Mirror Symmetry
Summer 2016 Seminar Notes

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Chapter 1

Mathematical Preliminaries

The aim of this chapter is to give a brief review of the required mathematical background for mirror symmetry.

1.1 Cohomology Theories

Throughout this section, X is a complex manifold, and H_{dR} is de Rham cohomology. We examine the relationships between some common cohomology theories on X.

1.1.1 Sheaf Cohomology

All of our sheaves take values in abelian groups. Let $\mathcal{F}$ be a presheaf on X.

Definition 1.1.1. Recall the definition of a **presheaf** $\mathcal{F}$ on X:

1. (presheaf) every open set U in X is assigned an abelian group $\mathcal{F}(U)$, such that if $V \subseteq U$ are two open sets, there is a restriction map $-|_{U\to V} : \mathcal{F}(U) \to \mathcal{F}(V)$ compatible with inclusion, i.e. $(-|_{U\to V})|_{V\to W} = -|_{U\to W}$ for any $W \subseteq V \subseteq U$.

If in addition $\mathcal{F}$ satisfies the following two properties, it is a **sheaf**:

2. (locality) if $\{U_\alpha\}$ is an open cover of X and $f, g \in \mathcal{F}(X)$ such that $f|_{U\to U_\alpha} = g|_{U\to U_\alpha}$ for every U_α, then $f = g$;

3. (gluing) if $\{U_\alpha\}$ is an open cover of X and $f_\alpha \in \mathcal{F}(U_\alpha)$ for every U_α are elements agreeing on overlaps, i.e. such that $f_\alpha|_{U_\alpha\to U_\alpha\cap U_\beta} = f_\beta|_{U_\beta\to U_\alpha\cap U_\beta}$, then we can glue the f_α together to get $f \in \mathcal{F}(X)$, i.e. $f|_{X\to U_\alpha} = f_\alpha$ for every U_α.

Definition 1.1.2. Let $\mathcal{U} = \{U_\alpha\}$ be an **ordered open cover** of X, i.e. with a partial order such that if α and β are incomparable then $U_\alpha \cap U_\beta$ is empty. A p**-simplex** σ of $\mathcal{U}$ is a totally ordered collection of open sets $U_{\alpha_0}, \dots, U_{\alpha_p} \in \mathcal{U}$; we call $U_{\alpha_0,\dots,\alpha_p} \coloneqq U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ its **support**, and often refer to σ by it instead. The k**-th boundary component** of a p-simplex $U_{\alpha_0,\dots,\alpha_p}$ is given by $\partial_k U_{\alpha_0,\dots,\alpha_p} \coloneqq U_{\alpha_0,\dots,\hat{\alpha}_k,\dots,\alpha_p}$. Cochains are maps from simplices to sheaf sections, and form a cochain complex:

$$C^p(\mathcal{U},\mathcal{F}) \coloneqq \prod_{\alpha_0<\cdots<\alpha_p} \mathcal{F}(U_{\alpha_0,\dots,\alpha_p}), \quad (\delta^p\omega)(\sigma) \coloneqq \sum_{k=0}^{p+1} (-1)^k \omega(\partial_k\sigma)|_\sigma : C^p(\mathcal{U},\mathcal{F}) \to C^{p+1}(\mathcal{U},\mathcal{F}).$$

The **Čech cohomology of** $\mathcal{U}$ with coefficients in $\mathcal{F}$, denoted $\check{H}^\bullet(\mathcal{U},\mathcal{F})$, is the cohomology of this complex.

Example 1.1.3. Let $\mathcal{F}$ be a sheaf. By the gluing condition for a sheaf, a global section $f \in \mathcal{F}(X)$ is defined by its values $f_\alpha := f|_{X \to U_\alpha} \in \mathcal{F}(U_\alpha)$ on every U_α in an open cover. These f_α form precisely the data for an element of $C^0(\mathcal{U}, \mathcal{F})$, and satisfy the gluing condition $f_\alpha = f_\beta$ on $U_\alpha \cap U_\beta$, which is precisely the statement $\delta_0 f = 0$. Hence $\check{H}^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$ for a sheaf $\mathcal{F}$.

Example 1.1.4. Let $\mathcal{O}$ denote the sheaf of holomorphic functions (and $\mathcal{O}^*$ the nowhere-zero ones) on $\mathbb{P}^1$. Recall that on $\mathbb{P}^1$ we have the charts $U = \mathbb{P}^1 \setminus \{S\}$ and $V = \mathbb{P}^1 \setminus \{N\}$, with coordinates u and v respectively. To look at sections of $\mathcal{O}(U)$ versus $\mathcal{O}(V)$, we use the transition map $v = u^{-1}$ on $U \cap V$. The cochains for this open cover are

$$C^0(\mathcal{U}, \mathcal{O}) = \mathcal{O}(U) \times \mathcal{O}(V), \quad C^1(\mathcal{U}, \mathcal{O}) = \mathcal{O}(U \cap V), \quad C^k(\mathcal{U}, \mathcal{O}) = 0 \;\forall k \geq 2.$$

We compute the sheaf cohomology.

1. ($\check{H}^0(\mathcal{U}, \mathcal{O})$) The boundary map δ_0 maps $(f, g) \in C^0(\mathcal{U}, \mathcal{O})$ to $g - f$. But

$$f = \sum_{k=0}^{\infty} f_k u^k, \quad g = \sum_{k=0}^{\infty} g_k v^k = \sum_{k=0}^{\infty} g_k u^{-k},$$

   so $g - f = 0$ iff $f_k = g_k = 0$ for all $k > 0$, and $f_0 = g_0$. Hence $\check{H}^0(\mathcal{U}, \mathcal{O}) \cong \mathbb{C}$, consisting of all constant functions.

2. ($\check{H}^1(\mathcal{U}, \mathcal{O})$) Given $h \in C^1(\mathcal{U}, \mathcal{O})$, rewrite its Laurent expansion:

$$h = \sum_{k=-\infty}^{\infty} h_k u^k = \sum_{k=0}^{\infty} h_k u^k + \sum_{k=1}^{\infty} h_k v^k = -f + g$$

   where $f \in \mathcal{O}(U)$ and $g \in \mathcal{O}(V)$. Hence $h \in \operatorname{im} \delta_0$, and $\check{H}^1(\mathcal{U}, \mathcal{O}) = 0$.

3. ($\check{H}^k(\mathcal{U}, \mathcal{O})$) Trivially, $\check{H}^k(\mathcal{U}, \mathcal{O}) = 0$ for $k \geq 2$.

Note that $\check{H}^0(\mathcal{U}, \mathcal{O}) \cong \mathbb{C}$ is consistent with what we know so far, since $\check{H}^0(\mathcal{U}, \mathcal{O}) = \mathcal{O}(\mathbb{P}^1)$, and Liouville's theorem shows that $\mathcal{O}(\mathbb{P}^1)$ can only contain constant functions.

Example 1.1.5. Recall the tautological line bundle $\mathcal{O}(-1)$ and its dual $\mathcal{O}(1)$ on $\mathbb{P}^n$; we have $\mathcal{O}(n) = \mathcal{O}(1)^n$. On the same charts on $\mathbb{P}^1$, since $\mathcal{O}(1)$ has transition function $u = v^{-1}$, we know $\mathcal{O}(n)$ has transition function $u^n = v^{-n}$. To construct a global section of $\mathcal{O}(n)$, given a monomial v^k on V, we require $u^n v^k = u^{n-k}$ to be well-defined on U, so $k \leq n$. In homogeneous coordinates $[x_0 : x_1]$, the global sections are therefore $x_0^n, x_0^{n-1} x_1, \ldots, x_1^n$, the homogeneous polynomials of degree n. The same story holds on $\mathbb{P}^N$. Hence $\dim H^0(\mathbb{P}^N, \mathcal{O}(n)) = \binom{N+n-1}{n-1}$. In particular, there are $\binom{9}{5} = 126$ independent global sections of $\mathcal{O}_{\mathbb{P}^4}(5)$.

Definition 1.1.6. The set of all open covers of X form a directed set under refinement. The **Čech cohomology of** X with coefficients in $\mathcal{F}$ is the direct limit $\check{H}^n(X, \mathcal{F}) := \varinjlim_{\mathcal{U}} \check{H}^n(\mathcal{U}, \mathcal{F})$.

Definition 1.1.7. An ordered open cover $\{U_\alpha\}$ is **good** if it is countable and every finite intersection $U_{\alpha_0, \ldots, \alpha_p}$ is either empty or contractible.

Theorem 1.1.8 ([1, Corollary of Theorem 5.4.1])**.** *The Čech cohomology of a good cover $\mathcal{U}$ is isomorphic to the Čech cohomology of X.*

One can define **sheaf cohomology** $H^n(X, \mathcal{F})$ as the right derived functors of the global sections functor Γ_X (i.e. $\mathcal{F} \mapsto \mathcal{F}(X)$). For us, Čech and sheaf cohomology are indistinguishable as long as we work with sheaves.

Theorem 1.1.9 ([1, Theorem 5.10.1])**.** *If X is a paracompact topological space, then Čech cohomology $\check{H}^n(X, \mathcal{F})$ and sheaf cohomology $H^n(X, \mathcal{F})$ are isomorphic for any sheaf $\mathcal{F}$.*

Čech cohomology is also directly related to de Rham cohomology, and, as we shall see, Dolbeault cohomology in the complex case. So we can think of Čech cohomology classes as forms.

Theorem 1.1.10 (Čech–de Rham isomorphism). *Let $\mathbb{R}$ denote the constant sheaf. There is a canonical isomorphism $\check{H}^k(X, \mathbb{R}) \cong H^k_{\mathrm{dR}}(X)$ for each k.*

Proof. By the Poincaré lemma, the **de Rham complex** of sheaves

$$0 \to \mathbb{R} \xrightarrow{\subseteq} \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \Omega^2(X) \xrightarrow{d} \cdots$$

is exact (by checking exactness on the stalks). Let $Z^n := \ker(d\colon \Omega^k \to \Omega^{k+1})$. The de Rham complex splits into a bunch of short exact sequences:

$$0 \to d\Omega^{k-1} \cong Z^k \xrightarrow{\subseteq} \Omega^k \xrightarrow{d} Z^{k+1} \to 0.$$

To each such short exact sequence is associated a long exact sequence of (sheaf) cohomology:

$$0 \to H^0(X, Z^k) \to H^0(X, \Omega^k) \to H^0(X, Z^{k+1}) \to H^1(X, Z^k) \to H^1(X, \Omega^k) \to H^1(X, Z^{k+1}) \to \cdots .$$

Fact: $H^i(X, \Omega^k) = 0$ for every k and $i > 0$ (since Ω^k is a fine sheaf). Hence we obtain isomorphisms

$$H^{i+1}(X, Z^0) \cong H^i(X, Z^1) \cong \cdots \cong H^1(X, Z^i).$$

But $Z^0 \cong \mathbb{R}$ and we more commonly write

$$H^1(X, Z^i) = \operatorname{coker}(H^0(X, \Omega^i) \to H^0(X, Z^{i+1})) = Z^{i+1}(X)/d\Omega^k(X) = H^{i+1}_{\mathrm{dR}}(X).$$

Hence $\check{H}^{i+1}(X, \mathbb{R}) \cong H^{i+1}_{\mathrm{dR}}(X)$. □

Definition 1.1.11. Let E be a holomorphic vector bundle on X and $\Omega^{0,q}(E) := \Omega^{0,q}(X) \otimes \Gamma(E)$ denote the space of E-valued $(0, q)$-forms. **Dolbeault cohomology** $H^q_{\bar\partial}(E)$ is the cohomology of the complex

$$\cdots \xrightarrow{\bar\partial} \Omega^{0,q}(E) \xrightarrow{\bar\partial} \Omega^{0,q+1}(E) \xrightarrow{\bar\partial} \Omega^{0,q+2}(E) \xrightarrow{\bar\partial} \cdots .$$

We write $H^{p,q}_{\bar\partial}(X)$ for $E = \Lambda^p T^{1,0} X$. The dimensions $h^{p,q}(X) := \dim_{\mathbb{C}} H^{p,q}_{\bar\partial}(X)$ are the **Hodge numbers** of X.

Lemma 1.1.12 ($\bar\partial$-Poincaré lemma). *$\bar\partial$-closed, i.e. holomorphic, implies $\bar\partial$-exact on $\mathbb{C}^n$.*

Theorem 1.1.13 (Čech–Dolbeault isomorphism). *Let $\Omega^{p,0}$ denote the sheaf of holomorphic p-forms on X. There is a natural isomorphism $\check{H}^q(X, \Omega^{p,0}) \cong H^{p,q}_{\bar\partial}(X)$.*

Proof. Analogous to the proof of the Čech–de Rham isomorphism, except now using the $\bar\partial$-Poincaré lemma to establish the exactness of the complex

$$0 \to \ker(\bar\partial\colon \Omega^{p,0}(X) \to \Omega^{p,1}(X)) \xrightarrow{\subseteq} \Omega^{p,0}(X) \xrightarrow{\bar\partial} \Omega^{p,1}(X) \xrightarrow{\bar\partial} \cdots .$$ □

Definition 1.1.14. Consider the double complex $\Omega^{\bullet,\bullet}$ with differentials ∂ and $\bar\partial$. The **Frölicher spectral sequence** is the spectral sequence of a double complex associated to $\Omega^{\bullet,\bullet}$. Since the total complex of $\Omega^{\bullet,\bullet}$ is $\Omega^\bullet(X)$, the Frölicher spectral sequence converges to complex de Rham cohomology $H^\bullet_{\mathrm{dR}}(X, \mathbb{C})$.

1.1.2 Morse Homology

Throughout this subsection, $f\colon X \to \mathbb{R}$ is a smooth function, and we equip X, viewed as a real manifold, with a Riemannian metric g. We also assume (f,g) is Morse–Smale, defined below.

Definition 1.1.15. A **critical point** of f is a point $p \in X$ with $df_p = 0$. Define the **Hessian**

$$H(f)_p\colon T_pX \to T_p^*X, \quad v \mapsto \nabla_v(df),$$

which is independent of the choice of connection ∇. (In coordinates, we recover the usual $\partial^2 f/\partial x_i \partial x_j$.) The critical point p is **non-degenerate** if the Hessian does not have zero eigenvalues. A non-degenerate critical point p has **Morse index** $\operatorname{ind}(p)$ the number of negative eigenvalues of the Hessian. The function f is **Morse** if all of its critical points are non-degenerate.

Definition 1.1.16. Recall that the **gradient** of f with respect to a metric g is the vector field $\operatorname{grad} f$ such that $g(\operatorname{grad} f, X) = Xf$. Equivalently, $\operatorname{grad} f = (df)^\sharp$. Let $\psi_t\colon X \to X$ be the one-parameter group of diffeomorphisms associated to the flow of $-\operatorname{grad} f$. The **descending manifold** $D(p)$ and **ascending manifold** $A(p)$ at a critical point p are

$$\begin{aligned} D(p) &\coloneqq \{x \in X : \lim_{t\to-\infty} \psi_t(x) = p\} \\ A(p) &\coloneqq \{x \in X : \lim_{t\to+\infty} \psi_t(x) = p\}. \end{aligned}$$

The pair (f,g) is **Morse–Smale** if f is Morse and $D(p)$ is transverse to $A(q)$ for every pair of critical points p and q (i.e. tangent spaces of $D(p)$ and $A(q)$ generate the tangent space at every intersection point).

Here are two useful and easy-to-prove facts: every flow line asymptotically approaches critical points, and $\dim D(p) = \operatorname{ind}(p)$ (so $\dim A(p) = \dim X - \operatorname{ind} p$ by the Morse–Smale condition).

Definition 1.1.17. Fix critical points p and q. A **flow line** from p to q is an integral curve $\gamma(t)$ of $-\operatorname{grad} f$ with $\lim_{t\to-\infty}\gamma(t) = p$ and $\lim_{t\to+\infty}\gamma(t) = q$. The **moduli space of flow lines** from p to q is

$$\begin{aligned} \mathcal{M}(p,q) &\coloneqq \{\text{flow lines from } p \text{ to } q\}/\sim, \quad \alpha \sim \beta \text{ if } \exists c \in \mathbb{R} : \alpha(t) = \beta(t+c) \\ &= (D(p) \cap A(q))/\mathbb{R}. \end{aligned}$$

A **broken flow line** consists, piecewise, of flow lines.

The Morse–Smale condition implies $D(p) \cap A(q)$ is a submanifold of X with dimension $\operatorname{ind}(p) - \operatorname{ind}(q)$. Since $\sim$ is a smooth, proper, free $\mathbb{R}$-action, $\mathcal{M}(p,q)$ is a manifold of dimension $\operatorname{ind}(p) - \operatorname{ind}(q) - 1$ when $p \neq q$ (otherwise the $\mathbb{R}$-action is trivial). Note that if $\operatorname{ind}(p) = k$ and $\operatorname{ind}(q) = k-1$ then $\mathcal{M}(p,q)$ is zero-dimensional. In fact, in this case, $\mathcal{M}(p,q)$ is compact as a corollary of the following theorem, and therefore is a finite set of points.

Theorem 1.1.18 ([2, Theorem 2.1])**.** *Let X be closed and (f,g) Morse–Smale. Then $\mathcal{M}(p,q)$ has a natural compactification to a smooth manifold with corners $\overline{\mathcal{M}(p,q)}$ where*

$$\overline{\mathcal{M}(p,q)} \setminus \mathcal{M}(p,q) = \bigcup_{k\geq 1} \bigcup_{\substack{p,r_1,\ldots,r_k,q \\ \text{distinct crit pts}}} \mathcal{M}(p,r_1) \times \mathcal{M}(r_1,r_2) \times \cdots \times \mathcal{M}(r_{k-1},r_k) \times \mathcal{M}(r_k,q).$$

Corollary 1.1.19. *If $\operatorname{ind}(p) - \operatorname{ind}(q) = 1$, then $\overline{\mathcal{M}(p,q)} = \mathcal{M}(p,q)$ is compact. If $\operatorname{ind}(p) - \operatorname{ind}(q) = 2$, then*

$$\partial\overline{\mathcal{M}(p,q)} = \bigcup_{\operatorname{ind}(r) = \operatorname{ind}(p)-1} \mathcal{M}(p,r) \times \mathcal{M}(r,q).$$

Proof. Since $\dim \mathcal{M}(r,s) = \operatorname{ind}(r) - \operatorname{ind}(s) - 1$, the space $\mathcal{M}(r,s)$ is non-empty only if $\operatorname{ind}(r) - \operatorname{ind}(s) \geq 1$. Hence $\overline{\mathcal{M}(p,q)} \setminus \mathcal{M}(p,q) = \emptyset$ when $\operatorname{ind}(p) - \operatorname{ind}(q) = 1$. Similar reasoning shows the $\operatorname{ind}(p) - \operatorname{ind}(q) = 2$ case. ☐

Definition 1.1.20. Fix orientations for $D(p)$ at every critical point p. There is an isomorphism at $x \in \gamma \in \mathcal{M}(p,q)$ given by

$$\begin{aligned} T_xD(p) &\cong T_x(D(p) \cap A(q)) \oplus (T_xX/T_xA(q)) && \text{transversality from Morse–Smale} \\ &\cong T_\gamma\mathcal{M}(p,q) \oplus T_x\gamma \oplus (T_xX/T_xA(q)) && \text{definition of } \mathcal{M}(p,q) \\ &\cong T_\gamma\mathcal{M}(p,q) \oplus T_x\gamma \oplus T_qD(q) && \text{translating } T_qD(q) \text{ along } \gamma. \end{aligned}$$

The **orientation** on $\mathcal{M}(p,q)$ is such that this isomorphism is orientation-preserving. Let C_k be the free abelian group generated by critical points of index k, and define the **Morse–Smale–Witten boundary map**

$$\partial_k^{\text{Morse}}\colon C_k \to C_{k-1}, \quad p \mapsto \sum_{\text{ind } q=k-1} \#\mathcal{M}(p,q)q$$

where $\#\mathcal{M}(p,q) \in \mathbb{Z}$ is counted with sign according to the orientation of $\mathcal{M}(p,q)$, which here is a discrete set of points.

Lemma 1.1.21. $(\partial_k^{\text{Morse}})^2 = 0$*, so* $(C_\bullet, \partial^{\text{Morse}})$ *is a chain complex.*

Proof. Let $\text{ind}(p) - \text{ind}(q) = 2$. The coefficient of q in $(\partial^{\text{Morse}})^2 p$ is

$$\sum_{\text{ind}(r)=\text{ind}(p)-1} \#\mathcal{M}(p,r) \cdot \#\mathcal{M}(r,q) = \# \bigcup_{\text{ind}(r)=\text{ind}(p)-1} \mathcal{M}(p,r) \times \mathcal{M}(r,q) = \#\partial\overline{\mathcal{M}(p,q)}.$$

Since $\overline{\mathcal{M}(p,q)}$ is an oriented 1-manifold with boundary, this quantity, the number of boundary points, is zero. □

Definition 1.1.22. Morse homology $H_\bullet^{\text{Morse}}(f,g)$ is the homology of the **Morse–Smale–Witten complex** $(C_\bullet, \partial^{\text{Morse}})$.

Example 1.1.23. The (upright) torus T^2 has four critical points with f the height function: p (index 2), q and r (index 1), and s (index 0). This choice of f is Morse, but with the induced metric g from $\mathbb{R}^3$, the pair (f,g) is not Morse–Smale: $D(q) \cap A(r)$ is non-empty, but transversality forces it to be. The solution is to tilt the torus a little; equivalently, perturb g. There are two flow lines, of opposite sign, for each relevant pair of critical points. Hence $\partial_k^{\text{Morse}} = 0$ for $k = 1, 2$. It follows that

$$H_2^{\text{Morse}}(f,g) = \mathbb{Z}, \quad H_1^{\text{Morse}}(f,g) = \mathbb{Z}^2, \quad H_0^{\text{Morse}}(f,g) = \mathbb{Z}.$$

Theorem 1.1.24 ([2, Theorem 3.1])**.** *Let X be a closed smooth manifold, $H_\bullet(X)$ denote singular homology on X, and (f,g) be a Morse–Smale pair on X. Then there is a canonical isomorphism $H_n^{\text{Morse}}(f,g) \cong H_n(X)$.*

Corollary 1.1.25. *The number of critical points of a Morse function is at least the sum $\sum_k \dim H_k(X)$ of the Betti numbers.*

Proof. The number of critical points is the sum of the dimensions of the Morse chain groups, which is at least the sum of the dimensions of the Morse homology groups, which is equal to the sum of the dimensions of the singular homology groups. □

The infinite-dimensional analogue of Morse homology is known as **Floer homology**. We shall primarily be concerned with Floer homology for mirror symmetry.

1.1.3 Equivariant Cohomology

In this subsection, G is a compact Lie group acting smoothly on X. Cohomology is taken with $\mathbb{Q}$ coefficients.

The usual cohomology theories fail to capture any information about this action of G on X. Equivariant cohomology is an extension of the usual cohomology for X/G in order to account for the action. For example, if G acts smoothly without fixed points, then X/G is a smooth manifold, and equivariant cohomology agrees with $H^*(X/G)$. On the other end of the spectrum, if $X = \{\mathrm{pt}\}$, then any G-action fixes pt, and we want equivariant cohomology to give a cohomological invariant of G. To construct equivariant cohomology we need the machinery of classifying spaces. So we first review some homotopy theory.

Definition 1.1.26. A continuous map $f\colon X \to Y$ of topological spaces is a **weak homotopy equivalence** if the induced maps $\pi(f)\colon \pi_*(X) \to \pi_*(Y)$ of homotopy groups are isomorphisms. A topological space X is **weakly contractible** if $X \to \mathrm{pt}$ is a **weak equivalence**. If in addition there is a continuous map $g\colon Y \to X$ such that $g \circ f$ and $f \circ g$ are homotopic to the identity maps id_X and id_Y respectively, then X and Y are **homotopy equivalent**.

Theorem 1.1.27 (Hurewicz, [3, Section 20.1])**.** *For any topological space X there exists a group homomorphism $h_k\colon \pi_k(X) \to H_k(X)$. If X is $(n-1)$-connected, i.e. $\pi_k(X) = 0$ for $1 \le k \le n-1$, then h_k is an isomorphism for $1 \le k \le n$, and h_1 is the abelianization map.*

Theorem 1.1.28 ([3, Section 10.8])**.** *For $\{X_\alpha\}$ a directed system of Hausdorff spaces, $\pi_i(\varinjlim X_\alpha) = \varinjlim \pi_i(X_\alpha)$*

Theorem 1.1.29 (Whitehead, [3, Theorem 20.1.5])**.** *If two connected CW complexes X and Y are weakly homotopy equivalent, then they are homotopy equivalent.*

(This theorem of Whitehead's provided some of the initial justification for working with CW complexes.) It is true that all smooth manifolds are homotopy equivalent to CW-complexes (see [4, page 220] for a proof for compact smooth manifolds, and discussion on the general case). The proof is Morse-theoretic.

Definition 1.1.30. If $P \to B$ is a principal G-bundle with P weakly contractible, then B is a **classifying space** for G, denoted BG, and P is a **universal G-bundle**, denoted EG.

Theorem 1.1.31 (Milnor, [3, Section 14.4])**.** *Let G be any topological group. Then there exists a classifying space for G (and a universal G-bundle) unique up to canonical homotopy equivalence.*

Proof sketch. We show that EG exists for closed subgroups of the Lie group $\mathrm{GL}(n)$; this is sufficient for our purposes. First, we find $E\,\mathrm{GL}(n)$. Define

$$\begin{aligned} V_n(\mathbb{R}^k) &\coloneqq \{(v_1, \ldots, v_n) \in (\mathbb{R}^k)^n \text{ linearly independent}\}, \\ V_n^O(\mathbb{R}^k) &\coloneqq \{(v_1, \ldots, v_n) \in (\mathbb{R}^k)^n \text{ orthonormal}\}. \end{aligned}$$

They are both referred to as the **Stiefel manifold** associated to $\mathrm{Gr}_n(\mathbb{R}^k)$. Note that $\pi\colon V_n(\mathbb{R}^k) \to \mathrm{Gr}_n(\mathbb{R}^k)$ is a principal $\mathrm{GL}(n)$-bundle, where π sends $(v_1, \ldots, v_n)$ to $\operatorname{span}\{v_1, \ldots, v_n\} \in \mathrm{Gr}_n(\mathbb{R}^k)$. We shall show that $V_n(\mathbb{R}^\infty) \coloneqq \varinjlim_k V_n(\mathbb{R}^k)$ is a universal $\mathrm{GL}(n)$-bundle over $\mathrm{Gr}_n(\mathbb{R}^\infty)$.

By Gram–Schmidt, $V_n^O(\mathbb{R}^k)$ is a deformation retract of $V_n(\mathbb{R}^k)$, so since homotopy commutes with direct limit (see 1.1.28), it suffices to show that $V_n^O(\mathbb{R}^k)$ is $(k-1)$-connected. There are fibrations

$$V_{n-1}^O(\mathbb{R}^{k-1}) \to V_n^O(\mathbb{R}^k) \xrightarrow{(v_1,\ldots,v_n)\mapsto v_n} S^{k-1}$$

so for $i < k-1$, we have $\pi_i(V_{n-1}^O(\mathbb{R}^{k-1})) \cong \pi_i(V_n^O(\mathbb{R}^k))$. Hence by induction these homotopy groups vanish, as desired, and we have shown $E\,\mathrm{GL}(n) = V_n(\mathbb{R}^\infty)$.

Now let $G \subset \mathrm{GL}(n)$ be a closed Lie subgroup. Take $EG \coloneqq V_n(\mathbb{R}^\infty)$ and let $BG \coloneqq EG/G$. Then $EG \to BG$ is a principal G-bundle, and by the same argument as above it is universal. ∎

The idea for equivariant cohomology is to start with the possibly not-free action of G on X, find a homotopy equivalent space $\tilde{X}$ on which G acts freely, and then look at $H^*(\tilde{X}/G)$. Here is an easy way to get such a space.

Definition 1.1.32. Let E be a G-bundle over X (which has a G-action). Define the product

$$E \times_G X = E \times X/\sim, \quad (eg, x) \sim (e, gx) \quad \forall e \in E, x \in X, g \in G.$$

The **homotopy quotient** X_G of X by G is $X_G \coloneqq EG \times_G X$. The **equivariant cohomology** of X is $H^*_G(X) \coloneqq H^*(X_G)$, the cohomology of the homotopy quotient.

Definition 1.1.33. The projection maps from $EG \times X$ induce maps $\sigma\colon X_G \to X/G$ and $\pi\colon X_G \to BG$ which fit into the **mixing diagram** of Borel and Cartan:

$$\begin{array}{ccccc} EG & \longleftarrow & EG \times X & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ BG & \xleftarrow{\sigma} & X_G & \xrightarrow{\pi} & X/G. \end{array}$$

So there are two ways to view X_G:

1. $\pi\colon X_G \to X/G$ is a fibered space with the fiber over $[Gx]$ being $EG/\{g : gx = x\}$;
2. $\sigma\colon X_G \to BG$ is a bundle with fiber X

Using the second perspective, the inclusion of the fiber $i\colon X \to X_G$ induces a map $i^*\colon H^*_G(X) \to H^*(X)$.

Example 1.1.34. If $X = \{\mathrm{pt}\}$, then $EG \times_G X = (EG \times \{\mathrm{pt}\})/G = EG/G = BG$, so the equivariant cohomology of a point with a G-action is the cohomology of BG, i.e. $H^*_G(\{\mathrm{pt}\}) = H^*(BG)$. For a general space X, the equivariant map $X \to \{\mathrm{pt}\}$ induces an $H^*(BG)$-module structure on $H^*_G(X)$ for any X. We use the notation $H^*_G \coloneqq H^*_G(\{\mathrm{pt}\}) = H^*(BG)$.

Example 1.1.35. If G acts freely on X, then $EG \times_G X = EG \times X/G$. By the Künneth theorem,

$$H^*_G(X) = H^*(EG \times X/G) = H^*(EG) \otimes H^*(X/G) = H^*(X/G)$$

since EG is weakly contractible, so its cohomology vanishes by the Hurewicz theorem. Hence when X/G is a smooth manifold, equivariant cohomology indeed agrees with regular cohomology. By the same reasoning, if G acts trivially, then $H^*_G(X) = H^*(X) \otimes H^*(BG)$.

Example 1.1.36. Let $G = \mathbb{C}^*$ act on $\mathbb{C}^n$ by scalar multiplication. The quotient map $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ is a principal $\mathbb{C}^*$-bundle. Since $\mathbb{C}^{n+1} \setminus \{0\}$ is homotopy equivalent to S^{2n+1}, the homotopy groups π_k for $k \leq 2n$ vanish, but the higher ones do not necessarily vanish. To fix this we take the direct limit of the inclusions $\mathbb{CP}^n \to \mathbb{CP}^{n+1} \to \cdots$ (which kills off the homotopy groups one by one; see 1.1.28) to get the universal principal $\mathbb{C}^*$-bundle $\mathbb{C}^\infty \to \mathbb{CP}^\infty$. Hence $BS^1 = \mathbb{CP}^\infty$, and

$$H^*_{S^1}(\{\mathrm{pt}\}) = H^*(BS^1) = H^*(\mathbb{CP}^\infty) = \mathbb{Q}[x],$$

generated by the Chern class $x = c_1(\mathcal{O}_{\mathbb{P}^\infty}(1))$. For a **torus action** by $\mathbb{T} \coloneqq (\mathbb{C}^*)^{m+1}$,

$$H^*_{\mathbb{T}}(\{\mathrm{pt}\}) = H^*((\mathbb{CP}^\infty)^{m+1}) = \mathbb{Q}[x_0, \ldots, x_m].$$

Since $H^*_{\mathbb{T}}(X)$ is an $H^*_{\mathbb{T}}(\{\mathrm{pt}\})$-module, we think of elements of $H^*_{\mathbb{T}}(X)$ as polynomial-valued forms.

It would be nice to have a de Rham-type construction in order to work with equivariant cohomology classes. This is possible for G a Lie group: we just have to work with G-equivariant forms.

Definition 1.1.37. Interpret the symmetric algebra $\mathrm{Sym}(\mathfrak{g}^*)$ as polynomials on $\mathfrak{g}$, and therefore $\mathrm{Sym}(\mathfrak{g}^*) \otimes \Omega^*(X)$ as polynomials on $\mathfrak{g}$ with coefficients in $\Omega^*(X)$. There is a natural G-action where G acts on $\mathfrak{g}^*$ by the **coadjoint action** $(g \cdot \xi)(Z) := X(\mathrm{Ad}_{g^{-1}} Z)$, and G acts on $\Omega^*(X)$ by pullback. Using this action, the space of smooth **G-equivariant forms** on X is

$$\Omega^*_G(X) := (\mathrm{Sym}(\mathfrak{g}^*) \otimes \Omega^*(X))^G,$$

i.e. G-invariant elements of $\mathrm{Sym}(\mathfrak{g}^*) \otimes \Omega^*(X)$. Equivalently, they are G-equivariant maps $\alpha \colon \mathfrak{g} \to \Omega^*(X)$, i.e. $\alpha(Z) = g^{-1} \cdot \alpha(\mathrm{Ad}(g)Z)$.

Definition 1.1.38. Define a grading on $\Omega^*_G(X)$ by

$$\deg \colon \Omega^*_G(X) \to \mathbb{Z}_{\geq 0}, \quad \theta \otimes \omega \mapsto 2 \deg_{\mathrm{poly}}(\theta) + \deg_{\mathrm{form}} \omega.$$

The **equivariant exterior derivative** $d_G \colon \Omega^*_G(X) \to \Omega^{*+1}_G(X)$ is given by extending the following map by linearity:

$$d_G|_{\mathfrak{g}^*} \alpha \colon \mathfrak{g} \to \Omega^*(X), \quad (d_{\mathfrak{g}}\alpha)(Z) := d(\alpha(Z)) - \iota_{Z^\#}(\alpha(Z))$$

where d is the usual exterior derivative, ι is contraction, and $Z^\#$ is the fundamental vector field associated to $Z \in \mathfrak{g}$. The complex $(\Omega^*_G(X), d_G)$ is called the **equivariant de Rham complex**.

Theorem 1.1.39 (Equivariant de Rham theorem, [5, Section 0.2])**.** *Let G be a compact Lie group acting on a compact smooth manifold X. Then there is an isomorphism between equivariant cohomology and the cohomology of the twisted de Rham complex:*

$$H^*_G(X) \cong H^*(\Omega^*_G(X), d_G).$$

This proof should be done from a supersymmetric perspective. In the literature the result is known as the equivalence between the **Borel model** (i.e. our topological definition of equivariant cohomology) and the **Cartan–Weil model** (i.e. the de Rham definition).

