Mirror Symmetry
Summer 2016 Seminar Notes

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Contents

Chapter 1

Mathematical Preliminaries

The aim of this chapter is to give a brief review of the required mathematical background for mirror symmetry.

1.1 Cohomology Theories

Throughout this section, X is a complex manifold, and H_{dR} is de Rham cohomology. We examine the relationships between some common cohomology theories on X.

1.1.1 Sheaf Cohomology

All of our sheaves take values in abelian groups. Let $\mathcal{F}$ be a presheaf on X.

Definition 1.1.1. Recall the definition of a **presheaf** $\mathcal{F}$ on X:

1. (presheaf) every open set U in X is assigned an abelian group $\mathcal{F}(U)$, such that if $V \subseteq U$ are two open sets, there is a restriction map $-|_{U\to V} : \mathcal{F}(U) \to \mathcal{F}(V)$ compatible with inclusion, i.e. $(-|_{U\to V})|_{V\to W} = -|_{U\to W}$ for any $W \subseteq V \subseteq U$.

If in addition $\mathcal{F}$ satisfies the following two properties, it is a **sheaf**:

2. (locality) if $\{U_\alpha\}$ is an open cover of X and $f, g \in \mathcal{F}(X)$ such that $f|_{U\to U_\alpha} = g|_{U\to U_\alpha}$ for every U_α, then $f = g$;

3. (gluing) if $\{U_\alpha\}$ is an open cover of X and $f_\alpha \in \mathcal{F}(U_\alpha)$ for every U_α are elements agreeing on overlaps, i.e. such that $f_\alpha|_{U_\alpha\to U_\alpha\cap U_\beta} = f_\beta|_{U_\beta\to U_\alpha\cap U_\beta}$, then we can glue the f_α together to get $f \in \mathcal{F}(X)$, i.e. $f|_{X\to U_\alpha} = f_\alpha$ for every U_α.

Definition 1.1.2. Let $\mathcal{U} = \{U_\alpha\}$ be an **ordered open cover** of X, i.e. with a partial order such that if α and β are incomparable then $U_\alpha \cap U_\beta$ is empty. A p**-simplex** σ of $\mathcal{U}$ is a totally ordered collection of open sets $U_{\alpha_0}, \dots, U_{\alpha_p} \in \mathcal{U}$; we call $U_{\alpha_0,\dots,\alpha_p} := U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ its **support**, and often refer to σ by it instead. The k**-th boundary component** of a p-simplex $U_{\alpha_0,\dots,\alpha_p}$ is given by $\partial_k U_{\alpha_0,\dots,\alpha_p} := U_{\alpha_0,\dots,\hat{\alpha}_k,\dots,\alpha_p}$. Cochains are maps from simplices to sheaf sections, and form a cochain complex:

$$C^p(\mathcal{U}, \mathcal{F}) := \prod_{\alpha_0<\cdots<\alpha_p} \mathcal{F}(U_{\alpha_0,\dots,\alpha_p}), \quad (\delta^p\omega)(\sigma) := \sum_{k=0}^{p+1} (-1)^k \omega(\partial_k\sigma)|_\sigma : C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F}).$$

The **Čech cohomology of** $\mathcal{U}$ with coefficients in $\mathcal{F}$, denoted $\check{H}^\bullet(\mathcal{U}, \mathcal{F})$, is the cohomology of this complex.

Example 1.1.3. Let $\mathcal{F}$ be a sheaf. By the gluing condition for a sheaf, a global section $f \in \mathcal{F}(X)$ is defined by its values $f_\alpha := f|_{X \to U_\alpha} \in \mathcal{F}(U_\alpha)$ on every U_α in an open cover. These f_α form precisely the data for an element of $C^0(\mathcal{U}, \mathcal{F})$, and satisfy the gluing condition $f_\alpha = f_\beta$ on $U_\alpha \cap U_\beta$, which is precisely the statement $\delta_0 f = 0$. Hence $\check{H}^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$ for a sheaf $\mathcal{F}$.

