Mirror Symmetry
Summer 2016 Seminar Notes

Anton Borissov, Henry Liu

May 15, 2016

Contents

Chapter 1

Mathematical Preliminaries

The aim of this chapter is to give a brief review of the required mathematical background for mirror symmetry.

1.1 Cohomology Theories

Throughout this section, X is a complex manifold, and H_{dR} is de Rham cohomology. We examine the relationships between some common cohomology theories on X.

1.1.1 Sheaf Cohomology

All of our sheaves take values in abelian groups. Let $\mathcal{F}$ be a presheaf on X.

Definition 1.1.1. Recall the definition of a **presheaf** $\mathcal{F}$ on X:

1. (presheaf) every open set U in X is assigned an abelian group $\mathcal{F}(U)$, such that if $V \subseteq U$ are two open sets, there is a restriction map $-|_{U\to V} : \mathcal{F}(U) \to \mathcal{F}(V)$ compatible with inclusion, i.e. $(-|_{U\to V})|_{V\to W} = -|_{U\to W}$ for any $W \subseteq V \subseteq U$.

If in addition $\mathcal{F}$ satisfies the following two properties, it is a **sheaf**:

2. (locality) if $\{U_\alpha\}$ is an open cover of X and $f, g \in \mathcal{F}(X)$ such that $f|_{U\to U_\alpha} = g|_{U\to U_\alpha}$ for every U_α, then $f = g$;
3. (gluing) if $\{U_\alpha\}$ is an open cover of X and $f_\alpha \in \mathcal{F}(U_\alpha)$ for every U_α are elements agreeing on overlaps, i.e. such that $f_\alpha|_{U_\alpha\to U_\alpha\cap U_\beta} = f_\beta|_{U_\beta\to U_\alpha\cap U_\beta}$, then we can glue the f_α together to get $f \in \mathcal{F}(X)$, i.e. $f|_{X\to U_\alpha} = f_\alpha$ for every U_α.

Definition 1.1.2. Let $\mathcal{U} = \{U_\alpha\}$ be an **ordered open cover** of X, i.e. with a partial order such that if α and β are incomparable then $U_\alpha \cap U_\beta$ is empty. A **p-simplex** σ of $\mathcal{U}$ is a totally ordered collection of open sets $U_{\alpha_0}, \ldots, U_{\alpha_p} \in \mathcal{U}$; we call $U_{\alpha_0,\ldots,\alpha_p} \coloneqq U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ its **support**, and often refer to σ by it instead. The **k-th boundary component** of a p-simplex $U_{\alpha_0,\ldots,\alpha_p}$ is given by $\partial_k U_{\alpha_0,\ldots,\alpha_p} \coloneqq U_{\alpha_0,\ldots,\hat{\alpha}_k,\ldots,\alpha_p}$. Cochains are maps from simplices to sheaf sections, and form a cochain complex:

$$C^p(\mathcal{U}, \mathcal{F}) \coloneqq \prod_{\alpha_0<\cdots<\alpha_p} \mathcal{F}(U_{\alpha_0,\ldots,\alpha_p}), \quad (\delta^p\omega)(\sigma) \coloneqq \sum_{k=0}^{p+1} (-1)^k \omega(\partial_k\sigma)|_\sigma : C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F}).$$

The **Čech cohomology of** $\mathcal{U}$ with coefficients in $\mathcal{F}$, denoted $\check{H}^\bullet(\mathcal{U}, \mathcal{F})$, is the cohomology of this complex.

Example 1.1.3. Let $\mathcal{F}$ be a sheaf. By the gluing condition for a sheaf, a global section $f \in \mathcal{F}(X)$ is defined by its values $f_\alpha := f|_{X \to U_\alpha} \in \mathcal{F}(U_\alpha)$ on every U_α in an open cover. These f_α form precisely the data for an element of $C^0(\mathcal{U}, \mathcal{F})$, and satisfy the gluing condition $f_\alpha = f_\beta$ on $U_\alpha \cap U_\beta$, which is precisely the statement $\delta_0 f = 0$. Hence $\check{H}^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$ for a sheaf $\mathcal{F}$.

Example 1.1.4. Let $\mathcal{O}$ denote the sheaf of holomorphic functions (and $\mathcal{O}^*$ the nowhere-zero ones) on $\mathbb{P}^1$. Recall that on $\mathbb{P}^1$ we have the charts $U = \mathbb{P}^1 \setminus \{S\}$ and $V = \mathbb{P}^1 \setminus \{N\}$, with coordinates u and v respectively. To look at sections of $\mathcal{O}(U)$ versus $\mathcal{O}(V)$, we use the transition map $v = u^{-1}$ on $U \cap V$. The cochains for this open cover are

$$C^0(\mathcal{U}, \mathcal{O}) = \mathcal{O}(U) \times \mathcal{O}(V), \quad C^1(\mathcal{U}, \mathcal{O}) = \mathcal{O}(U \cap V), \quad C^k(\mathcal{U}, \mathcal{O}) = 0 \; \forall k \geq 2.$$

We compute the sheaf cohomology.

1. ($\check{H}^0(\mathcal{U}, \mathcal{O})$) The boundary map δ_0 maps $(f, g) \in C^0(\mathcal{U}, \mathcal{O})$ to $g - f$. But

   $$f = \sum_{k=0}^{\infty} f_k u^k, \quad g = \sum_{k=0}^{\infty} g_k v^k = \sum_{k=0}^{\infty} g_k u^{-k},$$

   so $g - f = 0$ iff $f_k = g_k = 0$ for all $k > 0$, and $f_0 = g_0$. Hence $\check{H}^0(\mathcal{U}, \mathcal{O}) \cong \mathbb{C}$, consisting of all constant functions.

2. ($\check{H}^1(\mathcal{U}, \mathcal{O})$) Given $h \in C^1(\mathcal{U}, \mathcal{O})$, rewrite its Laurent expansion:

   $$h = \sum_{k=-\infty}^{\infty} h_k u^k = \sum_{k=0}^{\infty} h_k u^k + \sum_{k=1}^{\infty} h_k v^k = -f + g$$

   where $f \in \mathcal{O}(U)$ and $g \in \mathcal{O}(V)$. Hence $h \in \operatorname{im} \delta_0$, and $\check{H}^1(\mathcal{U}, \mathcal{O}) = 0$.

3. ($\check{H}^k(\mathcal{U}, \mathcal{O})$) Trivially, $\check{H}^k(\mathcal{U}, \mathcal{O}) = 0$ for $k \geq 2$.

Note that $\check{H}^0(\mathcal{U}, \mathcal{O}) \cong \mathbb{C}$ is consistent with what we know so far, since $\check{H}^0(\mathcal{U}, \mathcal{O}) = \mathcal{O}(\mathbb{P}^1)$, and Liouville's theorem shows that $\mathcal{O}(\mathbb{P}^1)$ can only contain constant functions.

Example 1.1.5. Recall the tautological line bundle $\mathcal{O}(-1)$ and its dual $\mathcal{O}(1)$ on $\mathbb{P}^n$; we have $\mathcal{O}(n) = \mathcal{O}(1)^n$. On the same charts on $\mathbb{P}^1$, since $\mathcal{O}(1)$ has transition function $u = v^{-1}$, we know $\mathcal{O}(n)$ has transition function $u^n = v^{-n}$. To construct a global section of $\mathcal{O}(n)$, given a monomial v^k on V, we require $u^n v^k = u^{n-k}$ to be well-defined on U, so $k \leq n$. In homogeneous coordinates $[x_0 : x_1]$, the global sections are therefore $x_0^n, x_0^{n-1} x_1, \ldots, x_1^n$, the homogeneous polynomials of degree n. The same story holds on $\mathbb{P}^N$. Hence $\dim H^0(\mathbb{P}^N, \mathcal{O}(n)) = \binom{N+n-1}{n-1}$. In particular, there are $\binom{9}{5} = 126$ independent global sections of $\mathcal{O}_{\mathbb{P}^4}(5)$.

Definition 1.1.6. The set of all open covers of X form a directed set under refinement. The **Čech cohomology of** X with coefficients in $\mathcal{F}$ is the direct limit $\check{H}^n(X, \mathcal{F}) := \varinjlim_{\mathcal{U}} \check{H}^n(\mathcal{U}, \mathcal{F})$.

Definition 1.1.7. An ordered open cover $\{U_\alpha\}$ is **good** if it is countable and every finite intersection $U_{\alpha_0, \ldots, \alpha_p}$ is either empty or contractible.

Theorem 1.1.8 ([1, Corollary of Theorem 5.4.1])**.** *The Čech cohomology of a good cover $\mathcal{U}$ is isomorphic to the Čech cohomology of X.*

One can define **sheaf cohomology** $H^n(X, \mathcal{F})$ as the right derived functors of the global sections functor Γ_X (i.e. $\mathcal{F} \mapsto \mathcal{F}(X)$). For us, Čech and sheaf cohomology are indistinguishable as long as we work with sheaves.

Theorem 1.1.9 ([1, Theorem 5.10.1])**.** *If X is a paracompact topological space, then Čech cohomology $\check{H}^n(X, \mathcal{F})$ and sheaf cohomology $H^n(X, \mathcal{F})$ are isomorphic for any sheaf $\mathcal{F}$.*

Čech cohomology is also directly related to de Rham cohomology, and, as we shall see, Dolbeault cohomology in the complex case. So we can think of Čech cohomology classes as forms.

Theorem 1.1.10 (Čech–de Rham isomorphism). *Let $\mathbb{R}$ denote the constant sheaf. There is a canonical isomorphism $\check{H}^k(X,\mathbb{R}) \cong H^k_{\mathrm{dR}}(X)$ for each k.*

Proof. By the Poincaré lemma, the **de Rham complex** of sheaves

$$0 \to \mathbb{R} \xrightarrow{\subseteq} \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \Omega^2(X) \xrightarrow{d} \cdots$$

is exact (by checking exactness on the stalks). Let $Z^n := \ker(d\colon \Omega^k \to \Omega^{k+1})$. The de Rham complex splits into a bunch of short exact sequences:

$$0 \to d\Omega^{k-1} \cong Z^k \xrightarrow{\subseteq} \Omega^k \xrightarrow{d} Z^{k+1} \to 0.$$

To each such short exact sequence is associated a long exact sequence of (sheaf) cohomology:

$$0 \to H^0(X,Z^k) \to H^0(X,\Omega^k) \to H^0(X,Z^{k+1}) \to H^1(X,Z^k) \to H^1(X,\Omega^k) \to H^1(X,Z^{k+1}) \to \cdots .$$

Fact: $H^i(X,\Omega^k) = 0$ for every k and $i > 0$ (since Ω^k is a fine sheaf). Hence we obtain isomorphisms

$$H^{i+1}(X,Z^0) \cong H^i(X,Z^1) \cong \cdots \cong H^1(X,Z^i).$$

But $Z^0 \cong \mathbb{R}$ and we more commonly write

$$H^1(X,Z^i) = \mathrm{coker}(H^0(X,\Omega^i) \to H^0(X,Z^{i+1})) = Z^{i+1}(X)/d\Omega^k(X) = H^{i+1}_{\mathrm{dR}}(X).$$

Hence $\check{H}^{i+1}(X,\mathbb{R}) \cong H^{i+1}_{\mathrm{dR}}(X)$. ∎

Definition 1.1.11. Let E be a holomorphic vector bundle on X and $\Omega^{0,q}(E) := \Omega^{0,q}(X) \otimes \Gamma(E)$ denote the space of E-valued $(0,q)$-forms. **Dolbeault cohomology** $H^q_{\bar\partial}(E)$ is the cohomology of the complex

$$\cdots \xrightarrow{\bar\partial} \Omega^{0,q}(E) \xrightarrow{\bar\partial} \Omega^{0,q+1}(E) \xrightarrow{\bar\partial} \Omega^{0,q+2}(E) \xrightarrow{\bar\partial} \cdots .$$

We write $H^{p,q}_{\bar\partial}(X)$ for $E = \Lambda^p T^{1,0}X$. The dimensions $h^{p,q}(X) := \dim_{\mathbb{C}} H^{p,q}_{\bar\partial}(X)$ are the **Hodge numbers** of X.

Lemma 1.1.12 ($\bar\partial$-Poincaré lemma). *$\bar\partial$-closed, i.e. holomorphic, implies $\bar\partial$-exact on $\mathbb{C}^n$.*

Theorem 1.1.13 (Čech–Dolbeault isomorphism). *Let $\Omega^{p,0}$ denote the sheaf of holomorphic p-forms on X. There is a natural isomorphism $\check{H}^q(X,\Omega^{p,0}) \cong H^{p,q}_{\bar\partial}(X)$.*

Proof. Analogous to the proof of the Čech–de Rham isomorphism, except now using the $\bar\partial$-Poincaré lemma to establish the exactness of the complex

$$0 \to \ker(\bar\partial\colon \Omega^{p,0}(X) \to \Omega^{p,1}(X)) \xrightarrow{\subseteq} \Omega^{p,0}(X) \xrightarrow{\bar\partial} \Omega^{p,1}(X) \xrightarrow{\bar\partial} \cdots .$$ ∎

Definition 1.1.14. Consider the double complex $\Omega^{\bullet,\bullet}$ with differentials ∂ and $\bar\partial$. The **Frölicher spectral sequence** is the spectral sequence of a double complex associated to $\Omega^{\bullet,\bullet}$. Since the total complex of $\Omega^{\bullet,\bullet}$ is $\Omega^\bullet(X)$, the Frölicher spectral sequence converges to complex de Rham cohomology $H^\bullet_{\mathrm{dR}}(X,\mathbb{C})$.

1.1.2 Morse Homology

Throughout this subsection, $f\colon X \to \mathbb{R}$ is a smooth function, and we equip X, viewed as a real manifold, with a Riemannian metric g. We also assume (f, g) is Morse–Smale, defined below.

Definition 1.1.15. A **critical point** of f is a point $p \in X$ with $df_p = 0$. Define the **Hessian**

$$H(f)_p\colon T_pX \to T_p^*X, \quad v \mapsto \nabla_v(df),$$

which is independent of the choice of connection ∇. (In coordinates, we recover the usual $\partial^2 f/\partial x_i \partial x_j$.) The critical point p is **non-degenerate** if the Hessian does not have zero eigenvalues. A non-degenerate critical point p has **Morse index** $\operatorname{ind}(p)$ the number of negative eigenvalues of the Hessian. The function f is **Morse** if all of its critical points are non-degenerate.

