String Theory and Supersymmetry
Winter 2016 Seminar Notes

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Chapter 1

Introduction to Strings

We asked "why fields?" when we started QFT; now we ask, why strings? Here are some potentially convincing reasons.

1. If we allow one more degree of freedom than particles, many IR/UV divergences disappear; we require less renormalization. If we allow more than one degree of freedom, new divergences arise from the increased internal degrees of freedom.
2. Every consistent string theory contains a massless spin-2 state, i.e. a graviton, whose interactions at low energies reduce to general relativity.
3. The Standard Model, based on QFT, has 25 adjustable constants. String theory has none, and leads to gauge groups big enough to include the Standard Model.
4. Consistent string theories force upon us supersymmetry and extra dimensions, which have arisen naturally from several different attempts to unify the Standard Model.

Regardless of whether they are convincing, we start in this chapter, as with any other physical model, by writing down an action. Specifically, we first write the action for a relativistic string by generalizing that of a relativistic point particle, and then we quantize the action. As with QFT there are different ways to quantize. For exposure and convenience, we use the analogue of canonical quantization for now, in order to quickly compute the spectrum of a string.

As usual, we take $\hbar = c = 1$, and use **Einstein summation convention**: indices that appear as both superscripts and subscripts are implicitly summed over.

1.1 Review of Relativity

We work in $\mathbb{R}^{D-1,1}$ where D is the **number of dimensions**. Recall that coordinates are written $x^\mu = (x^0, x^1, \ldots, x^D) = (ct, x^1, \ldots, x^D)$, and the metric is

$$-ds^2 \coloneqq \eta_{\mu\nu} dx^\mu dx^\nu, \quad \eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, \ldots, 1).$$

Note that $\eta^\mu{}_\mu = D$. We use the dot product to stand for the **Lorentz inner product**, e.g. $-ds^2 = dx \cdot dx$.

Definition 1.1.1. Define the **proper time** of a system as the time elapsed measured by a clock traveling in the same Lorentz frame as the system itself. In such a Lorentz frame, $dx^i = 0$ and dt is the proper time elapsed, so $-ds^2 = -dt_p^2$; define

$$ds \coloneqq \sqrt{ds^2} = dt_p \qquad \text{whenever } ds^2 > 0,$$

i.e. for timelike intervals. Hence ds is the **proper time interval**. The **relativistic momentum** is $p^\mu := m(dx^\mu/ds)$. Conveniently,

$$p^\mu p_\mu = m^2 \frac{dx^\mu}{ds}\frac{dx_\mu}{ds} = -m^2 \frac{ds^2}{ds^2} = -m^2.$$

Definition 1.1.2. A **Lorentz transformation** $\Lambda^\mu{}_\nu$ is an element of the Lorentz group, the collection of all linear isometries of $\mathbb{R}^{D-1,1}$. We say a^μ is a **vector** if under Lorentz transformations, it changes as $a'^\mu = \Lambda^\mu{}_\nu a^\nu$. A **Poincaré transformation** is a Lorentz transformation possibly followed by a translation.

Definition 1.1.3. The **world line** of a point particle is the path in spacetime $\mathbb{R}^{D-1,1}$ traced out by the particle as it evolves in time.

The underlying principle of relativity says that physical laws are independent of Lorentz frame. In other words, any action we write down that we want to be compatible with relativity must have external symmetries: it must be invariant under Lorentz transformations. We call this **Lorentz invariance**. As long as superscripts and subscripts match up, we do not have to worry about Lorentz invariance.

The **action** for a free relativistic **point particle** is obtained by writing down the simplest Lorentz invariant action, and then making sure dimensions work out. If γ is the path taken by the particle, the action is therefore

$$S_{\mathrm{pp}}[x] := -m\int_\gamma ds = -m\int_\gamma d\tau\,\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} = -m\int_\gamma d\tau\,\sqrt{-\dot{x}^\mu\cdot\dot{x}_\mu}$$

where a dot denotes a τ-derivative. Because ds is coordinate-independent, it does not what how we pick the parametrization τ. Physicists like to call this **reparametrization invariance**. This invariance is very important: without it, we have actually introduced a completely new parameter τ, thus increasing the number of degrees of freedom from $D-1$ to D.

