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Prob1：

设三点的坐标为：$(x_1, y_1)\quad(x_2, y_2)\quad(x_3, y_3)$

$f\left(x_1, y_1, x_2, y_2, x_3, y_3\right)=\sqrt{\left(x_1-x_2\right)^2+\left(y_1-y_2\right)^2}+\sqrt{\left(x_1-x_3\right)^2+\left(y_1-y_3\right)^2}+\sqrt{\left(x_2-x_3\right)^2+\left(y_2-y_3\right)^2}$
$0 \leqslant x_i \leq 2 \quad i=1,2,3\ 0 \leqslant y_i \leq 1 \quad i=1,2,3$

令$f_1\left(x_1, y_1, x_2, y_2, x_3, y_3\right)=\sqrt{\left(x_1-x_2\right)^2+\left(y_1-y_2\right)^2}$

易得其Hessian阵如下：

$$\begin{pmatrix}
-\frac{(x1-x2)^2}{\left((x1-x2)^2+(y1-y2)^2\right)^{3/2}}+\frac{1}{\sqrt{(x1-x2)^2+(y1-y2)^2}} & -\frac{(x1-x2)(y1-y2)}{\left((x1-x2)^2+(y1-y2)^2\right)^{3/2}} & \frac{(x1-x2)^2}{\left((x1-x2)^2+(y1-y2)^2\right)^{3/2}}-\frac{1}{\sqrt{(x1-x2)^2+(y1-y2)^2}} & \frac{(x1-x2)(y1-y2)}{\left((x1-x2)^2+(y1-y2)^2\right)^{3/2}} & 0 & 0 \\
-\frac{(x1-x2)(y1-y2)}{\left((x1-x2)^2+(y1-y2)^2\right)^{3/2}} & \frac{1}{\sqrt{(x1-x2)^2+(y1-y2)^2}}-\frac{(y1-y2)^2}{\left((x1-x2)^2+(y1-y2)^2\right)^{3/2}} & \frac{(x1-x2)(y1-y2)}{\left((x1-x2)^2+(y1-y2)^2\right)^{3/2}} & -\frac{1}{\sqrt{(x1-x2)^2+(y1-y2)^2}}+\frac{(y1-y2)^2}{\left((x1-x2)^2+(y1-y2)^2\right)^{3/2}} & 0 & 0 \\
\frac{(x1-x2)^2}{\left((x1-x2)^2+(y1-y2)^2\right)^{3/2}}-\frac{1}{\sqrt{(x1-x2)^2+(y1-y2)^2}} & \frac{(x1-x2)(y1-y2)}{\left((x1-x2)^2+(y1-y2)^2\right)^{3/2}} & -\frac{(x1-x2)^2}{\left((x1-x2)^2+(y1-y2)^2\right)^{3/2}}+\frac{1}{\sqrt{(x1-x2)^2+(y1-y2)^2}} & -\frac{(x1-x2)(y1-y2)}{\left((x1-x2)^2+(y1-y2)^2\right)^{3/2}} & 0 & 0 \\
\frac{(x1-x2)(y1-y2)}{\left((x1-x2)^2+(y1-y2)^2\right)^{3/2}} & -\frac{1}{\sqrt{(x1-x2)^2+(y1-y2)^2}}+\frac{(y1-y2)^2}{\left((x1-x2)^2+(y1-y2)^2\right)^{3/2}} & -\frac{(x1-x2)(y1-y2)}{\left((x1-x2)^2+(y1-y2)^2\right)^{3/2}} & \frac{1}{\sqrt{(x1-x2)^2+(y1-y2)^2}}-\frac{(y1-y2)^2}{\left((x1-x2)^2+(y1-y2)^2\right)^{3/2}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

得到此方阵的特征值为：

$$\left\{0,0,0,0,0, \frac{2}{\sqrt{\left(x_1-x_2\right)^2+\left(y_1-y_2\right)^2}}\right\}$$

显然Hessian阵是半正定的。

因此f_1是凸的，同理可得f_2, f_3是凸的，$f=f_1+f_2+f_3$是凸的（凸函数的非负加权求和仍然是凸函数）。

我们也可以通过定义来证明f是凸函数：

设$\mathbf{v}=\left(x_1, x_2, x_3, y_1, y_2, y_3\right)^T, \mathbf{x}=\left(x_1, y_1\right)^T, \mathbf{y}=\left(x_2, y_2\right)^T, \mathbf{z}=\left(x_3, y_3\right)^T$

$$f(v)=\|x-y\|_2+\|x-z\|_2+\|y-z\|_2$$

$\forall t \in(0,1)$:

$$\begin{gathered}
t f\left(\mathbf{v}^{(1)}\right)+(1-t) f\left(\mathbf{v}^{(2)}\right)=t\left\|x^{(1)}-y^{(1)}\right\|_2+t\left\|x^{(1)}-z^{(1)}\right\|_2+t\left\|y^{(1)}-z^{(1)}\right\|_2+ \\
(1-t)\left\|x^{(2)}-y^{(2)}\right\|_2+(1-t)\left\|x^{(2)}-z^{(2)}\right\|_2+(1-t)\left\|y^{(2)}-z^{(2)}\right\|_2 \\
\geq\left\|t\left(x^{(1)}-y^{(1)}\right)+(1-t)\left(x^{(2)}-y^{(2)}\right)\right\|+\left\|t\left(x^{(1)}-z^{(1)}\right)+(1-t)\left(x^{(2)}-z^{(2)}\right)\right\|+ \\
\left\|t\left(y^{(1)}-z^{(1)}\right)+(1-t)\left(y^{(2)}-z^{(2)}\right)\right\|=f\left(t v^{(1)}+(1-t) v^{(2)}\right)
\end{gathered}$$

因此这是一个凸优化问题！

通过 CVX 可以轻松得到最大值为5.236

最大值点$(0,0)\quad(2,0)\quad(2,1)$