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Quantum Field Theory
Fall 2015 Seminar Notes

Anton Borissov, Henry Liu

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Chapter 0

Review

Let's review some notation and concepts from quantum mechanics and special relativity. Note that not all these concepts carry over to quantum field theory. For example, in quantum mechanics, we are always working with states within some Hilbert space, but in quantum field theory, there is no suitable Hilbert space.

0.1 Quantum Mechanics

The fundamentals of quantum mechanics have had almost a century to be formalized, and indeed they have been! Here we give a somewhat axiomatic presentation of QM.

Axiom 1 (States)**.** Let $\mathcal{H}$ be a (complex) Hilbert space. Its projectivization $\mathbb{P}\mathcal{H}$ is the **state space** of our system.

1. An element of $\mathcal{H}$, i.e. a state, is called a **ket**, and is written $|x\rangle$.
2. An element of $\mathcal{H}^*$, i.e. a functional, is called a **bra**. The bra associated to $|x\rangle$ (under the identification $\mathcal{H} \cong \mathcal{H}^*$ given by the inner product) is denoted $\langle x|$.

Consequently, $\langle x|x\rangle = \|x\|^2$, which we usually want to normalize to be 1.

Note that the symbol inside the ket or bra is somewhat arbitrary. For example, the states of a quantum harmonic oscillator are written $|n\rangle$, for $n \in \mathbb{N}$.

Given a space of states, we can look at the operators that act on the states. These operators must be unitary, so that normalized states go to normalized states.

Axiom 2 (Observables)**.** To every classical observable (i.e. property of a system) is associated a quantum operator, called an **observable**. Observables are (linear) self-adjoint operators whose (real!) eigenvalues are possible values of the corresponding classical property of the system. For example,

1. $\hat{H}$ is the **Hamiltonian** of the system, which classically represents the "total energy" (kinetic + potential) in the system,
2. $\hat{x}$ is the **position operator**,
3. $\hat{p}$ is the **momentum operator**.

The convention in QM is that observables are denoted by symbols with hats on them. The process of "moving" from a classical picture of a system to a quantum picture by making classical observables into

operators is called **quantization**, because the possible values of the observables are often quantized, i.e. made discrete, whereas previously they formed a continuum.

A classical observable is simple: it is just a function f defined on the classical phase space, so in order to make a measurement of the observable, we simply apply f to the current state of the system. In QM it is not as simple, in most part due to its inherently probabilistic nature. But it is still straightforward.

Axiom 3 (Measurement)**.** If $\hat{A}$ is the observable and $\hat{A}\,|k\rangle = a_k\,|k\rangle$, i.e. $|k\rangle$ is an eigenstate with eigenvalue $a_k \in \mathbb{R}$, then the probability of obtaining a_k as the value of the measurement on $|\psi\rangle$ is $|\,\langle k|\psi\rangle\,|^2$. But not only is the outcome probabilistic, the state of the system after the measurement is $|k\rangle$. In other words, **measurement is projection**. This is fundamental to QM and cannot be emphasized enough.

There are some conventions for position and momentum eigenstates. Since $\hat{x}$ and $\hat{p}$ are conventional symbols to use for position and momentum respectively, the states $|x\rangle$ and $|p\rangle$ are position and momentum eigenstates with eigenvalues x and p respectively.

What about states that we don't measure? What are they doing as time passes? We need to specify the **dynamics** of our system, and this is where the quantum analog of the Hamiltonian comes into play.

Axiom 4 (Dynamics)**.** The time-evolution of the state $|\psi\rangle$ is specified by the Hamiltonian $\hat{H}$ of the system, and is given by the **Schrödinger equation**

$$i\hbar\frac{d\,|\psi\rangle}{dt} = \hat{H}\,|\psi\rangle\,,$$

where $\hbar$ is Planck's constant (later we will be working in units where $\hbar = 1$). Note that we can solve this first-order ODE:

$$|\psi(t)\rangle = \exp(-i\hat{H}t)\,|\psi(0)\rangle\,.$$

The operator $U(t) = \exp(-i\hat{H}t)$ is known as the **time-evolution operator**.