Definition 1.1.16. Recall that the **gradient** of f with respect to a metric g is the vector field $\operatorname{grad} f$ such that $g(\operatorname{grad} f, X) = Xf$. Equivalently, $\operatorname{grad} f = (df)^\sharp$. Let $\psi_t\colon X \to X$ be the one-parameter group of diffeomorphisms associated to the flow of $-\operatorname{grad} f$. The **descending manifold** $D(p)$ and **ascending manifold** $A(p)$ at a critical point p are

$$\begin{aligned} D(p) &\coloneqq \{x \in X : \lim_{t\to-\infty} \psi_t(x) = p\} \\ A(p) &\coloneqq \{x \in X : \lim_{t\to+\infty} \psi_t(x) = p\}. \end{aligned}$$

The pair (f, g) is **Morse–Smale** if f is Morse and $D(p)$ is transverse to $A(q)$ for every pair of critical points p and q (i.e. tangent spaces of $D(p)$ and $A(q)$ generate the tangent space at every intersection point).

Here are two useful and easy-to-prove facts: every flow line asymptotically approaches critical points, and $\dim D(p) = \operatorname{ind}(p)$ (so $\dim A(p) = \dim X - \operatorname{ind} p$ by the Morse–Smale condition).

Definition 1.1.17. Fix critical points p and q. A **flow line** from p to q is an integral curve $\gamma(t)$ of $-\operatorname{grad} f$ with $\lim_{t\to-\infty} \gamma(t) = p$ and $\lim_{t\to+\infty} \gamma(t) = q$. The **moduli space of flow lines** from p to q is

$$\begin{aligned} \mathcal{M}(p, q) &\coloneqq \{\text{flow lines from } p \text{ to } q\}/\sim, \quad \alpha \sim \beta \text{ if } \exists c \in \mathbb{R} : \alpha(t) = \beta(t + c) \\ &= (D(p) \cap A(q))/\mathbb{R}. \end{aligned}$$

A **broken flow line** consists, piecewise, of flow lines.

The Morse–Smale condition implies $D(p) \cap A(q)$ is a submanifold of X with dimension $\operatorname{ind}(p) - \operatorname{ind}(q)$. Since $\sim$ is a smooth, proper, free $\mathbb{R}$-action, $\mathcal{M}(p, q)$ is a manifold of dimension $\operatorname{ind}(p) - \operatorname{ind}(q) - 1$ when $p \neq q$ (otherwise the $\mathbb{R}$-action is trivial). Note that if $\operatorname{ind}(p) = k$ and $\operatorname{ind}(q) = k - 1$ then $\mathcal{M}(p, q)$ is zero-dimensional. In fact, in this case, $\mathcal{M}(p, q)$ is compact as a corollary of the following theorem, and therefore is a finite set of points.

Theorem 1.1.18 ([2, Theorem 2.1])**.** *Let X be closed and (f, g) Morse–Smale. Then $\mathcal{M}(p, q)$ has a natural compactification to a smooth manifold with corners $\overline{\mathcal{M}(p, q)}$ where*

$$\overline{\mathcal{M}(p, q)} \setminus \mathcal{M}(p, q) = \bigcup_{k\geq1} \bigcup_{\substack{p, r_1, \ldots, r_k, q \\ \text{distinct crit pts}}} \mathcal{M}(p, r_1) \times \mathcal{M}(r_1, r_2) \times \cdots \times \mathcal{M}(r_{k-1}, r_k) \times \mathcal{M}(r_k, q).$$

Corollary 1.1.19. *If $\operatorname{ind}(p) - \operatorname{ind}(q) = 1$, then $\overline{\mathcal{M}(p, q)} = \mathcal{M}(p, q)$ is compact. If $\operatorname{ind}(p) - \operatorname{ind}(q) = 2$, then*

$$\partial\overline{\mathcal{M}(p, q)} = \bigcup_{\operatorname{ind}(r)=\operatorname{ind}(p)-1} \mathcal{M}(p, r) \times \mathcal{M}(r, q).$$

Proof. Since $\dim \mathcal{M}(r, s) = \operatorname{ind}(r) - \operatorname{ind}(s) - 1$, the space $\mathcal{M}(r, s)$ is non-empty only if $\operatorname{ind}(r) - \operatorname{ind}(s) \geq 1$. Hence $\overline{\mathcal{M}(p, q)} \setminus \mathcal{M}(p, q) = \emptyset$ when $\operatorname{ind}(p) - \operatorname{ind}(q) = 1$. Similar reasoning shows the $\operatorname{ind}(p) - \operatorname{ind}(q) = 2$ case. $\square$

Definition 1.1.20. Fix orientations for $D(p)$ at every critical point p. There is an isomorphism at $x \in \gamma \in \mathcal{M}(p,q)$ given by

$$\begin{aligned} T_x D(p) &\cong T_x(D(p) \cap A(q)) \oplus (T_x X / T_x A(q)) && \text{transversality from Morse–Smale} \\ &\cong T_\gamma \mathcal{M}(p,q) \oplus T_x \gamma \oplus (T_x X / T_x A(q)) && \text{definition of } \mathcal{M}(p,q) \\ &\cong T_\gamma \mathcal{M}(p,q) \oplus T_x \gamma \oplus T_q D(q) && \text{translating } T_q D(q) \text{ along } \gamma. \end{aligned}$$

The **orientation** on $\mathcal{M}(p,q)$ is such that this isomorphism is orientation-preserving. Let C_k be the free abelian group generated by critical points of index k, and define the **Morse–Smale–Witten boundary map**

$$\partial_k^{\text{Morse}} \colon C_k \to C_{k-1}, \quad p \mapsto \sum_{\text{ind } q = k-1} \#\mathcal{M}(p,q) q$$

where $\#\mathcal{M}(p,q) \in \mathbb{Z}$ is counted with sign according to the orientation of $\mathcal{M}(p,q)$, which here is a discrete set of points.

Lemma 1.1.21. *$(\partial_k^{\text{Morse}})^2 = 0$, so $(C_\bullet, \partial^{\text{Morse}})$ is a chain complex.*

Proof. Let $\operatorname{ind}(p) - \operatorname{ind}(q) = 2$. The coefficient of q in $(\partial^{\text{Morse}})^2 p$ is

$$\sum_{\operatorname{ind}(r) = \operatorname{ind}(p) - 1} \#\mathcal{M}(p,r) \cdot \#\mathcal{M}(r,q) = \# \bigcup_{\operatorname{ind}(r) = \operatorname{ind}(p) - 1} \mathcal{M}(p,r) \times \mathcal{M}(r,q) = \#\partial\overline{\mathcal{M}(p,q)}.$$

Since $\overline{\mathcal{M}(p,q)}$ is an oriented 1-manifold with boundary, this quantity, the number of boundary points, is zero. □

Definition 1.1.22. Morse homology $H_\bullet^{\text{Morse}}(f,g)$ is the homology of the **Morse–Smale–Witten complex** $(C_\bullet, \partial^{\text{Morse}})$.

Example 1.1.23. The (upright) torus T^2 has four critical points with f the height function: p (index 2), q and r (index 1), and s (index 0). This choice of f is Morse, but with the induced metric g from $\mathbb{R}^3$, the pair (f,g) is not Morse–Smale: $D(q) \cap A(r)$ is non-empty, but transversality forces it to be. The solution is to tilt the torus a little; equivalently, perturb g. There are two flow lines, of opposite sign, for each relevant pair of critical points. Hence $\partial_k^{\text{Morse}} = 0$ for $k = 1, 2$. It follows that

$$H_2^{\text{Morse}}(f,g) = \mathbb{Z}, \quad H_1^{\text{Morse}}(f,g) = \mathbb{Z}^2, \quad H_0^{\text{Morse}}(f,g) = \mathbb{Z}.$$

Theorem 1.1.24 ([2, Theorem 3.1])**.** *Let X be a closed smooth manifold, $H_\bullet(X)$ denote singular homology on X, and (f,g) be a Morse–Smale pair on X. Then there is a canonical isomorphism $H_n^{\text{Morse}}(f,g) \cong H_n(X)$.*

Corollary 1.1.25. *The number of critical points of a Morse function is at least the sum $\sum_k \dim H_k(X)$ of the Betti numbers.*

Proof. The number of critical points is the sum of the dimensions of the Morse chain groups, which is at least the sum of the dimensions of the Morse homology groups, which is equal to the sum of the dimensions of the singular homology groups. □

The infinite-dimensional analogue of Morse homology is known as **Floer homology**. We shall primarily be concerned with Floer homology for mirror symmetry.

1.1.3 Equivariant Cohomology

In this subsection, G is a compact Lie group acting smoothly on X. Cohomology is taken with $\mathbb{Q}$ coefficients.

The usual cohomology theories fail to capture any information about this action of G on X. Equivariant cohomology is an extension of the usual cohomology for X/G in order to account for the action. For example, if G acts smoothly without fixed points, then X/G is a smooth manifold, and equivariant cohomology agrees with $H^*(X/G)$. On the other end of the spectrum, if $X = \{\mathrm{pt}\}$, then any G-action fixes pt, and we want equivariant cohomology to give a cohomological invariant of G. To construct equivariant cohomology we need the machinery of classifying spaces. So we first review some homotopy theory.

Definition 1.1.26. A continuous map $f\colon X \to Y$ of topological spaces is a **weak homotopy equivalence** if the induced maps $\pi(f)\colon \pi_*(X) \to \pi_*(Y)$ of homotopy groups are isomorphisms. A topological space X is **weakly contractible** if $X \to \mathrm{pt}$ is a **weak equivalence**. If in addition there is a continuous map $g\colon Y \to X$ such that $g \circ f$ and $f \circ g$ are homotopic to the identity maps id_X and id_Y respectively, then X and Y are **homotopy equivalent**.

Theorem 1.1.27 (Hurewicz, [3, Section 20.1])**.** *For any topological space X there exists a group homomorphism $h_k\colon \pi_k(X) \to H_k(X)$. If X is $(n-1)$-connected, i.e. $\pi_k(X) = 0$ for $1 \le k \le n-1$, then h_k is an isomorphism for $1 \le k \le n$, and h_1 is the abelianization map.*

Theorem 1.1.28 ([3, Section 10.8])**.** *For $\{X_\alpha\}$ a directed system of Hausdorff spaces, $\pi_i(\varinjlim X_\alpha) = \varinjlim \pi_i(X_\alpha)$*

Theorem 1.1.29 (Whitehead, [3, Theorem 20.1.5])**.** *If two connected CW complexes X and Y are weakly homotopy equivalent, then they are homotopy equivalent.*

(This theorem of Whitehead's provided some of the initial justification for working with CW complexes.) It is true that all smooth manifolds are homotopy equivalent to CW-complexes (see [4, page 220] for a proof for compact smooth manifolds, and discussion on the general case). The proof is Morse-theoretic.

Definition 1.1.30. If $P \to B$ is a principal G-bundle with P weakly contractible, then B is a **classifying space** for G, denoted BG, and P is a **universal G-bundle**, denoted EG.

Theorem 1.1.31 (Milnor, [3, Section 14.4])**.** *Let G be any topological group. Then there exists a classifying space for G (and a universal G-bundle) unique up to canonical homotopy equivalence.*

Proof sketch. We show that EG exists for closed subgroups of the Lie group $\mathrm{GL}(n)$; this is sufficient for our purposes. First, we find $E\,\mathrm{GL}(n)$. Define

$$V_n(\mathbb{R}^k) \coloneqq \{(v_1, \dots, v_n) \in (\mathbb{R}^k)^n \text{ linearly independent}\},$$
$$V_n^O(\mathbb{R}^k) \coloneqq \{(v_1, \dots, v_n) \in (\mathbb{R}^k)^n \text{ orthonormal}\}.$$

They are both referred to as the **Stiefel manifold** associated to $\mathrm{Gr}_n(\mathbb{R}^k)$. Note that $\pi\colon V_n(\mathbb{R}^k) \to \mathrm{Gr}_n(\mathbb{R}^k)$ is a principal $\mathrm{GL}(n)$-bundle, where π sends $(v_1, \dots, v_n)$ to $\mathrm{span}\{v_1, \dots, v_n\} \in \mathrm{Gr}_n(\mathbb{R}^k)$. We shall show that $V_n(\mathbb{R}^\infty) \coloneqq \varinjlim_k V_n(\mathbb{R}^k)$ is a universal $\mathrm{GL}(n)$-bundle over $\mathrm{Gr}_n(\mathbb{R}^\infty)$.

By Gram–Schmidt, $V_n^O(\mathbb{R}^k)$ is a deformation retract of $V_n(\mathbb{R}^k)$, so since homotopy commutes with direct limit (see 1.1.28), it suffices to show that $V_n^O(\mathbb{R}^k)$ is $(k-1)$-connected. There are fibrations

$$V_{n-1}^O(\mathbb{R}^{k-1}) \to V_n^O(\mathbb{R}^k) \xrightarrow{(v_1,\dots,v_n)\mapsto v_n} S^{k-1}$$

so for $i < k-1$, we have $\pi_i(V_{n-1}^O(\mathbb{R}^{k-1})) \cong \pi_i(V_n^O(\mathbb{R}^k))$. Hence by induction these homotopy groups vanish, as desired, and we have shown $E\,\mathrm{GL}(n) = V_n(\mathbb{R}^\infty)$.

Now let $G \subset \mathrm{GL}(n)$ be a closed Lie subgroup. Take $EG \coloneqq V_n(\mathbb{R}^\infty)$ and let $BG \coloneqq EG/G$. Then $EG \to BG$ is a principal G-bundle, and by the same argument as above it is universal. $\square$

The idea for equivariant cohomology is to start with the possibly not-free action of G on X, find a homotopy equivalent space $\tilde{X}$ on which G acts freely, and then look at $H^*(\tilde{X}/G)$. Here is an easy way to get such a space.

Definition 1.1.32. Let E be a G-bundle over X (which has a G-action). Define the product

$$E \times_G X = E \times X/\sim, \quad (eg, x) \sim (e, gx) \quad \forall e \in E, x \in X, g \in G.$$

The **homotopy quotient** X_G of X by G is $X_G \coloneqq EG \times_G X$. The **equivariant cohomology** of X is $H_G^*(X) \coloneqq H^*(X_G)$, the cohomology of the homotopy quotient.

Definition 1.1.33. The projection maps from $EG \times X$ induce maps $\sigma \colon X_G \to X/G$ and $\pi \colon X_G \to BG$ which fit into the **mixing diagram** of Borel and Cartan:

$$\begin{tikzcd}
EG \arrow[d] & EG \times X \arrow[l] \arrow[r] \arrow[d] & X \arrow[d] \\
BG & X_G \arrow[l, "\sigma"'] \arrow[r, "\pi"] & X/G.
\end{tikzcd}$$

So there are two ways to view X_G:

1. $\pi \colon X_G \to X/G$ is a fibered space with the fiber over $[Gx]$ being $EG/\{g : gx = x\}$;
2. $\sigma \colon X_G \to BG$ is a bundle with fiber X

Using the second perspective, the inclusion of the fiber $i \colon X \to X_G$ induces a map $i^* \colon H_G^*(X) \to H^*(X)$.

