MA 101 (Mathematics - I)
Tutorial Problems 5: (Differentiation 3,4,5a)

1. Find the following by using L'Hôpital's Rules, whenever needed. Do not forget to check the conditions needed for using L'Hôpital's Rules.

(i) $\lim\limits_{x\to 0+}\left(\dfrac{1}{x}-\dfrac{1}{\operatorname{Arctan} x}\right)$ (ii) $\lim\limits_{x\to\infty}\dfrac{x+\ln x}{x\ln x}$ (iii) $\lim\limits_{x\to 0+}(1+3/x)^x$ (iv) $\lim\limits_{x\to\infty} x^{1/x}$

Solution:
(i) $\lim\limits_{x\to 0+}\left(\dfrac{1}{x}-\dfrac{1}{\tan^{-1} x}\right)=\lim\limits_{x\to 0+}\dfrac{\tan^{-1}x-x}{x\tan^{-1}x}\left(\dfrac{0}{0}\right)=\lim\limits_{x\to 0+}\dfrac{\frac{1}{1+x^2}-1}{\tan^{-1}x+\frac{x}{1+x^2}}=\lim\limits_{x\to 0+}\dfrac{-x^2}{(1+x^2)\tan^{-1}x+x}$,

by LR1, if the last limit exists. The last limit is in the form $\left(\frac{0}{0}\right)$, and therefore equals

$$\lim_{x\to 0+}\frac{-2x}{(1+x^2)\frac{1}{1+x^2}+2x\tan^{-1}x+1}=\lim_{x\to 0+}\frac{-2x}{2+2x\tan^{-1}x}=\frac{0}{2}=0.$$

(ii) $\lim\limits_{x\to\infty}\dfrac{x+\ln x}{x\ln x}\left(\dfrac{\infty}{\infty}\right)=\lim\limits_{x\to\infty}\dfrac{1+\frac{1}{x}}{\ln x+1}$, by LR2, if the last limit exists. The last limit is 0, since for $\epsilon>0$ we have

$$\left|\frac{1+\frac{1}{x}}{\ln x+1}\right|<\frac{2}{\ln x}<\epsilon,$$

if $x>\max\{1,e^{2/\epsilon}\}=M$, say. Therefore the given limit is zero.

(iii) Suppose $f(x)=(1+3/x)^x$ for $x>0$. Then $\ln f(x)=x\ln(1+3/x)=\frac{\ln(1+3/x)}{1/x}$. Since $\ln(1+3/x)$ and $1/x$ are differentiable on $(0,1]$, and $\lim\limits_{x\to 0+}\ln(1+3/x)=\lim\limits_{x\to 0+}1/x=\infty$, by LH2

$$\lim_{x\to 0+}\ln f(x)=\lim_{x\to 0+}\frac{\frac{1}{1+3/x}\cdot\frac{-3}{x^2}}{\frac{-1}{x^2}}=\lim_{x\to 0+}\frac{3x}{x+3}=0.$$

Since Exp is continuous, $\lim\limits_{x\to 0}f(x)=\lim\limits_{x\to 0}e^{\ln f(x)}=e^{\left(\lim\limits_{x\to 0}\ln f(x)\right)}=e^0=1.$

(iv) Similar to (iii). Ans. 1.

2. Let $f:\mathbb{R}\to\mathbb{R}$ have second derivative at $c\in\mathbb{R}$. Prove that $\lim\limits_{h\to 0}\dfrac{f(c+h)-2f(c)+f(c-h)}{h^2}=f''(c)$.
Give example of a function $f:\mathbb{R}\to\mathbb{R}$ and a point $c\in\mathbb{R}$ for which the above limit exists, but $f''(c)$ does not exist.