Exercise 1.1.1. By computing $\delta(ds^2)$ in two different ways, show that

$$\delta S_{\mathrm{pp}}[x] = m\int_\gamma \delta(dx^\mu)\frac{dx_\mu}{ds} = \int_\gamma d\tau\left(\frac{d}{d\tau}\delta x^\mu\right)p_\mu = \delta x^\mu p_\mu\Big|_{\tau_i}^{\tau_f} - \int d\tau\,\delta x^\mu\frac{dp_\mu}{d\tau}.$$

Argue that the first term vanishes if we specify **initial and final conditions**. Hence deduce the equation of motion $dp_\mu/d\tau = 0$.

The action S_{pp} seems simple in the $\int_\gamma ds$ form, but is messy when parametrized. Later when we quantize using path integrals, S_{pp} is difficult to work with because of the derivatives under the square root. There is a different, classically-equivalent action we can work with. Introduce an additional field $\gamma_{\tau\tau}(\tau)$ (sometimes called an **einbein** in general relativity), which we can view as a metric on the world line, and take the action

$$S'_{\mathrm{pp}} := -\frac{1}{2}\int_\gamma d\tau\sqrt{-\gamma_{\tau\tau}}(\gamma^{\tau\tau}\dot{x}^\mu\dot{x}_\mu + m^2) = -\frac{1}{2}\int_\gamma d\tau\,(\eta^{-1}\dot{x}^\mu\dot{x}_\mu - \eta m^2), \quad \eta := \sqrt{-\gamma_{\tau\tau}(\tau)}.$$

It seems like we have arbitrarily added an extra degree of freedom, but in fact γ is completely specified by the equation of motion. The action S'_{pp} is much better to work with in a path integral, because it is **quadratic** in $\dot{x}^\mu$.

Exercise 1.1.2. Vary S'_{pp} with respect to $\gamma_{\tau\tau}$ to get the equation of motion $\gamma_{\tau\tau} = \dot{x}^\mu\dot{x}_\mu/m^2$. Substitute this expression back into S'_{pp} to obtain S_{pp}, and therefore conclude that the two actions are classically equivalent.

1.2 Nambu–Goto and Polyakov Actions

We graduate to **one-dimensional strings**; in this section we write down an action for them. There are two kinds of strings: those with two distinct endpoints, called **open strings**, and those which are loops, called **closed strings**. Because closed strings are just open strings with the extra constraint that the two endpoints match, we focus on open strings.

The action for the relativistic point particle is proportional to the proper time elapsed on the particle's world line. But the proper time, when multiplied by c, can be viewed as the "proper length" of the world line. The natural generalization, then, is to consider the surface in space-time traced out by the string as it evolves in time, called the **world sheet** M, and to define an action proportional to the "proper area" of the world sheet. The world sheet M is a two-dimensional surface, and therefore requires charts modeled on $\mathbb{R}^2$.

Definition 1.2.1. The **coordinates** we use on $\mathbb{R}^2$, the parameter space, are denoted (σ^0, σ^1), and so the **world sheet** M is locally a surface given by functions denoted $X^\mu(\sigma)$ (capitalized to disambiguate from the coordinates x^μ), called **string coordinates**. The lowercase Latin characters $a, b, \ldots$ are used to denote **indices** that run over values $0, 1$. Two notes:

1. The choice of parametrization (σ^0, σ^1) is, again, up to us, but usually we take the coordinate σ^0 to be the proper time, and σ^1 the position along the string.
2. For our purposes, $M = X^\mu$, i.e. the single chart X^μ describes the entire world sheet for the region of spacetime we care about.

Exercise 1.2.1. Show that the metric $\eta_{\mu\nu}$ on spacetime $\mathbb{R}^{D-1,1}$ induces a metric g on the world sheet via pullback along the inclusion $\iota\colon M \to \mathbb{R}^{D-1,1}$. Compute g and the area element:

$$g_{ab} = \partial_a X^\mu \partial_b X_\mu, \quad dA = d^2\sigma \sqrt{-\det g}.$$

A relativistic particle has a parameter we call mass. It turns out mass is not the appropriate physical interpretation of the corresponding parameter for strings. Instead, we interpret it as a **tension**, and denote it T_0.

Definition 1.2.2. The **Nambu–Goto action** for a relativistic string is given by

$$S_{\mathrm{NG}}[X] \coloneqq -T_0 \int_M dA = -T_0 \int_M d^2\sigma \sqrt{-\det g}.$$

Again, note that it satisfies **reparametrization invariance**, literally by construction.

