Mirror Symmetry
Summer 2016 Seminar Notes

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Contents

Chapter 1

Mathematical Preliminaries

The aim of this chapter is to give a brief review of the required mathematical background for mirror symmetry.

1.1 Cohomology Theories

Throughout this section, X is a complex manifold, and H_{dR} is de Rham cohomology. We examine the relationships between some common cohomology theories on X.

1.1.1 Sheaf Cohomology

All of our sheaves take values in abelian groups. Let $\mathcal{F}$ be a presheaf on X.

Definition 1.1.1. Recall the definition of a **presheaf** $\mathcal{F}$ on X:

1. (presheaf) every open set U in X is assigned an abelian group $\mathcal{F}(U)$, such that if $V \subseteq U$ are two open sets, there is a restriction map $-|_{U\to V} : \mathcal{F}(U) \to \mathcal{F}(V)$ compatible with inclusion, i.e. $(-|_{U\to V})|_{V\to W} = -|_{U\to W}$ for any $W \subseteq V \subseteq U$.

If in addition $\mathcal{F}$ satisfies the following two properties, it is a **sheaf**:

2. (locality) if $\{U_\alpha\}$ is an open cover of X and $f, g \in \mathcal{F}(X)$ such that $f|_{U\to U_\alpha} = g|_{U\to U_\alpha}$ for every U_α, then $f = g$;

3. (gluing) if $\{U_\alpha\}$ is an open cover of X and $f_\alpha \in \mathcal{F}(U_\alpha)$ for every U_α are elements agreeing on overlaps, i.e. such that $f_\alpha|_{U_\alpha\to U_\alpha\cap U_\beta} = f_\beta|_{U_\beta\to U_\alpha\cap U_\beta}$, then we can glue the f_α together to get $f \in \mathcal{F}(X)$, i.e. $f|_{X\to U_\alpha} = f_\alpha$ for every U_α.

Definition 1.1.2. Let $\mathcal{U} = \{U_\alpha\}$ be an **ordered open cover** of X, i.e. with a partial order such that if α and β are incomparable then $U_\alpha \cap U_\beta$ is empty. A p**-simplex** σ of $\mathcal{U}$ is a totally ordered collection of open sets $U_{\alpha_0}, \dots, U_{\alpha_p} \in \mathcal{U}$; we call $U_{\alpha_0,\dots,\alpha_p} := U_{\alpha_0} \cap \dots \cap U_{\alpha_p}$ its **support**, and often refer to σ by it instead. The k**-th boundary component** of a p-simplex $U_{\alpha_0,\dots,\alpha_p}$ is given by $\partial_k U_{\alpha_0,\dots,\alpha_p} := U_{\alpha_0,\dots,\hat{\alpha}_k,\dots,\alpha_p}$. Cochains are maps from simplices to sheaf sections, and form a cochain complex:

$$C^p(\mathcal{U}, \mathcal{F}) := \prod_{\alpha_0<\dots<\alpha_p} \mathcal{F}(U_{\alpha_0,\dots,\alpha_p}), \quad (\delta^p\omega)(\sigma) := \sum_{k=0}^{p+1} (-1)^k \omega(\partial_k\sigma)|_\sigma : C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F}).$$

The **Čech cohomology of** $\mathcal{U}$ with coefficients in $\mathcal{F}$, denoted $\check{H}^\bullet(\mathcal{U}, \mathcal{F})$, is the cohomology of this complex.

Example 1.1.3. Let $\mathcal{F}$ be a sheaf. By the gluing condition for a sheaf, a global section $f \in \mathcal{F}(X)$ is defined by its values $f_\alpha := f|_{X \to U_\alpha} \in \mathcal{F}(U_\alpha)$ on every U_α in an open cover. These f_α form precisely the data for an element of $C^0(\mathcal{U}, \mathcal{F})$, and satisfy the gluing condition $f_\alpha = f_\beta$ on $U_\alpha \cap U_\beta$, which is precisely the statement $\delta_0 f = 0$. Hence $\check{H}^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$ for a sheaf $\mathcal{F}$.

Example 1.1.4. Let $\mathcal{O}$ denote the sheaf of holomorphic functions (and $\mathcal{O}^*$ the nowhere-zero ones) on $\mathbb{P}^1$. Recall that on $\mathbb{P}^1$ we have the charts $U = \mathbb{P}^1 \setminus \{S\}$ and $V = \mathbb{P}^1 \setminus \{N\}$, with coordinates u and v respectively. To look at sections of $\mathcal{O}(U)$ versus $\mathcal{O}(V)$, we use the transition map $v = u^{-1}$ on $U \cap V$. The cochains for this open cover are

$$C^0(\mathcal{U}, \mathcal{O}) = \mathcal{O}(U) \times \mathcal{O}(V), \quad C^1(\mathcal{U}, \mathcal{O}) = \mathcal{O}(U \cap V), \quad C^k(\mathcal{U}, \mathcal{O}) = 0 \; \forall k \geq 2.$$

We compute the sheaf cohomology.

1. ($\check{H}^0(\mathcal{U}, \mathcal{O})$) The boundary map δ_0 maps $(f, g) \in C^0(\mathcal{U}, \mathcal{O})$ to $g - f$. But

$$f = \sum_{k=0}^{\infty} f_k u^k, \quad g = \sum_{k=0}^{\infty} g_k v^k = \sum_{k=0}^{\infty} g_k u^{-k},$$

   so $g - f = 0$ iff $f_k = g_k = 0$ for all $k > 0$, and $f_0 = g_0$. Hence $\check{H}^0(\mathcal{U}, \mathcal{O}) \cong \mathbb{C}$, consisting of all constant functions.

2. ($\check{H}^1(\mathcal{U}, \mathcal{O})$) Given $h \in C^1(\mathcal{U}, \mathcal{O})$, rewrite its Laurent expansion:

$$h = \sum_{k=-\infty}^{\infty} h_k u^k = \sum_{k=0}^{\infty} h_k u^k + \sum_{k=1}^{\infty} h_k v^k = -f + g$$

   where $f \in \mathcal{O}(U)$ and $g \in \mathcal{O}(V)$. Hence $h \in \operatorname{im} \delta_0$, and $\check{H}^1(\mathcal{U}, \mathcal{O}) = 0$.

3. ($\check{H}^k(\mathcal{U}, \mathcal{O})$) Trivially, $\check{H}^k(\mathcal{U}, \mathcal{O}) = 0$ for $k \geq 2$.

Note that $\check{H}^0(\mathcal{U}, \mathcal{O}) \cong \mathbb{C}$ is consistent with what we know so far, since $\check{H}^0(\mathcal{U}, \mathcal{O}) = \mathcal{O}(\mathbb{P}^1)$, and Liouville's theorem shows that $\mathcal{O}(\mathbb{P}^1)$ can only contain constant functions.

