Quantum Field Theory
Fall 2015 Seminar Notes

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Chapter 0

Review

Let's review some notation and concepts from quantum mechanics and special relativity. Note that not all these concepts carry over to quantum field theory. For example, in quantum mechanics, we are always working with states within some Hilbert space, but in quantum field theory, there is no suitable Hilbert space.

0.1 Quantum Mechanics

The fundamentals of quantum mechanics have had almost a century to be formalized, and indeed they have been! Here we give a somewhat axiomatic presentation of QM.

Axiom 1 (States)**.** Let $\mathcal{H}$ be a (complex) Hilbert space. Its projectivization $\mathbb{P}\mathcal{H}$ is the **state space** of our system.

- An element of $\mathcal{H}$, i.e. a state, is called a **ket**, and is written $|x\rangle$.
- An element of $\mathcal{H}^*$, i.e. a functional, is called a **bra**. The bra associated to $|x\rangle$ (under the identification $\mathcal{H} \cong \mathcal{H}^*$ given by the inner product) is denoted $\langle x|$.

Consequently, $\langle x|x\rangle = \|x\|^2$, which we usually want to normalize to be 1.

Note that the symbol inside the ket or bra is somewhat arbitrary. For example, the states of a quantum harmonic oscillator are written $|n\rangle$, for $n \in \mathbb{N}$.

Given a space of states, we can look at the operators that act on the states. These operators must be unitary, so that normalized states go to normalized states.

Axiom 2 (Observables)**.** To every classical observable (i.e. property of a system) is associated a quantum operator, called an **observable**. Observables are (linear) self-adjoint operators whose (real!) eigenvalues are possible values of the corresponding classical property of the system. For example,

- $\hat{H}$ is the **Hamiltonian** of the system, which classically represents the "total energy" (kinetic + potential) in the system,
- $\hat{x}$ is the **position operator**,
- $\hat{p}$ is the **momentum operator**.

The convention in QM is that observables are denoted by symbols with hats on them. The process of "moving" from a classical picture of a system to a quantum picture by making classical observables into operators is called **quantization**, because the possible values of the observables are often quantized, i.e. made discrete, whereas previously they formed a continuum.

A classical observable is simple: it is just a function f defined on the classical phase space, so in order to make a measurement of the observable, we simply apply f to the current state of the system. In QM it is not as simple, in most part due to its inherently probabilistic nature. But it is still straightforward.

Axiom 3 (Measurement)**.** If $\hat{A}$ is the observable and $\hat{A}\left|k\right\rangle = a_k \left|k\right\rangle$, i.e. $\left|k\right\rangle$ is an eigenstate with eigenvalue $a_k \in \mathbb{R}$, then the probability of obtaining a_k as the value of the measurement on $\left|\psi\right\rangle$ is $|\left\langle k|\psi\right\rangle|^2$. But not only is the outcome probabilistic, the state of the system after the measurement is $\left|k\right\rangle$. In other words, **measurement is projection**. This is fundamental to QM and cannot be emphasized enough.

There are some conventions for position and momentum eigenstates. Since $\hat{x}$ and $\hat{p}$ are conventional symbols to use for position and momentum respectively, the states $\left|x\right\rangle$ and $\left|p\right\rangle$ are position and momentum eigenstates with eigenvalues x and p respectively.

What about states that we don't measure? What are they doing as time passes? We need to specify the **dynamics** of our system, and this is where the quantum analog of the Hamiltonian comes into play.

Axiom 4 (Dynamics)**.** The time-evolution of the state $\left|\psi\right\rangle$ is specified by the Hamiltonian $\hat{H}$ of the system, and is given by the **Schrödinger equation**

$$i\hbar \frac{d\left|\psi\right\rangle}{dt} = \hat{H}\left|\psi\right\rangle,$$

where $\hbar$ is Planck's constant (later we will be working in units where $\hbar = 1$). Note that we can solve this first-order ODE:

$$\left|\psi(t)\right\rangle = \exp(-i\hat{H}t)\left|\psi(0)\right\rangle.$$

The operator $U(t) = \exp(-i\hat{H}t)$ is known as the **time-evolution operator**.

That's it! There are some quick consequences of these axioms we should explore before moving on. First, although measurement is probabilistic, we often work with states whose observables tend to take on values clumped around a certain value, which corresponds to the classical value of that observable for the system. So given a state $\left|\psi\right\rangle$ and observable $\hat{A}$, it is reasonable to define the **expectation value** and **standard deviation**

$$\langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle, \qquad \Delta\hat{A} = \sqrt{\langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2}.$$

Proposition 0.1.1 (Heisenberg's uncertainty principle)**.** *Let $\hat{A}$ and $\hat{B}$ be self-adjoint operators. Then*

$$\Delta\hat{A}\Delta\hat{B} \geq \frac{1}{2}\left|\langle[\hat{A},\hat{B}]\rangle\right|.$$

Proof. Note that the variance can also be written

$$\Delta\hat{A} = \langle\psi|(\hat{A} - \langle A\rangle)^2|\psi\rangle.$$

Without loss of generality, assume $\langle\hat{A}\rangle = \langle\hat{B}\rangle = 0$, since we can shift $\hat{A}$ and $\hat{B}$ by constants without affecting $\Delta\hat{A}$ and $\Delta\hat{B}$. Then an application of Cauchy-Schwarz (using braket notation) gives

$$\Delta\hat{A}\Delta\hat{B} = \|\hat{A}\left|\psi\right\rangle\|\|\hat{B}\left|\psi\right\rangle\| \geq \left|\langle\psi|\hat{A}\hat{B}|\psi\rangle\right|.$$

Now note that if $z = \langle\psi|\hat{A}\hat{B}|\psi\rangle$, then $|z| \geq |\operatorname{Im} z| = |z - z^*|/2$. Hence

$$\left|\langle\psi|\hat{A}\hat{B}|\psi\rangle\right| \geq \frac{1}{2}\left|\langle\psi|\hat{A}\hat{B}|\psi\rangle - \langle\psi|\hat{A}\hat{B}|\psi\rangle^*\right| = \frac{1}{2}\left|\langle\psi|\hat{A}\hat{B} - (\hat{A}\hat{B})^\dagger|\psi\rangle\right| = \frac{1}{2}\left|\langle\psi|[\hat{A},\hat{B}]|\psi\rangle\right|,$$

where the last equality follows from the observables being self-adjoint: $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger = \hat{B}\hat{A}$. ∎

For example, if we have a particle in $\mathbb{R}^n$, the Hilbert space underlying the state space is $\mathcal{H} = L^2(\mathbb{R}^n)$, and the position and momentum operators are given by

$$\hat{x} : \psi(x) \mapsto x\psi(x), \quad \hat{p} : \psi(x) \mapsto -i\hbar\nabla\psi(x).$$

A short calculation gives the **fundamental commutation relation** between $\hat{x}$ and $\hat{p}$:

$$[\hat{x}, \hat{p}] = i\hbar,$$

which we interpret as saying that we cannot know both the exact position and exact momentum of a particle at the same time.

0.2 Special Relativity

Special relativity describes the structure of spacetime. It says that spacetime is $\mathbb{R}^{1+3}$, known as **Minkowski space** (as opposed to $\mathbb{R}^4$, Euclidean space) and equipped with the **Minkowski metric**

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

where c is the speed of light (later we will work in units where $c = 1$). As with QM, there is a nice axiomatic presentation of SR, which is essentially just the following axiom.

Axiom 1 (Lorentz invariance)**.** The fundamental laws of physics must be invariant under isometries of Minkowski space. These isometries form the **Poincaré group** $\mathbb{R}^{1+3} \rtimes \mathrm{SO}(1,3)$. The subgroup $\mathrm{SO}(1,3)$ is known as the **Lorentz group**; its elements are called **Lorentz transformations**, and are precisely the isometries leaving the origin fixed.

So any Hamiltonian, Lagrangian, or physical expression we write down from now on had better be Lorentz invariant (we will usually work locally with nicely-behaved objects that are automatically invariant under the full Poincaré group if they are Lorentz invariant).

Along with special relativity, Einstein introduced his **summation notation** for tensors:

- Components of (contravariant) vectors $\vec{v}$ are written with superscripts, i.e. $\vec{v} = v^1e_1 + \cdots + v^ne_n$, and those of (covariant) covectors with subscripts;
- An index which appears both as a subscript and a superscript is implicitly summed over, i.e. $\vec{v} = v^ie_i$;
- Unbound indices (the ones not summed over) must appear on both sides of an equation.

For example, $T^{\mu\alpha} = g^{\mu\nu}T_\nu^\alpha$ demonstrates contraction with the metric tensor. When there is superscript that should be a subscript, or vice versa, the metric tensor is implicitly being used to raise and lower indices.

There are several conventions regarding Einstein's summation notation. Spacetime variables are indexed by Greek letters, e.g. μ or ν, which run from 0 to 3, while space-only variables are indexed by Roman letters, e.g. i or j, which run from 1 to 3. Given a 4-vector $v = v^\nu e_\nu$, we let $\vec{v} = v^ie_i$ be the space-only component, and v^2 generally denotes $v^\mu v_\mu$ whereas $\vec{v}^2$ generally denotes v^iv_i.

Chapter 1

Introduction: Klein Gordon and Dirac Fields

1.1 Klein Gordon Field

In this chapter, we will look at our first quantum field, called the Klein-Gordon field. This field arises from the Klein-Gordon equation

$$(\partial^2 + m^2)\phi = 0,$$

which came about as an attempt to make the Schrödinger equation compatible with special relativity, where time and space coordinates can be mixed by Lorentz transformations. Klein and Gordon first proposed it to describe wavefunctions of relativistic electrons, but that interpretation turned out to have some serious problems; nowadays we know it instead describes a quantum field. Although it is meaningless classically (i.e. it does not describe any classical system worth investigating), we will begin by examining Klein-Gordon fields classically, and then putting them through a process called canonical quantization to obtain the quantum Klein-Gordon field.

1.1.1 Why Fields?

Before we begin, let's motivate why we want to look at fields instead of wavefunctions. Why complicate things if we can do relativistic QM with wavefunctions, instead of QFT with quantum fields?

Volume 1 of Steven Weinberg's *Quantum Theory of Fields* is devoted to answering this question. A discussion of scattering experiments lead him to the S-matrix, and then to the local behaviour of experiments (which he calls the cluster decomposition principle), and then using Lorentz invariance, fields just practically fall out. Weinberg does a really good job of convincing us that QFT in some form or another really must exist if we assume Lorentz invariance and unitarity.

Peskin and Schroeder give a slightly different motivation, one that is closer to the historical reason of why fields were introduced. There are three main factors at play here.

- Single particle relativistic wave functions have unavoidable negative energy eigenstates. As an example, we can look at the Dirac equation. The Dirac equation comes from forcing the Schrödinger equation $i(d\Psi/dt) = \hat{H}\Psi$ to be Lorentz invariant. As it stands, it is first-order in time, but second-order in space. Suppose instead that

$$\hat{H} = \frac{1}{i}\alpha^j\partial_j + m\beta.$$

Since $E^2 = \vec{p}^2 + m^2$, we want $\hat{H}^2 = -\nabla^2 + m^2$, which gives

$$\alpha^j \alpha^k + \alpha^k \alpha^j = 2\delta^{jk}, \quad \alpha^j \beta + \beta \alpha^j = 0, \quad \beta^2 = 1.$$

Hence $\{\alpha^1, \alpha^2, \alpha^3, \beta\}$ are not scalars, but instead are the generators of a Clifford algebra; we take their simplest representation as matrices, which is as 4×4 complex matrices

$$\alpha^j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

where σ_j are the Pauli matrices. Now compute in momentum- space that

$$\widehat{H\psi}(\vec{p}) = (-i\vec{p} \cdot \vec{\alpha} + m\beta)\hat{\psi}(\vec{p}) = \begin{pmatrix} mI & \vec{p} \cdot \vec{\sigma} \\ \vec{p} \cdot \vec{\sigma} & -mI \end{pmatrix} \hat{\psi}(\vec{p}),$$

and a straightforward calculation shows that $\hat{H}$ has eigenvalues $\pm\sqrt{\vec{p}^2 + m^2}$. In particular, the energy can be negative!

Dirac attempted to resolve this issue by appealing to the Pauli exclusion principle and positing that there existed a whole "sea of negative-energy states" that were already occupied. Consequently, the holes in this sea would be antiparticles. This makes sense until we realize that that a particle falling into a negative-energy state would represent particle-antiparticle annihilation, but the Dirac equation is supposed to be modeling a single particle (an electron, actually). So philosophical issues aside, there are technical issues here. The field viewpoint will allow us to view particles as excitations of some field, and antiparticles of different types of excitations of the same field, but the key here is that these excitations all have positive energy, regardless of whether they represent particles or antiparticles. We will investigate this later on, when we see the Dirac field (which will contain the first non-trivial example of antiparticles).

