*Proof.* Suppose that $l \neq k$. Put $\epsilon = |l-k|/2$. As $a_n \to l$, $\exists n_0$ such that $a_{n_0}, a_{n_0+1}, \ldots \in B_\epsilon(l)$. So $a_{n_0}, a_{n_0+1}, \ldots \notin B_\epsilon(k)$, because $B_\epsilon(l) \cap B_\epsilon(k) = \emptyset$). That is, $B_\epsilon(k)$ misses infinitely many terms of $(a_n)$. So $a_n \not\to k$. A contradiction. ∎

EXERCISE 3.1.14 **(Nested interval theorem: another part)** *Let* $[a_1, b_1] \supseteq [a_2, b_2] \supseteq \cdots$ *be a sequence of intervals. If* $\lim(b_i - a_i) \to 0$, *then* $\cap[a_n, b_n]$ *is a singleton set.*

EXERCISE 3.1.15 *Consider the sequence* $(a_n = \frac{1}{n})$.

*a) Let* $a \neq 0$. *Then* $a_n \not\to a$ *as* $\exists \epsilon > 0$ *such that* $B_\epsilon(a)$ *misses infinitely many terms of* $(a_n)$. *Give a value for* $\epsilon$ *and argue how it is missing so many terms.*

*b) We know* $a_n \to 0$ *as each* $B_\epsilon(a)$ *contains a tail (which may depend on* $\epsilon$*) of* $(a_n)$. *Which tail?*

## 3.2 Bounded sequences

DEFINITION 3.2.1 We say a sequence $(a_n)$ in $\mathbb{R}$ is **bounded above** if $\exists x \in \mathbb{R}$ such that $a_n \leq x$, $\forall n$. The term **bounded below** is defined similarly. A sequence in $\mathbb{R}$ is called **bounded**, if it is bounded above and below. Essentially, this means the sequence is contained in a finite length interval. In a similar way, a sequence in $\mathbb{R}^n$ is called **bounded**, if it contained in some ball of finite radius.

EXAMPLE 3.2.2 *1. The sequence* $(2^n)$ *is bounded below but not not bounded above. Hence, it is not bounded.*

*2. The sequence* $((-1)^n)$ *is bounded.*

*3. Let* $(a_n)$ *be bounded and suppose that it lies in* $(a, b)$. *Similarly, let* $(b_n)$ *lie in* $(c, d)$. *Let* $t_n \in \mathbb{R}^2$ *be the point* $(a_n, b_n)$. *Then the sequence* $(t_n)$ *in* $\mathbb{R}^2$ *is a bounded sequence. In fact it lies in the ball* $B_\alpha(x, y)$, *where* $x = \frac{a+b}{2}$, $y =$ _____ *and* $\alpha =$ _____.

LEMMA 3.2.3 *Let* $(a_n)$ *be convergent. Then* $(a_n)$ *is bounded.*

*Proof.* Let $a_n \to a$. So each $B_\alpha(a)$ contains a tail of $(a_n)$. In particular, $B_1(a)$ contains a tail. That is, $\exists m$ such that $a_m, a_{m+1}, \ldots \in B_1(a)$. That is, $|a_n - a| < 1$ for each $n = m, m+1, \ldots$. So $|a_n| < |a| + 1$ for each $n \geq m$. Put $M := \max\{|a_1|, \ldots, |a_{n_0-1}|, |a| + 1\}$. Then $|a_n| \leq M$, for all $n$. This means $(a_n)$ lies inside $B_M(0)$. ∎

EXAMPLE 3.2.4 1. Is the converse of the previous result true? No. The sequence $((-1)^n)$ is bounded but nor convergent.

2. Is $(n^2 - n)$ convergent?. No, because it is not bounded.

EXERCISE 3.2.5 *Fix* $a > 1$. *Is* $(a^n)$ *convergent?*

THEOREM 3.2.6 **(LTs)** *Let* $a_n \to a$, $b_n \to b$, $k \in \mathbb{R}$. *Then the following statements hold true.*

*a)* $(a_n + b_n) \to (a + b)$ *and* $ka_n \to ka$.

*b)* $(a_n b_n) \to ab$.

*c If* $a > 0$, *then* $(a_n)$ *has a positive tail. Furthermore, if* $a_n$ *are nonzero, then* $\frac{1}{a_n} \to \frac{1}{a}$.

*d) If* $a_n \geq 0$, *then* $a \geq 0$.

*e) If $k \in \mathbb{N}$, $a_n \geq 0$, then $\sqrt[k]{a_n} \to \sqrt[k]{a}$.*

*f) If $a_n \to a$, then $|a_n| \to |a|$. The converse is not true.*

*Proof.*

a) To prove the first part, let $\epsilon > 0$. As $a_n \to a$, $\exists m$ such that $|a_n - a| < \frac{\epsilon}{2}$ for each $n \geq m$. As $b_n \to b$, $\exists m'$ such that $|b_n - b| < \frac{\epsilon}{2}$ for each $n \geq m'$. So, $|a_n + b_n - a - b| < \epsilon$, for all $n \geq \max\{m, m'\}$. The other part is already done.

b) As $(a_n)$ and $(b_n)$ are bounded, let $|a_n|, |b_n|, |a|, |b| \leq M$. As $a_n \to a$, $\exists n_0$ such that $|a_n - a| < \frac{\epsilon}{2M}$ for each $n \geq n_0$. As $b_n \to b$, $\exists n_1$ such that $|b_n - b| < \frac{\epsilon}{2M}$ for each $n \geq n_1$. Then

$$|a_n b_n - ab| \leq |a_n - a||b_n| + |a||b_n - b| \leq \frac{\epsilon}{2M} M + M\frac{\epsilon}{2M} = \epsilon,$$

for each $n \geq \max\{n_0, n_1\}$.

c) For the first part, put $\alpha = \frac{a}{2}$. As $a_n \to a$, $\exists m$ such that $\forall n \geq m$, we have $a_n \geq \frac{a}{2}$. So $(a_n)$ has a positive tail. To prove the next part, assume also that each $a_n > 0$. Now, for all $n \geq m$ we have

$$0 \leq \left|\frac{1}{a_n} - \frac{1}{a}\right| = \left|\frac{a - a_n}{a_n a}\right| \leq \frac{2}{a^2}|a - a_n|.$$

As $\frac{2}{a^2}|a - a_n| \to 0$, by sandwich lemma we are done.

d) If $a < 0$, then put $\alpha = \frac{|a|}{2} > 0$. Then $B_\alpha(a)$ does not contain any $a_n$. So $a$ is not the limit. $\Rightarrow\Leftarrow$. (This means a contradiction.)

e) We first show the statement for $a = 0$. For that take $\epsilon > 0$. Put $\alpha = \epsilon^k$. As $a_n \to 0$, $\exists m$ such that $a_n \leq \alpha$ for each $n \geq m$. Hence $\sqrt[k]{a_n} \leq \epsilon$ for each $n \geq m$. That is, $\sqrt[k]{a_n} \to 0$.

Now, assume $a > 0$. Then

$$|a_n - a| = \left|a_n^{\frac{1}{k}} - a^{\frac{1}{k}}\right|\left|a_n^{\frac{k-1}{k}} + a_n^{\frac{k-2}{k}} a^{\frac{1}{k}} + \cdots + a^{\frac{k-1}{k}}\right| \geq \left|a_n^{\frac{1}{k}} - a^{\frac{1}{k}}\right| a^{\frac{k-1}{k}} \geq 0$$

As $|a_n - a| \to 0$, by sandwich lemma, we are done.

f) Exercise. ■

**(LTs)(very important technique)**

Show that $a_n, b_n \to l$ iff $(a_1, b_1, a_2, b_2, \ldots) \to l$. You can also make a similar claim for $(a_1, b_1, c_1, a_2, b_2, c_2 \ldots)$. Can you make a similar claim for such a mix of 5 sequences?

Exercise 3.2.7 *What happens when we add and multiply, two divergent sequences? What happens when we add and multiply, a divergent sequence with a convergent sequence?*

Exercise 3.2.8 *Whether the following sequences are convergent? If yes, find the respective limits.*

1. $\frac{n^2+2n}{2n^2-5n+1}$.
2. $\frac{1}{\sqrt{n}+\sqrt{n+1}}$.
3. $\sqrt{n^2+2n} - \sqrt{n^2}$.
4. $(1+\epsilon)^{1/n}$ *for fixed* $\epsilon > 0$

5. $\alpha^{1/n}$ *for fixed* $\alpha > 0$.

6. $n^{1/n}$.

7. $((3^n + 5^n)^{1/n})$.

8. $(a_n b_n)$ *where* $(a_n)$ *is bounded and* $b_n \to 0$.

9. $\frac{1}{\sqrt{n^2+1}} + \cdots + \frac{1}{\sqrt{n^2+n}}$.