Example 1.1.34. If $X = \{\text{pt}\}$, then $EG \times_G X = (EG \times \{\text{pt}\})/G = EG/G = BG$, so the equivariant cohomology of a point with a G-action is the cohomology of BG, i.e. $H_G^*(\{\text{pt}\}) = H^*(BG)$. For a general space X, the equivariant map $X \to \{\text{pt}\}$ induces an $H^*(BG)$-module structure on $H_G^*(X)$ for any X. We use the notation $H_G^* \coloneqq H_G^*(\{\text{pt}\}) = H^*(BG)$.

Example 1.1.35. If G acts freely on X, then $EG \times_G X = EG \times X/G$. By the Künneth theorem,

$$H_G^*(X) = H^*(EG \times X/G) = H^*(EG) \otimes H^*(X/G) = H^*(X/G)$$

since EG is weakly contractible, so its cohomology vanishes by the Hurewicz theorem. Hence when X/G is a smooth manifold, equivariant cohomology indeed agrees with regular cohomology. By the same reasoning, if G acts trivially, then $H_G^*(X) = H^*(X) \otimes H^*(BG)$.

Example 1.1.36. Let $G = \mathbb{C}^*$ act on $\mathbb{C}^n$ by scalar multiplication. The quotient map $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ is a principal $\mathbb{C}^*$-bundle. Since $\mathbb{C}^{n+1} \setminus \{0\}$ is homotopy equivalent to S^{2n+1}, the homotopy groups π_k for $k \leq 2n$ vanish, but the higher ones do not necessarily vanish. To fix this we take the direct limit of the inclusions $\mathbb{CP}^n \to \mathbb{CP}^{n+1} \to \cdots$ (which kills off the homotopy groups one by one; see 1.1.28) to get the universal principal $\mathbb{C}^*$-bundle $\mathbb{C}^\infty \to \mathbb{CP}^\infty$. Hence $BS^1 = \mathbb{CP}^\infty$, and

$$H_{S^1}^*(\{\text{pt}\}) = H^*(BS^1) = H^*(\mathbb{CP}^\infty) = \mathbb{Q}[x],$$

generated by the Chern class $x = c_1(\mathcal{O}_{\mathbb{P}^\infty}(1))$. For a **torus action** by $\mathbb{T} \coloneqq (\mathbb{C}^*)^{m+1}$,

$$H_{\mathbb{T}}^*(\{\text{pt}\}) = H^*((\mathbb{CP}^\infty)^{m+1}) = \mathbb{Q}[x_0, \ldots, x_m].$$

Since $H_{\mathbb{T}}^*(X)$ is an $H_{\mathbb{T}}^*(\{\text{pt}\})$-module, we think of elements of $H_{\mathbb{T}}^*(X)$ as polynomial-valued forms.

It would be nice to have a de Rham-type construction in order to work with equivariant cohomology classes. This is possible for G a Lie group: we just have to work with G-equivariant forms.

Definition 1.1.37. Interpret the symmetric algebra $\mathrm{Sym}(\mathfrak{g}^*)$ as polynomials on $\mathfrak{g}$, and therefore $\mathrm{Sym}(\mathfrak{g}^*) \otimes \Omega^*(X)$ as polynomials on $\mathfrak{g}$ with coefficients in $\Omega^*(X)$. There is a natural G-action where G acts on $\mathfrak{g}^*$ by the **coadjoint action** $(g \cdot \xi)(Z) \coloneqq X(\mathrm{Ad}_{g^{-1}} Z)$, and G acts on $\Omega^*(X)$ by pullback. Using this action, the space of smooth G**-equivariant forms** on X is

$$\Omega_G^*(X) \coloneqq (\mathrm{Sym}(\mathfrak{g}^*) \otimes \Omega^*(X))^G,$$

i.e. G-invariant elements of $\mathrm{Sym}(\mathfrak{g}^*) \otimes \Omega^*(X)$. Equivalently, they are G-equivariant maps $\alpha \colon \mathfrak{g} \to \Omega^*(X)$, i.e. $\alpha(Z) = g^{-1} \cdot \alpha(\mathrm{Ad}(g)Z)$.

Definition 1.1.38. Define a grading on $\Omega_G^*(X)$ by

$$\deg \colon \Omega_G^*(X) \to \mathbb{Z}_{\geq 0}, \quad \theta \otimes \omega \mapsto 2 \deg_{\mathrm{poly}}(\theta) + \deg_{\mathrm{form}} \omega.$$

The **equivariant exterior derivative** $d_G \colon \Omega_G^*(X) \to \Omega_G^{*+1}(X)$ is given by extending the following map by linearity:

$$d_G|_{\mathfrak{g}^*}\alpha \colon \mathfrak{g} \to \Omega^*(X), \quad (d_{\mathfrak{g}}\alpha)(Z) \coloneqq d(\alpha(Z)) - \iota_{Z^\#}(\alpha(Z))$$

where d is the usual exterior derivative, ι is contraction, and $Z^\#$ is the fundamental vector field associated to $Z \in \mathfrak{g}$. The complex $(\Omega_G^*(X), d_G)$ is called the **equivariant de Rham complex**.

Theorem 1.1.39 (Equivariant de Rham theorem, [5, Section 0.2])**.** *Let G be a compact Lie group acting on a compact smooth manifold X. Then there is an isomorphism between equivariant cohomology and the cohomology of the twisted de Rham complex:*

$$H_G^*(X) \cong H^*(\Omega_G^*(X), d_G).$$

This proof should be done from a supersymmetric perspective. In the literature the result is known as the equivalence between the **Borel model** (i.e. our topological definition of equivariant cohomology) and the **Cartan–Weil model** (i.e. the de Rham definition).

1.2 Algebraic Topology

We stop distinguishing between isomorphic cohomology theories now. In particular, since X is always at least a smooth manifold, we think of singular cohomology $H^k(X)$ as de Rham cohomology. For a bundle E, $H^k(E)$ refers to sheaf cohomology.

1.2.1 Poincaré and Serre Duality

Unless otherwise stated, X in this section is a compact oriented n-manifold.

Theorem 1.2.1 (Poincaré duality, [4])**.** *Let X be a compact oriented n-manifold. The map*

$$\int_X \colon H^k(X) \otimes H^{n-k}(X) \to \mathbb{R}, \quad \omega \otimes \eta \mapsto \int_X \omega \wedge \eta$$

is a perfect pairing, and hence $H^k(X) \cong H^{n-k}(X)^*$.

If we relax the assumption that X is compact, then the issue is that $\int_X$ may not be well-defined. We work around this by using de Rham cohomology with compact support.

Definition 1.2.2. Let $\Omega_c^k(X)$ denote the k-forms on X with compact support. The **de Rham cohomology groups with compact support** $H_c^n(X)$ are the cohomology of the chain complex $(\Omega_c^\bullet(X), d)$.

Theorem 1.2.3 (Poincaré duality for non-compact manifolds, [4]). *Let X be an oriented n-manifold without boundary. The map*

$$\int_X : H^k(X) \otimes H_c^{n-k}(X) \to \mathbb{R}, \quad \omega \otimes \eta \mapsto \int_X \omega \wedge \eta$$

is a perfect pairing, and hence $H^k(X) \cong H_c^{n-k}(X)^*$.

Definition 1.2.4. Fix $C \subset X$ a closed $(n-k)$-submanifold. Then Poincaré duality identifies the map $\int_C : H^{n-k}(X) \to \mathbb{R}$ with a k-form $\eta_C \in H^k(X)$, called the **Poincaré dual class**. Explicitly, $\int_C \omega = \int_X \omega \wedge \eta_C$.

There is a relation between the Poincaré dual class and the Thom class, which we define below. Namely, the Poincaré dual class of C can be constructed as the Thom class of the normal bundle of C in X.

Theorem 1.2.5 ([6, Theorem 10.4]). *Let $\pi : E \to B$ be an oriented rank n real vector bundle and B is embedded into E as the zero section. Then*

1. *there exists a unique cohomology class $\Phi \in H^n(E, E \setminus B)$ called the* **Thom class** *such that for every $x \in B$, the restriction of Φ to $H^n(E_x, E_x \setminus \{0\})$ is the preferred generator specified by the orientation of E_x in E;*
2. *the* **Thom isomorphism** *: $H^k(E) \to H^{k+n}(E, E \setminus B)$, given by $\omega \mapsto \omega \wedge \Phi$, is an isomorphism for every k.*

Note that since B is a deformation retract of E, the rings $H^*(E)$ and $H^*(B)$ are isomorphic. Hence $\pi^*\Phi = 1 \in H^*(B)$.

Theorem 1.2.6 (Tubular neighborhood theorem, [6, Theorem 11.1]). *Let $C \subset X$ be a k-submanifold embedded in X. There exists an open neighborhood, called a* **tubular neighborhood***, of C in X diffeomorphic to the total space of the normal bundle of C. This diffeomorphism maps points in C to zero vectors.*

Proposition 1.2.7 ([4, Proposition 6.24a]). *Let $C \subset X$ be a closed $(n-k)$-submanifold. The Poincaré dual class $\eta_C \in H^k(X)$ of C is the Thom class of the normal bundle of C in X.*

Proof. Let NC denote the normal bundle of C in X, which has rank k because C is codimension k. Use the tubular neighborhood theorem to identify NC with an open neighborhood T of C in X, and then extend by zero to get $\Phi \in H^k(X)$ supported on T.

We shall show that $\int_X \omega \wedge \Phi = \int_C \omega$ for any $\omega \in H_c^{n-k}(X)$. The maps $\pi : T \to C$ and $\iota : C \to T$ induce isomorphisms of cohomology, so on forms ω and $\pi^*\iota^*\omega$ differ by at most an exact form $d\tau$. Then

$$\begin{aligned}\int_X \omega \wedge \Phi &= \int_T \omega \wedge \Phi = \int_T (\pi^*\iota^*\omega + d\tau) \wedge \Phi \\ &= \int_T \pi^*\iota^*\omega \wedge \Phi = \int_C \iota^*\omega \wedge \pi_*\Phi = \int_C \iota^*\omega\end{aligned}$$

where the last two steps involve the **projection formula** [4, Proposition 6.15] and noting that $\pi_*\Phi = 1$. □

Corollary 1.2.8. *Transverse intersection is Poincaré dual to the wedge product, i.e. for $C, D \subset X$ closed submanifolds intersecting transversally, $\eta_{C\cap D} = \eta_C \wedge \eta_D$.*

Proof. For transversal intersections, codimension is additive: $\operatorname{codim} C \cap D = \operatorname{codim} C + \operatorname{codim} D$. So the normal bundle of the intersection is $N(C \cap D) = NC \oplus ND$. Let $\Phi(E)$ denote the Thom class associated to the vector bundle E. By the characterization of the Thom class, for vector bundles E and F we have $\Phi(E \oplus F) = \Phi(E) \wedge \Phi(F)$; check that $\Phi(E) \oplus \Phi(F)$ restricts on each fiber to the preferred generator. Hence

$$\eta_{C\cap D} = \Phi(N_{C\cap D}) = \Phi(NC \oplus ND) = \Phi(NC) \wedge \Phi(ND) = \eta_C \wedge \eta_D.$$ □

Let X be a complex n-fold now. In the complex setting, we can refine Poincaré duality. The Čech–Dolbeault isomorphism 1.1.13 works for the more general setting in which we defined Dolbeault cohomology: if E is a holomorphic vector bundle over X, then $H^k(X, E) \cong H^k_{\bar\partial}(E)$. So we think of Čech cohomology classes $H^k(X, E)$ as E-valued $(0, k)$-forms.

Definition 1.2.9. The **canonical bundle** K_X of a complex n-fold X is the vector bundle of $(n, 0)$-forms. (Also commonly denoted $\Omega^n(X)$.)

Hence $H^{n-k}(X, E^* \otimes K_X)$ consists of E^*-valued $(n, n-k)$-forms. Given such a form ω and another form $\eta \in H^k(X, E)$, the form $\omega \wedge \eta$ is an (n, n)-form with complex coefficients. We can integrate it to get something in $\mathbb{C}$. At this point it is impossible not to wonder about whether the pairing $H^k(E) \otimes H^{n-k}(X, E^* \otimes K_X) \to \mathbb{C}$ given by wedging and then integrating is perfect.

Theorem 1.2.10 (Serre duality, [7, Corollary III.7.13])**.** *The pairing $H^k(X, E) \otimes H^{n-k}(X, E^* \otimes K_X) \to \mathbb{C}$ is perfect, so $H^k(X, E) \cong H^{n-k}(X, E^* \otimes K_X)^*$.*

Poincaré duality combined with Hodge decomposition gives

$$\bigoplus_{p+q=k} H^q(X, \Omega^p) = H^k(X, \mathbb{C}) \cong H^{2n-k}(X, \mathbb{C}) = \bigoplus_{p'+q'=2n-k} H^{q'}(X, \Omega^{p'}) = \bigoplus_{p+q=k} H^{n-q}(X, \Omega^{n-p}).$$

Serre duality says that in fact, each of the terms in the sum are isomorphic: $H^q(X, \Omega^p) \cong H^{n-q}(X, \Omega^{n-p})$.

1.2.2 Chern Classes via Chern–Weil Theory

For this subsection, we work over $\mathbb{C}$, and every vector bundle we consider is smooth and complex. We define Chern classes using the Chern–Weil approach. There are other equivalent approaches in more general settings. But for us, we take $\pi\colon E \to X$ to be a rank-n smooth complex vector bundle over a smooth manifold X. A connection $A \in \Omega^1(X, \operatorname{Ad} E)$ on E gives a curvature $F_A := dA + A \wedge A \in \Omega^2(X, \operatorname{Ad} E)$.

Definition 1.2.11. The **total Chern class** of E is

$$\begin{aligned} c(E) := \det\left(1 + \frac{i}{2\pi}F\right) &= 1 + \frac{i}{2\pi}\operatorname{tr}(F) + \frac{1}{8\pi}(\operatorname{tr}(F^2) - \operatorname{tr}(F)^2) + \cdots \\ &= 1 + c_1(E) + c_2(E) + \cdots \in H^0(X, \mathbb{R}) \oplus H^2(X, \mathbb{R}) \oplus \cdots . \end{aligned}$$

Its terms $c_k(E) \in H^{2k}(X, \mathbb{R})$ are the **Chern classes**. The total Chern class $c(X)$ of X is defined as $c(X) := c(T^{1,0}X)$.

Theorem 1.2.12 (Chern–Weil theorem, [8, Corollary 4.4.5, Lemma 4.4.6])**.** *The total Chern class $c(E)$ is closed and independent of the choice of connection A on E.*

Example 1.2.13. A magnetic monopole at the origin in $U(1)$ Maxwell theory is given by the trivial line bundle on $\mathbb{R}^3$ with connection $A = i\frac{1}{2r}\frac{1}{z-r}(xdy - ydx)$, where r is the coordinate on the fibers of the bundle. Then

$$F_A = i\frac{1}{2r^3}(xdy \wedge dz + ydz \wedge dx + zdx \wedge dy) = -\frac{i}{2r^2}(r^2 \sin\theta d\theta \wedge d\phi).$$

We easily check that $\int_{S^2} c_1 = \frac{i}{4\pi}\int_{S^2} F_A = 1$ for any 2-sphere around the origin.

