MA 101 (Mathematics - I)
Differentiation : Lecture Notes

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1 Differentiability and Derivative

Class 1

[1.1] DEFINITION Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$, and $c \in I$. We say that f is **differentiable** at c, if the limit

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists. In that case the limit is called the **derivative** of f at c, and is denoted by $f'(c)$. Further, f is said to be differentiable on I, if f is differentiable at each point in I.

[1.2] REMARK

1. The limits $\lim\limits_{x \to c^-} \frac{f(x)-f(c)}{x-c}$ and $\lim\limits_{x \to c^+} \frac{f(x)-f(c)}{x-c}$, if they exist, are called the **left hand derivative** $f'_-(c)$ and the **right hand derivative** $f'_+(c)$ of f at c, respectively. If $I = [a, b]$, then it follows that f is differentiable at a (resp. at b) means $f'(a) = f'_+(a)$ (resp. $f'(b) = f'_-(b)$) exists.
2. If $J \subseteq \mathbb{R}$ is a union of intervals, then we would say $f : J \to \mathbb{R}$ is **differentiable** if f is differentiable in every interval contained in J.
3. If $f : I \to \mathbb{R}$ is differentiable, the $x \mapsto f'(x)$ is a function $f' : I \to \mathbb{R}$, called the **derivative (function)** of f. If f' is differentiable on I, then we have the **second derivative** of f which is denoted by f'' or $f^{(2)}$. Similarly, for $n \in \mathbb{N}$, $f^{(n)}$, the n-th **derivative** of f is defined. It is also denoted by $\frac{d^n f}{dx^n}$ or $D^n f$, where D stands for $\frac{d}{dx}$.

[1.3] EXAMPLE $\frac{d}{dx} \sin x = \cos x$, $\frac{d}{dx} e^x = e^x$, $\frac{d}{dx} x^k = k x^{k-1}$ (for $x > 0, k \in \mathbb{Q}$).

[1.4] THEOREM *If $f : I \to \mathbb{R}$ is differentiable at $c \in I$, then f is continuous at c.*

Proof. By definition.

[1.5] EXAMPLE Discuss differentiability of $f : \mathbb{R} \to \mathbb{R}$, where

1. On $\mathbb{R}$ $f(x) = |x|$ is differentiable at every point other than 0.
2. On $\mathbb{R}$ $f(x) = |\sin x|$ is differentiable at every point other than $x = n\pi$, draw the graph. (Exercise)
3. The function $f(x) = \begin{cases} x^2, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$
   is continuous only at $x = 0$. Also differentiable at 0; use definition.

4. The function $n \in \mathbb{N}$ and $f(x) = \begin{cases} x^n \sin \frac{1}{x}, & \text{if } x \neq 0. \\ 0, & \text{if } x = 0. \end{cases}$
   is differentiable at 0, if and only if $n > 1$. (Exercise)

[**1.6**] REMARK Meaning of the derivative $f'(c)$:

1. Instantaneous rate of change at $x = c$
2. Slope of the tangent to the curve $y = f(x)$ at $(c, f(c))$
3. **Linear approximation** of f around c: Define

$$g(x) = \begin{cases} \frac{f(x)-f(c)}{x-c} - f'(c), & \text{if } x \neq c, \\ 0, & \text{if } x = c. \end{cases}$$

Then, g is continuous at c. Thus,

$$f(x) - f(c) - (x - c)f'(c) = (x - c)g(x), \text{where } \lim_{x\to c} g(x) = 0.$$

If you put $h = x - c$, you get

$$f(c + h) = f(c) + hf'(c) + hg(c + h), \text{where } \lim_{h\to 0} g(c + h) = 0.$$

If f is continuous around c, this gives an approximation, called **linear approximation** $f(c + h) \approx f(c) + hf'(c)$ of f on increment h at c.

[**1.7**] EXAMPLE We find an approximate value of $(8.3)^{1/3}$ using linear approximation. For $f(x) = x^{1/3}$. We have $f'(x) = \frac{1}{3}x^{-2/3}$. Therefore,

$$(8.3)^{1/3} = f(8 + 0.3) \approx f(8) + 0.3 \cdot f'(8) = 2 + 0.3 \cdot \frac{1}{3} \cdot \frac{1}{4} = 2 + 0.025 = 2.025.$$

[**1.8**] THEOREM **(Carathéodary's Theorem)** *Let f be defined on an interval I containing the point c. Then f is differentiable at c if and only if there is a function ϕ on I that is continuous at c and*

$$f(x) = f(c) + \phi(x)(x - c).$$

In that case, $\phi(c) = f'(c)$.

Proof. First, suppose f is differentiable at c. Define the function $\phi : I \to \mathbb{R}$ by

$$\phi(x) = \begin{cases} \frac{f(x)-f(c)}{x-c}, & x \neq c. \\ f'(c), & x = c, \end{cases}$$

Then, ϕ satisfies the conditions.
Conversely, suppose that such a function exists. Then for $x \neq c$, $\dfrac{f(x) - f(c)}{x - c} = \phi(x)$. Since ϕ is continuous at c, we have $\lim_{x\to c} \phi(x) = \phi(c)$. Thus, $\lim_{x\to c} \dfrac{f(x) - f(c)}{x - c}$ exists and equals $\phi(c)$. In other words, f is differentiable at c and $f'(c) = \phi(c)$.

[**1.9**] THEOREM **(Rules for derivatives)** *Let f, g be functions from I to $\mathbb{R}$, differentiable at $c \in I$, and $\alpha \in \mathbb{R}$. Then*

(1) αf is differentiable at c and $(\alpha f)'(c) = \alpha f'(c)$.

(2) (Sum Rule) $f + g$ is differentiable at c and $(f+g)'(c) = f'(c) + g'(c)$.

(3) (Product Rule) fg is differentiable at c and $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$.

(4) (Reciprocal Rule) If $g(c) \neq 0$, then $1/g$ is differentiable at c (in a suitable interval) and $(1/g)'(c) = -g'(c)/(g(c))^2$.

(5) (Quotient Rule) If $g(c) \neq 0$, then f/g is differentiable at c (in a suitable interval) and

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}.$$

Proof. Carathéodary's Theorem comes handy in proving these results. We provide proofs for (3) and (4). We leave the rest as exercises: (1) and (2) can be proved similarly, and (5) can be deduced from (3) and (4).

(3) By Carathéodary's Theorem, there are functions $\phi, \psi : I \to \mathbb{R}$ that are continuous at c and such that

$$f(x) = f(c) + \phi(x)(x-c), \; g(x) = g(c) + \psi(x)(x-c) \text{ and } \phi(c) = f'(c), \; \phi(c) = g'(c).$$

Thus,

$$\begin{aligned} (fg)(x) &= f(x)g(x) = \big(f(c) + \phi(x)(x-c)\big)\big(g(c) + \psi(x)(x-c)\big) \\ &= f(c)g(c) + (x-c)\big(f(c)\psi(x) + g(c)\phi(x) + (x-a)\phi(x)\psi(x)\big) \end{aligned}$$

Define $\eta : I \to \mathbb{R}$ by $\eta(x) = f(c)\psi(x) + g(c)\phi(x) + (x-a)\phi(x)\psi(x)$. Then η is continuous at c and $(fg)(x) = (fg)(c) + \eta(x)(x-c)$. Thus, By Carathéodary's Theorem fg is differentiable at c and

$$(fg)'(c) = \eta(c) = f(c)\psi(c) + g(c)\phi(c) = f(c)g'(c) + g(c)f'(c).$$

(4) Since g is continuous at c and $g(c) \neq 0$, there is $\delta > 0$ such that $g(x) \neq 0$ for all $x \in J = (c-\delta, c+\delta) \cap I$. Using the same ψ as in (3) above we have for $x \in J$

$$\frac{1}{g(x)} - \frac{1}{g(c)} = \frac{-1}{g(x)g(c)}(g(x) - g(c)) = \frac{-1}{g(x)g(c)}\psi(x)(x-c)$$

Define $\zeta : J \to \mathbb{R}$ by $\zeta(x) = \frac{-1}{g(x)g(c)}\psi(x)$. Then ζ is continuous at c. This gives $\frac{1}{g}$ is differentiable at c and $(\frac{1}{g})'(c) = \eta(c) = \dfrac{-\psi(c)}{(g(c))^2} = \dfrac{-g'(c)}{(g(c))^2}$. ■

[**1.10**] REMARK The sum rule and the product rule can be extended (by repeated application) to any finite number of functions $f_1, f_2, \ldots, f_n$ on I.