**(LTs)(important technique)** Let $a \in \mathbb{R}$. We want to show that there is a sequence of rationals converging to $a$ and a sequence of irrationals converging to $a$. There are two ways.

1. Recall that $(a, a+1/n)$ contains a rational $r_n$ and an irrational $i_n$. Argue that $r_n \to a$ and $i_n \to a$.

2. Alternately, note that $[10^n a] \leq 10^n a < [10^n a] + 1$. Divide by $10^n$. Apply sandwich to show that $(\frac{[10^n a]}{10^n})$ is a sequence of rationals and $(\frac{[10^n a]}{10^n} + \frac{\sqrt{2}}{n})$ is a sequence of irrationals converging to $a$. Did you realize the relationship between the given sequence of rational numbers and the decimal representation of $a$?

LEMMA 3.2.9 **(Ratio test for limits)** *Let* $a_n \neq 0$ *for each* $n$. *Suppose that* $\lim |\frac{a_{n+1}}{a_n}| = l$. *Then* $l < 1 \Rightarrow a_n \to 0$ *and* $l > 1 \Rightarrow (a_n)$ *is divergent.*

*Proof.* Let $l < 1$. As $|\frac{a_{n+1}}{a_n}| \to l$, $\exists n_0$ such that $|\frac{a_{n+1}}{a_n}| < (1+l)/2 = r$ (say), for each $n \geq n_0$. So $0 < |a_{n_0+k}| < |a_{n_0}| r^k$. So $|a_{n_0+k}| \overset{\text{sandwich}}{\to} 0$. So $a_{n_0+k} \to 0$. So $a_n \to 0$. The other part follows as the $n$th term does not go to 0. ∎

EXAMPLE 3.2.10 Take $a_n = \dfrac{5^n}{n!}$. Then $a_n \to 0$ as $\left|\frac{a_{n+1}}{a_n}\right| = \frac{5}{n+1} \to 0$. Alternately,

$$n > 5 \Rightarrow \frac{5^n}{n!} = \frac{5^5}{5!} \, \frac{5}{6} \, \frac{5}{7} \, \cdots \, \frac{5}{n} \leq \frac{5^5}{5!} \left(\frac{5}{6}\right)^{n-5} \to 0.$$

EXERCISE 3.2.11 1. *Give examples of convergent and divergent sequences with* $l = 1$.

2. *Test for convergence. Here* $a \in \mathbb{R}$ *and* $k \in \mathbb{N}$ *are fixed.*

   (a) $a_n = n^k a^n$.
   (b) $a_n = \frac{a^n}{n!}$.
   (c) $a_n = \frac{n^k}{n!}$.
   (d) $a_n = \frac{n!}{n^n}$.
   (e) $a_n = \frac{a^n}{n^k}$.

## 3.3 Monotone sequences

DEFINITION 3.3.1 We say a sequence $(a_n)$ is **increasing** if $a_n \leq a_{n+1}$ for each $n \in \mathbb{N}$. It is called **strictly increasing** if $a_n < a_{n+1}$ for each $n$. **Decreasing** and **strictly decreasing** sequences are defined similarly. A sequence is called **monotone** if it is either decreasing or increasing. Thus the sequence $(n)$ is monotone and $((-1)^n)$ is not.

EXAMPLE 3.3.2 Take $a_n = -\frac{1}{n}$. It is increasing and bounded above. Notice that $a_n \to 0 = \mathsf{lub}\{-1, -\frac{1}{2}, \cdots\}$. Is this true in general? Yes.

Theorem 3.3.3 **(Monotone convergence theorem (MCT))** *Let $(a_n)$ be increasing and bounded above. Put $A = \{a_1, a_2, \ldots\}$. Then $A$ is nonempty, bounded above and $a_n \to \mathsf{lub}\, A$.*

*Proof.* As $a_1 \in A$, it is nonempty. As $(a_n)$ is bounded above, we see that $A$ is bounded above. So $\mathsf{lub}\, A$ exists. Let $a = \mathsf{lub}\, A$. Take $\alpha > 0$. The number $a - \alpha$ is not an ub of $A$. So $\exists m \in \mathbb{N}$ such that $a_m > a - \alpha$. But then, $a - \alpha < a_m \leq a_{m+1} \leq a_{m+2} \leq \cdots \leq a$. Thus, each $B_\alpha(a)$ contains a tail of $(a_n)$. ∎

Exercise 3.3.4 *Is the converse of MCT is not true?*

Definition 3.3.5 At times we write $\boldsymbol{a_n \uparrow a}$ to mean that $(a_n)$ is increasing and $a_n \to a$. This means that $a = \mathsf{lub}\{a_1, a_2, \ldots\}$. The notation $\boldsymbol{a_n \downarrow a}$ has a similar meaning.

Corollary 3.3.6 *Let $(a_n)$ be decreasing, bounded below. Then $a_n \to \inf\{a_1, a_2, \ldots\}$. In fact, a monotone sequence is convergent iff it is bounded.*!!

Exercise 3.3.7 *Let $S \neq \emptyset$ and $(a_n)$ be a decreasing sequence of upper bounds of $S$. Suppose that $a_n \to a$. Show that $a$ is an upper bound of $S$.*

**Technique: application of MCT** Let $a_1 = 1$, $a_{n+1} = \frac{3a_n+7}{5}$. Is the sequence convergent?

1. If we can somehow show that the sequence is monotone and bounded, then we know the answer.
2. For that, notice that $a_1 < a_2$. Also $a_n \leq a_{n+1} \Rightarrow \frac{3a_n+7}{5} \leq \frac{3a_{n+1}+7}{5}$. So $a_n$ is increasing, by induction.
3. Also $a_n \leq 4$, by induction.
4. Hence by MCT, $(a_n)$ converges. Let $a_n \to l$.
5. Now using LTs, we see that $l = \frac{3l+7}{5}$. So $l = \frac{7}{2}$.

**It is important to prove that the sequence is convergent before applying the LTs.** For example, if we take $a_{n+1} = 3a_n + 7$ in the previous example, and directly apply LTss, then we get $l = 3l + 7$ and so $l = -7/2$. However, the sequence is not convergent.

Example 3.3.8 **(Geometric series)** Fix $0 < r < 1$. Put $a_n = \sum\limits_{i=0}^{n} r^n$. Then $a_n \to \frac{1}{1-r}$. To show this, note that $a_n = \frac{1-r^{n+1}}{1-r} < \frac{1}{1-r}$ and $a_n \uparrow$. Using MCT, let $a_n \to l$. Notice that $a_{n+1} = r + ra_n$. By LTs, $l = 1 + rl$. So $l = \frac{1}{1-r}$.

Example 3.3.9 **(Exponential function)** Fix $x > 0$. Take $a_n = \sum\limits_{k=0}^{n} \frac{x^k}{k!}$. It is clear that $a_n \uparrow$. Now put $m = [x] + 1$ and $r = \frac{x}{m}$. Note that $r < 1$. Then

$$a_{m+k} - a_m = \frac{x^{m+1}}{(m+1)!} + \cdots + \frac{x^{m+k}}{(m+k)!} \leq \frac{x^m}{m!} r + \cdots + \frac{x^m}{m!} r^k = \frac{x^m}{m!} \frac{r - r^{k+1}}{1-r} \leq \frac{x^m}{m!} \frac{r}{1-r}.$$

Hence

$$a_n \leq a_m + \frac{x^m}{m!} \frac{r}{1-r},$$

for each $n$. So $(a_n)$ is bounded above. Hence by MCT, the limit exists. This limit is called $\mathbf{exp}\, \boldsymbol{x}$.

Example 3.3.10 **(The number** $e$**)** Consider $a_n = (1+\frac{1}{n})^n$. Notice that

$$\frac{\binom{n}{k}}{n^k} = \frac{n(n-1)\cdots(n-k+1)}{n^k k!} = \frac{1(1-\frac{1}{n})\cdots(1-\frac{k-1}{n})}{k!} \leq \frac{\binom{n+1}{k}}{(n+1)^k}, \frac{1}{k!}.$$

Hence $(1+\frac{1}{n})^n$ is term wise smaller than that of $(1+\frac{1}{n+1})^{n+1}$ and the later has an extra term. Thus $a_n \uparrow$. Also, we have

$$1+\frac{\binom{n}{1}}{n}+\frac{\binom{n}{2}}{n^2}+\frac{\binom{n}{3}}{n^3}+\cdots+\frac{\binom{n}{n}}{n^n} \leq 1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}} < 3.$$

By MCT, $(a_n)$ is convergent. The limit is known as $\boldsymbol{e}$. Argue that $e \in (2,3)$.

Exercise 3.3.11 *As* $(1+\frac{1}{m})^m \to e$ *and* $\frac{m}{m+1} \to 1$, *we get* $(\frac{m+1}{m})^{m+1} \to$ ___. *Taking reciprocal, conclude that* $(1-\frac{1}{n})^n \to \frac{1}{e}$.