Example 1.1.5. Recall the tautological line bundle $\mathcal{O}(-1)$ and its dual $\mathcal{O}(1)$ on $\mathbb{P}^n$; we have $\mathcal{O}(n) = \mathcal{O}(1)^n$. On the same charts on $\mathbb{P}^1$, since $\mathcal{O}(1)$ has transition function $u = v^{-1}$, we know $\mathcal{O}(n)$ has transition function $u^n = v^{-n}$. To construct a global section of $\mathcal{O}(n)$, given a monomial v^k on V, we require $u^n v^k = u^{n-k}$ to be well-defined on U, so $k \leq n$. In homogeneous coordinates $[x_0 : x_1]$, the global sections are therefore $x_0^n, x_0^{n-1} x_1, \ldots, x_1^n$, the homogeneous polynomials of degree n. The same story holds on $\mathbb{P}^N$. Hence $\dim H^0(\mathbb{P}^N, \mathcal{O}(n)) = \binom{N+n-1}{n-1}$. In particular, there are $\binom{9}{5} = 126$ independent global sections of $\mathcal{O}_{\mathbb{P}^4}(5)$.

Definition 1.1.6. The set of all open covers of X form a directed set under refinement. The **Čech cohomology of** X with coefficients in $\mathcal{F}$ is the direct limit $\check{H}^n(X, \mathcal{F}) := \varinjlim_{\mathcal{U}} \check{H}^n(\mathcal{U}, \mathcal{F})$.

Definition 1.1.7. An ordered open cover $\{U_\alpha\}$ is **good** if it is countable and every finite intersection $U_{\alpha_0, \ldots, \alpha_p}$ is either empty or contractible.

Theorem 1.1.8 ([1, Corollary of Theorem 5.4.1])**.** *The Čech cohomology of a good cover $\mathcal{U}$ is isomorphic to the Čech cohomology of X.*

One can define **sheaf cohomology** $H^n(X, \mathcal{F})$ as the right derived functors of the global sections functor Γ_X (i.e. $\mathcal{F} \mapsto \mathcal{F}(X)$). For us, Čech and sheaf cohomology are indistinguishable as long as we work with sheaves.

Theorem 1.1.9 ([1, Theorem 5.10.1])**.** *If X is a paracompact topological space, then Čech cohomology $\check{H}^n(X, \mathcal{F})$ and sheaf cohomology $H^n(X, \mathcal{F})$ are isomorphic for any sheaf $\mathcal{F}$.*

Čech cohomology is also directly related to de Rham cohomology, and, as we shall see, Dolbeault cohomology in the complex case. So we can think of Čech cohomology classes as forms.

Theorem 1.1.10 (Čech–de Rham isomorphism)**.** *Let $\mathbb{R}$ denote the constant sheaf. There is a canonical isomorphism $\check{H}^k(X, \mathbb{R}) \cong H^k_{\mathrm{dR}}(X)$ for each k.*

Proof. By the Poincaré lemma, the **de Rham complex** of sheaves

$$0 \to \mathbb{R} \xrightarrow{\subseteq} \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \Omega^2(X) \xrightarrow{d} \cdots$$

is exact (by checking exactness on the stalks). Let $Z^n := \ker(d\colon \Omega^k \to \Omega^{k+1})$. The de Rham complex splits into a bunch of short exact sequences:

$$0 \to d\Omega^{k-1} \cong Z^k \xrightarrow{\subseteq} \Omega^k \xrightarrow{d} Z^{k+1} \to 0.$$

To each such short exact sequence is associated a long exact sequence of (sheaf) cohomology:

$$0 \to H^0(X, Z^k) \to H^0(X, \Omega^k) \to H^0(X, Z^{k+1}) \to H^1(X, Z^k) \to H^1(X, \Omega^k) \to H^1(X, Z^{k+1}) \to \cdots .$$

Fact: $H^i(X, \Omega^k) = 0$ for every k and $i > 0$ (since Ω^k is a fine sheaf). Hence we obtain isomorphisms

$$H^{i+1}(X, Z^0) \cong H^i(X, Z^1) \cong \cdots \cong H^1(X, Z^i).$$

But $Z^0 \cong \mathbb{R}$ and we more commonly write

$$H^1(X, Z^i) = \operatorname{coker}(H^0(X, \Omega^i) \to H^0(X, Z^{i+1})) = Z^{i+1}(X)/d\Omega^k(X) = H^{i+1}_{\mathrm{dR}}(X).$$

Hence $\check{H}^{i+1}(X, \mathbb{R}) \cong H^{i+1}_{\mathrm{dR}}(X)$. □

Definition 1.1.11. Let E be a holomorphic vector bundle on X and $\Omega^{0,q}(E) := \Omega^{0,q}(X) \otimes \Gamma(E)$ denote the space of E-valued $(0, q)$-forms. **Dolbeault cohomology** $H^q_{\bar\partial}(E)$ is the cohomology of the complex

$$\cdots \xrightarrow{\bar\partial} \Omega^{0,q}(E) \xrightarrow{\bar\partial} \Omega^{0,q+1}(E) \xrightarrow{\bar\partial} \Omega^{0,q+2}(E) \xrightarrow{\bar\partial} \cdots .$$

We write $H^{p,q}_{\bar\partial}(X)$ for $E = \Lambda^p T^{1,0}X$. The dimensions $h^{p,q}(X) := \dim_{\mathbb{C}} H^{p,q}_{\bar\partial}(X)$ are the **Hodge numbers** of X.

Lemma 1.1.12 ($\bar\partial$-Poincaré lemma)**.** *$\bar\partial$-closed, i.e. holomorphic, implies $\bar\partial$-exact on $\mathbb{C}^n$.*

Theorem 1.1.13 (Čech–Dolbeault isomorphism)**.** *Let $\Omega^{p,0}$ denote the sheaf of holomorphic p-forms on X. There is a natural isomorphism $\check{H}^q(X, \Omega^{p,0}) \cong H^{p,q}_{\bar\partial}(X)$.*

Proof. Analogous to the proof of the Čech–de Rham isomorphism, except now using the $\bar\partial$-Poincaré lemma to establish the exactness of the complex

$$0 \to \ker(\bar\partial\colon \Omega^{p,0}(X) \to \Omega^{p,1}(X)) \xrightarrow{\subseteq} \Omega^{p,0}(X) \xrightarrow{\bar\partial} \Omega^{p,1}(X) \xrightarrow{\bar\partial} \cdots .$$

□

Definition 1.1.14. Consider the double complex $\Omega^{\bullet,\bullet}$ with differentials ∂ and $\bar\partial$. The **Frölicher spectral sequence** is the spectral sequence of a double complex associated to $\Omega^{\bullet,\bullet}$. Since the total complex of $\Omega^{\bullet,\bullet}$ is $\Omega^\bullet(X)$, the Frölicher spectral sequence converges to complex de Rham cohomology $H^\bullet_{\mathrm{dR}}(X, \mathbb{C})$.