- $E = mc^2$ allows for particles to be created at high energies, and $\Delta E \Delta t = \hbar$ allows for virtual particles. This indicates we should really be looking at multi-particle instead of single-particle theories. While we can obtain multi-particle theories simply by looking at the tensor product of single-particle state spaces, the quantum mechanics arising from this construction do not permit the creation and annihilation of particles. We can't "destroy" or "create" a wavefunction; it exists for all time and space. Instead, the field viewpoint allows us to view particles as excitations of a field, which we can easily create or destroy.

- Wavefunctions and quantum mechanics don't care about special relativity. In particular, there is obvious causality violation in quantum mechanics! Set $H = \frac{\hat{p}^2}{2m}$ to be the free Hamiltonian, and let's compute the probability amplitude for propagation between two points x_0 and x in spacetime:

$$\begin{aligned} U(t) &= \langle \vec{x} | e^{-iHt} | \vec{x}_0 \rangle \\ &= \int \frac{d^3p}{(2\pi)^3} \langle \vec{x} | e^{-i(p^2/2m)t} | p \rangle \langle p | x \rangle \\ &= \int \frac{d^3p}{(2\pi)^3} e^{-i(p^2/2m)t} e^{i\vec{p} \cdot (\vec{x} - \vec{x}_0)} \\ &= \left(\frac{m}{2\pi i t} \right)^{3/2} e^{im(\vec{x} - \vec{x}_0)^2/2t} \end{aligned}$$

This last quantity is non-zero, even for x and x_0 that may be space-like separated, e.g. x inside the light cone, and x_0 outside it, which, in principle, allows faster-than-light transfer of information.

It is not clear immediately how field theory will help us here. But we will see that by rigorously enforcing Lorentz invariance when we write down field dynamics, the causality violation problem magically disappears.

Another important reason we want to do QFT is because, well, the theory predicts the outcome of numerous experiments to very high accuracy. In the end, physics is about constructing models: the fact that your model is giving good predictions is very strong evidence that it should be adopted, or at least seriously considered as a foundational theory. In particular, quantum electrodynamics (QED), which describes electromagnetism, is something we will see very soon that has been very well tested and agrees very well with experiments, up to the limits of what we can experimentally measure.

1.1.2 Elements of Classical Field Theory

Before we embark on the long journey through QFT, we need to review some tools from classical field theory first. This serves not only as a review, but as motivation for many calculations and objects we will be examining in the QFT world.

1.1.3 Lagrangian Field Theory

- Fundamental quantity in Lagrangian field theory is the action S. In high school, the Lagrangian is a function of time, positions, and velocities of a system: $L(t, x(t), \dot{x}(t))$. The action is given by $S = \int dt\, L$. Fields can also be described in a Lagrangian formalism, for instance by considering every point in space-time as a "particle" that wiggles back and forth with the amplitude of wiggling characterizing the strength of the field.

  Let $\varphi : M \to \mathbb{R}$, define a Lagrangian *density* $\mathcal{L}(t, \varphi, \partial_\mu \varphi)$, the honest Lagrangian $L = \int d^3x \mathcal{L}$, and finally define the action:

$$S = \int dt\, L = \int d^4x \mathcal{L}$$

> Four-vector notation:
>
> – Greek letters $\mu, \nu, \ldots \in \{0, 1, 2, 3\}$
>
> – Roman letters $i, g, \ldots \in \{1, 2, 3\}$.
>
> – $x^\mu = (x^0, x^1, x^2, x^3)$
>
> – Signature $(+ - - -)$
>
> – $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$
>
> – $\partial_\mu f = \frac{\partial f}{\partial x^\mu} = (\partial_0 f, \partial_1 f, \partial_2 f, \partial_3 f)$.

- Extremize the action. Let $\delta f = f(\varphi + \xi) - f(\varphi)$.

$$\begin{aligned} 0 = \delta S &= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \underbrace{\delta(\partial_\mu}_{\text{commute}} \varphi) \right) \\ &= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta\varphi \right) - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \delta\varphi \right] \end{aligned}$$

  By Stokes' theorem, we can break this integral up into two parts, one of which is called the boundary term. Taking a variation that is fixed along the boundary means $\delta\varphi \equiv 0$ on the boundary which means that the boundary term does not contribute to δS. Moreover, if we take $\delta S = 0$ for every variation, then we obtain the Euler Lagrange equations:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) - \frac{\partial \mathcal{L}}{\partial \varphi} = 0$$

Remark. The Lagragian formalism is useful for relativistic dynamics because all expressions are chosen to Lorentz invariant.

1.1.4 Hamiltonian Field Theory

- Introducing this makes the transition to the quantum theory easier.
- High school Hamiltonian formalism: $p = \frac{\partial L}{\partial \dot{q}}, H = \sum p\dot{q} - L$.
- Pretend that $\vec{x}$ enumerates points on the lattice of space-time:

$$\begin{aligned} p(\vec{x}) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}(\vec{x})} &= \frac{\partial}{\partial \dot{\varphi}(\vec{x})} \int d^3y \, \mathcal{L}(\varphi(y), \dot{\varphi}(y)) \\ &\sim \frac{\partial}{\partial \dot{\varphi}(\vec{x})} \sum \mathcal{L}(\varphi(y), \dot{\varphi}(y)) d^3y \\ &= \frac{\partial \mathcal{L}}{\partial \dot{\varphi}(\vec{x})} d^3x \\ &\equiv \pi(\vec{x}) d^3x \end{aligned}$$

since each point on the lattice represents a different variable, so the derivative just picks out the one at $\vec{x}$. We call $\pi(\vec{x})$ the momentum *density*. Therefore the Hamiltonian looks like:

$$H = \int d^3x \, [\pi(\vec{x})\dot{\varphi}(\vec{x}) - \mathcal{L}] .$$

(See the stress-energy tensor part for another derivation of the Hamiltonian which falls out of Noether's theorem for being the conserved quantity under time translations.)

One might ask why we are still singling out the time-parameter in the Hamiltonian formalism when we write $p(\vec{x}) = \partial\mathcal{L}/\partial\dot{\varphi}(\vec{x})$ instead of making it seem more Lorentz invariant by considering $\partial\mathcal{L}/\partial(\partial_\mu \varphi(\vec{x}))$ instead. This is because although special relativity dictates that time transforms with space, we still cannot treat them equally as coordinates. The Hamiltonian is, by definition, the infinitesimal generator of time translations, and hence is intrinsically associated with only the time coordinate. In fact, it is not true that the Hamiltonian density is always Lorentz invariant.

- **Important example:** Take $\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}m^2\varphi^2$. Euler-Lagrange equations become $\partial^\mu(\partial_\mu \varphi) + m^2\varphi = 0$ which is the Klein Gordon equation. The Hamiltonian becomes:

$$H = \int d^3x \mathcal{H} = \int d^3x \left[\underbrace{\frac{\pi^2}{2}}_{\text{moving in time}} + \underbrace{\frac{(\nabla\varphi)^2}{2}}_{\text{shearing in space}} + \underbrace{\frac{m^2\varphi^2}{2}}_{\text{existing at all}} \right]$$

1.1.5 Noether's Theorem - How to Compute Conserved Quantities

To every continuous transformation of the field we can assign an infinitesmal transformation:

$$\varphi(x) \to \varphi'(x) = \varphi(x) + \alpha \underbrace{\Delta\varphi(x)}_{\text{deformation}}$$

Transformations might also change the Lagrangians. The interplay between how the infinitesmal transformation changes the Lagrangian and the field is what gives rise to conserved quantities, or sometimes known as Noether charges.

$$\begin{aligned}\text{Symmetry} &\iff \text{Equations of motion} - \text{invariant}\\ &\iff \text{Action invariant (up to surface term)}\\ &\iff \mathcal{L}(x) \to \mathcal{L}(x) + \alpha\partial_m u\mathcal{J}^\mu(x)\end{aligned}$$

Taylor expanding the perturbation:

$$\begin{aligned}\Delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\varphi}\cdot\Delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\mu(\Delta\varphi)\\ &= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\Delta\varphi\right) + \left[\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right)\right]\\ &= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\Delta\varphi\right)\end{aligned}$$

Since we claimed that under the symmetry $\Delta\mathcal{L} = \partial_\mu\mathcal{J}^\mu$ we have the following relations:

$$\begin{aligned}j^\mu(x) &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\Delta\varphi - \mathcal{J}^\mu\\ \partial_\mu j^\mu &= 0\\ \frac{\partial}{\partial t}j^0 &= \partial_i j^i\end{aligned}$$

Define the charge $Q = \int d^3x\ j^0$. Then, if we assume that space does not have boundary, Stokes' theorem implies that $\partial Q/\partial t = 0$. Often, j^0 is called the charge density, and j^μ is called the current density.

Examples:

1. $\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)^2$ has the following field symmetry, $\varphi \to \varphi + \alpha$, ie. $\Delta\varphi \equiv$ const. There is no change to the Lagrangian, so $j^\mu = \partial^\mu\varphi$.
2. Space-time transformation, $x^\mu \to x^\mu - a^\mu$, implies

$$\begin{aligned}\varphi(x) \to \varphi(x+a) &= \varphi(x) + a^\nu\partial_\nu\varphi(x)\\ \mathcal{L}(x) \to \mathcal{L}(x+a) &= \mathcal{L}(x) + a^\mu\partial_\mu\mathcal{L}\\ . \qquad\qquad &= \mathcal{L}(x) + a^\nu\partial_\mu(\delta^\mu_\nu\mathcal{L})\end{aligned}$$

Therefore we write

$$T^\mu_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\nu\varphi - \delta^\mu_\nu\mathcal{L}$$

we get four separately conserved quantities.

This is called the **stress-energy tensor** or the **energy-momentum tensor** in various contexts. The $T^{\bullet 0}$ quantity gives rise to the Hamiltonian:

$$\int d^3xT^{00} = \int d^3x\mathcal{H} \equiv H$$

1.1.6 Quantizing the Klein-Gordon Field

Before we quantize, let's apply the classical theory to the classical Klein-Gordon field, which is defined by the Lagrangian
$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2,$$
where $\phi(\vec{x})$ is the real-valued **classical Klein-Gordon field**. We will interpret m as a mass later on, but for now it is just a parameter.

Exercise 1.1.1. By applying Euler-Lagrange, confirm that this Lagrangian for the classical Klein-Gordon field indeed gives the Klein-Gordon equation $(\partial^\mu\partial_\mu + m^2)\phi = 0$, and compute the Hamiltonian
$$H = \int d^3x\ \mathcal{H} = \int d^3x\ \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right).$$
(You should get that $\pi = \dot{\phi}$).

Now we enter the QFT world. For now we will work in the Schrödinger picture, where $\phi(\vec{x})$ and $\pi(\vec{y})$ are time-independent. We will take the classical Klein-Gordon field and **canonically quantize** it, which involves two steps:

1. promote ϕ and π to operators (i.e. $\phi(\vec{x})$ and $\phi(\vec{y})$ are now operators, not scalars), and
2. specify the commutation relations
$$[\phi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y})$$
$$[\phi(\vec{x}), \phi(\vec{y})] = [\pi(\vec{x}), \pi(\vec{y})] = 0.$$
This is in analogy with the QM of a multiparticle system, where if q_i and p_i are the momentum and position operators of the i-th particle, then
$$[q_i, p_j] = i\delta_{ij}$$
$$[q_i, q_j] = [p_i, p_j] = 0,$$
except now we have a continuum of particles, indexed by the continuous variable $\vec{x}$ instead of a discrete variable i.

Note that these commutation relations are taken to be **axioms**. At this point one may wonder why we treat ϕ and π as different operators when $\pi = \dot{\phi}$ for Klein-Gordon. This is for the same reason that x and $\dot{x}$ are treated independently in classical field theory: we abuse notation and write $(x, \dot{x})$ as coordinates on phase space, when really we should be writing (x, p). But we write $\dot{x}$ because we will always be evaluating objects on phase space at $(x, \dot{x})$.

But of course, imposing these axioms is easier said than done. What do ϕ and π look like?