Exercise 3.3.12 *Here we give an alternate argument that* $(n^{\frac{1}{n}})$ *converges to* 1. *We already have* $(1+\frac{1}{n})^n < 3 \leq n$. *Hence, if* $n \geq 3$, *then* $n^{n+1} > (n+1)^n$. *Taking* $n(n+1)$*th root both side, we get* $n^{\frac{1}{n}} > (n+1)^{\frac{1}{n+1}}$. *That is, the sequence* $(n^{\frac{1}{n}})$ *is decreasing from third term onwards. Is it bounded below? Is it convergent? How do you find the limit?*

Exercise 3.3.13 *Fix* $k \in \mathbb{N}$. *Take* $a_1 = 1$ *and* $a_{n+1} = \frac{a_n + \frac{k}{a_n}}{2}$. *Then* $a_n \to \sqrt{k}$.

## 3.4 Subsequences

Definition 3.4.1 Consider a sequence $(a_n)$. Let $n_1 < n_2 < n_3 < \cdots$ be some natural numbers. Then $(a_{n_k})$ is called a **subsequence** of $(a_n)$.

Example 3.4.2 Consider the sequence $(n)$. Then $2, 3, 5, 7, \cdots$ (prime numbers) and $(2n)$ are two subsequences. But $1, 1, 2, 3, 4, 5, \cdots$ is not a subsequence, as the original sequence does not have two 1's. Also $1, 2, 4, 3, 5, 6, \cdots$ is not a subsequence, as the order of 3 and 4 in the original sequence is not preserved. Is $(-1, -1, 1, 1, 1, 1, \ldots)$ a subsequence of $(-1)^n$)?

Lemma 3.4.3 **(LTs)(A subsequence of a convergent sequence converges to the same limit)** *Let* $a_n \to l$ *and* $(a_{n_k})$ *be a subsequence of* $(a_n)$. *(View* $(a_{n_k})$ *as a new sequence* $(b_k)$*). Then* $\lim\limits_{k\to\infty} b_k = l$.

*Proof.* Take $\alpha > 0$. Then $\exists m \in \mathbb{N}$ such that $a_n \in B_\alpha(l)$, $\forall n \geq m$. Thus, if $k \geq m$, we have $n_k \geq k \geq m$ and so $b_k = a_{n_k} \in B_\alpha(l)$. That is, $b_k \to l$. ∎

**Technique**

One of the following is sufficient to prove that $(a_n)$ is divergent.

1. $(a_n)$ has a divergent subsequence.
2. $(a_n)$ has two subsequences converging to two different limits.
3. $(a_n)$ is not bounded.

Example 3.4.4 Take $a_n = \sin n$. Does it converge? We will use the previous technique to show that it diverges. Assume that it converges to $l$. Then, using LTs on, $\sin(2n+1) = \sin 2n \cos 1 + (1 - 2\sin^2 n)\sin 1$, we see that $l \neq 0$. (Here we used that $\sin 2n$ also converges to $l$, etc.) Similarly, from $\sin(n+1) + \sin(n-1) = 2 \sin n \cos 1$, we get $l = 0$. $\Rightarrow\Leftarrow$

EXERCISE 3.4.5 *1. A sequence $(a_n)$ has a subsequence which does not converge. Then $(a_n)$ is* __________.

*2. A sequence $(a_n)$ has two subsequences with limits* $1$ *and* $2$. *Then $(a_n)$ is* __________.

*3. Put $a_n = \frac{\cos n}{n}$. Then $(a_n)$ is* __________.

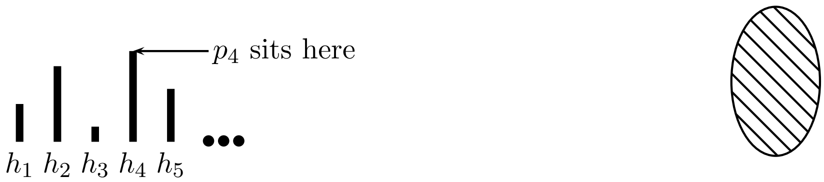
*4. Suppose $|a_n| \to 0$. Is $(a_n)$ necessarily convergent?*

*5. Give a sequence with* $5$ *convergent subsequences converging to* $1, 2, 3, 4, 5$, *respectively.*

*6. Let $(a_n)$ be a sequence with $a_{2n} \to l$ and $a_{2n-1} \to k$. Then $(a_n)$ is convergent iff* _____.

**Monotone subsequence theorem**

Persons $p_1, p_2, p_3, \ldots$ want to watch a movie. Their seats have heights $h_i$. All seats are in one line facing the screen and $p_1$ is farthest from the screen.




1. Can $p_2$ watch the movie? No, as $p_4$ has a higher seat.
2. Let $W$ be the set of persons who can watch the movie.
3. Assume that $W = \{p_{k_1}, \ldots, p_{k_n}\}$ is finite. Choose $n_0 > k_1, \cdots, k_n$.
4. Is $p_{n_0} \in W$? No. So, $\exists$ a higher seat in front. That is, $\exists n_1 > n_0$ such that $h_{n_1} > h_{n_0}$.
5. Is $p_{n_1} \in W$? No. So, $\exists n_2 > n_1$ such that $h_{n_2} > h_{n_1}$.
6. Continue to get $h_{n_0} < h_{n_1} < h_{n_2} < \cdots$, an increasing subsequence of $(h_n)$.
7. Assume $W = \{p_{n_1}, p_{n_2}, \ldots\}$ is infinite, where $n_1 < n_2 < \cdots$.
8. Then $h_{n_1} \geq h_{n_2} \geq \cdots$, as they all can watch the movie.

In either case, the sequence $(h_n)$ has a monotone subsequence.

THEOREM 3.4.6 **(Monotone Subsequence Theorem (MST))** *Every sequence of real numbers has a monotone subsequence.*

*Proof.* Let $(h_n)$ be the sequence. We first assume that all $a_n > 0$. Let $W = \{h_n \mid h_k \leq h_n$, for each $k > n\}$.

Case I. The set $W$ is finite. Then find a term $h_{n_0}$ with subscript $n_0$ higher than the subscripts of the elements of $W$. If $W$ is empty, take $n_0 = 1$. Then $h_{n_0} \notin W$. Hence, $\exists n_1 > n_0$ such that $h_{n_1} > h_{n_0}$. Again, $h_{n_1} \notin W$. Hence, $\exists n_2 > n_1$ such that $h_{n_2} > h_{n_1}$. Using induction, we get a subsequence $(h_{n_i})$ such that $h_{n_0} < h_{n_1} < \cdots$.

Case II. The set $W$ is infinite. Let $h_{m_1} \in W$. There exists $m_2 > m_1$ such that $h_{m_2} \in W$ (otherwise $W$ is finite). Then $h_{m_2} \leq h_{m_1}$. Repeat the argument. Use induction to get, $h_{m_1} \geq h_{m_2} \geq \cdots$.

Suppose that the sequence has both positive and negative numbers. Then either infinitely many terms are nonnegative or infinitely many terms are negative. If infinitely many terms are nonnegative, take the subsequence of all these terms and apply the above argument to it.

If infinitely many terms are negative, take the subsequence of all these terms, multiply by $-1$ and apply the above argument to it. ∎

Theorem 3.4.7 **(Bolzano-Weierstrass theroem (BWT))** *Every bounded sequence of real numbers has a convergent subsequence.*

*Proof.* Follows from MST and MCT. ∎

Example 3.4.8 The sequence $(\sin n)$ has a convergent subsequence.

Exercise 3.4.9 *Construct a sequence $(a_n)$ with infinitely many subsequential limits.*

Exercise 3.4.10 *Show that every bounded divergent sequence $(a_n)$ must have at least two distinct subsequential limits.*

## 3.5 Cauchy sequences

Example 3.5.1 What is the diameter (number) of a unit circle? What would you call as diameter of a unit square? Of $(1, 5)$?

Definition 3.5.2

1. The **diameter** of a nonempty set $S$ is defined as $\sup\{|x - y| \mid x, y \in S\}$. The diameter of a sequence $(a_n)$ is the diameter of the set $\{a_1, a_2, \cdots\}$.
2. We say $(a_n)$ is **Cauchy**, if for each $\alpha > 0$, there is a tail with diameter $\leq \alpha$. That is, if for each $\alpha > 0$, $\exists k$ such that $|a_n - a_m| < \alpha$ for each $n, m \geq k$. It does not matter, if you replace the '$<$' by '$\leq$'.

Example 3.5.3

1. The sequence $((-1)^n)$ is not Cauchy, as no tail has diameter less than 1.
2. The sequence $(\frac{1}{n})$ is Cauchy. Let $\epsilon > 0$. Choose $k \in \mathbb{N}$ such that $\frac{1}{k} < \epsilon$. Then, from $k$th term onwards, the distance among the terms is always less than $\epsilon$.


