Tutorial 3: Series2,3, Limitcontinuity1,2

A) If possible find examples with nonzero terms (wherever necessary).

1. Let l be a fixed real number. Give two series with distinct terms converging to l.

   Sol. For $l \neq 0$, take $l = \sum\limits_{n\geq 1} \frac{l}{2^n}$, $l = \sum\limits_{n\geq 1} \frac{l}{e-1}\frac{1}{n!}$.

   For $l = 0$, first take the above two series for $l = 1$ and then add one more term -1 at the beginning. As the above series have positive terms only, the new term will be distinct from them.

2. If $\sum a_n$ and $\sum b_n$ are divergent, is $\sum(a_n + b_n)$ necessarily divergent?

   Sol. No. Take $a_n = 1$, $b_n = -1$.

3. What happens to $\sum(a_n + b_n)$ when $\sum a_n = a$ and $\sum b_n$ diverges?

   Sol. Divergent. Put $c_n = a_n + b_n$. If $\sum c_n = c$, then $\sum b_n = \sum(c_n - a_n) = c - a$.

4. If $\sum a_n$ and $\sum b_n$ are convergent is $\sum(a_n b_n)$ necessarily convergent?

   Sol. No. Take $a_n = b_n = (-1)^n \frac{1}{\sqrt{n}}$.

5. Let $a_n, b_n \geq 0$. Suppose that $\sum a_n = l$ and $\sum b_n = t$. Is $\sum a_n b_n$ convergent?

   Sol. Yes. Let $S_k := \sum\limits_{n=1}^{k}(a_n b_n)$ (means 'let S_k denote this sum'). Then

   $$S_k \leq \left(\sum_{n=1}^{k} a_n\right)\left(\sum_{n=1}^{k} b_n\right) \leq lt.$$

   So the sequence (S_k) is bounded above by lt. As the sequence is increasing, by using MCT, we see that (S_k) is convergent. That is, $\sum a_n b_n$ is convergent.

   Alternately, as $\sum a_n$ is given to be convergent, we know that $a_n \to 0$. Hence $\exists k$ such that for each $n \geq k$, we have $a_n < 1$. As $a_n \geq 0$, we see that $a_n^2 < a_n$ for each $n \geq k$. Then the series $\sum\limits_{n\geq k} a_n^2$ is convergent by suing comparison test with $\sum\limits_{n\geq k} a_n$. Hence, the series $\sum_{n\geq 1} a_n^2$ also converges, by definition. Similarly, $\sum\limits_{n\geq 1} b_n^2$ converges. Then the series $\sum\limits_{n\geq 1} 2a_n b_n$ converges by using comparison test with $\sum\limits_{n\geq 1}(a_n^2 + b_n^2)$. Hence, the series $\sum\limits_{n\geq 1} a_n b_n$ also converges.

6. Let $a_n \geq 0$. If $\sum a_n^2$ converges, then is $\sum a_n$ convergent?

   Sol. No, for example consider $\sum \frac{1}{n}$

7. Can $\sum(a_n b_n)$ be convergent, given $\sum a_n, \sum b_n$ are divergent?

   Sol. Yes. Take $a_n = b_n = \frac{1}{n}$.

8. If $\sum a_n$ is convergent and (b_n) is bounded, is $\sum(a_n b_n)$ necessarily convergent?

   Sol. No. $a_n = (-1)^n \frac{1}{n}$ and $b_n = (-1)^n$.

9. Let $a_n \geq 0$ and $\sum a_n$ be convergent. Is $\sum \frac{\sqrt{a_n}}{n}$ necessarily convergent?

   Sol. Yes. As $ab \leq a^2 + b^2$, we have $\sqrt{a_n}\frac{1}{n} \leq a_n + \frac{1}{n^2}$. Use comparison test.

10. If $\sum a_n$ converges and $a_n \geq 0$ then is $\sum \frac{a_n}{n}$ necessarily convergent?

    Sol. Yes. By using comparison test with $\sum a_n$. Alternately, one can first show that $\sum a_n^2$ is convergent, and then use the previous item. Another alternate way is to apply Dirichlet's test.

11. Suppose that $a_n > 0$ and $\lim\limits_{n\to\infty} a_n = 0$. Is $\sum\limits_{n=1}^{\infty}(-1)^{n+1}a_n$ necessarily convergent?

    Sol. No. Consider the sequence

    $$(a_n) = (2, 1, 1, \frac{1}{2}, \frac{2}{3}, \frac{1}{3}, \frac{2}{4}, \frac{1}{4}, \cdots, \frac{2}{n}, \frac{1}{n}, \cdots).$$

    Here $a_{2n} = \frac{1}{n}$ and $a_{2n-1} = \frac{2}{n}$. Let s_n denote the sum of the first n elements of the alternating series for this sequence. Then

    $$s_{2n} = (2-1)+(1-\frac{1}{2})+(\frac{2}{3}-\frac{1}{3})+(\frac{2}{4}-\frac{1}{4})+\cdots+(\frac{2}{n}-\frac{1}{n}) = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n} \to \infty.$$

    The answer would have been 'yes' by Leibniz test, if $a_n \downarrow 0$.

