MA 101 (Mathematics I)

Multivariable Calculus : Hints / Solutions of Practice Problem Set - 2

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1. Examine whether the set $\{(x,x) : x \in \mathbb{R}\}$ is (a) open (b) closed in $\mathbb{R}^2$.

   Solution: We have $(0,0) \in S = \{(x,x) : x \in \mathbb{R}\}$. If possible, let $(0,0) \in S^0$. Then there exists $r > 0$ such that $B_r\big((0,0)\big) \subseteq S$. Since $(\frac{r}{2}, 0) \in B_r\big((0,0)\big)$ but $(\frac{r}{2}, 0) \notin S$, we get a contradiction. Hence $(0,0) \notin S^0$. Therefore S is not an open set in $\mathbb{R}^2$.
   Again, let $\big((x_n, x_n)\big)$ be any sequence in S such that $(x_n, x_n) \to (x,y) \in \mathbb{R}^2$. Then $x_n \to x$ and $x_n \to y$. Hence $x = y$ and so $(x,y) \in S$. Therefore S is a closed set in $\mathbb{R}^2$.

2. Examine whether the set $\{(x,y) \in \mathbb{R}^2 : 0 < x < y\}$ is (a) open (b) closed in $\mathbb{R}^2$.

   Solution: We have already shown in Ex.25 of Practice Problem Set - 1 that
   $S = \{(x,y) \in \mathbb{R}^2 : 0 < x < y\}$ is an open set in $\mathbb{R}^2$.
   Again, since $\left(\frac{1}{2n}, \frac{1}{n}\right) \in S$ for all $n \in \mathbb{N}$ and $\left(\frac{1}{2n}, \frac{1}{n}\right) \to (0,0) \notin S$, S is not a closed set in $\mathbb{R}^2$.

3. Examine whether the set $(0,1) \times \{0\}$ is (a) open (b) closed in $\mathbb{R}^2$.

   Solution: We have $(\frac{1}{2}, 0) \in (0,1) \times \{0\}$. If possible, let $(\frac{1}{2}, 0) \in \big((0,1) \times \{0\}\big)^0$. Then there exists $r > 0$ such that $B_r\big((\frac{1}{2}, 0)\big) \subseteq (0,1) \times \{0\}$. Since $(\frac{1}{2}, \frac{r}{2}) \in B_r\big((\frac{1}{2}, 0)\big)$ but $(\frac{1}{2}, \frac{r}{2}) \notin (0,1) \times \{0\}$, we get a contradiction. Hence $(\frac{1}{2}, 0) \notin \big((0,1) \times \{0\}\big)^0$. Therefore $(0,1) \times \{0\}$ is not an open set in $\mathbb{R}^2$.
   Again, since $\left(\frac{1}{n+1}, 0\right) \in (0,1) \times \{0\}$ for all $n \in \mathbb{N}$ and $\left(\frac{1}{n+1}, 0\right) \to (0,0) \notin (0,1) \times \{0\}$, $(0,1) \times \{0\}$ is not a closed set in $\mathbb{R}^2$.

4. If $f : \mathbb{R}^m \to \mathbb{R}$ is continuous, then show that $\{\mathbf{x} \in \mathbb{R}^m : f(\mathbf{x}) > 0\}$ is an open set in $\mathbb{R}^m$.

   Solution: Let $(\mathbf{x}_n)$ be any sequence in $\mathbb{R}^m \setminus S$, where $S = \{\mathbf{x} \in \mathbb{R}^m : f(\mathbf{x}) > 0\}$ and let $\mathbf{x}_n \to \mathbf{x} \in \mathbb{R}^m$. Since f is continuous at $\mathbf{x}$, $f(\mathbf{x}_n) \to f(\mathbf{x})$. Also, since $\mathbf{x}_n \in \mathbb{R}^m \setminus S$ for all $n \in \mathbb{N}$, $f(\mathbf{x}_n) \leq 0$ for all $n \in \mathbb{N}$ and hence it follows that $f(\mathbf{x}) \leq 0$. Thus $\mathbf{x} \in \mathbb{R}^m \setminus S$ and therefore $\mathbb{R}^m \setminus S$ is a closed set in $\mathbb{R}^m$. Consequently S is an open set in $\mathbb{R}^m$.