1.1.2 Morse Homology

Throughout this subsection, $f\colon X \to \mathbb{R}$ is a smooth function, and we equip X, viewed as a real manifold, with a Riemannian metric g. We also assume (f, g) is Morse–Smale, defined below.

Definition 1.1.15. A **critical point** of f is a point $p \in X$ with $df_p = 0$. Define the **Hessian**

$$H(f)_p\colon T_pX \to T_p^*X, \quad v \mapsto \nabla_v(df),$$

which is independent of the choice of connection ∇. (In coordinates, we recover the usual $\partial^2 f/\partial x_i \partial x_j$.) The critical point p is **non-degenerate** if the Hessian does not have zero eigenvalues. A non-degenerate critical point p has **Morse index** $\operatorname{ind}(p)$ the number of negative eigenvalues of the Hessian. The function f is **Morse** if all of its critical points are non-degenerate.

Definition 1.1.16. Recall that the **gradient** of f with respect to a metric g is the vector field $\operatorname{grad} f$ such that $g(\operatorname{grad} f, X) = Xf$. Equivalently, $\operatorname{grad} f = (df)^\sharp$. Let $\psi_t\colon X \to X$ be the one-parameter group of diffeomorphisms associated to the flow of $-\operatorname{grad} f$. The **descending manifold** $D(p)$ and **ascending manifold** $A(p)$ at a critical point p are

$$\begin{aligned} D(p) &\coloneqq \{x \in X : \lim_{t\to-\infty} \psi_t(x) = p\} \\ A(p) &\coloneqq \{x \in X : \lim_{t\to+\infty} \psi_t(x) = p\}. \end{aligned}$$

The pair (f, g) is **Morse–Smale** if f is Morse and $D(p)$ is transverse to $A(q)$ for every pair of critical points p and q (i.e. tangent spaces of $D(p)$ and $A(q)$ generate the tangent space at every intersection point).

Here are two useful and easy-to-prove facts: every flow line asymptotically approaches critical points, and $\dim D(p) = \operatorname{ind}(p)$ (so $\dim A(p) = \dim X - \operatorname{ind} p$ by the Morse–Smale condition).

Definition 1.1.17. Fix critical points p and q. A **flow line** from p to q is an integral curve $\gamma(t)$ of $-\operatorname{grad} f$ with $\lim_{t\to-\infty}\gamma(t) = p$ and $\lim_{t\to+\infty}\gamma(t) = q$. The **moduli space of flow lines** from p to q is

$$\begin{aligned} \mathcal{M}(p, q) &\coloneqq \{\text{flow lines from } p \text{ to } q\}/\sim, \quad \alpha \sim \beta \text{ if } \exists c \in \mathbb{R} : \alpha(t) = \beta(t + c) \\ &= (D(p) \cap A(q))/\mathbb{R}. \end{aligned}$$

A **broken flow line** consists, piecewise, of flow lines.

The Morse–Smale condition implies $D(p) \cap A(q)$ is a submanifold of X with dimension $\operatorname{ind}(p) - \operatorname{ind}(q)$. Since $\sim$ is a smooth, proper, free $\mathbb{R}$-action, $\mathcal{M}(p, q)$ is a manifold of dimension $\operatorname{ind}(p) - \operatorname{ind}(q) - 1$ when $p \neq q$ (otherwise the $\mathbb{R}$-action is trivial). Note that if $\operatorname{ind}(p) = k$ and $\operatorname{ind}(q) = k - 1$ then $\mathcal{M}(p, q)$ is zero-dimensional. In fact, in this case, $\mathcal{M}(p, q)$ is compact as a corollary of the following theorem, and therefore is a finite set of points.

Theorem 1.1.18 ([2, Theorem 2.1])**.** *Let X be closed and (f, g) Morse–Smale. Then $\mathcal{M}(p, q)$ has a natural compactification to a smooth manifold with corners $\overline{\mathcal{M}(p, q)}$ where*

$$\overline{\mathcal{M}(p, q)} \setminus \mathcal{M}(p, q) = \bigcup_{k\geq 1} \bigcup_{\substack{p, r_1, \ldots, r_k, q \\ \textit{distinct crit pts}}} \mathcal{M}(p, r_1) \times \mathcal{M}(r_1, r_2) \times \cdots \times \mathcal{M}(r_{k-1}, r_k) \times \mathcal{M}(r_k, q).$$

Corollary 1.1.19. *If $\operatorname{ind}(p) - \operatorname{ind}(q) = 1$, then $\overline{\mathcal{M}(p, q)} = \mathcal{M}(p, q)$ is compact. If $\operatorname{ind}(p) - \operatorname{ind}(q) = 2$, then*

$$\partial\overline{\mathcal{M}(p, q)} = \bigcup_{\operatorname{ind}(r) = \operatorname{ind}(p) - 1} \mathcal{M}(p, r) \times \mathcal{M}(r, q).$$

Proof. Since $\dim \mathcal{M}(r, s) = \operatorname{ind}(r) - \operatorname{ind}(s) - 1$, the space $\mathcal{M}(r, s)$ is non-empty only if $\operatorname{ind}(r) - \operatorname{ind}(s) \geq 1$. Hence $\overline{\mathcal{M}(p, q)} \setminus \mathcal{M}(p, q) = \emptyset$ when $\operatorname{ind}(p) - \operatorname{ind}(q) = 1$. Similar reasoning shows the $\operatorname{ind}(p) - \operatorname{ind}(q) = 2$ case. $\square$

Definition 1.1.20. Fix orientations for $D(p)$ at every critical point p. There is an isomorphism at $x \in \gamma \in \mathcal{M}(p,q)$ given by

$$\begin{aligned}
T_x D(p) &\cong T_x(D(p) \cap A(q)) \oplus (T_x X / T_x A(q)) && \text{transversality from Morse–Smale} \\
&\cong T_\gamma \mathcal{M}(p,q) \oplus T_x \gamma \oplus (T_x X / T_x A(q)) && \text{definition of } \mathcal{M}(p,q) \\
&\cong T_\gamma \mathcal{M}(p,q) \oplus T_x \gamma \oplus T_q D(q) && \text{translating } T_q D(q) \text{ along } \gamma.
\end{aligned}$$

The **orientation** on $\mathcal{M}(p,q)$ is such that this isomorphism is orientation-preserving. Let C_k be the free abelian group generated by critical points of index k, and define the **Morse–Smale–Witten boundary map**

$$\partial_k^{\text{Morse}} \colon C_k \to C_{k-1}, \quad p \mapsto \sum_{\text{ind } q = k-1} \#\mathcal{M}(p,q) q$$

where $\#\mathcal{M}(p,q) \in \mathbb{Z}$ is counted with sign according to the orientation of $\mathcal{M}(p,q)$, which here is a discrete set of points.