Example 1.1.4. Let $\mathcal{O}$ denote the sheaf of holomorphic functions (and $\mathcal{O}^*$ the nowhere-zero ones) on $\mathbb{P}^1$. Recall that on $\mathbb{P}^1$ we have the charts $U = \mathbb{P}^1 \setminus \{S\}$ and $V = \mathbb{P}^1 \setminus \{N\}$, with coordinates u and v respectively. To look at sections of $\mathcal{O}(U)$ versus $\mathcal{O}(V)$, we use the transition map $v = u^{-1}$ on $U \cap V$. The cochains for this open cover are

$$C^0(\mathcal{U}, \mathcal{O}) = \mathcal{O}(U) \times \mathcal{O}(V), \quad C^1(\mathcal{U}, \mathcal{O}) = \mathcal{O}(U \cap V), \quad C^k(\mathcal{U}, \mathcal{O}) = 0 \; \forall k \geq 2.$$

We compute the sheaf cohomology.

1. ($\check{H}^0(\mathcal{U}, \mathcal{O})$) The boundary map δ_0 maps $(f, g) \in C^0(\mathcal{U}, \mathcal{O})$ to $g - f$. But

   $$f = \sum_{k=0}^{\infty} f_k u^k, \quad g = \sum_{k=0}^{\infty} g_k v^k = \sum_{k=0}^{\infty} g_k u^{-k},$$

   so $g - f = 0$ iff $f_k = g_k = 0$ for all $k > 0$, and $f_0 = g_0$. Hence $\check{H}^0(\mathcal{U}, \mathcal{O}) \cong \mathbb{C}$, consisting of all constant functions.

2. ($\check{H}^1(\mathcal{U}, \mathcal{O})$) Given $h \in C^1(\mathcal{U}, \mathcal{O})$, rewrite its Laurent expansion:

   $$h = \sum_{k=-\infty}^{\infty} h_k u^k = \sum_{k=0}^{\infty} h_k u^k + \sum_{k=1}^{\infty} h_k v^k = -f + g$$

   where $f \in \mathcal{O}(U)$ and $g \in \mathcal{O}(V)$. Hence $h \in \operatorname{im} \delta_0$, and $\check{H}^1(\mathcal{U}, \mathcal{O}) = 0$.

3. ($\check{H}^k(\mathcal{U}, \mathcal{O})$) Trivially, $\check{H}^k(\mathcal{U}, \mathcal{O}) = 0$ for $k \geq 2$.

Note that $\check{H}^0(\mathcal{U}, \mathcal{O}) \cong \mathbb{C}$ is consistent with what we know so far, since $\check{H}^0(\mathcal{U}, \mathcal{O}) = \mathcal{O}(\mathbb{P}^1)$, and Liouville's theorem shows that $\mathcal{O}(\mathbb{P}^1)$ can only contain constant functions.

Example 1.1.5. Recall the tautological line bundle $\mathcal{O}(-1)$ and its dual $\mathcal{O}(1)$ on $\mathbb{P}^n$; we have $\mathcal{O}(n) = \mathcal{O}(1)^n$. On the same charts on $\mathbb{P}^1$, since $\mathcal{O}(1)$ has transition function $u = v^{-1}$, we know $\mathcal{O}(n)$ has transition function $u^n = v^{-n}$. To construct a global section of $\mathcal{O}(n)$, given a monomial v^k on V, we require $u^n v^k = u^{n-k}$ to be well-defined on U, so $k \leq n$. In homogeneous coordinates $[x_0 : x_1]$, the global sections are therefore $x_0^n, x_0^{n-1}x_1, \ldots, x_1^n$, the homogeneous polynomials of degree n. The same story holds on $\mathbb{P}^N$. Hence $\dim H^0(\mathbb{P}^N, \mathcal{O}(n)) = \binom{N+n-1}{n-1}$. In particular, there are $\binom{9}{5} = 126$ independent global sections of $\mathcal{O}_{\mathbb{P}^4}(5)$.

Definition 1.1.6. The set of all open covers of X form a directed set under refinement. The **Čech cohomology of** X with coefficients in $\mathcal{F}$ is the direct limit $\check{H}^n(X, \mathcal{F}) := \varinjlim_{\mathcal{U}} \check{H}^n(\mathcal{U}, \mathcal{F})$.