Solution: Since $f''(c)$ exists, there is an inverval $(c-r,c+r), r>0$, on which f is differentiable. For $h\in(-r,r)$, define $g(h)=f(c+h)-2f(c)+f(c-h)$. Then, $\lim\limits_{h\to 0}g(h)=0$, and $g'(h)=f'(c+h)-f'(c-h)$. By, LH1, the given limit is $\lim\limits_{h\to 0}\dfrac{g'(h)}{2h}=\lim\limits_{h\to 0}\dfrac{f'(c+h)-f'(c-h)}{2h}$, if it exists. Now,

$$\lim_{h\to 0}\frac{f'(c+h)-f'(c-h)}{2h}=\frac{1}{2}\left[\lim_{h\to 0}\frac{f'(c+h)-f'(c)}{h}+\lim_{h\to 0}\frac{f'(c-h)-f'(c)}{-h}\right]=\frac{1}{2}\left[f''(c)+f''(c)\right]=f''(c).$$

Hence the limit is $f''(c)$.

Example of a functions for which the limit exists, but $f''(c)$ does not exist:

(1) Define $f:\mathbb{R}\to\mathbb{R}$ by $f(x)=\begin{cases}1, & x>0,\\ 0, & x=0,\\ -1 & x<0.\end{cases}$

The corresponding limit with $c=0$ is 0, since $f(h)-2f(0)+f(-h)=0$. The function is not even continuous at 0.

(2) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$, if $x \geq 0$, and $f(x) = -x^2$, if $x < 0$. Then f has continuous derivative. Moreover, for any $h \in \mathbb{R}$, $f(-h) = -f(h)$ and therefore $f(h) - 2f(0) + f(-h) = 0$. However, $f''(0)$ does not exist.

3. For $x > 0$ show that $|(1+x)^{1/3} - (1 + \frac{1}{3}x - \frac{1}{9}x^2)| < (5/81)x^3$. Use this inequality to approximate $\sqrt[3]{1.2}$ and $\sqrt[3]{2}$, and find the bounds for errors in the estimations.

Solution: For $x > -1$, consider $f(x) = (1+x)^{1/3}$. Then f has derivates of all orders. We have

$$f'(x) = \frac{1}{3}(1+x)^{-2/3}, \; f''(x) = \frac{-2}{9}(1+x)^{-5/3}, \; f^{(3)}(x) = \frac{10}{27}(1+x)^{-8/3}.$$

Therefore, for $x > 0$, by Taylor's theorem, there is $c \in (0, x)$ such that

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(c)}{3!}x^3 = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}(1+c)^{-8/3}x^3.$$

Therefore, $|(1+x)^{1/3} - (1 + \frac{1}{3}x - \frac{1}{9}x^2)| < (5/81)x^3$, since $(1+c)^{-8/3} < 1$.

$\sqrt[3]{1.2} = f(1/5) = 1 + 1/15 - 1/225 = 239/225 \approx 1.06222$, Error $< \dfrac{1}{25 \cdot 81} < 0.0005$.

$\sqrt[3]{2} = f(1) = 1 + \frac{1}{3} - \frac{1}{9} = 11/9$, Error $< 5/81$.

4. Find the Taylor series of $\sin x \cos 3x$ about 0. What is the domain of convergence (the set in which f is given by the Taylor series)?

Solution: We have $f(x) = \sin x \cos 3x = \frac{1}{2}(\sin 4x - \sin 2x)$, and

$$f^{(n)}(x) = \frac{1}{2}\left[\frac{d^n}{dx^n}\sin 4x - \frac{d^n}{dx^n}\sin 2x\right].$$

Therefore

$$f^{(n)}(0) = \begin{cases} 0, & \text{if } n \text{ is even,} \\ \frac{1}{2}(4^n - 2^n) & \text{if } n = 4k+1, k \geq 0, \\ -\frac{1}{2}(4^n - 2^n) & \text{if } n = 4k+3, \; k \geq 0. \end{cases}$$

Thus, the Taylor series for f is

$$T(f, 0) = \frac{4-2}{2}x - \frac{4^3 - 2^3}{2 \cdot 3!}x^3 + \frac{4^5 - 2^5}{2 \cdot 5!}x^5 - \frac{4^7 - 2^7}{2 \cdot 7!}x^7 + \cdots.$$

Moreover, $|f^{(n)}(x)| < \frac{1}{2}(4^n + 2^n) < 4^n$. Therefore, for $x \in \mathbb{R}$, $|R_n| = \dfrac{|f^{(n+1)}(c)|}{(n+1)!}|x|^{n+1} \leq \dfrac{(4|x|)^{n+1}}{(n+1)!} \to 0$ (Use ratio test.) Thus, the doamin of convergence of the Taylor series is $\mathbb{R}$.