Let us try to motivate the form of the expression for ϕ and its conjugate π in terms of creation and annihilation operators. [1]

If we expand a solution to the Klein Gordon equation in a Fourier basis of plane waves, then we see that we naturally have some variables that we can quantize. What's more interesting is that the Klein Gordon equation gives rise to precisely the harmonic oscillator example from first year quantum mechanics. In terms

[1] Quibble: I don't like how it is done in Peskin. Why promote the coefficients in the Fourier transform, and why do *they* give rise to the creation and annihilation operators. I think there might be a good explanation out there already; Landau& Lifshitz, and Weinberg (Chapter 5) seem to take a good wack at the physics of this choice. Actually, in LL, the exposition seems to have avoided some of the integral manipulations that happened in Peskin and Schroeder.

of creation and annihilation operators, the first years wrote $\hat{q} = \frac{1}{\sqrt{2\omega}}(a + a^\dagger)$, $\hat{p} = \sqrt{\frac{\omega}{2}}(a - a^\dagger)$. Therefore we conjecture our fields have the following form: [2]

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}})$$

$$\pi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_p}{\sqrt{2}}} (a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} - a^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}})$$

1.2 Dirac Field

The formalism that we have built up so far tells that everything starts from a Lagrangian. In the theory of elementary particles and high energy physics there is one very special condition that we require of a Lagrangian: Lorentz invariance. To check whether a given expression of ϕ's and $\partial_\mu\phi$'s is Lorentz invariant we must understand how arbitrary field transform under the Lorentz group. Suppose a field has components ϕ_a, then a general transformation is given by

$$\phi'_a(x) = M(\Lambda)_{ab}\phi_b(\Lambda^{-1}x).$$

Thus, to solve the problem of constructing all Lagrangians we must first understand the representations of the Lorentz group, or at least of the Lorentz algebra. Taking a cue from the so(3) generators given by $J^{ij} = -i(x^i\partial^j - x^j\partial^i)$ it turns out that the generators for the Lorentz algebra are:

$$J^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu).$$

This gives commutation relations

$$[J^{\mu\nu}, J^{\rho\sigma}] = \dots.$$

The defining representation is given by the following$(\mathcal{J}^{\mu\nu})_{ab} = -i\delta^\mu_{[a}\delta^\nu_{b]} = -i(\delta^\mu_a\delta^\nu_b - \delta^\mu_b\delta^\nu_a)$ Dirac came up with another representation by taking 4 $n \times n$ matrices γ^μ satisfying $\gamma^\mu, \gamma^\nu = 2g^{\mu\nu} \times \mathbb{1}_{n\times n}$ and defining:

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu].$$

If we define $\sigma = (\mathbb{1}, \vec{\sigma})$ and $\bar{\sigma} = (\mathbb{1}, -\vec{\sigma})$ then

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

satisfies the commutation relations and gives rise to the **Dirac representation**. Objects that transform under that transform under this representation are called 4-component Dirac spinors, or just **Dirac spinors** for short.

Taking $\bar{\psi} = \gamma^0\psi^\dagger$, the Dirac equation, Lagrangian are given by:

$$(i\gamma^\mu\partial_\mu - m)\psi = 0$$

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

The conjugate variable to ψ is $i\psi^\dagger$. Before quantizing we solve the Dirac equation in plane waves, $u(p)e^{i\vec{p}\cdot\vec{x}}$ and $v(p)e^{-i\vec{p}\cdot\vec{x}}$. After rewriting the Dirac equation into a matrix equation it is not hard to see that arbitrary solutions $u(p)$ and $v(p)$ are given by the following expressions:

$$u^s(p) = \sqrt{m}\begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\sigma}\xi^s \end{pmatrix}$$

$$v^s(p) = \sqrt{m}\begin{pmatrix} \sqrt{p\cdot\sigma}\eta^s \\ -\sqrt{p\cdot\sigma}\eta^s \end{pmatrix}$$

[2]**Elaborate.**

where, for $s = 1, 2$, $\{\xi^s\}$ and $\{\eta^s\}$ are a basis for $\mathbb{C}^2$ and $\sqrt{p \cdot \sigma}$ is the square root of the positive eigenvalue of the associated matrix. (Phew, what a mouthful!)

Finally, we quantize this theory by introducing the **anticommutation relations** and rewriting ψ and $\bar{\psi}$ using raising and lowering operators

$$\begin{aligned}\{\psi(x), \bar{\psi}(y)\} &= \delta^{(3)}(\vec{x} - \vec{y}) \\ \psi(x) &= \int \frac{d^3p}{(2\pi)^3} a_{\vec{p}}^{s\dagger} u^s(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} - b_{\vec{p}}^{s\dagger} v^s(\vec{p}) e^{i\vec{p}\cdot\vec{x}} \\ \bar{\psi}(x) &= \int \frac{d^3p}{(2\pi)^3} b_{\vec{p}}^{s\dagger} \bar{v}^s(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} - a_{\vec{p}}^{s\dagger} \bar{u}^s(\vec{p}) e^{i\vec{p}\cdot\vec{x}}\end{aligned}$$

Using these cleverly chosen expressions we may write the Hamiltonian as

$$\begin{aligned}H &= \int \frac{d^3p}{(2\pi)^3} \sum_s E_{\vec{p}} (a_{\vec{p}}^{s\dagger} a_{\vec{p}}^s + b_{\vec{p}}^{s\dagger} b_{\vec{p}}^{s\dagger}) \\ Q &= \int \frac{d^3p}{(2\pi)^3} \sum_s (a_{\vec{p}}^{s\dagger} a_{\vec{p}}^{s\dagger} - b_{\vec{p}}^{s\dagger} b_{\vec{p}}^s)\end{aligned}$$

where Q is the conserved quantity coming from gauge invariance, $\psi'(x) = e^{\alpha(x)}\psi(x)$, of $\mathcal{L}_{\text{Dirac}}$.

Chapter 2

Path Integrals

So far, we have taken classical field theories and canonically quantized them to obtain the corresponding QFTs. In general, this canonical quantization process is difficult and tedious, but it motivates much of what we are about to do. The path integral approach to QFT will allow us to perform perturbative calculations more easily, and generalizes readily to other non-interacting theories. In for the entirety of this chapter, we will mostly be concerned with calculating **propagation amplitudes** for a perturbed theory.

2.1 Deriving the Path Integral

Suppose we have the Hamiltonian $\hat{H}$ for a quantum mechanical particle, and we want to compute the amplitude $\langle \vec{q}_b | e^{-i\hat{H}t} | \vec{q}_a \rangle$, i.e. the amplitude for the particle to travel from the point $\vec{q}_a$ to $\vec{q}_b$ in a given time t. Using the superposition principle, let's compute this by splitting up the time interval $[0, t]$ into n equal chunks of size $\delta t = t/n$:

$$\langle \vec{q}_b | e^{-i\hat{H}t} | \vec{q}_a \rangle = \int \cdots \int d\vec{q}_1 \cdots d\vec{q}_{n-1} \, \langle \vec{q}_b | e^{-i\hat{H}\delta t} | \vec{q}_{n-1} \rangle \, \langle \vec{q}_{n-1} | e^{-i\hat{H}\delta t} | \vec{q}_{n-2} \rangle \cdots \langle \vec{q}_1 | e^{-i\hat{H}\delta t} | \vec{q}_a \rangle \, .$$

What have we done? We are saying that the amplitude for propagation from $\vec{q}_a$ to $\vec{q}_b$ is equal to the amplitude for propagation from $\vec{q}_a$ to $\vec{q}_1$, then to $\vec{q}_2$, and so on, until $\vec{q}_b$, integrated over all possible $\vec{q}_j$. (Recall the double slit experiment and consider the case $n = 2$ if you are still confused.)

Now each of the terms needs to be evaluated. For convenience, let $\vec{q}_n = \vec{q}_b$ and $\vec{q}_0 = \vec{q}_a$. Let's do the simple case where $\hat{H} = \hat{p}^2/2m$, a free particle. A straightforward calculation shows:

$$\begin{aligned}
\langle \vec{q}_{j+1} | e^{-i(\hat{p}^2/2m)\delta t} | \vec{q}_j \rangle &= \int \frac{d^3p}{(2\pi)^3} \, \langle \vec{q}_{j+1} | e^{-i(\hat{p}^2/2m)\delta t} | p \rangle \, \langle p | \vec{q}_j \rangle \\
&= \int \frac{d^3p}{(2\pi)^3} \, e^{-i(p^2/2m)\delta t} \, \langle \vec{q}_{j+1} | p \rangle \, \langle p | \vec{q}_j \rangle \\
&= \int \frac{d^3p}{(2\pi)^3} \, e^{-i(p^2/2m)\delta t} e^{ip(\vec{q}_{j+1} - \vec{q}_j)}.
\end{aligned}$$

Ah, we know how to evaluate this integral: it's just a Gaussian! The final result, after some suggestive rearranging, is

$$\langle \vec{q}_{j+1} | e^{-i(\hat{p}^2/2m)\delta t} | \vec{q}_j \rangle = \left(\frac{m}{2\pi i \delta t} \right)^{3/2} \exp\left(i\delta t \frac{m}{2} \left(\frac{\vec{q}_{j+1} - \vec{q}_j}{\delta t} \right)^2 \right).$$

(The Gaussian integral itself is not trivial. [1]) Hence when we plug this back into our calculation for $\langle \vec{q}_b | e^{-i\hat{H}t} | \vec{q}_a \rangle$, we get

$$\langle \vec{q}_b | e^{-i\hat{H}t} | \vec{q}_a \rangle = \left(\frac{m}{2\pi i \delta t} \right)^{3n/2} \int d\vec{q}_1 \cdots d\vec{q}_{n-1} \exp\left(i\delta t \frac{m}{2} \sum_{j=1}^{n-1} \left(\frac{\vec{q}_{j+1} - \vec{q}_j}{\delta t} \right)^2 \right).$$

So far, everything we have done is rigorous. But now we make an intuitive leap: instead of approximating the propagation from $\vec{q}_a$ to $\vec{q}_b$ with a finite number of timesteps, we use infinitely many. In other words, we "integrate over paths" by letting $\delta t \to 0$ and $n \to \infty$, giving the formal expression

$$\langle \vec{q}_b | e^{-i\hat{H}t} | \vec{q}_a \rangle = \int D\vec{q}(t) \exp\left(i \int_0^t dt \, \frac{1}{2} m \vec{q}'(t)^2 \right)$$

where the **path integral** $\int D\vec{q}(t)$ is defined as

$$\int D\vec{q}(t) = \lim_{n \to \infty} \left(\frac{m}{2\pi i \delta t} \right)^{3n/2} \int \cdots \int d\vec{q}_1 \cdots d\vec{q}_{n-1}.$$

Exercise 2.1.1. Perform the same derivation of the path integral, but now starting with the Hamiltonian $\hat{H} = \hat{p}^2/2m + V(\hat{q})$. You should get

$$\langle \vec{q}_b | e^{-i\hat{H}t} | \vec{q}_a \rangle = \int D\vec{q}(t) \exp\left(i \int_0^t dt \, \frac{1}{2} m \vec{q}'(t)^2 - V(\vec{q}(t)) \right).$$

For now, let's not worry about the infinite constant in front of the path integral; it pales as an issue in comparison to the nonexistence of a Lebesgue measure on the space of paths. Actually, the constant will cancel out later.

Note that the integrand looks suspiciously like the Lagrangian corresponding to the Hamiltonian in both cases. This is indeed true, and can be demonstrated by plugging in a general Hamiltonian $\hat{H}(\hat{q}, \hat{p})$ and seeing how combinations of $\hat{q}$ and $\hat{p}$ act on the $|\vec{q}_i\rangle$.

Theorem 2.1.1. *Suppose $\hat{H}(\vec{q}, \vec{p})$ is a* **Weyl-ordered** *Hamiltonian, i.e. in a form where if there is a term $\vec{p}^{i_1} \vec{q}^{i_2} \cdots \vec{p}^{i_n}$, then there is a corresponding term $\vec{p}^{i_n} \vec{q}^{i_{n-1}} \cdots \vec{p}^{i_1}$. Then*

$$\langle \vec{q}_b | e^{-i\hat{H}t} | \vec{q}_a \rangle = \int D\vec{q}(t) D\vec{p}(t) \exp\left(i \int_0^t dt \, \vec{p}(t) \cdot \vec{q}(t) - H(\vec{q}(t), \vec{p}(t)) \right).$$

In particular, for Hamiltonians quadratic in $\vec{p}$, we can integrate away the $\int D\vec{p}(t)$, leaving only the Lagrangian in the integrand.

Proof. Details of the long calculation will not bring us much further enlightenment, so we omit them. See Peskin & Schroeder, pages 280-281 iff you like calculations and have some time to burn. □

[1] The relevant formula is as follows. For $A \in \mathrm{GL}(n, \mathbb{C})$ such that $A = A^T$ and $\mathrm{Re}\, A$ is positive semidefinite,

$$\int d^n x \, e^{-Ax \cdot x/2 + iy \cdot x} = \frac{(2\pi)^{n/2}}{\sqrt{\det A}} e^{-A^{-1} y^2/2}.$$

One proves this by showing it first for $n = 1$ and $A = I$, in which case it suffices to solve the DE

$$\frac{d}{dy} \int dx \, e^{-x^2/2 + iyx} = -y \int dx \, e^{-x^2/2 + iyx}.$$

Now suppose A is real and hence PSD. If we plug $x = \sqrt{A} v$ into the LHS of the formula, the RHS splits as a product of one-dimensional integrals, which we just calculated. Finally, since both sides are analytic and agree for real PSD matrices, they agree in general.

