Multiple View Geometry: Solution Exercise Sheet 3

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https://vision.in.tum.de/teaching/ss2016/mvg2016

Part I: Theory

1. (a)
$$M = \begin{pmatrix} I & T \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(b)
$$M = \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(c)
$$M = \begin{pmatrix} I & T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R & T \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{(d)} \ \ M = \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I & T \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R & RT \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & r_1t_x \\ r_{21} & r_{22} & r_{23} & r_2t_y \\ r_{31} & r_{32} & r_{33} & r_3t_z \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where r_1, r_2, r_3 are the row vectors of R: $R = \begin{pmatrix} - & r_1 & - \\ - & r_2 & - \\ - & r_3 & - \end{pmatrix}$.

2. Let
$$M := (M_1 - M_2) =: \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$$
.

"⇒"·

We show that M is skew-symmetric by distinguishing diagonal and off-diagonal elements of M:

(a)
$$\forall i : 0 = e_i^T M e_i = m_{ii}$$

where $e_i = i$ -th unit vector

(b)
$$\forall i \neq j : 0 = (e_i + e_j)^T M(e_i + e_j)$$

= $m_{ii} + m_{jj} + m_{ij} + m_{ji} \Rightarrow m_{ij} = -m_{ij}$

where $e_j = j$ -th unit vector

hence, $m_{ii} = 0$ and $m_{ij} = -m_{ij}$, i.e. M is skew-symmetric.

"⇐":

using $M = -M^T$, we directly calculate

$$\forall x \colon x^T M x = (x^T M x)^T = x^T M^T x = -(x^T M x)$$
$$\Rightarrow x^T M x = 0$$

3. We know:
$$\omega = (\omega_1 \ \omega_2 \ \omega_3)^T$$
 with $||\omega|| = 1$ and $\hat{\omega} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$.

(a)

$$\hat{\omega}^{2} = \begin{pmatrix} -(\omega_{2}^{2} + \omega_{3}^{2}) & \omega_{1}\omega_{2} & \omega_{1}\omega_{3} \\ \omega_{1}\omega_{2} & -(\omega_{1}^{2} + \omega_{3}^{2}) & \omega_{2}\omega_{3} \\ \omega_{1}\omega_{3} & \omega_{2}\omega_{3} & -(\omega_{1}^{2} + \omega_{2}^{2}) \end{pmatrix}$$

$$= \begin{pmatrix} \omega_{1}^{2} - (\omega_{1}^{2} + \omega_{2}^{2} + \omega_{3}^{2}) & \omega_{1}\omega_{2} & \omega_{1}\omega_{3} \\ \omega_{1}\omega_{2} & \omega_{2}^{2} - (\omega_{2}^{2} + \omega_{1}^{2} + \omega_{3}^{2}) & \omega_{2}\omega_{3} \\ \omega_{1}\omega_{3} & \omega_{2}\omega_{3} & \omega_{3}^{2} - (\omega_{3}^{2} + \omega_{1}^{2} + \omega_{2}^{2}) \end{pmatrix}$$

$$= \begin{pmatrix} \omega_{1}^{2} & \omega_{1}\omega_{2} & \omega_{1}\omega_{3} \\ \omega_{1}\omega_{2} & \omega_{2}^{2} & \omega_{2}\omega_{3} \\ \omega_{1}\omega_{3} & \omega_{2}\omega_{3} & \omega_{3}^{2} \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \omega\omega^{\top} - \mathbf{I}$$

$$\hat{\omega}^{3} = \hat{\omega} \, \hat{\omega}^{2}$$

$$= \hat{\omega} \, (\omega \omega^{T} - I)$$

$$= \hat{\omega} \, \omega \, (\omega^{T}) - \hat{\omega} I$$

$$= (\omega \times \omega) \, \omega^{T} - \hat{\omega}$$

$$= -\hat{\omega} \qquad (as \, \omega \times \omega = 0)$$

Alternative solution for $\hat{\omega}^3$:

$$\hat{\omega}^{3} = \begin{pmatrix}
-(\omega_{2}^{2} + \omega_{3}^{2}) & \omega_{1}\omega_{2} & \omega_{1}\omega_{3} \\
\omega_{1}\omega_{2} & -(\omega_{1}^{2} + \omega_{2}^{2}) & \omega_{2}\omega_{3} \\
\omega_{1}\omega_{3} & \omega_{2}\omega_{3} & -(\omega_{1}^{2} + \omega_{2}^{2})
\end{pmatrix} \cdot \begin{pmatrix}
0 & -\omega_{3} & \omega_{2} \\
\omega_{3} & 0 & -\omega_{1} \\
-\omega_{2} & \omega_{1} & 0
\end{pmatrix}$$