5. (a) Show that for $n \geq 0$, $\displaystyle\lim_{x \to 0} \frac{e^{-1/x^2}}{x^n} = 0$.

(b) Use induction to prove Leibniz's rule for the n-th derivative of a product:

$$(fg)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x) g^{(k)}(x).$$

(c) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} e^{-1/x^2}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Show that f is infinitely differentiable on $\mathbb{R}$ and find $f^{(n)}(0)$ for $n \in \mathbb{N}$.

(d) What is the Taylor series of f about 0, and on what interval is it defined?

(e) Does the remainder term for f at 0 in Taylor's theorem converge to zero as $n \to \infty$?

(f) On what set does the Taylor series converge to f?

Solution:

(a) Put $t = 1/x$. As $x \to 0+$, $t \to \infty$. Also $\dfrac{e^{-1/x^2}}{x^n} = \dfrac{t^n}{e^{t^2}}$, and $\lim\limits_{t\to\infty} \dfrac{t^n}{e^{t^2}}$ is of (∞/∞) form. BY LH2,

$$\lim_{t\to\infty} \frac{t^n}{e^{t^2}} = \lim_{t\to\infty} \frac{nt^{n-1}}{2te^{t^2}} = \frac{n}{2} \lim_{t\to\infty} \frac{t^{n-2}}{e^{t^2}}.$$

Since $\lim\limits_{t\to\infty} \dfrac{1}{e^{t^2}} = 0$ and $\lim\limits_{t\to\infty} \dfrac{t}{e^{t^2}} = 0$ (because $e^{t^2} > t^2$), by repeated application of the above, we get

$$\lim_{x\to 0+} \frac{e^{-1/x^2}}{x^n} = \lim_{t\to\infty} \frac{t^n}{e^{t^2}} = 0.$$

We also have $\lim\limits_{x\to 0-} \dfrac{e^{-1/x^2}}{x^n} = \lim\limits_{x\to 0+} \dfrac{e^{-1/x^2}}{(-x)^n} = 0.$

(b) (Note: here $f^{(0)}$ means f.) We have $(fg)' = f'g + fg'$, that is, the statement is true for $n = 1$. Assume

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}.$$

Then

$$\begin{aligned}
(fg)^{(n+1)} &= \sum_{k=0}^{n} \binom{n}{k} \left[f^{(n-k+1)} g^{(k)} + f^{(n-k)} g^{(k+1)} \right] \\
&= \sum_{k=0}^{n} \binom{n}{k} f^{(n-k+1)} g^{(k)} + \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k+1)} \\
&= f^{(n+1)} g^{(0)} + \sum_{k=1}^{n} \binom{n}{k} f^{(n-k+1)} g^{(k)} + \sum_{k=0}^{n-1} \binom{n}{k} f^{(n-k)} g^{(k+1)} + f^{(0)} g^{(n+1)} \\
&= f^{(n+1)} g^{(0)} + \sum_{k=1}^{n} \binom{n}{k} f^{(n-k+1)} g^{(k)} + \sum_{k=1}^{n} \binom{n}{k-1} f^{(n-(k-1))} g^{(k)} + f^{(0)} g^{(n+1)} \\
&= f^{(n+1)} g^{(0)} + \sum_{k=1}^{n} \left[\binom{n}{k} + \binom{n}{k-1} \right] f^{((n+1)-k))} g^{(k)} + f^{(0)} g^{(n+1)} \\
&= f^{(n+1)} g^{(0)} + \sum_{k=1}^{n} \binom{n+1}{k} f^{((n+1)-k))} g^{(k)} + f^{(0)} g^{(n+1)} \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} f^{((n+1)-k))} g^{(k)},
\end{aligned}$$

that is, the statement is true for $n + 1$. By induction, the result follows.

