Quantum Field Theory
Fall 2015 Seminar Notes

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Chapter 0

Review

Let's review some notation and concepts from quantum mechanics and special relativity. Note that not all these concepts carry over to quantum field theory. For example, in quantum mechanics, we are always working with states within some Hilbert space, but in quantum field theory, there is no suitable Hilbert space.

0.1 Quantum Mechanics

The fundamentals of quantum mechanics have had almost a century to be formalized, and indeed they have been! Here we give a somewhat axiomatic presentation of QM.

Axiom 1 (States)**.** Let $\mathcal{H}$ be a (complex) Hilbert space. Its projectivization $\mathbb{P}\mathcal{H}$ is the **state space** of our system.

- An element of $\mathcal{H}$, i.e. a state, is called a **ket**, and is written $|x\rangle$.
- An element of $\mathcal{H}^*$, i.e. a functional, is called a **bra**. The bra associated to $|x\rangle$ (under the identification $\mathcal{H} \cong \mathcal{H}^*$ given by the inner product) is denoted $\langle x|$.

Consequently, $\langle x|x\rangle = \|x\|^2$, which we usually want to normalize to be 1.

Note that the symbol inside the ket or bra is somewhat arbitrary. For example, the states of a quantum harmonic oscillator are written $|n\rangle$, for $n \in \mathbb{N}$.

Given a space of states, we can look at the operators that act on the states. These operators must be unitary, so that normalized states go to normalized states.

Axiom 2 (Observables)**.** To every classical observable (i.e. property of a system) is associated a quantum operator, called an **observable**. Observables are (linear) self-adjoint operators whose (real!) eigenvalues are possible values of the corresponding classical property of the system. For example,

- $\hat{H}$ is the **Hamiltonian** of the system, which classically represents the "total energy" (kinetic + potential) in the system,
- $\hat{x}$ is the **position operator**,
- $\hat{p}$ is the **momentum operator**.

The convention in QM is that observables are denoted by symbols with hats on them. The process of "moving" from a classical picture of a system to a quantum picture by making classical observables into operators is called **quantization**, because the possible values of the observables are often quantized, i.e. made discrete, whereas previously they formed a continuum.

A classical observable is simple: it is just a function f defined on the classical phase space, so in order to make a measurement of the observable, we simply apply f to the current state of the system. In QM it is not as simple, in most part due to its inherently probabilistic nature. But it is still straightforward.

Axiom 3 (Measurement)**.** If $\hat{A}$ is the observable and $\hat{A}\,|k\rangle = a_k\,|k\rangle$, i.e. $|k\rangle$ is an eigenstate with eigenvalue $a_k \in \mathbb{R}$, then the probability of obtaining a_k as the value of the measurement on $|\psi\rangle$ is $|\langle k|\psi\rangle|^2$. But not only is the outcome probabilistic, the state of the system after the measurement is $|k\rangle$. In other words, **measurement is projection**. This is fundamental to QM and cannot be emphasized enough.

There are some conventions for position and momentum eigenstates. Since $\hat{x}$ and $\hat{p}$ are conventional symbols to use for position and momentum respectively, the states $|x\rangle$ and $|p\rangle$ are position and momentum eigenstates with eigenvalues x and p respectively.

What about states that we don't measure? What are they doing as time passes? We need to specify the **dynamics** of our system, and this is where the quantum analog of the Hamiltonian comes into play.

Axiom 4 (Dynamics)**.** The time-evolution of the state $|\psi\rangle$ is specified by the Hamiltonian $\hat{H}$ of the system, and is given by the **Schrödinger equation**

$$i\hbar \frac{d\,|\psi\rangle}{dt} = \hat{H}\,|\psi\rangle\,,$$

where $\hbar$ is Planck's constant (later we will be working in units where $\hbar = 1$). Note that we can solve this first-order ODE:

$$|\psi(t)\rangle = \exp(-i\hat{H}t)\,|\psi(0)\rangle\,.$$

The operator $U(t) = \exp(-i\hat{H}t)$ is known as the **time-evolution operator**.

That's it! There are some quick consequences of these axioms we should explore before moving on. First, although measurement is probabilistic, we often work with states whose observables tend to take on values clumped around a certain value, which corresponds to the classical value of that observable for the system. So given a state $|\psi\rangle$ and observable $\hat{A}$, it is reasonable to define the **expectation value** and **standard deviation**

$$\langle \hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle\,, \qquad \Delta\hat{A} = \sqrt{\langle \hat{A}^2\rangle - \langle\hat{A}\rangle^2}.$$

Proposition 0.1.1 (Heisenberg's uncertainty principle)**.** *Let $\hat{A}$ and $\hat{B}$ be self-adjoint operators. Then*

$$\Delta\hat{A}\Delta\hat{B} \geq \frac{1}{2}\left|\langle[\hat{A},\hat{B}]\rangle\right|.$$

Proof. Note that the variance can also be written

$$\Delta\hat{A} = \langle\psi|(\hat{A} - \langle A\rangle)^2|\psi\rangle\,.$$

Without loss of generality, assume $\langle\hat{A}\rangle = \langle\hat{B}\rangle = 0$, since we can shift $\hat{A}$ and $\hat{B}$ by constants without affecting $\Delta\hat{A}$ and $\Delta\hat{B}$. Then an application of Cauchy-Schwarz (using braket notation) gives

$$\Delta\hat{A}\Delta\hat{B} = \|\hat{A}\,|\psi\rangle\,\|\|\hat{B}\,|\psi\rangle\,\| \geq \left|\langle\psi|\hat{A}\hat{B}|\psi\rangle\right|.$$

Now note that if $z = \langle\psi|\hat{A}\hat{B}|\psi\rangle$, then $|z| \geq |\operatorname{Im} z| = |z - z^*|/2$. Hence

$$\left|\langle\psi|\hat{A}\hat{B}|\psi\rangle\right| \geq \frac{1}{2}\left|\langle\psi|\hat{A}\hat{B}|\psi\rangle - \langle\psi|\hat{A}\hat{B}|\psi\rangle^*\right| = \frac{1}{2}\left|\langle\psi|\hat{A}\hat{B} - (\hat{A}\hat{B})^\dagger|\psi\rangle\right| = \frac{1}{2}\left|\langle\psi|[\hat{A},\hat{B}]|\psi\rangle\right|,$$

where the last equality follows from the observables being self-adjoint: $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger = \hat{B}\hat{A}$. □

For example, if we have a particle in $\mathbb{R}^n$, the Hilbert space underlying the state space is $\mathcal{H} = L^2(\mathbb{R}^n)$, and the position and momentum operators are given by

$$\hat{x} : \psi(x) \mapsto x\psi(x), \quad \hat{p} : \psi(x) \mapsto -i\hbar\nabla\psi(x).$$

A short calculation gives the **fundamental commutation relation** between $\hat{x}$ and $\hat{p}$:

$$[\hat{x}, \hat{p}] = i\hbar,$$

which we interpret as saying that we cannot know both the exact position and exact momentum of a particle at the same time.

0.2 Special Relativity

Special relativity describes the structure of spacetime. It says that spacetime is $\mathbb{R}^{1+3}$, known as **Minkowski space** (as opposed to $\mathbb{R}^4$, Euclidean space) and equipped with the **Minkowski metric**

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

where c is the speed of light (later we will work in units where $c = 1$). As with QM, there is a nice axiomatic presentation of SR, which is essentially just the following axiom.

Axiom 1 (Lorentz invariance)**.** The fundamental laws of physics must be invariant under isometries of Minkowski space. These isometries form the **Poincaré group** $\mathbb{R}^{1+3} \rtimes \mathrm{SO}(1, 3)$. The subgroup $\mathrm{SO}(1, 3)$ is known as the **Lorentz group**; its elements are called **Lorentz transformations**, and are precisely the isometries leaving the origin fixed.

So any Hamiltonian, Lagrangian, or physical expression we write down from now on had better be Lorentz invariant (we will usually work locally with nicely-behaved objects that are automatically invariant under the full Poincaré group if they are Lorentz invariant).

Along with special relativity, Einstein introduced his **summation notation** for tensors:

- Components of (contravariant) vectors $\vec{v}$ are written with superscripts, i.e. $\vec{v} = v^1e_1 + \cdots + v^ne_n$, and those of (covariant) covectors with subscripts;
- An index which appears both as a subscript and a superscript is implicitly summed over, i.e. $\vec{v} = v^ie_i$;
- Unbound indices (the ones not summed over) must appear on both sides of an equation.

For example, $T^{\mu\alpha} = g^{\mu\nu}T^{\alpha}_{\nu}$ demonstrates contraction with the metric tensor. When there is superscript that should be a subscript, or vice versa, the metric tensor is implicitly being used to raise and lower indices.

There are several conventions regarding Einstein's summation notation. Spacetime variables are indexed by Greek letters, e.g. μ or ν, which run from 0 to 3, while space-only variables are indexed by Roman letters, e.g. i or j, which run from 1 to 3. Given a 4-vector $v = v^\nu e_\nu$, we let $\vec{v} = v^ie_i$ be the space-only component, and v^2 generally denotes $v^\mu v_\mu$ whereas $\vec{v}^2$ generally denotes v^iv_i.

