MA 101 (Mathematics I)

Multivariable Calculus : Tutorial Problem Set - 1

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1. Let $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^m$. Show that $\|\mathbf{x}+\mathbf{y}\| = \|\mathbf{x}\| + \|\mathbf{y}\|$ iff $\mathbf{y} = \mathbf{0}$ or $\mathbf{x} = \alpha\mathbf{y}$ for some $\alpha \geq 0$.

2. Let $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^m$ and $r, s > 0$. Show that $B_r[\mathbf{x}] \cap B_s[\mathbf{y}] \neq \emptyset$ iff $\|\mathbf{x}-\mathbf{y}\| \leq r+s$.

3. Let $(\mathbf{x}_n)$ be a sequence in $\mathbb{R}^m$. Show that $(\mathbf{x}_n)$ converges in $\mathbb{R}^m$ iff for each $\mathbf{x} \in \mathbb{R}^m$, the sequence $(\mathbf{x}_n \cdot \mathbf{x})$ converges in $\mathbb{R}$.

4. State TRUE or FALSE with justification for each of the following statements.
   (a) If $(\mathbf{x}_n)$ is a sequence in $\mathbb{R}^m$ having no convergent subsequence, then it is necessary that $\lim\limits_{n\to\infty} \|\mathbf{x}_n\| = \infty$.
   (b) If $\big((x_n, y_n)\big)$ is a bounded sequence in $\mathbb{R}^2$ such that every convergent subsequence of $\big((x_n, y_n)\big)$ converges to $(0,1)$, then $\big((x_n, y_n)\big)$ must converge to $(0,1)$.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} \frac{xy}{x^2-y^2} & \text{if } x^2 \neq y^2, \\ 0 & \text{if } x^2 = y^2. \end{cases}$
   Determine all the points of $\mathbb{R}^2$ where f is continuous.

6. Let α, β be positive real numbers and let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by
   $f(x,y) = \begin{cases} \frac{|x|^\alpha |y|^\beta}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$
   Show that f is continuous iff $\alpha + \beta > 1$.

7. Let $f : S \subseteq \mathbb{R}^2 \to \mathbb{R}$ and let $(x_0, y_0) \in S$. Let $A = \{x \in \mathbb{R} : (x, y_0) \in S\}$ and $B = \{y \in \mathbb{R} : (x_0, y) \in S\}$. Define $\varphi(x) = f(x, y_0)$ for all $x \in A$ and $\psi(y) = f(x_0, y)$ for all $y \in B$. If f is continuous at (x_0, y_0), then show that $\varphi : A \to \mathbb{R}$ is continuous at x_0 and $\psi : B \to \mathbb{R}$ is continuous at y_0. Is the converse true? Justify.

8. If $S = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 3\}$, then determine (with justification) S^0.