MA 101 (Mathematics - I)
Tutorial Problems 4: (Differentiability 1, 2)

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1. Give an example of a continuous function on $\mathbb{R}$ which is not differentiable exactly at
(i) 1, (ii) $1, 2, 3$, (iii) every integer.

> **Solution:** (i) $|x-1|$, (ii) $|x-1|+|x-2|+|x-3|$, (iii) $f(x) = \begin{cases} x-[x] & \text{if } [x] \text{ is even,} \\ 1-(x-[x]) & \text{if } [x] \text{ is odd.} \end{cases}$
> or $f(x) = |\sin \pi x|$. Verify the claims.

2. Let $r > 0$ be a rational number, and $f : [0, \infty) \to \mathbb{R}$ be defined by $f(x) = x^r sin\frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$. Determine those values of r for which f is differentiable.

> **Solution:** For $x > 0$, f is differentiable at x, and
> $$\frac{d}{dx}x^r \sin\frac{1}{x} = rx^{r-1}\sin\frac{1}{x} - x^{r-2}\cos\frac{1}{x}.$$
> At the point 0: We have $\dfrac{f(x)-f(0)}{x-0} = x^{r-1}\sin\dfrac{1}{x}$ has the limit 0 as $x \to 0$, if $r > 1$.
> Let $0 < r \leq 1$. Then $\lim\limits_{x\to 0+} x^{r-1}\sin\dfrac{1}{x}$ does not exist.
> Thus, f is differentiable on $[0, \infty)$ if and only if $r > 1$.

3. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $c \in \mathbb{R}$, show that $f'(c) = \lim \Big(n\big(f(c+\frac{1}{n}) - f(c)\big)\Big)$.
Show by an example that the existence of the limit of this sequence does not imply the existence of $f'(c)$.

> **Solution:** Consider $g(x) = \dfrac{f(x)-f(c)}{x-c}$ for $x \neq c$. Then $\lim\limits_{x\to c} g(x) = f'(c)$. Since the sequence $(c + \frac{1}{n})$ converges to c, we must have $g(c+\frac{1}{n}) \to f'(c)$.
> Consider $f : \mathbb{R} \to \mathbb{R}$ given by
> $$f(x) = \begin{cases} 1, & x \in \{c+\frac{1}{n} : n \in \mathbb{N}\} \cup \{c\}, \\ 0, & \text{otherwise.} \end{cases}$$
> Then the given limit is 0, but f is not continuous at c.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Let $n \in \mathbb{N},\ a \in \mathbb{R}$. Find the limit $\lim\limits_{x\to a} \dfrac{a^n f(x) - x^n f(a)}{x-a}$.

> **Solution:** $\lim\limits_{x\to a} \dfrac{a^n f(x) - x^n f(a)}{x-a} = \lim\limits_{x\to a} \dfrac{a^n f(x) - a^n f(a) - (x^n f(a) - a^n f(a))}{x-a} = a^n f'(a) - f(a)na^{n-1}$.

5. Let $f : I \to \mathbb{R}$ be differentiable at $c \in (a, b)$, and $x_n < c < y_n$ in I such that $y_n - x_n \to 0$. Find $\lim\limits_{n\to\infty} \dfrac{f(y_n) - f(x_n)}{y_n - x_n}$, if it exists.

Solution: By Carathéodory's theorem, $f(x)-f(c)=\phi(x)(x-c)$ on I, where ϕ is continuous at c and $\phi(c)=f'(c)$. Thus,

$$\begin{aligned}
f(y_n)-f(x_n) &= \big(f(y_n)-f(c)\big)-\big(f(x_n)-f(c)\big)=\phi(y_n)(y_n-c)-\phi(x_n)(x_n-c)\\
&= (y_n-x_n)\phi(y_n)+x_n\big(\phi(y_n)-\phi(x_n)\big)-c\big(\phi(y_n)-\phi(x_n)\big)\\
&= (y_n-x_n)\left[\phi(y_n)-\frac{c-x_n}{y_n-x_n}\big(\phi(y_n)-\phi(x_n)\big)\right]
\end{aligned}$$

that is,

$$\begin{aligned}
\left|\frac{f(y_n)-f(x_n)}{(y_n-x_n)}-f'(c)\right| &= \left|(\phi(y_n)-f'(c))-\frac{c-x_n}{y_n-x_n}\big(\phi(y_n)-\phi(x_n)\big)\right|\\
&\le |\phi(y_n)-f'(c))|+\frac{c-x_n}{y_n-x_n}|\phi(y_n)-\phi(x_n)|\\
&\le |\phi(y_n)-f'(c))|+|\phi(y_n)-\phi(x_n)| \ \to\ 0,
\end{aligned}$$

as $0<\dfrac{c-x_n}{y_n-x_n}<1$, $\phi(y_n)-\phi(x_n)\to\phi(c)-\phi(c)=0$, and $\phi(y_n)\to\phi(c)=f'(c)$. Therefore, $\displaystyle\lim_{n\to\infty}\frac{f(y_n)-f(x_n)}{y_n-x_n}=f'(c)$.

