PROBLEMS

1. Find the following limits. (These limits all follow, after some algebraic manipulations, from the various parts of Theorem 2; be sure you know which ones are used in each case, but don't bother listing them.)

(i) $\displaystyle\lim_{x \to 1} \frac{x^2 - 1}{x + 1}$.

(ii) $\displaystyle\lim_{x \to 2} \frac{x^3 - 8}{x - 2}$.

(iii) $\displaystyle\lim_{x \to 3} \frac{x^3 - 8}{x - 2}$.

(iv) $\displaystyle\lim_{x \to y} \frac{x^n - y^n}{x - y}$.

(v) $\displaystyle\lim_{y \to x} \frac{x^n - y^n}{x - y}$.

(vi) $\displaystyle\lim_{h \to 0} \frac{\sqrt{a + h} - \sqrt{a}}{h}$.

2. Find the following limits.

(i) $\displaystyle\lim_{x \to 1} \frac{1 - \sqrt{x}}{1 - x}$.

(ii) $\displaystyle\lim_{x \to 0} \frac{1 - \sqrt{1 - x^2}}{x}$.

(iii) $\displaystyle\lim_{x \to 0} \frac{1 - \sqrt{1 - x^2}}{x^2}$.

3. In each of the following cases, find a δ such that $|f(x) - l| < \varepsilon$ for all x satisfying $0 < |x - a| < \delta$.

(i) $f(x) = x^4;\ l = a^4$.

(ii) $f(x) = \dfrac{1}{x};\ a = 1, l = 1$.

(iii) $f(x) = x^4 + \dfrac{1}{x};\ a = 1, l = 2$.

(iv) $f(x) = \dfrac{x}{1 + \sin^2 x};\ a = 0, l = 0$.

(v) $f(x) = \sqrt{|x|};\ a = 0, l = 0$.

(vi) $f(x) = \sqrt{x};\ a = 1, l = 1$.

4. For each of the functions in Problem 4-17, decide for which numbers a the limit $\lim_{x \to a} f(x)$ exists.

***5.** (a) Do the same for each of the functions in Problem 4-19.

(b) Same problem, if we use infinite decimals ending in a string of 0's instead of those ending in a string of 9's.

6. Suppose the functions f and g have the following property: for all $\varepsilon > 0$ and all x,

$$\text{if } 0 < |x-2| < \sin^2\left(\frac{\varepsilon^2}{9}\right) + \varepsilon, \text{ then } |f(x) - 2| < \varepsilon,$$

$$\text{if } 0 < |x-2| < \varepsilon^2, \text{ then } |g(x) - 4| < \varepsilon.$$

For each $\varepsilon > 0$ find a $\delta > 0$ such that, for all x,

(i) if $0 < |x-2| < \delta$, then $|f(x) + g(x) - 6| < \varepsilon$.

(ii) if $0 < |x-2| < \delta$, then $|f(x)g(x) - 8| < \varepsilon$.

(iii) if $0 < |x-2| < \delta$, then $\left|\dfrac{1}{g(x)} - \dfrac{1}{4}\right| < \varepsilon$.

(iv) if $0 < |x-2| < \delta$, then $\left|\dfrac{f(x)}{g(x)} - \dfrac{1}{2}\right| < \varepsilon$.

7. Give an example of a function f for which the following assertion is *false*: If $|f(x) - l| < \varepsilon$ when $0 < |x-a| < \delta$, then $|f(x) - l| < \varepsilon/2$ when $0 < |x-a| < \delta/2$.

8. (a) If $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ do not exist, can $\lim\limits_{x\to a}[f(x) + g(x)]$ or $\lim\limits_{x\to a} f(x)g(x)$ exist?

(b) If $\lim\limits_{x\to a} f(x)$ exists and $\lim\limits_{x\to a}[f(x)+g(x)]$ exists, must $\lim\limits_{x\to a} g(x)$ exist?