Chapter 1

Introduction: Klein Gordon and Dirac Fields

1.1 Klein Gordon Field

In this chapter, we will look at our first quantum field, called the Klein-Gordon field. This field arises from the Klein-Gordon equation

$$(\partial^2 + m^2)\phi = 0,$$

which came about as an attempt to make the Schrödinger equation compatible with special relativity, where time and space coordinates can be mixed by Lorentz transformations. Klein and Gordon first proposed it to describe wavefunctions of relativistic electrons, but that interpretation turned out to have some serious problems; nowadays we know it instead describes a quantum field. Although it is meaningless classically (i.e. it does not describe any classical system worth investigating), we will begin by examining Klein-Gordon fields classically, and then putting them through a process called canonical quantization to obtain the quantum Klein-Gordon field.

1.1.1 Why Fields?

Before we begin, let's motivate why we want to look at fields instead of wavefunctions. Why complicate things if we can do relativistic QM with wavefunctions, instead of QFT with quantum fields?

Volume 1 of Steven Weinberg's *Quantum Theory of Fields* is devoted to answering this question. A discussion of scattering experiments lead him to the S-matrix, and then to the local behaviour of experiments (which he calls the cluster decomposition principle), and then using Lorentz invariance, fields just practically fall out. Weinberg does a really good job of convincing us that QFT in some form or another really must exist if we assume Lorentz invariance and unitarity.

Peskin and Schroeder give a slightly different motivation, one that is closer to the historical reason of why fields were introduced. There are three main factors at play here.

- Single particle relativistic wave functions have unavoidable negative energy eigenstates. As an example, we can look at the Dirac equation. The Dirac equation comes from forcing the Schrödinger equation $i(d\Psi/dt) = \hat{H}\Psi$ to be Lorentz invariant. As it stands, it is first-order in time, but second-order in space. Suppose instead that
$$\hat{H} = \frac{1}{i}\alpha^j \partial_j + m\beta.$$

Since $E^2 = \vec{p}^2 + m^2$, we want $\hat{H}^2 = -\nabla^2 + m^2$, which gives

$$\alpha^j \alpha^k + \alpha^k \alpha^j = 2\delta^{jk}, \quad \alpha^j \beta + \beta \alpha^j = 0, \quad \beta^2 = 1.$$

Hence $\{\alpha^1, \alpha^2, \alpha^3, \beta\}$ are not scalars, but instead are the generators of a Clifford algebra; we take their simplest representation as matrices, which is as 4×4 complex matrices

$$\alpha^j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

where σ_j are the Pauli matrices. Now compute in momentum- space that

$$\widehat{H\psi}(\vec{p}) = (-i\vec{p} \cdot \vec{\alpha} + m\beta)\hat{\psi}(\vec{p}) = \begin{pmatrix} mI & \vec{p} \cdot \vec{\sigma} \\ \vec{p} \cdot \vec{\sigma} & -mI \end{pmatrix} \hat{\psi}(\vec{p}),$$

and a straightforward calculation shows that $\hat{H}$ has eigenvalues $\pm\sqrt{\vec{p}^2 + m^2}$. In particular, the energy can be negative!

Dirac attempted to resolve this issue by appealing to the Pauli exclusion principle and positing that there existed a whole "sea of negative-energy states" that were already occupied. Consequently, the holes in this sea would be antiparticles. This makes sense until we realize that that a particle falling into a negative-energy state would represent particle-antiparticle annihilation, but the Dirac equation is supposed to be modeling a single particle (an electron, actually). So philosophical issues aside, there are technical issues here. The field viewpoint will allow us to view particles as excitations of some field, and antiparticles of different types of excitations of the same field, but the key here is that these excitations all have positive energy, regardless of whether they represent particles or antiparticles. We will investigate this later on, when we see the Dirac field (which will contain the first non-trivial example of antiparticles).

- $E = mc^2$ allows for particles to be created at high energies, and $\Delta E \Delta t = \hbar$ allows for virtual particles. This indicates we should really be looking at multi-particle instead of single-particle theories. While we can obtain multi-particle theories simply by looking at the tensor product of single-particle state spaces, the quantum mechanics arising from this construction do not permit the creation and annihilation of particles. We can't "destroy" or "create" a wavefunction; it exists for all time and space. Instead, the field viewpoint allows us to view particles as excitations of a field, which we can easily create or destroy.

- Wavefunctions and quantum mechanics don't care about special relativity. In particular, there is obvious causality violation in quantum mechanics! Set $H = \frac{\hat{p}^2}{2m}$ to be the free Hamiltonian, and let's compute the probability amplitude for propagation between two points x_0 and x in spacetime:

$$\begin{aligned} U(t) &= \langle \vec{x} | e^{-iHt} | \vec{x}_0 \rangle \\ &= \int \frac{d^3p}{(2\pi)^3} \langle \vec{x} | e^{-i(p^2/2m)t} | p \rangle \langle p | x \rangle \\ &= \int \frac{d^3p}{(2\pi)^3} e^{-i(p^2/2m)t} e^{i\vec{p} \cdot (\vec{x} - \vec{x}_0)} \\ &= \left(\frac{m}{2\pi i t} \right)^{3/2} e^{im(\vec{x} - \vec{x}_0)^2/2t} \end{aligned}$$

This last quantity is non-zero, even for x and x_0 that may be space-like separated, e.g. x inside the light cone, and x_0 outside it, which, in principle, allows faster-than-light transfer of information.

It is not clear immediately how field theory will help us here. But we will see that by rigorously enforcing Lorentz invariance when we write down field dynamics, the causality violation problem magically disappears.

Another important reason we want to do QFT is because, well, the theory predicts the outcome of numerous experiments to very high accuracy. In the end, physics is about constructing models: the fact that your model is giving good predictions is very strong evidence that it should be adopted, or at least seriously considered as a foundational theory. In particular, quantum electrodynamics (QED), which describes electromagnetism, is something we will see very soon that has been very well tested and agrees very well with experiments, up to the limits of what we can experimentally measure.

1.1.2 Elements of Classical Field Theory

Before we embark on the long journey through QFT, we need to review some tools from classical field theory first. This serves not only as a review, but as motivation for many calculations and objects we will be examining in the QFT world.

1.1.3 Lagrangian Field Theory

- Fundamental quantity in Lagrangian field theory is the action S. In high school, the Lagrangian is a function of time, positions, and velocities of a system: $L(t, x(t), \dot{x}(t))$. The action is given by $S = \int dt\, L$. Fields can also be described in a Lagrangian formalism, for instance by considering every point in space-time as a "particle" that wiggles back and forth with the amplitude of wiggling characterizing the strength of the field.

  Let $\varphi : M \to \mathbb{R}$, define a Lagrangian *density* $\mathcal{L}(t, \varphi, \partial_\mu \varphi)$, the honest Lagrangian $L = \int d^3x \mathcal{L}$, and finally define the action:

$$S = \int dt\, L = \int d^4x \mathcal{L}$$

> Four-vector notation:
>
> - Greek letters $\mu, \nu, \ldots \in \{0, 1, 2, 3\}$
> - Roman letters $i, g, \ldots \in \{1, 2, 3\}$.
> - $x^\mu = (x^0, x^1, x^2, x^3)$
> - Signature $(+ - - -)$
> - $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$
> - $\partial_\mu f = \frac{\partial f}{\partial x^\mu} = (\partial_0 f, \partial_1 f, \partial_2 f, \partial_3 f)$.

- Extremize the action. Let $\delta f = f(\varphi + \xi) - f(\varphi)$.

$$\begin{aligned} 0 = \delta S &= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \underbrace{\delta(\partial_\mu}_{\text{commute}} \varphi) \right) \\ &= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta\varphi \right) - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \delta\varphi \right] \end{aligned}$$

  By Stokes' theorem, we can break this integral up into two parts, one of which is called the boundary term. Taking a variation that is fixed along the boundary means $\delta\varphi \equiv 0$ on the boundary which means that the boundary term does not contribute to δS. Moreover, if we take $\delta S = 0$ for every variation, then we obtain the Euler Lagrange equations:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) - \frac{\partial \mathcal{L}}{\partial \varphi} = 0$$

Remark. The Lagragian formalism is useful for relativistic dynamics because all expressions are chosen to Lorentz invariant.