But again, we have a square root and derivatives inside it, and now we know how to get rid of it: introduce an independent world sheet metric $\gamma_{ab}(\sigma)$. This time the metric is on a surface, so we need to specify the signature. We take Lorentzian signature $(-,+)$.

Definition 1.2.3. The **Polyakov action** for a relativistic string is given by

$$S_{\mathrm{P}}[X, \gamma] \coloneqq -\frac{T_0}{2} \int_M d^2\sigma \sqrt{-\gamma}\, \gamma^{ab} \partial_a X^\mu \partial_b X_\mu,$$

where γ without indices stands for $\det(\gamma_{ab})$. From now on, we always refer to γ_{ab} as the **metric**, and g_{ab} as the **induced metric**. Indices are raised/lowered using the metric γ_{ab}, not the induced metric g_{ab}. (In fact, from now on we basically forget about g_{ab}; we use it only to introduce the Nambu–Goto action, and the following exercise.)

Exercise 1.2.2. Show that $\delta\sqrt{-\gamma} = (1/2)\sqrt{-\gamma}\, \gamma^{ab} \delta\gamma_{ab}$, and therefore that

$$\delta_\gamma S_{\mathrm{P}}[X, \gamma] = -\frac{T_0}{2} \int_M d^2\sigma \sqrt{-\gamma}\, \delta\gamma^{ab} \left(g_{ab} - \frac{1}{2} \gamma_{ab} \gamma^{cd} g_{cd} \right).$$

Rearrange the obtained equation of motion and conclude that $g_{ab}\sqrt{-g} = \gamma_{ab}\sqrt{-\gamma}$. Hence replace γ in $S_{\mathrm{P}}[X, \gamma]$ with g, and obtain that $S_{\mathrm{P}}[X, \gamma] = S_{\mathrm{NG}}[X]$.

Definition 1.2.4. As in general relativity, define the **stress-energy tensor**

$$T^{ab}(\sigma) := -\frac{4\pi}{\sqrt{-\gamma}} \delta_\gamma S_{\mathrm{P}}[X, \gamma] = -2\pi T_0 \left(\partial^a X^\mu \partial^b X_\mu - \frac{1}{2} \gamma^{ab} \partial_c X^\mu \partial^c X_\mu \right),$$

so that the **equation of motion** arising from varying γ says $T_{ab} = 0$.

Exercise 1.2.3. (Important!) Now vary $S_{\mathrm{P}}[X, \gamma]$ with respect to X^μ to obtain

$$\begin{aligned} \delta_X S_{\mathrm{P}}[X, \gamma] &= -T_0 \int_M d^2\sigma \sqrt{-\gamma}\, \gamma^{ab} \left(\partial_a (\delta X^\mu \partial_b X_\mu) - \partial_a \partial_b X_\mu \delta X^\mu \right) \\ &= -T_0 \int_0^\ell d\sigma^1 \sqrt{-\gamma} \left[\delta X^\mu \partial^0 X_\mu \right]_{\sigma^0 = \tau_i}^{\sigma^0 = \tau_f} - T_0 \int_{\tau_i}^{\tau_f} d\sigma^0 \sqrt{-\gamma} \left[\delta X^\mu \partial^1 X_\mu \right]_{\sigma^1 = 0}^{\sigma^1 = \ell} \\ &\quad + T_0 \int_M d^2\sigma \sqrt{-\gamma}\, \delta X^\mu \nabla^2 X_\mu. \end{aligned}$$

A careful inspection of the terms in the variation $\delta_X S_{\mathrm{P}}[X, \gamma]$ yield interesting insights. For this variation to vanish, each of the terms must vanish independently, since they control different aspects of the string's behavior.

1. The last term is determined by the motion of the string in the domain $(0, \ell) \times (\tau_i, \tau_f)$, and therefore δX^μ is not constrained by any boundary conditions there. Hence we have the **equation of motion** $\sqrt{-\gamma}\, \nabla^2 X_\mu = 0$.
2. The first term is determined by the configuration of the string at times τ_i and τ_f. If we specify these configurations as **initial and final conditions**, then δX^μ is zero for the first term, so the term vanishes.
3. The second term is determined by the configuration of the endpoints of the string when $\sigma^0 \in (\tau_i, \tau_f)$. It does not vanish automatically, and we have to impose **boundary conditions** in order for it to do so.

