Homework 13

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1 Problem 19

1.1 Part a

EXACT INDSET = $\{(G, k) \mid$ The largest indep. set in G has size exactly $k\}$
EXACT INDSET = $\{(G, k) \mid \forall r \exists s$ (s is an indep. set in G and $|s| = k) \wedge$ (r is not an indep. set or $|r| \leq |s|)\}$

By definition of Π_2^p,
$T \leftarrow \{(G, k)$

s

r$\}$

where s is the advice tape and r is the disadvice tape.

A trivial check whether a set is independent will take n iterations of (n-1) comparisons, which is polynomial. Therefore, there does exist a k where $T(x, s, r)$ will accept in $|x|^k$ time.

Therefore, EXACT INDSET $\in \Pi_2^p$

1.2 Part b

SUCCINCT SET COVER = $\{(S, n, k) \mid \exists S' \ \forall i \ S' \subseteq \{1, 2, ..., |S|\} \wedge |S'| \leq k \wedge \vee_{i \in S'} \varphi_i\}$

By definition of Σ_2^p,
$T \leftarrow \{(S, n, k)$

S'

i$\}$

where S' is the advice tape and i is the disadvice tape.

Checking whether a statement is a tautology takes polynomial time. Therefore, there does exist a k where $T(x, S', i)$ will accept in $|x|^k$ time.

Therefore, SUCCINCT SET COVER $\in \Sigma_2^p$

1.3 Part c

VC-DIMENSION = $\{(C,k) \mid$ C is a collection S such that $VC(C) \geq$ k$\}$
VC-DIMENSION = $\{(C,k) \mid \exists X \forall s \forall X' \exists i$ where $X \subseteq U$, $s \subseteq U$, $|X| \geq |s|$, $X' \subseteq X$, $S_i \cap X = X'$ $\}$

By definition of Σ_3^p,
$T \leftarrow \{(C,k)$

X

(s, X')

$i\}$

All qualifiers take polynomial time, therefore there does exist a k where $T(x, X, (s, X'), i)$ will accept in $|x|^k$ time.
Therefore, VC-DIMENSION $\in \Sigma_3^p$

1.4 Part d

Let $L_1 = \{(G,k) \mid \exists s($s is an independent set in G and $|s| = k)\}$
Let $L_2 = \{(G,k) \mid \forall r($r is not an independent set in G or $|r| \leq k)\}$
Let $L = L_1 \cap L_2$

L = EXACT INDSET. Since $L_1 \in \Sigma_1^p$ and $L_2 \in \Pi_1^p$, $L \in DP$.

2 Problem 20

$3SAT$ is NP-complete and has been proven as such. $\overline{3SAT}$ is coNP-complete by property that negation of quantifiers in Σ_1^p yield the quantifiers in Π_1^p.

If $3SAT$ is reducible to $\overline{3SAT}$ under poly time reductions, then there exists polynomial time conversions between the two classes. As a result, $\Sigma_1^p \subseteq \Pi_1^p$ since all Σ_1^p problems can be converted into Π_1^p problems, and $\Pi_1^p \subseteq \Sigma_1^p$ since all Π_1^p problems can be converted into Σ_1^p problems. Therefore, $\Sigma_1^p = \Pi_1^p$.

For a fixed value of the first quantifier in Σ_n^p, there exists a Π_{n-1}^p, making $\Sigma_n^p = \Sigma_1^p \Pi_{n-1}^p$. This can be expanded further into the form $\Sigma_1^p \Pi_1^p ... \Sigma_1^p$. Further, since $\Sigma_1^p = \Pi_1^p$, $\Sigma_n^p = (\Sigma_1^p)^n$ Each Σ_1^p has its own machine with access to an oracle machine. We can construct an NDTM M which is the concatenation of all of these machines. If M accepts, then the Σ_n^p problem is accepted. M non-deterministically decides Σ_n^p in polynomial time. Therefore, Σ_n^p can be reduced to a Σ_1^p problem. Therefore, $\Sigma_n^p \subseteq \Sigma_1^p$. Therefore, $\Sigma_n^p = \Sigma_1^p$. This logic holds for Π_n^p.

The result is all problems in the PH can be reduced to an NP problem. Therefore, if $3SAT$ is reducible to $\overline{3SAT}$, $PH = NP$.