Definition 1.1.7. An ordered open cover $\{U_\alpha\}$ is **good** if it is countable and every finite intersection $U_{\alpha_0, \ldots, \alpha_p}$ is either empty or contractible.

Theorem 1.1.8 ([1, Corollary of Theorem 5.4.1])**.** *The Čech cohomology of a good cover $\mathcal{U}$ is isomorphic to the Čech cohomology of X.*

One can define **sheaf cohomology** $H^n(X, \mathcal{F})$ as the right derived functors of the global sections functor Γ_X (i.e. $\mathcal{F} \mapsto \mathcal{F}(X)$). For us, Čech and sheaf cohomology are indistinguishable as long as we work with sheaves.

Theorem 1.1.9 ([1, Theorem 5.10.1])**.** *If X is a paracompact topological space, then Čech cohomology $\check{H}^n(X, \mathcal{F})$ and sheaf cohomology $H^n(X, \mathcal{F})$ are isomorphic for any sheaf $\mathcal{F}$.*

Čech cohomology is also directly related to de Rham cohomology, and, as we shall see, Dolbeault cohomology in the complex case. So we can think of Čech cohomology classes as forms.

Theorem 1.1.10 (Čech–de Rham isomorphism)**.** *Let* $\mathbb{R}$ *denote the constant sheaf. There is a canonical isomorphism* $\check{H}^k(X,\mathbb{R}) \cong H^k_{\mathrm{dR}}(X)$ *for each* k*.*

Proof. By the Poincaré lemma, the **de Rham complex** of sheaves

$$0 \to \mathbb{R} \xrightarrow{\subseteq} \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \Omega^2(X) \xrightarrow{d} \cdots$$

is exact (by checking exactness on the stalks). Let $Z^n := \ker(d\colon \Omega^k \to \Omega^{k+1})$. The de Rham complex splits into a bunch of short exact sequences:

$$0 \to d\Omega^{k-1} \cong Z^k \xrightarrow{\subseteq} \Omega^k \xrightarrow{d} Z^{k+1} \to 0.$$

To each such short exact sequence is associated a long exact sequence of (sheaf) cohomology:

$$0 \to H^0(X,Z^k) \to H^0(X,\Omega^k) \to H^0(X,Z^{k+1}) \to H^1(X,Z^k) \to H^1(X,\Omega^k) \to H^1(X,Z^{k+1}) \to \cdots.$$

Fact: $H^i(X,\Omega^k) = 0$ for every k and $i > 0$ (since Ω^k is a fine sheaf). Hence we obtain isomorphisms

$$H^{i+1}(X,Z^0) \cong H^i(X,Z^1) \cong \cdots \cong H^1(X,Z^i).$$

But $Z^0 \cong \mathbb{R}$ and we more commonly write

$$H^1(X,Z^i) = \operatorname{coker}(H^0(X,\Omega^i) \to H^0(X,Z^{i+1})) = Z^{i+1}(X)/d\Omega^k(X) = H^{i+1}_{\mathrm{dR}}(X).$$

Hence $\check{H}^{i+1}(X,\mathbb{R}) \cong H^{i+1}_{\mathrm{dR}}(X)$. □

Definition 1.1.11. Let E be a holomorphic vector bundle on X and $\Omega^{0,q}(E) := \Omega^{0,q}(X) \otimes \Gamma(E)$ denote the space of E-valued $(0,q)$-forms. **Dolbeault cohomology** $H^{p,q}(E)$ is the cohomology of the complex

$$\cdots \xrightarrow{\bar\partial} \Omega^{0,q}(E) \xrightarrow{\bar\partial} \Omega^{0,q+1}(E) \xrightarrow{\bar\partial} \Omega^{0,q+2}(E) \xrightarrow{\bar\partial} \cdots.$$

We write $H^{p,q}(X)$ for $E = \Lambda^p T^{1,0}X$. The dimensions $h^{p,q}(E) := \dim_{\mathbb{C}} H^{p,q}(E)$ are the **Hodge numbers** of X with respect to E.

