# String Theory and Supersymmetry
# Winter 2016 Seminar Notes

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# Chapter 1

# Introduction to Strings

We asked "why fields?" when we started QFT; now we ask, why strings? Here are some potentially convincing reasons.

1. If we allow one more degree of freedom than particles, many IR/UV divergences disappear; we require less renormalization. If we allow more than one degree of freedom, new divergences arise from the increased internal degrees of freedom.
2. Every consistent string theory contains a massless spin-2 state, i.e. a graviton, whose interactions at low energies reduce to general relativity.
3. The Standard Model, based on QFT, has 25 adjustable constants. String theory has none, and leads to gauge groups big enough to include the Standard Model.
4. Consistent string theories force upon us supersymmetry and extra dimensions, which have arisen naturally from several different attempts to unify the Standard Model.

Regardless of whether they are convincing, we start in this chapter, as with any other physical model, by writing down an action. Specifically, we first write the action for a relativistic string by generalizing that of a relativistic point particle, and then we quantize the action. As with QFT there are different ways to quantize. We go through the analogue of canonical quantization in order to quickly compute the spectrum of a string, and then go through path integral quantization in preparation for studying string interactions.

As usual, we take $\hbar = c = 1$, and use **Einstein summation convention**: repeated indices are implicitly summed over.

## 1.1 Review of Relativity

We work in $\mathbb{R}^{D-1,1}$ where $D$ is the **number of dimensions**. Recall that coordinates are written $x^\mu = (x^0, x^1, \ldots, x^D) = (ct, x^1, \ldots, x^D)$, and the metric is

$$-ds^2 := \eta_{\mu\nu} dx^\mu dx^\nu, \quad \eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, \ldots, 1).$$

Note that $\eta^\mu{}_\mu = D$. We use the dot product to stand for the **Lorentz inner product**, e.g. $-ds^2 = dx \cdot dx$.

**Definition 1.1.1.** Define the **proper time** of a system as the time elapsed measured by a clock traveling in the same Lorentz frame as the system itself. In such a Lorentz frame, $dx^i = 0$ and $dt$ is the proper time elapsed, so $-ds^2 = -dt_p^2$; define

$$ds := \sqrt{ds^2} = dt_p \qquad \text{whenever } ds^2 > 0,$$

i.e. for timelike intervals. Hence $ds$ is the **proper time interval**. The **relativistic momentum** is $p^\mu := m(dx^\mu/ds)$. Conveniently,

$$p^\mu p_\mu = m^2 \frac{dx^\mu}{ds}\frac{dx_\mu}{ds} = -m^2 \frac{ds^2}{ds^2} = -m^2.$$

**Definition 1.1.2.** A **Lorentz transformation** $\Lambda^\mu{}_\nu$ is an element of the Lorentz group, the collection of all linear isometries of $\mathbb{R}^{D-1,1}$. We say $a^\mu$ is a **vector** if under Lorentz transformations, it changes as $a'^\mu = \Lambda^\mu{}_\nu a^\nu$. A **Poincaré transformation** is a Lorentz transformation possibly followed by a translation.

**Definition 1.1.3.** The **world line** of a point particle is the path in spacetime $\mathbb{R}^{D-1,1}$ traced out by the particle as it evolves in time.

The underlying principle of relativity says that physical laws are independent of Lorentz frame. In other words, any action we write down that we want to be compatible with relativity must have external symmetries: it must be invariant under Lorentz transformations. We call this **Lorentz invariance**. As long as superscripts and subscripts match up, we do not have to worry about Lorentz invariance.

The **action** for a free relativistic **point particle** is obtained by writing down the simplest Lorentz invariant action, and then making sure dimensions work out. If $\gamma$ is the path taken by the particle, the action is therefore

$$S_{\text{pp}}[x] := -m\int_\gamma ds = -m\int_\gamma d\tau\,\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} = -m\int_\gamma d\tau\,\sqrt{-\dot{x}^\mu\cdot\dot{x}_\mu}$$

where a dot denotes a $\tau$-derivative. Because $ds$ is coordinate-independent, it does not what how we pick the parametrization $\tau$. Physicists like to call this **reparametrization invariance**. This invariance is very important: without it, we have actually introduced a completely new parameter $\tau$, thus increasing the number of degrees of freedom from $D-1$ to $D$.

**Exercise 1.1.1.** By computing $\delta(ds^2)$ in two different ways, show that

$$\delta S_{\text{pp}}[x] = m\int_\gamma \delta(dx^\mu)\frac{dx_\mu}{ds} = \int_\gamma d\tau\left(\frac{d}{d\tau}\delta x^\mu\right)p_\mu = \delta x^\mu p_\mu\Big|_{\tau_i}^{\tau_f} - \int d\tau\delta x^\mu\frac{dp_\mu}{d\tau}.$$

Argue that the first term vanishes if we specify **initial and final conditions**. Hence deduce the equation of motion $dp_\mu/d\tau = 0$.

The action $S_{\text{pp}}$ seems simple in the $\int_\gamma ds$ form, but is messy when parametrized. Later when we quantize using path integrals, $S_{\text{pp}}$ is difficult to work with because of the derivatives under the square root. There is a different, classically-equivalent action we can work with. Introduce an additional field $\gamma_{\tau\tau}(\tau)$ (sometimes called an **einbein** in general relativity), which we can view as a metric on the world line, and take the action

$$S'_{\text{pp}} := -\frac{1}{2}\int_\gamma d\tau\sqrt{-\gamma_{\tau\tau}}(\gamma^{\tau\tau}\dot{x}^\mu\dot{x}_\mu + m^2) = -\frac{1}{2}\int_\gamma d\tau\,(\eta^{-1}\dot{x}^\mu\dot{x}_\mu - \eta m^2), \quad \eta := \sqrt{-\gamma_{\tau\tau}(\tau)}.$$

It seems like we have arbitrarily added an extra degree of freedom, but in fact $\gamma$ is completely specified by the equation of motion. The action $S'_{\text{pp}}$ is much better to work with in a path integral, because it is **quadratic** in $\dot{x}^\mu$.

**Exercise 1.1.2.** Vary $S'_{\text{pp}}$ with respect to $\gamma_{\tau\tau}$ to get the equation of motion $\gamma_{\tau\tau} = \dot{x}^\mu\dot{x}_\mu/m^2$. Substitute this expression back into $S'_{\text{pp}}$ to obtain $S_{\text{pp}}$, and therefore conclude that the two actions are classically equivalent.

## 1.2 Nambu–Goto and Polyakov Actions

We graduate to **one-dimensional strings**; in this section we write down an action for them. There are two kinds of strings: those with two distinct endpoints, called **open strings**, and those which are loops, called **closed strings**. Because closed strings are just open strings with the extra constraint that the two endpoints match, we focus on open strings.

The action for the relativistic point particle is proportional to the proper time elapsed on the particle's world line. But the proper time, when multiplied by $c$, can be viewed as the "proper length" of the world line. The natural generalization, then, is to consider the surface in space-time traced out by the string as it evolves in time, called the **world sheet** $\Sigma$, and to define an action proportional to the "proper area" of the world sheet. The world sheet $\Sigma$ is a two-dimensional surface, and therefore requires charts modeled on $\mathbb{R}^2$.

**Definition 1.2.1.** The **coordinates** we use on $\mathbb{R}^2$, the parameter space, are denoted $(\sigma^0, \sigma^1)$, and so the **world sheet** $\Sigma$ is locally a surface given by functions denoted $X^\mu(\sigma)$ (capitalized to disambiguate from the coordinates $x^\mu$), called **string coordinates**. The lowercase Latin characters $a, b, \ldots$ are used to denote **indices** that run over values $0, 1$. Two notes:

1. The choice of parametrization $(\sigma^0, \sigma^1)$ is, again, up to us, but usually we take the coordinate $\sigma^0$ to be the proper time, and $\sigma^1$ the position along the string.
2. For our purposes, $\Sigma = X^\mu$, i.e. the single chart $X^\mu$ describes the entire world sheet for the region of spacetime we care about.

**Exercise 1.2.1.** Show that the metric $\eta_{\mu\nu}$ on spacetime $\mathbb{R}^{D-1,1}$ induces a metric $g$ on the world sheet via pullback along the inclusion $\iota\colon \Sigma \to \mathbb{R}^{D-1,1}$. Compute $g$ and the area element:

$$g_{ab} = \partial_a X^\mu \partial_b X_\mu, \quad dA = d^2\sigma \sqrt{-\det g}.$$

A relativistic particle has a parameter we call mass. It turns out mass is not the appropriate physical interpretation of the corresponding parameter for strings. Instead, we interpret it as a **tension**, and denote it $T_0$. Old people write $T_0 = 1/2\pi\alpha'$ and call $\alpha'$ the **universal Regge slope**; we choose not to.

**Definition 1.2.2.** The **Nambu–Goto action** for a relativistic string is given by

$$S_{\mathrm{NG}}[X] \coloneqq -T_0 \int_\Sigma dA = -T_0 \int_\Sigma d^2\sigma \sqrt{-\det g}.$$

Again, note that it satisfies **reparametrization invariance**, literally by construction.

But again, we have a square root and derivatives inside it, and now we know how to get rid of it: introduce an independent world sheet metric $\gamma_{ab}(\sigma)$. This time the metric is on a surface, so we need to specify the signature. We take Lorentzian signature $(-,+)$.

**Definition 1.2.3.** The **Polyakov action** for a relativistic string is given by

$$S_{\mathrm{P}}[X, \gamma] \coloneqq -\frac{T_0}{2} \int_\Sigma d^2\sigma \sqrt{-\gamma}\, \gamma^{ab} \partial_a X^\mu \partial_b X_\mu,$$

where $\gamma$ without indices stands for $\det(\gamma_{ab})$. From now on, we always refer to $\gamma_{ab}$ as the **metric**, and $g_{ab}$ as the **induced metric**. World sheet indices are raised/lowered using the metric $\gamma_{ab}$, not the induced metric $g_{ab}$. (In fact, from now on we basically forget about $g_{ab}$; we use it only to introduce the Nambu–Goto action, and the following exercise.)

**Exercise 1.2.2.** Show that $\delta\sqrt{-\gamma} = (1/2)\sqrt{-\gamma}\, \gamma^{ab} \delta\gamma_{ab}$, and therefore that

$$\delta_\gamma S_{\mathrm{P}}[X, \gamma] = -\frac{T_0}{2} \int_\Sigma d^2\sigma \sqrt{-\gamma}\, \delta\gamma^{ab} \left( g_{ab} - \frac{1}{2} \gamma_{ab} \gamma^{cd} g_{cd} \right).$$

Rearrange the obtained equation of motion and conclude that $g_{ab}\sqrt{-g} = \gamma_{ab}\sqrt{-\gamma}$. Hence replace $\gamma$ in $S_{\mathrm{P}}[X, \gamma]$ with $g$, and obtain that $S_{\mathrm{P}}[X, \gamma] = S_{\mathrm{NG}}[X]$.

**Definition 1.2.4.** As in general relativity, define the **stress-energy tensor**

$$T_{ab}(\sigma) := -\frac{4\pi}{\sqrt{-\gamma}}\delta_\gamma S_{\mathrm{P}}[X,\gamma] = -2\pi T_0\left(\partial_a X^\mu \partial_b X_\mu - \frac{1}{2}\gamma_{ab}\partial_c X^\mu \partial^c X_\mu\right), \tag{1.1}$$

so that the equation of motion arising from varying $\gamma$ says $T_{ab} = 0$. We call $T_{ab} = 0$ a **constraint** on the equation of motion for $X^\mu$, which we derive soon.

**Exercise 1.2.3.** (Important!) Now vary $S_{\mathrm{P}}[X,\gamma]$ with respect to $X^\mu$ to obtain

$$\begin{aligned}
\delta_X S_{\mathrm{P}}[X,\gamma] &= -T_0 \int_\Sigma d^2\sigma\, \sqrt{-\gamma}\,\gamma^{ab}\left(\partial_a(\delta X^\mu \partial_b X_\mu) - \partial_a\partial_b X_\mu \delta X^\mu\right) \\
&= -T_0 \int_0^\ell d\sigma^1\, \sqrt{-\gamma}\left[\delta X^\mu \partial^0 X_\mu\right]_{\sigma^0=\tau_i}^{\sigma^0=\tau_f} - T_0 \int_{\tau_i}^{\tau_f} d\sigma^0\, \sqrt{-\gamma}\left[\delta X^\mu \partial^1 X_\mu\right]_{\sigma^1=0}^{\sigma^1=\ell} \\
&\quad + T_0 \int_\Sigma d^2\sigma\, \sqrt{-\gamma}\,\delta X^\mu \nabla^2 X_\mu.
\end{aligned}$$

A careful inspection of the terms in the variation $\delta_X S_{\mathrm{P}}[X,\gamma]$ yield interesting insights. For this variation to vanish, each of the terms must vanish independently, since they control different aspects of the string's behavior.

1. The last term is determined by the motion of the string in the domain $(0,\ell)\times(\tau_i,\tau_f)$, and therefore $\delta X^\mu$ is not constrained by any boundary conditions there. Hence we have the **equation of motion** $\sqrt{-\gamma}\,\nabla^2 X_\mu = 0$.
2. The first term is determined by the configuration of the string at times $\tau_i$ and $\tau_f$. If we specify these configurations as **initial and final conditions**, then $\delta X^\mu$ is zero for the first term, so the term vanishes.
3. The second term is determined by the configuration of the endpoints of the string when $\sigma^0 \in (\tau_i,\tau_f)$. It does not vanish automatically. We have to impose **boundary conditions** in order to make it vanish.

**Definition 1.2.5.** There are two different kinds of boundary conditions.

- The **free (Neumann) boundary condition** is $\partial^1 X_\mu(\sigma^0,0) = \partial^1 X_\mu(\sigma^0,\ell) = 0$.
- The **Dirichlet boundary condition** is $\delta X^\mu(\sigma^0,0) = \delta X^\mu(\sigma^0,\ell) = 0$.

Alternatively, if the string is **closed**, i.e. we have the **periodicity** conditions

$$X^\mu(\sigma^0,0) = X^\mu(\sigma^0,\ell), \quad \partial^a X^\mu(\sigma^0,0) = \partial^a X^\mu(\sigma^0,\ell), \quad \gamma_{ab}(\sigma^0,0) = \gamma_{ab}(\sigma^0,\ell),$$

no additional boundary conditions are necessary.

For a long time, string theorists did not seriously consider the Dirichlet boundary condition. Why should the endpoints of an open string be fixed, and if they were, where would they be fixed onto? In particular, this fixing of endpoints would violate momentum conservation. Then Polchinski, in the 1990s, suggested that the endpoints are attached to **D-branes**, which should themselves be thought of as dynamical objects alongside strings. Conceptually, then,

1. a D0-brane is a particle, a D1-brane is a string, and so on, and they interact non-trivially;
2. the Dirichlet boundary condition says that a given D1-brane has fixed endpoints on a higher D$p$-brane;
3. any momentum lost by the D1-brane is absorbed by the D$p$-brane; and
4. the Neumann boundary condition is just saying there is a D-dimensional D-brane permeating all of space-time, i.e. the string endpoints are not fixed at all.

We return to this D-brane perspective much later on. It is hard enough to quantize strings without more dynamical objects floating around. We take **Neumann boundary conditions** for now.

## 1.3 Gauge Freedom and Gauge Fixing

There is another reason the Polyakov action is preferable over the Nambu–Goto action: it has more symmetries, and these symmetries make it easier to gauge fix (using Faddeev–Popov or otherwise) when we try to quantize. The Polyakov action is invariant under the following symmetries:

1. $D$-dimensional **Poincaré transformations**:
$$X^\mu(\sigma) \mapsto \Lambda^\mu{}_\nu X^\nu(\sigma) + a^\mu, \quad \gamma_{ab}(\sigma) \mapsto \gamma_{ab}(\sigma);$$
2. **Reparametrization** (i.e. diffeomorphisms): for new coordinates $\tilde{\sigma}^a(\sigma)$,
$$X^\mu(\sigma) \mapsto X^\mu(\tilde{\sigma}), \quad \gamma_{ab}(\sigma) \mapsto \frac{\partial \sigma^c}{\partial \tilde{\sigma}^a}\frac{\partial \sigma^d}{\partial \tilde{\sigma}^b}\gamma_{cd}(\sigma);$$
3. 2-dimensional **Weyl transformations**: for arbitrary $\omega(\sigma)$,
$$X^\mu(\sigma) \mapsto X^\mu(\sigma), \quad \gamma_{ab}(\sigma) \mapsto \exp(2\omega(\sigma))\gamma_{ab}(\sigma).$$

   The Nambu–Goto action is not invariant under Weyl transformations.

**Exercise 1.3.1.** Verify all these statements. (This should be quite straightforward.)

**Definition 1.3.1.** Let diff denote the group of diffeomorphisms acting on $\Sigma$, and Weyl the group of Weyl transformations acting on $\Sigma$; these are **internal symmetries**, while Poincaré transformations are **external symmetries**. The product diff $\times$ Weyl is the **gauge group**. The orbit, in the space of all possible fields and metrics, of a particular $(X, \gamma)$ under the action of the gauge group is the **gauge orbit**.

A good exercise in working with the gauge and external symmetries is to make sure Polyakov action is as general as possible. This also reduces future work when we need the additional terms in the Polyakov action. Note that here, contrary to the case in QFT, the symmetries are very demanding. Weyl invariance in particular is very odd: it prevents us from adding terms such as
$$\int_\Sigma d^2\sigma\,\sqrt{-\gamma}\,V(X), \quad \mu\int_\Sigma d^2\sigma\,\sqrt{-\gamma}.$$

**Exercise 1.3.2.** Convince yourself that the action must contain one more $\gamma^{ab}$ than $\gamma_{ab}$ in order to satisfy Weyl invariance and counteract the change in $\sqrt{-\gamma}$. Since such a $\gamma^{ab}$ can only pair up indices with derivatives, we need a second-order Lorentz-invariant term that is coordinate-independent. Convince yourself that other than $\partial_a X^\mu \partial_b X_\mu$, this term can only involve $\gamma^{ab}$ and $\gamma_{ab}$, and that in fact it must be the **scalar curvature** $R$. Show that under a Weyl transformation,
$$\sqrt{-\gamma}\,R \mapsto \sqrt{-\gamma}(R - 2\nabla^2\omega).$$
Hence argue that we need another term integrated over $\partial\Sigma$ to counteract $\nabla^2(\sqrt{-\gamma}\,\omega)$. Putting everything together, conclude that
$$\chi \coloneqq \frac{1}{4\pi}\int_\Sigma d^2\sigma\,\sqrt{-\gamma}\,R + \frac{1}{2\pi}\int_{\partial\Sigma} ds\,k$$
is Weyl invariant, and that it is essentially the only term we can add to the Polyakov action. Here $ds$ is proper time along $\partial\Sigma$ using the metric $\gamma_{ab}$, and $k \coloneqq \pm t^a n_b \nabla_a t^b$ is the **geodesic curvature** of the boundary, where $t^a$ is a unit vector tangent to the boundary, and $n_b$ an outward-pointing unit vector, and we choose $\pm$ depending on whether the boundary is timelike or spacelike.

Let's explore a few choices of gauge, some which use up all the gauge freedom, and some which do not. We commonly use reparametrization invariance to simplify expressions, so let's explore some choices of gauge using reparametrization invariance first.

**Definition 1.3.2.** We can reparametrize $(\sigma^0, \sigma^1)$ such that $\sigma^0$ corresponds to the time coordinate $x^0$, i.e. $X^0 = R\sigma^0$ for some dimensionful constant $R$. This is **static gauge**, named as such because then lines of constant $\sigma^0$ correspond to the string at fixed moments in time, i.e. the string is static. Another choice is **light cone gauge**, given by $X^+ = R\sigma^0$, where

$$X^\pm := \frac{1}{\sqrt{2}}(X^0 \pm X^1), \quad \sigma^\pm := \frac{1}{\sqrt{2}}(\sigma^0 \pm \sigma^1)$$

are **light cone coordinates** on Minkowski space and the world sheet respectively. When in light cone gauge, the indices $i, j, \ldots$ range over $\{2, \ldots, D\}$.

Clearly neither static gauge nor light cone gauge exhausts the gauge freedom: we haven't done anything with the metric! But it is hard to transform the metric in a useful way while staying in static or light cone gauge. Let's take a different approach and try to transform the metric first.

The transformation of the scalar curvature computed in the exercise above says we can use Weyl invariance to locally set the scalar curvature to zero, by solving $2\nabla^2\omega = R$ and then applying the Weyl transformation $\exp(2\omega)$. But we are in two dimensions, where the symmetries of the Riemann curvature tensor determine it from $R$:

$$R_{abcd} = R_{cdab},\ R_{abcd} = -R_{bacd} = -R_{abdc} \implies R_{abcd} = (1/2)(\gamma_{ac}\gamma_{bd} - \gamma_{ad}\gamma_{bc})R.$$

Hence we can always locally get a flat metric, which, possibly after applying a coordinate transformation, gives $\gamma_{ab} = \eta_{ab}$, the flat Minkowski metric.

**Definition 1.3.3.** If we consider only reparametrization and not Weyl transformations, the metric $\gamma_{ab}$ can always be brought to the form $\exp(2\omega)\eta_{ab}$. Forcing the metric to be of that form is known as **conformal gauge**. Performing the additional Weyl transformation to obtain $\gamma_{ab} = \eta_{ab}$ is known as **unit gauge**. In general, the form of the metric we choose to put $\gamma_{ab}$ in is called the **fiducial metric**.

**Exercise 1.3.3.** (Important!) Show that in unit gauge, the equation of motion and its constraints become

$$\partial_a\partial^a\vec{X} = 0, \quad \partial_0\vec{X}\cdot\partial_1\vec{X} = 0, \quad (\partial_0\vec{X})^2 + (\partial_1\vec{X})^2 = R^2.$$

In this form, the constraints are called **Virasoro conditions**. Argue that by tensoriality, the Virasoro conditions still hold in static gauge, where $X^\mu = (R\sigma^0, \vec{X})$. Hence show in static gauge that at the (free) endpoints an open string, i.e. endpoints satisfying the Neumann boundary condition, $|\partial_t\vec{X}| = 1$. (**Be careful**: $\partial_t$ is not $\partial_0$. What is $\partial_t$?) Conclude that string endpoints always move at the speed of light.

How many internal degrees of freedom have we used up if we put the metric $\gamma_{ab}$ in unit gauge? Well, diff has two degrees of freedom, one for each coordinate, and Weyl has one, for the scale of the metric. But the metric itself has three independent components, being symmetric. Hence we expect to be done with choosing a representative of each gauge orbit.

But, perhaps unexpectedly, there is more gauge freedom: there are non-trivial transformations in diff $\times$ Weyl that preserve unit gauge! The key to finding these transformations is to realize that $\Sigma$ is actually a **Riemann surface**: let $z := \sigma^0 + i\sigma^1$, so that $ds^2 = dzd\bar{z}$. Now if $f(z)$ is a holomorphic change of coordinates, then

$$z \mapsto f(z), \quad ds^2 \mapsto |\partial_z f|^{-2}dzd\bar{z},$$

so now applying the Weyl transformation $\exp(2\ln|\partial_z f|)$ recovers $ds^2$. Clearly the composition of the two transformations is non-trivial.

What went wrong? Well, just because dimensions match up does not mean we have spanned the whole space of gauge transformations! The holomorphic diffeomorphisms above actually have **measure zero** in diff. When we stop working locally and work globally instead, these extra bits of freedom are removed by boundary conditions.

**Definition 1.3.4.** When we successfully pick a unique and continuously-varying choice of representative in each gauge orbit, our theory is **gauge-fixed**. When such a choice is impossible due to topological obstructions, our theory has **Gribov ambiguity**. (For us, there is no Gribov ambiguity; we are just failing to consider boundary conditions.)

## 1.4 Quantization via Canonical Commutation Relations

When we did QFT, we started by **canonically quantizing** the Klein-Gordon and Dirac fields, which allowed us to immediately investigate some aspects of the quantized free theories, such as that Klein-Gordon fields represent bosons and Dirac fields represent fermions, and to obtain the spectrum and Hilbert space of states. On the other hand, **path integral quantization** gave us an easy way to compute interactions in perturbative QFT, such as scattering amplitudes. We do the same for string theory: first, in this section, we canonically quantize in order to write down the spectrum and Hilbert space of states, and then, in the next section, we quantize using the path integral to work with interactions.

In string theory, canonical quantization no different from what we saw in QFT. The procedure is the same: take the classical object (e.g. Lagrangian, Hamiltonian, solutions) you want to quantize, and impose **canonical commutation relations** modeled on $[x,p]=i$ on dynamical variables, by promoting them all to operators.

### 1.4.1 Classical Solutions

We take classical solutions and quantize them in light cone gauge as well as two more gauge-fixing conditions for the metric: set

$$X^+ = \sigma^0, \quad \partial_1\gamma_{11} = 0, \quad \det\gamma_{ab} = -1.$$

Note that we have dispensed with the dimensionful constant $R$; it can be reinserted via dimensional analysis. The first thing to do right after picking a gauge is to rewrite all the relevant objects in that gauge. To do so, we need some formulas.

**Exercise 1.4.1.** Show that in this gauge, $\gamma_{11}(\sigma^0)$ depends only on $\sigma^0$, and we have

$$\begin{pmatrix} \gamma^{00} & \gamma^{01} \\ \gamma^{10} & \gamma^{11} \end{pmatrix} = \begin{pmatrix} -\gamma_{11}(\sigma^0) & \gamma_{01}(\sigma) \\ \gamma_{01}(\sigma) & \gamma_{11}^{-1}(\sigma^0)(1-\gamma_{01}^2(\sigma)) \end{pmatrix}.$$

Furthermore, show that $\partial_a X^\mu \partial_a X_\mu = 2\partial_a X^+ \partial_a X^- - \partial_a X^i \partial_a X^i$. (Recall that indices $i,j,\dots$ range over $\{2,\dots,D\}$).

**Definition 1.4.1.** Given a dynamical variable $V(\sigma)$, define its associated **center of mass** (conceptually at a fixed time) variables

$$v(\sigma^0) = \frac{1}{\ell}\int_0^\ell d\sigma^1\, V(\sigma), \quad \tilde{V}(\sigma) = V(\sigma) - v(\sigma^0),$$

i.e. we split $V = v + \tilde{V}$ where $v$ is the mean value of $V$, and $\tilde{V}$ has mean zero.

For example, using that $\partial_1 X^+ = 0$, we have

$$\partial_1 \tilde{X}^- = \partial_1 X^- = \frac{1}{\sqrt{2}}(\partial_1 X^0 - \partial_1 X^1) = \sqrt{2}\partial_1 X^0,$$

and using that $\partial_0 X^+ = 1$, we have

$$\partial_0 X^0 \partial_1 X^0 - \partial_0 X^1 \partial_1 X^1 = (\partial_0 X^0 + \partial_0 X^1)\partial_1 X^0 - \partial_0 X^1(\partial_1 X^0 + \partial_1 X^1) = \sqrt{2}\partial_1 X^0 - 0.$$

**Exercise 1.4.2.** Using all these calculations, show that the Polyakov Lagrangian in this gauge is

$$L = -\frac{T_0}{2}\int_0^\ell d\sigma^1 \left[\gamma_{11}(2\partial_0 X^- - \partial_0 X^i \partial_0 X^i) - 2\gamma_{01}(\partial_1 \tilde{X}^- - \partial_0 X^i \partial_1 X^i) + \gamma_{11}^{-1}(1-\gamma_{01}^2)\partial_1 X^i \partial_1 X^i\right].$$

Argue that because $\tilde{X}^-$ does not appear with time derivatives, it is not a dynamical variable, and therefore when we vary $S_{\mathrm{P}}$ with respect to $\gamma$, it constrains $\partial_1\gamma_{01}$ to be zero. Show that the Neumann boundary condition, in this gauge, gives $\gamma_{01} = 0$ at the endpoints $\sigma = 0, \ell$, and conclude that $\gamma_{01} = 0$ everywhere. Therefore write down the simplified **Lagrangian**:

$$L = -T_0 \ell \gamma_{11} \partial_0 x^- + \frac{T_0}{2}\int_0^\ell d\sigma^1 \left(\gamma_{11}\partial_0 X^i \partial_0 X^i - \gamma_{11}^{-1}\partial_1 X^i \partial_1 X^i\right),$$

The next step is to write down the Hamiltonian, which is the **Legendre transform** of the Lagrangian. Recall that this means we write down momenta $\Pi_\mu$ corresponding to $X^\mu$, and then define

$$H := \int_0^\ell \Pi_\mu \partial_0 X^\mu - L = \int_0^\ell d\sigma^1 \left(\Pi_+\partial_0 X^+ + \Pi_-\partial_0 X^- + \Pi_i \partial_0 X^i\right) - L = p_-\partial_0 x^- + \int_0^\ell \Pi_i \partial_0 X^i - L,$$

where $p_-$ is the momentum conjugate to $x^-$, and $\Pi^i$ is the momentum density conjugate to $X^i$:

$$p_- := \frac{\partial L}{\partial \partial_0 x^-} = -T_0 \ell \gamma_{11}, \quad \Pi^i := \frac{\delta L}{\delta \partial_0 X^i} = T_0 \gamma_{11} \partial_0 X^i = \frac{p^+}{\ell}\partial_0 X^i.$$

Note that $p_- = -p^+$. Simplifying, we get the Hamiltonian

$$H = \frac{\ell T_0}{2p^+}\int_0^\ell d\sigma^1 \left(\frac{1}{T_0}\Pi^i \Pi^i + T_0 \partial_1 X^i \partial_1 X^i\right),$$

which is precisely the **Hamiltonian** for $D-2$ free fields $X^i$, with $p^+ \propto \gamma_{11}$ a conserved quantity.