6. Suppose $f:[a,b]\to\mathbb{R}$ is differentiable on (a,b) and $\displaystyle\lim_{x\to a+}f'(x)=\ell$. Show that f is differentiable at a and $f'(a)=\ell$ if and only if f is continuous at a.

Solution: If f is differentiable at a, then it must be continuous at a, that is, continuity at a is necessary.

Conversely, suppose f is continuous at a. Let $\epsilon>0$ be given. Then there exists $0<\delta\le b-a$ such that $|f'(c)-\ell|<\epsilon$ for $c\in(a,a+\delta)$. Now, let $x\in(a,a+\delta)$. Then, f is continuous on $[a,x]$ and differentiable on (a,x). By MVT, there is $c\in(a,x)$ such that $\dfrac{f(x)-f(a)}{x-a}=f'(c)$. Thus,

$$\left|\frac{f(x)-f(a)}{x-a}-\ell\right|=|f'(c)-\ell|<\epsilon.$$

Since this is true for $x\in(a,a+\delta)$, f is differentiable at a and $f'(a)=\ell$.

7. Let $f:[a,b]\to[a,b]$ be differentiable. Assume that $f'(x)\neq 1$ for $x\in[a,b]$. Prove that f has a unique fixed point in $[a,b]$.

Solution: Since f is continuous, we know that f has a fixed point (using IVT). Suppose, if possible, $c,d\in[a,b],\ c<d,\ f(c)=c,\ f(d)=d$. Since f is differentiable on $[c,d]$, by MVT we get $t\in(c,d)$ such that $f'(t)=\dfrac{f(d)-f(c)}{d-c}=1$, a contradiction.

8. Let $f:[a,b]\to\mathbb{R}$ be differentiable. Assume that there exists no $x\in[a,b]$ such that $f(x)=0=f'(x)$. Prove that the number of zeroes of f in $[a,b]$ is finite.

Solution: Suppose there are infinitely many zeros of f in $[a,b]$. Then we get a sequence (x_n) of distinct points in $[a,b]$ such that $f(x_n)=0$ for every n. By Bolzano-Weierstrass Theorem, we get a convergent subsequence, say (y_n) of (x_n). Let $c=\lim y_n$. Then $c\in[a,b]$. Since f is continuous at c, we have

$f(c) = \lim f(y_n) = 0$.
(Note that at most one y_n can be c. Delete it from the sequence if there is one.)
Let $g(x) = \dfrac{f(x) - f(c)}{x - c}$ for $x \neq c$. Then, $\lim_{x \to c} g(x) = f'(c)$. Since $y_n \to c$, and $g(y_n) = 0$, we get $f'(c) = 0$, that is, $f(c) = 0 = f'(c)$.

9. Show that $\frac{\sin x}{x}$ is strictly decreasing on $(0, \pi/2)$.
(There was a typo: decreasing, not increasing.)

Solution: Differentiate. Differentiate the numerator again.

10. Consider the function $h : \mathbb{R} \to \mathbb{R}$ given by $h(x) = x^3 + 2x + 1$. Show that h is a bijection, and therefore has an inverse h^{-1} on $\mathbb{R}$. Find $(h^{-1})'(y)$ at the points y corresponding to $x = 0, 1, -1$.

Solution: First, show that $h(x) = h(y) \implies x = y$, so that h is one-one. As $\lim_{x \to \infty} h(x) = \infty$ and $\lim_{x \to -\infty} h(x) = -\infty$, h is onto, by intermediate value theorem. Since $h'(0) = 2, h'(\pm 1) = 5$, we have $(h^{-1})'(y)$ at the points corresponding to $x = 0, 1, -1$ are $1/2, 1/5, 1/5$, respectively.

11. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable such that $f(0) = f(1) = 0$ and $f'(0) > 0, f'(1) > 0$. Show that there are distinct $c_1, c_2 \in (0, 1)$ such that $f'(c_1) = f'(c_2) = 0$.

Solution: Since $f'(0) > 0$, there is $0 < \delta_1 < 1$ such that $f(x) > f(0) = 0$ for all $x \in (0, \delta_1)$. Similarly, since $f'(1) > 0$ there is $0 < \delta_2 < 1$ such that $f(x) < f(1) = 0$ for all $x \in (\delta_2, 1)$. Thus, $f(\frac{\delta_1}{2}) > 0$ and $f(\frac{1+\delta_2}{2}) < 0$. By, IVT, there is c between $\frac{\delta_1}{2}$ and $\frac{1+\delta_2}{2}$ such that $f(c) = 0$. Now use Rolle's theorem for $[a, c]$ and $[c, b]$.