That's it! There are some quick consequences of these axioms we should explore before moving on. First, although measurement is probabilistic, we often work with states whose observables tend to take on values clumped around a certain value, which corresponds to the classical value of that observable for the system. So given a state $|\psi\rangle$ and observable $\hat{A}$, it is reasonable to define the **expectation value** and **standard deviation**

$$\langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle\,, \qquad \Delta\hat{A} = \sqrt{\langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2}.$$

Proposition 0.1.1 (Heisenberg's uncertainty principle)**.** *Let $\hat{A}$ and $\hat{B}$ be self-adjoint operators. Then*

$$\Delta\hat{A}\Delta\hat{B} \geq \frac{1}{2}\left|\langle[\hat{A},\hat{B}]\rangle\right|.$$

Proof. Note that the variance can also be written

$$\Delta\hat{A} = \langle\psi|(\hat{A} - \langle A\rangle)^2|\psi\rangle\,.$$

Without loss of generality, assume $\langle\hat{A}\rangle = \langle\hat{B}\rangle = 0$, since we can shift $\hat{A}$ and $\hat{B}$ by constants without affecting $\Delta\hat{A}$ and $\Delta\hat{B}$. Then an application of Cauchy-Schwarz (using braket notation) gives

$$\Delta\hat{A}\Delta\hat{B} = \|\hat{A}\,|\psi\rangle\,\|\|\hat{B}\,|\psi\rangle\,\| \geq \left|\langle\psi|\hat{A}\hat{B}|\psi\rangle\right|.$$

Now note that if $z = \langle\psi|\hat{A}\hat{B}|\psi\rangle$, then $|z| \geq |\operatorname{im} z| = |z - z^*|/2$. Hence

$$\left|\langle\psi|\hat{A}\hat{B}|\psi\rangle\right| \geq \frac{1}{2}\left|\langle\psi|\hat{A}\hat{B}|\psi\rangle - \langle\psi|\hat{A}\hat{B}|\psi\rangle^*\right| = \frac{1}{2}\left|\langle\psi|\hat{A}\hat{B} - (\hat{A}\hat{B})^\dagger|\psi\rangle\right| = \frac{1}{2}\left|\langle\psi|[\hat{A},\hat{B}]|\psi\rangle\right|,$$

where the last equality follows from the observables being self-adjoint: $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger = \hat{B}\hat{A}$. □

For example, if we have a particle in $\mathbb{R}^n$, the Hilbert space underlying the state space is $\mathcal{H} = L^2(\mathbb{R}^n)$, and the position and momentum operators are given by

$$\hat{x} : \psi(x) \mapsto x\psi(x), \quad \hat{p} : \psi(x) \mapsto -i\hbar\nabla\psi(x).$$

A short calculation gives the **fundamental commutation relation** between $\hat{x}$ and $\hat{p}$:

$$[\hat{x}, \hat{p}] = i\hbar,$$

which we interpret as saying that we cannot know both the exact position and exact momentum of a particle at the same time.

0.2 Special Relativity

Special relativity describes the structure of spacetime. It says that spacetime is $\mathbb{R}^{1+3}$, known as **Minkowski space** (as opposed to $\mathbb{R}^4$, Euclidean space) and equipped with the **Minkowski metric**

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

where c is the speed of light (later we will work in units where $c = 1$). As with QM, there is a nice axiomatic presentation of SR, which is essentially just the following axiom.

Axiom 1 (Lorentz invariance)**.** The fundamental laws of physics must be invariant under isometries of Minkowski space. These isometries form the **Poincaré group** $\mathbb{R}^{1+3} \rtimes \mathrm{SO}(1,3)$. The subgroup $\mathrm{SO}(1,3)$ is known as the **Lorentz group**; its elements are called **Lorentz transformations**, and are precisely the isometries leaving the origin fixed.

So any Hamiltonian, Lagrangian, or physical expression we write down from now on had better be Lorentz invariant (we will usually work locally with nicely-behaved objects that are automatically invariant under the full Poincaré group if they are Lorentz invariant).

Along with special relativity, Einstein introduced his **summation notation** for tensors:

1. Components of (contravariant) vectors $\vec{v}$ are written with superscripts, i.e. $\vec{v} = v^1e_1 + \cdots + v^ne_n$, and those of (covariant) covectors with subscripts;
2. An index which appears both as a subscript and a superscript is implicitly summed over, i.e. $\vec{v} = v^ie_i$;
3. Unbound indices (the ones not summed over) must appear on both sides of an equation.