We can also directly write down **classical solutions**: the equation of motion in this gauge is $\partial_+\partial_- X^i = 0$, which has the general solution

$$X^i(\sigma) = X_L^i(\sigma^+) + X_R^i(\sigma^-)$$

for arbitrary functions $X_L^i$ and $X_R^i$, describing **left-moving** and **right-moving** waves respectively, which we can expand as Fourier series:

$$X_L^i(\sigma^+) = \frac{1}{2}x^i(0) + \frac{1}{2T_0}p^i(0)\sigma^+ + i\frac{\ell}{\pi}\sqrt{\frac{1}{2T_0}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^i e^{-in\pi\sigma^+/\ell},$$

$$X_R^i(\sigma^-) = \frac{1}{2}x^i(0) + \frac{1}{2T_0}p^i(0)\sigma^- + i\frac{\ell}{\pi}\sqrt{\frac{1}{2T_0}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^i e^{-in\pi\sigma^-/\ell}.$$

(Here $p^i$ are center of mass variables for $\Pi^i$ with an extra factor of $\ell$. We've also mucked around with the normalization factors for Fourier coefficients for later convenience.) Because the $X^i$ are real fields, we have the **constraints** $\tilde{\alpha}_n^i = (\tilde{\alpha}_{-n}^i)^*$ and $\alpha_n^i = (\alpha_{-n}^i)^\dagger$ on the Fourier coefficients.

**Exercise 1.4.3.** Show that the Neumann boundary condition forces $\tilde{\alpha}_n^i = \alpha_n^i$, so that the general form of a **classical solution for an open string** is

$$X^i(\sigma) = x^i(0) + \frac{1}{2T_0}p^i(0)\sigma^0 + i\frac{\ell}{\pi}\sqrt{\frac{1}{2T_0}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^i e^{-in\pi\sigma^0/\ell}\cos\frac{n\pi\sigma^1}{\ell}.$$

Finally, we must write down the **constraints**, i.e. the Virasoro conditions in this gauge. They become $(\partial_+ X)^2 = (\partial_- X)^2 = 0$, which give conditions on the momenta $p^i$ and Fourier coefficients $\alpha_n^i$. Both $\partial_+$ and $\partial_-$ give the same result, so we compute

$$\partial_+ X^i = \partial_+ X_L^i = \frac{1}{2T_0} p^i(0) + \sqrt{\frac{1}{2T_0}} \sum_{n \neq 0} \alpha_n^i e^{-in\pi\sigma^+/\ell}.$$

Hence, writing $\alpha_0^i \coloneqq \sqrt{1/2T_0}\, p^i(0)$,

$$0 = (\partial_+ X)^2 = \frac{1}{T_0} \sum_n L_n e^{-i\pi n\sigma^+/\ell}, \quad L_n \coloneqq \frac{1}{2} \sum_m \alpha_m \cdot \alpha_{n-m}.$$

So the $L_n$ are the Fourier coefficients of the constraints. By the linear independence of the Fourier basis, $L_n = 0$ for all $n \in \mathbb{Z}$. In particular, since $p_\mu p^\mu = -M^2$ is the effective mass and $L_0$ contains the momentum, $L_0 = 0$ implies that the **effective mass** of the string is

$$M^2 = -p \cdot p = 4T_0 \sum_{m>0} \alpha_m \cdot \alpha_{-m}.$$

### 1.4.2 Canonical Quantization

Quantization is now trivial: we impose the canonical **equal-time commutation relations**

$$[x^-, p^+] = i\eta^{-+} = -i, \quad [X^i(\sigma), \Pi^j(\sigma')] = i\delta^{ij}\delta(\sigma - \sigma'),$$

with all other commutators vanishing. In terms of Fourier components,

$$[x^-, p^+] = -i, \quad [x^i, p^j] = i\delta^\mu{}_\nu, \quad [\alpha_m^i, \alpha_n^j] = m\delta^{ij}\delta_{m+n,0},$$

with all other commutators vanishing. So as in QFT, we can treat $\alpha_n^i$ as creation/annihilation operators ($\alpha$ is annihilation, $\alpha^\dagger$ is creation), and build up our state space using them. Note that instead of just a single creation/annihilation operator, we have an infinite tower of them!

**Definition 1.4.2.** The **creation/raising operators** are $\alpha_{-m}^i$ and the **annihilation/lowering operators** are $\alpha_n^i$. The **ground state of a string with momentum** $k$ is defined as the eigenstate $|0;k\rangle$ of $p^i$, the center of mass momenta, annihilated by the annihilation operators, i.e.

$$p^+ |0;k\rangle = k^+ |0;k\rangle, \quad p^i |0;k\rangle = k^i |0;k\rangle, \quad \alpha_m^i |0;k\rangle = 0 \quad \forall m > 0.$$

Note that the zero-momentum ground state $|0;0\rangle$ of a string is not the true **vacuum state**, which consists of no strings at all; we denote the true vacuum state $|\text{vacuum}\rangle$.

Unlike QFT, each raising operator $\alpha_{-m}^i$ (for varying $m$) creates a different mode. So the **independent states** are labeled using center of mass momenta $k = (k^+, k^i)$, and occupation numbers $N_{i,n}$ for $i = 2, \ldots, D$ and $n = 1, 2, \ldots$:

$$|N;k\rangle \coloneqq \left( \prod_{i=2}^{D} \prod_{n=1}^{\infty} \frac{(\alpha_{-n}^i)^{N_{i,n}}}{\sqrt{n^{N_{i,n}} N_{i,n}!}} \right) |0;k\rangle.$$

(The normalization is chosen for convenience.) Hence there are an infinite number of different first excitations of a single string. Let $\mathcal{H}_1$ denote the space of all possible single-string states:

$$\mathcal{H}_1 \coloneqq \operatorname{span}\{|N;k\rangle : \text{all possible } N, k\}.$$

**Definition 1.4.3.** The **state space**, of any number of strings, is a **bosonic Fock space**

$$\mathrm{Sym}(\mathcal{H}_1) \coloneqq |\text{vacuum}\rangle \oplus \mathcal{H}_1 \oplus (\mathcal{H}_1 \odot \mathcal{H}_1) \oplus (\mathcal{H}_1 \odot \mathcal{H}_1 \odot \mathcal{H}_1) \oplus \cdots \oplus \cdots,$$

where $\odot$ is the symmetrized tensor product

$$v_1 \odot \cdots \odot v_n \coloneqq \frac{1}{n!} \sum_{\sigma \in S_k} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}.$$

The $n$-th term in the sum $\mathrm{Sym}(\mathcal{H}_1)$ is the state space of $n$ strings. We symmetrize because it turns out the strings we are working are bosonic, i.e. they have integer spin, i.e. they commute, instead of anticommuting. ($\mathrm{Sym}(\mathcal{H}_1)$ is known by us mathematicians as a **symmetric algebra**; a fermionic Fock space, for objects with half-integer spins, is an exterior algebra.)

We still need to impose the constraints $L_n = 0$, coming from the Virasoro conditions. Naively one might just insist that as operators, $L_n = 0$, but this quickly runs into problems (cf. Gupta–Bleuler quantization of QED). Instead, we impose $L_n |\text{phys}\rangle = 0$ for any physical state $|\text{phys}\rangle$.

### 1.4.3 Spectrum and Critical Dimension

By mass-energy equivalence, to find the spectrum of our quantized string is equivalent to finding its effective mass, i.e. we must look at the quantized version of $M^2 = 4T_0 \sum_{m>0} |\alpha_m|^2$. But when we quantize, $\alpha_m$ and $\alpha_{-m}$ no longer commute, so there is an operator ordering ambiguity here. There are two choices: either we quantize $\alpha_m \cdot \alpha_{-m}$, or we quantize $\alpha_{-m} \cdot \alpha_m$. They both give

$$M^2 = 4T_0 \sum_{m>0} (N_m + a), \quad N_m \coloneqq \alpha_{-m} \cdot \alpha_m$$

(where by analogy with the harmonic oscillator, we've defined the **number operators** $N_m$), but the first with $a = m(D-2)/2$, using the commutation relation $[a^i_m, a^i_{-m}] = m$, and the second with $a = 0$. There are some physical arguments for why we pick the former during the quantization of the simple harmonic oscillator (Heisenberg uncertainty principle, etc.), but it boils down to the assertion that we want the ground state of the system to have non-zero energy. Hence we pick $a = m(D-2)/2$.

**Exercise 1.4.4.** Recall/review from QFT that $\sum_{m>0} m = -1/12$, and therefore conclude that the ground state and first excited states, i.e. $\alpha^i_{-m} |0; k\rangle$ for any $m$, have energies

$$M_0^2 = 4T_0 \frac{2-D}{24}, \quad M_1^2 = 4T_0 \frac{26-D}{24}.$$

Fix an $m$. The first excited state $\alpha^i_{-m} |0; k\rangle$ acts as a vector because it has a vector index $i$, so it better be Lorentz invariant. In particular, in the rest frame, the (spatial rotation subgroup of the) Lorentz group can act on a vector to make it point in any spatial direction, so vectors better have $D-1$ states. But $\alpha^i_{-m}$ lives in the standard representation of $\mathrm{SO}(D-2)$: it only has $D-2$ states contained in it. This is not good!

Here is the solution: we posit that $M_1^2 = 0$. Then there is no rest frame! Consequently, we are only free to rotate around the direction of motion, giving only $D-2$ states, exactly the number that we have. But this implies $D = 26$, known as the **critical dimension** of bosonic string theory. This entire argument is sketchy, and we (hopefully) give a more rigorous argument later that $D = 26$ is the only dimension that works, based on enforcing Weyl invariance.

There is another problem: $M_0^2 < 0$ for $D > 2$, especially for $D = 26$. We have **negative energy states**, known as **tachyons**! This is explained from a field-theoretic perspective: given a field $\phi$, its mass squared is just $\partial^2 V(\phi)/\partial\phi^2|_{\phi=0}$. We are actually expanding around a critical point of the potential that is a maximum, i.e. an **unstable** point, therefore resulting in a negative mass-squared. Currently it is unknown whether there are stable points in the purely bosonic theory. However, with the addition of fermions and **supersymmetry**, giving the **superstring**, the problem disappears. This is content for much later on.

## 1.5 Quantization via Path Integral

Now it is time to develop a different tool. Recall from QFT that we have a giant machine for quantizing classical theories and studying their interactive pictures: the path integral. However, before we begin plugging the Polyakov action into the machine, we need to make a modification. From now on, the world sheet is equipped with a **Euclidean metric** $g_{ab}$, instead of a Lorentzian one $\gamma_{ab}$. This is so that the path integral over metrics is better defined. The transition from Euclidean to Minkowski is, formally, done via **Wick rotation**: $x^0 \mapsto ix^0$ and similarly for the metric. The **Euclidean path integral**, and the Euclidean action (with the additional terms on top of the Wick-rotated Polyakov action), is therefore

$$Z := \frac{1}{\mathrm{Vol}} \int \mathcal{D}g\, \mathcal{D}X\, \exp(-S_{\mathrm{P}}[X,g]),$$

$$S_{\mathrm{P}}[X,g] = \frac{T_0}{2} \int_\Sigma d^2\sigma\, \sqrt{g}\, g^{ab} \partial_a X^\mu \partial_b X_\mu + \lambda \left( \frac{1}{4\pi} \int_\Sigma d^2\sigma\, \sqrt{g}\, R + \frac{1}{2\pi} \int_{\partial\Sigma} ds\, k \right)$$

where Vol is the volume of the gauge action on the **configuration space** consisting of all possible $X^\mu$ and $g$). More explicitly, we can imagine partitioning configuration space into gauge orbits; we actually want to integrate on a path through these gauge orbits. But now recall from QFT that we have another giant machine for doing so: the Faddeev-Popov method.

### 1.5.1 The Faddeev-Popov Method

Let's first recall that the idea behind Faddeev-Popov is very natural: we want to do a change of coordinates in configuration space so that instead of integrating over a mish-mash of $g$ and $X$, we integrate such that one variable goes along gauge orbits, and the other goes along the gauge-fixed path. Although this sounds technical, we perform procedures like this quite often without realizing it! For example, consider the calculation

$$\iint dx\, dy\, e^{-x^2-y^2} = \int d\theta \int dr\, re^{-r^2} = 2\pi \int dr\, re^{-r^2} = \pi.$$

What is really happening here is that we recognized the $U(1)$ symmetry of the original integrand, and changed variables in order to factor out that symmetry. Instead of integrating over $(x, y)$, we integrated over $(r, \theta)$, with $\theta$ parametrizing the gauge orbits. Furthermore, we picked out the $y = 0$ representative of each gauge orbit for the remaining integral.

Armed with this motivation, we can proceed. Let $\hat{g}_{ab}$ be the fiducial metric; it represents our choice of gauge fixing, just like the choice $y = 0$. Let $\zeta$ be shorthand for a combined coordinate and Weyl transformation:

$$\zeta\colon g_{ab} \mapsto g^\zeta_{ab} := \exp(2\omega(\sigma)) \frac{\partial\sigma^c}{\partial\sigma'^a} \frac{\partial\sigma^d}{\partial\sigma'^b} g_{cd}(\sigma).$$

**Definition 1.5.1.** Let $\mathcal{D}\zeta$ be a gauge invariant measure on diff $\times$ Weyl. (Whether such a measure exists is very relevant for us, but we disregard it for now.) Define the **Faddeev-Popov determinant** $\Delta_{\mathrm{FP}}$ by

$$\Delta^{-1}_{\mathrm{FP}}(g) := \int \mathcal{D}\zeta\, \delta[\hat{g}^\zeta - g].$$

Here the $\delta$ is the **Dirac functional**, i.e. $\hat{g}^\zeta$ and $g$ must agree at every point $\sigma$.

**Exercise 1.5.1.** Show that $\Delta_{\mathrm{FP}}(g)$ is gauge-invariant by computing that $\Delta_{\mathrm{FP}}(g^\zeta)^{-1} = \Delta_{\mathrm{FP}}(g)^{-1}$.

Now it is time to do the calculation to factor out the integral over the gauge orbits. The first step is to add a 1 to the integral:

$$Z = \int \frac{\mathcal{D}g\, \mathcal{D}X}{\mathrm{Vol}} \exp(-S_{\mathrm{P}}[X,g]) = \int \frac{\mathcal{D}g\, \mathcal{D}X\, \mathcal{D}\zeta}{\mathrm{Vol}} \Delta_{\mathrm{FP}}(g) \delta[\hat{g}^\zeta - g] \exp(-S_{\mathrm{P}}[X,g]).$$

The second step is to do the integral over $g$, which, due to the $\delta[\hat{g}^\zeta - g]$, amounts to replacing $g$ with $\hat{g}^\zeta$:

$$Z = \int \frac{\mathcal{D}X\,\mathcal{D}\zeta}{\text{Vol}} \Delta_{\text{FP}}(\hat{g}^\zeta) \exp(-S_{\text{P}}[X, \hat{g}^\zeta]).$$

Finally, since both $\Delta_{\text{FP}}$ and $S_{\text{P}}$ are gauge-invariant, we can replace $\hat{g}^\zeta$ with $\hat{g}$. Then nothing in the integrand depends on $\zeta$ anymore, so it factors out and cancels the volume normalization:

$$Z = \int \frac{\mathcal{D}\zeta}{\text{Vol}} \int \mathcal{D}X\, \Delta_{\text{FP}}(\hat{g}) \exp(-S_{\text{P}}[X, \hat{g}]) = \int \mathcal{D}X\, \Delta_{\text{FP}}(\hat{g}) \exp(-S_{\text{P}}[X, \hat{g}]).$$

**Exercise 1.5.2.** Evaluate $\iint dx\, dy\, e^{-x^2-y^2}$ by applying the Faddeev-Popov method to its $U(1)$ symmetry and the gauge-fixing condition $y = 0$. Conclude that the Faddeev-Popov method is completely rigorous in finite dimensions, and that $\Delta_{\text{FP}}$ is actually a Jacobian (hence the name Faddeev-Popov determinant).

### 1.5.2 Computing the Faddeev-Popov Determinant

It remains to compute the Faddeev-Popov determinant $\Delta_{\text{FP}}$ for the diff $\times$ Weyl action on world sheet metrics. To do so, we make the simplifying assumption that diff $\times$ Weyl actually acts freely on metrics $g$, i.e. for each $g$, there is exactly one $\zeta$ such that $\delta[\hat{g}^\zeta - g] = 0$. Obviously this assumption is false: we showed earlier that the action has fixed points (albeit a measure zero set of them). But it is true locally, so we deal with the global issues later. The reason we make this assumption is so that we can compute $\Delta_{\text{FP}}(\hat{g})^{-1}$ by integrating only around a small neighborhood of $\zeta = 0$. In this neighborhood, we can take infinitesimal Weyl transformations $\omega(\sigma)$ and infinitesimal diffeomorphisms $\delta\sigma^\alpha = v^\alpha(\sigma)$, and write

$$\Delta_{\text{FP}}^{-1}(\hat{g}) = \int \mathcal{D}\omega\,\mathcal{D}v\,\delta[2\omega\hat{g}_{ab} + \nabla_a v_b + \nabla_b v_a].$$

Note that now we are integrating over the Lie algebra of diff $\times$ Weyl. We want to get rid of the delta functional.

**Exercise 1.5.3.** For a function $\phi\colon \mathbb{R}^D \to \mathbb{R}$, derive the integral form

$$\delta[\phi] = \int_{j\colon \mathbb{R}^D \to \mathbb{R}} \mathcal{D}j(x)\, \exp\left(2\pi i \int d^D x\, j(x)\phi(x)\right)$$

by applying the one-dimensional identity $\delta(x) = \int dp\, \exp(2\pi i p x)$ to piecewise linear paths, and then taking the limit as the number of path segments goes to infinity.

In our case, the function inside the delta functional lives on the world sheet $\Sigma$, whose integration measure is $d^2\sigma\,\sqrt{\hat{g}}$ (remember we fixed the fiducial metric). Hence, if $\beta$ ranges over symmetric 2-tensors on $\Sigma$, then

$$\Delta_{\text{FP}}^{-1}(\hat{g}) = \int \mathcal{D}\omega\,\mathcal{D}v\,\mathcal{D}\beta\, \exp\left(2\pi i \int d^2\sigma\,\sqrt{\hat{g}}\,\beta^{ab}(2\omega\hat{g}_{ab} + \nabla_a v_b + \nabla_b v_a)\right).$$

But we can directly do the integral over $\omega$. The one and only term containing an $\omega$ factors out to give a delta functional:

$$\int \mathcal{D}\omega\, \exp\left(2\pi i \int d^2\sigma\,\sqrt{\hat{g}}\,\beta^{ab}(2\omega\hat{g}_{ab})\right) = \delta[2\beta^{ab}\hat{g}_{ab}],$$

i.e. in the remaining integral, $\beta^{ab}$ is traceless:

$$\Delta_{\text{FP}}^{-1}(\hat{g}) = \int \mathcal{D}v\,\mathcal{D}\beta\, \exp\left(2\pi i \int d^2\sigma\,\sqrt{\hat{g}}\,\beta^{ab}(\nabla_a v_b + \nabla_b v_a)\right).$$

**Recap**: we are integrating over vector fields $v$ and symmetric 2-tensors $\beta$ such that $\beta^{ab}$ is traceless, both living on $\Sigma$.

### 1.5.3 Faddeev-Popov Ghosts

We are not done: the path integral above is for $\Delta_{\mathrm{FP}}^{-1}$, but we want $\Delta_{\mathrm{FP}}$ itself. There is a general procedure for inverting $\Delta_{\mathrm{FP}}^{-1}$. To understand it, we must first clarify what $\Delta_{\mathrm{FP}}$ really is. Let $F$ is the gauge-fixing condition. (For us, $F$ is a function of $g$ and $\zeta$ and takes values in symmetric 2-tensors.) Note that via a change of variables from $\zeta$ to $F$,

$$\Delta_{\mathrm{FP}}^{-1} = \int D\zeta\, \delta(F) = \int DF \,\det\left[\frac{\delta\zeta}{\delta F}\right]\delta(F) = \det\left[\frac{\delta\zeta}{\delta F}\right]_{F=0}.$$

This change of variables is valid again because we assume $\zeta$ acts freely on gauge orbits, and $F$ is supposed to pick a unique representative from each gauge orbit, so $\zeta$ and $F$ "have the same number of degrees of freedom" as physicists like to say. Now all we have to do is invert the determinant. For this, we use a clever trick, which is developed in the following two exercises.

**Exercise 1.5.4.** Show by analogy from the finite dimensional case for two real fields $\phi^1$ and $\phi^2$ that

$$\int \mathcal{D}\phi^1\,\mathcal{D}\phi^2 \exp\left(i\int d^Dx\,\phi^1 A\phi^2\right) = (\det A)^{-1}.$$

**Exercise 1.5.5.** Recall from QFT that we defined **Grassmann numbers**: they are anti-commuting formal variables, i.e. $\theta\eta = -\eta\theta$, that form an algebra. We also worked out the **Berezin integral** for Grassmann-valued quantities, with the convention that $\int d\theta \int d\eta\,\eta\theta = 1$. If $\theta$ and $\eta$ are Grassmann variables, i.e. taking values in the Grassmann algebra, and $b \in \mathbb{R}$, review/show (in order) that

$$\theta^2 = 0, \qquad \int d\theta f(\theta) = \frac{\partial f}{\partial\theta}, \qquad \int d\theta\, d\eta\, \exp(-\theta b\eta) = b$$

Hence show by analogy with the finite dimensional case that for Grassmann-valued fields $\chi^1$ and $\chi^2$,

$$\int \mathcal{D}\chi^1\,\mathcal{D}\chi^2 \exp\left(-\int d^Dx\,\chi^1 A\chi^2\right) = \det A.$$

So here's the trick: if we have a path integral expression for $(\det A)^{-1}$, to get $\det A$ we simply replace ordinary variables with Grassmann variables! In particular, to get $\Delta_{\mathrm{FP}}(\hat g)$ from $\Delta_{\mathrm{FP}}(\hat g)^{-1}$, we replace $(\beta_{ab}, v^a)$ with Grassmann-valued fields $(b_{ab}, c^a)$, with $b^{ab}$, like $\beta^{ab}$, being traceless:

$$\Delta_{\mathrm{FP}}(\hat g) = \int \mathcal{D}b\,\mathcal{D}c\,\exp(S_{\mathrm{G}}), \quad S_{\mathrm{G}} \coloneqq \frac{1}{2\pi}\int d^2\sigma\,\sqrt{\hat g}\,b_{ab}\nabla^a c^b.$$

Note that we've implicitly made a few cosmetic changes:

1. Because $b$ is a symmetric 2-tensor (do **not** confuse the fact that $b$ is symmetric, i.e. $b_{ab} = b_{ba}$, with $b$ being anti-commutative, e.g. $b_{ab}\theta = -\theta b_{ab}$), we can rewrite
$$b^{ab}(\nabla_a c_b + \nabla_b c_a) = b^{ab}\nabla_a c_b + b^{ab}\nabla_a c_b = 2b^{ab}\nabla_a c_b = 2b_{ab}\nabla^a c^b.$$
2. We chose slightly different normalization factors to make later computations cleaner.

The quantity $S_{\mathrm{G}}$ is called the **ghost action**: when we plug $\Delta_{\mathrm{FP}}(\hat g)$ back into the path integral, we get

$$Z = \int \mathcal{D}X\,\mathcal{D}b\,\mathcal{D}c\,\exp(-S_{\mathrm{P}}[X,\hat g] - S_{\mathrm{G}}[b,c]),$$

i.e. $S_{\mathrm{G}}$ becomes part of the action. The fields $b$ and $c$, which do not correspond physically to anything, are **Faddeev-Popov ghost fields**. The price of gauge fixing is the introduction of these unphysical ghosts.

**Exercise 1.5.6.** Repeat the computation of $\Delta_{\mathrm{FP}}$ for QED, and show that for QED, $\Delta_{\mathrm{FP}}$ is independent of any fields. Hence conclude that QED has no Faddeev-Popov ghosts. (That's why quantizing QED went a lot faster. QCD has ghosts, however.)

# Chapter 2

# Conformal Field Theory

String theory as we have defined it so far is a 2 dimensional theory where the fields are parameterized by two coordinates $(\sigma^1, \sigma^2)$. We shall now explore the conformal symmetry of the Polyakov action and deduce a number of important technical tools that will enable us to say a lot about the properties of this quantum field theory. This conformal symmetry is especially large in two dimensions and provides significant constraints.

The technical tool that will drive this whole chapter is the **operator product expansion** (OPE). This is a canonical form for the product of two local operators:

$$\mathcal{A}_i(\sigma_1)\mathcal{A}_j(\sigma_2) = \sum_k c_{ij}^k(\sigma_1 - \sigma_2)\mathcal{A}_k(\sigma_2). \tag{2.1}$$

This will turn out to be much like a Laurent expansion, however, in a conformally invariant theory the form of the $c_{ij}^k(\sigma_1 - \sigma_2)$ is severely restricted.

There are many reasons why it is useful for us to learn about CFT. One reason is that certain critical phase transitions can be described by a CFT and using the AdS/CFT correspondence we may be able to take a highly correlated system and rewrite it in terms of a weakly coupled theory of supergravity. To understand this sentence will require lots of time, so let's not waste anymore!

The plan for this chapter as of January 1, 2016 will be to showcase important details of chapter 2 from Polchinski's Volume 1 leaving out some technical details for as exercises. The introductory section is based off of Ginsparg's Applied CFT, while the Operator Technqiues is based off of DiFrancesco et al. In the future it would be nice to include $d$-dimensional CFT and it's application to condensed matter systems especially because this is one of the most immediate applications of string theory.

## 2.1 Introduction to CFT

Let's be frank. The most important objects in quantum field theory are correlation functions: every kind of interaction is expressed in terms of how fields are correlated with each other. Therefore it is big news that when a system admits conformal symmetry, the correlation functions are so severely constrained that plenty of analytical tools can be applied to solve problems. In this introduction we will only glimpse the power of conformal symmetry to a 2d CFT.

TALK ABOUT THIS!

- First we begin with some terminology. In our adventures on QFT, we defined fields from some base manifold into a vector space, $\phi : M \to V$. The manifold $M$ in our case was locally Minkowski, which in fancy terms meant that we were building a QFT on spacetime. In string theory, what we are doing is building a QFT on a 2D worldsheet with values being space-time with (possibly) more than one spatial dimension: $X^\mu : S^1 \times \mathbb{R} \to \mathbb{R}^{1,d-1}$.

- Second, when we write expressions like $X^\mu X_\mu$ we are referring explicitly to an underlying metric, $g$, which assigns meaning to contractions: $g_{\mu\nu} X^\mu X^\nu$.

Now let's introduce the main object in CFT: the **conformal transformations**. To do this we begin with a question: on a Riemannian manifold, locally $(\mathbb{R}^d, \eta)$, what are diffeomorphisms that fix the metric up to scaling:

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x).$$

If we define an infinitesimal coordinate transformation $x^\mu \to x^\mu + \epsilon^\mu$ then the line element transforms as:

$$ds^2 \to ds^2 + (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) dx^\mu dx^\nu.$$

Therefore we must have $\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu$ is proportional to $\eta_{\mu\nu}$ which imposes the following constraint:

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d} (\partial \cdot \epsilon) \eta_{\mu\nu}. \tag{2.2}$$

For $d > 2$ the constraints give rise to the following possible transformations[1]

1. Inhomogeneous Poincaré group: $\epsilon^\mu = \omega^\mu_\nu x^\nu + a^\mu$, with $\omega^{\mu\nu}$ antisymmetric,
2. Scalings: $\epsilon^\mu = \lambda x^\mu$,
3. Special conformal: $\epsilon^\mu = b^\mu x^2 - 2x^\mu b \cdot x$.

When $d = 2$ something special happens. The constraint (2.2) reduces to the Cauchy-Riemann conditions, which means that the finite transformation corresponding to the conformal transformations are simply holomorphic functions.

**RESTRICTIONS ON CORRELATION FUNCTIONS**

Two point functions $C = \langle \phi_1(x_1) \phi_2(x_2) \rangle$

- Translational invariance $\implies C(x_1 - x_2)$.
- Rotational invariance $\implies C(|x_1 - x_2|)$
- Scale invariance $\implies C(r_{ij}/r_{kl})$
- Special conformal $\implies C(\frac{r_{ij} r_{kl}}{r_{ik} r_{jl}})$

*Remark* (**Conformal Bootstrap**). We may express the OPEs (ie. the correlation functions) for a product of any two or more fields, using only the quasi-primary fields. (Cf. Chapter 15 – Vol 2 Polchinski)

## 2.2 Conformal Normal Order and Operator Product Expansions

In our QFT adventures we focused on computing correlation functions since every physical quantity could be expressed in terms of them. However, in our journey we focused a lot on operators of the form $\langle \phi_1 \phi_2 \cdots \phi_n \rangle$. We shall now generalize this ever so slightly.

**Definition 2.2.1.** Let $\sigma_0$ be a fixed point and consider a classical world-sheet field theory with fields $X_1(\sigma), \ldots, X_n(\sigma)$. A **local functional** is a function $\mathscr{F}[X]$ taking in, as arguments, $X_i(\sigma_0)$ and $\partial_a X_i(\sigma_0)$ which are taken at $\sigma_0$. A **local operator** is the quantized version of a local functional that has well defined expectation values. Often a local operator is given by the normal ordering of a local functional.

[1] see Ginsparg pg 5

Here are some examples of local operators $X^\mu(0,0), X^\mu(a,b)X_\mu(a,b), \partial_{\bar{z}}X^3(z_1,\bar{z}_1)$. However, $X^\mu(z_1,\bar{z}_1)+X^\mu(z_2,\bar{z}_2)$ is not a local operator. Before introducing the operator product expansion we need to introduce conformal normal ordering which will be used in the definition of OPE.