Lemma 1.1.12 ($\bar\partial$-Poincaré lemma)**.** *$\bar\partial$-closed, i.e. holomorphic, implies $\bar\partial$-exact on* $\mathbb{C}^n$*.*

Theorem 1.1.13 (Čech–Dolbeault isomorphism)**.** *Let* $\Omega^{p,0}$ *denote the sheaf of holomorphic p-forms on* X*. There is a natural isomorphism* $\check{H}^q(X,\Omega^{p,0}) \cong H^{p,q}(X)$*.*

Proof. Analogous to the proof of the Čech–de Rham isomorphism, except now using the $\bar\partial$-Poincaré lemma to establish the exactness of the complex

$$0 \to \ker(\bar\partial\colon \Omega^{p,0}(X) \to \Omega^{p,1}(X)) \xrightarrow{\subseteq} \Omega^{p,0}(X) \xrightarrow{\bar\partial} \Omega^{p,1}(X) \xrightarrow{\bar\partial} \cdots.$$ □

Definition 1.1.14. Consider the double complex $\Omega^{\bullet,\bullet}$ with differentials ∂ and $\bar\partial$. The **Frölicher spectral sequence** is the spectral sequence of a double complex associated to $\Omega^{\bullet,\bullet}$. Since the total complex of $\Omega^{\bullet,\bullet}$ is $\Omega^\bullet(X)$, the Frölicher spectral sequence converges to complex de Rham cohomology $H^\bullet_{\mathrm{dR}}(X,\mathbb{C})$.

1.1.2 Morse Homology

Throughout this subsection, $f\colon X \to \mathbb{R}$ is a smooth function, and we equip X, viewed as a real manifold, with a Riemannian metric g. We also assume (f, g) is Morse–Smale, defined below.

Definition 1.1.15. A **critical point** of f is a point $p \in X$ with $df_p = 0$. Define the **Hessian**

$$H(f)_p\colon T_pX \to T_p^*X, \quad v \mapsto \nabla_v(df),$$

which is independent of the choice of connection ∇. (In coordinates, we recover the usual $\partial^2 f/\partial x_i \partial x_j$.) The critical point p is **non-degenerate** if the Hessian does not have zero eigenvalues. A non-degenerate critical point p has **Morse index** $\operatorname{ind}(p)$ the number of negative eigenvalues of the Hessian. The function f is **Morse** if all of its critical points are non-degenerate.

Definition 1.1.16. Recall that the **gradient** of f with respect to a metric g is the vector field $\operatorname{grad} f$ such that $g(\operatorname{grad} f, X) = Xf$. Equivalently, $\operatorname{grad} f = (df)^\sharp$. Let $\psi_t\colon X \to X$ be the one-parameter group of diffeomorphisms associated to the flow of $-\operatorname{grad} f$. The **descending manifold** $D(p)$ and **ascending manifold** $A(p)$ at a critical point p are

$$\begin{aligned} D(p) &\coloneqq \{x \in X : \lim_{t\to-\infty} \psi_t(x) = p\} \\ A(p) &\coloneqq \{x \in X : \lim_{t\to+\infty} \psi_t(x) = p\}. \end{aligned}$$

The pair (f, g) is **Morse–Smale** if f is Morse and $D(p)$ is transverse to $A(q)$ for every pair of critical points p and q (i.e. tangent spaces of $D(p)$ and $A(q)$ generate the tangent space at every intersection point).

Here are two useful and easy-to-prove facts: every flow line asymptotically approaches critical points, and $\dim D(p) = \operatorname{ind}(p)$ (so $\dim A(p) = \dim X - \operatorname{ind} p$ by the Morse–Smale condition).

Definition 1.1.17. Fix critical points p and q. A **flow line** from p to q is an integral curve $\gamma(t)$ of $-\operatorname{grad} f$ with $\lim_{t\to-\infty} \gamma(t) = p$ and $\lim_{t\to+\infty} \gamma(t) = q$. The **moduli space of flow lines** from p to q is

$$\begin{aligned} \mathcal{M}(p, q) &\coloneqq \{\text{flow lines from } p \text{ to } q\}/\sim, \quad \alpha \sim \beta \text{ if } \exists c \in \mathbb{R} : \alpha(t) = \beta(t + c) \\ &= (D(p) \cap A(q))/\mathbb{R}. \end{aligned}$$

A **broken flow line** consists, piecewise, of flow lines.

