MA 101 (Mathematics I)

Multivariable Calculus : Practice Problem Set - 3

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1. If $f(x,y) = e^x(x\cos y - y\sin y)$ for all $(x,y) \in \mathbb{R}^2$, then show that $f_{xx}(x,y) + f_{yy}(x,y) = 0$ for all $(x,y) \in \mathbb{R}^2$.

2. If $f(x,y) = x^2 \tan^{-1}\left(\frac{y}{x}\right)$ for all $(x,y) \in \mathbb{R}^2 \setminus \{(x,y) \in \mathbb{R} : x \neq 0\}$, then find $\frac{\partial^2 f}{\partial x \partial y}(1,1)$.

3. If $f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$ for all $(x,y,z) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$, then show that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$ at each point of $\mathbb{R}^3 \setminus \{(0,0,0)\}$.

4. Find all $\mathbf{u} \in \mathbb{R}^2$ with $\|\mathbf{u}\| = 1$ for which the directional derivative $D_{\mathbf{u}}f(0,0)$ exists, if for all $(x,y) \in \mathbb{R}^2$,
   (a) $f(x,y) = \sqrt{|x^2 - y^2|}$.
   (b) $f(x,y) = \big|\,|x| - |y|\,\big| - |x| - |y|$.
   (c) $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$
   (d) $f(x,y) = \begin{cases} \frac{x}{y} & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$

5. State TRUE or FALSE with justification for each of the following statements.
   (a) If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous such that $f_x(0,0)$ exists (in $\mathbb{R}$), then $f_y(0,0)$ must exist (in $\mathbb{R}$).
   (b) If $f : \mathbb{R}^2 \to \mathbb{R}$ is such that for each $\mathbf{u} \in \mathbb{R}^2$ with $\|\mathbf{u}\| = 1$, the directional derivative of f at $(0,0)$ along $\mathbf{u}$ is 0, then f must be continuous at $(0,0)$.

6. Let the height $H(x,y)$ of a hill from the ground (considered as the xy-plane) at the point (x,y) be given by $H(x,y) = 1000 - 0.005x^2 - 0.01y^2$. We assume that the positive x-axis points east and the positive y-axis points north. Consider a person situated at the point $(60, 40, 966)$ on the hill.
   (a) If the person starts walking due south, then will (s)he start to ascend or descend the hill?
   (b) If the person starts walking north-west, then will (s)he start to ascend or descend the hill?
   (c) If the person starts climbing further, in which direction will (s)he find it most difficult to climb?

7. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} \frac{x^2y(x-y)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$
   Examine whether $f_{xy}(0,0) = f_{yx}(0,0)$.

8. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$
   Determine all the points of $\mathbb{R}^2$ where $f_{xy} : \mathbb{R}^2 \to \mathbb{R}$ and $f_{yx} : \mathbb{R}^2 \to \mathbb{R}$ are continuous.

9. Let $f(x, y) = x + y^2 + xy$ for all $(x, y) \in \mathbb{R}^2$. Using directly the definition of differentiability, show that $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable and also find $f'(x_0, y_0)$, where $(x_0, y_0) \in \mathbb{R}^2$.

10. Let S be a nonempty open subset of $\mathbb{R}^m$ and let $g : S \to \mathbb{R}^m$ be continuous at $\mathbf{x}_0 \in S$. If $f : S \to \mathbb{R}$ is such that $f(\mathbf{x}) - f(\mathbf{x}_0) = g(\mathbf{x}) \cdot (\mathbf{x} - \mathbf{x}_0)$ for all $\mathbf{x} \in S$, then show that f is differentiable at $\mathbf{x}_0$.

11. The directional derivatives of a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ at $(0, 0)$ in the directions of $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$ and $\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$ are 1 and 2 respectively. Find $f_x(0, 0)$ and $f_y(0, 0)$.

12. Examine the differentiability of f at $\mathbf{0}$, where
    (a) $f : \mathbb{R}^m \to \mathbb{R}$ satisfies $|f(\mathbf{x})| \le \|\mathbf{x}\|^2$ for all $\mathbf{x} \in \mathbb{R}^n$.
    (b) $f : \mathbb{R}^m \to \mathbb{R}$ is defined by $f(\mathbf{x}) = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$.
    (c) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x, y) = \sqrt{|xy|}$ for all $(x, y) \in \mathbb{R}^2$.
    (d) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x, y) = \big|\, |x| - |y| \,\big| - |x| - |y|$ for all $(x, y) \in \mathbb{R}^2$.
    (e) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x, y) = \begin{cases} \frac{x^3}{x^2+y^2} & \text{if } (x, y) \ne (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$
    (f) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x, y) = \begin{cases} \frac{y}{|y|}\sqrt{x^2 + y^2} & \text{if } y \ne 0, \\ 0 & \text{if } y = 0. \end{cases}$
    (g) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x, y) = \begin{cases} \sqrt{x^2 + y^2} & \text{if } y > 0, \\ x & \text{if } y = 0, \\ -\sqrt{x^2 + y^2} & \text{if } y < 0. \end{cases}$
    (h) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x, y) = \begin{cases} 1 & \text{if } y < x^2 < 2y, \\ 0 & \text{otherwise.} \end{cases}$
    (i) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x, y) = \begin{cases} x & \text{if } |x| < |y|, \\ -x & \text{if } |x| \ge |y|. \end{cases}$
    (j) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x, y) = \begin{cases} \frac{\sin(x^2y^2)}{x^2+y^2} & \text{if } (x, y) \ne (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$
    (k) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x, y) = \begin{cases} \sin^2 x + x^2 \sin \frac{1}{x} & \text{if } x \ne 0, \\ 0 & \text{if } x = 0. \end{cases}$

13. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \begin{cases} (x^2 + y^2) \sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) & \text{if } (x, y) \ne (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$
    Show that f is differentiable at $(0, 0)$ although neither $f_x : \mathbb{R}^2 \to \mathbb{R}$ nor $f_y : \mathbb{R}^2 \to \mathbb{R}$ is continuous at $(0, 0)$.

14. Let $f(x, y) = \begin{cases} (x^2 + y^2) \cos\left(\frac{1}{x^2+y^2}\right) & \text{if } (x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}, \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$
    Examine whether $f : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable.

15. Let $\alpha \in \mathbb{R}$ and $\alpha > 0$. If $f(x, y) = |xy|^\alpha$ for all $(x, y) \in \mathbb{R}^2$, then determine all values of α for which $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(0, 0)$.

16. Determine all the points of $\mathbb{R}^2$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable, where for all $(x, y) \in \mathbb{R}^2$,
(a) $f(x, y) = |xy|$ (b) $f(x, y) = (xy)^{\frac{2}{3}}$ (c) $f(x, y) = |x| \sin(x^2 + y^2)$
(d) $f(x, y) = \begin{cases} x^2 + y^2 & \text{if both } x, y \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$

17. State TRUE or FALSE with justification for each of the following statements.
(a) If $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ and if $f(x, y) = |xy|$ for all $(x, y) \in S$, then $f : S \to \mathbb{R}$ is differentiable.
(b) There exists a function $f : \mathbb{R}^2 \to \mathbb{R}$ which is differentiable only at $(1, 0)$.

18. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable at $(0, 0)$ and let $\lim\limits_{x \to 0} \frac{f(x,x) - f(x,-x)}{x} = 1$. Find $f_y(0, 0)$.

19. Let $f : \mathbb{R}^m \to \mathbb{R}$ be differentiable at $\mathbf{0}$ and let $f(\alpha \mathbf{x}) = \alpha f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^m$ and for all $\alpha \in \mathbb{R}$. Show that $f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$.

20. Let $f : \mathbb{R}^m \to \mathbb{R}$ be differentiable at $\mathbf{0}$ and $f(\mathbf{0}) = 0$. Show that there exist $\alpha > 0$ and $r > 0$ such that $|f(\mathbf{x})| \le \alpha \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^m$ with $\|\mathbf{x}\| < r$.

21. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that f_x exists (in $\mathbb{R}$) at all points of $B_\delta\big((x_0, y_0)\big)$ for some $(x_0, y_0) \in \mathbb{R}^2$ and $\delta > 0$, f_x is continuous at (x_0, y_0) and $f_y(x_0, y_0)$ exists (in $\mathbb{R}$). Show that f is differentiable at (x_0, y_0).

22. Let $f,\, g : S \subseteq \mathbb{R}^m \to \mathbb{R}$ be differentiable at $\mathbf{x}_0 \in S^0$. Show that
(a) $f + g : S \to \mathbb{R}$ is differentiable at $\mathbf{x}_0$ and $\nabla(f + g)(\mathbf{x}_0) = \nabla f(\mathbf{x}_0) + \nabla g(\mathbf{x}_0)$.
(b) $fg : S \to \mathbb{R}$ is differentiable at $\mathbf{x}_0$ and $\nabla(fg)(\mathbf{x}_0) = g(\mathbf{x}_0)\nabla f(\mathbf{x}_0) + f(\mathbf{x}_0)\nabla g(\mathbf{x}_0)$.
(c) if $g(\mathbf{x}_0) \ne 0$, then $\frac{f}{g} : S \to \mathbb{R}$ is differentiable at $\mathbf{x}_0$ and $\nabla\big(\frac{f}{g}\big)(\mathbf{x}_0) = \frac{g(\mathbf{x}_0)\nabla f(\mathbf{x}_0) - f(\mathbf{x}_0)\nabla g(\mathbf{x}_0)}{g(\mathbf{x}_0)^2}$.

23. Using the linearization of a suitable function at a suitable point, find an approximate value of $\big((3.8)^2 + 2(2.1)^3\big)^{\frac{1}{5}}$.

24. Show that the maximum error in calculating the volume of a right circular cylinder is approximately $\pm\, 8\%$ if its radius can be measured with a maximum error of $\pm\, 3\%$ and its height can be measured with a maximum error of $\pm\, 2\%$.