1.2 Algebraic Topology

We stop distinguishing between isomorphic cohomology theories now. In particular, since X is always at least a smooth manifold, we think of singular cohomology $H^k(X)$ as de Rham cohomology. For a bundle E, $H^k(E)$ refers to sheaf cohomology.

1.2.1 Poincaré and Serre Duality

Unless otherwise stated, X in this section is a compact oriented n-manifold.

Theorem 1.2.1 (Poincaré duality, [4])**.** *Let X be a compact oriented n-manifold. The map*

$$\int_X \colon H^k(X) \otimes H^{n-k}(X) \to \mathbb{R}, \quad \omega \otimes \eta \mapsto \int_X \omega \wedge \eta$$

is a perfect pairing, and hence $H^k(X) \cong H^{n-k}(X)^*$.

If we relax the assumption that X is compact, then the issue is that $\int_X$ may not be well-defined. We work around this by using de Rham cohomology with compact support.

Definition 1.2.2. Let $\Omega^k_c(X)$ denote the k-forms on X with compact support. The **de Rham cohomology groups with compact support** $H^n_c(X)$ are the cohomology of the chain complex $(\Omega^\bullet_c(X), d)$.

Theorem 1.2.3 (Poincaré duality for non-compact manifolds, [4]). *Let X be an oriented n-manifold without boundary. The map*

$$\int_X : H^k(X) \otimes H_c^{n-k}(X) \to \mathbb{R}, \quad \omega \otimes \eta \mapsto \int_X \omega \wedge \eta$$

is a perfect pairing, and hence $H^k(X) \cong H_c^{n-k}(X)^*$.

Definition 1.2.4. Fix $C \subset X$ a closed $(n-k)$-submanifold. Then Poincaré duality identifies the map $\int_C : H^{n-k}(X) \to \mathbb{R}$ with a k-form $\eta_C \in H^k(X)$, called the **Poincaré dual class**. Explicitly, $\int_C \omega = \int_X \omega \wedge \eta_C$.

There is a relation between the Poincaré dual class and the Thom class, which we define below. Namely, the Poincaré dual class of C can be constructed as the Thom class of the normal bundle of C in X.

Theorem 1.2.5 ([6, Theorem 10.4]). *Let $\pi : E \to B$ be an oriented rank n real vector bundle and B is embedded into E as the zero section. Then*

1. *there exists a unique cohomology class $\Phi \in H^n(E, E \setminus B)$ called the* **Thom class** *such that for every $x \in B$, the restriction of Φ to $H^n(E_x, E_x \setminus \{0\})$ is the preferred generator specified by the orientation of E_x in E;*
2. *the* **Thom isomorphism** *: $H^k(E) \to H^{k+n}(E, E \setminus B)$, given by $\omega \mapsto \omega \wedge \Phi$, is an isomorphism for every k.*

Note that since B is a deformation retract of E, the rings $H^*(E)$ and $H^*(B)$ are isomorphic. Hence $\pi^*\Phi = 1 \in H^*(B)$.

Theorem 1.2.6 (Tubular neighborhood theorem, [6, Theorem 11.1]). *Let $C \subset X$ be a k-submanifold embedded in X. There exists an open neighborhood, called a* **tubular neighborhood***, of C in X diffeomorphic to the total space of the normal bundle of C. This diffeomorphism maps points in C to zero vectors.*

Proposition 1.2.7 ([4, Proposition 6.24a]). *Let $C \subset X$ be a closed $(n-k)$-submanifold. The Poincaré dual class $\eta_C \in H^k(X)$ of C is the Thom class of the normal bundle of C in X.*

Proof. Let NC denote the normal bundle of C in X, which has rank k because C is codimension k. Use the tubular neighborhood theorem to identify NC with an open neighborhood T of C in X, and then extend by zero to get $\Phi \in H^k(X)$ supported on T.

We shall show that $\int_X \omega \wedge \Phi = \int_C \omega$ for any $\omega \in H_c^{n-k}(X)$. The maps $\pi : T \to C$ and $\iota : C \to T$ induce isomorphisms of cohomology, so on forms ω and $\pi^*\iota^*\omega$ differ by at most an exact form $d\tau$. Then

$$\begin{aligned}\int_X \omega \wedge \Phi &= \int_T \omega \wedge \Phi = \int_T (\pi^*\iota^*\omega + d\tau) \wedge \Phi \\ &= \int_T \pi^*\iota^*\omega \wedge \Phi = \int_C \iota^*\omega \wedge \pi_*\Phi = \int_C \iota^*\omega\end{aligned}$$

where the last two steps involve the **projection formula** [4, Proposition 6.15] and noting that $\pi_*\Phi = 1$. ☐

Corollary 1.2.8. *Transverse intersection is Poincaré dual to the wedge product, i.e. for $C, D \subset X$ closed submanifolds intersecting transversally, $\eta_{C \cap D} = \eta_C \wedge \eta_D$.*

Proof. For transversal intersections, codimension is additive: $\operatorname{codim} C \cap D = \operatorname{codim} C + \operatorname{codim} D$. So the normal bundle of the intersection is $N(C \cap D) = NC \oplus ND$. Let $\Phi(E)$ denote the Thom class associated to the vector bundle E. By the characterization of the Thom class, for vector bundles E and F we have $\Phi(E \oplus F) = \Phi(E) \wedge \Phi(F)$; check that $\Phi(E) \oplus \Phi(F)$ restricts on each fiber to the preferred generator. Hence

$$\eta_{C\cap D} = \Phi(N_{C\cap D}) = \Phi(NC \oplus ND) = \Phi(NC) \wedge \Phi(ND) = \eta_C \wedge \eta_D.$$

☐

Let X be a complex n-fold now. In the complex setting, we can refine Poincaré duality. The Čech–Dolbeault isomorphism 1.1.13 works for the more general setting in which we defined Dolbeault cohomology: if E is a holomorphic vector bundle over X, then $H^k(X,E) \cong H^k_{\bar\partial}(E)$. So we think of Čech cohomology classes $H^k(X,E)$ as E-valued $(0,k)$-forms.

Definition 1.2.9. The **canonical bundle** K_X of a complex n-fold X is the vector bundle of $(n,0)$-forms. (Also commonly denoted $\Omega^n(X)$.)

Hence $H^{n-k}(X, E^* \otimes K_X)$ consists of E^*-valued $(n, n-k)$-forms. Given such a form ω and another form $\eta \in H^k(X,E)$, the form $\omega \wedge \eta$ is an (n,n)-form with complex coefficients. We can integrate it to get something in $\mathbb{C}$. At this point it is impossible not to wonder about whether the pairing $H^k(E) \otimes H^{n-k}(X, E^* \otimes K_X) \to \mathbb{C}$ given by wedging and then integrating is perfect.

Theorem 1.2.10 (Serre duality, [7, Corollary III.7.13])**.** *The pairing $H^k(X,E) \otimes H^{n-k}(X, E^* \otimes K_X) \to \mathbb{C}$ is perfect, so $H^k(X,E) \cong H^{n-k}(X, E^* \otimes K_X)^*$.*

Poincaré duality combined with Hodge decomposition 1.3.14 gives

$$\bigoplus_{p+q=k} H^q(X, \Omega^p) = H^k(X,\mathbb{C}) \cong H^{2n-k}(X,\mathbb{C}) = \bigoplus_{p'+q'=2n-k} H^{q'}(X, \Omega^{p'}) = \bigoplus_{p+q=k} H^{n-q}(X, \Omega^{n-p}).$$

Serre duality says that in fact, each of the terms in the sum are isomorphic: $H^q(X,\Omega^p) \cong H^{n-q}(X, \Omega^{n-p})$.

1.2.2 Chern Classes via Chern–Weil Theory

For this subsection, we work over $\mathbb{C}$, and every vector bundle we consider is smooth and complex. We define Chern classes using the Chern–Weil approach. There are other equivalent approaches in more general settings. But for us, we take $\pi\colon E \to X$ to be a rank-n smooth complex vector bundle over a smooth manifold X. A connection $A \in \Omega^1(X, \operatorname{Ad} E)$ on E gives a curvature $F_A := dA + A \wedge A \in \Omega^2(X, \operatorname{Ad} E)$.

Definition 1.2.11. The **total Chern class** of E is

$$\begin{aligned} c(E) := \det\left(1 + \frac{i}{2\pi}F\right) &= 1 + \frac{i}{2\pi}\operatorname{tr}(F) + \frac{1}{8\pi}(\operatorname{tr}(F^2) - \operatorname{tr}(F)^2) + \cdots \\ &= 1 + c_1(E) + c_2(E) + \cdots \in H^0(X,\mathbb{R}) \oplus H^2(X,\mathbb{R}) \oplus \cdots . \end{aligned}$$

Its terms $c_k(E) \in H^{2k}(X,\mathbb{R})$ are the **Chern classes**. The total Chern class $c(X)$ of X is defined as $c(X) := c(T^{1,0}X)$.

Theorem 1.2.12 (Chern–Weil theorem, [8, Corollary 4.4.5, Lemma 4.4.6])**.** *The total Chern class $c(E)$ is closed and independent of the choice of connection A on E.*

Example 1.2.13. A magnetic monopole at the origin in $U(1)$ Maxwell theory is given by the trivial line bundle on $\mathbb{R}^3$ with connection $A = i\frac{1}{2r}\frac{1}{z-r}(xdy - ydx)$, where r is the coordinate on the fibers of the bundle. Then

$$F_A = i\frac{1}{2r^3}(xdy \wedge dz + ydz \wedge dx + zdx \wedge dy) = -\frac{i}{2r^2}(r^2 \sin\theta d\theta \wedge d\phi).$$

We easily check that $\int_{S^2} c_1 = \frac{i}{4\pi}\int_{S^2} F_A = 1$ for any 2-sphere around the origin.

Theorem 1.2.14. *The Chern classes satisfy and are uniquely determined by the following properties:*

1. *$c_0(E) = 1$ and $c_k(E) = 0$ if $k > \dim E$;*
2. *(Naturality) if $f\colon Y \to X$ is continuous, then $f^*c(E) = c(f^*E)$;*
3. *(Whitney product formula) $c(E \oplus F) = c(E) \wedge c(F)$;*

4. $c_1(\mathcal{O}_{\mathbb{P}^1}(-1))$ *is minus the preferred generator (given by the orientation) of* $H^2(\mathbb{P}^1)$.

Proof. Property 1 is clear from the definition of $c_k(E)$. Property 2 follows from the multiplicative property of the determinant for block diagonal matrices; the curvature on $E \oplus F$ splits as a curvature on E and a curvature on F. Property 3 follows from pulling back a connection A on E to a connection f^*A on f^*E, and then using that pullbacks commute with everything.

Property 4 is more tedious and serves as our second explicit calculation of a Chern class. On $\mathbb{P}^1$, take the usual charts (U, u) and (V, v) with $u = v^{-1}$ on $U \cap V$. The local 1-forms

$$A_U = \frac{\bar{u}du}{1 + u\bar{u}}, \quad A_V = \frac{\bar{v}dv}{1 + v\bar{v}}$$

form the globally-defined **Chern connection** on $\mathcal{O}_{\mathbb{P}^1}(-1)$, the tautological bundle. We cheat a little and work only over U instead of all of $\mathbb{P}^1$. The curvature is

$$F_{A_U} = \frac{(1 + u\bar{u})d\bar{u} \wedge du - \bar{u}(\bar{u}du + ud\bar{u}) \wedge du}{(1 + u\bar{u})^2} = -\frac{du \wedge d\bar{u}}{(1 + u\bar{u})^2}.$$

Hence, in real coordinates, $c_1(\mathcal{O}_{\mathbb{P}^1}(-1)) = -\frac{dx \wedge dy}{\pi(1+x^2+y^2)^2}$. To compare this with the preferred generator, we simply integrate both and compare the result. (This is valid since $\dim H^2(\mathbb{P}^1) = 1$.) We know the preferred generator integrates to 1, whereas

$$\int_{\mathbb{P}^1} c_1(\mathcal{O}_{\mathbb{P}^1}(-1)) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dx \wedge dy}{(1 + x^2 + y^2)^2} = -\frac{1}{\pi}\int_0^{\infty}\int_0^{2\pi} \frac{r d\theta \wedge dr}{(1 + r^2)^2} = -1. \qquad \square$$

Proposition 1.2.15. *Let E be a complex vector bundle, and $\bar{E}$ be the same underlying real vector bundle as E but with conjugate complex structure. Then $c_k(\bar{E}) = (-1)^k c_k(E)$. If E has a Hermitian metric, then $E^* \cong \bar{E}$ canonically.*

Proof. Straightforward exercise: given a connection on E, what is the connection on $\bar{E}$? $\square$

Proposition 1.2.16 (Splitting principle)**.** *If $0 \to A \to B \to C \to 0$ is a short exact sequence, then $c(B) = c(A) \wedge c(C)$.*

Proof. Short exact sequences of smooth vector bundles always split: pick a metric on B and show that $C \cong A^\perp$. Hence $c(B) = c(A \oplus C)$, and then we use the Whitney product formula. $\square$

Note: this is **not** the usual "splitting principle". The usual splitting principle says that to prove an identity on Chern classes, it suffices to pretend that the bundle completely splits into line bundles and prove the identity for that case. For more detail, see [4, Section 21].

Example 1.2.17. We compute the total Chern class of $\mathbb{P}^n$. We first construct the **Euler sequence** on $\mathbb{P}^n$, given by

$$0 \to \mathbb{C} \to \mathcal{O}_{\mathbb{P}^n}(1)^{n+1} \to T^{1,0}\mathbb{P}^n \to 0.$$

Since $X = \mathbb{P}^n$ is $Y = \mathbb{C}^{n+1} \setminus \{0\}$ mod a $\mathbb{C}^*$ action, given $n+1$ linear functionals v_i on $\mathbb{C}^{n+1}$, the vector field $\sum_i v_i \partial_i$ on X is invariant under this $\mathbb{C}^*$ action and descends to a vector field on $\mathbb{P}^n$. The v_i are sections of $\mathcal{O}_{\mathbb{P}^n}(1)$, so this construction is the map $\mathcal{O}_{\mathbb{P}^n}(1)^{n+1} \to T^{1,0}\mathbb{P}^n$. Its kernel is the line bundle associated to $Z = \sum_i x_i \partial_i$ (here x_i are the coordinates on Y): for homogeneous polynomials f, we have $\frac{1}{d}\sum_i x_i \partial_i f = f$. Another way to see this is to visualize $\mathbb{P}^n$ as a sphere in Y, so when we project, the radial vector field Z and its multiples are precisely the kernel.

Clearly $c(\mathbb{C}) = 1$, so by the splitting principle, $c(\mathbb{P}^n) = c(\mathcal{O}_{\mathbb{P}^n}(1)^{n+1}) = c(\mathcal{O}_{\mathbb{P}^n}(1))^{n+1}$. Let $x = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$. Then $c(\mathbb{P}^n) = (1 + x)^{n+1}$.

Using the symbol x to stand for $c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ is fairly common. We shall do so from now on. (The reason is that x generates the cohomology ring $H^*(\mathbb{P}^n)$.)

Definition 1.2.18. If we formally factorize the total Chern class as $c(E) = \prod_{k=1}^{r}(1 + a_k)$, then the **Chern character class** is

$$\mathrm{ch}(E) \coloneqq \sum_{k=1}^{r} \exp(a_i) = r + c_1(E) + \frac{1}{2}(c_1(E)^2 - 2c_2(E)) + \cdots .$$

Proposition 1.2.19. *In the Chern–Weil setting,* $\mathrm{ch}(E) = \operatorname{tr}\exp(iF/2\pi)$.

Of course, the Chern character class, being a combination of Chern classes, does not contain more information than the Chern class. The reason we work with it instead of the Chern class is the following proposition.

Proposition 1.2.20. *The Chern character satisfies*

$$\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F), \quad \mathrm{ch}(E \otimes F) = \mathrm{ch}(E) \wedge \mathrm{ch}(F).$$

Example 1.2.21. Let $X = V(p)$ be a smooth projective variety in $\mathbb{P}^n$ with p a degree d homogeneous polynomial, i.e. a section of $\mathcal{O}_{\mathbb{P}^n}(d)$. To compute the Chern class of X, we use the **adjunction formula** $NX \cong \mathcal{O}(d)|_X$ (see [8, Proposition 2.2.17] for details), so that

$$0 \to TX \to T\mathbb{P}^n|_X \to NX \cong \mathcal{O}(d)|_X \to 0$$

is a short exact sequence. Since $\mathcal{O}(d) = \mathcal{O}(1)^{\otimes d}$, we can't use the Whitney sum property of the Chern class, but we can use the Chern character:

$$\mathrm{ch}(\mathcal{O}(d)) = \mathrm{ch}(\mathcal{O}(1))^d = \exp(x)^d = \exp(dx),$$

so $c(\mathcal{O}(d)) = 1 + dx$. Hence $c(X) = (1 + x)^{n+1}/(1 + dx)$.

Finally, we need to connect the theory of Chern classes with sheaf cohomology. Consider the short exact sequence of sheaves given by

$$0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0.$$

Its associated long exact sequence of cohomology contains

$$\cdots \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}) \to H^1(X, \mathcal{O}^*) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \to \cdots .$$

Definition 1.2.22. The **Picard group** of X is the group of isomorphism classes of holomorphic line bundles under tensor product.

Theorem 1.2.23 ([8, Corollary 2.2.10])**.** *Let $\mathcal{O}^*$ be the sheaf of nowhere-zero holomorphic functions. Then* $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}^*)$.

Theorem 1.2.24 ([8, Proposition 4.4.12])**.** *Under the identification of elements of $H^1(X, \mathcal{O}^*)$ with isomorphism classes of holomorphic line bundles, the connecting map $\delta\colon H^1(X, \mathcal{O}^*) \to H^2(X, \mathbb{Z})$ is the first Chern class c_1.*

1.2.3 The Euler Class and Euler Characteristic

Recall that the (holomorphic) Euler characteristic of a sheaf $\mathcal{F}$ is $\chi(\mathcal{F}) \coloneqq \sum_k (-1)^k \dim H^k(X, \mathcal{F})$.

Definition 1.2.25. Let $\pi\colon E \to B$ be an oriented rank n vector bundle over a smooth n-fold B. In 1.2.5 we defined its Thom class $\Phi \in H^n(E, E\setminus B)$. The inclusion $(E,\emptyset) \subset (E, E\setminus B)$ gives a homomorphism $H^k(E, E\setminus B) \to H^k(E)$ which we denote by $\omega \mapsto \omega|_E$. The **Euler class** $e(E)$ of E is the image of the Thom class Φ under the composition

$$H^n(E, E\setminus B) \xrightarrow{-|_E} H^n(E) \xrightarrow{(\pi^*)^{-1}} H^n(B)$$

where the last isomorphism is canonical and comes from B being a deformation retract of E. Again, if X is a manifold, $e(X) \coloneqq e(TX)$.

Proposition 1.2.26. *Whenever both are defined, $e(E) = c_n(E)$ for E of rank n.*

Proof sketch. Given E a smooth complex vector bundle, $e(E)$ is well-defined because the complex structure on E induces an orientation. We can use the Euler class to construct the Chern classes [6, Section 14]. Then it suffices to verify that the Chern classes we constructed this way satisfy the four axioms 1.2.14 of Chern classes. For example, the Whitney sum formula comes from $\Phi(E\oplus F) = \Phi(E)\wedge\Phi(F)$ being preserved throughout the construction. $\square$

We like to distinguish between the Euler class and the top Chern class for several reasons. One is that the Euler class is topological, whereas the Chern classes are differential geometric. Another is that it is sometimes easier to prove properties of the Euler class using properties of the Thom class rather than all the Chern classes, especially from the Chern–Weil approach.

Proposition 1.2.27 ([6, Property 9.3, Property 9.7])**.** *Properties of the Euler class $e(E)$ that do not directly follow from $e(E) = c_n(E)$:*

1. *if the orientation of E is flipped, $e(E)$ changes sign;*
2. *if E has a nowhere zero global section, then $e(E) = 0$.*

Proof. Property 1 is obvious: flipping the orientation of E flips the sign of the Thom class Φ, since $\Phi|_{E_x}$ is the preferred generator. Property 2 comes from $B \xrightarrow{s} E\setminus B \subset E \xrightarrow{\pi} B$ being the identity for a non-zero global section s. Then

$$H^n(B) \xrightarrow{\pi^*} H^n(E) \to H^n(E\setminus B) \xrightarrow{s^*} H^n(B)$$

is the identity on $H^n(B)$. But $\pi^* e(E) = \Phi|_E$, the restriction of the Thom class, by definition, so we have $s^*((\Phi|_E)|_{E\setminus B}) = e(E)$. Since $\Phi \in H^n(E, E\setminus B)$, the composition of these two restrictions is zero. Hence $e(E) = s^*0 = 0$. $\square$

Proposition 1.2.28. *Let $E \to M$ be a smooth oriented real vector bundle of rank r over the smooth compact oriented manifold M of dimension $n \geq r$. Let Z be the zero set of a smooth section $s\colon M \to E$ that is transversal to the zero section $\iota\colon M \to E$. Then Z is a smooth submanifold of M of codimension r and there is a natural bundle isomorphism $NZ \cong E|_Z$. Consequently, $e(E)$ is Poincaré dual to Z.*

Proof. A straightforward exercise. Hint: remember that $e(E)$ is the restriction of the Thom class, and then use 1.2.7 and 1.2.8. $\square$

The main purpose of this subsection is the following generalization of the Gauss–Bonnet theorem. We shall use it extensively when calculating Euler characteristic. To determine the Euler class explicitly, we often use many of the preceding results identifying it with various other objects.

Theorem 1.2.29 (Generalized Gauss–Bonnet)**.** *Let X be a compact complex manifold. Then*

$$\int_X e(X) = \chi(X).$$

Proof. We shall prove this in the next subsection, as a consequence of the Hirzebruch–Riemann–Roch formula. (There are much easier proofs, though.) □

Example 1.2.30. We can continue 1.2.17 to compute the Euler characteristic of $\mathbb{P}^n$. Note that every hyperplane $H \cong \mathbb{P}^{n-1} \subset \mathbb{P}^n$ is Poincaré dual to x. So x^n is Poincaré dual to the intersection of n generic hyperplanes, which is a point. In other words, x^n is the preferred generator given by the orientation, and hence $\int_{\mathbb{P}^n} x^n = 1$. (For a more explicit calculation of this, see [6, Theorem 14.10]. The explicit form for x is the obvious generalization of $-c_1(\mathcal{O}_{\mathbb{P}^1}(-1))$, which we computed in 1.2.14.)

Since $c(\mathbb{P}^n) = (1+x)^{n+1}$, we have $c_n = (n+1)x^n$. Hence $\int_{\mathbb{P}^n} c_n = n+1$. By the generalized Gauss–Bonnet theorem, $\chi(\mathbb{P}^n) = n+1$.

Example 1.2.31. Recall that the **degree** of a curve in $\mathbb{P}^2$ is just the degree of the defining homogeneous polynomial. (For a more general definition of the degree of a variety, see [7, Section I.7].) Fact: A degree d curve X in $\mathbb{P}^2$ has Chern class $1 + (3-d)x$. (Remember we write x for $c_1(\mathcal{O}_{\mathbb{P}^n}(1))$.) Then

$$\chi(X) = \int_X c_1(X) = \int_{\mathbb{P}^2} c_1(X)(xd) = \int_{\mathbb{P}^2} d(3-d)x^2 = d(3-d).$$

But for nonsingular curves X, we have $\chi(X) = 2 - 2g$. Hence $g = (d-1)(d-2)/2 = \binom{d-1}{2}$.

Example 1.2.32. By 1.2.21, a quintic hypersurface Q in $\mathbb{P}^4$ has total Chern class $c(Q) = (1+x)^5/(1+5x) = 1 + 10x^2 - 40x^3$. (Note that $c_1(Q) = 0$, so Q is Calabi–Yau.) We want to compute its Euler characteristic using HRR, but integrating over Q is hard. Instead, we use 1.2.28: Q is defined as the zero set of a section of $\mathcal{O}(5) \to \mathbb{P}^4$, so $e(\mathcal{O}(5))$ is Poincaré dual to Q. The Euler class is just the top Chern class (see 1.2.26), so $e(\mathcal{O}(5)) = c_1(\mathcal{O}(5))$,

$$\chi(Q) = \int_Q e(Q) = \int_Q c_3(Q) = \int_{\mathbb{P}^4} c_3(Q) \wedge c_1(\mathcal{O}(5)) = \int_{\mathbb{P}^4} (-40x^3)(5x) = -200 \int_{\mathbb{P}^4} x^4 = -200.$$

1.2.4 The Hirzebruch–Riemann–Roch Formula

Here E is a rank r holomorphic vector bundle over a compact complex n-fold X. The Hirzebruch–Riemann–Roch formula is part of a long sequence of generalizations of Gauss–Bonnet, relating geometric quantities to topological quantities.

Definition 1.2.33. Again, formally factor $c(E) = \prod_{k=1}^r (1 + a_k)$. The **Todd class** is

$$\operatorname{td}(E) \coloneqq \prod_{i=1}^r \frac{a_i}{1 - \exp(-a_i)} = 1 + \frac{1}{2}c_1(E) + \frac{1}{2}(c_1(E)^2 + c_2(E)) + \cdots.$$

The Todd class $\operatorname{td}(X)$ of X is defined as $\operatorname{td}(X) \coloneqq \operatorname{td}(TX)$.

Theorem 1.2.34 (Hirzebruch–Riemann–Roch, [7, Theorem A.4.1])**.** *Let E be a holomorphic vector bundle over a compact complex manifold X. Then*

$$\chi(E) = \int_X \operatorname{ch}(E) \wedge \operatorname{td}(X).$$

where on the right hand side we only integrate the top form, i.e. $\sum_k \operatorname{ch}_k \wedge \operatorname{td}_{n-k}$.

We can often use the Hirzebruch–Riemann–Roch (HRR) formula to compute the dimension of a specific cohomology group, either because some other cohomology groups vanish, or because we know their dimensions. Before we begin calculating anything, we need the following helpful result.

Theorem 1.2.35 (Grothendieck's vanishing theorem, [7, Theorem 2.7])**.** *Let X be a Noetherian topological space of dimension n. Then $H^i(X, \mathcal{F}) = 0$ for $i > n$ and any sheaf of abelian groups $\mathcal{F}$.*

The remainder of this section is examples of the HRR formula. Keep in mind that we are always working with sheaf cohomology, not de Rham cohomology. For example, $H^0(TX)$ is by no means equal to $H^0(X)$, and is not freely generated by connected components of TX. Instead, $H^0(TX) = \Gamma(TX)$, and since global sections generate automorphisms, $H^0(TX)$ consists of holomorphic automorphisms of X.

Example 1.2.36. Let $\mathcal{M}_g$ denote the **moduli space of complex structures** on a genus g closed surface. We shall see later (or recall from Teichmüller theory) that $\dim \mathcal{M}_g = \dim_{\mathbb{C}} H^1(TX)$ where X is a genus g closed Riemann surface. HRR gives

$$\begin{aligned}\dim_{\mathbb{C}} H^0(TX) - \dim_{\mathbb{C}} H^1(TX) = \chi(TX) = \int_X \mathrm{ch}(TX) \wedge \mathrm{td}(TX)& \\ = \int_X (1 + c_1(TX)) \wedge (1 + (1/2)c_1(TX)) = \frac{3}{2}\int_X c_1(TX) = 3 - 3g.&\end{aligned}$$

The last equality comes from c_1 being the top Chern class for X, i.e. the Euler class, so applying generalized Gauss–Bonnet gives $\int_X c_1(TX) = \chi(X) = 2 - 2g$. For $g \geq 2$, the Riemann surface X has no non-trivial automorphisms, so $\dim_{\mathbb{C}} H^0(TX) = 0$. Hence $\dim \mathcal{M}_g = 3g - 3$.

Example 1.2.37. An important object in mirror symmetry is the space of holomorphic maps from a Riemann surface Σ to a Calabi–Yau n-fold M, i.e. a Kähler n-fold M with $c_1(M) = 0$. (We'll be more careful about the definition of Calabi–Yau later.) An infinitesimal deformation of a holomorphic map, given by a vector field χ^i, must satisfy $\bar{\partial}\chi^i = 0$ if we want the deformed map to still be holomorphic. Hence $\chi \in H^0_{\bar{\partial}}(\phi^* TM)$, the space of such deformations. By HRR,

$$\begin{aligned}\dim_{\mathbb{C}} H^0(\phi^* TM) - \dim_{\mathbb{C}} H^1(\phi^* TM) = \int_X \mathrm{ch}(\phi^* TM) \wedge \mathrm{td}(\Sigma)& \\ = \int_X (n + \phi^* c_1(TM)) \wedge (1 + (1/2)c_1(\Sigma)) = n(1-g).&\end{aligned}$$

We assume for now that $H^1(\phi^* TM) = 0$, so the space of deformations is $n(1-g)$-dimensional. For $n = 3$ and $g = 0$, the dimension is 3, but there is also a 3-dimensional group of automorphisms of the genus-zero Riemann surface $\mathbb{P}^1$ which does not affect the image curve. Hence the space of genus 0 holomorphic curves inside a Calabi–Yau 3-fold is zero, and we may be able to count them!

Example 1.2.38. Let X be a connected compact curve and L a holomorphic line bundle on X. Fact: the natural isomorphism $H^2(X, \mathbb{Z}) = \mathbb{Z}$ is given by integration over X, and under this isomorphism we have $c_1(L) \mapsto \deg(L)$ ([8, Exercise 4.4.1]). By HRR,

$$\chi(L) = \int_X c_1(L) + \frac{1}{2}c_1(X) = \deg(L) + 1 - g.$$

For the case $L = \mathcal{O}(D)$ for a divisor D on X, we know $\dim_{\mathbb{C}} H^0(L) = \ell(D)$, and by Serre duality $\dim_{\mathbb{C}} H^1(L) = \dim_{\mathbb{C}} H^0(\mathcal{O}(K - D)) = \ell(K - D)$ where K is the canonical divisor. Hence we recover the classical **Riemann–Roch formula** $\ell(D) - \ell(K - D) = \deg D + 1 - g$.

We now use HRR to derive two important and powerful formulas: the Hirzebruch signature theorem, and, as previously promised, the generalized Gauss–Bonnet theorem.

Definition 1.2.39. Let X be a compact complex n-fold (and $h^{p,q} := \dim_{\mathbb{C}} H^{p,q}_{\bar{\partial}}(X)$ be its Betti numbers). The **Hirzebruch χ_y-genus** is

$$\chi_y := \sum_{p=0}^{n} \chi(X, \Omega^p) y^p = \sum_{p=0}^{n}\sum_{q=0}^{n} (-1)^q h^{p,q}(X) y^p.$$

Corollary 1.2.40. *Factorize* $c(X) = \prod_{k=1}^{n}(1 + a_k)$. *Then*

$$\chi_y = \int_X \prod_{k=1}^{n}(1 + y\exp(-a_k))\frac{a_k}{1 - \exp(-a_k)}.$$

Proof. If $TX = \bigoplus_{p=0}^{n} L_i$, then $\bigoplus_{p=0}^{n} \Omega^p(X)y^p = \bigotimes \Omega(yL_i)$. But $\Omega(yL_i)$ is just yT^*L_i, so we use the fact 1.2.15 that $c_1(T^*L_i) = -c_1(TL_i) = a_i$ to get

$$\operatorname{ch}\left(\bigoplus_{p=0}^{n} \Omega^p y^p\right) = \prod_{p=0}^{n} \operatorname{ch}(\Omega(yL_i)) = \prod_{p=0}^{n}(1 + y\exp(-a_p)).$$

Hence by HRR and the definition of the Todd class, we are done. $\square$

We first prove the generalized Gauss–Bonnet theorem using this formula for χ_y. To simplify the proof we restrict to the case of compact Kähler manifolds (in order to use Hodge decomposition), which shall be the main setting we work in anyway.

Theorem 1.2.41 (Generalized Gauss–Bonnet). *Let X be a compact Kähler manifold. Then*

$$\int_X e(X) = \chi(X).$$

Proof. Note that $\chi(X) = \chi_{-1}$ since by Hodge decomposition 1.3.14, $\sum_{p+q=k} h^{p,q}(X) = b_k(X)$, the k-th Betti number of X. Applying the χ_y formula 1.2.40, we get

$$\chi_{-1} = \int_X \prod_{k=1}^{n} a_k = \int_X c_n(X) = \int_X e(X)$$

where the last equality is from 1.2.26. $\square$

Now we prove the Hirzebruch signature theorem. The big picture is that on X there is an operator D called the **signature operator** whose index we want to compute, and Hodge theory tells us that $D) = \operatorname{sgn}(X)$. The signature theorem says the right hand side is a topological quantity.

Definition 1.2.42. The **signature** $\operatorname{sgn}(X)$ of a compact Kähler n-fold X is the signature of the bilinear form given by the intersection pairing $H^n(X, \mathbb{R}) \times H^n(X, \mathbb{R}) \to \mathbb{R}$.

Proposition 1.2.43 ([8, Corollary 3.3.18]). *If a compact Kähler n-fold has $n = 2m$, then* $\operatorname{sgn}(X) = \sum_{p,q=0}^{2m}(-1)^p h^{p,q}(X)$.

Theorem 1.2.44 (Hirzebruch signature theorem). *Let* $L(X) \coloneqq \prod_{k=1}^{n} a_k \coth(a_k/2)$ *be the* **Hirzebruch L-genus**. *Then* $\operatorname{sgn}(X) = \int_X L(X)$.

Proof. By 1.2.43, setting $y = 1$ in the Hirzebruch χ_y-genus gives the signature $\operatorname{sgn}(X)$. But using the formula 1.2.40 for the χ_y-genus, we get $\operatorname{sgn}(X) = \chi_1 = \int_X L(X)$. $\square$

1.2.5 Fixed-Point Theorems and Localization

Theorem 1.2.45 (Poincaré–Hopf index theorem). *Let M be a compact differentiable manifold, and X a vector field on M with isolated zeros x_i and pointing in the outward normal direction along ∂M. Then* $\sum_i \operatorname{ind}_X(x_i) = \chi(M)$.

