MA 101 (Mathematics - I)
Integration : Exercise set 2 : Hints and Solutions

Fundamental Theorems of Calculus

1. Let $f : [a, b] \to \mathbb{R}$ be continuous. Assume that there exist distinct constants α and β such that for all $c \in [a, b]$, we have $\alpha \displaystyle\int_a^c f(x)dx + \beta \int_c^b f(x)dx = 0$. Show that $f = 0$.

> **Solution:** Define $F(x) = \int_a^x f$. Then $\alpha F(c) + \beta(F(b) - F(c)) = 0$ for every $c \in [a, b]$, that is, $F(x) = \frac{\beta}{\beta - \alpha} F(b)$ for all $x \in [a, b]$. Hence, $f(x) = F'(x) = 0$.

2. Let $g : \mathbb{R} \to \mathbb{R}$ be differentiable. Let $F(x) = \int_0^{g(x)} t^2 dt$. Prove that $F'(x) = g^2(x)g'(x)$ for all $x \in \mathbb{R}$. If $G(x) = \int_{h(x)}^{g(x)} t^2 dt$, then what is $G'(x)$?

> **Solution:** By Leibniz Rule **[1.41]** $G'(x) = (g(x))^2 g'(x) - (h(x))^2 h'(x)$.

3. Let $f : [a, b] \to \mathbb{R}$ be continuous and $g : [c, d] \to [a.b]$ be differentiable. Define $\phi(x) = \int_a^{g(x)} f(t)dt$. Prove that ϕ is differentiable and compute its derivative.

> **Solution:** As in **[1.41]**, show that $\phi'(x) = f(g(x))g'(x)$.

4. If f'' is continuous on $[a, b]$, show that $\int_a^b xf''(x)dx = [bf'(b) - f(b)] - [af'(a) - f(a)]$.

> **Solution:** Use intigration by parts, $\int_a^b xf''(x)dx = (xf'(x))\Big|_a^b - \int_a^b f'(x)dx = \cdots$.

5. Let $f > 0$ be continuous on $[1, \infty)$ and suppose that for $x > 0$, $\displaystyle\int_1^x f(t)dt \le (f(x))^2$. Prove that $f(x) \ge \frac{1}{2}(x - 1)$.

> **Solution:** Let $g(x) := \int_1^x f(t)dt$. Then $0 \le g(x) \le (f(x))^2$ and, so $g'(x) = f(x) \ge (g(x))^{1/2}$, i.e., $(g(x))^{-1/2} g'(x) \ge 1$. Thus,
>
> $$x - 1 = \int_1^x 1 \le \int_1^x (g(t))^{-1/2} g'(t)dt = 2(g(t))^{1/2}\Big|_1^x = 2(g(x))^{1/2} \le 2f(x).$$

6. If $f : [a, b] \to \mathbb{R}$ is continuously differentiable, show that $\displaystyle\lim_{n\to\infty} \int_a^b f(x) \cos nx \, dx = 0$.

Solution: Since f' is continuous on $[a, b]$, f' is bounded. Let $M = \max\{|f'(x)| : x \in [a, b]\}$. Now, applying integration by parts, we find that

$$\int_a^b f(x)\cos nx\,dx = \frac{1}{n}\Big(f(x)\sin nx\Big)\Big|_a^b - \frac{1}{n}\int_a^b f'(x)\sin nx\,dx$$

$$\le \frac{1}{n}(|f(a)| + |f(b)| + M(b-a)) \to 0, \text{ as } n \to \infty.$$

Hence the result follows.

7. If $f : [0; 1] \to \mathbb{R}$ is continuous, then show that $\int_0^x \big(\int_0^u f(t)dt\big)du = \int_0^x (x-u)f(u)du$.

Solution: Let $F(u) = \int_0^u f(t)dt$ for $u \in [0, 1]$. Then LHS $= \int_0^x F(u) \cdot 1du$. Integration by parts gives the RHS. (Note that $F'(u) = f(u)$ for $u \in [0, 1]$.)
(Alternative: We differentiate both sides with respect to x.)

8. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and let $g(x) = \int_0^x (x-t)f(t)dt$ for $x \in \mathbb{R}$. Prove that $g''(x) = f(x)$ for all $x \in \mathbb{R}$.

Solution: We have $g(x) = x\int_0^x f(t)dt - \int_0^x tf(t)dt$ for $x \in \mathbb{R}$. Since f is continuous, by the Â¯first fundamental theorem of calculus, $g : \mathbb{R} \to \mathbb{R}$ is differentiable and $g'(x) = \int_0^x f(t)dt + xf(x) - xf(x) = \int_0^x f(t)dt$ for $x \in \mathbb{R}$. Again, since f is continuous, by the first fundamental theorem of calculus, $g' : \mathbb{R} \to \mathbb{R}$ is differentiable and $g''(x) = f(x)$ for $x \in \mathbb{R}$.

