THU-70250403, Convex Optimization (Fall 2020) **Homework: 5**

Dual Problems and Classificiation

Lecturer: Li Li `li-li@tsinghua.edu.cn`

Student:

Problem 1

Considering the following optimization problem

$$\min_{\boldsymbol{x},\boldsymbol{z}\in\mathbb{R}^n} \quad \frac{1}{2}\left|\boldsymbol{x}\right|_2^2 + \frac{1}{2}\left|A\boldsymbol{z}-\boldsymbol{b}\right|_2^2 \tag{1}$$

$$\text{s.t.} \quad \boldsymbol{x}-\boldsymbol{z}=\boldsymbol{c} \tag{2}$$

where $A \in \mathbb{R}^{m\times n}$ are known constant matrix with $\text{rank}(A) = n$, $\boldsymbol{b} \in \mathbb{R}^m$, $\boldsymbol{c} \in \mathbb{R}^n$ are known constant vectors.

Please derive the corresponding dual problem.

Problem 2

The sum of the largest elements of a vector.

Define $f : \mathbb{R}^n \to \mathbb{R}$ as

$$f(\boldsymbol{x}) = \sum_{i=1}^{r} \boldsymbol{x}_{[i]},$$

where r is an integer between 1 and n , and $\boldsymbol{x}_{[1]} \geqslant \boldsymbol{x}_{[2]} \geqslant \cdots \geqslant \boldsymbol{x}_{[r]}$ are the components of x sorted in decreasing order. In other words, $f(x)$ is the sum of the r largest elements of x . In this problem we study the constraint

$$f(\boldsymbol{x}) \leqslant \alpha.$$

This is a convex constraint, and equivalent to a set of $n!/(r!(n-r)!)$ linear inequalities

$$\boldsymbol{x}_{i_1} + \cdots + \boldsymbol{x}_{i_r} \leqslant \alpha, \quad 1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n.$$

The purpose of this problem is to derive a more compact representation.

1. Given a vector $\boldsymbol{x} \in \mathbb{R}^n$, show that $f(\boldsymbol{x})$ is equal to the optimal value of the LP

   $$\begin{array}{ll} \text{maximize} & \boldsymbol{x}^T\boldsymbol{y} \\ \text{subject to} & \mathbf{0} \preceq \boldsymbol{y} \preceq \mathbf{1} \\ & \mathbf{1}^T\boldsymbol{y} = r \end{array}$$

   with $\boldsymbol{y} \in \mathbb{R}^n$ as variable.

2. Derive the dual of the LP in part (a). Show that it can be written as

$$\begin{array}{ll} \text{minimize} & rt + \mathbf{1}^T \boldsymbol{u} \\ \text{subject to} & t\mathbf{1} + \boldsymbol{u} \succeq \boldsymbol{x} \\ & \boldsymbol{u} \succeq \mathbf{0}, \end{array}$$

where the variables are $t \in \mathbb{R}$, $\boldsymbol{u} \in \mathbb{R}^n$. By duality this LP has the same optimal value as the LP in (a), *i.e.*, $f(x)$. We therefore have the following result: $\boldsymbol{x}$ satisfies $f(\boldsymbol{x}) \leqslant \alpha$ if and only if there exist $t \in \mathbb{R}$, $\boldsymbol{u} \in \mathbb{R}^n$ such that

$$rt + \mathbf{1}^T \boldsymbol{u} \leqslant \alpha, \qquad t\mathbf{1} + \boldsymbol{u} \succeq \boldsymbol{x}, \qquad \boldsymbol{u} \succeq \mathbf{0}.$$

These conditions form a set of $2n+1$ linear inequalities in the $2n+1$ variables $\boldsymbol{x}, \boldsymbol{u}, t$.

Problem 3

The so called *Support Vector Data Description* (SVDD) [1] is a one-class classification method. Given a set of data $(\boldsymbol{x}_i)$, $i = 1, \ldots, l$, $\boldsymbol{x}_i \in \mathbb{R}^n$, SVDD tries to find a closed boundary (indeed an hypersphere centered at $\boldsymbol{a} \in \mathbb{R}^n$ and has a radius $r \in \mathbb{R}$) around the data by solving the following problem

$$\min_{\boldsymbol{a}, r} \quad r^2 + C \sum_{i=1}^{L} \xi_i \tag{3}$$

$$\text{s.t.} \quad |\boldsymbol{x}_i - \boldsymbol{a}|_2^2 \leq r^2 + \xi_i, \ i = 1, \ldots, L \tag{4}$$

$$\xi_i \geq 0, \ i = 1, \ldots, L \tag{5}$$

where $C > 0$ is a penalty parameter.

Please explain the geometry meaning of ξ_i above and derive the dual problem.

References

[1] D. M. J. Tax, R. P. W. Duin, "Support Vector Data Description," *Machine Learning*, vol. 54, no. 1, pp. 45-66, 2004.