Lemma 1.1.21. *$(\partial_k^{\text{Morse}})^2 = 0$, so $(C_\bullet, \partial^{\text{Morse}})$ is a chain complex.*

Proof. Let $\text{ind}(p) - \text{ind}(q) = 2$. The coefficient of q in $(\partial^{\text{Morse}})^2 p$ is

$$\sum_{\text{ind}(r) = \text{ind}(p)-1} \#\mathcal{M}(p,r) \cdot \#\mathcal{M}(r,q) = \# \bigcup_{\text{ind}(r)=\text{ind}(p)-1} \mathcal{M}(p,r) \times \mathcal{M}(r,q) = \#\partial\overline{\mathcal{M}(p,q)}.$$

Since $\overline{\mathcal{M}(p,q)}$ is an oriented 1-manifold with boundary, this quantity, the number of boundary points, is zero. □

Definition 1.1.22. **Morse homology** $H_\bullet^{\text{Morse}}(f,g)$ is the homology of the **Morse–Smale–Witten complex** $(C_\bullet, \partial^{\text{Morse}})$.

Example 1.1.23. The (upright) torus T^2 has four critical points with f the height function: p (index 2), q and r (index 1), and s (index 0). This choice of f is Morse, but with the induced metric g from $\mathbb{R}^3$, the pair (f,g) is not Morse–Smale: $D(q) \cap A(r)$ is non-empty, but transversality forces it to be. The solution is to tilt the torus a little; equivalently, perturb g. There are two flow lines, of opposite sign, for each relevant pair of critical points. Hence $\partial_k^{\text{Morse}} = 0$ for $k = 1, 2$. It follows that

$$H_2^{\text{Morse}}(f,g) = \mathbb{Z}, \quad H_1^{\text{Morse}}(f,g) = \mathbb{Z}^2, \quad H_0^{\text{Morse}}(f,g) = \mathbb{Z}.$$

Theorem 1.1.24 ([2, Theorem 3.1])**.** *Let X be a closed smooth manifold, $H_\bullet(X)$ denote singular homology on X, and (f,g) be a Morse–Smale pair on X. Then there is a canonical isomorphism $H_n^{\text{Morse}}(f,g) \cong H_n(X)$.*

Corollary 1.1.25. *The number of critical points of a Morse function is at least the sum $\sum_k \dim H_k(X)$ of the Betti numbers.*

Proof. The number of critical points is the sum of the dimensions of the Morse chain groups, which is at least the sum of the dimensions of the Morse homology groups, which is equal to the sum of the dimensions of the singular homology groups. □

The infinite-dimensional analogue of Morse homology is known as **Floer homology**. We shall primarily be concerned with Floer homology for mirror symmetry.

1.1.3 Equivariant Cohomology

1.2 Algebraic Topology

We stop distinguishing between isomorphic cohomology theories now. In particular, since X is always at least a smooth manifold, we think of singular cohomology $H^k(X)$ as de Rham cohomology. For a bundle E, $H^k(E)$ refers to sheaf cohomology.

1.2.1 Poincaré and Serre Duality

Unless otherwise stated, X in this section is a compact oriented n-manifold.

Theorem 1.2.1 (Poincaré duality, [3]). *Let X be a compact oriented n-manifold. The map*

$$\int_X : H^k(X) \otimes H^{n-k}(X) \to \mathbb{R}, \quad \omega \otimes \eta \mapsto \int_X \omega \wedge \eta$$

is a perfect pairing, and hence $H^k(X) \cong H^{n-k}(X)^*$.

If we relax the assumption that X is compact, then the issue is that $\int_X$ may not be well-defined. We work around this by using de Rham cohomology with compact support.

Definition 1.2.2. Let $\Omega_c^k(X)$ denote the k-forms on X with compact support. The **de Rham cohomology groups with compact support** $H_c^n(X)$ are the cohomology of the chain complex $(\Omega_c^\bullet(X), d)$.

Theorem 1.2.3 (Poincaré duality for non-compact manifolds, [3]). *Let X be an oriented n-manifold without boundary. The map*

$$\int_X : H^k(X) \otimes H_c^{n-k}(X) \to \mathbb{R}, \quad \omega \otimes \eta \mapsto \int_X \omega \wedge \eta$$

is a perfect pairing, and hence $H^k(X) \cong H_c^{n-k}(X)^*$.

Definition 1.2.4. Fix $C \subset X$ a closed $(n-k)$-submanifold. Then Poincaré duality identifies the map $\int_C : H^{n-k}(X) \to \mathbb{R}$ with a k-form $\eta_C \in H^k(X)$, called the **Poincaré dual class**. Explicitly, $\int_C \omega = \int_X \omega \wedge \eta_C$.

There is a relation between the Poincaré dual class and the Thom class, which we define below. Namely, the Poincaré dual class of C can be constructed as the Thom class of the normal bundle of C in X.

Theorem 1.2.5 ([4, Theorem 10.4]). *Let $\pi : E \to B$ be an oriented rank n real vector bundle and B is embedded into E as the zero section. Then*

1. *there exists a unique cohomology class $\Phi \in H^n(E, E \setminus B)$ called the* **Thom class** *such that for every $x \in B$, the restriction of Φ to $H^n(E_x, E_x \setminus \{0\})$ is the preferred generator specified by the orientation of E_x in E;*
2. *the* **Thom isomorphism** *: $H^k(E) \to H^{k+n}(E, E \setminus B)$, given by $\omega \mapsto \omega \wedge \Phi$, is an isomorphism for every k.*

Note that since B is a deformation retract of E, the rings $H^*(E)$ and $H^*(B)$ are isomorphic. Hence $\pi^*\Phi = 1 \in H^*(B)$, which shall be very important in the upcoming proof.

Theorem 1.2.6 (Tubular neighborhood theorem, [4, Theorem 11.1]). *Let $C \subset X$ be a k-submanifold embedded in X. There exists an open neighborhood, called a* **tubular neighborhood***, of C in X diffeomorphic to the total space of the normal bundle of C. This diffeomorphism maps points in C to zero vectors.*

Proposition 1.2.7 ([3, Proposition 6.24a]). *Let $C \subset X$ be a closed $(n-k)$-submanifold. The Poincaré dual class $\eta_C \in H^k(X)$ of C is the Thom class of the normal bundle of C in X.*

Proof. Let NC denote the normal bundle of C in X, which has rank k because C is codimension k. Use the tubular neighborhood theorem to identify NC with an open neighborhood T of C in X, and then extend by zero to get $\Phi \in H^k(X)$ supported on T.

