Syllabus

Single variable calculus: Convergence of sequences and series of real numbers; Continuity of functions; Differentiability, Rolle's theorem, mean value theorem, Taylor's theorem; Power series; Riemann integration, fundamental theorem of calculus, improper integrals; Application to length, area, volume and surface area of revolution.

Multivariable calculus: Vector functions of one variable - continuity and differentiability; Scalar valued functions of several variables, continuity, partial derivatives, directional derivatives, gradient, differentiability, chain rule; Tangent planes and normals, maxima and minima, Lagrange multiplier method; Repeated and multiple integrals with applications to volume, surface area; Change of variables; Vector fields, line and surface integrals; Greens, Gauss and Stokes theorems and their applications.

Texts: G. B. Thomas, Jr. and R. L. Finney, Calculus and Analytic Geometry, Pearson India, 9th Edition, 2006

References: R. G. Bartle and D. R. Sherbert, Introduction to Real Analysis, Wiley India, 4th Edition, 2014.

S. R. Ghorpade and B. V. Limaye, An Introduction to Calculus and Real Analysis, Springer India, 2006.

T. M. Apostol, Calculus, Volume-2, Wiley India, 2003.

J. E. Marsden, A. J. Tromba and A. Weinstein, Basic Multivariable Calculus, Springer India, 2002.

Instructions

- Till continuity, the course coverage will happen in the following style. The number of 55m lectures shows the number of lectures we would take in a normal situation.

Topic	No of 55m lectures	videos	total video time
Real numbers	2	Realnumbers1,2,3	94m
Sequences	3	Sequences1,2,3	140m
Series	3	Series1,2,3	148m
Limit/continuity	3	Limitcontinuity1,2,3,4	174m

- Accordingly videos for the next week (or three lectures) will be uploaded on Monday mornings, 10 am.

- For the initial part we will also provide you some lecture notes. That contains some extra discussions which will not be covered in the classes. They are marked 'self-study'. If you are not already aware of these topics, then go through them. We expect you to be aware of the statements here, at the least.

- The notes contains many exercises. There are some $*$-marked questions. They are challenging in nature (according to me). Many of the exercises are guided exercises, which contain steps to solve the problem.

- It is possible that there are some errors in the notes. You may observe some 'slip of tongue' in the videos. Please try to make the best sense out of them and proceed. You can report the errors found in the notes to me at 'pati@iitg.ac.in'.

- There are some concepts like 'interior points', 'closure' and 'boundary points' in the notes. At present you may avoid them. But learn till the limit points.

- As time progresses, we expect you to get adjusted to a text/reference book (the choice depends on you; for me Bartle Sherbert is good enough) and there will not be any extra topics that we would like you to be aware of. Hence there may not be any notes available for the later part of the course.

- There will be a quiz based on the topics till continuity (included), on 12 Dec, Saturday, Time:10-10:50 am. You will get 10m extra for submission. The portal will automatically close the submission at 11:00. No complains will be entertained. The quiz will be of a total mark 25. Keep checking the course web-page for other information.

Chapter 1

Understanding ‘for each’ and ‘there exists’ statements

After this chapter, I should be able to answer the following.

1. Tell the negation of ‘if there is a living being on Mars, then it is a human being’.
2. Tell the negation of ‘Each living being on Mars is a human being’.
3. Suppose that there are no living beings on Mars. Tell the truth value of the previous statements.
4. Suppose that there are some birds living on Mars and no other living beings. Tell the truth value of the first two statements.

This is a very short chapter to familiarize the readers with the statements involving the symbols $\forall$, $\exists$ and their uses.

1. Consider the statement P: ‘$\forall x \in A$, x is a zyx’. We read ‘$\forall$’ as ‘for each’ or at times ‘for all’ or ‘for every’. So the meaning of the statement P is ‘each element in A is a zyx’.
   (a) The statement P is considered true, if each element of A is a zyx.
   (b) The statement P is considered false, if there is an element of A which is not a zyx.
   (c) Hence, the statement P is true for $A = \emptyset$.
2. Consider the statement Q: ‘$\exists x \in A$ such that x is a zyx’. We read ‘$\exists$’ as ‘there exists’ or ‘there is’. So the meaning of the statement Q is ‘there is an element in A which is a zyx’.
   (a) The statement Q is considered true, if at least one element of A is a zyx.
   (b) The statement Q is considered false, if no element of A is a zyx.
   (c) Thus, the statement Q is false for $A = \emptyset$.
3. Notice the style of writing. We always write ‘for each x, (comma) something’ and ‘there exists x such that (in place of comma) something’. Other variations in English exist. For example, ‘$\forall x \in A$, $x \in B$’ may also be written as ‘$\forall x \in A$, we have $x \in B$’.
4. Sometimes we see a statement like ‘if $x \in A$, then $x \in B$’. This actually means ‘$\forall x \in A$, $x \in B$’.
5. The negation of ‘$\forall x \in A$, $x \in B$’ is the statement ‘$\exists x \in A$ such that $x \notin B$.
6. The negation of ‘$\exists x \in A$ such that $x \in B$’ is the statement ‘$\forall x \in A$, $x \notin B$.

Chapter 2

The set of real numbers

After this chapter, I should be able to answer the following.

1. Is it possible to create a list $x_1, x_2, \ldots$ that contains all the real numbers?
2. Can I express the set of real numbers as a union of pairwise disjoint sets (it means a collection of sets where each pair is disjoint) containing five elements each?
3. Argue that, if $x^9 < 3$, there there is a natural number n such that $(x + \frac{1}{n})^9 < 3$.
4. Let $x = \sqrt{2}^{\sqrt{3}^{\sqrt{5}}}$. Supply a rational number y such that $|x - y| < 10^{-100}$. Also supply a z such that $|y - z| < 10^{-1000}$.
5. Let $x = \sqrt{2}^{\sqrt{3}^{\sqrt{5}}}$. Does there exist integers m, n such that $|x - (m + n\sqrt{2})| < 10^{-100}$?
6. Can I find a bijection from $[0, 1)$ to $(0, 1)$?
7. Let S be a nonempty set. Can I find a bijection from S to the power set $P(S)$?
8. A real number in $(0, 1)$ has a nonterminating nonrecurring decimal representation. Must its representation in base-2 be nonterminating and nonrecurring?
9. Can I express $(0, 1)$ as a union of pairwise disjoint nontrivial closed intervals?

2.1 Real numbers

We all have some idea about the real numbers. Bellow we give a formal definition.

Definition 2.1.1 **(Real numbers)**It is a set F of symbols with two binary operations $+, *$ and a binary relation $\leq$ defined on it, which satisfy the following axioms.

1. **(Closure)** $a + b \in F$ and $a * b \in F$ for each $a, b \in F$.
2. **(Cssociative)** $a + (b + c) = (a + b) + c$ and $a * (b * c) = (a * b) * c$for each $a, b, c \in F$.
3. **(Commutative)** $a + b = b + a$ and $a * b = b * a$ for each $a, b \in F$.
4. **(Additive identity)** There is an element $0 \in F$ such that $a + 0 = a$ for each $a \in F$.
5. **(Additive inverse)** For each $a \in F$, there is an element $b \in F$ such that $a + b = 0$.

6. **(Multiplicative identity)** There is an element $1 \in F$, $1 \neq 0$, such that $a * 1 = a$ for each $a \in F$.

7. **(Multiplicative inverse)** For each $a \in F$, $a \neq 0$, there is an element $b \in F$ such that $a * b = 1$.

8. **(distributive)** $a * (b + c) = a * b + a * c$ for each $a, b, c \in F$.

9. **(Trichotomy)** For each $a, b \in F$, exactly one of $a \lneq b$, $a = b$, $a \gneq b$ holds.

10. **(Transitive)** '$a \leq b$ and $b \leq c$' imply $a \leq c$ for each a, b, c.

11. **(Positivity)** For each $a, b, c \in F$ we have $a \leq b$ implies $a + c \leq b + c$ and $a, b \geq 0$ implies $a * b \geq 0$.

12. **(Special axiom)** For each nonempty subset $A, B \subseteq F$ with the property that $a \leq b$ for each $a \in A$ and $b \in B$, there is an element $m \in F$ such that $a \leq m \leq b$ for each $a \in A$ and $b \in B$.

It is known that any two such sets have the same structure. Thus, there is only one such set. It is called the set of **real numbers**. We denote it by $\mathbb{R}$. Normally we write ab in place of $a * b$.

Below we are listing down some basic properties of real numbers that can be derived from the definition. Note that there are many other properties that can also be derived but we have not listed them here, as our scope does not permit us.

LEMMA 2.1.2 *a) The zero element in $\mathbb{R}$ is unique.*

b) The additive inverse of an element a in $\mathbb{R}$ is unique. It is denoted by $-a$.

c) We have $-0 = 0$.

d) The multiplicative identity element in $\mathbb{R}$ is unique.

e) The multiplicative inverse of a nonzero element a in $\mathbb{R}$ is unique. It is denoted by a^{-1} or $\frac{1}{a}$.

f) We have $1^{-1} = 1$.

g) We have $x0 = 0$ for each x.

h) If $ab = 0$ then either $a = 0$ or $b = 0$.

i) We have $-a = (-1)a$ and $-(-a) = a$ for each a. So $(-1)(-1) = -(-1) = 1$.

j) We have $(a^{-1})^{-1} = a$ for each nonzero a.

k) If $a \geq 0$, then $-a \leq 0$.

l) We have $1 > 0$. (This means $1 \geq 0$ and $1 \neq 0$.)

m) If $a > 0$, then $a^{-1} > 0$.

n) If $a \geq b$ and $c \geq 0$, then $ac \geq bc$.

Proof. a) Suppose there are two elements 0_1 and 0_2 which satisfy the property of the additive identity. Then $0_1 = 0_1 + 0_2 = 0_2$.

b) Let b and c be two elements which satisfy the property of an additive inverse of a. Then $b = b + 0 = b + (a + c) = (b + a) + c = 0 + c = c$.

g) We have $x0 = x(0 + 0) = x0 + x0$. So $x0 = 0$.

h) If $a = 0$, then we have nothing to show. Let $a \neq 0$. Then a^{-1} exists. So $0 = a^{-1}0 = a^{-1}(ab) = (a^{-1}a)b = 1b = b$.

i) Note that $0 = [1 + (-1)]a = 1a + (-1)a = a + (-1)a$. So $-a = -a + 0 = -a + a + (-1)a = 0 + (-1)a = (-1)a$.

j) Note that $a^{-1}a = 1 = a^{-1}(a^{-1})^{-1}$. Multiply by a from left.

k) Add $-a$ to both sides.

l) If possible let $1 \leq 0$. Then $-1 \geq 0$. So $1 = (-1)^2 \geq 0$. So $1 = 0$. A contradiction.

m) Suppose that $a^{-1} \leq 0$. So $-a^{-1} \geq 0$. So $a(-a^{-1}) \geq 0$. But $a(-a^{-1}) = a(-1)a^{-1} = -1$ and we know $-1 < 0$.

n) Let $a \geq b$. Adding $-b$ both sides, we get $a - b \geq 0$. As $c \geq 0$, we get $c(a - b) \geq 0$. So $ca - cb \geq 0$. So $ca \geq cb$. ■

REMARK 2.1.3

1. Once we know $1 > 0$, we inductively define the natural numbers $\mathbb{N}$ as $\{1, 1+1, 1+1+1, \ldots\}$. By our definition this is a subset of $\mathbb{R}$. Of course, we have developed the symbols $1, 2, 3, \ldots$ for them.