**Definition 2.2.2.** Write $z_{12} = z_1 - z_2$. Let $\mathscr{F}$ be an arbitrary function of $X$. Define the (free-field) **normal order** of $\mathscr{F}$ to be the functional:

$$:\mathscr{F}: = \mathscr{F} + \sum (\text{subtractions}) \tag{2.3}$$

$$= \exp\left(\frac{\alpha'}{2}\int d^2z_1 d^2z_2 \ln|z_{12}|^2 \frac{\delta}{\delta X^\mu(z_1,\bar{z}_1)}\frac{\delta}{\delta X_\mu(z_2,\bar{z}_2)}\right)\mathscr{F} \tag{2.4}$$

This is very analogous to the normal ordering that we saw in QFT where the coefficient $\eta^{\mu\nu}\ln|z_{12}|^2$ is replaced by the corresponding propogator $\Delta(z_1,z_2)$. The QFT version of normal ordering is useful for calculating matrix elements, while this version is useful for computing the OPE.

Here are two examples: $:X^\mu(z,\bar{z}): = X^\mu(z,\bar{z})$, and

$$:X^\mu(z_1,\bar{z}_2)X^\nu(z_2,\bar{z}_2): = X^\mu(z_1,\bar{z}_2)X^\nu(z_2,\bar{z}_2) + \frac{\alpha'}{2}\eta^{\mu\nu}\ln|z_{12}|^2. \tag{2.5}$$

The reason why this ordering is useful is because of the following:

**Conjecture 2.2.3** (Fundamental Property of Normal Ordering)**.** *Normal ordered expressions satisfy the classical equations of motion averaged over paths. In the case of classical bosonic field this amounts to saying:*

$$\langle \partial\bar{\partial} :\mathscr{F}: \rangle = 0.$$

This is true for instance in the case of $\langle \partial\bar{\partial} :X^\mu(z_1,\bar{z}_1)X^\nu(z_2,\bar{z}_2): \rangle = 0$ (Polchinski Vol 1 Page 36).

**Proposition 2.2.4.** *Let $\mathscr{F},\mathscr{G}$ be two local operators. Then,*

$$:\mathscr{F}::\mathscr{G}: =:\mathscr{F}\mathscr{G}: + \sum (\textit{cross-contractions}) \tag{2.6}$$

$$= \exp\left(-\frac{\alpha'}{2}\int d^2z_1 d^2z_2 \ln|z_{12}|^2 \frac{\delta}{\delta^\mu_F(z_1,\bar{z}_1)}\frac{\delta}{\delta_{\mu G(z_2,\bar{z}_2)}}\right) :\mathscr{F}G: \tag{2.7}$$

*Remark.* The operator product expansion, as given by (2.1), is our definition of the OPE. However, it turns out, just like in complex analysis, that the singular part of this expansion is the one that plays the most crucial role. What does the singular part of the OPE do for us? It turns out that it gives us a way to compute the variation of local operators under conformal transformations. Here's how. Using the Ward identity in $d = 2$ we relate $\delta\mathscr{A}(z_0,\bar{z}_0)$ with the residue of $j(z)\mathscr{A}(z_0,\bar{z}_0)$ at $z_0$. Rewrite $j$ in terms of the energy-momentum tensor and use the singular part of the OPE of $T\mathscr{A}$ to calculate the residue.

Now we describe how to compute the singular part of the OPE in free field theory. In particular, this means that the following derivation only works for the bosonic non-interacting string. We will revise and possibly generalize our method later, if need be. The key observation is that harmonic functions can locally be written as a sum of a holomorphic and antiholomorphic part.

**Lemma 2.2.5.** *Let $f : \mathbb{C} \to \mathbb{C}$ be harmonic. Then $\partial\bar{\partial}f = 0$ and so $f = a(z) + b(\bar{z})$ where $a$ is holomorphic and $b$ is antiholomorphic.*

Using Prop. 2.2.4 we notice that, since $:\mathscr{F}:$ is non-singular, the singular part in the OPE of $:\mathscr{F}::\mathscr{G}:$ is given by the coefficient functions in the cross-contractions.

**Example 2.2.6.** Using the definition of normal ordering (2.3), we may write (cf. (2.5))

$$X^\mu(z_1,\bar z_1)X^\nu(z_2,\bar z_2) = -\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z_{12}|^2 + \sum_{k=1}^{\infty}\frac{1}{k!}(z_{12})^k :X^\nu\partial^k X^\mu(z_2,\bar z_2): +(\bar z_{12})^k :X^\nu\bar\partial^k X^\mu(z_2,\bar z_2):$$
$$\sim -\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z_{12}|^2$$

The first equation is the full operator product expansion and the equivalence (up to singular terms) shows that $X^\mu X^\nu$ behaves like $\ln|z_{12}|^2$ for $z_1 \to z_2$.

**Example 2.2.7.** Let's suppose we have a product of two composite operators. $\mathscr{F}(z) = \partial X^\mu(z)\partial X_\mu(z)$ and $\mathscr{G}(z') = \partial' X^\nu(z')\partial' X_\nu(z')$. Using the harmonicity of normal ordering we obtain:

$$:\mathscr{F}(z)::\mathscr{G}: =:\mathscr{F}(z)\mathscr{G}(z'): -4\frac{\alpha'}{2}(\partial\partial'\ln|z-z'|^2) :\partial X^\mu\partial' X_\mu(z'): +2\eta^\mu_\mu\left(-\frac{\alpha'}{2}\partial\partial'\ln|z-z'|^2\right)^2$$
$$\sim \frac{D\alpha'^2}{2(z-z')^4} - \frac{2\alpha'}{(z-z')^2} :\partial X^\mu\partial' X_\mu(z'): -\frac{2\alpha'}{z-z'} :\partial X^\mu\partial' X_\mu(z'):$$

In general CFTs we require the basis in which we expand operator products to transform like a tensor under conformal transformations. Moreover, the conformal invariance then puts even more restrictions on the coefficient functions rendering them unique up to a constant.

### 2.2.1 Ward Identity

Although the idea of the OPE is what drives this chapter, the Ward identity is the oil that makes the engine turn (cf. Remark 2.2). Suppose we are given a coordinate transformation $\sigma' = \sigma + \delta\sigma$, that is a symmetry of the theory, how do operators transform under this transformation? Denote the transformation of fields as follows: $X'_\mu(\sigma) = X_\mu(\sigma) + \delta X_\mu(\sigma)$. Now we consider a slightly more general transformation:

$$X'_\mu(\sigma) = X_\mu(\sigma) + \rho(\sigma)\delta X_\mu(\sigma).$$

Such a general transformation might not be a symmetry of the action. However, the path integral *is* invariant under change of coordinates, which means:

$$0 = \delta\left(\int \mathcal{D}X\, e^{-S[X]}\mathscr{A}(\sigma_0)\right) = \int \mathcal{D}X\,\delta(e^{-S[X]})\mathscr{A}(\sigma_0) + e^{-S[X]}\delta\mathscr{A}(\sigma_0)$$
$$= \int \mathcal{D}X\,(d^d\sigma\sqrt{g})e^{-S[X]}j^a(\sigma)\partial_a\rho(\sigma)\mathscr{A}(\sigma_0) + e^{-S[X]}\delta\mathscr{A}(\sigma_0)$$

Applying Stoke's theorem:

$$\langle\delta\mathscr{A}(\sigma_0)\rangle = \frac{i\epsilon}{2\pi}\int d^d\sigma\sqrt{g}\,\langle\partial_a j^a(\sigma)\rangle$$
$$= \frac{i\epsilon}{2\pi}\langle\oint_{\partial R} dA\, n^a j_a(\sigma)\mathscr{A}(\sigma_0)\rangle$$

In operator form and in $d=2$ this looks like

$$\frac{2\pi}{i\epsilon}\delta\mathscr{A}(\sigma_0) = \oint_{\partial R}(j_z dz - j_{\bar z}d\bar z)\mathscr{A}(z_0,\bar z_0) \qquad (2.8)$$

In the case that $j_z$ and $j_{\bar z}$ are (anti)holomorphic then we have the following relation:

$$\mathrm{Res}_{z\to z_0}\, j(z)\mathscr{A}(z_0,\bar z_0) + \overline{\mathrm{Res}}_{\bar z\to\bar z_0}\tilde j(\bar z)\mathscr{A}(z_0,\bar z_0) = \frac{1}{i\epsilon}\delta\mathscr{A}(z_0,\bar z_0).$$

### 2.2.2 Applications of OPE

Let us show that the $X^\mu$-theory is conformally invariant. This amounts to showing if $z' = f(z)$, for some holomorphic $f$, then $X'^\mu(z', \bar{z}') = X(z, \bar{z})$. For our purposes it will be easier to check this infinitesimally. consider

$$z' = z + \epsilon v(z) \tag{2.9}$$

for holomorphic $v$ (and similarly for the $\bar{z}'$). We want to show that such a transformation gives rise to the following variation:

$$X'^\mu(z', \bar{z}') = X^\mu(z, \bar{z}) - \epsilon v^a(z)\partial_a X^\mu(z, \bar{z}) - \epsilon v^a(z)^* \bar{\partial} X^\mu$$

because this is the infinitesimal version of $X'^\mu(z', \bar{z}') = X(z, \bar{z})$. The idea will be to use the Ward identity,

$$\mathrm{Res}_{z\to z_0}\, j(z)\mathscr{A}(z_0, \bar{z}_0) + c.c. = \frac{1}{i\epsilon}\delta\mathscr{A}(z_0, \bar{z}_0),$$

to compute the variation of $\mathscr{A}$. Therefore, we must first compute the current $j^a(z)$ corresponding to $v^a(z)$, then compute the OPE $j(z)\mathscr{A}(z_0, \bar{z}_0)$ to understand the asymptotics around $(z_0, \bar{z}_0)$, and finally compute the residue to obtain the symmetry that we are interested in.

**Exercise 2.2.1.** Show that the Noether current, corresponding to the symmetry (2.9), is given by $j_a = iv^b T_{ab}$ where $T_{ab}$ is the normal ordered version of the stress-energy tensor

$$T_{ab} = -\frac{1}{\alpha'} :\!\left(\partial_a X^\mu \partial_b X_\mu - \frac{1}{2}\delta_{ab}\partial_c X^\mu \partial^c X_\mu\right)\!: \;.$$

Moreover, show that $T^a_a = 0$, that is the tensor is traceless. Rewriting this in complex coordinates, show this is equivalent to $T_{z\bar{z}} = 0$. Also, using $\partial^a T_{ab} = 0 = T^a_a$, we have $\bar{\partial} T_{zz} = \partial T_{\bar{z}\bar{z}} = 0$, showing that $T = T_{zz}, \tilde{T} = T_{\bar{z}\bar{z}}$ are holomorphic and anti-holomorphic. Next, show that the OPEs of $T\mathscr{A}$ and $\tilde{T}\mathscr{A}$ have the following asymptotics:

$$T(z)X^\mu(0) \sim \frac{1}{z}\partial_z X^\mu(0), \qquad \tilde{T}(\bar{z})X^\mu(0) \sim \frac{1}{\bar{z}}\bar{\partial} X^\mu(0).$$

### 2.2.3 TODO: $bc$ theory – an extended example

The $bc$ ghost theory appears often in string theory. We start with an action

$$S = \frac{1}{2\pi}\int d^2z\, b\bar{\partial}c$$

and we posit that $b$ and $c$ are anticommutators that are holomorphic primary fields $(h_b, 0) = (\lambda, 0)$ and $(h_c, 0) = (1 - \lambda, 0)$.

### 2.2.4 Primary and Quasi-Primary Fields

In a CFT, we would like to use a particular basis for the OPE (2.1). This is a set of local operators which transform under conformal transformations similar to a tensor:

$$\mathscr{O}'(z', \bar{z}') = (\partial z')^{-h}(\bar{\partial}\bar{z}')^{-\tilde{h}}\mathscr{O}(z, \bar{z}). \tag{2.10}$$

We call such a local operator a **primary field** or **conformal tensor** of weight $(h, \tilde{h})$. These quasi-primary fields, by definition, play nice with conformal transformations, thus we may expect that the OPE of $T\mathscr{O}$ will be particularly nice. In fact, this does turn out to be the case:

$$T(z)\mathscr{O}(0,0) = \frac{h}{z^2}\mathscr{O}(0,0) + \frac{1}{z}\partial\mathscr{O}(0,0) + \cdots \tag{2.11}$$

**Example 2.2.8.** The operator $:(\prod_i \partial^{m_i} X^{\mu_i})(\prod_j \partial^{n_j} X^{\nu_j})e^{ik\cdot X}:$ has weight $\left(\frac{\alpha' k^2}{4} + \sum_i m_i, \frac{\alpha' k^2}{4} + \sum_j n_j\right)$.

**Proposition 2.2.9** (Refined OPE)**.** *Using rigid translations, scaling and rotations to both sides of an OPE we can write, for any two primary operators $\mathscr{A}_i, \mathscr{A}_j$:*

$$\mathscr{A}_i(z_1,\bar{z}_1)\mathscr{A}_j(z_2,\bar{z}_2) = \sum_k z_{12}^{h_k-h_i-h_j}\,\bar{z}_{12}^{\tilde{h}_k-\tilde{h}_i-\tilde{h}_j}\mathscr{A}_k(z_2,\bar{z}_2) \tag{2.12}$$

**Example 2.2.10.** *bc* CFT There are many different free conformal field theories. We have, in fact, already met with two in the first chapter. The first is the $X^\mu$ theory, which we have gotten to know quite well. The second comes from § 1.5.3: Faddeev-Popov ghosts $b_{ab}, c^a$ with action

$$S_G = \frac{1}{2\pi}\int d^2\sigma b_{ab}\partial^a c^b \tag{2.13}$$

is a free CFT where $b, c$ are primary fields (conformal tensors) with weights $(h_b, 0) = (\lambda, 0)$ and $(h_c, 0) = (1-\lambda, 0)$. For this theory we can compute the OPEs, and all of the other quantities in a similar manner to what we did above. All of these important facts are left as exercises with answers in Polchinski pg 50-51.

## 2.3 Hilbert Spaces and Operator Techniques

Much of what we covered above can be described solely using the path integral formulation, however, canonical quantization allows powerful algebraic techniques to be employed to study CFTs. Let us now attempt to describe the Hilbert space.

We are going to construct a Hilbert space on $S^1 \times \mathbb{R} \cong \mathbb{C}$ where the isomorphism is a conformal mapping. The isomorphism takes $(x,t) \mapsto e^{2\pi(t+ix)/L}$. Next, we assume that there exists a ground state $|0\rangle$ in the theory and to every operator $\phi$ we define the incoming state:

$$|\phi_{\text{in}}\rangle = \phi(0,0)\,|0\rangle\,.$$

This identification is one direction of the state-operator correspondence and we will discuss at length before we get to calculating string amplitudes.

What we can do next, is consider expanding out the $z, \bar{z}$ dependence out of a conformal field:

$$\phi(z,\bar{z}) = \sum_{m,n\in\mathbb{Z}} \frac{1}{z^{m+h}}\frac{1}{\bar{z}^{n+h}}\phi_{m,n}$$
$$\phi_{m,n} = \frac{1}{(2\pi i)^2}\oint dz \oint d\bar{z}\, z^{m+h-1}\bar{z}^{n+h-1}\phi(z,\bar{z})$$

In conformal field theories the holomorphic and antiholomorphic part are often decoupled so that we may write only one and recover the other when needed.

### 2.3.1 Radial Ordering, OPEs $\Leftrightarrow$ Commutation Relations

Our next goal is to write down an expression for the Hamiltonian. This will happen in these next two subsections. Read carefully, otherwise you might miss it!

Recall that correlation functions defined in QFT are always interpreted to be time-ordered. In our case, since time corresponds to the radial part of the complex numbers we are interested in radial order of operators:

$$\mathcal{R}\Phi(z)\Phi(w) = \begin{cases}\Phi(z)\Phi(w) & |z|>|w| \\ \Phi(w)\Phi(z) & |w|>|z|\end{cases}.$$

From now on, then, we will assume that all expressions involving operators will be radially ordered. Now, we switch over to describing the relations between OPEs and commutators. If $a(z), b(z)$ are two holomorphic fields, Notice that radial ordering on the expression $\oint_w dz\, a(z)b(w)$, implies that we must have:

$$\begin{aligned}\oint_w dz\, a(z)b(w) &= \oint_{C_1} dz\, a(z)b(w) - \oint_{C_2} dz\, b(w)a(z) \\ &= [A, b]\end{aligned}$$

where $A = \oint_C a(z)dz$ and the counterclockwise contours, $C_1, C_2$, are circles centred around the origin with radii $|w| + \epsilon, |w| - \epsilon$. Therefore we can define the commutator of two operators which are integrals of holomorphic fields as:

$$[A, B] = \oint_0 dw \oint_w dz\, a(z)b(w) \tag{2.14}$$

This formula is very important because it translates results from the OPE language into operator language. The generators for the Virasoro algebra are one example of operators that can be described as integrals of (anti)holomorphic fields. In this way, we are able to calculate the commutator relations and from there we will derive the raising and lowering operators for the Hilbert space.

### 2.3.2 Virasoro Algebra

First, let $z' = z + \epsilon(z)$ denote the infinitesimal *conformal* transformation and define the conformal charge by:

$$Q_\epsilon = \frac{1}{2\pi i} \oint dz\, \epsilon(z) T(z)$$

Now using the conformal Ward identity (2.8) with $j_z = \epsilon(z)T(z)$, we get:

$$\begin{aligned}\delta_\epsilon X &= \frac{1}{2\pi i} \oint dz \epsilon(z) \langle T(z) X \rangle \\ &= [Q_\epsilon, X]\end{aligned}$$

Next, we define the generators $\{L_n, \bar{L}_n\}$ of the Virasoro algebra by the modes of the holomorphic and antiholomorphic parts of the energy momentum tensor:

$$T(z) = \sum_{n \in \mathbb{Z}} \frac{1}{z^{n+2}} L_n \qquad\qquad L_n = \frac{1}{2\pi i} \oint dz\, z^{n+1} T(z)$$

Using the results of the previous section, we may translate our knowledge of the OPEs of the energy momentum tensor to obtain commutation relations between the $L_n$'s. Applying (2.14) we get the following commutation relations:

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12} n(n^2 - 1)\delta_{n+m,0} \tag{2.15}$$

But there's more! If we expand the infinitesimal conformal change, $\epsilon(z) = \sum_{n \in \mathbb{Z}} z^{n+1} \epsilon_n$, where the $\epsilon_n$'s are c-numbers, then the conformal charge is given by the expression:

$$Q_\epsilon = \sum_{n \in \mathbb{Z}} \epsilon_n L_n. \tag{2.16}$$

If we take a look at what happens when only $\epsilon_k = \lambda \delta_{k,0}$, then the transformation is just a scaling by $\lambda$ which in our case means time translations! Thus,

$$\delta_\epsilon X = [Q_\epsilon, X] \tag{2.17}$$
$$= [L_0 + \tilde{L}_0, X] \tag{2.18}$$

In particular, because we identified the left hand side of the above equation as time-translations, this means that $Q_\epsilon$ is the conserved charge corresponding to time translations. Therefore $L_0 + \tilde{L}_0$ is proportional to the Hamiltonian of the CFT.

### 2.3.3 Verma Modules

This section will be a very quick introduction to a "representation" of the Virasoro algebra, that is, the Verma modules.

- The vacuum state needs to be invariant under global conformal transformations.
- Primary fields define asymptotic states that will turn out to be the highest weight states: $|\phi_{in}\rangle = \phi(0,0)|0\rangle$ with $\dim \phi = (h, \bar{h})$ implies

$$L_0 |h, \bar{h}\rangle = h |h, \bar{h}\rangle \,,$$
$$\bar{L}_0 |h, \bar{h}\rangle = \bar{h} |h, \bar{h}\rangle \,,$$
$$L_n |h, \bar{h}\rangle = 0 \;\; \forall n > 0$$

- By mode expanding the primary field we obtain raising and lowering operators: $\phi(z, \bar{z}) = \sum_{n \in \mathbb{Z}} \frac{1}{z^{n+h}} \phi_n$. It can be shown (using your favourite OPE) that

$$[L_0, \phi_{-m}] = m\phi_{-m}.$$

It turns out that the $L_{-m}$, $m > 0$ also increase the conformal dimension:

$$[L_0, L_{-m}] = mL_{-m}.$$

Repeated application of the $L_{-m}$'s will generate a Verma module of the Virasoro algebra.

## 2.4 Vertex Operators

Initial states correspond to equal time objects with $\tau \to -\infty$. Under our identification with the complex plane means that this state at $\tau = -\infty$ should correspond to an object local object, where local here refers to the same usage in the definition of local operator. Thus, local operators correspond to initial states and vice versa. Let us understand this correspondence through examples. Consider the identity operator $\mathbb{1}$, what should the state $|\mathbb{1}\rangle$ correspond to? Well, notice that the charges $\alpha^\mu_m$ , for $m > 0$, annihilate the state $|\mathbb{1}\rangle$ since:

$$\sqrt{\frac{2}{\alpha'}} \int \frac{dz}{2\pi}\, z^m \partial X^\mu \cdot \mathbb{1} = 0$$

since the argument of the integral is holomorphic and similarly for $\tilde{\alpha}^\mu_m |\mathbb{1}\rangle$. Therefore, $|\mathbb{1}\rangle$ is annihilated by all lowering operators which means that it is proportional to the vacuum, $|0, 0; 0\rangle$. In fact, we may define it this way:

$$|\mathbb{1}\rangle \equiv |0, 0; 0\rangle \,.$$

Now that we have the vacuum we can go the other way: given excitations what are the vertex operators corresponding to them?

$$\begin{aligned}\alpha^\mu_{-m} \left|\mathbb{1}\right\rangle &= \sqrt{\frac{2}{\alpha'}} \int \frac{dz}{2\pi}\, z^{-m} \partial X^\mu \\ &= \sqrt{\frac{2}{\alpha'}} \frac{i}{(m-1)!} \partial^m X^\mu(0), m \geq 1\end{aligned}$$

In particular,

$$\alpha^\mu_{-1} \left|\mathbb{1}\right\rangle = i\sqrt{2/\alpha'}\partial X^\mu(0), \qquad \tilde{\alpha}^\mu_{-1} \left|\mathbb{1}\right\rangle = i\sqrt{2/\alpha'}\bar{\partial} X^\mu(0).$$

# Chapter 3

# BRST Quantization

We now have enough background in conformal field theory to continue our exploration into string theory. However, for completeness, there are a few loose ends we must tie up first, and a few more tools we must develop.

The first problem we tackle is that our canonical and path integral quantizations of the Polyakov action were somewhat unsatisfactory.

1. We were vague about how canonical quantization imposes the quantized Virasoro conditions: how can we tell which states are physical using our theory?
2. We did not derive the spectrum or state space using the path integral. With the introduction of ghost fields, not every configuration that is integrated over is a valid state.

**Becchi-Rouet-Stora-Tyutin (BRST) quantization** cures these problems simultaneously. It is a much more advanced method for quantizing a field theory with gauge symmetries and constraints. Instead of applying it directly to the open bosonic string, we first develop it in general.

The setting is any $D$-dimensional field theory with action $S$, and fields $\phi_r$, infinitesimal gauge symmetries $K_\alpha$, and gauge-fixing conditions $F^A[\phi]$. The infinitesimal gauge symmetries form a Lie algebra $G$, i.e. $[K_\alpha, K_\beta] = f_{\alpha\beta}{}^\gamma K_\gamma$ where $f_{\alpha\beta}{}^\gamma$ are the **structure constants** of $G$. Importantly, the $f_{\alpha\beta}{}^\gamma$ must be independent of the fields $\phi_r$. This condition does not hold for all gauge theories: when it does not, we must rely on some even more sophisticated machinery known as the **Batalin-Vilkovisky (BV) formalism**. Fortunately, we do not need to for the bosonic string.

**Note**: indices become messy in this section. We use $A$ (and not $B$, which appears as a non-indexing subscript later) to index the gauge-fixing conditions, i.e. the degrees of gauge freedom, $\alpha, \beta, \ldots$ to index the gauge symmetries, and $r, s, \ldots$ to index the fields. This avoids conflict with $a, b, \ldots$, which index worldsheet coordinates, and $i, j, \ldots$ and $\mu, \nu, \ldots$, which index space and spacetime coordinates.

**More important note**: throughout this section we use **deWitt notation**, where indices on fields not only index field components, but also spacetime. For example, $\phi_r$ is really $\phi_i(x)$ where $r$ ranges over all possible values of $i$ and $x$. Consequently, when contracting two deWitt indices, we must also integrate over the spacetime variable(s), with the appropriate measure.

**Exercise 3.0.1.** Apply the Faddeev-Popov method to this more general setting to obtain the gauge-fixed path integral

$$\int \mathcal{D}\phi_r \, \mathcal{D}B_A \, \mathcal{D}b_A \, \mathcal{D}c^\alpha \, \exp(-S - S_{\mathrm{gf}} - S_{\mathrm{G}})$$

Here, as for the bosonic string,

- $S$ is the original gauge-invariant action,
- $S_{\text{gf}} := -iB_A F^A[\phi]$ is the **gauge fixing action**, and
- $S_{\text{G}} := b_A c^\alpha K_\alpha F^A[\phi]$ is the **Faddeev-Popov ghost action**.

(We did not explicitly see $S_{\text{gf}}$ for the bosonic string because we immediately integrated it away: there, $\int \mathcal{D}B_A \exp(-S_{\text{gf}}) = \delta[\hat{g}^\zeta - g]$. If we had written the $\delta$ functional as a path integral, its variable of integration would have been $B_A$.) The auxiliary fields $B_A$ are sometimes called **Nakanishi-Lautrup fields**.

The resulting action $S + S_{\text{gf}} + S_{\text{G}}$ is not gauge-invariant anymore, but it has a very important symmetry called **BRST symmetry**.

## 3.1 BRST Symmetry

**Definition 3.1.1.** The infinitesimal **BRST transformation** $\delta_B$ is given by

$$\begin{aligned}
\delta_B \phi_r &:= -i\theta c^\alpha K_\alpha \phi_r, \\
\delta_B B_A &:= 0, \\
\delta_B b_A &:= \theta B_A, \\
\delta_B c^\alpha &:= \frac{i\theta}{2} f_{\beta\gamma}{}^\alpha c^\beta c^\gamma.
\end{aligned}$$

(This is not the cleanest way of writing the BRST transformation; be assured that this definition can actually be very well-motivated, as we see soon.)

**Proposition 3.1.2.** *The BRST transformation $\delta_B$ is a symmetry of the action $S + S_{gf} + S_G$, i.e.*

$$\delta_B(S + S_{gf} + S_G) = 0.$$

*Proof/Exercise.* First we establish two small identities, both of which are fairly straightforward:

$$\delta_B(F^A[\phi]) = -i\theta c^\alpha K_\alpha F^A[\phi], \qquad c^\alpha c^\beta K_\alpha K_\beta = \frac{1}{2} c^\alpha c^\beta f_{\alpha\beta}{}^\gamma K_\gamma.$$

Using these two identities and that $c^\alpha$ and $c^\beta$ anti-commute,

$$\begin{aligned}
\delta_B(c^\alpha K_\alpha F^A[\phi]) &= \left(\frac{i}{2}\theta f_{\beta\gamma}{}^\alpha c^\beta c^\gamma\right) K_\alpha F^A[\phi] + c^\alpha K_\alpha(-i\theta c^\beta K_\beta F^A[\phi]) \\
&= \frac{i}{2}\theta f_{\beta\gamma}{}^\alpha c^\beta c^\gamma K_\alpha F^A[\phi] - \frac{i\theta}{2} c^\alpha c^\beta f_{\alpha\beta}{}^\gamma K_\gamma F^A[\phi] = 0.
\end{aligned}$$

Hence when we compute $\delta_B(S_{\text{G}})$, the only non-vanishing term comes from $\delta_B b_\lambda$. Similarly, since $\delta_B(B_A) = 0$, the only non-vanishing term in $\delta_B(S_{\text{gf}})$ comes from $\delta_B(F^A[\phi])$. But then again using the first of the two identities,

$$\delta_B(S_{\text{gf}} + S_{\text{G}}) = -iB_A \delta_B(F^A[\phi]) + \theta B_A c^\alpha K_\alpha F^A[\phi] = 0.$$

Finally, $\delta_B(S) = 0$ since $S$ is the original gauge-invariant action and is only a function of $\phi_r$, but the transformation $\delta_B \phi_r$ is no more than an infinitesimal gauge transformation. ☐

Note that $\delta_B$ mixes commuting (e.g. $\phi_r$ and $B_A$) and anti-commuting (e.g. $b_A$ and $c^\alpha$) objects. For example, $b_A$ is supposed to anti-commute, but $\delta_B b_A = \theta B_A$, and $B_A$ commutes. Hence $\theta$ must be anti-commuting, i.e. a Grassmann variable. Because of this mixing, physicists say that the BRST symmetry is a **supersymmetry**. The field $c^\alpha$ has ghost number $+1$, the field $b_A$ and parameter $\theta$ have ghost number $-1$, and all other fields have ghost number 0.

**Exercise 3.1.1.** Let $\delta$ be an infinitesimal symmetry of the fields $\phi_r$. Review/show that by Noether's theorem, associated to the symmetry $\delta$ is a conserved charge

$$Q \coloneqq \int d^{D-1}\vec{x}\, \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)}\delta\phi_r - \delta\mathcal{L}\right) = \int d^{D-1}\vec{x}\, (\Pi^r \delta\phi_r - \delta\mathcal{L})\,.$$

Suppose that $\delta\mathcal{L} = 0$. When quantized, $Q = :\!\int d^{D-1}\vec{x}\, \Pi^r \delta\phi_r\!:$ is a **generator** of the symmetry $\delta$: show that

$$\delta\phi_r = [Q, \phi_r],$$

and more generally, $\delta G = [Q, G]$ for any function $G$ depending only on the fields $\phi_r$ and not $x$. Finally, argue that $\delta\theta_r = \{Q, \theta_r\}$ for fermionic, i.e. Grassmann-valued, fields $\theta_r$. Let

$$[\cdot,\cdot]_- \coloneqq [\cdot,\cdot], \quad [\cdot,\cdot]_+ \coloneqq \{\cdot,\cdot\},$$

so that for a general field $G$, we can write $\delta G = [Q, G]_\pm$.

