[D] A series in $\mathbb{R}$ is an expression $a_1 + a_2 + a_3 + \cdots$, where (a_n) is a sequence. Notations: $\sum_{n=1}^{\infty} a_n$, $\sum_{n\geq 1} a_n$ and $\sum a_n$.

- a_n is called the nth term of the series $\sum a_n$.
- $A_n := a_1 + \cdots + a_n$ is called the nth partial sum of the series $\sum a_n$.
- If terms are not labeled, we use S_n for the partial sum.

[D] We say $\sum a_n$ converges to l, if $A_n \to l$. Notation: $\sum a_n = l$.

- $\sum a_n$ is called divergent if it is not convergent.
- The number l is called the limit/value/sum of $\sum a_n$.

[Eg] $\sum (-1)^n$ is divergent, as $(S_n) = (-1, 0, -1, 0, \cdots)$ is divergent.

- $\sum_{n\geq 1} 1$ is divergent, as $(S_n) = (1, 2, 3, \cdots)$ diverges.

- The harmonic series $\sum \frac{1}{n}$. It diverges, as $(1 + \cdots + \frac{1}{n})$ is divergent.
- $\sum \frac{1}{n(n+1)} = 1$, as $S_n = 1 - \frac{1}{n+1} \to 1$.
- The series $\sum\limits_{n \geq k} (b_n - b_{n+1})$ is called a telescoping series of the sequence (b_n). It converges to $b_k - l$ iff $b_n \to l$. !!
- The series $\sum_{n\geq 1} a_n$ is convergent iff $\sum_{n \geq k} a_n$ is convergent.
- Consider $\sum\limits_{n \geq 0} ar^n$. If $|r| < 1$, then $S_n = a\frac{1-r^n}{1-r} \to \frac{a}{1-r}$. So the series converges. It is called the geometric series

[R: nth term test] Let $\sum a_n$ converge to l. Then $a_n \to 0$.

Po We have $S_{n+1}, S_n \to l$. So $a_{n+1} = S_{n+1} - S_n \to 0$. □

[Eg] Is $\sum (-1)^n$ convergent? No, as $a_n \not\to 0$.

[Eg] Fix $p > 1$. Then the series $\sum_{n\geq 3} \frac{1}{n^p}$ converges. In fact

$$\begin{aligned} S_{2^n} &= \left(\tfrac{1}{3^p} + \tfrac{1}{4^p}\right) + \left(\tfrac{1}{5^p} + \cdots + \tfrac{1}{8^p}\right) + \cdots + \left(\tfrac{1}{(2^{n-1}+1)^p} + \cdots + \tfrac{1}{(2^n)^p}\right) \\ &\leq \frac{2}{2^p} + \frac{4}{4^p} + \cdots + \frac{2^{n-1}}{(2^{n-1})^p} \\ &= \frac{1}{2^{(p-1)}} + \left(\frac{1}{2^{(p-1)}}\right)^2 + \cdots + \left(\frac{1}{2^{(p-1)}}\right)^{n-1} \\ &= r + r^2 + \cdots + r^{n-1} \leq \frac{1}{1-r}. \end{aligned}$$

As $S_n \uparrow$ and bounded above, by MCT, (S_n) converges.

[R: algebra] Let $\sum a_n = a$, $\sum b_n = b$, and $\alpha \in \mathbb{R}$. Then $\sum(a_n + \alpha b_n)$ converges to $a+\alpha b$. !! So, $\sum(a_n + \alpha b_n) = a + \alpha b = \sum a_n + \alpha \sum b_n$.

[Eg] $\left(\frac{1}{2} + \frac{1}{3^2}\right) + \left(\frac{1}{2^3} + \frac{1}{3^4}\right) + \cdots$

$$= \left(\frac{1}{2} + \frac{1}{2^3} + \cdots\right) + \left(\frac{1}{3^2} + \frac{1}{3^4} + \cdots\right) = \frac{2}{3} + \frac{1}{8}.$$

[R: comparison test] Let $0 \leq a_n \leq b_n$. Then

a) $\sum b_n$ converges $\Rightarrow$ $\sum a_n$ converges.

b) $\sum a_n$ diverges $\Rightarrow$ $\sum b_n$ diverges. !!

