MA 101 (Mathematics I)

Multivariable Calculus : Practice Problem Set - 2

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1. Examine whether the following sets are (a) open (b) closed in $\mathbb{R}^2$.
   (a) $\{(x,y) \in \mathbb{R}^2 : 0 < x < y\}$ (b) $\{(x,x) : x \in \mathbb{R}\}$ (c) $\{(x,y) \in \mathbb{R}^2 : y \in \mathbb{Z}\}$ (d) $(0,1) \times \{0\}$

2. If $f : \mathbb{R}^m \to \mathbb{R}$ is continuous, then show that
   (a) $\{\mathbf{x} \in \mathbb{R}^m : f(\mathbf{x}) > 0\}$ is an open set in $\mathbb{R}^m$.
   (b) $\{\mathbf{x} \in \mathbb{R}^m : f(\mathbf{x}) \geq 0\}$ and $\{\mathbf{x} \in \mathbb{R}^m : f(\mathbf{x}) = 0\}$ are closed sets in $\mathbb{R}^m$.

3. Using Ex.2 above, show that $\{(x,y,z) \in \mathbb{R}^3 : x^2 + 2z < 3|y|\,\}$ is an open set in $\mathbb{R}^3$ and $\{(x,y,z) \in \mathbb{R}^3 : \sin(xyz) = |xy|\,\}$ is a closed set in $\mathbb{R}^3$.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} \frac{\sin(xy)}{xy} & \text{if } xy \neq 0, \\ 1 & \text{if } xy = 0. \end{cases}$
   Show that f is continuous.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that $f(x,y) = e^{-\frac{x^2-2xy+y^2}{|x-y|}}$ for all $(x,y) \in \mathbb{R}^2$ with $x \neq y$. If $x \in \mathbb{R}$, then find $f(x,x)$ such that f is continuous on $\mathbb{R}^2$.

6. Let $f : S \subseteq \mathbb{R}^m \to \mathbb{R}^k$ be continuous and let $g : \mathbb{R}^m \to \mathbb{R}^k$ be such that $g(\mathbf{x}) = f(\mathbf{x})$ for all $\mathbf{x} \in S$.
   (a) Show that g need not be continuous on S.
   (b) If S is an open set in $\mathbb{R}^m$, then show that g is continuous on S.

7. Let $S_1 = \{(x,y) \in \mathbb{R}^2 : (x-1)^2 + y^2 < 4\}$ and $S_2 = \{(x,y) \in \mathbb{R}^2 : x^2 + (y-1)^2 < 9\}$. Does there exist a continuous function from S_1 onto S_2? Justify.

8. If $S = \{\mathbf{x} \in \mathbb{R}^m : \|\mathbf{x}\| < 1\}$, then does there exist a non-constant continuous function $f : \mathbb{R}^m \to \mathbb{R}$ such that $f(\mathbf{x}) = 5$ for all $\mathbf{x} \in S$? Justify.

9. Let $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{x} \neq \mathbf{y}$. Find a continuous function $f : \mathbb{R}^m \to \mathbb{R}$ such that $f(\mathbf{x}) = 1$, $f(\mathbf{y}) = 0$ and $0 \leq f(\mathbf{z}) \leq 1$ for all $\mathbf{z} \in \mathbb{R}^m$.

10. Let $f : \mathbb{R}^m \to \mathbb{R}$ be continuous such that $\lim\limits_{\|\mathbf{x}\| \to \infty} f(\mathbf{x}) = 1$. Show that f is bounded on $\mathbb{R}^m$.

11. State TRUE or FALSE with justification for each of the following statements.
    (a) There exists $r > 0$ such that $\sin(xy) < \cos(xy)$ for all $x,\, y \in [-r,\, r]$.
    (b) There exists a continuous function $f : \mathbb{R} \to \mathbb{R}^2$ such that $f(\cos n) = (n, \frac{1}{n})$ for all $n \in \mathbb{N}$.
    (c) There exists a continuous function from $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ onto $\mathbb{R}^2$.
    (d) There exists a one-one continuous function from $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ onto $\mathbb{R}^2$.

12. If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is continuous, then does there exist a sequence $\big((x_n, y_n)\big)$ in $\mathbb{R}^2$ such that $x_n^2 + y_n^2 = \frac{1}{2}$ and $f(x_n, y_n) = (n, \frac{1}{n})$ for all $n \in \mathbb{N}$? Justify.