Any Hamiltonian can be Weyl-ordered by commuting $\hat{p}$ and $\hat{q}$, so this theorem is very general. In fact, it is general enough that from now on, we will work directly with the Lagrangian and almost completely ignore the Hamiltonian formalism. There is one major advantage in doing so: the Lagrangian makes symmetries and conservation laws very clear. For example, when we write down a Lorentz-invariant Lagrangian, the path integral is automatically Lorentz-invariant.

In fact, the quantum system we are considering is very general as well. In our entire derivation of the path integral, we did not use anything beyond the relationship between $\hat{q}$ and $\hat{p}$. So in particular, our derivation holds not only for quantum mechanical systems, but also for QFTs. For example, if we take the Lagrangian $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - V(\phi)$ for a real scalar field, then

$$\langle\phi_b(\vec{x})|e^{-i\hat{H}t}|\phi_a(\vec{x})\rangle = \int D\phi(x)\, \exp\left(i\int_0^t d^4x\, \frac{1}{2}(\partial_\mu\phi)^2 - V(\phi)\right),$$

where here $D\phi(x)$ indicates that we are integrating over a path taking values in fields. In particular, $\phi(0,\vec{x})$ is constrained to be $\phi_a(\vec{x})$, and $\phi(t,\vec{x})$ is constrained to be $\phi_b(\vec{x})$.

2.2 Correlation Functions

Okay, what good is the path integral? The answer is they are useful when we apply perturbations to free field theory. Most of the QFT we will be looking at is perturbative, so path integrals will give us a good deal of physics.

Suppose we have the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_{int}$, where $\hat{H}_0$ is a Hamiltonian we are supposed to have understood well already, and $\hat{H}_{int}$ is a perturbation known as the **interaction Hamiltonian**. Usually $\hat{H}_0$ will be the Hamiltonian for the free field theory, i.e. the Klein-Gordon Hamiltonian. Let $|\Omega\rangle$ be the ground state of $\hat{H}$. We are interested in computing the probability amplitude of a propagation from $\vec{x}$ to $\vec{y}$, i.e.

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle\,.$$

How do we compute this quantity? Let's start with a seemingly-unrelated quantity:

$$\int D\phi(x)\,\phi(x_1)\phi(x_2)\exp\left(i\int_{-t}^t d^4x\ \mathcal{L}(\phi)\right),$$

where the path $\phi(x)$ starts at some $\phi_a(\vec{x})$ at time $-t$ and ends at some $\phi_b(\vec{x})$ at time t. Suppose $x_1^0 < x_2^0$. Then we are going to divide up this path integral into three components:

1. from $\phi_a(\vec{x})$ at time $-t$ to $\phi_1(\vec{x})$ at time x_1^0,
2. from $\phi_1(\vec{x})$ at time x_1^0 to $\phi_2(\vec{x})$ at time x_2^0,
3. from $\phi_2(\vec{x})$ at time x_2^0 to $\phi_b(\vec{x})$ at time t.

Note that here, $-t < x_1^0 < x_2^0 < t$, and since the intermediate field configurations $\phi_1(\vec{x})$ and $\phi_2(\vec{x})$ are arbitrary, we must integrate over them as well. Hence the integral becomes

$$\int D\phi_1(\vec{x})\int D\phi_2(\vec{x})\,\phi_1(\vec{x}_1)\phi_2(\vec{x}_2)\,\langle\phi_b|e^{-i\hat{H}(t-x_2^0)}|\phi_2\rangle\,\langle\phi_2|e^{-i\hat{H}(x_2^0-x_1^0)}|\phi_1\rangle\,\langle\phi_1|e^{-i\hat{H}(x_1^0-(-t))}|\phi_a\rangle\,.$$

Now we use completeness: $\int D\phi_1\,\phi_1(\vec{x}_1)\,|\phi_1\rangle\,\langle\phi_1| = \phi_1(\vec{x}_1)$, where the ϕ_1 on the LHS is a scalar field, and on the RHS is an operator. Doing the same for for ϕ_2, the integrals disappear, and some rearrangement gives

$$\langle\phi_b(\vec{x})|\,e^{-i\hat{H}(t-x_2^0)}\phi(\vec{x}_2)e^{-i\hat{H}(x_2^0-x_1^0)}\phi(\vec{x}_1)e^{-i\hat{H}(x_1^0-(-t))}\,|\phi_a(\vec{x})\rangle\,. \tag{2.1}$$

Aha, but $\phi(x_2) = e^{i\hat{H}x_2^0}\phi(\vec{x}_2)e^{-i\hat{H}x_2^0}$ in the Heisenberg picture, so this simplifies further to

$$\langle\phi_b(\vec{x})|\, e^{-i\hat{H}t}\phi(x_2)\phi(x_1)e^{-i\hat{H}t}\,|\phi_a(\vec{x})\rangle\,.$$

We're not done yet! During this calculation, we had to assume x_1 came before x_2 in time, so that the path integral split well. If x_1 actually came after x_2, then we simply exchange x_1 and x_2 in the final result. This motivates the following definition.

Definition 2.2.1. Given two operators $\phi(x_1)$ and $\phi(x_2)$, the **time-ordering operator** T applies them in the correct temporal order, i.e.

$$T\{\phi(x_1)\phi(x_2)\} = \begin{cases} \phi(x_1)\phi(x_2) & x_1^0 > x_2^0 \\ \phi(x_2)\phi(x_1) & x_2^0 > x_1^0. \end{cases}$$

Hence we should really be looking to calculate $\langle\Omega|T\phi(x_1)\phi(x_2)|\Omega\rangle$, while right now we have the quantity $\langle\phi_b(\vec{x})|\, e^{-i\hat{H}t}T\phi(x_2)\phi(x_1)e^{-i\hat{H}t}\,|\phi_a(\vec{x})\rangle$. In other words, our problem is to obtain $|\Omega\rangle$ from $e^{-i\hat{H}t}|\phi_a(\vec{x})\rangle$. Physicists have a hilarious trick for doing so. First expand $|\phi_a\rangle$ in the eigenbasis $\{|\Omega\rangle, |1\rangle, \dots\}$ of $\hat{H}$:

$$e^{-i\hat{H}t}\,|\phi_a\rangle = e^{-iE_\Omega t}\,|\Omega\rangle\,\langle\Omega|\phi_a\rangle + \sum_{n>0} e^{-iE_n t}\,|n\rangle\,\langle n|\phi_a\rangle\,.$$

Now remember that the ground state energy is the lowest energy, i.e. $E_\Omega < E_n$ for all $n > 0$. So here's what we do to keep the $|\Omega\rangle$ term while getting rid of everything else: we take the limit $t \to \infty(1 - i\epsilon)$. Since $e^{-iE_nT(1-i\epsilon)}$ will decay faster than $e^{-iE_\Omega T(1-i\epsilon)}$, because e^{-E_nt} decays faster than $e^{-E_\Omega t}$, it follows that when $T \to \infty$, every other term except the $|\Omega\rangle$ term vanishes.

$$\lim_{t\to\infty(1-i\epsilon)} e^{-i\hat{H}t}\,|\phi_a\rangle = \langle\Omega|\phi_a\rangle\, e^{-E_\Omega\infty(1-i\epsilon)}\,|\Omega\rangle\,.$$

It remains to get rid of the extraneous factors in the final expression. Well that's easy, we just divide out by

$$\langle\phi_b|\, e^{-i\hat{H}t}e^{-i\hat{H}t}\,|\phi_a\rangle = \int D\phi(x)\, \exp\left(i\int_{-t}^{t} d^4x\; \mathcal{L}(\phi)\right).$$

Theorem 2.2.2. *The amplitude for a propagation between spacetime points x_1 and x_2 is*

$$\langle\Omega|\, T\phi(x_1)\phi(x_2)\,|\Omega\rangle = \lim_{t\to\infty(1-i\epsilon)} \frac{\int D\phi(x)\,\phi(x_1)\phi(x_2)\exp\left(i\int_{-t}^{t} d^4x\;\mathcal{L}\right)}{\int D\phi(x)\,\exp\left(i\int_{-t}^{t} d^4x\;\mathcal{L}\right)}.$$

This quantity is important enough to have a name: it is called the **two-point correlation function**. It is usually denoted $\langle\phi(x_1)\phi(x_2)\rangle$ for convenience. Analogously, we have n**-point correlation functions** $\langle\phi(x_1)\cdots\phi(x_n)\rangle$.

Since $\pm\infty(1 - i\epsilon)$ is "a finite distance" away from $\pm\infty$, we usually write $\int_{-\infty}^{\infty} d^4x\;\mathcal{L}$ in the exponential. Better yet, we write $\int d^4x\;\mathcal{L}$ and take it to be understood that we are integrating over all spacetime now.

2.3 The Generating Functional

This formula for the propagation amplitude may not seem like much of an improvement. But it is, and it will be obvious by the end of this section how. Let's begin with the **free-field Lagrangian**

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2.$$

(Later on we will see why this is called the free field Lagrangian.) If we plug this Lagrangian into the path integral, the integral is directly computable; the end result is a Klein-Gordon field, which we are already familiar with. So let's add a general perturbation term:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 + J\phi,$$

where here $J(x)$ is a function of x, representing an **excitation**, and usually called a **source function**. The resulting path integral is written

$$Z[J] = \int D\phi \exp\left(i \int d^4x \, \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 + J\phi\right),$$

and called the **generating functional**. In this notation, we want to find $Z[J]/Z[0]$.

Note: adding a source function is not the same thing as adding an interaction. Source functions merely allow us to create sources and sinks, whereas interactions allow the excitations generated by the sources and sinks to interact with themselves. Here we are still working within a free Klein-Gordon theory.

We can write $Z[J]$ in a very explicit form. First, let's rewrite the Lagrangian a little via integration by parts:

$$\int d^4x \, \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 + J\phi = \int_0^t d^4x \, \frac{1}{2}\phi(-\partial^2 - m^2)\phi + J\phi.$$

We will evaluate this a little informally, but the entire argument can be made formal once we introduce the Green's function (the definition of which will be motivated by this argument). Imagine that the integral above is actually a giant sum, $\phi = (\phi_1, \ldots, \phi_n)$ is merely a vector, and $(-\partial^2 - m^2) = A$ merely an $n \times n$ matrix. Then $Z[J]$ becomes

$$\int d\phi_1 \cdots \int d\phi_n \, \exp\left(\frac{i}{2}\phi^T A \phi + iJ\phi\right) = \left(\frac{(2\pi i)^n}{\det A}\right)^{\frac{1}{2}} \exp\left(-\frac{i}{2}JA^{-1}J\right).$$

Physicists call this process "discretizing spacetime," which sounds cooler.

Now we want to pass back into the continuum limit, i.e. replace ϕ as a vector with ϕ as a field, and A with $(-\partial^2 - m^2)$. But what should be replace A^{-1} by? In the discretized case, we had $AA^{-1} = I$, so by analogy, we should replace A^{-1} by a function $G(x - y)$ satisfying

$$(-\partial^2 - m^2)G(x - y) = \delta(x - y).$$

Such a function $G(x-y)$ is a **Green's function** for the linear differential operator $(-\partial^2 - m^2)$. So we pause quickly to introduce Green's functions and related objects.

2.3.1 Green's Functions and Propagators

Definition 2.3.1. Given a linear differential operator $L(x)$ (acting on distributions), its **Green's function** $G(x-y)$ satisfies $L(x)G(x-y) = -i\delta(x-y)$. Hence given a differential equation of the form $L(x)u(x) = f(x)$, we can compute

$$L(x)\int dy \, G(x-y)f(y) = \int L(x)G(x-y)f(y)\,dy = -i\int \delta(x-y)f(y)\,dy = -if(x),$$

so that $u(x) = i\int dy \, G(x-y)f(y)$ is a solution.

For example, the defining property of the Green's function $G(x-y)$ for the Klein-Gordon operator (∂^2+m^2) can be written in momentum space:

$$(\partial^2+m^2)\int \frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}\tilde{G}(p) = \int \frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}.$$

Then it is easy to solve for $\tilde{G}(p)$. Equating the two integrands,

$$(\partial^2+m^2)e^{-ip(x-y)}\tilde{G}(p) = (-p^2+m^2)e^{-ip(x-y)}\tilde{G}(p) = -ie^{-ip(x-y)},$$

so we can directly write

$$G(x-y) = \int \frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}\tilde{G}(p) = \int \frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}\frac{i}{p^2-m^2}.$$

Here we must pause for a moment: there is something wrong with this integral. When we integrate over p, there are two singularities at $p^0 = \pm E_{\vec{p}}$, so this integral diverges. That's okay, say the physicists, let's just specify how we treat the poles, and write down the following version of the Green's function:

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}\tilde{G}(p) = \int \frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}\frac{i}{p^2-m^2+i\epsilon}.$$

Now the poles are displaced above and below the real p^0 axis, at $p^0 = \pm(E_{\vec{p}}-i\epsilon)$ in the "complex p^0 plane", and we don't have divergence issues anymore. This version of the Green's function is called the **Feynman propagator**.