$$= \begin{pmatrix}
0 & \omega_{3} \cdot (\omega_{1}^{2} + \omega_{2}^{2} + \omega_{3}^{2}) & -\omega_{2} \cdot (\omega_{1}^{2} + \omega_{2}^{2} + \omega_{3}^{2}) \\
-\omega_{3} \cdot (\omega_{1}^{2} + \omega_{2}^{2} + \omega_{3}^{2}) & 0 & \omega_{1} \cdot (\omega_{1}^{2} + \omega_{2}^{2} + \omega_{3}^{2}) \\
\omega_{2} \cdot (\omega_{1}^{2} + \omega_{2}^{2} + \omega_{3}^{2}) & -\omega_{1} \cdot (\omega_{1}^{2} + \omega_{2}^{2} + \omega_{3}^{2}) & 0
\end{pmatrix}$$

$$= -\hat{\omega}$$

(b) The formulas for n even and odd can be found by writing down the solutions for $n = 1, \dots, 6$:

$$\begin{array}{lll} \hat{\omega} \\ \hat{\omega}^2 \\ \hat{\omega}^3 &=& -\hat{\omega} \\ \hat{\omega}^4 &=& -\hat{\omega}^2 \\ \hat{\omega}^5 &=& \hat{\omega} \\ \hat{\omega}^6 &=& \hat{\omega}^2 \end{array} \qquad \begin{array}{ll} \mathrm{as:} \ \hat{\omega}^4 = \hat{\omega}^3 \hat{\omega} = -\hat{\omega} \hat{\omega} = -\hat{\omega}^2 \\ \mathrm{as:} \ \hat{\omega}^5 = \hat{\omega}^4 \hat{\omega} = -\hat{\omega}^2 \hat{\omega} = -\hat{\omega}^3 = -(-\hat{\omega}) = \hat{\omega} \\ \mathrm{as:} \ \hat{\omega}^6 = \hat{\omega}^5 \hat{\omega} = \hat{\omega} \hat{\omega} = \hat{\omega}^2 \end{array}$$

For even numbers:

$$\hat{\omega}^2
\hat{\omega}^4 = -\hat{\omega}^2
\hat{\omega}^6 = \hat{\omega}^2$$

For odd numbers:

$$\begin{array}{ccc} \hat{\omega} & & \\ \hat{\omega}^3 & = & -\hat{\omega} \\ \hat{\omega}^5 & = & \hat{\omega} \end{array}$$

$$n \text{ even:} \quad \hat{\omega}^n = (-1)^{\frac{n}{2}+1} \hat{\omega}^2$$

$$n \text{ odd:} \quad \hat{\omega}^n = (-1)^{\frac{n-1}{2}} \hat{\omega}$$

Proof via complete induction:

i. For even numbers n:

-
$$n=2$$
: $\hat{\omega}^2=(-1)^{\frac{2}{2}+1}\hat{\omega}^2$

- Induction step $n \to n+2$:

$$\begin{array}{rcl} \hat{\omega}^{n+2} & = & \hat{\omega}^n \cdot \hat{\omega}^2 \\ & = & (-1)^{\frac{n}{2}+1} \cdot \hat{\omega}^2 \cdot \hat{\omega}^2 \\ & = & (-1)^{\frac{n}{2}+1} \cdot \hat{\omega}^3 \cdot \hat{\omega} \\ & \stackrel{(a)}{=} & (-1)^{\frac{(n+2)}{2}+1} \cdot \hat{\omega}^2 \end{array} \tag{assumption}$$

ii. For odd numbers n:

-
$$n = 3$$
: $\hat{\omega}^3 = -\hat{\omega} = (-1)^{\frac{3-1}{2}}\hat{\omega}$

- Induction step $n \to n+2$:

$$\hat{\omega}^{n+2} = \hat{\omega}^n \cdot \hat{\omega}^2$$

$$= (-1)^{\frac{n-1}{2}} \cdot \hat{\omega} \cdot \hat{\omega}^2 \qquad \text{(assumption)}$$

$$= (-1)^{\frac{n-1}{2}} \cdot \hat{\omega}^3$$

$$\stackrel{(a)}{=} (-1)^{\frac{n-1}{2}+1} \cdot \hat{\omega}$$

$$= (-1)^{\frac{(n+2)-1}{2}} \cdot \hat{\omega}$$

(c) We know: $\omega \in \mathbb{R}^3$. Let $v = \frac{\omega}{\|\omega\|}$ and $t = \|\omega\|$. Hence, w = vt, $\hat{\omega} = v\hat{\nu}$.

$$\begin{split} e^{\hat{\omega}} &= e^{\hat{\nu}t} \\ &= \sum_{n=0}^{\infty} \frac{(\hat{\nu}t)^n}{n!} \\ \stackrel{(b)}{=} I + \underbrace{\sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^{2n}}{(2n)!}}_{1-\cos(t)} \hat{\nu}^2 + \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!}}_{\sin(t)} \hat{\nu} \\ \stackrel{(\text{def.})}{=} I + \frac{\hat{\omega}^2}{\|\omega\|^2} (1 - \cos(\|\omega\|)) + \frac{\hat{\omega}}{\|\omega\|} \sin(\|\omega\|) \end{split}$$