5. If $f : \mathbb{R}^m \to \mathbb{R}$ is continuous, then show that $\{\mathbf{x} \in \mathbb{R}^m : f(\mathbf{x}) \geq 0\}$ and $\{\mathbf{x} \in \mathbb{R}^m : f(\mathbf{x}) = 0\}$ are closed sets in $\mathbb{R}^m$.

   Solution: Let $(\mathbf{x}_n)$ be any sequence in $S_1 = \{\mathbf{x} \in \mathbb{R}^m : f(\mathbf{x}) \geq 0\}$ and let $\mathbf{x}_n \to \mathbf{x} \in \mathbb{R}^m$. Since f is continuous at $\mathbf{x}$, $f(\mathbf{x}_n) \to f(\mathbf{x})$. Also, since $\mathbf{x}_n \in S_1$ for all $n \in \mathbb{N}$, $f(\mathbf{x}_n) \geq 0$ for all $n \in \mathbb{N}$ and hence it follows that $f(\mathbf{x}) \geq 0$. Thus $\mathbf{x} \in S_1$ and therefore S_1 is a closed set in $\mathbb{R}^m$.
   Again, let $(\mathbf{x}_n)$ be any sequence in $S_2 = \{\mathbf{x} \in \mathbb{R}^m : f(\mathbf{x}) = 0\}$ and let $\mathbf{x}_n \to \mathbf{x} \in \mathbb{R}^m$. Since f is continuous at $\mathbf{x}$, $f(\mathbf{x}_n) \to f(\mathbf{x})$. Also, since $\mathbf{x}_n \in S_2$ for all $n \in \mathbb{N}$, $f(\mathbf{x}_n) = 0$ for all $n \in \mathbb{N}$ and hence it follows that $f(\mathbf{x}) = 0$. Thus $\mathbf{x} \in S_2$ and therefore S_2 is a closed set in $\mathbb{R}^m$.

6. Using Ex.2 in the Practice Problem Set - 2, show that $\{(x,y,z) \in \mathbb{R}^3 : x^2 + 2z < 3|y|\,\}$ is an open set in $\mathbb{R}^3$ and $\{(x,y,z) \in \mathbb{R}^3 : \sin(xyz) = |xy|\,\}$ is a closed set in $\mathbb{R}^3$.

   Solution: If $f(x,y,z) = 3|y| - x^2 - 2z$ and $g(x,y,z) = \sin(xyz) - |xy|$ for all $(x,y,z) \in \mathbb{R}^3$, then we know that both $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}$ are continuous. Hence by Ex.2(a) of Practice Problem Set - 2, $\{(x,y,z) \in \mathbb{R}^3 : x^2 + 2z < 3|y|\,\} = \{(x,y,z) \in \mathbb{R}^3 : f(x,y,z) > 0\}$ is an open set in $\mathbb{R}^3$ and by Ex.2(b) of Practice Problem Set - 2,
   $\{(x,y,z) \in \mathbb{R}^3 : \sin(xyz) = |xy|\} = \{(x,y,z) \in \mathbb{R}^3 : g(x,y,z) = 0\}$ is a closed set in $\mathbb{R}^3$.

7. Let $f : S \subseteq \mathbb{R}^m \to \mathbb{R}^k$ be continuous and let $g : \mathbb{R}^m \to \mathbb{R}^k$ be such that $g(\mathbf{x}) = f(\mathbf{x})$ for all $\mathbf{x} \in S$.
   (a) Show that g need not be continuous on S.
   (b) If S is an open set in $\mathbb{R}^m$, then show that g is continuous on S.