Hence there is a conserved charge $Q_B$ associated to the BRST symmetry $\delta_B$. There are two important properties of $Q_B$. To establish both of them, we first need to calculate that

$$\delta_B(b_A F^A) = (\theta B_A)F^A - b_A(i\theta c^\alpha K_\alpha F^A) = i\theta(S_{\text{gf}} + S_{\text{G}}).$$

**Proposition 3.1.3.** *Physical states satisfy* $Q_B\,|\text{phys}\rangle = 0$.

*Proof/Exercise.* Let $\delta$ be an infinitesimal transformation of the gauge-fixing functionals $F^A$, i.e. $F^A \mapsto F^A + \delta F^A$. Physical amplitudes $\langle f|i\rangle$ should be independent of our choice of gauge, so we require

$$0 = \delta\,\langle f|i\rangle = i\theta\,\langle f|\,\delta_B(b_A\delta F^A)\,|i\rangle = -\,\langle f|\,\left\{Q_B, b_A\delta F^A\right\}|i\rangle\,.$$

(The second equality follows from writing $\langle f|i\rangle$ as a path integral.) For this equality to hold for all variations $\delta F^A$, we must have $Q_B^\dagger\,|f\rangle = Q_B\,|i\rangle = 0$. (It must be that $Q_B^\dagger = Q_B$, otherwise there would be another symmetry associated with $Q_B^\dagger$.) Since $|f\rangle$ and $|i\rangle$ are arbitrary physical states, we are done. □

**Proposition 3.1.4.** $Q_B$ *is nilpotent, i.e.* $Q_B^2 = 0$.

*Proof/Exercise.* A few calculations show that $\delta_B^2 = 0$. A few more calculations show that $[Q_B, [Q_B, G]_\pm]_\mp = [Q_B^2, G]$ for any formal variable $G$. Hence for any function $G$ of the fields,

$$0 = \delta_B\delta_B G = i\theta'[Q_B, i\theta[Q_B, G]_\pm]_\pm = \theta'\theta[Q_B, [Q_B, G]_\pm]_\mp = \theta'\theta[Q_B^2, G].$$

Since $G$ is arbitrary, $Q_B^2$ must be a scalar multiple of the identity. But $Q_B$ increases ghost number by 1, as one can verify from the definition of $\delta_B$, and the identity operator does not do this. Hence $Q_B^2 = 0$. □

These two properties allow us to directly construct the **physical (BRST) state space** $\mathcal{H}_{\text{BRST}}$. First, note that states of the form $Q_B\,|\chi\rangle$, called **null states**, are automatically physical states, since $Q_B^2 = 0$. But they are orthogonal to all physical states, since $\langle\psi|\,(Q_B\,|\chi\rangle) = (\langle\psi|\,Q_B)\,|\chi\rangle = 0$. So their presence is never measurable: two physical states differing by a null state are physically equivalent. Hence

$$\mathcal{H}_{\text{BRST}} \coloneqq \frac{\{|\psi\rangle : Q_B\,|\psi\rangle = 0\}}{\{|\psi\rangle : |\psi\rangle = Q_B\,|\chi\rangle\}},$$

i.e. the **cohomology** of the operator $Q_B$.

## 3.2 Mathematical Formalism

**Note**: this subsection is **very optional**: it can be completely skipped. But if you are curious about the mathematical underpinnings of BRST, and what we actually calculated in the previous section, read on! This subsection contains a very rough outline of BRST quantization as Lie algebra cohomology, in the context of symplectic reduction. For more detail, see José Figueroa-O'Farrill's PhD thesis.[1]

What is BRST quantization really doing? Let's first restate the underlying problem of quantizing a constrained system: we have a phase space $M$ and a Hamiltonian action of a Lie group $G$ on $M$. For us, $M$ is the space of field configurations $\{\phi_r, B_A, b_A, c^\alpha\}$ and $G$ is the gauge group diff $\times$ Weyl. But for now, let's imagine $M$ is finite-dimensional.

There is a correspondence, which can be made into a diffeomorphism, between $M$ and the maximal closed ideals of $C^\infty(M)$, so it suffices to study $C^\infty(M)$. But we actually want to understand $M$ after modding out by the action of $G$. Since the action of $G$ on $M$ is Hamiltonian, we get via **symplectic reduction** the "quotient" symplectic manifold $\tilde{M}$, sometimes denoted $M//G$. The symplectic reduction is done via the **moment map** $\Phi\colon M \to \mathfrak{g}^*$, where $\mathfrak{g}$ is the Lie algebra of $G$:

$$M_0 \coloneqq \Phi^{-1}(0), \quad \tilde{M} \coloneqq M_0/G.$$

The Marsden-Weinstein theorem says that if $G$ acts freely on $\Phi^{-1}(0)$, then $\tilde{M}$ is a symplectic manifold, inheriting the symplectic form from $M$. Note that this reduction is often, more precisely, called **coisotropic reduction**. [2] We want to understand $C^\infty(\tilde{M})$.

The Hamiltonian action of $G$ on $M$ induces a Hamiltonian action of $\mathfrak{g}$ on $C^\infty(M)$, and therefore on $C^\infty(M_0)$. Quite unsurprisingly, it turns out that

$$C^\infty(\tilde{M}) = C^\infty(M_0)^{\mathfrak{g}} = H^0(\mathfrak{g}; C^\infty(M_0)),$$

where the second equality is the definition of **Lie algebra cohomology**: it is the derived functor associated with the **invariants functor** $(-)^{\mathfrak{g}}$. But even $M_0$ is hard to understand; we really want $C^\infty(\tilde{M})$ in terms of $C^\infty(M)$. A standard tool called the **Koszul complex** provides a resolution of $C^\infty(M_0)$ in terms $C^\infty(M)$-modules:

$$\cdots \xrightarrow{\delta} \Lambda^2\mathfrak{g} \otimes C^\infty(M) \xrightarrow{\delta} \mathfrak{g} \otimes C^\infty(M) \xrightarrow{\delta} C^\infty(M) \to C^\infty(M_0) \to 0.$$

Here $\delta$ extends the action $\mathfrak{g} \otimes C^\infty(M) \to C^\infty(M)$ as an odd derivation, i.e.

$$\delta(X_1 \wedge \cdots \wedge X_n \otimes f) \coloneqq \sum_{k=1}^n X_1 \wedge \cdots \wedge \hat{X}_k \wedge \cdots \wedge X_n \otimes \delta(X_k \otimes f).$$

Given this Koszul complex, denoted $K^\bullet$, we can now construct its **Chevalley-Eilenberg resolution**

$$K^{\bullet,\bullet} \coloneqq \operatorname{Hom}(\Lambda^\bullet\mathfrak{g}, \Lambda^\bullet\mathfrak{g} \otimes C^\infty(M)) = \Lambda^\bullet\mathfrak{g}^* \otimes \Lambda^\bullet\mathfrak{g} \otimes C^\infty(M)$$

in order to compute Lie algebra cohomology. The horizontal sequences come from the Koszul complex, while the vertical differentials $d$ come from Lie algebra cohomology. [3] Form the **total complex** $\operatorname{Tot}(K)^\bullet \coloneqq \bigoplus_{p-q=n} K^{p,q}$ with differential $D = d + \delta$. (Note that this is a little different than usual, since the horizontal complex is homological while the vertical complex is cohomological.)

[1] http://www.maths.ed.ac.uk/~jmf/Research/PVBLICATIONS/Thesis.pdf.

[2] The terminology describes how the zero locus $M_0 \coloneqq \Phi^{-1}(0)$ arising in symplectic reduction sits inside $M$: if $T_pM_0^\perp \subset T_pM_0$, then $M_0$ is **coisotropic**. This is in contrast with when $T_0M_0^\perp \cap T_pM_0 = \{0\}$, in which case we call $M_0$ **symplectic**. It turns out that for an equivariant moment map $\Phi\colon M \to \mathfrak{g}^*$, the zero locus $M_0$ is always coisotropic.

[3] If $\rho\colon \mathfrak{g} \to \operatorname{End}(V)$ is a representation, then $C^p(\mathfrak{g}; V) \coloneqq \operatorname{Hom}(\Lambda^p\mathfrak{g}, V)$ are the **$p$-cochains**, and the differential $d$ is given by extending

$$d\colon C^0(\mathfrak{g}; V) = V \to C^1(\mathfrak{g}; V), \quad (dv)(X) = \rho(X)v$$

to $\Lambda^\bullet\mathfrak{g}^*$ as an anti-derivation, and to $\Lambda^\bullet\mathfrak{g}^* \otimes V$ by $d(\alpha \otimes v) = d\alpha \otimes v + (-1)^{|\alpha|}\alpha \otimes dv$. In this case, $V = \Lambda^\bullet\mathfrak{g} \otimes C^\infty(M)$.

**Proposition 3.2.1.** *$H^p(\mathrm{Tot}(K)) = C^\infty(\tilde{M})$ for $p = 0$ and is zero otherwise.*

*Proof.* Let $E_n^{i,j}$ be the (cohomological, vertical) spectral sequence associated to $K^{\bullet,\bullet}$, and recall that it converges to $H^\bullet(\mathrm{Tot}(K))$.

1. The first page is the cohomology of the horizontal differentials $\delta$, so $E_1^{p,q} = H^q(K^{p,\bullet}) = \Lambda^p \mathfrak{g}^* \otimes C^\infty(M_0)$ if $q = 0$, and is zero otherwise.
2. The second page is the cohomology of the vertical differentials $d$ on $E_1^{\bullet,\bullet}$, so $E_2^{p,q} = H^p(E_1^{\bullet,q}) = C^\infty(M_0)^{\mathfrak{g}} = C^\infty(\tilde{M})$ if $p = q = 0$, and is zero otherwise.

Hence the spectral sequence collapses at the second page, and we conclude that $H^0(\mathrm{Tot}(K)) = C^\infty(\tilde{M})$. $\square$

So now we have a direct way to compute $C^\infty(\tilde{M})$: just form the total complex $\mathrm{Tot}(K)$ and compute its cohomology. The total complex $\mathrm{Tot}(K)$ has another name: it is usually called the **BRST complex**, and its differential $D$ is the **BRST symmetry**.

**Proposition 3.2.2.** *The complex $\mathrm{Tot}(K)$ has the structure of a* **graded Poisson superalgebra***, which for us just means that it has a* **Poisson bracket** *$\{\cdot,\cdot\}$ that respects the grading. Let $b_i$ be a basis for $\mathfrak{g}$, and $c^i$ be a dual basis for $\mathfrak{g}^*$. If $\phi_i := \Phi(b_i)$ are the components of the moment map, then the differential $D$ on $\mathrm{Tot}(K)$ arises as $D = \{Q, -\}$, with*

$$Q = c^i \phi_i - \frac{1}{2} f^i_{jk} c^j c^k b_i \in \mathrm{Tot}(K)^1,$$

*where $f^i_{jk}$ are the structure constants of $\mathfrak{g}$.*

*Proof.* Check that $\{Q, -\}$ acts as $D$ on the generators, i.e. on $f \in C^\infty(M)$, $X \in \mathfrak{g}$, and $\alpha \in \mathfrak{g}^*$. $\square$

Physicists call $c^i$ and $b_i$ **ghosts** and **anti-ghosts** respectively. The total degree in $\mathrm{Tot}(K)$ is called the **ghost number** (so note that the action of $Q$ increases the ghost number by 1). In fact, this is how BRST quantization works in general for the physicists: given a system with symmetries $K_i$ forming a Lie algebra $\mathfrak{g}$, we

1. introduce a basis $\{b_i\}$ for $\mathfrak{g}$ and $\{c^i\}$ for $\mathfrak{g}^*$,
2. write down the BRST operator $Q = c^i K_i - (1/2) f^i_{jk} c^j c^k b_i$, and
3. compute cohomology, treating everything as operators, to find physical states.

The underlying hypothesis is, of course, that the following diagram commutes:

$$\begin{tikzcd}
C^\infty(M) \arrow[r, "\text{quantization}"] \arrow[d, "\text{classical BRST}"'] & \mathcal{H} \arrow[d, "\text{quantum BRST}"] \\
C^\infty(\tilde{M}) \arrow[r, "\text{quantization}"] & \mathcal{H}_{\text{gauge-fixed}},
\end{tikzcd}$$

so that instead of having to quantize $C^\infty(\tilde{M})$ directly, which we have no idea how to do in general, we quantize $C^\infty(M)$ and then apply the machinery of BRST, which is easy, quantized or not. In fact, the quantization of $C^\infty(\tilde{M})$ is usually defined using this diagram.

## 3.3 BRST Quantization of the Bosonic String

We can now apply the general machinery of BRST quantization and our knowledge of CFT to the open bosonic string. (We need CFT to apply Noether's theorem, to get the BRST charge $Q_B$.) This involves quite a lot of calculation that can be safely skipped. We record them here in grisly detail for reference.

### 3.3.1 BRST Symmetry and Critical Dimension

First, in order to apply CFT, we need to rewrite the action in complex coordinates $(z, \bar{z})$. Pick **unit gauge**, i.e. $g_{ab} = \delta_{ab}$, so that $\nabla = \partial$, and

$$S_{\text{X}} = T_0 \int_\Sigma d^2z\, \partial X \cdot \bar{\partial} X, \quad S_{\text{G}} = \frac{1}{2\pi} \int_m d^2z\, \left( b_{zz} \partial_{\bar{z}} c^z + b_{\bar{z}\bar{z}} \partial_z c^{\bar{z}} \right).$$

**Notation**: we write $S_{\text{X}}$ instead of $S_{\text{P}}$ for the Polyakov action from now on. Similarly, $T_{\text{X}}$ is the holomorphic part of the energy-momentum tensor for $S_{\text{X}}$, and $T_{\text{G}}$ for $S_{\text{G}}$. In general, X as a subscript refers to quantities involving the **matter fields** $X^\mu$, and G as a subscript refers to quantities involving the **ghost fields** $b_{ab}$ and $c^a$. Remember that tildes indicate the anti-holomorphic part.

The equations of motion for the ghosts give $\partial_{\bar{z}} b_{zz} = \partial_z b_{\bar{z}\bar{z}} = 0$ and $\partial_{\bar{z}} c^z = \partial_z c^{\bar{z}} = 0$, so define

$$b(z) \coloneqq b_{zz}(z, \bar{z}), \quad \tilde{b}(\bar{z}) \coloneqq b_{\bar{z}\bar{z}}(z, \bar{z}), \quad c(z) \coloneqq c^z(z, \bar{z}), \quad \tilde{c}(\bar{z}) \coloneqq c^{\bar{z}}(z, \bar{z}).$$

Using this new notation, the ghost action is $S_{\text{G}} = (1/2\pi) \int d^2z\, (b\bar{\partial}c + \tilde{b}\partial\tilde{c})$. It is straightforward now to write down the BRST symmetry, since we only need to consider diffeomorphisms and not Weyl transformations: nothing depends on the metric anymore.

**Proposition 3.3.1.** *The BRST symmetry for the bosonic string is*

$$\begin{aligned} \delta_B X^\mu &= i\theta(c\partial X^\mu + \tilde{c}\bar{\partial} X^\mu), \\ \delta_B b &= i\theta(T_X + T_G), \qquad & \delta_B \tilde{b} &= i\theta(\tilde{T}_X + \tilde{T}_G), \\ \delta_B c &= i\theta c\partial c, & \delta_B \tilde{c} &= i\theta \tilde{c}\bar{\partial}\tilde{c}, \end{aligned}$$

*where* $T_X = -(2\pi T_0)(\partial X)^2$ *and* $T_G = -2 :b\partial c: + :c\partial b:$.

*Proof.* The gauge transformations here are diffeomorphisms and Weyl transformations, but Weyl transformations leave $X^\mu$ invariant, so the expression for $\delta_B X^\mu$ comes directly from the definition.

To calculate $\delta_B b = \theta B$, we need to find the equation of motion for $B$, which we integrated out, so that we can rewrite it in terms of the other fields. Well it is easy to find the equation of motion: simply write down the total non-gauge-fixed action and vary it with respect to the metric to get

$$S = S_{\text{X}} - iB^{ab}(\delta_{ab} - g_{ab}) + S_{\text{G}} \implies \delta_g S = \frac{\sqrt{g}}{4\pi} \delta g_{ab}(T_{\text{X}} - iB^{ab} + T_{\text{G}}) \implies B^{ab} = -i(T_{\text{X}} + T_{\text{G}}).$$

Finally, to calculate $\delta_B c^a = (i\theta/2) f_{bc}{}^a c^b c^c$, we need the structure constants $f_{bc}{}^a$. Since $[K_\alpha, K_\beta] X^\mu = 0$ when either of $K_\alpha$ or $K_\beta$ is a Weyl transformation, again we only need to consider diffeomorphisms. Since infinitesimal diffeomorphisms have as a basis $\delta_w X^\mu(z) = \delta(z - w)\partial_z X^\mu(z)$, we can compute

$$f_{bc}{}^a = -\delta^a_c \delta(z_1 - z_2)\partial_b \delta(z_1 - z_3) + \delta^a_b \delta(z_1 - z_3)\partial_a \delta(z_1 - z_2),$$

where $z_1, z_2, z_3$ are the coordinates corresponding to $a, b, c$ respectively. Plugging this into the definition of $\delta_B c^a$ and rewriting in complex coordinates gives the desired result. □

Note that we disregard the Nakanishi-Lautrup fields $B_A$ in the BRST construction here, because we could and did integrate them away when we gauge-fixed the action.

**Proposition 3.3.2.** *The holomorphic part* $j_B(z)$ *of the conserved BRST current is*

$$j_B(z) \coloneqq cT_X + \frac{1}{2} :cT_G: + \frac{3}{2}\partial^2 c = cT_X + :bc\partial c: + \frac{3}{2}\partial^2 c,$$

*where the* $\partial^2 c$ *term is added manually to make* $j_B(z)$ *a tensor.*

*Proof.* Let $\mathcal{J}^x$ denote the variation in the action, in standard coordinates, i.e.

$$\frac{\theta}{2}\int d^2z\,\partial_x\mathcal{J}^x := \delta_B(S_{\mathrm{X}}+S_{\mathrm{G}}) = iT_0\theta\int d^2z\,\partial_x(c^x\partial X\cdot\bar{\partial}X).$$

Hence, by the standard Noether procedure, the conserved current is

$$(j_B)^x(z,\bar{z}) = :\frac{\delta(\mathcal{L}_{\mathrm{X}}+\mathcal{L}_{\mathrm{G}})}{\delta(\partial_x\Phi_i(z,\bar{z}))}\delta_B\Phi_i(z,\bar{z}) - \mathcal{J}^x:$$

where $\Phi_i$ ranges over all dynamical fields, i.e. $X, b, c$. Switching to complex coordinates,

$$\begin{aligned} j_B(z) = 2\pi i\left(\frac{1}{2}(j_B)^{\bar{z}}\right) &= \pi i:\left(\frac{T_0}{2}\partial X\cdot\delta_B X - \frac{1}{\pi}b\delta_B c - \frac{iT_0}{2}\tilde{c}\partial X\cdot\bar{\partial}X\right): \\ &= -2\pi T_0 :c\partial X\cdot\partial X: + :bc\partial c: =:cT_{\mathrm{X}}: + :bc\partial c: \ . \end{aligned}$$

□

The analogous formula clearly holds for $\tilde{j}_B$. Now recall that the BRST charge, by definition, is the conserved charge associated with this conserved current: $Q_B = (1/2\pi i)\int(dz\,j_B - d\bar{z}\,\tilde{j}_B)$.

**Proposition 3.3.3.** *$Q_B^2 = 0$ if and only if $D = 26$.*

*Proof.* We directly compute the OPE, ignoring the total derivative term:

$$\begin{aligned} 2Q_B^2 = \{Q_B, Q_B\} &= \int dz\,dw :c(z)\left(T_{\mathrm{X}}(z)+\frac{1}{2}T_{\mathrm{G}}(z)\right)::c(w)\left(T_{\mathrm{X}}(w)+\frac{1}{2}T_{\mathrm{G}}(w)\right): \\ &= -\frac{1}{12}\int dw\,\partial_w^3c(w)c(w)(D-26), \end{aligned}$$

which vanishes if and only if $D = 26$. □

This calculation is the rigorous derivation of the **critical dimension**. The failure of $Q_B^2$ to be zero automatically for any $D$ is known as the **Weyl anomaly**. Alternatively, we could compute this result from the mode expansion of $Q_B$, which we record here for use in the next subsection:

$$Q_B = 2\sum_{n=-\infty}^{\infty}c_nL_{-n}^X + \sum_{m,n=-\infty}^{\infty}(m-n)\,\circ\!\!\!\circ\, c_mc_nb_{-m-n}\,\circ\!\!\!\circ\, - 2c_0$$

where $L_{-n}^X$ are the modes of $T_X$, and $\circ\!\!\!\circ\ \circ\!\!\!\circ$ denotes **creation-annihilation normal ordering**.

### 3.3.2 Physical State Space and the No-Ghost Theorem

We can now find the **physical state space** by calculating the BRST cohomology of $Q_B$ acting on the entire state space. First, we need to write down the state space. This is entirely analogous to when we canonically quantized naively, using the mode expansion coefficients $\alpha_m^i$ as **raising/lowering operators**. But now we are not in lightcone gauge anymore, and we have two additional ghost fields. Hence we now have three sets $\{\alpha_m^\mu\}$, $\{b_m\}$, $\{c_m\}$ of raising/lowering operators, arising from the mode expansions

$$\partial X^\mu(z) = -i\sqrt{\frac{1}{2\pi T_0}}\sum_{m=-\infty}^{\infty}\frac{\alpha_m^\mu}{z^{m+1}},\quad b = \sum_{m=-\infty}^{\infty}\frac{b_m}{z^{m+2}},\quad c = \sum_{m=-\infty}^{\infty}\frac{c_m}{z^{m-1}}.$$

We also define, as usual, the **number operators** for each set of operators, and the total number operator:

$$N_n^\mu := \alpha_{-n}^\mu\alpha_n^\mu,\quad N_n^b := nb_{-n}c_n,\quad N_n^c := nc_{-n}b_n,\quad N := \sum_{n=1}^{\infty}\left(\sum_{\mu=0}^{25}N_n^\mu + N_n^b + N_n^c\right).$$

The associated eigenvalue of $N$ is called the **level** of the string. The most general level-$N$ open bosonic string is, as expected,

$$|N;k\rangle = \sum_{N_n^\mu, N_n^b, N_n^c} C_{N_n^\mu, N_n^b, N_n^c} \left( \prod_{\mu=0}^{25} \prod_{n=1}^{\infty} \frac{(\alpha_{-n}^\mu)^{N_n^\mu}}{\sqrt{n^{N_n^\mu} N_n^\mu!}} \right) \left( \prod_{n=1}^{\infty} (b_{-n})^{N_n^b} \right) \left( \prod_{n=0}^{\infty} (c_{-n})^{N_n^c} \right) |0;k\rangle \,,$$

where the sum is of course over the quantum numbers $\{N_n^\mu, N_n^b, N_n^c\}$ giving a level-$N$ string, and $C_{N_n^\mu, N_n^b, N_n^c}$ are normalization constants. What are the ground states $|0;k\rangle$? From canonical quantization, we have a definition of the ground states $|0;k\rangle_{\mathrm{X}}$ for the matter fields, but not for the ghost fields. Well, ground states should always be annihilated by lowering operators, so now we define the **ground state** $|0;k\rangle$ such that

$$p^\mu |0;k\rangle = k^\mu |0;k\rangle \,, \quad b_0 |0;k\rangle = 0, \quad \alpha_n^\mu |0;k\rangle = b_n |0;k\rangle = c_n |0;k\rangle_{\mathrm{G}} = 0 \quad \forall n > 0.$$

**Exercise 3.3.1.** Note that $c_0$ is a creation operator while $b_0$ is not: this exercise explains why. Show that if $b_0$ were also a creation operator, the system generated by $b_0$ and $c_0$ has a two-fold degeneracy at every state $|0;k\rangle$, which splits into states $|0;k,\uparrow\rangle$ and $|0;k,\downarrow\rangle$ satisfying

$$\begin{aligned} b_0 |0;k,\uparrow\rangle &= |0;k,\downarrow\rangle \,, \quad b_0 |0;k,\downarrow\rangle = 0 \\ c_0 |0;k,\downarrow\rangle &= |0;k,\uparrow\rangle \,, \quad c_0 |0;k,\uparrow\rangle = 0. \end{aligned}$$

Using the mode expansion of $Q_B$ from the previous subsection, show that $Q_B$ acts on either of these states as $2c_0((L_X)_0 - 1)$, and therefore that

$$Q_B |0;k,\downarrow\rangle = 2\left(\frac{k^2}{2\pi T_0} - 1\right) |0;k,\uparrow\rangle \,, \quad Q_B |0;k,\uparrow\rangle = 0.$$

Hence if $k^2 \neq 2\pi T_0$, the states $|0;k,\uparrow\rangle$ are BRST-exact. But not all the states $|0;k,\downarrow\rangle$ are BRST-closed, hence $\langle \mathrm{phys}|0;k,\downarrow\rangle \propto \delta(k^2 - 2\pi T_0)$. Since amplitudes in QFTs cannot have delta functions, conclude that $|0;k,\uparrow\rangle$ for $k^2 = 2\pi T_0$ are the real physical states, which therefore satisfy $b_0 |0;k,\uparrow\rangle = 0$.

This also explains why we require the ground state $|0;k\rangle$ to be annihilated by $b_0$: when we write $|0;k\rangle$, we implicitly mean $|0;k,\uparrow\rangle$.

**Definition 3.3.4.** The **physicality condition** for states $|N;k\rangle$ to be physical is the criteria $Q_B |N;k\rangle = b_0 |N;k\rangle = 0$. Note that $L_0 = \{Q_B, b_0\}$, so $L_0 = (1/2\pi T_0)(k^2 + m^2)$ annihilates all physical states, i.e. $k^2 = -m^2$ for all physical states. Hence we index using $\vec{k}$ instead of $k$, since given $\vec{k}$ we can solve for $k^0$.

For the ground state, then, $m^2 = -k^2 = -2\pi T_0$, which is, of course, the same result that we got via canonical quantization. We essentially worked out in the exercise that $|0;\vec{k}\rangle$ for $k^2 = 2\pi T_0$ is the unique ground state. (There are no BRST-exact states here to remove, since $Q_B |0;\vec{k}\rangle = 0$.)

For $N = 1$, the most general state is of the form

$$|1;\vec{k}\rangle = (e \cdot \alpha_{-1} + \beta b_{-1} + \gamma c_{-1}) |0;\vec{k}\rangle \,, \quad k^2 = 0$$

where $e^\mu$ is a 26-vector and $\beta, \gamma \in \mathbb{C}$ are scalars. The physicality condition requires

$$\begin{aligned} 0 = Q_B |1;\vec{k}\rangle &= 2(c_1 (L_X)_{-1} + c_0 (L_X)_0 + c_{-1}(L_X)_1 + c_0 c_{-1} b_1 + c_1 c_0 b_{-1} - c_0) |1;\vec{k}\rangle \\ &= 2\sqrt{\frac{1}{\pi T_0}} (\beta k \cdot \alpha_{-1} + k \cdot e c_{-1}) |0;\vec{k}\rangle \,, \end{aligned}$$

i.e. $k \cdot e = \beta = 0$. (Note that $b_0 |1;\vec{k}\rangle$ is automatically satisfied.) Also, note that this calculation also gives us the BRST-exact states, $c_{-1} |0;\vec{k}\rangle$ and $k \cdot \alpha_{-1} |0;\vec{k}\rangle$. Hence, after removing these BRST-exact states, the remaining **physical states** are

$$|1;\vec{k}\rangle = e \cdot \alpha_{-1} |0;\vec{k}\rangle \,, \quad k^2 = k \cdot e = 0, \quad e^\mu \sim e^\mu + \zeta k^\mu.$$

A basis for these remaining physical states is $\{\alpha^i_{-1}\,|1;\vec{k}\rangle : i = 2, \ldots, 25\}$. Note that all these results for $N = 1$ matches up exactly with our heuristic derivation a long time ago when we applied canonical quantization. But now that we have obtained these results rigorously, it is time to reveal that an open string at $N = 1$ excitation is usually identified with a **photon** with **polarization** $e^\mu$.

**Theorem 3.3.5.** *The BRST state space, i.e. the cohomology of $Q_B$, is isomorphic to the state space obtained from the subspace (of the Hilbert space arising from canonical quantization) annihilated by the Virasoro algebra.*

*Proof.* See Polchinski, section 4.4, pg. 141. □

**Exercise 3.3.2.** By repeating the BRST calculation for $N = 1$, conclude that at $N = 2$ there are a total of 324 proper physical states, 300 of which come from $\alpha^i_{-1}\alpha^j_{-1}\,|0;\vec{k}\rangle$ and 24 of which come from $\alpha^i_{-2}\,|0;\vec{k}\rangle$.

It turns out, quite fortunately, that we do not need to go through the tedious BRST procedure to actually find what the proper physical states are; for the bosonic string, at least, BRST quantization is just a tool used to establish the following very important and practical theorem.

**Theorem 3.3.6** (No-ghost theorem)**.** *The BRST state space is isomorphic to the subspace (of the Hilbert space arising from canonical quantization) with no $X^0$, $X^1$, $b$, or $c$ excitations.*

*Proof.* See Polchinski, section 4.4, pg. 137. □

**Exercise 3.3.3.** Show, as a corollary of the no-ghost theorem, that the generating function for the number of states at excitation $N$ is

$$\prod_{n=1}^{\infty} \frac{1}{(1-x^n)^{24}} = 1 + 24x + 324x^2 + 3200x^3 + 25650x^4 + \cdots = \eta(x)^{-24} \propto \Delta(x),$$

where $\eta(q)$ is the **Dedekind eta function** and $\Delta(x)$ is the **modular discriminant**. Conclude that there is something magical about the number 24, and that string theory is worth studying for its mathematics (even if the physicists laugh at it.)