Proof. Assume M is oriented. (Otherwise take the double cover $\tilde{M} \to M$, satisfying $\chi(\tilde{M}) = 2\chi(M)$, and proceed with $\tilde{M}$.) From 1.2.28 we see that $e(TM)$ is Poincaré dual to the zeros of any vector field, in particular, X. Hence $\int_M e(TM) = \sum_i \mathrm{ind}_X(x_i)$ where x_i are the isolated zeros of X. But generalized Gauss–Bonnet gives $\int_M e(TM) = \chi(M)$. ☐

Example 1.2.46. Consider the holomorphic vector field $u\frac{\partial}{\partial u}$ on $\mathbb{P}^1$. Since $\frac{\partial}{\partial u} = -v^2\frac{\partial}{\partial v}$, we have $u\frac{\partial}{\partial u} = -v\frac{\partial}{\partial v}$. Hence this vector field has two zeros at $u = 0$ and $v = 0$, and these zeros both have index 1. Note that $\chi(\mathbb{P}^1) = 2$.

Writing Poincaré–Hopf like $\int_M e(M) = \sum_i \mathrm{ind}_X(x_i)$ for a vector field tells us that some global integrals can be computed using local information at a finite set of (fixed) points. This kind of equation is called a **localization formula**. We shall develop a localization formula for equivariant cohomology with far-reaching consequences.

Definition 1.2.47. If $\iota\colon V \to X$ is a map of compact manifolds, then there is a **pushforward** or **wrong way map** on cohomology $\iota_*\colon H^*(V) \to H^{*+k}(X)$: use Poincaré duality on V, push forward the homology cycle, then use Poincaré duality on X.

Example 1.2.48. Let $\iota\colon V \to X$ be an inclusion of compact manifolds and let k be the codimension of V. Then $\iota_* 1 = \Phi(NV)$, the Thom form of the normal bundle of V. This is straightforward to check: the Poincaré dual of 1 is all of V, which sits in X by the inclusion i, which is dual, fiberwise, to the preferred generator on each fiber, which is precisely the Thom class. The Thom form $\Phi(NV)$ can be extended by zero so that it lives in $H^k(X)$, and the pullback $\iota^*\colon H^k(X) \to H^k(V)$ then gives $\iota^*\Phi(NV) = e(NV)$ (c.f. the definition 1.2.25 of the Euler class). Hence

$$\iota^*\iota_* 1 = e(NV).$$

The composition $\iota^*\iota_*$ is called the **Gysin operation**.

Now we move to the setting for equivariant cohomology: X has an action by a topological group G. The idea is that, compared to ordinary cohomology, the equivariant cohomology has a larger coefficient ring, namely the polynomial ring $H_G^* := H_G^*(\{\mathrm{pt}\})$. In ordinary cohomology we cannot invert $e(NV)$, but in equivariant cohomology we can. However we first need to define characteristic classes in equivariant cohomology.

Definition 1.2.49. A vector bundle $\pi\colon E \to X$ is **equivariant** under the G-action on X if the action $G \times X \to X$ lifts to $G \times E \to E$ in a way such that π is equivariant. An equivariant vector bundle E induces a vector bundle $E_G := EG \times_G E$ on the homotopy quotient $EG \times_G X$. An **equivariant characteristic class** of E is an ordinary characteristic class of E_G, which is an element of $H^*(EG \times_G X) = H_G^*(X)$. A cohomology class $\alpha \in H^*(X)$ has a **equivariant extension** if there exists $\alpha^G \in H_G^*(X)$ with $\iota^*\alpha^G = \alpha$ where $\iota\colon X \to X_G$ is fiber-wise inclusion.

Example 1.2.50. The **equivariant Chern classes** of a G-equivariant rank n complex vector bundle $\pi\colon E \to X$ are the regular Chern classes of $E_G \to X_G$, i.e.

$$c_k^G(E) := c_k(E_G) = c_k(EG \times_G E) \in H^*(X_G) = H_G^*(X).$$

In particular the **equivariant Euler class** is the top equivariant Chern class $e^G(E) := c_n^G(E)$.

For the remainder of this subsection we are always working with equivariant cohomology, so we omit the superscript G on characteristic classes. Since we are usually concerned with a torus action for equivariant cohomology, let $\mathbb{T} := (\mathbb{C}^*)^m$ be an algebraic torus acting on X. Recall from 1.1.36 that in this case we can view equivariant cohomology classes as represented by $H_{\mathbb{T}}^* = \mathbb{Q}[x_0, \ldots, x_m]$-valued forms.

Theorem 1.2.51 (Atiyah–Bott, [9]). *Let $\mathbb{T}$ be a torus acting on X, and split the locus of fixed points F into connected components F_i. The equivariant Euler classes $e_{\mathbb{T}}(NF_i) \in H^*(F_i) \otimes H_{\mathbb{T}}^*$ are invertible in the localized ring $H^*(F_i) \otimes_{\mathbb{Q}} \mathrm{Frac}(H_{\mathbb{T}}^*)$.*

Proof sketch. Terms of positive degree in $H^*(F_i)$ are nilpotent, so an element in $H^*(F_i)$ is invertible if its H^0 component is nonzero. So $e(NF_i)$ is invertible if its component in $H^0(F_i) \otimes H^*_{\mathbb{T}}$ is a non-zero polynomial. Some work shows this is always the case; see page 9 in the cited paper. $\square$

View $X_{\mathbb{T}}$ as a bundle over $B\mathbb{T}$ with fiber X. Then the projection $\pi^X \colon X_{\mathbb{T}} \to B\mathbb{T}$ induces a wrong-way map $\pi^X_* \colon H^*_{\mathbb{T}}(X) \to H^*_{\mathbb{T}}$ which is precisely integration over the fibers. To disambiguate, we write integration over the fibers as $\int_{X_{\mathbb{T}}/B\mathbb{T}}$ and integration over X as $\int_X$. The relation between the two is given by

$$\begin{array}{ccc} X & \xrightarrow{\iota} & X_{\mathbb{T}} \\ \downarrow & & \pi\downarrow \\ p & \xrightarrow{\iota_p} & B\mathbb{T} \end{array} \qquad\qquad \iota_p^* \int_{X_{\mathbb{T}}/B\mathbb{T}} \tilde{\alpha} = \int_X \iota^* \tilde{\alpha} = \int_X \alpha.$$

Here $\iota_p^* \colon H^*_{\mathbb{T}} \cong \mathbb{Q}[x_0, \ldots, x_m] \to H^*(\{\mathrm{pt}\}) = \mathbb{Q}$ is essentially the evaluation at zero map.

Corollary 1.2.52 (Atiyah–Bott localization formula)**.** *Let $\iota^{F_i} \colon F_i \to X$ be the inclusions of the connected components of the fixed point locus. For $\phi \in H^*_{\mathbb{T}}(X)$ and working in $H^*_{\mathbb{T}}(X)[(H^*_{\mathbb{T}})^{-1}]$,*

$$\int_{X_{\mathbb{T}}/B\mathbb{T}} \phi = \sum_i \int_{(F_i)_{\mathbb{T}}/B\mathbb{T}} \frac{\iota^*\phi}{e(NF_i)} \in \mathrm{Frac}(H^*_{\mathbb{T}}).$$

Proof. From the identity $(\iota^{F_i})^* \iota^{F_i}_* 1 = e(NF_i)$ and the Atiyah–Bott theorem, we can invert $e(NF_i)$ and write $\phi = \sum_i \iota_*(\iota^*\phi/e(NF_i))$. We can apply π^X_* to both sides. But $\pi^X_* \circ \iota^{F_i}_* = \pi^{F_i}_*$, i.e. integration over F_i. Hence

$$\int_{X_{\mathbb{T}}/B\mathbb{T}} \phi = \pi^X_* \phi = \sum_i \pi^{F_i}_* \frac{\iota^*\phi}{e(NF_i)} = \sum_i \int_{(F_i)_{\mathbb{T}}/B\mathbb{T}} \frac{\iota^*\phi}{e(NF_i)}. \qquad \square$$

The idea is that the Atiyah–Bott localization formula simplifies many equivariant cohomology integrals by letting us only calculate over fixed points. It also allows us prove many fixed-point theorems. We do one such theorem as an example of localization.

Proposition 1.2.53. *Let $\mathbb{T}$ act on X with fixed point locus $\bigcup_i F_i$. Then*

$$\chi(X) = \sum_i \chi(F_i).$$

Proof. From the generalized Gauss–Bonnet theorem 1.2.41 we know $\chi(X) = \int_X e(X)$. The key is that the action of $\mathbb{T}$ on X lifts to an action of $\mathbb{T}$ on TX, so the same statement holds in equivariant cohomology with $\int_{X_{\mathbb{T}}/B\mathbb{T}} \colon H^*_{\mathbb{T}}(X) \to H^*_{\mathbb{T}}$ being integration over the fiber (or the wrong-way map π^X_*). Localizing,

$$\int_{X_{\mathbb{T}}/B\mathbb{T}} e(X) = \sum_i \int_{(F_i)_{\mathbb{T}}/B\mathbb{T}} \frac{\iota^* e(X)}{e(NF_i)}.$$

Remember here $\iota \colon F_i \to X$ are inclusions. Recall that $e(X) = c_n(X)$ and note that there is an exact sequence of equivariant bundles $TF_i \to \iota^* TX \to NF_i$, so by naturality 1.2.14 and the splitting principle,

$$\iota^* e(X) = \iota^* c_n(X) = c_n(\iota^* TX) = c_n(TF_i) c_n(NF_i).$$

Hence when we plug this back into the localized expression,

$$\chi(X) = \int_{X_{\mathbb{T}}/B\mathbb{T}} e(X) = \sum_i \int_{(F_i)_{\mathbb{T}}/B\mathbb{T}} c_n(TF_i) = \sum_i \chi(F_i). \qquad \square$$

1.3 Calabi–Yau Manifolds

In this section we develop the theory of Calabi–Yau manifolds, the main objects of mirror symmetry.

1.3.1 Kähler Geometry

We briefly review Kähler geometry in this subsection. Let X be a complex n-fold with complex structure J and coordinates $(z^1, \ldots, z^n, \bar{z}^1, \ldots, \bar{z}^n)$. Let $g = g_{i\bar{j}}(z)$ be a Hermitian metric on X.

Definition 1.3.1. The 2-form $\omega(X, Y) \coloneqq g(X, JY)$ is the **associated Kähler form**. We say g is a **Kähler metric** and (M, g) is a **Kähler manifold** if $d\omega = 0$. In coordinates, if $g = g_{i\bar{j}}(z) dz^i \odot d\bar{z}^{\bar{j}}$, then $\omega = (i/2) g_{i\bar{j}}(z) dz^i \wedge d\bar{z}^{\bar{j}}$.

Proposition 1.3.2. *The volume form associated to a Hermitian metric g is equal to $\omega^n/n!$. Hence if X is a closed Kähler manifold, $\omega^k \in H^{2k}_{\mathrm{dR}}(X)$ is non-zero for every $1 \le k \le n$.*

Proof. Let $e_1, \ldots, e_n$ be a complex orthonormal basis of $T_{X,x}$, so that the corresponding real basis is $e_1, Je_1, \ldots, e_n, Je_n$ and it suffices to check $(\omega^n/n!)(e_1 \wedge Je_1 \wedge \cdots \wedge e_n \wedge Je_n) = 1$. But this is obvious, since $\omega = (i/2) \sum_i dz^i \wedge d\bar{z}^{\bar{i}}$ in orthonormal coordinates.

If $\omega^k = d\eta$, then $\omega^n = d(\omega^{n-k} \wedge \eta)$ by the Kähler condition. By Stokes', $\int_X \omega^n = 0$ since X is closed. But this is supposed to be the volume of X. $\square$

So it is easy to give examples of non-Kähler complex manifolds: just pick a closed complex n-fold whose H^{2k}_{dR} is zero for some $1 \le k \le n$, e.g. Hopf surfaces.

Definition 1.3.3. Let E be a Hermitian vector bundle over X with Hermitian metric h. A connection ∇ on E is **compatible** with h if

$$d(h(\sigma, \tau)) = h(\nabla\sigma, \tau) + h(\sigma, \nabla\tau).$$

Here we define $h(\sigma \otimes \eta, \tau) \coloneqq h(\sigma, \tau)\eta$ where $\eta \in \Omega^*(X)$. But h is sesquilinear, so $h(\sigma, \tau \otimes \eta) = h(\sigma, \tau)\overline{\eta}$.

Proposition 1.3.4. *Let E be a holomorphic vector bundle with holomorphic structure $\bar{\partial}_E$. Then there exists a unique connection ∇ on E, called the* **Chern connection**, *compatible with h and with $\nabla^{0,1} = \bar{\partial}_E$.*

Proof. Writing down the $(1,0)$-part of the condition for compatibility, we get

$$\partial(h(\sigma, \tau)) = h(\nabla^{1,0}\sigma, \tau) + h(\sigma, \nabla^{0,1}\tau) = h(\nabla^{1,0}\sigma, \tau)$$

where the last equality follows from $\nabla^{0,1} = \bar{\partial}_E$ and σ, τ being holomorphic sections. Since h is non-degenerate, this equality uniquely specifies $\nabla^{1,0}$. $\square$

Theorem 1.3.5 ([10, Theorem 3.13, Proposition 3.14], [8, Proposition 4.A.17])**.** *The following are equivalent:*

1. *the metric h is Kähler, i.e. $\partial_i h_{j\bar{k}} = \partial_j h_{i\bar{k}}$,*
2. *the Christoffel symbols Γ^i_{jk} of the Levi–Civita connection are* **pure**, *i.e. Γ^i_{jk} and $\Gamma^{\bar{i}}_{\bar{j}\bar{k}}$ may be non-zero, but "mixed" symbols like $\Gamma^{\bar{i}}_{jk}$ vanish,*
3. *the complex structure J is* **parallel** *with respect to the Levi–Civita connection of h, i.e. $\nabla J = 0$,*
4. *the Chern connection and the Levi–Civita connection coincide,*
5. *there exists* **normal coordinates** *in a neighborhood of any $x \in X$, i.e. holomorphic coordinates $z_1, \ldots, z_n$ such that $h_{i\bar{j}} = \delta_{ij} + O(\sum_i |z_i|^2)$,*
6. *X is a $2n$-dimensional Riemannian manifold with holonomy in $U(n)$.*

Definition 1.3.6. The Kähler condition $d\omega = 0$ implies that for any point $x \in X$ there exists an open neighborhood U and a function $\Phi\colon U \to \mathbb{R}$ on U with $\omega|_U = (i/2)\partial\bar{\partial}\Phi$. Namely, it suffices to pick the domain $U \cong \mathbb{C}^n$ of a chart: by the $\bar{\partial}$-Poincaré lemma, ω is $\bar{\partial}$-exact on U, and then we apply the global $\partial\bar{\partial}$-lemma 1.3.16 (which we shall use Hodge theory to prove in the next subsection). The function Φ is the **Kähler potential**. Note that Φ is not uniquely determined: $\Phi + f(z) + g(\bar{z})$ defines the same ω.

The Kähler potential is local and cannot be defined globally! For if $\omega = (i/2)\partial\bar{\partial}\Phi$ globally, then ω is exact, violating 1.3.2.

Definition 1.3.7. Let X be a compact Kähler manifold. The **Kähler class** of a Kähler structure on X is the cohomology class $[\omega] \in H^{1,1}(X)$ of its Kähler form. The **Kähler cone**

$$\mathcal{K}_X \subset H^{1,1}(X) \cap H^2(X, \mathbb{R})$$

is the set of all Kähler classes associated to any Kähler structure on X. (Clearly $\mathcal{K}_X$ is closed under positive rescaling, so this is indeed a cone.)

Later when we look at deformations of Calabi–Yau manifolds, we shall see that any class in $H^{1,1}(X)$ can be used to deform the metric slightly. So $h^{1,1}$ classifies infinitesimal deformations of the metric (c.f. 1.3.25).

1.3.2 Hodge Theory of Kähler Manifolds

We now look at the Hodge theory of Kähler manifolds, which is more interesting than the Hodge theory of Riemannian manifolds. For completeness we first review the Riemannian case.

Definition 1.3.8. Let X be a compact oriented Riemannian n-fold with Riemannian metric g. Define the L^2 **inner product** on $\Omega^k(X)$ by $(\alpha, \beta)_{L^2} \coloneqq \int \alpha \wedge \star\beta$ where $\star$ is the **Hodge star**. The (formal) adjoint of the exterior derivative $d\colon \Omega^k(X) \to \Omega^{k+1}(X)$ under $(\cdot,\cdot)_{L^2}$ is denoted $d^*\colon \Omega^{k+1}(X) \to \Omega^k(X)$ and called the **codifferential**. A quick calculation shows $d^* = (-1)^{n(k-1)+1} \star d\star = (-1)^k \star^{-1} d\star$, which commutes with d. Define the space $\mathcal{H}^k_\Delta(X)$ of **harmonic forms** to be the kernel of the **Laplacian** Δ:

$$\mathcal{H}^k_\Delta(X) \coloneqq \{\alpha \in \Omega^k(X) : \Delta\alpha = 0\}, \quad \Delta \coloneqq dd^* + d^*d\colon \Omega^k(X) \to \Omega^k(X).$$

The **Hodge decomposition** is an orthogonal decomposition (with respect to the L^2 inner product)

$$\Omega^k(X) \cong \operatorname{im}(\Omega^{k-1} \xrightarrow{d} \Omega^k) \oplus \operatorname{im}(\Omega^{k+1} \xrightarrow{d^*} \Omega^k) \oplus \mathcal{H}^k_\Delta.$$

Since d^* and therefore Δ commutes with d, we have $d\mathcal{H}^k_\Delta = 0$, so there is a canonical map $\varphi\colon \mathcal{H}^k_\Delta(X) \to H^k_{\mathrm{dR}}(X)$. The **Hodge theorem** says φ is an isomorphism of vector spaces, i.e. every cohomology class has a unique harmonic representative.

Definition 1.3.9. For X a complex manifold now with a Hermitian metric, $d = \partial + \bar{\partial}$. Denoting the L^2 adjoint of ∂ and $\bar{\partial}$ by ∂^* and $\bar{\partial}^*$ respectively, define the **$\bar{\partial}$-laplacian** $\Delta_{\bar{\partial}} \coloneqq \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ and similarly for the ∂-laplacian. Similarly if E is a holomorphic vector bundle with holomorphic structure $\bar{\partial}_E$ then $\Delta_E \coloneqq \bar{\partial}_E\bar{\partial}_E^* + \bar{\partial}_E^*\bar{\partial}_E$. Define the **$\bar{\partial}$-harmonic forms**

$$\mathcal{H}^{p,q}_{\bar{\partial}}(X) \coloneqq \{\alpha \in \Omega^{p,q}(X) : \Delta_{\bar{\partial}}\alpha = 0\}$$

and similarly for **∂-harmonic forms**.

Lemma 1.3.10. *Let X be a compact complex manifold with Hermitian metric. Then α is $\bar{\partial}$-harmonic iff α is both $\bar{\partial}$- and $\bar{\partial}^*$-closed.*

Proof. Note that $(\Delta_{\bar{\partial}}\alpha, \alpha) = \|\bar{\partial}^*\alpha\|^2 + \|\bar{\partial}\alpha\|^2 = 0$ implies $\bar{\partial}^*\alpha = \bar{\partial}\alpha = 0$. The converse is trivial. □

Definition 1.3.11. Let ω be the associated Kähler form. The **Lefschetz operator** $L\colon \Omega^{p,q} \to \Omega^{p+1,q+1}$ is given by $\alpha \mapsto \alpha \wedge \omega$. The **dual Lefschetz operator** $\Lambda\colon \Omega^{p+1,q+1} \to \Omega^{p,q}$ is the L^2 adjoint of the Lefschetz operator L. Note that by 1.3.2, $L^k \neq 0$ for $1 \leq k \leq n$.

Theorem 1.3.12 (Kähler identities, [10, Section 6.1])**.** *If (X, h) is a Kähler manifold, then*

$$[\bar{\partial}^*, L] = i\partial, \quad \bar{\partial}^*\partial + \partial\bar{\partial}^* = 0, \quad \Delta_d = 2\Delta_\partial = 2\Delta_{\bar{\partial}}.$$

Proof sketch. Bash the first equation out locally in $\mathbb{C}$, and then use (holomorphic) normal coordinates to put it onto X. Then bash some more for the other two equations. □

Corollary 1.3.13. *If $\alpha \in \Omega^k$ is harmonic, then its components $\alpha^{p,q} \in \Omega^{p,q}$ are harmonic.*

Note that by the Kähler identities, we do not need to specify the Laplacian with respect to which a form is harmonic. In particular, (p, q)-forms harmonic for Δ_d are also harmonic for $\Delta_{\bar{\partial}}$.

Theorem 1.3.14 (Hodge decomposition, [8, Theorem 3.2.8])**.** *Let X be a compact Kähler manifold. There exists orthogonal* **Hodge decompositions**

$$\Omega^{p,q} = \partial\Omega^{p-1,q} \oplus \partial^*\Omega^{p+1,q} \oplus \mathcal{H}^{p,q}_\partial = \bar{\partial}\Omega^{p,q-1} \oplus \bar{\partial}^*\Omega^{p,q+1} \oplus \mathcal{H}^{p,q}_{\bar{\partial}}.$$

Furthermore $\mathcal{H}^{p,q}_\partial = \mathcal{H}^{p,q}_{\bar{\partial}}$, and the canonical projection $\mathcal{H}^{p,q}_{\bar{\partial}} \to H^{p,q}(X, \mathbb{C})$ given by 1.3.10 is an isomorphism. Hence by 1.3.13, $H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}$.

Corollary 1.3.15. *The odd Betti numbers $b_{2k+1} \coloneqq \dim_{\mathbb{C}} H^{2k+1}(X, \mathbb{C})$ of a compact Kähler manifold X are even.*

Proof. Complex conjugation acts naturally on $H^{p+q}(X, \mathbb{C})$ with $\overline{H^{p,q}} = H^{q,p}$. □

Corollary 1.3.16 (Global $\partial\bar{\partial}$ lemma, [8, Corollary 3.2.10])**.** *Let X be a compact Kähler manifold. If $\alpha \in \Omega^{p,q}(X)$ and $d\alpha = 0$ then*

$$\alpha \text{ is } d\text{-exact} \iff \alpha \text{ is } \partial\text{-exact} \iff \alpha \text{ is } \bar{\partial}\text{-exact} \iff \alpha \text{ is } \partial\bar{\partial}\text{-exact.}$$

Proof. By the Hodge decomposition theorem 1.3.14, any of the four conditions above implies α is orthogonal to $\mathcal{H}^{p,q}$ (where since X is Kähler we do not need to specify which Laplacian we are considering for harmonicity). We shall show this implies α is $\partial\bar{\partial}$-exact.

Since α is d-closed and therefore ∂-closed, $(\partial\alpha, \beta) = (\alpha, \partial^*\beta) = 0$, so α is also orthogonal to $\partial^*\Omega^{p+1,q}$. By the Hodge decomposition with respect to ∂, it follows that $\alpha \in \partial\Omega^{p-1,q}$, i.e. $\alpha = \partial\gamma$ for some $\gamma \in \Omega^{p-1,q}$. By Hodge decomposition with respect to $\bar{\partial}$ on γ,

$$\gamma = \bar{\partial}\beta + \bar{\partial}^*\beta' + \beta'', \quad \alpha = \partial\gamma = \partial\bar{\partial}\beta + \partial\bar{\partial}^*\beta'$$

where 1.3.10 implies that $\partial\beta'' = 0$. We use the Kähler identity $\partial\bar{\partial}^* = -\bar{\partial}^*\partial$ from 1.3.12 to get

$$0 = \bar{\partial}\alpha = \bar{\partial}\partial\bar{\partial}\beta + \bar{\partial}\partial\bar{\partial}^*\beta' = 0 - \bar{\partial}\bar{\partial}^*\partial\beta'.$$

But $0 = (\bar{\partial}\bar{\partial}^*\partial\beta', \partial\beta') = \|\bar{\partial}^*\partial\beta'\|^2$, so the term $\partial\bar{\partial}^*\beta'$ in α vanishes and we are left with $\alpha = \partial\bar{\partial}\beta$. □

There is a different orthogonal decomposition of cohomology that is useful in computations. This decomposition arises from the Lefschetz operator L rather than Hodge theory.

Theorem 1.3.17 (Hard Lefschetz theorem, [10, Theorem 6.25])**.** *Let X be a compact Kähler manifold. For $k \leq n$, the $(n-k)$-th power of the Lefschetz operator gives an isomorphism*

$$L^{n-k}\colon H^k(X, \mathbb{C}) \xrightarrow{\sim} H^{2n-k}(X, \mathbb{C}).$$

Equivalently, since ω is a $(1,1)$-form, $H^{p,q}(X, \mathbb{C}) \cong H^{n-q,n-p}(X, \mathbb{C})$.

Theorem 1.3.18 (Lefschetz hyperplane theorem, [11, Theorem 1.23])**.** *Let $X \subseteq \mathbb{P}^N$ be a (not necessarily smooth) n-dimensional algebraic subvariety, and $Y = \mathbb{P}^{N-1} \cap X$ be a hyperplane section such that $X \setminus Y$ is smooth. Then the restriction $H^k(X, \mathbb{Z}) \to H^k(Y, \mathbb{Z})$ is an isomorphism for $k < n - 1$ and injective for $k = n - 1$.*

Theorem 1.3.19 (Lefschetz decomposition, [10, Corollary 6.26, Lemma 6.31])**.** *Writing $H^r(X, \mathbb{C})_{prim} \coloneqq \ker L^{n-r+1} \subset H^r(X, \mathbb{C})$ for the* **primitive forms***, we have the (orthogonal with respect to the L^2 inner product) Lefschetz decomposition*

$$H^k(X, \mathbb{C}) \cong \bigoplus_{k-2r\geq 0} L^r H^{k-2r}(X, \mathbb{C})_{prim}.$$

The **Hodge diamond** is the collection of Hodge numbers $h^{p,q}$ arranged in a diamond with rows of increasing $p + q$. Its symmetries for a compact Kähler manifold are:

1. reflection across the vertical axis $h^{p,q} = h^{q,p}$, from complex conjugation;
2. reflection across the horizontal axis $h^{p,q} = h^{n-q,n-p}$, from the Hodge star or the hard Lefschetz theorem;
3. $180°$ rotation $h^{p,q} = h^{n-p,n-q}$, from Serre duality 1.2.10;

1.3.3 Calabi–Yau Manifolds

Throughout this subsection, X is a compact Kähler manifold with all the notation of the previous subsections, and also with Levi–Civita connection ∇. We use ∇^K to denote the induced connection on the canonical bundle $K_X \coloneqq \Lambda^{n,0}X$.

Definition 1.3.20. Recall the **Riemann curvature tensor** and **Ricci curvature tensor**

$$R(X, Y)Z \coloneqq \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad \mathrm{Ric}(X, Y) \coloneqq \mathrm{tr}(Z \mapsto R(Z, X)Y).$$

In coordinates, if ${R^l}_{ijk}$ is the Riemann curvature tensor, then $R_{ij} \coloneqq {R^k}_{ikj}$ is the Ricci curvature tensor. By 1.3.5, $\nabla J = 0$, so $R(X, Y)JZ = JR(X, Y)Z$, which gives $\mathrm{Ric}(JX, JY) = \mathrm{Ric}(X, Y)$ after some work. Hence $\mathrm{Ric}(JX, Y)$ is skew-symmetric in X and Y. The **Ricci form** ρ of X is defined by $\rho(X, Y) \coloneqq \mathrm{Ric}(JX, Y)$.

Proposition 1.3.21 ([12, Proposition 12.2])**.** *The Ricci tensor of a Kähler manifold satisfies*

$$\mathrm{Ric}(X, Y) = (1/2)\,\mathrm{tr}(R(X, JY) \circ J).$$

It follows that the Ricci form ρ is d-closed.

Proof sketch. The identity for the Ricci tensor is just a short calculation using the first Bianchi identity $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$. Then

$$2d\rho(X, Y, Z) = 2((\nabla_X \rho)(Y, Z) + \text{cyclic perms}) = \mathrm{tr}(((\nabla_X R)(Y, Z) + \text{cyclic perms}) \circ J) = 0$$

by the second Bianchi identity $(\nabla_X R)(Y, Z) + \text{cyclic perms} = 0$. □

Proposition 1.3.22 ([12, Proposition 17.4])**.** *Let X be a Kähler manifold. Then $c_1(X) = (1/2\pi)[\rho]$.*

Proof sketch. By Chern–Weil, $c_1(X) = -c_1(K_X)$. Let F_∇ be the curvature operator $F_\nabla \in \Gamma(\Lambda^2 X \otimes \mathrm{End}(TX))$ of the Chern connection on X and R be the Riemannian curvature tensor of the Levi–Civita connection on X. Then a straightforward calculation shows $F_\nabla(X, Y) = R(X, Y)$ on TX. Writing ∇^K for the induced connection on K_X, we have $F_{\nabla^K} = -\,\mathrm{tr}\, F_\nabla = -\,\mathrm{tr}\, R$. But then using 1.3.21,

$$i\rho(X, Y) = i\,\mathrm{Ric}(JX, Y) = \frac{i}{2}\,\mathrm{tr}_{\mathbb{R}}(R(X, Y) \circ J) = \frac{i}{2}(2i\,\mathrm{tr}_{\mathbb{C}}(R(X, Y))) = -\,\mathrm{tr}_{\mathbb{C}}(R(X, Y)) = F_{\nabla^K}(X, Y). \quad □$$

Theorem 1.3.23 ([12, Theorem 17.5]). *Let X be a compact Kähler manifold. The following are equivalent:*

1. *X is Ricci-flat, i.e.* $\mathrm{Ric} = 0$,
2. *the Chern connection of the canonical bundle K_X is flat,*
3. *there exists a (local) section of the canonical bundle parallel to the Levi–Civita connection ∇,*
4. *X has holonomy in* $\mathrm{SU}(n)$,

If X satisfies any of these conditions, we say X is a **Calabi–Yau manifold**.

Proof sketch. (2) $\iff$ (4) Let ∇ be the Levi–Civita connection on X. The holonomy of the induced connection ∇^K on K_X satisfies $\mathrm{Hol}_0(\nabla^K) = \det(\mathrm{Hol}_0(\nabla))$, so $\mathrm{Hol}_0(\nabla^K) = \{1\}$ iff the original holonomy is in $\mathrm{SU}(n)$. By the Ambrose–Singer theorem, $\mathrm{Hol}_0(\nabla^K) = \{1\}$ iff ∇^K is flat.

(2) $\iff$ (1) By 1.3.22, the vanishing of the curvature of ∇^K is equivalent to the vanishing of ρ.

(2) $\iff$ (3) This is the general principle that a connection on a line bundle is flat iff there exists a (locally defined) parallel section (which is global if $\pi_1(X) = 0$). ☐

Note that by 1.3.22, $c_1(X) = 0$ is a necessary condition for there to be a Ricci-flat Kähler metric on X. The following deep theorem of Calabi and Yau shows this condition is also sufficient when X is compact, i.e. c_1 is the only topological obstruction.

Theorem 1.3.24 (Calabi–Yau, [12, Theorem 18.1]). *Let X be a compact Kähler manifold with Kähler form ω and Ricci form ρ. For every ρ_1 in the cohomology class of $2\pi c_1(X) \in H^{1,1}(X) \cap H^2(X, \mathbb{R})$, there exists a unique Kähler metric with Kähler form ω_1 in the cohomology class of ω and with Ricci form ρ_1.*

Corollary 1.3.25. *Let M be a compact complex manifold with $c_1(X) = 0 \in H^{1,1}(X) \cap H^2(X, \mathbb{R})$. Then there is a unique Ricci-flat Kähler metric in each Kähler class on X. The family of Ricci-flat Kähler metrics on X is therefore isomorphic to the Kähler cone of X.*

The Hodge numbers of Calabi–Yau manifolds are special. Since the canonical bundle K_X of a Calabi–Yau manifold X is trivial, it is generated by a nowhere-vanishing global holomorphic $(n, 0)$-form $f(z)dz^1 \wedge \cdots \wedge dz^n$. But M is compact, so f is therefore constant, and $h^{n,0} = 1$. By the symmetries of the Hodge diamond for Kähler manifolds, $h^{n,0} = h^{0,n} = h^{0,0} = h^{n,n} = 1$.

Proposition 1.3.26. *Let X be Calabi–Yau. If $\chi(X) \neq 0$, then $b_1 = 0$, i.e. $h^{1,0} = h^{0,1} = 0$.*

Proof. It suffices to find b_1 for the Ricci-flat metric since b_1 is a topological invariant. Let α be a harmonic 1-form. The Weitzenböck formula says $\Delta\alpha = \nabla^*\nabla\alpha + \tilde{R}\alpha$ where $\tilde{R}$ depends only on the curvature. Hence $\nabla^*\nabla\alpha = 0$, which when integrated by parts gives $\nabla\alpha = 0$. In other words, α is constant. But $\chi(X) \neq 0$, so α must have a zero somewhere (c.f. 1.2.28). Hence $\alpha = 0$. By the Hodge theorem 1.3.14, $H^1 = \mathcal{H}^1_\Delta = 0$. ☐

We are generally interested in Calabi–Yau 3-folds. It follows by the discussion above that the Hodge diamond of Calabi–Yau 3-folds is almost fully determined:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

By 1.3.25, we interpret $h^{1,1}$ as the number of linearly independent infinitesimal deformations of the Kähler structure of X. The next subsection interprets $h^{2,1}$.

1.3.4 Calabi–Yau Moduli Space

In this subsection we explore the moduli space of Calabi–Yau manifolds following [13]. A lot of bashing is omitted.

Let X be a Calabi–Yau 3-fold with (holomorphically) trivial canonical bundle K_X. (**Note**: this is stronger than just being Calabi–Yau according to our definition 1.3.23, and is our setting from now on). Let g and $g + \delta g$ be two Ricci-flat metrics. Then, working in the gauge $\nabla^j g_{ij} = 0$, some work shows

$$\mathrm{Ric}(g + \delta g) = 0 + O((\delta g)^2) \implies \nabla^a \nabla_a \delta g_{ij} + 2R_i{}^a{}_j{}^b \delta g_{ab} = 0,$$

Definition 1.3.27. The equation $\nabla^a \nabla_a \delta g_{ij} + 2R_i{}^a{}_j{}^b \delta g_{ab} = 0$ is called the **Lichnerowicz equation**.

