String Theory and Supersymmetry
Winter 2016 Seminar Notes

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Chapter 1

Introduction to Strings

We asked “why fields?” when we started QFT; now we ask, why strings? Here are some potentially convincing reasons.

1. If we allow one more degree of freedom than particles, many IR/UV divergences disappear; we require less renormalization. If we allow more than one degree of freedom, new divergences arise from the increased internal degrees of freedom.
2. Every consistent string theory contains a massless spin-2 state, i.e. a graviton, whose interactions at low energies reduce to general relativity.
3. The Standard Model, based on QFT, has 25 adjustable constants. String theory has none, and leads to gauge groups big enough to include the Standard Model.
4. Consistent string theories force upon us supersymmetry and extra dimensions, which have arisen naturally from several different attempts to unify the Standard Model.

Regardless of whether they are convincing, we start in this chapter, as with any other physical model, by writing down an action. Specifically, we first write the action for a relativistic string by generalizing that of a relativistic point particle, and then we quantize the action. As with QFT there are different ways to quantize. For exposure and convenience, we use the analogue of canonical quantization for now, in order to quickly compute the spectrum of a string.

As usual, we take $\hbar = c = 1$, and use **Einstein summation convention**: indices that appear as both superscripts and subscripts are implicitly summed over.

1.1 Review of Relativity

We work in $\mathbb{R}^{D-1,1}$ where D is the **number of dimensions**. Recall that coordinates are written $x^\mu = (x^0, x^1, \dots, x^D) = (ct, x^1, \dots, x^D)$, and the metric is

$$-ds^2 := \eta_{\mu\nu} dx^\mu dx^\nu, \quad \eta_{\mu\nu} = \operatorname{diag}(-1, 1, 1, \dots, 1).$$

Note that $\eta^\mu{}_\mu = D$. We use the dot product to stand for the **Lorentz inner product**, e.g. $-ds^2 = dx \cdot dx$.

Definition 1.1.1. Define the **proper time** of a system as the time elapsed measured by a clock traveling in the same Lorentz frame as the system itself. In such a Lorentz frame, $dx^i = 0$ and dt is the proper time elapsed, so $-ds^2 = -dt_p^2$; define

$$ds := \sqrt{ds^2} = dt_p \qquad \text{whenever } ds^2 > 0,$$

i.e. for timelike intervals. Hence ds is the **proper time interval**. The **relativistic momentum** is $p^\mu := m(dx^\mu/ds)$. Conveniently,

$$p^\mu p_\mu = m^2 \frac{dx^\mu}{ds}\frac{dx_\mu}{ds} = -m^2 \frac{ds^2}{ds^2} = -m^2.$$

Definition 1.1.2. A **Lorentz transformation** $\Lambda^\mu{}_\nu$ is an element of the Lorentz group, the collection of all linear isometries of $\mathbb{R}^{D-1,1}$. We say a^μ is a **vector** if under Lorentz transformations, it changes as $a'^\mu = \Lambda^\mu{}_\nu a^\nu$. A **Poincaré transformation** is a Lorentz transformation possibly followed by a translation.

Definition 1.1.3. The **world line** of a point particle is the path in spacetime $\mathbb{R}^{D-1,1}$ traced out by the particle as it evolves in time.

The underlying principle of relativity says that physical laws are independent of Lorentz frame. In other words, any action we write down that we want to be compatible with relativity must have external symmetries: it must be invariant under Lorentz transformations. We call this **Lorentz invariance**. As long as superscripts and subscripts match up, we do not have to worry about Lorentz invariance.

The **action** for a free relativistic **point particle** is obtained by writing down the simplest Lorentz invariant action, and then making sure dimensions work out. If γ is the path taken by the particle, the action is therefore

$$S_{\text{pp}}[x] := -m \int_\gamma ds = -m \int_\gamma d\tau \, \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} = -m \int_\gamma d\tau \, \sqrt{-\dot{x}^\mu \cdot \dot{x}_\mu}$$

where a dot denotes a τ-derivative. Because ds is coordinate-independent, it does not what how we pick the parametrization τ. Physicists like to call this **reparametrization invariance**. This invariance is very important: without it, we have actually introduced a completely new parameter τ, thus increasing the number of degrees of freedom from $D-1$ to D.