Theorem 1.2.14. *The Chern classes satisfy and are uniquely determined by the following properties:*

1. *$c_0(E) = 1$ and $c_k(E) = 0$ if $k > \dim E$;*
2. *(Naturality) if $f\colon Y \to X$ is continuous, then $f^*c(E) = c(f^*E)$;*
3. *(Whitney product formula) $c(E \oplus F) = c(E) \wedge c(F)$;*

4. $c_1(\mathcal{O}_{\mathbb{P}^1}(-1))$ *is minus the preferred generator (given by the orientation) of* $H^2(\mathbb{P}^1)$*.*

Proof. Property 1 is clear from the definition of $c_k(E)$. Property 2 follows from the multiplicative property of the determinant for block diagonal matrices; the curvature on $E \oplus F$ splits as a curvature on E and a curvature on F. Property 3 follows from pulling back a connection A on E to a connection f^*A on f^*E, and then using that pullbacks commute with everything.

Property 4 is more tedious and serves as our second explicit calculation of a Chern class. On $\mathbb{P}^1$, take the usual charts (U, u) and (V, v) with $u = v^{-1}$ on $U \cap V$. The local 1-forms

$$A_U = \frac{\bar{u}du}{1 + u\bar{u}}, \quad A_V = \frac{\bar{v}dv}{1 + v\bar{v}}$$

form the globally-defined **Chern connection** on $\mathcal{O}_{\mathbb{P}^1}(-1)$, the tautological bundle. We cheat a little and work only over U instead of all of $\mathbb{P}^1$. The curvature is

$$F_{A_U} = \frac{(1 + u\bar{u})d\bar{u} \wedge du - \bar{u}(\bar{u}du + ud\bar{u}) \wedge du}{(1 + u\bar{u})^2} = -\frac{du \wedge d\bar{u}}{(1 + u\bar{u})^2}.$$

Hence, in real coordinates, $c_1(\mathcal{O}_{\mathbb{P}^1}(-1)) = -\frac{dx \wedge dy}{\pi(1+x^2+y^2)^2}$. To compare this with the preferred generator, we simply integrate both and compare the result. (This is valid since $\dim H^2(\mathbb{P}^1) = 1$.) We know the preferred generator integrates to 1, whereas

$$\int_{\mathbb{P}^1} c_1(\mathcal{O}_{\mathbb{P}^1}(-1)) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dx \wedge dy}{(1 + x^2 + y^2)^2} = -\frac{1}{\pi}\int_0^{\infty}\int_0^{2\pi} \frac{r d\theta \wedge dr}{(1 + r^2)^2} = -1. \qquad \square$$

Proposition 1.2.15. *Let E be a complex vector bundle, and $\bar{E}$ be the same underlying real vector bundle as E but with conjugate complex structure. Then $c_k(\bar{E}) = (-1)^k c_k(E)$. If E has a Hermitian metric, then $E^* \cong \bar{E}$ canonically.*

Proof. Straightforward exercise: given a connection on E, what is the connection on $\bar{E}$? $\square$

Proposition 1.2.16 (Splitting principle)**.** *If $0 \to A \to B \to C \to 0$ is a short exact sequence, then $c(B) = c(A) \wedge c(C)$.*

Proof. Short exact sequences of smooth vector bundles always split: pick a metric on B and show that $C \cong A^\perp$. Hence $c(B) = c(A \oplus C)$, and then we use the Whitney product formula. $\square$

Note: this is **not** the usual "splitting principle". The usual splitting principle says that to prove an identity on Chern classes, it suffices to pretend that the bundle completely splits into line bundles and prove the identity for that case. For more detail, see [4, Section 21].

Example 1.2.17. We compute the total Chern class of $\mathbb{P}^n$. We first construct the **Euler sequence** on $\mathbb{P}^n$, given by

$$0 \to \mathbb{C} \to \mathcal{O}_{\mathbb{P}^n}(1)^{n+1} \to T^{1,0}\mathbb{P}^n \to 0.$$

Since $X = \mathbb{P}^n$ is $Y = \mathbb{C}^{n+1} \setminus \{0\}$ mod a $\mathbb{C}^*$ action, given $n+1$ linear functionals v_i on $\mathbb{C}^{n+1}$, the vector field $\sum_i v_i \partial_i$ on X is invariant under this $\mathbb{C}^*$ action and descends to a vector field on $\mathbb{P}^n$. The v_i are sections of $\mathcal{O}_{\mathbb{P}^n}(1)$, so this construction is the map $\mathcal{O}_{\mathbb{P}^n}(1)^{n+1} \to T^{1,0}\mathbb{P}^n$. Its kernel is the line bundle associated to $Z = \sum_i x_i \partial_i$ (here x_i are the coordinates on Y): for homogeneous polynomials f, we have $\frac{1}{d}\sum_i x_i \partial_i f = f$. Another way to see this is to visualize $\mathbb{P}^n$ as a sphere in Y, so when we project, the radial vector field Z and its multiples are precisely the kernel.

Clearly $c(\mathbb{C}) = 1$, so by the splitting principle, $c(\mathbb{P}^n) = c(\mathcal{O}_{\mathbb{P}^n}(1)^{n+1}) = c(\mathcal{O}_{\mathbb{P}^n}(1))^{n+1}$. Let $x = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$. Then $c(\mathbb{P}^n) = (1 + x)^{n+1}$.

Using the symbol x to stand for $c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ is fairly common. We shall do so from now on. (The reason is that x generates the cohomology ring $H^*(\mathbb{P}^n)$.)

Definition 1.2.18. If we formally factorize the total Chern class as $c(E) = \prod_{k=1}^r (1 + a_k)$, then the **Chern character class** is

$$\mathrm{ch}(E) := \sum_{k=1}^{r} \exp(a_i) = r + c_1(E) + \frac{1}{2}(c_1(E)^2 - 2c_2(E)) + \cdots .$$

Proposition 1.2.19. *In the Chern–Weil setting,* $\mathrm{ch}(E) = \mathrm{tr}\exp(iF/2\pi)$.

Of course, the Chern character class, being a combination of Chern classes, does not contain more information than the Chern class. The reason we work with it instead of the Chern class is the following proposition.

Proposition 1.2.20. *The Chern character satisfies*

$$\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F), \quad \mathrm{ch}(E \otimes F) = \mathrm{ch}(E) \wedge \mathrm{ch}(F).$$

Example 1.2.21. Let $X = V(p)$ be a smooth projective variety in $\mathbb{P}^n$ with p a degree d homogeneous polynomial, i.e. a section of $\mathcal{O}_{\mathbb{P}^n}(d)$. To compute the Chern class of X, we use the **adjunction formula** $NX \cong \mathcal{O}(d)|_X$ (see [8, Proposition 2.2.17] for details), so that

$$0 \to TX \to T\mathbb{P}^n|_X \to NX \cong \mathcal{O}(d)|_X \to 0$$

is a short exact sequence. Since $\mathcal{O}(d) = \mathcal{O}(1)^{\otimes d}$, we can't use the Whitney sum property of the Chern class, but we can use the Chern character:

$$\mathrm{ch}(\mathcal{O}(d)) = \mathrm{ch}(\mathcal{O}(1))^d = \exp(x)^d = \exp(dx),$$

so $c(\mathcal{O}(d)) = 1 + dx$. Hence $c(X) = (1 + x)^{n+1}/(1 + dx)$.

Finally, we need to connect the theory of Chern classes with sheaf cohomology. Consider the short exact sequence of sheaves given by

$$0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0.$$

Its associated long exact sequence of cohomology contains

$$\cdots \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}) \to H^1(X, \mathcal{O}^*) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \to \cdots .$$

Definition 1.2.22. The **Picard group** of X is the group of isomorphism classes of holomorphic line bundles under tensor product.

Theorem 1.2.23 ([8, Corollary 2.2.10])**.** *Let $\mathcal{O}^*$ be the sheaf of nowhere-zero holomorphic functions. Then* $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}^*)$.

Theorem 1.2.24 ([8, Proposition 4.4.12])**.** *Under the identification of elements of $H^1(X, \mathcal{O}^*)$ with isomorphism classes of holomorphic line bundles, the connecting map $\delta\colon H^1(X, \mathcal{O}^*) \to H^2(X, \mathbb{Z})$ is the first Chern class c_1.*

1.2.3 The Euler Class and Euler Characteristic

Recall that the (holomorphic) Euler characteristic of a sheaf $\mathcal{F}$ is $\chi(\mathcal{F}) := \sum_k (-1)^k \dim H^k(X, \mathcal{F})$.

Definition 1.2.25. Let $\pi\colon E \to B$ be an oriented rank n vector bundle over a smooth n-fold B. In 1.2.5 we defined its Thom class $\Phi \in H^n(E, E \setminus B)$. The inclusion $(E, \emptyset) \subset (E, E \setminus B)$ gives a homomorphism $H^k(E, E \setminus B) \to H^k(E)$ which we denote by $\omega \mapsto \omega|_E$. The **Euler class** $e(E)$ of E is the image of the Thom class Φ under the composition

$$H^n(E, E \setminus B) \xrightarrow{-|_E} H^n(E) \xrightarrow{(\pi^*)^{-1}} H^n(B)$$

where the last isomorphism is canonical and comes from B being a deformation retract of E. Again, if X is a manifold, $e(X) \coloneqq e(TX)$.

Proposition 1.2.26. *Whenever both are defined, $e(E) = c_n(E)$ for E of rank n.*

Proof sketch. Given E a smooth complex vector bundle, $e(E)$ is well-defined because the complex structure on E induces an orientation. We can use the Euler class to construct the Chern classes [6, Section 14]. Then it suffices to verify that the Chern classes we constructed this way satisfy the four axioms 1.2.14 of Chern classes. For example, the Whitney sum formula comes from $\Phi(E \oplus F) = \Phi(E) \wedge \Phi(F)$ being preserved throughout the construction. ☐

We like to distinguish between the Euler class and the top Chern class for several reasons. One is that the Euler class is topological, whereas the Chern classes are differential geometric. Another is that it is sometimes easier to prove properties of the Euler class using properties of the Thom class rather than all the Chern classes, especially from the Chern–Weil approach.

Proposition 1.2.27 ([6, Property 9.3, Property 9.7])**.** *Properties of the Euler class $e(E)$ that do not directly follow from $e(E) = c_n(E)$:*

1. *if the orientation of E is flipped, $e(E)$ changes sign;*
2. *if E has a nowhere zero global section, then $e(E) = 0$.*

Proof. Property 1 is obvious: flipping the orientation of E flips the sign of the Thom class Φ, since $\Phi|_{E_x}$ is the preferred generator. Property 2 comes from $B \xrightarrow{s} E \setminus B \subset E \xrightarrow{\pi} B$ being the identity for a non-zero global section s. Then

$$H^n(B) \xrightarrow{\pi^*} H^n(E) \to H^n(E \setminus B) \xrightarrow{s^*} H^n(B)$$

is the identity on $H^n(B)$. But $\pi^* e(E) = \Phi|_E$, the restriction of the Thom class, by definition, so we have $s^*((\Phi|_E)|_{E \setminus B}) = e(E)$. Since $\Phi \in H^n(E, E \setminus B)$, the composition of these two restrictions is zero. Hence $e(E) = s^* 0 = 0$. ☐

Proposition 1.2.28. *Let $E \to M$ be a smooth oriented real vector bundle of rank r over the smooth compact oriented manifold M of dimension $n \geq r$. Let Z be the zero set of a smooth section $s\colon M \to E$ that is transversal to the zero section $\iota\colon M \to E$. Then Z is a smooth submanifold of M of codimension r and there is a natural bundle isomorphism $NZ \cong E|_Z$. Consequently, $e(E)$ is Poincaré dual to Z.*

Proof. A straightforward exercise. Hint: remember that $e(E)$ is the restriction of the Thom class, and then use 1.2.7 and 1.2.8. ☐

The main purpose of this subsection is the following generalization of the Gauss–Bonnet theorem. We shall use it extensively when calculating Euler characteristic. To determine the Euler class explicitly, we often use many of the preceding results identifying it with various other objects.

Theorem 1.2.29 (Generalized Gauss–Bonnet)**.** *Let X be a compact complex manifold. Then*

$$\int_X e(X) = \chi(X).$$

Proof. We shall prove this in the next subsection, as a consequence of the Hirzebruch–Riemann–Roch formula. (There are much easier proofs, though.) □

Example 1.2.30. We can continue 1.2.17 to compute the Euler characteristic of $\mathbb{P}^n$. Note that every hyperplane $H \cong \mathbb{P}^{n-1} \subset \mathbb{P}^n$ is Poincaré dual to x. So x^n is Poincaré dual to the intersection of n generic hyperplanes, which is a point. In other words, x^n is the preferred generator given by the orientation, and hence $\int_{\mathbb{P}^n} x^n = 1$. (For a more explicit calculation of this, see [6, Theorem 14.10]. The explicit form for x is the obvious generalization of $-c_1(\mathcal{O}_{\mathbb{P}^1}(-1))$, which we computed in 1.2.14.)

Since $c(\mathbb{P}^n) = (1+x)^{n+1}$, we have $c_n = (n+1)x^n$. Hence $\int_{\mathbb{P}^n} c_n = n+1$. By the generalized Gauss–Bonnet theorem, $\chi(\mathbb{P}^n) = n+1$.

Example 1.2.31. Recall that the **degree** of a curve in $\mathbb{P}^2$ is just the degree of the defining homogeneous polynomial. (For a more general definition of the degree of a variety, see [7, Section I.7].) Fact: A degree d curve X in $\mathbb{P}^2$ has Chern class $1 + (3-d)x$. (Remember we write x for $c_1(\mathcal{O}_{\mathbb{P}^n}(1))$.) Then

$$\chi(X) = \int_X c_1(X) = \int_{\mathbb{P}^2} c_1(X)(xd) = \int_{\mathbb{P}^2} d(3-d)x^2 = d(3-d).$$

But for nonsingular curves X, we have $\chi(X) = 2 - 2g$. Hence $g = (d-1)(d-2)/2 = \binom{d-1}{2}$.

