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Problem 1

$$\min_{x,z\in R^n} \ \frac{1}{2}\|x\|_2^2 + \frac{1}{2}\|Az-b\|_2^2 \qquad A\in R^{m\times n} \quad rank(A)=n$$

$$\text{st} \quad x-z=c \qquad b\in R^m \quad c\in R^n$$

原问题等价于：

$$\min_z \ \frac{1}{2}\|c+z\|_2^2 + \frac{1}{2}\|Az-b\|_2^2$$

等价于 $\min_z \ \frac{1}{2}\|c+z\|_2^2 + \frac{1}{2}\|y\|_2^2$

$$\text{st} \quad Az-b=y$$

$$L(z,y,u) = \frac{1}{2}\|c+z\|_2^2 + \frac{1}{2}\|y\|_2^2 + u^T(Ax-b-y)$$

$$g(u) = \inf_{z,y}\left\{\frac{1}{2}(c+z)^T(c+z) + \frac{1}{2}y^Ty + u^T(Ax-b-y)\right\}$$

$$= \inf_{z,y}\left\{\frac{1}{2}z^Tz + c^Tz + \frac{1}{2}y^Ty - u^Ty\right\} + \frac{1}{2}c^Tc + u^TAx - u^Tb$$

$$\frac{\partial(\frac{1}{2}z^Tz + c^Tz + \frac{1}{2}y^Ty - u^Ty)}{\partial z} = z+c=0 \Rightarrow z=-c$$

$$\frac{\partial(\frac{1}{2}z^Tz + c^Tz + \frac{1}{2}y^Ty - u^Ty)}{\partial y} = y-u=0 \Rightarrow y=u$$

$$g(u) = \frac{1}{2}c^Tc - c^Tc + \frac{1}{2}u^Tu - u^Tu + \frac{1}{2}c^Tc + u^TAx - u^Tb$$

$$= u^TAx - \frac{1}{2}u^Tu - u^Tb$$

对偶问题

$$\text{maximize} \quad g(u)$$

Problem 2.

(a) $$\begin{aligned} &\text{maximize} && x^T y \\ &\text{s.t} && 0 \le y \le 1 \\ &&& 1^T y = r \end{aligned}$$

y是优化变量　r为正整数

不妨假设 $x_1 \geq x_2 \geq \cdots \geq x_n$

显然 最优值为 $x_1 + x_2 + \cdots + x_r$　此时, $y_1 = y_2 = \cdots = y_r = 1$, $y_{r+1} = y_{r+2} = \cdots = y_n = 0$

(b) (a)问题等价于

$$\begin{aligned} &\text{minimize} && -x^T y \\ &\text{s.t} && -y \le 0 \\ &&& y \le 1 \\ &&& 1^T y = r \end{aligned}$$

$$\begin{aligned} L(y,u,v,w) &= -x^T y - u^T y + v^T(y-1) + w(1^T y - r) \\ &= -v^T 1 - wr + (-x - u + v + w1)^T y \end{aligned}$$

$$g(u,v,w) = \begin{cases} -v^T 1 - wr & -x-u+v+w1 = 0 \\ -\infty & \text{otherwise} \end{cases}$$

对偶问题

$$\begin{aligned} &\text{maximize} && -v^T 1 - wr \\ &\text{s.t} && -u + v + w1 = x \\ &&& u \geq 0,\ v \geq 0 \end{aligned}$$

去掉u将等式约束变为不等式约束, 改下变量名 ($v \to u$, $w \to t$)

得到

$$\begin{aligned} &\text{minimize} && rt + 1^T u \\ &\text{s.t} && t1 + u \geq x \\ &&& u \geq 0 \end{aligned}$$

证毕

Problem 3.

$$\min_{a,r} \quad r^2 + C\sum_{i=1}^{l} \xi_i \qquad a\in R^n \quad r\in R$$

$$\text{s.t} \quad |x_i - a|_2^2 \le r^2 + \xi_i \quad i=1,\cdots l \qquad C>0$$

$$\xi_i \ge 0 \qquad i=1,\cdots l$$

C是一个权衡超球体体积和误差率的惩罚参数

上述问题是非凸的 $|x_i-a|_2^2 - r^2 - \xi_i$ 关于r是凹的，令 $R=r^2$

原问题等价于:

$$\min_{a,R} \quad R + C\sum_{i=1}^{l}\xi_i$$

$$\text{s.t} \quad R+\xi_i - |x_i - a|_2^2 \ge 0, \; i=1,\cdots l$$

$$\xi_i \ge 0 \qquad i=1,\cdots l$$

$$R \ge 0$$

$$L(a,R,\xi,u,v,w) = R + C\sum_{i=1}^{l}\xi_i - \sum_{i=1}^{l} u_i(R+\xi_i - |x_i-a|_2^2) - \sum_{i=1}^{l} v_i\xi_i - wR$$

$$= R(1-\sum_{i=1}^{l}u_i - w) + \sum_{i=1}^{l}\xi_i(C-u_i-v_i) + \sum_{i=1}^{l}u_i(|x_i-a|_2^2)$$

$$\frac{\partial L}{\partial R}=0 \Rightarrow \quad 1=\sum_{i=1}^{l}u_i - w$$

$$\frac{\partial L}{\partial \xi_i}=0 \Rightarrow \quad C-u_i-v_i=0 \quad i=1,2\cdots l$$

$$\frac{\partial L}{\partial a}=0 \Rightarrow \quad a\sum_{i=1}^{l}u_i = \sum_{i=1}^{l}u_i x_i \Rightarrow a = \frac{\sum_{i=1}^{l}u_i x_i}{1+w}$$

$$g(u,v,w) = \sum_{i=1}^{l} u_i \left| x_i - \frac{\sum_{i=1}^{l}u_i x_i}{1+w}\right|_2^2$$

对偶问题

$$\max \quad g(u,v,w)$$

$$\text{s.t} \quad u \geq 0,\ v \geq 0,\ w \geq 0$$

$$1 = 1^T u - w$$

$$c = u_i - v_i$$

其等价于 $\max \quad \sum_{i=1}^{L} u_i \| x_i - \sum_{i=1}^{L} u_i x_i \|_2^2$

$$\text{s.t} \quad 0 \leq u_i \leq C$$

$$1^T u = 1$$