Lemma 3.5.4 *Every Cauchy sequence is bounded.*

*Proof.* Let $(a_n)$ be a Cauchy sequence. So $\exists k$ such that then onwards terms have distance less than 1 among them. Put $M = \max\{|a_1|, \ldots, |a_k|\}$. Then $a_1, \ldots, a_k$ are within a distance $M$ from 0. As all other terms are within a distance 1 from $a_k$, we see that, each $a_n$ is within a distance $M + 1$ from 0. ∎

Theorem 3.5.5 **(Cauchy criterion)** *A sequence in $\mathbb{R}$ is convergent iff it is Cauchy.*

*Proof.* $\Rightarrow$ Let $a_n \to l$. We want to show that it is Cauchy. For that let $\epsilon > 0$. As $a_n \to l$, $\exists n_0$ such that $|a_n - l| < \epsilon/2$, $\forall n \geq n_0$. Hence, $|a_n - a_m| \leq |a_n - l| + |l - a_m| \leq \epsilon$, $\forall n, m \geq n_0$.

$\Leftarrow$ As $(a_n)$ is Cauchy, it is bounded. Using BWT, let $a_{n_k} \to l$. We want to show that $a_n \to l$. For that, let $\epsilon > 0$. As $(a_n)$ is Cauchy, there is a tail $T$ in which terms have distance less than $\epsilon/2$ among them. As $a_{n_k} \to l$, there is a tail $T'$ contained in $(l - \frac{\epsilon}{2}, l + \frac{\epsilon}{2})$. The tail $T'$ must contain a term from $T$. Let it be $a_{n_r}$. So, each element of the tail $T$ have a distance less than $\epsilon$ from $l$. ∎

Example 3.5.6 Take $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Is it convergent? No. How? Because, it is not Cauchy. How? Well, to see that, take $\epsilon = \frac{1}{3}$. Can we find a $n_0$ such that $|a_n - a_m| < \epsilon$ for each $m, n \geq n_0$? Suppose that, we can. Then, in particular, we should have $|a_{2n_0} - a_{n_0}| < \frac{1}{3}$. But, $|a_{2n_0} - a_{n_0}| = \frac{1}{n_0+1} + \cdots + \frac{1}{2n_0} > \frac{1}{2}$. $\Rightarrow\Leftarrow$

DEFINITION 3.5.7 A sequence $(a_n)$ is called **contractive** if $\exists c \in (0,1)$ such that $|a_{n+2} - a_{n+1}| \leq c|a_{n+1} - a_n|$, $\forall n$.

LEMMA 3.5.8 *A contractive sequence is Cauchy. Hence it is convergent.*

*Proof.* Assume that $(a_n)$ is contractive and $|a_{n+2} - a_{n+1}| \leq c|a_{n+1} - a_n|$. Let $\epsilon > 0$. Take $n_0 > 3$ such that $|a_2 - a_1|c^{n_0-1}\frac{1}{1-c} \leq \epsilon$. Then for each $n \geq n_0$ and $k \in \mathbb{N}$, we have

$$\begin{aligned}
|a_{n+k} - a_n| &\leq |a_{n+k} - a_{n+k-1}| + |a_{n+k-1} - a_{n+k-2}| + \cdots + |a_{n+1} - a_n| \\
&\leq |a_2 - a_1|(c^{n+k-2} + c^{n+k-1} + \cdots + c^{n-1}) \\
&\leq |a_2 - a_1|c^{n-1}\frac{1}{1-c} \leq \epsilon.
\end{aligned}$$

So $(a_n)$ is Cauchy. ■

EXAMPLE 3.5.9 Take $a_1 = 2$, $a_{n+1} = 2 + \frac{1}{a_n}$ for each $n$. Is it convergent? To see this, note that each $a_n > 2$ and

$$|a_{n+2} - a_{n+1}| = \left|\frac{1}{a_{n+1}} - \frac{1}{a_n}\right| = \left|\frac{|a_{n+1} - a_n|}{a_{n+1}a_n}\right| \leq \frac{1}{4}|a_{n+1} - a_n|.$$

So, it is contractive and hence it is convergent. Let the limit be $l$. Then the limit must satisfy $l^2 - 2l - 1 = 0$. So either $l = 1 + \sqrt{2}$ or $l = 1 - \sqrt{2}$. But as each $a_n \geq 2$, we must have $l \geq 2$. So $l = 1 + \sqrt{2}$.

EXERCISE* 3.5.10 *Let $a_1 = 1$, $a_2 = 3$, $a_3 = 7$ and $a_{n+3} = \frac{a_n + a_{n+1} + a_{n+2}}{3}$ for $n \in \mathbb{N}$. Is $(a_n)$ convergent?*

EXERCISE* 3.5.11 *It is given that the distance between any two terms of the sequence $(a_n)$ is at most 1. My friend claims that, in that case all the terms of the sequence can be fitted in an interval of length 1. Is that correct?*

EXERCISE* 3.5.12 *There are two particles $A$ and $B$, placed at 0 and 1, on day 1, respectively. On day $n+1$, particle $A$ moves right by one tenth of the distance between the particles on the $n$th day and particle $B$ moves left by two tenth of the distance between the particles on $n$th day. Do you think they will meet eventually? If so, where?*

*Let us randomize it a little bit. On $n+1$th day a coin is tossed.*

1. *If it is 'head', then $A$ moves right by one tenth of the distance between the particles on the $n$th day and particle $B$ moves left by two tenth of the distance between the particles on $n$th day.*

2. *If it is 'tail', then $A$ moves right by two tenth of the distance between the particles on the $n$th day and particle $B$ moves left by four tenth of the distance between the particles on $n$th day.*

*Now, what is your answer and how do you argue?*

## 3.6 Length of the unit circle and angles (self-study)

DEFINITION 3.6.1

1. Let $\widehat{AB}$ be an arc on a circle. How do we define its length? We select a few points $A = A_0, A_1, \ldots, A_n = B$ on the arc, in that order from $A$ to $B$. Let us call this set of points 'a point set $P^*$ on the curve'.

2. If $P^*$ is a point set, define $l(P^*)$ as the sum of the lengths $A_0A_1 + A_1A_2 + \cdots + A_{n-1}A_n$.

3. The set $\{l(P^*) \mid P^* \text{ is a point set of } \widehat{AB}\}$ is bounded above. To see this, for simplicity, consider the arc of the unit circle in the first quadrant. For any point set $P^*$, it is easy to see that $l(P^*) \leq 2$. We show that below.

4. If $(A = (x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n) = B)$ is a point set, here $A = (0, 1)$ and $B = (1, 0)$, then

$$\begin{aligned} l(P^*) &= \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} + \cdots + \sqrt{(x_n - x_{n-1})^2 + (y_n - y_{n-1})^2} \\ &\le (x_1 - x_0) + (y_0 - y_1) + \cdots (x_n - x_{n-1}) + (y_{n-1} - y_n) \\ &= x_1 - x_n + y_0 - y_n = 1 + 1. \end{aligned}$$

5. The **length** of the arc is defined as $\sup_{P^* \text{ is a point set}} l(P^*)$.

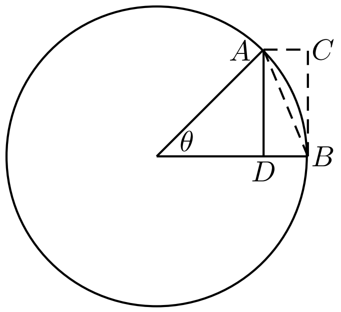
6. The length of the half unit circle is denoted by the number $\boldsymbol{\pi}$.

7. It is easy to see that, by the definition, the perimeter of a circle of radius $r$ is $2\pi r$.

8. Hence, the angle made by an arc of the unit circle at the center is also denoted by the length of that arc.

9. We also take $2\pi = 360°$.

10. Consider the angle $\theta < 90$ and the unit circle shown below.




With an argument similar to the above, we see that $\widehat{AB} \le AC + BC$. That is,

$$AD \le AB \le \widehat{AB} \le AC + BC,$$

from which we get

$$\sin\theta \le \theta \le \sin\theta + (1 - \cos\theta). \tag{3.1}$$

EXERCISE* 3.6.2 *Let $A = (1, 0)$ and $B(0, 1)$ and consider the arc $\widehat{AB}$ of the unit circle (in the first quadrant). A set $\{0 = \theta_0, \theta_1, \ldots, \theta_n = \pi/2\}$, where $\theta_0 < \theta_1 < \cdots < \theta_n$, is called a* **partition of $[0, \pi/2]$**. *Corresponding to each partition $P$ of $[0, \pi/2]$, we get a point set $P^*$ of points $(\cos\theta_i, \sin\theta_i)$ on the arc $\widehat{AB}$. Let $P_n$ denote the partition $\{0, \pi/2n, 2\pi/2n, \ldots, n\pi/2n\}$. Then*

$$\sup_{P \text{ partition of } [0,\pi/2]} l(P^*) = \sup_n l(P_n^*).$$

EXERCISE* 3.6.3 *Consider an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $b > a$. Show that its length $C$ satisfies $2\pi a < C < 2\pi b$.*

## 3.7 Limsup and liminf (self study, read it when you need)

DEFINITION 3.7.1 If $(a_n)$ is not bounded above, we define $\limsup a_n = \infty$. If $(a_n)$ is bounded above, note that

$$\sup\{a_1, a_2, \cdots\} \ge \sup\{a_2, a_3, \cdots\} \ge \sup\{a_3, a_4, \cdots\} \ge \cdots$$

is a monotone sequence of real numbers. If it is bounded below, then the limit is called the $\mathbf{\limsup a_n}$. We write $\limsup a_n = -\infty$, if the above sequence is not bounded below. The term $\mathbf{\liminf a_n}$ is defined similarly.