[**1.11**] THEOREM **(The Chain Rule)** *Let I, J be intervals in $\mathbb{R}$, $f : I \to \mathbb{R}$, $g : J \to \mathbb{R}$, $f(I) \subseteq J$. Let $c \in I$, f is differentiable at c and g is differentiable at $f(c)$. Then, $g \circ f : I \to \mathbb{R}$ is differentiable at c and $(g \circ f)'(c) = g'(f(c))f'(c)$.*

Proof. There is ϕ on I with $f(x) = f(c)+\phi(x)(x-c)$, where ϕ is continuous at c and $\phi(c) = f'(c)$. Again, there is ψ on J with $g(y) = g(d) + \psi(y)(y - d)$, where ψ is continuous at $d = f(c)$ and $\psi(d) = g'(d)$.
Let $h = g \circ f$. Then

$$\begin{aligned} h(x) &= g(f(x)) = g\big(f(c) + \phi(x)(x - c)\big) \\ &= g(d + \phi(x)(x - c)) = g(y), \quad (\text{where } y = d + \phi(x)(x - c) \in J) \\ &= g(d) + \psi(y)\underline{(y - d)} = g(d) + \psi(y)\underline{\phi(x)(x - c)} \end{aligned}$$

that is, $h(x) - h(c) = (x - c)\psi(y)\phi(x) = (x - c)\big[\psi(d + \phi(x)(x - c))\phi(x)\big]$. Take $\eta(x) = \psi(d + \phi(x)(x - c))\phi(x)$. Since ϕ and ψ are continuous at c, η is continuous at c and $\eta(c) = \psi(d)\phi(c) = g'(d)f'(c)$. Therefore, by Carathéodary's Theorem, $h = g \circ f$ is differentiable at c and $h'(c) = g'(f(c))f'(c)$. ∎

Class 2

[**1.12**] THEOREM **(The Inverse Function Theorem)** *Let I be an interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be strictly monotone and continuous on I. Let $J = f(I)$ and $g : J \to \mathbb{R}$ be the (strictly monotone and continuous) inverse of f. If f is differentiable at $c \in I$ and $f'(c) \neq 0$, then g is differentiable at $d := f(c) \in J$, and*

$$g'(d) = \frac{1}{f'(c)} = \frac{1}{f'(g(d))}.$$

Proof. By Carathéodary's theorem, there is $\phi : I \to \mathbb{R}$, continuous at c with $f(x) - f(c) = \phi(x)(x - c)$ and $\phi(c) = f'(c)$.

Since $\phi(c) \neq 0$, $\phi(x) \neq 0$ in some $V = (c - \delta, c + \delta) \cap I$. Note that $U = f(V)$ is an interval and $d \in U$. For $y \in U$ we have

$$y - d = f(g(y)) - f(c) = \phi(g(y))(g(y) - c) = \phi(g(y)(g(y) - g(d)$$

that is, $g(y) - g(d) = \dfrac{1}{\phi(g(y))}(y - d)$. Since $\phi(g(y)) \neq 0$, g is continuous at d and ϕ is continuous at $c = g(d)$ we get $\dfrac{1}{\phi \circ g}$ is continuous at d. Thus g is differentiable ate d and $g'(d) = \dfrac{1}{(\phi \circ g)(d)} = \dfrac{1}{f'(g(d))} = \dfrac{1}{f'(c)}$. ∎

[**1.13**] EXAMPLE For the differentiable function $f : \mathbb{R} \to (0, \infty)$, $f(x) = e^x$, the inverse function is $g : (0, \infty) \to \mathbb{R}$ given by $g(y) = \ln y$. Also, $f'(c) = e^c$. At $y = d$, $c := g(d) = \ln d$, and by IFT,

$$g'(d) = \frac{1}{f'(c)} = \frac{1}{e^c} = \frac{1}{e^{\ln d}} = \frac{1}{d}.$$

In other words, $\frac{d}{dx} \ln x = \frac{1}{x}$.

[**1.14**] EXAMPLE Let $r \in \mathbb{R}$, $f(x) = x^r := e^{r \ln x}, x > 0$. Use Chain Rule to deduce that $f'(x) = rx^{r-1}$.

2 (Lagrange's) Mean Value Theorem

[2.1] DEFINITION The function $f : I \to \mathbb{R}$ is said to have a **local (relative) maximum** at $c \in I$, if there exists $\delta > 0$ such that $f(x) \le f(c)$ for all $x \in (c-\delta, c+\delta) \cap I$. **Local (relative) minimum** is defined similarly. A **local (relative) extremum** means either a local maximum or a local minimum.

[2.2] THEOREM *If $f : I \to \mathbb{R}$ has a local extremum at an interior point $c \in I$, and f is differentiable at c, then $f'(c) = 0$.*

Proof. Suppose f has a local minimum at c. Since c is an interior point of I, there exists an interval $J = (c-\delta, c+\delta) \subseteq I$ such that for all $x \in J$ we have $f(x) \ge f(c)$. Thus,

$$\frac{f(x)-f(c)}{x-c} \le 0, \text{ for } x \in (c-\delta, c), \text{ and } \frac{f(x)-f(c)}{x-c} \ge 0, \text{ for } x \in (c, c+\delta).$$

Therefore, $f'_+(c) \ge 0$ and $f'_-(c) \le 0$. Since $f'(c) = f'_-(c) = f'_+(c)$, we get $f'(c) = 0$. ■

[2.3] THEOREM **(Rolle's theorem)** *If $f : [a, b] \to \mathbb{R}$ is continuous, differentiable on (a, b) and $f(a) = f(b)$, then there is a point $c \in (a, b)$ such that $f'(c) = 0$.*

Proof. Since f is continuous on $[a, b]$, there are $x_1, x_2 \in [a, b]$ such that $f(x_1) \le f(x) \le f(x_2)$, for all $x \in I$. If $f(x_1) = f(x_2)$, then f is a constant function, and therefore $f'(c) = 0$ for any $c \in (a, b)$, e.g., we can take $c = (a + b)/2$.

Suppose $f(x_1) \ne f(x_2)$. Then, at least one of x_1 and x_2 must be in (a, b), because $f(a) = f(b)$. Thus, there is a local extremum $c \in \{x_1, x_2\}$ of f in (a, b). By **[2.2]**, $f'(c) = 0$. ■

[2.4] COROLLARY *Between two real zeroes of a differentiable function f, there is a zero of f'.*

[2.5] EXAMPLE The equation $x^2 = x \sin x + \cos x$ has exactly two real roots. To see this, put $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2 - x \sin x - \cos x$. Then, f is differentiable, $f'(x) = x(2 - \cos x)$. Thus, $f'(x) = 0$ exactly at $x = 0$, and therefore, f cannot have more than two distinct zeroes. Note that $f(0) = -1 < 0, f(2) > 0, f(-2) > 0$. Thus, f has a zero in $(-2, 0)$ and a zero in $(0, 2)$.