Proposition 1.3.28. *The solutions of the Lichnerowicz equation are in one-to-one correspondence with $H^{1,1}(X) \oplus H^{2,1}(X)$, where $H^{1,1}$ corresponds to variations of Kähler structure, and $H^{2,1}$ corresponds to variations of complex structure.*

Proof. Since X is Kahler, 1.3.5 tells us the Christoffel symbols are "pure", i.e. only Γ^i_{jk} and $\Gamma^{\bar{i}}_{\bar{j}\bar{k}}$ are nonzero, and other symbols with mixed indices are zero. This fact and some more work then shows that $\delta g_{i\bar{j}}$, which we call variations of **mixed type**, and δg_{ij}, which we call variations of **pure type**, individually satisfy the Lichnerowicz equation.

Let $\Omega_{abc} \in H^{3,0}$ be the preferred holomorphic $(3,0)$-form. Both types of variations can be associated to forms:

$$i\delta g_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}} \in H^{1,1}(X, \mathbb{R}), \quad \Omega_{ab}{}^{\bar{j}} \delta g_{\bar{i}\bar{j}} dz^a \wedge dz^b \wedge d\bar{z}^{\bar{i}} \in H^{2,1}(X, \mathbb{R}).$$

Using a Weitzenböck formula, these forms are harmonic iff their associated variations $g_{i\bar{j}}$ and g_{ij} satisfy the Lichnerowicz equation.

The mixed variation $\delta g_{i\bar{j}}$ gives a new Kähler form since $i\delta g_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}}$ is harmonic and therefore d-closed:

$$\tilde{\omega} := i(g_{i\bar{j}} + \delta g_{i\bar{j}}) dz^i \wedge d\bar{z}^{\bar{j}} = \omega + i\delta g_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}}.$$

The pure variation δg_{ij} gives a new metric $\tilde{g}_{ij} = g_{ij} + \delta g_{ij}$. We want the pure indices to vanish in some new coordinate system $x^i \mapsto x'^i := x^i + f^i(x)$, where

$$\tilde{g}'_{ij} = \tilde{g}_{ij} - (\partial_i f^k) g_{kj} - (\partial_j f^k) g_{ik}.$$

Hence $\delta g_{\bar{i}\bar{j}} - (\partial_{\bar{i}} f^k) g_{k\bar{j}} - (\partial_{\bar{j}} f^k) g_{\bar{i}k} = 0$. Since $\delta g_{\bar{i}\bar{j}} \neq 0$ is our variation, f cannot be holomorphic. So pure variations change the complex structure. $\square$

It is not surprising that pure variations, represented by $H^{2,1}(X)$, correspond to variations in complex structure: Kodaira–Spencer deformation theory tells us infinitesimal deformations of complex structure for a compact complex manifold are represented by $H^1(TX)$ [10, Section 9.1.2], and for Calabi–Yau manifolds we have the identification $H^1(TX) \cong H^1(\Lambda^2 T^*X) = H^{2,1}(X)$ via the pairing $\wedge\colon TM \otimes \Lambda^2 TM \to \Lambda^3 TM = 1$.

Theorem 1.3.29 (Tian–Todorov, [14])**.** *Deformations of Calabi–Yau manifolds X with $H^0(X, TX) = 0$, i.e. automorphisms are discrete, are* **unobstructed***, i.e. every infinitesimal deformation of a Calabi–Yau manifold lifts to an actual deformation. In particular, the family of Ricci-flat Kähler metrics, i.e. the Kähler cone, is smooth of dimension $h^{1,1}$.*

Example 1.3.30. We saw in 1.2.32 that the first Chern class $c_1(Q)$ of the quintic hypersurface Q in $\mathbb{P}^4$ vanishes, so Q can be given a Calabi–Yau structure. By the Lefschetz hyperplane theorem 1.3.18, $h^{1,1}(Q) = h^{1,1}(\mathbb{P}^4)$, which is just 1 since the cohomology ring of $\mathbb{P}^N$ is generated by x, a generic hyperplane. We also know $\chi(Q) = -200$, but from the Hodge diamond of Calabi–Yau manifolds we see $\chi(Q) = 2(h^{1,1}(Q) - h^{1,2}(Q))$. Hence $h^{1,2}(Q) = 101$.

A quick sanity check for $h^{1,2}(Q) = 101$ goes as follows. From 1.1.5, there are 126 ways to deform the defining equation for Q inside $\mathcal{O}_{\mathbb{P}^4}(5)$. One of these vanishes on Q, being Q itself. So there are 125 infinitesimal deformations of complex structure, but $\mathbb{P}^4$ has automorphism group PGL(5), which is of dimension $5^2 - 1$, leaving $125 - 24 = 101$ distinct deformations.

Definition 1.3.31. The **moduli space of complex structures** for the Calabi–Yau 3-fold X is denoted $\mathcal{M}_X$. We write X_z for the Calabi–Yau 3-fold at z in $\mathcal{M}_X$ and set $X_0 \coloneqq X$. (The coordinates z are coming up soon.)

For the rest of this subsection, we focus on the moduli space of complex structures $\mathcal{M}_X$. The moduli space of Kähler structures is much more involved, and we shall investigate it later. To understand the structure of $\mathcal{M}_X$, we investigate how the generating $(3,0)$-form $\Omega \coloneqq (1/3!)\Omega_{ijk} dz^i \wedge dz^j \wedge dz^k$ of X moves inside $H^{3,0}(X)$ as we move around in $\mathcal{M}_X$. First we need to define coordinates on $\mathcal{M}_X$.

Definition 1.3.32. Let (A^a, B_b) with $a, b = 0, 1, \ldots, h^{2,1}$ be a symplectic homology basis of $H_3(X)$, i.e. under the intersection pairing, $\langle A^a, B_b \rangle = \delta^a_b$. Let (α_a, β^b) the Poincaré dual cohomology basis. (Such a basis is unique up to a $\mathrm{Sp}(b_3, \mathbb{Z})$ transformation, i.e. up to preservation of the intersection form that defines Poincaré duality.) Writing $\Omega = z^a \alpha_a - w_b \beta^b$, we have $z^a = \int_{A^a} \Omega$ and $w_b = \int_{B_b} \Omega$, called the **periods** of Ω.

Theorem 1.3.33 (Bryant–Griffiths, [15])**.** *Locally, $\{z^a\}$ is sufficient to determine the complex structure of X, and the $\{w_b\}$ are redundant, i.e. they can be computed as functions of z^a.*

The $\{z^a\}$ are **projective coordinates**, since under $\Omega \mapsto \lambda\Omega$ they scale as $z^a \mapsto \lambda z^a$, but we only care about Ω up to scalar multiples. So we view $(z^0, \ldots, z^{h^{2,1}}) \in \mathbb{P}^{h^{2,1}}$, and the scaling $\Omega \mapsto e^{f(z)}\Omega$ is a gauge redundancy. (**Notation**: we use a, b, c to index coordinates on the moduli space, and i, j, k to index coordinates on each X_z.)

Definition 1.3.34. The **Hodge bundle** $\mathcal{H}$ is the bundle over $\mathcal{M}_X$ with fiber $H^3(X_z; \mathbb{C})$ at the point $z \in \mathcal{M}_X$. On $\mathcal{H}$, the intersection pairing $(\theta, \eta) = i\int_{X_z} \theta \wedge \bar{\eta}$ provides a natural Hermitian metric. There is a natural flat connection on $\mathcal{H}$ called the **Gauss–Manin connection**. (We shall see it in more detail later when we discuss the tt^* equations.) The 3-form Ω (and its scalar multiples) defines the **vacuum line bundle** inside the Hodge bundle $\mathcal{H}$. The intersection pairing on $\mathcal{H}$ induces an $h \coloneqq i\int_{X_z} \Omega \wedge \bar{\Omega}$. Note that under $\Omega \mapsto e^{f(z)}\Omega$, the quantity h transforms as $h \mapsto h e^f e^{\bar{f}}$.

Definition 1.3.35. On $\mathcal{M}_X$, fix local affine coordinates $\{t^a\}$ with $a = 1, \ldots, h^{2,1}$. The **Kähler potential** K on $\mathcal{M}_X$ is given by

$$K \coloneqq -\log \int_{X_z} \Omega \wedge \bar{\Omega},$$

which does indeed transform as a Kähler potential: under $\Omega \mapsto e^{f(z)}\Omega$, we have $K \mapsto K - f - \bar{f}$, defining the same Kähler form. The **Weil–Petersson metric** on $\mathcal{M}_X$ is the metric associated to the Kähler potential: $g_{a\bar{b}} \coloneqq \partial_a \bar{\partial}_{\bar{b}} K$.

Proposition 1.3.36 (Griffiths transversality)**.** *The variation $\partial_a \Omega$ of Ω with respect to the coordinate z^a satisfies*

$$\partial_a \Omega = k_a \Omega + \chi_a \in H^{3,0}(X_z) \oplus H^{2,1}(X_z)$$

where k_a may depend on z^a but not on the coordinates of X_z. Consequently $\int_{X_z} \Omega \wedge \partial_a \Omega = 0$.

Proof. Let $\zeta^i(x, z)$ be a system of holomorphic coordinates on X_z varying with z, with $\zeta^i(x, 0) = x^i$, the original coordinates on X_0. Then $\Omega = (1/3!)\Omega_{ijk}(\zeta) d\zeta^i \wedge d\zeta^j \wedge d\zeta^k$, and

$$\partial_a \Omega = \frac{1}{3!} \partial_a \Omega_{ijk} d\zeta^i \wedge d\zeta^j \wedge d\zeta^k + \frac{1}{2!} \Omega_{ijk} d\zeta^i \wedge d\zeta^j \wedge \partial_a(d\zeta^k).$$

Clearly $d\zeta^k$ is a $(1,0)$-form with respect to the complex structure at z^a. With a variation of complex structure, it moves into $\Omega^{1,0}(X_z) \oplus \Omega^{0,1}(X_z)$. Hence $\partial_a \Omega \in \Omega^{3,0}(X_z) \oplus \Omega^{2,1}(X_z)$. Since d is independent of complex structure, $\partial_a \Omega$ is still d-closed, and therefore in $H^{3,0}(X_z) \oplus H^{2,1}(X_z)$. ∎

Corollary 1.3.37. *If $\Omega = z^a\alpha_a - w_b\beta^b$ in the symplectic cohomology basis (α_a, β^b), then $2w_c = \partial_c(z^a w_a)$.*

Proof. By Griffiths transversality, $\int_{X_z} \Omega \wedge \partial_c\Omega = 0$. Writing this out in coordinates,

$$(z^a\alpha_a - w_b\beta^b, \alpha_c - \partial_c w_b\beta^b) = w_c - z^a\partial_c w_a = 0.$$

Then $w_c = z^a\partial_c w_a = \partial_c(z^a w_a) - w_c$. □

Definition 1.3.38. Define the **prepotential** $\mathcal{G} \coloneqq z^a w_a$. Since $2w_c = \partial_c\mathcal{G}$, summing with z^c on both sides gives $z^c\partial_c\mathcal{G} = 2\mathcal{G}$, so $\mathcal{G}$ is homogeneous of degree 2 in the z^a. The prepotential $\mathcal{G}$ fully specifies the Kähler structure of $\mathcal{M}_X$ by

$$i\int_{X_z} \Omega \wedge \bar{\Omega} = i(\bar{z}^a\partial_a\mathcal{G} - z^a\bar{\partial}_a\bar{\mathcal{G}}).$$

We call a Kähler structure with holomorphic prepotential **special Kähler**. The prepotential encodes all the important data for mirror symmetry and is quite an important object!

It may be useful at this point to get used to all these new objects by verifying the following useful formulas. Writing $\partial_a\Omega = k_a\Omega + \chi_a$, we have:

1. $g_{a\bar{b}} = -ie^K \int_{X_z} \chi_a \wedge \bar{\chi}_{\bar{b}}$;
2. $k_a = -\partial_a K$, the Kähler potential.

Proposition 1.3.39. *By the same arguments as for Griffiths transversality in 1.3.36,*

$$\partial_a\partial_b\Omega \in H^{3,0} \oplus H^{2,1} \oplus H^{1,2}, \qquad \int_{X_z} \Omega \wedge \partial_a\partial_b\Omega = 0, \qquad \int_{X_z} \partial_a\Omega \wedge \partial_b\Omega = 0.$$

However these equalities give no new relations on the prepotential $\mathcal{G}$.

Definition 1.3.40. Note that the same argument fails for $\partial_a\partial_b\partial_c\Omega$. The **Yukawa coupling**

$$\kappa_{abc} \coloneqq -\int_{X_z} \Omega \wedge \partial_a\partial_b\partial_c\Omega = \partial_a\partial_b\partial_c\mathcal{G}$$

is in general non-zero. We usually **normalize** the Yukawa coupling by normalizing Ω to satisfy $z^0 = \int_{A^0} \Omega = 1$, i.e. replacing Ω with Ω/z^0. Consequently, the **normalized Yukawa coupling** is $\kappa_{abc}/(z^0)^2$.

Proposition 1.3.41. *Identify $\chi_a \in H^{2,1}(X_z)$ in the expansion $\partial_a\Omega = k_a\Omega + \chi_a$ with $\chi_a^i \in H^1(TX_z) \cong H^{2,1}(X_z)$. Then*

$$\kappa_{abc} = -\int_{X_z} \Omega \wedge (\Omega_{ijk}\chi_a^i \wedge \chi_b^j \wedge \chi_c^k).$$

Proof. Note that, up to the identification $H^1(TX_z) \cong H^{2,1}(X_z)$, the form $\chi_a \wedge \chi_b \wedge \chi_c$ is precisely the $(0,3)$-component of $\partial_a\partial_b\partial_c\Omega$ by Griffiths transversality 1.3.36. The identification, however, is easily worked out as $\chi_a^i = (1/2\|\Omega\|^2)\bar{\Omega}^{ijk}(\chi_a)_{ij\bar{k}}d\bar{z}^{\bar{k}}$. (Note here $\bar{z}^{\bar{k}}$ is a coordinate on X_z, not a moduli space coordinate.) □

1.3.5 Mirror Symmetry for the Quintic Hypersurface

We now have the tools necessary to talk about mirror symmetry in its simplest and most well-understood case: the quintic hypersurface. One main characteristic of mirror symmetry is that it produces corresponding Calabi–Yau manifolds whose Hodge diamond is flipped. In the quintic case, this means we want $h^{1,1} = 101$ and $h^{2,1} = 1$ (c.f. 1.3.30). Hence we start with a one-parameter (for the complex structure) family of quintics. Along the way we shall see more predictions of mirror symmetry.

Definition 1.3.42. Consider the one-parameter family of quintics

$$X_\psi \coloneqq \{[x_0 : x_1 : x_2 : x_3 : x_4] \in \mathbb{P}^4 : f_\psi \coloneqq x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5\psi x_0 x_1 x_2 x_3 x_4 = 0\}.$$

Note that f_ψ is invariant under the action by

$$G \coloneqq \{(a_0, a_1, a_2, a_3, a_4) \in (\mathbb{Z}/5\mathbb{Z})^5 : \sum a_i = 0\}/\{(a, a, a, a, a)\}$$

where $(a_i) \in G$ acts on X_ψ by $x_j \mapsto x_j \lambda^{a_j}$ with λ a fifth root of unity. We remove the trivial action (a, a, a, a, a) from G because of the scale invariance of $\mathbb{P}^4$. The quotient X_ψ/G is definitely not smooth, since the action of G still has fixed points. Also, for non-generic ψ, the space X_ψ may not even be smooth.

Example 1.3.43. Let $g_{01} \in G$ be the map $[x_0 : x_1 : x_2 : x_3 : x_4] \mapsto [\lambda x_0 : \lambda^4 x_1 : x_2 : x_3 : x_4]$. Then g_{01} generates a $\mathbb{Z}/5\mathbb{Z}$ subgroup of G, which is the stabilizer of the quintic

$$C_{01} \coloneqq \{x_0 = x_1 = 0,\ x_2^5 + x_3^5 + x_4^5 = 0\}.$$

There are a total of $\binom{5}{2} = 10$ such quintics C_{ij} that are fixed points of the G-action with stabilizer $\mathbb{Z}/5\mathbb{Z}$. In X_ψ/G they correspond to curves $\tilde{C}_{ij} \coloneqq C_{ij}/G \cong \mathbb{P}^1$.

Example 1.3.44. By the same reasoning, the $\binom{5}{3} = 10$ quintics analogous to

$$C_{012} \coloneqq \{x_0 = x_1 = x_2 = 0,\ x_3^5 + x_4^5 = 0\}$$

have stabilizer $(\mathbb{Z}/5\mathbb{Z})^2$, and correspond to points $\tilde{C}_{ijk} \coloneqq C_{ijk}/G \cong \{\mathrm{pt}\}$ in X_ψ/G. Note that such a point $\tilde{C}_{ijk}$ is the intersection $\tilde{C}_{ij} \cap \tilde{C}_{jk} \cap \tilde{C}_{ik}$.

Example 1.3.45. If $f_\psi = df_\psi = 0$, then X_ψ has a singularity. Since

$$\partial_i f_\psi = 0 \implies x_i^5 = \psi x_0 x_1 x_2 x_3 x_4,$$

at such a singularity we have $x_0^5 \cdots x_4^5 = \psi^5 x_0^5 \cdots x_4^5$ from multiplying the equations together, i.e. $\psi^5 = 1$.

Theorem 1.3.46 ([16, Appendix: Proposition 4])**.** *There exists a* **crepant resolution** *of the quotient X_ψ/G, i.e. a Calabi–Yau 3-fold $\tilde{X}_\psi$ with a map $\pi\colon \tilde{X}_\psi \to X_\psi/G$ which*

1. *(resolution) is an isomorphism outside $\pi^{-1}(\bigcup \tilde{C}_{ij})$, and*
2. *(crepant) preserves the trivial canonical bundle with $K_{\tilde{X}_\psi} = \pi^* K_{X_\psi/G}$.*

Definition 1.3.47. For any fifth root of unity λ, there is a natural isomorphism between $\tilde{X}_\psi$ and $\tilde{X}_{\lambda\psi}$ induced by $[x_0 : x_1 : \cdots : x_4] \mapsto [\lambda^{-1} x_0 : x_1 : \cdots x_4]$ (assuming we resolved singularities in a compatible way). So ψ^5 is a more natural parameter, and we view it as parameterizing the complex structure. The family $\{\tilde{X}_\psi\} \to \{\psi^5\} \cong \mathbb{C}$ is the **quintic mirror family**.

We can keep track of the exceptional divisors introduced when blowing up the singularities (which is what we do to get the crepant resolution):

1. blowing up (twice) the 10 curves $\tilde{C}_{ij}$ of A_4 singularities introduces $2 \cdot 10 + 2 \cdot 10 = 40$ divisors,
2. blowing up (twice) the 10 points $\tilde{C}_{ijk}$ introduces $3 \cdot 10 + 1 \cdot 30 = 60$ divisors.

Hence there are 100 new divisors in $\tilde{X}_\psi$ aside from the hyperplane coming from X_ψ. We have shown the following.

Proposition 1.3.48. $h^{1,1}(\tilde{X}_\psi) = 101$.

However, mirror symmetry makes stronger predictions than just the symmetry of the Hodge diamond. The remainder of this section outlines these predictions. We want to understand the moduli space $\mathcal{M}_{\tilde{X}_\psi}$. To do so we first explicitly write down the generator Ω of $H^{3,0}(\tilde{X}_\psi)$, and then find its periods.

Definition 1.3.49. Define the form Ξ on $\mathbb{C}^5$ by

$$\Xi := \sum_{k=0}^{4} (-1)^k x_k dx_0 \wedge \cdots \wedge \widehat{dx_k} \wedge \cdots \wedge dx_4.$$

For a homogeneous degree-5 polynomial P, the form Ξ/P descends to $\mathbb{P}^4$. If γ_P is a small loop around $P = 0$ in $\mathbb{P}^4$, then

$$\Omega := \int_{\gamma_P} \frac{\Xi}{P}$$

is a well-defined holomorphic $(3,0)$-form on $P = 0$. By considering P as a local coordinate normal to $P = 0$, we can write $dx_4 = (\partial_P x_4)dP$. Since $\int_{\gamma_P} dP/P = 2\pi i$, in $\mathbb{P}^4$ we have

$$\int_{\gamma_P} \frac{\Xi}{P} = 2\pi i \sum_{k=0}^{3} (-1)^k x_k (\partial_P x_4) dx_0 \wedge \cdots \wedge \widehat{dx_k} \wedge \cdots \wedge dx_3.$$

In affine coordinates, we can set $x_0 = 1$ and take x_1, x_2, x_3 to be coordinates on $P = 0$. Note that x_4 is fully determined by $P = 0$. For $P = f_\psi$, we get Ω for the mirror quintic.

Proposition 1.3.50 (Picard–Fuchs equation)**.** *Let $z := 1/(5\psi)^5$ and $\Theta := z\partial_z$. The periods Ω_i of Ω satisfy the* **Picard–Fuchs equation**

$$[\Theta^4 - 5z(5\Theta + 4) \cdots (5\Theta + 1)](5\psi\Omega_i) = 0.$$

Proof. If $\{\Gamma_i\}$ is a basis for $H_3(\tilde{X}_\psi)$, then $\Omega_i = \int_{\Gamma_i} \int_{\gamma_P} \Xi/P$. Introduce new variables a_i into f_ψ, the defining equation for the mirror quintic: $f_\psi = \sum_i a_i x_i^5 - 5\psi \prod_i x_i$. Then Ω_i is a function of $a_1, \ldots, a_5$ and ψ.

1. Let $(s_i) = (a_1, \ldots, a_5, \psi)$. For λ a fifth root of unity, using the equation for Ω_i above,

   $$\Omega_i(\lambda a_1, \ldots, \lambda a_5, \lambda\psi) = \lambda^{-1} \Omega_i(a_1, \ldots, a_5, \psi).$$

   Taking ∂_λ at $\lambda = 1$ gives $(\sum_i s_i \partial_{s_i} + 1)\Omega_i = 0$, i.e. Ω_i is homogeneous of degree -1 in $a_1, \ldots, a_5, \psi$.

2. Under the $\mathrm{PGL}(5, \mathbb{C})$ transformation $x_j \mapsto \lambda x_j$ and $x_5 \mapsto \lambda^{-1} x_5$,

   $$\Omega_i(a_1, \ldots, \lambda^5 a_j, \ldots, \lambda^{-5} a_5, \psi) = \Omega_i(a_1, \ldots, a_5, \psi).$$

   Now ∂_λ at $\lambda = 1$ gives $(a_i \partial_{a_i} - a_5 \partial_{a_5})\Omega = 0$, i.e. Ω_i is a function of $a_1 \cdots a_5$.

We conclude that $\Omega_i = \frac{1}{5\psi}\omega_i(a_1 \cdots a_5/(5\psi)^5)$. Hence we put $z = a_1 \cdots a_5/(5\psi)^5$. Note that if $\Theta := z\partial_z$ then the identities

$$\partial_{a_i} = \frac{\Theta}{a_i}, \quad \frac{1}{5}\partial_\psi \frac{1}{(5\psi)^N} f(z) = -\frac{1}{(5\psi)^{N+}} (5\Theta + N) f(z), \quad z(\Theta + 1) = \Theta z$$

hold. Hence, from $f_\psi = 0$, we get

$$\begin{aligned} 0 &= \left(\prod \partial_{a_i} - (\partial_\psi/5)^5 \right) \Omega_i \\ &= \left(\frac{1}{a_1 \cdots a_5} \Theta^5 + \frac{1}{(5\psi)^6} (5\Theta + 5) \cdots (5\Theta + 1) \right) \omega_i \\ &= \Theta(\Theta^4 - 5z(5\Theta + 4) \cdots (5\Theta + 1))\omega_i. \end{aligned}$$

The fourth-order equation $(\Theta^4 - 5z(5\Theta + 4) \cdots (5\Theta + 1))\omega_i = 0$ is equivalent. $\square$

Corollary 1.3.51. *The Yukawa coupling $Y := \kappa_{000} = -\int_{\tilde{X}_\psi} \Omega \wedge \Theta\Theta\Theta\Omega$ satisfies the differential equation*

$$\Theta Y = \frac{-5^5 z}{1 + 5^5 z} Y.$$

Proof. Expand the Picard–Fuchs equation in terms of z, wedge it with Ω, and apply Griffiths transversality $\int \Omega \wedge \Theta\Omega = \int \Omega \wedge \Theta\Theta\Omega = 0$. $\square$

If we look for a holomorphic solution $y_0(z) = \sum_{n=0}^{\infty} c_n z^n$ of the Picard–Fuchs equation, then we get the recursion

$$n^4 c_n = -5(5(n-1)+4)\cdots(5(n-1)+1)c_{n-1}$$

and therefore $c_n = (-1)^n(5n)!/(n!)^5$. General methods from the theory of hypergeometric DEs assures us this is the unique (up to a constant) holomorphic solution. There is also the solution

$$y_1(z) = y_0(z)\log(-z) + 5\sum_{n=1}^{\infty}(-1)^n \frac{(5n)!}{(n!)^5}\left(\sum_{j=n+1}^{5n}\frac{1}{j}\right)x^n$$

which has a logarithmic singularity.

We can also solve the DE for the Yukawa coupling to get $Y = 5/(1+5^5 z)$. Note that the constant 5 is not determined by the DE and is obtained through other methods. The normalized Yukawa coupling (c.f. 1.3.40) is therefore

$$Y = \frac{5}{(1+5^5 x)y_0(x)^2}.$$

This is called the **B-model correlation function** by physicists.

Mirror symmetry predicts an equality of the B-model correlation function, which comes from the complex structure moduli, with the **A-model correlation function**, which comes from the Kähler moduli (which we have not discussed yet), via the **mirror map** between the complex and Kähler moduli of the Calabi–Yau manifold and its mirror. The A-model correlation function is much harder to compute and involves objects called Gromov–Witten invariants; suffice it to say for now that they are numbers n_d counting rational curves of degree d in the Calabi–Yau manifold. In the case of the mirror quintic, the map from the complex moduli of the mirror, given by the coordinate z, and the Kähler moduli of the quintic, given by the coordinate q, is

$$q = \exp(y_1(z)/y_0(z)) = -z\exp\left(\frac{5}{y_0(z)}\sum_{n=1}^{\infty}\frac{(5n)!}{(n!)^5}\left(\sum_{j=n+1}^{5n}\frac{1}{j}\right)x^n\right).$$

The equality of the B-model and A-model correlation functions is expressed by

$$\begin{aligned} 5 + \sum_{d=1}^{\infty} n_d d^3 \frac{q^d}{1-q^d} &= \frac{5}{(1+5^5 z)y_0(z)^2}\left(\frac{q}{z}\frac{dz}{dq}\right)^3 \\ &= 5 + 2875\frac{q}{1-q} + 609250\cdot 2^3\frac{q^2}{1-q^2} + 317206375\cdot 3^3\frac{q^3}{1-q^3} + \cdots. \end{aligned}$$

The enumerative prediction $n_3 = 317206375$ (and of course the higher coefficients) in [17] sparked much of the initial mathematical interest in mirror symmetry.

1.4 Toric Geometry

Chapter 2

Physics Preliminaries

It is difficult to rigorously define what a quantum field theory (QFT) is. Instead, we provide some examples of the main classes of QFTs and their defining characteristics. In general, a **quantum field theory** (QFT) consists of three ingredients:

1. a choice of manifold M of dimension d (usually Riemannian or Lorentzian), possibly with boundary,
2. objects living on M, e.g.
   (a) **gauge fields**: connections on a principal bundle over M,
   (b) **matter fields**: sections of a vector bundle over M,
3. a complex-valued functional S depending on the objects of the theory.

For example, a **quantum gauge theory** has fields which are sections of associated vector bundles. A **quantum gravity theory** is obtained by making the (choice of) metric on M an object of the theory as well.

If $\partial M = \bigcup_i B_i$ then the set of field configurations on the boundary give rise to Hilbert spaces $\mathcal{H}_i$. The **path integral** can be viewed as a map $\bigotimes_i \mathcal{H}_i \to \mathbb{C}$, and encodes the dynamics of the theory by weighing each possible field configuration by e^{-S}. Alternatively, suppose $M = N \times [0,t]$ where $\partial N = \varnothing$. This corresponds to a multi-linear map $\mathcal{H}^* \otimes \mathcal{H} \to \mathbb{C}$ or equivalently a linear map $U(t)\colon \mathcal{H} \to \mathcal{H}$. Moreover, by gluing two manifolds together we get the relation $U(t_1 + t_2) = U(t_1)U(t_2)$. By a theorem from functional analysis there exists a Hermitian operator H, which we call the **Hamiltonian**, which is given by $U(t) = e^{-tH}$ or $U(t) = e^{-itH}$ in the Euclidean or Minkowski case, respectively.

From string theory we saw that many quantized theories require a certain dimensionality which is often larger than the four spacetime dimensions we observe. **Compactification** is one way to achieve the effective 4-dimensional space-time that we see. The general idea is to write $M = N \times K$ for some compact K and N 4-dimensional in such a way that the action S is very large for field configurations not constant over K. Then the weighing e^{-S} allows us to consider an "effective" path integral over N, over constant modes on K.

Luckily, mirror symmetry entails studying QFTs with $d = 2$, so we focus mainly on low-dimensional QFTs. We begin with $d = 0$, where by introducing fermionic fields and supersymmetry we can already discuss localization and deformation invariance. Then at $d = 1$, which corresponds to quantum mechanics, we shall look at supersymmetric σ-models, and Landau–Ginzburg theories. Finally at $d = 2$ we get to see T-duality, and σ-models on a Kähler manifold and their connections to Landay–Ginzburg theories.

We shall focus more on supersymmetry (abbreviated **susy**) and its consequences than non-susy aspects. Non-susy quantum theory is covered in both the QFT and string theory notes.

2.1 QFT with $d = 0$

First, let us discuss the fields. For a real-valued theory with $\dim M = 0$, we may identify functions $X : M \to \mathbb{R}$ with just an element $X \in \mathbb{R}$. To model fermions we shall ask our variables to anti-commute: $\psi_i \psi_j = -\psi_j \psi_i$.

The path-integral, or what Hori et al. calls the **partition function**, is given by the expression

$$Z := \int e^{-S[X,\psi]} dX d\psi.$$

Whether it's for correlation appearing in statistical mechanics or scattering amplitudes, many physical quantities can be expressed in terms of **correlation functions**. Using the path integral approach, a correlation function is an expectation value of an operator weighted by e^{-S}:

$$\langle F(X,\psi)\rangle = \int F(X,\psi) e^{-S} dX d\psi.$$

Let's be clever. For the moment, assume that F is a polynomial in X and ψ. Modifying the argument of the exponential $\exp(-S + J_1 X + J_2 \psi)$ and applying the correct number of derivatives we can write:

$$\langle F \rangle = \frac{\partial}{\partial J_1} \frac{\partial}{\partial J_2} \cdots \Big|_{J_1 = J_2 = 0} \left(\int e^{-S + J_1 X + J_2 \psi} \, dX d\psi \right).$$

The integral on the right side may be denoted by $Z[X, \psi; J_1, J_2]$. Then we may expand the exponential as a power series in J_1 and J_2, and use perturbative techniques to iteratively compute higher and higher-order corrections to $\langle F \rangle$. Each term in such an expansion can be encoded as a **Feynman diagram**. For more on the path integral approach, including the machinery of Feynman diagrams, refer to the QFT notes.

2.1.1 Supersymmetry and a Toy Model

Supersymmetry is both mathematically and physically a useful tool. From a mathematical point of view, "the classical field equations have non-trivial odd symmetries: eg. gradient flow lines in Morse theory, holomorphic curves, gauge theory instantons, monopoles, Seiberg–Witten solutions, hyperKahler structures, Calabi-Yau metrics, metrics of G2 and Spin7 holonomy." From a physical point of view, "[supersymmetric] theories offer a possible way of solving the 'hierarchy problem,' the mystery of the enormous ratio of the Planck mass to the 300 GeV energy scale of electroweak symmetry breaking. Supersymmetry also has the quality of uniqueness that we search for in fundamental physical theories. There is an infinite number of Lie groups that can be used to combine particles of the same spin in ordinary symmetry multiplets, but there are only eight kinds of supersymmetry in four spacetime dimensions, of which only one, the simplest, could be directly relevant to observed particles." **To do** (1)

Let us consider a simple QFT with one bosonic field X and two fermionic fields ψ_1, ψ_2 with an equally simple action:

$$S = S_0(X) - \psi_1 \psi_2 S_1(X). \tag{2.1}$$

Recall that in **fermionic integration**, i.e. integrating over fermionic variables, imposing linearity and a normalization condition gives the rules $\int d\psi \, 1 = 0$ and $\int d\psi \, \psi = 1$. Taylor expanding the path-integral, and using the fermionic integration rules, we

$$\int e^{-S_0(X) + \psi_1 \psi_2 S_1(X)} dX d\psi_1 d\psi_2 = \int e^{-S_0} S_1(X).$$

For the rest of the section, we are going to pick a convenient choice for the functions $S_0(X), S_1(X)$ so that we get a feeling for what a supersymmetric theories have to offer. Fix a function $h\colon \mathbb{R} \to \mathbb{R}$, called the **superpotential**, and let

$$S_0(X) = \frac{1}{2}(\partial h)^2, \qquad S_1(X) = \partial^2 h.$$

We use the notation $\partial h = h'(X)$ to refer to the derivative of h. Together with the form of the action (2.1), we get a theory that is invariant under the transformations:

$$\delta X = \epsilon^1\psi_1 + \epsilon^2\psi_2, \qquad \delta\psi_1 = \epsilon^2\partial h, \qquad \delta\psi_2 = -\epsilon^1\partial h. \tag{2.2}$$

Let's check this explicitly. Notice that the variational operator δ can be treated as if it were a derivation: $\delta(AB) = (\delta A)B + A(\delta B)$. Then,

$$\begin{aligned}
\delta(S) &= \delta(S_0) - \delta(\psi_1)\psi_2 S_1 - \psi_1\delta(\psi_2)S_1 - \psi_1\psi_2\delta(S_1)\\
&= (\delta h)(\partial^2 h(\epsilon^1\psi_1 + \epsilon^2\psi_2)) - (\epsilon^2\partial h)\psi_2\partial^2 h - \psi_1(-\epsilon^1\partial h)\partial^2 h - \psi_1\psi_2(\partial^3 h(\epsilon^1\psi_1 + \epsilon^2\psi_2))\\
&= \partial h\partial^2 h(\epsilon^1\psi_1 + \epsilon^2\psi_2 - \epsilon^2\psi_2 + \psi_1\epsilon^1) + \partial^3 h(-\psi_1\psi_1\psi_2\epsilon^1 + \psi_1\epsilon^2\psi_2\psi_2)\\
&= 0
\end{aligned}$$

To show that the measure is also invariant, we proceed analogously except we must keep in mind that we *pullback* the measure and do not push it forward:

$$\begin{aligned}
f^*(dX \wedge d\psi_1 \wedge d\psi_2) &= df^*(X) \wedge df^*(\psi_1) \wedge df^*(\psi_2)\\
&= dX \wedge d\psi_1 \wedge d\psi_2 - d(\epsilon^1\psi_1 + \epsilon^2\psi_2) \wedge d\psi_1 \wedge d\psi_2\\
&\quad - dX \wedge d(\epsilon^2\partial h) \wedge d\psi_2\\
&\quad - dX \wedge d\psi_1 \wedge d(-\epsilon^1\partial h)\\
&= 0
\end{aligned}$$

The last two terms are 0 because $d(\partial h) = (\partial^2 h)dX$ and by wedgeing with another dX zeros out the entire term.

