MA 101 (Mathematics - I)
Tutorial 6: Differentiation 5b, Integration 1, 2

1. (a) Find an explicit formula for the function represented by the power series $\displaystyle\sum_{n=1}^{\infty} nx^n$ and indicate its domain of convergence.

   (b) Find an explicit formula for the function represented by the power series $\displaystyle\sum_{n=1}^{\infty} n^2x^n$ in its interval of convergence. Use it to find the sum of $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{2^n}$ and $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{3^n}$.

Solution: (a) The geometric series $\displaystyle\sum_{n=0}^{\infty} x^n$ converges to $g(x) = \dfrac{1}{1-x}$ with domain of convergence $(-1, 1)$. The term by term differentiation gives for $x \in (-1, 1)$

$$\sum_{n=1}^{\infty} nx^{n-1} = g'(x) = \frac{1}{(1-x)^2}, \quad \text{and therefore} \quad \sum_{n=1}^{\infty} nx^n = x\sum_{n=1}^{\infty} nx^{n-1} = \frac{x}{(1-x)^2}.$$

The domain of convergence is $(-1, 1)$.

(b) Again, by term by term differentiation to $\dfrac{x}{(1-x)^2} = \displaystyle\sum_{n=1}^{\infty} nx^n$ gives

$$\sum_{n=1}^{\infty} n^2x^{n-1} = \frac{1+x}{(1-x)^3}, \quad \text{and therefore} \quad \sum_{n=1}^{\infty} n^2x^n = \frac{x(1+x)}{(1-x)^3} = f(x) \text{ (say)}$$

on $(-1, 1)$. We have $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{2^n} = f(1/2) = 6$ and $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{3^n} = f(1/3) = 3/2$.

2. Find the Taylor series of the function $f(x) = (1+x)e^{-x} - (1-x)e^x$ about 0. Using this, find the sum of the series

$$\frac{1}{3!} + \frac{2}{5!} + \cdots + \frac{n}{(2n+1)!} + \cdots.$$

Solution: Note: Suppose that the power series $\sum a_nx^n$ has radius of convergence R. Define $f(x) = \sum a_nx^n$ on $(-R, R)$. Then $\sum a_nx^n$ is the Taylor series of f about 0 (because $a_n = \frac{f^{(n)}(0)}{n!}$) and the Taylor series converges to f on $(-R, R)$.

For $x \in \mathbb{R}$

$$\begin{aligned} f(x) &= x(e^x + e^{-x}) - (e^x - e^{-x}) = 2x\sum_{k=0}^{\infty}\frac{x^{2k}}{(2k)!} - 2\sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)!} \\ &= 2\sum_{k=0}^{\infty}\frac{((2k+1)-1)x^{2k+1}}{(2k+1)!} = 4\sum_{k=1}^{\infty}\frac{kx^{2k+1}}{(2k+1)!}, \end{aligned}$$

because the involved series are convergent. This is the Taylor series for f with domain of convergence $\mathbb{R}$.

We have

$$\frac{1}{3!} + \frac{2}{5!} + \cdots + \frac{n}{(2n+1)!} + \cdots = \frac{1}{4}f(1) = \frac{1}{2e}.$$

3. Let $f : [a, b] \to R$ be a bounded function. If there is a partition P of $[a, b]$ such that $L(f, P) = U(f, P)$, then prove that f is a constant function.

Solution: Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$. Let

$$m_i = \operatorname{glb}\{f(x) : x \in [x_{i-1}, x_i]\} \text{ and } M_i = \operatorname{lub}\{f(x) : x \in [x_{i-1}, x_i]\}.$$

Since $0 = U(f,P) - L(f,P) = \displaystyle\sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1})$, we must have $m_i = M_i$ for $1 \le i \le n$. Therefore, there are constants $c_1, \ldots, c_n$ such that $f(x) = c_i$ for all $x \in [x_{i-1}, x_i]$. For $1 \le i \le n-1$, $c_i = c_{i+1}$, since $x_i \in [x_{i-1}, x_i] \cap [x_i, x_{i+1}]$. Hence f is a constant function.

4. Define $f : [-1,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Show that $f \in \mathcal{R}([-1,1])$ and that $\int_{-1}^{1} f = 0$.

(b) Define $F(x) = \int_{-1}^{x} f$ on $[-1,1]$. Show that F is differentiable. In particular, $F'(0) = f(0)$ although f is not continuous at 0.

Solution: (a) Note that for any partition P of $[-1,1]$, $L(f,P) = 0$. Let $\epsilon > 0$. We find that there is a partition P of $[-1,1]$ such that $U(f,P) < \epsilon$.

If $\epsilon > 2$, then for any P, $U(f,P) \le M(b-a) = M(1-(-1)) = 2 < \epsilon$. (Here $M = \operatorname{lub} f = 1$.) Suppose $\epsilon \le 2$.

Choose $n \in \mathbb{N}$ such that $1/n < \epsilon/2$. Choose $\delta > 0$ such that we get a partition

$$P = \{-1 < 0 < \frac{1}{n} + \delta < \frac{1}{n-1} - \delta < \frac{1}{n-1} + \delta < \cdots < \frac{1}{2} - \delta < \frac{1}{2} + \delta < 1 - \delta < 1\}$$

of $[-1,1]$. Since f has maximum value 1 on the intervals $[0, \frac{1}{n} + \delta], [\frac{1}{n-1} - \delta, \frac{1}{n-1} + \delta], \ldots, [\frac{1}{2} - \delta, \frac{1}{2} + \delta], [1-\delta, 1]$ and zero on the others, we have

$$U(f,P) = \frac{1}{n} + \delta + 2\delta(n-2) + \delta < \epsilon/2 + 2(n-1)\delta.$$

Therefore, if $\delta < \epsilon/(4(n-1))$, then $U(f,P) < \epsilon$. For this P, $U(f,P) - L(f,P) < \epsilon$. Hence, f is integrable. Further, $\int_{-1}^{1} f = \operatorname{lub}\{L(f,P) : P \text{ is a partition of } [-1,1]\} = 0$.