We shall show that $\int_X \omega \wedge \Phi = \int_C \omega$ for any $\omega \in H_c^{n-k}(X)$. The maps $\pi\colon T \to C$ and $\iota\colon C \to T$ induce isomorphisms of cohomology, so on forms ω and $\pi^*\iota^*\omega$ differ by at most an exact form $d\tau$. Then

$$\begin{aligned} \int_X \omega \wedge \Phi = \int_T \omega \wedge \Phi &= \int_T (\pi^*\iota^*\omega + d\tau) \wedge \Phi \\ &= \int_T \pi^*\iota^*\omega \wedge \Phi = \int_C \iota^*\omega \wedge \pi^*\Phi = \int_C \iota^*\omega. \end{aligned}$$ □

Corollary 1.2.8. *Transverse intersection is Poincaré dual to the wedge product, i.e. for $C, D \subset X$ closed submanifolds intersecting transversally, $\eta_{C\cap D} = \eta_C \wedge \eta_D$.*

Proof. For transversal intersections, codimension is additive: $\operatorname{codim} C \cap D = \operatorname{codim} C + \operatorname{codim} D$. So the normal bundle of the intersection is $N(C \cap D) = NC \oplus ND$. Let $\Phi(E)$ denote the Thom class associated to the vector bundle E. By the characterization of the Thom class, for vector bundles E and F we have $\Phi(E \oplus F) = \Phi(E) \wedge \Phi(F)$; check that $\Phi(E) \oplus \Phi(F)$ restricts on each fiber to the preferred generator. Hence

$$\eta_{C\cap D} = \Phi(N_{C\cap D}) = \Phi(NC \oplus ND) = \Phi(NC) \wedge \Phi(ND) = \eta_C \wedge \eta_D.$$ □

Let X be a complex n-fold now. In the complex setting, we can refine Poincaré duality. The Čech–Dolbeault isomorphism 1.1.13 works for the more general setting in which we defined Dolbeault cohomology: if E is a holomorphic vector bundle over X, then $H^k(X, E) \cong H^k_{\bar\partial}(E)$. So we think of Čech cohomology classes $H^k(X, E)$ as E-valued $(0, k)$-forms.

Definition 1.2.9. The **canonical bundle** K_X of a complex n-fold X is the vector bundle of $(n, 0)$-forms. (Also commonly denoted $\Omega^n(X)$.)

Hence $H^{n-k}(X, E^* \otimes K_X)$ consists of E^*-valued $(n, n-k)$-forms. Given such a form ω and another form $\eta \in H^k(X, E)$, the form $\omega \wedge \eta$ is an (n, n)-form with complex coefficients. We can integrate it to get something in $\mathbb{C}$. At this point it is impossible not to wonder about whether the pairing $H^k(E) \otimes H^{n-k}(X, E^* \otimes K_X) \to \mathbb{C}$ given by wedging and then integrating is perfect.

Theorem 1.2.10 (Serre duality, [5, Corollary III.7.13]). *The pairing $H^k(X, E) \otimes H^{n-k}(X, E^* \otimes K_X) \to \mathbb{C}$ is perfect, so $H^k(X, E) \cong H^{n-k}(X, E^* \otimes K_X)^*$.*

Poincaré duality combined with Hodge decomposition gives

$$\bigoplus_{p+q=k} H^q(X, \Omega^p) = H^k(X, \mathbb{C}) \cong H^{2n-k}(X, \mathbb{C}) = \bigoplus_{p'+q'=2n-k} H^{q'}(X, \Omega^{p'}) = \bigoplus_{p+q=k} H^{n-q}(X, \Omega^{n-p}).$$

Serre duality says that in fact, each of the terms in the sum are isomorphic: $H^q(X, \Omega^p) \cong H^{n-q}(X, \Omega^{n-p})$.

1.2.2 Chern Classes via Chern–Weil Theory

For this subsection, we work over $\mathbb{C}$, and every vector bundle we consider is smooth and complex. We define Chern classes using the Chern–Weil approach. There are other equivalent approaches in more general settings. But for us, we take $\pi\colon E \to X$ to be a rank-n smooth complex vector bundle over a smooth manifold X. A connection $A \in \Omega^1(X, \operatorname{Ad} E)$ on E gives a curvature $F_A \coloneqq dA + A \wedge A \in \Omega^2(X, \operatorname{Ad} E)$.

Definition 1.2.11. The **total Chern class** of E is

$$\begin{aligned} c(E) &\coloneqq \det\left(1 + \frac{i}{2\pi} F\right) = 1 + \frac{i}{2\pi}\operatorname{tr}(F) + \frac{1}{8\pi}(\operatorname{tr}(F^2) - \operatorname{tr}(F)^2) + \cdots \\ &= 1 + c_1(E) + c_2(E) + \cdots \in H^0(X, \mathbb{R}) \oplus H^2(X, \mathbb{R}) \oplus \cdots . \end{aligned}$$

Its terms $c_k(E) \in H^{2k}(X, \mathbb{R})$ are the **Chern classes**. The total Chern class $c(X)$ of X is defined as $c(X) \coloneqq c(T^{1,0}X)$.

Theorem 1.2.12 (Chern–Weil theorem, [6, Corollary 4.4.5, Lemma 4.4.6])**.** *The total Chern class $c(E)$ is closed and independent of the choice of connection A on E.*

Example 1.2.13. A magnetic monopole at the origin in $U(1)$ Maxwell theory is given by the trivial line bundle on $\mathbb{R}^3$ with connection $A = i\frac{1}{2r}\frac{1}{z-r}(xdy - ydx)$, where r is the coordinate on the fibers of the bundle. Then

$$F_A = i\frac{1}{2r^3}(xdy \wedge dz + ydz \wedge dx + zdx \wedge dy) = -\frac{i}{2r^2}(r^2 \sin\theta d\theta \wedge d\phi).$$

We easily check that $\int_{S^2} c_1 = \frac{i}{4\pi}\int_{S^2} F_A = 1$ for any 2-sphere around the origin.

Theorem 1.2.14. *The Chern classes satisfy and are uniquely determined by the following properties:*

1. $c_0(E) = 1$ *and* $c_k(E) = 0$ *if* $k > \dim E$;
2. *(Naturality) if* $f\colon Y \to X$ *is continuous, then* $f^*c(E) = c(f^*E)$;
3. *(Whitney product formula)* $c(E \oplus F) = c(E) \wedge c(F)$;
4. $c_1(\mathcal{O}_{\mathbb{P}^1}(-1))$ *is minus the preferred generator (given by the orientation) of* $H^2(\mathbb{P}^1)$.

Proof. Property 1 is clear from the definition of $c_k(E)$. Property 2 follows from the multiplicative property of the determinant for block diagonal matrices; the curvature on $E \oplus F$ splits as a curvature on E and a curvature on F. Property 3 follows from pulling back a connection A on E to a connection f^*A on f^*E, and then using that pullbacks commute with everything.