[2.6] THEOREM **(Mean value theorem)** *If $f : [a, b] \to \mathbb{R}$ is continuous, and if f is differentiable on (a, b), then there is a point $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.*

Proof. Take $\ell : \mathbb{R} \to \mathbb{R}$ to be defined by $\ell(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x - a)$.
Note that $y = \ell(x)$ is the straight line passing through the points $(a, f(a))$ and $(b, f(b))$.
Define $\phi : [a, b] \to \mathbb{R}$ by $\phi(x) = f(x) - \ell(x)$. Then ϕ is continuous on $[a, b]$ and differentiable on (a, b). Moreover, $\phi(a) = \phi(b) = 0$. By Rolle's Theorem, there is $c \in (a, b)$ such that $\phi'(c) = 0$, i.e., $f'(c) = \frac{f(b)-f(a)}{b-a}$. ■

[2.7] REMARK Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then for $a + h \in (a, b]$,

$$f(a + h) = f(a) + hf'(c)$$

for some $c \in (a, a + h)$. Compare with Linear Approximation.

`Q`: If f is a constant function on $J \subseteq \mathbb{R}$, then $f' = 0$. Is the converse true? Not in general. Example?

[2.8] COROLLARY *Let $f : I \to \mathbb{R}$ be differentiable. (Note that I is an interval.)*

(1) If $f'(x) = 0$ for all $x \in I$, then f is a constant function.

(2) If $f'(x) \geq 0$ for all $x \in I$, then f is increasing on I (strict if $f(x) > 0$.)

(3) If $f'(x) \leq 0$ for all $x \in I$, then f is decreasing on I (strict if $f(x) < 0$.)

Proof. Let $r, s \in I$ with $r < s$. Then, f differentiable (and so continuous also) on $[r, s]$. By MVT, there is $c \in (r, s)$ such that $f(s) - f(r) = f'(c)(s - r)$.

In case $f'(x) = 0$ for all $x \in I$, we get $f(s) = f(r)$. Similarly, in case $f'(x) \geq 0$ (resp. $f'(x) \leq 0$) for all $x \in I$, we get $f(s) \geq f(r)$ (resp. $f(s) \leq f(r)$). Since r, s are arbitrary, we get the results. ∎

[2.9] EXAMPLE For $x \in [0, \frac{\pi}{2}]$, $\sin x \geq x - \frac{x^3}{6}$.

To see this Put $f(x) = \sin x - x + \frac{x^3}{6}$. Then $f'(x) = \cos x - 1 + x^2/2 = 2[(x/2)^2 - (\sin(x/2))^2] \geq 0$ for all $x \in [0, \frac{\pi}{2}]$. Since $f(0) = 0$, we get $f(x) \geq 0$.

[2.10] REMARK True or false? If f is a differentiable function on $[a, b]$ and $c \in (a, b)$ is such that $f'(c) > 0$, then there is $(c - \delta, c + \delta) \subseteq (a, b)$ on which f is increasing.

False. Take $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + 2x^2 \sin \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$. Then, f is differentiable with $f'(x) = 1 + 4x \sin \frac{1}{x} - 2 \cos \frac{1}{x}$ for $x \neq 0$ and $f'(0) = 1$. For $n \in \mathbb{N}$, $f'\left(\frac{1}{(2n+1)\pi}\right) = 3$ and $f'\left(\frac{1}{2n\pi}\right) = -1$. Since f' is continuous on $(0, \infty)$, for any $n \in \mathbb{N}$, there is an interval around $\frac{1}{(2n+1)\pi}$ on which $f' > 0$, and so f is increasing there. Similarly, there is an interval around $\frac{1}{2n\pi}$ on which $f' < 0$ and so f is decreasing. Therefore, for any $\delta > 0$, f is not increasing on $(0, \delta)$, and therefore on $(-\delta, \delta)$. (Note that for such an example, f' should not be continuous at c. Why?)

However, the following result holds from definition.

[2.11] PROPOSITION *Let $f : I \to \mathbb{R}$ be differentiable at $c \in I$, and $f'(c) > 0$. Then, there exists $\delta > 0$ such that*

$$f(x) > f(c) \text{ for } x \in (c, c + \delta) \cap I, \text{ and } f(x) < f(c) \text{ for } x \in (c - \delta, c) \cap I.$$

Proof. Since $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c} = f'(c)$, there exists $\delta > 0$ such that for $x \in (c - \delta, c + \delta) \cap I$

$$0 < f'(c) - f'(c)/2 < \frac{f(x) - f(c)}{x - c} < f'(c) + f'(c)/2,$$

(taking $\epsilon = f'(c)/2$). The result holds for this δ. ∎

[2.12] EXERCISE Write and prove a similar statement for the case when $f'(c) < 0$.

[**2.13**] THEOREM **(Intermediate value property of derivatives)** *Let $f : [a, b] \to \mathbb{R}$ be differentiable and let $f'(a) < k < f'(b)$. Then there exists $c \in (a, b)$ such that $f'(c) = k$.*
[Here, $f'(a) < k < f'(b)$ may be replaced by $f'(b) < k < f'(a)$.]

Proof. Consider $g : [a, b] \to \mathbb{R}$ defined by $g(x) = kx - f(x)$. Then g is differentiable on $[a, b]$, and $g'(x) = k - f'(x)$. Since $g'(a) = k - f'(a) > 0$, there is x in (a, b) such that $g(x) > g(a)$ (by [**2.11**]). Similarly, since $g'(b) = k - f'(b) < 0$, there is $y \in (a, b)$ such that $g(y) > g(b)$ (by [**2.12**]). Since g is continuous on $[a, b]$, it assumes a maximum at some $c \in [a, b]$. By the above discussion, $c \notin \{a, b\}$. So, c is an interior point in $[a, b]$ and a point of local maximum for g. We therefore get $g(c) = 0$, that is, $f'(c) = k$. ■

[**2.14**] QUESTION Can $f(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0, \end{cases}$ be the derivative of some function on $\mathbb{R}$?

[**2.15**] EXERCISE Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable such that $f(-1) = 5, f(0) = 0$ and $f(1) = 10$. Prove that there exist $c_1, c_2 \in (-1, 1)$ such that $f'(c_1) = -3$ and $f'(c_2) = 3$.

[**2.16**] REMARK **Sufficient conditions for local extremum:**

(1) **First derivative test:** Let f be a continuous function on $[a, b]$ and $\delta > 0$ such that $(c - \delta, c + \delta) \subseteq (a, b)$. Suppose f is differentiable on $(c - \delta, c)$ and $(c, c + \delta)$.

   (i) If $f' \geq 0$ on $(c - \delta, c)$ and $f' \leq 0$ on $(c, c + \delta)$, then f has a local maximum at c.
   (ii) If $f' \leq 0$ on $(c - \delta, c)$ and $f' \geq 0$ on $(c, c + \delta)$, then f has a local minimum at c.

(2) **Second derivative test:** Let f be a continuous function on $[a, b]$, and $c \in (a, b)$, and f is twice differentiable at c.

   (i) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.
   (ii) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

Proof. To prove (1) use [**2.8**]. To prove (2) use [**2.11**] and [**2.12**] for f'. ■

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3 A few solved examples

[3.1] EXAMPLE Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 \sin(1/x^2)$ for $x \neq 0$ and $f(0) = 0$. Then f is differentiable: At $x \neq 0$, $f'(x) = 2x\sin(1/x^2) - \frac{2}{x}\cos(1/x^2)$.
At 0, $\lim_{x\to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} x\sin(1/x^2) = 0$, because $-|x| \leq x\sin(1/x^2) \leq |x|$ for $x \neq 0$.
However, f' is not bounded in any interval $[-t, t]$ for $t > 0$. Let $M > 0$. We produce $x \in [-t, t]$ such that $f'(x) > M$ (producing x with $f'(x) < -M$ would also be fine). Now,

$$f'(x) > M \quad \text{if} \quad -\tfrac{2}{x}\cos(1/x^2) > M - 2x\sin(1/x^2), \text{ and so if } \; -\tfrac{2}{x}\cos(1/x^2) \geq M + 2t.$$

Thus, we are looking for $x \in [-t, t]$ such that $-\frac{1}{x}\cos(1/x^2) > \frac{M}{2} + t$. Choose $n \in \mathbb{N}$ such that $x := 1/\sqrt{(2n+1)\pi} < \min\{t, 1/(\frac{m}{2} + t)\}$. Then $x \in [-t, t]$, and $\cos(1/x^2) = -1$ and so $-\frac{1}{x}\cos(1/x^2) = 1/x > \frac{M}{2} + t$. Thus, f' is not bounded above on $[-t, t]$. (Can you now show that f' is not bounded below also?) In particular, f' is not continuous at 0.