Po a) Suppose that $\sum b_n = b$. As $0 \leq A_n \leq B_n \leq b$, and $A_n \uparrow$, by MCT, (A_n) must converge. $\square$

[Eg] a) $\sum \frac{1}{\sqrt{n}2^n}$ is convergent as $\frac{1}{\sqrt{n}2^n} \leq \frac{1}{2^n}$ and $\sum \frac{1}{2^n}$ is convergent.

b) $\sum \frac{1}{n^p}$ for $0 < p < 1$ is divergent as $\frac{1}{n^p} \geq \frac{1}{n}$ and $\sum \frac{1}{n}$ is divergent.

[R: sandwich] Let $a_n \leq b_n \leq c_n$. Then $\sum a_n$, $\sum c_n$ conv $\Rightarrow \sum b_n$ conv.

Po $\sum a_n$, $\sum c_n$ conv $\Rightarrow \sum (c_n - a_n)$ conv. As $0 \leq b_n - a_n \leq c_n - a_n$, $\sum (b_n - a_n)$ conv (comp test). So $\sum ((b_n - a_n) + a_n)$ conv (algebra).

[Eg] Is $\frac{1}{2}-\frac{1}{3^2}-\frac{1}{2^3}+\frac{1}{3^4}+\frac{1}{2^5}+\frac{1}{3^6}-\frac{1}{2^7}-\frac{1}{3^8}-\frac{1}{2^9}-\frac{1}{3^{10}}+\cdots$ convergent?

Yes, as $-\frac{1}{2^n} \le a_n \le \frac{1}{2^n}$ (sandwich).

- Is $\frac{1}{2}-\frac{1}{2^2}+\frac{1}{2^3}-\frac{1}{2^4}+\cdots$ convergent? Yes, as $-\frac{1}{2^n} \le a_n \le \frac{1}{2^n}$.

- Arbitrarily give $+$ or $-$ signs to the terms in the above series. Then? Yes. The same argument.

- Let $\sum |a_n|$ be conv. Must $\sum a_n$ conv? Yes. As $-|a_n| \le a_n \le |a_n|$.

[Eg] A series $\sum a_n$ is called absolutely convergent, if $\sum |a_n|$ is convergent. If $\sum a_n$ is convergent, but $\sum |a_n|$ is not, then $\sum a_n$ is called conditionally convergent.

[R: abs-conv-test] If $\sum a_n$ converges absolutely, then $\sum a_n$ is convergent. !!

[Eg] Consider $1 - \frac{1}{2} + \frac{1}{3} - \cdots$. Then $S_{2n} = 1 - \frac{1}{2} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}$

$$= \frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(2n-1)(2n)} \quad \le \quad \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)}.$$

Hence(?) (S_{2n}) converges. Let $S_{2n} \to l$. Then $S_{2n+1} = S_{2n} + \frac{1}{2n+1} \to l$. So $S_n \to l$. So the series is convergent. Converges absolutely?

[R: Limit comparison test] Let $a_n, b_n > 0$, and $\lim \frac{a_n}{b_n} = c$.

a) If $c > 0$, then $\sum a_n$ converges $\Leftrightarrow \sum b_n$ converges.

b) If $c = 0$, then $\sum b_n$ converges $\Rightarrow \sum a_n$ converges.

Po a) Let $c > 0$. Then $\exists k \in \mathbb{N}$ s.t. $\forall n \ge k$, we have $\frac{c}{2} \le \frac{a_n}{b_n} \le \frac{3c}{2}$.

That is, $0 \le \frac{c}{2} b_n \le a_n \le \frac{3c}{2} b_n$ holds $\forall n \ge k$. Apply sandwich twice.

b) $c = 0$. So $\exists m \in \mathbb{N}$ s.t. $\forall n \ge m$, we have $\frac{a_n}{b_n} \le 1$. Apply sandwich.

[Eg] $\sum_{n\geq 25} \frac{1}{2n-10\sqrt{n}+5}$ is divergent (lim comp test with $\sum \frac{1}{n}$).

[R: condensation test] Let $a_n \geq 0$ be decreasing. Then

$$\sum a_n \text{ is convergent} \qquad \text{iff} \qquad \sum 2^n a_{2^n} \text{ is convergent.}$$

Po Let $\sum a_n$ be convergent. Notice that

$$\begin{aligned} & a_2 + (a_3 + a_4) + (a_5 + \cdots + a_8) + \cdots + (a_{2^{n-1}+1} + \cdots + a_{2^n}) \\ \geq\ & a_2 + \quad 2a_4 \quad + \quad 4a_8 \quad + \cdots + \quad 2^{n-1} a_{2^n}. \end{aligned}$$

Multiply by 2. So $\sum 2^n a_{2^n}$ is conv. Converse: if $\sum 2^n a_{2^n}$ is conv, then

$$2a_2 + 4a_4 + \cdots + 2^n a_{2^n} \geq (a_2 + a_3) + (a_4 + \cdots + a_7) + \cdots + a_{2^{n+1}-1}.$$

Hence $\sum a_n$ converges. $\square$

[Eg] Fix $p>1$. Then $\sum_{n\geq 2} \frac{1}{n(\ln n)^p}$ conv as $2^n a_{2^n} = \frac{1}{(\ln 2)^p} \frac{1}{n^p}$ and $\sum \frac{1}{n^p}$ conv.

