Tutorial 2: Sequence2,3, Series 1

1. Let $a_1 = 1$, $a_2 = 3$, $a_3 = 7$, and define $a_{n+3} = \dfrac{a_n + a_{n+1} + a_{n+2}}{3}$, for $n \geq 1$. Is (a_n) convergent?

2. Let $S \neq \emptyset$ and (a_n) be a decreasing sequence of upper bounds of S. Let $a_n \to a$. Show that a is an upper bound.

3. Test for convergence: (a_n) where $a_1 = 2$, $a_{n+1} = \sqrt{2a_n - 1}$ for $n \in \mathbb{N}$.

4. There are two particles A and B, placed at 0 and 1, on day 1, respectively. On day $n+1$, particle A moves right by one tenth of the distance between the particles on the nth day and particle B moves left by two tenth of the distance between the particles on nth day. Do you think they will meet eventually? If so, where?

   Let us randomize it a little bit. On $n+1$th day a coin is tossed.

   (a) If it is 'head', then A moves right by one tenth of the distance between the particles on the nth day and particle B moves left by two tenth of the distance between the particles on nth day.

   (b) If it is 'tail', then A moves right by two tenth of the distance between the particles on the nth day and particle B moves left by four tenth of the distance between the particles on nth day.

   Now, what is your answer and how do you argue?

5. Is $\sum \frac{n! \ln n}{n^n}$ convergent?

6. Is $\sum \frac{n!^{2n}}{n^{n^2}}$ convergent?

7. Is $\sum \frac{e^{n\pi}}{\pi^{ne}}$ convergent?

8. Is $\sum\limits_{n=2}^{\infty} \frac{1}{(\ln n)^2}$ convergent?

9. Is $\sum\limits_{n=2}^{\infty} (-1)^n \frac{\ln n}{\ln n^2}$ convergent?

10. Is $\sum \frac{\cos(n\pi)}{n\sqrt{n}}$ convergent?

11. Fix $x \in \mathbb{R}$. We already know that $a_n = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$ converges and the limit is defined as $\exp x$. We also know that $a_n = (1 + \frac{1}{n})^n$ converges and the limit is e.

(a) Let (n_k) be a sequence of natural numbers diverging to ∞. Then $\lim_{k\to\infty}(1 + \frac{1}{n_k})^{n_k} = e$.

(b) Let $a_k > 0$ be a sequence of rationals diverging to ∞. Show that $\lim(1 + \frac{1}{a_k})^{a_k} \to e$.

(c) Show that, for any rational number $x = \frac{p}{q} > 0$, $e^x = \lim \left(1 + \frac{x}{n}\right)^n$.

(d) Let $x > 0$ be a rational. Show that $\exp x = e^x$.

(Put $s_n = \left(1 + \frac{x}{n}\right)^n$, $t_n = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$ and $r_k = \binom{n}{0} + \binom{n}{1}\frac{x}{n} + \cdots + \binom{n}{k}\frac{x^k}{n^k}$. Let $\alpha > 0$. Show that $\exists k$ such that $\forall\, n > k$ we have $r_{k-1} \le s_n \le r_{k-1} + \alpha$.)

(e) Conclude that $\lim \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right) = e$.

(f) [irrationality of e] Conclude that e is irrational.

(Assume that $e = \frac{p}{q}$, $\gcd(p, q) = 1$. Then $q!e - q!(1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{q!})$ must be a positive integer. Can it be?)