[3.2] EXAMPLE Suppose $f(x) = x^3 + x^2 - 5x + 3$ for $x \in \mathbb{R}$. We show that f is one-one on $[1, 5]$ but not one-one on $\mathbb{R}$.
We have $f'(x) = 3x^2 + 2x - 5 = (3x+5)(x-1)$. Since $f'(x) > 0$ for $x > 1$, f is one-one on $[1, 5]$ (in fact on any subset of $[1, \infty)$). However, f is not one-one on $\mathbb{R}$: $f(1) = 0, f(0) = 3, f(-5) = -72$. IVT, there is $t \in (-5, 0)$ such that $f(t) = f(1) = 0$.

[3.3] EXAMPLE For $0 < x < y$, $\dfrac{y - x}{y} < \ln\dfrac{y}{x} < \dfrac{y - x}{x}$.

To see this let $f(t) = \ln t$ on $[x, y]$. Then f is differentiable on $[x, y]$ and $f'(t) = 1/t$. By MVT, there is $c \in (x, y)$ such that

$$\ln y - \ln x = \frac{1}{c}(y - x), \text{ i.e, } \ln\frac{y}{x} = \frac{1}{c}(y - x).$$

Since $\dfrac{1}{y} < \dfrac{1}{c} < \dfrac{1}{x}$, we have

$$\frac{y - x}{y} < \ln\frac{y}{x} < \frac{y - x}{x}.$$

From the above let us deduce that if $e \leq x < y$, then $x^y > y^x$. Since $x\ln(y/x) < y - x$, we have $\ln\frac{y^x}{x^x} = x\ln(y/x) < y - x$, i.e., $\dfrac{y^x}{x^x} < e^{y-x} \leq x^{y-x} = \dfrac{x^y}{x^x}$ (since $e \leq x$ implies $e^t \leq x^t$ for any t).
Thus, $y^x < x^y$.
In particular, we have $e^\pi > \pi^e$, since $e < \pi$.

[3.4] EXAMPLE Suppose $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable at 0 and given that $f(\frac{1}{n}) = 0$ for all $n \in \mathbb{N}$. Let us find $f'(0)$ and $f''(0)$.
First, since f is twice differentiable at 0, f must be differentiable in an interval $[-r, r]$, $r > 0$. In particular, it is differentiable at 0, and so continuous at 0. Since $\frac{1}{n} \to 0$, have $f(\frac{1}{n}) \to f(0)$ yielding $f(0) = 0$.
Next, $f'(0) = \lim_{x\to 0} \dfrac{f(x) - f(0)}{x - 0}$, and the sequence $(\frac{1}{n})$ converges to 0, we have

$$f'(0) = \lim_{n\to\infty} \frac{f(1/n) - f(0)}{1/n - 0} = 0.$$

Finally, choose $m \in \mathbb{N}$ such that $\frac{1}{m} \le r$. For $n \ge m$, f is differentiable on $[0, 1/n]$ with $f(0) = f(1/n) = 0$. By MVT, there is $x_n \in [0, 1/n]$ such that $f'(x_n) = 0$. Then $x_n \to 0$ and therefore

$$f''(0) = \lim_{n\to\infty} \frac{f'(x_n) - f'(0)}{x_n - 0} = 0.$$

Class 3

4 L'Hôpital's Rules

[**4.1**] THEOREM *Let $f, g : (a, b) \to \mathbb{R}$, $c \in (a, b)$, $f(c) = g(c) = 0$, $f'(c), g'(c)$ exist, and $g'(c) \neq 0$. Then,* $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{f'(c)}{g'(c)}$.

Proof. Since $g'(c) \neq 0$, for $x \neq c$ we have

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(c)}{g(x) - g(c)} = \frac{\frac{f(x)-f(c)}{x-a}}{\frac{g(x)-g(c)}{x-a}} \to \frac{f'(c)}{g'(c)}, \quad \text{as } x \to c.$$ ■

[**4.2**] REMARK Similar results hold for left/right hand limit at an end point in the domain.

[**4.3**] EXAMPLE Consider $h(x) = \dfrac{\ln \cos x}{x}$ on $(0, \pi/2)$. The functions $f(x) = \ln \cos x$ and $g(x) = x$ are defined on $[0, \pi/2)$ and $f(0) = g(0) = 0$. Moreover, $f'(0) = -\tan 0 = 0$ and $g'(0) = 1 \neq 0$. Therefore, $\lim\limits_{x\to 0+} h(x) = f'(0)/g'(0) = 0$.

[**4.4**] EXAMPLE Find the limit $\lim\limits_{x\to\infty} \left(1 + \dfrac{1}{x^2}\right)^x$, if it exists.

Putting $y = 1/x$, we see that the limit will be equal to $\lim\limits_{y\to 0+} f(y)$, where $f(y) = (1 + y^2)^{1/y}$. We have $\ln f(y) = \dfrac{\ln(1 + y^2)}{y} = \dfrac{g(x)}{h(x)}$. Since $g(0) = h(0) = 0$, $g'(0) = 0$ and $h'(0) = 1 \neq 0$, we have $\lim\limits_{y\to 0+} \ln f(y) = \dfrac{g'(0)}{h'(0)} = 0$. Since Exp is continuous, we have $\lim\limits_{y\to 0+} f(y) = 1$.

[**4.5**] THEOREM **(Cauchy's Mean Value Theorem (CMVT))** *Let f and g be continuous on $[a, b]$ and differentiable on (a, b), and assume that $g'(x) \neq 0$ for all $x \in (a, b)$. Then there exists $c \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof. We use Rolle's theorem to a function $\phi = f - \lambda g$ on $[a, b]$, where $\lambda \in \mathbb{R}$ is a constant. Clearly ϕ is continuous on $[a, b]$ and differentiable on (a, b). To hold $\phi(a) = \phi(b)$ we have $f(a) - \lambda g(a) = f(b) - \lambda g(b)$, i.e., $\lambda = \frac{f(b)-f(a)}{g(b)-g(a)}$. For this value of λ, by Rolle's theorem, there is $c \in (a, b)$ such that $\phi'(c) = 0$, i.e., $f'(c) = \lambda g'(c)$. Thus, $\frac{f(b)-f(a)}{g(b)-g(a)} = \lambda = \frac{f'(c)}{g'(c)}$. ■

[**4.6**] REMARK

1. CMVT is not derived by using MVT to f and g and taking ratios.
2. Geometrically, CMVT states that for the differentiable curve $\gamma : [a,b] \to \mathbb{R}^2$ given by $\gamma(t) = (g(t), f(t))$, there is a point $\gamma(c)$ where the tangent is parallel to the chord joining $\gamma(a)$ and $\gamma(b)$.

[**4.7**] EXAMPLE Here is a typical example how CMVT is effectively used. Suppose $0 < a < b$ and ϕ is differentiable on $[a,b]$. The claim is that there is $c \in [a,b]$ such that

$$\frac{b\phi(a) - a\phi(b)}{b-a} = \phi(c) - c\phi'(c).$$

To see this, define $f(x) = \frac{\phi(x)}{x}$ and $g(x) = \frac{1}{x}$ on $[a,b]$. Verify that all conditions of CMVT are satisfied by f and g. The existence of $c \in (a,b)$ with $\dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(c)}{g'(c)}$ will amount to the stated result (Verify this).