Definition 1.2.5. There are two different kinds of boundary conditions.

- The **free (Neumann) boundary condition** is $\partial^1 X^\mu(\sigma^0, 0) = \partial^1 X^\mu(\sigma^0, \ell) = 0$.
- The **Dirichlet boundary condition** is $\delta X^\mu(\sigma^0, 0) = \delta X^\mu(\sigma^0, \ell) 0$.

Alternatively, if the string is **closed**, i.e. we have the **periodicity** conditions

$$X^\mu(\sigma^0, 0) = X^\mu(\sigma^0, \ell), \quad \partial^a X^\mu(\sigma^0, 0) = \partial^a X^\mu(\sigma^0, \ell), \quad \gamma_{ab}(\sigma^0, 0) = \gamma_{ab}(\sigma^0, \ell),$$

no additional boundary conditions are necessary.

For a long time, string theorists did not seriously consider the Dirichlet boundary condition. Why should the endpoints of an open string be fixed, and if they were, where would they be fixed onto? In particular, this fixing of endpoints would violate momentum conservation. Then Polchinski, in the 1990s, suggested that the endpoints are attached to **D-branes**, which should themselves be thought of as dynamical objects alongside strings. Conceptually, then,

1. a D0-brane is a particle, a D1-brane is a string, and so on, and they interact non-trivially;
2. the Dirichlet boundary condition says that a given D1-brane has fixed endpoints on a higher Dp-brane;
3. any momentum lost by the D1-brane is absorbed by the Dp-brane; and
4. the Neumann boundary condition is just saying there is a D-dimensional D-brane permeating all of space-time, i.e. the string endpoints are not fixed at all.

We return to this D-brane perspective much later on. It is hard enough to quantize strings without more dynamical objects floating around.

1.3 Gauge Fixing

There is another reason the Polyakov action is preferable over the Nambu–Goto action: it has more symmetries, and these symmetries make it easier to gauge fix (using Faddeev–Popov or otherwise) when we try to quantize. The Polyakov action is invariant under the following symmetries:

1. D-dimensional **Poincaré transformations**:
$$X^\mu(\sigma) \mapsto \Lambda^\mu{}_\nu X^\nu(\sigma) + a^\mu, \quad \gamma_{ab}(\sigma) \mapsto \gamma_{ab}(\sigma);$$
2. **Reparametrization** (i.e. diffeomorphisms): for new coordinates $\tilde{\sigma}^a(\sigma)$,
$$X^\mu(\sigma) \mapsto X^\mu(\sigma), \quad \gamma_{ab}(\sigma) \mapsto \frac{\partial \sigma^c}{\partial \tilde{\sigma}^a} \frac{\partial \sigma^d}{\partial \tilde{\sigma}^b} \gamma_{cd}(\sigma);$$
3. 2-dimensional **Weyl transformations**: for arbitrary $\omega(\sigma)$,
$$X^\mu(\sigma) \mapsto X^\mu(\sigma), \quad \gamma_{ab}(\sigma) \mapsto \exp(2\omega(\sigma))\gamma_{ab}(\sigma).$$

The Nambu–Goto action is not invariant under Weyl transformations.

Exercise 1.3.1. Verify all these statements. (This should be quite straightforward.)

Definition 1.3.1. Let diff denote the group of diffeomorphisms acting on Σ, and Weyl the group of Weyl transformations acting on Σ.

Before we continue, let's make sure our Lagrangian is as general as possible, to reduce future work. If $\mathcal{L}$ has more terms, they need to satisfy all the above symmetries. In particular, Weyl invariance is very odd: it prevents us from adding terms such as
$$\int_M d^2\sigma \sqrt{-\gamma}\, V(X), \quad \mu \int_M d^2\sigma \sqrt{-\gamma}.$$