2. Their negatives ($-a$ is called the 'negative of a') also exist in $\mathbb{R}$. The set $\mathbb{Z}$ of integers is defined as $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$. This is also a subset of $\mathbb{R}$.

3. When a is an integer and b is a nonzero natural number, then $\frac{a}{b}$ is defined as ab^{-1}. The set $\mathbb{Q}$ of rational numbers is defined as $\{\frac{a}{b} \mid a \in \mathbb{Z}, b \in \mathbb{N}, \gcd(a, b) = 1\}$. This is also a subset of $\mathbb{R}$.

4. The set $\mathbb{R}$ has more numbers. Any number in $\mathbb{R}$ which is not rational, is called an **irrational** number. We will very soon show that there are many irrational numbers.

DEFINITION 2.1.4

1. If $a < b$ are two real numbers, by (a, b) we mean the set $\{x \in \mathbb{R} \mid a < x < b\}$. It is read as the **open interval** (a, b).

2. If $a \leq b$ are two real numbers, by $[a, b]$ we mean the set $\{x \in \mathbb{R} \mid a \leq x \leq b\}$. It is read as the **closed interval** $[a, b]$.

3. The interval $[a, a] = \{a\}$ is called a **trivial interval**.

4. The sets $[a, b)$ and $(a, b]$ are defined similarly.

5. By $(-\infty, a)$ we mean the set $\{x \in \mathbb{R} \mid x < a\}$.

6. Intervals like $(-\infty, a]$, (a, ∞) and $[a, \infty)$ are defined similarly.

7. We also denote $\mathbb{R}$ by $(-\infty, \infty)$.

8. We call all these above type of subsets **intervals**.

2.1.1 Exercises

EXERCISE 2.1.5

1. *Show that there is no positive rational number x satisfying $x^2 = 2$. (We have not used the symbol $\sqrt{2}$, as we are yet to show that there are two elements in $\mathbb{R}$ which satisfy $x^2 = 2$. One of them is negative. The other one positive, and we shall denote that by $\sqrt{2}$.)*

2. *More generally, let n be a natural number which not a square (of a natural number). Show that, there is no positive rational number x satisfying $x^2 = n$. (We will see later that, there is a unique positive real number x satisfying $x^2 = n$.)*

3. *Even more generally, let n and k be natural numbers and suppose that there is no natural number m for which $n = m^k$. Show that Show that, there is no positive rational number x satisfying $x^k = n$.*

EXERCISE 2.1.6 *Prove Gauss theorem: Let* $a_1, \cdots, a_n \in \mathbb{Z}$ *and* $r = \frac{p}{q}$, $\gcd(p,q) = 1$, *be a rational root of* $x^n + a_1x^{n-1} + \cdots + a_n = 0$. *Then* $q = 1$.

EXERCISE 2.1.7

1. *Show that the sum and product of rational numbers is a rational number.*
2. *Show that the sum and product of irrational numbers could be a rational number.*

EXERCISE* 2.1.8 *Let* $S \subseteq \mathbb{R}$ *be nonempty and suppose that for each* $a, b \in S$, *all points in between* a *and* b *are also in* S. *Show that* S *is an interval. (It is challenging to prove this at this stage. However, one can give a simple proof later using the concepts of a least upper bound and a greatest lower bound.)*

2.2 The concept of bounds

DEFINITION 2.2.1 Let $A \subseteq \mathbb{R}$ and $b \in \mathbb{R}$. We say b is an **upper bound** of A, if $\forall x \in A, x \leq b$. That is each element of A is less than or equal to b.

EXAMPLE 2.2.2

1. Take $A = \{1, 2, 3\}$. Then 5 is an upper bound. So are $3, 4$.
2. If b is an upper bound of A and $c \geq b$, then c is also an upper bound of A.
3. For $A = \{1, 2, 3\}$, the number 2.9 is not an upper bound, as the set has an element $3 > 2.9$.
4. For $A = \emptyset$, the number -2 is an upper bound. This is because, if we want to say that -2 is not an upper bound, then we have to find an element of A which is larger than -2.

DEFINITION 2.2.3 The term **lower bound** is defined similarly. An element $a \in A$ is called a **maximum** of A if it is an upper bound of A. The term **minimum** is defined similarly.

EXAMPLE 2.2.4

1. The set $A = \{1, 2, 3\}$ has a maximum 3.
2. Let $A = \mathbb{Q}_-$ be the set of negative (zero not allowed) rational numbers. It does not have a maximum. The number 0 is an upper bound of A, but it does not belong to A.
3. An empty set cannot have a maximum. This is so, as a maximum must be an element of the set and empty set has no elements.

EXERCISE 2.2.5 *Show that if* a *and* b *are two maximums of* A, *then* $a = b$. *(Thus, we write 'the' maximum element of a set', when it exists.) Similarly, a set can have at most one minimum.*

LEMMA 2.2.6 *The set* $\mathbb{N}$ *does not have a maximum.*

Proof. On the contrary, suppose that n is the maximum of $\mathbb{N}$. Then by definition, $n \in \mathbb{N}$ and hence $n+1 \in \mathbb{N}$. But as n is a maximum of $\mathbb{N}$, it must be an upper bound of $\mathbb{N}$. In particular, we should have $n \geq n + 1$, as $n + 1 \in \mathbb{N}$. But that is not true. ■

LEMMA 2.2.7 **(Well ordering principle)** *Let* A *be a nonempty subset of natural numbers. Then* A *has a minimum.*

Proof. We first prove by induction that if $|A| = n$, then A has a minimum.

The statement is true for $n = 1$. Assume that the statement is true for $n = 1, 2, \ldots, k$. Now let $A = \{a_1, a_2, \ldots, a_{k+1}\} \subseteq \mathbb{N}$. Then $B = \{a_1, \ldots, a_k\}$ has size k and by induction hypothesis it has a maximum. Let it be a_p. Compare a_p and a_{k+1}. If $a_p \leq a_{k+1}$, then $a_p \leq a_i$ for each $i = 1, \ldots, k + 1$. If

$a_p > a_{k+1}$, then $a_{k+1} \le a_i$ for each $i = 1, \ldots, k+1$. In either case, A has a minimum. So the statement is true for $n = k+1$ and hence it is true for each n.

Next, let A be any nonempty subset of natural numbers. As $A \neq \emptyset$, it contains a natural number, say, k. Take $B = A \cap J_k$, where $\boldsymbol{J_k} := \{1, 2, \ldots, k\}$. Then B is nonempty and $|B| \le k$. Hence, B has a minimum, say b. Hence, $b \in B \subseteq A$, b is less than or equal to each element of B, and $b \le k$. As the remaining elements of A are more than k, we see that $b \le a$ for each $a \in A$. As $b \in A$, it is the minimum of A. ■

DEFINITION 2.2.8 We say $A \subseteq \mathbb{R}$ is **bounded above** in $\mathbb{R}$, if an upper bound of A exists in $\mathbb{R}$. The term **bounded below** is defined similarly. A set which is both bounded below and bounded above is called **bounded**.

EXAMPLE 2.2.9

1. The set $A = \{1, 2, 3\}$ is bounded in $\mathbb{R}$. Write three upper bounds and three lower bounds.
2. The set $\mathbb{N}$ is bounded below in $\mathbb{R}$, as -1 is a lower bound.
3. The set $A = \emptyset$ is bounded above by -2. We have already argued this. In fact each real number is an upper bound of $\emptyset$. Similarly, each real number is a lower bound of $\emptyset$.

LEMMA 2.2.10 *The set $\mathbb{N}$ is not bounded above in $\mathbb{Q}$. (The meaning of this sentence is similar to the definition, if we use $\mathbb{Q}$ in place of $\mathbb{R}$.)*

Proof. Suppose that $\frac{p}{q}$ is an upper bound of $\mathbb{N}$ in $\mathbb{Q}$. Then $\frac{p}{q} \ge 1$. So it is positive number. So $p \in \mathbb{N}$. In that case $\frac{p}{q} \le p < p+1 \in \mathbb{N}$. But this means that $\frac{p}{q}$ cannot be an upper bound. A contradiction. ■

2.3 Least upper bound

- Is the set $A = (0, 1)$ bounded above in $X = (-5, 1)$? No, as there is no element in X which is an upper bound of A.
- Is $A = (0, 1)$ bounded above in $X = (-5, 10)$? Yes. The number $5 \in X$ is an upper bound of A.
- So, a set A may not have an upper bound in X but if take a superset Y of X, then A may have an upper bound in Y.
- We know that $\mathbb{N}$ is not bounded above in $\mathbb{Q}$. But as $\mathbb{R}$ is a superset of $\mathbb{Q}$ (in fact contains a lot of extra elements, we will show this), we may wonder whether $\mathbb{R}$ might contain an upper bound of $\mathbb{N}$. We shall answer this a little later.

Recall that a set A is bounded above in X means that the set $U = \{\text{upper bounds of } A \text{ in } X\} \neq \emptyset$.

DEFINITION 2.3.1

1. Let $\emptyset \neq A \subseteq X$ be bounded above. Put $U = \{\text{upper bounds of } A \text{ in } X\}$. <u>If U has a minimum u</u>, then u is called the **least upper bound** (also called **supremum**) of A in X. Notation: $\mathsf{lub}\, A$, also $\sup A$.
2. The term **greatest lower bound** (also called **infimum**) is defined similarly. Notation: $\mathsf{glb}\, A$, also $\inf A$.
3. When the set X is not mentioned, our understanding will be that $X = \mathbb{R}$.

EXAMPLE 2.3.2

1. Take $X = \mathbb{Q}$ and $A = \{0, 1, 2\}$. Then $\sup A = 2$ and $\inf A = 0$.

2. Take $A = (-\infty, 1)$. (Here $X = \mathbb{R}$.) Then $\sup A = 1$ and $\inf A$ does not exist in $\mathbb{R}$.

3. Notice that 'supremum' is a generalization of 'maximum'. That is, if S has a maximum s, then $s = \sup S$.

Lemma 2.3.3 **(Another meaning of lub)** *Let $\emptyset \neq S \subseteq \mathbb{R}$ be bounded above. TFAE (The following are equivalent.)*

a) The number $s = \mathsf{lub}\, S$.

b) The number s is an upper bound of S and for each $\epsilon > 0$ there is an element $y \in S$ such that $y > s - \epsilon$.

Proof. a)$\Rightarrow$b). Suppose that $s = \mathsf{lub}\, S$. Then s is an upper bound of S and it is the least. Let $\epsilon > 0$. Then the number $s - \epsilon$ is not an upper bound of S. That is, $\exists y \in S$ such that $y > s - \epsilon$.

b)$\Rightarrow$a). Let s be an upper bound of S. Suppose that for each $\epsilon > 0$, $\exists y \in S$ such that $y > s - \epsilon$. Thus no $s - \epsilon$ is an upper bound of S. So, as s is given to be an upper bound, we see that s is the least of all upper bounds. ∎

Theorem 2.3.4 **(Lub property of $\mathbb{R}$)** *Let $\emptyset \neq A \subseteq \mathbb{R}$ be bounded above in $\mathbb{R}$. Then* $\sup A$ *exists in* $\mathbb{R}$.