[**4.8**] THEOREM **L'Hôpital's Rule 1 ($\frac{0}{0}$ form)** *Let $f, g : (a,b) \to \mathbb{R}$ be differentiable such that*

(1) $\lim\limits_{x\to b-} f(x) = \lim\limits_{x\to b-} g(x) = 0,$

(2) $g'(x) \neq 0$ *for all* $x \in (a,b)$, *and*

(3) $\lim\limits_{x\to b-} \dfrac{f'(x)}{g'(x)} = \ell,$

Then $\lim\limits_{x\to b-} \dfrac{f(x)}{g(x)} = \ell$. *(Here, b can be ∞ and ℓ can be $\pm\infty$.)*

Note: Similar results hold for right hand limit at a and two sided limit at $c \in (a,b)$.

Proof. (**Case** $b, \ell \in \mathbb{R}$) Set $f(b) = g(b) = 0$ so that f, g are continuous on $(a,b]$. Then, for $x \in (a,b)$, by CMVT, there is $t \in [x,b]$ such that

$$\frac{f(x)}{g(x)} = \frac{f(b)-f(x)}{g(b)-g(x)} = \frac{f'(t)}{g'(t)}.$$

Suppose $\epsilon > 0$. From (3) there exists $\delta > 0$ such that for $x \in (b-\delta, b)$

$$\left|\frac{f'(x)}{g'(x)} - \ell\right| < \epsilon.$$

Thus, for $x \in (b-\delta, b)$, $\left|\dfrac{f(x)}{g(x)} - \ell\right| = \left|\dfrac{f'(t)}{g'(t)} - \ell\right| < \epsilon$, as $t \in (b-\delta)$. Hence the result.

(**Case** $\ell \in \mathbb{R},\ b = \infty$.) Choose positive R with $R \geq a$ and define F, G on $(0, 1/R)$ by

$$F(t) = f(1/t),\ G(t) = g(1/t),$$

and use the above case for $t \to 0+$.

(Case $\ell = \infty$.) Suppose $M > 0$.

(If $b \in \mathbb{R}$) there exists $\delta > 0$ such that for $x \in (b-\delta, b)$
(If $b = \infty$) there exists $K > 0$ such that for $x \geq K$

$$\frac{f'(x)}{g'(x)} > M.$$

Now proceed as in the previous cases. Similarly the case when $\ell = -\infty$ can be proved. ■

[**4.9**] EXAMPLE Find the limit $\lim\limits_{x\to 1}\left[\frac{x}{x-1} - \frac{1}{\ln x}\right]$, if it exists. We have on $(0,2)$

$$\frac{x}{x-1} - \frac{1}{\ln x} = \frac{x\ln x - (x-1)}{(x-1)\ln x} = \frac{f(x)}{g(x)},$$

where f, g are differentiable and $f(1) = g(1) = 0$. Moreover, $f'(x) = \ln x$, $g'(x) = \frac{x-1}{x} + \ln x$, and $g'(x) \neq 0$ in $(0,2)\setminus\{1\}$. Thus, by L'Hôpital's Rule 1, the required limit equals

$$\lim_{x\to 1}\frac{f'(x)}{g'(x)} = \lim_{x\to 1}\frac{x\ln x}{x-1+x\ln x} = \lim_{x\to 1}\frac{\phi(x)}{\psi(x)},$$

if it exists. Now, $\phi(1) = \psi(1) = 0$, $\phi'(1) = 1$ and $\psi'(1) = 2$. By [**4.1**] (not [**4.10**]) we have $\lim\limits_{x\to 1}\frac{\phi(x)}{\psi(x)} = \frac{1}{2}$. Therefore the required limit is $1/2$.

[**4.10**] THEOREM **L'Hôpital's Rule 2 ($\frac{\infty}{\infty}$ form)** *Let $f, g : (a,b) \to \mathbb{R}$ be differentiable such that*

(1) $\lim\limits_{x\to b-} f(x) = \lim\limits_{x\to b-} g(x) = \infty$,

(2) $g'(x) \neq 0$ for all $x \in (a,b)$, and

(3) $\lim\limits_{x\to b-}\frac{f'(x)}{g'(x)} = \ell$,

Then $\lim\limits_{x\to b-}\frac{f(x)}{g(x)} = \ell$. (Here, b can be ∞ and ℓ can be $\pm\infty$.)

Proof. We prove the case when $b = \infty$, $\ell \in \mathbb{R}$, and leave the others as exercises.

Suppose $\epsilon > 0$ be given. From (3), there is $R \geq a$ such that for $x > R$

$$\left|\frac{f'(x)}{g'(x)} - \ell\right| < \epsilon/2. \tag{4.1}$$

Next, in view of (1), we can choose $R_1 \geq R$ such that for all $x \geq R_1$, $f(x) > \max\{f(R), 0\}$, $g(x) > \max\{g(R), 0\}$. Then for $x \geq R_1$, $f(x)/g(x)$ is defined.

Next, for $x > R_1$, by CMVT, there is $c \in (R, x)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(x)-f(R)}{g(x)-g(R)} = \frac{f(x)\left(1-\frac{f(R)}{f(x)}\right)}{g(x)\left(1-\frac{g(R)}{g(x)}\right)} \quad \text{(defined, since } f(x) > f(R), g(x) > g(R)\text{)}$$

Therefore, for $x \geq R_1$ there is $c \in (R, x)$ such that $\dfrac{f(x)}{g(x)} = \dfrac{f'(c)}{g'(c)}\psi(x)$, where $\psi(x) = \dfrac{1 - \frac{g(R)}{g(x)}}{1 - \frac{f(R)}{f(x)}}$. Note that $\psi(x) \to 1$ as $x \to \infty$. For $x \geq R_1$ we have

$$\begin{aligned}
\left|\frac{f(x)}{g(x)} - \ell\right| &= \left|\frac{f'(c)}{g'(c)}\psi(x) - \ell\right| \\
&= \left|\frac{f'(c)}{g'(c)}(\psi(x) - 1) + \frac{f'(c)}{g'(c)} - \ell\right| \\
&\leq \left|\frac{f'(c)}{g'(c)}\right| |\psi(x) - 1| + \left|\frac{f'(c)}{g'(c)} - \ell\right| \\
&< (|\ell| + \epsilon/2)|\psi(x) - 1| + \left|\frac{f'(c)}{g'(c)} - \ell\right|
\end{aligned}$$

because (4.1) implies that $\left|\left|\dfrac{f'(x)}{g'(x)}\right| - |\ell|\right| < \epsilon/2$, yielding $\left|\dfrac{f'(c)}{g'(c)}\right| < |\ell| + \epsilon/2$. Now, as $\lim\limits_{x\to\infty} \psi(x) = 1$, we can choose $R_2 \geq R_1$ such that for $x > R_2$, $|\psi(x) - 1| < \dfrac{\epsilon}{2(|\ell| + \epsilon/2)}$. Then, for $x > R_2$ we have

$$\left|\frac{f(x)}{g(x)} - \ell\right| < (|\ell| + \epsilon/2)\frac{\epsilon}{2(|\ell| + \epsilon/2)} + \epsilon/2 = \epsilon.$$

■

[**4.11**] EXAMPLE Find the limit $\lim\limits_{x\to\infty} x^n e^{-x}$, $n \in \mathbb{N}$, if it exists.
We write $x^n e^{-x}$ as $\frac{x^n}{e^x} = \frac{f(x)}{g(x)}$, where $f(x) = x^n \to \infty, g(x) = e^x \to \infty$. Moreover, f and g are differentiable on $\mathbb{R}$ and $g'(x) \neq 0$ for any x. Since $\lim\limits_{x\to\infty} \dfrac{1}{e^x} = 0$, by repeated application of L'Hôpital's Rule 2, we have

$$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{nx^{n-1}}{e^x} = \cdots = \lim_{x\to\infty} \frac{n!}{e^x} = 0.$$

[**4.12**] REMARK If $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} g(x) = \infty$ and $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = 0$, then we say that g **grows much faster** than f. From the above example, we see that e^x grows much faster than any polynomial $a_0 + a_1 x + \cdots + a_n x^n,\ a_n > 0$.

