HW5 Problem1

Let $\boldsymbol{y} = A\boldsymbol{z} - \boldsymbol{b}$, then the original problem becomes

$$\begin{array}{ll} \min_{\boldsymbol{x},\boldsymbol{y}} & \frac{1}{2}|\boldsymbol{x}|_2^2 + \frac{1}{2}|\boldsymbol{y}|_2^2 \\ \text{s.t.} & \boldsymbol{y} = A\boldsymbol{z} - \boldsymbol{b} \\ & \boldsymbol{x} - \boldsymbol{z} = \boldsymbol{c} \end{array}$$

Now we introduce $\boldsymbol{v}_1 \in \mathbb{R}^n, \boldsymbol{v_2} \in \mathbb{R}^m$ and formulate the lagrange function as follows:

$$\begin{aligned} L\left(\boldsymbol{v_1}, \boldsymbol{v_2}\right) &= \frac{1}{2}|\boldsymbol{x}|_2^2 + \frac{1}{2}|\boldsymbol{y}|_2^2 + \boldsymbol{v}_1^T(\boldsymbol{x} - \boldsymbol{z} - \boldsymbol{c}) + \boldsymbol{v_2^T}(A\boldsymbol{z} - \boldsymbol{b} - \boldsymbol{y}) \\ &= \frac{1}{2}\boldsymbol{x}^T\boldsymbol{x} + \boldsymbol{v}_1^T\boldsymbol{x} + \frac{1}{2}\boldsymbol{y}^T\boldsymbol{y} - \boldsymbol{v_2^T}\boldsymbol{y} + \left(\boldsymbol{v_2}^TA - \boldsymbol{v}_1\right)\boldsymbol{z} - \boldsymbol{v_1^T}\boldsymbol{c} - \boldsymbol{v_2^T}\boldsymbol{b} \\ &\Rightarrow \\ g\left(\boldsymbol{v_1}, \boldsymbol{v_2}\right) &= \inf_{\boldsymbol{x},\boldsymbol{y}}\left(\frac{1}{2}\boldsymbol{x}^T\boldsymbol{x} + \boldsymbol{v}_1^T\boldsymbol{x} + \frac{1}{2}\boldsymbol{y}^T\boldsymbol{y} - \boldsymbol{v_2^T}\boldsymbol{y} + \left(\boldsymbol{v_2^T}A - \boldsymbol{v}_1\right)z - \boldsymbol{v_1^T}\boldsymbol{c} - \boldsymbol{v_2^T}\boldsymbol{b}\right) \\ &= -\boldsymbol{v_1^T}\boldsymbol{c} - \boldsymbol{v_2}^T\boldsymbol{b} + \inf_{\boldsymbol{x}}\left(\frac{1}{2}\boldsymbol{x}^T\boldsymbol{x} + \boldsymbol{v}_1^T\boldsymbol{x}\right) + \inf_{\boldsymbol{y}}\left(\frac{1}{2}\boldsymbol{y}^T\boldsymbol{y} - \boldsymbol{v_2^T}\boldsymbol{y}\right) + \inf_{\boldsymbol{z}}\left(\left(\boldsymbol{v_2}^TA - \boldsymbol{v}_1^T\right)\boldsymbol{z}\right) \end{aligned}$$

Then we can get the following result,

$$g(\boldsymbol{v}) = \begin{cases} -\frac{1}{2}\boldsymbol{v_1^T}\boldsymbol{v_1} - \boldsymbol{v}_1^T\boldsymbol{c} - \frac{1}{2}\boldsymbol{v_2^T}\boldsymbol{v_2} - \boldsymbol{v_2^T}\boldsymbol{b} & \boldsymbol{v_2}^TA = \boldsymbol{v}_1^T \\ -\infty & \text{Otherwise} \end{cases}$$

Therefore, the dual problem of the original problem can be formulated as follows:

$$\begin{array}{cc} \max_{\boldsymbol{v_1},\boldsymbol{v_2}} & -\frac{1}{2}\boldsymbol{v_1^T}\boldsymbol{v_1} - \boldsymbol{v_1^T}\boldsymbol{c} - \frac{1}{2}\boldsymbol{v_2}^T\boldsymbol{v_2} - \boldsymbol{v_2^T}\boldsymbol{b} \\ \text{s.t.} & \boldsymbol{v_2^T}A = \boldsymbol{v}_1^T \end{array}$$

Also, we can use the subject condition to eliminate v_1 and get,

$$\max_{\boldsymbol{v_2}} -\frac{1}{2}\boldsymbol{v_2^T}AA^T\boldsymbol{v_2} - \boldsymbol{v_2^T}Ac - \frac{1}{2}\boldsymbol{v_2}^T\boldsymbol{v_2} - \boldsymbol{v_2}^T\boldsymbol{b}$$