# Chapter 4

# Amplitudes

In QFT, after we studied the spectrum of a free particle, we proceeded to add an interaction term in the form of $(\lambda/4!)\phi^4$. Then we studied how to compute $n$-point functions in $\phi^4$ theory. We shall do the same now for the bosonic string. However two things are different.

1. We do not need to add an interaction term to the Polyakov action: interactions are implicitly encoded.
2. Computing amplitudes is a lot harder unless the string sources are taken to infinity, which corresponds to computing S-matrix elements.

For simplicity, this chapter works out the theory for **closed strings** only. The simplicity arises from the worldsheet of interacting closed strings being a **closed surface**, i.e. we do not need to deal with boundary components. Henceforth when we say surface, we mean closed surface.

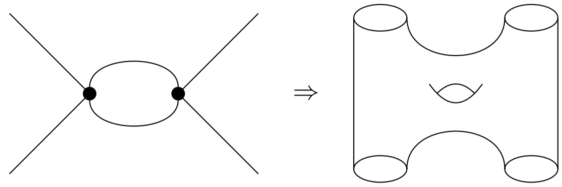
**Theorem 4.0.1** (Classification of closed surfaces)**.** *Any closed surface $\Sigma$ is homeomorphic to the connected sum of a 2-sphere with $p \geq 0$ tori, and has Euler characteristic $\chi(\Sigma) = 2 - 2p$.*

## 4.1 Computing Scattering Amplitudes

As a first example, let's look at the scattering amplitude for four closed strings, with sources approaching $X^0 = \pm\infty$. Diagrammatically, this looks like four infinitely long cylinders attached to a sphere in the middle. On these cylinders, we have the complex coordinate $w = x + iy$ where $x \cong x + 2\pi$ goes around the string, and $-\infty < y \leq 0$ is identified with the time coordinate $X^0$.

Consider the new coordinate $z = \exp(-iw)$. In this coordinate, the cylinders taper off conically as $y \to -\infty$, and therefore as the cylinders become infinitely long, we end up with a sphere with 4 **marked points**. This is essentially the **state-operator correspondence** from CFT: at each of these 4 marked points is now a local **vertex operator** $\mathcal{V}_j(k)$. (Incoming and outgoing states are distinguished by the sign of $k^0$.) Generally, we look at ground state, i.e. **tachyon**, vertex operators $:\!\exp(ik \cdot X)\!:$.

The same argument can be repeated for open strings to obtain the closed disk with marked points on it. In both cases, the resulting (conformally-equivalent) worldsheet is compact, with some number of marked points on it. Note that the **genus** of the worldsheet encodes different interactions! **Tree-level amplitudes** in QFT correspond to a sphere with marked points; **one-loop amplitudes** correspond to a torus with marked points; and so on. For example,



$\cong$ $T^2$ with 4 marked points.

In QFT, to obtain the 4-point correlation function, we summed over all tree-level, i.e. zero-loop, Feynman diagrams, followed by one-loop diagrams, etc. Similarly, to obtain the 4-point scattering amplitude here, we must sum over all genus zero surfaces with four marked points and their metrics, then genus one surfaces with four marked points and their metrics, etc.

Do surfaces of different genus contribute differently to the path integral? Recall that when we first discussed the gauge symmetries of the Polyakov action, we found that aside from the already-existing term, there could only be one other term satisfying all the gauge symmetries, given by

$$\chi := \frac{1}{4\pi}\int_\Sigma d^2\sigma\,\sqrt{-\gamma}\,R + \frac{1}{2\pi}\int_{\partial\Sigma} ds\,k$$

Of course, by Gauss-Bonnet, $\chi$ is nothing more than the **Euler characteristic** of the worldsheet. It acts as the **coupling constant** for worldsheets of different genus.

**Definition 4.1.1.** An $n$**-string scattering amplitude** involving $n$ strings with momenta $(k_i)^\mu$ and states $j_i$ is given by

$$S_{j_1\cdots j_n}(k_1,\ldots,k_n) := \sum_{\substack{\text{closed}\\ \text{surfaces } \Sigma}} \int \frac{\mathcal{D}X\,\mathcal{D}g}{\text{diff}\times\text{Weyl}} \exp(-S_{\text{X}} - \lambda\chi)\prod_{i=1}^n \int d^2\sigma_i\,\sqrt{g(\sigma_i)}\,\mathcal{V}_{j_i}(k_i,\sigma_i)$$

where the $\mathcal{V}_{j_i}$ are the local operators arising from the state-operator correspondence. Note that we integrate them over the worldsheet to preserve diffeomorphism invariance. Also, since $\exp(-\lambda\chi)$ is a topological invariant, it does not depend on the **embedding** $X^\mu$ of $\Sigma$ into spacetime, nor the metric $g$ on $\Sigma$, so we define $g_s := \exp(-\lambda)$, the **string coupling constant**, and write

$$S_{j_1\cdots j_n}(k_1,\ldots,k_n) = \sum_{\text{genus } p=0}^{\infty} g_s^{-(2-2p)} \int \frac{\mathcal{D}X\,\mathcal{D}g}{\text{diff}\times\text{Weyl}} \exp(-S_{\text{X}})\prod_{i=1}^n \int d^2\sigma_i\,\sqrt{g(\sigma_i)}\,\mathcal{V}_{j_i}(k_i,\sigma_i)$$

**Definition 4.1.2.** For a fixed genus $p$, let $\text{Met}(\Sigma_p)$ be the space of all metrics on any surface $\Sigma_p$ of genus $p$. Then the configuration space we are integrating over is

$$\frac{\text{Met}(\Sigma_p)\times C^\infty(\Sigma_p,\mathbb{R}^D)\times(\Sigma_p)^n}{\text{diff}\times\text{Weyl}} \cong \mathcal{M}(\Sigma_p)\times C^\infty(\Sigma_p,\mathbb{R}^D)\times(\Sigma_p)^n$$

where $\mathcal{M}(\Sigma_p)$ is the **moduli space of genus-$p$ Riemann surfaces**. Here, the action of Weyl on $\text{Met}(\Sigma_p)$ simply defines the **conformal classes** $\text{Conf}(\Sigma_p)$ of metrics on $\Sigma_p$, i.e. $\text{Met}(\Sigma_p)/\text{Weyl} \cong \text{Conf}(\Sigma_p)$.

Recall that $\mathcal{M}(\Sigma_p)$ is actually finite-dimensional, so these is some hope of ending up with a well-defined integral! There are a few things we must do in order to have this happen: we must

1. completely understand the action of diff $\times$ Weyl on $\text{Met}(\Sigma_p)$ (including the remnant gauge symmetry we encountered, in the form of holomorphic diffeomorphisms, even after gauge-fixing),
2. use Faddeev-Popov to write $S_{j_1\cdots j_n}(k_1,\ldots,k_n)$ as a finite-dimensional integral over the moduli space $\mathcal{M}(\Sigma_p)$, and
3. define a measure on $\mathcal{M}(\Sigma_p)$, hopefully descending from $\text{Met}(\Sigma_p)$, so that this integral is well-defined.

Physicists call tangent vectors of the moduli space **metric moduli**, or just **moduli** for short.

### 4.1.1 Integration Measure on the Moduli Space

We shall begin with items 1 and 3; they are related. But the former is surprisingly non-trivial for genus $p < 2$. (Usually, people avoid such difficulties by defining $\mathcal{M}(\Sigma_p)$ differently for $p < 2$. But we cannot.)

1. For genus $p < 2$, there exist **conformal Killing vectors** (CKVs) on $\Sigma_p$, i.e. non-trivial transformations in diff $\times$ Weyl that act as the identity on the metric. We examined some, given by holomorphic diffeomorphisms, when we introduced gauge-fixing.
2. The CKVs form a group, called the **conformal Killing group** $\mathrm{CKG}(\Sigma_p)$. If $\mathrm{CKG}(\Sigma_p)$ has real dimension $k$, then we can remove these extra degrees of gauge freedom by fixing $k$ (real) coordinates of marked points on $\Sigma_p$.

The CKG causes us problems; we must understand how to identify which infinitesimal diff $\times$ Weyl transformations are in the CKG. So consider an infinitesimal diff $\times$ Weyl transformation

$$\delta g_{ab} = 2\delta\omega g_{ab} - \nabla_a \delta\sigma_b - \nabla_b \delta\sigma_a = (2\delta\omega - \nabla_c \delta\sigma^c) g_{ab} - 2(P_1 \delta\sigma)_{ab}$$

where we define a differential operator $P_1$ taking vectors into traceless symmetric 2-tensors:

$$(P_1 \delta\sigma)_{ab} = \frac{1}{2}(\nabla_a \delta\sigma_b + \nabla_b \delta\sigma_a - g_{ab} \nabla_c \delta\sigma^c).$$

By definition, $\delta$ is a CKV iff $\delta g_{ab} = 0$, which, from the variation $\delta g_{ab}$, happens iff $(P_1 \delta\sigma)_{ab} = 0$, called the **conformal Killing equation**. (To get this, just take the trace of $\delta g_{ab} = 0$, which enforces $2\delta\omega = \nabla \cdot \delta\sigma$.)

In much the same way, we can identify infinitesimal transformations of $\mathrm{Met}(\Sigma_p)$ that correspond to moduli: they are variations $\delta' g_{ab}$ that are **orthogonal** (hang on, we'll define the metric soon) to all diff $\times$ Weyl variations, i.e.

$$\begin{aligned} 0 &= \int d^2\sigma \sqrt{g}\, \delta' g_{ab} \left( (2\delta\omega - \nabla \cdot \delta\sigma) g^{ab} - 2(P_1 \delta\sigma)^{ab} \right) \\ &= \int d^2\sigma \sqrt{g} \left( \delta' g_{ab} g^{ab} (2\delta\omega - \nabla \cdot \delta\sigma) - 2(P_1^\dagger \delta' g)_a \delta\sigma^a \right), \end{aligned}$$

where we define $(P_1^\dagger u)_a = -\nabla^b u_{ab}$. We see from this calculation that $\delta'$ is a modulus iff

$$g^{ab} \delta' g_{ab} = 0 \text{ and } (P_1^\dagger \delta' g)_a = 0.$$

**Exercise 4.1.1.** Show that in conformal gauge, the conformal Killing equation and the condition that $\delta'$ is a modulus become, respectively,

$$\partial_{\bar z} \delta z = \partial_z \delta \bar z = 0, \quad \partial_{\bar z} \delta' g_{zz} = \partial_z \delta' g_{\bar z \bar z} = 0,$$

i.e. CKVs are **holomorphic vector fields** and moduli are **holomorphic quadratic differentials**.

**Definition 4.1.3.** Define an inner product on symmetric tensors (of any rank) on a surface with metric $g$ by

$$(A, B)_g := \int d^2\sigma \sqrt{g}\, A \cdot B,$$

where the dot denotes contraction on all indices. Using this inner product, define a **metric** $\langle \cdot, \cdot \rangle_{\mathrm{Met}}$ on $\mathrm{Met}(\Sigma)$ as follows. Fix $g \in \mathrm{Met}(\Sigma)$, and let $(g^I)$ be coordinates in an open neighborhood of $g$. Given $X, Y \in T_g \mathrm{Met}(\Sigma)$, we can expand them in coordinates as

$$X = X^I \frac{\partial}{\partial g^I} = X^I_{a_1 b_1} \frac{\partial}{\partial g^I_{a_1 b_1}} \in T_g \mathrm{Met}(\Sigma), \quad Y = Y^J \frac{\partial}{\partial g^J} = Y^J_{a_2 b_2} \frac{\partial}{\partial g^J_{a_2 b_2}} \in T_g \mathrm{Met}(\Sigma).$$

Then define

$$\langle X, Y\rangle_{\text{Met}} \coloneqq (X^I, Y^I)_g = \int_\Sigma d^2\sigma\,\sqrt{g}\,g^{a_1a_2}(\sigma)g^{b_1b_2}(\sigma)X^I_{a_1b_1}(\sigma)Y^I_{a_2b_2}(\sigma).$$

Consequently, we get a well-defined measure on $\text{Met}(\Sigma)$.

**Exercise 4.1.2.** Check that, under this metric, moduli are indeed orthogonal to the diff $\times$ Weyl action. Indeed, check that we have an **orthogonal decomposition** of metric variations

$$\delta g = \{\text{Weyl}\} \oplus \{\text{diff}\} \oplus \{\text{moduli}\} = \{\text{Weyl}\} \oplus \operatorname{Im} P_1 \oplus \ker P_1^\dagger$$

by showing that $P_1^\dagger$ is the adjoint of $P_1$ under $(\cdot,\cdot)_{g(x)}$, and recalling that $\ker A^\dagger = (\operatorname{Im} A)^\perp$ for an operator $A$. Also, verify that $\ker P_1$ is the CKG. $P_1$ is an important operator!

It is fairly easy to check that $\langle\cdot,\cdot\rangle_{\text{Met}}$ is invariant under the diff $\times$ Weyl action, so it descends from $\text{Met}(\Sigma_p)$ to a well-defined metric on $\mathcal{M}(\Sigma_p)$, known as the **Weil-Petersson metric**. Hence we also obtain a well-defined measure on $\mathcal{M}(\Sigma_p)$.

### 4.1.2 Calculating the Measure Using Faddeev-Popov

Now we shall apply Faddeev-Popov in order to compute what this measure on $\mathcal{M}(\Sigma_p)$ is. Essentially, we shall perform a change of variables, from integrating over metrics and vertex positions to integrating over variables corresponding to the orthogonal decomposition, i.e. integrating over the gauge group, moduli (of dimension $\tau$), and unfixed vertex positions. In other words, we wish to find the Jacobian for the transformation

$$\frac{1}{\text{diff}\times\text{Weyl}}\int_{\text{Met}(\Sigma_p)}\mathcal{D}g\,d^{2n}\sigma \to \frac{1}{\text{diff}\times\text{Weyl}}\int_{\text{diff}\times\text{Weyl}}\mathcal{D}\zeta\int_{\mathcal{M}(\Sigma_p)}d^\tau m\int_{(\Sigma_p)^{n-\kappa/2}}d^{2n-\kappa}\sigma.$$

Note that because of the additional gauge freedom given by the CKG, we can choose the gauge-fixing conditions now to also fix $\kappa = \dim \text{CKG}(\Sigma_p)$ of the vertex operator coordinates, so that $\sigma_i^a \to \hat{\sigma}_i^a$ for some set, denoted Fixed, of fixed coordinates $(a,i)$.

To find the appropriate Jacobian for this change of variables, we use Faddeev-Popov. We can directly write down the **Faddeev-Popov determinant**:

$$\Delta_{\text{FP}}(g,\sigma)^{-1} = \int_{\mathcal{M}(\Sigma)}d^\tau m\int_{\text{diff}\times\text{Weyl}}\mathcal{D}\zeta\,\delta(\hat{g}(m)^\zeta - g)\prod_{(a,i)\in\text{Fixed}}\delta((\hat{\sigma}_i^\zeta)^a - \sigma_i^a).$$

Previously, we rewrote the right hand side as an integration over infinitesimal transformations under the false assumption that the gauge group acted freely on the configuration space of embeddings and metrics. This was OK because we did not explicitly compute anything that depended on getting factors correct in numerical results. However, now we care. Note that:

1. an integral over infinitesimal transformations, i.e. the Lie algebra, only captures behavior in the connected component of the identity in diff $\times$ Weyl;
2. by homogeneity, the value of the integral in the connected component of the identity is the same as the value in any other connected component;
3. after removing the CKG, the action of the connected component of the identity $(\text{diff}\times\text{Weyl})_0 = \text{diff}_0\times\text{Weyl}$ is free.

Hence we can, again, write the integral as an integral over the Lie algebra, but now we have a constant, finite factor $n_R$ in front:

$$\Delta_{\text{FP}}(\hat{g},\sigma)^{-1} = n_R\int d^\tau(\delta m)\,\mathcal{D}(\delta\omega)\,\mathcal{D}(\delta\sigma)\,\delta(\delta g_{ab})\prod_{(a,i)\in\text{Fixed}}\delta(\delta\sigma^a(\hat{\sigma}_i)),$$

where $n_R$ is the size of the **mapping class group** $\mathrm{Mod}(\Sigma) = |\mathrm{diff}(\Sigma)/\mathrm{diff}_0(\Sigma)|$. (We don't care about Weyl, since it is trivially connected.)

Now we follow the exact same procedure as we did for inverting the Faddeev-Popov determinant for the Polyakov action: compute $\delta g_{ab}$, introduce auxiliary variables of integration $x$ and $\beta_{ab}$ to write the delta functionals as exponentials, integrate out $\delta\omega$ to obtain the constraint that $\beta_{ab}$ is traceless, and replace the bosonic variables $(\delta\sigma^a, \beta_{ab}, x_{ai}, \delta m^t)$ with fermionic variables $(c^a, b_{ab}, \eta_{ai}, \zeta^t)$, to get

$$\Delta_{\mathrm{FP}}(\hat{g}, \sigma) = \frac{1}{n_R} \int \mathcal{D}b\, \mathcal{D}c\, d^\tau \zeta\, d^\kappa \eta\, \exp\left( -\frac{1}{4\pi}(b, 2P_1 c - \zeta^j \partial_j \hat{g})_{\hat{g}} + \sum_{(a,i)\in\mathrm{Fixed}} \eta_{ai} c^a(\hat{\sigma}_i) \right).$$

(We've added in some convenient normalization factors.) Finally, we do the integratation over the Grassmann parameters $\eta_{ai}$ and $\zeta^i$. The (somewhat elegant) result is

$$\Delta_{\mathrm{FP}}(\hat{g}, \sigma) = \frac{1}{n_R} \int \mathcal{D}b\, \mathcal{D}c\, \exp(-S_{\mathrm{G}}) \prod_{j=1}^{\tau} \frac{(b, \partial_j \hat{g})_{\hat{g}}}{4\pi} \prod_{(a,i)\in\mathrm{Fixed}} c^a(\hat{\sigma}_i)$$

But there is a nicer way to write $\Delta_{\mathrm{FP}}$. As a Jacobian, it is a determinant, and as a Jacobian for a change of variables into an orthogonal decomposition, we expect it to decompose as a product of determinants. Indeed, it does. First, obtain real eigenbases $\{\mathcal{C}_J\}$ and $\{\mathcal{B}_K\}$ for the operators $P_1^\dagger P_1$ and $P_1 P_1^\dagger$, i.e.

$$P_1^\dagger P_1 (\mathcal{C}_J)^a = v_J^2 (\mathcal{C}_J)^a, \quad P_1 P_1^\dagger (\mathcal{B}_K)_{ab} = w_K^2 (\mathcal{B}_K)_{ab},$$

normalized such that $(\mathcal{C}_J, \mathcal{C}_{J'}) = \delta_{JJ'}$ and $(\mathcal{B}_K, \mathcal{B}_{K'}) = \delta_{KK'}$. These eigenbases are related. Note that $P_1\mathcal{C}_J$ is an eigenfunction of $P_1 P_1^\dagger$, and similarly, $P_1\mathcal{B}_K$ is an eigenfunction of $P_1^\dagger P_1$. Hence there is a one-to-one correspondence between the eigenbases $\{\mathcal{C}_J\}$ and $\{\mathcal{B}_K\}$ except for when $P_1\mathcal{C}_J = 0$ or $P_1\mathcal{B}_K = 0$, i.e. when the eigenfunction has a zero eigenvalue. The $\mathcal{C}_J$ with zero eigenvalue correspond to $\kappa$ CKVs, and the $\mathcal{B}_K$ with zero eigenvalue correspond to $\tau$ moduli. Denote these eigenfunctions of zero eigenvalue $\{(\mathcal{C}_{0j})^a\}_{j=1}^{\kappa}$ and $\{(\mathcal{B}_{0k})_{ab}\}_{k=1}^{\tau}$. The rest (of non-zero eigenvalue) are indexed as normal with $J, K = 1, \ldots,$ and satisfy $(\mathcal{B}_J)_{ab} = (1/w_J)(P_1\mathcal{C}_J)_{ab}$.

If we rewrite the integral for $\Delta_{\mathrm{FP}}$ in terms of these eigenbases, we get

$$\begin{aligned}\Delta_{\mathrm{FP}} = \int \prod_{k=1}^{\tau} db_{0k} \prod_{j=1}^{\kappa} dc_{0j} \prod_J db_J\, dc_J\, \exp\left(-\frac{w_J b_J c_J}{2\pi}\right) \\ \times \left( \prod_{k'=1}^{\tau} \sum_{k''=1}^{\tau} \frac{b_{0k''}}{4\pi} (\mathcal{B}_{0k''}, \partial_{k'} \hat{g}) \right) \left( \prod_{(a,i)\in\mathrm{Fixed}} \sum_{j'=1}^{\kappa} c_{0j'} (\mathcal{C}_{0j'})^a(\sigma_i) \right),\end{aligned}$$

which is just a product of three Gaussian integrals. Calculating these integrals over Grassmann-valued fields, we get

$$\Delta_{\mathrm{FP}} = \left[ \det\left( \frac{(\mathcal{B}_{0k}, \partial_{k'} \hat{g})}{4\pi} \right)_{k,k'=1}^{\tau} \right] \left[ \det\left( (\mathcal{C}_{0j})^a(\sigma_i) \right)_{j=1,(a,i)\in\mathrm{Fixed}}^{\kappa} \right] \left[ \det'\left( \frac{P_1^\dagger P_1}{2\pi} \right)^{1/2} \right],$$

where $\det'$ indicates that we omit zero eigenvalues. Otherwise the whole expression is trivially zero.

For tree-level and one-loop amplitudes, evaluating this expression for $\Delta_{\mathrm{FP}}$ is not too bad. For higher-loop amplitudes, there is more work involved. But in that case we must put in more work anyway: to even integrate on the moduli space requires us to put **Fenchel-Nielsen** coordinates on it, which is not an easy task.

### 4.1.3 The $X^\mu$ Integration

It remains to handle the $X^\mu$ integration

$$\left\langle \prod_{i=1}^{n} \mathcal{V}_{j_i}(k_i, \sigma_i) \right\rangle := \int \mathcal{D}X \, \exp(-S_X) \prod_{i=1}^{n} \mathcal{V}_{j_i}(k_i, \sigma_i)$$

for a fixed moduli $g$. To evaluate this integral, we use the same trick we used in QFT: introduce a formal variable $J$, write down the **generating functional**

$$Z[J; g] = \left\langle \exp\left( i \int d^2\sigma \sqrt{g} \, J_\mu X^\mu \right) \right\rangle$$

and do tricks with $J^\mu(\sigma)$ in order to get the vertex operator insertions we want. We can directly compute $Z[J]$ by writing the Polyakov action as

$$S_X = \frac{T_0}{2} \int d^2\sigma \sqrt{g} \, X_\mu \Delta_g X^\mu, \quad \Delta_g := \nabla^2 = \frac{1}{\sqrt{g}} \partial_a \sqrt{g} \, g^{ab} \partial_b,$$

which is a Gaussian. Expand $X$ and $J$ in terms of an eigenbasis of the Laplacian:

$$\Delta_g \mathcal{X}_I = -\omega_I^2 \mathcal{X}_I, \quad X^\mu = \sum_I x_I^\mu \mathcal{X}_I, \quad J^\mu = \sum_I J_I^\mu \mathcal{X}_I,$$

where the eigenfunctions $\psi_I$ are chosen to be orthogonal with respect to $(\cdot, \cdot)_g$. Then

$$Z[J; g] = \prod_{I,\mu} \int \mathcal{D}x_I^\mu \, \exp\left( -\frac{T_0}{2} \omega_I^2 x_I \cdot x_I + i J_I \cdot x_I \right).$$

Before doing the Gaussian integrals, note that the zero mode $\mathcal{X}_0$ is special: since $\nabla^2 \mathcal{X}_0 = 0$, it has an extremum in the interior of the Riemann surface $\Sigma$. By the maximum modulus principle, $\mathcal{X}_0$ must be constant. Hence its corresponding Gaussian integral becomes $\int \mathcal{D}X^\mu \exp(iJ_0 \cdot x_0) = \delta^D(J_0)$.

**Exercise 4.1.3.** Review how to do Gaussian integrals, in particular the $Z[J]$ integral from QFT, and calculate that for a fixed moduli $g$,

$$Z[J; g] = i(2\pi)^D \delta^D(J_0) \left( \frac{\det'((T_0/\pi)\Delta_g)}{\int d^2\sigma \sqrt{g}} \right)^{-D/2} \exp\left( -\frac{1}{2} \int d^2\sigma_1 \, d^2\sigma_2 \, J(\sigma_1) G'(\sigma_1, \sigma_2) J(\sigma_2) \right),$$

where the **Green's function** (excluding zero modes) is

$$G'(\sigma_1, \sigma_2) := \sum_{I \neq 0} \frac{1}{T_0} \frac{1}{\omega_I^2} \mathcal{X}_I(\sigma_1) \mathcal{X}_I(\sigma_2).$$

Verify that $G'(\sigma_1, \sigma_2)$ satisfies

$$-T_0 \Delta_g G'(\sigma_1, \sigma_2) = \sum_{I \neq 0} \mathcal{X}_I(\sigma_1) \mathcal{X}_I(\sigma_2) = \frac{1}{\sqrt{g}} \delta^2(\sigma_1 - \sigma_2) - \mathcal{X}_0^2$$

using the completeness relation for the eigenbasis $\{\mathcal{X}_I\}$.

We shall use this differential equation for the Green's function in order find out what it is. Note that it depends on $\Delta_g$, which changes with different moduli, and, in particular, different genus. Hence for every $n$, to compute the $n$-loop corrections to the amplitude, we need to find $G'$ for every element in the moduli space.

## 4.2 Tree-Level Amplitudes

In this section, we compute tree-level amplitudes $S_{S^2}(k_1, \ldots, k_n)$ for **$n$-tachyon scattering**, which, for closed strings, corresponds to $\Sigma = S^2$ with $n$ marked points with **tachyon vertex operators** $\mathcal{V}_{j_i}(k_i, \sigma_i) =:e^{ik\cdot X}:$. Hence we must first understand $\mathcal{M}(S^2)$ and $\mathrm{CKG}(S^2)$.

**Proposition 4.2.1.** *$\mathcal{M}(S^2)$ is a single point, and $\mathrm{CKG}(S^2) \cong \mathrm{PSL}(2, \mathbb{C})$, with real dimension 6.*

*Proof.* Take the standard atlas for $S^2$ given by stereographic projection: one coordinate $z$ on $S^2 \setminus \{N\}$, i.e. everywhere except the north pole, and another coordinate $u$ on $S^2 \setminus \{S\}$, i.e. everywhere except the south pole. In conformal gauge, we know from a previous exercise that moduli are holomorphic quadratic differentials $\delta g_{zz}(z)$ and CKVs are holomorphic vector fields $\delta z(z)$. But these objects must be well-defined globally, so we need to consider them on the $u$ patch:

$$\delta u = \frac{\partial u}{\partial z}\delta z = -z^{-2}\delta z, \quad \delta g_{uu} = \left(\frac{\partial u}{\partial z}\right)^{-2} \delta g_{zz} = z^4 \delta g_{zz}.$$

If $\delta g_{uu}$ is well-defined at $u = 0$, i.e. the north pole, then $\delta g_{zz} \propto z^{-4}$ as $|z| \to \infty$. Hence $\delta g_{zz}$ is a bounded entire function, which, by Liouville's theorem, is constant. In the moduli space, we have already modded out by Weyl transformations, so all constant functions are equivalent. (In fact, by the uniformization theorem, every metric is equivalent to the constant-curvature metric induced by the inclusion $S^2 \to \mathbb{R}^3$.) However, for CKVs, we only need $\delta z \propto z^2$ as $|z| \to \infty$, so in general,

$$\delta z = a_0 + a_1 z + a_2 z^2, \quad \delta \bar{z} = a_0^* + a_1^* \bar{z} + a_2^* \bar{z}^2.$$

Hence $\dim_{\mathbb{R}} \mathrm{CKG}(S^2) = 6$. It is well-known that these are precisely the infinitesimal transformations of the Möbius group

$$\mathrm{PSL}(2, \mathbb{C}) = \left\{\frac{az+b}{cz+d} : ad - bc = 1\right\} / ((a, b, c, d) \sim (-a, -b, -c, -d)). \qquad \square$$

Because the moduli space is trivial, we take $g$ to be the constant curvature metric on the sphere. Most things that involve $g$ now become constants, and the integral over moduli space disappears. We write $Z[J; g]$ as $Z[J]$.