B) True or false?

1. Let $f : \mathbb{N} \to \mathbb{R}$ be defined as $f(x) = 5$. Then $\lim\limits_{x\to 4} f(x) = 5$.

   Sol. No. Limits are defined at cluster points of A.

2. Let $f : A \to \mathbb{R}$ and a be a cluster point of A. Suppose that for each sequence (a_n) of points from A converging to a, we have $f(a_n) \to l$ (here we are not putting the restriction that $a_n \neq a$). Then $\lim\limits_{x\to a} f(x) = l$.

Sol. T. If it is satisfied for all these sequences, it will be satisfied for a subset of these sequences, namely, those for which terms are not a. The given hypothesis will give us continuity if we assume $a \in A$.

3. Take $f(x) = x^5 + 7x^4 - 10x^3 + 5$. We want to show that $\lim_{x \to 1} f(x) = 3$. For that we start with 'Let $\epsilon > 0$'. Then

$$\delta = \min\{\sqrt[5]{1+\frac{\epsilon}{3}} - 1,\ \sqrt[4]{1+\frac{\epsilon}{21}} - 1,\ \sqrt[3]{1+\frac{\epsilon}{30}} - 1\}$$

is an appropriate value.

Sol. T. For $f_1(x) = x^5$, with $\epsilon/3$, and $\delta_1 = \sqrt[5]{1+\frac{\epsilon}{3}} - 1$, we see that

'for each $x \in \{x \mid 0 < |x-1| < \delta_1$, we have $|f_1(x) - 1| < \epsilon/3$.'

(Refer to notes or video, if you did note understand the above.)

For $f_2(x) = 7x^4$, with $\epsilon/21$ and $\delta_2 = \sqrt[4]{1+\frac{\epsilon}{21}} - 1$ we see that

'for each $x \in \{x \mid 0 < |x-1| < \delta_2$, we have $|x^4 - 1| < \epsilon/21$ and so $|f_2(x) - 7| < \epsilon/3$.'

Similarly, with $\delta_3 = \sqrt[3]{1+\frac{\epsilon}{30}} - 1$ we see that

'for each $x \in \{x \mid 0 < |x-1| < \delta_3$, we have $|f_3(x) + 10| < \epsilon/3$.'

Hence for $0 < |x-1| < \delta = \min\{\delta_1, \delta_2, \delta_3\}$, we have

$$\begin{aligned} |f(x) - 3| &= |x^5 + 7x^4 - 10x^3 + 5 - 1 - 7 + 10 - 5| \\ &\leq |x^5 - 1| + |7x^4 - 7| + |-10x^3 + 10| < \epsilon. \end{aligned}$$

C) Other questions.

1. Define $\lim_{x \to c} f(x) = \infty$ in both ways. Compare with texts.

   Sol. We say $\lim_{x \to c} f(x) = \infty$ if for each $n \in \mathbb{N}$, $\exists \delta > 0$ such that $f(D_\delta(c)) \subseteq [n, \infty)$.

   We say $\lim_{x \to c} f(x) = \infty$ if for each sequence $x_n \to c$, $x_n \neq c$, we have $f(x_n) \to \infty$.

2. Define $\lim_{x \to \infty} f(x) = l$ in both ways. Compare with the texts. Did you find it similar to that of $\lim_{n \to \infty} a_n = l$, where $a_n = f(n)$?

   Sol. We say $\lim_{x \to \infty} f(x) = l$ if for each $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $f([n, \infty)) \subseteq (l - \epsilon, l + \epsilon)$.

We say $\lim_{x\to\infty} f(x) = l$ if for each $x_n \to \infty$, we have $f(x_n) \to l$.

Yes, the first one is similar to the definition of convergence, if we consider $f : \mathbb{N} \to \mathbb{R}$.

3. Define f on $\mathbb{R}$ as

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{1}{n},\ n \in \mathbb{N} \\ 0 & \text{else.} \end{cases}$$

Find the points a at which $\lim_{x\to a} f(x)$ exists.

Sol. Limit exists and it is 0 at all $a \in \mathbb{R}$. Notice that $f(x) \leq x$ on $(0,1]$ and 0 outside.

Case I. $a = 0$. Let $x_n \to 0$, $x_n \neq 0$. Then we have $0 \leq f(x_n) \leq |x_n|$. Hence, by sandwich lemma $f(x_n) \to 0$. So limit of f exists at $a = 0$ and it is 0

Case II. Let $a \neq 0$. We show that $\lim_{x\to a} f(x) = 0$. For that, let $\epsilon > 0$. We will find a delta such that $f(D_\delta(a)) \subseteq B_\epsilon(0)$.

Recall that the set $A = \{1/n \mid n \in \mathbb{N}\}$ has only one limit point and it is 0. So a is not a limit point of A. Hence there exists $\delta > 0$ such that $D_\delta(a)$ does not contain any point of A. Hence $f(D_\delta(a)) = \{0\} \subseteq B_\epsilon(0)$.