Proof. Let U be the set of upper bounds of A in $\mathbb{R}$. As A is bounded above, U is nonempty. So, we have two sets A and U such that $a \leq u$, for each $a \in A$ and $u \in U$. Then by the special axiom mentioned in the definition, we have an element $m \in \mathbb{R}$ such that $a \leq m \leq u$ for each $a \in A$ and $u \in U$. So m is an upper bound of A and m is a lower bound of U. But as m is an upper bound of A, we see that $m \in U$. Thus $m = \min U$. Hence, m being the least of the upper bound set, we have $m = \sup A$. ∎

Corollary 2.3.5 **(Glb property of $\mathbb{R}$)** *Let $\emptyset \neq A \subseteq \mathbb{R}$ be bounded below in $\mathbb{R}$. Then* $\inf A$ *exists in* $\mathbb{R}$.

Well ordering principle: alternate proof using a very useful technique

We have already proved the well ordering principle 'every nonempty subset of $\mathbb{N}$ contains a minimum'. Here we supply an alternate proof.

Let $\emptyset \neq A \subseteq \mathbb{N}$. As A is bounded below in $\mathbb{R}$, let $k = \mathsf{glb}\, A$. As $k + \frac{1}{n}$ is not a lower bound of A, $\exists k_n \in A$, such that $k + \frac{1}{n} > k_n \geq k$. Since, the integers $k_2, k_3, \cdots, k_n, \cdots \in [k, k + \frac{1}{2})$, we get $k_2 = k_3 = \cdots = m \in A$ (say). So, $|m - k| = |k_n - k| \leq \frac{1}{n}$, $\forall n$. Thus $k = m \in S$ and so $k = \min A$.

Recall that we knew that $\mathbb{N}$ is not bounded above in $\mathbb{Q}$. But as $\mathbb{R}$ was a superset of $\mathbb{Q}$, we did not know whether the same is true in $\mathbb{R}$.

Theorem 2.3.6 *The set $\mathbb{N}$ is not bounded above in $\mathbb{R}$.*

Proof. Suppose $\mathbb{N}$ is bounded above. Using lub property, let $x = \sup \mathbb{N}$. So $x - \frac{1}{2}$ is not an upper bound of $\mathbb{N}$. So $\exists n_0 \in \mathbb{N}$ such that $x - \frac{1}{2} < n_0$. But then $x < x + \frac{1}{2} < n_0 + 1$, a natural number. So x cannot be an upper bound of $\mathbb{N}$, a contradiction. ∎

Theorem 2.3.7 **(Archimedean property)** *Let $a > 0$, $b \in \mathbb{R}$. Then $\exists\, n \in \mathbb{N}$ such that $na > b$.*

Proof. Follows from the fact that the set $\{na \mid n \in \mathbb{N}\}$ is not bounded above in $\mathbb{R}$. ∎

Exercise 2.3.8 *If a number $x \geq 0$ satisfies $x \leq \frac{1}{n}$ for each n, then $x =$ ___.*

Example 2.3.9 **(Application of Archimedean property)** Take $A = \{x > 0 \mid x^6 < 3\}$. Does A have a maximum? No. If it does, let it be a. So $a^6 < 3$ and $\frac{3-a^6}{(1+a)^6} > 0$. By Archimedean principle $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{3-a^6}{(1+a)^6}$. We have

$$\begin{aligned} \left(a+\frac{1}{n}\right)^6 &= a^6 + \binom{6}{1}a^5\frac{1}{n} + \binom{6}{2}a^4\frac{1}{n^2} + \cdots + \binom{6}{6}\frac{1}{n^6} \\ &\leq a^6 + \binom{6}{1}a^5\frac{1}{n} + \binom{6}{2}a^4\frac{1}{n} + \cdots + \binom{6}{6}\frac{1}{n} \\ &< a^6 + \frac{1}{n}\left[a^6 + \binom{6}{1}a^5 + \binom{6}{2}a^4 + \cdots + \binom{6}{6}\right] \\ &= a^6 + \frac{1}{n}(1+a)^6 < 3. \end{aligned}$$

Thus $a + \frac{1}{n} \in A$, a contradiction.

Exercise 2.3.10 **(Existence of qth root)** *Take $A = \{x > 0 \mid x^6 < 3\}$. We know that A does not have a maximum.*

a) *Take $B = \{x > 0 \mid x^6 > 3\}$. Show that B does not contain a minimum.*

b) *Are both A and B nonempty? Is $a < b$ for each $a \in A$ and $b \in B$?*

c) *By the special axiom, there exists $t \in \mathbb{R}$ such that $a \leq t \leq b$ for each $a \in A$ and $b \in B$. Should $t^6 < 3$? Should $t^6 > 3$?*

d) *Is $t^6 = 3$?*

e) *Prove the general statement. Let $a \in \mathbb{R}_+$ and $q \in \mathbb{N}$ be fixed. Then there exists a unique $b \in \mathbb{R}_+$ such that $b^q = a$. This number $b > 0$ is denoted by $\boldsymbol{a^{\frac{1}{q}}}$ or $\sqrt[q]{a}$.*

Example 2.3.11 **(Existence of irrational numbers)** We know that there is a positive real number x such that $x^2 = 2$. We also know that, this x is not a rational number. Hence this x, which is denoted by $\sqrt{2}$ is an irrational number. It is easy to show that if $n \in \mathbb{N}$ is not a square then $\sqrt{n}$ is an irrational number. Also if a is a nonzero rational number and b is an irrational number, then ab is an irrational number. In that way, one can see that there are many irrational numbers in $\mathbb{R}$.

Theorem 2.3.12 **(Greatest integer function)** *Let $a \in \mathbb{R}$ be fixed. Then there exists a unique integer k such that $k \leq a < k+1$.*

Proof. Consider $S = \{n \in \mathbb{Z} \mid n \leq a\}$. Then $S \neq \emptyset$ (otherwise the set of integers is bounded below by a, which is not true) and bounded above by a. Put $k = \sup S$. We can argue (in way similar to that in the well ordering principle) that k is an integer. As k is the lub and a is an upper bound of S, we get $k \leq a$. Also $a < k+1$ because if $k+1 \leq a$, then $k+1 \in S$, and so k cannot be the lub of S.

To prove the uniqueness of k, note that any integer $l < k$ satisfies $l+1 \leq k \leq a$ and any integer $l > k$ satisfies $l \geq k+1 > a$. ∎

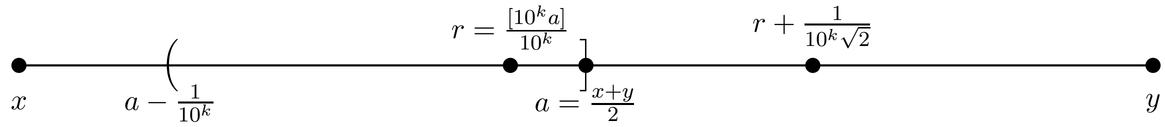
Definition 2.3.13 The integer k in the previous theorem is called the **greatest integer less than or equal to $\boldsymbol{a}$**. It is denoted by $[\boldsymbol{a}]$.

Remark 2.3.14 The above theorem essentially means that $\mathbb{R} = \bigcup\limits_{z\in\mathbb{Z}} [z, z+1)$.

Theorem 2.3.15 **(Density theorem)** *There is a rational and an irrational in any interval (x, y).*

Proof. First we need to observe something. Take $a \in \mathbb{R}$. Then $[10^5 a] \le 10^5 a < [10^5 a] + 1$. So $10^5 a - 1 < [10^5 a] \le 10^5 a$. So $\frac{[10^5 a]}{10^5}$ is a rational number in the interval $(a - \frac{1}{10^5}, a]$.

Now we start the proof. Consider an interval (x, y). By Archimedean property, there is a natural number k such that $\frac{1}{10^k} < \frac{y-x}{2}$. Taking $a = \frac{x+y}{2}$, we get $(a - \frac{1}{10^k}, a] \subseteq (x, y)$.



As $\frac{[10^k a]}{10^k} \in (a - \frac{1}{10^k}, a]$, we see that $\frac{[10^k a]}{10^k} + \frac{1}{10^k\sqrt{2}} \in (x, y)$, which is an irrational number. ∎

Theorem 2.3.16 **(Nested interval theorem)** *Let* $[a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3] \supseteq \cdots$. *Then* $\bigcap\limits_{n=1}^{\infty} [a_n, b_n] \neq \emptyset$.

Proof. Let $L = \{a_1, a_2, \cdots\}$ be the left endpoints and $R = \{b_1, b_2, \cdots\}$ be the right endpoints. Then $a_n \le a_{n+m} \le b_{n+m} \le b_m$, for each $n, m \in \mathbb{N}$. Now applying the special property of $\mathbb{R}$ (in the definition), we see that, there exists a real number z such that $a_i \le z \le b_j$ for each $i, j \in \mathbb{N}$. In particular, $z \in [a_n, b_n]$ for each $n \in \mathbb{N}$. Hence z is in the intersection. ∎

Remark 2.3.17 In the previous proof, we could proceed to show that the intersection is actually a closed interval. We show that below.

As L is bounded above by each element of R, we see that $\mathsf{lub}\, L$ exists in $\mathbb{R}$. Let it be a. Similarly, put $b = \mathsf{glb}\, R$.

Note that if $z \in [a, b]$, then $a_n \le a \le z \le b \le b_n$ for each $n \in \mathbb{N}$. So $[a, b]$ is contained in the intersection.

If z is a number less than $a = \mathsf{lub}\, L$, then there exists $a_n \in L$ such that $a_n > z$. In that case $z \notin [a_n, b_n]$. So it will not be in the intersection.

Similarly, if $z > b$, then z is not in the intersection. Hence, the intersection is precisely $[a, b]$, which is a closed interval.

2.3.1 Exercises

Exercise 2.3.18 **(Irrationals)**

1. *Consider the equation* $p(x) := x^3 - 5 = 0$.

   (a) *Is there an integer* $z \le 1$ *which is a root? Is there an integer* $z \ge 2$ *which is a root?*

   (b) *Are* $1, 2$ *roots of this equation? Is there a rational root of this equation?*

   (c) *Should there be a root of this equation in* $\mathbb{R}$*? What do you conclude about it?*

2. *Show that* $\sqrt{n}$ *is irrational when* $n \in \mathbb{N}$ *is not a perfect square.*

3. *Generalize it for* k*th roots of natural numbers.*

4. *Is* $\sqrt[3]{\frac{5-2\sqrt{3}}{7}}$ *a rational number?*

Exercise 2.3.19 **(least upper bounds)**

1. $\mathsf{lub}\{1,2,3\} = \_\_$; $\mathsf{lub}(-\mathbb{N}) = \_\_$; $\mathsf{lub}\,\mathbb{N} = \_\_$; $\mathsf{lub}(-1,0) = \_\_$.
2. *Let A and B be nonempty and bounded sets such that $A \cap B \neq \emptyset$.*
   (a) *Assuming $A \subseteq B$, give a relation between $\mathsf{lub}\,A$ and $\mathsf{lub}\,B$.*
   (b) *Order lub's of $A \cup B$, A and $A \cap B$.*
   (c) *Express $\mathsf{lub}(A \cup B)$ using $\mathsf{lub}\,A$ and $\mathsf{lub}\,B$.*
   (d) *Relate $\mathsf{lub}(A + B)$, $\mathsf{lub}(A)$ and $\mathsf{lub}(B)$.*
   (e) *Is there is relationship between lub of A and lub of $10A$?*
3. *Ask the above questions for glb's.*
4. *Relate lub and glb of A and $-A$ (this is the set of negatives of elements of A).*
5. *Let $A \subseteq \mathbb{R}_+$ be bounded. Relate lub and glb of A with those of $\frac{1}{A}$ (this is the set of reciprocals of elements of A).*
6. *Find the lub and glb of the following sets.*
   (a) $\left\{1 + \frac{(-1)^n}{n} \mid n \in \mathbb{N}\right\}$
   (b) $\left\{\frac{3n+2}{2n+1} \mid n \in \mathbb{N}\right\}$
   (c) $\left\{\frac{2n^2+1}{3n+2} \mid n \in \mathbb{N}\right\}$
   (d) $\left\{x \in \mathbb{R} \mid 3x^2 - 10x + 3 < 0\right\}$.
7. *Give 5 pairwise disjoint subsets of irrational numbers whose lub is 1 and glb is 0.*

Exercise 2.3.20 **($\mathbb{Q}$ does not have lub property)** *Take $A = \{x \in \mathbb{Q} \mid x > 0, x^2 < 2\}$. Suppose that $\mathsf{lub}\,A$ exists in $\mathbb{Q}$ and let it be u.*

a) *Argue that A does not have a maximum. Hence $u \notin A$.*

b) *We already know from high school that $u^2 \neq 2$.*

c) *Thus $u \in B = \{x \in \mathbb{Q}_+ \mid x^2 > 2\}$. Argue that each element of B is an upper bound of A and that B does not have a minimum. Arrive at a contradiction.*

Hence conclude that $\mathsf{lub}\,A$ cannot exist in $\mathbb{Q}$.