   Solution: (a) Let $f(x,y) = 1$ for all $(x,y) \in S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ and $g(x,y) = \begin{cases} 1 & \text{if } (x,y) \in S, \\ 2 & \text{if } (x,y) \in \mathbb{R}^2 \setminus S. \end{cases}$
   Then $f : S \to \mathbb{R}$ is continuous (as a constant function) and $f(x,y) = g(x,y)$ for all $(x,y) \in S$. However, g is not continuous at $(1,0) \in S$, since $\left(1 + \frac{1}{n}, 0\right) \to (1,0)$ but
   $g\left(1 + \frac{1}{n}, 0\right) = 2 \to 2 \neq 1 = g(1,0)$.
   (b) Let $\mathbf{x}_0 \in S$ and $\varepsilon > 0$. Since S is an open set in $\mathbb{R}^m$, there exists $r > 0$ such that $B_r(\mathbf{x}_0) \subseteq S$. Since f is continuous at $\mathbf{x}_0$, there exists $s > 0$ such that $\|f(\mathbf{x}) - f(\mathbf{x}_0)\| < \varepsilon$ for all $\mathbf{x} \in S \cap B_s(\mathbf{x}_0)$. If $\delta = \min\{r, s\} > 0$, then $B_\delta(\mathbf{x}_0) \subseteq B_r(\mathbf{x}_0) \subseteq S$ and $B_\delta(\mathbf{x}_0) \subseteq B_s(\mathbf{x}_0)$. Hence for all $\mathbf{x} \in B_\delta(\mathbf{x}_0)$, we have $g(\mathbf{x}) = f(\mathbf{x})$ and $\|g(\mathbf{x}) - g(\mathbf{x}_0)\| < \varepsilon$. Therefore g is continuous at $\mathbf{x}_0$. Since $\mathbf{x}_0 \in S$ is arbitrary, g is continuous on S.

8. Let $f : \mathbb{R}^m \to \mathbb{R}$ be continuous such that $\lim\limits_{\|\mathbf{x}\| \to \infty} f(\mathbf{x}) = 1$. Show that f is bounded on $\mathbb{R}^m$.

   Solution: Since $\lim\limits_{\|\mathbf{x}\| \to \infty} f(\mathbf{x}) = 1$, there exists $r > 0$ such that $|f(\mathbf{x}) - 1| < 1$ for all $\mathbf{x} \in \mathbb{R}^m$ with $\|\mathbf{x}\| > r$. Hence $|f(\mathbf{x})| = |f(\mathbf{x}) - 1 + 1| \leq |f(\mathbf{x}) - 1| + 1 < 2$ for all $\mathbf{x} \in \mathbb{R}^m$ with $\|\mathbf{x}\| > r$. Again, since $S = \{\mathbf{x} \in \mathbb{R}^m : \|\mathbf{x}\| \leq r\}$ is a closed and bounded subset of $\mathbb{R}^m$ and since $f : \mathbb{R}^m \to \mathbb{R}$ is continuous, $f(S)$ is a bounded subset of $\mathbb{R}$. Hence there exists $K > 0$ such that $|f(\mathbf{x})| \leq K$ for all $\mathbf{x} \in S$. If $M = \max\{2, K\}$, then $M > 0$ and $|f(\mathbf{x})| \leq M$ for all $\mathbf{x} \in \mathbb{R}^m$. Consequently f is bounded on $\mathbb{R}^m$.

9. State TRUE or FALSE with justification: There exists a continuous function $f : \mathbb{R} \to \mathbb{R}^2$ such that $f(\cos n) = (n, \frac{1}{n})$ for all $n \in \mathbb{N}$.

   Solution: Since $(\cos n)$ is a bounded sequence in $\mathbb{R}$, by Bolzano-Weierstrass theorem in $\mathbb{R}$, there exists a strictly increasing sequence (n_k) in $\mathbb{N}$ and $\alpha \in \mathbb{R}$ such that $\cos n_k \to \alpha$. If $f : \mathbb{R} \to \mathbb{R}^2$ is continuous, then $\left(n_k, \frac{1}{n_k}\right) = f(\cos n_k) \to f(\alpha)$ in $\mathbb{R}^2$ and consequently the sequence (n_k) converges in $\mathbb{R}$, which is not true, since (n_k) is unbounded. Hence it follows that no continuous function $f : \mathbb{R} \to \mathbb{R}^2$ can exist satisfying $f(\cos n) = (n, \frac{1}{n})$ for all $n \in \mathbb{N}$. Therefore the given statement is FALSE.

10. State TRUE or FALSE with justification: There exists a continuous function from $\{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$ onto $\mathbb{R}^2$.

Solution: We know that $\{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\} = B_1\big[(0,0)\big]$ is a closed and bounded set in $\mathbb{R}^2$ and $\mathbb{R}^2$ is not bounded. Hence there cannot exist any continuous function from $B_1\big[(0,0)\big]$ onto $\mathbb{R}^2$.

11. If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is continuous, then does there exist a sequence $\big((x_n,y_n)\big)$ in $\mathbb{R}^2$ such that $x_n^2+y_n^2 = \frac{1}{2}$ and $f(x_n,y_n) = (n, \frac{1}{n})$ for all $n \in \mathbb{N}$? Justify.

