THU-70250403, Convex Optimization (Fall 2020)	**Homework: 1**

Convex Set

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Student:

Problem 1

A problem appeared in the 2011 Graduate Entrance Examination of Department of Math, Peking University.

Suppose $T(x) : D \to \mathbb{R}^n$ is a function with $x \in D$, D is a convex set in $\mathbb{R}^n$. The first and second order partial derivatives $T(x)$ are all continuous on D, and the corresponding Jacobian matrix $J(x)$ of $T(x)$ is always positive definite on D. Please prove that $T(x)$ is an injective function.

Problem 2

Which of the following sets S are polyhedra? If possible, express S in the form $S = \{\boldsymbol{x} \mid A\boldsymbol{x} \leq b, F\boldsymbol{x} = g\}$.

- $S = \{y_1 a_1 + y_2 a_2 \mid -1 \leqslant y_1 \leqslant 1, -1 \leqslant y_2 \leqslant 1\}$, where a_1, $a_2 \in \mathbb{R}^n$.
- $S = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} \geq \boldsymbol{0}, \boldsymbol{1}^T\boldsymbol{x} = 1, \sum_{i=1}^n x_i a_i = b_1, \sum_{i=1}^n x_i a_i^2 = b_2\}$, where $a_1, \cdots, a_n \in \mathbb{R}$ and $b_1, b_2 \in \mathbb{R}$.
- $S = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} \geq \boldsymbol{0}, \boldsymbol{x}^T\boldsymbol{y} \leqslant 1$ for all y with $|y|_2 = 1\}$.
- $S = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} \geq \boldsymbol{0}, \boldsymbol{x}^T\boldsymbol{y} \leqslant 1$ for all y with $\sum_{i=1}^n |y_i| = 1\}$.

Problem 3

Suppose all the following sets are not empty. Please explain whether they always have extreme points, respectively.

a) $\Omega_1 : \{\boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}, A \in \mathbb{R}^{m\times n}\}$

b) $\Omega_2 : \{\boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} \geq \boldsymbol{b}, A \in \mathbb{R}^{m\times n}, \operatorname{rank}(A) = m\}$

c) $\Omega_3 : \{\boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}, A \in \mathbb{R}^{m\times n}, \operatorname{rank}(A) = n\}$

Problem 4

Let $\Omega \in \mathbb{R}^n$ be the polyhedral cone defined as $\Omega = \{\boldsymbol{x} \mid A\boldsymbol{x} \geq \boldsymbol{0}\}$. Please prove that the following are equivalent:

1. $\boldsymbol{0}$ is an extreme point of Ω.

2. The cone Ω does not contain a line.
3. The rows of A span $\mathbb{R}^n$.

References