Exercise 1.3.2. Convince yourself that $\mathcal{L}$ must contain one more γ^{ab} than γ_{ab} in order to satisfy Weyl invariance and counteract the change in $\sqrt{-\gamma}$. Since such a γ^{ab} can only pair up indices with derivatives, we need a second-order Lorentz-invariant term that is coordinate-independent. Convince yourself that other than $\partial_a X^\mu \partial_b X_\mu$, this term can only involve γ^{ab} and γ_{ab}, and that in fact it must be the **scalar curvature** R. Show that under a Weyl transformation,
$$\sqrt{-\gamma}\, R \mapsto \sqrt{-\gamma}(R - 2\nabla^2\omega).$$
Hence argue that we need another term integrated over ∂M to counteract $\nabla^2(\sqrt{-\gamma}\,\omega)$. Putting everything together, conclude that
$$\chi \coloneqq \frac{1}{4\pi} \int_M d^2\sigma \sqrt{-\gamma}\, R + \frac{1}{2\pi} \int_{\partial M} ds\, k$$
is Weyl invariant. Here ds is proper time along ∂M using the metric γ_{ab}, and $k \coloneqq \pm t^a n_b \nabla_a t^b$ is the **geodesic curvature** of the boundary, where t^a is a unit vector tangent to the boundary, and n_b an outward-pointing unit vector, and we choose $\pm$ depending on whether the boundary is timelike or spacelike.

Let's proceed with gauge fixing: we need to use up the internal degrees of freedom in diff $\times$ Weyl. The transformation of the scalar curvature computed in the exercise above says we can use Weyl invariance to locally set the scalar curvature to zero, by solving $2\nabla^2\omega = R$ and then applying the Weyl transformation $\exp(2\omega)$. But we are in two dimensions, where the symmetries of the Riemann curvature tensor determine it from R:
$$R_{abcd} = R_{cdab},\ R_{abcd} = -R_{bacd} = -R_{abdc} \implies R_{abcd} = \frac{1}{2}(\gamma_{ac}\gamma_{bd} - \gamma_{ad}\gamma_{bc})R.$$
Hence not only can we locally set $R = 0$, we can locally set $\gamma_{ab} = \eta_{ab}$, the flat Minkowski metric.

Definition 1.3.2. If we consider only reparametrization and not Weyl transformations, the metric γ_{ab} can always be brought to the form $\exp(2\omega)\eta_{ab}$. Forcing the metric to be of that form is known as **conformal gauge**. Performing the additional Weyl transformation to obtain $\gamma_{ab} = \eta_{ab}$ is known as **unit gauge**. In general, the form of the metric we choose to put γ_{ab} in is called the **fiducial metric**.

How many internal degrees of freedom have we used up if we put the metric γ_{ab} in unit gauge? Well, diff has two degrees of freedom, one for each coordinate, and Weyl has one, for the scale of the metric. But the metric itself has three independent components, being symmetric. Hence we expect to be done: we have canonically chosen a representative of each gauge equivalence class.

But, perhaps unexpectedly, there is more gauge freedom: there are non-trivial transformations in diff $\times$ Weyl that preserve unit gauge! The key to finding these transformations is to realize that Σ is actually a **Riemann surface**: let $z \coloneqq \sigma^0 + i\sigma^1$, so that $ds^2 = dzd\bar{z}$. Now if $f(z)$ is a holomorphic change of coordinates, then

$$z \mapsto f(z), \quad ds^2 \mapsto |\partial_z f|^{-2} dzd\bar{z},$$

so now applying the Weyl transformation $\exp(2\ln|\partial_z f|)$ recovers ds^2. Clearly the composition of the two transformations is non-trivial.

What went wrong? Well, just because dimensions match up does not mean we have spanned the whole space of gauge transformations! The holomorphic diffeomorphisms above actually have **measure zero** in diff. When we stop working locally and work globally instead, these extra bits of freedom are canceled by boundary conditions.

1.4 Quantization via Canonical Commutation Relations

1.4.1 Spectrum of Open String

1.5 Quantization via Path Integral

Recall from QFT that we have a giant machine for quantizing classical theories: the path integral. Before we begin plugging the Polyakov action into the machine, however, we need to make a modification. From now on, the world sheet is equipped with a **Euclidean metric** g_{ab}, instead of a Lorentzian one γ_{ab}. This is so that the path integral over metrics is better defined. The transition from Euclidean to Minkowski is, formally, done via **Wick rotation**: $x^0 \mapsto ix^0$ and similarly for the metric. The **Euclidean path integral**, and the Euclidean action (with the additional terms on top of the Wick-rotated Polyakov action), is therefore

$$Z \coloneqq \frac{1}{\mathrm{Vol}} \int \mathcal{D}g\, \mathcal{D}X\, \exp(-S_{\mathrm{P}}[X,g]),$$

$$S_{\mathrm{P}}[X,g] = \frac{T_0}{2} \int_M d^2\sigma\, \sqrt{g}\, g^{ab} \partial_a X^\mu \partial_b X_\mu + \lambda \left(\frac{1}{4\pi} \int_M d^2\sigma\, \sqrt{g}\, R + \frac{1}{2\pi} \int_{\partial M} ds\, k \right)$$

where Vol is the volume of the gauge action on the **configuration space** consisting of all possible X^μ and g). More explicitly, we can imagine partitioning configuration space into gauge orbits; we actually want to integrate on a path through these gauge orbits. But now recall from QFT that we have another giant machine for doing so: the Faddeev-Popov method.