We first compute the $X^\mu$ integral using the results of the previous section. Using the fact that $\partial\bar{\partial} \ln |z| = 2\pi\delta(z)$, by solving the DE, we get the Green's function

$$G'(\sigma_1, \sigma_2) = -\frac{1}{4\pi T_0} \ln |z_1 - z_2|^2 + f(z_1, \bar{z}_1) + f(z_2, \bar{z}_2), \quad f(z, \bar{z}) \coloneqq \frac{\mathcal{X}_0^2}{8\pi T_0} \int d^2 w \, e^{2\omega(z,\bar{z})} \ln |z - w|^2 + C$$

where the function $f$ acts as a **regulator** that cancels out in the final amplitudes, and $C$ is just a constant to make sure $G'$ is orthogonal to $\mathcal{X}_0$, a constant. The tachyon vertex operators $:e^{ik_i\cdot X}:$ correspond to $J(\sigma) = \sum_i \delta(\sigma - \sigma_i)$. Consequently,

$$Z[J] = C^{\mathrm{X}}_{S^2} \delta^D(\sum_i k_i) \exp\left(-\sum_{i<j} k_i \cdot k_j G'(\sigma_i, \sigma_j) - \sum_i k_i^2 G'_r(\sigma_i, \sigma_i)\right), \; C^{\mathrm{X}}_{S^2} \coloneqq \frac{i(2\pi)^D}{\mathcal{X}_0^D} \left(\frac{\det'(\frac{T_0}{\pi}\Delta_g)}{\int d^2\sigma \sqrt{g}}\right)^{-\frac{D}{2}},$$

where the $G'_r \neq G'$ because of the normal ordering. In fact, for normal-ordered vertex operators, we get $G'_r(\sigma, \sigma) = 2f(z, \bar{z})$, so that using momentum conservation $\sum_i k_i = 0$, the regulator $f$ cancels and we get

$$Z[J] = C^{\mathrm{X}}_{S^2} \delta^D\left(\sum_i k_i\right) \prod_{i<j} |z_i - z_j|^{k_i\cdot k_j/2\pi T_0}.$$

Since the sphere has no moduli, we are free to ignore the integral over moduli space. Most of the Faddeev-Popov measure becomes constant, except the CKG determinant. From our computation of the CKG, we know a basis for CKVs is given by $\mathcal{C}^z = 1, z, z^2$ and $\mathcal{C}^{\bar{z}} = 1, \bar{z}, \bar{z}^2$, so

$$\Delta_{\mathrm{FP}} = C^{\mathrm{G}}_{S^2} \det(\mathcal{C}^a_{0j}(\sigma_i)) = C^{\mathrm{G}}_{S^2} \det((z_i)^{j-1})^3_{i,j=1} \det((\bar{z}_i)^{j-1})^3_{i,j=1} = C^{\mathrm{G}}_{S^2} |z_1 - z_2|^2 |z_1 - z_3|^2 |z_2 - z_3|^2,$$

absorbing the two other (now constant) determinants in $C^{\mathrm{G}}_{S^2}$. Assume that we are scattering $n > 2$ tachyons, so that the $\dim_{\mathbb{C}} \mathrm{CKG}(S^2) = 3$ degrees of freedom can fix the positions $\hat{z}_1, \hat{z}_2, \hat{z}_3$ of the first three local operators: it is well-known that the Möbius transformations $\mathrm{PSL}(2, \mathbb{C})$ act transitively on triplets of points, i.e. is 3-transitive.

**Proposition 4.2.2.** *The closed string, tree-level, n-tachyon scattering amplitude is given by*

$$S_{S^2}(k_1, \ldots, k_n) = g_s^{-2+n} C^X_{S^2} C^G_{S^2} \delta^D\left(\sum_{i=1}^n k_i\right) |\hat{z}_1 - \hat{z}_2|^2 |\hat{z}_1 - \hat{z}_3|^2 |\hat{z}_2 - \hat{z}_3|^2 \int \prod_{i=4}^n d^2 z_i \prod_{i<j} |z_i - z_j|^{k_i \cdot k_j / 2\pi T_0}$$

*for any choice of $\hat{z}_1, \hat{z}_2, \hat{z}_3 \in \mathbb{C}^2$.*

The closed string, four-tachyon scattering amplitude is the first computed non-trivial string scattering amplitude, and therefore has a name: the **Virasoro-Shapiro amplitude**. We shall compute it explicitly.

**Example 4.2.3** (Virasoro-Shapiro amplitude)**.** Write $\alpha' = 1/2\pi T_0$. Pick $\hat{z}_1 = 1$, $\hat{z}_2 = 0$, and $\hat{z}_3 = \infty$. Then any term $|\hat{z}_3 - z_i|$ can be treated as $|\hat{z}_3|$, so

$$|\hat{z}_1 - \hat{z}_3|^{2+\alpha' k_1 \cdot k_3} |\hat{z}_2 - \hat{z}_3|^{2+\alpha' k_2 \cdot k_3} |\hat{z}_3 - z_4|^{\alpha' k_3 \cdot k_4} = |\hat{z}_3|^{4+\alpha' k_3 \cdot (k_1 + k_2 + k_4)} = |\hat{z}_3|^0 = 1,$$

by recalling that on-shell tachyons satisfy $k^2 = 4/\alpha'$, and therefore, using momentum conservation,

$$4 + \alpha' k_3 (k_1 + k_2 + k_4) = 4 - \alpha' k_3^2 = 4 - 4 = 0.$$

Introduce **Mandelstam variables** like we did for 4-point scattering in QFT:

$$s \coloneqq -(k_1 + k_2)^2, \quad t \coloneqq -(k_1 + k_3)^2, \quad u \coloneqq -(k_1 + k_4)^2,$$

which satisfy $s + t + u = -16/\alpha'$. Again using the on-shell condition and momentum conservation, we can show that the remaining part of the integral is

$$\int d^2 z_4 \, |1 - z_4|^{\alpha' k_1 \cdot k_4} |z_4|^{\alpha' k_2 \cdot k_4} = \int d^2 z_4 |1 - z_4|^{-(\alpha' u)/2 - 4} |z_4|^{-(\alpha' t)/2 - 4}.$$

Now recall that using the **Euler beta function**,

$$\int d^2 z \, |z|^{2a-2} |1 - z|^{2b-2} = 2\pi \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b)\Gamma(a+c)\Gamma(b+c)}, \quad a + b + c = 1,$$

so that our final expression is

$$S_{S^2}(k_1, \ldots, k_4) = g_s^2 C^{\mathrm{X}}_{S^2} C^{\mathrm{G}}_{S^2} \delta^D(k_1 + \cdots + k_4) 2\pi \frac{\Gamma(-1 - \frac{\alpha'}{4}s)\Gamma(-1 - \frac{\alpha'}{4}t)\Gamma(-1 - \frac{\alpha'}{4}u)}{\Gamma(2 + \frac{\alpha'}{4}s)\Gamma(2 + \frac{\alpha'}{4}t)\Gamma(2 + \frac{\alpha'}{4}u)}.$$

The analogous amplitude for open strings, called the **Veneziano amplitude**, led to the birth of string theory in the 1970s. Note that any expectation value on the sphere $S^2$ must be invariant under $\mathrm{CKG}(S^2) = \mathrm{PSL}(2, \mathbb{C})$, the Möbius group. This invariance is known as **Möbius invariance**, and holds in particular for amplitudes.

## 4.3 One-Loop Corrections

In this section, we outline[1] the computation for one-loop corrections $S_{T^2}(k_1, \dots, k_n)$ to amplitudes for $n$-tachyon scattering, which, for closed strings, corresponds to $\Sigma = T^2$, the torus, with $n$ marked points. Hence we must understand $\mathcal{M}(T^2)$ and $\mathrm{CKG}(T^2)$. Two notes:

1. recall that $\mathbb{H}^2$ is the **hyperbolic plane**, which can be identified with the **upper half plane**; and
2. for this section, for simplicity, we set $2\pi T_0 = 1$.

**Proposition 4.3.1.** *$\mathcal{M}(T^2) = \mathbb{H}^2/\operatorname{PSL}(2,\mathbb{Z})$, and $\mathrm{CKG}(T^2) = U(1) \times U(1)$.*

*Proof sketch.* By the uniformization theorem, $T^2$ has universal cover $\mathbb{C}^2$, and we obtain $T^2$ from $\mathbb{C}^2$ by quotienting out by a lattice, i.e. $\mathbb{C}^2/\sim$ where $z \sim z + 2\pi\tau$. To eliminate overcounting, we take $\tau \in \mathbb{H}^2$. The claim is that two torii are equivalent if and only if their $\tau$'s equal. Note however that $\operatorname{diff}(T^2)/\operatorname{diff}_0(T^2) = \operatorname{PSL}(2,\mathbb{Z})$ is no longer trivial, and we must quotient it out. Hence $\mathcal{M}(T^2) = \mathbb{H}^2/\operatorname{PSL}(2,\mathbb{Z})$.

The CKG is generated by the translations

$$\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}, \quad \tau\frac{\partial}{\partial z} + \bar{\tau}\frac{\partial}{\partial \bar{z}}.$$

Since translation across the fundamental domain is actually the identity on the resulting torus, two distinct translations differing only by fundamental domains are actually equivalent. Hence we obtain $U(1) \times U(1)$ as the CKG. □

Recall that the action of the **modular group** $\operatorname{PSL}(2,\mathbb{Z})$ on $\mathbb{H}^2$ is generated by the transformations $\tau \mapsto \tau + 1$ and $\tau \mapsto -1/\tau$. Hence we must take a **fundamental domain** of $\mathbb{H}^2/\operatorname{PSL}(2,\mathbb{Z})$, which usually is chosen to be

$$\mathfrak{F} := \{\tau \in \mathbb{H}^2 : \Re(\tau) \in [-1/2, 1/2], \, |\tau| \geq 1\}.$$

Now we want to compute the Green's function $G'(\sigma_1, \sigma_2)$. Note that as a function on $\mathbb{C}^2$, it is doubly periodic and is (mostly) the sum of holomorphic and antiholomorphic functions (this comes from the PDE for the Green's function). These conditions indicate that $G'$ may be related to **theta functions**.

### 4.3.1 Theta Functions

For reference, we briefly write down some definitions regarding theta functions. They are important for not just the one-loop calculation, but also for superstring amplitudes later on.

**Definition 4.3.2.** The **Jacobi theta function** $\vartheta(v, \tau)$ is the unique function, up to normalization, satisfying

$$\vartheta(v+1, \tau) = \vartheta(v, \tau), \quad \vartheta(v+\tau, \tau) = \exp(-\pi i\tau - 2\pi i v)\vartheta(v, \tau).$$

Explicitly, it is given by

$$\begin{aligned}\vartheta(v, \tau) &:= \sum_{n=-\infty}^{\infty} \exp\left(\pi i n^2\tau + 2\pi i n v\right) \\ &= \prod_{n=1}^{\infty}(1-q^n)(1+zq^{n-1/2})(1+z^{-1}q^{m-1/2}), \quad q = e^{2\pi i\tau}, \, z = e^{2\pi i v}.\end{aligned}$$

[1]For details on this computation, see http://ccdb5fs.kek.jp/cgi-bin/img/allpdf?198803041.

**Proposition 4.3.3.** *The Jacobi theta satisfies the* **Jacobi identities**

$$\vartheta(v, \tau+1) = \vartheta(v+1/2, \tau), \quad \vartheta(v/\tau, -1/\tau) = (-i\tau)^{1/2} \exp(\pi i v^2/\tau)\vartheta(v, \tau).$$

**Definition 4.3.4.** The **auxiliary theta functions** are

$$\vartheta(v, \tau; a, b) := \exp\left(\pi i a^2 \tau + 2\pi i a(v+b)\right) \vartheta(v + a\tau + b, \tau),$$

and common notations for special cases (which we write as sums using the **Jacobi triple product**) are

$$\begin{aligned} \vartheta_{00}(v, \tau) &:= \vartheta_3(v|\tau) := \vartheta(v, \tau; 0, 0) = \sum_{m=-\infty}^{\infty} q^{m^2/2} z^m \\ \vartheta_{11}(v, \tau) &:= -\vartheta_1(v|\tau) := \vartheta(v, \tau; 1/2, 1/2) = -i \sum_{m=-\infty}^{\infty} (-1)^m q^{(m-1/2)^2} z^{m-1/2}. \end{aligned}$$

For completeness, $\vartheta_{01} := \vartheta_4 := \vartheta(v, \tau; 0, 1/2)$ and $\vartheta_{10} := \vartheta_2 := \vartheta(v, \tau; 1/2, 0)$.

**Exercise 4.3.1.** Show that the **Dedekind eta function** $\eta(\tau)$ is given by

$$\eta(\tau) := q^{1/24} \prod_{m=1}^{\infty} (1 - q^m) = \left(-\frac{\vartheta'_{11}(0|\tau)}{2\pi}\right)^{1/3},$$

and that it has the modular transformations

$$\eta(\tau+1) = \exp(i\pi/12)\eta(\tau), \quad \eta(-1/\tau) = (-i\tau)^{1/2}\eta(\tau).$$

### 4.3.2 The Amplitude Calculation

Using theta functions, we take a guess for $G'(z_1, \bar{z}_1, z_2, \bar{z}_2)$: it should be proportional to $\ln|\vartheta_{11}(z_1 - z_2|\tau)|^2$, so that it satisfies all the desired properties. This is mostly correct, except that the theta functions are not completely doubly periodic, so we need an extra term to counteract the extra factors:

$$G'(z_1, \bar{z}_1, z_2, \bar{z}_2) = -\frac{1}{2} \ln \left| \vartheta_{11}\left(\frac{z_1 - z_2}{2\pi} \middle| \tau\right)\right|^2 + \frac{\operatorname{Im}(z_1 - z_2)^2}{4\pi \operatorname{Im}(\tau)} + k(\tau, \bar{\tau}),$$

where $k(\tau, \bar{\tau})$ is orthogonal to $\mathcal{X}_0$ and acts as a regulator.

**Exercise 4.3.2.** Using the same procedure as for $S^2$, conclude that

$$Z[J] = C^{\mathrm{X}}_{T^2}(\tau)\delta^D\left(\sum_{i=1}^{n} k_i\right) \prod_{i<j} \exp\left(-\pi \frac{(\operatorname{Im}(z_i - z_j))^2}{\tau_2}\right) \left|\frac{\vartheta_{11}(z_i - z_j)}{\vartheta'_{11}(0)}\right|^{k_i \cdot k_j}.$$

Now we must be careful, because unlike $S^2$, the moduli space for $T^2$ is non-trivial. Hence terms such as $\det'(\Delta_g/2\pi^2)$ and $\int d^2\sigma\sqrt{g}$, which are hidden in $C^{\mathrm{X}}_{T^2}(\tau)$, are no longer constant, and instead depend on the modulus $\tau$. In particular, note that $\sqrt{g} = \tau$.

**Exercise 4.3.3.** Write $\tau = \tau_1 + i\tau_2$, where $\tau_1, \tau_2 \in \mathbb{R}$. Show that the eigen-decomposition of $\Delta_g$ on the torus, for a fixed modulus $\tau$, is

$$\lambda_{n,m} := \frac{4\pi^2}{\tau_2^2}(m + \tau n)(m + \bar{\tau} n), \quad \psi_{n,m}(\sigma_1, \sigma_2) = \exp\left(2\pi i \left(n\sigma_1 - \frac{\tau_1}{\tau_2} n\sigma^2 - \frac{1}{\tau_2} m\sigma^2\right)\right),$$

and therefore, using zeta regularization,

$$\det{}'\left(\frac{\Delta_g}{2\pi^2}\right) = \prod_{(m,n)\neq(0,0)} \frac{2}{\tau_2^2}(m+\tau n)(m+\bar{\tau}n) = \tau_2^2|\eta(\tau)|^4.$$

Hence we can write $C^{\mathrm{X}}_{T^2}(\tau) = C^{\mathrm{X}}_{T^2}\tau_2^{-13}|\eta(\tau)|^{-52}$ where here the constant $C^{\mathrm{X}}_{T^2}$ is no longer a function of $\tau$. The extra factor of $\tau_2^{13}$ comes from $(\int d^2\sigma\sqrt{g})^{13}$.

**Exercise 4.3.4.** Instead of gauge-fixing the degrees of freedom in the CKG, we explicitly divide out by its volume this time. Show that the volume of the CKG satisfies

$$\mathrm{Vol(CKG)}^2 = \det\begin{pmatrix} \tau_2 & \tau_1\tau_2 \\ \tau_1\tau_2 & \tau_2|\tau|^2 \end{pmatrix} = \tau_2^2|\tau|^2 - \tau_1^2\tau_2^2 = \tau_2^4.$$

Also, compute that the moduli and CKG determinants give $1/\tau_2$. Finally, compute $\det'(P_1^\dagger P_1/2\pi)^{1/2} \propto \tau_2^2|\eta(\tau)|^4$, absorbing the constant factors in $C^{\mathrm{G}}_{T^2}$.

**Proposition 4.3.5.** *The closed string, one-loop correction to the $n$-tachyon scattering amplitude is given by*

$$S_{T^2}(k_1,\dots,k_n) = g_s^n C^X_{T^2} C^G_{T^2} \int_{\mathfrak{F}} d^2\tau \, \frac{1}{\tau_2^2}\tau_2^{-12}|\eta(\tau)|^{-48} \int \prod_{i=1}^n dz_i \prod_{i<j}\chi(z_i - z_j;\tau)^{k_i\cdot k_j}$$

*where $\eta(\tau)$ is the Dedekind eta function, and*

$$\chi(z;\tau) = \exp\left(-\pi\frac{(\mathrm{Im}\, z)^2}{\tau_2}\right)\left|\frac{\vartheta_{11}(z)}{\vartheta_{11}'(0)}\right|$$

*is a modular invariant.*

There are a few things to note about this amplitude.

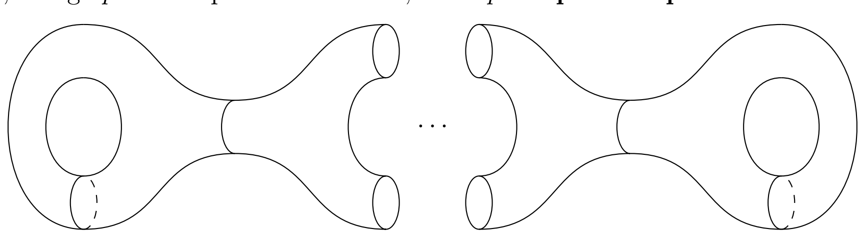
1. It is invariant under the action of the modular group, as it should be: the measure $d^2\tau/\tau_2^2$ and $\tau_2|\eta(\tau)|^4$ are individually invariant, the latter using properties of theta functions.
2. The exponent $-48$ comes from the 24 possible excitations; in the calculations, the additional factor $|\eta(\tau)|^4$ from $\det'(P_1^\dagger P_1/2\pi)^{1/2}$ can be traced back to the removal of ghost excitations.

This **modular invariance** plays a key role later on in the construction of the heterotic string theory.

## 4.4 Higher-Order Corrections

For all genera $p \geq 2$, a general derivation of the $n$-tachyon scattering amplitudes exists; the genus 0 and 1 cases are special because of the existence of CKVs. But every component of this derivation becomes more elaborate. From our calculations on $S^2$ and $T^2$, we see that there are three main components to calculating amplitudes, so in this section we merely jot down some notes outlining how one goes about calculating each of these components.

First, to integrate over moduli space, we require coordinates on it. The usual way to put coordinates on $\mathcal{M}(\Sigma_p)$ to first put coordinates on the **Teichmüller space** $\mathcal{T}(\Sigma_p)$ of metrics. The general idea is to take $\Sigma_p$ and cut it up, along $3p-3$ simple closed curves, into $2p-2$ **pairs of pants**:



Each of the $3p-3$ simple closed curves is homotopic to a geodesic. The $3p-3$ **lengths** of these $3p-3$ geodesics form one half of the coordinates. There are $3p-3$ more coordinates measuring **twists** in the legs of each pair of pants. Together, these length and twist parameters form **Fenchel-Nielsen coordinates** for Teichmüller space, which descend to the moduli space.

We also need to compute the various determinants: the two (no more CKVs) determinants forming the Faddeev-Popov measure, and the determinant of $\Delta_g$ coming from the $X^\mu$ integral. Define the **Selberg zeta function**

$$Z(s) \coloneqq \prod_{\gamma \text{ primitive}} \prod_{k=1}^{\infty} (1 - \exp(-(s+k)l(\gamma)),$$

where $\gamma$ ranges over simple closed geodesics of $\Sigma_p$ and $l(\gamma)$ is the length of $\gamma$. Then

$$\det(\Delta_g) = \exp(-c_0(2-2p))Z'(1), \quad \det(P_1^\dagger P_1) = \exp(-c_1(2-2p))Z(2),$$

with a giant formula for $c_n$:

$$c_n \coloneqq \sum_{0 \le m < n-1/2} (2n-2m-1)\ln(2n-m) - \left(n+\frac{1}{2}\right)^2 + 2(n-[n])\left(n+\frac{1}{2}\right)\ln 2\pi + 2\zeta'(-1).$$

The remaining determinant depends on specific details of Fenchel-Nielsen coordinates, and so we say no more about it.

Finally, we must find the Green's function. For $T^2$, exponentiating the Green's function gave us $\chi(z;\tau)$. It turns out we can generalize $\chi(z;\tau)$ to higher genus with a lot more work. Define the **Riemann theta function**

$$\vartheta(\vec{z}, \Omega; \vec{a}, \vec{b}) \coloneqq \sum_{\vec{n} \in \mathbb{Z}^p} \exp\left(i\pi(\vec{a}+\vec{n})^T\Omega(\vec{a}+\vec{n}) + 2\pi i(\vec{a}+\vec{n})^T(\vec{z}+\vec{b})\right)$$

for $\vec{z} \in \mathbb{C}^p$ and $\vec{a}, \vec{b} \in \mathbb{Q}^p$ and $\Omega \in M_p(\mathbb{C})$ symmetric with $\operatorname{Im}\Omega > 0$; this is just a higher-dimensional generalization of the Jacobi theta function, and also satisfies many periodicity properties.

Recall that a homology basis of $\Sigma_p$ consists $2p$ cycles $\alpha_i$ and $\beta_j$, to which we can associate $p$ holomorphic 1-forms $\vec{\omega} = (\omega^i)_{i=1}^p$, called **abelian differentials**. The **period matrix** is then $\Omega_{ij} \coloneqq \int_{\beta_i} \omega_j$, and it is a fact that $\Omega_{ij}$ is symmetric with $\operatorname{Im}\Omega > 0$. We can now state the generalization of $\chi$:

$$\chi(z_1, z_2) \coloneqq \exp\left(-\pi\left(\operatorname{Im}\int_{z_1}^{z_2}\omega\right)^T(\operatorname{Im}\Omega)^{-1}\left(\operatorname{Im}\int_{z_1}^{z_2}\omega\right)\right)|E(z_1,z_2)|,$$

where $E(z_1,z_2)$ is the **Schottky-Klein prime form** of $\Sigma_p$:

$$E(z_1,z_2) \coloneqq \frac{\vartheta\left(\int_{z_1}^{z_2}\omega, \Omega; \vec{a}, \vec{b}\right)}{\sqrt{\partial_i\vartheta(0,\Omega;\vec{a},\vec{b})\omega^i(z_1)}\sqrt{\partial_j\vartheta(0,\Omega;\vec{a},\vec{b})\omega^j(z_2)}}.$$

The vectors $\vec{a}$ and $\vec{b}$ must have entries in $(1/2)\mathbb{Z}$, such that $4\vec{a}\cdot\vec{b}$ is odd, and the prime form $E(z_1,z_2)$ does not actually depend on the choice of $\vec{a}$ and $\vec{b}$.

In the end, the goal of calculating higher-loop corrections for the purely bosonic string is to, hopefully, carry the techniques over to the superstring, and therefore show that superstring theory is finite at all loops. For more details, see, in order, the following papers.


1. A. A. Belavin, V. Knizhnik, A. Morozov and A. Perelomov, Two and three loop amplitudes in the bosonic string theory, Phys. Lett. B 177 (1986), 324-328.

2. M. Matone, Extending the Belavin-Knizhnik "wonderful formula" by the characterization of the Jacobian, JHEP 1210 (2012) 175.

# Chapter 5

# Toroidal Compactification

Goal of this chapter is to introduce the basic role of compactifications in field/string theories. We shall first investigate Kaluza–Klein's idea of how to add a theory of electromagnetism to a theory of gravity, then proceed to see how this idea ports over to a theory of strings. All the while we shall quotient the last dimension by a translational symmetry. In the closed string picture this gives two features for the states: roughly speaking nonzero momenta give a $\frac{1}{R}$ additional mass and when strings wind around the compact dimension they get an additional term contributing to the mass that is proportional to $R$. Exploring the consequences of the mass spectrum we will uncover a lot of additional physics that comes about from this procedure of compactifying dimensions. In particular, this will be our first exposure to T-duality and D-branes.

## 5.1 Toroidal compactifications in field theory

The Kaluza–Klein mechanism attempts to incorporate gauge theory and gravity into one theory. Let us suppose that we have $D = d + 1$ dimensions, with the extra dimension compact: $x^d \equiv x^2 + 2\pi R$. Now let's suppose we have a metric $G^D_{MN}$, with $M, N \in \{0, \ldots, d\}$.

**Proposition 5.1.1.** *The metric $G^D_{MN}$ separates into three parts $G_{\mu\nu}, G_{\mu d}, G_{dd}$, that depend only on the non-compact dimensions. In particular the most general metric invariant under translations of $x^d$ will be given by:*

$$ds^2 = G_{MN} dx^M dx^N = G_{\mu\nu} dx^\mu dx^\nu + G_{dd}(dx^d + A_\mu dx^\mu)^2$$

Reparametrizations in the $d$-th component, $x'^d = x^d + \epsilon^d(x^\mu)$ give rise to a gauge transformation $A'_\mu = A_\mu - \partial_\mu \epsilon^d$.

Now let us take an aside and describe what we mean by a "small" compact extra dimension reducing a theory. Consider a massless scalar field $\phi$. This means the Lagrangian is given by $\mathcal{L} = \partial_M \phi \partial^M \phi$ and thus gives an equation of motion $\partial_M \partial^M \phi = 0$. Now, by periodicity of $x^d$ we expect the conjugate momentum to be discrete: $|p_d\rangle = \sum_{x_d} e^{ip_d x_d} |x_d\rangle = \sum_{x_d} e^{ip_d x_d} |x_d + 2\pi R\rangle = e^{-ip_d 2\pi R} |p_d\rangle$, which implies that $p_d = n/R$ for some $n \in \mathbb{Z}$. This discretization means that we can expand the massless scalar $\phi$ as a power series:

$$\phi(x^M) = \sum_{n\in\mathbb{Z}} \phi_n(x^\mu) \exp(inx^d/R)$$

Plugging this into the equation of motion for a massless scalar in $D$ dimensions gives a tower of massive scalars in $d$ dimensions. Taking $G_{dd} = 1$ simplifies this equation to become:

$$\partial_\mu \partial^\mu \phi_n(x^\mu) = \frac{n^2}{R^2}\phi_n(x^\mu).$$

In particular, $\phi_n$, is a scalar with mass $\frac{n}{R}$. When energies $\Lambda \ll \frac{1}{R}$, these massive scalars do not appear in the theory and thus we say that our theory is reduced to a $d$-dimensional one.

## 5.2 Toroidal Compactification for Strings

We showed that

$$X^\mu(z,\bar{z}) = \frac{x^\mu}{2} + \frac{\tilde{x}^\mu}{2} - i\sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu + \tilde{\alpha}_0^\mu)\tau + \sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu - \tilde{\alpha}_0^\mu)\sigma + \text{oscillators}.$$

Suppose that we take $\sigma^1$, the space parameter of the string and run it around by $2\pi$: $\sigma^1 \rightsquigarrow \sigma^1 + 2\pi$. This will get us:

$$X^\mu(z,\bar{z}) \rightsquigarrow X^\mu(z,\bar{z}) + 2\pi\sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu - \tilde{\alpha}_0^\mu).$$

In the non-compact case, we have single-valuedness of $X^\mu$ this gives means that $\alpha_0^\mu = \tilde{\alpha}_0^\mu \propto p^\mu$. In the compact case, $X^{25} \rightsquigarrow X^{25} + 2\pi w R$, where $w$ corresponds to the number of times the string wound around the compact dimension. Recall, that the momentum of the string can be given by

$$p^\mu = \frac{1}{\sqrt{2\alpha'}}(\alpha_0^\mu + \tilde{\alpha}_0^\mu),$$

which means that we have two restrictions on the pair $\alpha_0^{25}$ and $\tilde{\alpha}_0^{25}$. Since $p^d = \frac{n}{R}$, we have the following equations for $\alpha_0^{25}$ and $\tilde{\alpha}_0^{25}$:

$$\alpha_0^{25} = \left(\frac{n}{R} + \frac{wR}{\alpha'}\right)\sqrt{\frac{\alpha'}{2}}$$
$$\tilde{\alpha}_0^{25} = \left(\frac{n}{R} - \frac{wR}{\alpha'}\right)\sqrt{\frac{2}{\alpha'}}$$

When we compactify the dimension $d = 25$ we can ask what happens to the mass spectrum of the string in the remaining $0, \ldots, 24$ dimensions.

$$m^2 = -p^\mu p_\mu = \frac{2}{\alpha'}(\alpha_0^{25})^2 + \frac{4}{\alpha'}(N-1) \tag{5.1}$$
$$= \frac{2}{\alpha'}(\tilde{\alpha}_0^{25})^2 + \frac{4}{\alpha'}(\tilde{N}-1) \tag{5.2}$$

Here $N, \tilde{N}$ refer to the sum over $i$ of the $\alpha_i^{25}$ and $\tilde{\alpha}_i^{25}$ terms. These two equations are equivalent to the following:

$$m^2 = \frac{n^2}{R^2} + \frac{w^2R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \tilde{N} - 2) \tag{5.3}$$
$$0 = nw + N - \tilde{N} \tag{5.4}$$

Note that the tower of Kaluza–Klein states reappears in the stringy-setting. In addition to this we obtain our first stringy phenomenon which is a contribution to the mass from the winding number.

Now that we have a mass spectrum for the theory, we can look at a special case that is of importance to the theory: the massless states. In field theory or string theory as we've seen it before we know that certain massless states, vector bosons or photons, admit a $U(1)$ gauge group.