[Eg] Let $\sum(a_n+b_n)$ be convergent. Must $a_1+b_1+a_2+b_2+\cdots$ converge? No. $\sum(1-1)=0$ but $1-1+1-1+\cdots$ is <u>divergent</u>.

- So, removal of brackets may change convergence. However,

[R] If $\sum a_n = a$ and $\sum b_n = b$, then

$$\sum(a_n+b_n) = a+b = a_1+b_1+a_2+b_2+\cdots .$$

- In the above example, both $\sum a_n$ and $\sum b_n$ were divergent.

[Ex] Show that insertion of brackets (grouping consecutive terms) into a convergent series keeps it convergent. However, insertion of brackets into a divergent series can make it convergent.

[Root test-I]

a) If $|a_n|^{\frac{1}{n}} \leq r < 1$, for all $n \geq$ some n_0, then $\sum |a_n|$ converges.

b) If $|a_n|^{\frac{1}{n}} \geq 1$, for all $n \geq$ some n_0, then $\sum a_n$ diverges.

[Root test-II] Suppose that $|a_n|^{\frac{1}{n}} \to r$.

a) If $r < 1$, then $\sum |a_n|$ converges.

b) If $r > 1$, then $\sum a_n$ diverges.

c) If $r = 1$, then $\sum a_n$ may or may not converge.

[Eg] $\sum \frac{n!^n}{n^{n^2}}$ converges as $\quad a_n^{1/n} = \frac{n!}{n^n} \leq \frac{1}{n} \to 0.$

[Ratio test-I] Let $a_n \neq 0$.

a) If $\frac{|a_{n+1}|}{|a_n|} \leq r < 1$, for $n \geq$ some n_0, then $\sum |a_n|$ converges.

b) If $\frac{|a_{n+1}|}{|a_n|} \geq 1$, for all $n \geq$ some n_0, then $\sum a_n$ diverges.

[Ratio test-II] Suppose that $\frac{|a_{n+1}|}{|a_n|} \to r$. (Assumed $a_n \neq 0$).

a) If $r < 1$, then $\sum |a_n|$ is convergent.

b) If $r > 1$, then $\sum a_n$ diverges.

c) If $r = 1$, then $\sum a_n$ may converge or diverge.

[Eg] a) $\sum \frac{2^n+5}{3^n}$ converges as $\frac{a_{n+1}}{a_n} \to \frac{2}{3} < 1$.

b) $\sum \frac{x^n}{n!}$ converges as $\frac{|a_{n+1}|}{|a_n|} = \frac{|x|}{n+1} \to 0 < 1$.

[Raabe's test-I] Let $a_n \neq 0$ and $a > 1$.

a) If $\frac{|a_{n+1}|}{|a_n|} \leq 1 - \frac{a}{n}$, for all $n \geq$ some k, then $\sum |a_n|$ converges.

b) If $\frac{|a_{n+1}|}{|a_n|} \geq 1 - \frac{1}{n}$, for all $n \geq$ some n_0, then $\sum |a_n|$ diverges.

Po a) So $n|a_{n+1}| \leq (n-a)|a_n|$ or $(a-1)|a_n| \leq (n-1)|a_n| - n|a_{n+1}|$.
So $(a-1)(|a_k| + \cdots + |a_n|) \leq (k-1)|a_k| - n|a_{n+1}| < (k-1)|a_k|$.

[Raabe's test-II] Let $n\left(1 - \frac{|a_{n+1}|}{|a_n|}\right) \to a$.

a) If $a > 1$, then $\sum |a_n|$ is convergent.

b) If $a < 1$ then $\sum |a_n|$ is divergent.

c) If $a = 1$ then no conclusion. Think of $\sum \frac{1}{n}$ and $\sum \frac{1}{n(\ln n)^2}$.

Po a) $\exists k$ s.t. $n(1 - \frac{a_{n+1}}{a_n}) > b > 1$, for all $n \geq k$.