(c) If $\lim\limits_{x\to a} f(x)$ exists and $\lim\limits_{x\to a} g(x)$ does not exist, can $\lim\limits_{x\to a}[f(x)+g(x)]$ exist?

(d) If $\lim\limits_{x\to a} f(x)$ exists and $\lim\limits_{x\to a} f(x)g(x)$ exists, does it follow that $\lim\limits_{x\to a} g(x)$ exists?

9. Prove that $\lim\limits_{x\to a} f(x) = \lim\limits_{h\to 0} f(a+h)$. (This is mainly an exercise in understanding what the terms mean.)

10. (a) Prove that $\lim\limits_{x\to a} f(x) = l$ if and only if $\lim\limits_{x\to a}[f(x) - l] = 0$. (First see why the assertion is obvious; then provide a rigorous proof. In this chapter most problems which ask for proofs should be treated in the same way.)

(b) Prove that $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to a} f(x-a)$.

(c) Prove that $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} f(x^3)$.

(d) Give an example where $\lim\limits_{x\to 0} f(x^2)$ exists, but $\lim\limits_{x\to 0} f(x)$ does not.

11. Suppose there is a $\delta > 0$ such that $f(x) = g(x)$ when $0 < |x-a| < \delta$. Prove that $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x)$. In other words, $\lim\limits_{x\to a} f(x)$ depends only on the values of $f(x)$ for x near a—this fact is often expressed by saying that limits are a "local property." (It will clearly help to use δ', or some other letter, instead of δ, in the definition of limits.)

12. (a) Suppose that $f(x) \le g(x)$ for all x. Prove that $\lim\limits_{x\to a} f(x) \le \lim\limits_{x\to a} g(x)$, provided that these limits exist.

(b) How can the hypotheses by weakened?

(c) If $f(x) < g(x)$ for all x, does it necessarily follow that $\lim_{x \to a} f(x) < \lim_{x \to a} g(x)$?

13. Suppose that $f(x) \le g(x) \le h(x)$ and that $\lim_{x \to a} f(x) = \lim_{x \to a} h(x)$. Prove that $\lim_{x \to a} g(x)$ exists, and that $\lim_{x \to a} g(x) = \lim_{x \to a} f(x) = \lim_{x \to a} h(x)$. (Draw a picture!)

***14.** (a) Prove that if $\lim_{x \to 0} f(x)/x = l$ and $b \neq 0$, then $\lim_{x \to 0} f(bx)/x = bl$. Hint: Write $f(bx)/x = b[f(bx)/bx]$.

(b) What happens if $b = 0$?

(c) Part (a) enables us to find $\lim_{x \to 0} (\sin 2x)/x$ in terms of $\lim_{x \to 0} (\sin x)/x$. Find this limit in another way.

15. Evaluate the following limits in terms of the number $\alpha = \lim_{x \to 0} (\sin x)/x$.

(i) $\displaystyle \lim_{x \to 0} \frac{\sin 2x}{x}$.

(ii) $\displaystyle \lim_{x \to 0} \frac{\sin ax}{\sin bx}$.

(iii) $\displaystyle \lim_{x \to 0} \frac{\sin^2 2x}{x}$.

(iv) $\displaystyle \lim_{x \to 0} \frac{\sin^2 2x}{x^2}$.

(v) $\displaystyle \lim_{x \to 0} \frac{1 - \cos x}{x^2}$.

(vi) $\displaystyle \lim_{x \to 0} \frac{\tan^2 x + 2x}{x + x^2}$.

(vii) $\displaystyle \lim_{x \to 0} \frac{x \sin x}{1 - \cos x}$.

(viii) $\displaystyle \lim_{h \to 0} \frac{\sin(x + h) - \sin x}{h}$.

(ix) $\displaystyle \lim_{x \to 1} \frac{\sin(x^2 - 1)}{x - 1}$.

(x) $\displaystyle \lim_{x \to 0} \frac{x^2(3 + \sin x)}{(x + \sin x)^2}$.

