IIT GUWAHATI

MA-102 – MATHEMATICS – II (LINEAR ALGEBRA) **Instructors:** K V KRISHNA and VINAY WAGH

TEST-2 EXAM DATE: 8^{th} MAY 2021

DURATION: 1 HOUR 30 MINUTES TOTAL MARKS: 25

INSTRUCTIONS

1. It is expected that you are solving these questions **on your own**, and without any external help. If any kind of malpractice is found, you will get **ZERO** marks, for the entire exam.
2. Please submit the assignment in the **PDF** format only.
3. While submitting, make sure to write your name, roll number and page-number **on all the pages**.
4. Solve each question on a separate page.

1. (2 points) Consider the map $T : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_6(\mathbb{R})$ given by $p(X) \mapsto (2X^3 - X^2 + X + 1) \cdot p(X)$. Verify whether T is a linear transformation or not. If it is linear then find its standard matrix.

Solution: $T : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_6(\mathbb{R})$ given by $p(X) \mapsto (2X^3 - X^2 + X + 1) \cdot p(X)$.

T is a linear transformation if:

1. $T(p+q) = T(p) + T(q)$
2. $T(\alpha p) = \alpha T(p)$

$$\begin{aligned} T(p(X) + q(X)) &= (2X^3 - X^2 + X + 1) \cdot \big(p(X) + q(X)\big) \\ &= (2X^3 - X^2 + X + 1) \cdot p(X) + (2X^3 - X^2 + X + 1) \cdot q(X) \\ &= T(p(X)) + T(q(X)). \end{aligned}$$

Similarly,

$$\begin{aligned} T(\alpha p(X)) &= (2X^3 - X^2 + X + 1) \cdot \big(\alpha p(X)\big) \\ &= (2X^3 - X^2 + X + 1) \cdot \alpha p(X) \\ &= \alpha \cdot (2X^3 - X^2 + X + 1) \cdot p(X) \\ &= \alpha T(p(X)). \end{aligned}$$

One mark for linearity

The standard basis for $\mathcal{P}_3(\mathbb{R})$ is $\mathcal{B} = \{X^3, X^2, X, 1\}$, and that of $\mathcal{P}_6(\mathbb{R})$ is $\mathcal{C} = \{X^6, X^5, X^4, X^3, X^2, X, 1\}$. Computing the image of each of the basis vector from $\mathcal{B}$ and express it as a vector with respect to the basis $\mathcal{C}$:

$$\begin{aligned} T(X^3) &= 2X^6 - X^5 + X^4 + X^3 &\rightsquigarrow\quad & [2, -1, 1, 1, 0, 0, 0]^T \\ T(X^2) &= 2X^5 - X^4 + X^3 + X^2 &\rightsquigarrow\quad & [0, 2, -1, 1, 1, 0, 0]^T \\ T(X) &= 2X^4 - X^3 + X^2 + X &\rightsquigarrow\quad & [0, 0, 2, -1, 1, 1, 0]^T \\ T(1) &= 2X^3 - X^2 + X + 1 &\rightsquigarrow\quad & [0, 0, 0, 2, -1, 1, 1]^T \end{aligned}$$

Thus the standard matrix of T is

No marks for a wrong matrix.

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & -1 & 2 & 0 \\ 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

One mark for the correct matrix.

Note: If the student takes different order of the basis elements (but the same basis), then the matrix may change!

2. (2 points) Let V, W be finite-dimensional vector spaces. Let $T : V \to W$ be a linear transformation If $\dim V < \dim W$ then show that T cannot be surjective (onto).

Solution: Note that, from the rank-nullity theorem:

$$\dim(\mathfrak{R}(T)) + \dim(\ker(T)) = \dim(V) < \dim(W)$$

Since $\dim(\ker(T)) \geq 0$,

$$\dim(\mathfrak{R}(T)) < \dim(W).$$

Thus there exists $v \in W$ such that $v \notin \mathfrak{R}(T)$. Hence, T cannot be surjective.

— 0 or 2 marks. No part marking

3. (2 points) Does there exist an injective linear transformation $\Phi : \mathbb{Z}_5^7 \to \mathcal{L}(\mathbb{Z}_5, \mathbb{Z}_5^7)$, where $\mathcal{L}(V, W)$ denote the space of all linear transformations from V to W? Justify your answer.