1.1.4 Hamiltonian Field Theory

- Introducing this makes the transition to the quantum theory easier.
- High school Hamiltonian formalism: $p = \frac{\partial L}{\partial \dot{q}}, H = \sum p\dot{q} - L$.
- Pretend that $\vec{x}$ enumerates points on the lattice of space-time:

$$\begin{aligned} p(\vec{x}) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}(\vec{x})} &= \frac{\partial}{\partial \dot{\varphi}(\vec{x})} \int d^3y \, \mathcal{L}(\varphi(y), \dot{\varphi}(y)) \\ &\sim \frac{\partial}{\partial \dot{\varphi}(\vec{x})} \sum \mathcal{L}(\varphi(y), \dot{\varphi}(y)) d^3y \\ &= \frac{\partial \mathcal{L}}{\partial \dot{\varphi}(\vec{x})} d^3x \\ &\equiv \pi(\vec{x}) d^3x \end{aligned}$$

since each point on the lattice represents a different variable, so the derivative just picks out the one at $\vec{x}$. We call $\pi(\vec{x})$ the momentum *density*. Therefore the Hamiltonian looks like:

$$H = \int d^3x \, [\pi(\vec{x})\dot{\varphi}(\vec{x}) - \mathcal{L}] \, .$$

(See the stress-energy tensor part for another derivation of the Hamiltonian which falls out of Noether's theorem for being the conserved quantity under time translations.)

One might ask why we are still singling out the time-parameter in the Hamiltonian formalism when we write $p(\vec{x}) = \partial\mathcal{L}/\partial\dot{\varphi}(\vec{x})$ instead of making it seem more Lorentz invariant by considering $\partial\mathcal{L}/\partial(\partial_\mu\varphi(\vec{x}))$ instead. This is because although special relativity dictates that time transforms with space, we still cannot treat them equally as coordinates. The Hamiltonian is, by definition, the infinitesimal generator of time translations, and hence is intrinsically associated with only the time coordinate. In fact, it is not true that the Hamiltonian density is always Lorentz invariant.

- **Important example:** Take $\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)^2 - \frac{1}{2}m^2\varphi^2$. Euler-Lagrange equations become $\partial^\mu(\partial_\mu\varphi) + m^2\varphi = 0$ which is the Klein Gordon equation. The Hamiltonian becomes:

$$H = \int d^3x \mathcal{H} = \int d^3x \left[\underbrace{\frac{\pi^2}{2}}_{\text{moving in time}} + \underbrace{\frac{(\nabla\varphi)^2}{2}}_{\text{shearing in space}} + \underbrace{\frac{m^2\varphi^2}{2}}_{\text{existing at all}} \right]$$

1.1.5 Noether's Theorem - How to Compute Conserved Quantities

To every continuous transformation of the field we can assign an infinitesmal transformation:

$$\varphi(x) \to \varphi'(x) = \varphi(x) + \alpha \underbrace{\Delta\varphi(x)}_{\text{deformation}}$$

Transformations might also change the Lagrangians. The interplay between how the infinitesmal transformation changes the Lagrangian and the field is what gives rise to conserved quantities, or sometimes known as Noether charges.

$$\begin{aligned}\text{Symmetry} &\iff \text{Equations of motion} - \text{invariant}\\ &\iff \text{Action invariant (up to surface term)}\\ &\iff \mathcal{L}(x) \to \mathcal{L}(x) + \alpha\partial_m u\mathcal{J}^\mu(x)\end{aligned}$$

Taylor expanding the perturbation:

$$\begin{aligned}\Delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\varphi}\cdot\Delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\mu(\Delta\varphi)\\ &= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\Delta\varphi\right) + \left[\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right)\right]\\ &= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\Delta\varphi\right)\end{aligned}$$

Since we claimed that under the symmetry $\Delta\mathcal{L} = \partial_\mu\mathcal{J}^\mu$ we have the following relations:

$$\begin{aligned}j^\mu(x) &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\Delta\varphi - \mathcal{J}^\mu\\ \partial_\mu j^\mu &= 0\\ \frac{\partial}{\partial t}j^0 &= \partial_i j^i\end{aligned}$$

Define the charge $Q = \int d^3x\; j^0$. Then, if we assume that space does not have boundary, Stokes' theorem implies that $\partial Q/\partial t = 0$. Often, j^0 is called the charge density, and j^μ is called the current density.

Examples:

1. $\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)^2$ has the following field symmetry, $\varphi \to \varphi + \alpha$, ie. $\Delta\varphi \equiv \text{const}$. There is no change to the Lagrangian, so $j^\mu = \partial^\mu\varphi$.

2. Space-time transformation, $x^\mu \to x^\mu - a^\mu$, implies

$$\begin{aligned}\varphi(x) \to \varphi(x+a) &= \varphi(x) + a^\nu\partial_\nu\varphi(x)\\ \mathcal{L}(x) \to \mathcal{L}(x+a) &= \mathcal{L}(x) + a^\mu\partial_\mu\mathcal{L}\\ . \qquad\qquad &= \mathcal{L}(x) + a^\nu\partial_\mu(\delta^\mu_\nu\mathcal{L})\end{aligned}$$

Therefore we write

$$T^\mu_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\nu\varphi - \delta^\mu_\nu\mathcal{L}$$

we get four separately conserved quantities.

This is called the **stress-energy tensor** or the **energy-momentum tensor** in various contexts. The $T^{\bullet 0}$ quantity gives rise to the Hamiltonian:

$$\int d^3x T^{00} = \int d^3x\mathcal{H} \equiv H$$

1.1.6 Quantizing the Klein-Gordon Field

Before we quantize, let's apply the classical theory to the classical Klein-Gordon field, which is defined by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2,$$

where $\phi(\vec{x})$ is the real-valued **classical Klein-Gordon field**. We will interpret m as a mass later on, but for now it is just a parameter.

Exercise 1.1.1. By applying Euler-Lagrange, confirm that this Lagrangian for the classical Klein-Gordon field indeed gives the Klein-Gordon equation $(\partial^\mu\partial_\mu + m^2)\phi = 0$, and compute the Hamiltonian

$$H = \int d^3x\; \mathcal{H} = \int d^3x \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right).$$

(You should get that $\pi = \dot{\phi}$).

Now we enter the QFT world. For now we will work in the Schrödinger picture, where $\phi(\vec{x})$ and $\pi(\vec{y})$ are time-independent. We will take the classical Klein-Gordon field and **canonically quantize** it, which involves two steps:

1. promote ϕ and π to operators (i.e. $\phi(\vec{x})$ and $\phi(\vec{y})$ are now operators, not scalars), and
2. specify the commutation relations

$$[\phi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y})$$
$$[\phi(\vec{x}), \phi(\vec{y})] = [\pi(\vec{x}), \pi(\vec{y})] = 0.$$

   This is in analogy with the QM of a multiparticle system, where if q_i and p_i are the momentum and position operators of the i-th particle, then

$$[q_i, p_j] = i\delta_{ij}$$
$$[q_i, q_j] = [p_i, p_j] = 0,$$

   except now we have a continuum of particles, indexed by the continuous variable $\vec{x}$ instead of a discrete variable i.

Note that these commutation relations are taken to be **axioms**. At this point one may wonder why we treat ϕ and π as different operators when $\pi = \dot{\phi}$ for Klein-Gordon. This is for the same reason that x and $\dot{x}$ are treated independently in classical field theory: we abuse notation and write $(x, \dot{x})$ as coordinates on phase space, when really we should be writing (x, p). But we write $\dot{x}$ because we will always be evaluating objects on phase space at $(x, \dot{x})$.

But of course, imposing these axioms is easier said than done. What do ϕ and π look like?

Let us try to motivate the form of the expression for ϕ and its conjugate π in terms of creation and annihilation operators. [1]

If we expand a solution to the Klein Gordon equation in a Fourier basis of plane waves, then we see that we naturally have some variables that we can quantize. What's more interesting is that the Klein Gordon equation gives rise to precisely the harmonic oscillator example from first year quantum mechanics. In terms

[1] Quibble: I don't like how it is done in Peskin. Why promote the coefficients in the Fourier transform, and why do *they* give rise to the creation and annihilation operators. I think there might be a good explanation out there already; Landau& Lifshitz, and Weinberg (Chapter 5) seem to take a good wack at the physics of this choice. Actually, in LL, the exposition seems to have avoided some of the integral manipulations that happened in Peskin and Schroeder.

of creation and annihilation operators, the first years wrote $\hat{q} = \frac{1}{\sqrt{2\omega}}(a + a^\dagger)$, $\hat{p} = \sqrt{\frac{\omega}{2}}(a - a^\dagger)$. Therefore we conjecture our fields have the following form: [2]

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}})$$
$$\pi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_p}{\sqrt{2}}} (a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} - a^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}})$$

1.2 Dirac Field

The formalism that we have built up so far tells that everything starts from a Lagrangian. In the theory of elementary particles and high energy physics there is one very special condition that we require of a Lagrangian: Lorentz invariance. To check whether a given expression of ϕ's and $\partial_\mu\phi$'s is Lorentz invariant we must understand how arbitrary field transform under the Lorentz group. Suppose a field has components ϕ_a, then a general transformation is given by

$$\phi'_a(x) = M(\Lambda)_{ab}\phi_b(\Lambda^{-1}x).$$

Thus, to solve the problem of constructing all Lagrangians we must first understand the representations of the Lorentz group, or at least of the Lorentz algebra. Taking a cue from the so(3) generators given by $J^{ij} = -i(x^i\partial^j - x^j\partial^i)$ it turns out that the generators for the Lorentz algebra are:

$$J^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu).$$

This gives commutation relations

$$[J^{\mu\nu}, J^{\rho\sigma}] = \dots.$$

The defining representation is given by the following$(\mathcal{J}^{\mu\nu})_{ab} = -i\delta^\mu_{[a}\delta^\nu_{b]} = -i(\delta^\mu_a\delta^\nu_b - \delta^\mu_b\delta^\nu_a)$ Dirac came up with another representation by taking 4 $n \times n$ matrices γ^μ satisfying $\gamma^\mu, \gamma^\nu = 2g^{\mu\nu} \times \mathbb{1}_{n\times n}$ and defining:

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu].$$

If we define $\sigma = (\mathbb{1}, \vec{\sigma})$ and $\bar{\sigma} = (\mathbb{1}, -\vec{\sigma})$ then

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

satisfies the commutation relations and gives rise to the **Dirac representation**. Objects that transform under that transform under this representation are called 4-component Dirac spinors, or just **Dirac spinors** for short.