Exercise 2.3.1. Let $\theta(x-y)$ be the **Heaviside step function**, i.e. it is 1 when $x>y$, and 0 otherwise. Compute that

$$D_F(x-y) = \theta(x^0-y^0)\int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}e^{-ip(x-y)}\Big|_{p^0=E_{\vec{p}}} + \theta(y^0-x^0)\int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}e^{-ip(x-y)}\Big|_{p^0=-E_{\vec{p}}}$$

by analytic continuation into the complex p^0 plane, and by closing the contour either in the upper half plane when $x^0>y^0$, or in the lower half plane when $x^0<y^0$.

Why is the Green's function $D_F(x-y)$ called a propagator? The answer lies in computing the amplitude $\langle 0|\phi(\vec{x})\phi(\vec{y})|0\rangle$ for a particle to propagate from a point $\vec{x}$ in space to another point $\vec{y}$ in space:

$$\begin{aligned}\langle 0|\phi(\vec{x})\phi(\vec{y})|0\rangle &= \int \frac{d^3\vec{p}_1}{(2\pi)^3\sqrt{2E_{\vec{p}_1}}}\int \frac{d^3\vec{p}_2}{(2\pi)^3\sqrt{2E_{\vec{p}}}}e^{-i\vec{p}_1\vec{x}}e^{i\vec{p}_2\vec{y}}\,\langle 0|[a_{\vec{p}_1},a^\dagger_{\vec{p}_2}]|0\rangle \\ &= \int \frac{d^3\vec{p}}{(2\pi)^3(2E_{\vec{p}})}e^{-ip(x-y)}\Big|_{p^0=E_{\vec{p}}}.\end{aligned}$$

Hence we have

$$D_F(x-y) = \theta(x^0-y^0)\,\langle 0|\phi(\vec{x})\phi(\vec{y})|0\rangle + \theta(y^0-x^0)\,\langle 0|\phi(\vec{y})\phi(\vec{x})|0\rangle = \langle 0|T\phi(\vec{x})\phi(\vec{y})|0\rangle\,.$$

So the Feynman propagator is the two-point correlation function for free field theory: it represents the amplitude for an excitation to propagate between $\vec{x}$ and $\vec{y}$.

2.3.2 Computing the Generating Functional

Now that we know about Green's functions, let's return to computing the generating functional $Z[J]$. Recall that we left off at passing back into the continuum limit from the discretized path integral

$$\int d\phi_1\cdots\int d\phi_n\,\exp\left(\frac{i}{2}\phi^TA\phi+iJ\phi\right) = \left(\frac{(2\pi i)^n}{\det A}\right)^{\frac{1}{2}}\exp\left(-\frac{i}{2}JA^{-1}J\right).$$

Now we know what to replace A^{-1} with: the Green's function $-iD_F(x-y)$. Hence

$$Z[J] = C \exp\left(-\frac{1}{2}\int d^4x\, d^4y\, J(x)D_F(x-y)J(y)\right)$$

for some constant C. What is C? It is $Z[0]$. We have proved the following result.

Proposition 2.3.2. *Let*

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{e^{ik(x-y)}}{k^2 - m^2 + i\epsilon}$$

be the Feynman propagator. Then

$$Z[J] = Z[0] \exp\left(-\frac{1}{2}\int d^4x\, d^4y\, J(x)D_F(x-y)J(y)\right).$$

Using this formula, let's compute some of the terms in $Z[J]/Z[0]$. Write

$$W[J] = -\frac{1}{2}\int\int d^4x\, d^4y\, J(x)D_F(x-y)J(y),$$

so that

$$Z[J]/Z[0] = \exp(iW[J]) = \sum_{n=0}^{\infty}\frac{(iW[j])^n}{n!}.$$

The $n=2$ term is therefore proportional to

$$\int\int\int\int d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4\, D_F(x_1-x_2)D_F(x_3-x_4)J(x_1)J(x_2)J(x_3)J(x_4).$$

How can we interpret this term physically? Well, recall that $D_F(x-y)$ is the propagation amplitude between x and y. So this integral is, up to a constant, the amplitude for an excitation at x_2 to propagate to x_1, and an excitation at x_4 to propagate to x_3, where x_1, x_2, x_3, x_4 can range over all space. The point is that the propagation from x_2 to x_1 does not affect the propagation from x_4 to x_3 whatsoever: we can completely separate the integrals. This is why we say $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2$ is a **free field theory**: there are no terms that create interactions between different excitations of the field!

For the free field Lagrangian, sometimes we can even explicitly compute $W[J]$. For example, let's take $J(x) = J_1(x) + J_2(x)$ where $J_i(\vec{x}) = \delta^{(3)}(\vec{x} - \vec{x}_i)$, to represent two distinct time-independent excitations. Then $W[J]$ will contain terms for J_1J_1, J_2J_2, and J_1J_2 and J_2J_1. We neglect the first two, since J_1J_1 would be present in $W[J]$ regardless of whether J_2 is present or not, and similarly for J_2; they correspond to "self-interaction" and are not interesting. Let's look at the other two terms:

$$\begin{aligned}
&-\frac{1}{2}\iint d^4x\, d^4y\, J_1(x)D_F(x-y)J_2(y) + J_2(x)D_F(x-y)J_1(y) \\
&= -\frac{1}{2}\iint dx_1^0\, dx_2^0\, D_F(x_1-x_2) + D_F(x_2-x_1) \\
&= -\iint dx_1^0\, dx_2^0 \int\frac{dp^0}{2\pi}\, e^{ip^0(x_1^0-x_2^0)}\int\frac{d^3p}{(2\pi)^3}\frac{e^{i\vec{p}(\vec{x}_1-\vec{x}_2)}}{p^2-m^2+i\epsilon} \\
&= \int dx_1^0 \int\frac{d^3p}{(2\pi)^3}\frac{e^{i\vec{p}(\vec{x}_1-\vec{x}_2)}}{\vec{p}^2+m^2+i\epsilon}.
\end{aligned}$$

Now we can do three things. First, isolate the $\int dx_1^0$; this evaluates to t, the time over which we do our path integral. Second, get rid of the $i\epsilon$ in the denominator; $\vec{p}^2 + m^2$ is always positive, so there are no poles. Third, remember that $Z[J]$ for the free, **unperturbed** theory is just

$$Z[J] = \langle 0|e^{-i\hat{H}_0 t}|0\rangle = e^{-iE_0 t},$$

so up to the constant $Z[0]$, we can equate $e^{-iE_0 t}$ with $e^{iW[J]}$. We already have a factor of t in $W[J]$, so that cancels, and we are left with

$$E_0 = -\int \frac{d^3p}{(2\pi)^3} \frac{e^{i\vec{p}(\vec{x}_1-\vec{x}_2)}}{\vec{p}^2+m^2} < 0.$$

Whoa. What happened? We put two time-independent excitations on a field, and the ground state energy decreased. There is an attractive force between the two excitations!

Exercise 2.3.2. We didn't finish the computation of E_0: do the integral to obtain $E_0 = -e^{-mr}/4\pi r$ where r is the distance between $\vec{x}_1$ and $\vec{x}_2$.

Note: it may be confusing that earlier, we said the free-field theory is non-interacting, i.e. excitations do not interact, whereas here we clearly have an interaction (an attractive force) between two excitations. We must be careful what we mean by "excitation". An **excitation of the field** ϕ is represented by a propagator $D_F(x-y)$: we think of it as the exchange of a virtual particle between x and y. It is true that in free-field theory, two such field excitations do not interact, as we showed earlier with the $n=2$ term of $Z[J]$. However, two excitations in the form of **sources** and **sinks** placed on the field, i.e. terms in $J(x)$, are of course allowed to interact, via excitations of the field ϕ.

2.4 Feynman Diagrams

Okay, enough of free field theory; while it is interesting, it is unphysical to expect that excitations do not interact with each other. Let's move on to ϕ^4 **theory**, where the Lagrangian looks like

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4.$$

The λ is a **coupling constant**, and dictates how strongly the ϕ^4 term impacts the free field theory. The generating functional is now

$$Z[J,\lambda] = \int D\phi \exp\left(i\int d^4x\, \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 + J\phi\right).$$

How shall we compute $Z[J,\lambda]$? We have no idea whether it can be written in closed form. So the physicists do it perturbatively.

Let's consider a much easier problem to gain some insight: let q be a one-dimensional variable, and evaluate

$$Z = \int_{-\infty}^{\infty} dq\, \exp\left(-\frac{1}{2}q^2 + \lambda q^4 + Jq\right).$$

We know how to do this integral for $\lambda = 0$: it would just be a Gaussian. So expand it as a series

$$Z = \int_{-\infty}^{\infty} dq\, e^{-q^2/2+Jq}\left(1 + \lambda q^4 + \lambda^2 q^8 + \cdots\right).$$

How do we evaluate each individual term? Here's a trick:

$$\int_{-\infty}^{\infty} dq\, e^{-q^2/2+Jq} q^{4n} = \left(\frac{d}{dJ}\right)^{4n} \int_{-\infty}^{\infty} dq\, e^{-q^2/2+Jq},$$

and we know how to evaluate the remaining Gaussian integral! So

$$Z = \left(1 + \lambda\left(\frac{d}{dJ}\right)^4 + \lambda^2\left(\frac{d}{dJ}\right)^8 + \cdots\right)\int_{-\infty}^{\infty} dq\, e^{-q^2/2+Jq}.$$

We can do the original path integral for $Z[J]$ using this trick, but first we need to make sense of what d/dJ means when $J(x)$ is a function. Fortunately, mathematicians have done this for us already.

Definition 2.4.1. Let X be a space of functions, and $\Phi : X \to \mathbb{C}$ a functional (not necessarily linear). The **functional derivative** $\delta\Phi(f)/\delta f(x)$ is formally defined as

$$\frac{\delta\Phi(f)}{\delta f(x)} = \lim_{\epsilon\to 0} \frac{\Phi(f+\epsilon\delta_x) - \Phi(f)}{\epsilon},$$

where δ_x is a delta function with its pole at x. An important property is that $\delta f(x)/\delta f(y) = \delta^{(4)}(x-y)$.

Exercise 2.4.1. Show that

$$Z[J,\lambda] = Z[0,0] \exp\left(-\frac{i}{4!}\lambda \int d^4w \left(\frac{\delta}{\delta J(w)}\right)^4\right) \exp\left(-\frac{1}{2}\int d^4x\, d^4y\, J(x) D_F(x-y) J(y)\right)$$

by first extending the solution to the easier problem to multiple dimensions, and then "infinite dimensions". Then use the previous theorem, where we computed $Z[J,0]$.

At last, we arrive at the reason for which we have been doing all these calculations. Let's expand $Z[J,\lambda]$ in another way:

$$\begin{aligned} Z[J,\lambda] &= \int D\phi\, e^{i\int d^4x\, \mathcal{L}} \sum_{n=0}^{\infty} \frac{\left(i\int d^4x\, J(x)\phi(x)\right)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \left(\int dx_1 \cdots dx_n\, J(x_1)\cdots J(x_n)\right)\left(\int D\phi\, \phi(x_1)\cdots\phi(x_n) e^{i\int d^4x\, \mathcal{L}}\right). \end{aligned}$$

The second term looks oddly familiar. Indeed, it is (up to normalization), the n**-point correlation function** $\langle \phi(x_1)\cdots\phi(x_n)\rangle$ that we wanted to compute from a long time ago! So what the path integral has really given us is an extremely easy way to perturbatively compute the n-point correlation functions: we simply need to compute the coefficient of J^n, which is a series in λ. Taking the first few terms $\lambda^0, \lambda^1, \cdots$ will give an approximation to the correlation function.

Theorem 2.4.2. *The generating functional gives the following formula for n-point correlation functions:*

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle = (-i)^n \frac{\delta^n}{\delta J(x_1)\cdots\delta J(x_n)} \frac{Z[J,\lambda]}{Z[0,\lambda]}\bigg|_{J=0}.$$

For example, let's compute the λ^1 (first-order) terms in the 4-point correlation function. This is equivalent to computing the $\lambda^1 J^4$ term in $Z[J,\lambda]$. (**Note**: J^4 here really stands for $\left(\int dx\, J(x)\right)^4$, but we will abuse notation a little.) The $\lambda^1 J^4$ term comes from a J^8 term in $\exp(-(1/2)W[J])$ being differentiated by a λ^1 term in the other exponential:

$$\left(-\frac{i}{4!}\lambda\int d^4w \left(\frac{\delta}{\delta J(w)}\right)^4\right)\left(\frac{1}{4!2^4}\int d^4x_1\cdots d^4x_8\, J_1J_2J_3J_4J_5J_6J_7J_8D_{12}D_{34}D_{56}D_{78}\right),$$

where J_n stands for $J(x_n)$, and D_{ij} stands for $D_F(x_i - x_j)$.