Example 3.7.2 1. Let $a_n = (-1)^n$. Then $\limsup a_n = 1$ and $\liminf a_n = -1$.

2. Let $a_n = n^2 \sin^2(\frac{n\pi}{2})$. Then $\limsup a_n = \infty$ and $\liminf a_n = 0$.

3. Let $a_n = -n$. Then $\limsup a_n = -\infty$ and $\liminf a_n = -\infty$.

4. Let $a_n = \frac{1}{n}$. Then $\limsup a_n = 0$ and $\liminf a_n = 0$.

Lemma 3.7.3 *For a sequence* $(a_n)$, $\liminf a_n \leq \limsup a_n$.

*Proof.* Let $(a_n)$ be bounded. Put $T_n =$ the $n$th tail. Note that $x_n := \inf T_n \leq y_n := \sup T_n$. So $\liminf a_n = \lim x_n \leq \lim y_n = \limsup a_n$. If $(a_n)$ is not bounded above then $\limsup a_n = \infty$, so $\liminf a_n \leq \limsup a_n$. If $(a_n)$ is not bounded below then $\liminf a_n = -\infty$, so $\liminf a_n \leq \limsup a_n$. ∎

Exercise 3.7.4 *A sequence* $(a_n)$ *is convergent iff* $\liminf a_n = \limsup a_n = l \in \mathbb{R}$.

## 3.8 Diverging to infinity (self study)

Definition 3.8.1 We say $\boldsymbol{a_n \to \infty}$ or $\boldsymbol{\lim a_n = \infty}$ or '$a_n$ diverges to $\infty$', if for each $k \in \mathbb{N}$, there is a tail with each term $a_n \geq k$. We define $\boldsymbol{a_n \to -\infty}$ or $\boldsymbol{\lim a_n = -\infty}$ similarly. Some texts may write 'converges to infinity' in place of 'diverges to infinity' as the definition is in a sense similar to the convergence.

Example 3.8.2 1. The sequence $(n)$ diverges to $\infty$ and $(-n)$ diverges to $-\infty$.

2. The sequence $1, 0, 2, 0, 3, 0, 4, 0, \cdots$ is unbounded and does not diverge to $\infty$. Can you write a definition of $(a_n)$ not diverging to $\infty$.

3. We have $a_n \to \infty$ iff $\liminf a_n = \infty$. Of course, it means $\liminf a_n = \limsup a_n = \infty$.

Exercise 3.8.3 **(LTs)** *Let* $a_n \to \infty$ *and* $(a_{n_k})$ *be a subsequence of* $(a_n)$. *Show that* $\lim\limits_{k\to\infty} a_{n_k} = \infty$.

## 3.9 The exponential function (self study)

Fix $x \in \mathbb{R}$. We already know that $a_n = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$ converges and the limit is defined as $\exp x$. We also know that $a_n = (1 + \frac{1}{n})^n$ converges and the limit is $e$.

Exercise 3.9.1 *1. Let* $(n_k)$ *be a sequence of natural numbers diverging to* $\infty$. *Then* $\lim\limits_{k\to\infty} (1 + \frac{1}{n_k})^{n_k} = e$.

*2. Let* $a_k > 0$ *be a sequence of rationals diverging to* $\infty$. *Show that* $\lim (1 + \frac{1}{a_k})^{a_k} \to e$.

*3. Show that, for any rational number* $x = \frac{p}{q} > 0$, $e^x = \lim \left(1 + \frac{x}{n}\right)^n$.

*4. Let* $x > 0$ *be a rational. Show that* $\exp x = e^x$. [Hint: Put $s_n = \left(1 + \frac{x}{n}\right)^n$, $t_n = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$ and $r_k = \binom{n}{0} + \binom{n}{1}\frac{x}{n} + \cdots + \binom{n}{k}\frac{x^k}{n^k}$. Let $\alpha > 0$. Show that $\exists k$ such that $\forall n > k$ we have $r_{k-1} \leq s_n \leq r_{k-1} + \alpha$.]

*5. Conclude that* $\lim \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right) = e$.

*6.* **(irrationality of** $e$**)** *Conclude that* $e$ *is irrational.* [Hint: Assume that $e = \frac{p}{q}$, $\gcd(p, q) = 1$. Then $q!e - q!(1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{q!})$ must be a positive integer. Can it be?]

7. *Let $x, y \geq 0$. Show that*

$$\begin{aligned} 1+(x+y)+\frac{(x+y)^2}{2!}+\cdots+\frac{(x+y)^n}{n!} &\leq \left(1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}\right)\left(1+y+\frac{y^2}{2!}+\cdots+\frac{y^n}{n!}\right) \\ &\leq 1+(x+y)+\frac{(x+y)^2}{2!}+\cdots+\frac{(x+y)^{2n}}{(2n)!}. \end{aligned}$$

   *Hence show that* $\exp(x)\exp(y) = \exp(x+y)$.

8. *Let $x, y \in \mathbb{R}$. Put $S_n(x) = 1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}$. Then, we have $S_n(x)S_n(y)-S_n(x+y) = r_1+\cdots+r_m$, where $r_i$ are some terms in $x$ and $y$. Show that*

$$|S_n(x)S_n(y) - S_n(x+y)| \leq |r_1| + \cdots + |r_m| = S_n(|x|)S_n(|y|) - S_n(|x|+|y|).$$

   *Hence, show that*

$$\exp(x)\exp(y) = \exp(x+y) \qquad \forall x, y. \tag{3.2}$$

9. *Show that* $\exp(x) \neq 0$.

10. *Show that* $\exp(x) = \exp(\frac{x}{2}+\frac{x}{2}) > 0$. *Also show that* $\exp(-x)$ *is the reciprocal of* $\exp(x)$.

11. *Show that if $x > 0$, then* $\exp(x) > 1$ *and if $x < 0$, then* $\exp(x) < 1$.

12. *Show that* $\exp(x)$ *is a strictly increasing (hence one-one) function from $\mathbb{R}$ to $(0, \infty)$.*

13. *Let $r \in (1, \infty)$. Show that there is a number $x > 0$ such that* $1 < \exp(x) < r$.

14. *Let $r \in (0, \infty)$. Show that $\exists x$ such that* $\exp(x) = r$ *(onto function).*
   [Hint: For $r > 1$, put $L = \{x \in \mathbb{R} \mid \exp(x) < r\}$ and consider $l = \mathsf{lub}\, L$.
   If $\exp l < r$. Then $1 < \frac{r}{\exp l}$. Use the previous item to get a contradiction.
   If $\exp l > r$, then use the previous item and and the increasing property to get a contradiction.
   For $r < 1$, use the reciprocal.]

DEFINITION 3.9.2 1. For a positive real number $x$ we define $\mathbf{\ln x}$ as the unique number $a$ which satisfies $\exp a = x$.

2. For $x > 0$, we define $\boldsymbol{x^b}$ as the number $\exp(b \ln x)$. So, $x^0 = 1$.

EXERCISE 3.9.3 **(Log function)** *Is $\ln x$ a strictly increasing function? Show that for small positive $x$, we have*

$$\frac{x}{2} < \ln(1+x) < x.$$

EXERCISE 3.9.4 **(Exponential)** *Suppose $e \leq x < y$. Show that $x^y > y^x$. In particular, $e^\pi > \pi^e$.*

EXERCISE 3.9.5 **(Exponential)** *Let $M$ be a fixed natural number.*

1. *Show that there is a natural number $n$ such that for all real number $x > n$ we have $e^x > x^M$.*
   [Hint: Focus on $\frac{x^{M+1}}{(M+1)!}$.]

2. *Hence show that for all large $y$, we have $y > (\ln y)^M$.*

3. *Let $e > p > 1$ be fixed. Show that there is a natural number $n$ such that for all real number $x > n$ we have $p^x > x^M$.* [Hint: Use previous item, focus on $e^{\alpha x}$.]

## 3.10 Exercises

EXERCISE 3.10.1 *Discuss the convergence of* $(\frac{\sin(1/n)}{(1/n)})$. *Let* $P_n$ *be a regular polygon with* $2(2^n)$ *vertices which are on the circle. Show that the area* $A(P_n)$ *of* $P_n$ *increases to* $\pi r^2$.