Taking $\bar{\psi} = \gamma^0\psi^\dagger$, the Dirac equation, Lagrangian are given by:

$$(i\gamma^\mu\partial_\mu - m)\psi = 0$$
$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

The conjugate variable to ψ is $i\psi^\dagger$. Before quantizing we solve the Dirac equation in plane waves, $u(p)e^{i\vec{p}\cdot\vec{x}}$ and $v(p)e^{-i\vec{p}\cdot\vec{x}}$. After rewriting the Dirac equation into a matrix equation it is not hard to see that arbitrary solutions $u(p)$ and $v(p)$ are given by the following expressions:

$$u^s(p) = \sqrt{m}\begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\sigma}\xi^s \end{pmatrix}$$
$$v^s(p) = \sqrt{m}\begin{pmatrix} \sqrt{p\cdot\sigma}\eta^s \\ -\sqrt{p\cdot\sigma}\eta^s \end{pmatrix}$$

[2]**Elaborate.**

where, for $s = 1, 2$, $\{\xi^s\}$ and $\{\eta^s\}$ are a basis for $\mathbb{C}^2$ and $\sqrt{p \cdot \sigma}$ is the square root of the positive eigenvalue of the associated matrix. (Phew, what a mouthful!)

Finally, we quantize this theory by introducing the **anticommutation relations** and rewriting ψ and $\bar{\psi}$ using raising and lowering operators

$$\{\psi(x), \bar{\psi}(y)\} = \delta^{(3)}(\vec{x} - \vec{y}) \tag{1.1}$$

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} a^s_{\vec{p}} u^s(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} + b^{s\dagger}_{\vec{p}} v^s(\vec{p}) e^{i\vec{p}\cdot\vec{x}} \tag{1.2}$$

$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} b^s_{\vec{p}} \bar{v}^s(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} + a^{s\dagger}_{\vec{p}} \bar{u}^s(\vec{p}) e^{i\vec{p}\cdot\vec{x}} \tag{1.3}$$

Using these cleverly chosen expressions we may write the Hamiltonian as

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s E_{\vec{p}} (a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} + b^{s\dagger}_{\vec{p}} b^{s\dagger}_{\vec{p}})$$

$$Q = \int \frac{d^3p}{(2\pi)^3} \sum_s (a^{s\dagger}_{\vec{p}} a^{s\dagger}_{\vec{p}} - b^{s\dagger}_{\vec{p}} b^s_{\vec{p}})$$

where Q is the conserved quantity coming from gauge invariance, $\psi'(x) = e^{\alpha(x)}\psi(x)$, of $\mathcal{L}_{\text{Dirac}}$.

Chapter 2

Path Integrals

So far, we have taken classical field theories and canonically quantized them to obtain the corresponding QFTs. In general, this canonical quantization process is difficult and tedious, but it motivates much of what we are about to do. The path integral approach to QFT will allow us to perform perturbative calculations more easily, and generalizes readily to other non-interacting theories. In for the entirety of this chapter, we will mostly be concerned with calculating **propagation amplitudes** for a perturbed theory.

2.1 Deriving the Path Integral

Suppose we have the Hamiltonian $\hat{H}$ for a quantum mechanical particle, and we want to compute the amplitude $\langle \vec{q}_b | e^{-i\hat{H}t} | \vec{q}_a \rangle$, i.e. the amplitude for the particle to travel from the point $\vec{q}_a$ to $\vec{q}_b$ in a given time t. Using the superposition principle, let's compute this by splitting up the time interval $[0, t]$ into n equal chunks of size $\delta t = t/n$:

$$\langle \vec{q}_b | e^{-i\hat{H}t} | \vec{q}_a \rangle = \int \cdots \int d\vec{q}_1 \cdots d\vec{q}_{n-1} \, \langle \vec{q}_b | e^{-i\hat{H}\delta t} | \vec{q}_{n-1} \rangle \, \langle \vec{q}_{n-1} | e^{-i\hat{H}\delta t} | \vec{q}_{n-2} \rangle \cdots \langle \vec{q}_1 | e^{-i\hat{H}\delta t} | \vec{q}_a \rangle \, .$$

What have we done? We are saying that the amplitude for propagation from $\vec{q}_a$ to $\vec{q}_b$ is equal to the amplitude for propagation from $\vec{q}_a$ to $\vec{q}_1$, then to $\vec{q}_2$, and so on, until $\vec{q}_b$, integrated over all possible $\vec{q}_j$. (Recall the double slit experiment and consider the case $n = 2$ if you are still confused.)

Now each of the terms needs to be evaluated. For convenience, let $\vec{q}_n = \vec{q}_b$ and $\vec{q}_0 = \vec{q}_a$. Let's do the simple case where $\hat{H} = \hat{p}^2/2m$, a free particle. A straightforward calculation shows:

$$\begin{aligned}
\langle \vec{q}_{j+1} | e^{-i(\hat{p}^2/2m)\delta t} | \vec{q}_j \rangle &= \int \frac{d^3p}{(2\pi)^3} \, \langle \vec{q}_{j+1} | e^{-i(\hat{p}^2/2m)\delta t} | p \rangle \, \langle p | \vec{q}_j \rangle \\
&= \int \frac{d^3p}{(2\pi)^3} \, e^{-i(p^2/2m)\delta t} \, \langle \vec{q}_{j+1} | p \rangle \, \langle p | \vec{q}_j \rangle \\
&= \int \frac{d^3p}{(2\pi)^3} \, e^{-i(p^2/2m)\delta t} e^{ip(\vec{q}_{j+1} - \vec{q}_j)}.
\end{aligned}$$

Ah, we know how to evaluate this integral: it's just a Gaussian! The final result, after some suggestive rearranging, is

$$\langle \vec{q}_{j+1} | e^{-i(\hat{p}^2/2m)\delta t} | \vec{q}_j \rangle = \left(\frac{m}{2\pi i \delta t} \right)^{3/2} \exp \left(i\delta t \frac{m}{2} \left(\frac{\vec{q}_{j+1} - \vec{q}_j}{\delta t} \right)^2 \right).$$

(The Gaussian integral itself is not trivial. [1]) Hence when we plug this back into our calculation for $\langle \vec{q}_b | e^{-i\hat{H}t} | \vec{q}_a \rangle$, we get

$$\langle \vec{q}_b | e^{-i\hat{H}t} | \vec{q}_a \rangle = \left(\frac{m}{2\pi i \delta t} \right)^{3n/2} \int d\vec{q}_1 \cdots d\vec{q}_{n-1} \exp \left(i \delta t \frac{m}{2} \sum_{j=1}^{n-1} \left(\frac{\vec{q}_{j+1} - \vec{q}_j}{\delta t} \right)^2 \right).$$

So far, everything we have done is rigorous. But now we make an intuitive leap: instead of approximating the propagation from $\vec{q}_a$ to $\vec{q}_b$ with a finite number of timesteps, we use infinitely many. In other words, we "integrate over paths" by letting $\delta t \to 0$ and $n \to \infty$, giving the formal expression

$$\langle \vec{q}_b | e^{-i\hat{H}t} | \vec{q}_a \rangle = \int D\vec{q}(t) \, \exp \left(i \int_0^t dt \, \frac{1}{2} m \vec{q}'(t)^2 \right)$$

where the **path integral** $\int D\vec{q}(t)$ is defined as

$$\int D\vec{q}(t) = \lim_{n \to \infty} \left(\frac{m}{2\pi i \delta t} \right)^{3n/2} \int \cdots \int d\vec{q}_1 \cdots d\vec{q}_{n-1}.$$

Exercise 2.1.1. Perform the same derivation of the path integral, but now starting with the Hamiltonian $\hat{H} = \hat{p}^2/2m + V(\hat{q})$. You should get

$$\langle \vec{q}_b | e^{-i\hat{H}t} | \vec{q}_a \rangle = \int D\vec{q}(t) \, \exp \left(i \int_0^t dt \, \frac{1}{2} m \vec{q}'(t)^2 - V(\vec{q}(t)) \right).$$

For now, let's not worry about the infinite constant in front of the path integral; it pales as an issue in comparison to the nonexistence of a Lebesgue measure on the space of paths. Actually, the constant will cancel out later.