Improper Integrals

9. Compute the following improper integrals or prove their divergence.

(a) $\displaystyle\int_1^\infty \frac{dx}{x^4}$ (b) $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}}$ (c) $\displaystyle\int_0^\infty e^{-ax}dx,\ a > 0$ (d) $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

(e) $\displaystyle\int_0^1 \frac{dx}{x\ln x}$ (f) $\displaystyle\int_0^1 \frac{dx}{x\ln|\ln x|}$ (g) $\displaystyle\int_0^1 \frac{dx}{x|\ln x|^{3/2}}$ (h) $\displaystyle\int_0^1 \frac{\sqrt{x}\,dx}{(1+x)^2}$

Solution: (a) $\displaystyle\int_1^\infty \frac{dx}{x^4} = \lim_{t\to\infty}\int_1^t \frac{dx}{x^4} = \frac{1}{3}\lim_{t\to\infty}\left(1 - \frac{1}{t^3}\right) = 1/3.$

(b) Diverges.

(c) Converges. $\displaystyle\int_0^\infty e^{-ax}dx = \lim_{t\to\infty}\int_0^t e^{-ax}dx = \frac{-1}{a}\lim_{t\to\infty}(e^{-at} - e^0) = 1/a$

(d) Convrges. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to 1-}\int_0^t \frac{dx}{\sqrt{1-x^2}}$. Change variable: $y = \sin t$, intgrate and take limit. Ans. $\pi/2$

(e) Diverges. Note that $x\ln x \to 0$ as $x \to 1$. Also, $x \ln x \to 0$ as $x \to 0$ (Use L'Hópital rule). So, we consider the two improper integrals $\displaystyle\int_0^{1/2} \frac{dx}{x\ln x}$ and $\displaystyle\int_{1/2}^1 \frac{dx}{x\ln x}$. The given integral converges if and only if these two integrals converge. Now,

$$\int_0^{1/2} \frac{dx}{x\ln x} = \lim_{t\to 0+}\int_t^{1/2} \frac{dx}{x\ln x} = \lim_{t\to 0+}\int_{\ln t}^{\ln(1/2)} \frac{dy}{y} = \lim_{t\to 0+}(\ln|\ln(1/2)| - \ln|\ln t|) = -\infty.$$

Similarly, $\displaystyle\int_{1/2}^1 \frac{dx}{x\ln x} = \lim_{t\to 1-}\int_{1/2}^t \frac{dx}{x\ln x} = \lim_{t\to 0+}(\ln|\ln t| - \ln|\ln t|) = -\infty.$

(f) Diverges: Camprare $\displaystyle\frac{1}{x|\ln|\ln x||} \geq \frac{1}{x|\ln x|}$ and use (e)

(g) Diverges: as in (f)

(h) This is in fact a Riemann integral: With the variable change $x = \tan^2 t$, the integral becomes $\displaystyle\int_0^{\pi/4} 2\sin^2 t\,dt = \frac{1}{2} + \frac{\pi}{4}$.

10. Test the following improper integrals for convergence.

(a) $\displaystyle\int_0^1 \frac{\sqrt{x}}{\sqrt{1-x^4}}\,dx$ (b) $\displaystyle\int_0^{\pi/2} \frac{\ln\sin x}{\sqrt{x}}\,dx$

Solution: (a) Converges. We have $f(x) = \displaystyle\frac{\sqrt{x}}{\sqrt{(1+x^2)(1+x)}} \cdot \frac{1}{\sqrt{1-x}}$.

Let $g(x) = \displaystyle\frac{1}{\sqrt{1-x}}$. Then $\displaystyle\lim_{x\to 1-}\frac{f(x)}{g(x)} = 1/2$. Since $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x}}\,dx$ is convergent, the result follows.

(b) Converges: Let $f(x) = \displaystyle\frac{\ln\sin x}{\sqrt{x}}$, $g(x) = \displaystyle\frac{1}{x^{3/4}}$. Show that $\frac{f(x)}{g(x)} \to 0$ as $x \to 0+$. Since $\int_0^1 g(x)dx$ is covergent, the result follows.

11. Determine the values of k for which the following integrals converge.

(a) $\displaystyle\int_a^b \frac{dx}{(b-x)^k}$, $(b < a)$ (b) $\displaystyle\int_0^\pi \frac{dx}{\sin^k x}$

Solution: (a) $k < 1$ (b) $k < 1$.

12. For what values of k and t the integral $\displaystyle\int_0^\infty \frac{x^k}{1+x^t}\,dx$ is convergent?

Solution: If both $k > -1$ and $t > k+1$ hold. Note: $\displaystyle\int_0^1 \frac{x^k}{1+x^t}\,dx$ converges if and only if $k > -1$ (Compare with $1/x^{-k}$), and $\displaystyle\int_1^\infty \frac{x^k}{1+x^t}\,dx$ if and only if $t-k > 1$ (compare with $1/x^{t-k}$).

13. Examine whether the following integrals are convergent

(a) $\displaystyle\int_0^\infty \sin(x^2)dx$, (b) $\displaystyle\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$.