Property 4 is more tedious and serves as our second explicit calculation of a Chern class. On $\mathbb{P}^1$, take the usual charts (U, u) and (V, v) with $u = v^{-1}$ on $U \cap V$. The local 1-forms

$$A_U = \frac{\bar{u}du}{1 + u\bar{u}}, \quad A_V = \frac{\bar{v}dv}{1 + v\bar{v}}$$

form the globally-defined **Chern connection** on $\mathcal{O}_{\mathbb{P}^1}(-1)$, the tautological bundle. We cheat a little and work only over U instead of all of $\mathbb{P}^1$. The curvature is

$$F_{A_U} = \frac{(1 + u\bar{u})d\bar{u} \wedge du - \bar{u}(\bar{u}du + ud\bar{u}) \wedge du}{(1 + u\bar{u})^2} = -\frac{du \wedge d\bar{u}}{(1 + u\bar{u})^2}.$$

Hence, in real coordinates, $c_1(\mathcal{O}_{\mathbb{P}^1}(-1)) = -\frac{dx \wedge dy}{\pi(1+x^2+y^2)^2}$. To compare this with the preferred generator, we simply integrate both and compare the result. (This is valid since $\dim H^2(\mathbb{P}^1) = 1$.) We know the preferred generator integrates to 1, whereas

$$\int_{\mathbb{P}^1} c_1(\mathcal{O}_{\mathbb{P}^1}(-1)) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dx \wedge dy}{(1 + x^2 + y^2)^2} = -\frac{1}{\pi}\int_0^{\infty}\int_0^{2\pi} \frac{rd\theta \wedge dr}{(1 + r^2)^2} = -1. \qquad \square$$

Proposition 1.2.15 (Splitting principle)**.** *If $0 \to A \to B \to C \to 0$ is a short exact sequence, then $c(B) = c(A) \wedge c(C)$.*

Proof. Short exact sequences of smooth vector bundles always split: pick a metric on B and show that $C \cong A^\perp$. Hence $c(B) = c(A \oplus C)$, and then we use the Whitney product formula. $\square$

Note: this is **not** the usual "splitting principle". The usual splitting principle says that to prove an identity on Chern classes, it suffices to pretend that the bundle completely splits into line bundles and prove the identity for that case. For more detail, see [3, Section 21].

Example 1.2.16. We compute the total Chern class of $\mathbb{P}^n$. We first construct the **Euler sequence** on $\mathbb{P}^n$, given by

$$0 \to \mathbb{C} \to \mathcal{O}_{\mathbb{P}^n}(1)^{n+1} \to T^{1,0}\mathbb{P}^n \to 0.$$

Since $X = \mathbb{P}^n$ is $Y = \mathbb{C}^{n+1} \setminus \{0\}$ mod a $\mathbb{C}^*$ action, given $n+1$ linear functionals v_i on $\mathbb{C}^{n+1}$, the vector field $\sum_i v_i \partial_i$ on X is invariant under this $\mathbb{C}^*$ action and descends to a vector field on $\mathbb{P}^n$. The v_i are sections of $\mathcal{O}_{\mathbb{P}^n}(1)$, so this construction is the map $\mathcal{O}_{\mathbb{P}^n}(1)^{n+1} \to T^{1,0}\mathbb{P}^n$. Its kernel is the line bundle associated to $Z = \sum_i x_i \partial_i$ (here x_i are the coordinates on Y): for homogeneous polynomials f, we have $\frac{1}{d}\sum_i x_i \partial_i f = f$. Another way to see this is to visualize $\mathbb{P}^n$ as a sphere in Y, so when we project, the radial vector field Z and its multiples are precisely the kernel.

Clearly $c(\mathbb{C}) = 1$, so by the splitting principle, $c(\mathbb{P}^n) = c(\mathcal{O}_{\mathbb{P}^n}(1)^{n+1}) = c(\mathcal{O}_{\mathbb{P}^n}(1))^{n+1}$. Let $x = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$. Then $c(\mathbb{P}^n) = (1+x)^{n+1}$.

Using the symbol x to stand for $c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ is fairly common. We shall do so from now on. (The reason is that x generates the cohomology ring $H^*(\mathbb{P}^n)$.)

Example 1.2.17. Let $X = V(p)$ be a smooth projective variety in $\mathbb{P}^n$ with p a degree d homogeneous polynomial, i.e. a section of $\mathcal{O}_{\mathbb{P}^n}(d)$. To compute the Chern class of X, we use the **adjunction formula** $NX \cong \mathcal{O}(d)|_X$ (see [6, Proposition 2.2.17] for details), so that

$$0 \to TX \to T\mathbb{P}^n|_X \to NX \cong \mathcal{O}(d)|_X \to 0$$

is a short exact sequence. Since $\mathcal{O}(d) = \mathcal{O}(1)^{\otimes d}$, we can't use the Whitney sum property of the Chern class, but we can use the Chern character:

$$\mathrm{ch}(\mathcal{O}(d)) = \mathrm{ch}(\mathcal{O}(1))^d = \exp(x)^d = \exp(dx),$$

so $c(\mathcal{O}(d)) = 1 + dx$. Hence $c(X) = (1+x)^{n+1}/(1+dx)$.

Definition 1.2.18. If we formally factorize the total Chern class as $c(E) = \prod_{k=1}^r (1 + a_k)$, then the **Chern character class** is

$$\mathrm{ch}(E) := \sum_{k=1}^r \exp(a_i) = r + c_1(E) + \frac{1}{2}(c_1(E)^2 - 2c_2(E)) + \cdots.$$

Proposition 1.2.19. *In the Chern–Weil setting,* $\mathrm{ch}(E) = \operatorname{tr}\exp(iF/2\pi)$.

Of course, the Chern character class, being a combination of Chern classes, does not contain more information than the Chern class. The reason we work with it instead of the Chern class is the following proposition.

Proposition 1.2.20. *The Chern character satisfies*

$$\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F), \quad \mathrm{ch}(E \otimes F) = \mathrm{ch}(E) \wedge \mathrm{ch}(F).$$

Finally, we need to connect the theory of Chern classes with sheaf cohomology. Consider the short exact sequence of sheaves given by

$$0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0.$$

Its associated long exact sequence of cohomology contains

$$\cdots \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}) \to H^1(X, \mathcal{O}^*) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \to \cdots.$$

Definition 1.2.21. The **Picard group** of X is the group of isomorphism classes of holomorphic line bundles under tensor product.

Theorem 1.2.22 ([6, Corollary 2.2.10])**.** *Let $\mathcal{O}^*$ be the sheaf of nowhere-zero holomorphic functions. Then* $\operatorname{Pic}(X) \cong H^1(X, \mathcal{O}^*)$.