Exercise 2.3.21 **(Not bounded)** *Show that the set $\{10^n \mid n \in \mathbb{N}\}$ is not bounded above in $\mathbb{R}$. To show this suppose that this set is bounded above with a as an upper bound. Then argue that a becomes an upper bound for $\mathbb{N}$. Arrive at a contradiction.*

Exercise 2.3.22 **(Irrationals)** *Consider an interval (a,b). Write expressions for many rationals and irrationals in it using the greatest integer function, a, b and k for a natural number.*

Exercise* 2.3.23 **(Density of $\{m\alpha + n \mid m \in \mathbb{Z}, n \in \mathbb{N}\}$, when α is irrational)** *Let α be an irrational number. Fix an interval (a,b). Follow the following steps to show that it contains a number of the form $m\alpha + n$, where $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. Without loss, assume $a > 0$ (otherwise add a positive integer).*

1. *Take a positive number $r < b - a$. Show that (a,b) contains an integer multiple of r.*

2. *Consider an interval (x, y). Choose a natural number $k > 5$ such that $y - x > \frac{1}{k}$ and $\frac{1}{k} < \alpha$. Consider the numbers $x_i = i\alpha - [i\alpha]$, $i = 1, 2, \cdots k$. Show that $\{x_0 = 0, x_1, \cdots, x_k, x_{k+1} = 1\}$ is a set of $k+2$ distinct numbers in in $[0, 1]$.*

3. *Conclude that there is a number $r_0 = m_0\alpha + n_0$ such that $0 < m_0\alpha + n_0 < \frac{1}{k}$, where $m_0, n_0 \in \mathbb{Z}$.*

4. *If the n_0 in $r_0 = m_0\alpha + n_0$ is positive, we have nothing to argue. Otherwise, noting that $n_0 \neq 0$, let $r = m_0\alpha - k_0$, where $k_0 \in \mathbb{N}$. Now consider all the multiples $r_0, 2r_0, \ldots, ir_0$ below 5. Notice the distance from ir_0 to 5. Is $0 < 5 - ir_0 < r_0 < \frac{1}{k}$? Does it have the form $m\alpha + n$, where $n \in \mathbb{N}$?*

5. *Conclude that there is a number $r_0 = m_0\alpha + n_0$ such that $0 < m_0\alpha + n_0 < \frac{1}{k}$, where $m_0 \in \mathbb{Z}$ and $n_0 \in \mathbb{N}$.*

6. *Consider integer multiples of r_0 to conclude that the interval (x, y) contains a number of the form $m\alpha + n$, where $m \in \mathbb{Z}$, $n \in \mathbb{N}$.*

EXERCISE* 2.3.24 **(Two useful inequalities)**

1. **(Cauchy-Bunyakovskii-Schwartz)** *Let $a_i, b_i \in \mathbb{R}$, $i = 1, \cdots, n$. Show, by induction that $(\sum_{i=1}^n a_i b_i)^2 \leq (\sum_{i=1}^n a_i^2)(\sum_{i=1}^n b_i^2)$.*

2. **(Bernoulli's inequality)** *Take $a_1 > -1, \cdots, a_n > -1$ with the same sign. Prove by induction that $\prod\limits_{i=1}^{n}(1 + a_i) \geq 1 + \sum\limits_{i=1}^{n} a_i$. In particular, show that for $x > -1$, we have $(1 + x)^n \geq 1 + nx$.*

EXERCISE 2.3.25 **(Verify nested interval theorem)** *Determine the following sets.*
a) $\bigcap\limits_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n})$ b) $\bigcap\limits_{n=1}^{\infty}[-\frac{1}{n}, \frac{1}{n}]$ c) $\bigcap\limits_{n=1}^{\infty}(1 - \frac{2}{n}, 2 + \frac{1}{n}]$ d) $\bigcap\limits_{n=1}^{\infty}(0, \frac{1}{n}]$.

2.4 Bijections (self study)

2.4.1 Relation, function, bijection

DEFINITION 2.4.1 1. A **relation** from X to Y means a subset of $X \times Y$. That is, a relation f from X to Y is a collections of some pairs (a, b) where $a \in X$ and $b \in Y$. It can be empty.

2. There are 2^6 different relations from $\{1, 2\}$ to $\{1, 2, 3\}$.

3. A function f from X to Y is a relation for which satisfies

   (a) $\{a \mid (a, b) \in f\} = X$ and

   (b) for each $x \in X$, there is a unique $y \in Y$ such that $(x, y) \in f$.

   We read '$f : X \to Y$' as 'f from X to Y'.

4. Let $f : X \to Y$ be a function. For $x \in X$, by $f(x)$ we denote that unique element of Y for which $(x, f(x)) \in f$. The element $f(x)$ is called the **image** of x. For a set $S \subseteq X$, we define $\boldsymbol{f(S)}$:= $\{f(x) \mid x \in S\}$.

5. If $f : X \to Y$ is function, then we know that $f(x)$ is uniquely determined by f and x. This is why many a times, we define a function by stating the rule of association of x with $f(x)$. For example, we can define a function $f : \mathbb{R} \to \mathbb{R}$ as $f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q}. \end{cases}$

EXERCISE 2.4.2 *Let $Y = \{a, b, c, d\}$. How many of the 2^{20} relations from J_5 to Y are functions? How many of them have $f(3) = a$?*

DEFINITION 2.4.3 A function $f : X \to Y$ is called **one-one** if distinct elements of X have distinct images. A function $f : X \to Y$ is called **onto** if each element of Y is the image of some element of X. A function $f : X \to Y$ is called a **bijection** if f is both one-one and onto.

EXERCISE 2.4.4 *Using J_5 and J_6 give a function which is (a) one-one but not onto (b) onto but not one-one (c) neither one-one nor onto. (d) Does there exist a function $f : J_5 \to J_5$ which is one-one but not onto? (e) Does there exist a function $f : J_5 \to J_5$ which is onto but not one-one?*

EXERCISE 2.4.5 *Give two functions from $\mathbb{N}$ to $\mathbb{N}$ in each of the following cases. Write their set (of pairs) forms and draw pictures representing them.*
(a) One-one but not onto (b) one-one and onto (c) onto but not one-one (d) neither one-one and onto.

EXERCISE 2.4.6 *We describe $\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, 8, \cdots\}$ and $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$.*

1. *Can you write a similar description of $\mathbb{Z}$ without using the first few dots? Mention at least 8 terms.*
2. *Write the above description of $\mathbb{N}$ just below your description of $\mathbb{Z}$.*
3. *Draw some vertical line. Does this represent a bijection (in picture) from $\mathbb{Z}$ to $\mathbb{N}$?*
4. *Express the above bijection by specifying a rule of association.*
5. *Can you give another bijection (in picture)? One more?*

EXERCISE 2.4.7 *a) Give a bijection $f : \{1, 2, 3\} \to \{a, b, c\}$ and a bijection $g : \{4, 5, 6, 7\} \to \{P, I, T, A\}$.*

b) Use f and g of a) to give a bijection from $\{1, 2, 3\} \cup \{4, 5, 6, 7\}$ to $\{a, b, c\} \cup \{P, I, T, A\}$.

c) Prove the general statement. Let $f : A_1 \to B_1$ and $g : A_2 \to B_2$ be bijections. Suppose that $A_1 \cap A_2 = \emptyset = B_1 \cap B_2$. Then there is a bijection from $A_1 \cup A_2$ to $B_1 \cup B_2$.

EXERCISE 2.4.8 *Let $f : A \to B$ be a bijection. Give a bijection from A^2 to B^2. From A^5 to B^5.*

EXERCISE 2.4.9 *Supply bijections.*

a) From $(0, 1)$ to $(0, 1)$. *b) From $(0, 1)$ to $(2, 3)$.* *c) From $(0, 1)$ to (c, d).*

d) From $(0, b)$ to (c, d). *e) From (a, b) to (c, d).* *f) From $(0, 1)$ to $(0, \infty)$.*

g) From (a, b) to (c, ∞). *h) From (a, b) to $(-\infty, \infty)$.* *i) From $(0, 1)^2$ to $\mathbb{R}^2$.*

2.4.2 Train-seat argument

Train-seat argument

Imagine elements $P = (0, 1)$ as persons and elements of $S = (3, 4)$ as seats in a train. Consider a bijection $f : P = (0, 1) \to T = (3, 4)$. So, each person has been assigned a seat by f and the train is full.

1. Now suppose a new person 0 is arriving. He wants a seat.
2. To manage it, ask two persons $\frac{1}{2}, \frac{1}{3}$ to vacate the seats. So two seats $f(\frac{1}{2}), f(\frac{1}{3})$ are vacant. But we have 3 persons to take those seats. Not possible.
3. Suppose we make $\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{30}$ vacate their seats? Can we manage it?

4. Suppose we make $\frac{1}{2}, \frac{1}{3}, \dots$ vacate their seats? Can we manage it now?

5. What do we do, if we had two new persons arriving? Fifty new persons arriving? A set $\{a_1, a_2, \cdots\}$ of new persons arriving?

EXERCISE 2.4.10 *Give bijections.*

a) From $[0,1)$ *to* $(0,1)$. *b) From* $(0,1) \cup \{1,2,3,4\}$ *to* $(0,1)$. *c) From* $(0,1) \cup \mathbb{N}$ *to* $(0,1)$.

d) From $[0,1]$ *to* $[0,1] \setminus \{\frac{1}{1}, \frac{1}{3}, \frac{1}{5}, \cdots\}$. *e) From* $\mathbb{R}$ *to* $\mathbb{R} \setminus \mathbb{N}$. *f) From* $(0,1)$ *to* $\mathbb{R} \setminus \mathbb{N}$.

g) From $[0,1]$ *to* $\mathbb{R} \setminus \mathbb{N}$. *h) From* $(0,1)$ *to* $(1,2) \cup (3,4)$.

EXERCISE 2.4.11 *Supply a bijection from* $(0,1)$ *to* $(1,2) \cup (3,4) \cup (5,6) \cup (7,8) \cup \cdots$.

2.4.3 Finite, infinite, countable, uncountable

Finite-infinite

1. A set S is said to be **infinite** if there a bijection from $T \subsetneq S$ to S. All other sets are **finite**. Show that $\mathbb{N}$ is infinite.