Note that the integrand looks suspiciously like the Lagrangian corresponding to the Hamiltonian in both cases. This is indeed true, and can be demonstrated by plugging in a general Hamiltonian $\hat{H}(\hat{q}, \hat{p})$ and seeing how combinations of $\hat{q}$ and $\hat{p}$ act on the $|\vec{q}_i\rangle$.

Theorem 2.1.1. *Suppose $\hat{H}(\vec{q}, \vec{p})$ is a* **Weyl-ordered** *Hamiltonian, i.e. in a form where if there is a term $\vec{p}^{i_1} \vec{q}^{i_2} \cdots \vec{p}^{i_n}$, then there is a corresponding term $\vec{p}^{i_n} \vec{q}^{i_{n-1}} \cdots \vec{p}^{i_1}$. Then*

$$\langle \vec{q}_b | e^{-i\hat{H}t} | \vec{q}_a \rangle = \int D\vec{q}(t) D\vec{p}(t) \, \exp \left(i \int_0^t dt \, \vec{p}(t) \cdot \vec{q}(t) - H(\vec{q}(t), \vec{p}(t)) \right).$$

In particular, for Hamiltonians quadratic in $\vec{p}$, we can integrate away the $\int D\vec{p}(t)$, leaving only the Lagrangian in the integrand.

Proof. Details of the long calculation will not bring us much further enlightenment, so we omit them. See Peskin & Schroeder, pages 280-281 iff you like calculations and have some time to burn. □

[1] The relevant formula is as follows. For $A \in \mathrm{GL}(n, \mathbb{C})$ such that $A = A^T$ and $\mathrm{Re}\, A$ is positive semidefinite,

$$\int d^n x \, e^{-Ax \cdot x/2 + iy \cdot x} = \frac{(2\pi)^{n/2}}{\sqrt{\det A}} e^{-A^{-1} y^2/2}.$$

One proves this by showing it first for $n = 1$ and $A = I$, in which case it suffices to solve the DE

$$\frac{d}{dy} \int dx \, e^{-x^2/2 + iyx} = -y \int dx \, e^{-x^2/2 + iyx}.$$

Now suppose A is real and hence PSD. If we plug $x = \sqrt{A} v$ into the LHS of the formula, the RHS splits as a product of one-dimensional integrals, which we just calculated. Finally, since both sides are analytic and agree for real PSD matrices, they agree in general.

Any Hamiltonian can be Weyl-ordered by commuting $\hat{p}$ and $\hat{q}$, so this theorem is very general. In fact, it is general enough that from now on, we will work directly with the Lagrangian and almost completely ignore the Hamiltonian formalism. There is one major advantage in doing so: the Lagrangian makes symmetries and conservation laws very clear. For example, when we write down a Lorentz-invariant Lagrangian, the path integral is automatically Lorentz-invariant.

In fact, the quantum system we are considering is very general as well. In our entire derivation of the path integral, we did not use anything beyond the relationship between $\hat{q}$ and $\hat{p}$. So in particular, our derivation holds not only for quantum mechanical systems, but also for QFTs. For example, if we take the Lagrangian $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - V(\phi)$ for a real scalar field, then

$$\langle\phi_b(\vec{x})|e^{-i\hat{H}t}|\phi_a(\vec{x})\rangle = \int D\phi(x)\,\exp\left(i\int_0^t d^4x\,\frac{1}{2}(\partial_\mu\phi)^2 - V(\phi)\right),$$

where here $D\phi(x)$ indicates that we are integrating over a path taking values in fields. In particular, $\phi(0,\vec{x})$ is constrained to be $\phi_a(\vec{x})$, and $\phi(t,\vec{x})$ is constrained to be $\phi_b(\vec{x})$.

2.2 Correlation Functions

Okay, what good is the path integral? The answer is they are useful when we apply perturbations to free field theory. Most of the QFT we will be looking at is perturbative, so path integrals will give us a good deal of physics.

Suppose we have the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_{int}$, where $\hat{H}_0$ is a Hamiltonian we are supposed to have understood well already, and $\hat{H}_{int}$ is a perturbation known as the **interaction Hamiltonian**. Usually $\hat{H}_0$ will be the Hamiltonian for the free field theory, i.e. the Klein-Gordon Hamiltonian. Let $|\Omega\rangle$ be the ground state of $\hat{H}$. We are interested in computing the probability amplitude of a propagation from $\vec{x}$ to $\vec{y}$, i.e.

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle\,.$$

How do we compute this quantity? Let's start with a seemingly-unrelated quantity:

$$\int D\phi(x)\,\phi(x_1)\phi(x_2)\,\exp\left(i\int_{-t}^t d^4x\;\mathcal{L}(\phi)\right),$$

where the path $\phi(x)$ starts at some $\phi_a(\vec{x})$ at time $-t$ and ends at some $\phi_b(\vec{x})$ at time t. Suppose $x_1^0 < x_2^0$. Then we are going to divide up this path integral into three components:

1. from $\phi_a(\vec{x})$ at time $-t$ to $\phi_1(\vec{x})$ at time x_1^0,
2. from $\phi_1(\vec{x})$ at time x_1^0 to $\phi_2(\vec{x})$ at time x_2^0,
3. from $\phi_2(\vec{x})$ at time x_2^0 to $\phi_b(\vec{x})$ at time t.

Note that here, $-t < x_1^0 < x_2^0 < t$, and since the intermediate field configurations $\phi_1(\vec{x})$ and $\phi_2(\vec{x})$ are arbitrary, we must integrate over them as well. Hence the integral becomes

$$\int D\phi_1(\vec{x})\int D\phi_2(\vec{x})\,\phi_1(\vec{x}_1)\phi_2(\vec{x}_2)\,\langle\phi_b|e^{-i\hat{H}(t-x_2^0)}|\phi_2\rangle\,\langle\phi_2|e^{-i\hat{H}(x_2^0-x_1^0)}|\phi_1\rangle\,\langle\phi_1|e^{-i\hat{H}(x_1^0-(-t))}|\phi_a\rangle\,.$$

Now we use completeness: $\int D\phi_1\,\phi_1(\vec{x}_1)\,|\phi_1\rangle\,\langle\phi_1| = \phi_1(\vec{x}_1)$, where the ϕ_1 on the LHS is a scalar field, and on the RHS is an operator. Doing the same for for ϕ_2, the integrals disappear, and some rearrangement gives

$$\langle\phi_b(\vec{x})|\,e^{-i\hat{H}(t-x_2^0)}\phi(\vec{x}_2)e^{-i\hat{H}(x_2^0-x_1^0)}\phi(\vec{x}_1)e^{-i\hat{H}(x_1^0-(-t))}\,|\phi_a(\vec{x})\rangle\,. \tag{2.1}$$

Aha, but $\phi(x_2) = e^{i\hat{H}x_2^0}\phi(\vec{x}_2)e^{-i\hat{H}x_2^0}$ in the Heisenberg picture, so this simplifies further to

$$\langle \phi_b(\vec{x})| \, e^{-i\hat{H}t}\phi(x_2)\phi(x_1)e^{-i\hat{H}t} \, |\phi_a(\vec{x})\rangle \, .$$

We're not done yet! During this calculation, we had to assume x_1 came before x_2 in time, so that the path integral split well. If x_1 actually came after x_2, then we simply exchange x_1 and x_2 in the final result. This motivates the following definition.

Definition 2.2.1. Given two operators $\phi(x_1)$ and $\phi(x_2)$, the **time-ordering operator** T applies them in the correct temporal order, i.e.

$$T\{\phi(x_1)\phi(x_2)\} = \begin{cases} \phi(x_1)\phi(x_2) & x_1^0 > x_2^0 \\ \phi(x_2)\phi(x_1) & x_2^0 > x_1^0. \end{cases}$$

Hence we should really be looking to calculate $\langle \Omega | T\phi(x_1)\phi(x_2)|\Omega\rangle$, while right now we have the quantity $\langle \phi_b(\vec{x})| \, e^{-i\hat{H}t}T\phi(x_2)\phi(x_1)e^{-i\hat{H}t} \, |\phi_a(\vec{x})\rangle$. In other words, our problem is to obtain $|\Omega\rangle$ from $e^{-i\hat{H}t}|\phi_a(\vec{x})\rangle$. Physicists have a hilarious trick for doing so. First expand $|\phi_a\rangle$ in the eigenbasis $\{|\Omega\rangle, |1\rangle, \dots\}$ of $\hat{H}$:

$$e^{-i\hat{H}t} \, |\phi_a\rangle = e^{-iE_\Omega t} \, |\Omega\rangle \, \langle \Omega|\phi_a\rangle + \sum_{n>0} e^{-iE_n t} \, |n\rangle \, \langle n|\phi_a\rangle \, .$$