(xi) $\displaystyle \lim_{x \to 1} (x^2 - 1)^3 \sin\left(\frac{1}{x - 1}\right)^3$.

16. (a) Prove that if $\lim_{x \to a} f(x) = l$, then $\lim_{x \to a} |f|(x) = |l|$.

(b) Prove that if $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$, then $\lim_{x \to a} \max(f, g)(x) = \max(l, m)$ and similarly for min.

17. (a) Prove that $\lim_{x\to 0} 1/x$ does not exist, i.e., show that $\lim_{x\to 0} 1/x = l$ is false for every number l.

(b) Prove that $\lim_{x\to 1} 1/(x-1)$ does not exist.

18. Prove that if $\lim_{x\to a} f(x) = l$, then there is a number $\delta > 0$ and a number M such that $|f(x)| < M$ if $0 < |x-a| < \delta$. (What does this mean pictorially?) Hint: Why does it suffice to prove that $l-1 < f(x) < l+1$ for $0 < |x-a| < \delta$?

19. Prove that if $f(x) = 0$ for irrational x and $f(x) = 1$ for rational x, then $\lim_{x\to a} f(x)$ does not exist for any a.

***20.** Prove that if $f(x) = x$ for rational x, and $f(x) = -x$ for irrational x, then $\lim_{x\to a} f(x)$ does not exist if $a \neq 0$.

21. (a) Prove that if $\lim_{x\to 0} g(x) = 0$, then $\lim_{x\to 0} g(x)\sin 1/x = 0$.

(b) Generalize this fact as follows: If $\lim_{x\to 0} g(x) = 0$ and $|h(x)| \le M$ for all x, then $\lim_{x\to 0} g(x)h(x) = 0$. (Naturally it is unnecessary to do part (a) if you succeed in doing part (b); actually the statement of part (b) may make it easier than (a)—that's one of the values of generalization.)

22. Consider a function f with the following property: if g is any function for which $\lim_{x\to 0} g(x)$ does not exist, then $\lim_{x\to 0}[f(x) + g(x)]$ also does not exist. Prove that this happens if and only if $\lim_{x\to 0} f(x)$ *does* exist. Hint: This is actually very easy: the assumption that $\lim_{x\to 0} f(x)$ does not exist leads to an immediate contradiction if you consider the right g.

****23.** This problem is the analogue of Problem 22 when $f+g$ is replaced by $f\cdot g$. In this case the situation is considerably more complex, and the analysis requires several steps (those in search of an especially challenging problem can attempt an independent solution).

(a) Suppose that $\lim_{x\to 0} f(x)$ exists and is $\neq 0$. Prove that if $\lim_{x\to 0} g(x)$ does not exist, then $\lim_{x\to 0} f(x)g(x)$ also does not exist.

(b) Prove the same result if $\lim_{x\to 0} |f(x)| = \infty$. (The precise definition of this sort of limit is given in Problem 37.)

(c) Prove that if neither of these two conditions holds, then there is a function g such that $\lim_{x\to 0} g(x)$ does not exist, but $\lim_{x\to 0} f(x)g(x)$ does exist. Hint: Consider separately the following two cases: (1) for some $\varepsilon > 0$ we have $|f(x)| > \varepsilon$ for all sufficiently small x. (2) For every $\varepsilon > 0$, there are arbitrarily small x with $|f(x)| < \varepsilon$. In the second case, begin by choosing points x_n with $|x_n| < 1/n$ and $|f(x_n)| < 1/n$.

***24.** Suppose that A_n is, for each natural number n, some *finite* set of numbers in $[0, 1]$, and that A_n and A_m have no members in common if $m \neq n$. Define

f as follows:

$$f(x) = \begin{cases} 1/n, & x \text{ in } A_n \\ 0, & x \text{ not in } A_n \text{ for any } n. \end{cases}$$

Prove that $\lim_{x \to a} f(x) = 0$ for all a in $[0, 1]$.