For example, $T^{\mu\alpha} = g^{\mu\nu}T^{\alpha}_{\nu}$ demonstrates contraction with the metric tensor. When there is superscript that should be a subscript, or vice versa, the metric tensor is implicitly being used to raise and lower indices.

There are several conventions regarding Einstein's summation notation. Spacetime variables are indexed by Greek letters, e.g. μ or ν, which run from 0 to 3, while space-only variables are indexed by Roman letters, e.g. i or j, which run from 1 to 3. Given a 4-vector $v = v^\nu e_\nu$, we let $\vec{v} = v^ie_i$ be the space-only component, and v^2 generally denotes $v^\mu v_\mu$ whereas $\vec{v}^2$ generally denotes v^iv_i.

Chapter 1

Klein-Gordon Field

1.1 Why Fields?

Volume 1 of Steven Weinberg's *Quantum Theory of Fields* is devoted to answering this question. A discussion of scattering experiments lead him to the S-matrix, and then to the local behaviour of experiments (which he calls the cluster decomposition principle), and then using Lorentz invariance, fields just practically fall out. Weinberg does a really good job of convincing us that QFT in some form or another really must exist if we assume Lorentz invariance and unitarity.

Peskin and Schroeder give a slightly different motivation, one that is closer to the historical reason of why fields were introduced:

- Single particle relativistic wave functions $\implies$ inconsistencies in theory (negative energy eigenstates) **ELABORATE**

  The Dirac equation comes from forcing the Schrödinger equation $i(d\Psi/dt) = \hat{H}\Psi$ to be Lorentz invariant. As it stands, it is first-order in time, but second-order in space. Suppose instead that

  $$\hat{H} = \frac{1}{i}\alpha^j \partial_j + m\beta.$$

  Since $E^2 = \vec{p}^2 + m^2$, we want $\hat{H}^2 = -\nabla^2 + m^2$, which gives

  $$\alpha^j\alpha^k + \alpha^k\alpha^j = 2\delta^{jk}, \quad \alpha^j\beta + \beta\alpha^j = 0, \quad \beta^2 = 1.$$

  Hence $\{\alpha^1, \alpha^2, \alpha^3, \beta\}$ are not scalars, but instead are the generators of a Clifford algebra; we take their simplest representation as matrices, which is as 4×4 complex matrices

  $$\alpha^j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

  where σ_j are the Pauli matrices. Now compute in momentum-space that

  $$\widehat{H\psi}(\vec{p}) = (-i\vec{p}\cdot\vec{\alpha} + m\beta)\hat{\psi}(\vec{p}) = \begin{pmatrix} mI & \vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -mI \end{pmatrix}\hat{\psi}(\vec{p}),$$

  and a straightforward calculation shows that $\hat{H}$ has eigenvalues $\pm\sqrt{\vec{p}^2 + m^2}$. In particular, the energy can be negative!

Dirac attempted to resolve this issue by appealing to the Pauli exclusion principle and positing that there existed a whole "sea of negative-energy states" that were already occupied. Consequently, the holes in this sea would be antiparticles. This makes sense until we realize that that a particle falling into a negative-energy state would represent particle-antiparticle annihilation, but the Dirac equation is supposed to be modeling a single particle (an electron, actually). So philosophical issues aside, there are technical issues here.

- $E = mc^2$ allows for particles to be created at high energies
- $\Delta E \cdot \Delta t = \hbar$ allows for virtual particles
- Causality violation. Set $H = \frac{\hat{p}^2}{2m}$

$$\begin{aligned} U(t) &= \langle \vec{x} | e^{-iHt} | \vec{x}_0 \rangle \\ &= \int \frac{d^3p}{(2\pi)^3} \langle \vec{x} | e^{-i(p^2/2m)t} | p \rangle \langle p | x \rangle \\ &= \int \frac{d^3p}{(2\pi)^3} e^{-i(p^2/2m)t} e^{i\vec{p}\cdot(\vec{x}-\vec{x}_0)} \\ &= \left(\frac{m}{2\pi i t}\right)^{3/2} e^{im(\vec{x}-\vec{x}_0)^2/2t} \end{aligned}$$

This last quantity is non-zero, even for arbitariry x that may by space-like separated.