Example 1.2.32. By 1.2.18, a quintic hypersurface Q in $\mathbb{P}^4$ has total Chern class $c(Q) = (1+x)^5/(1+5x) = 1 + 10x^2 - 40x^3$. (Note that $c_1(Q) = 0$, so Q is Calabi–Yau.) We want to compute its Euler characteristic using HRR, but integrating over Q is hard. Instead, we use 1.2.28: Q is defined as the zero set of a section of $\mathcal{O}(5) \to \mathbb{P}^4$, so $e(\mathcal{O}(5))$ is Poincaré dual to Q. The Euler class is just the top Chern class (see 1.2.26), so $e(\mathcal{O}(5)) = c_1(\mathcal{O}(5))$,

$$\chi(Q) = \int_Q e(Q) = \int_Q c_3(Q) = \int_{\mathbb{P}^4} c_3(Q) \wedge c_1(\mathcal{O}(5)) = \int_{\mathbb{P}^4} (-40x^3)(5x) = -200 \int_{\mathbb{P}^4} x^4 = -200.$$

1.2.4 The Hirzebruch–Riemann–Roch Formula

Here E is a rank r holomorphic vector bundle over a compact complex n-fold X. The Hirzebruch–Riemann–Roch formula is part of a long sequence of generalizations of Gauss–Bonnet, relating geometric quantities to topological quantities.

Definition 1.2.33. Again, formally factor $c(E) = \prod_{k=1}^r (1 + a_k)$. The **Todd class** is

$$\operatorname{td}(E) \coloneqq \prod_{i=1}^r \frac{a_i}{1 - \exp(-a_i)} = 1 + \frac{1}{2}c_1(E) + \frac{1}{2}(c_1(E)^2 + c_2(E)) + \cdots .$$

The Todd class $\operatorname{td}(X)$ of X is defined as $\operatorname{td}(X) \coloneqq \operatorname{td}(TX)$.

Theorem 1.2.34 (Hirzebruch–Riemann–Roch, [7, Theorem A.4.1])**.** *Let E be a holomorphic vector bundle over a compact complex manifold X. Then*

$$\chi(E) = \int_X \operatorname{ch}(E) \wedge \operatorname{td}(X).$$

where on the right hand side we only integrate the top form, i.e. $\sum_k \operatorname{ch}_k \wedge \operatorname{td}_{n-k}$.

We can often use the Hirzebruch–Riemann–Roch (HRR) formula to compute the dimension of a specific cohomology group, either because some other cohomology groups vanish, or because we know their dimensions. Before we begin calculating anything, we need the following helpful result.

Theorem 1.2.35 (Grothendieck's vanishing theorem, [7, Theorem 2.7])**.** *Let X be a Noetherian topological space of dimension n. Then $H^i(X, \mathcal{F}) = 0$ for $i > n$ and any sheaf of abelian groups $\mathcal{F}$.*

The remainder of this section is examples of the HRR formula. Keep in mind that we are always working with sheaf cohomology, not de Rham cohomology. For example, $H^0(TX)$ is by no means equal to $H^0(X)$, and is not freely generated by connected components of TX. Instead, $H^0(TX) = \Gamma(TX)$, and since global sections generate automorphisms, $H^0(TX)$ consists of holomorphic automorphisms of X.

Example 1.2.36. Let $\mathcal{M}_g$ denote the **moduli space of complex structures** on a genus g closed surface. We shall see later (or recall from Teichmüller theory) that $\dim \mathcal{M}_g = \dim_{\mathbb{C}} H^1(TX)$ where X is a genus g closed Riemann surface. HRR gives

$$\begin{aligned}\dim_{\mathbb{C}} H^0(TX) - \dim_{\mathbb{C}} H^1(TX) = \chi(TX) = \int_X \mathrm{ch}(TX) \wedge \mathrm{td}(TX) \\ = \int_X (1 + c_1(TX)) \wedge (1 + (1/2)c_1(TX)) = \frac{3}{2}\int_X c_1(TX) = 3 - 3g.\end{aligned}$$

The last equality comes from c_1 being the top Chern class for X, i.e. the Euler class, so applying generalized Gauss–Bonnet gives $\int_X c_1(TX) = \chi(X) = 2 - 2g$. For $g \geq 2$, the Riemann surface X has no non-trivial automorphisms, so $\dim_{\mathbb{C}} H^0(TX) = 0$. Hence $\dim \mathcal{M}_g = 3g - 3$.

Example 1.2.37. An important object in mirror symmetry is the space of holomorphic maps from a Riemann surface Σ to a Calabi–Yau n-fold M, i.e. a Kähler n-fold M with $c_1(M) = 0$. (We'll be more careful about the definition of Calabi–Yau later.) An infinitesimal deformation of a holomorphic map, given by a vector field χ^i, must satisfy $\bar{\partial}\chi^i = 0$ if we want the deformed map to still be holomorphic. Hence $\chi \in H^0_{\bar{\partial}}(\phi^* TM)$, the space of such deformations. By HRR,

$$\begin{aligned}\dim_{\mathbb{C}} H^0(\phi^* TM) - \dim_{\mathbb{C}} H^1(\phi^* TM) = \int_X \mathrm{ch}(\phi^* TM) \wedge \mathrm{td}(\Sigma) \\ = \int_X (n + \phi^* c_1(TM)) \wedge (1 + (1/2)c_1(\Sigma)) = n(1-g).\end{aligned}$$

We assume for now that $H^1(\phi^* TM) = 0$, so the space of deformations is $n(1-g)$-dimensional. For $n = 3$ and $g = 0$, the dimension is 3, but there is also a 3-dimensional group of automorphisms of the genus-zero Riemann surface $\mathbb{P}^1$ which does not affect the image curve. Hence the space of genus 0 holomorphic curves inside a Calabi–Yau 3-fold is zero, and we may be able to count them!

Example 1.2.38. Let X be a connected compact curve and L a holomorphic line bundle on X. Fact: the natural isomorphism $H^2(X, \mathbb{Z}) = \mathbb{Z}$ is given by integration over X, and under this isomorphism we have $c_1(L) \mapsto \deg(L)$ ([8, Exercise 4.4.1]). By HRR,

$$\chi(L) = \int_X c_1(L) + \frac{1}{2}c_1(X) = \deg(L) + 1 - g.$$

For the case $L = \mathcal{O}(D)$ for a divisor D on X, we know $\dim_{\mathbb{C}} H^0(L) = \ell(D)$, and by Serre duality $\dim_{\mathbb{C}} H^1(L) = \dim_{\mathbb{C}} H^0(\mathcal{O}(K - D)) = \ell(K - D)$ where K is the canonical divisor. Hence we recover the classical **Riemann–Roch formula** $\ell(D) - \ell(K - D) = \deg D + 1 - g$.

We now use HRR to derive two important and powerful formulas: the Hirzebruch signature theorem, and, as previously promised, the generalized Gauss–Bonnet theorem.

Definition 1.2.39. Let X be a compact complex n-fold (and $h^{p,q} := \dim_{\mathbb{C}} H^{p,q}_{\bar{\partial}}(X)$ be its Betti numbers). The **Hirzebruch χ_y-genus** is

$$\chi_y := \sum_{p=0}^n \chi(X, \Omega^p) y^p = \sum_{p=0}^n \sum_{q=0}^n (-1)^q h^{p,q}(X) y^p.$$

Corollary 1.2.40. *Factorize $c(X) = \prod_{k=1}^n (1 + a_k)$. Then*

$$\chi_y = \int_X \prod_{k=1}^n (1 + y \exp(-a_k)) \frac{a_k}{1 - \exp(-a_k)}.$$

Proof. If $TX = \bigoplus_{p=0}^n L_i$, then $\bigoplus_{p=0}^n \Omega^p(X) y^p = \bigotimes \Omega(yL_i)$. But $\Omega(yL_i)$ is just yT^*L_i, so we use the fact 1.2.15 that $c_1(T^*L_i) = -c_1(TL_i) = a_i$ to get

$$\operatorname{ch}\left(\bigoplus_{p=0}^n \Omega^p y^p\right) = \prod_{p=0}^n \operatorname{ch}(\Omega(yL_i)) = \prod_{p=0}^n (1 + y \exp(-a_p)).$$

Hence by HRR and the definition of the Todd class, we are done. □

We first prove the generalized Gauss–Bonnet theorem using this formula for χ_y. To simplify the proof we restrict to the case of compact Kähler manifolds (in order to use Hodge decomposition), which shall be the main setting we work in anyway.

Theorem 1.2.41 (Generalized Gauss–Bonnet). *Let X be a compact Kähler manifold. Then*

$$\int_X e(X) = \chi(X).$$

Proof. Note that $\chi(X) = \chi_{-1}$ since by Hodge decomposition, $\sum_{p+q=k} h^{p,q}(X) = b_k(X)$, the k-th Betti number of X. Applying the χ_y formula 1.2.40, we get

$$\chi_{-1} = \int_X \prod_{k=1}^n a_k = \int_X c_n(X) = \int_X e(X)$$

where the last equality is from 1.2.26. □

Now we prove the Hirzebruch signature theorem. The big picture is that on X there is an operator D called the **signature operator** whose index we want to compute, and Hodge theory tells us that $D) = \operatorname{sgn}(X)$. The signature theorem says the right hand side is a topological quantity.

Definition 1.2.42. The **signature** $\operatorname{sgn}(X)$ of a compact Kähler n-fold X is the signature of the bilinear form given by the intersection pairing $H^n(X, \mathbb{R}) \times H^n(X, \mathbb{R}) \to \mathbb{R}$.

Proposition 1.2.43 ([8, Corollary 3.3.18]). *If a compact Kähler n-fold has $n = 2m$, then $\operatorname{sgn}(X) = \sum_{p,q=0}^{2m} (-1)^p h^{p,q}(X)$.*

Theorem 1.2.44 (Hirzebruch signature theorem). *Let $L(X) \coloneqq \prod_{k=1}^n a_k \coth(a_k/2)$ be the* **Hirzebruch L-genus**. *Then $\operatorname{sgn}(X) = \int_X L(X)$.*

Proof. By 1.2.43, setting $y = 1$ in the Hirzebruch χ_y-genus gives the signature $\operatorname{sgn}(X)$. But using the formula 1.2.40 for the χ_y-genus, we get $\operatorname{sgn}(X) = \chi_1 = \int_X L(X)$. □

1.2.5 Fixed-Point Theorems and Localization

Theorem 1.2.45 (Poincaré–Hopf index theorem). *Let M be a compact differentiable manifold, and X a vector field on M with isolated zeros x_i and pointing in the outward normal direction along ∂M. Then $\sum_i \operatorname{ind}_X(x_i) = \chi(M)$.*

Proof. Assume M is oriented. (Otherwise take the double cover $\tilde{M} \to M$, satisfying $\chi(\tilde{M}) = 2\chi(M)$, and proceed with $\tilde{M}$.) From 1.2.28 we see that $e(TM)$ is Poincaré dual to the zeros of any vector field, in particular, X. Hence $\int_M e(TM) = \sum_i \mathrm{ind}_X(x_i)$ where x_i are the isolated zeros of X. But generalized Gauss–Bonnet gives $\int_M e(TM) = \chi(M)$. $\square$

Example 1.2.46. Consider the holomorphic vector field $u\frac{\partial}{\partial u}$ on $\mathbb{P}^1$. Since $\frac{\partial}{\partial u} = -v^2\frac{\partial}{\partial v}$, we have $u\frac{\partial}{\partial u} = -v\frac{\partial}{\partial v}$. Hence this vector field has two zeros at $u = 0$ and $v = 0$, and these zeros both have index 1. Note that $\chi(\mathbb{P}^1) = 2$.

Writing Poincaré–Hopf like $\int_M e(M) = \sum_i \mathrm{ind}_X(x_i)$ for a vector field tells us that some global integrals can be computed using local information at a finite set of (fixed) points. This kind of equation is called a **localization formula**. We shall develop a localization formula for equivariant cohomology with far-reaching consequences.

Definition 1.2.47. If $\iota\colon V \to X$ is a map of compact manifolds, then there is a **pushforward** or **wrong way map** on cohomology $\iota_*\colon H^*(V) \to H^{*+k}(X)$: use Poincaré duality on V, push forward the homology cycle, then use Poincaré duality on X.

Example 1.2.48. Let $\iota\colon V \to X$ be an inclusion of compact manifolds and let k be the codimension of V. Then $\iota_* 1 = \Phi(NV)$, the Thom form of the normal bundle of V. This is straightforward to check: the Poincaré dual of 1 is all of V, which sits in X by the inclusion i, which is dual, fiberwise, to the preferred generator on each fiber, which is precisely the Thom class. The Thom form $\Phi(NV)$ can be extended by zero so that it lives in $H^k(X)$, and the pullback $\iota^*\colon H^k(X) \to H^k(V)$ then gives $\iota^*\Phi(NV) = e(NV)$ (c.f. the definition 1.2.25 of the Euler class). Hence

$$\iota^*\iota_* 1 = e(NV).$$

The composition $\iota^*\iota_*$ is called the **Gysin operation**.

Now we move to the setting for equivariant cohomology: X has an action by a topological group G. The idea is that, compared to ordinary cohomology, the equivariant cohomology has a larger coefficient ring, namely the polynomial ring $H_G^* \coloneqq H_G^*(\{\mathrm{pt}\})$. In ordinary cohomology we cannot invert $e(NV)$, but in equivariant cohomology we can. However we first need to define characteristic classes in equivariant cohomology.

Definition 1.2.49. A vector bundle $\pi\colon E \to X$ is **equivariant** under the G-action on X if the action $G \times X \to X$ lifts to $G \times E \to E$ in a way such that π is equivariant. An equivariant vector bundle E induces a vector bundle $E_G \coloneqq EG \times_G E$ on the homotopy quotient $EG \times_G X$. An **equivariant characteristic class** of E is an ordinary characteristic class of E_G, which is an element of $H^*(EG \times_G X) = H_G^*(X)$. A cohomology class $\alpha \in H^*(X)$ has a **equivariant extension** if there exists $\alpha^G \in H_G^*(X)$ with $\iota^*\alpha^G = \alpha$ where $\iota\colon X \to X_G$ is fiber-wise inclusion.

Example 1.2.50. The **equivariant Chern classes** of a G-equivariant rank n complex vector bundle $\pi\colon E \to X$ are the regular Chern classes of $E_G \to X_G$, i.e.

$$c_k^G(E) \coloneqq c_k(E_G) = c_k(EG \times_G E) \in H^*(X_G) = H_G^*(X).$$

In particular the **equivariant Euler class** is the top equivariant Chern class $e^G(E) \coloneqq c_n^G(E)$.

For the remainder of this subsection we are always working with equivariant cohomology, so we omit the superscript G on characteristic classes. Since we are usually concerned with a torus action for equivariant cohomology, let $\mathbb{T} \coloneqq (\mathbb{C}^*)^m$ be an algebraic torus acting on X. Recall from 1.1.36 that in this case we can view equivariant cohomology classes as represented by $H_{\mathbb{T}}^* = \mathbb{Q}[x_0, \ldots, x_m]$-valued forms.