1.5.1 The Faddeev-Popov Method

Let's first recall that the idea behind Faddeev-Popov is very natural: we want to do a change of coordinates in configuration space so that instead of integrating over a mish-mash of g and X, we integrate such that one variable goes along gauge orbits, and the other goes along the gauge-fixed path. Although this sounds technical, we perform procedures like this quite often without realizing it! For example, consider the calculation

$$\iint dx\, dy\, e^{-x^2-y^2} = \int d\theta \int dr\, re^{-r^2} = 2\pi \int dr\, re^{-r^2} = \pi.$$

What is really happening here is that we recognized the $U(1)$ symmetry of the original integrand, and changed variables in order to factor out that symmetry. Instead of integrating over (x, y), we integrated over (r, θ), with θ parametrizing the gauge orbits. Furthermore, we picked out the $y = 0$ representative of each gauge orbit for the remaining integral.

Armed with this motivation, we can proceed. Let $\hat{g}_{ab}$ be the fiducial metric; it represents our choice of gauge fixing, just like the choice $y = 0$. Let ζ be shorthand for a combined coordinate and Weyl transformation:

$$\zeta\colon g_{ab} \mapsto g^{\zeta}_{ab} \coloneqq \exp(2\omega(\sigma)) \frac{\partial\sigma^c}{\partial\sigma'^a} \frac{\partial\sigma^d}{\partial\sigma'^b} g_{cd}(\sigma).$$

Definition 1.5.1. Let $\mathcal{D}\zeta$ be a gauge invariant measure on diff $\times$ Weyl. (Whether such a measure exists is very relevant for us, but we disregard it for now.) Define the **Faddeev-Popov determinant** Δ_{FP} by

$$\Delta_{\mathrm{FP}}^{-1}(g) \coloneqq \int \mathcal{D}\zeta\, \delta[\hat{g}^{\zeta} - g].$$

Here the δ is the **Dirac functional**, i.e. $\hat{g}^{\zeta}$ and g must agree at every point σ.

Exercise 1.5.1. Show that $\Delta_{\mathrm{FP}}(g)$ is gauge-invariant by computing that $\Delta_{\mathrm{FP}}(g^{\zeta})^{-1} = \Delta_{\mathrm{FP}}(g)^{-1}$.

Now it is time to do the calculation to factor out the integral over the gauge orbits. The first step is to add a 1 to the integral:

$$Z = \int \frac{\mathcal{D}g\, \mathcal{D}X}{\mathrm{Vol}} \exp(-S_{\mathrm{P}}[X, g]) = \int \frac{\mathcal{D}g\, \mathcal{D}X\, \mathcal{D}\zeta}{\mathrm{Vol}} \Delta_{\mathrm{FP}}(g) \delta[\hat{g}^{\zeta} - g] \exp(-S_{\mathrm{P}}[X, g]).$$

The second step is to do the integral over g, which, due to the $\delta[\hat{g}^{\zeta} - g]$, amounts to replacing g with $\hat{g}^{\zeta}$:

$$Z = \int \frac{\mathcal{D}X\, \mathcal{D}\zeta}{\mathrm{Vol}} \Delta_{\mathrm{FP}}(\hat{g}^{\zeta}) \exp(-S_{\mathrm{P}}[X, \hat{g}^{\zeta}]).$$

Finally, since both Δ_{FP} and S_{P} are gauge-invariant, we can replace $\hat{g}$ with $\hat{g}^{\zeta}$. Then nothing in the integrand depends on ζ anymore, so it factors out and cancels the volume normalization:

$$Z = \int \frac{\mathcal{D}\zeta}{\mathrm{Vol}} \int \mathcal{D}X\, \Delta_{\mathrm{FP}}(\hat{g}) \exp(-S_{\mathrm{P}}[X, \hat{g}]) = \int \mathcal{D}X\, \Delta_{\mathrm{FP}}(\hat{g}) \exp(-S_{\mathrm{P}}[X, \hat{g}]).$$

Exercise 1.5.2. Evaluate $\iint dx\, dy\, e^{-x^2-y^2}$ by applying the Faddeev-Popov method to its $U(1)$ symmetry and the gauge-fixing condition $y = 0$. Conclude that the Faddeev-Popov method is completely rigorous in finite dimensions, and that Δ_{FP} is actually a Jacobian (hence the name Faddeev-Popov determinant).