Solution: Define $\Phi : \mathbb{Z}_5^7 \to \mathcal{L}(\mathbb{Z}_5, \mathbb{Z}_5^7)$ by $v \mapsto T_v$, where $T_v : \mathbb{Z}_5 \to \mathbb{Z}_5^7$ given by $T_v(\lambda) = \lambda v$.

Let $v, w \in \mathbb{Z}_5^7$. Suppose $v = w$. Then $\Phi(v) = T_v$ and $\Phi(w) = T_w$, where $T_v(\lambda) = \lambda v$ and $T_w(\lambda) = \lambda w$. If $v = w$, then

$\lambda v = \lambda w$ for all $\lambda \in \mathbb{Z}_5$ and hence $T_v = T_w$. Thus Φ is **well-defined**.

Defining the map correctly — ① mark

$\Phi(\alpha v + \beta w) = T_{\alpha v + \beta w}$.

$$\begin{aligned} T_{\alpha v+\beta w}(\lambda) &= \lambda(\alpha v + \beta w) \\ &= \lambda\alpha v + \lambda\beta w) \\ &= \alpha\lambda v + \beta\lambda w) \\ &= \alpha T_v(\lambda) + \beta T_w(\lambda) \end{aligned}$$

(Well-definedness is optional — i.e. Do not deduct mark if not shown.)

Thus, $T_{\alpha v+\beta w} = \alpha T_v + \beta T_w$, i.e. $\Phi(\alpha v + \beta w) = \alpha\Phi(v) + \beta\Phi(w)$. Hence Φ is **linear**.

To show Φ is injective, we will show that $\ker(\Phi) = \{0\}$.

Alternately, student may show via defn. i.e. $\Phi(v) = \Phi(w) \Rightarrow v = w$.

$$\begin{aligned} \ker(\Phi) &= \{v \in \mathbb{Z}_5^7 \mid T_v(\lambda) = 0 \quad \forall\lambda \in \mathbb{Z}_5\} \\ &= \{v \in \mathbb{Z}_5^7 \mid \lambda v = 0 \quad \forall\lambda \in \mathbb{Z}_5\} \\ &= \{0\} \end{aligned}$$

Thus, Φ is **injective**. — ① mark.

Alternately,

Some students may show that $\dim(\mathcal{L}(\mathbb{Z}_5, \mathbb{Z}_5^7)) = 7 = \dim(\mathbb{Z}_5^7)$. And then define a map between the two bases. In such case, the map needs to be **linearly** extended to whole space. Further, a justification for the injectivity should also be given.

If no justification, no marks!

No need to insist on the proof of this step.

4. (5 points) Let $\mathcal{M}_{n\times n}$ be the vector space of $n\times n$ matrices. Define $S:\mathcal{M}_{3\times 3}\to\mathcal{M}_{3\times 3}$ by $T(A) = 3A+3A^T$. Then

   (a) Show S is a linear transformation

   (b) Find a basis for $\ker(S)$.

   (c) Find a basis for $\Re(S)$.

   (d) Construct an orthonormal basis for $\ker(S)$, from the above basis.

Solution: Without loss of generality, let us assume that $M_{3\times 3}$ consists of real matrices.

(a) For every $A, B\in M_{3\times 3}, \alpha\in\mathbb{R}$ we have

$$\begin{aligned} S(\alpha A+\beta B) &= 3(\alpha A+\beta B)+3(\alpha A+\beta B)^T = 3\alpha A+3\beta B+3\alpha A^T+3\beta B^T \\ &= 3\alpha A+3\alpha A^T+3\beta B+3\beta B^T \\ &= 3\alpha(A+A^T)+3\beta(B+B^T) \\ &= \alpha S(A)+\beta S(B). \end{aligned}$$

① mark.

Thus S is a linear transformation.

(b) To compute $\ker(S)$:

Suppose $A=(a_{ij})\in\ker(S)\subset M_{3\times 3}$. Thus we have

$$S(A)=0 \iff 3(A+A^T)=0 \iff A^T=-A \iff a_{ji}=-a_{ij} \quad \forall i,j.$$

In particular we have $a_{ii}=0$ for every i.

It follows that

$$\ker S = \left\{ \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} : \ a,b,c\in\mathbb{R} \right\},$$

span & lin. indep both are necessary

hence $\ker T = \operatorname{span}\{A_1, A_2, A_3\}$, where

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},\ A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

① mark.