(b) For $x \in [-1,1]$, f is integrable on $[-1,x]$ and

$$F(x) = \int_{-1}^{x} f = \operatorname{lub}\{L(f,P) : P \text{ is a partition of } [-1,x]\} = 0.$$

F being a constant function, is differentiable. In particular, $F'(0) = f(0)$ although f is not continuous at 0.

(This exhibits that the converse of the First Fundamental Theorem of Calculus is not true.)

5. Show that (a) $\displaystyle\int_0^1 \frac{x^4}{\sqrt{1+4x^{90}}} \ge \frac{1}{5\sqrt{5}}$. (b) $\displaystyle\left|\int_0^3 \frac{x^3(x-4)}{1+x^{10}} \sin(2020x)dx\right| \le 81.$

Solution: (a) For $x \in [0,1]$ we have $1 + 4x^{90} \le 5$. Therefore,

$$\int_0^1 \frac{x^4}{\sqrt{1+4x^{90}}} \ge \frac{1}{\sqrt{5}} \int_0^1 x^4 = \frac{1}{5\sqrt{5}}.$$

(b) For $x \in [0,3]$,

$$|f(x)| = \left| \frac{x^3(x-4)}{1+x^{10}} \sin(2020x) \right| \le |x^3(x-4)|.$$

Now, $x^3(x-4)$ decreases from 0 to -27 on $[0,3]$. Hence $|\int_0^3 f| \le \int_0^3 |f| \le 27 \int_0^3 1 = 81$.

6. (1) If $f \in \mathcal{R}[a,b]$, $f \ge 0$ and $\int_a^b f = 0$, then show that $f = 0$ at each point of continuity of f.
(2) If f is continuous, $f \ge 0$ and $\int_a^b f = 0$, then conclude that $f = 0$ on $[a,b]$.
(3) Show that the results need not hold if $f \ge 0$ is not assumed.

Solution:
(1) Suppose f is continuous at $c \in [a,b]$. Suppose, if possible, $f(c) = \alpha > 0$. Then there exists $a_1, b_1 \in [a,b], a_1 < b_1$, such that $c \in [a_1, b_1]$ and $f(x) > \alpha/2$ for $x \in [a_1, b_1]$. Let P be a partition of $[a,b]$ with $[a_1, b_1]$ as one of the subintervals produced by P. Since $\text{glb}\{f(x) : x \in [a_1, b_1]\} \ge f(c)/2$, we get $\displaystyle\int_a^b f \ge L(f,P) \ge \frac{(b_1-a_1)f(c)}{2} > 0$, a contradiction. Hence $f(c) = 0$.

(2) Follows immediately from Part (1).

(3) Consider $f : [-1,1] \to \mathbb{R}$ defined by $f(x) = x$. Then f is continuous on $[-1,1]$ and we can show that $\int_{-1}^1 f = 0$. However, $f(x) \ne 0$ for any $x \ne 0$.

7. Let $f > 0$ and continuous on $[a,b]$. Let $M = \max f$ on $[a,b]$. Show that

$$\lim_{n\to\infty} \left(\int_a^b (f(x))^n dx \right)^{1/n} = M.$$

Solution: First note that

$$\left(\int_a^b (f(x))^n dx \right)^{1/n} \le \left(\int_a^b M^n dx \right)^{1/n} = M(b-a)^{1/n}$$

Let $0 < \epsilon < M$. Since $(b-a)^{1/n} \to 1$, there is $m_0 \in \mathbb{N}$ such that $1 - \epsilon/M < (b-a)^{1/n} < 1 + \epsilon/M$ for all $n \ge m_0$. Then, for $n \ge m_0$ we have

$$\left(\int_a^b (f(x))^n dx \right)^{1/n} \le M + \epsilon.$$

Let $c \in [a,b]$ be such that $f(c) = M$. There are $a_1, b_1 \in [a,b]$, $a_1 < b_1$, such that $c \in [a_1, b_1]$ and $f(x) > M - \epsilon/2$ for $x \in [a_1, b_1]$. Then,

$$\left(\int_a^b (f(x))^n dx \right)^{1/n} \ge \left(\int_{a_1}^{b_1} (f(x))^n dx \right)^{1/n} \ge \left(\int_{a_1}^{b_1} (M-\epsilon/2)^n dx \right)^{1/n} \ge (b_1-a_1)^{1/n}(M-\epsilon/2)$$

Since $(b_1-a_1)^{1/n} \to 1$, there is $m_1 \in \mathbb{N}$ such that $1 - \epsilon/(2M) < (b_1-a_1)^{1/n} < 1 + \epsilon/(2M)$ for $n \ge m_1$. Then for $n \ge m_1$ we have

$$\left(\int_a^b (f(x))^n dx \right)^{1/n} \ge (b_1-a_1)^{1/n}(M-\epsilon/2) \ge \Big(1-\epsilon/(2M)\Big)(M-\epsilon/2) = M-\epsilon/2-\epsilon/2+\epsilon^2/(4M) > M-\epsilon.$$

Hence, for $n \geq \max\{m_0, m_1\}$ we have

$$\left| \left(\int_a^b (f(x))^n dx \right)^{1/n} - M \right| < \epsilon.$$

Therefore, the result follows.