[**4.13**] EXERCISE Find the following by using L'Hôpital's Rules, whenever needed. Do not forget to check the conditions needed for using L'Hôpital's Rules.

(i) $\lim\limits_{x\to 0+} \dfrac{\sqrt{1+x} - 1}{\sqrt{x}}$ (ii) $\lim\limits_{x\to \frac{\pi}{2}} \dfrac{1 - \sin x}{1 + \cos 2x}$ (iii) $\lim\limits_{x\to 0} \dfrac{x^2 \sin \frac{1}{x}}{\sin x}$ (iv) $\lim\limits_{x\to 0+} \left(\dfrac{\sin x}{x}\right)^{1/x}$

(v) $\lim\limits_{x\to 0+} \dfrac{e^{-1/x^2}}{x}$ (vi) $\lim\limits_{x\to 0} \left(\dfrac{1}{\sin x} - \dfrac{1}{x}\right)$ (vii) $\lim\limits_{x\to\infty} \dfrac{x - \sin x}{2x + \sin x}$

Class 4

5 Taylor's Theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous. If f is differentiable at a, then (recall linear approximation)

$$f(b) \approx f(a) + f'(a)(b - a).$$

In other words, f is approximated by a linear polynomial $f(a) + f'(a)(x - a)$. If f has higher derivatives, do we have better approximations?

Suppose p is a polynomial of degree k. Then

$$p(x) = p(0) + p'(0)x + \frac{p^{(2)}(0)}{2!}x^2 + \cdots + \frac{p^{(k)}(0)}{k!}x^k.$$

In fact, for any $a \in \mathbb{R}$

$$p(x) = p(a) + p'(a)(x - a) + \frac{p^{(2)}(a)}{2!}(x - a)^2 + \cdots + \frac{p^{(k)}(a)}{k!}(x - a)^k.$$

For example, $p(x) = 1 + 2x^2 + x^3$ can be written as

$$\begin{aligned} p(x) &= 1 + 2(x - 1 + 1)^2 + (x - 1 + 1)^3 = 4 + 7(x - 1) + 5(x - 1)^2 + (x - 1)^3 \\ &= p(1) + p'(1)(x - 1) + \frac{p^{(2)}(1)}{2!}(x - 1)^2 + \frac{p^{(3)}(1)}{3!}(x - 1)^3, \end{aligned}$$

since $p(1) = 4,\ p'(1) = 7,\ p^{(2)}(1) = 10$ and $p^{(3)}(1) = 6$.

[5.1] THEOREM **(Taylor)** *Let $f : [\alpha, \beta] \to \mathbb{R}$ be such that $f', f^{(2)}, \ldots, f^{(n)}$ are continuous on $[\alpha, \beta]$ and $f^{(n+1)}$ exists on (α, β). Let $a \in [\alpha, \beta]$. Then for $x \in [\alpha, \beta]$ there exists c between x and a such that*

$$f(x) = f(a) + f'(a)(x - a) + \frac{f^{(2)}(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - a)^{n+1}. \quad (5.2)$$

Proof. The idea is to use Rolle's theorem to a suitable function. Look at

$$\begin{aligned} F(t) &:= f(t) + f'(t)(x - t) + \frac{f^{(2)}(t)}{2!}(x - t)^2 + \cdots + \frac{f^{(n)}(t)}{n!}(x - t)^n + M(x - t)^{n+1}, \\ &= f(t) + \sum_{k=1}^{n} \frac{f^{(k)}(t)}{k!}(x - t)^k \ + M(x - t)^{n+1}. \end{aligned}$$

where M is chosen so that $F(x) = F(a)$. This will be so, when M satisfies

$$f(x) = f(a) + \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k \ + M(x - a)^{n+1}. \quad (5.3)$$

Let I be the closed interval with endpoints a and x. Then, F is continuous on I and differentiable on the interior of I. By Rolle's theorem, there is c in the interior of I such that $F'(c) = 0$. Note that

$$\begin{aligned} F'(t) &= f'(t) + \sum_{k=1}^{n} \left(\frac{f^{(k+1)}(t)}{k!}(x-t)^k - \frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1} \right) - (n+1)M(x-t)^n \\ &= \frac{f^{(n+1)}(t)}{(n)!}(x-t)^n - (n+1)M(x-t)^n \quad \left[\text{Note: } \frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1} = f'(t) \text{ when } k = 1.\right] \end{aligned}$$

Thus, $F(c) = 0$ gives $M = \dfrac{f^{(n+1)}(c)}{(n+1)!}$. In view of (5.3), we get (5.2). ■

[**5.2**] DEFINITION The polynomial

$$T_n(f,a)(x) := f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is called the **Taylor polynomial** of f of degree n about a, and $R_n := \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ the **remainder** after n terms.

[**5.3**] EXAMPLE For $x > 0$, show that $1 + \dfrac{x}{2} - \dfrac{x^2}{8} < \sqrt{1+x} < 1 + \dfrac{x}{2}$.

Let $f(x) = \sqrt{1+x}, x \geq 0$. Taylor's theorem (with $n = 1$) gives $f(x) = 1 + \dfrac{x}{2} - \dfrac{1}{4}(1+c)^{-3/2}\dfrac{x^2}{2!}$ for some $0 < c < x$. Since $0 < (1+c)^{-3/2} < 1$, we have $1 + \dfrac{x}{2} - \dfrac{x^2}{8} < \sqrt{1+x} < 1 + \dfrac{x}{2}$.

[**5.4**] EXAMPLE For $x > 0$, show that $\sin x < x - \dfrac{x^3}{6} + \dfrac{x^5}{120}$.

Using Taylor's theorem for the function $f(x) = \sin x,\ 0 < x < \pi/2$, about $x = 0$ (with $n = 4$) we have

$$\sin x = \sin 0 + (\cos 0)x + \frac{-\sin 0}{2!}x^2 + \frac{-\cos 0}{3!}x^3 + \frac{\sin 0}{4!}x^4 + \frac{\cos c}{5!}x^5 = x - \frac{x^3}{6} + \frac{x^5 \sin c}{120}$$

for some $c \in (0, x)$. Since $\cos c \leq 1$, we have $\sin x \leq x - \dfrac{x^3}{6} + \dfrac{x^5}{120}$.

[**5.5**] EXERCISE Show that for $x \in [-1, 1]$, $\sin x$ can be approximated by $x - \frac{x^3}{3!} + \frac{x^5}{5!}$ with error less than 0.001.
[Hint: Use Taylor's Theorem for $\sin x$ about 0 and $n = 6$. Show that for $|x| \leq 1,\ |R_6| < \frac{1}{5040} < 0.001$.]

[**5.6**] EXERCISE Show that $\cos x \geq 1 - \frac{1}{2}x^2$ for all $x \in \mathbb{R}$.