25. Explain why the following definitions of $\lim_{x \to a} f(x) = l$ are all correct: For every $\delta > 0$ there is an $\varepsilon > 0$ such that, for all x,

(i) if $0 < |x - a| < \varepsilon$, then $|f(x) - l| < \delta$.
(ii) if $0 < |x - a| < \varepsilon$, then $|f(x) - l| \leq \delta$.
(iii) if $0 < |x - a| < \varepsilon$, then $|f(x) - l| < 5\delta$.
(iv) if $0 < |x - a| < \varepsilon/10$, then $|f(x) - l| < \delta$.

***26.** Give examples to show that the following definitions of $\lim_{x \to a} f(x) = l$ are *not* correct.

(a) For all $\delta > 0$ there is an $\varepsilon > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - l| < \varepsilon$.
(b) For all $\varepsilon > 0$ there is a $\delta > 0$ such that if $|f(x) - l| < \varepsilon$, then $0 < |x - a| < \delta$.

27. For each of the functions in Problem 4-17 indicate for which numbers a the one-sided limits $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ exist.

***28.** (a) Do the same for each of the functions in Problem 4-19.
(b) Also consider what happens if decimals ending in 0's are used instead of decimals ending in 9's.

29. Prove that $\lim_{x \to a} f(x)$ exists if $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$.

30. Prove that

(i) $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(-x)$.
(ii) $\lim_{x \to 0} f(|x|) = \lim_{x \to 0^+} f(x)$.
(iii) $\lim_{x \to 0} f(x^2) = \lim_{x \to 0^+} f(x)$.

(These equations, and others like them, are open to several interpretations. They might mean only that the two limits are equal if they both exist; or that if a certain one of the limits exists, the other also exists and is equal to it; or that if either limit exists, then the other exists and is equal to it. Decide for yourself which interpretations are suitable.)

***31.** Suppose that $\lim_{x \to a^-} f(x) < \lim_{x \to a^+} f(x)$. (Draw a picture to illustrate this assertion.) Prove that there is some $\delta > 0$ such that $f(x) < f(y)$ whenever $x < a < y$ and $|x - a| < \delta$ and $|y - a| < \delta$. Is the converse true?

*32. Prove that $\lim_{x\to\infty}(a_nx^n+\cdots+a_0)/(b_mx^m+\cdots+b_0)$ exists if and only if $m \geq n$. What is the limit when $m = n$? When $m > n$? Hint: the one easy limit is $\lim_{x\to\infty} 1/x^k = 0$; do some algebra so that this is the only information you need.

33. Find the following limits.

(i) $\displaystyle\lim_{x\to\infty} \frac{x+\sin^3 x}{5x+6}$.

(ii) $\displaystyle\lim_{x\to\infty} \frac{x\sin x}{x^2+5}$.

(iii) $\displaystyle\lim_{x\to\infty} \sqrt{x^2+x}-x$.

(iv) $\displaystyle\lim_{x\to\infty} \frac{x^2(1+\sin^2 x)}{(x+\sin x)^2}$.

34. Prove that $\lim_{x\to 0^+} f(1/x) = \lim_{x\to\infty} f(x)$.

35. Find the following limits in terms of the number $\alpha = \lim_{x\to 0}(\sin x)/x$.

(i) $\displaystyle\lim_{x\to\infty} \frac{\sin x}{x}$.

(ii) $\displaystyle\lim_{x\to\infty} x\sin\frac{1}{x}$.

36. Define "$\lim_{x\to -\infty} f(x) = l$."

(a) Find $\lim_{x\to -\infty}(a_nx^n+\cdots+a^0)/(b_mx^m+\cdots+b_0)$.
(b) Prove that $\lim_{x\to\infty} f(x) = \lim_{x\to -\infty} f(-x)$.
(c) Prove that $\lim_{x\to 0^-} f(1/x) = \lim_{x\to -\infty} f(x)$.