Exercise 2.4.2. Do this computation. It is really not as bad as it looks: think of the action of $\partial/\partial J(w)$ as selecting one of the J_i's, and setting its variable, i.e. x_i, to w. So applying $(\partial/\partial J(w))^4$ really just picks four different x_i's and sets them to w. For example, we can pick x_2, x_4, x_6, x_8 to get a term proportional to

$$\lambda \int dw \int d^4x_1\, d^4x_3\, d^4x_5\, d^4x_7\, J_1J_3J_5J_7D_{1w}D_{3w}D_{5w}D_{7w}.$$

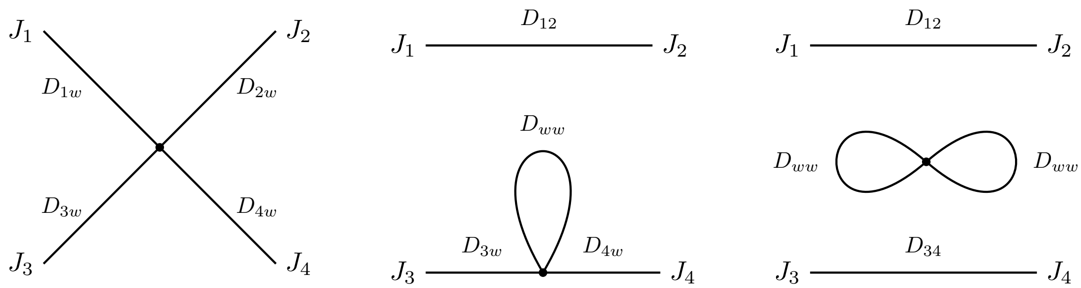
Note, however, there are many ways to get a term of this form. We could have picked any of the 4! permutations of x_2, x_4, x_6, x_8. Or any of the 4! permutations of x_1, x_3, x_5, x_7. Or we can substitute 1 for 2, or 3 for 4, etc. in any such choice. There are **symmetry factors** for each term. Compute these carefully. You should get the result

$$\int_w \int \int \int \int J_1 J_2 J_3 J_4 \left(-i\lambda D_{1w} D_{2w} D_{3w} D_{4w} - \frac{i\lambda}{2} D_{12} D_{3w} D_{4w} D_{ww} - \frac{i\lambda}{8} D_{12} D_{34} D_{ww} D_{ww} \right),$$

from which you can conclude that the first-order term in $\langle \phi(x_1) \cdots \phi(x_4) \rangle$ is

$$-i\lambda \int dw \left(D_{1w} D_{2w} D_{3w} D_{4w} + \frac{1}{2} D_{12} D_{3w} D_{4w} D_{ww} + \frac{1}{8} D_{12} D_{34} D_{ww} D_{ww} \right).$$

As with the free field case, we can physically interpret each of these terms. For example, in the first term, excitations from x_1, x_2, x_3, x_4 are propagating from/to an interaction point w. In the second term, there is a **self-interaction** from w to w, and interactions between x_1 and w, x_2 and w, and x_3 and x_4. We can similarly interpret the third term. The point is that to each term we can associate a little pictorial diagram of what is physically happening:



These diagrams both represent what is physically happening, in position space, and the terms in the integral. They are also known as **Feynman diagrams**!

When one does more of these types of calculations for other $\lambda^k J^n$ terms, one can see many patterns. In each of the three Feynman diagrams above, there are:

- one **internal vertex** (the black dot), corresponding to the comes from the variable w introduced by the λ^1 term (if we were to compute a λ^k term, there would be k internal vertices);
- four **external vertices** coming from the four J's (if we were to compute a J^n term, there would be n external vertices);
- four **propagators** coming from the four D_F's (if we were to compute a J^n term, there would be n external vertices).

Exercise 2.4.3. There are also rules for what possible diagrams can arise in the ϕ^4 theory. Convince yourself that any diagram has:

- four propagator ends at each internal vertex;
- an even number of external vertices;
- ((number of internal propagators) - (number of internal vertices) + 1) loops.

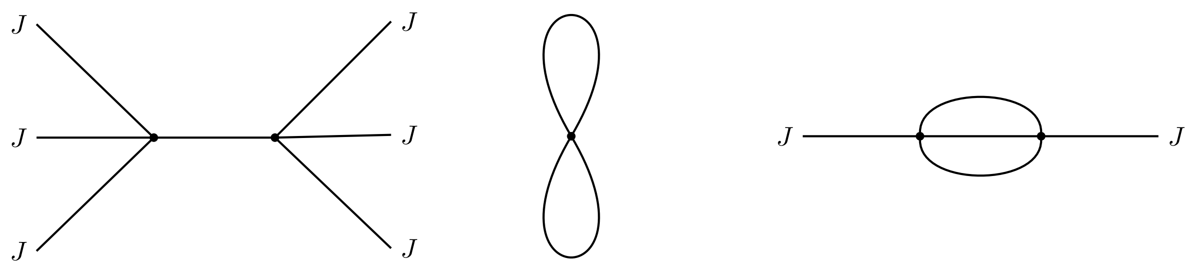
So we see that Feynman diagrams are good, intuitive, easy-to-manipulate representations of the terms of $Z[J, \lambda]$. In fact, instead of going from a term in $Z[J, \lambda]$ to a Feynman diagram, we can go the other way: given a Feynman diagram that can arise from the ϕ^4 theory, we can directly write down the term associated to it. This greatly simplifies the calculation of the terms, since now we can just draw pictures instead of taking $4k$ derivatives $\delta/\delta J(w)$ acting on $2n$ terms $J(x)$.

Theorem 2.4.3. *Given a Feynman diagram, one can write down its corresponding value using the* **position-space Feynman rules for ϕ^4 theory***:*

- *for each propagator between vertices x and y, add a $D_F(x-y)$ term;*
- *for each internal vertex, add a $(-i\lambda)\int d^4z$ term;*
- *for each external vertex, add a factor of 1;*
- *divide by the* **symmetry factor***.*

The symmetry factor is the number of ways of interchanging internal components of the diagram without changing the diagram itself.

Exercise 2.4.4. Symmetry factors take some getting used to. They arise from the combinatorics of how the derivatives $\delta/\delta J$ can hit different J, and the symmetry $D_F(x-y) = D_F(y-x)$. For example, when a diagram has a loop in it, we naively overcount by a factor of two because of the symmetry in D_F. Compute the symmetry factors of the following three diagrams using similar reasoning:



You should get 1 and $2 \cdot 2 \cdot 2 = 8$ and $3! = 6$ respectively.

Exercise 2.4.5. Rewrite the position-space Feynman rules in momentum-space, to obtain the **momentum-space Feynman rules for ϕ^4 theory**:

- for each internal propagator, label it with a momentum p and add a $i/(p^2 - m^2 + i\epsilon)$ term;
- for each external propagator (i.e. source), label it with a momentum p and add a e^{-ipx} term;
- for each internal vertex, add a $-i\lambda$ term;
- for each vertex, impose momentum conservation by adding a $\delta^{(4)}(p_1 + p_2 - p_3 - p_4)$ term, where p_1, p_2, p_3, p_4 are the four propagators entering/leaving the vertex;
- integrate over every undetermined momentum p with $\int d^4p/(2\pi)^4$;
- divide by the symmetry factor.

Note that in momentum space, we must orient each propagator. The orientation is arbitrary, since $D_F(x-y) = D_F(y-x)$, but necessary for imposing momentum conservation.

2.5 Connected vs Disconnected

Chapter 3

Quantum Electrodynamics

3.1 Functional Quantization of Spinor Fields

Now that we have understood the essentials of the path integral formulation we would now like to apply this formalism to understand how fermions interact in an electromagnetic field. Before we can get anywhere let us revisit the Dirac field and attempt to quantize it using the path integral formalism.

Recall that in our discussion of the Dirac field we realized that fermionic quantization required anti-commutation relations. This suggests that the eigenfunctions of these operators also need to anticommute. These sorts of variables are called Grassmann numbers: $\theta\eta = -\eta\theta$. To proceed, we first need to understand how to integrate over such variables. It will turn out that integration over these variables is much easier than regular integration! Consider the Taylor expansion $f(\theta) = A + B\theta + C\theta^2 + D\theta^3 + \cdots$. If θ is a Grassmann variable then $\theta^2 = 0$ and so $f(\theta) = A + B\theta$. The two properties that we would like of integration is that it is linear and invariant under a translation of variables, $\theta \to \theta + \eta_0$. Thus:

$$\begin{aligned}\int d\theta\, f(\theta) = \int d\theta\, A + B\theta = \int d(\theta + \eta_0)\, A + B(\theta + \eta_0) \\ = \int d\theta\, (A + B\eta_0) + B\theta\end{aligned}$$

By linearity, we expect that this integral be a linear function of the constant term, A, and the linear term, B. The last equality implies that The integral does not depend on the constant term. Therefore, we may assume that the integral evaluates to:[1]

$$\int d\theta\, A + B\theta = B$$

If we integrate more than one Grassmann number we need the following convention that $\int d\eta \int d\theta\, \theta\eta = 1$. This definition makes some integrals very easy to evaluate. We compile a few that will be useful for us later on:

$$\int d\theta^* d\theta e^{-\theta^* b\theta} = b^{-1}, \quad \left(\prod_i \int d\theta_i^* d\theta_i\right) e^{-\theta_i^* B_{ij}\theta_j} = \prod_i b_i = \det B, \quad \left(\prod_i \int d\theta_i^* d\theta_i\right) \theta_k \theta_l^* e^{-\theta_i^* B_{ij}\theta_j} = (\det B)(B^{-1})_{kl} \tag{3.1}$$

[1] Integration by differentiation!

Recall that the Lagrangian for the Dirac field is given by $\mathcal{L}_{Dirac} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$ where $\partial\!\!\!/ = \gamma^\mu \partial_\mu$. Using the integrals given in (3.1) it is possible to immediately calculate the correlation functions for the free field theory directly. Instead, we will get some more practice with our generating functional method.

Take the generating functional,

$$Z[\bar{\eta}, \eta] = \int D\bar{\psi} D\psi \exp\left[i \int d^4x\, \bar{\psi}(i\partial\!\!\!/ - m)\psi + \bar{\eta}\psi + \bar{\psi}\eta\right]$$

So that if you make the substitution $\psi \to \psi + (i\partial\!\!\!/ - m)^{-1}\bar{\eta}$ then we can evaluate Z explicitly:

$$Z[\bar{\eta}, \eta] = Z_0 \cdot \exp\left[-\int d^4x d^4y \bar{\eta}(x) S_F(x-y)\eta(y)\right]$$

Therefore we are in position to evaluate arbitrary n-point correlation functions. This is where the generating functional method shines, on the one hand it is easy to compute explicitly, and on the other hand by differentiating it in a special way it gives rise to the correlation functions! In the free field case, here is an example of such a correlation function calculation:

$$\langle 0|T\psi(x_1)\psi(x_2)|0\rangle = Z_0^{-1} \cdot \left(\frac{-i\delta}{\delta\bar{\eta}(x_1)}\right)\left(\frac{+i\delta}{\delta\eta(x_2)}\right) Z[\bar{\eta}, \eta] = \left(\frac{-i\delta}{\delta\bar{\eta}(x_1)}\right)\left(\frac{+i\delta}{\delta\eta(x_2)}\right) \exp\left[-\int d^4x d^4y \bar{\eta}(x) S_F(x-y)\eta(y)\right]$$

What are possible interactions that we can add to the Dirac field? Since we've already discussed the scalar field, we may start by coupling the Dirac field to the scalar field and trying to compute the correlation functions.

To talk about QED we need the quantized version of the electromagnetic field. Here we go!

3.2 Functional Quantization of Electromagnetic Field

The electromagnetic field is described by a 4-vector, $A_\mu = (\phi, \vec{A})$, which packages the information about the electric field into ϕ and the magnetic field into $\vec{A}$. By taking the Lagrangian: $\mathcal{L}_{EM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, we recover Maxwell's equations (exercise!). We may write the Lagrangian explicitly in terms of the A_μ:

3.3 Aside: Scattering Amplitudes

Let us briefly try to understand the big picture – or, in fact, I guess it's the small picture! A **cross section** is the fundamental physical quantity that an experimental particle physicist is interested in, here's roughly what it is. Suppose you want to understand the inner workings of protons. You will devise an experiment to smash some number of protons together and try to investigate the result. In fact, your theorist friends suggest to you some possible types of particles that can come out. Ideally, if we're not in one already, we will consider two protons colliding to produce other species: $pp \to abc \ldots$.