Solution: If possible, let there exist a sequence $\big((x_n,y_n)\big)$ in $\mathbb{R}^2$ such that $x_n^2+y_n^2 = \frac{1}{2}$ and $f(x_n,y_n) = (n, \frac{1}{n})$ for all $n \in \mathbb{N}$. Then $\|(x_n,y_n)\| = \sqrt{x_n^2+y_n^2} = \frac{1}{\sqrt{2}}$ for all $\mathbb{N}$ and so $\big((x_n,y_n)\big)$ is a bounded sequence in $\mathbb{R}^2$. Hence by the Bolzano-Weierstrass theorem in $\mathbb{R}^2$, there exist $(x,y) \in \mathbb{R}^2$ and a convergent subsequence $\big((x_{n_k},y_{n_k})\big)$ of $\big((x_n,y_n)\big)$ such that $\big(x_{n_k},y_{n_k}\big) \to (x,y)$. Since f is continuous at (x,y), $\big(n_k, \frac{1}{n_k}\big) = f\big(x_{n_k},y_{n_k}\big) \to f(x,y) \in \mathbb{R}^2$. Consequently the sequence (n_k) converges in $\mathbb{R}$, which is not true, since (n_k) is unbounded. Hence it follows that there cannot exist any sequence $\big((x_n,y_n)\big)$ in $\mathbb{R}^2$ such that $x_n^2+y_n^2 = \frac{1}{2}$ and $f(x_n,y_n) = (n, \frac{1}{n})$ for all $n \in \mathbb{N}$.

12. Examine whether $\lim\limits_{(x,y)\to(0,0)} \frac{x^3y}{x^4+y^2}$ exist (in $\mathbb{R}$) and find its value if it exist (in $\mathbb{R}$).

Solution: Let $\big((x_n,y_n)\big)$ be any sequence in $\mathbb{R}^2 \setminus \{(0,0)\}$ such that $(x_n,y_n) \to (0,0)$. Then $x_n \to 0$ and $y_n \to 0$. Since $\left|\frac{x_n^3y_n}{x_n^4+y_n^2}\right| = \left|\frac{x_n^2y_n}{x_n^4+y_n^2}\right||x_n| \leq \frac{1}{2}|x_n| \to 0$, it follows that $\frac{x_n^3y_n}{x_n^4+y_n^2} \to 0$. Therefore $\lim\limits_{(x,y)\to(0,0)} \frac{x^3y}{x^4+y^2} = 0$.

13. Examine whether $\lim\limits_{(x,y)\to(0,0)} \frac{|x|}{y^2}e^{-|x|/y^2}$ exists (in $\mathbb{R}$) and find its value if it exists (in $\mathbb{R}$).

Solution: Let $f(x,y) = \frac{|x|}{y^2}e^{-|x|/y^2}$ for all $(x,y) \in \mathbb{R}^2$ with $y \neq 0$. We have $(0, \frac{1}{n}) \to (0,0)$ and $(\frac{1}{n^2}, \frac{1}{n}) \to (0,0)$. Also, $f(0, \frac{1}{n}) \to 0$ and $f(\frac{1}{n^2}, \frac{1}{n}) \to \frac{1}{e}$. Since $\lim\limits_{n\to\infty} f\big(0, \frac{1}{n}\big) \neq \lim\limits_{n\to\infty} f\big(\frac{1}{n^2}, \frac{1}{n}\big)$, $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist (in $\mathbb{R}$).

14. Examine whether $\lim\limits_{(x,y)\to(0,0)} \frac{x^3+y^2}{x^2+y}$ exists (in $\mathbb{R}$) and find its value if it exists 9in $\mathbb{R}$).

Solution: Let $f(x,y) = \frac{x^3+y^2}{x^2+y}$ for all $(x,y) \in \mathbb{R}^2$ with $x^2+y \neq 0$. We have $\big(\frac{1}{n}, 0\big) \to (0,0)$ and $\big(\frac{1}{n}, \frac{1}{n^3} - \frac{1}{n^2}\big) \to (0,0)$. Also, $f\big(\frac{1}{n}, 0\big) = \frac{1}{n} \to 0$ and $f\big(\frac{1}{n}, \frac{1}{n^3} - \frac{1}{n^2}\big) = 1 + \frac{1}{n}(\frac{1}{n} - 1)^2 \to 1$. Since $f\big(\frac{1}{n}, 0\big) \neq f\big(\frac{1}{n}, \frac{1}{n^3} - \frac{1}{n^2}\big)$, $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist (in $\mathbb{R}$).