Now remember that the ground state energy is the lowest energy, i.e. $E_\Omega < E_n$ for all $n > 0$. So here's what we do to keep the $|\Omega\rangle$ term while getting rid of everything else: we take the limit $t \to \infty(1 - i\epsilon)$. Since $e^{-iE_n T(1-i\epsilon)}$ will decay faster than $e^{-iE_\Omega T(1-i\epsilon)}$, because $e^{-E_n t}$ decays faster than $e^{-E_\Omega t}$, it follows that when $T \to \infty$, every other term except the $|\Omega\rangle$ term vanishes.

$$\lim_{t\to\infty(1-i\epsilon)} e^{-i\hat{H}t} \, |\phi_a\rangle = \langle \Omega|\phi_a\rangle \, e^{-E_\Omega \infty(1-i\epsilon)} \, |\Omega\rangle \, .$$

It remains to get rid of the extraneous factors in the final expression. Well that's easy, we just divide out by

$$\langle \phi_b| \, e^{-i\hat{H}t}e^{-i\hat{H}t} \, |\phi_a\rangle = \int D\phi(x) \, \exp\left(i\int_{-t}^{t} d^4x \, \mathcal{L}(\phi)\right).$$

Theorem 2.2.2. *The amplitude for a propagation between spacetime points x_1 and x_2 is*

$$\langle \Omega| \, T\phi(x_1)\phi(x_2) \, |\Omega\rangle = \lim_{t\to\infty(1-i\epsilon)} \frac{\int D\phi(x) \, \phi(x_1)\phi(x_2) \exp\left(i\int_{-t}^{t} d^4x \, \mathcal{L}\right)}{\int D\phi(x) \, \exp\left(i\int_{-t}^{t} d^4x \, \mathcal{L}\right)}.$$

This quantity is important enough to have a name: it is called the **two-point correlation function**. It is usually denoted $\langle \phi(x_1)\phi(x_2)\rangle$ for convenience. Analogously, we have n-**point correlation functions** $\langle \phi(x_1)\cdots\phi(x_n)\rangle$.

Since $\pm\infty(1 - i\epsilon)$ is "a finite distance" away from $\pm\infty$, we usually write $\int_{-\infty}^{\infty} d^4x \, \mathcal{L}$ in the exponential. Better yet, we write $\int d^4x \, \mathcal{L}$ and take it to be understood that we are integrating over all spacetime now.

2.3 The Generating Functional

This formula for the propagation amplitude may not seem like much of an improvement. But it is, and it will be obvious by the end of this section how. Let's begin with the **free-field Lagrangian**

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2.$$

(Later on we will see why this is called the free field Lagrangian.) If we plug this Lagrangian into the path integral, the integral is directly computable; the end result is a Klein-Gordon field, which we are already familiar with. So let's add a general perturbation term:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 + J\phi,$$

where here $J(x)$ is a function of x, representing an **excitation**, and usually called a **source function**. The resulting path integral is written

$$Z[J] = \int D\phi \exp\left(i\int d^4x\, \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 + J\phi\right),$$

and called the **generating functional**. In this notation, we want to find $Z[J]/Z[0]$.

Note: adding a source function is not the same thing as adding an interaction. Source functions merely allow us to create sources and sinks, whereas interactions allow the excitations generated by the sources and sinks to interact with themselves. Here we are still working within a free Klein-Gordon theory.

We can write $Z[J]$ in a very explicit form. First, let's rewrite the Lagrangian a little via integration by parts:

$$\int d^4x\, \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 + J\phi = \int d^4x\, \frac{1}{2}\phi(-\partial^2 - m^2)\phi + J\phi.$$

We will evaluate this a little informally, but the entire argument can be made formal once we introduce the Green's function (the definition of which will be motivated by this argument). Imagine that the integral above is actually a giant sum, $\phi = (\phi_1, \ldots, \phi_n)$ is merely a vector, and $(-\partial^2 - m^2) = A$ merely an $n \times n$ matrix. Then $Z[J]$ becomes

$$\int d\phi_1 \cdots \int d\phi_n \, \exp\left(\frac{i}{2}\phi^T A\phi + iJ\phi\right) = \left(\frac{(2\pi i)^n}{\det A}\right)^{\frac{1}{2}} \exp\left(-\frac{i}{2}JA^{-1}J\right).$$

Physicists call this process "discretizing spacetime," which sounds cooler.

Now we want to pass back into the continuum limit, i.e. replace ϕ as a vector with ϕ as a field, and A with $(-\partial^2 - m^2)$. But what should be replace A^{-1} by? In the discretized case, we had $AA^{-1} = I$, so by analogy, we should replace A^{-1} by a function $G(x-y)$ satisfying

$$(-\partial^2 - m^2)G(x-y) = \delta(x-y).$$

Such a function $G(x-y)$ is a **Green's function** for the linear differential operator $(-\partial^2 - m^2)$. So we pause quickly to introduce Green's functions and related objects.

2.3.1 Green's Functions and Propagators

Definition 2.3.1. Given a linear differential operator $L(x)$ (acting on distributions), its **Green's function** $G(x-y)$ satisfies $L(x)G(x-y) = -i\delta(x-y)$. Hence given a differential equation of the form $L(x)u(x) = f(x)$, we can compute

$$L(x)\int dy\, G(x-y)f(y) = \int L(x)G(x-y)f(y)\, dy = -i\int \delta(x-y)f(y)\, dy = -if(x),$$

so that $u(x) = i\int dy\, G(x-y)f(y)$ is a solution.

For example, the defining property of the Green's function $G(x-y)$ for the Klein-Gordon operator (∂^2+m^2) can be written in momentum space:

$$(\partial^2+m^2)\int\frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}\tilde{G}(p)=\int\frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}.$$

Then it is easy to solve for $\tilde{G}(p)$. Equating the two integrands,

$$(\partial^2+m^2)e^{-ip(x-y)}\tilde{G}(p)=(-p^2+m^2)e^{-ip(x-y)}\tilde{G}(p)=-ie^{-ip(x-y)},$$

so we can directly write

$$G(x-y)=\int\frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}\tilde{G}(p)=\int\frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}\frac{i}{p^2-m^2}.$$

Here we must pause for a moment: there is something wrong with this integral. When we integrate over p, there are two singularities at $p^0=\pm E_{\vec{p}}$, so this integral diverges. That's okay, say the physicists, let's just specify how we treat the poles, and write down the following version of the Green's function:

$$D_F(x-y)=\int\frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}\tilde{G}(p)=\int\frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}\frac{i}{p^2-m^2+i\epsilon}.$$

Now the poles are displaced above and below the real p^0 axis, at $p^0=\pm(E_{\vec{p}}-i\epsilon)$ in the "complex p^0 plane", and we don't have divergence issues anymore. This version of the Green's function is called the **Feynman propagator**.

Exercise 2.3.1. Let $\theta(x-y)$ be the **Heaviside step function**, i.e. it is 1 when $x>y$, and 0 otherwise. Compute that

$$D_F(x-y)=\theta(x^0-y^0)\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}e^{-ip(x-y)}\Big|_{p^0=E_{\vec{p}}}+\theta(y^0-x^0)\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}e^{-ip(x-y)}\Big|_{p^0=-E_{\vec{p}}}$$

by analytic continuation into the complex p^0 plane, and by closing the contour either in the upper half plane when $x^0>y^0$, or in the lower half plane when $x^0<y^0$.

Why is the Green's function $D_F(x-y)$ called a propagator? The answer lies in computing the amplitude $\langle 0|\phi(\vec{x})\phi(\vec{y})|0\rangle$ for a particle to propagate from a point $\vec{x}$ in space to another point $\vec{y}$ in space:

$$\begin{aligned}\langle 0|\phi(\vec{x})\phi(\vec{y})|0\rangle&=\int\frac{d^3\vec{p}_1}{(2\pi)^3\sqrt{2E_{\vec{p}_1}}}\int\frac{d^3\vec{p}_2}{(2\pi)^3\sqrt{2E_{\vec{p}}}}e^{-i\vec{p}_1\vec{x}}e^{i\vec{p}_2\vec{y}}\,\langle 0|[a_{\vec{p}_1},a^\dagger_{\vec{p}_2}]|0\rangle\\&=\int\frac{d^3\vec{p}}{(2\pi)^3(2E_{\vec{p}})}e^{-ip(x-y)}\Big|_{p^0=E_{\vec{p}}}.\end{aligned}$$

Hence we have

$$D_F(x-y)=\theta(x^0-y^0)\,\langle 0|\phi(\vec{x})\phi(\vec{y})|0\rangle+\theta(y^0-x^0)\,\langle 0|\phi(\vec{y})\phi(\vec{x})|0\rangle=\langle 0|T\phi(\vec{x})\phi(\vec{y})|0\rangle\,.$$

So the Feynman propagator is the two-point correlation function for free field theory: it represents the amplitude for an excitation to propagate between $\vec{x}$ and $\vec{y}$.