(c) Let $x \neq 0$. Then $f'(x) = e^{-1/x^2} \cdot 2x^{-3}$ and $f''(x) = e^{-1/x^2}\big(-6x^{-4} + 4x^{-6}\big)$. Observe that for $n = 1, 2$, $f^{(n)}(x) = f(x)P_n(\frac{1}{x})$, where $P_n(t)$ is a polynomial. Suppose $f^{(k)}(x) = f(x)P_k(\frac{1}{x})$ for some polinomial $P_k(t)$. Then

$$f^{(k+1)}(x) = f'(x)P_k(1/x) + f(x)P'(1/x) \cdot \frac{-1}{x^2} = f(x)P_{k+1}(1/x),$$

where $P_{k+1}(t) = 2t^3P_k(t) - t^2P'(t)$, a polynomial. Therefore, for $n \geq 1$, $f^{(n)}(x) = f(x)P_n(\frac{1}{x})$ for some polynomial $P_n(t)$.

Claim: for $n \geq 1$, $f^{(n)}(0) = 0$. Clearly, $f'(0) = \lim\limits_{x\to 0} \dfrac{e^{-1/x^2}}{x} = 0$, by (a). Suppose $f^{(n-1)}(0) = 0$. Now, for $x \neq 0$, $f^{(n-1)}(x) = f(x)\big(a_0 + \frac{a_1}{x} + \cdots + \frac{a_m}{x^m}\big)$ for some $m \in \mathbb{N}, a_0, \ldots, a_m \in \mathbb{R}$. Therefore, $f^{(n)}(0) = \lim\limits_{x\to 0} \dfrac{f^{(n-1)}(x)}{x} = \lim\limits_{x\to 0} \Big[f(x)\Big(\dfrac{a_0}{x} + \dfrac{a_1}{x^2} + \cdots + \dfrac{a_m}{x^{m+1}}\Big)\Big] = 0$, by (a).
Hence, f is infinitely differentiable, and $f^{(n)}(0) = 0$, for all n.

(d) The Taylor series for f about is $T(f, 0) = \sum a_n x^n$, where $a_n = \dfrac{f^{(n)}(0)}{n!} = 0$.

(e) Since the Taylor polynomial $T_n(f, 0)(x)$ is the zero polynomial, the remainder term of f at $x \neq 0$ is $R_n(x) = f(x) - T_n(f, 0)(x) = e^{-1/x^2}$, which does not converge to 0 as $n \to \infty$.

(f) The Taylor series converges to f only on the set $\{0\}$.

6. Suppose $a_n \to a$, $b_n \to b$, and $a < b$. Suppose (c_n) is given by $c_{2n-1} = a_n, c_{2n} = b_n$ for $n \in \mathbb{N}$. What can you say about $\limsup c_n$ and $\liminf c_n$? Justify your claim. (Note that b can be ∞ and a can be $-\infty$.)

Solution: The sequence (c_n) is $(a_1, b_1, a_2, b_2, \ldots)$. Let $u = \limsup c_n$ and $\ell = \liminf c_n$. If $b = \infty$, then (c_n) is not bounded above, and so $u = \infty = b$.

Let $b \in \mathbb{R}$. Then (c_n) is bounded above. For $n \in \mathbb{N}$, let $u_n = \text{lub}\,\{c_k : k \geq n\}$. Let $0 < \epsilon \leq \frac{b-a}{2}$ be given. Then there exists $K \in \mathbb{N}$ such that $a_n < b - \epsilon$ and $b_n \in (b - \epsilon, b + \epsilon)$ for $n \geq K$. Then for $n \geq 2K$, $u_n \in [b - \epsilon, b + \epsilon]$, i.e., $|u_n - b| \leq \epsilon$. Thus, $u_n \to b$. Since $u_n \to \limsup c_n$, we have $u = \limsup c_n$.

Similarly, [or considering $(-c_n)$] we can show $\ell = a$.