No mark if either is not shown

The latter also shows that $\dim\ker S=3$, since the set $\{A_1, A_2, A_3\}$ is linearly independent. For it,

$$aA_1+bA_2+cA_3 = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} = 0 \iff a=b=c=0.$$

Applicable to both parts (b) & (c)

(c) To compute range of S: Given $A=(a_{ij})\in M_{3\times 3}$ we set $A'=S(A)=(a'_{ij})$. Then $a'_{ij}=a_{ij}+a_{ji}$ for every i,j. It follows that

$$\begin{aligned} A' &= \begin{bmatrix} 2a_{11} & a_{12}+a_{21} & a_{13}+a_{31} \\ a_{21}+a_{12} & 2a_{22} & a_{23}+a_{32} \\ a_{31}+a_{13} & a_{32}+a_{23} & 2a_{33} \end{bmatrix} \\ &= 2a_{11}E_{11}+2a_{22}E_{22}+2a_{33}E_{33}+(a_{12}+a_{21})E+(a_{13}+a_{31})F+(a_{23}+a_{32})G, \end{aligned}$$

① mark.

with

$$E_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\quad E_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\quad E_{33} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\quad F = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},\quad G = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Now the set $\{E_{11}, E_{22}, E_{33}, E, F, G\}$ is linearly independet. For if,

$$\sum_{i=1}^{3} a_i E_{ii} + bE + cF + dG = \begin{bmatrix} a_1 & b & c \\ b & a_2 & d \\ c & d & a_3 \end{bmatrix} = 0 \iff a_1 = a_2 = a_3 = b = c = d = 0,$$

Thus, the set $\{E_{11}, E_{22}, E_{33}, E, F, G\}$ forms a basis for the range of S, and therefore $\dim \Re(S) = 6$.

(d) For orthonormal basis, let us use the DOT product defined by the **sum of component-wise product**.

Applying Gram-Schmidt process to $\{A_1, A_2, A_3\}$.

Take $\mathbf{u_1} = A_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

2 marks.

Compute $\mathbf{u_2} = \mathbf{A_2} - \text{proj}_{\mathbf{u_1}}(\mathbf{A_2}) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{bmatrix}$

Then compute $\mathbf{u_3} = \mathbf{A_3} - \text{proj}_{\mathbf{u_1}}(\mathbf{A_3}) - \text{proj}_{\mathbf{u_2}}(\mathbf{A_3}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$

Now set $\mathbf{v_1} = \frac{A_1}{\|A_1\|} = \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\mathbf{v_2} = \frac{u_2}{\|u_2\|} = \begin{bmatrix} 0 & 0 & \frac{\sqrt{3}}{3} \\ 0 & 0 & 0 \\ -\frac{\sqrt{3}}{3} & 0 & -\frac{\sqrt{3}}{3} \end{bmatrix}$ and $\mathbf{v_3} = \frac{u_3}{\|u_3\|} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & 0 \end{bmatrix}$.

Thus the set $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is an orthomormal basis.

No marks if 'inner product' is not mentioned.

Student may chk orthogonality of A_1, A_2, A_3 directly. — Acceptable.

If orthonormal basis is not computed (i.e. only orthogonal) then only 1 mark.

5. (5 points) Let $A = \begin{bmatrix} -18 & 0 & 0 & -42 \\ 8 & -4 & 0 & 24 \\ -14 & -8 & 4 & -18 \\ 10 & 0 & 0 & 26 \end{bmatrix}$ be a **complex** matrix.

(a) Compute the characteristic polynomial of A.

(b) Compute all eigenvalues of A and the corresponding eigenvectors.

(c) Write down the algebraic and geometric multiplicities of each of the eigenvalue.

(d) Diagonalize A.

(e) Verify that A satisfies its characteristic polynomial.

Solution:

(a) To compute characteristic polynomial:

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} -18 & 0 & 0 & -42 \\ 8 & -4 & 0 & 24 \\ -14 & -8 & 4 & -18 \\ 10 & 0 & 0 & 26 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)$$
$$= \lambda^4 - 8\lambda^3 - 64\lambda^2 + 128\lambda + 768$$

1 mark

(b) Note that $P(\lambda) = \lambda^4 - 8\lambda^3 - 64\lambda^2 + 128\lambda + 768 = (\lambda+4)^2(\lambda-4)(\lambda-12)$.