Theorem 1.2.51 (Atiyah–Bott, [9])**.** *Let $\mathbb{T}$ be a torus acting on X, and split the locus of fixed points F into connected components F_i. The equivariant Euler classes $e_{\mathbb{T}}(NF_i) \in H^*(F_i) \otimes H_{\mathbb{T}}^*$ are invertible in the localized ring $H^*(F_i) \otimes_{\mathbb{Q}} \mathrm{Frac}(H_{\mathbb{T}}^*)$.*

Proof sketch. Terms of positive degree in $H^*(F_i)$ are nilpotent, so an element in $H^*(F_i)$ is invertible if its H^0 component is nonzero. So $e(NF_i)$ is invertible if its component in $H^0(F_i)\otimes H^*_{\mathbb{T}}$ is a non-zero polynomial. Some work shows this is always the case; see page 9 in the cited paper. $\square$

View $X_{\mathbb{T}}$ as a bundle over $B\mathbb{T}$ with fiber X. Then the projection $\pi^X\colon X_{\mathbb{T}}\to B\mathbb{T}$ induces a wrong-way map $\pi^X_*\colon H^*_{\mathbb{T}}(X)\to H^*_{\mathbb{T}}$ which is precisely integration over the fibers. To disambiguate, we write integration over the fibers as $\int_{X_{\mathbb{T}}/B\mathbb{T}}$ and integration over X as $\int_X$. The relation between the two is given by

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[d] & X_{\mathbb{T}} \arrow[d, "\pi"'] \\ p \arrow[r, "\iota_p"] & B\mathbb{T} \end{tikzcd} \qquad \iota_p^*\int_{X_{\mathbb{T}}/B\mathbb{T}}\tilde{\alpha}=\int_X\iota^*\tilde{\alpha}=\int_X\alpha.$$

Here $\iota_p^*\colon H^*_{\mathbb{T}}\cong\mathbb{Q}[x_0,\dots,x_m]\to H^*(\{\mathrm{pt}\})=\mathbb{Q}$ is essentially the evaluation at zero map.

Corollary 1.2.52 (Atiyah–Bott localization formula)**.** *Let $\iota^{F_i}\colon F_i\to X$ be the inclusions of the connected components of the fixed point locus. For $\phi\in H^*_{\mathbb{T}}(X)$ and working in $H^*_{\mathbb{T}}(X)[(H^*_{\mathbb{T}})^{-1}]$,*

$$\int_{X_{\mathbb{T}}/B\mathbb{T}}\phi=\sum_i\int_{(F_i)_{\mathbb{T}}/B\mathbb{T}}\frac{\iota^*\phi}{e(NF_i)}\in\mathrm{Frac}(H^*_{\mathbb{T}}).$$

Proof. From the identity $(\iota^{F_i})^*\iota^{F_i}_*1=e(NF_i)$ and the Atiyah–Bott theorem, we can invert $e(NF_i)$ and write $\phi=\sum_i\iota_*(\iota^*\phi/e(NF_i))$. We can apply π^X_* to both sides. But $\pi^X_*\circ\iota^{F_i}_*=\pi^{F_i}_*$, i.e. integration over F_i. Hence

$$\int_{X_{\mathbb{T}}/B\mathbb{T}}\phi=\pi^X_*\phi=\sum_i\pi^{F_i}_*\frac{\iota^*\phi}{e(NF_i)}=\sum_i\int_{(F_i)_{\mathbb{T}}/B\mathbb{T}}\frac{\iota^*\phi}{e(NF_i)}.\qquad\square$$

The idea is that the Atiyah–Bott localization formula simplifies many equivariant cohomology integrals by letting us only calculate over fixed points. It also allows us prove many fixed-point theorems. We do one such theorem as an example of localization.

Proposition 1.2.53. *Let $\mathbb{T}$ act on X with fixed point locus $\bigcup_i F_i$. Then*

$$\chi(X)=\sum_i\chi(F_i).$$

Proof. From the generalized Gauss–Bonnet theorem 1.2.41 we know $\chi(X)=\int_X e(X)$. The key is that the action of $\mathbb{T}$ on X lifts to an action of $\mathbb{T}$ on TX, so the same statement holds in equivariant cohomology with $\int_{X_{\mathbb{T}}/B\mathbb{T}}\colon H^*_{\mathbb{T}}(X)\to H^*_{\mathbb{T}}$ being integration over the fiber (or the wrong-way map π^X_*). Localizing,

$$\int_{X_{\mathbb{T}}/B\mathbb{T}}e(X)=\sum_i\int_{(F_i)_{\mathbb{T}}/B\mathbb{T}}\frac{\iota^*e(X)}{e(NF_i)}.$$

Remember here $\iota\colon F_i\to X$ are inclusions. Recall that $e(X)=c_n(X)$ and note that there is an exact sequence of equivariant bundles $TF_i\to\iota^*TX\to NF_i$, so by naturality 1.2.14 and the splitting principle,

$$\iota^*e(X)=\iota^*c_n(X)=c_n(\iota^*TX)=c_n(TF_i)c_n(NF_i).$$

Hence when we plug this back into the localized expression,

$$\chi(X)=\int_{X_{\mathbb{T}}/B\mathbb{T}}e(X)=\sum_i\int_{(F_i)_{\mathbb{T}}/B\mathbb{T}}c_n(TF_i)=\sum_i\chi(F_i).\qquad\square$$

1.3 Calabi–Yau Manifolds

1.4 Toric Geometry

Chapter 2

Physics Preliminaries

2.1 Overview

To do (1)

1. Choose Manifold (M, g) with or without boundary. If with, then specify additional data. g may be Riemannian or Lorentzian.
2. **Objects:** fields. Here are some examples:
   - Gauge fields: connections over a principal bundle over M.
   - Matter fields: sections of a vector bundle over M.

   A **quantum gauge theory** has fields which are sections of associated vector bundles.

   - Sigma-model fields: $\phi\colon M \to X$, for some target manifold X.

   A **quantum gravity theory** is obtained by integrating over various choices of metrics on M.
3. The **action** allows us to define the path-integral by weighting the fields with e^{-S} or e^{iS} depending on whether or not M is Riemannian or Lorentzian.
4. Boundaries. If $\partial M = \bigcup_i B_i$ then the set of field configurations on the boundary give rise to Hilbert spaces: $\mathcal{H}_i$. The path integral can be viewed as a map $\bigotimes_i \mathcal{H}_i \to \mathbb{C}$.
5. Boundaries (evolution). Suppose $M = N \times [0, t]$ where $\partial N = \varnothing$. This corresponds to a multi-linear map $\mathcal{H}^* \otimes \mathcal{H} \to \mathbb{C}$ or equivalently a linear map $U(t) : \mathcal{H} \to \mathcal{H}$. Moreover, by gluing two manifolds together we get the following relation:

   $$U(t_1 + t_2) = U(t_1)U(t_2).$$

   By a theorem from functional analysis there exists a Hermitian operator H, which we call the Hamiltonian, which is given by $U(t) = e^{-tH}$ or $U(t) = e^{-itH}$ in the Euclidean or Minkowski case, respectively.
6. Dimensionality.
   (a) Kaluza–Klein reduction is one way to acheive the effective 4-dimensional space-time that we see.
   (b) Damping non-constant modes with action e^{-S} effective path integral on the larger component.

   "Luckily for us, the study of mirror symmetry entails studying QFTs with $d = 2$, so our aim is to study mainly low-dimensional QFTs."

7. $d = 0$.
   (a) Old ingredients: fermionic fields, supersymmetry.
   (b) New ingredients: localization and deformation invariance.
8. $d = 1$, Quantum Mechanics.
   (a) New ingredients: SUSY σ-models, Landau–Ginzburg (σ-models with extra potential functions on target)
9. $d = 2$
   (a) Almost free theories: σ-models with a flat torus being the target. We review T-duality here.
   (b) σ-models on Kähler manifold: Gauge theoretic description, connection to Landau–Ginzburg. (Toric geometry is needed here.)
   (c) Superspace review and a tickle of Mirror Symmetry.

2.2 QFT with $d = 0$

First, let us discuss the fields. For a real-valued theory with $\dim M = 0$, we may identify functions $X : M \to \mathbb{R}$ with just an element $X \in \mathbb{R}$. To model fermions we shall ask our variables to anti-commute: $\psi_i \psi_j = -\psi_j \psi_i$.

The path-integral, or what Hori et al. calls the **partition function**, is given by the expression

$$Z := \int e^{-S[X,\psi]} dX d\psi.$$

Whether it's for correlation appearing in statistical mechanics or scattering amplitudes, many physical quantities can be expressed in terms of **correlation functions**. Using the path integral approach, a correlation function is an expectation value of an operator weighted by e^{-S}:

$$\langle F(X,\psi) \rangle = \int F(X,\psi) e^{-S} dX d\psi.$$

Let's be clever. For the moment, assume that F is a polynomial in X and ψ. Modifying the argument of the exponential $\exp(-S + J_1 X + J_2 \psi)$ and applying the correct number of derivatives we can write:

$$\langle F \rangle = \frac{\partial}{\partial J_1} \frac{\partial}{\partial J_2} \cdots \bigg|_{J_1 = J_2 = 0} \left(\int e^{-S + J_1 X + J_2 \psi} \, dX d\psi \right).$$

The integral on the right side may be denoted by $Z[X, \psi; J_1, J_2]$.

To do (2) **To do** (3)

2.2.1 Supersymmetry

Supersymmetry is both mathematically and physically a useful tool. From a mathematical point of view, "the classical field equations have non-trivial odd symmetries: eg. gradient flow lines in Morse theory, holomorphic curves, gauge theory instantons, monopoles, Seiberg–Witten solutions, hyperKahler structures, Calabi-Yau metrics, metrics of G2 and Spin7 holonomy." From a physical point of view, "[supersymmetric] theories offer a possible way of solving the 'hierarchy problem,' the mystery of the enormous ratio of the Planck mass to the 300 GeV energy scale of electroweak symmetry breaking. Supersymmetry also has the quality of uniqueness that we search for in fundamental physical theories. There is an infinite number of Lie groups that can be used to combine particles of the same spin in ordinary symmetry multiplets, but there are only eight kinds of supersymmetry in four spacetime dimensions, of which only one, the simplest, could be directly relevant to observed particles." **To do** (4)

Let us consider a simple QFT with one bosonic field X and two fermionic fields ψ_1, ψ_2 with an equally simple action:

$$S = S_0(X) - \psi_1\psi_2 S_1(X). \tag{2.1}$$

Taylor expanding the path-integral, and using the fermionic path-integral rules (3), we get the following expression for the integral:

$$\int e^{-S_0(X)+\psi_1\psi_2 S_1(X)} dX d\psi_1 d\psi_2 = \int e^{-S_0} S_1(X).$$

For the rest of the section, we are going to pick a convenient choice for the functions $S_0(X), S_1(X)$ so that we get a feeling for what a supersymmetric theories have to offer. Fix a function $h : \mathbb{R} \to \mathbb{R}$, and let

$$S_0(X) = \frac{1}{2}(\partial h)^2, \qquad\qquad S_1(X) = \partial^2 h.$$

We use the notation $\partial h = h'(X)$ to refer to the derivative of h. Together with the form of the action (2.1), we get a theory that is invariant under the transformations:

$$\delta X = \epsilon^1\psi_1 + \epsilon^2\psi_2, \qquad \delta\psi_1 = \epsilon^2\partial h, \qquad \delta\psi_2 = -\epsilon^1\partial h. \tag{2.2}$$

Let's check this explicitly. Notice that the variational operator δ can be treated as if it were a derivation: $\delta(AB) = (\delta A)B + A(\delta B)$. **To do** (5) Then,

$$\begin{aligned}
\delta(S) &= \delta(S_0) - \delta(\psi_1)\psi_2 S_1 - \psi_1\delta(\psi_2)S_1 - \psi_1\psi_2\delta(S_1)\\
&= (\delta h)(\partial^2 h(\epsilon^1\psi_1 + \epsilon^2\psi_2)) - (\epsilon^2\partial h)\psi_2\partial^2 h - \psi_1(-\epsilon^1\partial h)\partial^2 h - \psi_1\psi_2(\partial^3 h(\epsilon^1\psi_1 + \epsilon^2\psi_2))\\
&= \partial h\partial^2 h(\epsilon^1\psi_1 + \epsilon^2\psi_2 - \epsilon^2\psi_2 + \psi_1\epsilon^1) + \partial^3 h(-\psi_1\psi_1\psi_2\epsilon^1 + \psi_1\epsilon^2\psi_2\psi_2)\\
&= 0
\end{aligned}$$

To show that the measure is also invariant, we proceed analogously except we must keep in mind that we *pullback* the measure and do not push it forward:

$$\begin{aligned}
f^*(dX \wedge d\psi_1 \wedge d\psi_2) &= df^*(X) \wedge df^*(\psi_1) \wedge df^*(\psi_2)\\
&= dX \wedge d\psi_1 \wedge d\psi_2 - d(\epsilon^1\psi_1 + \epsilon^2\psi_2) \wedge d\psi_1 \wedge d\psi_2\\
&\qquad - dX \wedge d(\epsilon^2\partial h) \wedge d\psi_2\\
&\qquad - dX \wedge d\psi_1 \wedge d(-\epsilon^1\partial h)\\
&= 0
\end{aligned}$$

The last two terms are 0 because $d(\partial h) = (\partial^2 h)dX$ and by wedgeing with another dX zeros out the entire term.

2.2.2 Localization

The group of supersymmetry transformations is surprisingly large. We continue with the model described in the previous section and in particular show that if ∂h is nowhere zero then the path integral vanishes. The idea is to use a supersymmetry transformation to make one of the fermionic fields vanish: $S(X, \psi_1, \psi_2) = S(\hat{X}, 0, \hat{\psi}_2)$ which will allow us to simplify the path-integral. Choose,

$$\hat{X} = X - \frac{\psi_1 \psi_2}{\partial h(X)}, \qquad \hat{\psi}_1 = \alpha(X)\psi_1, \qquad \hat{\psi}_2 = \psi_1 + \psi_2.$$

In particular, taking $\alpha(X) = 0$ this is a susy transformation with $\epsilon^1 = \epsilon^2 = -\frac{\psi_1}{\partial h}$. In the new coordinates, $\hat{\psi}_1 = 0$ just as we were looking for. The path integral can be evaluated:

$$\begin{aligned}\int e^{-S(X,\psi_1,\psi_2)}\, dX d\psi_1 d\psi_2 &= \int e^{-S(\hat{X}(X),0,\hat{\psi}_2(\psi_1+\psi_2))}\, dX d\psi_1 d\psi_2 \\ &= \int e^{-S(\hat{X},0,\hat{\psi}_2)} \left(\alpha - (\partial^2 h)(\partial h)^{-2}\hat{\psi}_1\hat{\psi}_2\right) d\hat{X} d\hat{\psi}_1 d\hat{\psi}_2\end{aligned}$$

The first term in the sum, $\int d\hat{\psi}_1 \left(\int e^{-\hat{S}}\alpha\right) = 0$ because the argument is independent of $\hat{\psi}_1$. The second term vanishes because **To do** (6)

This transformation only is valid away from the critical points of h. Taking the path integral as a sum over critical points and expanding $h(X) = h(X_c) + \frac{\alpha_c}{2}(X - X_c)^2 + \cdots$, we get

$$\begin{aligned}Z &= \sum \int \frac{dX d\psi_1 d\psi_2}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\alpha_c^2 (X - X_c)^2 + \alpha_c \psi_1\psi_2\right) \\ &= \sum_{X_c} \frac{1}{\sqrt{2\pi}} \int dX \exp\left(-\frac{1}{2}\alpha_c^2(X - X_c)^2\right)\alpha_c \\ &= \sum_{X_c} \alpha_c \sqrt{\frac{1}{\alpha_c^2}}\end{aligned}$$

The factor of $\sqrt{2\pi}$ can be factored out by normalizing the measure appropriately. This normalization corresponds to defining the correlation functions as $\frac{Z[\lambda,0]}{Z[0,0]}$, which in our case, $Z[0,0] = \sqrt{2\pi}$. Therefore the path-integral becomes an integer:

$$Z = \sum_{X:\, h'(X)=0} \frac{h''(X)}{|h''(X)|}.$$

Remark (Localization). The path integral is localized at loci where the fermionic variation under supersymmetry vanishes.