Your detectors cover only a portion of 4π, as the experimentalists call it, that is, the sphere surrounding the collision point. Therefore, we would like to compute the amount of scattering events in the direction of the detector. This is given the symbol:

$$\left(\frac{d\sigma}{d\Omega}\right)$$

This is a very technical object and it depends on a lot of parameters. Qualitatively, we may simplify this object by considering just the **scattering events** which are partly characterized by momenta (particle type, spin, etc. are also valid quantum numbers). The scattering events can be analyzed by looking at the **S-matrix**:

$$\begin{aligned}\langle \vec{p}_1 \cdots |S|k_A k_B\rangle &:= {}_{\text{out}}\langle \vec{p}_1 \vec{p}_2 \cdots |\vec{k}_A \vec{k}_B\rangle_{\text{in}} \\ &= \lim_{T\to\infty} \langle \vec{p}_1 \cdots |e^{iH(2T)}|\vec{k}_A \vec{k}_B\rangle\end{aligned}$$

In words, this means that we assume that our particles are idealized momentum eigenstates and long after they interact they again will be idealized momentum eigenstates. We apply a trick, that sometimes particles don't interact at all, and we can take that out of the S matrix:

$$S = \mathbb{1} + iT$$

Assuming unitarity of S (which is a big assumption!) T is a Hermitian operator (think of $e^{i\theta}$) and it contains all of the interesting dynamics. By also considering energy conservation can be written as:

$$\langle p_1 \dots |iT|k_A k_B\rangle = (2\pi)^4 \delta^{(4)}(k_A + k_B - \sum p_f)\mathcal{M}(k_A, k_B \to \{p_f\})$$

What this means is that the scattering amplitude is really a distribution and the density is given by the expression $\mathcal{M}$. To briefly connect this back to the cross section, in the case that there are four identical particles (two incoming and two outgoing) then the cross section will be (PS 4.85)

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{cm}^2}.$$

Computing scattering amplitude using cross sections

Now we interpret the scattering amplitude as something very similar to a correlation function. Recall a correlation function for us was

$$\langle\Omega|T\{\phi(x_1)\phi(x_2)\}|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\int D\phi\, \phi(x_1)\phi(x_2) \exp[i \int_{-T}^{T} \mathcal{L}]}{\int D\phi\, \exp[i \int_{-T}^{T} \mathcal{L}]} \tag{3.2}$$

$$= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\{\phi(x_1)\phi(x_2) \exp[-i \int_{-T}^{T} dt\, H_I(t)]\}|0\rangle}{\langle 0|T\{\exp[-i \int_{-T}^{T} dt\, H_I(t)]\}|0\rangle} \tag{3.3}$$

Although we haven't explicitly mentioned this in our lectures, the formula in line (3.3) has appeared in the derivation of (3.2) (see (2.1) to compare).

Here's a quick review of how we derived a path integral representation for the correlation functions:

- Using a leap of intuition we define (or is it prove?)

$$\langle\phi(a)|e^{-iHt}|\phi(b)\rangle = \int \mathcal{D}\phi \exp\left[i \int d^4x \mathcal{L}\right]$$

- Investigate the expression

$$\int \mathcal{D}\phi\, \phi(x_1)\phi(x_2) \exp\left[i \int d^4x \mathcal{L}\right]$$

  by splitting up the path into three sections.
- Simplify this to get

$$\langle\phi_b(\vec{x})|\, e^{-i\hat{H}(t-x_2^0)}\phi(\vec{x}_2)e^{-i\hat{H}(x_2^0-x_1^0)}\phi(\vec{x}_1)e^{-i\hat{H}(x_1^0-(-t))}\,|\phi_a(\vec{x})\rangle \tag{3.4}$$

- The Canonical Quantizers prefer to rewrite this expression up to prefactors (which come from approximating $\phi_{a,b}(\vec{x})$ with $|\Omega\rangle$):

$$\langle 0|T\left\{\phi_I(x_1)\phi_I(x_2)\exp[-i\int_{-T}^{T} dt\, H_I(t)]\right\}|0\rangle$$

  The extra I's on the ϕ's absorb the free evolution into the original ϕ's.

Now we may interpret the scattering amplitude as the following:

$$\langle\vec{p}_1\ldots\vec{p}_n|iT|\vec{p}_A\vec{p}_B\rangle = \lim_{T\to\infty(1-i\epsilon)} {}_0\langle\vec{p}_1\ldots\vec{p}_n|e^{-iH(2T)}|\vec{p}_A\vec{p}_B\rangle_0 \tag{3.5}$$

$$\propto \lim_{T\to\infty(1-i\epsilon)}\left({}_0\langle\vec{p}_1\ldots\vec{p}_n|T\exp\left[-i\int_{-T}^{T} dt\, H_I(t)\right]|\vec{p}_A\vec{p}_B\rangle_0\right)_{\text{connected, amputated}} \tag{3.6}$$

The zeros on the end of the bracket refer to idealistic wavepackets. Our next goal will be to interpret this amplitude as a sum of Feynman diagrams. We will now introduce **Wick's theorem**, and thus **Wick contractions**, using which we'll be able to write down the Feynman diagrams very easily.

Wick's Theorem

In canonical quantization there are raising and lowering operators. We introduce a particular order for expressions involving operators: raising on the left and lowering on the right – **Normal Order**. A Wick contraction is defined by:

$$\phi(\overset{\sqcap}{x)\phi(}y) = D_F(x-y)$$

Finally Wick's theorem states the following:

Theorem 3.3.1.

$$T\{\phi(x_1)\phi(x_2)\cdots\phi(x_n)\} = N\{\phi(x_1)\phi(x_2)\cdots\phi(x_n) + \textit{all possible contractions}\}$$

This is a theorem with very major *computational* consequences for physics. In particular, this makes the Feynman diagrams immediate! Let's go through an example from Peskin and Schroeder:

$$\langle 0|T\left\{\phi(x)\phi(y) + \phi(x)\phi(y)\left[-i\int dt\int d^3z\frac{\lambda}{4!}\phi(z)^4\right] + \cdots\right\}|0\rangle$$

$$= \langle 0|T\{\phi(x)\phi(y)\}|0\rangle + \frac{(-i\lambda}{4!}\langle 0|T\{\phi(x)\phi(y)\phi(z)\phi(z)\phi(z)\phi(z)|0\rangle$$

Apply Wick's Theorem:

$$
\begin{aligned}
&\langle 0|N\{\phi(x)\phi(y) + \phi(x)\phi(y)\}|0\rangle \\
&+ \frac{(-i\lambda)}{4!}\langle 0|N\{\phi(x)\phi(y)\phi(z)\phi(z)\phi(z)\phi(z)\}|0\rangle \\
&+ \frac{(-i\lambda)}{4!}\langle 0|N\{\phi(x)\phi(y)\phi(z)\phi(z)\phi(z)\phi(z)\}|0\rangle \\
&+ \frac{(-i\lambda)}{4!}\langle 0|N\{\phi(x)\phi(y)\phi(z)\phi(z)\phi(z)\phi(z)\}|0\rangle + \cdots \\
&+ \frac{(-i\lambda)}{4!}\langle 0|N\{\phi(x)\phi(y)\phi(z)\phi(z)\phi(z)\phi(z)\}|0\rangle + \cdots \\
&+ \frac{(-i\lambda)}{4!}\langle 0|N\{\phi(x)\phi(y)\phi(z)\phi(z)\phi(z)\phi(z)\}|0\rangle + \cdots
\end{aligned}
$$

Now you can probably see where the Feynman Diagrams will be appearing! They In fact, using this approach I would hazard that it's even easier to see where the Feynman diagrams are coming from! Because of the normal ordering most of these are zero, in fact only the Fuchsia coloured terms are non-zero.

Let's apply these rules to calculate the scattering amplitude (finally!). One last rule before we do this, since $|\vec{p}\rangle \propto a^\dagger_{\vec{p}}|0\rangle$, it makes sense to define:

$$\phi(x)|p\rangle = e^{-ip\cdot x} \qquad\qquad \langle p|\phi(x) = e^{+ip\cdot x}$$

Now, we compute a few examples. NOTE: the first amplitude does **not** have a time ordering so we compute it directly.

$$\langle p_1p_2|p_Ap_B\rangle = \sqrt{2E_1 2E_2 2E_A 2E_B}\,\langle 0|a_1a_2a^\dagger_A a^\dagger_B|0\rangle = (diagram)$$

$$
\begin{aligned}
\langle p_1p_2|\mathcal{T}\left\{\frac{(-i\lambda)}{4!}\int d^4x\phi_x\phi_x\phi_x\phi_x\right\}|p_Ap_B\rangle = \frac{(-i\lambda)}{4!}\int d^4x\Big(&(\ )\times\langle p_1p_2|\phi_x\phi_x\phi_x\phi_x|p_Ap_B\rangle \\
&+(\ )\times\langle p_1p_2|\phi_x\phi_x\phi_x\phi_x|p_Ap_B\rangle \\
&+(4!)\times\langle p_1p_2|\phi_x\phi_x\phi_x\phi_x|p_Ap_B\rangle
\end{aligned}
$$

The blue terms are not interesting. The very first term is just zero because they contribute to the $\mathbb{1}$ in $S = \mathbb{1} + iT$; the second blue term is also not interesting because it also contributes to the $\mathbb{1}$. The fuchsia term is very interesting because by applying the rules that we've expressed above we get:

$$\langle p_1p_2|iT|p_Ap_B\rangle \approx \frac{-i\lambda}{4!}\int d^4x\,(4!)\times e^{i(p_1+p_2-p_A-p_B)\cdot x} = (-i\lambda)\cdot(2\pi)^4\delta^{(4)}(p_1+p_2-p_A-p_B)$$

But we also had,

$$\langle p_1p_2|iT|p_Ap_B\rangle = i\mathcal{M}(p_1p_2\to p_Ap_B)\cdot(2\pi)^4\delta^{(4)}(p_1+p_2-p_A-p_B)$$

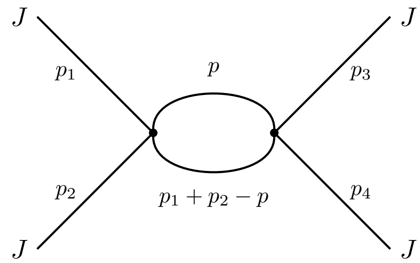
To be continued...

- (4.96-4.103)
- Trick: exponentiation of vacuum bubbles $\equiv$ to finding the phase factor for $|\Omega\rangle \propto |0\rangle$. Similarly the amputations on external legs comes from the 0 in the formular for the S-matrix, (3.6).
- Write down Feynman rules for Bosons, Fermions, EM.
- Compute one example, if running out of time, end off with:
- Do Coulomb potential (4.133-4.136)

Chapter 4

Renormalization

There is a problem with the machinery we have been developing that we have been sweeping under the rug. This problem is best illustrated by explicitly computing the following Feynman diagram for ϕ^4 theory:



Here we've labeled each source and propagator with its momentum, where since p is an internal momentum, we must integrate over it. The value of the diagram is

$$\mathcal{M} = \frac{(-i\lambda)^2}{2} \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} \frac{i}{(p_1 + p_2 - p)^2 - m^2 + i\epsilon}.$$

As $|p| \to \infty$, the integrand is $O(1/p^4)$, which diverges logarithmically when integrated over! This divergence cannot simply be ignored, since it does not get absorbed into the $Z[0,0]$ constant that we can ignore; it is part of a J^4 term. In fact, there are many such divergences. An easy observation is that any diagram with this diagram as a sub-diagram will have the same divergence. So we must develop a way to remove these divergences, or, rather, explain why they come about in our naive theory, and how to avoid them in a more sophisticated theory. In this chapter, we will develop such a theory and systematically remove these divergences using **renormalization theory**.

4.1 Counting Divergences

The first step to getting rid of divergences is to understand what kind of divergences occur, and this we will do by analyzing the divergent Feynman diagrams. For this section, we will work in d dimensions (instead of 4) to make the general structure of the results clearer.

Why did the above diagram diverge? The obvious answer is that there weren't enough powers of p in the denominator.

Definition 4.1.1. The **superficial degree of divergence** D is the difference

$$D = (\text{power of } p \text{ in numerator}) - (\text{power of } p \text{ in denominator}).$$

Every integration dp counts as a power of p in the numerator. If $D \geq 0$, we say the integral is **superficially divergent**.