[**5.7**] THEOREM **(Application to Extremum)** *Let $f^{(n)}$ be continuous on $I = (\alpha, \beta)$, $a \in I$ and $n \geq 2$. Suppose $f'(a) = f''(a) = \cdots = f^{(n-1)}(a) = 0$ and $f^{(n)}(a) \neq 0$.*

1. *If n is even and $f^{(n)}(a) < 0$, then f has a local maximum at a.*

2. If n is even and $f^{(n)}(a) > 0$, then f has a local minimum at a.

3. If n is odd, then f does not have a local extremum at a.

Proof. Since $f^{(n)}$ is continuous and $f^{(n)}(a) \neq 0$, $f^{(n)}$ has same sign as $f^{(n)}(a)$ in a neighbourhood J of a. With the given conditions, for $x \in J$, we have by Taylor's theorem

$$f(x) = f(a) + \frac{f^{(n)}(c)}{n!}(x-a)^n,$$

for some $c \in J$. Now, look at the signs of $f(x) - f(a)$ in various cases. ∎

[5.8] EXAMPLE Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \cos x + \frac{1}{2}x^2 - \frac{1}{24}x^4$. Then,

$$f'(x) = -\sin x + x - \tfrac{1}{6}x^3,\ f''(x) = -\cos x + 1 - \tfrac{1}{2}x^2,$$
$$f^{(3)}(x) = \sin x - x,\ f^{(4)}(x) = \cos x - 1,\ f^{(5)}(x) = -\sin x,\ f^{(6)}(x) = -\cos x.$$

We have $f^{(k)}(0) = 0$ for $1 \le k \le 5$ and $f^{(6)}(0) = -1 < 0$. By (1) of **[5.7]**, f has a local maximum at $x = 0$.

[5.9] DEFINITION Suppose $f : I \to \mathbb{R}$ is infinitely differentiable and $a \in J$. Then

$$T(f,a)(x) := \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

(with $f^{(0)} = f$) is called the **Taylor series** of f about a. When $a = 0$, it is called the **Maclaurin series**. If the remainder $R_n := \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \to 0$ for given x, then the sequence $z_n = T_n(f,a)(x) \to f(x)$, i.e., $f(x) = T(f)(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$. We say that the Taylor series converges to $f(x)$ at x.

[5.10] EXAMPLE The Maclaurin series for $f(x) = e^x,\ x \in \mathbb{R}$: As $f^{(n)}(0) = e^0 = 1$ for all $n \in \mathbb{N}$,

$$T(f,0)(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

We have $R_n = e^c \frac{x^{n+1}}{(n+1)!} \to 0$ for any $x \in \mathbb{R}$. Thus, $e^x = T(f,0)(x),\ x \in \mathbb{R}$, that is, e^x is given by its Maclaurin series.

[5.11] EXERCISE Verify that $\sin x$ and $\cos x$ are given by their Maclaurin series:

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \end{aligned}$$

Class 5a

6 Revisiting limit superior and inferior

Let (a_n) be a bounded sequence and $|a_n| \leq M$ for all n. For $n \in \mathbb{N}$ let

$$b_n =: \text{lub}\{a_k : k \geq n\}, \ c_n =: \text{glb}\{a_k : k \geq n\}.$$

Then $b_{n+1} = \text{lub}\{a_{k+1}, a_{k+2}, \ldots\} \geq \text{lub}\{a_k, a_{k+1}, \ldots\} = b_n$, that is, (b_n) is a decreasing sequence bounded below by $-M$. Therefore (b_n) converges to some point in $[-M, M]$. Similarly, (c_n) is increasing and converges to some point in $[-M, M]$.

[6.1] DEFINITION For a real sequence (a_n) define the **limit superior** of (a_n) as follows:

$$\limsup a_n = \begin{cases} \lim b_n, & \text{if } (a_n) \text{ is bounded above,} \\ \infty, & \text{if } (a_n) \text{ is not bounded above.} \end{cases}$$

Similarly, **limit inferior** of (a_n) is defined to be

$$\limsup a_n = \begin{cases} \lim c_n, & \text{if } (a_n) \text{ is bounded below,} \\ \infty, & \text{if } (a_n) \text{ is not bounded below.} \end{cases}$$

[6.2] RESULT *For a sequence (a_n), $\liminf a_n \leq \limsup a_n$.*

Proof. Suppose (a_n) is bounded. Then $b_n := \text{glb}\{a_k : k \geq n\} \leq \text{lub}\{a_k : k \geq n\} =: c_n$. Therefore,

$$\liminf a_n = \lim_{n\to\infty} b_n \leq \lim_{n\to\infty} c_n = \limsup a_n.$$

If (a_n) is not bounded above, then $\limsup a_n = \infty$ and if (a_n) is not bounded below, then $\liminf a_n = -\infty$. So, the result follows. ∎

[6.3] RESULT *For a sequence (a_n), $a_n \to \ell$ if and only if $\limsup a_n = \liminf a_n = \ell$. (Here, $\ell \in \mathbb{R}$ or $\ell = \pm\infty$.)*

Proof. Suppose $a_n \to \ell$. First, let $\ell \in \mathbb{R}$. Let $\epsilon > 0$. There is $m \in \mathbb{N}$ such that $a_n \in (\ell - \epsilon, \ell + \epsilon)$ for $n \geq m$. We will have $b_n, c_n \in (\ell - \epsilon, \ell + \epsilon)$ for all $n \geq m$. Consequently, $\limsup a_n, \liminf a_n \in (\ell - \epsilon, \ell + \epsilon)$. Thus, $|\limsup a_n - \ell| < \epsilon, |\liminf a_n| < \epsilon$. Since $\epsilon > 0$ is arbitrary, we have $\limsup a_n = \liminf a_n = \ell$. Next, let $\ell = \infty$. Let $K > 0$. Then, there is $m \in \mathbb{N}$ such that $a_n \geq K$ for $n \geq m$. Thus, for $n \geq m$, $c_n := \text{glb}\{a_k : k \geq n\} \geq K$. Therefore, $c_n \to \infty$. Thus, $\liminf an = \infty$. Moreover, since (a_n) is not bounded above, $\limsup a_n = \infty$. Similarly, if $\ell = -\infty$, then we can show as above that $\limsup a_n = \liminf a_n = -\infty$.

Conversely, suppose $\limsup a_n = \liminf a_n = \ell$. In case $\ell \in \mathbb{R}$, (a_n) is bounded. We have $c_n \leq a_n \leq b_n$ and the sandwich theorem gives $a_n \to \ell$. If $\ell = \infty$, $c_n \to \infty$ and $c_n \leq a_n$ give $a_n \to \infty$. If $\ell = -\infty$, then $b_n \to \infty$ and $a_n \leq b_n$ give $a_n \to \infty$. ∎

[6.4] EXAMPLE

(i)	$a_n = (-1)^n$	:	$\limsup a_n = 1$	$\liminf a_n = -1$.
(ii)	$a_n = (-1)^n n$	:	$\limsup a_n = \infty$	$\liminf a_n = -\infty$.
(ii)	$a_n = -n$	:	$\limsup a_n = -\infty$	$\liminf a_n = -\infty$.
(iii)	$a_n = \frac{1}{n}$	:	$\limsup a_n = 0$	$\liminf a_n = 0$.

[6.5] RESULT *For $n \in \mathbb{N}$, let $a_n \geq 0, x_n > 0$ such that $x_n \to x > 0$ and $\ell = \limsup a_n$. Then* $\limsup a_n x_n = \infty$*, if $\ell = \infty$ and* $\limsup a_n x_n = \ell x$*, if $\ell \in \mathbb{R}$.*

Proof. If $\ell = \infty$, then (a_n) is not bounded above, and so is $(a_n x_n)$. Therefore $\limsup a_n x_n = \infty$. Suppose $\ell \in \mathbb{R}$. Let $b_n := \text{glb}\{a_k : k \geq n\}$ and $b'_n := \text{glb}\{a_k x_k : k \geq n\}$. Let $x > \epsilon > 0$. Since $x_n \to x > 0$, there is $m \in \mathbb{N}$ such that $x - \epsilon < x_n < x + \epsilon$. Then $a_n(x - \epsilon) \leq a_n x_n \leq a_n(x + \epsilon)$ for $n \leq m$. Therefore, $b_n(x - \epsilon \leq b'_n \leq b_n(x + \epsilon)$. Taking limits we have

$$\ell(x - \epsilon) \leq \limsup a_n x_n \leq \ell(x + \epsilon), \text{ i.e., } |\limsup a_n x_n - \ell x| \leq \ell\epsilon.$$

Since ϵ is arbitrary, we must have $\limsup a_n x_n = \ell x$. ∎

[6.6] EXERCISE Find $\limsup a_n x_n$, if

$$x_n = n^{1/n} \text{ and } a_n = \begin{cases} \frac{n-1}{n^2} & \text{if } n \text{ is odd,} \\ \frac{n}{n-1} & \text{if } n \text{ is even.} \end{cases}$$

Class 5b

7 Power series

[7.1] DEFINITION A **power series** about a is an expression $P(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$, where $a_n \in \mathbb{R}$. For given $x \in \mathbb{R}$, $P(x)$ is an infinite series. The **domain of convergence** of the power series is $\{x \in \mathbb{R} : P(x) \text{ is convergent}\}$.