QFT seems to solve all of these mysteries. One very good feature of the theory is that it predicts a lot of experiments to very high accuracy. QED is something we will see very soon that has been very well tested and agrees very well with experiments.

1.2 Elements of Classical Field Theory

1.2.1 Lagrangian Field Theory

- Fundamental quantity in Lagrangian field theory is the action S. In high school, the Lagrangian is a function of time, positions, and velocities of a system: $L(t, x(t), \dot{x}(t))$. The action is given by $S = \int dt\, L$. Fields can also be described in a Lagrangian formalism, for instance by considering every point in space-time as a "particle" that wiggles back and forth with the amplitude of wiggling characterizing the strength of the field.

  Let $\varphi : M \to \mathbb{R}$, define a Lagrangian *density* $\mathcal{L}(t, \varphi, \partial_\mu \varphi)$, the honest Lagrangian $L = \int d^3x \mathcal{L}$, and finally define the action:

$$S = \int dt\, L = \int d^4x \mathcal{L}$$

Four-vector notation:

- Greek letters $\mu, \nu, \ldots \in \{0, 1, 2, 3\}$
- Roman letters $i, g, \ldots \in \{1, 2, 3\}$.
- $x^\mu = (x^0, x^1, x^2, x^3)$
- Signature $(+ - - -)$
- $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$

– $\partial_\mu f = \frac{\partial f}{\partial x^\mu} = (\partial_0 f, \partial_1 f, \partial_2 f, \partial_3 f)$.

- Extremize the action. Let $\delta f = f(\varphi + \xi) - f(\varphi)$.

$$\begin{aligned} 0 = \delta S &= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \underbrace{\delta(\partial_\mu}_{\text{commute}} \varphi) \right) \\ &= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi \right) - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \delta\varphi \right] \end{aligned}$$

By Stokes' theorem, we can break this integral up into two parts, one of which is called the boundary term. Taking a variation that is fixed along the boundary means $\delta\varphi \equiv 0$ on the boundary which means that the boundary term does not contribute to δS. Moreover, if we take $\delta S = 0$ for every variation, then we obtain the Euler Lagrange equations:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) - \frac{\partial \mathcal{L}}{\partial \varphi} = 0$$

Remark. The Lagragian formalism is useful for relativistic dynamics because all expressions are chosen to Lorentz invariant.

1.2.2 Hamiltonian Field Theory

- Introducing this makes the transition to the quantum theory easier.
- High school Hamiltonian formalism: $p = \frac{\partial L}{\partial \dot{q}}, H = \sum p\dot{q} - L$.
- Pretend that $\vec{x}$ enumerates points on the lattice of space-time:

$$\begin{aligned} p(\vec{x}) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}(\vec{x})} &= \frac{\partial}{\partial \dot{\varphi}(\vec{x})} \int d^3y \, \mathcal{L}(\varphi(y), \dot{\varphi}(y)) \\ &\sim \frac{\partial}{\partial \dot{\varphi}(\vec{x})} \sum \mathcal{L}(\varphi(y), \dot{\varphi}(y)) d^3y \\ &= \frac{\partial \mathcal{L}}{\partial \dot{\varphi}(\vec{x})} d^3x \\ &\equiv \pi(\vec{x}) d^3x \end{aligned}$$

since each point on the lattice represents a different variable, so the derivative just picks out the one at $\vec{x}$. We call $\pi(\vec{x})$ the momentum *density*. Therefore the Hamiltonian looks like:

$$H = \int d^3x \, \left[\pi(\vec{x}) \dot{\varphi}(\vec{x}) - \mathcal{L} \right].$$

(See the stress-energy tensor part for another derivation of the Hamiltonian which falls out of Noether's theorem for being the conserved quantity under time translations.)