2. If S is infinite and $S \subseteq S'$, then S' is infinite. Prove it!

3. Suppose that $f : J_n \to S$ is a bijection, where $S \subsetneq J_n$ (it is the set $\{1, 2, \dots, n\}$). Assume that $f(n) = r$. Define g on J_{n-1} as $g(i) = f(i)$ if $f(i) < r$; otherwise $g(i) = f(i) - 1$. Then g is a bijection from J_{n-1} to a proper subset of J_{n-1}.

4. As $\emptyset$ is finite, apply induction to show that J_n are finite for each $n \in \mathbb{N}$.

Countability

1. A nonempty set S is called **countable** if there exists a one-one map from S to $\mathbb{N}$.

2. Is $f(x) = x$ is a one-one map from J_n to $\mathbb{N}$?

3. Give a one-one map from $\mathbb{Z}$ to $\mathbb{N}$.

4. Is $f(m,n) = 2^m 3^n$ is a one-one map from $\mathbb{N}^2$ to $\mathbb{N}$?

5. Consider the following list: $\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{3}{1}, \frac{2}{2}, \frac{1}{3}, \frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \frac{5}{1}, \frac{4}{2}, \frac{3}{3}, \frac{2}{4}, \frac{1}{5}, \frac{6}{1}, \cdots$. Notice that each positive rational number appears at least once in the list. In which position $\frac{p}{q}$ appears in the list? (Look at the positions in the table.)

$$\begin{array}{cccc} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \cdots \\ \frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \cdots \\ \frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

6. Define $f : \mathbb{Q}_+ \to \mathbb{N}$ as $f(\frac{p}{q})$ = position of the rational number $\frac{p}{q}$ (here $\gcd(p,q) = 1$) in the above list. Is it one-one?

7. Let S be an infinite set. Choose an element of S and call it $f(1)$. Notice that $S \setminus \{f(1)\}$ is infinite. Choose an element of $S \setminus \{f(1)\}$ and call it $f(2)$. Continue in this way to find a one-one map $f : \mathbb{N} \to S$.

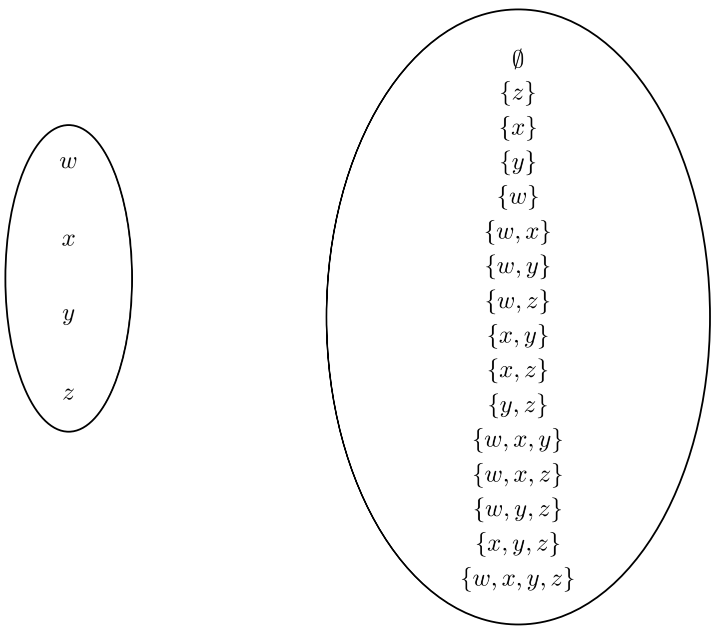
8. Conclude that an infinite set S is countable iff there is a bijection from $g : \mathbb{N} \to S$. In that case the list $g(1), g(2), g(3), \cdots$ is called an **enumeration** of S.

9. **(countable union of countable sets is countable)** Let $A_1, A_2, A_3, \cdots$ is a list of nonempty countable sets. Let $f_i : A_i \to \mathbb{N}$ be the corresponding one-one maps. By p_n denote the nth prime number. Define $f : \bigcup\limits_{i=1}^{\infty} A_i \to \mathbb{N}$ as $f(x) = p_i^{f_i(x)}$, where i is the smallest such that $x \in A_i$. Show that f is one-one.

Uncountable

1. Look at the following picture. On the left we have a set $S = \{x, y, z, w\}$. On the right we have $P(S)$, the power set of S.

   Draw four lines (of your choice) depicting a one-one function. Call it f. Consider the set $A = \{s \in S : s \notin f(s)\}$. Notice that $A \in P(S)$. I guarantee that you did not draw a line to A. How?



2. Prove **Cantor's Theorem**. *There is no bijection from a nonempty S to $P(S)$.*

3. Conclude that $P(\mathbb{N})$ is an uncountable set.

$\mathbb{R}$ is uncountable

There are two more simpler arguments in the next section. Suppose that $\mathbb{R}$ is countable and let $x_1, x_2, \ldots$ be an enumeration of $\mathbb{R}$.

1. Find $a_1 < b_1$ such that $x_1 \notin [a_1, b_1]$.

2. Find $[a_2, b_2] \subseteq [a_1, b_1]$, $a_2 < b_2$, so that $x_2 \notin [a_2, b_2]$ and $b_2 - a_2 = \frac{b_1 - a_1}{3}$. [Hint: divide $[a_1, b_1]$ into three parts.]

3. Find $[a_3, b_3] \subseteq [a_2, b_2]$, $a_3 < b_3$, so that $x_3 \notin [a_3, b_3]$ and $b_3 - a_3 = \frac{b_2 - a_2}{3}$.

4. Use nested interval theorem to show that $\bigcap\limits_{n=1}^{\infty}[a_n, b_n]$ contains a real number not in the list.

5. This is a contradiction. So $\mathbb{R}$ is uncountable.

$\blacksquare$

Exercise 2.4.12 *Give a finite set, a countable infinite set and an uncountable infinite set $S \subseteq \mathbb{R}$ such that $\mathsf{lub}\, S \in S$. Give a finite set, a countable infinite bounded set and an uncountable infinite set $S \subseteq \mathbb{R}$ such that $\mathsf{lub}\, S \notin S$.*

Exercise 2.4.13 *True or False? If $S \subseteq \mathbb{R}$ is an infinite set which contains its lub, then $S \setminus \{\mathsf{lub}\, S\}$ cannot contain its lub.*

Exercise* 2.4.14 *Let S be an uncountable subset of $\mathbb{R}$. Can you find two disjoint subsets $A, B \subseteq S$ such that there is a bijection from $A \to B$ and $A \cup B = S$? For example, if $S = [0, 2]$, then I could take $A = [0, 1]$ and $B = (1, 2]$.*

2.5 Decimal and Binary Representations (self study)

Throughout this section we assume that $x_i, y_j \in \{0, 1, 2, \cdots, 9\}$. Throughout we shall discuss only about positive real numbers and their decimal representation. For the negatives, put a $-$ sign and for zero, it is 0.

Definition 2.5.1 1. We write $a = x_k \cdots x_1.y_1 \cdots y_m$, to mean that

$$a = x_k 10^{k-1} + \cdots + x_1 + \frac{y_1}{10} + \cdots + \frac{y_m}{10^m}.$$

Notice that $y_j = [10^j a] - 10[10^{j-1}a]$.

2. Suppose that $a = x_k x_{k-1} \cdots x_1.y_1 y_2 \cdots y_m$, where $x_k, y_m \neq 0$. Then we call the expression $x_k x_{k-1} \cdots x_1.y_1 y_2 \cdots y_m$ a **terminating decimal representation**(TDR) of a.

Exercise 2.5.2 *Verify that the only positive real numbers with a TDR are the rationals $\frac{p}{q}$, where* $\gcd(p, q) = 1$ *and $q = 2^i 5^j$, where $i, j \in \{0, 1, 2, 3 \cdots\}$.*

Exercise 2.5.3 *Verify that a TDR of a number, if exists, is unique.*

Definition 2.5.4 Consider an expression $x_k x_{k-1} \cdots x_1.y_1 y_2 \cdots$, where y_is are nonzero infinitely often (that is, infinitely many y_is are nonzero). Then the set of rational numbers

$$F = \{x_k \cdots x_1.y_1, \quad x_k \cdots x_1.y_1 y_2, \quad x_k \cdots x_1.y_1 y_2 y_3, \quad \cdots\}$$

is nonempty and bounded above by $n + 1$, where $n = x_k \cdots x_1$. We shall call F as the **finite decimal set** (just a name). Put $a = \mathsf{lub}\, F$. Then we call the expression

$$x_k x_{k-1} \cdots x_1.y_1 y_2 \cdots$$

a **nonterminating decimal representation**(NTDR) of a.

Example 2.5.5 **(Finding NTDR)**

1. Let $a > 0$ be different from the numbers considered in Exercise 2.5.2. Then the NTDR of a is

$$x_k x_{k-1} \cdots x_1.y_1 y_2 \cdots$$

where $x_k \ldots x_1$ is the TDR of $[a]$ and $y_i = [10^i a] - 10[10^{i-1}a]$.

2. In fact, due to the property of the greatest integer function, we have

$$x_k x_{k-1} \cdots x_1 . y_1 y_2 \cdots y_m = \frac{[10^m a]}{10^m} \leq a.$$

3. Note that $F := \{x_k \cdots x_1.y_1, \quad x_k \cdots x_1.y_1y_2, \quad x_k \cdots x_1.y_1y_2y_3, \quad \cdots\}$ has lub a.

   Proof. Let $b = \mathsf{lub}\, F$. So $b \leq a$. Suppose that $b < a$. Then $\exists m$ such that $b < a - \frac{1}{10^m} < a$. We have

$$b < a - \frac{1}{10^m} < \frac{[10^m a]}{10^m} = x_k x_{k-1} \cdots x_1 . y_1 y_2 \cdots y_m$$

   and so b cannot be an ub of S, a contradiction. Thus $b = a$. In the above, the second inequality follows as $10^m a < [10^m a] + 1$.

4. Let $a > 0$ be a number considered in Exercise 2.5.2. If $a \notin \mathbb{N}$, to find its NTDR, we take its TDR, decrease the last digit by 1 and put the next digits 9. For example,

$$\frac{1}{10} = .1 = .0999$$

   If $a \in \mathbb{N}$, its NTDR is obtained as

$$x_k \cdots x_1.9999 \cdots$$

   where $x_k \cdots x_1$ is the TDR of $a - 1$. For example,

$$20080 = 20079.999 \cdots .$$

Exercise 2.5.6 *Show that each positive real number has a unique NTDR. Thus, each positive real number is the lub of a naturally defined set of rational numbers.*

Definition 2.5.7 Let $a > 0$ have an NTDR

$$x_k \cdots x_1 \;.\; y_1 \cdots y_i \; \overbrace{y_{i+1} \cdots y_j} \; y_{i+1} \cdots y_j \; y_{i+1} \cdots y_j \cdots$$

In this case we say that a has a **recurring** NTDR.