If $D \geq 0$ then surely the integral is divergent. However, this is not an if and only if. For example, take $\iint dx\, dy\, 1/(1+x^2)^2$ in $\mathbb{R}^2$, for which $D = -2$, yet the integral still diverges because it does not decay in y. The good news is that this is the only sort of pathological divergence we will get.

Theorem 4.1.2 (Weinberg-Dyson)**.** *A connected Feynman diagram is convergent if and only if the it and all of its connected subdiagrams are not superficially divergent.*

Suppose now that we are given a Feynman diagram for some process in ϕ^4 theory, with V internal vertices, E external propagators (from sources J to internal vertices), and I internal propagators. Using the momentum-space Feynman rules, each internal propagator decreases D by 2, and every momentum being integrated over increases D by 4. How many such momenta are there? Well, each internal propagator is associated to one, but every internal vertex adds a single linear constraint on them. So there are $I - V + 1$ independent momenta (since global momentum conservation frees one constraint), and

$$D = -2I + d(I - V + 1) = (d-2)I - dV + d.$$

Now note that every external propagator connects to one internal vertex, every internal propagator to two, and every vertex connects four propagators, so $2I + E = 4V$. Then

$$D = \frac{1}{2}(d-2)(4V - E) - dV + d = (d-4)V - \frac{1}{2}(d-2)E + d.$$

The behavior of D depends on the dimensionality d.

- If $d = 2$ or $d = 3$, as the number of vertices increases, D decreases. Hence there can only be a finite number of superficially divergent diagrams. QFTs with this property are **super-renormalizable**.
- If $d = 4$ (the case we care about), $D = 4 - E$. Hence the only superficially divergent diagrams are those with at most four sources, but there are infinitely many of them, and $D \leq d$. QFTs with this sort of property are **renormalizable**.
- If $d \geq 5$, there are diagrams of all orders that are superficially divergent (due to the differing signs of V and E). QFTs with this property are **non-renormalizable**.

Exercise 4.1.1. Consider QED. Let V be the number of internal vertices, E_e, E_γ be the number of external electron and photon propagators, respectively, and I_e, I_γ the number of internal ones, respectively. Compute that

$$D = \frac{d-4}{2}V - \frac{d-2}{2}E_\gamma - \frac{d-1}{2}E_e,$$

and thus conclude that QED is renormalizable in $d = 4$.

Exercise 4.1.2. In general, show that a QFT in d dimensions with several different fields f and several different interactions i has

$$D = d - \sum_f a_f E_f - \sum_i b_i V_i,$$

where

- E_f is the number of external propagators of field type f,
- V_i is the number of internal vertices of interaction type i,
- $-d + 2a_f$ is the degree of the propagator for field type f,

- $[m^{b_i}]$ is the dimension (in natural units, where $[l] = [t] = [m^{-1}]$) on the coupling constant for interaction type i.

Exercise 4.1.3 (Optional, for those who know general relativity or Riemannian geometry)**.** The **Einstein-Hilbert action** in general relativity is

$$S = \int d^4x \left(\frac{R}{16\pi} + \frac{\Lambda}{8\pi} \right) \sqrt{-g}$$

where R is the Ricci curvature, Λ is the cosmological constant, and g the metric. Compute the dimension of R and therefore conclude that gravity is non-renormalizable.

4.2 Regularization

Obviously, we want to call a QFT renormalizable only if we can really make the divergences vanish in a systematic way. Of course, it turns out we can. There are two distinct steps in doing so: **regularization**, and **renormalization**. We first tackle regularization.

The idea behind regularization is simple: if we have a physical theory, there are limitations when it is valid. For example, classical mechanics applied to particles is a good approximation to quantum mechanics in low energy regimes, where we can pretend $\hbar = 0$, but it is by no means valid at, say, energies where mass-energy equivalence allows for the creation of new particles. (In fact, we need QFT for those regimes.) At those energies, we must incorporate more subtle corrections to the dynamics that come from QM, or QFT. Here's an example of regularization in action, alongside some cool physics.

Example 4.2.1 (Casimir effect)**.** Take a large $L \times L \times L$ vacuum cavity and insert another $L \times L$ plate parallel to one of the walls at a distance $d \ll L$. The plate disturbs the electromagnetic field in the vacuum cavity, and create some energy E relative to the ground state energy 0. Then we expect to detect a force $F = -\partial E/\partial d$ on the plate. Since this is just an example of regularization, let's simplify a bit. Instead of the EM field, which is a spinor field, we work with a massless scalar field, and in $1+1$ dimensions. From EM/QM, we know the EM modes in the cavity are quantized, with wave vector $k = n\pi/d$, i.e. modes $\sin(n\pi x/d)$. Naively, then, $E = f(d) + f(L-d)$ where

$$f(d) = \frac{1}{2} \sum_{n=1}^{\infty} \omega_n = \frac{\pi}{2d} \sum_{n=1}^{\infty} n \to \infty.$$

But we are neglecting the physics: high-frequency waves are not "seen" by the plates, whose electrons move at a finite speed. So introduce a parameter s and a factor $e^{-sn/d}$ to dampen the energy contribution of higher-frequency modes:

$$f(d) = \frac{1}{2d} \sum_{n=1}^{\infty} n e^{-sn/d} = -\frac{\pi}{2} \frac{\partial}{\partial s} \sum_{n=1}^{\infty} e^{-sn/d} = -\frac{\pi}{2} \frac{\partial}{\partial s} \frac{1}{1 - e^{-s/d}} = \frac{\pi}{2d} \frac{e^{s/d}}{(e^{s/d} - 1)^2}.$$

As a series,

$$f(d) = \frac{\pi d}{2s^2} - \frac{\pi}{24d} + \frac{\pi s^2}{480 d^3} + O(s^4).$$

It remains to compute

$$F = -\frac{\partial E}{\partial d} = -(f'(d) - f'(L-d)) \xrightarrow{s \to 0} -\frac{\pi}{24} \left(\frac{1}{d^2} - \frac{1}{(L-d)^2} \right).$$

For $d \ll L$, this is approximately $F = -\pi/24d^2$, the attractive **Casimir force**. In particular, it is finite. In addition, there are other things to note here.

- The general technique used here is known as **zeta function regularization**.
- One may object to this calculation, because it depended on a choice of **regularization** term $e^{-sn/d}$. However, it can be shown that F is independent of the choice of regularization.
- This calculation, which is often repeated in a different context in string theory, is the source of all the silliness surrounding $\sum_{n=1}^{\infty} n = -1/12$: the "equality" arises from matching the $\pi/2d$ terms in the unregularized and regularized $f(d)$ (since other terms cancel or vanish as $s \to 0$ in F).

Let's look at the divergent diagram at the beginning of this chapter. The integral is over the momentum p. As $p \to \infty$, the energy of the internal propagator also goes to infinity. But we don't expect QFT to be valid up to arbitrarily high energies! The solution, as with the Casimir effect, is to impose a cutoff: instead of integrating to infinity, we integrate up to some value Λ. To do the resulting integral, we introduce **Mandelstam variables** $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$, and $u = (p_1 - p_4)^2$. Then it turns out the amplitude is

$$\mathcal{M} = -i\lambda + \frac{i\lambda^2}{32\pi^2}\left[\log\frac{\Lambda^2}{s} + \log\frac{\Lambda^2}{t} + \log\frac{\Lambda^2}{u}\right] + O(\lambda^3).$$

This method is essentially **Pauli-Villars regularization**. But we are not going to regularize this way, because it is not gauge covariant: later on, when we have a non-abelian gauge group, this kind of regularization is going to upset our choice of gauge. Instead, we are going to use **dimensional regularization**.

Dimensional regularization is exactly what it sounds like. We notice that some integrals are divergent at $d = 4$, but not necessarily divergent at lower dimensions. So we perform analytic continuation in the number of dimensions! This is really not as stupid as it sounds. To save some time, we'll do an example, and along the way, introduce the necessary machinery.

4.2.1 Basic One-Loop Diagram in ϕ^4

At the beginning of this chapter, we considered the basic one-loop diagram in ϕ^4 theory, whose amplitude (rewritten a little) is given by

$$\mathcal{M} = \frac{(-i\lambda)^2}{2}\int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - m^2 + i\epsilon}\frac{i}{(p+q)^2 - m^2 + i\epsilon}.$$

Here q is some combination of external momenta. This integral is has superficial degree of divergence 0; we are going to regularize it and get the same result as we would have gotten using Pauli-Villars regularization.

We begin by doing a **Wick rotation**. This is where we change from integrating in Minkowski space to Euclidean space, by substituting q^0 with iq^0, and p^0 with ip^0:

$$\mathcal{M} = \frac{(-i\lambda)^2}{2}\int \frac{d^4p}{(2\pi)^4}\frac{-i}{(|p|^2 + m^2)(|p+q|^2 + m^2)},$$

where here $|p|^2$ denotes the Euclidean (as opposed to Minkowski) norm. Note that we can remove the $i\epsilon$ now that there are no poles.

The next step is to apply the following trick, called **Feynman's formula** to turn the product of quadratics in the denominator into a power of a single quadratic.

Proposition 4.2.2 (Feynman's formula). *Suppose $c_1, \ldots, c_n \in \mathbb{C}$ such that their convex hull does not contain the origin. Then*

$$\frac{1}{c_1 \cdots c_n} = (n-1)!\int_{[0,1]^n} d^n x\, \frac{\delta(1 - \sum x_j)}{\left(\sum c_j x_j\right)^n}.$$

Exercise 4.2.1. Prove Feynman's formula by induction on n using the following steps. First prove the base case $n = 2$:

$$\int_0^1 \frac{dx}{(c_1 x + c_2(1-x))^2} = \frac{1}{c_1 c_2}.$$

Then differentiate both sides $n-1$ times with respect to c_1 to get a formula for $1/c_1^n c_2$. Finally, do the inductive step using this formula.

To apply Feynman's formula to our Wick-rotated integral, we let c_j be the j-th quadratic term in the denominator, to get

$$\mathcal{M} = \frac{(-i\lambda)^2}{2} \int \frac{d^4p}{(2\pi)^4} \int_0^1 dx \, \frac{i}{(|p|^2 + 2xp \cdot q + x|q|^2 + m^2)^2}.$$

The third step is to do a **linear change of variables** so that the denominator looks like $(|p|^2 + c(q,x))^{I+1}$ for some positive quadratic function c. In our case, the substitution $k = p + xq$ suffices:

$$\mathcal{M} = \frac{(-i\lambda)^2}{2} \int_0^1 dx \int \frac{d^4p}{(2\pi)^4} \frac{i}{(|k|^2 + x(1-x)|q|^2 + m^2)^2}.$$

The final step is to **evaluate the inner integral over d dimensions**, instead of 4, and Wick-rotate back to Minkowski space. This is not hard to do, since now the inner integral is guaranteed to be a radial function, so we can work in polar coordinates. The key to evaluating the polar integral is the following formula.

Proposition 4.2.3. *Let $k \in \mathbb{Z}$ and $0 < d < 2n - k$. Then*

$$\int_0^\infty dr \, \frac{r^{2k+d-1}}{(r^2+c^2)^n} = \frac{c^{2k+d-2n}}{2} \frac{\Gamma(k + d/2)\Gamma(n-k-d/2)}{\Gamma(n)}.$$

Proof. Substitute $t = (r/c)^2$ and evaluate the integral in terms of the beta function $B(x,y)$. Then recall that $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$. $\square$

Let's carry through with the calculation:

$$\begin{aligned}
\int \frac{d^dp}{(2\pi)^d} \frac{1}{(|k|^2 + x(1-x)|q|^2 + m^2)^2} &= \frac{2\pi^{d/2}}{\Gamma(d/2)(2\pi)^d} \int_0^\infty \frac{r^{d-1}\, dr}{(r^2 + x(1-x)|q|^2 + m^2)^2} \\
&= \frac{2}{\Gamma(d/2)(4\pi)^{d/2}} \frac{1}{2(x(1-x)|q|^2 + m^2)^{(4-d)/2}} \frac{\Gamma(d/2)\Gamma(2-d/2)}{\Gamma(2)} \\
&= \frac{\Gamma(2-d/2)}{(4\pi)^{d/2}(x(1-x)|q|^2 + m^2)^{(4-d)/2}}.
\end{aligned}$$

Wick-rotating back and plugging this result back into the outer integral for $\mathcal{M}$, which we call $I_d(q)$,

$$\mathcal{M} = I_d(q) = -i\frac{(-i\lambda)^2}{2} \frac{\Gamma(2-d/2)}{(4\pi)^{d/2}} \int_0^1 \frac{dx}{(m^2 - x(1-x)q^2)^{(4-d)/2}}.$$

If we carefully keep track of convergence issues, we will discover that as long as $d \in (3,4)$, we are okay in this case. For later convenience, define a new function $J_d(s)$ such that $I_d(q) = (-i\lambda)^2 J_d(q^2)$.