2.1.2 Localization and Deformation Invariance

The invariance of the action and partition function for this particular action under the susy transformations is very powerful: we shall show that the path integral $Z = \int e^{-S(X,\psi_1,\psi_2)}dXd\psi_1 d\psi_2$ depends only very weakly on the choice of h.

Proposition 2.1.1. *Suppose ρ is a function such that ρ and $\partial\rho$ vanishes at infinity. Then $h \to h + \rho$ leaves the action invariant.*

Proof. In general, let $\delta\mathcal{O}$ be the susy variation of some operator $\mathcal{O}(X, \psi_1, \psi_2)$. We often want to compute the **correlation function**

$$\langle\delta\mathcal{O}\rangle := \int dXd\psi_1 d\psi_2\, e^{-S(X,\psi_1,\psi_2)}\delta\mathcal{O} = \int dXd\psi_1 d\psi_2\, \delta(e^{-S(X,\psi_1,\psi_2)}\mathcal{O})$$

where the equality follows from the susy invariance of the action S. As long as $\mathcal{O}$ does not interfere at infinity with the decay of e^{-S}, this is zero.

In our case, take $g = \partial\rho(X)\psi_1$ and consider the susy transformation 2.2 with $\epsilon^1 = \epsilon^2 = \epsilon$:

$$\delta_\epsilon g = \partial^2\rho\delta X\psi_1 + \partial\rho\delta\psi_1 = \epsilon(\partial\rho\partial h - \partial^2\rho\psi_1\psi_2).$$

Now consider the deformation of h given by $h \to h + \rho$, and note that the action $S = \frac{1}{2}(\partial h)^2 - \partial^2 h \psi_1 \psi_2$ transforms exactly as $\delta_\rho S = \partial\rho\partial h - \partial^2 \rho \psi_1 \psi_2$. It follows that

$$\langle \delta_\rho S \rangle = \epsilon^{-1} \langle \delta_\epsilon g \rangle = 0. \qquad \square$$

Corollary 2.1.2. *Let $\mathcal{O}$ be an operator invariant under the susy transformations, i.e. $\delta\mathcal{O} = 0$. Then the correlation function $\langle \mathcal{O} \rangle$ is* **localized***, i.e. non-zero, only around the critical points of h, where $\partial h = 0$. In particular, this is true for the path integral $Z = \langle 1 \rangle$.*

Proof. Take $\rho = \lambda h$ and iterate the deformation $h \mapsto h + \rho$, i.e. a rescaling of h. As $\lambda \to \infty$, the weighing e^{-S} exponentially suppresses contributions to Z except in infinitesimal neighborhoods of $\partial h = 0$, i.e. the critical points of h. $\square$

Hence we may take the path integral as a sum over critical points X_c. Expanding $h(X) = h(X_c) + \frac{\alpha_c}{2}(X - X_c)^2 + \cdots$, we get

$$\begin{aligned} Z &= \sum \int \frac{dX d\psi_1 d\psi_2}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\alpha_c^2 (X - X_c)^2 + \alpha_c \psi_1 \psi_2\right) \\ &= \sum_{X_c} \frac{1}{\sqrt{2\pi}} \int dX \exp\left(-\frac{1}{2}\alpha_c^2 (X - X_c)^2\right) \alpha_c = \sum_{X_c} \alpha_c \sqrt{\frac{1}{\alpha_c^2}} \end{aligned}$$

The factor of $\sqrt{2\pi}$ can be factored out by normalizing the measure appropriately. This normalization corresponds to defining the correlation functions as $\frac{Z[\lambda,0]}{Z[0,0]}$, which in our case is $Z[0,0] = \sqrt{2\pi}$. Therefore the path-integral becomes an integer:

$$Z = \sum_{X\,:\,\partial h(X) = 0} \frac{\partial^2 h(X)}{|\partial^2 h(X)|}.$$

In particular, if h is a polynomial, then:

1. if h is of odd degree, then $\partial h = 0$ has an even number of roots with $\partial^2 h$ alternating in sign, so $Z = 0$;
2. if h is of even degree, then $Z = \pm 1$ depending on whether $h \to \pm\infty$ as $|X| \to \infty$.

One of the advantages of having such a simple model in $d = 0$ is that checking results is very easy: simply do the path integral for Z. We have

$$Z = \int \frac{dX d\psi_1 d\psi_2}{\sqrt{2\pi}} e^{-\frac{1}{2}(\partial h)^2 + (\partial^2 h)\psi_1\psi_2} = \int \frac{dX}{\sqrt{2\pi}} \partial^2 h e^{-\frac{1}{2}(\partial h)^2}.$$

Make the change of variables $y = \partial h$. Since $X \mapsto y = \partial h$ is not necessarily one-to-one, changing variables gives

$$Z = D \int \frac{dy}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} = D$$

where D is the degree of the map $X \mapsto y = \partial h$. For polynomials, this precisely recovers the results from deformation invariance and localization.

2.1.3 Landau–Ginzburg Theory

After delving into the details of the previous model it is comforting to note that the complex analogue is what is known as the Landau–Ginzburg theory: $(X, \psi_1, \psi_2) \rightsquigarrow (z, \bar{z}, \psi_1, \psi_2, \bar{\psi}_1, \bar{\psi}_2)$, and action:

$$S = |\partial W|^2 - (\partial^2 W)\psi_1\psi_2 - \overline{\partial^2 W}\bar{\psi}_1\bar{\psi}_2. \tag{2.3}$$

We call the holomorphic function $W(z)$ the **superpotential** and it will play the role h did in the previous sections. The complex susy transformations then take the form of:

$$\delta z = \epsilon^1\psi_1 + \epsilon^2\psi_2, \qquad \delta\psi_1 = \epsilon^2\overline{\partial W}, \qquad \delta\psi_2 = -\epsilon^1\overline{\partial W}, \qquad \delta\bar{z} = \delta\bar{\psi}_1 = \delta\bar{\psi}_2 = 0 \tag{2.4}$$

$$\bar{\delta}\bar{z} = \bar{\epsilon}^1\bar{\psi}_1 + \bar{\epsilon}^2\bar{\psi}_2, \qquad \bar{\delta}\bar{\psi}_1 = \bar{\epsilon}^2\partial W, \qquad \bar{\delta}\bar{\psi}_2 = -\bar{\epsilon}^1\partial W, \qquad \bar{\delta}z = \bar{\delta}\psi_1 = \bar{\delta}\psi_2 = 0 \tag{2.5}$$

Once again, the localization principle applies by choosing $\epsilon^1 = \epsilon^2 = -\frac{\psi_1}{\partial W}$, and writing $W(z) = W(z_c) + \frac{\alpha_c}{2}(z - z_c)^2 + \cdots$, then with proper normalization, the path-integral integrates to:

$$\begin{aligned} Z &= \frac{1}{2\pi}\int e^{-|\alpha(z-z_c)|^2}(\alpha\psi^1\psi^2 + \bar{\alpha}\bar{\psi}_1\bar{\psi}_2)\, dz d\bar{z}\, d\psi_1 d\psi_2\, d\bar{\psi}_1 d\bar{\psi}_2 \\ &= \frac{|\alpha|^2}{2\pi}\sum_{z_c}\int e^{-|\alpha(z-z_c)|^2}\, dz d\bar{z} \\ &= \frac{|\alpha|^2}{2\pi}\sum_{z_c}\int e^{-|\alpha|^2(x^2+y^2)}\, dx dy = \sum_{z_c} 1 = \#\text{ of critical points of } W \end{aligned}$$

The existence of supersymmetry does not allow us to trivially compute arbitrary correlation functions but it does help with some. In particular, if we have an observable $\mathcal{O}$ that is invariant under one of the supersymmetry transformations then we can use localization. For example, let $f(z)$ be a holomorphic observable. Then $\bar{\delta}f = 0$, so by the localization principle 2.1.2,

$$\langle f(z)\rangle = \int f(z)e^{-S} = \int f(z)|\partial^2 W|^2 e^{-\frac{1}{2}|\partial W|^2} = \sum_{z_c} f(z_c).$$

Definition 2.1.3. The set of all fields that vanish under $\bar{\delta}$ are called **chiral fields**. In particular because $\bar{\delta}$ is a derivation it follows that this set is closed under multiplication. If we take $\epsilon^1 = \epsilon^2$ then $\bar{\delta}^2 = 0$, so $\bar{\delta}$-exact fields are automatically chiral and not very interesting. Hence we can consider the cohomology ring $\ker\bar{\delta}/\operatorname{im}\bar{\delta}$, often called the **chiral ring**.

Example 2.1.4 (Chiral Ring for $d = 0$ Landau–Ginzburg)**.** Since $\bar{\delta}(f(z)\bar{\psi}_1) = f(z)\partial W(z)$, bosonic chiral fields, i.e. holomorphic functions, are trivially chiral if they have ∂W as a factor. The converse is true too. Hence the chiral ring is $\mathcal{R} := \mathbb{C}[z]/\mathcal{I}$ where $\mathcal{I}$ is the ideal generated by ∂W.

2.2 QFT with $d = 1$

Let M be a 1-dimensional manifold. We know that there are only two diffeomorphism classes of 1-manifolds without boundary: $\mathbb{R}$ and S^1. For simplicity, the only manifold with boundary that we will consider is the unit interval $[0, 1]$.

2.2.1 Quantum Mechanics

For $d \geq 1$ there are two different formalisms for quantum theory: the **operator formalism**, and the **path integral formalism**. We used the path integral formalism for $d = 0$; now it suffices to generalize to integrals over $\mathrm{Hom}(M, \mathbb{C})$ where $\dim M = 1$. The operator formalism arises when we consider manifolds M with boundaries. Think of M as spacetime, and boundaries as space. To each boundary we associate a **Hilbert space of states** $\mathcal{H}$. For $d = 1$ QFT, a boundary is a point. We take $\mathcal{H} = L^2(\mathbb{R}, \mathbb{C})$, i.e. wavefunctions, from quantum mechanics.

Let $Z_{t_2;t_1}$ denote the **time evolution** operator taking a state at time t_1 to a state at time t_2. Physically we expect $Z_{t_3;t_2}Z_{t_2;t_1} = Z_{t_3;t_1}$. By Stone's theorem, there exists a self-adjoint **Hamiltonian** H such that $Z_{t_2;t_1} = e^{-i(t_2-t_1)H}$. This is usually written $U(t) \coloneqq e^{-itH}$, so that $U(t)$ is the operator that "evolves for time t". We generally consider the action and corresponding Hamiltonian

$$S = \int L\, dt = \int \left[\frac{1}{2}\left(\frac{dX}{dt}\right)^2 - V(X)\right] dt, \quad H = \frac{1}{2}p^2 + V(X).$$

Canonical quantization in the operator formalism involves promoting classical scalar-valued quantities to operator-valued quantities, and then imposing the commutation relation $[X, p] = i$.

Example 2.2.1. When compactifying we are often interested in $M = S^1_\beta$, the circle of circumference β. We view M as $[0, \beta]$ with endpoints identified. So on M, the Euclidean (via Wick-rotation) path integral $Z_E(\beta)$ looks like

$$Z_E(\beta) = \int dX_1 Z_{E,\beta}(X_1, X_1) = \operatorname{tr}\exp(-\beta H),$$

where $Z_{E,\beta}(X, Y)$ is the Euclidean path integral on $[0, \beta]$.

2.2.2 Supersymmetric Quantum Mechanics

Example 2.2.2. Consider a supersymmetric theory with a single bosonic variable x and a complex superpartner ψ, with Lagrangian

$$L \coloneqq \frac{1}{2}\dot{x}^2 - \frac{1}{2}h'(x)^2 + \frac{i}{2}\left(\overline{\psi}\dot{\psi} - \dot{\overline{\psi}}\psi\right) - h''(x)\overline{\psi}\psi,$$

where $\overline{\psi} = \psi^\dagger$, and the second term $h'(x)^2$ plays the role of the potential $-V(x)$. The following susy transformations change the Lagrangian by a total time derivative:

$$\delta x = \epsilon\overline{\psi} - \overline{\epsilon}\psi, \qquad \delta\psi = \epsilon(i\dot{x} + h'(x)), \qquad \delta\overline{\psi} = \overline{\epsilon}(-i\dot{x} + h'(x)), \tag{2.6}$$

where $\epsilon = \epsilon_1 + i\epsilon_2$ and $\overline{\epsilon} = \epsilon^*$ is the complex conjugate. As long as the boundary variation vanishes (which will be the case on an open manifold) then the action is invariant. Let δ_1, δ_2 be variations as in (2.6) but with $\epsilon_2 = 0$ and $\epsilon_1 = 0$ respectively. Then:

$$[\delta_1, \delta_2]x = 2i(\epsilon_1\overline{\epsilon}_2 - \epsilon_2\overline{\epsilon}_1)\dot{x}, \qquad [\delta_1, \delta_2]\psi = 2i(\epsilon_1\overline{\epsilon}_2 - \epsilon_2\overline{\epsilon}_1)\dot{\psi}.$$

Applying the Noether procedure with $\epsilon = \epsilon(t)$ to get: $\delta L = -i\dot{\epsilon}Q - i\dot{\overline{\epsilon}}Q^\dagger$ where **To do** (2)

$$Q = \overline{\psi}(i\dot{x} + h'(x)), \qquad Q^\dagger = \psi(-i\dot{x} + h'(x)).$$

We can quantize by computing conjugate variables, promoting scalar-valued fields to operator-valued fields, and then imposing commutation relations $[x, p] = i$ and $\{\psi, \overline{\psi}\} = 1$. The Hamiltonian is

$$H = \frac{1}{2}p^2 + \frac{1}{2}h'(x)^2 + \frac{1}{2}h''(x)(\overline{\psi}\psi - \psi\overline{\psi}).$$

(Sanity check: $[H, Q] = [H, Q^\dagger] = 0$, so indeed the Q and $Q^\dagger$ are conserved charges.) Note that $\psi, \overline{\psi}$ along with the fermionic number operator $F \coloneqq \overline{\psi}\psi$ form a raising/lowering operator algebra. The state space consists of a representation of the bosonic part x, p and the fermionic part $\psi, \overline{\psi}$ of the theory. Defining $|0\rangle$ by $\psi|0\rangle = 0$, the fermionic states are $|0\rangle, \overline{\psi}|0\rangle$. (There are no more since $\psi^2 = 0$.) The bosonic representation is $L^2(\mathbb{R}, \mathbb{C})$, with $\hat{x}\Psi(x) \coloneqq x\Psi(x)$ and $\hat{p}\Psi(x) = -i\Psi'(x)$. Hence

$$\mathcal{H} \coloneqq \mathcal{H}^B \oplus \mathcal{H}^F, \quad \mathcal{H}^B \coloneqq L^2(\mathbb{R}, \mathbb{C})|0\rangle, \quad \mathcal{H}^F \coloneqq L^2(\mathbb{R}, \mathbb{C})\overline{\psi}|0\rangle$$

with a $\mathbb{Z}/2\mathbb{Z}$-grading given by $(-1)^F$.

Definition 2.2.3. A **supersymmetric quantum mechanical system** is $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space $\mathcal{H}$ with an even operator H (the **Hamiltonian**) and two odd (wrt. the grading) operators $Q, Q^\dagger$ called **supercharges** satisfying:

$$Q^2 = (Q^\dagger)^2 = 0, \quad \{Q, Q^\dagger\} = 2H.$$

A list of consequences (which we have seen in the example already) follows from this axiomatic definition.

1. The supercharges Q and $Q^\dagger$ are conserved, since

$$2[H, Q] = [\{Q, Q^\dagger\}, Q] = 0.$$

2. Let $(-1)^F$ denote the operator defining the $\mathbb{Z}/2\mathbb{Z}$-grading. Then

$$Q(-1)^F = -(-1)^F Q, \quad H(-1)^F = (-1)^F H.$$

3. If we define $Q_1 = Q + Q^\dagger$ then

$$2H = \{Q, Q^\dagger\} = Q_1^2 \geq 0,$$

and $H |\alpha\rangle = 0$ iff $Q |\alpha\rangle = Q^\dagger |\alpha\rangle = 0$, i.e. zero-energy states are annihilated by both Q and $Q^\dagger$. Since Q and $Q^\dagger$ are charges of the susy transformations, $Q |\alpha\rangle = 0$ means $|\alpha\rangle$ is susy invariant, i.e. supersymmetric. So supersymmetric is equivalent to zero-energy.

4. The Hilbert space can be graded by energy levels, i.e. eigen-decomposition of the Hamiltonian H: write

$$\mathcal{H} := \bigoplus_{n=0}^{\infty} \mathcal{H}_n, \quad E_0 = 0 < E_1 < E_2 < \cdots.$$

Since $Q, Q^\dagger, (-1)^F$ commute with H, they preserve energy levels.

5. Each energy level $\mathcal{H}_n$ splits into bosonic and fermionic parts $\mathcal{H}_n^B$ and H_n^F. The charges $Q, Q^\dagger$ swap $\mathcal{H}_n^B$ and $\mathcal{H}_n^F$. For $E_n > 0$, $Q_1^2 = 2E_n$, so Q_1 is invertible and defines an isomorphism $\mathcal{H}_n^B \cong \mathcal{H}_n^F$. (Such an isomorphism does not hold at the zero energy level.)

It follows that under continuous (adiabatic) deformations of the theory, i.e. the spectrum of the Hamiltonian deforms continuously, the **supersymmetric index** or **Witten index**

$$\mathrm{tr}(-1)^F := \dim \mathcal{H}_0^B - \dim \mathcal{H}_0^F$$

is an invariant.

To touch base with mathematics let us consider Q as a coboundary operator of the complex $\cdots \to \mathcal{H}^B \to \mathcal{H}^F \to \mathcal{H}^B \to \mathcal{H}^F \to \cdots$. This complex is naturally graded by the energy levels, and Q preserves this grading. Let $H^B(Q)$ and $H^F(Q)$ be the two distinct cohomology groups of the complex. At excited energies, the cohomology is trivial: take a Q-closed state $Q |\alpha\rangle = 0$ and write

$$|\alpha\rangle = \frac{1}{2E_n} H |\alpha\rangle = \frac{1}{2E_n} \{Q, Q^\dagger\} |\alpha\rangle = Q\left(\frac{1}{2E_n} Q^\dagger |\alpha\rangle\right) \in \operatorname{im} Q.$$

Therefore the cohomology group on the bosonic and fermionic Hilbert spaces gets a contribution only from the (supersymmetric) ground states:

$$H^B(Q) \cong \mathcal{H}_{(0)}^B, \qquad H^F(Q) \cong \mathcal{H}_{(0)}^F. \tag{2.7}$$

Note this is an isomorphism not an equality as $H^\bullet(Q)$ is obtained by a quotient, where as $\mathcal{H}_{(0)}^B$ is an honest subgroup of the Hilbert space.

Example 2.2.4. Let's find the (susy) ground states of the susy potential theory in 2.2.2. In the basis $\{|0\rangle, \overline{\psi}|0\rangle\}$, the supercharges are

$$Q = \overline{\psi}(ip + h'(x)) = \begin{pmatrix} 0 & 0 \\ \frac{d}{dx} + h'(x) & 0 \end{pmatrix}, \qquad \overline{Q} = \psi(-ip + h'(x)) = \begin{pmatrix} 0 & -\frac{d}{dx} + h'(x) \\ 0 & 0 \end{pmatrix}.$$

Therefore we look for f_1, f_2 so that $\Psi = f_1(x)|0\rangle + f_2(x)\overline{\psi}|0\rangle$ is annihilated by Q and $\overline{Q}$. This leads to two differential equations with exact solutions:

$$f_1(x) = c_1 e^{-h(x)}, \qquad f_2(x) = c_2 e^{h(x)}.$$

We also want $f_1, f_2 \in L^2$ which means that they are either 0 or exponentially decaying at *both* $x = \pm\infty$. In one case $e^{-h(x)}|0\rangle$ is the only ground state, and $\operatorname{tr}(-1)^F = 1 - 0 = 1$; in the other, $e^{h(x)}\overline{\psi}|0\rangle$ is the only ground state, and $\operatorname{tr}(-1)^F = 0 - 1 = -1$.

Example 2.2.5. Taking $h(x) = (\omega/2)x^2$ in the susy potential theory 2.2.2 gives the **susy harmonic oscillator** with potential $V(x) = \frac{1}{2}(h'(x))^2$. The Hamiltonian here is $H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 x^2 + \frac{1}{2}\omega[\overline{\psi}, \psi]$, and by the same analysis as the previous example, there is exactly one susy ground state for both $\omega > 0$ (bosonic) and $\omega < 0$ (fermionic).

If instead of a $\mathbb{Z}/2\mathbb{Z}$-grading we had a $\mathbb{Z}$-grading and the operator Q had charge 1, then $\mathcal{H} = \bigoplus_{p\in\mathbb{Z}} \mathcal{H}^p$ forms a complex $\cdots \xrightarrow{Q} \mathcal{H}^{p-1} \xrightarrow{Q} \mathcal{H}^p \xrightarrow{Q} \cdots$. By an analogous argument,

$$\mathcal{H}_0^B = \bigoplus_{p \text{ even}} H^p(Q), \quad \mathcal{H}_0^F = \bigoplus_{p \text{ odd}} H^p(Q), \quad H^p(Q) \coloneqq \frac{\ker Q\colon \mathcal{H}^p \to \mathcal{H}^{p+1}}{\operatorname{im} Q\colon \mathcal{H}^{p-1} \to \mathcal{H}^p}.$$

The Witten index $\operatorname{tr}(-1)^F$ is therefore the Euler characteristic for the complex:

$$\operatorname{tr}(-1)^F = \dim \mathcal{H}_0^B - \dim \mathcal{H}_0^F = \sum_{p\in\mathbb{Z}} (-1)^p \dim H^p(Q).$$

Because of the appearance of the trace we may write down path integral expressions for $\operatorname{tr}(e^{\beta H})$ and $\operatorname{tr}\left[(-1)^F e^{-\beta H}\right]$. **To do** (3)

2.2.3 Perturbative Semi-Classical Analysis

We now switch to the Hamiltonian formulation, and at the same time rescale $h \to \lambda h$ for a large $\lambda \gg 1$ using the deformation invariance result 2.1.1. The Hamiltonian is therefore

$$H = \frac{1}{2}p^2 + \frac{\lambda^2}{2}(h'(x))^2 + \frac{\lambda}{2}h''(x)[\overline{\psi}, \psi].$$

The ground states will be localized at the minimum of $(h'(x))^2$. Expanding h around a critical point, and at the same time rescale the coordinates $x - x_i = \frac{1}{\sqrt{\lambda}}(\tilde{x} - \tilde{x}_i)$ the expansion becomes

$$h(x) = \frac{1}{2\lambda}h''(x_i)(\tilde{x} - \tilde{x}_i)^2 + \frac{1}{6\lambda^{3/2}}h'''(x_i)(\tilde{x} - \tilde{x}_i)^3 + O(\lambda^{-2}).$$

The Hamiltonian then becomes:

$$H = \lambda\left(\frac{1}{2}\tilde{p}^2 + \frac{1}{2}h''(x_i)^2(\tilde{x} - \tilde{x}_i)^2 + \frac{1}{2}h''(x_i)\left[\overline{\psi}, \psi\right]\right) + \lambda^{1/2}(\cdots) + \mathcal{O}(\lambda^{-1/2}),$$

where $\tilde{p} = -i\frac{d}{d\tilde{x}}$. The $\mathcal{O}(\lambda)$ term is a supersymmetric harmonic oscillator (c.f. 2.2.5) with $\omega = h''(x_i)$. Hence there is exactly one supersymmetric ground state associated to every critical point. For N critical points $x_1, \ldots, x_N$, there are N approximate susy ground states $\Psi_1, \ldots, \Psi_N$, and so the Witten index is given by

$$\mathrm{tr}(-1)^F = \sum_{i=1}^{N} \mathrm{sign}(h''(x_i)).$$

Since the Witten index is invariant under smooth deformations of the theory, this perturbative calculation is valid non-perturbatively. However, the perturbative ground states we found are not true ground states: we know from 2.2.4 that there is at most one ground state, yet we seem to have N. Later we shall see non-perturbative corrections, known as instantons.

If the target space is $\mathbb{R}^n$, with n bosonic and $2n$ fermionic variables then the Hamiltonian becomes a sum $H = \frac{1}{2}\sum_I p_I^2 + \frac{1}{2}(\partial_I h(x))^2 + \frac{1}{2}(\partial_I\partial_J h)[\overline{\psi}^I, \psi^J]$, and supercharges: $Q = \overline{\psi}^I(ip_I + \partial_I h)$, $\overline{Q} = \psi^I(-ip_I + \partial_I h)$. In this case, $h : \mathbb{R}^n \to \mathbb{R}$. Expanding about a critical point of h and choosing the right coordinate system $\xi^I = \xi^I_{(i)}$:

$$h(x) = h(x_i) + \sum c_I (\xi^I)^2 + \cdots$$

In particular, this means that the Hamiltonian simplifies and we again pick out the $\mathcal{O}(\lambda)$ term to get the ground state:

$$\Psi_i = \left(\bigotimes_{I\,:\,c_I>0} \exp(-\lambda c_I \xi^I)\,|0\rangle \right) \otimes \left(\bigotimes_{I\,:\,c_I<0} \exp(-\lambda |c_I| \xi^I)\bar{\psi}^I\,|0\rangle \right).$$

In particular, if we denote the Morse indices of h by μ_i, the Witten index is given by $\mathrm{tr}(-1)^F = \sum_{i=1}^{N}(-1)^{\mu_i}$.

Example 2.2.6 (Landau–Ginzburg, $n = 2m$)**.** The Landau–Ginzburg model in $d = 1$ is the complex analogue of our discussion above. The target space is now $\mathbb{C}^m \equiv \mathbb{R}^{2m}$ and we take the Lagrangian

$$L = \sum_{i=1}^{m} \left(|\dot{z}_i|^2 + i\bar{\psi}^i \partial_t \psi^{\bar{i}} + i\bar{\psi}^{\bar{i}} \partial_t \psi^i - \frac{1}{4}|\partial_i W|^2 \right) - \frac{1}{2}\sum_{ij}(\partial_i\partial_j W \psi^i \bar{\psi}^j + \partial_{\bar{i}}\partial_{\bar{j}} \overline{W} \psi^{\bar{i}} \bar{\psi}^{\bar{j}}).$$

Suppose W has N nondegenerate critical points $p_1, \ldots, p_N$. That is, $\det \partial_i\partial_j W(p_a) \neq 0$. Expanding W around these critical points and rewriting in real coordinates we get

$$W(z) = \sum_{k=1}^{m} (z^k)^2 + \mathcal{O}((z^k)^3) = \sum_{k=1}^{m} \left[(x^k)^2 - (y^k)^2 \right] + \cdots.$$

Therefore the Morse indices for *any* critical point are equal to the complex dimension m, i.e. they all have $(-1)^F = (-1)^m$, so they are either all bosonic or all fermionic, and therefore will never be lifted to higher energies.

To do (4)

2.2.4 Sigma Models

We explore an extended example of a supersymmetric quantum theory that will be very important later on. The **sigma model** has as a target manifold an oriented compact Riemannian manifold (M, g). Denote by $\mathcal{T}$ the one dimensional manifold on which our QFT lives. The bosonic and fermionic **fields** are maps of the form

$$\phi\colon \mathcal{T} \to M, \qquad \psi, \bar{\psi} \in \Gamma(\mathcal{T}, \phi^* TM \otimes \mathbb{C}) \tag{2.8}$$

Let D_t be the covariant derivative given by the Levi–Civita connection, and Γ^I_{JK} its Christoffel symbols. Let R_{IJKL} be the Riemannian curvature tensor. The **Lagrangian** is given by:

$$\begin{aligned} L &= \frac{1}{2}g_{IJ}\dot{\phi}^I\dot{\phi}^J + \frac{i}{2}(\bar{\psi}^I D_t\psi^J - D_t\bar{\psi}^I\psi^J) - \frac{1}{2}R_{IJKL}\psi^I\bar{\psi}^J\psi^K\bar{\psi}^L \\ &= \frac{1}{2}\langle\dot{\phi},\dot{\phi}\rangle + \frac{i}{2}(\langle\bar{\psi},\nabla^{LC}_t\psi\rangle - \langle\nabla^{LC}_t\bar{\psi},\psi\rangle) - \frac{1}{2}R(\psi,\bar{\psi},\psi,\bar{\psi}), \end{aligned} \tag{2.9}$$

The **susy transformations** are given by:

$$\delta\phi^I = \epsilon\bar{\psi}^I - \bar{\epsilon}\psi^I, \qquad \delta\psi^I = \epsilon(i\dot{\phi}^I - \Gamma^I_{JK}\bar{\psi}^J\psi^K), \qquad \delta\bar{\psi}^I = \bar{\epsilon}(-i\dot{\phi}^I - \Gamma^I_{JK}\bar{\psi}^J\psi^K).$$

The corresponding **supercharges** are given by:**To do** (5)

$$Q = ig_{IJ}\bar{\psi}^I\dot{\phi}^J = i\langle\bar{\psi},\dot{\psi}\rangle, \qquad \bar{Q} = -ig_{IJ}\psi^I\dot{\psi}^J = -i\langle\psi,\dot{\bar{\psi}}\rangle.$$

There is an extra symmetry, **phase invariance**, of this free Lagrangian: $(\psi,\bar{\psi}) \to (e^{-i\theta}\psi, e^{i\theta}\bar{\psi})$. The corresponding Noether charge, sometimes called the **F-charge**, is

$$F = g_{IJ}\bar{\psi}^I\psi^J = \langle\bar{\psi},\psi\rangle.$$

Quantizing this system requires the **conjugate momenta**:

$$p_I = \frac{\partial L}{\partial\dot{\phi}^I} = g_{IJ}\dot{\phi}^J, \qquad \pi_{\psi I} = ig_{IJ}\bar{\psi}^J.$$

Finally, imposing **commutation relations**

$$[\phi^I, p_J] = i\delta^I{}_J, \qquad \{\psi^I,\bar{\psi}^J\} = g^{IJ},$$

with all other commutators vanishing. The supercharges are conveniently write $Q = i\bar{\psi}^I p_I, \bar{Q} = -i\psi^I p_I$. The ambiguity in the choice of Hamiltonian is fixed if we impose $\{Q,\bar{Q}\} = 2H$. Finally, note that $[F,Q] = Q$ and $[F,\bar{Q}] = -\bar{Q}$ from which we get $[H,F] = 0$.

There exists a natural representation of the observables on the space

$$\mathcal{H} = \Omega^*(M,\mathbb{C}), \quad \text{with} \quad \langle\omega_1,\omega_2\rangle_{\mathcal{H}} = \int_M \bar{\omega}_1 \wedge *\omega_2.$$

The observables can be assigned the following operators:

$$\phi^I = x^I\cdot, \qquad p_I = -i\nabla_I, \qquad \bar{\psi}^I = dx^I\wedge, \qquad \psi^I = g^{IJ}\frac{\partial}{\partial x^J}\lrcorner \tag{2.10}$$

The ground state, corresponding to the intersection of all $\ker\psi^I$, is naturally identified with $1 \in \Omega^0(M)$. The F-charge, or fermion number, is the degree of the form and so the natural grading on the Hilbert space $\mathcal{H} = \bigoplus_p \Omega^p(M,\mathbb{C})$ corresponds to the fermion number grading. The supercharge $Q = i\bar{\psi}^I p_I = dx^I \wedge \frac{\partial}{\partial x^I}$ is just the exterior derivative d, and its Hermitian conjugate is $\bar{Q} = Q^\dagger = d^\dagger$, which we identify with the adjoint. Finally, the Hamiltonian H is given by

$$H = \frac{1}{2}\{Q,\bar{Q}\} = \frac{1}{2}(dd^\dagger + d^\dagger d) = \frac{1}{2}\Delta,$$

the Laplace–Beltrami operator. The ground states are then just the harmonic forms:

$$\mathcal{H}_{(0)} = \mathrm{Harm}(M,g) = \bigoplus_{p=0}^{n}\mathrm{Harm}^p(M,g).$$

Now we are starting to see the correspondence between supersymmetric quantum mechanics and differential geometry. In fact, Hodge theory is not too far away! As we showed before, the ground states can be characterized by cohomology (see (2.7)). Since $[F, Q] = Q$ the Q-complex may be graded by the fermion number, or in more mathematical language, the de Rham cohomology is graded by the degree of the forms. In fact, since we know the ground states correspond to the the harmonic forms, we use the same proof as for (2.7) to show that

$$\mathcal{H}_{(0)} = \mathrm{Harm}(M, g) \cong H^\bullet(Q) = H^\bullet_{dR}(M),$$

and even more that with respect to the fermion number grading,

$$\mathrm{Harm}^p(M, g) = H^p(Q) \cong H^p_{dR}(M).$$

The Witten index, $\mathrm{tr}(-1)^F$, counting the parity of the fermion numbers becomes:

$$\mathrm{tr}(-1)^F = \sum_{p=0}^{n} (-1)^p \dim H^p_{dR}(M) = \chi(M).$$

To do (6)

Other interesting mathematics can be recovered by modifying the structure of the target manifold.

1. We can **deform** the Lagrangian by adding a potential $h\colon M \to \mathbb{R}$:
$$\delta L := -\frac{1}{2} g^{IJ} \partial_I h \partial_J h - D_I \partial_J h \overline{\psi}^I \psi^J, \quad \text{giving } Q = \overline{\psi}^I (ip_I + \partial_I h) \tag{2.11}$$
where $D_I \partial_J h := \partial_I \partial_J h - \Gamma^K_{IJ} \partial_K h$. Then Q becomes identified with $d_h := d + dh\wedge$, giving the **twisted de Rham complex** (Ω^*, d_h). However this complex is isomorphic to the original (untwisted) de Rham complex via e^{-h}.