Solution: (a) Converges. Use Dirichlet test: $\sin x^2 = (2x\sin x^2)\frac{1}{2x}$.
(b) Converges. Take $f(x) = \frac{\ln x}{\sqrt{x}}$ and $g(x) = 1/(x^{3/4})$. Then $\frac{f(x)}{g(x)} \to 0$ as $x \to 0+$. Since $\int_0^1 g$ is convergent, $\int_0^1 f$ converges.
Note: Comparison test and limit comparison test are valid also in Improper integrals on bounded intervals.

14. The integral $\int_a^\infty f(x)dx$ is said converge **absolutely** if $\int_a^\infty |f(x)|dx$ is convergent. If the integral $\int_a^\infty f(x)dx$ converges, but not absolutely, then it is said to converge **conditionally**. Show that $\displaystyle\int_1^\infty \frac{\sin x}{x^p}dx$ converges absolutely if $p > 1$ and conditionally if $0 < p \le 1$.

Solution: For $p > 1$, use comparison $|\frac{\sin x}{x^p}|$ with $\frac{1}{x^p}$. For $0 < p \le 1$, use Dirichlet test with $f(x) = \frac{1}{x^p}$ and $g(x) = \sin x$ to show $\displaystyle\int_1^\infty \frac{\sin x}{x^p}dx$ converges.

Finally, For $0 < p \le 1$, $\displaystyle\int_1^\infty \frac{|\sin x|}{x^p}dx$ diverges, because if $\displaystyle\int_1^\infty \frac{|\sin x|}{x^p}dx$ converges, then $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^p}dx$ converges (as $\sin^2 x \le |\sin x|$). By Dirichlet test $\displaystyle\int_1^\infty \frac{\cos 2x}{x^p}dx$ converges.
Thus, $\displaystyle\int_1^\infty \frac{1}{x^p}dx = \int_1^\infty \frac{2\sin^2 x + \cos 2x}{x^p}dx$ converges, a contradiction.

Solution: More from Notes:
[2.21](c) Examine for convergence: $\displaystyle\int_1^\infty e^{-t}t^p\,dt$.
If $p \le 0$, then $0 < e^{-t}t^p \le e^{-t}$ and $\int_0^\infty e^{-t}dt = e$. By comparison, the integral converges.
If $p > 0$, let $m = [p]$. Then, $e^t > \dfrac{t^{m+3}}{(m+3)!}$, that is, $e^{-t} < \dfrac{(m+3)!}{t^{m+3}}$. Therefore, $0 < e^{-t}t^p < e^{-t}t^{m+1} < \dfrac{(m+3)!}{t^2}$. Use camparison test again.

Solution: **[2.26]** Determine all real numbers p for which the integral $\displaystyle\int_0^\infty \frac{e^{-x}-1}{x^p}\,dx$ is convergent.

Let $f(x) = -\dfrac{e^{-x}-1}{x^p}\,dx$. Let $g(x) = \frac{1}{x^{p-1}}$ for $x \in (0,1]$. Then $\frac{f(x)}{g(x)} \to 1$. Therefore, $\displaystyle\int_0^1 \frac{e^{-x}-1}{x^p}\,dx$ is convergent if and only if $p-1<1$, i.e., $p<1$. Similarly, comapring with $1/x^p$, show that $\displaystyle\int_1^\infty \frac{e^{-x}-1}{x^p}\,dx$ is convergent if and only if $p>1$. Hence, convergent if and only if $1<p<2$.

Applications of integrals

15. Find the area of the region enclosed by the curve $y=\sqrt{|x+1|}$ and the line $5y=x+7$.
16. Find the area enclosed by the curve $r=a\cos 3\theta,\ -\pi/6 \le \theta \le \pi/6$.

Solution: The curve will form a loop.

17. Sketch the curve $x=a\sin 2t,\ y=a\sin t$ and find the area of one of its loops.

Solution: $4a^2/3$

18. Find the length of the curve $y=\int_0^x \sqrt{\cos 2t}\,dt,\ 0\le x\le \frac{\pi}{4}$.

Solution: 1

19. A curve is given by the equation

$$x=a(\cos\theta+\theta\sin\theta),\ y=a(\sin\theta-\theta\cos\theta).$$

Find the length of the arc from $\theta=0$ to $\theta=\alpha$.

Solution: $\frac{1}{2}a\alpha^2$.

20. Consider the funnel formed by revolving the curve $y=\frac{1}{x}$ about the x-axis, between $x=1$ and $x=a$, where $a>1$. If V_a and S_a denote respectively the volume and the surface area of the funnel, then show that $\lim\limits_{a\to\infty} V_a=\pi$ and $\lim\limits_{a\to\infty} S_a=\infty$.

21. Find the volume and area of the curved surface of a paraboloid of revolution formed by revolving the parabola $y^2 = 4ax$ about the x-axis, and bounded by the section $x = x_1$.

22. Show that the area of the surface obtained by revolving the cardioid $r = 1 + \cos\theta$ about the x-axis is $\frac{32}{5}\pi a^2$.