Thus the roots are -4, with multiplicity 2, 4 and 12, each with multiplicity 1, hence the eigenvalues are $4, -4$ and 12. To compute eigenvectors:

1 mark

For $\lambda = -4$:

$$\operatorname{null}(A + 4I) = \operatorname{null}\left(\begin{bmatrix} -18 & 0 & 0 & -42 \\ 8 & -4 & 0 & 24 \\ -14 & -8 & 4 & -18 \\ 10 & 0 & 0 & 26 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)$$
$$= \operatorname{null}\left(\begin{bmatrix} -14 & 0 & 0 & -42 \\ 8 & 0 & 0 & 24 \\ -14 & -8 & 8 & -18 \\ 10 & 0 & 0 & 30 \end{bmatrix}\right)$$

One mark if all 3 eigenvalues & all eigenvectors are correct.

The RREF is

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

To find the null space, solve the system:

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

If we take $x_3 = t$, $x_4 = s$, then $x_1 = -3s$, $x_2 = 3s + t$.

Thus null space is the span of

$$\begin{bmatrix} -3s \\ 3s+t \\ t \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} -3 \\ 3 \\ 0 \\ 1 \end{bmatrix} s.$$

Thus, the basis for the eigenspace is $\left\{ \begin{bmatrix} -3 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

For $\lambda = 4$, eigenspace is the null$(A - 4I)$ and the basis for the eigenspace is: $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

For $\lambda = 12$, eigenspace is the null$(A - 12I)$ and the basis for the eigenspace is: $\begin{bmatrix} -7 \\ 4 \\ -3 \\ 5 \end{bmatrix}$,

(c)

$$\left.\begin{array}{lll} \lambda = -4 & \text{alg. mult.} = 2 & \text{geom. mult.} = 2 \\ \lambda = 4 & \text{alg. mult.} = 1 & \text{geom. mult.} = 1 \\ \lambda = 12 & \text{alg. mult.} = 1 & \text{geom. mult.} = 1 \end{array}\right\}$$

1 mark.

(d) To diagonalize A, use the matrix P formed by the eigenvectors: $P = \begin{bmatrix} -3 & 0 & -7 & 0 \\ 3 & 1 & 4 & 0 \\ 0 & 1 & -3 & 1 \\ 1 & 0 & 5 & 0 \end{bmatrix}$.

1 mark (only if both correct)

Compute $P^{-1} = \begin{bmatrix} -\frac{5}{8} & 0 & 0 & -\frac{7}{8} \\ \frac{11}{8} & 1 & 0 & \frac{9}{8} \\ \frac{1}{8} & 0 & 0 & \frac{3}{8} \\ -1 & -1 & 1 & 0 \end{bmatrix}$.

Writing of P&D is essential. P^{-1} is optional.

Thus $P^{-1}AP = D = \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$.

(e) Charateristic polynomial of A is:

$$\lambda^4 - 8\lambda^3 - 64\lambda^2 + 128\lambda + 768.$$

Thus

$$
\begin{aligned}
&A^4 - 8A^3 - 64A^2 + 128A + 768I \\
&= \begin{bmatrix} -17664 & 0 & 0 & -53760 \\ 10240 & 256 & 0 & 30720 \\ -7680 & 0 & 256 & -23040 \\ 12800 & 0 & 0 & 38656 \end{bmatrix} - 8 \begin{bmatrix} -18 & 0 & 0 & -42 \\ 8 & -4 & 0 & 24 \\ -14 & -8 & 4 & -18 \\ 10 & 0 & 0 & 26 \end{bmatrix}^3 \\
&\quad - 64 \begin{bmatrix} -18 & 0 & 0 & -42 \\ 8 & -4 & 0 & 24 \\ -14 & -8 & 4 & -18 \\ 10 & 0 & 0 & 26 \end{bmatrix}^2 + 128 \begin{bmatrix} -18 & 0 & 0 & -42 \\ 8 & -4 & 0 & 24 \\ -14 & -8 & 4 & -18 \\ 10 & 0 & 0 & 26 \end{bmatrix} + \begin{bmatrix} 768 & 0 & 0 & 0 \\ 0 & 768 & 0 & 0 \\ 0 & 0 & 768 & 0 \\ 0 & 0 & 0 & 768 \end{bmatrix} \\
&= \begin{bmatrix} -17664 & 0 & 0 & -53760 \\ 10240 & 256 & 0 & 30720 \\ -7680 & 0 & 256 & -23040 \\ 12800 & 0 & 0 & 38656 \end{bmatrix} - \begin{bmatrix} -13056 & 0 & 0 & -37632 \\ 7168 & -512 & 0 & 21504 \\ -6400 & -1024 & 512 & -16128 \\ 8960 & 0 & 0 & 26368 \end{bmatrix} \\
&\quad - \begin{bmatrix} -6144 & 0 & 0 & -21504 \\ 4096 & 1024 & 0 & 12288 \\ -3072 & 0 & 1024 & -9216 \\ 5120 & 0 & 0 & 16384 \end{bmatrix} + \begin{bmatrix} -2304 & 0 & 0 & -5376 \\ 1024 & -512 & 0 & 3072 \\ -1792 & -1024 & 512 & -2304 \\ 1280 & 0 & 0 & 3328 \end{bmatrix} + \begin{bmatrix} 768 & 0 & 0 & 0 \\ 0 & 768 & 0 & 0 \\ 0 & 0 & 768 & 0 \\ 0 & 0 & 0 & 768 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