2. We can make M Kähler. Then there is an **extended supersymmetry** as in 2.2.6 giving four supercharges $Q_+, Q_+^\dagger, Q_-, Q_-^\dagger$ such that the original Riemannian supercharges satisfy $Q = i(Q_- + Q_+^\dagger)$. There are also two F-charges F_A and F_V that commute, i.e. $[F_A, F_V] = 0$, so $\mathcal{H} = \Omega^*(M, \mathbb{C})$ decomposes with respect to both charges into $\bigoplus_{p,q} \Omega^{p,q}(M)$. Then $Q_+ = -i\overline{\partial}$ and $Q_- = -i\partial$, the Dolbeault operators, and $F_A = q + p$ and $F_V = q - p$. We recover the Hodge decomposition and the Dolbeault isomorphism $\mathcal{H}^{p,q}(M) = H^{p,q}_{\overline{\partial}}(M)$.

2.2.5 Instantons

Adding a deformation $h\colon M \to \mathbb{R}$ to the Riemannian manifold case is analogous to the introduction of a potential h in the susy potential theory; we repeat the perturbative analysis in subsection 2.2.3. When we scale $h \mapsto \lambda h$, the number of susy ground states is preserved (c.f. 2.1.1), and the Hamiltonian is

$$H_\lambda := \frac{1}{2}\Delta + \frac{1}{2}\lambda^2 g^{IJ} \partial_J h \partial_J h + \frac{1}{2}\lambda D_I \partial_J h [\overline{\psi}^I, \psi^J].$$

In general, we can do any operations that preserve the Q-cohomology, and therefore the number of susy ground states. Again, at large λ, low-energy states are localized around critical points x_i of h. So choose coordinates x^I around x_i such that $h = h(x_i) + c_I (x^I)^2 + O((x^I)^3)$ and c_I are the eigenvalues of the Hessian, and for simplicity, deform either h or the metric g_{IJ} so that $g_{IJ} = \delta_{IJ}$. The leading-order term in the Hamiltonian is

$$H_0(x_i) := \sum_{I=1}^{n} \frac{1}{2} p_I^2 + \frac{1}{2}\lambda^2 c_I^2 (x^I)^2 + \frac{1}{2}\lambda c_I [\overline{\psi}^I, \psi^I],$$

which is n independent susy harmonic oscillators. To leading order, the susy ground state is therefore

$$\Psi_i^{(0)} := \exp\left(-\lambda |c_I| (x^I)^2\right) \prod_{J : c_J < 0} \overline{\psi}^J |0\rangle . \tag{2.12}$$

Hence $(-1)^F = (-1)^{\#\{J : c_J < 0\}}$, and the exponent is just the **Morse index** μ_i of h at x_i.

As discussed earlier, it is not necessarily the case that each Ψ_i is actually a susy ground state in the full theory, i.e. it may be that $Q\Psi_i \neq 0$. So let's look at the theory non-perturbatively: we want to compute the matrix elements of Q

$$\langle \Psi_j, Q\Psi_i \rangle = \int_M \overline{\Psi}_j \wedge \star(d + dh\wedge)\Psi_i.$$

(Here we are implicitly using the identifications in (2.10).) Since Ψ_j is a μ_j-form and $Q\Psi_i = (d + dh\wedge)\Psi_i$ is a $(\mu_i + 1)$-form, the matrix element is potentially nonzero only when $\mu_j = \mu_i + 1$. We evaluate it via path integral.

Proposition 2.2.7. *The desired matrix element as a path integral is*

$$\langle \Psi_j, Q\Psi_i \rangle = \lim_{t\to\infty} \langle \Psi_j, e^{-tH}[Q, h]e^{-tH}\Psi_i \rangle = \int_{\phi(-\infty)=x_i, \phi(+\infty)=x_j} \mathcal{D}\phi\, \mathcal{D}\psi\, \mathcal{D}\overline{\psi}\, e^{-S_E} \overline{\psi}^I \partial_I h|_{\tau=0}$$

where the Euclidean (Wick-rotated) action S_E is

$$S = \int_{-\infty}^{\infty} d\tau\, \frac{1}{2} g_{IJ} \partial_\tau \phi^I \partial_\tau \phi^J + \frac{\lambda^2}{2} g^{IJ} \partial_I h \partial_J h + g_{IJ} \overline{\psi}^I D_\tau \psi^J + \lambda D_I \partial_J h \overline{\psi}^I \psi^J + \frac{1}{2} R_{IJKL} \psi^I \overline{\psi}^J \psi^K \overline{\psi}^L .$$

Proof. Not trivial at all. See pg. 224 in Hori. □

There is a Euclidean susy transformation δ_ϵ on S_E and its boundary conditions $\phi(-\infty) = x_i$ and $\phi(+\infty) = x_j$ that preserves the integrand $[Q, h] = \overline{\psi}^I \partial_I h$. By the same argument as for the localization principle in $d = 0$ (c.f. 2.1.2), the path integral receives contributions only from δ_ϵ-fixed points. In this particular case, such fixed points satisfy $\partial_\tau \phi^I = \lambda g^{IJ} \partial_J h$, and are minimizers of the (bosonic part of the) action S_E. Such minimizers are **instantons**. Looking at this equation, we see it describes an **ascending gradient flow** from x_i to x_j, where $\mu_i = \mu_j + 1$ (c.f. 1.1.17).

For "generic" h (c.f. 1.1.16), we can do the calculation explicitly using localization, and get

$$\langle \Psi_j, Q\Psi_i \rangle = \sum_\gamma n_\gamma \exp(-\lambda(h(x_j) - h(x_i)))$$

where $n_\gamma = \pm 1$ depending on the orientation of the instanton γ (c.f. 1.1.20). (The exponential can be eliminated by rescaling Ψ_i.) Note that Q looks suspiciously like the **Morse boundary map** 1.1.20. Since originally $Q^2 = 0$, it is also nilpotent when acting on the perturbative ground states. Hence if we define the **graded space of perturbative ground states** $C^\mu := \bigoplus_{\mu_i=\mu} \mathbb{C}\Psi_i$, we get the **Morse–Witten complex** $0 \to C^0 \xrightarrow{Q} C^1 \xrightarrow{Q} \cdots \xrightarrow{Q} C^n \xrightarrow{Q} 0$, and the space of susy ground states is of course its cohomology. Hence we have proved the Morse homology theorem 1.1.24.

2.3 QFT with $d = 1 + 1$

A **free** field theory has action quadratic in the field variables; recall from QFT that by adding in higher powers, we add vertices into Feynman diagrams, and therefore non-trivial interactions between fields. In $d = 1 + 1$ (where we write $1 + 1$ instead of 2 to emphasize that there is 1 space dimension and 1 time dimension), we shall primarily look at free theories.

2.3.1 Free Bosonic Scalar Field Theory

The ingredients of a free bosonic scalar field theory are given by:

1. let $\Sigma = \mathbb{R} \times S^1$ be the **world sheet**, where $\mathbb{R}$ is parametrized by the time t and S^1 by the spatial coordinate s of period 2π;
2. let $x = x(t, s) \in \mathbb{R}$ be the **scalar field** (so that we can also consider this theory as a sigma model, where $x\colon \Sigma \to \mathbb{R}$);
3. the action is
$$S \coloneqq \frac{1}{2\pi} \int_\Sigma L \, dt \, ds = \frac{1}{4\pi} \int_\Sigma \left((\partial_t x)^2 - (\partial_s x)^2\right) \, dt \, ds.$$

Euler–Lagrange gives the **equation of motion** $(\partial_t^2 - \partial_s^2)x = 0$, so a general solution is $x(t, s) = f(t - s) + g(t + s)$, i.e. a linear combination of **left movers** and **right movers**.

Noether's procedure applied to a translation $\delta x = \alpha(t, s)$ on the target space $\mathbb{R}$ gives

$$\delta S = \frac{1}{2\pi} \int_\Sigma \partial_\mu \alpha j^\mu \, dt \, ds, \quad j^t = \partial_t x, \; j^s = -\partial_s x.$$

Hence j^μ is a current and has a conservation law $\partial_\mu j^\mu = 0$, and the charge $p \coloneqq (1/2\pi) \int_{S^1} j^t \, ds$ is conserved; physically it is the **target space momentum**. Similarly, Noether's procedure applied to a translation $\delta_\alpha x = \alpha^\mu \partial_\mu x$ on the worldsheet Σ gives the energy-momentum tensor, and conserved charges

$$H \coloneqq \frac{1}{4\pi} \int_{S^1} \left((\partial_t x)^2 + (\partial_s x)^2\right) \, ds, \quad P \coloneqq \frac{1}{2\pi} \int_{S^1} \partial_t x \partial_s x \, ds$$

which are the **Hamiltonian** and **momentum** of the system itself.

To help us find the dynamics of the system, write the Fourier expansion of $x(t, s)$ on S^1:

$$x(t, s) = x_0(t) + \sum_{n \neq 0} x_n(t) e^{ins}, \quad x_0(t) \in \mathbb{R}, \; x_{-n}(t) = x_n(t)^*.$$

The action therefore **decouples** into

$$S = \int dt \left(\frac{1}{2} (\dot{x}_0)^2 + \sum_{n=1}^\infty (|\dot{x}_n|^2 - n^2 |x_n|^2) \right),$$

i.e. infinitely many decoupled harmonic oscillators, the n-th one with potential $U = n^2 |x_n|^2$, along with a real scalar x_0. This scalar x_0 has conjugate momentum p_0 and a momentum eigenstate $|k\rangle_0$ for each k (with momentum $k^2/2$), with Hamiltonian $H_0 = (1/2) p_0^2$. We also know the Hamiltonians H_n of the harmonic oscillators:

$$H_n = \alpha_{-n} \alpha_n + \tilde{\alpha}_{-n} \tilde{\alpha}_n + n$$

where $\alpha_{-n}, \tilde{\alpha}_{-n}$ are **raising operators** and $\alpha_n, \tilde{\alpha}_n$ are **lowering operators**. Let $|0\rangle_n$ be the state annihilated by α_n and $\tilde{\alpha}_n$, so that

$$|k\rangle \coloneqq |k\rangle_0 \otimes \bigotimes_{n=1}^\infty |0\rangle_n$$

is a ground state of momentum $k^2/2$ that can be excited by combinations of α_{-n} and $\tilde{\alpha}_n$. The **total Hamiltonian** is

$$H = H_0 + \sum_{n=1}^\infty H_n = \frac{1}{2} p_0^2 + \sum_{n=1}^\infty \alpha_{-n} \alpha_n + \sum_{n=1}^\infty \tilde{\alpha}_{-n} \tilde{\alpha}_n - \frac{1}{12}$$

where we used $\sum_{n=1}^{\infty} n = -1/12$ via zeta regularization (to eliminate the conformal anomaly).

In this choice of basis many expressions become nicer. The target space and worldsheet momentum are

$$p = \frac{1}{2\pi}\int_{S^1} \dot{x}\, ds = \dot{x}_0 = p_0$$
$$P = \sum_{n+m=0} im\dot{x}_n x_m = -\sum_{n=1}^{\infty} \alpha_{-n}\alpha_n + \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}\tilde{\alpha}_n.$$

(Note that P is essentially a **number operator**.) Using commutation relations we obtain

$$x_0(t) = e^{iHt}x_0e^{-iHt} = x_0 + tp_0, \quad \alpha_n(t) = e^{iHt}\alpha_n e^{-iHt} = e^{-int}\alpha_n.$$

Since $x_n = (\tilde{\alpha}_{-n} - \alpha_n)/(in\sqrt{2})$ we get

$$x(t,s) = x_0 + tp_0 + \frac{i}{\sqrt{2}}\sum_{n\neq 0}\frac{1}{n}(\alpha_n e^{-in(t-s)} + \tilde{\alpha}_n e^{-in(t+s)})$$

which is the most general solution compatible with the periodicity $x(t, s+2\pi) = x(t,s)$.

We now want to compute the partition function for this theory. In $d = 1$, for a circle of circumference β, we wrote $Z(\beta) = \operatorname{tr} e^{-\beta H}$. Now we have a cylinder of length $2\pi\tau_2$ and circumference 2π, but there are inequivalent ways of gluing the ends, e.g. first shifting one end by $2\pi\tau_1$ and then gluing. (This corresponds to inequivalent **complex structures** on the torus parametrized by τ_2/τ_1.) In operator language, this shift corresponds to inserting the **translation operator** $e^{-2\pi i\tau_1 P}$ into the trace:

$$Z(\tau_1,\tau_2) = \operatorname{tr} e^{-2\pi i\tau_1 P}e^{-2\pi\tau_2 H}.$$

Write $H_R \coloneqq (1/2)(H-P)$ and $H_L \coloneqq (1/2)(H+P)$ to decouple the left and right movers, and write $\tau \coloneqq \tau_1 + i\tau_2$. Explicitly,

$$H_R = \frac{1}{2}(H-P) = \frac{1}{4}p_0^2 + \sum_{n=1}^{\infty}\alpha_{-n}\alpha_n - \frac{1}{24}.$$

The total Hilbert space of states factors as $\mathcal{H} = \mathcal{H}_0 \otimes \bigotimes_n (\mathcal{H}_n^L \otimes \mathcal{H}_n^R)$, i.e. zero modes, and left/right-movers at level n, so

$$\begin{aligned} Z(\tau,\overline{\tau}) = \operatorname{tr} e^{2\pi i\tau H_R}e^{-2\pi i\overline{\tau}H_L} &= (q\overline{q})^{-1/24}\operatorname{tr}_{\mathcal{H}_0}(q\overline{q})^{p_0^2/4}\prod_{n=1}^{\infty}\operatorname{tr}_{\mathcal{H}_n^L}\overline{q}^{\tilde{\alpha}_{-n}\tilde{\alpha}_n}\operatorname{tr}_{\mathcal{H}_n^R}q^{\alpha_{-n}\alpha_n} \\ &= (q\overline{q})^{-1/24}\left(V\int_{-\infty}^{\infty}\frac{dp}{2\pi}e^{-2\pi\tau_2(p^2/2)}\right)\prod_{n=1}^{\infty}\frac{1}{1-\overline{q}^n}\frac{1}{1-q^n} \\ &= \frac{V}{2\pi}\frac{1}{\sqrt{\tau_2}}|\eta(\tau)|^{-2} \end{aligned} \tag{2.13}$$

where $\eta(\tau) \coloneqq q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ is the **Dedekind eta function**, and V is a **regularization cutoff**. The normalization $(q\overline{q})^{-1/24} = e^{-2\pi\tau_2(-1/12)}$ comes from the regularized **zero-point energy** $\sum_{n=1}^{\infty} n = -1/12$. Note that by the modular invariance of the Dedekind eta, the partition function is diffeomorphism-invariant (on T^2, acting on τ), as expected.

2.3.2 Sigma Models and T-Duality

Now consider a sigma model with target space S^1 of radius R. The difference is that in doing the Fourier decomposition, the circle has discrete Fourier modes, and the target space momentum is quantized: $p = l/R$

for $l \in \mathbb{Z}$. Also, there are topologically non-trivial scalar field configurations $x(s,t)$ classified by the **winding number** m given by $x(s+2\pi) = x(s) + 2\pi mR$. There is a "current" (not arising as an equation of motion) associated to the winding number: modify the conserved momentum current j^μ by

$$\begin{cases} j^t = \partial_t x \\ j^s = -\partial_s x \end{cases} \quad \Rightarrow \quad \begin{cases} j^t_w = \partial_s x \\ j^s_w = -\partial_t x. \end{cases}$$

Clearly $\partial_\mu j^\mu_w = 0$, and the corresponding "charge" is

$$w := \frac{1}{2\pi}\int_{S^1} j^t_w\, ds = \frac{1}{2\pi}(x(2\pi) - x(0)) = mR.$$

States therefore have two quantum numbers: **momentum** l and **winding number** m. The **Hilbert space** of states decomposes as $\mathcal{H} = \bigoplus_{(l,m)\in\mathbb{Z}\oplus\mathbb{Z}} \mathcal{H}_{(l,m)}$, where $\mathcal{H}_{(l,m)}$ has $p = l/R$ and $w = mR$, with basic element $|l,m\rangle$ annihilated by α_n and $\tilde{\alpha}_n$ for $n > 0$. Let p_0 and w_0 be the operators giving the momentum and winding number:

$$p_0\,|l,m\rangle = \frac{l}{R}\,|l,m\rangle\,, \quad w_0\,|l,m\rangle = mR\,|l,m\rangle\,.$$

Via commutation relations, we see that $e^{i(l/R)x_0}$ shifts the momentum (as it does for the free scalar boson). We can also create an operator to shift the winding number; call it $e^{imR\hat{x}_0}$, so that

$$e^{i(l_1/R)x_0}\,|l,m\rangle = |l+l_1,m\rangle\,, \quad e^{im_1R\hat{x}_0}\,|l,m\rangle = |l,m+m_1\rangle\,.$$

If we separate left movers from right movers again and write new operators

$$p_R := \frac{1}{\sqrt{2}}(p_0 - w_0), \quad p_L := \frac{1}{\sqrt{2}}(p_0 + w_0),$$

then again we can write down the left-movers' Hamiltonian and the right-movers' Hamiltonian:

$$H_R = \frac{1}{2}(H-P) = \frac{1}{2}p_R^2 + \sum_{n=1}^{\infty}\alpha_{-n}\alpha_n - \frac{1}{24}.$$

Proposition 2.3.1. *Let $q = e^{2\pi i\tau}$. The* **partition function** *is*

$$Z(\tau,\overline{\tau};R) = |\eta(\tau)|^{-2}\sum_{(l,m)\in\mathbb{Z}\oplus\mathbb{Z}} q^{\frac{1}{4}(l/R-mR)^2}\,\overline{q}^{\frac{1}{4}(l/R+mR)^2}.$$

Proof. The factor $|\eta(\tau)|^{-2}$ comes from the oscillator modes as in the free scalar boson case. But now instead of a zero mode integral over $|k\rangle$ for $k\in\mathbb{R}$, we have a discrete sum over $|l,m\rangle$ for $(l,m)\in\mathbb{Z}\oplus\mathbb{Z}$. From the expressions for the left and right Hamiltonians, the zero modes are precisely $(1/2)p_R^2$ and $(1/2)p_L^2$. $\square$

Note that the partition function is invariant under $R \mapsto 1/R$, and the full spectrum is invariant after switching winding number and momentum via $l \leftrightarrow m$ (since the spectrum of p_0 goes up in increments of $1/R$ and w_0 in increments of R).

Definition 2.3.2. There is an isomorphism, called **T-duality**, of the Hilbert space $\mathcal{H}$ on radius R with the Hilbert space $\hat{\mathcal{H}}$ on radius $1/R$ under which $\mathcal{H}_{(l,m)} \mapsto \hat{\mathcal{H}}_{(m,l)}$. This corresponds to the exchange of operators $(p_R, p_L) \mapsto (-\hat{p}_R, \hat{p}_L)$.

This is why we called the operator associated to a shift in the winding number $\hat{x}_0$: in the T-dual theory, it is the zero mode of the coordinate $\hat{x}$, which shifts momentum. The other Fourier modes are mapped as $\alpha_n \mapsto -\hat{\alpha}_n$ and $\tilde{\alpha}_n \mapsto \hat{\tilde{\alpha}}_n$.

Remark. There is a nice path integral derivation of T-duality. For generality, let Σ be a Riemann surface of genus g. Set $\phi := x/R$, which is 2π-periodic, put local coordinates $\sigma^\mu = (\sigma^1, \sigma^2)$ on Σ and a Euclidean metric $h = h_{\mu\nu}$ on Σ, so that the action is

$$S_\phi = \frac{1}{4\pi} \int_\Sigma R^2 \sqrt{h}\, h^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \, d^2\sigma.$$

Introduce an **auxiliary field** (of one-forms) B_μ and consider the action

$$S' := \frac{1}{2\pi} \int_\Sigma \frac{1}{2R^2} \sqrt{h}\, h^{\mu\nu} B_\mu B_\nu \, d^2\sigma + \frac{i}{2\pi} \int_\Sigma B \wedge d\phi.$$

When we plug this into the path integral $\int D\phi\, DB\, e^{-S'}$, the auxiliary field B_μ acts as a Lagrange multiplier. Completing the square and integrating it out, we get a factor of $\delta[B - iR^2 \star d\phi]$. When we plug this back into S' we recover S exactly, so S and S' are equivalent.

Now we change the order of integration: integrate out ϕ first instead of B. The second term in S' immediately gives $\delta[dB]$, so the general form of B is

$$B = d\theta_0 + \sum_{i=1}^{2g} a_i \omega^i$$

for a symplectic cohomology basis ω^i of $H^1(\Sigma, \mathbb{R}) \cong \mathbb{R}^{2g}$, i.e. there is an associated homology basis γ_i such that $\int_{\gamma_i} \omega_j = \delta_{ij}$. Hence $J^{ij} := \int_\Sigma \omega^i \wedge \omega^j$ is a non-degenerate integral matrix with inverse an integral matrix too. Integrating over ϕ puts constraints on the a_i because $d\phi$ also has an expansion in the basis ω^i, but with coefficients $2\pi n_i$ where $n_i \in \mathbb{Z}$.

$$\int_\Sigma B \wedge d\phi = 2\pi \sum_{i,j} a_i J^{ij} n_j.$$

Integrating over ϕ is a summation over n_i, and since $\sum_n e^{ian} = 2\pi \sum_m \delta(a - 2\pi m)$, the integration sets $a_i = 2\pi m_i$ for $m_i \in \mathbb{Z}$. It follows that $B = d\theta$ for θ a 2π-periodic field.

Hence after integrating out ϕ in S', we get

$$S_\theta = \frac{1}{4\pi} \frac{1}{R^2} \sqrt{h}\, h^{\mu\nu} \partial_\mu \theta \partial_\nu \theta \, d^2\sigma$$

which is a sigma model with target S^1 of radius $1/R$. In particular, comparing the two expressions for B gives

$$R d\phi = i \frac{1}{R} \star d\theta,$$

but $R d\phi$ and $iR \star d\phi$ are the conserved currents in the original system that measure momentum and winding number respectively.

Remark. Consider a sigma model with target $T^2 = S^1_{R_1} \times S^1_{R_2}$. Replace the parameters R_1, R_2 with the area and complex structure of the torus (supposing for simplicity that the complex structure is purely imaginary):

$$A := R_1 R_2, \quad \sigma := i R_1 / R_2.$$

Using T-duality on R_2, we get

$$(A, \operatorname{im} \sigma) = (R_1 R_2, R_1/R_2) \mapsto (A', \operatorname{im} \sigma') = (R_1/R_2, R_1 R_2),$$

i.e. the shape (complex structure) and the size (Kähler structure) of the target T^2 have been exchanged under this duality. This is a hint toward mirror symmetry being a refinement of T-duality.

2.4 Superspace Formalism

So far we have used supersymmetric Lagrangians without specifying how they arise. In this section we discuss how to systematically come up with susy theories in $d = 1 + 1$ with two complex supercharges, generally denoted Q_+, Q_-.

2.4.1 Superspace and Superfields

Definition 2.4.1. The $(2, 2)$**-superspace** is the space with two real (commuting) bosonic coordinates x^0, x^1 carrying a Minkowski metric and two complex (anticommuting) fermionic coordinates θ^+, θ^-, which transform under the Lorentz group as

$$\begin{pmatrix} x^0 \\ x^1 \end{pmatrix} \mapsto \begin{pmatrix} \cosh\gamma & \sinh\gamma \\ \sinh\gamma & \cosh\gamma \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix}, \quad \theta^\pm \mapsto e^{\pm\gamma/2}\theta^\pm.$$

A **superfield** is a $\mathbb{C}$-valued function $\mathcal{F}(x^0, x^1, \theta^+, \theta^-, \overline{\theta}^+, \overline{\theta}^-)$ on superspace. Since $(\theta^\pm)^2 = 0$, there are at most $2^4 = 16$ non-zero terms when $\mathcal{F}$ is expanded in the fermionic coordinates. A superfield is **bosonic** if $[\theta^\alpha, \mathcal{F}] = 0$ and **fermionic** if $\{\theta^\alpha, \mathcal{F}\} = 0$.

Definition 2.4.2. Let $x^\pm \coloneqq x^0 \pm x^1$ and $\partial_\pm \coloneqq (1/2)(\partial_0 \pm \partial_1)$. Define **supercharges** $\mathcal{Q}_\pm$ and **covariant derivatives** $D_\pm$ on superspace:

$$\mathcal{Q}_\pm \coloneqq \partial_{\theta^\pm} + i\overline{\theta}^\pm\partial_\pm, \quad \overline{\mathcal{Q}}_\pm \coloneqq -\partial_{\overline{\theta}^\pm} - i\theta^\pm\partial_\pm,$$

$$D_\pm \coloneqq \partial_{\theta^\pm} - i\overline{\theta}^\pm\partial_\pm, \quad \overline{D}_\pm \coloneqq -\partial_{\overline{\theta}^\pm} + i\theta^\pm\partial_\pm.$$

They satisfy the commutation relations

$$\{\mathcal{Q}_\pm, \overline{\mathcal{Q}}_\pm\} = -2i\partial_\pm, \quad \{D_\pm, \overline{D}_\pm\} = 2i\partial_\pm, \quad \{D_\pm, \mathcal{Q}_\pm\} = 0.$$

Definition 2.4.3. A **chiral superfield** Φ is a superfield satisfying $\overline{D}_\pm\Phi = 0$; the complex conjugate of a chiral superfield satisfies $D_\pm\overline{\Phi} = 0$, and is called an **anti-chiral superfield**. A **twisted chiral superfield** U is a superfield satisfying $\overline{D}_+U = D_-U = 0$; its complex conjugate is an **twisted anti-chiral superfield**.

Proposition 2.4.4. *Chiral superfields and twisted chiral superfields have the expansions*

$$\Phi(x^\mu, \theta^\pm, \overline{\theta}^\pm) = \phi(y^\pm) + \theta^\alpha\psi_\alpha(y^\pm) + \theta^+\theta^- F(y^\pm), \quad y^\pm \coloneqq x^\pm - i\theta^\pm\overline{\theta}^\pm,$$
$$U(x^\mu, \theta^\pm, \overline{\theta}^\pm) = v(\tilde{y}^\pm) + \theta^+\overline{\chi}_+(\tilde{y}^\pm) + \overline{\theta}^-(\tilde{y}^\pm) + \theta^+\overline{\theta}^- E(\tilde{y}^\pm), \quad \tilde{y}^\pm \coloneqq x^\pm \mp i\theta^\pm\overline{\theta}^\pm.$$

Proof. First establish that $y^\pm$ and $\tilde{y}^\pm$ are chiral and twisted chiral respectively. Then do casework on the 16 terms in the expansion of a general superfield. $\square$

Definition 2.4.5. The **supersymmetry transformation** we shall consider is

$$\delta \coloneqq \epsilon_+\mathcal{Q}_- - \epsilon_-\mathcal{Q}_+ - \overline{\epsilon}_+\overline{\mathcal{Q}}_- + \overline{\epsilon}_-\overline{\mathcal{Q}}_+.$$

There are two kinds of terms invariant under δ (for a proof of this, see Hori, section 12.1):

1. **D-terms** $\int d^2x\, d^4\theta\, K[\mathcal{F}_i]$ where K is an arbitrary functional of arbitrary superfields $\mathcal{F}_i$ (here $d^4\theta \coloneqq d\theta^+ d\theta^- d\overline{\theta}^- d\overline{\theta}^+$);
2. **F-terms** $\int d^2x\, d^2\theta\, W[\Phi_i]|_{\overline{\theta}^\pm=0}$ where W is a holomorphic functional of chiral superfields Φ_i (here $d^2\theta \coloneqq d\theta^- d\theta^+$).

Proposition 2.4.6. *Superfield calculus is very much like regular calculus:*

1. *(Poincaré lemma)* $D_\pm$*-closed implies* $D_\pm$ *exact, and similarly for* $\overline{D}_\pm$*;*
2. *(*$\overline{D}_+\overline{D}_-$*-lemma) a chiral superfield* Φ *can be written as* $\Phi = \overline{D}_+\overline{D}_-\mathcal{E}$ *for some superfield* $\mathcal{E}$*, and a twisted chiral superfield* U *as* $U = \overline{D}_+D_-\mathcal{V}$ *for some superfield* $\mathcal{V}$*;*
3. $\overline{D}_+\overline{D}_-\mathcal{F} = 0$ *implies* $\mathcal{F} = \mathcal{G}_+ + \mathcal{G}_-$ *for* $\overline{D}_+\mathcal{G}_+ = \overline{D}_-\mathcal{G}_- = 0$*, and similarly for* $D_+D_-, \overline{D}_+D_-, D_+\overline{D}_-$*;*
4. $\overline{D}_+\overline{D}_-\mathcal{F} = D_+D_-\mathcal{F} = 0$ *implies* $\mathcal{F} = U_1 + \overline{U}_2$ *for some twisted chiral superfields* U_i*;*
5. $\overline{D}_+D_-\mathcal{F} = D_+\overline{D}_-\mathcal{F} = 0$ *implies* $\mathcal{F} = \Phi_1 + \overline{\Phi}_2$ *for some chiral superfields* Φ_i*.*

Definition 2.4.7. The vector and axial **R-rotations** of a superfield are given by operators

$$e^{i\alpha F_V} : \mathcal{F}(x^\mu, \theta^\pm, \overline{\theta}^\pm) \mapsto e^{i\alpha q_V}\mathcal{F}(x^\mu, e^{-i\alpha}\theta^\pm, e^{i\alpha}\overline{\theta}^\pm),$$

$$e^{i\alpha F_A} : \mathcal{F}(x^\mu, \theta^\pm, \overline{\theta}^\pm) \mapsto e^{i\alpha q_A}\mathcal{F}(x^\mu, e^{\mp i\alpha}\theta^\pm, e^{\pm i\alpha}\overline{\theta}^\pm).$$

The numbers q_V and q_A are the vector and axial **R-charges**.

Example 2.4.8. Let Φ be a chiral superfield. Using the expansion 2.4.4 we can compute the D-term

$$S_{\text{kin}} \coloneqq \int d^2x\, d^4\theta\, \overline{\Phi}\Phi = \int d^2x \left(|\partial_0\phi|^2 - |\partial_1\phi|^2 + i\overline{\psi}_-(\partial_0 + \partial_1)\psi_- + i\overline{\psi}_+(\partial_0 - \partial_1)\psi_+ + |F|^2\right),$$

which is the susy action for a complex scalar field ϕ and two **Dirac fermion fields** $\psi_\pm, \overline{\psi}_\pm$. The field F has no kinetic term: it is an **auxiliary field**. The F-term for the functional W, called the **superpotential**, is

$$S_W \coloneqq \int d^2x\, d^2\theta\, W(\Phi) + \text{c.c.} = \int d^2x\, (W'[\phi]F - W''[\phi]\psi_+\psi_- + \text{c.c.})\,.$$

Adding the D-term and the F-term, we get the action for the scalar ϕ and the Dirac fermion $\psi_\pm, \overline{\psi}_\pm$, with a potential $|W'[\phi]|^2$ for ϕ and the **Yukawa interaction** (or fermion mass term) $W''[\phi]\psi_+\psi_-$. There is a remaining term $|F + \overline{W}'[\overline{\phi}]|^2$, which when integrated out gives a $\delta[F + \overline{W}'[\overline{\phi}]]$ term, i.e. setting $F = -\overline{W}'[\overline{\phi}]$. Explicitly, the supersymmetry looks like

$$\delta\phi = \epsilon_+\psi_- - \epsilon_-\psi_+, \quad \delta\psi_\pm = \pm 2i\overline{\epsilon}_\mp\partial_\pm\phi + \epsilon_\pm F, \quad \delta F = -2i\overline{\epsilon}_+\partial_-\psi_+ - 2i\overline{\epsilon}_-\partial_+\psi_-.$$

Hence this theory of a single chiral superfield is precisely the $d = 2$ generalization of the 1d susy potential theory 2.2.2. We can check that the **supercharges** $Q_\pm, \overline{Q}_\pm$ are

$$Q_\pm = \int dx^1\, G^0_\pm, \quad G^0_\pm \coloneqq 2\partial_\pm\overline{\phi}\psi_\pm \mp i\overline{\psi}_\mp\overline{W}'[\overline{\phi}],$$

and that they transform as **spinors**, i.e. $Q_\pm \mapsto e^{\mp\gamma/2}Q_\pm$.