2.3.2 Computing the Generating Functional

Now that we know about Green's functions, let's return to computing the generating functional $Z[J]$. Recall that we left off at passing back into the continuum limit from the discretized path integral

$$\int d\phi_1\cdots\int d\phi_n\,\exp\left(\frac{i}{2}\phi^TA\phi+iJ\phi\right)=\left(\frac{(2\pi i)^n}{\det A}\right)^{\frac{1}{2}}\exp\left(-\frac{i}{2}JA^{-1}J\right).$$

Now we know what to replace A^{-1} with: the Green's function $-iD_F(x-y)$. Hence

$$Z[J] = C \exp\left(-\frac{1}{2}\int d^4x\, d^4y\, J(x)D_F(x-y)J(y)\right)$$

for some constant C. What is C? It is $Z[0]$. We have proved the following result.

Proposition 2.3.2. *Let*

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{e^{ik(x-y)}}{k^2-m^2+i\epsilon}$$

be the Feynman propagator. Then

$$Z[J] = Z[0] \exp\left(-\frac{1}{2}\int d^4x\, d^4y\, J(x)D_F(x-y)J(y)\right).$$

Using this formula, let's compute some of the terms in $Z[J]/Z[0]$. Write

$$W[J] = -\frac{1}{2}\int\int d^4x\, d^4y\, J(x)D_F(x-y)J(y),$$

so that

$$Z[J]/Z[0] = \exp(iW[J]) = \sum_{n=0}^{\infty}\frac{(iW[j])^n}{n!}.$$

The $n=2$ term is therefore proportional to

$$\int\int\int\int d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4\, D_F(x_1-x_2)D_F(x_3-x_4)J(x_1)J(x_2)J(x_3)J(x_4).$$

How can we interpret this term physically? Well, recall that $D_F(x-y)$ is the propagation amplitude between x and y. So this integral is, up to a constant, the amplitude for an excitation at x_2 to propagate to x_1, and an excitation at x_4 to propagate to x_3, where x_1, x_2, x_3, x_4 can range over all space. The point is that the propagation from x_2 to x_1 does not affect the propagation from x_4 to x_3 whatsoever: we can completely separate the integrals. This is why we say $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2$ is a **free field theory**: there are no terms that create interactions between different excitations of the field!

For the free field Lagrangian, sometimes we can even explicitly compute $W[J]$. For example, let's take $J(x) = J_1(x) + J_2(x)$ where $J_i(\vec{x}) = \delta^{(3)}(\vec{x}-\vec{x}_i)$, to represent two distinct time-independent excitations. Then $W[J]$ will contain terms for J_1J_1, J_2J_2, and J_1J_2 and J_2J_1. We neglect the first two, since J_1J_1 would be present in $W[J]$ regardless of whether J_2 is present or not, and similarly for J_2; they correspond to "self-interaction" and are not interesting. Let's look at the other two terms:

$$\begin{aligned}
&-\frac{1}{2}\iint d^4x\, d^4y\, J_1(x)D_F(x-y)J_2(y) + J_2(x)D_F(x-y)J_1(y)\\
&= -\frac{1}{2}\iint dx_1^0\, dx_2^0\, D_F(x_1-x_2) + D_F(x_2-x_1)\\
&= -\iint dx_1^0\, dx_2^0 \int \frac{dp^0}{2\pi}\, e^{ip^0(x_1^0-x_2^0)} \int \frac{d^3p}{(2\pi)^3}\frac{e^{i\vec{p}(\vec{x}_1-\vec{x}_2)}}{p^2-m^2+i\epsilon}\\
&= \int dx_1^0 \int \frac{d^3p}{(2\pi)^3}\frac{e^{i\vec{p}(\vec{x}_1-\vec{x}_2)}}{\vec{p}^2+m^2+i\epsilon}.
\end{aligned}$$

Now we can do three things. First, isolate the $\int dx_1^0$; this evaluates to t, the time over which we do our path integral. Second, get rid of the $i\epsilon$ in the denominator; $\vec{p}^2+m^2$ is always positive, so there are no poles. Third, remember that $Z[J]$ for the free, **unperturbed** theory is just

$$Z[J] = \langle 0|e^{-i\hat{H}_0t}|0\rangle = e^{-iE_0t},$$

so up to the constant $Z[0]$, we can equate $e^{-iE_0 t}$ with $e^{iW[J]}$. We already have a factor of t in $W[J]$, so that cancels, and we are left with

$$E_0 = -\int \frac{d^3p}{(2\pi)^3} \frac{e^{i\vec{p}(\vec{x}_1-\vec{x}_2)}}{\vec{p}^2+m^2} < 0.$$

Whoa. What happened? We put two time-independent excitations on a field, and the ground state energy decreased. There is an attractive force between the two excitations!

Exercise 2.3.2. We didn't finish the computation of E_0: do the integral to obtain $E_0 = -e^{-mr}/4\pi r$ where r is the distance between $\vec{x}_1$ and $\vec{x}_2$.

Note: it may be confusing that earlier, we said the free-field theory is non-interacting, i.e. excitations do not interact, whereas here we clearly have an interaction (an attractive force) between two excitations. We must be careful what we mean by "excitation". An **excitation of the field** ϕ is represented by a propagator $D_F(x-y)$: we think of it as the exchange of a virtual particle between x and y. It is true that in free-field theory, two such field excitations do not interact, as we showed earlier with the $n=2$ term of $Z[J]$. However, two excitations in the form of **sources** and **sinks** placed on the field, i.e. terms in $J(x)$, are of course allowed to interact, via excitations of the field ϕ.

2.4 Feynman Diagrams

Okay, enough of free field theory; while it is interesting, it is unphysical to expect that excitations do not interact with each other. Let's move on to ϕ^4 **theory**, where the Lagrangian looks like

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4.$$

The λ is a **coupling constant**, and dictates how strongly the ϕ^4 term impacts the free field theory. The generating functional is now

$$Z[J,\lambda] = \int D\phi \exp\left(i\int d^4x\, \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 + J\phi\right).$$

How shall we compute $Z[J,\lambda]$? We have no idea whether it can be written in closed form. So the physicists do it perturbatively.

Let's consider a much easier problem to gain some insight: let q be a one-dimensional variable, and evaluate

$$Z = \int_{-\infty}^{\infty} dq\, \exp\left(-\frac{1}{2}q^2 + \lambda q^4 + Jq\right).$$

We know how to do this integral for $\lambda = 0$: it would just be a Gaussian. So expand it as a series

$$Z = \int_{-\infty}^{\infty} dq\, e^{-q^2/2+Jq}\left(1+\lambda q^4 + \lambda^2 q^8 + \cdots\right).$$

How do we evaluate each individual term? Here's a trick:

$$\int_{-\infty}^{\infty} dq\, e^{-q^2/2+Jq} q^{4n} = \left(\frac{d}{dJ}\right)^{4n} \int_{-\infty}^{\infty} dq\, e^{-q^2/2+Jq},$$

and we know how to evaluate the remaining Gaussian integral! So

$$Z = \left(1+\lambda\left(\frac{d}{dJ}\right)^4 + \lambda^2\left(\frac{d}{dJ}\right)^8 + \cdots\right)\int_{-\infty}^{\infty} dq\, e^{-q^2/2+Jq}.$$

We can do the original path integral for $Z[J]$ using this trick, but first we need to make sense of what d/dJ means when $J(x)$ is a function. Fortunately, mathematicians have done this for us already.

Definition 2.4.1. Let X be a space of functions, and $\Phi : X \to \mathbb{C}$ a functional (not necessarily linear). The **functional derivative** $\delta\Phi(f)/\delta f(x)$ is formally defined as

$$\frac{\delta\Phi(f)}{\delta f(x)} = \lim_{\epsilon\to 0} \frac{\Phi(f+\epsilon\delta_x) - \Phi(f)}{\epsilon},$$

where δ_x is a delta function with its pole at x. An important property is that $\delta f(x)/\delta f(y) = \delta^{(4)}(x-y)$.

Exercise 2.4.1. Show that

$$Z[J,\lambda] = Z[0,0] \exp\left(-\frac{i}{4!}\lambda \int d^4w \left(\frac{\delta}{\delta J(w)}\right)^4\right) \exp\left(-\frac{1}{2}\int d^4x\, d^4y\, J(x) D_F(x-y) J(y)\right)$$

by first extending the solution to the easier problem to multiple dimensions, and then "infinite dimensions". Then use the previous theorem, where we computed $Z[J,0]$.

At last, we arrive at the reason for which we have been doing all these calculations. Let's expand $Z[J,\lambda]$ in another way:

$$\begin{aligned} Z[J,\lambda] &= \int D\phi\, e^{i\int d^4x\, \mathcal{L}} \sum_{n=0}^{\infty} \frac{\left(i\int d^4x\, J(x)\phi(x)\right)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \left(\int dx_1 \cdots dx_n\, J(x_1)\cdots J(x_n)\right) \left(\int D\phi\, \phi(x_1)\cdots\phi(x_n) e^{i\int d^4x\, \mathcal{L}}\right). \end{aligned}$$

The second term looks oddly familiar. Indeed, it is (up to normalization), the **n-point correlation function** $\langle \phi(x_1)\cdots\phi(x_n)\rangle$ that we wanted to compute from a long time ago! So what the path integral has really given us is an extremely easy way to perturbatively compute the n-point correlation functions: we simply need to compute the coefficient of J^n, which is a series in λ. Taking the first few terms $\lambda^0, \lambda^1, \cdots$ will give an approximation to the correlation function.