EXERCISE 3.10.2 *Let* $(a_n)$ *be a convergent sequence of integers. Show that its limit is an integer by showing that the sequence is* **eventually** *constant (this means* $a_k = a_{k+1} = a_{k+2} = \cdots$ *for some* $k$*).*

EXERCISE 3.10.3

1. *Let* $x_n \to 0$. *Use sandwich lemma to show that* $\sin(x_n) \to 0$.
2. *Let* $x_n \to l$. *Use* $\sin(x_n) - \sin(l)$ *formula to show that* $\sin(x_n) \to \sin(l)$.
3. *Take* $a \in [-1, 1]$. *Put* $l = \sin^{-1}(a) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. *Show that there exists a sequence* $(a_n)$ *converging to* $l$, *where* $a_n = m_n\pi + k_n \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $m_n \in \mathbb{Z}$ *and* $k_1 < k_2 < \cdots$ *are natural numbers.*
4. *What is the limit of* $\sin(a_n)$?
5. *Conclude that* $(\sin(n))$ *has a subsequence converging to* $a$.

EXERCISE 3.10.4 *Test for convergence:* $\{a_n\}$ *where* $a_1 = 2$, $a_{n+1} = \sqrt{2a_n - 1}$ *for* $n \in \mathbb{N}$.

EXERCISE 3.10.5 *Test for convergence.*

1. $a_1 = \sqrt{2}$ *and* $a_{n+1} = \sqrt{2 + \sqrt{a_n}}$ *for* $n \in \mathbb{N}$.
2. $x_1, x_2 \in \mathbb{R}$, $x_n = \frac{x_{n-1}+x_{n-2}}{2}$ *for* $n \geq 3$.
3. $a_n = \left(2^{\frac{1}{2}} - 2^{\frac{1}{3}}\right)\left(2^{\frac{1}{2}} - 2^{\frac{1}{4}}\right)\cdots\left(2^{\frac{1}{2}} - 2^{\frac{1}{n+2}}\right)$.
4. $\{(1 - \frac{1}{n})\sin(\frac{n\pi}{2})\}$.
5. $\{\frac{a^{bn}}{b^{an}}\}$, *where* $e \leq a < b$ *and* $a, b \in \mathbb{Q}$.
6. $(1 + \frac{1}{n^2})^n$.
7. $\frac{\ln n}{\sqrt{n}}$, $n \geq 2$.
8. *Fix* $p > 0$ *and take* $a_n = \frac{\ln n}{n^p}$, $n \geq 2$.

EXERCISE 3.10.6 *For a sequence of points* $(x_n)$ *in* $\mathbb{R}^k$, *we say* $x_n \to l \in \mathbb{R}^k$, *if* $|x_n - l| \to 0$. *Geometrically,* $x_n \to l$ *means each* $B_\alpha(l)$ *contains a tail of the sequence. If* $x_n \to l$, *then* $l$ *is called the* **limit** *of the sequence.*

1. *Show that a sequence* $x_n = (x_{n1}, x_{n2}) \to l = (l_1, l_2)$ *iff* $x_{n1} \to l_1$ *and* $x_{n2} \to l_2$.
2. *Take any point* $a \in \mathbb{R}^2$. *Give a sequence* $x_n \to a$ *such that each* $x_n$ *has both coordinates rational. Give a sequence* $x_n \to a$ *such that each* $x_n$ *has only the first coordinate rational.*

EXERCISE 3.10.7 *Let* $(a_n)$ *be a sequence of positive numbers and* $a_n \to \inf\{a_1, a_2, \cdots\}$. *Is it necessary that* $(a_n)$ *should have a decreasing tail?*

EXERCISE 3.10.8 **(Fibonacci sequence)** *Let* $F_1 = F_2 = 1$, *and define* $F_n = F_{n-1} + F_{n-2}$, $\forall n > 2$. *Show that the sequence* $(\frac{F_n}{F_{n-1}})$ *converges to* $\frac{\sqrt{5}+1}{2}$. *This number is known as the* **golden ratio**.

EXERCISE 3.10.9 *Let* $a_n \to l$. *Show that* $\frac{a_1+a_2+\cdots+a_n}{n} \to l$. *Show that the converse is not true.*

Exercise 3.10.10 *Let* $a_1 = 0$, $a_{2n} = \frac{a_{2n-1}}{2}$, *and* $a_{2n+1} = \frac{1}{2} + a_{2n}$, *for* $n = 1, 2, \cdots$. *Write first* 15 *terms of the sequence. What are* $\limsup a_n$ *and* $\liminf a_n$*? Justify your answer.*

Exercise* 3.10.11 *Let* $X$ *be a nonempty subset of* $\mathbb{R}$. *Then the following are equivalent.*

a) *Every Cauchy sequence of elements of* $X$ *converges to a point in* $X$.

b) *The set* $X$ *is closed.*

Exercise* 3.10.12 **(completeness)** *The following statements are equivalent.*

a) *Every nonempty subset* $S$ *which is bounded above has the lub in* $\mathbb{R}$.

b) *Every nonempty subset* $S$ *which is bounded below has the glb in* $\mathbb{R}$.

c) *Every Cauchy sequence of elements of* $\mathbb{R}$ *converge to some point in* $\mathbb{R}$.

Exercise 3.10.13 *There was a sequence of distinct positive numbers written on a paper. I have not seen that. My friend sees it and says that it is created by taking an increasing sequence* $(a_n)$ *and a decreasing sequence* $(b_n)$ *following the rule*

*'insert* $b_1$ *wherever you want in the sequence* $(a_n)$ *and* $b_{n+1}$ *can only be inserted after* $b_n$ *and to the right of* $b_n$*'.*

*How do I verify, whether he is correct?*

Exercise 3.10.14 *Write True/False.*

1. *A sequence can have exactly two limits.*
2. *A sequence must have at least one limit.*
3. *A bounded sequence must have a limit.*
4. *An unbounded sequence will never have a limit.*
5. *A monotone sequence must have a limit.*
6. *A monotone sequence which is bounded above, must have a limit.*
7. *A bounded monotone sequence must have a limit.*
8. *If* $a_n \to a$, *then* $|a_n| \to |a|$.
9. *If* $|a_n| \to a$, *then* $(a_n)$ *must converge.*
10. *If* $a_n \neq 0$ *and* $(a_n)$ *is convergent, then* $(\frac{1}{a_n})$ *must be convergent.*
11. *If* $a_n \neq 0$ *and* $a_n \to a \neq 0$, *then* $(\frac{1}{a_n})$ *must be convergent.*
12. *If* $a_n \to 5$, *then* $(a_{p_n})$ *may be divergent, where* $p_n = n$*th prime number.*
13. *If* $a_n \to 5$ *and* $(a_{n_k})$ *is a subsequence, then* $(a_{n_k})$ *must be convergent.*
14. *If* $a_{2n} \to 5$ *and* $a_{2n+1} \to 5$, *then* $a_n \to 5$.
15. *If* $a_{2n} \to 5$ *and* $a_{3n} \to 5$, *then* $a_n \to 5$.
16. *If* $a_{3n} \to 5$, $a_{3n+1} \to 5$ *and* $a_{3n+2} \to 5$, *then* $a_n \to 5$.
17. *If* $(a_n)$ *is a Cauchy sequence, then* $(a_n)$ *has a constant tail.*

18. *If* $(a_n)$ *is a Cauchy sequence of integers, then* $(a_n)$ *has a constant tail.*

19. $(a_n)$ *is convergent iff a tail of* $(a_n)$ *is convergent.*

20. $(a_n)$ *is divergent iff a tail of* $(a_n)$ *is divergent.*

21. *If* $(a_n)$ *and* $(b_n)$ *are divergent, then* $(a_n + b_n)$ *is necessarily divergent.*

22. *If* $(a_n)$ *and* $(b_n)$ *are divergent, then* $(a_n b_n)$ *is necessarily divergent.*

23. *If* $(a_n)$ *is convergent and* $(b_n)$ *is divergent, then* $(a_n + b_n)$ *is necessarily divergent.*

24. *If* $(a_n)$ *is convergent and* $(b_n)$ *is divergent, then* $(a_n b_n)$ *is necessarily divergent.*

25. $(\frac{[10^n\pi]}{10^n})$ *is a monotone increasing sequence which is bounded above.*

26. $0 \leq \pi - \frac{[10^n\pi]}{10^n} < \frac{1}{10^n}$.

27. *Let* $a_n = 1 + \cdots + \frac{1}{n}$. *Then* $a_n \uparrow \infty$.

28. *If* $(a_n)$ *is a sequence of real numbers such that the difference of consecutive terms goes to* $0$, *i.e.,* $|a_{n+1} - a_n| \to 0$, *then* $(a_n)$ *is convergent.*

29. *Every Cauchy sequence of elements of* $\mathbb{N}$ *converges to a point in* $\mathbb{N}$.