Theorem 1.2.23 ([6, Proposition 4.4.12])**.** *Under the identification of elements of $H^1(X, \mathcal{O}^*)$ with isomorphism classes of holomorphic line bundles, the connecting map $\delta\colon H^1(X, \mathcal{O}^*) \to H^2(X, \mathbb{Z})$ is the first Chern class c_1.*

1.2.3 The Euler Class and Euler Characteristic

Recall that the (holomorphic) Euler characteristic of a sheaf $\mathcal{F}$ is $\chi(\mathcal{F}) \coloneqq \sum_k (-1)^k \dim H^k(X, \mathcal{F})$.

Definition 1.2.24. Let $\pi\colon E \to B$ be an oriented rank n vector bundle over a smooth n-fold B. In 1.2.5 we defined its Thom class $\Phi \in H^n(E, E \setminus B)$. The inclusion $(E, \emptyset) \subset (E, E \setminus B)$ gives a homomorphism $H^k(E, E \setminus B) \to H^k(E)$ which we denote by $\omega \mapsto \omega|_E$. The **Euler class** $e(E)$ of E is the image of the Thom class Φ under the composition

$$H^n(E, E \setminus B) \xrightarrow{-|_E} H^n(E) \xrightarrow{(\pi^*)^{-1}} H^n(B)$$

where the last isomorphism is canonical and comes from B being a deformation retract of E. Again, if X is a manifold, $e(X) \coloneqq e(TX)$.

Proposition 1.2.25. *Whenever both are defined, $e(E) = c_n(E)$ for E of rank n.*

Proof sketch. Given E a smooth complex vector bundle, $e(E)$ is well-defined because the complex structure on E induces an orientation. We can use the Euler class to construct the Chern classes [4, Section 14]. Then it suffices to verify that the Chern classes we constructed this way satisfy the four axioms 1.2.14 of Chern classes. For example, the Whitney sum formula comes from $\Phi(E \oplus F) = \Phi(E) \wedge \Phi(F)$ being preserved throughout the construction. □

We like to distinguish between the Euler class and the top Chern class for several reasons. One is that the Euler class is topological, whereas the Chern classes are differential geometric. Another is that it is sometimes easier to prove properties of the Euler class using properties of the Thom class rather than all the Chern classes, especially from the Chern–Weil approach.

Proposition 1.2.26 ([4, Property 9.3, Property 9.7])**.** *Properties of the Euler class $e(E)$ that do not directly follow from $e(E) = c_n(E)$:*

1. *if the orientation of E is flipped, $e(E)$ changes sign;*
2. *if E has a nowhere zero global section, then $e(E) = 0$.*

Proof. Property 1 is obvious: flipping the orientation of E flips the sign of the Thom class Φ, since $\Phi|_{E_x}$ is the preferred generator. Property 2 comes from $B \xrightarrow{s} E \setminus B \subset E \xrightarrow{\pi} B$ being the identity for a non-zero global section s. Then

$$H^n(B) \xrightarrow{\pi^*} H^n(E) \to H^n(E \setminus B) \xrightarrow{s^*} H^n(B)$$

is the identity on $H^n(B)$. But $\pi^* e(E) = \Phi|_E$, the restriction of the Thom class, by definition, so we have $s^*((\Phi|_E)|_{E \setminus B}) = e(E)$. Since $\Phi \in H^n(E, E \setminus B)$, the composition of these two restrictions is zero. Hence $e(E) = s^* 0 = 0$. □

Proposition 1.2.27. *Let $E \to M$ be a smooth oriented real vector bundle of rank r over the smooth compact oriented manifold M of dimension $n \geq r$. Let Z be the zero set of a smooth section $s\colon M \to E$ that is transversal to the zero section $\iota\colon M \to E$. Then Z is a smooth submanifold of M of codimension r and there is a natural bundle isomorphism $T_Z M \cong E|_M$. Consequently, $e(E)$ is Poincaré dual to Z.*

Proof. A straightforward exercise. Hint: remember that $e(E)$ is the restriction of the Thom class, and then use 1.2.7 and 1.2.8. □

The main purpose of this subsection is the following generalization of the Gauss–Bonnet theorem. We shall use it extensively when calculating Euler characteristic. To determine the Euler class explicitly, we often use many of the preceding results identifying it with various other objects.

Theorem 1.2.28 (Generalized Gauss–Bonnet)**.** *Let X be a compact complex manifold. Then*

$$\int_X e(X) = \chi(X).$$

Proof. We shall prove this in the next subsection, as a consequence of the Hirzebruch–Riemann–Roch formula. (There are much easier proofs, though.) □

Example 1.2.29. We can continue 1.2.16 to compute the Euler characteristic of $\mathbb{P}^n$. Note that every hyperplane $H \cong \mathbb{P}^{n-1} \subset \mathbb{P}^n$ is Poincaré dual to x. So x^n is Poincaré dual to the intersection of n generic hyperplanes, which is a point. In other words, x^n is the preferred generator given by the orientation, and hence $\int_{\mathbb{P}^n} x^n = 1$. (For a more explicit calculation of this, see [4, Theorem 14.10]. The explicit form for x is the obvious generalization of $-c_1(\mathcal{O}_{\mathbb{P}^1}(-1))$, which we computed in 1.2.14.)

Since $c(\mathbb{P}^n) = (1+x)^{n+1}$, we have $c_n = (n+1)x^n$. Hence $\int_{\mathbb{P}^n} c_n = n+1$. By the generalized Gauss–Bonnet theorem, $\chi(\mathbb{P}^n) = n+1$.

Example 1.2.30. Recall that the **degree** of a curve in $\mathbb{P}^2$ is just the degree of the defining homogeneous polynomial. (For a more general definition of the degree of a variety, see [5, Section I.7].) Fact: A degree d curve X in $\mathbb{P}^2$ has Chern class $1 + (3-d)x$. (Remember we write x for $c_1(\mathcal{O}_{\mathbb{P}^n}(1))$.) Then

$$\chi(X) = \int_X c_1(X) = \int_{\mathbb{P}^2} c_1(X)(xd) = \int_{\mathbb{P}^2} d(3-d)x^2 = d(3-d).$$

But for nonsingular curves X, we have $\chi(X) = 2 - 2g$. Hence $g = (d-1)(d-2)/2 = \binom{d-1}{2}$.