Now that there is a possibility for the string to wind around $d = 25$, we saw that the holomorphic and anti-holomorphic components of the theory do not quite equal each other. This gives rise to two distinct vector bosons and thus a $U(1) \times U(1)$ gauge group. Explicitly, using the symmetric tensor and the antisymmetric tensor, we can form the following fields:

$$A_{\mu(R)} \equiv \frac{1}{2}(G - B)_{\mu,25}; \qquad A_{\mu(L)} \equiv \frac{1}{2}(G + B)_{\mu,25}.$$

**Exercise 5.2.1.** (Optional, but informative.) It is useful to tie together the various different approaches that we have for describing the string fields/states/vertex operators. Here is a table, taken from Charles Johnson's, *D-branes*, which relates these three concepts:

| $field$ | $state$ | $operator$ |
|---|---|---|
| $G_{\mu\nu}$ | $(\alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1} + \alpha^{\nu}_{-1}\tilde{\alpha}^{\mu}_{-1})\,\lvert 0; k\rangle$ | $\partial X^{\mu}\partial X^{\nu} + \partial X^{\mu}\partial X^{\nu}$ |
| $B_{\mu\nu}$ | $(\alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1} - \alpha^{\nu}_{-1}\tilde{\alpha}^{\mu}_{-1})\,\lvert 0; k\rangle$ | $\partial X^{\mu}\bar{\partial} X^{\nu} + \partial X^{\mu}\bar{\partial} X^{\nu}$ |
| $A_{\mu(R)}$ | $\alpha^{\mu}_{-1}\tilde{\alpha}^{25}_{-1}\,\lvert 0; k\rangle$ | $\partial X^{\mu}\bar{\partial} X^{25}$ |
| $A_{\mu(L)}$ | $\tilde{\alpha}^{\mu}_{-1}\alpha^{25}_{-1}\,\lvert 0; k\rangle$ | $\partial X^{25}\bar{\partial} X^{\mu}$ |
| $\phi \equiv \frac{1}{2}\log G_{25,25}$ | $\alpha^{25}_{-1}\tilde{\alpha}^{25}_{-1}\,\lvert 0; k\rangle$ | $\partial X^{25}\bar{\partial} X^{\mu}$ |

As an extra exercise, determine the state and associated vertex operator at zero momentum for the dilaton $\Phi$.

## 5.3 T-Duality and Corollaries

The mass spectrum (5.4) has curious symmetries. Associated to the compactness we the Kaluza–Klein states that correspond to $w = 0$. As $R \to \infty$ these states become massless; however, the states for which the winding number is non-zero suddenly become extremely massive. This is precisely what we expect in the non-compact case.

Conversely, if $R \to 0$, all of the Kaluza–Klein states (which traverse the compact direction) become massive. Roughly speaking, this means that open strings, or strings with $w = 0$ tend to prefer to move in the non-compact dimensions. In closed-string theory, however, there is an extra twist: if the closed string manages to wind itself around the compact dimension then it becomes massless. This corresponds, again, to an effective non-compact dimension.

This is precisely the T-duality:

$$n \leftrightarrow w, \quad R \leftrightarrow R' = \frac{\alpha'}{R}.$$

In particular, this duality exchanges the mode operators by: $\alpha_0^{25} \to \alpha_0^{25}$ and $\tilde{\alpha}_0^{25} \to -\tilde{\alpha}_0^{25}$. In terms of the $X(z, \bar{z})$, the transformations take $X^{25}(z, \bar{z}) = X^{25}(z) + X^{25}(\bar{z}) \to X'^{25}(z, \bar{z}) = X^{25}(z) - X^{25}(\bar{z})$. The action, energy-momentum tensor, conformal invariance, etc. are all preserved under this duality and hence these theories are completely identical.

### 5.3.1 Enhanced Gauge Symmetry

Let us take $R = \sqrt{\alpha'}$. This is interesting because this is a fixed point of the T-duality with respect to the radius. Solving for $\alpha_0^{25}, \tilde{\alpha}_0^{25}$ we get:

$$\alpha_0^{25} = \frac{n + w}{\sqrt{2}}, \qquad \tilde{\alpha}_0^{25} = \frac{n - w}{\sqrt{2}}$$

In exactly the same manner as before, we may compute the two restrictions corresponding to the massless spectrum:

$$(n+w)^2 + 4N = 4, \qquad (n-w)^2 + 4\tilde{N} = 4.$$

There is an immediate solution which we have already encountered: $n = w = 0$ and $N = \tilde{N} = 1$. These are the first excited states which include the vectors for the $U(1) \times U(1)$ gauge symmetry for the compactified theory. However, there are more solutions. Among other ones, the four solutions of interest to us are:

$$\begin{aligned} n = -w = \pm 1, &\quad N = 1, \tilde{N} = 0 \\ n = w = \pm 1, &\quad N = 0, \tilde{N} = 1 \end{aligned}$$

The first pair of solutions are left-moving excitations of the strings living in the compact direction, the second pair represent right-moving excitations. These states add give rise to an $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ gauge symmetry. Let us explore this is in more detail.

For illustrative purposes, we show the form of the vertex operators that correspond to the pair of solutions listed above:

$$:\bar{\partial} X^\mu e^{ik\cdot X} \exp\left[\frac{\pm 2i}{\sqrt{\alpha'}} X_L^{25}\right]: \qquad\qquad :\partial X^\mu e^{ik\cdot X} \exp\left[\frac{\pm 2i}{\sqrt{\alpha'}} X_R^{25}\right]: \ .$$

These vertex operators carry with them both the plane-wave factors which give a Kaluza–Klein charge and antisymmetric-tensor gauge charge, corresponding to the $\exp\left[\frac{\pm 2i}{\sqrt{\alpha'}} X_{L,R}^{25}\right]$. The charges that act on these vertex operators can be written as:

$$j^1(z) =:\cos\left[\frac{2}{\sqrt{\alpha'}} X_L^{25}\right]: \qquad\qquad j^2(z) =:\sin\left[\frac{2}{\sqrt{\alpha'}} X_L^{25}\right]: \qquad\qquad j^3(z) = \frac{i}{\sqrt{\alpha'}} \partial X_L^{25}(z),$$

normalized to obtain:

$$j^i(z) j^j(0) \sim \frac{\delta^{ij}}{2z^2} + i\frac{\epsilon^{ijk}}{z} j^k(0).$$

This gives the $\mathrm{SU}(2)_L$ lie algebra, with a similar one for the $\mathrm{SU}(2)_R$. The Laurent coefficients of these holomorphic currents give rise to the infinite dimensional algebra:

$$\begin{aligned} j^i(z) &= \sum_{m\in\mathbb{Z}} \frac{j_m^i}{z^{m+1}} \\ [j_m^i, j_n^j] &= \frac{m}{2}\delta_{m,-n}\delta^{ij} + i\epsilon^{ijk} j_{m+n}^k. \end{aligned}$$

This algebra goes by the names current algebra, affine Lie algebra, Kac–Moody algebra. In this particular case, this realization is called a *level one $SU(2)$* current algebra, where the level refers to the quantization of the $\frac{1}{z^2}$ coefficient. We will see this later in chapter 11 which discusses the heterotic string.

## 5.4 Open strings and D-branes

Up to now we have only discussed T-duality in closed string theory. If we consider open strings then we get additional phenomenon that provide a very important generalization of the concept of a "string." These are D-branes and will play a fundamental role in the future.

### 5.4.1 T-duality for Open Strings: Preview

First, we must recall the open string spectrum. The only difference from closed strings is the lack of a winding number, which roughly means that the mass should naïvely just involve the Kaluza–Klein states $m^2 = \frac{n^2}{R^2} + \frac{4}{\alpha'}(N-1)$. Looking at the $R \to 0$ limit these extra Kaluza–Klein states become extremely massive and thus do not appear in our theory. In a physical sense, this means that the strings unwind out of the compactified dimension and only move in the 25 other dimensions; thus we say the theory has become an effective 25-dimensional theory. But wait! A theory of open strings should contain closed strings as well, for example, the ends of the strings should be able to merge under the interactions. We saw before, because of T-duality, the closed strings still live in an effective 26 dimensional theory! Where has the extra dimension gone?

This is where T-duality comes in. First, using the respresentation from the compactification, we have $X^{25}$ and $X'^{25}$ given by:

$$X^{25}(z,\bar{z}) = X_L^{25}(z) + X_R^{25}(\bar{z})$$
$$X'^{25}(z,\bar{z}) = X_L^{25}(z) - X_R^{25}(\bar{z})$$

Using the identification $\partial_1 = \partial_z + \partial_{\bar{z}}$, and $\partial_2 = i(\partial_z - \partial_{\bar{z}})$, we arrive at the Neumann-to-Dirichlet boundary condition swap:

$$\partial_n X^{25} = -i\partial_t X'^{25}.$$

In fact, the endpoints of the string must lie on the same hyperplane:

$$X'^{25}(\pi) - X'^{25}(0) = \int_0^\pi d\sigma^1 \partial_1 X'^{25} = -i\int_0^\pi d\sigma^1 \partial_2 X^{25} = -2\pi\alpha' v^{25} = \frac{-2\pi\alpha'\ell}{R} = -2\pi\ell R'.$$

Recall the compactification, $X'^{25}(z) \equiv X'^{25}(z) + 2\pi m R'$, where $m \in \mathbb{Z}$. This means that $X'^{25}(\pi) = X'^{25}(0)$, or in a more physical language, the ends of the string are fixed by the Dirichlet condition to a 25-dimensional hyperplane $\{X'^{25} - X'^{25}(0) = 0\}$. By considering an exchange of graviton between any two arbitrary strings, it can be shown that all strings have their endpoints fixed to the same hyperplane.

### 5.4.2 Interlude: Chan–Paton factors and Wilson lines

**Chan–Paton factors.** Historically, strings were meant to describe two quarks connected by a flux tube. Quarks at either end have a colour degree of freedom, and in a similar way (one that in fact is the most general way to preserve Poincaré and conformal invariance), we may assign an additional degree of freedom to the ends of an open string. If $i,j$ are indices for this degree of freedom, with $1 \le i,j \le n$, then the new states look like: $|N;k;a\rangle = \sum_{i,j=1}^n |N;k;ij\rangle\, \lambda_{ij}^a$; $\lambda_{ij}^a \in \mathfrak{u}(n)$ where the $i,j$ indices give the complete basis of $n^2$ matrices for $\mathfrak{u}(n)$ and the $a$ simply refers to a linear combination of them. This enlarging of the Hilbert space is expected because we have added new degrees of freedom, but these degrees of freedom affect the other quantities that we have seen, in particular, $A_\mu^{25} = A_\mu^a \lambda^a$

**Wilson lines.** The choice of gauge potential $A_\mu^{25}$ can significantly affect the physics. Suppose that $A_\mu^{25}$ is flat: $F^{25} = dA^{25} + iA^{25} \wedge A^{25} = 0$ which gives a Wilson line for some charge $q$:

$$W_q = \exp\left(iq\int_0^{2\pi R} A^{25}(x^{25})dx^{25}\right) = e^{iq\boldsymbol{\theta}}, \qquad \boldsymbol{\theta} = \mathrm{diag}(\theta_1,\ldots,\theta_n).$$

In particular, it is possible to choose a gauge in which $A^{25}$ is constant and equal to

$$A^{25} = -\frac{\boldsymbol{\theta}}{2\pi R}.$$

Classically, the momentum shifts to $\tilde{p} = p - qA = p - q\tilde{\theta}$, and indeed, by writing down a gauge fixed action (see Polchinski for the point-particle case), the momentum for the state $|ij\rangle$ shifts to:

$$v^{25} = \frac{2\pi\ell - \theta_j + \theta_i}{2\pi R}.$$

Immediately (c.f. (5.2)) we can write down the open string spectrum:

$$m^2 = \frac{(2\pi\ell - \theta_j + \theta_i)^2}{4\pi^2 R^2} + \frac{1}{\alpha'}(N-1).$$

This means that the difference between the endpoints of the string are given by:

$$\Delta X'^{25} = X'^{25}(\pi) - X'^{25}(0) = -(2\pi\ell - \theta_j + \theta_i)R'.$$

Up to an additive constant, we may say that the endpoint $i$ lives on the hyperplane at $X'^{25} = \theta_i R'$. We can rewrite the spectrum in a suggestive way, $m^2 = \left(\frac{\Delta X'^{25}}{2\pi\alpha'}\right)^2 + \frac{1}{\alpha'}(N-1)$, which shows that stretched strings contribute more mass.

**Question:** Why do we need to introduce $W_q$?

## 5.5 D-branes: A First Look

The hyperplanes that we observed above were a consequence of a constant background field $A^d$. If we add a $x^\mu$ dependence to $A^d$ then the hyperplane will become a hypersurface. In general, if there are $d - p \geq 1$ compactification directions, then we say the resulting surface is a D$p$-brane. In fact, D1-branes are just strings and so D$p$-branes are the higher-dimensional generalization.

These D-branes are dynamical objects, they may interact with the strings and with one another. In order to understand this interaction we must write down a plausible action for them. Let $\xi^a, a = 0, \ldots, p$.

$$\boldsymbol{S}_p = -T_p \int d^{p+1}\xi \; e^{-Phi} \sqrt{\det\left[G_{ab} + B_{ab} + 2\pi\alpha' F_{ab}\right]}.$$

$G_{ab} = \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b} G_{\mu\nu}(X(\xi))$ and $B_{ab} = \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b} B_{\mu\nu}(X(\xi))$ are the induced metric and antisymmetric tensors on the brane.

It turns out that $T_p$, the tension of the D-brane, is completely determined in terms of the string tensions. The computation of $T_p$ comes from a gravitational coupling between two D-branes an a string propogating between it.

# Chapter 6

# Introduction to Superstrings

So far we have looked at a purely bosonic string theory: from the physicality condition $L_0 |\psi\rangle = 0$ we get the mass-shell condition $k^2 + m^2 = 0$, which is nothing more than the Klein-Gordon equation, describing spin-zero bosons. But consequently we have tachyons, and no fermions, and we want to fix both issues. The idea is to extend the Virasoro algebra to get the **superconformal algebra**, which has both a bosonic and fermionic component. The bosonic component shall have the Polyakov action we know and love, and the fermionic component shall give a new energy-momentum tensor $T_F$, whose zero modes give a new physicality condition, this time enforcing the Dirac equation $i\not{k} + m = 0$.

**Definition 6.0.1.** For convenience, we use Feynman's **slash notation**, defined as follows. Let $\Gamma^\mu_{(D)}$ be a basis for the Clifford algebra $\mathrm{Cl}_{D-1,1}$, i.e. with

$$\{\Gamma^\mu_{(D)}, \Gamma^\nu_{(D)}\} = 2\eta^{\mu\nu}_{(D)} I_n.$$

Then if $A$ is a covector, we write $\not{A} \coloneqq \Gamma^k_{(D)} A_k$. (Usually we omit the subscript, since it is clear from context whether we are working on spacetime or the worldsheet.) The $\Gamma^\mu$ are not uniquely specified, but there is a commonly-used choice: for $D = 2$ and $D = 4$, we take the **Pauli matrices**

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and define

$$\Gamma^0_{(2)} \coloneqq \sigma^2, \quad \Gamma^1_{(2)} \coloneqq i\sigma^1, \quad \Gamma^0_{(4)} \coloneqq \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \Gamma^i_{(4)} \coloneqq \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$$

There are two known ways to add these new fermionic modes: the **Green–Schwarz** (GS) formulation, and the **Ramond–Neveu–Schwarz** (RNS) formulation. They are equivalent after quantization, but the RNS formulation turns out to be easier for us to study.

## 6.1 The Ramond–Neveu–Schwarz Superstring

To introduce the superstring, we go through the same steps as we did for the bosonic string. But now we have the benefit of having developed several tools, including CFT and BRST. Hence we can proceed in a much more structured manner. We shall

1. write down the superstring action and identify its supersymmetry,

2. identify and investigate the constraint algebra for physical superstring states,
3. discuss superstring boundary conditions and corresponding spectra,
4. find the superstring ghosts, their CFT and central charge, and the superstring critical dimension, and
5. apply covariant quantization to obtain physical states.

**Notation:** a subscript $B$ indicates the bosonic part of an object, and a subscript $F$ indicates the fermionic part. For example, $S_B$ denotes the Polyakov action, while $S_F$ denotes the fermionic action, which we define now. However, we re-interpret what $T_B$ means: instead of seeing it as the energy-momentum tensor, we see it as the operator encoding the Virasoro algebra, i.e. the bosonic constraint algebra. Hence $T_F$ is not, a priori, the "fermionic energy-momentum tensor"; it is the operator encoding the fermionic constraint algebra.

**Definition 6.1.1.** Let $\psi^\mu(z), \tilde{\psi}^\mu(\bar{z})$ be $D$ new anticommuting (Majorana) spinor fields on the worldsheet. The **supersymmetric worldsheet action** in conformal gauge is given by

$$S_{\mathrm{X}} = \int d^2\sigma \left( T_0 \partial_a X^\mu \partial^a X_\mu - \frac{i}{4\pi} \bar{\psi}^\mu \partial\!\!\!/ \psi_\mu \right) = \frac{1}{4\pi} \int d^2 z \left( 4\pi T_0 \partial X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial} \psi_\mu + \tilde{\psi}^\mu \partial \tilde{\psi}_\mu \right),$$

where, as usual, $\psi^\mu$ and $\tilde{\psi}^\mu$ in complex coordinates denote the holomorphic and antiholomorphic part of $\psi^\mu(\sigma)$ respectively.

Why do we enforce that there are $D$ new fields $\psi^\mu$, and why are they Majorana spinors as opposed to, say, Dirac spinors or Majorana–Weyl spinors? The answer lies in the resulting **supersymmetry**, which is crucial to making this new theory work.

### 6.1.1 Supersymmetry and the Super-Virasoro Algebra

**Definition 6.1.2.** Let $\epsilon(\sigma)$ be an anticommuting two-component spinor. The **worldsheet supersymmetry** is given by the infinitesimal transformation

$$\delta X^\mu(\sigma) \coloneqq \bar{\epsilon}(\sigma) \psi^\mu(\sigma), \quad \delta \psi^\mu(\sigma) \coloneqq -i \partial\!\!\!/ X^\mu(\sigma) \epsilon(\sigma).$$

(As a side note, introducing spinor fields $\psi^\mu$ such that there is an automatic worldsheet supersymmetry is the RNS formalism. In contrast, the GS formalism introduces new spinor fields such that there is an automatic **spacetime supersymmetry**.)

**Exercise 6.1.1.** Use the chirality condition $\Gamma^i \Gamma_j \partial_i \epsilon = 0$ to show that the worldsheet supersymmetry is indeed a symmetry. Then rewrite it in complex coordinates as

$$\delta X^\mu(z, \bar{z}) = -\epsilon(z) \psi^\mu(z) - \epsilon^*(z) \tilde{\psi}^\mu(z), \quad \delta \psi^\mu(z) = \epsilon(z) \partial X^\mu(z), \quad \delta \tilde{\psi}^\mu(\bar{z}) = \epsilon(\bar{z})^* \bar{\partial} X^\mu(\bar{z}).$$

Show that the **worldsheet supercurrent** $T_F^{\mathrm{m}}$ generating this supersymmetry and the **matter energy-momentum tensor** $T_B^{\mathrm{m}}$ of the new action are given by

$$T_F^{\mathrm{m}}(z) \coloneqq i\sqrt{4\pi T_0}\, \psi^\mu(z) \partial X_\mu(z), \quad T_B^{\mathrm{m}}(z) \coloneqq -2\pi T_0 \partial X^\mu(z) \partial X_\mu(z) - \frac{1}{2} \psi^\mu(z) \partial \psi_\mu(z).$$

(The superscripts are there because later on we have the same objects again, but for ghosts.)

Importantly, the commutator of two worldsheet supersymmetries is a translation on the worldsheet. The energy-momentum tensor is the current associated with translations, and the Virasoro algebra arises from the modes of the energy-momentum tensor. Hence it is reasonable to suspect that the worldsheet supersymmetry gives an **extension** of the Virasoro algebra, i.e. a bigger Lie algebra containing the Virasoro algebra as a subalgebra. This is indeed true: the resulting Lie (super)algebra is known as the **super-Virasoro algebra**. It is the two-dimensional case of a **superconformal algebra**.

It is crucial that we have the super-Virasoro algebra, as opposed to just the Virasoro algebra, as the constraints on physical states: the Virasoro algebra itself knows nothing about the fermionic fields $\psi^\mu$, and therefore cannot eliminate unphysical excitations of $\psi^\mu$.

**Exercise 6.1.2.** (Important!) First compute the following OPEs by playing around with the identity $0 = \int \mathcal{D}X\mathcal{D}\psi \, \frac{\delta}{\delta A}(e^{-S}B)$ for appropriate operators $A$ and $B$:

$$\psi^\mu(z)\psi^\nu(0) \sim \frac{\eta^{\mu\nu}}{z}, \quad \partial X^\mu(z)\partial X^\nu(0) \sim -\frac{1}{4\pi T_0}\frac{\eta^{\mu\nu}}{z^2}, \quad \psi^\mu(z)\partial X^\nu(0) \sim \text{regular terms.}$$

Using these OPEs, verify the following OPEs (this is just an exercise in Wick contraction):

$$\begin{aligned} T_B(z)T_B(0) &\sim \frac{3D/2}{2z^4} + \frac{2}{z^2}T_B(0) + \frac{1}{z}\partial T_B(0) \\ T_B(z)T_F(0) &\sim \frac{3/2}{z^2}T_F(0) + \frac{1}{z}\partial T_F(0) \\ T_F(z)T_F(0) &\sim \frac{D}{z^3} + \frac{2}{z}T_B(0). \end{aligned}$$

For example, the third OPE is calculated as follows:

$$\begin{aligned} {:}\psi^\mu(z)\partial X_\mu(z){:}\,{:}\psi^\nu(0)\partial X_\nu(0){:} &\sim \frac{\eta^{\mu\nu}}{z}\left(-\frac{1}{4\pi T_0}\frac{\eta_{\mu\nu}}{z^2}\right) + \frac{{:}\partial X_\mu(z)\partial X^\mu(0){:}}{z} - \frac{1}{4\pi T_0}\frac{{:}\psi^\mu(z)\psi_\mu(0){:}}{z^2} \\ &\sim -\frac{1}{4\pi T_0}\frac{D}{z^3} + \frac{{:}\partial X^\mu(0)\partial X_\mu(0){:}}{z} + \frac{1}{4\pi T_0}\frac{{:}\psi^\mu(0)\partial\psi_\mu(0){:}}{z} \\ &\sim -\frac{1}{4\pi T_0}\left(\frac{D}{z^3} + \frac{2}{z}T_B(0)\right), \end{aligned}$$

and then multiply by $(i\sqrt{4\pi T_0})^2 = -4\pi T_0$.

From the $T_BT_B$ OPE, the central charge of the super-Virasoro algebra is $c = 3D/2$, and from the $T_BT_F$ OPE, $T_F$ is a weight $(3/2, 0)$ tensor. Now obviously there is an antiholomorphic copy of the same algebra, so altogether we have a $(N, \tilde{N}) = (1, 1)$-**superconformal field theory** (SCFT). Here $N = 1$ refers to the number of $(3/2, 0)$ currents.

### 6.1.2 Ramond and Neveu–Schwarz Sectors

We shall investigate the **closed RNS superstring**, so as usual we use the cylindrical coordinate $w = \sigma^1 + i\sigma^2$, with $w \sim w + 2\pi$. Then $X^\mu$ must be $2\pi$-periodic.

**Definition 6.1.3.** Even with periodicity conditions and Poincaré invariance enforced, there are two possible periodicity conditions for $\psi^\mu$:

1. **Ramond** (R): $\psi^\mu(w + 2\pi) = \psi^\mu(w)$, and
2. **Neveu–Schwarz** (NS): $\psi^\mu(w + 2\pi) = -\psi^\mu(w)$.

Alternatively, we write $\psi^\mu(w + 2\pi) = \exp(2\pi i v)\psi^\mu(w)$ with $v = 0$ or $1/2$. The four possibilities for $(v, \tilde{v})$ give rise to four different closed superstring theories, denoted **NS–NS**, **NS–R**, **R–NS**, and **R–R**. We shall see that NS–NS and R–R are bosons, and NS–R and R–NS are fermions.

Taking $(v, \tilde{v})$ into account, then, we write the relevant mode expansions as

$$\psi^\mu(w) = i^{-1/2}\sum_{r\in\mathbb{Z}+v}\psi^\mu_r \exp(irw), \quad T_B(w) = \sum_{m\in\mathbb{Z}} L_m \exp(imw), \quad T_F(w) = \sum_{r\in\mathbb{Z}+v} G_r \exp(irw).$$

Note that, from the explicit expressions for $T_F$ and $T_B$, we have $T_F(w+2\pi) = \exp(2\pi i v) T_F(w)$ as well, and hence the sums are over $\mathbb{Z}+v$ (and $\mathbb{Z}+\tilde{v}$ for the antiholomorphic part). But $T_B(w+2\pi) = T_B(w)$ is untwisted, so there the sum is over $\mathbb{Z}$. The mode expansion for $\partial X^\mu$ is of course unchanged, with its modes denoted $\alpha_m^\mu$.

**Exercise 6.1.3.** Show that imposing the canonical commutation relations

$$\{\psi^\mu(w), \psi^\nu(w')\} = \eta^{\mu\nu}\delta(w-w'), \quad [\partial X^\mu(w), \partial X^\nu(w')] = -i\eta^{\mu\nu}\delta(w-w'),$$

and likewise for the antiholomorphic part, results in the following commutation relations for modes:

$$\{\psi_r^\mu, \psi_s^\nu\} = \{\tilde{\psi}_r^\mu, \tilde{\psi}_s^\nu\} = \eta^{\mu\nu}\delta_{r,-s}, \quad [\alpha_m^\mu, \alpha_n^\nu] = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m\eta^{\mu\nu}\delta_{m,-n},$$

**Exercise 6.1.4.** (Optional) Show, using the OPEs for the super-Virasoro algebra, that

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n} \\
\{G_r, G_s\} &= 2L_{r+s} + \frac{c}{12}(4r^2 - 1)\delta_{r,-s} \\
[L_m, G_r] &= \frac{m-2r}{2} G_{m+r}.
\end{aligned}$$

These modes form the **Ramond algebra** for integer $r, s$, and the **Neveu–Schwarz algebra** for half-integer $r, s$; these are the two $N = 1$ supersymmetric (minimal) extensions of the Virasoro algebra.

Unsurprisingly, the **ground state** $|0;k\rangle_{\mathrm{R}}$ of the R sector differs from the ground state $|0;k\rangle_{\mathrm{NS}}$ of the NS sector. Both are defined the same way:

$$\alpha_m^\mu |0;k\rangle_{\mathrm{R}} = \psi_r^\mu |0;k\rangle_{\mathrm{R}} = 0, \quad \alpha_m^\mu |0;k\rangle_{\mathrm{NS}} = \psi_m^\mu |0;k\rangle_{\mathrm{NS}} = 0 \quad \forall m, r > 0.$$

But $|0;k\rangle_{\mathrm{R}}$ is degenerate because of the existence of the zero modes $\psi_0^\mu$ in the R sector, by the same argument as for $b_0$ and $c_0$ in the $bc$ CFT (c.f. BRST state space). Instead of two zero modes $b_0$ and $c_0$, however, we now have $D$ of them, satisfying $\{\psi_0^\mu, \psi_0^\nu\} = \eta^{\mu\nu}$, the Clifford algebra relation (up to scalars)! Hence $|0;k\rangle_{\mathrm{R}}$ lives in an irreducible representation of the Clifford algebra $\mathrm{Cl}_{D-1,1}$, and is a **spinor**, representing a **fermion**.

In contrast, $|0;k\rangle_{\mathrm{NS}}$ is non-degenerate because there is no $\psi_0^\mu$ in the NS sector, and therefore is a **scalar**, representing a **boson**.

**Exercise 6.1.5.** (Important!) Verify, using the super-Virasoro algebra relations, the mode expansions

$$G_r = \sum_{n\in\mathbb{Z}} \alpha_n \cdot \psi_{r-n}, \quad L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} :\alpha_{m-n}\cdot\alpha_n: + \frac{1}{4}\sum_{r\in\mathbb{Z}+v} (2r-m) :\psi_{m-r}\cdot\psi_r: + a^X \delta_{m,0}$$

where $a^X$ is a normal ordering constant equal to $D/16$ for the R sector and 0 for the NS sector.

Using the mode expansions of $L_m$ and $G_r$, we can say more about the ground states. For the R ground state, the constraint

$$G_0 |0\rangle_{\mathrm{R}} = \left(\alpha_0 \cdot \psi_0 + \sum_{n\neq 0} \alpha_{-n}\cdot\psi_n\right)|0\rangle_{\mathrm{R}} = 0$$

is equivalent to $\alpha_0 \cdot \psi_0 |0\rangle_{\mathrm{R}} = 0$ since $\psi_n^\mu |0\rangle_{\mathrm{R}} = 0$ for $n > 0$ by definition. But

1. recall from the quantization of the bosonic string that $\alpha_0^\mu \propto p^\mu = -i\partial^\mu$, and
2. since $\{\psi_0^\mu, \psi_0^\nu\} = \eta^{\mu\nu}$, they act as Dirac matrices $\Gamma^\mu$.

Hence $\alpha_0 \cdot \psi_0 |0\rangle_{\mathrm{R}} \propto -i\not{\partial}|0\rangle_{\mathrm{R}} = 0$ is precisely the **massless Dirac equation**, and it follows that $|0\rangle_{\mathrm{R}}$ is a massless spinor.

### 6.1.3 Superconformal Ghosts

We added anti-commuting ghost fields $b$ and $c$ to the Polyakov action in order to invert the Faddeev–Popov determinant for a commuting variable $X^\mu$. But now we have an anti-commuting variable $\psi^\mu$, and so to invert its Faddeev–Popov determinant, we introduce **commuting** ghost fields $\beta$ and $\gamma$. By analogy with the $bc$ CFT, the new **ghost action** is

$$S_{\mathrm{g}} \coloneqq \frac{1}{2\pi}\int d^2z(b\bar{\partial}c + \beta\bar{\partial}\gamma).$$

(The new fields $\beta$ and $\gamma$ are called **superconformal ghosts**.) However, we must be a little careful when deriving the ghost energy-momentum tensor.