30. *Every Cauchy sequence of elements of* $\mathbb{Z}$ *converges to a point in* $\mathbb{Z}$.

31. *Every Cauchy sequence of elements of a nonempty finite set* $S \subseteq \mathbb{R}$ *converges to a point in* $S$.

32. *Every Cauchy sequence of elements of* $[0, 1]$ *converges to a point in* $[0, 1]$.

33. *Every Cauchy sequence of elements of* $(0, 1]$ *converges to a point in* $(0, 1]$.

34. *Every Cauchy sequence of elements of* $\mathbb{Q}$ *converges to a point in* $\mathbb{Q}$.

35. *The sequence* $(\sin n)$ *has more than* $50,000$ *Cauchy subsequences with distinct terms.*

36. *If a sequence of rational numbers converges to* $a$ *and a sequence of irrational numbers also converges to* $a$ *then* $a$ *must be* $0$.

37. *Let* $\alpha > 0$ *be fixed. Then there exists a sequence* $(a_n)$ *with* $\limsup a_n = \alpha \cdot \liminf a_n$.

38. *Let* $a_n > 0$ *be decreasing with limit* $a$. *Then* $(\sqrt[n]{a_n})$ *is convergent.*

39. *Let* $a_n > 0$ *be decreasing with limit* $a$ *such that* $(\sqrt[n]{a_n})$ *is convergent to* $l$. *Then* $0 \leq l \leq 1$.

40. *Let* $a_n > 0$ *be decreasing with limit* $a$ *such that* $(\sqrt[n]{a_n})$ *is convergent. Then the limit of the later could only be* $0$ *or* $1$.

41. *Let* $a_n > 0$. *Suppose that* $a_n \to 0$ *is FALSE. Then there exists* $\alpha > 0$ *such that* $a_n > \alpha$ *for infinitely many values of* $n$.

# Chapter 4

# Infinite series

**After this chapter, I should be able to answer the following.**

1. A robot is at the origin. It moves 1 unit right on the first day, $\frac{1}{2}$ unit left on the second day, $\frac{1}{3}$ unit right on the third day, $\frac{1}{4}$ unit left on the fourth day, and so on. Eventually, is it approaching to some real number?

2. I know how to define the sum of finitely many numbers. How can I define the sum of a infinite set of numbers?

3. Does it make sense to say 'sum of elements of $(0, 1)$'?

4. Does it make sense to write sum of elements of $\{1, \frac{1}{2}, \frac{1}{4}, \ldots\}$?

5. Argue that, it only makes sense to talk about the sum of a countably infinite ordered set of numbers, that is, a sequence $(a_1, a_2, \ldots)$.

   The sum, in that case is, evaluated by first adding $a_2$ to $a_1$ (call the result $A_2$), then adding $a_3$ to $A_2$ (call the result $A_3$), then adding $a_4$ to $A_3$, and so on.

   In that case, the value of the sum is the limiting value of the $A_n$.

6. Argue that if the value of the sum of a sequence is $a$, then the value of a rearrangement of the sequence can be different.

7. How to make sense of the sum of the terms of a sequence, without even knowing the terms (just by knowing that the terms are between this and that)?

8. Consider $f(x) = 20x^9 - 50x^8 - 17x^7 + 37x^6 - 500x^5 + 10^{25}x^4 - 23x + 2020$. Show that outside some interval $[-M, M]$, the value of $x^{10}$ dominates (is more than) $|f(x)|$.

## 4.1 Convergence and standard examples

REMARK 4.1.1 **(Why don't we define the sum of elements of an arbitrary set?)**

1. Let $S \subseteq \mathbb{R}$ be infinite. Does it make sense to define sum of all the elements of $S$? Not in general. Take for example $S = (0, 1)$. Then there are infinitely many numbers in $S$ which are more than .5. So, if we are going to take the sum we will not get a real number.

2. In general, given a set $S$ of positive numbers, define $S_n = \{s \in S \mid s > \frac{1}{n}$. If any of the $S_n$ is infinite, then it is not possible to define the sum.

3. So assume that each $S_n$ is finite. Then notice that $S = \cup S_n$ and so $S$ is countable.

4. So, let $S$ be a countably infinite set of real numbers. Is it possible to define the sum. Not in general. For example, consider the set $S = \{1, \frac{-1}{2}, \frac{1}{3}, \frac{-1}{4}, \ldots, \}$. One person may add the terms one by one just as it is shown in that list. However another person may take some term as the first, and some other term as the second and so on. In that case the limiting values they will get can be different. We will see that soon.

5. So, we fix the order. A countably infinite ordered set of real numbers is nothing but a sequence $(a_1, a_2, \ldots)$. To make it more general, we also allow repetition of numbers. So they are actually functions from $\mathbb{N}$ to $\mathbb{R}$.

6. And, we define the value of the sum as the limiting value of the sequence $(A_n)$, where $A_n = a_1 + a_2 + \cdots + a_n$, if it is a real number.

DEFINITION 4.1.2 1. An **infinite series** is an expression $a_1 + a_2 + a_3 + \cdots$, where $a_n \in \mathbb{R}$. We use $\sum \boldsymbol{a_n}$ to denote it.

2. We use $\sum\limits_{\boldsymbol{n\geq k}} \boldsymbol{a_n}$ to denote $a_k + a_{k+1} + \cdots$. A simple mention of $\sum a_n$ would mean that $n \geq 1$.

3. The term $a_n$ is normally called the $n$th term, even if the series starts with $a_k$.

4. For a series $\sum_{n\geq k} a_n$, put $A_n = a_k + \cdots + a_n$. The sequence $(A_n)_{n\geq k}$ is called the **sequence of partial sums**.

5. Normally, for a series $\sum a_n$ we use the $A_n$ for the partial sum and for $\sum b_n$ we use $B_n$. If the terms are not labeled, mostly we use $S_n$. Other texts may not use these notations.

6. We say $\sum a_n$ is **convergent** with **limit/value** $l$ if the sequence of partial sums $(A_k)$ converges to $l$ and denote this by $\sum a_n = l$.

7. A series is **divergent** if it is not convergent.

8. We say $\sum a_n$ **diverges to $\infty$** to mean that $(A_n)$ diverges to $\infty$ and denote this by $\sum a_n = \infty$. The notation $\sum a_n = -\infty$ is defined similarly.

EXAMPLE 4.1.3 1. The series $\sum (-1)^n$ is divergent, as $(S_n) = (-1, 0, -1, 0, \ldots)$ is divergent.

2. We have $\sum\limits_{n\geq 1} 1 = \infty$. That is, the series diverges to $\infty$. This is because, $(S_n) = (n)$ diverges to $\infty$.

3. The series $\sum \frac{1}{2^n} = 1$, as $S_n \to 1$.**!!**

4. Fix $k \in \mathbb{N}$. Then $\sum\limits_{n\geq 1} a_n$ is convergent iff $\sum\limits_{n\geq k} a_n$ is convergent.

5. Fix $a \neq 0$ and $r$ with $|r| < 1$. The series $\sum_{n\geq 0} ar^n$ converges to the value is $\frac{a}{1-r}$.**!!** This is called the **geometric series**.

6. The series $\sum \frac{\boldsymbol{1}}{\boldsymbol{n}}$ is called the **harmonic series**. It diverges to $\infty$, as we know from Example 3.5.6. Another way of seeing it, is the following. The partial sum

$$S_{2^n} = 1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \cdots + \frac{1}{8}) + \cdots + (\frac{1}{2^{n-1}+1} + \cdots + \frac{1}{2^n}) \geq 1 + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} = 1 + \frac{n}{2}.$$

Since $S_n \uparrow$, we see that $S_n \to \infty$.

7. The series $\sum\limits_{n\geq 2} \frac{1}{n\ln n} = \infty$. To see this do the grouping again. Then the partial sum $S_{2^n}$

$$\begin{aligned} &= \frac{1}{2\ln 2} + (\frac{1}{3\ln 3} + \frac{1}{4\ln 4}) + (\frac{1}{5\ln 5} + \cdots + \frac{1}{8\ln 8}) + \cdots + (\frac{1}{(2^{n-1}+1)\ln(2^{n-1}+1)} + \cdots + \frac{1}{2^n \ln 2^n}) \\ &\geq \frac{1}{2\ln 2} + \frac{2}{4\ln 4} + \frac{4}{8\ln 8} + \cdots + \frac{2^{n-1}}{2^n \ln 2^n} = \frac{1}{2\ln 2}\left[1 + \frac{1}{2} + + \frac{1}{3} + \cdots + \frac{1}{n}\right]. \end{aligned}$$

Since $S_n \uparrow$, we see that $S_n \to \infty$.

8. We have $\sum \frac{1}{n(n+1)} = 1$, as $S_k = \sum\limits_{n=1}^{k} (\frac{1}{n} - \frac{1}{n+1}) = 1 - \frac{1}{k+1} \to 1$, as $k \to \infty$.