Exercise 1.1.1. By computing $\delta(ds^2)$ in two different ways, show that

$$\delta S_{\text{pp}}[x] = m \int_\gamma \delta(dx^\mu)\frac{dx_\mu}{ds} = \int_\gamma d\tau \left(\frac{d}{d\tau}\delta x^\mu\right) p_\mu = \delta x^\mu p_\mu \Big|_{\tau_i}^{\tau_f} - \int d\tau \delta x^\mu \frac{dp_\mu}{d\tau}.$$

Argue that the first term vanishes if we specify **initial and final conditions**. Hence deduce the equation of motion $dp_\mu/d\tau = 0$.

The action S_{pp} seems simple in the $\int_\gamma ds$ form, but is messy when parametrized. Later when we quantize using path integrals, S_{pp} is difficult to work with because of the derivatives under the square root. There is a different, classically-equivalent action we can work with. Introduce an additional field $\gamma_{\tau\tau}(\tau)$ (sometimes called an **einbein** in general relativity), which we can view as a metric on the world line, and take the action

$$S'_{\text{pp}} := -\frac{1}{2}\int_\gamma d\tau \sqrt{-\gamma_{\tau\tau}}(\gamma^{\tau\tau}\dot{x}^\mu \dot{x}_\mu + m^2) = -\frac{1}{2}\int_\gamma d\tau \, (\eta^{-1}\dot{x}^\mu\dot{x}_\mu - \eta m^2), \quad \eta := \sqrt{-\gamma_{\tau\tau}(\tau)}.$$

It seems like we have arbitrarily added an extra degree of freedom, but in fact g is completely specified by the equation of motion. The action S'_{pp} is much better to work with in a path integral, because it is **quadratic** in $\dot{x}^\mu$.

Exercise 1.1.2. Vary S'_{pp} with respect to $\gamma_{\tau\tau}$ to get the equation of motion $\gamma_{\tau\tau} = \dot{x}^\mu\dot{x}_\mu/m^2$. Substitute this expression back into S'_{pp} to obtain S_{pp}, and therefore conclude that the two actions are classically equivalent.

1.2 Nambu–Goto and Polyakov Actions

We graduate to **one-dimensional strings**; in this section we write down an action for them. There are two kinds of strings: those with two distinct endpoints, called **open strings**, and those which are loops, called **closed strings**. Because closed strings are just open strings with the extra constraint that the two endpoints match, we focus on open strings.

The action for the relativistic point particle is proportional to the proper time elapsed on the particle's world line. But the proper time, when multiplied by c, can be viewed as the "proper length" of the world line. The natural generalization, then, is to consider the surface in space-time traced out by the string as it evolves in time, called the **world sheet** M, and to define an action proportional to the "proper area" of the world sheet. The world sheet M is a two-dimensional surface, and therefore requires charts modeled on $\mathbb{R}^2$.

Definition 1.2.1. The **coordinates** we use on $\mathbb{R}^2$, the parameter space, are denoted (τ, σ), and so the **world sheet** M is locally a surface given by functions denoted $X^\mu(\tau, \sigma)$ (capitalized to disambiguate from the coordinates x^μ), called **string coordinates**. The lowercase Latin characters $a, b, \ldots$ are used to denote **indices** that run over values τ, σ. Two notes:

1. The choice of parametrization (τ, σ) is, again, up to us, but usually we take the coordinate τ to be the proper time, and σ the position along the string.
2. For our purposes, $M = X^\mu$, i.e. the single chart X^μ describes the entire world sheet for the region of spacetime we care about.

Exercise 1.2.1. Show that the metric $\eta_{\mu\nu}$ on spacetime $\mathbb{R}^{D-1,1}$ induces a metric g on the world sheet via pullback along the inclusion $\iota\colon M \to \mathbb{R}^{D-1,1}$. Compute g and the area element:

$$g_{ab} = \partial_a X^\mu \partial_b X_\mu, \quad dA = d\tau\, d\sigma\, \sqrt{-\det g}.$$

A relativistic particle has a parameter we call mass. It turns out mass is not the appropriate physical interpretation of the corresponding parameter for strings. Instead, we interpret it as a **tension**, and denote it T_0.

