g(x) - g(f(0))}{x - 0} = \begin{cases} \frac{\sin x}{x}, & \text{if } x \in \mathbb{Q}, \\ 1, & \text{otherwise,} \end{cases} \quad \to 1 \;\text{ as } x \to 0.$$
>
> Thus, $g'(f(0)) = 1$. Finally,
>
> $$\frac{(g \circ f)(x) - (g \circ f)(0))}{x - 0} = \begin{cases} \frac{\sin^2 x}{x}, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{otherwise,} \end{cases} \quad \to 0 \;\text{ as } x \to 0.$$
>
> Thus, $(g \circ f)'(0)) = 0 = 1 \cdot 0 = g'(f(0))f'(0)$.

8. Find the number of real roots of the equation $x^4 + 2x^2 - 6x + 2 = 0$.

> **Solution:** (Hint.) Define $p(x) = x^4 + 2x^2 - 6x + 2$ on $\mathbb{R}$. Show that $p''(x) > 0$ for all $x \in \mathbb{R}$. So, p' cannot vanish at more than one distinct points, and thereofre p cannot vanish at more than two distinct points. Use IVT to show that p vanishes at least at two distinct points.

9. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $|f(x) - f(y)| \leq (x-y)^2$ for all $x, y \in \mathbb{R}$. Show that f is a constant function.

> **Solution:** Sorry, question 4 repeated.

10. Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable at 0. If $f(\frac{1}{n}) = 0$ for all $n \in \mathbb{N}$, then find $f'(0)$ and $f''(0)$.

Solution: First, since f is twice differentiable at 0, f must be differentiable in an interval $[-r, r]$, $r > 0$. In particular, it is differentiable at 0, and so continuous at 0. Since $\frac{1}{n} \to 0$, have $f(\frac{1}{n}) \to f(0)$ yielding $f(0) = 0$.

Next, $f'(0) = \lim_{x \to 0} \dfrac{f(x) - f(0)}{x - 0}$, and the sequence $(\frac{1}{n})$ converges to 0, we have

$$f'(0) = \lim_{n\to\infty} \frac{f(1/n) - f(0)}{1/n - 0} = 0.$$

Finally, choose $m \in \mathbb{N}$ such that $\frac{1}{m} \le r$. For $n \ge m$, f is differentiable on $[0, 1/n]$ with $f(0) = f(1/n) = 0$. By MVT, there is $x_n \in [0, 1/n]$ such that $f'(x_n) = 0$. Then $x_n \to 0$ and therefore

$$f''(0) = \lim_{n\to\infty} \frac{f'(x_n) - f'(0)}{x_n - 0} = 0.$$

11. Let f be differentiable on $(0, \infty)$ and $\lim_{x\to\infty} f'(x) = 0$. Put $g(x) = f(x+1) - f(x)$. Show that $\lim_{x\to\infty} g(x) = 0$.

Solution: Let $\epsilon > 0$. Since $\lim_{x\to\infty} f'(x) = 0$, there is $M > 0$ such that $|f'(x)| < \epsilon$ for all $x \ge M$. Let $x \ge M$. Since f is differentiable on $[x, x+1]$, by MVT, there is $y \in (x, x+1)$ such that

$$\frac{f(x+1) - f(x)}{(x+1) - x} = f'(y),$$

that is, $g(x) = f'(y)$. Then $y > M$ and therefore, $|g(x)| = |f'(y)| < \epsilon$. Hence, $\lim_{x\to\infty} g(x) = 0$.

12. If $f(x) = x^3 + x^2 - 5x + 3$ for $x \in \mathbb{R}$, then show that f is one-one on $[1, 5]$ but not one-one on $\mathbb{R}$.

Solution: We have $f'(x) = 3x^2 + 2x - 5 = (3x+5)(x-1)$. Since $f'(x) > 0$ for $x > 1$, f is one-one on $[1, 5]$ (in fact on any subset of $[1, \infty)$). However, f is not one-one on $\mathbb{R}$: $f(1) = 0, f(0) = 3, f(-5) = -72$. IVT, there is $t \in (-5, 0)$ such that $f(t) = f(1) = 0$.

13. Prove that for $x \ge -1$ and $\alpha > 1$, $(1+x)^\alpha \ge 1 + \alpha x$.

Solution: Let $f : [-1, \infty) \to \mathbb{R}$ be defined by $f(x) = (1+x)^\alpha - (1+\alpha x)$, $x \ge -1$. Then f is differentiable and $f'(x) = \alpha[(1+x)^{\alpha-1} - 1]$. Now, $f'(x) \le 0$ for all $x \in [-1, 0]$ and $f'(x) \ge 0$ for all $x \in [0, \infty)$. Hence f is decreasing on $[-1, 0]$ and increasing on $[0, \infty)$. So $f(x) \ge f(0) = 0$ for all $x \in \mathbb{R}$.

14. (1) For $0 < x < y$, show that $\dfrac{y-x}{y} < \ln\dfrac{y}{x} < \dfrac{y-x}{x}$.
(2) Deduce that if $e \le a < b$, then $a^b > b^a$. (In particular $e^\pi > \pi^e$.)

Solution: (1) Let $f(t) = \ln t$ on $[x, y]$. Then f is differentiable on $[x, y]$ and $f'(t) = 1/t$. By MVT, there is $c \in (x, y)$ such that

$$\ln y - \ln x = \frac{1}{c}(y - x), \text{ i.e, } \ln\frac{y}{x} = \frac{1}{c}(y-x).$$

Since $\dfrac{1}{y} < \dfrac{1}{c} < \dfrac{1}{x}$, we have

$$\frac{y-x}{y} < \ln\frac{y}{x} < \frac{y-x}{x}.$$

(2) From the above let us deduce that if $e \le x < y$, then $x^y > y^x$. Since $x \ln(y/x) < y - x$, we have $\ln \frac{y^x}{x^x} = x \ln(y/x) < y - x$, i.e., $\dfrac{y^x}{x^x} < e^{y-x} \le x^{y-x} = \dfrac{x^y}{x^x}$ (since $e \le x$ implies $e^t \le x^t$ for any t). Thus, $y^x < x^y$.
In particular, we have $e^\pi > \pi^e$, since $e < \pi$.

15. Show that $0 < \frac{1}{x} \ln\left(\frac{e^x - 1}{x}\right) < 1$ for $x > 0$.

Solution: (Hint.) Show that $0 < \ln\left(\frac{e^x-1}{x}\right) < x$ for $x > 0$. $e^x > 1 + x$. So take $a = x, b = e^x - 1$ and apply MVT on $f(t) = \ln t$ on $[a, b]$.

16. Find the points of local maximum and local minimum for $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = 1 - x^{2/3}$.

Solution: The function is differentiable everywhere, except at 0. For $x \neq 0$, $f'(x) = -2/(3x^{1/3})$. Now, $f'(x) > 0$ for $x < 0$, and $f'(x) < 0$ for $x > 0$. Hence, f is increasing on $(\infty, 0)$ and decreasing on $(0, \infty)$. Since f is continuous at 0, f has a local maximum at $x = 0$.