Homework 3 Solution

June 8, 2021

1 Problem 1

1.1 Part (a)

The adjacency matrix is:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

(if according to the definition in the textbook, the adjacency matrix should be A^T instead, here we consider both forms correct)

1.2 Part (b)

The incidence matrix is:

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

(The transpose of A is also a correct answer)

1.3 Part (c)

The projection onto black nodes is:

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

2 Problem 2

2.1 Part (a)

$$\mathbf{k} = A\mathbf{1}$$

2.2 Part (b)

$$m = \frac{1}{2}\mathbf{1}^T A\mathbf{1}$$

2.3 Part (c)

$$N = A^2$$

2.4 Part (d)

$$t = \mathbf{tr}(A^3)/6$$

Here $\mathbf{tr}(\cdot)$ is the trace of the matrix (sum of diagonal entries).

3 Problem 3

3.1 Part (a)

Ingoing edges:

$$\sum_{i=1}^{r} k_i^{in}$$

Outgoing edges:

$$\sum_{i=1}^{r} k_i^{out}$$

3.2 Part (b)

The number of such edges:

$$\sum_{i=1}^{r} k_i^{in} - \sum_{i=1}^{r} k_i^{out}$$

3.3 Part (c)

By part (b) (since all edges of k_r^{in} belong to such category),

$$k_r^{in} \leq \sum_{i=1}^{r} k_i^{in} - \sum_{i=1}^{r} k_i^{out}$$

and since $\sum_{i=1}^{n} k_i^{in} = \sum_{i=1}^{n} k_i^{out}$, we have

$$k_r^{in} \leq \sum_{i=r+1}^{n} k_i^{out} - \sum_{i=r+1}^{n} k_i^{in}$$

Similarly by part (b),

$$k_{r+1}^{out} \leq \sum_{i=1}^{r} k_i^{in} - \sum_{i=1}^{r} k_i^{out}$$

4 Problem 4

The total number of paths of length r is given by the corresponding entry in the matrix A^r, thus the total weight can be computed by the following expression:

$$Z = I + \alpha A + \alpha^2 A^2 + \dots$$

Note that if $\lim_{n\to\infty} \alpha^n A^n = 0$, then we have

$$Z(I - \alpha A) = I$$

otherwise the sum does not converge. If the sum converges, then $Z = (I - \alpha A)^{-1}$.

5 Problem 5

5.1 Part (a)

It is easy to check that $A\mathbf{1} = \mathbf{d}$ where $\mathbf{d}$ is the degree vector, since the network is k-regular, $\mathbf{d} = k\mathbf{1}$, therefore $\mathbf{1}$ is an eigenvector of A with eigenvalue k.

5.2 Part (b)

The Katz centrality vector x satisfies:

$$x = \alpha A x + \beta \mathbf{1}$$

Let $x = c\mathbf{1}$, then we have the following equation:

$$c = \alpha c k + \beta$$

Solving it, we get

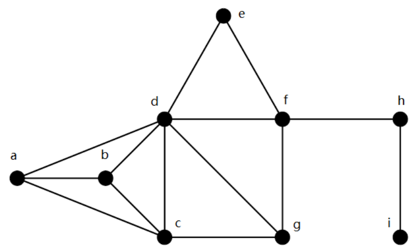
$$c = \beta/(1 - \alpha k)$$

5.3 Part (c)

Betweenness/Closeness centrality can be different for nodes in a regular graph.

6 Problem 6

6.1 Part (a)



A 3-core is $\{a, b, c, d\}$.

6.2 Part (b)

The reciprocity is 3/4.

6.3 Part (c)

The cosine similarity is $\frac{1}{\sqrt{5}}$

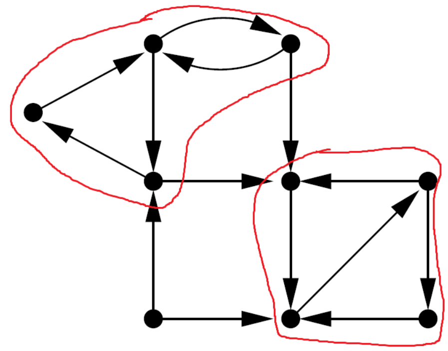
7 Problem 7

7.1 Part (a)

For the first network, the entire network is a 3-core. For the second network, there is no 3-core.

7.2 Part (b)

There are 3 strongly connected components: (a) the top-left 4 nodes; (b) the bottom-right 4 nodes; (c) the bottom-left node. (We only concern about maximal strongly connected components, it is fine to point out smaller ones but not necessary)



7.3 Part (c)

The top-left node has local clustering coefficient equal to 0. The node at center has local clustering coefficient 1/3. The bottom-right node has local clustering coefficient 2/3. Other nodes has local clustering coefficient 1.

7.4 Part (d)

The modularity is defined as

$$M = \frac{1}{2}\sum_{k}\sum_{v_i,v_j\in S_k}(A_{ij} - \frac{k_i k_j}{2m})$$

(You can also use the normalized version with $\frac{1}{2m}$ factor instead)

The first cluster has total degree 8 and 3 internal edges, the second cluster has total degree 12 and 5 internal edges.

Therefore, we can get $M = (16 - \frac{8^2+12^2}{20})/2 = 2.8$

7.5 Part (e)

The unnormalized betweenness centrality is $(n-1)(n-2)+2(n-1)+1 = n^2-n+1$