1.5.2 Computing the Faddeev-Popov Determinant

It remains to compute the Faddeev-Popov determinant Δ_{FP} for the diff $\times$ Weyl action on world sheet metrics. To do so, we make the simplifying assumption that diff $\times$ Weyl actually acts freely on metrics g, i.e. for each g, there is exactly one ζ such that $\delta[\hat{g}^{\zeta} - g] = 0$. Obviously this assumption is false: we showed earlier that the action has fixed points (albeit a measure zero set of them). But it is true locally, so we deal with the global issues later. The reason we make this assumption is so that we can compute $\Delta_{\mathrm{FP}}(\hat{g}) = \delta[\hat{g}^{\zeta} - \hat{g}]$

by integrating only around a small neighborhood of $\zeta = 0$. In this neighborhood, we can take infinitesimal Weyl transformations $\omega(\sigma)$ and infinitesimal diffeomorphisms $\delta\sigma^\alpha = v^\alpha(\sigma)$, and write

$$\Delta_{\mathrm{FP}}^{-1}(\hat{g}) = \int \mathcal{D}\omega\, \mathcal{D}v\, \delta[2\omega\hat{g}_{ab} + \nabla_a v_b + \nabla_b v_a].$$

Note that now we are integrating over the Lie algebra of diff $\times$ Weyl. We want to get rid of the delta functional.

Exercise 1.5.3. For a function $\phi\colon \mathbb{R}^D \to \mathbb{R}$, derive the integral form

$$\delta[\phi] = \int_{j\colon \mathbb{R}^D \to \mathbb{R}} \mathcal{D}j(x)\, \exp\left(2\pi i \int d^D x\, j(x)\phi(x)\right)$$

by applying the one-dimensional identity $\delta(x) = \int dp\, \exp(2\pi i p x)$ to piecewise linear paths, and then taking the limit as the number of path segments goes to infinity.

In our case, the function inside the delta functional lives on the world sheet Σ, whose integration measure is $d^2\sigma\, \sqrt{\hat{g}}$ (remember we fixed the fiducial metric). Hence, if β ranges over symmetric 2-tensors on Σ, then

$$\Delta_{\mathrm{FP}}^{-1}(\hat{g}) = \int \mathcal{D}\omega\, \mathcal{D}v\, \mathcal{D}\beta\, \exp\left(2\pi i \int d^2\sigma\, \sqrt{\hat{g}}\, \beta^{ab}(2\omega\hat{g}_{ab} + \nabla_a v_b + \nabla_b v_a)\right).$$

But we can directly do the integral over ω. The one and only term containing an ω factors out to give a delta functional:

$$\int \mathcal{D}\omega\, \exp\left(2\pi i \int d^2\sigma\, \sqrt{\hat{g}}\, \beta^{ab}(2\omega\hat{g}_{ab})\right) = \delta[2\beta^{ab}\hat{g}_{ab}],$$

i.e. in the remaining integral, β^{ab} is traceless:

$$\Delta_{\mathrm{FP}}^{-1}(\hat{g}) = \int \mathcal{D}v\, \mathcal{D}\beta\, \exp\left(2\pi i \int d^2\sigma\, \sqrt{\hat{g}}\, \beta^{ab}(\nabla_a v_b + \nabla_b v_a)\right).$$

Recap: we are integrating over vector fields v and symmetric 2-tensors β such that β^{ab} is traceless, both living on Σ.