Remark. The function h in our model is called the **superpotential**, the critical points of which characterize the path-integral.

2.2.3 Deformation Invariance

Proposition 2.2.1. *Suppose ρ is a function such that ρ and $\partial\rho$ vanishes at infinity. Then $h \to h + \rho$ leaves the action invariant.*

Proof. Take $g = \partial\rho(X)\psi_1$ and consider susy variation $\delta_\epsilon g$, with $\epsilon^1 = \epsilon^2 = \epsilon$. Now consider a deformation of the superpotential: $h \to h + \rho$. Then the action $S = \frac{1}{2}(\partial h)^2 - \partial^2 h \psi_1\psi_2$ transforms with $\epsilon\delta_\rho S = \delta_\epsilon g$. Therefore $\langle \delta_\rho S \rangle = \epsilon^{-1}\langle \delta_\epsilon g \rangle = 0$. □

2.2.4 Landau–Ginzburg

After delving into the details of the previous model it is comforting to note that the complex analogue is what is known as the Landau–Ginzburg theory: $(X, \psi_1, \psi_2) \rightsquigarrow (z, \bar{z}, \psi_1, \psi_2, \bar{\psi}_1, \bar{\psi}_2)$, and action:

$$S = |\partial W|^2 - (\partial^2 W)\psi_1\psi_2 - \overline{\partial^2 W}\bar{\psi}_1\bar{\psi}_2. \tag{2.3}$$

We call the holomorphic function $W(z)$ the **superpotential** and it will play the role h did in the previous sections. The complex susy transformations then take the form of:

$$\delta z = \epsilon^1\psi_1 + \epsilon^2\psi_2, \qquad \delta\psi_1 = \epsilon^2\overline{\partial W}, \qquad \delta\psi_2 = -\epsilon^1\overline{\partial W}, \qquad \delta\bar{z} = \delta\bar{\psi}_1 = \delta\bar{\psi}_2 = 0 \tag{2.4}$$

$$\bar{\delta}\bar{z} = \bar{\epsilon}^1\bar{\psi}_1 + \bar{\epsilon}^2\bar{\psi}_2, \qquad \bar{\delta}\bar{\psi}_1 = \bar{\epsilon}^2\partial W, \qquad \bar{\delta}\bar{\psi}_2 = -\bar{\epsilon}^1\partial W, \qquad \bar{\delta}z = \bar{\delta}\psi_1 = \bar{\delta}\psi_2 = 0 \tag{2.5}$$

Once again, the localization principle applies by choosing $\epsilon^1 = \epsilon^2 = -\frac{\psi_1}{\overline{\partial W}}$, and writing $W(z) = W(z_c) + \frac{\alpha_c}{2}(z - z_c)^2 + \cdots$, then with proper normalization, the path-integral integrates to:

$$\begin{aligned}
Z &= \frac{1}{2\pi}\int e^{-|\alpha(z-z_c)|^2}(\alpha\psi^1\psi^2 + \bar{\alpha}\bar{\psi}_1\bar{\psi}_2)\, dz d\bar{z}\, d\psi_1 d\psi_2\, d\bar{\psi}_1 d\bar{\psi}_2 \\
&= \frac{|\alpha|^2}{2\pi}\sum_{z_c}\int e^{-|\alpha(z-z_c)|^2}\, dz d\bar{z} \\
&= \frac{|\alpha|^2}{2\pi}\sum_{z_c}\int e^{-|\alpha|^2(x^2+y^2)}\, dx dy = \sum_{z_c} 1 = \#\text{ of critical points of } W
\end{aligned}$$

2.2.5 Holomorphicity and Correlation functions

To do (7) Let $f(z)$ be a holomorphic observable. Then by the localization principle,

$$\begin{aligned}
\langle f(z)\rangle &= \int f(z)e^{-S} \\
&= \int f(z)|\partial^2 W|^2 e^{-\frac{1}{2}|\partial W|^2} \\
&= \sum_{z_c} f(z_c)
\end{aligned}$$

The set of all fields that vanish under $\bar{\delta}$ are called **chiral fields**. In particular because $\bar{\delta}$ is a derivation it follows that this set is closed under multiplication. If we take $\epsilon^1 = \epsilon^2$ then $\bar{\delta}^2 = 0$ and so we can consider then cohomology ring $\frac{\ker\bar{\delta}}{\operatorname{im}\bar{\delta}}$ is often called the **chiral ring**.

2.3 QFT with $d = 1$

Let M be a 1-dimensional manifold. We know that there are only two diffeomorphism classes of 1-manifolds without boundary: $\mathbb{R}$ and S^1. For simplicity, the only manifold with boundary that we will consider is the unit interval $[0, 1]$.

2.3.1 Quantum Mechanics

1. $S = \int L dt = \int \left[\frac{1}{2}\left(\frac{dX}{dt}\right)^2 - V(X)\right] dt$

2. Hilbert space of states as $L^2(\mathbb{R}, \mathbb{C})$.
3. $f \mapsto Z_{t_2;t_1} f$ given by $(Z_{t_2;t_1} f)(Y) = \int Z(Y, t_2; X, t_1) f(X) dX$
4. Time evolution operator $Z_t = e^{-itH}$.
5. On the circle: $Z_E(\beta) = \int dX_1 Z_{E,\beta}(X_1, X_1) = \operatorname{tr}\exp(-\beta H)$.
6. Example: Harmonic oscillator
   (a) To compute partition function in two different way: operator formalism $\operatorname{tr} e^{-\beta H}$ and directly using path-integral requires zeta function regularization.
7. Example: Sigma model with target space: S^1 with radius R.
$$S(X) = \int \frac{1}{2}\dot{X}^2 dt.$$
8. Example: Sigma model with target space: $\mathbb{R}$.
9. Example: Sigma model with target space: (M, g).
$$S = \frac{1}{2}\int dt g_{ij}(X)\frac{dX^i}{dt}\frac{dX^j}{dt}, g_{ij} = \delta_i j + C_{ijkl}X^k X^l + \cdots .$$
   (a) Operator formalism
   (b) Ambiguity of Hamiltonian: related to the measure in the path integral.
   (c) Supersymmetry imposes constraint: fixes a particular Hamiltonian
10. Complicated actions require approximations to evaluate. The semi-classical approximations take the classical solution: $\delta S/\delta X\big|_{X=X_{cl}} = 0$ and expand the action around that critical point:
$$S(X) = S(X_{cl}) + \frac{(\delta X)^2}{2}\frac{\delta^2 S}{\delta X^2}\bigg|_{X=X_{cl}} + \mathcal{O}((\delta X)^3).$$
    Then the partition function becomes
$$\begin{aligned} Z = \int \mathcal{D}X e^{iS(X)} &= e^{iS(X_{cl})}\int \mathcal{D}(\delta X)\exp\left(i\frac{(\delta X)^2}{2}\frac{\delta^2 S}{\delta X^2}\right) \\ &= \frac{\exp\left(iS(X_{cl})\right)}{\sqrt{\det\left(\frac{\delta^2 S(X_{cl})}{\delta X^2}\right)}} \end{aligned}$$

2.3.2 Structure of Hilbert Space

Definition 2.3.1. A **supersymmetric quantum mechanical system** is $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H}$ with an even operator H and two odd operators $Q, Q^\dagger$ called supercharges satisfying: $Q^2 = (Q^\dagger)^2 = 0$, $\{Q, Q^\dagger\} = 2H$.

A consequence is $[H, Q] = [H, Q^\dagger] = 0$. The operator defining the $\mathbb{Z}_2$ grading is denoted $(-1)^F$. Moreover, $H = \frac{1}{2}\{Q, Q^\dagger\} \geq 0$ with equality if and only if the state is annihilated by both supercharges. If a state is annihlated by a charge then it is invariant under that symmetry, which means that ground states of a SUSY quantum mechanical system are automatically supersymmetric.

Since $Q, \overline{Q}, (-1)^F$ commute with the Hamiltonian, we first put a grading on the Hilbert space, by energy levels, and then grade the resulting subspaces by the $\mathbb{Z}_2$ grading: $\mathcal{H} = \bigoplus \mathcal{H}_n^+ \oplus \mathcal{H}_n^-$. If we define $Q_1 = Q + Q^\dagger$ then

$$Q_1^2 = 2H.$$

For $E_n > 0$, $Q_1^2 = 2E_n$ so Q_1 is invertible and defines an isomorphism

$$\mathcal{H}_n^+ \cong \mathcal{H}_n^-.$$

This has an important consequence for adiabatic/continuous deformations. Degenerate energy levels may split but the dimension of positive energy bosonic states is the same as the fermionic states. Some of the degenerate ground states may split off to form positive energy states, but they again must come in pairs. The invariant that we should consider is the difference:

$$\dim \mathcal{H}_0^B - \dim \mathcal{H}_0^F = \mathrm{tr}\big[(-1)^F e^{-\beta H}\big]. \tag{2.6}$$

This is called the **supersymmetric index** or the **Witten index** and is often written as $\mathrm{tr}(-1)^F$.

To touch base with mathematics let us consider Q as a coboundary operator of the complex: $\cdots \to \mathcal{H}^B \to \mathcal{H}^F \to \mathcal{H}^B \to \mathcal{H}^F \to \cdots$. This complex is naturally graded by the energy levels, and Q preserves this grading. However, the cohomology is trivial: take a Q-closed state $Q\,|\alpha\rangle = 0$ and apply $|\alpha\rangle = \frac{1}{2E_n} H\,|\alpha\rangle = \frac{1}{2E_n}\{Q, Q^\dagger\}\,|\alpha\rangle = Q\left(\frac{1}{2E_n} Q^\dagger\,|\alpha\rangle\right) \in \mathrm{im}\,Q$. Therefore the cohomology group on the bosonic and fermionic Hilbert spaces gets a contribution only from the (supersymmetric) ground states:

$$H^B(Q) \cong \mathcal{H}^B_{(0)}, \qquad H^F(Q) \cong \mathcal{H}^F_{(0)}. \tag{2.7}$$

Note this is an isomorphism not an equality as $H^\bullet(Q)$ is obtained by a quotient, where as $\mathcal{H}^B_{(0)}$ is an honest subgroup of the Hilbert space.

In this form, the Witten index $\mathrm{tr}(-1)^F$, (2.6), resembles the Euler characteristic for the complex. It is also not surprising that because of the appearance of the trace we may write down path integral expressions for $\mathrm{tr}(e^{\beta H})$ and $\mathrm{tr}\left[(-1)^F e^{-\beta H}\right]$. **To do** (8)

2.3.3 Supersymmetric Hilbert space: Example

Example 2.3.2. Let us consider a supersymmetric theory with a single bosonic variable x and a complex superpartner ψ. The Lagrangian is given by:

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}h'(x)^2 + \frac{i}{2}\left(\overline{\psi}\dot{\psi} - \dot{\overline{\psi}}\psi\right) - h''(x)\overline{\psi}\psi,$$

where $\overline{\psi} = \psi^\dagger$, and the second term plays the role of the potential $-V(x)$. Now the following transformations change the Lagrangian by a total time derivative:

$$\delta x = \epsilon\overline{\psi} - \overline{\epsilon}\psi, \qquad \delta\psi = \epsilon(i\dot{x} + h'(x)), \qquad \delta\overline{\psi} = \overline{\epsilon}(-i\dot{x} + h'(x)), \tag{2.8}$$

where $\epsilon = \epsilon_1 + i\epsilon_2$ and $\overline{\epsilon} = \epsilon^*$ is the complex conjugate. As long as the boundary variation vanishes (which will be the case on an open manifold) then the action is invariant. Let δ_1, δ_2 be variations as in (2.8) but with $\epsilon_2 = 0$ and $\epsilon_1 = 0$ respectively. Then:

$$[\delta_1, \delta_2]x = 2i(\epsilon_1\overline{\epsilon}_2 - \epsilon_2\overline{\epsilon}_1)\dot{x}, \qquad [\delta_1, \delta_2]\psi = 2i(\epsilon_1\overline{\epsilon}_2 - \epsilon_2\overline{\epsilon}_1)\dot{\psi}.$$

Applying the Noether procedure with $\epsilon = \epsilon(t)$ to get: $\delta L = -i\dot{\epsilon}Q - i\dot{\overline{\epsilon}}\overline{Q}$ where **To do** (9)

$$Q = \overline{\psi}(i\dot{x} + h'(x)), \qquad \overline{Q} = \psi(-i\dot{x} + h'(x)).$$

Quantizion: Compute conjugate variables, promote to operators, impose commutation relations.

$H = \frac{1}{2}p^2 + \frac{1}{2}h'(x)^2 + \frac{1}{2}h''(x)(\overline{\psi}\psi - \psi\overline{\psi})$.

Check:

1. $[H, Q] = [H, \overline{Q}] = 0$, note $Q^\dagger = \overline{Q}$.
2. Let $\mathcal{O}(x, \psi, \overline{\psi})$ be any observable (self-adjoint operator). Show that $\delta\mathcal{O} = [\hat{\delta}, \mathcal{O}]$ where $\hat{\delta} = \epsilon Q + \overline{\epsilon}\overline{Q}$.
3. The Hilbert space decomposes as $L^2(\mathbb{R}, \mathbb{C}) \otimes \mathbb{C}^2$ as a result of the bosonic and fermionic degrees of freedom.