Definition 1.2.2. The **Nambu–Goto action** for a relativistic string is given by

$$S_{\mathrm{NG}}[X] \coloneqq -T_0 \int_M dA = -T_0 \int_M d\tau\, d\sigma\, \sqrt{-\det g}.$$

Again, note that it satisfies **reparametrization invariance**, literally by construction.

But again, we have a square root and derivatives inside it, and now we know how to get rid of it: introduce an independent world sheet metric $\gamma_{ab}(\tau, \sigma)$. This time the metric is on a surface, so we need to specify the signature. We take Lorentzian signature $(-, +)$.

Definition 1.2.3. The **Polyakov action** for a relativistic string is given by

$$S_{\mathrm{P}}[X, \gamma] \coloneqq -\frac{T_0}{2} \int_M d\tau\, d\sigma\, \sqrt{-\gamma}\, \gamma^{ab} \partial_a X^\mu \partial_b X_\mu,$$

where γ without indices stands for $\det(\gamma_{ab})$. From now on, we always refer to γ_{ab} as the **metric**, and g_{ab} as the **induced metric**. Indices are raised/lowered using the metric γ_{ab}, not the induced metric g_{ab}. (In fact, from now on we basically forget about g_{ab}; we use it only to introduce the Nambu–Goto action, and the following exercise.)

Exercise 1.2.2. Show that $\delta\sqrt{-\gamma} = (1/2)\sqrt{-\gamma}\, \gamma^{ab} \delta\gamma_{ab}$, and therefore that

$$\delta_\gamma S_{\mathrm{P}}[X, \gamma] = -\frac{T_0}{2} \int_M d\tau\, d\sigma\, \sqrt{-\gamma}\, \delta\gamma^{ab} \left(g_{ab} - \frac{1}{2} \gamma_{ab} \gamma^{cd} g_{cd} \right).$$

Rearrange the obtained equation of motion and conclude that $g_{ab}\sqrt{-g} = \gamma_{ab}\sqrt{-\gamma}$. Hence replace γ in $S_{\mathrm{P}}[X, \gamma]$ with g, and obtain that $S_{\mathrm{P}}[X, \gamma] = S_{\mathrm{NG}}[X]$.

Definition 1.2.4. As in general relativity, define the **stress-energy tensor**

$$T^{ab}(\tau,\sigma) := -\frac{4\pi}{\sqrt{-\gamma}}\delta_\gamma S_{\mathrm{P}}[X,\gamma] = -2\pi T_0\left(\partial^a X^\mu \partial^b X_\mu - \frac{1}{2}\gamma^{ab}\partial_c X^\mu \partial^c X_\mu\right),$$

so that the **equation of motion** arising from varying γ says $T_{ab} = 0$.

Exercise 1.2.3. (Important!) Now vary $S_{\mathrm{P}}[X,\gamma]$ with respect to X^μ to obtain

$$\begin{aligned}\delta_X S_{\mathrm{P}}[X,\gamma] &= -T_0 \int_M d\tau\, d\sigma\, \sqrt{-\gamma}\,\gamma^{ab}\left(\partial_a(\delta X^\mu \partial_b X_\mu) - \partial_a\partial_b X_\mu \delta X^\mu\right) \\ &= -T_0\int_0^\ell d\sigma\,\sqrt{-\gamma}\,[\delta X^\mu \partial^\tau X_\mu]_{\tau=\tau_i}^{\tau=\tau_f} - T_0\int_{\tau_i}^{\tau_f} d\tau\,\sqrt{-\gamma}\,[\delta X^\mu \partial^\sigma X_\mu]_{\sigma=0}^{\sigma=\ell} \\ &\quad + T_0\int_M d\tau\, d\sigma\,\sqrt{-\gamma}\,\delta X^\mu \nabla^2 X_\mu.\end{aligned}$$

A careful inspection of the terms in the variation $\delta_X S_{\mathrm{P}}[X,\gamma]$ yield interesting insights. For this variation to vanish, each of the terms must vanish independently, since they control different aspects of the string's behavior.