1.5.3 Faddeev-Popov Ghosts

We are not done: the path integral above is for $\Delta_{\mathrm{FP}}^{-1}$, but we want Δ_{FP} itself. There is a general procedure for inverting $\Delta_{\mathrm{FP}}^{-1}$. To understand it, we must first clarify what Δ_{FP} really is. Let F is the gauge-fixing condition. (For us, F is a function of g and ζ and takes values in symmetric 2-tensors.) Note that via a change of variables from ζ to F,

$$\Delta_{\mathrm{FP}}^{-1} = \int D\zeta\, \delta(F) = \int DF\, \det\left[\frac{\delta\zeta}{\delta F}\right]\delta(F) = \det\left[\frac{\delta\zeta}{\delta F}\right]_{F=0}.$$

This change of variables is valid again because we assume ζ acts freely on gauge orbits, and F is supposed to pick a unique representative from each gauge orbit, so ζ and F "have the same number of degrees of freedom" as physicists like to say. Now all we have to do is invert the determinant. For this, we use a clever trick, which is developed in the following two exercises.

Exercise 1.5.4. Show by analogy from the finite dimensional case for two real fields ϕ^1 and ϕ^2 that

$$\int \mathcal{D}\phi^1\, \mathcal{D}\phi^2 \exp\left(i \int d^d x\, \phi^1 A \phi^2\right) = (\det A)^{-1}.$$

Exercise 1.5.5. Recall from QFT that we defined **Grassmann numbers**: they are anti-commuting formal variables, i.e. $\theta\eta = -\eta\theta$, that form an algebra. We also worked out the **Berezin integral** for Grassmann-valued quantities, with the convention that $\int d\theta \int d\eta\, \eta\theta = 1$. If θ and η are Grassmann variables, i.e. taking values in the Grassmann algebra, and $b \in \mathbb{R}$, review/show (in order) that

$$\theta^2 = 0, \qquad \int d\theta\, f(\theta) = \frac{\partial f}{\partial \theta}, \qquad \int d\theta\, d\eta\, \exp(-\theta b \eta) = b$$

Hence show by analogy with the finite dimensional case that for Grassmann-valued fields χ^1 and χ^2,

$$\int \mathcal{D}\chi^1\, \mathcal{D}\chi^2 \exp\left(-\int d^d x\, \chi^1 A \chi^2\right) = \det A.$$

So here's the trick: if we have a path integral expression for $(\det A)^{-1}$, to get $\det A$ we simply replace ordinary variables with Grassmann variables! In particular, to get $\Delta_{\mathrm{FP}}(\hat{g})$ from $\Delta_{\mathrm{FP}}(\hat{g})^{-1}$, we replace (β_{ab}, v^a) with Grassmann-valued fields (b_{ab}, c^a), with b^{ab}, like β^{ab}, being traceless:

$$\Delta_{\mathrm{FP}}(\hat{g}) = \int \mathcal{D}b\, \mathcal{D}c\, \exp(S_{\mathrm{G}}), \qquad S_{\mathrm{G}} \coloneqq \frac{1}{2\pi} \int d^2\sigma\, \sqrt{\hat{g}}\, b_{ab} \nabla^a c^b.$$

Note that we've implicitly made a few cosmetic changes:

1. Because b is a symmetric 2-tensor (do **not** confuse the fact that b is symmetric, i.e. $b_{ab} = b_{ba}$, with b being anti-commutative, e.g. $b_{ab}\theta = -\theta b_{ab}$), we can rewrite

$$b^{ab}(\nabla_a c_b + \nabla_b c_a) = b^{ab}\nabla_a c_b + b^{ab}\nabla_a c_b = 2b^{ab}\nabla_a c_b.$$

2. We also rewrote $b^{ab}\nabla_a c_b = b_{ab}\nabla^a c^b$.
3. We chose slightly different normalization factors to make later computations cleaner. These factors make absolutely no difference, because we can absorb them into the fields b and c.

The quantity S_{G} is called the **ghost action**, because when we plug $\Delta_{\mathrm{FP}}(\hat{g})$ back into the path integral, we get

$$Z = \int \mathcal{D}X\, \mathcal{D}b\, \mathcal{D}c\, \exp(-S_{\mathrm{P}}[X, \hat{g}] - S_{\mathrm{G}}[b, c]),$$

i.e. S_{G} becomes part of the action. The fields b and c, which do not correspond physically to anything, are known as **Faddeev-Popov ghost fields**. The price of gauge fixing is the introduction of these unphysical ghosts.

Exercise 1.5.6. Repeat the computation of Δ_{FP} for QED, and show that for QED, Δ_{FP} is independent of any fields. Hence conclude that QED has no Faddeev-Popov ghosts. (That's why quantizing QED went a lot faster. QCD has ghosts, however.)

Chapter 2

Conformal Field Theory