37. We define $\lim_{x\to a} f(x) = \infty$ to mean that for all N there is a $\delta > 0$ such that, for all x, if $0 < |x-a| < \delta$, then $f(x) > N$. (Draw an appropriate picture!)

(a) Show that $\lim_{x\to 3} 1/(x-3)^2 = \infty$.
(b) Prove that if $f(x) > \varepsilon > 0$ for all x, and $\lim_{x\to a} g(x) = 0$, then

$$\lim_{x\to a} f(x)/|g(x)| = \infty.$$

38. (a) Define $\lim_{x\to a^+} f(x) = \infty$, $\lim_{x\to a^-} f(x) = \infty$, and $\lim_{x\to a} f(x) = \infty$. (Or at least convince yourself that you could write down the definitions if you had the energy. How many other such symbols can you define?)
(b) Prove that $\lim_{x\to 0^+} 1/x = \infty$.
(c) Prove that $\lim_{x\to 0^+} f(x) = \infty$ if and only if $\lim_{x\to\infty} f(1/x) = \infty$.

39. Find the following limits, when they exist.

(i) $\displaystyle\lim_{x\to\infty} \frac{x^3+4x-7}{7x^2-x+1}$

(ii) $\lim\limits_{x \to \infty} x(1 + \sin^2 x)$.

(iii) $\lim\limits_{x \to \infty} x \sin^2 x$.

(iv) $\lim\limits_{x \to \infty} x^2 \sin \dfrac{1}{x}$.

(v) $\lim\limits_{x \to \infty} \sqrt{x^2 + 2x} - x$.

(vi) $\lim\limits_{x \to \infty} x(\sqrt{x+2} - \sqrt{x})$.

(vii) $\lim\limits_{x \to \infty} \dfrac{\sqrt{|x|}}{x}$.

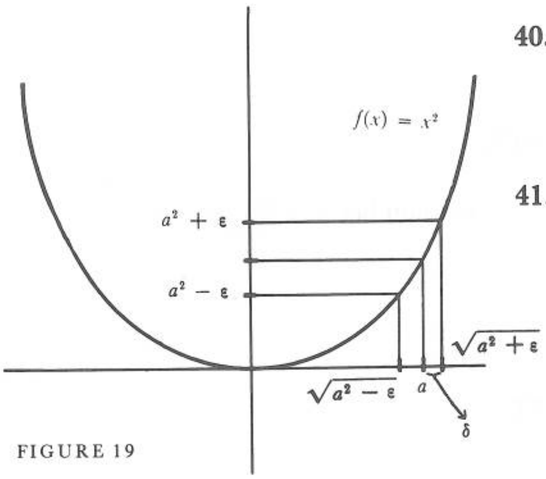

FIGURE 19

40. (a) Find the perimeter of a regular n-gon inscribed in a circle of radius r; use radian measure for any trigonometric functions involved. [Answer: $2rn \sin(\pi/n)$.]

(b) What value does this perimeter approach as n becomes very large?

41. After sending the manuscript for the first edition of this book off to the printer, I thought of a much simpler way to prove that $\lim\limits_{x \to a} x^2 = a^2$ and $\lim\limits_{x \to a} x^3 = a^3$, without going through all the factoring tricks on page 95. Suppose, for example, that we want to prove that $\lim\limits_{x \to a} x^2 = a^2$, where $a > 0$. Given $\varepsilon > 0$, we simply let δ be the minimum of $\sqrt{a^2 + \varepsilon} - a$ and $a - \sqrt{a^2 - \varepsilon}$ (see Figure 19); then $|x - a| < \delta$ implies that $\sqrt{a^2 - \varepsilon} < x < \sqrt{a^2 + \varepsilon}$, so $a^2 - \varepsilon < x^2 < a^2 + \varepsilon$, or $|x^2 - a^2| < \varepsilon$. It is fortunate that these pages had already been set, so that I couldn't make these changes, because this "proof" is completely fallacious. Wherein lies the fallacy?