1. The last term is determined by the motion of the string in the domain $(0,\ell)\times(\tau_i,\tau_f)$, and therefore δX^μ is not constrained by any boundary conditions there. Hence we have the **equation of motion** $\sqrt{-\gamma}\,\nabla^2 X_\mu = 0$.
2. The first term is determined by the configuration of the string at times τ_i and τ_f. If we specify these configurations as **initial and final conditions**, then δX^μ is zero for the first term, so the term vanishes.
3. The second term is determined by the configuration of the endpoints of the string when $\tau \in (\tau_i,\tau_f)$. It does not vanish automatically, and we have to impose **boundary conditions** in order for it to do so.

Definition 1.2.5. Let σ_* denote the σ-coordinate of an endpoint, i.e. either $\sigma_* = 0$ or $\sigma_* = \ell$.

- The **free (Neumann) boundary condition** is $\partial^\sigma X^\mu(\tau,\sigma_*) = 0$.
- The **Dirichlet boundary condition** is $\delta X^\mu(\tau,\sigma_*) = 0$.

Alternatively, if the string is **closed**, i.e. we have the **periodicity** conditions

$$X^\mu(\tau,0) = X^\mu(\tau,\ell), \quad \partial^a X^\mu(\tau,0) = \partial^a X^\mu(\tau,\ell), \quad \gamma_{ab}(\tau,0) = \gamma_{ab}(\tau,\ell),$$

no additional boundary conditions are necessary.

For a long time, string theorists did not seriously consider the Dirichlet boundary condition. Why should the endpoints of an open string be fixed, and if they were, where would they be fixed onto? In particular, this fixing of endpoints would violate momentum conservation. Then Polchinski, in the 1990s, suggested that the endpoints are attached to **D-branes**, which should themselves be thought of as dynamical objects alongside strings. Conceptually, then,

1. a D0-brane is a particle, a D1-brane is a string, and so on, and they interact non-trivially;
2. the Dirichlet boundary condition says that a given D1-brane has fixed endpoints on a higher Dp-brane;
3. any momentum lost by the D1-brane is absorbed by the Dp-brane; and
4. the Neumann boundary condition is just saying there is a D-dimensional D-brane permeating all of space-time, i.e. the string endpoints are not fixed at all.

We return to this D-brane perspective much later on. It is hard enough to quantize strings without more dynamical objects floating around.

1.3 Gauge Fixing

There is another reason the Polyakov action is preferable over the Nambu–Goto action: it has more symmetries, and these symmetries make it easier to gauge fix (using Faddeev–Popov or otherwise) when we try to quantize. The Polyakov action is invariant under the following symmetries:

1. D-dimensional **Poincaré transformations**:

$$X^\mu(\tau,\sigma) \mapsto \Lambda^\mu{}_\nu X^\nu(\tau,\sigma) + a^\mu, \quad \gamma_{ab}(\tau,\sigma) \mapsto \gamma_{ab}(\tau,\sigma);$$

2. **Reparametrization**: for new coordinates $\tilde{\sigma}^a(\tau,\sigma)$,

$$X'^\mu(\tau',\sigma') = X^\mu(\tau,\sigma), \quad \gamma_{ab}(\tau,\sigma) \mapsto \frac{\partial\sigma^c}{\partial\tilde{\sigma}^a}\frac{\partial\sigma^d}{\partial\tilde{\sigma}^b}\gamma_{cd}(\tau,\sigma);$$

3. 2-dimensional **Weyl transformations**: for arbitrary $\omega(\tau,\sigma)$,

$$X^\mu(\tau,\sigma) \mapsto X^\mu(\tau,\sigma), \quad \gamma_{ab}(\tau,\sigma) \mapsto \exp(2\omega(\tau,\sigma))\gamma_{ab}(\tau,\sigma).$$

The Nambu–Goto action is not invariant under Weyl transformations.

Exercise 1.3.1. Verify all these statements. (This should be quite straightforward.)