The Morse–Smale condition implies $D(p) \cap A(q)$ is a submanifold of X with dimension $\operatorname{ind}(p) - \operatorname{ind}(q)$. Since $\sim$ is a smooth, proper, free $\mathbb{R}$-action, $\mathcal{M}(p, q)$ is a manifold of dimension $\operatorname{ind}(p) - \operatorname{ind}(q) - 1$ when $p \neq q$ (otherwise the $\mathbb{R}$-action is trivial). Note that if $\operatorname{ind}(p) = k$ and $\operatorname{ind}(q) = k - 1$ then $\mathcal{M}(p, q)$ is zero-dimensional. In fact, in this case, $\mathcal{M}(p, q)$ is compact as a corollary of the following theorem, and therefore is a finite set of points.

Theorem 1.1.18 ([2, Theorem 2.1])**.** *Let X be closed and (f, g) Morse–Smale. Then $\mathcal{M}(p, q)$ has a natural compactification to a smooth manifold with corners $\overline{\mathcal{M}(p, q)}$ where*

$$\overline{\mathcal{M}(p, q)} \setminus \mathcal{M}(p, q) = \bigcup_{k\geq 1} \bigcup_{\substack{p, r_1, \ldots, r_k, q \\ \textit{distinct crit pts}}} \mathcal{M}(p, r_1) \times \mathcal{M}(r_1, r_2) \times \cdots \times \mathcal{M}(r_{k-1}, r_k) \times \mathcal{M}(r_k, q).$$

Corollary 1.1.19. *If $\operatorname{ind}(p) - \operatorname{ind}(q) = 1$, then $\overline{\mathcal{M}(p, q)} = \mathcal{M}(p, q)$ is compact. If $\operatorname{ind}(p) - \operatorname{ind}(q) = 2$, then*

$$\partial\overline{\mathcal{M}(p, q)} = \bigcup_{\operatorname{ind}(r)=\operatorname{ind}(p)-1} \mathcal{M}(p, r) \times \mathcal{M}(r, q).$$

Proof. Since $\dim \mathcal{M}(r, s) = \operatorname{ind}(r) - \operatorname{ind}(s) - 1$, the space $\mathcal{M}(r, s)$ is non-empty only if $\operatorname{ind}(r) - \operatorname{ind}(s) \geq 1$. Hence $\overline{\mathcal{M}(p, q)} \setminus \mathcal{M}(p, q) = \emptyset$ when $\operatorname{ind}(p) - \operatorname{ind}(q) = 1$. Similar reasoning shows the $\operatorname{ind}(p) - \operatorname{ind}(q) = 2$ case. ∎

Definition 1.1.20. Fix orientations for $D(p)$ at every critical point p. There is an isomorphism at $x \in \gamma \in \mathcal{M}(p,q)$ given by

$$\begin{aligned}
T_x D(p) &\cong T_x(D(p) \cap A(q)) \oplus (T_x X / T_x A(q)) && \text{transversality from Morse–Smale} \\
&\cong T_\gamma \mathcal{M}(p,q) \oplus T_x\gamma \oplus (T_x X / T_x A(q)) && \text{definition of } \mathcal{M}(p,q) \\
&\cong T_\gamma \mathcal{M}(p,q) \oplus T_x\gamma \oplus T_q D(q) && \text{translating } T_q D(q) \text{ along } \gamma.
\end{aligned}$$

The **orientation** on $\mathcal{M}(p,q)$ is such that this isomorphism is orientation-preserving. Let C_k be the free abelian group generated by critical points of index k, and define the **Morse–Smale–Witten boundary map**

$$\partial_k^{\mathrm{Morse}} \colon C_k \to C_{k-1}, \quad p \mapsto \sum_{\operatorname{ind} q = k-1} \#\mathcal{M}(p,q) q$$

where $\#\mathcal{M}(p,q) \in \mathbb{Z}$ is counted with sign according to the orientation of $\mathcal{M}(p,q)$, which here is a discrete set of points.

