Multiple View Geometry: Exercise Sheet 3

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Part I: Theory

The following exercises should be **solved at home**. You do not have to hand in your solutions, however, writing it down will help you present your answer during the tutorials.

1. Indicate the matrices $M \in SE(3) \subset \mathbb{R}^{4\times 4}$ representing the following transformations:
   (a) Translation by the vector $T \in \mathbb{R}^3$.
   (b) Rotation by the rotation matrix $R \in \mathbb{R}^{3\times 3}$.
   (c) Rotation by R followed by the translation T.
   (d) Translation by T followed by the rotation R.

2. Let $M_1, M_2 \in \mathbb{R}^{3\times 3}$. Please prove the following:

$$\begin{array}{ccc} \mathbf{x}^T M_1 \mathbf{x} = \mathbf{x}^T M_2 \mathbf{x} & \text{iff} & M_1 - M_2 \text{ is skew-symmetric} \\ \text{for all} \quad \mathbf{x} \in \mathbb{R}^3 & & (\text{i.e. } M_1 - M_2 \in so(3)) \end{array}$$

   Info: The group $SO(3)$ is called a **Lie group**.
   The space $so(3) = \left\{ \hat{\omega} \mid \omega \in \mathbb{R}^3 \right\}$ of skew-symmetric matrices is called its **Lie algebra**.

3. Consider a vector $\omega \in \mathbb{R}^3$ with $\|\omega\| = 1$ and its corresponding skew-symmetric matrix $\hat{\omega}$.
   (a) Show that $\hat{\omega}^2 = \omega\omega^\top - I$ and $\hat{\omega}^3 = -\hat{\omega}$.
   (b) Following the result of (a), find simple rules for the calculation of $\hat{\omega}^n$ and proof your result. Distinguish between odd and even numbers n.
   (c) Derive the Rodrigues' formula for a skew-symmetric matrix $\hat{\omega}$ corresponding to an arbitrary vector $\omega \in \mathbb{R}^3$ (i.e. $\|\omega\|$ does not have to be equal to 1):

$$e^{\hat{\omega}} = I + \frac{\hat{\omega}}{\|\omega\|} \sin(\|\omega\|) + \frac{\hat{\omega}^2}{\|\omega\|^2} (1 - \cos(\|\omega\|))$$

   Hint: Combine your result from (b) with

$$e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!} \quad \text{and} \quad \sin(t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!} \quad \text{and} \quad 1 - \cos(t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^{2n}}{(2n)!}$$

Part II: Practical Exercises

This exercise is to be solved during the tutorial.

1. Homogeneous transformation matrices

   (a) Download the package `ex3.zip` and use `openOFF.m` to load the 3D model `model.off`.

   (b) Write a function that rotates the model around its *center* (i.e. the mean of its vertices) for given rotation angles α, β and γ around the x-, y- and z-axis. Use homogeneous coordinates and describe the overall transformation by a single matrix. The rotation matrices around the respective axes are as follows:

rotation matrix (x-axis)	rotation matrix (y-axis)	rotation matrix (z-axis)
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$	$\begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}$	$\begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$

   (c) Rotate the model first 5 degrees around the x-axis and then 25 degrees around the z-axis. Now start again by doing the same rotation around the z-axis first followed by the x-axis rotation. What do you observe?

   (d) Perform a translation in addition to the rotation. Find a suitable matrix from $SE(3)$ for this purpose and add it to your function from (c). Translate the model by the vector $(0.5\ \ 0.2\ \ 0.1)^\top$.

2. Twist-coordinates

   (a) Write a function which takes a vector $w \in \mathbb{R}^3$ as input and returns its corresponding element $R = e^{\hat{w}} \in SO(3) \subset \mathbb{R}^{3\times 3}$ from the Lie group. Hence, the function will be a concatenation of the hat operator $\hat{\ }: \mathbb{R}^3 \to so(3)$ and the exponential mapping.

   (b) Implement another function which performs the corresponding inverse transformation and test the two functions on some examples.

   (c) Implement similar functions which calculate the transformation for twists.
   I.e. from $\xi \in \mathbb{R}^6$ to $e^{\hat{\xi}} \in SE(3) \subset \mathbb{R}^{4x4}$ and the other way around.

   (d) How can you use Matlab's built-in functions `expm` and `logm` to achieve the same functionality (your solutions to (a)-(c) should *not* use these functions)?