15. Examine whether $\lim\limits_{(x,y)\to(0,0)} \frac{\sqrt{x^2y^2+1}-1}{x^2+y^2}$ exist (in $\mathbb{R}$) and find its values if it exists (in $\mathbb{R}$).

Solution: Let $\big((x_n,y_n)\big)$ be any sequence in $\mathbb{R}^2 \setminus \{(0,0)\}$ such that $(x_n,y_n) \to (0,0)$. Then $x_n \to 0$ and $y_n \to 0$. Since $0 \leq \frac{\sqrt{x_n^2y_n^2+1}-1}{x_n^2+y_n^2} = \frac{x_n^2y_n^2}{(x_n^2+y_n^2)\big(\sqrt{x_n^2y_n^2+1}+1\big)} \leq \frac{x_n^2y_n^2}{x_n^2+y_n^2} \leq y_n^2 \to 0$, it follows that $\frac{\sqrt{x_n^2y_n^2+1}-1}{x_n^2+y_n^2} \to 0$. Therefore $\lim\limits_{(x,y)\to(0,0)} \frac{\sqrt{x^2y^2+1}-1}{x^2+y^2} = 0$.

16. Examine whether $\lim\limits_{(x,y)\to(0,0)} \frac{x^3y^2+y^6}{x^6+y^4}$ exists (in $\mathbb{R}$) and find its value if it exists (in $\mathbb{R}$).

Solution: Let $f(x,y) = \frac{x^3y^2+y^6}{x^6+y^4}$ for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$. We have $\left(\frac{1}{n},0\right) \to (0,0)$ and $\left(\frac{1}{\sqrt[3]{n}},\frac{1}{\sqrt{n}}\right) \to (0,0)$. Also, $f\left(\frac{1}{n},0\right) \to 0$ and $f\left(\frac{1}{\sqrt[3]{n}},\frac{1}{\sqrt{n}}\right) \to \frac{1}{2}$.
Since $\lim\limits_{(x,y)\to(0,0)} f\left(\frac{1}{n},0\right) \neq \lim\limits_{(x,y)\to(0,0)} f\left(\frac{1}{\sqrt[3]{n}},\frac{1}{\sqrt{n}}\right)$, $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist (in $\mathbb{R}$).

17. Examine whether $\lim\limits_{(x,y,z)\to(0,0,0)} \frac{(x+y+z)^2}{x^2+y^2+z^2}$ exists (in $\mathbb{R}$) and find its value if it exists (in $\mathbb{R}$).

Solution: Let $f(x,y,z) = \frac{(x+y+z)^2}{x^2+y^2+z^2}$ for all $(x,y,z) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$. We have $\left(\frac{1}{n},0,0\right) \to (0,0,0)$ and $\left(\frac{1}{n},\frac{1}{n},0\right) \to (0,0,0)$. Also, $f\left(\frac{1}{n},0,0\right) = 1 \to 1$ and $f\left(\frac{1}{n},\frac{1}{n},0\right) = 2 \to 2$.
Since $\lim\limits_{n\to\infty} f\left(\frac{1}{n},0,0\right) \neq \lim\limits_{n\to\infty} f\left(\frac{1}{n},\frac{1}{n},0\right)$, $\lim\limits_{(x,y,z)\to(0,0,0)} f(x,y,z)$ does not exist (in $\mathbb{R}$).

18. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} x+y & \text{if } x \neq y, \\ 1 & \text{if } x = y. \end{cases}$
Examine whether $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists (in $\mathbb{R}$).

Solution: We have $\left(\frac{1}{n},0\right) \to (0,0)$ and $\left(\frac{1}{n},\frac{1}{n}\right) \to (0,0)$. Also, $f\left(\frac{1}{n},0\right) = \frac{1}{n} \to 0$ and $f\left(\frac{1}{n},\frac{1}{n}\right) = 1 \to 1$. Since $\lim\limits_{n\to\infty} f\left(\frac{1}{n},0\right) \neq \lim\limits_{n\to\infty} f\left(\frac{1}{n},\frac{1}{n}\right)$, $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist (in $\mathbb{R}$).