Lemma 1.1.21. *$(\partial_k^{\mathrm{Morse}})^2 = 0$, so $(C_\bullet, \partial^{\mathrm{Morse}})$ is a chain complex.*

Proof. Let $\operatorname{ind}(p) - \operatorname{ind}(q) = 2$. The coefficient of q in $(\partial^{\mathrm{Morse}})^2 p$ is

$$\sum_{\operatorname{ind}(r) = \operatorname{ind}(p)-1} \#\mathcal{M}(p,r) \cdot \#\mathcal{M}(r,q) = \# \bigcup_{\operatorname{ind}(r) = \operatorname{ind}(p)-1} \mathcal{M}(p,r) \times \mathcal{M}(r,q) = \#\partial\overline{\mathcal{M}(p,q)}.$$

Since $\overline{\mathcal{M}(p,q)}$ is an oriented 1-manifold with boundary, this quantity, the number of boundary points, is zero. ☐

Definition 1.1.22. **Morse homology** $H_\bullet^{\mathrm{Morse}}(f,g)$ is the homology of the **Morse–Smale–Witten complex** $(C_\bullet, \partial^{\mathrm{Morse}})$.

Example 1.1.23. The (upright) torus T^2 has four critical points with f the height function: p (index 2), q and r (index 1), and s (index 0). This choice of f is Morse, but with the induced metric g from $\mathbb{R}^3$, the pair (f,g) is not Morse–Smale: $D(q) \cap A(r)$ is non-empty, but transversality forces it to be. The solution is to tilt the torus a little; equivalently, perturb g. There are two flow lines, of opposite sign, for each relevant pair of critical points. Hence $\partial_k^{\mathrm{Morse}} = 0$ for $k = 1, 2$. It follows that

$$H_2^{\mathrm{Morse}}(f,g) = \mathbb{Z}, \quad H_1^{\mathrm{Morse}}(f,g) = \mathbb{Z}^2, \quad H_0^{\mathrm{Morse}}(f,g) = \mathbb{Z}.$$

Theorem 1.1.24 ([2, Theorem 3.1])**.** *Let X be a closed smooth manifold, $H_\bullet(X)$ denote singular homology on X, and (f,g) be a Morse–Smale pair on X. Then there is a canonical isomorphism $H_n^{\mathrm{Morse}}(f,g) \cong H_n(X)$.*

Corollary 1.1.25. *The number of critical points of a Morse function is at least the sum $\sum_k \dim H_k(X)$ of the Betti numbers.*

Proof. The number of critical points is the sum of the dimensions of the Morse chain groups, which is at least the sum of the dimensions of the Morse homology groups, which is equal to the sum of the dimensions of the singular homology groups. ☐

The infinite-dimensional analogue of Morse homology is known as **Floer homology**. We shall primarily be concerned with Floer homology for mirror symmetry.

1.1.3 Equivariant Cohomology

1.2 Differential Topology

We stop distinguishing between isomorphic cohomology theories now. In particular, since X is always at least a smooth manifold, we think of singular cohomology $H^k(X)$ as de Rham cohomology.

1.2.1 Poincaré Duality

Unless otherwise stated, X in this section is a compact oriented n-manifold.

Theorem 1.2.1 (Poincaré duality, [4]). *Let X be a compact oriented n-manifold. The map*

$$\int_X : H^k(X) \otimes H^{n-k}(X) \to \mathbb{R}, \quad \omega \otimes \eta \mapsto \int_X \omega \wedge \eta$$

is a perfect pairing, and hence $H^k(X) \cong H^{n-k}(X)^*$.

If we relax the assumption that X is compact, then the issue is that $\int_X$ may not be well-defined. We work around this by using de Rham cohomology with compact support.

Definition 1.2.2. Let $\Omega_c^k(X)$ denote the k-forms on X with compact support. The **de Rham cohomology groups with compact support** $H_c^n(X)$ are the cohomology of the chain complex $(\Omega_c^\bullet(X), d)$.

Theorem 1.2.3 (Poincaré duality for non-compact manifolds, [4]). *Let X be an oriented n-manifold without boundary. The map*

$$\int_X : H^k(X) \otimes H_c^{n-k}(X) \to \mathbb{R}, \quad \omega \otimes \eta \mapsto \int_X \omega \wedge \eta$$

is a perfect pairing, and hence $H^k(X) \cong H_c^{n-k}(X)^*$.