**Exercise 6.1.6.** Start with the action $\int d^2z\, b\bar{\partial}c$ for the $bc$ CFT and:

1. derive its energy-momentum tensor $T = (\partial b)c - \partial(bc)$ using Noether's theorem;
2. argue that we can add any total derivative term $\lambda\partial(bc)$ to the action without affecting the equations of motion, so the energy-momentum tensor should really be $T = (\partial b)c - \lambda\partial(bc)$;
3. calculate the OPE of $T$ with $b$ and $c$ to show that $b$ has weight $(\lambda, 0)$, and $c$ has weight $(1-\lambda, 0)$.

Now recall that $b$ comes from $b_{ab}$ and $c$ comes from $c^a$. So under a conformal transformation, $b$ has weight $(2, 0)$, and $c$ has weight $(-1, 0)$. Hence conclude that $\lambda = 2$.

We can repeat this procedure for the $\beta\gamma$ CFT to get the analogous energy-momentum tensor involving $\beta$ and $\gamma$, but now with $\lambda = 3/2$. This weight comes from the weight of $T_F^{\mathrm{m}}$, which is the generator of **superconformal transformations**. (The half-integer corroborates that $\psi$ transforms as a spinor, i.e. with half-integer spin; $\beta$ and $\gamma$ are associated with $\psi$.) Putting this together with the $bc$ CFT, the total **ghost energy-momentum tensor** is

$$T_B^{\mathrm{g}} \coloneqq -(\partial b)c - 2b\partial c - \frac{1}{2}(\partial\beta)\gamma - \frac{3}{2}\beta\partial\gamma.$$

There is also a ghost version $T_F^{\mathrm{g}}$ of $T_F^{\mathrm{m}}$ arising from the supersymmetry:

$$T_F^{\mathrm{g}} \coloneqq -(\partial\beta)c - \frac{3}{2}\beta\partial c - 2b\gamma.$$

We shall examine where it comes from later on.

**Exercise 6.1.7.** Verify, by computing the $TT$ OPEs, that

1. the **central charge** of the $bc$ CFT is $-3(2\lambda-1)^2 + 1$;
2. the **central charge** of the $\beta\gamma$ CFT is $3(2\lambda-1)^2 - 1$.

Hence the central charge for the ghost super-CFT is $-26 + 11 = -15$. Recalling that the central charge for the matter CFT is $3D/2$, conclude that the **superstring critical dimension** is $D = 10$.

Now, in preparation for finding the physical state space, we write down the relevant mode expansions and the commutator for the modes:

$$\beta(z) \coloneqq \sum_{r\in\mathbb{Z}+v} \frac{\beta_r}{z^{r+3/2}}, \quad \gamma(z) \coloneqq \sum_{r\in\mathbb{Z}+v} \frac{\gamma_r}{z^{r-1/2}}, \quad [\gamma_r, \beta_s] = \delta_{r,-s}.$$

**Definition 6.1.4.** The **ground states** $|0\rangle_{\mathrm{R}}$ and $|0\rangle_{\mathrm{NS}}$ must satisfy the additional requirements

$$\beta_r\,|0\rangle_{\mathrm{NS}} = \gamma_s\,|0\rangle_{\mathrm{NS}} = 0 \quad \forall r, s \geq 1/2$$
$$\beta_r\,|0\rangle_{\mathrm{R}} = \gamma_s\,|0\rangle_{\mathrm{NS}} = 0 \quad \forall r \geq 0,\ s \geq 1$$

Note that, as with the $bc$ CFT, we group $\beta_0$ with the lowering operators and $\gamma_0$ with the raising operators.

### 6.1.4 The Physical State Space

By analogy with the derivation for the purely bosonic theory, the mass formula for a given state is

$$\frac{M^2}{2\pi T_0} = \begin{cases} \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n + \sum_{r=1/2}^{\infty} r\psi_{-r} \cdot \psi_r - \frac{1}{2} & \text{NS sector} \\ \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n + \sum_{r=1}^{\infty} r\psi_{-r} \cdot \psi_r + 0 & \text{R sector.} \end{cases}$$

(It is a small exercise to compute the normal-ordering constants $-1/2$ and 0.)

Canonical quantization or BRST quantization proceeds as usual to obtain the Fock space of states. Instead of outlining those calculations, we just state the following result and move on to details about the physical state space, which contains more subtleties than for the bosonic theory.

**Theorem 6.1.5** (Superstring no-ghost theorem)**.** *The physical state space is isomorphic to the subspace (of the Hilbert space arising from canonical quantization) with no $X^0$, $X^1$, $\psi^0$, $\psi^1$, $b$, $c$, $\beta$ or $\gamma$ excitations.*

Let's use the no-ghost theorem to explicitly write down where the various low-level states live. The R ground state is a massless spinor, and therefore lives in a spinor representation of SO(8). But there are two such representations, denoted $\mathbf{8}$ and $\mathbf{8}'$; they come from the decomposition of the representation $\mathbf{16}$ of Spin(8) into the $\pm 1$ eigenspaces of $\Gamma^9$. Similarly, the NS first excited state lives in a vector representation of SO(8), of which there is only one, denoted $\mathbf{8}_v$.

**Exercise 6.1.8.** (Optional) Show in light-cone gauge that the representations $\mathbf{8}$ and $\mathbf{8}'$ have spin $s = 1/2$ and $s = -1/2$ respectively.

There is some conventional notation for the **low-level state sectors**, summed up in the following table:

| notation | state | SO(8) spin | $M^2$ |
|---|---|---|---|
| NS+ | $\psi^i_{-1/2}\,\lvert 0\rangle_{\text{NS}}$ | $\mathbf{8}_v$ | 0 |
| NS− | $\lvert 0\rangle_{\text{NS}}$ | $\mathbf{1}$ | $-\pi T_0$ |
| R+ | $\lvert 0\rangle_{\text{R}}^{+}$ | $\mathbf{8}$ | 0 |
| R− | $\lvert 0\rangle_{\text{R}}^{-}$ | $\mathbf{8}'$ | 0 |

This table reminds us of the remaining issue: there is still a tachyon! Even more worryingly, it it actually violating supersymmetry: the tachyon has no fermionic counterpart. The idea now is to say with iron certainty that we want to enforce supersymmetry, and therefore we want to find some consistent way to remove the violating states.

**Definition 6.1.6.** Define the **worldsheet fermion number operator**

$$(-1)^F = \begin{cases} (-1)^{\sum_{r=1/2}^{\infty} \psi_{-r}\cdot\psi_r + 1} & \text{NS sector} \\ \Gamma^9 (-1)^{\sum_{r=1}^{\infty} \psi_{-r}\cdot\psi_r} & \text{R sector} \end{cases}$$

and similarly for $(-1)^{\tilde{F}}$. The **Gliozzi–Scherk–Olive (GSO) projection** is a truncation of the superstring spectrum to produce a consistent, supersymmetric theory:

1. the GSO projection of the NS sector consists of the states with

$$(-1)^F \lvert N\rangle_{\text{NS},L} = \lvert N\rangle_{\text{NS},L}\,, \quad (-1)^{\tilde{F}} \lvert N\rangle_{\text{NS},R} = \lvert N\rangle_{\text{NS},R}\,;$$

2. the GSO projection of the R sector consists of the states with

$$(-1)^F \lvert N\rangle_{\text{R},L} = \lvert N\rangle_{\text{R},L}\,, \quad (-1)^{\tilde{F}} \lvert N\rangle_{\text{R},R} = \chi \lvert N\rangle_{\text{R},R}\,,$$

where the **chirality** $\chi = \pm 1$ arises from the two different (opposite-chirality) spinor representations $\mathbf{8}$ and $\mathbf{8}'$.

**Exercise 6.1.9.** Show that GSO projection is compatible with the physical state conditions, i.e. that $[(-1)^F, L_m] = \{(-1)^F, G_r\} = 0$ for all $m, r$.

There are many ways [1] to motivate or "derive" GSO projection. The basic idea is that we want to enforce (worldsheet and spacetime) supersymmetry, modular invariance, and some technical conditions involving how we don't want anti-commuting operators to be able to take bosons to bosons. (Precisely why we need to rule that out is complicated, but at an intuitive level it makes sense.) Instead of outlining the technicalities, we shall just take the pragmatic approach: it works. The following exercise provides additional support.

**Exercise 6.1.10.** Write down the generating function for the number of fermionic states (in the R sector):

$$f_{\mathrm{R}}(q) \coloneqq 8 \prod_{m=1}^{\infty} (1-q^{2m})^{-8}(1+q^{2m})^8$$

where the 8 comes from the eight-fold degeneracy of $|0\rangle_{\mathrm{R}}$. Then write down the generating function for the number of bosonic states:

$$f_{\mathrm{NS}}(q) \coloneqq \prod_{m=1}^{\infty} (1-q^{2m})^{-8} \frac{1}{2q} \left( \prod_{n=1}^{\infty} (1+q^{2n-1})^8 - \prod_{n=1}^{\infty} (1-q^{2n-1})^8 \right).$$

Now rewrite $f_{\mathrm{R}}$ and $f_{\mathrm{NS}}$ in terms of theta functions (see the relevant subsection in the one-loop amplitudes section) and use "**Jacobi's abstruse identity**" $\vartheta_4(0;\tau)^4 = \vartheta_3(0;\tau)^4 - \vartheta_2(0;\tau)^4$ to prove that the GSO-projected state space has the same number of fermions as bosons at every level, i.e. $f_{\mathrm{R}}(q) = f_{\mathrm{NS}}(q)$.

The GSO projection removes the tachyon, which has $(-1)^F|0\rangle_{\mathrm{NS}} = -|0\rangle_{\mathrm{NS}}$, and all states that arise from an odd number of $\psi$ excitations. It also removes one chiral branch of the R sector:

1. **type IIA** superstring theory has $\chi = 1$, giving a non-chiral theory;
2. **type IIB** superstring theory has $\chi = -1$, giving a chiral theory.

### 6.1.5 Superspace and Gravitinos

There is a loose end to tie up: we have been working in (super)conformal gauge this whole time (check how we defined the supersymmetric action). For completeness, we state the action and supersymmetry for the full, un-gauged theory.

**Definition 6.1.7.** The full **supersymmetric worldsheet action** is

$$\frac{1}{4\pi}\int d^2\sigma\sqrt{g}\left(4\pi T_0 g^{ab}\partial_a X^\mu \partial_b X_\mu - i\bar{\psi}^\mu \nabla\!\!\!\!/\, \psi_\mu - i(\bar{\chi}_a \Gamma^b \Gamma^a \psi_\mu)\left(\partial_b X^\mu - \frac{1}{4} i \bar{\chi}_b \psi^\mu\right)\right)$$

with $\chi_a$, called the **worldsheet gravitino**, the supersymmetric partner of the metric $g$. The full **local supersymmetry** is

$$\delta X^\mu \coloneqq \bar{\epsilon}\psi^\mu, \quad \delta\psi^\mu \coloneqq -i\partial\!\!\!/ X^\mu \epsilon,$$

$$\delta g_{ab} \coloneqq \bar{\epsilon}(\gamma_a \chi_b + \gamma_b \chi_a), \quad \delta\chi_a \coloneqq 2\nabla_a \epsilon.$$

The analogue of the conformal gauge, called **superconformal gauge**, is given by

$$g_{ab} = e^\phi \delta_{ab}, \quad \chi_a = \gamma_a \zeta$$

where $\zeta$ is a constant Majorana spinor. Then $\phi$ and $\zeta$ decouple from the action.

[1] See Polchinski 10.6, or Green–Schwarz–Witten 4.3.3

If we were to start, properly, at the beginning, we would take this un-gauged action and gauge fix it via Faddeev–Popov. For arbitrary genus, this is rather hard due to the anticommuting fields: the (super)moduli space now contains anticommuting coordinates, and becomes a **supermanifold**. Hence we started with the action in conformal gauge and proceeded from there, under the assumption that the whole gauge-fixing procedure does still work when anticommuting fields are added.

It is nice, however, to introduce anticommuting coordinates anyway, in order to motivate the full form of the action. The idea is that instead of formulating the action over a two-dimensional space $\Sigma$, we formulate it on a two-dimensional **superspace** $\hat{\Sigma}$, which has two new anti-commuting coordinates $\theta^A$ forming a two-component Majorana spinor.

**Definition 6.1.8.** Define **supersymmetry on superspace** $\delta$ as the transformation with generator

$$Q_A \coloneqq \frac{\partial}{\partial \bar{\theta}^A} + i(\Gamma^a \theta)_A \partial_a,$$

i.e. $\delta Y^\mu = [\bar{\epsilon} Q, Y^\mu]$ for general functions $Y^\mu$.

**Exercise 6.1.11.** Check that $[\delta_1, \delta_2]$ is a translation by first calculating that

$$\delta\theta^A = [\bar{\epsilon}Q, \theta^A] = \epsilon^A, \quad \delta\sigma^a = [\bar{\epsilon}Q, \sigma^a] = i\bar{\epsilon}\Gamma^a\theta.$$

Using the two-dimensional **Fierz identity** $\theta_A \bar{\theta}_B = (-1/2)\delta_{AB}\bar{\theta}_C\theta^C$, show that if we expand $Y^\mu(\sigma, \theta) = X^\mu + \bar{\theta}\psi^\mu + (1/2)\bar{\theta}\theta B^\mu$ (and similarly for $X^\mu$), then

$$\delta X^\mu = \bar{\epsilon}\psi^\mu, \quad \delta\psi^\mu = -i\partial\!\!\!/ X^\mu \epsilon + B^\mu \epsilon, \quad \delta B^\mu = -i\partial\!\!\!/ \psi^\mu \bar{\epsilon}.$$

Hence recover worldsheet supersymmetry in conformal gauge by setting $\partial\!\!\!/ \psi^\mu = 0$, which is the equation of motion for $\psi^\mu$.

**Exercise 6.1.12.** Write down all the gauge symmetries of the un-gauged supersymmetric action:

1. diffeomorphism and Weyl actions on $\sigma^a$ and $g^{ab}$;
2. diffeomorphism and Weyl actions on $\theta^A$ and $\chi^a$;
3. worldsheet supersymmetry.

Apply Faddeev–Popov to get the un-gauged ghost action

$$\int d^2\sigma\, d^2\theta\, B\bar{D}C + \bar{B}D\bar{C}, \quad B \coloneqq \beta + \theta b,\ C \coloneqq c + \theta\gamma.$$

The usual derivative $\partial$ is not invariant under supersymmetry, so we need a "covariant derivative" invariant under supersymmetry. A little bit of searching gives the following definition.

**Definition 6.1.9.** The **superspace covariant derivative** is $D \coloneqq \partial/\partial\bar{\theta} - i\Gamma^a\theta\partial_a$.

Using the covariant derivative we can write down the **supersymmetric worldsheet action** in a different way:

$$S_{\mathrm{X}} \coloneqq \frac{i}{4\pi}\int d^2\sigma\, d^2\theta\, \bar{D}Y^\mu DY_\mu, \quad Y^\mu \coloneqq X^\mu + \bar{\theta}\psi^\mu + \frac{1}{2}\bar{\theta}\theta B^\mu.$$

**Exercise 6.1.13.** Show that the supersymmetric worldsheet action written this way is equal to the one we wrote previously.

**Exercise 6.1.14.** Show that the energy-momentum tensors $T^{\mathrm{m}}_B, T^{\mathrm{m}}_F$ and $T^{\mathrm{g}}_B, T^{\mathrm{g}}_F$ arise as the components of the **super-energy-momentum tensors**

$$T^{\mathrm{m}} = T^{\mathrm{m}}_F + \theta T^{\mathrm{m}}_B = -\frac{1}{2}DX^\mu\partial X_\mu, \quad T^{\mathrm{g}} = T^{\mathrm{g}}_F + \theta T^{\mathrm{g}}_B = -C\partial B + \frac{1}{2}DCDB - \frac{3}{2}\partial CB.$$

## 6.2 The Heterotic String

We have found a consistent superstring theory by extending the constraint algebra from the Virasoro algebra to the $(1, 1)$-superconformal algebra. The natural question is: can we find more consistent superstring theories by imposing different constraint algebras? The main reason behind wanting to do so is that type II theory is not satisfactory as a **grand unified theory**; it doesn't even contain the $\mathrm{SU}(3) \times \mathrm{SU}(2) \times U(1)$ of the standard model!

One natural idea is to look at $(N, \tilde{N})$-superconformal algebras for $N > 1$. But using these algebras as constraint algebras fails for non-trivial reasons; [2] it turns out only $N = 0$ and $N = 1$ work. So the other slightly less natural idea is to look at the case where $N \neq \tilde{N}$, i.e. the $(0, 1)$**-superconformal algebra**. Such a constraint algebra is only possible for closed strings, since level matching in open strings enforces $N = \tilde{N}$.

**Definition 6.2.1.** The **heterotic string** is the closed superstring theory with the $(0, 1)$-superconformal algebra as its constraint algebra.

The interpretation is that the left-moving (holomorphic) sector of the theory acts like a purely bosonic string with 26 left-moving coordinates, whereas the right-moving sector acts as a superstring with 10 right-moving coordinates.

To enforce the $(0, 1)$-superconformal algebra as the constraint algebra, we must first decide how many spacetime dimensions we actually have: 10 or 26? The simplest approach postulates that we still have 10 spacetime dimensions in both sectors, so there are

1. 10 left- and right-moving bosonic fields $X^\mu(z, \bar{z})$ for $\mu = 0, \ldots, 9$, and
2. 10 right-moving fermionic (Majorana–Weyl spinor) fields $\tilde{\psi}^\mu(\bar{z})$ for $\mu = 0, \ldots, 9$.

These have total central charge $(c, \tilde{c}) = (10, 15)$ and ghost central charge $(-26, -15)$, so we need to balance out the remaining $(16, 0)$ charge. We could add sixteen more left-moving bosonic coordinates, but that is not the easiest way to get to the heterotic theory. Instead, the simplest solution is to add

3. 32 left-moving fermionic (Majorana–Weyl spinor) fields $\lambda^A(z)$ for $A = 1, \ldots, 32$.

By doing so, we are exploring the **fermionic realization of heterotic strings**, as opposed to the **bosonic realization**, which we shall sketch later.

**Definition 6.2.2.** In conformal gauge, the **action** for the heterotic theory is

$$S := \frac{1}{4\pi} \int d^2z (4\pi T_0 \partial X^\mu \bar{\partial} X_\mu + \lambda^A \bar{\partial} \lambda_A + \tilde{\psi}^\mu \partial \tilde{\psi}_\mu).$$

**Definition 6.2.3.** Let $\epsilon(\bar{z})$ be an anticommuting spinor with only a right-moving component. The **worldsheet supersymmetry** is given by the infinitesimal transformation

$$\delta X^\mu(z, \bar{z}) := \bar{\epsilon}(z) \tilde{\psi}^\mu(\bar{z}), \quad \delta \tilde{\psi}^\mu(\bar{z}) := -i \not{\partial} X^\mu(z, \bar{z}) \epsilon(\bar{z}).$$

We need to impose boundary conditions on the fields $\lambda^A$. There are already some conditions imposed by the periodicity of $T_B$.

**Exercise 6.2.1.** Compute the energy-momentum tensors

$$T_B^{\mathrm{m}}(z) = -2\pi T_0 \partial X^\mu(z) \partial X_\mu(z) - \frac{1}{2} \lambda^A \partial \lambda_A, \quad T_F^{\mathrm{m}}(\bar{z}) = i\sqrt{4\pi T_0} \tilde{\psi}^\mu(\bar{z}) \partial X_\mu(z, \bar{z}).$$

Since $T_B^{\mathrm{m}}$ must be periodic, conclude that the fields $\lambda^A$ are periodic up to some $O \in O(32)$:

$$\lambda^A(w + 2\pi) = O^A_B \lambda^B(w).$$

[2] See Polchinski 11.1 or Green–Schwarz–Witten 4.5.1. One issue is that the critical dimension of the $N = 2$ theory is $D = 2$, and for $N > 2$ it becomes negative!

Systematically searching for all possible consistent heterotic theories is technically difficult, [3] so instead we state the known results.

**Theorem 6.2.4** (Schellekens)**.** *There are* 9 *possible ten-dimensional heterotic theories:*

| *Gauge group* | *# Ground state bosons* | *# Ground state fermions* | *# Tachyons* | *Spacetime SUSY* |
|---|---|---|---|---|
| SO(32) | *496* | *496* | *0* | *Yes* |
| $E_8 \times E_8$ | *496* | *496* | *0* | *Yes* |
| SO(16) × SO(16) | *240* | *512* | *0* | *No* |
| SO(32) | *496* | *0* | *32* | *No* |
| SO(16) × $E_8$ | *368* | *256* | *16* | *No* |
| SO(8) × SO(24) | *304* | *384* | *8* | *No* |
| $(E_7 \times \mathrm{SU}(2))^2$ | *272* | *448* | *4* | *No* |
| $U(16)$ | *256* | *480* | *2* | *No* |
| $E_8$ | *248* | *496* | *1* | *No* |

Out of these theories, only the first two are relevant to us, because they are the only ones that exhibit spacetime supersymmetry. The next two subsections focus on their construction. The SO(32) theory is workable, but the $E_8 \times E_8$ theory of practical interest because it comfortably fits the current Standard Model gauge group SU(3) × SU(2) × $U(1)$.

### 6.2.1 The $SO(32)$ Heterotic String

The SO(32) symmetry is the easier one to construct. We postulate that each of the $\lambda^A$ fields have the same boundary conditions, and therefore that there is an **internal** SO(32) **symmetry** of the $\lambda^A$.

**Definition 6.2.5.** In the right-moving sector, we have analogues of the R and NS sectors:

1. the **P sector** is defined by $\lambda^A(w + 2\pi) = \lambda^A(w)$;
2. the **A sector** is defined by $\lambda^A(w + 2\pi) = -\lambda^A(w)$.

This is equivalent to requiring that $O^A_B$ is diagonalizable. We let $\nu = 0$ in the P sector and $\nu = 1/2$ in the A sector. States are denoted $|N\rangle_{\mathrm{P},L}$ and $|N\rangle_{\mathrm{A},L}$ respectively. (The subscript $L$ reminds us that these states are in the left-moving sector.)

**Exercise 6.2.2.** Compute the normal ordering constants

$$a^{\mathrm{m}}_{\mathrm{P}} = \frac{8}{24} - \frac{32}{24} = -1, \quad a^{\mathrm{m}}_{\mathrm{A}} = \frac{8}{24} + \frac{32}{48} = 1$$

in the A and P sectors and show by analogy to the R and NS sectors of the superstring that the masses for the left-moving sector are

$$\frac{M^2}{8\pi T_0} = \begin{cases} \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n + \sum_{r=1}^{\infty} r\lambda_{-r} \cdot \lambda_r + 1 = N + 1 & \text{P sector} \\ \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n + \sum_{r=1/2}^{\infty} r\lambda_{-r} \cdot \lambda_r - 1 = N - 1 & \text{A sector.} \end{cases}$$

Show, or recall, the constraint that $N - a^{\mathrm{m}} = \tilde{N} - \tilde{a}^{\mathrm{m}}$, i.e. $N \pm 1 = \tilde{N}$ depending on which of the P and A sectors we are in.

The zero modes in the P sector live in a spinor representation of Spin(32), by analogy with the R sector zero modes, so $|0\rangle_{\mathrm{P}}$ is a spinor. (We usually omit the subscript $L$ because the P and A sectors are only in the left-moving sector.)

[3] See Schellekens' "Classification of Ten-Dimensional Heterotic Strings"

**Definition 6.2.6.** The **GSO projection operator** on the $\lambda^A$ fields is given by

$$(-1)^F := \begin{cases} \lambda_0 (-1)^{\sum_{n=1}^{\infty} \lambda_{-n}\cdot\lambda_n} & \text{P sector} \\ (-1)^{\sum_{r=1/2}^{\infty} \lambda_{-r}\cdot\lambda_r} & \text{A sector,} \end{cases}$$

where $\lambda_0 := \lambda_0^1 \cdots \lambda_0^{32}$ is the analogue of the chirality operator $\Gamma^9$. The **GSO projection** is given by $(-1)^F |\psi\rangle_L = |\psi\rangle_L$.

The reason why we need the GSO projection for the internal $\lambda^A$ fields is unclear so far; the real reason is to preserve modular invariance at the one-loop level for amplitudes, but this demonstration is computationally involved and we shall skip it. However, we of course need to GSO project the right-moving sector, where we also require $(-1)^F |\psi\rangle_R = |\psi\rangle_R$.

Let's make a nice little table for the **low-level states** again. The states of the type II superstring are specified by their SO(8) quantum numbers; here,

1. elements in the left-moving sector are specified by their $\mathrm{SO}(8) \times \mathrm{SO}(32)$ quantum numbers (the SO(8) for the usual bosonic string, and the SO(32) for the $\lambda^A$), and
2. elements in the right-moving sector are specified by their SO(8) quantum numbers (the SO(8) for the usual superstring).

The GSO projection removes the tachyon $|0\rangle_{\mathrm{A},L}$, so we look at massless states in the left-moving and right-moving sectors. The right-moving sector still has the same states ($\mathbf{8}_v$ and $\mathbf{8}$) as the type II superstring, so the interesting spectrum is that of the left-moving sector.

Recall that $M^2 \propto N - a^{\mathrm{m}}$ in the left-moving sector. In the A sector, we computed that $a^{\mathrm{m}}_{\mathrm{A}} = 1$, so massless states have $N = 1$. They are as follows:

1. $\alpha^i_{-1} |0\rangle_{\mathrm{A},L}$, which transforms as $(\mathbf{8}_v, \mathbf{1})$, i.e. it lives in the vector representation of SO(8), and the trivial representation of SO(32);
2. $\lambda^A_{-1/2}\lambda^B_{-1/2} |0\rangle_{\mathrm{A},L}$, which transforms as $(\mathbf{1}, \mathbf{496})$, where $496 = \binom{32}{2}$.

On the other hand, in the P sector, we have $M^2 \propto N + 1$. Since $N$ is a non-negative operator, we cannot have $M^2 = 0 \propto N + 1$, and therefore the P sector has no massless states. Nonetheless, it is important to keep the P sector around to preserve unitarity and modular invariance.

Hence we get the following table of low-level states in the left-moving and right-moving sectors:

| $M^2$ | P | A | $\tilde{\mathrm{NS}}$ | $\tilde{\mathrm{R}}$ |
|---|---|---|---|---|
| $-8\pi T_0$ | - | $(\mathbf{1}, \mathbf{1})$ | - | - |
| 0 | - | $(\mathbf{8}_v, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{496})$ | $\mathbf{8}_v$ | $\mathbf{8}$ |

where we include the tachyon for completeness, though it is not a physical state. By tensoring together physical states in the left-moving sector with states in the right-moving sector, we get actual physical states of the heterotic string:

$$\begin{aligned} (\mathbf{8}_v, \mathbf{1}) \otimes (\mathbf{8}_v \oplus \mathbf{8}) &= (\mathbf{1}, \mathbf{1}) \oplus (\mathbf{28}, \mathbf{1}) \oplus (\mathbf{35}, \mathbf{1}) \oplus (\mathbf{56}, \mathbf{1}) \oplus (\mathbf{8}', \mathbf{1}) \\ (\mathbf{1}, \mathbf{496}) \otimes (\mathbf{8}_v \oplus \mathbf{8}) &= (\mathbf{8}_v, \mathbf{496}) \oplus (\mathbf{8}, \mathbf{496}). \end{aligned}$$

The latter is a $N = 1$ **super Yang–Mills multiplet** in the adjoint of SO(32). It is a gauge symmetry in spacetime, confirming that the $\lambda^A$ should be viewed as internal (gauge) fields, and is the underlying reason we call this theory the SO(32) heterotic string. (Aside: the other product is the **type I supergravity multiplet**.)

### 6.2.2 The $E_8 \times E_8$ Heterotic String

We got the SO(32) heterotic string by imposing identical boundary conditions for all 32 fields $\lambda^A$. Specially, we said they are either all in the A sector, or all in the P sector. The natural alternative is to look at an $\mathrm{SO}(n) \times \mathrm{SO}(32-n)$ symmetry, i.e. $n$ of the $\lambda^A$ fields are in the P sector, and $32-n$ are in the A sector. Then there are **four possible sectors**: PP, AP, PA and AA.

**Exercise 6.2.3.** Compute the normal ordering constants

$$\begin{aligned}
a_{\mathrm{PP}} &= \frac{8}{24} - \frac{n}{24} - \frac{32-n}{24} = -1 \\
a_{\mathrm{PA}} &= \frac{8}{24} + \frac{32-n}{48} - \frac{n}{24} = 1 - \frac{n}{16} \\
a_{\mathrm{AP}} &= \frac{8}{24} + \frac{n}{48} - \frac{32-n}{24} = \frac{n}{16} - 1 \\
a_{\mathrm{AA}} &= \frac{8}{24} + \frac{n}{48} + \frac{32-n}{48} = 1.
\end{aligned}$$

We still have the constraint $N + a^{\mathrm{m}} = \tilde{N}$, but $N$ is half-integer valued and $\tilde{N}$ is integer-valued. If we are to have any states in the PA or AP sectors, $a^{\mathrm{m}}_{\mathrm{PA}} = -a^{\mathrm{m}}_{\mathrm{AP}}$ must be a half-integer, leaving the possibilities $n = 8, 16, 24$. (If $n = 0$ or $n = 32$, we recover SO(32) heterotic theory.)

1. If $n = 8$ or $n = 24$, then we have an $\mathrm{SO}(8) \times \mathrm{SO}(24)$ theory which suffers from one-loop anomalies.
2. If $n = 16$, then we have (a priori) an $\mathrm{SO}(16) \times \mathrm{SO}(16)$ theory.

Hence we focus on the $n = 16$ case.