Define the fermionic number operator, $F = \overline{\psi}\psi$ which measures the number of fermions of a state (either 0 or 1). A more convenient operator that (does not annihilate the state without fermions) is given by $(-1)^F$. Check:

1. $[F, Q] = Q, [F, \overline{Q}] = -\overline{Q}$
2. $\left\{Q, (-1)^F\right\} = \left\{\overline{Q}, (-1)^F\right\} = 0$
3. $\{Q, Q\} = \left\{\overline{Q}, \overline{Q}\right\} = 0$.
4. $\left\{Q, \overline{Q}\right\} = 2H$

2.3.4 Supersymmetric Ground States: Towards Morse Theory

As discussed before, the grounds states of a supersymmetric Hamiltonian are precisely the states which span the intersection of $\ker Q$ and $\ker \overline{Q}$. In the basis $\{|0\rangle, \overline{\psi}|0\rangle\}$, the supercharges are

$$Q = \overline{\psi}(ip + h'(x)) = \begin{pmatrix} 0 & 0 \\ \frac{d}{dx} + h'(x) & 0 \end{pmatrix}, \qquad \overline{Q} = \psi(-ip + h'(x)) = \begin{pmatrix} 0 & -\frac{d}{dx} + h'(x) \\ 0 & 0 \end{pmatrix}.$$

Therefore we look for f_1, f_2 so that $\Psi = f_1(x)|0\rangle + f_2(x)\overline{\psi}|0\rangle$ is annihilated by Q and $\overline{Q}$. This leads to two differential equations with exact solutions:

$$f_1(x) = c_1 e^{-h(x)}, \qquad f_2(x) = c_2 e^{h(x)}.$$

We also want $f_1, f_2 \in L^2$ which means that they are either 0 or exponentially decaying at *both* $x = \pm\infty$. Therefore we have two solutions:

$$e^{-h(x)}|0\rangle, \text{if } h \to \infty, \text{ as } x \to \pm\infty, \qquad e^{h(x)}\overline{\psi}|0\rangle, \text{if } h \to -\infty, \text{ as } x \to \pm\infty.$$

In the case of a harmonic oscillator, $h(x) = \frac{\omega}{2}x^2$, with potential $V(x) = \frac{1}{2}(h'(x))^2 = \frac{\omega^2 x^2}{2}$. Therefore the ground states become:

$$\Psi_{\omega>0} = e^{-\frac{1}{2}\omega x^2}|0\rangle, \qquad \Psi_{\omega<0} = e^{-\frac{1}{2}|\omega|x^2}\overline{\psi}|0\rangle.$$

2.3.5 Semi-classical analysis

We now switch to the Hamiltonian formulation, and at the same time rescale $h \to \lambda h$ for a large $\lambda \gg 1$. This goes back to the deformation invariance result that the action is invarinat under deformations of this form:

$$H = \frac{1}{2}p^2 + \frac{\lambda^2}{2}(h'(x))^2 + \frac{\lambda}{2}h''(x)[\overline{\psi}, \psi].$$

The ground states will be localized at the minimum of $(h'(x))^2$. Expanding h around a critical point, and at the same time rescale the coordinates $x - x_i = \frac{1}{\sqrt{\lambda}}(\tilde{x} - \tilde{x}_i)$ the expansion becomes

$$h(x) = \frac{1}{2\lambda}h''(x_i)(\tilde{x} - \tilde{x}_i)^2 + \frac{1}{6\lambda^{3/2}}h'''(x_i)(\tilde{x} - \tilde{x}_i)^3 + O(\lambda^{-2}).$$

The Hamiltonian the becomes:

$$H = \lambda\left(\frac{1}{2}\tilde{p}^2 + \frac{1}{2}h''(x_i)^2(\tilde{x} - \tilde{x}_i)^2 + \frac{1}{2}h''(x_i)\left[\overline{\psi}, \psi\right]\right) + \lambda^{1/2}(\cdots) + (\cdots) + \mathcal{O}(\lambda^{-1/2}),$$

where $\tilde{p} = -i\frac{d}{d\tilde{x}}$. The $\mathcal{O}(\lambda)$ term is a supersymmetric harmonic oscillator with $\omega = h''(x_i)$. The ground state around x_i are:

$$\Psi_i = e^{-\frac{\lambda}{2}h''(x_i)(x-x_i)^2}\left|0\right\rangle + \mathcal{O}(\lambda^{-1/2}), \quad \text{if } h''(x_i) > 0,$$
$$\Psi_i = e^{-\frac{\lambda}{2}|h''(x_i)|(x-x_i)^2}\left|0\right\rangle + \mathcal{O}(\lambda^{-1/2}), \ \text{if } h''(x_i) < 0$$

This perturbation theory, tells us there is exactly one supersymmetric ground state for every critical point and so the Witten index is given by

$$\operatorname{tr}(-1)^F = \sum_{i=1}^{N}\operatorname{sign}(h''(x_i)).$$

If the target space is $\mathbb{R}^n$, with n bosonic and $2n$ fermionic variables then the Hamiltonian becomes a sum $H = \frac{1}{2}\sum_I p_I^2 + \frac{1}{2}(\partial_I h(x))^2 + \frac{1}{2}(\partial_I\partial_J h)[\overline{\psi}^I, \psi^J]$, and supercharges: $Q = \overline{\psi}^I(ip_I + \partial_I h)$, $\overline{Q} = \psi^I(-ip_I + \partial_I h)$. In this case, $h : \mathbb{R}^n \to \mathbb{R}$. Expanding about a critical point of h and choosing the right coordinate system $\xi^I = \xi^I_{(i)}$:

$$h(x) = h(x_i) + \sum c_I(\xi^I)^2 + \cdots$$

In particular, this means that the Hamiltonian simplifies and we again pick out the $\mathcal{O}(\lambda)$ term to get the ground state:

$$\Psi_i = \left(\bigotimes_{I\,:\,c_I>0}\exp(-\lambda c_I\xi^I)\left|0\right\rangle\right)\otimes\left(\bigotimes_{I\,:\,c_I<0}\exp(-\lambda|c_I|\xi^I)\bar{\psi}^I\left|0\right\rangle\right).$$

In particular, if we denote the Morse indices of h by μ_i, the Witten index is given by $\operatorname{tr}(-1)^F = \sum_{i=1}^N(-1)^{\mu_i}$.

2.3.6 Landau–Ginzburg, $n = 2m$

The Landau–Ginzburg model is complex analogue of our discussion above. The target space will now be $\mathbb{C}^m \equiv \mathbb{R}^{2m}$ and we will take the Lagrangian to be:

$$L = \sum_{i=1}^{m}\left(|\dot{z}_i|^2 + i\bar{\psi}^i\partial_t\psi^{\bar{i}} + i\bar{\psi}^{\bar{i}}\partial_t\psi^i - \frac{1}{4}|\partial_i W|^2\right) - \frac{1}{2}\sum_{ij}(\partial_i\partial_j W\psi^i\bar{\psi}^j + \partial_{\bar{i}}\partial_{\bar{j}}\overline{W}\psi^{\bar{i}}\bar{\psi}^{\bar{j}}).$$

Suppose W has N nondegenerate critical points $p_1, \ldots, p_N$. That is, $\det\partial_i\partial_j W(p_a) \neq 0$. Expanding W around these critical points and rewriting in real coordinates we get:

$$\begin{aligned} W(z) &= \sum_{k=1}^{m}(z^k)^2 + \mathcal{O}((z^k)^3) \\ &= \sum_{k=1}^{m}\left[(x^k)^2 - (y^k)^2\right] + \cdots \end{aligned}$$

Therefore the Morse indices for *any* critical point are equal to the complex dimension m.

To do (10)

2.3.7 Sigma Models

Let (M, g) be a Riemannian manifold and denote by $\mathcal{T}$ the one dimensional manifold one which our QFT lives. The bosonic and fermionic variables are maps of the form

$$\phi\colon \mathcal{T} \to M, \qquad \psi, \bar{\psi} \in \Gamma(\mathcal{T}, \phi^* TM \otimes \mathbb{C}) \tag{2.9}$$

The Lagrangian is given by:

$$L = \frac{1}{2} g_{IJ} \dot{\phi}^I \dot{\phi}^J + \frac{i}{2}(\bar{\psi}^I D_t \psi^J - D_t \bar{\psi}^I \psi^J) - \frac{1}{2} R_{IJKL} \psi^I \bar{\psi}^J \psi^K \bar{\psi}^L \tag{2.10}$$

$$= \frac{1}{2} \langle \dot{\phi}, \dot{\phi} \rangle + \frac{i}{2}(\langle \bar{\psi}, \nabla_t^{LC} \psi \rangle - \langle \nabla_t^{LC} \bar{\psi}, \psi \rangle) - \frac{1}{2} R(\psi, \bar{\psi}, \psi, \bar{\psi}), \tag{2.11}$$

The supersymmetry transformations are given by:

$$\delta \phi^I = \epsilon \bar{\psi}^I - \bar{\epsilon} \psi^I, \qquad \delta \psi^I = \epsilon (i \dot{\phi}^I - \Gamma^I_{JK} \bar{\psi}^J \psi^K), \qquad \delta \bar{\psi}^I = \bar{\epsilon}(-i \dot{\phi}^I - \Gamma^I_{JK} \bar{\psi}^J \psi^K) s.$$

The corresponding supercharges are given by:**To do** (11)

$$Q = i g_{IJ} \bar{\psi}^I \dot{\phi}^J = i \langle \bar{\psi}, \dot{\psi} \rangle, \qquad \bar{Q} = -i g_{IJ} \psi^I \dot{\psi}^J = -i \langle \psi, \dot{\bar{\psi}} \rangle .$$

There is an extra symmetry of this free Lagrangian: $(\psi, \bar{\psi}) \to (e^{-i\theta} \psi, e^{i\theta} \bar{\psi})$. The corresponding Noether charge is

$$F = g_{IJ} \bar{\psi}^I \psi^J = \langle \bar{\psi}, \psi \rangle .$$

Quantizing this system, requires the conjugate momenta:

$$p_I = \frac{\partial L}{\partial \dot{\phi}^I} = g_{IJ} \dot{\phi}^J, \qquad \pi_{\psi\, I} = i g_{IJ} \bar{\psi}^J.$$

Finally, imposing commutation relations:

$$[\phi^I, p_J] = i \delta^I{}_J, \qquad \{\psi^I, \bar{\psi}^J\} = g^{IJ},$$

with all other commutators vanishing. The supercharges are conveniently write $Q = i \bar{\psi}^I p_I, \bar{Q} = -i \psi^I p_I$. The ambiguitiy in the choice of Hamiltonian is fixed if we impose $\{Q, \bar{Q}\} = 2H$. Finally, note that $[F, Q] = Q$ and $[F, \bar{Q}] = -\bar{Q}$ from which we get $[H, F] = 0$.

There exists a natural representation of the observables on the space:

$$\mathcal{H} = \Omega(M) \otimes \mathbb{C}, \quad \text{with} \quad \langle \omega_1, \omega_2 \rangle_{\mathcal{H}} = \int_M \bar{\omega}_1 \wedge * \omega_2.$$

The observables can be assigned the following operators:

$$\phi^I = x^I \cdot, \qquad p_I = -i \nabla_I, \qquad \bar{\psi}^I = dx^I \wedge, \qquad \psi^I = g^{IJ} \frac{\partial}{\partial x^J} \lrcorner$$

The ground state, corresponding to the intersection of all $\ker \psi^I$. The F-charge, or fermion number, is the degree of the form and so the natural grading on the Hilbert space

$$\mathcal{H} = \bigoplus \Omega^p(M) \otimes \mathbb{C}$$

corresponds to the fermion number grading. The supercharge Q is the extrior derivative, $Q = i \bar{\psi}^I p_I = dx^I \wedge \frac{\partial}{\partial x^I} = d$, and Hermitian conjugate is $\bar{Q} = Q^\dagger = d^\dagger$. Finally, we compute the Hamiltonian H by the supersymmetry relation

$$H = \frac{1}{2}\{Q, \bar{Q}\} = \frac{1}{2}(dd^\dagger + d^\dagger d) = \frac{1}{2}\Delta,$$

the Laplace–Beltrami operator. The ground states are then just the harmonic forms:

$$\mathcal{H}_{(0)} = \mathrm{Harm}(M, g) = \bigoplus_{p=0}^{n} \mathrm{Harm}^p(M, g).$$

Now we are starting to see the correspondence between supersymmetric quantum mechanics and differential geometry. In fact, Hodge theory is not too far away! As we showed before, the ground states can be characterized by cohomology (see (2.7)). Since $[F, Q] = Q$ the Q-complex may be graded by the fermion number, or in more mathematical language, the de Rham cohomology is graded by the degree of the forms. In fact, since we know the ground states correspond to the the harmonic forms we use the same proof as for (2.7) to show that

$$\mathcal{H}_{(0)} = \mathrm{Harm}(M, g) \cong H^\bullet(Q) = H^\bullet_{dR}(M),$$

and even more that with respect to the fermion number grading,

$$\mathrm{Harm}(M, g) \cong H^p_{dR}(M).$$

The Witten index, $\mathrm{tr}(-1)^F$, counting the parity of the fermion numbers becomes:

$$\mathrm{tr}(-1)^F = \sum_{p=0}^{n} (-1)^p \dim H^p_{dR}(M) = \chi(M).$$

To do (12)

To do...

☐ 1 (p. 21): Make a shorter introduction.

☐ 2 (p. 22): For completeness: feynman diagrams?

☐ 3 (p. 22): Recall: Fermionic path integral rules .

☐ 4 (p. 23): Revamp this introduction.

☐ 5 (p. 23): Proof that variation is a derivation.

☐ 6 (p. 24): Show that the integral vanishes because the argument is a total derivative.

☐ 7 (p. 25): Prove that the Localization principle applies to holomorphic observables.

☐ 8 (p. 27): Understand the periodic boundary conditions in path integral for fermions.

☐ 9 (p. 27): Derive this Noether charge!

☐ 10 (p. 29): Show that this Landau–Ginzburg model is $\mathcal{N} = 2$ supersymmetric.

☐ 11 (p. 30): Work out the Noether charge for Riemannian manifold sigma model.

☐ 12 (p. 31): Prove Gauss Bonnet using the path integral: Compute the Witten index $\mathrm{tr}[(-1)^F e^{-\beta H}]$ in terms of the Riemann curvature tensor and use the localization principle in the limit of $\beta \to 0$.

Bibliography

[1] Roger Godement. *Topologie algébrique et théorie des faisceaux*. Hermann, 1998.

[2] Michael Hutchings. Lecture notes on morse homology (with an eye toward floer theory and pseudoholomorphic curves). Course notes, Fall 2012.

[3] Tammo tom Dieck. *Algebraic Topology*. American Mathematical Society, 2008.

[4] Raoul Bott and Loring Tu. *Differential Forms in Algebraic Topology*. Springer–Verlag, 1982.

[5] Victor Guillemin and Shlomo Sternberg. *Supersymmetry and equivariant de Rham theory*. Springer–Verlag, 1999.

[6] John Milnor and James Stasheff. *Characteristic Classes*. Princeton University Press, 1974.

[7] Robin Hartshorne. *Algebraic Geometry*. Springer–Verlag, 1997.

[8] Daniel Huybrechts. *Complex Geometry*. Springer–Verlag, 2005.

[9] Michael Atiyah and Raoul Bott. The moment map and equivariant cohomology. *Topology*, 23(1):1–28, 1984.