19. Show that $\lim\limits_{x\to0}\left(\lim\limits_{y\to0} \frac{x^2y^2}{x^2y^2+(x-y)^2}\right) = 0 = \lim\limits_{y\to0}\left(\lim\limits_{x\to0} \frac{x^2y^2}{x^2y^2+(x-y)^2}\right)$ but that $\lim\limits_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2y^2+(x-y)^2}$ does not exist (in $\mathbb{R}$).

Solution: Let $f(x,y) = \frac{x^2y^2}{x^2y^2+(x-y)^2}$ for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$.
Then $\lim\limits_{y\to0} f(x,y) = \frac{0}{x^2} = 0$ for each $x \in \mathbb{R} \setminus \{0\}$ and $\lim\limits_{x\to0} f(x,y) = \frac{0}{y^2} = 0$ for each $y \in \mathbb{R} \setminus \{0\}$.
Consequently $\lim\limits_{x\to0}\left(\lim\limits_{y\to0} f(x,y)\right) = 0 = \lim\limits_{y\to0}\left(\lim\limits_{x\to0} f(x,y)\right)$.
Again, we have $\left(\frac{1}{n},0\right) \to (0,0)$ and $\left(\frac{1}{n},\frac{1}{n}\right) \to (0,0)$. Also, $f\left(\frac{1}{n},0\right) = 0 \to 0$ and $f\left(\frac{1}{n},\frac{1}{n}\right) = 1 \to 1$. Since $\lim\limits_{n\to\infty} f\left(\frac{1}{n},0\right) \neq \lim\limits_{n\to\infty} f\left(\frac{1}{n},\frac{1}{n}\right)$, $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist (in $\mathbb{R}$).

20. Show that $\lim\limits_{(x,y)\to(0,0)} \frac{1}{3x^2+y^4} = \infty$.

Solution: Let $\big((x_n,y_n)\big)$ be any sequence in $\mathbb{R}^2 \setminus \{(0,0)\}$ such that $(x_n,y_n) \to (0,0)$. Then $x_n \to 0$, $y_n \to 0$ and hence $3x_n^2 + y_n^4 \to 0$. If $r > 0$, then there exists $n_0 \in \mathbb{N}$ such that $3x_n^2 + y_n^4 < \frac{1}{r}$ for all $n \geq n_0$ and so $\frac{1}{3x_n^2+y_n^4} > r$ for all $n \geq n_0$. Therefore $\frac{1}{3x_n^2+y_n^4} \to \infty$ and consequently $\lim\limits_{(x,y)\to(0,0)} \frac{1}{3x^2+y^4} = \infty$.

21. Let I be an open interval in $\mathbb{R}$ and let $F : I \to \mathbb{R}^m$ be a differentiable function such that $F(t) \cdot F'(t) = 0$ for all $t \in I$. Show that $\| F(t) \|$ is constant for all $t \in I$.

Solution: Since F is differentiable, the function $t \mapsto \|F(t)\|^2 = F(t) \cdot F(t)$ from I to $\mathbb{R}$ is also differentiable and $\frac{d}{dt}\left(\|F(t)\|^2\right) = F'(t) \cdot F(t) + F(t) \cdot F'(t) = 2\,F(t) \cdot F'(t) = 0$ for all $t \in I$. Hence there exists $c \in \mathbb{R}$ such that $\|F(t)\|^2 = c$ for all $t \in I$. Clearly $c \geq 0$ and so $\|F(t)\| = \sqrt{c}$ for all $t \in I$.

MA 101 (Mathematics I)

Multivariable Calculus : Hints / Solutions of Practice Problem Set - 3

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1. If $f(x,y) = e^x(x\cos y - y\sin y)$ for all $(x,y) \in \mathbb{R}^2$, then show that $f_{xx}(x,y) + f_{yy}(x,y) = 0$ for all $(x,y) \in \mathbb{R}^2$.

2. If $f(x,y) = x^2 \tan^{-1}\left(\frac{y}{x}\right)$ for all $(x,y) \in \mathbb{R}^2 \setminus \{(x,y) \in \mathbb{R} : x \neq 0\}$, then find $\frac{\partial^2 f}{\partial x \partial y}(1,1)$.

3. If $f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$ for all $(x,y,z) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$, then show that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$ at each point of $\mathbb{R}^3 \setminus \{(0,0,0)\}$.