Exercise 2.5.8 *Let $a > 0$. Show that the NTDR of a is recurring iff a is rational. Thus, a real number is irrational iff it has a* **nonrecurring** *NTDR.*

Exercise 2.5.9 *Decimal representations are some times called* **base 10 representations**. *Define and discuss* **binary (base 2) representations** *of real numbers.*

Exercise 2.5.10 *If x is an irrational number, should its base 6 representation necessarily be nonrecurring and nonterminating?*

Exercise 2.5.11 *In base 6, the expression $.123123123\cdots$ represents* lub *of a naturally defined set? Write the first (smallest) four elements of the set.*

Exercise 2.5.12 *In base 6, the expression $.123123123\cdots$ represents a real number. Evaluate that.*

2.6 Two special arguments (self-study)

Uncountability of $\mathbb{R}$

1. If A is uncountable, then any superset is uncountable. So, it is enough to show that $(0,1)$ is uncountable.
2. Consider the function $f : P(\mathbb{N}) \to (0,1)$ defined as $f(A)$ is the real number $.y_1y_2\cdots$, where $y_i = 3$ if $i \in A$ and $y_i = 6$ if $i \notin A$. Show that f is one-one.
3. We know that $P(\mathbb{N})$ is uncountable. Conclude that $f(P(\mathbb{N}))$ is uncountable.
4. As $(0,1)$ is a superset of $f(P(\mathbb{N}))$, it is uncountable.
5. Hence, conclude that any nontrivial interval is uncountable.

Cantor's diagonalization argument
Here is another famous alternate way to show that $(0,1)$ is uncountable.

1. Suppose that $(0,1)$ is countable and let $x_1, x_2, \ldots$ be an enumeration of these real numbers.
2. Let $.x_{k1}x_{k2}\cdots$ be the NTDR of the real number x_k.
3. Define a new NTDR as $.y_1y_2\cdots$, where $y_i = 3$ if $x_{ii} > 5$ and $y_i = 7$ if $x_{ii} \le 5$.
4. We already know that this NTDR represents a unique real number. Argue that it is in $(0,1)$.
5. Argue that it is not in the list, to get a contradiction.

2.7 Open and closed sets

Definition 2.7.1 1. In $\mathbb{R}^n$ by 0 we mean the origin $(0,0,\cdots,0)$. For $x = (x_1,\cdots,x_n), y = (y_1,\cdots,y_n) \in \mathbb{R}^n$, we define

$$|x-y| = \sqrt{(x_1-y_1)^2+\cdots+(x_n-y_n)^2}. \tag{2.1}$$

2. Let $\alpha > 0$ and $a \in \mathbb{R}^n$. Then by the **open ball $\boldsymbol{B_\alpha(a)}$** we mean the set $\{x \in \mathbb{R}^n : |x-a| < \alpha\}$. It is centered at a and has the radius α.
3. It is also called a **neighbourhood** of a. (Some texts also call any set which contains a $B_\alpha(a)$, a neighbourhood of a.)
4. The set $\boldsymbol{D_\alpha(a)}$:$= B_\alpha(a) \setminus \{a\}$, is called a **punctured ball** at a or a **deleted neighbourhood** of a.

Remark 2.7.2 Henceforth, whenever we mention any ball $B_\epsilon(a)$ or a punctured ball $D_\alpha(a)$, we shall understand that ϵ, α are some positive numbers.

Example 2.7.3 1. In $\mathbb{R}$, $B_\alpha(a) = (a-\alpha, a+\alpha)$. So $B_1(2) = (1,3)$.

2. In $\mathbb{R}$, $D_\alpha(a) = (a-\alpha, a) \cup (a, a+\alpha)$. So $D_1(2) = (1,2) \cup (2,3)$.
3. In $\mathbb{R}^2$, $B_1(0)$ is the unit disc $\{(x,y) \mid x^2+y^2 < 1\}$.

4. In $\mathbb{R}^2$, $D_1(0)$ is the punctured unit disc $\{(x, y) \mid x^2 + y^2 < 1\} \setminus \{0\}$.

5. In $\mathbb{R}^3$, the set $B_1(0)$ is a solid sphere.

REMARK 2.7.4 Henceforth, when we mention $B_\alpha(a)$ or $D_\alpha(a)$ we understand that $\alpha > 0$.

EXAMPLE 2.7.5 1. Take $A = (0, 1)$ and $a = .9$. Can you find a $B_\delta(a) \subseteq A$? Draw picture and tell.

2. Take $A = (0, 1)$ and $a \in (0, 1)$. Can you find a $B_\delta(a) \subseteq A$?

3. Take $A = B_1(0)$ in $\mathbb{R}^2$ and $a = (.9, .9)$. Find a $B_\delta(a) \subseteq A$? [Hint: look at the line passing through 0 and a.]

DEFINITION 2.7.6 A set $A \subseteq \mathbb{R}^n$ is **open** if A contains a neighbourhood of <u>each</u> point $a \in A$. A set A is **closed** if A^c is open.

EXAMPLE 2.7.7 1. The set $(0, 1)$ is open in $\mathbb{R}$. We have seen it in the previous example.

2. The sets (a, b), $(a, b) \cup (c, d)$, $\emptyset$, $\mathbb{R}$, are open in $\mathbb{R}$.

3. Any $B_\alpha(a)$ is open in $\mathbb{R}^n$.

4. The set $[0, 1)$ is not open in $\mathbb{R}$, as the condition fails at 0.

5. Any nonempty countable subset $S \subseteq \mathbb{R}$ is not open. The condition fails at each point. This is so, as $B_\alpha(x)$ is the interval $(x - \alpha, x + \alpha)$ which we know is uncountable.

LEMMA 2.7.8 *1. Let $\{A_\alpha \mid \alpha \in I\}$ be a class of open subsets of $\mathbb{R}^n$. Then $\bigcup\limits_{\alpha \in I} A_\alpha$ is open. This follows from the definition.*

2. Let $\{A_\alpha \mid \alpha \in I\}$ be a class of closed subsets of $\mathbb{R}^n$. Then $\bigcap\limits_{\alpha \in I} A_\alpha$ is closed. This follows by taking complements.

3. Let A and B be open sets. Then $A \cap B$ is open. This follows from the definition.

Try a proof by contradiction to see that sometimes proofs by contradictions can be more difficult than a direct proof.

4. If A and B are closed, then $A \cup B$ is closed. This follows by taking complements.

REMARK 2.7.9 1. The definition of an open set can be rewritten as '$A \subseteq \mathbb{R}^n$ is open if $\forall x \in A$, $\exists \delta > 0$ such that $B_\delta(x) \subseteq A$'.

2. So, A is not open if $\exists x \in A$ such that $\forall \delta > 0$, $B_\delta(x) \cap A^c \neq \emptyset$.

3. So A is not open iff it has a point at which each neighbourhood contains points outside A.

DEFINITION 2.7.10 When $A \subseteq \mathbb{R}$ does not contain 0, we shall use $\frac{1}{A}$ to denote $\{\frac{1}{x} \mid x \in A\}$. This is not a common notation, but it is useful for us.

For $A, B \subseteq \mathbb{R}^n$, and $c \in \mathbb{R}$, we define $A + B = \{a + b \mid a \in A, b \in B\}$ and $cB = \{cb \mid b \in B\}$.

EXERCISE 2.7.11 *Write 'Y' for yes and 'N' for no, against the given sets, for being open or closed in their respective $\mathbb{R}^n$.*

	$\emptyset$	$\lvert S\rvert = 5$	$\mathbb{N}$	$\mathbb{Q}$	$\mathbb{R}$	$[a, b]$	$[0, 1)$	$(0, 1) \cup \{2\}$	$\frac{1}{\mathbb{N}}$	$\frac{1}{\mathbb{N}} \cup \{0\}$	$\mathbb{Q}^c$	$\mathbb{N}^2$	$(0, 1)^2$
open													
closed													

DEFINITION 2.7.12 A point $a \in \mathbb{R}^n$ is called a **limit point** of S if each $D_\alpha(a)$ contains a point of S. It is also called a **cluster point** of S. The **closure $\overline{S}$** of S is defined as $\overline{S} = S \cup \{\text{all limit points of } S\}$.

EXAMPLE 2.7.13

1. The set $\emptyset$ does not have any limit point in $\mathbb{R}^n$. By definition. So $\overline{\emptyset} = \emptyset$.
2. The singleton set $S = \{a\}$ does not have any limit point in $\mathbb{R}^n$. So $\overline{S} = S$.

   Proof. If $x \neq a$, then take $\alpha = |x - a|$. In that case $D_\alpha(x)$ does not contain any point of S. If $x = a$, then $D_1(x)$ does not contain any point of S.
3. Let $a \neq b$ be points in $\mathbb{R}^n$. Then the set $\{a, b\}$ does have any limit point. So $\overline{S} = S$.
4. A finite set S does not have a limit point in $\mathbb{R}^n$. So $\overline{S} = S$.
5. The set $S = \mathbb{N}$ does not have a limit point in $\mathbb{R}$. So $\overline{S} = S$.
6. The set $S = \frac{1}{\mathbb{N}}$ has exactly one limits point, namely, 0. So $\overline{S} = S \cup \{0\}$.

   Proof. To show that 0 is a limit point, let $\alpha > 0$ be arbitrary. By Archimedean principle, $D_\alpha(0)$ contains some $\frac{1}{n_0}$.

   Now, let $x < 0$. Take $\alpha = |x|/2$. Then $D_\alpha(x) \cap S = \emptyset$. So x is not a limit point. Similarly, argue that if $x > 1$, then it is not a limit point. Argue that if $x \in (\frac{1}{n+1}, \frac{1}{n})$, then x is not a limit point of S. Argue that if $x = \frac{1}{n}$, then also it is not a limit point of S.
7. A closed S set contains all its limit points. Thus $\overline{S} = S$.

   Proof. If $t \in S^c$, as S^c is open, some $B_\delta(t) \subseteq S^c$. So $D_\delta(t) \cap S = \emptyset$. So t is not a limit point of S.
8. The set $S = \frac{1}{\mathbb{N}} \cup (1 + \frac{1}{\mathbb{N}})$ has exactly two limits points in $\mathbb{R}$, namely, 0 and 1. So $\overline{S} = S \cup \{0, 1\}$.
9. A limit point of S need not be in S.
10. $[0, 1] = \overline{[0, 1)} = \overline{(0, 1)} = \overline{\mathbb{Q} \cap (0, 1)}$.
11. $\overline{R} = \overline{\mathbb{Q}} = \overline{\mathbb{Q}^c} = \mathbb{R}$.
12. $\overline{(0, 1) \times (0, 1)} = [0, 1] \times [0, 1]$.
13. Each point of an open set is a limit point, by definition.

EXERCISE 2.7.14

1. *Give some sets S which have some limit points in S and some outside S.*
2. *Give three types of examples of sets S which have limit points but all of them are in S.*
3. *Let $S \subseteq T$. Show that $\overline{S} \subseteq \overline{T}$.*
4. *Let A be a finite subset of S. Show that $\overline{S} = \overline{S \setminus A}$.*
5. *Show that $\overline{S}$ is always closed.*
6. *Show that $\overline{S} = S$ iff S is closed.*
7. *A set S is said to be* **dense** *in $\mathbb{R}^n$ if $\overline{S} = \mathbb{R}^n$. Give five proper dense subsets of $\mathbb{R}$.*
8. *What is the relationship between $\overline{\overline{S}}$ and $\overline{S}$?*
9. *Let $S \subseteq \mathbb{R}^n$. Let $F = \{C \mid S \subseteq C, C \text{ closed}\}$. Show that $\overline{S} = \bigcap\limits_{C \in F} C$.*
10. *Do you have a set S for which the set of limit points is precisely $\mathbb{N}$?*

DEFINITION 2.7.15 A point $a \in S$ is called an **interior point** of S if a $B_\alpha(a) \subseteq S$. The set of all interior points of S is denoted by S°.

