MA 101 (Mathematics - I)
Differentiation : Exercise set 2

CMVT/L'Hôpital's Rules

1. Use CMVT to derive the following: Suppose f, g are differentiable on $[a, b]$ and $|f'(x)| \geq |g'(x)| > 0$ for all x. Show that for $a \leq x < y \leq b$, $|f(y) - f(x)| \geq |g(y) - g(x)|$.

2. Find the following by using L'Hôpital's Rules, whenever needed. Do not forget to check the conditions needed for using L'Hôpital's Rules.
(i) $\displaystyle\lim_{x\to 0+} \frac{\sqrt{1+x}-1}{\sqrt{x}}$ (ii) $\displaystyle\lim_{x\to \frac{\pi}{2}} \frac{1-\sin x}{1+\cos 2x}$ (iii) $\displaystyle\lim_{x\to 0} \frac{x^2 \sin \frac{1}{x}}{\sin x}$ (iv) $\displaystyle\lim_{x\to 0+} \left(\frac{\sin x}{x}\right)^{1/x}$
(v) $\displaystyle\lim_{x\to 0+} \frac{e^{-1/x^2}}{x}$ (vi) $\displaystyle\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$ (vii) $\displaystyle\lim_{x\to \infty} \frac{x - \sin x}{2x + \sin x}$ (viii) $\displaystyle\lim_{x\to \pi/2-} (\sec x - \tan x)$.

3. Let f be a differentiable on $(0, \infty)$ and suppose that $\displaystyle\lim_{x\to\infty} \big(f(x) + f'(x)\big) = L$. Show that $\displaystyle\lim_{x\to\infty} f(x) = L$ and $\displaystyle\lim_{x\to\infty} f'(x) = 0$. [Hint. $f(x) = \frac{e^x f(x)}{e^x}$.]

4. Try to use L'Hôpital's Rule to find the limit of $\dfrac{\tan x}{\sec x}$ as $x \to (\pi/2)-$. Also, evaluate it directly by changing to sines and cosines.

Taylor's Theorem

5. Let x_0 be a fixed in $\mathbb{R}$. Find the n-th Taylor polynomial and the remainder for the following functions f about x_0, and check for $x \in \mathbb{R}$ whether the remainder term converges to zero as $n \to \infty$.
(i) $f(x) := e^x$ on $\mathbb{R}$, (ii) $f(x) := \sin x$ on $\mathbb{R}$,

6. Show that for any $k \in \mathbb{N}$ and for all $x > 0$
$$x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots - \frac{x^{2k}}{2k} < \ln(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{x^{2k+1}}{2k+1}.$$

7. For a differentiable function $f : [a, b] \to \mathbb{R}$, a point $c \in (a, b)$ is called a **point of inflection** of f if $f(x) - f(c) - f'(c)(x - c)$ changes sign as x increases through c in an interval containing c.
Suppose $n \in \mathbb{N}$ is odd, $f'(c) = \cdots = f^{(n-1)}(c) = 0$ and $f^{(n)}(c) \neq 0$. Show that c is a point of inflection for f.

8. What is the Taylor series for a polynomial?

9. Consider the function
$$f(t) = \begin{cases} e^{-1/t}, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0. \end{cases}$$
Show that
(1) f is infinitely differentiable on $\mathbb{R}$.
(2) f has a Taylor series about the point 0.
(3) the Taylor series converges to a function different from f.

10. Determine whether $x = 0$ is a point of local maximum/minimum of the following functions defined on $\mathbb{R}$:
(i) $f(x) := x^4 - x^3 + 2$, (ii) $g(x) := x - \sin x$, (iii) $h(x) = \sin x + \frac{1}{6}x^3$, (iv) $k(x) := \cos x - 1 + \frac{1}{2}x^2$.

Limit superior/inferior

11. Find limit superior and limit inferior of the following sequences.

 (1) $a_n = \frac{n}{n+1}$, if n is odd, and $a_n = \frac{1}{n}$, if n is even.
 (2) $a_n = (-1)^n(1 - \frac{1}{n})$.
 (3) $a_n = (-1)^n(n + \frac{1}{2^n})$
 (4) $(1, -1, \frac{1}{2}, -2, \frac{1}{3}, -3, \ldots)$
 (5) $(-1)^n(1 - \frac{1}{n})n^{1/n}$