4. Find all $\mathbf{u} \in \mathbb{R}^2$ with $\|\mathbf{u}\| = 1$ for which the directional derivative $D_{\mathbf{u}}f(0,0)$ exists, if for all $(x,y) \in \mathbb{R}^2$,
   (a) $f(x,y) = \sqrt{|x^2 - y^2|}$.
   (b) $f(x,y) = \big|\,|x| - |y|\,\big| - |x| - |y|$.
   (c) $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$
   (d) $f(x,y) = \begin{cases} \frac{x}{y} & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$

5. State TRUE or FALSE with justification for each of the following statements.
   (a) If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous such that $f_x(0,0)$ exists (in $\mathbb{R}$), then $f_y(0,0)$ must exist (in $\mathbb{R}$).
   (b) If $f : \mathbb{R}^2 \to \mathbb{R}$ is such that for each $\mathbf{u} \in \mathbb{R}^2$ with $\|\mathbf{u}\| = 1$, the directional derivative of f at $(0,0)$ along $\mathbf{u}$ is 0, then f must be continuous at $(0,0)$.

6. Let the height $H(x,y)$ of a hill from the ground (considered as the xy-plane) at the point (x,y) be given by $H(x,y) = 1000 - 0.005x^2 - 0.01y^2$. We assume that the positive x-axis points east and the positive y-axis points north. Consider a person situated at the point $(60,40,966)$ on the hill.
   (a) If the person starts walking due south, then will (s)he start to ascend or descend the hill?
   (b) If the person starts walking north-west, then will (s)he start to ascend or descend the hill?
   (c) If the person starts climbing further, in which direction will (s)he find it most difficult to climb?

7. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} \frac{x^2y(x-y)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$
   Examine whether $f_{xy}(0,0) = f_{yx}(0,0)$.

8. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$
   Determine all the points of $\mathbb{R}^2$ where $f_{xy} : \mathbb{R}^2 \to \mathbb{R}$ and $f_{yx} : \mathbb{R}^2 \to \mathbb{R}$ are continuous.

9. Let $f(x,y) = x + y^2 + xy$ for all $(x,y) \in \mathbb{R}^2$. Using directly the definition of differentiability, show that $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable and also find $f'(x_0, y_0)$, where $(x_0, y_0) \in \mathbb{R}^2$.

10. Let S be a nonempty open subset of $\mathbb{R}^m$ and let $g : S \to \mathbb{R}^m$ be continuous at $\mathbf{x}_0 \in S$. If $f : S \to \mathbb{R}$ is such that $f(\mathbf{x}) - f(\mathbf{x}_0) = g(\mathbf{x}) \cdot (\mathbf{x} - \mathbf{x}_0)$ for all $\mathbf{x} \in S$, then show that f is differentiable at $\mathbf{x}_0$.

11. The directional derivatives of a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ at $(0,0)$ in the directions of $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$ and $\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$ are 1 and 2 respectively. Find $f_x(0,0)$ and $f_y(0,0)$.

12. Examine the differentiability of f at $\mathbf{0}$, where
(a) $f : \mathbb{R}^m \to \mathbb{R}$ satisfies $|f(\mathbf{x})| \le \|\mathbf{x}\|^2$ for all $\mathbf{x} \in \mathbb{R}^n$.
(b) $f : \mathbb{R}^m \to \mathbb{R}$ is defined by $f(\mathbf{x}) = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$.
(c) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x,y) = \sqrt{|xy|}$ for all $(x,y) \in \mathbb{R}^2$.
(d) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x,y) = \big|\,|x| - |y|\,\big| - |x| - |y|$ for all $(x,y) \in \mathbb{R}^2$.
(e) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$
(f) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x,y) = \begin{cases} \frac{y}{|y|}\sqrt{x^2+y^2} & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$
(g) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x,y) = \begin{cases} \sqrt{x^2+y^2} & \text{if } y > 0, \\ x & \text{if } y = 0, \\ -\sqrt{x^2+y^2} & \text{if } y < 0. \end{cases}$
(h) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x,y) = \begin{cases} 1 & \text{if } y < x^2 < 2y, \\ 0 & \text{otherwise.} \end{cases}$
(i) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x,y) = \begin{cases} x & \text{if } |x| < |y|, \\ -x & \text{if } |x| \ge |y|. \end{cases}$
(j) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x,y) = \begin{cases} \frac{\sin(x^2y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$
(k) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x,y) = \begin{cases} \sin^2 x + x^2 \sin\frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$

13. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$
Show that f is differentiable at $(0,0)$ although neither $f_x : \mathbb{R}^2 \to \mathbb{R}$ nor $f_y : \mathbb{R}^2 \to \mathbb{R}$ is continuous at $(0,0)$.