1 mark.

Some calculations are necessary. Just substituting A in the poly is not sufficient.

If using D for the calculation then it is essential to mention that A & D have the same char. poly.

6. (4 points) Compute the basis for the four fundamental spaces of the following matrix. Also state and verify the rank-nullity theorem in terms of these spaces.

NOTE

$$
\begin{bmatrix} 1 & 0 & 2 & 3 & 4 \\ 1 & 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}
$$

Solution: $A = \begin{bmatrix} 1 & 0 & 2 & 3 & 4 \\ 1 & 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$.

Step 1: Compute RREF.

Step 2: Convert the matrix back to an equivalent system of equations and solve the system in terms of the free variables, to obtain the basis for the **null space** of A. Using similar calculations, obtain the basis for the **null space** of A^T.

Step 3: Identify the pivot elements and hence get the corresponding columns of A, to obtain the basis for the **column space**.

Step 4: **Row space** of A = row space of the RREF. Thus $row(A)$ is spanned by the rows of the RREF.

The $RREF(A) = \begin{bmatrix} 1 & 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Null space of A: $null(A)$:

$$span\left\{\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}-2\\0\\1\\0\\0\end{bmatrix},\begin{bmatrix}2\\0\\0\\-2\\1\end{bmatrix}\right\}.$$

Column space of A: $col(A)$:

$$span\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}3\\4\\1\end{bmatrix}\right\}.$$

Row space of A: $row(A)$:

$$span\{[1,0,2,4,6],[1,0,2,3,4]\}.$$

Null space of A^T: $null(A^T)$:

$$span\left\{\begin{bmatrix}1\\-1\\1\end{bmatrix}\right\}.$$

(2) marks
Full marks if at least 3 spaces are correct.
(1) mark if two spaces are correct.
No mark if only one is correct!

Recall: `Rank-nullity theorem`: If A is an $m \times n$ matrix, then

$$\text{rank}(A) + \text{nullity}(A) = n.$$

rank(A) = the number of leading variables in the general solution of $AX = 0$

nullity(A) = the number of parameters in the general solution of $AX = 0$.

Thus,

$$\dim(\text{row}(A)) + \dim(\text{null}(A)) = \text{num columns} = 5$$
$$\dim(\text{col}(A)) + \dim(\text{null}(A^T)) = \text{num rows} = 3$$

(2) marks
one mark each.

7. (5 points) Verify that the following defines an inner product on $\mathcal{P}_2(\mathbb{R})$:

NOTE

$$\langle p(X), q(X)\rangle = a_0b_0 + 2a_0b_1 + 3a_0b_2 + 2a_1b_0 + 2a_1b_1 + 4a_1b_2 + 3a_2b_0 + 4a_2b_1 + 8a_2b_2,$$

where $p(X) = a_0 + a_1X + a_2X^2$ and $q(X) = b_0 + b_1X + b_2X^2$.

With respect to this inner product, compute $\|1 - 2X + 3X^2\|$. Further, compute the distance and angle between $1 - 2X + 3X^2$ and $3 - 2X - X^2$.

Solution: Everybody gets 5 marks for this questions.

End of the questions for Test-2