- **Important example:** Take $\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2} m^2 \varphi^2$. Euler-Lagrange equations become $\partial^\mu(\partial_\mu \varphi) + m^2\varphi = 0$ which is the Klein Gordon equation. The Hamiltonian becomes:

$$H = \int d^3x \mathcal{H} = \int d^3x \left[\underbrace{\frac{\pi^2}{2}}_{\text{moving in time}} + \underbrace{\frac{(\nabla\varphi)^2}{2}}_{\text{shearing in space}} + \underbrace{\frac{m^2\varphi^2}{2}}_{\text{existing at all}} \right]$$

1.2.3 Noether's Theorem - How to Compute Conserved Quantities

To every continuous transformation of the field we can assign an infinitesmal transformation:

$$\varphi(x) \to \varphi'(x) = \varphi(x) + \alpha \underbrace{\Delta\varphi(x)}_{\text{deformation}}$$

Transformations might also change the Lagrangians. The interplay between how the infinitesmal transformation changes the Lagrangian and the field is what gives rise to conserved quantities, or sometimes known as Noether charges.

$$\begin{aligned} \text{Symmetry} &\iff \text{Equations of motion – invariant} \\ &\iff \text{Action invariant (up to surface term)} \\ &\iff \mathcal{L}(x) \to \mathcal{L}(x) + \alpha\partial_m u\mathcal{J}^\mu(x) \end{aligned}$$

Taylor expanding the perturbation:

$$\begin{aligned} \Delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\varphi}\cdot\Delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\mu(\Delta\varphi) \\ &= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\Delta\varphi\right) + \left[\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right)\right] \\ &= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\Delta\varphi\right) \end{aligned}$$

Since we claimed that under the symmetry $\Delta\mathcal{L} = \partial_\mu\mathcal{J}^\mu$ we have the following relations:

$$\begin{aligned} j^\mu(x) &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\Delta\varphi - \mathcal{J}^\mu \\ \partial_\mu j^\mu &= 0 \\ \frac{\partial}{\partial t}j^0 &= \partial_i j^i \end{aligned}$$

Define the charge $Q = \int d^3x\ j^0$. Then, if we assume that space does not have boundary, Stokes' theorem implies that $\partial Q/\partial t = 0$. Often, j^0 is called the charge density, and j^μ is called the current density.

> Therefore to compute a conserved quantity we compare the deformation of the Lagrangian due to the φ changing with the deformation of the Lagrangian due to the symmetry transformation.

Examples:

1. $\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)^2$ has the following field symmetry, $\varphi \to \varphi + \alpha$, ie. $\Delta\varphi \equiv$ const. There is no change to the Lagrangian, so $j^\mu = \partial^\mu\varphi$.
2. Space-time transformation, $x^\mu \to x^\mu - a^\mu$, implies

$$\begin{aligned} \varphi(x) \to \varphi(x+a) &= \varphi(x) + a^\nu\partial_\nu\varphi(x) \\ \mathcal{L}(x) \to \mathcal{L}(x+a) &= \mathcal{L}(x) + a^\mu\partial_\mu\mathcal{L} \\ . \qquad &= \mathcal{L}(x) + a^\nu\partial_\mu(\delta^\mu_\nu\mathcal{L}) \end{aligned}$$

Therefore we write

$$T^\mu_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\nu\varphi - \delta^\mu_\nu\mathcal{L}$$

we get four separately conserved quantities. **CHECK**

This is called the stress-energy tensor or the energy-momentum tensor in various contexts. The $T^{\bullet 0}$ quantity gives rise to the Hamiltonian:

$$\int d^3x T^{00} = \int d^3x \mathcal{H} \equiv H$$

1.2.4 Summary of Computing Noether Charges

Field or coordinate transformation $\rightsquigarrow \{\Delta\phi, \Delta\mathcal{L}\} \rightsquigarrow j^\mu(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\Delta\varphi - \mathcal{J}^\mu \rightsquigarrow Q = \int j^0 d^3x$ conserved charge.

1.3 Quantizing Klein Gordon Field

Quantization of field theories involves two steps:

1. Promote ϕ and π to operators,
2. Specify commutation relations:

$$[\phi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y})$$
$$[\phi(\vec{x}), \phi(\vec{y})] = [\pi(\vec{x}), \pi(\vec{y})] = 0$$

Note that here we are in the Schrodinger picture, so that ϕ, π are operators independent of time. Next, we are going to introduce a basis that will diagonalize the Hamiltonian, and we shall express ϕ and π in terms of these operators.

ELABORATE

The exact form of the following is a little tricky to motivate (but this is done in Weinberg, chapter 5). One way to motivate this is to look at the Klein-Gordon equation in the Fourier representation. This is exactly the harmonic oscillator equation and so it is reasonable to assume that the operators ϕ, π can be manipulated in a similar way to obtain the following ansatz:

$$\phi(\vec{x}) =$$
$$\pi(\vec{x}) =$$