[7.2] EXAMPLE (i) Every polynomial is a power series (with $a_n = 0$ for all large n's).
(ii) Taylor series of a function is a power series.

[7.3] REMARK Suppose $r \in \mathbb{R}$ and $P(r)$ is absolutely convergent. Then $P(x)$ is absolutely convergent for all $x \in [a - d, a + d]$, where $d = |a - r|$. How does the domain of convergence look like?

We will consider $a = 0$, without any loss of generality.

[7.4] THEOREM *Consider a power series $P(x) = \sum_{n=0}^{\infty} a_n x^n$. There exist $R \in [0, \infty) \cup \{\infty\}$ such that $P(x)$ converges absolutely if $|x| < R$ and diverges if $|x| > R$.*

Wait for a while for a proof.

[7.5] DEFINITION The theorem means that the domain of convergence is an interval with end-points $-R$ and R. This R is called the **radius of convergence** of $P(x)$.

Proof. of **[7.5]** Let $\rho = \limsup |a_n|^{1/n}$ and define

$$R = \begin{cases} \infty, & \text{if } \rho = 0, \\ 1/\rho, & \text{if } 0 < \rho < \infty, \\ 0, & \text{if } \rho = \infty. \end{cases}$$

Take the case $0 < R < \infty$. Let $0 < |x| < R$. Then $\dfrac{1}{|x|} > \dfrac{1}{R} = \rho$. Choose $0 < r < 1$ such that $\dfrac{1}{|x|} > \dfrac{r}{|x|} > \rho$. Thus, there exists k such that $\sup\limits_{n \geq k} |a_n|^{1/n} < \dfrac{r}{|x|}$, that is, $|a_n x^n| < r^n$ for $n \geq k$. Hence

$$\sum |a_n x^n| \leq \sum_{n=0}^{k-1} |a_n x^n| + \sum_{n=k}^{\infty} r^n \ < \ \infty,$$

that is, $P(x)$ is absolutely convergent.

Next, let $|x| > R$, so that $\rho > \frac{1}{|x|}$. Then $\sup\limits_{n \geq k} |a_n|^{1/n} > \dfrac{1}{|x|}$ for every k. So, there are infinitely many n such that $|a_n|^{1/n} > \frac{1}{|x|}$, that is, $|a_n x^n| > 1$. Thus, $a_n x^n \not\to 0$, and so $\sum a_n x^n$ is divergent.

The cases $R = \infty$ and $R = 0$ can be proved similarly. ■

[7.6] RESULT *For a power series* $\sum a_n x^n$, $R = \lim |a_n / a_{n+1}|$, *if it exists.*

Proof. Let $S = \lim\limits_{n \to \infty} |a_n / a_{n+1}|$ First, let $0 \leq S < \infty$. For $x \neq 0$ we have

$$\lim_{n \to \infty} \frac{|a_n x^n|}{|a_{n+1} x^{n+1}|} = \frac{1}{|x|} \lim_{n \to \infty} |a_n / a_{n+1}| = \frac{S}{|x|}.$$

By D'Alemberts ratio test, $\sum |a_n x^n|$ converges if $\frac{S}{|x|} > 1$, i.e., if $|x| < S$, and diverges if $|x| > S$. We therefore must have $S = R$.

The case $S = \infty$ can be handled similarly. ■

[7.7] EXAMPLE Find radius of convergence R and domain of convergence D:

1. $\sum \dfrac{x^n}{n!}$, $\quad R = \infty, \quad D = \mathbb{R}$.
2. $\sum \dfrac{x^n}{n}$, $\quad R = 1, \quad D = [-1, 1)$.
3. $\sum n^2 x^n$, $\quad R = 1, \quad D = (-1, 1)$.
4. $\sum n! x^n$, $\quad R = 0, \quad D = \{0\}$.
5. $1 + x^2 + \dfrac{x^4}{4!} + x^6 + \dfrac{x^8}{8!} + \cdots$, $\quad R = 1, \quad D = (-1, 1)$.

[**7.8**] THEOREM **(Term by term differentiation)** *Suppose* $\sum a_n x^n$ *has radius of convergence* $R > 0$, *and* $f(x) = \sum a_n x^n$ *for* $x \in (-R, R)$. *Then,* f *is differentiable on* $(-R, R)$ *and* $f'(x) = \sum n a_n x^{n-1}$.

Proof. Since $\limsup |na_n|^{1/n} = \limsup |a_n|^{1/n}$, the series $\sum n a_n x^{n-1}$ converges in $(-R, R)$. Now, for $x, x+h \in (-R, R)$ we have

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{\sum a_n(x+h)^n - \sum a_n x^n}{h} \\ &= \frac{\sum a_n\big((x+h)^n - x^n\big)}{h} && \text{(as both the series are convergent)} \\ &= \sum a_n n(x+\theta_n h)^{n-1} && \text{(for some } 0<\theta_n<1\text{, by MVT)}\end{aligned}$$

Now, choose $K < R$ such that $x, x+h \in [-K, K]$. Then

$$\begin{aligned} & \left|\sum a_n n(x+\theta_n h)^{n-1} - \sum a_n n x^{n-1}\right| \\ = & \left|\sum a_n n\big[(x+\theta_n h)^{n-1} - x^{n-1}\big]\right| && \text{(as both the series are convergent)} \\ = & \left|\sum a_n n\big[(\theta_n h)(n-1)(x+\beta_n h)^{n-2}\big]\right| && \text{(for some } 0<\beta_n<\theta_n\text{, by MVT.)} \\ \le & \sum \left|a_n n\big[(\theta_n h)(n-1)(x+\beta_n h)^{n-2}\big]\right| \\ \le & |h| \sum \left|a_n n(n-1)K^{n-2}\right| \to 0, \quad \text{as } h \to 0.\end{aligned}$$

Hence $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \sum a_n n x^{n-1}$, as desired. ■

[**7.9**] COROLLARY *If* $f(x) = \sum a_n x^n$ *with* $R > 0$, *then* $a_n = \frac{f^{(n)}(0)}{n!}$. *In particular, if* $f(x) = \sum a_n x^n = \sum b_n x^n$ *on some nonempty interval* $(-r, r)$, *then* $a_n = b_n$ *for all* n.

[**7.10**] EXAMPLE For $-1 < x < 1$,

$$\frac{d}{dx}\ln(1+x) = \frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots .$$

The power series $P(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ converges in $(-1, 1)$ and is such that its term by term derivative is the series for $\frac{1}{1+x}$. Thus, $P'(x) = \frac{d}{dx}\ln(1+x)$. Since $P(0) = 0 = \ln(1+0)$, we get

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots .$$

[**7.11**] EXERCISE Assume the Maclaurin's series for $e^x, \sin x$ and $\cos x$, and verify the following:

$$\frac{d}{dx}e^x = e^x,\ \frac{d}{dx}\sin x = \cos x, \text{ and } \frac{d}{dx}\cos x = -\sin x \text{ on } \mathbb{R}.$$

End of Differentiation