In addition to the supersymmetry, if we assign axial R-charge 0 for Φ, there is obviously an axial R-symmetry

$$\Phi(x^\pm, \theta^\pm, \overline{\theta}^\pm) \mapsto \Phi(x^\pm, e^{\mp i\alpha}\theta^\pm, e^{\pm i\alpha}\overline{\theta}^\pm)$$

since both θ^4 (in the D-term) and θ^2 (in the F-term) are invariant under such a phase change. Things transform as

$$\phi \mapsto \phi, \quad \psi_\pm \mapsto e^{\mp i\alpha}\psi_\pm, \quad Q_\pm \mapsto e^{\mp i\alpha}Q_\pm,$$

and we have the conserved charge $F_A = \int dx^1\, J^0_A$, where the current J is

$$J^0_A = \overline{\psi}_+\psi_+ - \overline{\psi}_-\psi_-, \quad J^1_A = -\overline{\psi}_+\psi_+ - \overline{\psi}_-\psi_-.$$

Similarly, we can look at vector R-rotation. Clearly the D-term is invariant under an arbitrary choice of vector R-charge, but θ^2 has vector R-charge -2, so the F-term is invariant iff the vector R-charge of Φ can be assigned so that $W[\phi]$ is charge 2. In the special case that $W[\phi] = c\Phi^k$, this is obviously possible: assign vector R-charge $2/k$ to Φ, so that

$$\phi \mapsto e^{(2/k)i\alpha}\phi, \quad \psi_\pm \mapsto e^{((2/k)-1)i\alpha}\psi_\pm, \quad Q_\pm \mapsto e^{-i\alpha}Q_\pm,$$

with conserved charge $F_V = \int dx^0\, J_V^0$ where the current is

$$J_V^0 = \frac{2i}{k}(\partial_0\overline{\phi}\phi - \overline{\phi}\partial_0\phi) - \left(\frac{2}{k}-1\right)(\overline{\psi}_+\psi_+ + \overline{\psi}_-\psi_-), \quad J_V^1 = \frac{2i}{k}(-\partial_1\overline{\phi}\phi + \overline{\phi}\partial_1\phi) + \left(\frac{2}{k}-1\right)(\overline{\psi}_+\psi_+ - \overline{\psi}_-\psi_-).$$

2.4.2 $\mathcal{N} = (2,2)$ Supersymmetric QFTs

Definition 2.4.9. Suppose we have a susy field theory with two complex supercharges $Q_\pm, \overline{Q}_\pm$. The theory must be Poincaré invariant, so we have Noether charges for time translations ∂_0, spatial translations ∂_1, and Lorentz rotations $x^0\partial_1 + x^1\partial_0$, corresponding to H, P, M, the Hamiltonian, momentum, and angular momentum, respectively. If the action is also invariant under vector and axial R-rotations, then there are also Noether charges F_V, F_A. Assuming no anomalies when quantizing, we get the relations

$$\begin{gathered}
Q_+^2 = Q_-^2 = \overline{Q}_+^2 = \overline{Q}_-^2 = 0, \\
\{Q_\pm, \overline{Q}_\pm\} = H \pm P, \\
\{\overline{Q}_+, \overline{Q}_-\} = Z, \quad \{Q_+, Q_-\} = Z^*, \\
\{Q_-, \overline{Q}_+\} = \tilde{Z}, \quad \{Q_+, \overline{Q}_-\} = \tilde{Z}^*, \\
[iM, Q_\pm] = \mp Q_\pm, \quad [iM, \overline{Q}_\pm] = \mp\overline{Q}_\pm, \\
[iF_V, Q_\pm] = -iQ_\pm, \quad [iF_V, \overline{Q}_\pm] = i\overline{Q}_\pm, \\
[iF_A, Q_\pm] = \mp iQ_\pm, \quad [iF_A, \overline{Q}_\pm] = \pm i\overline{Q}_\pm.
\end{gathered} \tag{2.14}$$

where $Z, \tilde{Z}$ are **central charges** and must commute with all the operators in the theory. For example, $Z = 0$ if F_V is conserved, and $\tilde{Z} = 0$ if F_A is conserved. These relations define the $\mathcal{N} = (2,2)$ **supersymmetry Lie algebra**. Representations of this algebra are called **supermultiplets**. Examples follow.

Example 2.4.10. The **chiral multiplet** $(\phi, \psi_\pm, F)$ consists of the components of a chiral superfield and is an example of a supermultiplet. Its **lowest component** ϕ is a scalar, and satisfies

$$[\overline{Q}_\pm, \phi] = \overline{\mathcal{Q}}\mathcal{F}|_{\theta^\pm = \overline{\theta}^\pm = 0} = (\overline{D}_\pm - 2i\theta^\pm\partial_\pm)\mathcal{F}|_{\theta^\pm=\overline{\theta}^\pm=0} = 0.$$

Conversely, given an operator ϕ such that $[\overline{Q}_\pm, \phi] = 0$, we can construct a chiral multiplet $(\phi, \psi_\pm, F)$ by

$$\psi_\pm \coloneqq [iQ_\pm, \phi], \quad F \coloneqq \{Q_+, [Q_-, \phi]\}.$$

In other words, a chiral multiplet is defined by ϕ satisfying $[\overline{Q}_\pm, \phi] = 0$. We can do a similar procedure for **twisted chiral multiplets** $(v, \overline{\chi}_+, \chi_-, \tilde{F})$, and get that they are characterized by

$$[\overline{Q}_+, v] = [Q_-, v] = 0, \quad \overline{\chi}_+ \coloneqq [iQ_+, v], \quad \chi_- \coloneqq -[i\overline{Q}_-, v], \quad \tilde{F} \coloneqq -\{Q_+, [\overline{Q}_-, v]\}.$$

Definition 2.4.11. There is an order-2 automorphism of the $\mathcal{N} = (2,2)$ susy algebra given by

$$Q_- \longleftrightarrow \overline{Q}_-, \quad F_V \longleftrightarrow F_A, \quad Z \longleftrightarrow \tilde{Z} \tag{2.15}$$

with all other generators fixed. Two $\mathcal{N} = (2,2)$ susy QFTs are **mirror** to each other if they are equivalent as QFTs where the Hilbert state space isomorphism maps the generators of the $\mathcal{N} = (2,2)$ susy algebra according to (2.15). Hence, for example, a chiral multiplet of one theory is mapped to a twisted chiral multiplet of the mirror (c.f. the defining equations of chiral and twisted chiral multiplets in 2.4.10). (There are, of course, other automorphisms, e.g. swap Q_+ with Q_-, but they are not physically interesting.)

2.4.3 Non-linear Sigma Models and Landau–Ginzburg Models

Definition 2.4.12. Consider n chiral multiplets $\Phi^1, \dots, \Phi^n$ on the worldsheet Σ. We can interpret this as a theory $\Sigma \to M$ for an n-fold M (after appropriate gluing). Replace the D-term $\overline{\Phi}\Phi$ of the chiral superfield theory with a general real function $K(\Phi^i, \overline{\Phi}^i)$, and write

$$\begin{aligned}\mathcal{L}_{\text{kin}} := \int d^4\theta\, K(\Phi^i, \overline{\Phi}^i) = -g_{i\bar{j}}\partial^\mu \phi^i \partial_\mu \overline{\phi}^{\bar{j}} + i g_{i\bar{j}} \overline{\psi}_-^{\bar{j}} (D_0 + D_1)\psi_-^i + i g_{i\bar{j}} \overline{\psi}_+^{\bar{j}} (D_0 - D_1)\psi_+^i \\ + R_{i\bar{j}k\bar{l}} \psi_+^i \psi_-^k \overline{\psi}_-^{\bar{j}} \overline{\psi}_+^{\bar{l}} + g_{i\bar{j}}(F^i - \Gamma^i_{jk}\psi_+^j \psi_-^k)(\overline{F}^{\bar{j}} - \Gamma^{\bar{j}}_{\bar{k}\bar{l}} \overline{\psi}_-^{\bar{k}} \overline{\psi}_+^{\bar{j}}).\end{aligned} \tag{2.16}$$

To ensure non-degeneracy, assume that $g_{i\bar{j}} := \partial_i \partial_j K(\Phi^i, \overline{\Phi}^i)$ is positive definite. Since it satisfies condition 1 of 1.3.5, it determines a Kähler metric g on M with Christoffel symbols Γ^i_{jk}. (Note that this is really a local construction, but using the invariance of the action under coordinate change and shifts of Kähler potential $K(\Phi^i, \overline{\Phi}^i) \mapsto K(\Phi^i, \overline{\Phi}^i) + f(\Phi^i) + \overline{f}(\overline{\Phi}^i)$, the gluing process is valid.) This is called a **supersymmetric non-linear sigma model**, with fields

$$\phi\colon \Sigma \to M, \quad \psi_\pm \in \Gamma(\Sigma, \phi^* T^{1,0} M \otimes S_\pm), \quad \overline{\psi}_\pm \in \Gamma(\Sigma, \phi^* T^{0,1} M \otimes S_\pm)$$

where $S_\pm$ is the spinor bundle.

We can add an F-term $\mathcal{L}_W := \frac{1}{2}\int d^2\theta\, W(\Phi^i) + \text{c.c.}$ for the superpotential $W(\Phi^i)$, which we view as a holomorphic function on M. We can repeat the process in 2.4.8 to get the explicit form of the Lagrangian density $\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_W$. The components $F^i, \overline{F}^{\bar{i}}$ of the chiral multiplet are again auxiliary fields, and are integrated out to give deltas setting $F^i = \Gamma^i_{jk}\psi_+^j \psi_-^k - \frac{1}{2} g^{i\bar{l}} \partial_{\bar{l}} \overline{W}$. By construction, the theory is $\mathcal{N} = (2,2)$ supersymmetric, with supercharges $Q_\pm, \overline{Q}_\pm$.

Proposition 2.4.13. *Let $U(1)_V$ and $U(1)_A$ denote the vector and axial R-rotations 2.4.7 of the classical non-linear sigma model with superpotential W. Then $U(1)_A$ is always a symmetry, but $U(1)_V$ is only if W is* **quasi-homogeneous***, i.e. $W(\lambda^{q^i}\Phi^i) = \lambda^2 W(\Phi^i)$ for some q^i.*

Proof. See the discussion at the end of 2.4.8. ☐

Definition 2.4.14. From the proof of 2.1.1, we see that a classical symmetry $\delta S = 0$ gives a symmetry of the resulting quantized theory $\langle \delta \mathcal{O} \rangle = \int \mathcal{D}X\, e^{iS}\mathcal{O} = 0$ iff the measure $\mathcal{D}X$ is also invariant under δ. When it is not, i.e. $\delta \mathcal{D}X \neq 0$, the symmetry δ is **anomalous**.

Proposition 2.4.15. *Let $U(1)_V$ and $U(1)_A$ denote the vector and axial R-rotations 2.4.7 of the quantized non-linear sigma model without superpotential, i.e. $W = 0$. Then $U(1)_V$ is never anomalous, but $U(1)_A$ is if $c_1(M) \neq 0$, in which case the $U(1)_A$ symmetry is broken, but its $\mathbb{Z}/2k\mathbb{Z}$ subgroup generated by $e^{2\pi i/2k}$ remains a symmetry, where $k = \int_\Sigma \phi^* c_1(T^{1,0}M)$.*

Proof. Note that the vector and axial R-rotations act on the Dirac fermions $\psi^i_\pm$ only, so it suffices to look only at the fermions, and only one pair $\psi_\pm$ instead of n of them. From the general form (2.16), the fermionic part of the action in this special case is just

$$S = \int d^2 z (i\overline{\psi}_+ D_z \psi_+ + i\overline{\psi}_- D_{\bar{z}} \psi_-)$$

where $D_z, D_{\bar{z}}$ is the covariant derivative (wrt the Levi–Civita connection on $\phi^* TM^{1,0}$, where the spinor ψ lives).

A standard procedure now is to look at the spectral decomposition of $D_{\bar{z}}^\dagger D_{\bar{z}}$ and $D_z^\dagger D_z$. The Atiyah–Singer index theorem tells us that

$$\dim\ker D_{\bar{z}} - \dim\ker D_z = \int_\Sigma \phi^* c_1(T^{1,0}M) = k,$$

so there are k more zero modes of $D_{\bar{z}}^{\dagger}D_{\bar{z}}$ than $D_z^{\dagger}D_z$. (If $k<0$, flip the roles of $D_{\bar{z}}$ and D_z.) Let $\varphi_{\pm}^n, \overline{\varphi}_{\pm}^n$ be the nonzero modes with eigenvalues λ_n, and $\varphi_-^{0\alpha}, \overline{\varphi}_+^{0\alpha}$ be the zero modes. Then

$$\begin{aligned}
\overline{\psi}_- &= \sum_{n=1}^{\infty} b_n \overline{\varphi}_-^n, \quad \psi_- = \sum_{\alpha=1}^{k} c_{0\alpha}\varphi_-^{0\alpha} + \sum_{n=1}^{\infty} c_n \varphi_-^n \\
\psi_+ &= \sum_{n=1}^{\infty} \tilde{b}_n \varphi_+^n, \quad \overline{\psi}_- = \sum_{\alpha=1}^{k} \tilde{c}_{0\alpha}\overline{\varphi}_+^{0\alpha} + \sum_{n=1}^{\infty} \tilde{c}_n \overline{\varphi}_+^n
\end{aligned} \tag{2.17}$$

$$\mathcal{D}\psi\mathcal{D}\overline{\psi}e^{-S} = \prod_{\alpha=1}^{k} dc_{0\alpha}d\tilde{c}_{0\alpha} \prod_{n=1}^{\infty} db_n dc_n d\tilde{b}_n d\tilde{c}_n \exp\left(-\sum_{n=1}^{\infty}\lambda_n(b_n c_n + \tilde{c}_n\tilde{b}_n)\right).$$

(We did the same kind of procedure much more explicitly in section 4.1.2 of the string theory notes, where we computed the Faddeev–Popov determinant.) Now we see that $db_n dc_n d\tilde{b}_n d\tilde{c}_n$ is both $U(1)_V$ and $U(1)_A$ invariant, but $dc_{0\alpha}d\tilde{c}_{0\alpha}$ has vector charge 0 and axial charge 2, breaking the $U(1)_A$ symmetry unless $k=0$. In total we get an anomalous axial charge of $2k$. $\square$

Compactify the spatial direction x^1, so that $\Sigma = \mathbb{R}\times S^1$ and impose periodic (as opposed to anti-periodic) boundary conditions for all fields. To find the susy ground states of the non-linear sigma model, we use a clever trick. If we let $Q \coloneqq \overline{Q}_+ + Q_-$, then using the susy algebra relations (2.14), we get $\{Q,\overline{Q}\} = 2H$ and $Q^2 = \overline{Q}^2 = 0$, precisely the relations 2.2.3 defining a supersymmetric QM system. The insight is that

$$\begin{aligned}
Q &= -i\int_{S^1} dx^1\, ig_{i\bar{j}}\overline{\psi}_+^{\bar{j}}(\partial_0+\partial_1)\phi^i + ig_{i\bar{j}}\psi_-^i(\partial_0-\partial_1)\overline{\phi}^{\bar{j}} - \frac{1}{2}\psi_-^i\partial_i W - \frac{1}{2}\overline{\psi}_+^{\bar{i}}\bar{\partial}_{\bar{i}}\overline{W} \\
&= \int_{S^1} dx^1\, \overline{\psi}^I(x^1)\left(ig_{IJ}(x^1)\partial_0\phi^J(x^1) + \frac{\delta h}{\delta\phi^I(x^1)}\right), \quad \overline{\psi}^i \coloneqq -i\psi_-^I,\ \overline{\psi}^{\bar{j}} = -i\overline{\psi}_+^{\bar{j}}
\end{aligned}$$

if there exists a functional $h[\phi(x^1)]$ satisfying

$$\frac{\delta h}{\delta\phi^i} = -ig_{i\bar{j}}\partial_1\overline{\phi}^{\bar{j}} - \frac{1}{2}\partial_i W, \quad \frac{\delta h}{\delta\overline{\phi}^{\bar{j}}} = ig_{i\bar{j}}\partial_1\phi^i - \frac{1}{2}\partial_{\bar{j}}W, \tag{2.18}$$

and that Q in this form is exactly the supercharge (2.11) for a $d=1$ SQM deformed by a potential h, with target space $LM \coloneqq \{\phi\colon S^1\to M\}$, i.e. the **loop space** of M. Hence if we can find such a functional h, every result we had in section 2.3 holds for our $d=1+1$ non-linear sigma model.

Proposition 2.4.16. *Fix a* **base loop** *in each connected component of the loop space LM. Given a loop ϕ, let ϕ_0 be the base loop in its connected component, and let $\hat{\phi}$ be the homotopy connecting ϕ_0 to ϕ. The desired functional h is then given by*

$$h[\phi] \coloneqq \int_{S^1\times[0,1]}\hat{\phi}^*\omega - \int_{S^1} dx^1\,\mathrm{Re}(W[\phi^i])$$

with ω the Kähler form of M.

Proof. For a variation δ of $\hat{\phi}$, we have

$$\begin{aligned}
\delta\int_{S^1\times[0,1]}\hat{\phi}^*\omega &= \int_{S^1\times[0,1]} d\left(ig_{i\bar{j}}\left(\delta\hat{\phi}^i d\overline{\hat{\phi}^{\bar{j}}} - d\hat{\phi}^i\delta\overline{\hat{\phi}^{\bar{j}}}\right)\right) \\
&= \int_{S^1} ig_{i\bar{j}}\left(\delta\hat{\phi}^i d\overline{\hat{\phi}^{\bar{j}}} - d\hat{\phi}^i\delta\overline{\hat{\phi}^{\bar{j}}}\right)\Big|_{\tau=0}^{\tau=1} = \int_{S^1} ig_{i\bar{j}}\left(\delta\phi^i d\overline{\phi}^{\bar{j}} - d\phi^i\delta\overline{\phi}^{\bar{j}}\right)
\end{aligned}$$

where the last equality uses that $\hat{\phi}|_{\tau=0} = \phi_0$, so $\delta\hat{\phi}|_{\tau=0} = 0$. Hence the variation of the first term in h produces the first terms in (2.18). The variation of the second term $\int_{S^1} dx^1\,\mathrm{Re}(W[\phi^i])$ of h clearly produces the desired second terms. $\square$

Corollary 2.4.17. *For the non-linear sigma model on a compact connected Kähler manifold M and superpotential $W = 0$, the ground states are in one-to-one correspondence with harmonic forms on M.*

Proof. Using the deformation invariance of the potential h, we can rescale $h \mapsto \lambda h$, under which the ground states are localized around critical points of h, i.e. ϕ such that

$$\frac{\delta h}{\delta \phi^i} = -i g_{i\bar{j}} \partial_i \overline{\phi}^{\bar{j}} = 0, \quad \frac{\delta h}{\delta \overline{\phi}^{\bar{j}}} = -i g_{i\bar{j}} \partial_i \overline{\phi}^{i} = 0.$$

Clearly these are the trivial loops $\phi\colon S^1 \to \{\text{pt}\} \in M$, so the set of critical points is isomorphic to M itself. Now apply equation (2.12) to see that perturbative ground states are harmonic forms on M. (For details, see page 303 in Hori.)

It remains to verify that there are no instanton corrections to these perturbative ground states. A short calculation (see page 304 in Hori) shows that all critical points have the same Morse index, and therefore there cannot be any instantons. □

2.5 Renormalization

Chapter 3

Gromov–Witten Theory

3.1 Setting

The main objects of study in Gromov–Witten theory are the moduli spaces $\overline{\mathcal{M}}_{g,n}(X,\beta)$ of stable maps. They are the result of starting with the moduli space of Riemann surfaces $\mathcal{M}_g$ and enriching the problem by adding more data. In particular, since $\mathcal{M}_g$ is not compact, we shall compactify it by adding nodal curves, which is where we begin.

3.1.1 Nodal Curves

Definition 3.1.1. A **nodal point** of a curve is a point that is locally analytically given by $xy = 0$ (in $\mathbb{C}^2$).

Definition 3.1.2. A domain A is **normal** if it is integrally closed in its fraction field. A variety X is **normal** if $\mathcal{O}(X)_p$ is normal for every point p. The **normalization** of X is a normal variety $\tilde{X}$ with a **normalization map** $\nu\colon \tilde{X} \to X$ with the following universal property: for every normal variety Z, every dominant (image is dense) morphism $Z \to X$ factors through $\tilde{X}$.

Proposition 3.1.3 ([7, Exercise II.3.8])**.** *Let X be a variety. For each open affine subset $U \subset X$, let $\tilde{U}$ be the affine variety associated to the integral closure of $\mathbb{C}[U]$ in its quotient field. Then the $\tilde{U}$ glue to form a variety $\tilde{X}$, which is precisely the normalization of X.*

Corollary 3.1.4. *The normalization of a nodal curve is the Riemann surface obtained by "ungluing", or "pulling apart" its nodes.*

Proof. Note that the normalization of $\mathbb{C}[x,y]/(xy)$ is isomorphic to two copies of $\mathbb{C}[x]$, the affine line. Hence locally the normalization pulls the two branches of a node apart, and then we can glue the local normalizations together. □

Definition 3.1.5. Let Σ be a nodal curve and $\nu\colon \tilde{\Sigma} \to \Sigma$ be the normalization map. The preimages in $\tilde{\Sigma}$ of the nodes in Σ are the **node branches**. If $\{\tilde{\Sigma}_i\}$ are the connected components of $\tilde{\Sigma}$, then $\nu(\tilde{\Sigma}_i)$ are the **irreducible components** of Σ.

Proposition 3.1.6. *[7, Exercise IV.1.8] Suppose the node branches of a normalization $\nu\colon \tilde{\Sigma} \to \Sigma$ are $b_1, \ldots, b_{2k}$. Then there is a* **normalization short exact sequence**

$$0 \to \mathcal{O}_\Sigma \to \nu_* \mathcal{O}_{\tilde{\Sigma}} \to \bigoplus_i \mathcal{O}_{b_i} \to 0.$$

Definition 3.1.7. The **dual graph** of a nodal curve Σ has vertices corresponding to the components of Σ (and are labeled with their genera), and edges corresponding to nodes.

Proposition 3.1.8 ([7, Corollary V.5.6])**.** *The* **arithmetic genus** $p_a(\Sigma) \coloneqq 1 - \chi(\Sigma, \mathcal{O}_\Sigma)$ *of a nodal curve* Σ *is the (geometric) genus of a "smoothing" of* Σ*, where each node is replaced with a smooth tube.*

Corollary 3.1.9. *If* Σ *is a curve with* δ *nodes and* $\tilde{\Sigma}$ *has* n *components with genera* $g_1, \dots, g_n$*, then* $p_a(\Sigma) = \sum_i (g_i - 1) + \delta + 1$.

3.1.2 Moduli Spaces of Stable Curves

From this subsection onward, when we talk about the genus of a nodal curve, we mean its arithmetic genus, i.e. the geometric genus of its "smoothing" (c.f. 3.1.8).

Definition 3.1.10. The **moduli space of Riemann surfaces of genus** g is denoted $\mathcal{M}_g$. From Hurwitz's automorphisms theorem we know that the automorphism group of each point in $\mathcal{M}_g$ is finite, and so in fact $\mathcal{M}_g$ is a non-singular **Deligne–Mumford stack** for $g > 1$. From 1.2.36, $\dim \mathcal{M}_g = 3g - 3$.

$\mathcal{M}_g$ is not compact: imagine a sequence of Riemann surfaces degenerating at a point by pinching. The resulting singularity is nodal, and the idea is to compactify $\mathcal{M}_g$ by adding in these degenerate surfaces.

Definition 3.1.11. A (connected) nodal curve is **stable** if

1. every irreducible component of geometric genus 0 has at least three node branches,
2. every irreducible component of geometric genus 1 has at least one node branch.

The **moduli space of stable curves of genus** g is denoted $\overline{\mathcal{M}}_g$.

Proposition 3.1.12. *For a (connected) nodal curve, the condition of being stable is equivalent to the property of having a finite automorphism group.*

Proof. Recall that $\dim \mathrm{Aut}(S^2) = 3$ and $\dim \mathrm{Aut}(T^2) = 1$, and Hurwitz's theorem ensures that higher-genus components have finite automorphism group. It is straightforward to check that marking three (resp. one) points on each S^2 (resp. T^2), i.e. requiring that automorphisms fix those points, leaves a finite automorphism group G. Automorphisms of nodal curves fix node branches. □

Theorem 3.1.13. *[18]* $\overline{\mathcal{M}}_g$ *is a connected, irreducible, compact, non-singular Deligne–Mumford stack of dimension* $3g - 3$*, and is the compactification of* $\mathcal{M}_g$.

We can both generalize and enrich the structure by considering nodal curves with marked points, just like how we look at Riemann surfaces with n marked points in Teichmüller theory.

Definition 3.1.14. An n**-pointed curve** $(\Sigma, p_1, \dots, p_n)$ is a nodal curve Σ with n distinct labeled non-singular **marked points** $p_1, \dots, p_n$. A **special point** of a pointed curve is a point on the normalization that is either a node branch or the preimage of a marked point p_i. In the dual graph, we use n labeled **tails** or **half edges** (generally labeled from 1 to n) to represent the marked points. A pointed curve is **stable** if the combinatorial condition 3.1.11 holds when considering special points in general instead of just node branches. The **moduli space of stable** n**-pointed curves of genus** g is denoted $\overline{\mathcal{M}}_{g,n}$.

Proposition 3.1.15. *There are no stable* n*-pointed genus* g *curves if* $2g - 2 + n \leq 0$.

Proof. Every stable nodal curve (without marked points) is clearly of genus at least 2, so by 3.1.12, we require at least three (resp. one) marked points on genus 0 (resp. genus 1) nodal curves. This rules out $(g, n) = (0,0), (0,1), (0,2), (1,0)$. □

Theorem 3.1.16. *[19]* $\overline{\mathcal{M}}_{g,n}$ *is a connected, irreducible, compact, non-singular Deligne–Mumford stack of dimension* $3g - 3 + n$*, and is the compactification of* $\mathcal{M}_{g,n}$.

Example 3.1.17. Recall that given any triple of distinct points (p_1, p_2, p_3) of $\mathbb{P}^1$, there exists a unique automorphism $\phi \in \mathrm{Aut}(\mathbb{P}^1) \cong \mathrm{PGL}(2)$ sending $(p_1, p_2, p_3) \mapsto (0, 1, \infty)$. If $p = (p_1, p_2, p_3, p_4)$ is a quadruple instead, let $\lambda(p) := \phi(p_4)$, called the **cross ratio** of p because

$$\lambda([1 : x_1], [1 : x_2], [1 : x_3], [1 : x_4]) = \frac{(x_2 - x_3)(x_4 - x_1)}{(x_2 - x_1)(x_4 - x_3)} \in \mathbb{P}^1 \setminus \{0, 1, \infty\}.$$

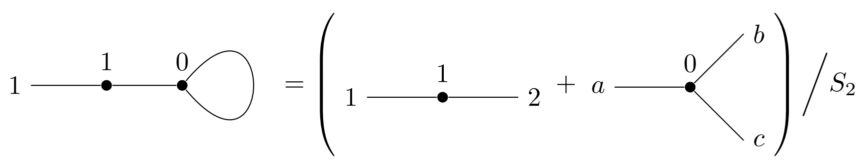
Hence λ bijects $\mathcal{M}_{0,4}$ with $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. Compactifying, $\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$.

Definition 3.1.18. If $n_1 \geq n_2$ and $2g - 2 + n_2 \geq 0$, there is a **forgetful morphism**

$$\overline{\mathcal{M}}_{g,n_1} \to \overline{\mathcal{M}}_{g,n_2}$$

given by removing all but the first n_2 marked points. The resulting curve may not be stable, but the only "destabilizing" components are genus 0, and we can contract those away. (A "destabilizing" genus 1 component cannot appear because it would be a disconnected vertex in the dual graph.) The morphism $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ can be identified with the **universal curve** over $\overline{\mathcal{M}}_{g,n}$. For more detail, see [20, section 1.3].

Definition 3.1.19. There is a **stratification** of $\overline{\mathcal{M}}_{g,n}$: to each stable graph Γ of genus g with n marked points, we associate the subset $\mathcal{M}_\Gamma \subset \overline{\mathcal{M}}_{g,n}$ of all curves with dual graph Γ. From the dual graph picture, we obtain elements of $\mathcal{M}_\Gamma$ by taking each vertex with its edges and gluing the edges together in various ways, e.g.



where here we quotient out by $S_2 = \mathrm{Sym}\,\Gamma$ because given that we glue 2 to a, the labels for b and c could be swapped without affecting the result. In general, $S_\Gamma = (\prod_i \mathcal{M}_{g_i,n_i}) / \mathrm{Sym}\,\Gamma$ if the i-th vertex of Γ has genus g_i and n_i edges.

Proposition 3.1.20. *The codimension of the stratum defined by S_Γ in $\overline{\mathcal{M}}_{g,n}$ is precisely the number of edges in Γ, i.e. the number of nodes in a generic curve in S_Γ.*

Proof. Let Γ have m vertices of genera $g_1, \ldots, g_m$ and δ edges. By 3.1.9, S_Γ lives in $\overline{\mathcal{M}}_{g,n}$ where $g = \sum_i (g_i - 1) + \delta + 1$, so

$$\dim \overline{\mathcal{M}}_{g,n} = 3g - 3 + n = 3 \sum_i (g_i - 1) + 3\delta + n.$$

Every node comes from gluing two marked points together, so in $S_\Gamma = (\prod_i \mathcal{M}_{g_i,n_i}) / \mathrm{Sym}\,\Gamma$ we require an extra 2δ marked points in addition to the n marked points. By 3.1.16 we get

$$\dim S_\Gamma = \sum_i (3g_i - 3) + (n + 2\delta).$$

Subtracting gives $\mathrm{codim}_{\overline{\mathcal{M}}_{g,n}} S_\gamma = \delta$. ◻

Definition 3.1.21. The closure of each stratum is a subvariety of $\overline{\mathcal{M}}_{g,n}$, called a **boundary cycle**. Boundary cycles of codimension 1, i.e. nodal curves with one node, are called **boundary divisors**. If $A = \{1, \ldots, n\}$ are the marked points, then for $A = A_1 \sqcup A_2$ and $g_1 + g_2 = g$, let $D(g_1, A_1 | g_2, A_2)$ denote the boundary divisor with one component of genus g_1 containing the marked points A_1, and another component of genus g_2 containing the marked points A_2. In genus 0, we often omit the genus and write $D(A_1 | A_2) := D(0, A_1 | 0, A_2)$.

Proposition 3.1.22. *Every boundary cycle is naturally isomorphic to a product of $\overline{\mathcal{M}}_{g,n}$ of lower dimensions. In particular, every boundary cycle is smooth and irreducible.*

Proof. For boundary divisors $D(g_1, A_1|g_2, A_2)$, it suffices to normalize at a node with node branch labeled x, to create a curve of genus g_1 with marked points $A_1 \cup \{x\}$ and another curve of genus g_2 with marked points $A_2 \cup \{x\}$, and so $D(g_1, A_1|g_2, A_2) \cong \overline{\mathcal{M}}_{g_1,A_1\cup\{x\}} \times \overline{\mathcal{M}}_{g_2,A_2\cup\{x\}}$. Smoothness and irreducibility follow by induction on the dimension of $\overline{\mathcal{M}}_{g,n}$. The same argument applies to boundary cycles in general. $\square$

Example 3.1.23. Let $\epsilon\colon \overline{\mathcal{M}}_{0,n+1} \to \overline{\mathcal{M}}_{0,n}$ be the forgetful morphism, which forgets the last marked point x. If $D(A|B)$ is a boundary divisor of $\overline{\mathcal{M}}_{0,n}$, then

$$\epsilon^* D(A|B) = D(A \cup \{x\}|B) + D(A|B \cup \{x\})$$

since the extra marked point x must come from somewhere. (The coefficients of 1 come from the geometric fibers being reduced.) In general, $\epsilon\colon \overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,4}$ satisfies

$$\epsilon^* D(ij|kl) = \sum_{i,j\in A,\, k,l\in B} D(A|B)$$

for marked points $\{i, j, k, l\}$ of $\overline{\mathcal{M}}_{0,4}$. But $\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$, so boundary divisors are just points on $\mathbb{P}^1$, and are therefore all linearly equivalent. In particular, fixing $\{i, j, k, l\}$, we have $D(ij|kl) \sim D(ik|jl) \sim D(il|jk)$. Under pullback to $\overline{\mathcal{M}}_{0,n}$, which preserves linear equivalence, we get the $\overline{\mathcal{M}}_{0,n}$ **divisor relations**

$$\sum_{i,j\in A,\, k,l\in B} D(A|B) \;\sim \sum_{i,k\in A,\, j,l\in B} D(A|B) \;\sim \sum_{i,l\in A,\, j,k\in B} D(A|B). \tag{3.1}$$

(Here A and B are varying, not $\{i, j, k, l\} \subset \{1, \dots, n\}$, which is fixed.) These relations are crucial to the associativity of the product structure on the big quantum cohomology ring.

3.1.3 Moduli Spaces of Stable Maps

Throughout this subsection, X is a non-singular projective variety. We generally use Σ to denote a pointed nodal curve.

Definition 3.1.24. Two morphisms $f\colon \Sigma \to X$ and $f'\colon \Sigma' \to X$ from pointed nodal curves $(\Sigma, p_1, \dots, p_n)$ and $(\Sigma', p'_1, \dots, p'_n)$ to X are **isomorphic** when there is an isomorphism $\tau\colon \Sigma \to \Sigma'$ with $\tau(p_i) = p'_i$ and $f' \circ \tau = f$. A morphism $f\colon \Sigma \to X$ is **stable** if the combinatorial condition 3.1.11 holds when considering special points on **contracted components** of Σ, i.e. components that map to a single point of X under f, instead of just node branches on every irreducible component. As usual, stability here is equivalent to having a finite automorphism group.

Definition 3.1.25. A stable map $f\colon \Sigma \to X$ **represents** a homology class $\beta \in H_2(X, \mathbb{Z})$ if $f_*[\Sigma] = \beta$. The **moduli space of stable maps** from n-pointed genus g nodal curves to X representing the class β is denoted $\overline{\mathcal{M}}_{g,n}(X, \beta)$. The subscript n may be omitted if $n = 0$. If $X \cong \mathbb{P}^r$, then we write $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ where here d means xd with x the preferred hyperplane generator of $H_2(\mathbb{P}^r, \mathbb{Z}) \cong \mathbb{Z}$ (c.f. 1.2.30).

Example 3.1.26. Since $f_*[\Sigma] = 0$ iff f maps everything to a point, $\overline{\mathcal{M}}_{g,n}(X, 0) \cong \overline{\mathcal{M}}_{g,n} \times X$. In particular, if $X = \{\mathrm{pt}\}$, then we recover the moduli space of stable curves. Slightly less trivially, $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^m, 1) = \mathrm{Gr}(1, m)$, parameterizing lines in $\mathbb{P}^m$.

Example 3.1.27. Consider $\overline{\mathcal{M}}_{1,0}(\mathbb{P}^2, 3)$. It has one component consisting generically of non-singular cubic curves in $\mathbb{P}^2$. The dimension of rational cubics in $\mathbb{P}^2$ is 8, but there is one more degree of freedom corresponding to the moduli of the contracted genus 1 component, so this component is of dimension 9. However there is another component consisting generically of reducible curves with a genus zero component and a genus one component, and now there is an extra degree of freedom corresponding to the point of attachment between the two components, so this component is of dimension 10. This example shows that $\overline{\mathcal{M}}_{g,n}(X,\beta)$ in general has **impure dimension**.

Definition 3.1.28. For each marked point p_i, there is an **evaluation map**

$$\mathrm{ev}_i \colon \overline{\mathcal{M}}_{g,n}(X,\beta) \to X, \quad (\Sigma, p_1, \ldots, p_n, f) \mapsto f(p_i).$$

If $n_1 \geq n_2$ and both spaces exist, there is a **forgetful morphism**

$$\overline{\mathcal{M}}_{g,n_1}(X,\beta) \to \overline{\mathcal{M}}_{g,n_2}(X,\beta).$$

Finally, given a morphism $g \colon X \to Y$, there is an **induced morphism**

$$\overline{\mathcal{M}}_{g,n}(X,\beta) \to \overline{\mathcal{M}}_{g,n}(Y, g_*\beta).$$

When $Y = \{\mathrm{pt}\}$ this is the structure morphism to $\overline{\mathcal{M}}_{g,n}$.