Example 1.2.31. By 1.2.17, a quintic hypersurface Q in $\mathbb{P}^4$ has total Chern class $c(Q) = (1+x)^5/(1+5x) = 1 + 10x^2 - 40x^3$. (Note that $c_1(Q) = 0$, so Q is Calabi–Yau.) We want to compute its Euler characteristic using HRR, but integrating over Q is hard. Instead, we use 1.2.27: Q is defined as the zero set of a section of $\mathcal{O}(5) \to \mathbb{P}^4$, so $e(\mathcal{O}(5))$ is Poincaré dual to Q. The Euler class is just the top Chern class (see 1.2.25), so $e(\mathcal{O}(5)) = c_1(\mathcal{O}(5))$,

$$\chi(Q) = \int_Q e(Q) = \int_Q c_3(Q) = \int_{\mathbb{P}^4} c_3(Q) \wedge c_1(\mathcal{O}(5)) = \int_{\mathbb{P}^4} (-40x^3)(5x) = -200 \int_{\mathbb{P}^4} x^4 = -200.$$

1.2.4 The Hirzebruch–Riemann–Roch Formula

Here E is a rank r holomorphic vector bundle over a compact complex n-fold X. The Hirzebruch–Riemann–Roch formula is part of a long sequence of generalizations of Gauss–Bonnet, relating geometric quantities to topological quantities.

Definition 1.2.32. Again, formally factor $c(E) = \prod_{k=1}^{r}(1 + a_k)$. The **Todd class** is

$$\operatorname{td}(E) \coloneqq \prod_{i=1}^{r} \frac{a_i}{1 - \exp(-a_i)} = 1 + \frac{1}{2}c_1(E) + \frac{1}{2}(c_1(E)^2 + c_2(E)) + \cdots .$$

The Todd class $\operatorname{td}(X)$ of X is defined as $\operatorname{td}(X) \coloneqq \operatorname{td}(TX)$.

Theorem 1.2.33 (Hirzebruch–Riemann–Roch, [5, Theorem A.4.1])**.** *Let E be a holomorphic vector bundle over a compact complex manifold X. Then*

$$\chi(E) = \int_X \operatorname{ch}(E) \wedge \operatorname{td}(X).$$

where on the right hand side we only integrate the top form, i.e. $\sum_k \operatorname{ch}_k \wedge \operatorname{td}_{n-k}$.

We can often use the Hirzebruch–Riemann–Roch (HRR) formula to compute the dimension of a specific cohomology group, either because some other cohomology groups vanish, or because we know their dimensions. Before we begin calculating anything, we need the following helpful result.

Theorem 1.2.34 (Grothendieck's vanishing theorem, [5, Theorem 2.7])**.** *Let X be a Noetherian topological space of dimension n. Then $H^i(X, \mathcal{F}) = 0$ for $i > n$ and any sheaf of abelian groups $\mathcal{F}$.*

The remainder of this section is examples of the HRR formula. Keep in mind that we are always working with sheaf cohomology, not de Rham cohomology. For example, $H^0(TX)$ is by no means equal to $H^0(X)$, and is not freely generated by connected components of TX. Instead, $H^0(TX) = \Gamma(TX)$, and since global sections generate automorphisms, $H^0(TX)$ consists of holomorphic automorphisms of X.

Example 1.2.35. Let $\mathcal{M}_g$ denote the **moduli space of complex structures** on a genus g closed surface. We shall see later (or recall from Teichmüller theory) that $\dim \mathcal{M}_g = \dim_{\mathbb{C}} H^1(TX)$ where X is a genus g closed Riemann surface. HRR gives

$$\begin{aligned}\dim_{\mathbb{C}} H^0(TX) - \dim_{\mathbb{C}} H^1(TX) = \chi(TX) &= \int_X \operatorname{ch}(TX) \wedge \operatorname{td}(TX) \\ &= \int_X (1 + c_1(TX)) \wedge (1 + (1/2)c_1(TX)) = \frac{3}{2}\int_X c_1(TX) = 3 - 3g.\end{aligned}$$

The last equality comes from c_1 being the top Chern class for X, i.e. the Euler class, so applying generalized Gauss–Bonnet gives $\int_X c_1(TX) = \chi(X) = 2 - 2g$. For $g \geq 2$, the Riemann surface X has no non-trivial automorphisms, so $\dim_{\mathbb{C}} H^0(TX) = 0$. Hence $\dim \mathcal{M}_g = 3g - 3$.

Example 1.2.36. An important object in mirror symmetry is the space of holomorphic maps from a Riemann surface Σ to a Calabi–Yau n-fold M, i.e. a Kähler n-fold M with $c_1(M) = 0$. (We'll be more careful about the definition of Calabi–Yau later.) An infinitesimal deformation of a holomorphic map, given by a vector field χ^i, must satisfy $\bar{\partial}\chi^i = 0$ if we want the deformed map to still be holomorphic. Hence $\chi \in H^0_{\bar{\partial}}(\phi^* TM)$, the space of such deformations. By HRR,

$$\begin{aligned}\dim_{\mathbb{C}} H^0(\phi^* TM) - \dim_{\mathbb{C}} H^1(\phi^* TM) &= \int_X \operatorname{ch}(\phi^* TM) \wedge \operatorname{td}(\Sigma) \\ &= \int_X (n + \phi^* c_1(TM)) \wedge (1 + (1/2)c_1(\Sigma)) = n(1 - g).\end{aligned}$$

We assume for now that $H^1(\phi^* TM) = 0$, so the space of deformations is $n(1 - g)$-dimensional. For $n = 3$ and $g = 0$, the dimension is 3, but there is also a 3-dimensional group of automorphisms of the genus-zero Riemann surface $\mathbb{P}^1$ which does not affect the image curve. Hence the space of genus 0 holomorphic curves inside a Calabi–Yau 3-fold is zero, and we may be able to count them!

Example 1.2.37. Let X be a connected compact curve and L a holomorphic line bundle on X. Fact: the natural isomorphism $H^2(X,\mathbb{Z}) = \mathbb{Z}$ is given by integration over X, and under this isomorphism we have $c_1(L) \mapsto \deg(L)$ ([6, Exercise 4.4.1]). By HRR,

$$\chi(L) = \int_X c_1(L) + \frac{1}{2}c_1(X) = \deg(L) + 1 - g.$$

For the case $L = \mathcal{O}(D)$ for a divisor D on X, we know $\dim_{\mathbb{C}} H^0(L) = \ell(D)$, and by Serre duality $\dim_{\mathbb{C}} H^1(L) = \dim_{\mathbb{C}} H^0(\mathcal{O}(K - D)) = \ell(K - D)$ where K is the canonical divisor. Hence we recover the classical **Riemann–Roch formula** $\ell(D) - \ell(K - D) = \deg D + 1 - g$.

To do (1)

To do (2)

1.3 Fixed-Point Theorems

1.4 Calabi–Yau Manifolds

1.5 Toric Geometry

To do…

- ☐ 1 (p. 14): Huybrecht def 5.1.3 of Hirzebruch χ_y-genus
- ☐ 2 (p. 14): Prove Huybrecht cor 5.1.4 in detail to get generalized Gauss–Bonnet

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