Definition 1.2.4. Fix $C \subset X$ a closed $(n-k)$-submanifold. Then Poincaré duality identifies the map $\int_C : H^{n-k}(X) \to \mathbb{R}$ with a k-form $\eta_C \in H^k(X)$, called the **Poincaré dual class**. Explicitly, $\int_C \omega = \int_X \omega \wedge \eta_C$.

There is a relation between the Poincaré dual class and the Thom class, which we define below. Namely, the Poincaré dual class of C can be constructed as the Thom class of the normal bundle of C in X.

Theorem 1.2.5 ([3, Theorem 10.4]). *Let $\pi : E \to B$ be an oriented rank-n real vector bundle and B is embedded into E as the zero section. Then*

1. *there exists a unique cohomology class $\Phi \in H^n(E, E \setminus B)$ called the* **Thom class** *such that for every $x \in B$, the restriction of u to $H^n(E_x, E_x \setminus \{0\})$ is the preferred generator specified by the orientation of E_x in E;*
2. *the* **Thom isomorphism** *: $H^k(E) \to H^{k+n}(E, E \setminus B)$, given by $\omega \mapsto \omega \wedge \Phi$, is an isomorphism for every k.*

Note that since B is a deformation retract of E, the rings $H^*(E)$ and $H^*(B)$ are isomorphic. Hence $\pi^*\Phi = 1 \in H^*(B)$, which shall be very important in the upcoming proof.

Theorem 1.2.6 (Tubular neighborhood theorem, [3, Theorem 11.1]). *Let $C \subset X$ be a k-submanifold embedded in X. There exists an open neighborhood, called a* **tubular neighborhood***, of C in X diffeomorphic to the total space of the normal bundle of C. This diffeomorphism maps points in C to zero vectors.*

Proposition 1.2.7 ([4, Proposition 6.24a]). *Let $C \subset X$ be a closed $(n-k)$-submanifold. The Poincaré dual class $\eta_C \in H^k(X)$ of C is the Thom class of the normal bundle of C in X.*

Proof. Let NC denote the normal bundle of C in X, which has rank k because C is codimension k. Use the tubular neighborhood theorem to identify NC with an open neighborhood T of C in X, and then extend by zero to get $\Phi \in H^k(X)$ supported on T.

We shall show that $\int_X \omega \wedge \Phi = \int_C \omega$ for any $\omega \in H_c^{n-k}(X)$. The maps $\pi\colon T \to C$ and $\iota\colon C \to T$ induce isomorphisms of cohomology, so on forms ω and $\pi^*\iota^*\omega$ differ by at most an exact form $d\tau$. Then

$$\begin{aligned}\int_X \omega \wedge \Phi &= \int_T \omega \wedge \Phi = \int_T (\pi^*\iota^*\omega + d\tau) \wedge \Phi \\ &= \int_T \pi^*\iota^*\omega \wedge \Phi = \int_C \iota^*\omega \wedge \pi^*\Phi = \int_C \iota^*\omega. \qquad \square\end{aligned}$$

Corollary 1.2.8. *Transverse intersection is Poincaré dual to the wedge product, i.e. for $C, D \subset X$ closed submanifolds intersecting transversally, $\eta_{C\cap D} = \eta_C \wedge \eta_D$.*

Proof. For transversal intersections, codimension is additive: $\operatorname{codim} C \cap D = \operatorname{codim} C + \operatorname{codim} D$. So the normal bundle of the intersection is $N(C \cap D) = NC \oplus ND$. Let $\Phi(E)$ denote the Thom class associated to the vector bundle E. By the characterization of the Thom class, for vector bundles E and F we have $\Phi(E \oplus F) = \Phi(E) \wedge \Phi(F)$; check that $\Phi(E) \oplus \Phi(F)$ restricts on each fiber to the preferred generator. Hence

$$\eta_{C\cap D} = \Phi(N_{C\cap D}) = \Phi(NC \oplus ND) = \Phi(NC) \wedge \Phi(ND) = \eta_C \wedge \eta_D. \qquad \square$$

1.2.2 Serre Duality

1.2.3 Characteristic Classes

1.2.4 The Grothendieck–Riemann–Roch Formula

1.3 Calabi–Yau Manifolds

1.4 Toric Geometry

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