Definition 3.1.29. Let $f \colon \Sigma \to X$ be an immersion of a non-singular curve into a non-singular variety. The short exact sequence $0 \to T\Sigma \to f^*TX \to N\Sigma \to 0$ gives the **deformation long exact sequence**

$$0 \to H^0(\Sigma, T\Sigma) \to H^0(\Sigma, f^*TX) \to H^0(\Sigma, N\Sigma) \to H^1(\Sigma, T\Sigma) \to H^1(\Sigma, f^*TX) \to H^1(\Sigma, N\Sigma) \to 0$$

since by Grothendieck's vanishing criterion 1.2.35 we have $H^2(\Sigma, T\Sigma) = 0$.

1. Interpret $H^0(\Sigma, T\Sigma)$ as infinitesimal automorphisms of Σ, and $H^1(\Sigma, T\Sigma)$ as infinitesimal deformations of Σ, and $H^2(\Sigma, T\Sigma) = 0$ as obstructions to such deformations.
2. Interpret $H^0(\Sigma, N\Sigma)$ as infinitesimal deformations of the map f, and $H^1(\Sigma, N\Sigma)$ as obstructions to such deformations.
3. Interpret $H^0(\Sigma, f^*TX)$ as infinitesimal deformations of f where the structure of Σ itself is held fixed, and $H^1(\Sigma, f^*TX)$ as obstructions to such deformations.

Hence it is common to write the deformation long exact sequence as

$$0 \to \mathrm{Aut}(\Sigma) \to \mathrm{Def}(f) \to \mathrm{Def}(\Sigma, f) \to \mathrm{Def}(\Sigma) \to \mathrm{Ob}(f) \to \mathrm{Ob}(\Sigma, f) \to 0.$$

In general, for a map f from pointed nodal curves, we must add a term in front since there may be non-trivial automorphisms of $(\Sigma, p_1, \ldots, p_n, f)$ (which is zero iff f is stable):

$$\begin{aligned} 0 \to \mathrm{Aut}(\Sigma, p_1, \ldots, p_n, f) &\to \mathrm{Aut}(\Sigma, p_1, \ldots, p_n) \to \\ \mathrm{Def}(f) \to \mathrm{Def}(\Sigma, p_1, \ldots, p_n, f) &\to \mathrm{Def}(\Sigma, p_1, \ldots, p_n) \to \\ \mathrm{Ob}(f) \to \mathrm{Ob}(\Sigma, p_1, \ldots, p_n, f) &\to 0. \end{aligned} \tag{3.2}$$

(See Hori Remark 24.4.1 for a cohomological description of these terms for pointed nodal curves in general.)

Proposition 3.1.30. *If $\mathrm{Ob}(\Sigma, p_1, \ldots, p_n, f) = 0$, then the dimension of $\overline{\mathcal{M}}_{0,n}(X,\beta)$ at a generic point, called the* **expected dimension** *or* **virtual dimension**, *is*

$$\mathrm{vdim}\, \overline{\mathcal{M}}_{g,n}(X,\beta) \coloneqq \int_\beta c_1(TX) + \dim X + n - 3.$$

Proof. Since we are dealing with stable maps, the first term in the deformation long exact sequence (3.2) vanishes. By hypothesis, the last term also vanishes. So

$$\begin{aligned} \dim \overline{\mathcal{M}}_{0,n}(X,\beta) &= \dim \mathrm{Def}(\Sigma, p_1, \ldots, p_n, f) \\ &= -\dim \mathrm{Aut}(\Sigma, p_1, \ldots, p_n) + h^0(\Sigma, f^*TX) + \dim \mathrm{Def}(\Sigma, p_1, \ldots, p_n) - h^1(\Sigma, f^*TX). \end{aligned}$$

By Hirzebruch–Riemann–Roch (c.f. 1.2.34 and the calculation in 1.2.37),

$$h^0(\Sigma, f^*TX) - h^1(\Sigma, f^*TX) = \int_\Sigma (\dim X + f^*c_1(TX))(1 + (1/2)c_1(\Sigma)) = \int_\beta c_1(TX) + (\dim X)(1-g).$$

Similarly, $\dim \mathrm{Def}(\Sigma, p_1, \dots, p_n) - \dim \mathrm{Aut}(\Sigma, p_1, \dots, p_n) = 3g - 3 + n$. Adding the two, we are done. $\square$

Definition 3.1.31. The target space X is **convex** if $h^1(\Sigma, f^*TX) = 0$ for every genus 0 stable map $f\colon \Sigma \to X$. Consequently, convex spaces have $\overline{\mathcal{M}}_{g,n}(X, \beta)$ of expected dimension.

Proposition 3.1.32. *If $f\colon \Sigma \to X$ is a genus 0 stable map and f^*TX is generated by global sections, then X is convex.*

Proof. Sheaf cohomology is the derived functor of the global sections functor. $\square$

Corollary 3.1.33. *All algebraic homogeneous spaces are convex; in particular, projective spaces are convex. They therefore have $\overline{\mathcal{M}}_{0,n}(X, \beta)$ of the expected dimension.*

3.2 Gromov–Witten Invariants

3.2.1 Enumerative Geometry via Stable Maps

Definition 3.2.1. Let N_d denote the number of **rational plane curves of degree** d that pass through $3d-1$ given points in general position. (For us, rational is equivalent to genus zero.)

Example 3.2.2. Clearly through any two distinct points on the plane there is a unique line, so $N_1 = 1$. It is also well-known that five generic points on the plane suffice to uniquely specify a conic, so $N_2 = 1$. Classically, it was also known that $N_3 = 12$.

Remark. Why $3d-1$ points? From 1.2.31, we get that the genus for a nodal plane curve of degree d is $g = \binom{d-1}{2} - \delta$ where δ is the number of nodes. Hence to get rational curves, we require $\delta = \binom{d-1}{2}$ nodes. The space of irreducible degree d curves is $\mathbb{P}\Gamma(\mathcal{O}_{\mathbb{P}^2}(d))$, which has dimension $\binom{d+2}{2} - 1 = d(d+3)/2$, but each node is a codimension 1 condition, so the space of rational plane curves of degree d is dimension $d(d+3)/2 - \binom{d-1}{2} = 3d-1$, and to get a finite number of curves we must specify $3d-1$ more conditions.

Remark. We would like to work with $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, d)$, i.e. stable maps with n marked points, instead of rational plane curves passing through n points in $\mathbb{P}^2$. There are some subtleties that need to be addressed, e.g. it could be that a rational plane curve f passes through a point $p \in \mathbb{P}^2$ twice, and so there is more than one way to mark the original curve in order to obtain f. Thankfully it turns out these subtleties never occur and are not issues; a proof of this statement is in [20, Section 3.5]. From this perspective, it is clear we need to specify $3d-1$ points to get a finite number of curves: $\dim \overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, d) = 3d + 2 - 3 = 3d - 1$ by 3.1.30.

Theorem 3.2.3 (Kontsevich)**.** *The following recursive relation holds:*

$$N_d = \sum_{\substack{d_A + d_B = d \\ d_A, d_B > 0}} N_{d_A} N_{d_B} \left(d_A^2 d_B^2 \binom{3d-4}{3d_A - 2} - d_A^3 d_B \binom{3d-4}{3d_A - 1} \right).$$

Proof. Let $n = 3d$, and consider $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, d)$ with marked points $\{p_1, \dots, p_{n-2}, \ell_1, \ell_2\}$. Let $P_1, \dots, P_{n-2}$ be general points and L_1, L_2 be general lines in $\mathbb{P}^2$. Define

$$Y \coloneqq \mathrm{ev}_1^{-1}(P_1) \cap \cdots \cap \mathrm{ev}_{n-2}^{-1}(P_{n-2}) \cap \mathrm{ev}_{n-1}^{-1}(L_1) \cap \mathrm{ev}_n^{-1}(L_2) \subset \overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, d),$$

By the Kleiman–Bertini transversality theorem [20, Theorem 3.4.2], we can always choose the P_i and L_1, L_2 such that Y is a curve intersecting the boundary of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, d)$ transversally. Hence by analogy with (or pullback from) (3.1),

$$Y \cap \sum_{\substack{\ell_1,\ell_2 \in A, \\ p_1,p_2 \in B, \\ d_A+d_B=d}} D(A, d_A|B, d_B) \sim Y \cap \sum_{\substack{\ell_1,p_1 \in A, \\ \ell_2,p_2 \in B, \\ d_A+d_B=d}} D(A, d_A|B, d_B).$$

We shall do some casework to count the number of points on both sides. Equating the resulting expressions will give Kontsevich's recursion.

1. Suppose $d_A = 0$. Degree 0 maps are constant, but the stability criteria 3.1.24 then requires at least three special points on the component Σ_A. There is a node branch on Σ_A already, so we need to put at least two marked points on Σ_A. But maps in Y take distinct marked points to distinct points on $\mathbb{P}^2$, so the only way $Y \cap D(A, 0|B, d)$ is non-empty is for $A = \{\ell_1, \ell_2\}$, with the component Σ_A mapping to the point $L_1 \cap L_2$. The remaining $3d-2$ markings on Σ_B must therefore be mapped to the $3d-2$ points $p_1, \ldots, p_{n-2}$, so
$$\#(Y \cap D(\{\ell_1, \ell_2\}, 0|\{p_1, \ldots, p_{n-2}\}, d)) = N_d.$$
Terms of this form can only occur on the lhs, where $\ell_1, \ell_2 \in A$. So terms with $d_B = 0$ are zero.

2. Consider the lhs and suppose $1 \leq d_A \leq d-1$. Note that the degree d_A curve Σ_A can only be required to pass through at most $3d_A - 1$ points, and similarly for the degree d_B curve Σ_B. But $\Sigma_A \cup \Sigma_B$ must pass through the $n-2 = (3d_1 - 1) + (3d_2 - 1)$ points $P_1, \ldots, P_{n-2}$, so we must have $|A| = 3d_A - 1 + 2$ and $|B| = 3d_B - 1$. (The extra 2 points on Σ_A are ℓ_1, ℓ_2, which are only required to map to the lines L_1, L_2.) Since
$$\#\{\ell_1, \ell_2 \in A,\, p_1, p_2 \in B,\, |A| = 3d_A + 1\} = \binom{3d-4}{3d_A - 1},$$
it suffices now to count the number of points in $Y \cap D(A, d_A|B, d_B)$ for a fixed set A.
   (a) There are N_{d_A} and N_{d_B} choices for the images $f(\Sigma_A)$ and $f(\Sigma_B)$ respectively.
   (b) The mark ℓ_1 must map to a point on $f(\Sigma_A) \cap L_1$, which by Bézout's theorem contains d_A points, and similarly for ℓ_2.
   (c) Analogously, there are $d_A d_B$ choices for the image of the intersection point $\Sigma_A \cap \Sigma_B$, which must lie in $f(\Sigma_A) \cap f(\Sigma_B)$.

3. The analogous argument for the rhs gives that $|A| = 3d_A$ are the only non-zero terms. There are $\binom{3d-4}{3d_A-2}$ such partitions, and for each partition there are $N_{d_A} N_{d_B} d_A^2 d_B^2$ points in $Y \cap D(A, d_A|B, d_B)$.

Equating the total for the lhs and the total for the rhs, we get

$$N_d + \sum_{\substack{d_A+d_B=d \\ d_A,d_B>0}} N_{d_A} N_{d_B} d_A^3 d_B \binom{3d-4}{3d_A - 1} = \sum_{\substack{d_A+d_B=d \\ d_A,d_B>0}} N_{d_A} N_{d_B} d_A^2 d_B^2 \binom{3d-4}{3d_A - 2}. \qquad \square$$

3.2.2 Gromov–Witten and Descendant Invariants

There is a different way to interpret the numbers N_d which indicates the correct way to generalize them:

$$N_d = \int_{\overline{\mathcal{M}}_{0,3d-1}(\mathbb{P}^2,d)} \mathrm{ev}_1^*(P) \cup \mathrm{ev}_2^*(P) \cup \cdots \cup \mathrm{ev}_{3d-1}^*(P)$$

where P is Poincaré dual to the point class. Then we can ask the obvious question: what if we replace the point class with some other cohomology classes, and do this for an arbitrary $\overline{\mathcal{M}}_{g,n}(X, \beta)$?

The problem is that from 3.1.27 we know $\overline{\mathcal{M}}_{g,n}(X,\beta)$ has impure dimension in general, so we need to be very careful about what $\int_{\overline{\mathcal{M}}_{g,n}(X,\beta)}$ means. Instead of trying to define the integral, we shall instead find a **virtual fundamental class** $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}$ in the expected dimension of $H_*(\overline{\mathcal{M}}_{g,n}(X,\beta),\mathbb{Q})$, pairing with which gives the desired integral. Of course, if the fundamental class $[\overline{\mathcal{M}}_{g,n}(X,\beta)]$ is well-defined, then we also want $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}} = [\overline{\mathcal{M}}_{g,n}(X,\beta)]$. For now, we assume that such a virtual fundamental class exists. We shall outline its construction in the next subsection.

Definition 3.2.4. At each point $(\Sigma, p_1, \ldots, p_n, f)$ of $\overline{\mathcal{M}}_{g,n}(X,\beta)$, the cotangent space to Σ at p_i is a one-dimensional $\mathbb{C}$-vector space. These spaces patch together to give the **i-th tautological line bundle** $\mathbb{L}_i$ over $\overline{\mathcal{M}}_{g,n}(X,\beta)$. Formally, if ω_π is the relative dualizing sheaf of the forgetful morphism $\pi\colon \overline{\mathcal{M}}_{g,n+1}(X,\beta) \to \overline{\mathcal{M}}_{g,n}(X,\beta)$ and σ_i is the section corresponding to the mark p_i, then the i-th tautological line bundle is just $\mathbb{L}_i \coloneqq \sigma_i^*\omega_\pi$.

Definition 3.2.5. Define the **ψ-classes** $\psi_i \coloneqq c_1(\sigma_i^*\omega_\pi)$. Let $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}$ be the virtual fundamental class. Given classes $\gamma_1, \ldots, \gamma_n \in H^*(X)$, the corresponding **(gravitational) descendant invariant** is

$$\langle \tau_{a_1}(\gamma_1)\cdots\tau_{a_n}(\gamma_n)\rangle_{g,\beta}^X \coloneqq \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\gamma_1) \cup \psi_1^{a_1} \cup \cdots \cup \mathrm{ev}_n^*(\gamma_n) \cup \psi_n^{a_n}.$$

When all the a_i are zero, we call $\langle \gamma_1 \cdots \gamma_n\rangle_{g,\beta}^X \coloneqq \langle \tau_0(\gamma_1)\cdots\tau_0(\gamma_n)\rangle_{g,\beta}^X$ the **Gromov–Witten invariant** corresponding to $\gamma_1, \ldots, \gamma_n$.

Notation: from this point onward, we work in the Chow ring $A^*(\overline{\mathcal{M}}_{g,n}(X,\beta))$ (for various n) unless otherwise indicated. Since elements of A^i are codimension-i cycles, we will not distinguish between boundary divisors $D(g_1, A, \beta_1|g_2, B, \beta_2)$ and their images in the Chow ring.

Lemma 3.2.6 (Comparison lemma for convex X)**.** *The ψ-classes do not commute with forgetful morphisms $\pi\colon \overline{\mathcal{M}}_{g,n+1}(X,\beta) \to \overline{\mathcal{M}}_{g,n}(X,\beta)$, but satisfy*

$$\psi_i - \pi^*\psi_i = D_{i,n+1} \coloneqq D(\{i, n+1\}, 0|\{1, \ldots, \hat{i}, \ldots, n\}, \beta) \in A^1(\overline{\mathcal{M}}_{g,n+1}(X,\beta)). \tag{3.3}$$

Proof. We prove this for $X = \{\mathrm{pt}\}$; the general case uses the same idea but requires more technical analysis. (To avoid technicalities with the virtual fundamental class, we required X to be convex, c.f. 3.1.31.) On $\overline{\mathcal{M}}_{g,n+1} \setminus D_{i,n+1}$, there is no problem (with respect to $\mathbb{L}_i$) with forgetting the marked point p_{n+1}; the resulting components are stable. So $\pi^*\mathbb{L}_i = \mathbb{L}_i$ except on $D_{i,n+1}$, i.e. $\mathbb{L}_i \cong \pi^*\mathbb{L}_i \otimes \mathcal{O}(rD_{i,n+1})$ for some $r \in \mathbb{Z}$. But pulling back along $\sigma_i\colon \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n+1}$ shows that

$$\mathcal{O} = \sigma_i^*\mathbb{L}_i = \sigma_i^*\pi^*\mathbb{L}_i \otimes \sigma_i^*\mathcal{O}(D_{i,n+1})^{\otimes r} = \mathbb{L}_i \otimes (\mathbb{L}_i^*)^{\otimes r}.$$

Hence $r = 1$. Taking cohomology, we are done. □

Corollary 3.2.7. *In the Chow ring $A^*(\overline{\mathcal{M}}_{g,n+1}(X,\beta))$,*

$$D_{i,n+1} \cdot D_{j,n+1} = \begin{cases} -\pi^*\psi_i \cdot D_{i,n+1} & i = j \\ 0 & i \neq j. \end{cases} \tag{3.4}$$

It follows by induction that

$$\psi_i^a = \pi^*\psi_i^a + \pi^*\psi_i^{a-1} D_{i,n+1}. \tag{3.5}$$

Proof. First note that $\sigma_i^*\psi_i = 0$, since the i-th marked point on the section σ_i is on a genus zero component with three marks, on which $\psi_i = 0$. But σ_i^* is equivalent to intersecting with $D_{i,n+1}$ and then pushing down, so $\pi_*(\psi_i \cap D_{i,n+1}) = 0$. Since $\pi|_{D_{i,n+1}}$ is an isomorphism, we get that $\psi_i \cdot D_{i,n+1} = 0$ in $A^*(\overline{\mathcal{M}}_{g,n+1}(X,\beta)$.

Now it suffices to apply the comparison lemma (3.3):

$$-\pi^*\psi_i \cdot D_{i,n+1} = (-\psi_i + D_{i,n+1}) \cdot D_{i,n+1} = D_{i,n+1}^2.$$

For $i \neq j$, note that $D_{i,n+1}$ and $D_{j,n+1}$ are disjoint, and therefore $D_{i,n+1} \cdot D_{j,n+1} = 0$. □

The comparison lemma is a very useful tool, since it lets us relate descendant invariants of $\overline{\mathcal{M}}_{g,n+1}(X,\beta)$ with those of $\overline{\mathcal{M}}_{g,n}(X,\beta)$.

Proposition 3.2.8. *The descendant invariants satisfy the* **string**, **dilaton**, *and* **divisor equations***:*

$$\langle\tau_0(1)\prod_i \tau_{a_i}(\gamma_i)\rangle^X_{g,\beta} = \sum_{j=1}^n \langle\tau_{a_j-1}(\gamma_j)\prod_{i\neq j}\tau_{a_i}(\gamma_i)\rangle^X_{g,\beta}, \tag{3.6}$$

$$\langle\tau_1(1)\prod_i \tau_{a_i}(\gamma_i)\rangle^X_{g,\beta} = (2g-2+n)\langle\prod_i \tau_{a_i}(\gamma_i)\rangle^X_{g,\beta}, \tag{3.7}$$

$$\langle\gamma\prod_i \tau_{a_i}(\gamma_i)\rangle^X_{g,\beta} = (\int_\beta \gamma)\langle\prod_i \tau_{a_i}(\gamma_i)\rangle^X_{g,\beta} + \sum_{i=1}^n \langle\tau_{a_i-1}(\gamma_i\cup\gamma)\prod_{j\neq i}\tau_{a_j}(\gamma_j)\rangle^X_{g,\beta}, \quad \gamma\in H^2(X). \tag{3.8}$$

In these equations, any term containing $\tau_k(\gamma)$ where $k<0$ is taken to be zero.

Proof. Since the evaluation maps ev_i for $i=1,\dots,n$ are unaffected by the forgetful map $\overline{\mathcal{M}}_{g,n+1}(X,\beta)\to\overline{\mathcal{M}}_{g,n}(X,\beta)$, it suffices to prove these identities for $\overline{\mathcal{M}}_{g,n+1}\to\overline{\mathcal{M}}_{g,n}$ and the ψ-classes.

The main tool is the comparison lemma. We prove the string equation for convex X as an example. Applying (3.5) followed by (3.4),

$$\begin{aligned}\langle\tau_0\prod_{i=1}^n\tau_{a_i}\rangle_g &\coloneqq \int_{\overline{\mathcal{M}}_{g,n+1}}\prod_{i=1}^n\psi_i^{a_i} = \int_{\overline{\mathcal{M}}_{g,n+1}}\prod_{i=1}^n(\pi^*\psi_i^{a_i}+\pi^*\psi_i^{a_i-1}\cdot D_{i,n+1}) \\ &= \int_{\overline{\mathcal{M}}_{g,n+1}}\pi^*\left(\prod_{i=1}^n\psi_i^{a_i}\right)+\sum_{j=1}^n\pi^*\left(\psi_j^{a_j-1}\prod_{i\neq j}\psi_i^{a_i}\right)\cdot D_{j,n+1}.\end{aligned}$$

The first term integrates to zero: its dimension is too high for $\overline{\mathcal{M}}_{g,n}$. The second term is precisely the rhs of the string equation, after pushing everything down to $\overline{\mathcal{M}}_{g,n}$ from $\overline{\mathcal{M}}_{g,n+1}$.

The dilaton and divisor equations come from similar applications of the comparison lemma. □

3.2.3 The Virtual Fundamental Class

The construction of the virtual fundamental class is technical and relies on the deformation theory of stacks; for details, see [21]. We outline the construction and focus on a few relevant special cases only.

Definition 3.2.9. Let E be a rank r vector bundle on a non-singular scheme X of dimension n. The zero section $Z\coloneqq Z(s)\subset X$ of a section $s\in\Gamma(E)$ is in general very badly behaved, and of impure dimension. The procedure here if we want to integrate over Z is well-known: if I is the ideal sheaf of $Z\to X$, define the **normal cone**

$$C_{Z/X}\coloneqq \operatorname{Spec}\left(\bigoplus_{k=0}^\infty I^k/I^{k+1}\right),$$

which we should think of as an analogue of the normal bundle. (Note that the normal cone is distinct from the **normal sheaf** $\operatorname{Spec}(\operatorname{Sym}(I/I^2))$.)

Definition 3.2.10. In general, for X a scheme, a **cone** on X is the affine X-scheme $\operatorname{Spec}S$, where $S=\bigoplus_{i=0}^\infty S^i$ be a graded sheaf of $\mathcal{O}_X$-algebras generated by S^1 as an $\mathcal{O}_X$-algebra, with $S^0\cong\mathcal{O}_X$.

Lemma 3.2.11. *Let $X\to Y$ be a closed embedding of schemes with ideal sheaf I. If Y has pure dimension k, then so does $C_{X/Y}$.*

Proof. Compose the embedding $X \to Y$ with $Y \to Y \times \mathbb{A}^1$ at $0 \in \mathbb{A}^1$. The blowup $\mathrm{Bl}_X(Y \times \mathbb{A}^1)$ has pure dimension k. In it is the exceptional divisor is $P(C_{X/Y} \oplus 1)$, which also has pure dimension k. But $C_{X/Y}$ is an open subscheme of $P(C_{X/Y} \oplus 1)$, and therefore must also have pure dimension k. ☐

Definition 3.2.12. By the lemma, the normal cone $C_{Z/X}$ for the zero section Z defines a cycle class $[C_{Z/X}] \in A_n(E|_Z)$. Hence $s^*[C_{Z/X}] \in A_{n-r}(Z)$ is a suitable **fundamental class**. The following lemma shows that this is really just a refinement of the Euler class of E.

Lemma 3.2.13 ([22, Chapter 4]). *If $i\colon Z \to X$ is the inclusion, then $i_*(s^*[C_{Z/X}]) \in A_{n-r}(X)$ is the Euler class $c_r(E) \cap [X]$ of E.*

This construction fails in general for $\overline{\mathcal{M}}_{g,n}(X,\beta)$ because of obstruction-theoretic reasons: we have $\mathrm{Ob}(\Sigma, p_1, \ldots, p_n, f) \neq 0$ in general; c.f. (3.2). The appropriate construction for this general case is much more technical; only the special cases 3.2.20, 3.2.21, and 3.2.22 will be relevant to us.

Definition 3.2.14. Let X be a Deligne–Mumford stack. (It suffices to think about this construction on its atlas $U \to M$, where U is just a scheme.) An **obstruction theory** on X is a complex of sheaves $\mathcal{E}^\bullet$ on X and a morphism $\phi\colon \mathcal{E}^\bullet \to L_X^\bullet$ in the derived category to the cotangent complex $L_X^\bullet$, such that $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is surjective. We view $h^0(L_X^\bullet)$ as controlling deformations, and $h^{-1}(L_X^\bullet)$ as controlling obstructions.

Definition 3.2.15. Define the stack quotient $h^1/h^0(\mathcal{E}^\bullet) \coloneqq \ker(\mathcal{E}^1 \to \mathcal{E}^2)/\operatorname{coker}(\mathcal{E}^{-1} \to \mathcal{E}^0)$. An obstruction theory $\mathcal{E}^\bullet$ is **perfect** if $h^1/h^0((\mathcal{E}^\bullet)^\vee)$ is smooth over X, where the dual is taken in the derived category.

Definition 3.2.16. Take the atlas $U \to X$ of the Deligne–Mumford stack X and embed U in a smooth scheme W, with I the ideal sheaf of the embedding. The differential $d\colon I \to \Omega_W^1$ induces a map

$$\bigoplus_k I^k/I^{k+1} \to \bigoplus_k \mathrm{Sym}^k(\Omega_W^1/I\Omega_W^1).$$

Applying Spec we get a map $T_W|_U \to C_{U/W}$. The **intrinsic normal cone** $\mathcal{C}_U$ is the stack quotient of $\mathcal{C}_{U/W}$ by $T_W|_U$, and these glue to give a global intrinsic normal cone $\mathcal{C}_X$. Similarly we can get an **intrinsic normal sheaf** $\mathcal{N}_X$.

Proposition 3.2.17 ([21, pp. 66–69]). $h^1/h^0((L_X^\bullet)^\vee) \cong \mathcal{N}_X$.

Definition 3.2.18. Let $\mathcal{E}^\bullet$ be a perfect obstruction theory on X. By the proposition, $\mathcal{N}_X$ embeds in $h^1/h^0((\mathcal{E}^\bullet)^\vee)$, and therefore so does $\mathcal{C}_X$. Let E^{-1} be the vector bundle associated to $\mathcal{E}^{-1}$, i.e. $\mathcal{O}(E^{-1}) = \mathcal{E}^{-1}$, and take its fiber product $C \coloneqq (E^{-1})^* \times_{h^1/h^0((\mathcal{E}^\bullet)^\vee)} \mathcal{C}_X$, which is just a cone. The **virtual fundamental class** of X is then $[X]^{\mathrm{vir}} \coloneqq C \cap Z((E^{-1})^*)$.

Example 3.2.19. Let $Z \subset X$ be the zero locus of a section $s \in \Gamma(E)$ again. Let $\mathcal{E}^\bullet = [\mathcal{O}(E^*)|_Z \to (\Omega_X^1)|_Z]$, which is a perfect obstruction theory because Y is smooth. Then $h^1/h^0((\mathcal{E}^\bullet)^\vee)$ is just the stack quotient of $E|_Z$ by $TX|_Z$. The cone C is precisely the normal cone $C_{Z/X}$. Hence the virtual fundamental class is $s^*[C_{Z/X}]$, the same as in 3.2.12.

Example 3.2.20. Consider $\overline{\mathcal{M}}_{g,n}(X,\beta)$ in the **unobstructed** case, when $\mathrm{Ob}(\Sigma, p_1, \ldots, p_n, f) = 0$ for every f in the moduli space. Then any perfect obstruction theory on $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is trivial, i.e. $h^1(\mathcal{E}^\bullet) = 0$. The resulting virtual fundamental class is just the usual fundamental class $[\overline{\mathcal{M}}_{g,n}(X,\beta)]$. Examples of this unobstructed case include: $g = 0$ and X is convex (c.f. 3.1.31); $g = 0$ and $\beta = 0$;

Example 3.2.21. Consider $\overline{\mathcal{M}}_{g,n}(X,\beta)$ in the **non-singular** case, when the moduli space is smooth. Then define the canonical obstruction bundle Ob with fiber $\mathrm{Ob}(\Sigma, p_1, \ldots, p_n, f)$ at the point $(\Sigma, p_1, \ldots, p_n, f)$. Since the moduli space is nonsingular, each fiber has the same dimension and Ob is indeed a vector bundle. The perfect obstruction theory $\mathcal{E}^\bullet$ on $\overline{\mathcal{M}}_{g,n}(X,\beta)$ has $h^0(\mathcal{E}^\bullet) = \mathrm{Ob}$, which is locally free. Then by [21, Proposition 5.6], the virtual fundamental class is $e(\mathrm{Ob}) \cap [\overline{\mathcal{M}}_{g,n}(X,\beta)]$.

Example 3.2.22. Consider $\overline{\mathcal{M}}_{g,n}(X,\beta)$ when $g = 0$ and X is a degree l hypersurface in $\mathbb{P}^m$. Write $\overline{\mathcal{M}}_{0,n}(X,d)$ to stand for the union of $\overline{\mathcal{M}}_{0,n}(X,\beta)$ for all β that push forward to the class xd in $\mathbb{P}^m$. The natural map $i\colon \overline{\mathcal{M}}_{0,n}(X,d) \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^m,d)$ is an embedding: let $\mathcal{V}_d$ be the rank $dl+1$ vector bundle on $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m,d)$ whose fiber over a stable map f is $\Gamma(\Sigma, f^*\mathcal{O}_{\mathbb{P}^m}(l))$, so that the section s defining X defines a section $\tilde{s}$ of $\mathcal{V}_d$ such that $\overline{\mathcal{M}}_{0,n}(X,d)$ is the zero locus of $\tilde{s}$.

This is precisely the situation of 3.2.12 with $Z(\tilde{s}) = \overline{\mathcal{M}}_{0,n}(X,d)$ and $X = \overline{\mathcal{M}}_{0,n}(\mathbb{P}^m,d)$. The normal cone construction gives $\tilde{s}^*[C_{Z/X}]$. It takes some work to check that this is the virtual fundamental class in the sense of 3.2.18; we omit this check. For our calculational purposes, we only need the class

$$i_*([\overline{\mathcal{M}}_{0,n}(X,d)]^{\mathrm{vir}}) = i_*(\tilde{s}^*[C_{Z/X}]) = e(\mathcal{V}_d) \cap [\overline{\mathcal{M}}_{0,n}(\mathbb{P}^4,d)],$$

where the last equality used 3.2.13.

3.2.4 The Quantum Cohomology Ring

To do...

- ☐ 1 (p. 33): Revamp this introduction.
- ☐ 2 (p. 37): Derive this Noether charge!
- ☐ 3 (p. 39): Understand the periodic boundary conditions in path integral for fermions.
- ☐ 4 (p. 40): Show that this Landau–Ginzburg model is $\mathcal{N} = 2$ supersymmetric.
- ☐ 5 (p. 41): Work out the Noether charge for Riemannian manifold sigma model.
- ☐ 6 (p. 42): Prove Gauss Bonnet using the path integral: Compute the Witten index $\mathrm{tr}[(-1)^F e^{-\beta H}]$ in terms of the Riemann curvature tensor and use the localization principle in the limit of $\beta \to 0$.

Bibliography

[1] Roger Godement. *Topologie algébrique et théorie des faisceaux.* Hermann, 1998.

[2] Michael Hutchings. Lecture notes on morse homology (with an eye toward floer theory and pseudoholomorphic curves). Course notes, Fall 2012.

[3] Tammo tom Dieck. *Algebraic Topology.* American Mathematical Society, 2008.

[4] Raoul Bott and Loring Tu. *Differential Forms in Algebraic Topology.* Springer–Verlag, 1982.

[5] Victor Guillemin and Shlomo Sternberg. *Supersymmetry and equivariant de Rham theory.* Springer–Verlag, 1999.

[6] John Milnor and James Stasheff. *Characteristic Classes.* Princeton University Press, 1974.

[7] Robin Hartshorne. *Algebraic Geometry.* Springer–Verlag, 1997.

[8] Daniel Huybrechts. *Complex Geometry.* Springer–Verlag, 2005.

[9] Michael Atiyah and Raoul Bott. The moment map and equivariant cohomology. *Topology*, 23(1):1–28, 1984.

[10] Claire Voisin. *Hodge Theory and Complex Algebraic Geometry, I.* Cambridge University Press, 2002.

[11] Claire Voisin. *Hodge Theory and Complex Algebraic Geometry II.* Cambridge University Press, 2003.

[12] Andrei Moroianu. *Lectures on Kähler Geometry.* Cambridge University Press, 2007.

[13] Philip Candelas and Xenia de la Ossa. Moduli space of calabi–yau manifolds. *Nuclear Physics B*, 355:455–481, 1991.

[14] Gang Tian. Smoothness of the universal deformation space of compact calabi–yau manifolds and its petersson–weil metric. *Mathematical Aspects of String Theory*, 1, 1987.

[15] Robert L. Bryant and Phillip A. Griffiths. *Arithmetic and Geometry: Papers Dedicated to I.R. Shafarevich on the Occasion of His Sixtieth Birthday. Volume II: Geometry*, chapter Some Observations on the Infinitesimal Period Relations for Regular Threefolds with Trivial Canonical Bundle, pages 77–102. Birkhäuser Boston, 1983.

[16] D. G. Markushevich, M. A. Olshanetsky, and A. M. Perelomov. Description of a class of superstring compactifications related to semi-simple lie algebras. *Communications in Mathematical Physics*, 111(2):247–274, 1987.

[17] Philip Candelas, Xenia de la Ossa, Paul Green, and Linda Parkes. A pair of calabi–yau manifolds as an exactly soluble superconformal theory. *Nuclear Physics B*, 359(1):21–74, 1991.

[18] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. *Publ. Math. IHES*, 36:75–109, 1969.

[19] Finn Knudsen. The projectivity of the moduli space of stable curves, ii. *Math. Scand.*, 52:1225–1265, 1983.

[20] Joachim Kock and Israel Vainsencher. *An Invitation to Quantum Cohomology.* Birkhäuser Boston, 2007.

[21] K. Behrend and B. Fantechi. The intrinsic normal cone. *Inventiones mathematicae*, 128(1):45–88, 1997.

[22] William Fulton. *Intersection Theory.* Springer–Verlag New York, 1998.