EXAMPLE 2.7.16

1. $(1,2)^\circ = (1,2) = [1,2]^\circ$.
2. $\emptyset^\circ = \emptyset$, $\mathbb{R}^\circ = \mathbb{R}$, $\mathbb{Q}^\circ = \emptyset$.
3. Is there a countable set with an interior point? No. If the set has an interior point, then by definition, it contains an open ball, which is uncountable.
4. Can you have a uncountable subset of $\mathbb{R}$ with empty interior? Yes, $\mathbb{Q}^c$.
5. Each interior point of S is a limit point of S, by definition.
6. For any set S, by definition, S° is either empty or uncountable.

EXERCISE 2.7.17

1. *Take a set $S \subseteq \mathbb{R}^n$. Let $G = \{O \mid O \subseteq S, O \text{ open}\}$. Show that $S^\circ = \underset{O \in G}{\cup} O$.*
2. *Show that S is open iff $S = S^\circ$.*
3. *Show that $S^\circ = S^{\circ\circ}$.*

DEFINITION 2.7.18 The **boundary ∂S** of S is the set $\overline{S} \cap \overline{S^c}$.

EXAMPLE 2.7.19

1. Let $A = (0,1)$, $B = [0,1)$, $C = [0,1]$, and $D = \{0,1\}$. Then $\partial A = \{0,1\} = \partial B = \partial C = \partial D$.
2. Take $A = \{(x,y) : x^2 + y^2 < 1\}$, $B = \{(x,y) : x^2 + y^2 \leq 1\}$, $C = \{(x,y) : x^2 + y^2 = 1\}$, and $D = \{(x,y) : x^2 + y^2 > 1\}$. Then $\partial A = \partial B = \partial C = \partial D = \{(x,y) : x^2 + y^2 = 1\}$.
3. Take $A = (0,1) \cap \mathbb{Q}$. Then $\partial A = [0,1]$.

2.8 Exercises

EXERCISE 2.8.1 *Show that $\partial S = S$ iff S is closed and $S^\circ = \emptyset$.*

EXERCISE 2.8.2 *Fill the blanks with always/sometimes/never. All sets are subsets of $\mathbb{R}$.*

1. *A nonempty finite set is _____ open and _____ closed.*
2. *A nonempty subset of $\mathbb{Z}$ is _____ open and _____ closed.*
3. *An infinite subset of $\mathbb{Q}$ is _____ open and _____ closed.*
4. *An infinite subset of $\mathbb{Q}^c$ is _____ open and _____ closed.*
5. *Nontrivial intervals are _____ open and _____ closed.*
6. *Let $A_1, \cdots, A_n$, $n > 1$, be nonempty closed sets. Their union is _____ open and _____ closed.*
7. *Let $A_1, \cdots, A_n$, $n > 1$, be nonempty open sets. Their union is _____ open and _____ closed.*
8. *Union of infinitely many nonempty open sets is _____ open and _____ closed.*
9. *Union of infinitely many nonempty closed sets are _____ open and _____ closed.*
10. *Intersection of infinitely many nonempty closed sets are _____ open and _____ closed.*

11. *Intersection of infinitely many nonempty open sets are _____ open and _____ closed.*

Exercise 2.8.3 *Which of these are open/closed in $\mathbb{R}$? Give their closures and interiors.*

1. $\{x \in \mathbb{R} \mid |x| < 2\}$.
2. $B_1(0)$ *in* $\mathbb{R}^2$.
3. $\{x \in \mathbb{R} \mid 1 \leq x^2 < 2\}$.
4. $\{x \in \mathbb{R}^2 \mid 1 \leq |x|^2 < 2\}$.

Exercise 2.8.4 **(a property of a polynomial)**

1. *Take* $p(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$. *Is the set* $\{x \mid p(x) < 0\}$ *open? Is the set* $\{x : p(x) \leq 0\}$ *closed? Is the later the closure of the former?*
2. *Give a function* $f : \mathbb{R} \to \mathbb{R}$ *to show that* $\{x \mid f(x) < 0\}$ *need not be open.*

Exercise 2.8.5 *Express* $(0,1) = \bigcup\limits_{\alpha \in I} A_\alpha$, *where* A_α *are closed, in two ways, first by taking* I *countable and then by taking* I *uncountable.*

Exercise 2.8.6

1. *True or False? The set of points on the coordinate axes of* $\mathbb{R}^2$ *is closed.*
2. *True or False? The set of points on the coordinate axes of* $\mathbb{R}^2$ *except the origin is closed.*
3. *Let* $O \subseteq \mathbb{R}$ *be a bounded nonempty open set and* $a = \operatorname{lub} O$. *Can it happen that* $a \in O$?
4. *Is it true: a bounded infinite subset of* $\mathbb{R}$ *which does not have a maximum and a minimum must be open?*

Exercise 2.8.7 **(Optional: structure of open sets in $\mathbb{R}$)**

1. *Show that the only sets which are both open and closed in* $\mathbb{R}$ *are:* $\emptyset$ *and* $\mathbb{R}$. [Hint: Let S be a proper open subset and $x \in \S$. Then S contains an open interval containing x. Let (a,b) be a maximal open interval contained in S (this means there is no open interval (c,d) such that $(a,b) \subsetneq (c,d) \subseteq S$). Argue that both $a = -\infty$ and $b = \infty$ cannot be true. Suppose that $b < \infty$. Argue that $b \in S^c$ and b is not an interior point of S^c.]
2. *Hence conclude that every nonempty open subset of* $\mathbb{R}$ *is a countable union of open intervals.* [Hint: Let O be a nonempty open set. If $O = \mathbb{R}$ or $\emptyset$, then we have nothing to show. So assume that O is a nonempty proper subset. By (a), each $x \in S$ is contained in a maximal open interval $I_x = (a,b)$. Argue that for $x, y \in S$, either $I_x = I_y$ or $I_x \cap I_y = \emptyset$. Do you see that union of all these maximal intervals is S? Relate rational numbers to show that there are countably many such intervals.]

Exercise 2.8.8 **(Optional: G_δ sets)**

1. *A subset* A *of* $\mathbb{R}$ *is called a* $\boldsymbol{G_\delta}$ *set if it can be expressed as intersection of countably many open sets. Is* $\{1,2,3\}$ *a* G_δ *set? Is every finite set* G_δ? *Show that* $\mathbb{N}$ *and* $\{\frac{1}{n} : n \in \mathbb{N}\}$ *are* G_δ.
2. *Show that the set* $\mathbb{Q}$ *is not a* G_δ *set.*

Exercise* 2.8.9 *Can you express* $\mathbb{R}$ *as a union of sets of size* 5? *As a union of pairwise disjoint sets of size* 5?

EXERCISE* 2.8.10 *Can you express* $(0,1)$ *as union of pairwise disjoint nontrivial closed intervals? What about* $\mathbb{R}$*?*

EXERCISE 2.8.11 *Do you have a set* S *for which the set of limit points is precisely* $\mathbb{Q}$*?*

EXERCISE 2.8.12 *Let* T *be the set of limit points of* S *and* U *be the set of limit points of* T. *Show that* $U \subseteq T$. *Can it be strict?*

EXERCISE 2.8.13 *Let* $S \subseteq [1,2]$ *be an infinite set. Show that it has a limit point.*

EXERCISE 2.8.14 *Let* S *be an uncountable subset of* $\mathbb{R}$. *Show that it has a limit point.*

EXERCISE* 2.8.15 *Let* $S \subseteq \mathbb{R}$ *have* 9 *limit points. Must it be a countably infinite set?*

Chapter 3

Sequences

After this chapter, I should be able to answer the following.

1. Let a be a fixed real number. Supply a sequence $a_n \to a$, such that the only the odd (position) terms are rational numbers.

2. Let $a_n = \frac{1}{\sqrt{n^2+1}} + \cdots + \frac{1}{\sqrt{n^2+n}}$. Is (a_n) convergent?

3. Let $A_1 = (5, 0)$, $A_2 = (7, 9)$ and $A_3 = (1, 23)$. Let A_4 be the centroid of the triangle $A_1A_2A_3$, A_5 be the centroid of the triangle $A_2A_3A_4$, and so on. If you continue visiting these centroids, ultimately where do you arrive?

4. Show that the sequences $(1 + \frac{1}{n})^n$ and $(1 + \frac{2}{n})^n$ converge. There a relation between their limits. Can you prove that?

5. It is given that the distance between each pair of terms of a sequence is less than 1. My friend claims that, in that case, all the terms can be fitted in an interval of length 1? Is he correct?

6. Let $x = \sqrt{2}^{\sqrt{15}}$ and $y = \frac{1}{x}$. Can I find a natural number n such that $|y - \sin(n)| < 10^{-100}$?

7. Let $p > 0$ be fixed. Define $a_n = \frac{\ln n}{n^p}$. Does it converge?

8. Let $a_n = (1 + \frac{1}{n^2})^n$. What is the limit of (a_n)?

9. How do I define the length of an arc of a circle? (Obviously, we do not measure that with a thread or by a process which would give different values with different trials.)

10. Show that $a_n = \frac{\sin(1/n)}{1/n}$ converges to 1.

11. There is an infinite queue in front of the gate to heaven. Can you remove some persons (finitely or infinitely many) from the queue so that the remaining is an infinite queue and the persons in this queue are arranged according to their heights?

12. If Earth collects $\frac{1}{n!}$ units of a particular compound on nth year (from external sources), then eventually, how much of weight will it gain?

13. If Earth collects $\frac{1}{n}$ units of a particular compound on nth year (from external sources), then will it will become heavier than the Sun one day?

3.1 Initial concepts

DEFINITION 3.1.1 Let $S \neq \emptyset$. A **sequence** of elements of S is a function $f : \mathbb{N} \to S$. We can view it as the ordered list $(a_1, a_2, \ldots)$, where $a_n = f(n)$. Mostly we shall concentrate on sequences of real numbers.

EXAMPLE 3.1.2 Here we supply a list of some sequences.

a) $(1, 2, 3, 4, 5, \cdots)$ is a sequence. It is better to write it as (n), so that one knows precisely, how the nth term is defined. However, at times we use the three dots, when the terms are naturally understandable.

b) $(\frac{1}{n}) = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots)$.

c) $(0) = (0, 0, 0, 0, 0, \cdots)$.

d) $((-1)^n)$, that is, the sequence $(-1, 1, -1, 1, -1, \cdots)$.

e) (a_n), where $a_1 = a_2 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for $n > 2$. It is the sequence of **Fibonacci numbers**: $1, 1, 2, 3, 5, 8, 13, 21, \cdots$. This way of defining a sequence is called **recursive**.

f) $(1 + \frac{(-1)^n}{n})$, that is, $(0, 1 + \frac{1}{2}, 1 - \frac{1}{3}, 1 + \frac{1}{4}, 1 - \frac{1}{5}, \cdots)$.

g) (a_n) where $a_n = \sum\limits_{i=1}^{n} \frac{1}{2^i}$.

h) (r_n), where r_n is the remainder while dividing the nth prime by 7. We do not know what is $r_{10^{1000}}$ exactly. But we know that it is well-defined. That is, it exists and it is unique.

i) $(\sin n)$.

j) (x_n), where $x_n = [10^n\sqrt{2}] - 10[10^{n-1}\sqrt{2}]$ is the digit at nth decimal place in the decimal expansion of $\sqrt{2}$.