13. Examine whether the following limits exist (in $\mathbb{R}$) and find their values if they exist (in $\mathbb{R}$).
   (a) $\lim\limits_{(x,y)\to(0,0)} \frac{x^3 y}{x^4+y^2}$ (b) $\lim\limits_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$ (c) $\lim\limits_{(x,y)\to(0,0)} \frac{|x|}{y^2} e^{-|x|/y^2}$
   (d) $\lim\limits_{(x,y)\to(0,0)} \frac{x^3+y^2}{x^2+y}$ (e) $\lim\limits_{(x,y)\to(0,0)} \frac{\sqrt{x^2y^2+1}-1}{x^2+y^2}$ (f) $\lim\limits_{(x,y)\to(0,0)} \frac{x^3y^2+y^6}{x^6+y^4}$
   (g) $\lim\limits_{(x,y,z)\to(0,0,0)} \frac{(x+y+z)^2}{x^2+y^2+z^2}$

14. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} x+y & \text{if } x \neq y, \\ 1 & \text{if } x = y. \end{cases}$
   Examine whether $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists (in $\mathbb{R}$).

15. Let $S \subseteq \mathbb{R}^2$, $(x_0, y_0) \in \mathbb{R}^2$ and $r > 0$ be such that $\big(B_r(x_0) \times B_r(y_0)\big) \setminus \{(x_0, y_0)\} \subseteq S$. Let $\lim\limits_{x\to x_0} f(x,y)$ exist (in $\mathbb{R}$) for each $y \in B_r(y_0) \setminus \{y_0\}$, $\lim\limits_{y\to y_0} f(x,y)$ exist (in $\mathbb{R}$) for each $x \in B_r(x_0) \setminus \{x_0\}$ and $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = \ell \in \mathbb{R}$.
   Show that $\lim\limits_{x\to x_0} \Big(\lim\limits_{y\to y_0} f(x,y) \Big) = \lim\limits_{y\to y_0} \Big(\lim\limits_{x\to x_0} f(x,y) \Big) = \ell$.
   $\Big[\lim\limits_{x\to x_0} \Big(\lim\limits_{y\to 0} f(x,y) \Big)$ and $\lim\limits_{x\to x_0} \Big(\lim\limits_{y\to 0} f(x,y) \Big)$ are called the iterated limits of f at (x_0, y_0).$\Big]$

16. Show that $\lim\limits_{x\to 0} \Big(\lim\limits_{y\to 0} \frac{x^2}{x^2+y^2} \Big) \neq \lim\limits_{y\to 0} \Big(\lim\limits_{x\to 0} \frac{x^2}{x^2+y^2} \Big)$ and hence conclude that $\lim\limits_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ does not exist (in $\mathbb{R}$).

17. Show that $\lim\limits_{x\to 0} \Big(\lim\limits_{y\to 0} \frac{x^2y^2}{x^2y^2+(x-y)^2} \Big) = 0 = \lim\limits_{y\to 0} \Big(\lim\limits_{x\to 0} \frac{x^2y^2}{x^2y^2+(x-y)^2} \Big)$ but that $\lim\limits_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2y^2+(x-y)^2}$ does not exist (in $\mathbb{R}$).

18. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} x \sin\frac{1}{y} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0. \end{cases}$
   Show that $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$ and $\lim\limits_{y\to 0} \Big(\lim\limits_{x\to 0} f(x,y) \Big) = 0$ but that $\lim\limits_{y\to 0} f(x,y)$ does not exist (in $\mathbb{R}$) if $x \in \mathbb{R} \setminus \{0\}$ and so $\lim\limits_{x\to 0} \Big(\lim\limits_{y\to 0} f(x,y) \Big) = 0$ is not defined.

19. Show that $\lim\limits_{(x,y)\to(0,0)} \frac{1}{3x^2+y^4} = \infty$.

20. Let I be an open interval in $\mathbb{R}$ and let $F : I \to \mathbb{R}^m$ be a differentiable function such that $F(t) \cdot F'(t) = 0$ for all $t \in I$. Show that $\| F(t) \|$ is constant for all $t \in I$.