[Eg] $a_n = \frac{3\cdot 6\cdot 9\cdots(3n)}{7\cdot 10\cdot 13\cdots(3n+4)} r^n$, $r > 0$. Then $\frac{a_{n+1}}{a_n} = \frac{3n+3}{3n+7} \to r$.

If $r < 1$, convergent by ratio test. If $r > 1$, divergent by ratio test.

If $r = 1$, ratio test is inconclusive. Apply Raabe's test.

$n(1 - \frac{a_{n+1}}{a_n}) = n(1 - \frac{3n+3}{3n+7}) = \frac{4n}{3n+7} \to \frac{4}{3}$. Convergent.

- Avoiding Raabe's test, we could use the ideas of geometric and harmonic series (including the p-series), to work it out. See the notes.

[R: Leibniz test] Let $a_n \downarrow 0$. Then $a_1 - a_2 + a_3 - a_4 + \cdots$ conv. alternating series

Po Notice: $a_1 \geq a_1 - a_2 + a_3 \geq a_1 - a_2 + a_3 - a_4 + a_5 \geq \cdots \geq 0$.

That is, $A_{2n+1} \downarrow$ and bounded below. So $A_{2n+1} \to$ some l (MCT).

As $a_n \to 0$, $A_{2n+2} \to l$. So $A_n \to l$. □

[Eg.] Fix $p > 0$. Then $\sum (-1)^n \frac{1}{n^p}$ is convergent, as $a_n \downarrow 0$.

- I have written 9 positive numbers: $a_1, \ldots, a_9$. Without my knowledge my friend rearranges them: $b_1, \ldots, b_9$. Do you think both will have the same sum? What if, I had $a_1, a_2, \ldots$ where $\sum a_n = a$?

[R] Let $a_n \geq 0$ with $\sum a_n = a$. Let $b_1, b_2, \ldots$ be a rearrangement (bijective image) of $a_1, a_2, \ldots$. Then $\sum b_n$ converges to a.

Po Given n, there exists n' s.t. the terms $b_1, \ldots, b_n$ are in $a_1, a_2, \ldots, a_{n'}$. Hence, $B_n \leq A_{n'} \leq a$. But as $B_n \uparrow$, by MCT $B_n \to b$ (say). So $\sum b_n = b$ and $b \leq a$.

Now, $\sum b_n = b$ and (a_n) is a rearrangement of (b_n). So $a \leq b$. $\square$

[R] Let $\sum a_n$ be abs conv, (b_n) a rearrangement of (a_n). Then $\sum b_n = \sum a_n$.

- Under absolute convergence, a series behaves like the sum of finitely many numbers. You can rearrange terms and still have the same value.

- What happens if you rearrange terms in a conditionally convergent series?

- Let $$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\cdots=S.$$

$\frac{1}{2}-\frac{1}{4}+\frac{1}{6}+\cdots=\frac{S}{2}.$

Insert zeros: $$0+\frac{1}{2}+0-\frac{1}{4}+0+\frac{1}{6}+0-\frac{1}{8}+0+\cdots=\frac{S}{2}.$$

Add them: $$1+0+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+0+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}\cdots=\frac{3S}{2}.$$

- Drop the zeros. It is a rearrangement of the series (top) of value $\frac{3S}{2}$.

[Riemann rearrangement theorem.] Let $\sum a_n$ be conditionally convergent. Then it can be rearranged to converge to any fixed real number. It can also be rearranged to be divergent.

[R: Abel's partial sum formula] Put $A_n = a_1 + \cdots + a_n$. Put $A_0 = 0$. Then

$$\sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n} (A_i - A_{i-1}) b_i = \sum_{i=1}^{n} A_i b_i - \sum_{i=1}^{n-1} A_i b_{i+1}$$

$$= \sum_{i=1}^{n-1} A_i (b_i - b_{i+1}) + A_n b_n.$$

[R: Dirichlet's test] If $|A_n| < M$ and $b_i \downarrow 0$, then $\sum (a_i b_i)$ converges.

[Eg.] Is $|\sum_{i=1}^{n} \sin k|$ bounded? Yes. By $\frac{2}{\sin(1)}$, as

$2 \sin 1 \sin k = \cos(k-1) - \cos(k+1)$. So $\sum \frac{\sin(n)}{n}$ is convergent.

[R: Abel's test] Take $\sum a_n$ convergent and (b_n) monotone convergent. Then

$\sum (a_n b_n)$ is convergent.

Po Let $b_n \to b$. Consider $c_n = b_n - b$ or $c_n = b - b_n$.