9. Let $p > 1$ and consider $\sum \frac{1}{n^p}$. Then the partial sum $S_{2^n - 1}$

$$\begin{aligned} &= \frac{1}{1} + (\frac{1}{2^p} + \frac{1}{3^p}) + (\frac{1}{4^p} + \cdots + \frac{1}{7^p}) + \cdots + (\frac{1}{(2^{n-1})^p} + \cdots + \frac{1}{(2^n - 1)^p}) \\ &\leq \frac{1}{1} + \frac{2}{2^p} + \frac{4}{4^p} \cdots + \frac{2^{n-1}}{(2^{n-1})^p} \\ &= 1 + \frac{1}{r} + \frac{1}{r^2} + \cdots + \frac{1}{r^{n-1}}, \qquad (r = 2^{p-1}) \\ &\leq \frac{1}{1-r}. \end{aligned}$$

Since $S_n \uparrow$, by MCT, $(S_n)$ converges. (Try this for $\sum \frac{1}{n(\ln n)^p}$, where $p > 1$ is fixed.)

Exercise 4.1.4 **(Cauchy condition)** *The series $\sum a_n$ converges iff $(S_n)$ is Cauchy. That is, iff*

$$\forall \alpha > 0, \exists k \in \mathbb{N} \text{ such that } |a_n + a_{n+1} + \cdots + a_m| < \alpha, \text{ for all } m > n \geq k.$$

Lemma 4.1.5 **($n$th term test)** *Suppose that $\sum a_n$ is convergent. Then $a_n \to 0$. Converse is not true.*

*Proof.* Let $\sum a_n = l$. So the partial sum $S_n \to l$. So $b_n = S_n - S_{n-1} \to 0$. It is easy to find an example to show that the converse is not true. ∎

Exercise 4.1.6 *Test for convergence.*

1. $\sum n^2$
2. $\sum \frac{(2n)!}{n!n!}$
3. *geometric series for* $|r| \geq 1$
4. $\sum(a + nb)$, *for fixed* $a, b \in \mathbb{R}$.

## 4.2 Limit theorems

Lemma 4.2.1 **(LTs)** *Let $\sum a_n = l$, $\sum b_n = k$ and $\alpha \in \mathbb{R}$. Then $\sum(a_n \pm \alpha b_n) = l \pm \alpha k = \sum a_n \pm \alpha \sum b_n$.***!!**

Example 4.2.2 We have

$$\left(\frac{1}{2} + \frac{1}{3^2}\right) + \left(\frac{1}{2^3} + \frac{1}{3^4}\right) + \cdots = \left(\frac{1}{2} + \frac{1}{2^3} + \cdots\right) + \left(\frac{1}{3^2} + \frac{1}{3^4} + \cdots\right) = \frac{2}{3} + \frac{1}{8}.$$

Definition 4.2.3 The series $\sum(b_n - b_{n+1})$ is called the **telescoping series** of $(b_n)$.

Theorem 4.2.4 *Telescoping series of $(b_n)$ is convergent iff $(b_n)$ is convergent.***!!**

**Removal of brackets** Let $\sum(a_n - b_n)$ be convergent. Is the series $a_1 - b_1 + a_2 - b_2 + a_3 - b_3 + \cdots$ necessarily convergent? No. Notice that $\sum(1-1) = (1-1) + (1-1) + (1-1) + \cdots$ is convergent, but the series $1 - 1 + 1 - 1 + 1 - 1 + \cdots$ is not convergent. So removing brackets may destroy convergence. However, if $\sum a_n$ and $\sum b_n$ are convergent, then $\sum(a_n - b_n) = a_1 - b_1 + a_2 - b_2 + a_3 - b_3 + \cdots = a - b$. (Use LTs.)

EXERCISE 4.2.5 *Show that insertion of brackets (grouping consecutive terms) into a convergent series keeps it convergent. However, insertion of brackets into a divergent series can make it convergent.*

EXERCISE 4.2.6 *Suppose that* $(a_1 + a_2 + \cdots + a_{n_1}) + (a_{n_1+1} + a_{n_1+2} + \cdots + a_{n_2}) + (a_{n_2+1} + a_{n_2+2} + \cdots + a_{n_3}) + \cdots$ *is convergent with limit* $l$. *Suppose also that* $\sum a_n$ *is convergent. Is it necessary that* $\sum a_n = l$*?*

## 4.3 Comparison test, sandwich and absolute convergence

LEMMA 4.3.1 **(Comparison test)** *Let* $0 \le a_n \le b_n$.

a) *If* $\sum b_n = l$, *then* $\sum a_n$ *converges and* $0 \le \sum a_n \le \sum b_n$.

b) *If* $\sum a_n = \infty$, *then* $\sum b_n = \infty$.

*Proof.* a) Put $A_n = \sum_{i=1}^{n} a_i$ and $B_n = \sum\limits_{i=1}^{n} b_i$. As $B_n \uparrow$, we see that $B_n \le l$. As $A_n \le B_n \le l$ and as $A_n \uparrow$, by MCT, $(A_n)$ must converge to $\mathsf{lub}\{A_1, A_2, \ldots\} \le \mathsf{lub}\{B_1, B_2, \ldots\} = l = \sum b_n$.

b) Exercise. ∎

EXAMPLE 4.3.2 1. The series $\sum \frac{1}{\sqrt{n}2^n}$ is convergent as $\frac{1}{\sqrt{n}2^n} \le \frac{1}{2^n}$ and $\sum \frac{1}{2^n}$ is convergent.

2. The series $\sum \frac{1}{n^p}$ for $0 < p < 1$ is divergent as $\frac{1}{n^p} \ge \frac{1}{n}$ and $\sum \frac{1}{n}$ is divergent.

LEMMA 4.3.3 **(sandwich)** *Let* $a_n \le b_n \le c_n$. *If* $\sum a_n$ *and* $\sum c_n$ *are convergent, then* $\sum b_n$ *is convergent.*

*Proof.* As $0 \le (b_n - a_n) \le (c_n - a_n)$ and $\sum(c_n - a_n)$ is convergent, by comparison test $\sum(b_n - a_n)$ is convergent. As $\sum a_n$ is convergent, we have $\sum b_n = \sum[(b_n - a_n) + a_n]$ is convergent. ∎

EXAMPLE 4.3.4 1. Is $\frac{1}{2} \overbrace{-\frac{1}{3^2} - \frac{1}{2^3}} \overbrace{+\frac{1}{3^4} + \frac{1}{2^5} + \frac{1}{3^6}} \overbrace{-\frac{1}{2^7} - \frac{1}{3^8} - \frac{1}{2^9} - \frac{1}{3^{10}}} + \cdots$ convergent? Here the terms are alternately $\frac{1}{2^n}$ or $\frac{1}{3^n}$ with the sign of the first term positive and then the next two terms are negative and so on. Yes, this is convergent, as $-\frac{1}{2^n} \le a_n \le \frac{1}{2^n}$.

2. Is $\frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} - \frac{1}{2^4} + \cdots$ convergent? Yes, as $-\frac{1}{2^n} \le a_n \le \frac{1}{2^n}$.

3. Arbitrarily give $+$ or $-$ signs to the terms in the above series. Then? Yes, it is convergent with the same argument.

4. Suppose that $\sum |a_n|$ is convergent. Must $\sum a_n$ be convergent? Yes. As $-|a_n| \le a_n \le |a_n|$.

DEFINITION 4.3.5 A series $\sum a_n$ is called **absolutely convergent**, if $\sum |a_n|$ is convergent. If $\sum a_n$ is convergent, but $\sum |a_n|$ is not, then $\sum a_n$ is called **conditionally convergent**.

EXAMPLE 4.3.6 a) The series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$ is absolutely convergent as $1 + \frac{1}{2} + \frac{1}{4} + \cdots$ is convergent.

b) The series $1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{2^3} + \frac{1}{4^2} - \frac{1}{2^4} + \cdots$ converges absolutely.

c) Let $a_n, b_n > 0$. Suppose that $\sum a_n$ and $\sum b_n$ are convergent. Then the series $a_1 - b_1 + a_2 - b_2 + a_3 - b_3 + \cdots$ converges absolutely.

THEOREM 4.3.7 **(absolute convergence test)** *If $\sum a_n$ is absolutely convergent, then $\sum a_n$ is convergent.***!!**

EXAMPLE 4.3.8 Consider $1 - \frac{1}{2} + \frac{1}{3} - \cdots$. Then

$$S_{2n} = 1 - \frac{1}{2} + \cdots + \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(2n-1)(2n)} \leq \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)}.$$

Hence $(S_{2n})$ is convergent (why?). Let $l$ be the limit. Then $S_{2n+1} = S_{2n} + \frac{1}{2n+1} \to l$. So $S_n \to l$. So the series is convergent. Does it converge absolutely?