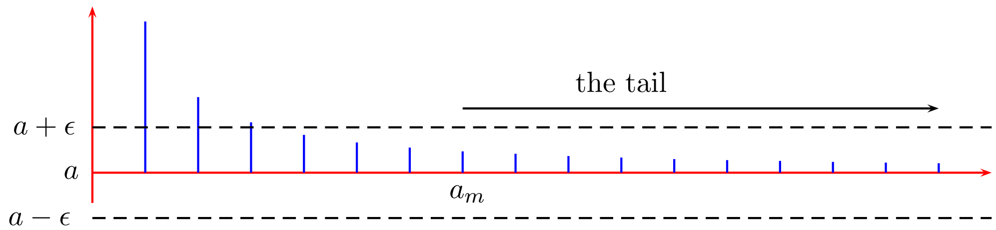
DEFINITION 3.1.3 Let (a_n) be a sequence, $k \in \mathbb{N}$. Then sequence $a_k, a_{k+1}, a_{k+2}, \cdots$ as a **tail** of (a_n). Some texts also write a sequence using $\boldsymbol{\{a_n\}}$.

Meaning of convergence

Suppose $a_n = \frac{1}{n}$ is the purchase capacity of 1 rupee on nth day.

1. What will be the purchase capacity of the rupee ultimately? 0.
2. It means, given any positive number ϵ, there comes a day such that, then onwards, the purchase capacity of the rupee is less than ϵ.
3. That is, $\forall \epsilon > 0$, $\exists m$ such that $a_n \leq \epsilon$, for all $n \geq m$.
4. That is, $B_\epsilon(0)$ contains a tail of (a_n), namely, $a_m, a_{m+1}, \ldots$.
5. In short, each $B_\epsilon(0)$ contains a tail of (a_n).

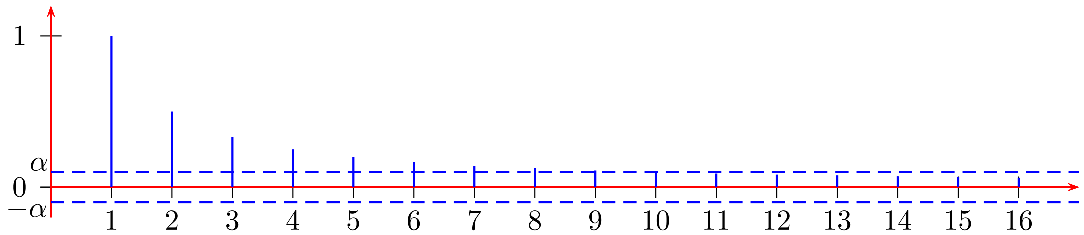
DEFINITION 3.1.4 We say (a_n) **converges** to a if each $B_\epsilon(a)$ contains a tail of (a_n). In other other words, if 'for each $\epsilon > 0$, there exists $m \in \mathbb{N}$ such that for each $n \geq m$ we have $a_n \in B_\epsilon(a)$.' Viewing a_m as the height of the blue lines, we have the following picture to help our understanding.



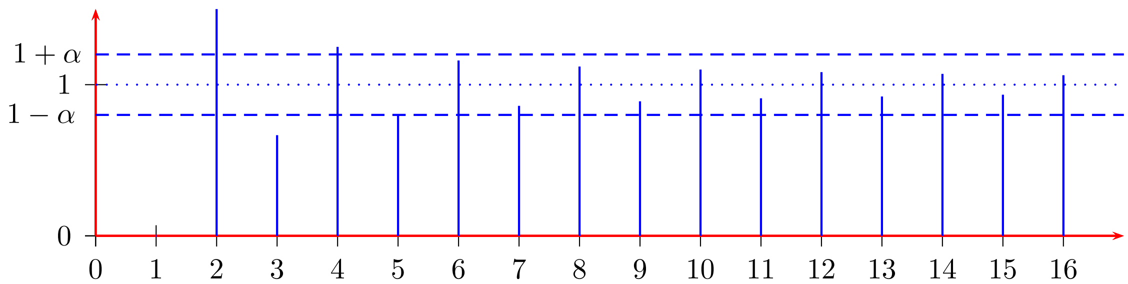
We write $\boldsymbol{a_n \to a}$ to mean that (a_n) converges to a. In this case a is called a **limit** of (a_n). We also write $\boldsymbol{\lim\limits_{n\to\infty} a_n = l}$ for this. We say (a_n) is **divergent** if it is not convergent. It is important to note that $a_n \in B_\epsilon(a)$ means $|a_n - a| < \epsilon$.

EXAMPLE 3.1.5

1. Consider (a_n) where $a_n = \frac{1}{n}$. Give a tail that is contained in $B_\alpha(0)$. The answer is $(a_m, a_{m+1}, \ldots)$ where $m = [\frac{1}{\alpha}] + 1$. (See, if $m = [\frac{1}{\alpha}] + 1$, then $\frac{1}{\alpha} < m$, by definition. So $\frac{1}{m} < \alpha$.) So $a_n \to 0$.



2. Consider (a_n) where $a_n = 1 + \frac{(-1)^n}{n}$. Is it true that each $B_\alpha(1)$ contains a tail? Yes, it contains $(a_m, a_{m+1}, \ldots)$ for $m = [\frac{1}{\alpha}] + 1$. So $a_n \to 1$.



3. Consider $a_n = \sqrt{2} + \frac{2}{n}$. Can you find a $a \in \mathbb{R}$ such that each $B_\alpha(a)$ contains a tail? Yes, $a = \sqrt{2}$. For each $\alpha > 0$, we see that $a_n \in B_\alpha(a)$ for each $n \geq m := [\frac{2}{\alpha}] + 1$. This is so, as $n > \frac{2}{\alpha}$ and so $\frac{2}{n} < \alpha$, implying that $\frac{2}{n} \in B_\alpha(0)$. So $\sqrt{2} + \frac{2}{n} \in B_\alpha(\sqrt{2})$ for each $n \geq m$.

4. Consider $a_n = n$. Can you find a $a \in \mathbb{R}$ such that each $B_\alpha(a)$ contains a tail? No. Suppose there is such an a. Then each $B_\alpha(a)$ should contain a tail of (a_n). In particular, $B_1(a)$ should contain a tail. That is, it should contain infinitely many integers. But that cannot happen as $B_1(a)$ has length 2.

5. Consider $a_n = (-1)^n$. Can you find a $a \in \mathbb{R}$ such that each $B_\alpha(a)$ contains a tail? No. The ball $B_{0.1}(a)$ has length 0.2. It cannot contain a tail, as any tail contains two distinct integers.

6. Is the constant sequence $5, 5, 5, \cdots$ convergent? Yes. The limit is 5. This is so, as each $B_\alpha(5)$ contains a tail (here it is the whole sequence).

Technique to estimate

Let $a_n = \frac{2n+5}{3n+1}$. Then $a_n \to \frac{2}{3}$. This is so, as

$$\left|\frac{2}{3} - \frac{2n+5}{3n+1}\right| = \frac{13}{9n+3} < \frac{13}{9n} < \frac{2}{n} < \epsilon$$

for $n > m := [\frac{2}{\epsilon}]$. This means $a_m, a_{m+1}, \ldots \in B_\epsilon(\frac{2}{3})$.

Below we are using 'LTs' to mean 'Limit theorem for sequences'. It becomes easier to refer to these results.

LEMMA 3.1.6 **(LTs)** *The sequence $a_n \to 0$ iff the sequence $|a_n| \to 0$.*

Proof. This is so, because, $a_n \in B_\epsilon(0) \Leftrightarrow |a_n| \in B_\epsilon(0)$. ∎

LEMMA 3.1.7 **(LTs)** *Fix $0 < a < 1$. Take $a_n = a^n$. Then $a_n \to 0$.*

Proof. Let $\epsilon > 0$. Write $\frac{1}{a} = 1 + b$, where $b > 0$. So

$$|a^n - 0| = a^n = \frac{1}{(1+b)^n} \overset{\text{binomial}}{\leq} \frac{1}{nb} \leq \epsilon$$

for $n > [\frac{1}{b\epsilon}]$. ∎

LEMMA 3.1.8 **(LTs)(Sandwich lemma/squeeze theorem)** *Let $a_n \leq b_n \leq c_n$. (These sequences are in $\mathbb{R}$.) If $a_n \to l$ and $c_n \to l$, then $b_n \to l$.*

Proof. Let $\epsilon > 0$. As $a_n \to l$, there exists m such that $a_n \in B_\epsilon(l)$ for each $n \geq m$. As $c_n \to l$, there exists m' such that $c_n \in B_\epsilon(l)$ for each $n \geq m'$. So $a_n, c_n \in B_\epsilon(l)$ for each $n \geq N := \max\{m, m'\}$. Thus $b_n \in B_\epsilon(l)$ for each $n \geq N$. ∎

EXAMPLE 3.1.9 Note that $0 \leq |\frac{\sin n}{n}| \leq \frac{1}{n}$. So $|\frac{\sin n}{n}| \to 0$. So $\frac{\sin n}{n} \to 0$.

LEMMA 3.1.10 **(LTs)** *Let $a_n \to a$ and fix $c \in \mathbb{R}$. Then $ca_n \to ca$.*

Proof. Let $\alpha > 0$. Put $\epsilon = \frac{\alpha}{1+|c|}$. As $a_n \to a$, there exists m such that $|a_n - a| < \epsilon$ for each $n \geq m$. So, as $|ca_n - ca| < |c|\epsilon < (1 + |c|)\epsilon = \alpha$ for each $n \geq m$. ∎

LEMMA 3.1.11 *A sequence (a_n) converges to a iff a tail of (a_n) converges to a.*

Proof. Exercise. ∎

Understanding $a_n \not\to a$

We write $a_n \not\to a$ to mean that $\lim\limits_{n\to\infty} a_n \neq a$. Negating the definition of limit, we see that $\lim\limits_{n\to\infty} a_n \neq a$ means

$$\exists \alpha > 0 \text{ such that } B_\alpha(a) \text{ does not contain a tail.}$$

That is, $\exists \alpha > 0$ such that $B_\alpha(a)$ misses infinitely many terms of (a_n).

EXAMPLE 3.1.12 Does $\frac{1}{n} \to .1$? No. As $B_{.05}(.1) = (.05, .15)$ misses infinitely many a_n's, namely, each a_n, $n \geq 20$.

LEMMA 3.1.13 **(Uniqueness of limit)** *Let $a_n \to l$ and $a_n \to k$. Then $l = k$. So the limit of a convergent sequence is unique.*

Proof. Suppose that $l \neq k$. Put $\epsilon = |l-k|/2$. As $a_n \to l$, $\exists n_0$ such that $a_{n_0}, a_{n_0+1}, \ldots \in B_\epsilon(l)$. So $a_{n_0}, a_{n_0+1}, \ldots \notin B_\epsilon(k)$, because $B_\epsilon(l) \cap B_\epsilon(k) = \emptyset$). That is, $B_\epsilon(k)$ misses infinitely many terms of (a_n). So $a_n \nrightarrow k$. A contradiction. ■

EXERCISE 3.1.14 **(Nested interval theorem: another part)** *Let* $[a_1, b_1] \supseteq [a_2, b_2] \supseteq \cdots$ *be a sequence of intervals. If* $\lim(b_i - a_i) \to 0$*, then* $\cap[a_n, b_n]$ *is a singleton set.*

EXERCISE 3.1.15 *Consider the sequence* $(a_n = \frac{1}{n})$.

a) *Let* $a \neq 0$. *Then* $a_n \nrightarrow a$ *as* $\exists \epsilon > 0$ *such that* $B_\epsilon(a)$ *misses infinitely many terms of* (a_n). *Give a value for* ϵ *and argue how it is missing so many terms.*

b) *We know* $a_n \to 0$ *as each* $B_\epsilon(a)$ *contains a tail (which may depend on* ϵ*) of* (a_n). *Which tail?*