

INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI

MA 102 Mathematics-II (Test-3)

Duration: **90** Minutes (10:00 AM to 11:30 AM)

Total Marks: **25** Marks

Important Instructions:

- There are **SEVEN** questions in this paper. Answer **all** questions. Answer to subparts of a question should appear together.
- Write your Roll Number on all pages. Also, put page number on every page.
- Submit your answer in a single PDF file. Arrange your answers for Q.1 to Q.7 in the order and create the PDF file.
- The PDF file name should be your Roll Number.
- No clarification will be given during the examination.

1. Prove that there is a unique continuous function $f : (0, \infty) \rightarrow \mathbb{R}$ such that

$$f(x) = 3 + \frac{1}{x} \int_1^x f(s) ds, \quad \text{for all } x \in (0, \infty).$$

Find the function f .

[3 marks]

2. Solve the initial-value problem

[3 marks]

$$(1 + e^x y + x e^x y) dx + (x e^x + 2) dy = 0; \quad y(1) = 1.$$

3. Discuss the existence and uniqueness of the following initial-value problem

$$\frac{dy}{dx} = (x^3 + 1) \sin^2(y); \quad y(0) = 0,$$

on the strip $S = \{(x, y) \in \mathbb{R}^2 \mid |x| \leq \alpha, |y| < \infty\}$, where $\alpha > 0$ is a fixed real number.

[3 marks]

4. The half-life for radium is approximately 1600 years. Find what percentage of a given quantity of radium disintegrates in 100 years.

[3 marks]

5. (a) If the roots of the auxiliary equation associated with a certain 10^{th} -order homogeneous linear differential equation with constant coefficients are

$$1, 1, 1, 2, 2, 2, 1 + 2i, 1 - 2i, 1 + 2i, 1 - 2i,$$

then write the general solution.

[1 mark]

- (b) Let $y_1(x)$, $y_2(x)$ and $y_3(x)$ be three linearly independent solutions of

$$x^3 \frac{d^3 y}{dx^3} + x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 2y = 0, \quad x > 0.$$

Let $W(y_1, y_2, y_3)(x)$ denote the Wronskian of y_1, y_2, y_3 . If $W(y_1, y_2, y_3)(1) = -6$, then find the value of $W(y_1, y_2, y_3)(4)$.

[1 mark]

(c) Find a differential operator that annihilates the function

[1 mark]

$$x^3 e^{-x} + x^2 e^{-5x} \sin(3x).$$

6. Find a basis for the subspace of real solutions of

$$\sum_{k=1}^7 \frac{d^k y}{dx^k} + y = 0$$

that satisfy $\lim_{x \rightarrow +\infty} y(x) = 0$.

[5 marks]

7. Find the general solution of

$$\frac{d^2 y}{dx^2} - y = \frac{1}{1 + e^x}$$

on the interval $(-\infty, \infty)$.

[5 marks]

***** Paper Ends *****