14. Let $f(x,y) = \begin{cases} (x^2+y^2)\cos\left(\frac{1}{x^2+y^2}\right) & \text{if } (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}, \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$
Examine whether $f : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable.

15. Let $\alpha \in \mathbb{R}$ and $\alpha > 0$. If $f(x,y) = |xy|^\alpha$ for all $(x,y) \in \mathbb{R}^2$, then determine all values of α for which $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(0,0)$.

16. Determine all the points of $\mathbb{R}^2$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable, where for all $(x, y) \in \mathbb{R}^2$,

(a) $f(x, y) = |xy|$ (b) $f(x, y) = (xy)^{\frac{2}{3}}$ (c) $f(x, y) = |x| \sin(x^2 + y^2)$

(d) $f(x, y) = \begin{cases} x^2 + y^2 & \text{if both } x, y \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$

17. State TRUE or FALSE with justification for each of the following statements.

(a) If $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ and if $f(x, y) = |xy|$ for all $(x, y) \in S$, then $f : S \to \mathbb{R}$ is differentiable.

(b) There exists a function $f : \mathbb{R}^2 \to \mathbb{R}$ which is differentiable only at $(1, 0)$.

18. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable at $(0, 0)$ and let $\lim\limits_{x \to 0} \frac{f(x,x) - f(x,-x)}{x} = 1$. Find $f_y(0, 0)$.

19. Let $f : \mathbb{R}^m \to \mathbb{R}$ be differentiable at $\mathbf{0}$ and let $f(\alpha\mathbf{x}) = \alpha f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^m$ and for all $\alpha \in \mathbb{R}$. Show that $f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$.

20. Let $f : \mathbb{R}^m \to \mathbb{R}$ be differentiable at $\mathbf{0}$ and $f(\mathbf{0}) = 0$. Show that there exist $\alpha > 0$ and $r > 0$ such that $|f(\mathbf{x})| \leq \alpha\|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^m$ with $\|\mathbf{x}\| < r$.

21. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that f_x exists (in $\mathbb{R}$) at all points of $B_\delta\big((x_0, y_0)\big)$ for some $(x_0, y_0) \in \mathbb{R}^2$ and $\delta > 0$, f_x is continuous at (x_0, y_0) and $f_y(x_0, y_0)$ exists (in $\mathbb{R}$). Show that f is differentiable at (x_0, y_0).

22. Let $f,\ g : S \subseteq \mathbb{R}^m \to \mathbb{R}$ be differentiable at $\mathbf{x}_0 \in S^0$. Show that

(a) $f + g : S \to \mathbb{R}$ is differentiable at $\mathbf{x}_0$ and $\nabla(f + g)(\mathbf{x}_0) = \nabla f(\mathbf{x}_0) + \nabla g(\mathbf{x}_0)$.

(b) $fg : S \to \mathbb{R}$ is differentiable at $\mathbf{x}_0$ and $\nabla(fg)(\mathbf{x}_0) = g(\mathbf{x}_0)\nabla f(\mathbf{x}_0) + f(\mathbf{x}_0)\nabla g(\mathbf{x}_0)$.

(c) if $g(\mathbf{x}_0) \neq 0$, then $\frac{f}{g} : S \to \mathbb{R}$ is differentiable at $\mathbf{x}_0$ and $\nabla\big(\frac{f}{g}\big)(\mathbf{x}_0) = \frac{g(\mathbf{x}_0)\nabla f(\mathbf{x}_0) - f(\mathbf{x}_0)\nabla g(\mathbf{x}_0)}{g(\mathbf{x}_0)^2}$.

23. Using the linearization of a suitable function at a suitable point, find an approximate value of $\big((3.8)^2 + 2(2.1)^3\big)^{\frac{1}{5}}$.

24. Show that the maximum error in calculating the volume of a right circular cylinder is approximately $\pm 8\%$ if its radius can be measured with a maximum error of $\pm 3\%$ and its height can be measured with a maximum error of $\pm 2\%$.