Theorem 2.4.2. *The generating functional gives the following formula for n-point correlation functions:*

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle = (-i)^n \frac{\delta^n}{\delta J(x_1)\cdots\delta J(x_n)} \frac{Z[J,\lambda]}{Z[0,\lambda]}\bigg|_{J=0}.$$

For example, let's compute the λ^1 (first-order) terms in the 4-point correlation function. This is equivalent to computing the $\lambda^1 J^4$ term in $Z[J,\lambda]$. (**Note**: J^4 here really stands for $\left(\int dx\, J(x)\right)^4$, but we will abuse notation a little.) The $\lambda^1 J^4$ term comes from a J^8 term in $\exp(-(1/2)W[J])$ being differentiated by a λ^1 term in the other exponential:

$$\left(-\frac{i}{4!}\lambda \int d^4w \left(\frac{\delta}{\delta J(w)}\right)^4\right) \left(\frac{1}{4!2^4}\int d^4x_1 \cdots d^4x_8\, J_1 J_2 J_3 J_4 J_5 J_6 J_7 J_8 D_{12} D_{34} D_{56} D_{78}\right),$$

where J_n stands for $J(x_n)$, and D_{ij} stands for $D_F(x_i - x_j)$.

Exercise 2.4.2. Do this computation. It is really not as bad as it looks: think of the action of $\partial/\partial J(w)$ as selecting one of the J_i's, and setting its variable, i.e. x_i, to w. So applying $(\partial/\partial J(w))^4$ really just picks four different x_i's and sets them to w. For example, we can pick x_2, x_4, x_6, x_8 to get a term proportional to

$$\lambda \int dw \int d^4x_1\, d^4x_3\, d^4x_5\, d^4x_7\, J_1 J_3 J_5 J_7 D_{1w} D_{3w} D_{5w} D_{7w}.$$

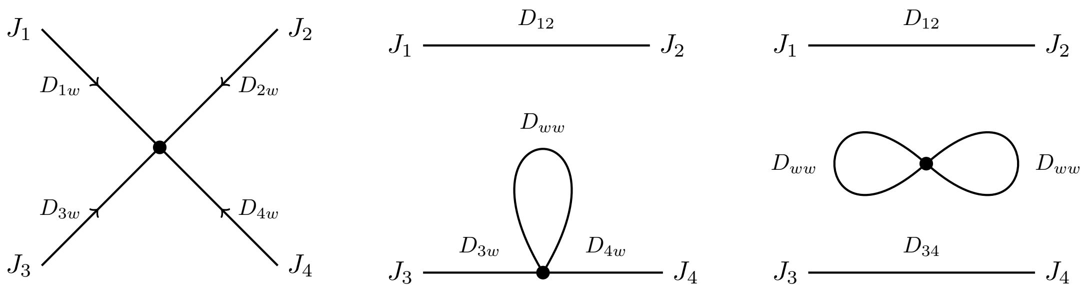
Note, however, there are many ways to get a term of this form. We could have picked any of the 4! permutations of x_2, x_4, x_6, x_8. Or any of the 4! permutations of x_1, x_3, x_5, x_7. Or we can substitute 1 for 2, or 3 for 4, etc. in any such choice. There are **symmetry factors** for each term. Compute these carefully. You should get the result

$$\int_w \int \int \int \int J_1 J_2 J_3 J_4 \left(-i\lambda D_{1w} D_{2w} D_{3w} D_{4w} - \frac{i\lambda}{2} D_{12} D_{3w} D_{4w} D_{ww} - \frac{i\lambda}{8} D_{12} D_{34} D_{ww} D_{ww} \right),$$

from which you can conclude that the first-order term in $\langle \phi(x_1) \cdots \phi(x_4) \rangle$ is

$$-i\lambda \int dw \left(D_{1w} D_{2w} D_{3w} D_{4w} + \frac{1}{2} D_{12} D_{3w} D_{4w} D_{ww} + \frac{1}{8} D_{12} D_{34} D_{ww} D_{ww} \right).$$

As with the free field case, we can physically interpret each of these terms. For example, in the first term, excitations from x_1, x_2, x_3, x_4 are propagating from/to an interaction point w. In the second term, there is a **self-interaction** from w to w, and interactions between x_1 and w, x_2 and w, and x_3 and x_4. We can similarly interpret the third term. The point is that to each term we can associate a little pictorial diagram of what is physically happening:



These diagrams both represent what is physically happening, in position space, and the terms in the integral. They are also known as **Feynman diagrams**!

When one does more of these types of calculations for other $\lambda^k J^n$ terms, one can see many patterns. In each of the three Feynman diagrams above, there are:

- one **internal vertex** (the black dot), corresponding to the comes from the variable w introduced by the λ^1 term (if we were to compute a λ^k term, there would be k internal vertices);
- four **external vertices** coming from the four J's (if we were to compute a J^n term, there would be n external vertices);
- four **propagators** coming from the four D_F's (if we were to compute a J^n term, there would be n external vertices).

Exercise 2.4.3. There are also rules for what possible diagrams can arise in the ϕ^4 theory. Convince yourself that any diagram has:

- four propagator ends at each internal vertex;
- an even number of external vertices;
- ((number of internal propagators) - (number of internal vertices) + 1) loops.

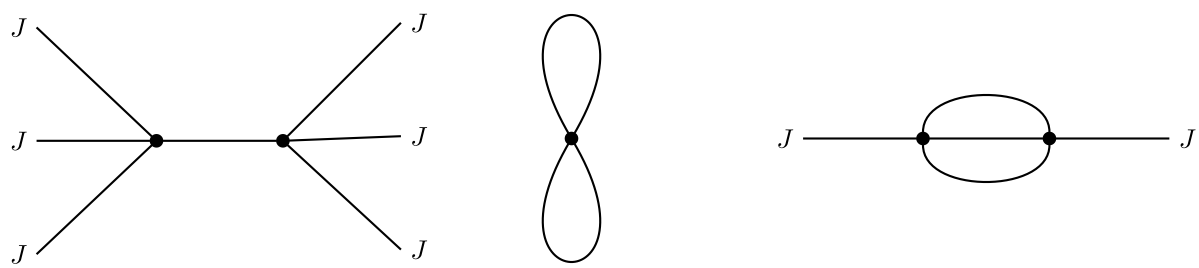
So we see that Feynman diagrams are good, intuitive, easy-to-manipulate representations of the terms of $Z[J, \lambda]$. In fact, instead of going from a term in $Z[J, \lambda]$ to a Feynman diagram, we can go the other way: given a Feynman diagram that can arise from the ϕ^4 theory, we can directly write down the term associated to it. This greatly simplifies the calculation of the terms, since now we can just draw pictures instead of taking $4k$ derivatives $\delta/\delta J(w)$ acting on $2n$ terms $J(x)$.

Theorem 2.4.3. *Given a Feynman diagram, one can write down its corresponding value using the* **position-space Feynman rules for ϕ^4 theory***:*

- *for each propagator between vertices x and y, add a $D_F(x-y)$ term;*
- *for each internal vertex, add a $(-i\lambda)\int d^4z$ term;*
- *for each external vertex, add a factor of 1;*
- *divide by the* **symmetry factor***.*

The symmetry factor is the number of ways of interchanging internal components of the diagram without changing the diagram itself.

Exercise 2.4.4. Symmetry factors take some getting used to. They arise from the combinatorics of how the derivatives $\delta/\delta J$ can hit different J, and the symmetry $D_F(x-y) = D_F(y-x)$. For example, when a diagram has a loop in it, we naively overcount by a factor of two because of the symmetry in D_F. Compute the symmetry factors of the following three diagrams using similar reasoning:



You should get 1 and $2 \cdot 2 \cdot 2 = 8$ and $3! = 6$ respectively.

Exercise 2.4.5. Rewrite the position-space Feynman rules in momentum-space, to obtain the **momentum-space Feynman rules for ϕ^4 theory**:

- for each internal propagator, label it with a momentum p and add a $i/(p^2 - m^2 + i\epsilon)$ term;
- for each external propagator (i.e. source), label it with a momentum p and add a e^{-ipx} term;
- for each internal vertex, add a $-i\lambda$ term;
- for each vertex, impose momentum conservation by adding a $\delta^{(4)}(p_1 + p_2 - p_3 - p_4)$ term, where p_1, p_2, p_3, p_4 are the four propagators entering/leaving the vertex;
- integrate over every undetermined momentum p with $\int d^4p/(2\pi)^4$;
- divide by the symmetry factor.

Note that in momentum space, we must orient each propagator. The orientation is arbitrary, since $D_F(x-y) = D_F(y-x)$, but necessary for imposing momentum conservation.

2.5 Connected vs Disconnected

Chapter 3

Quantum Electrodynamics

3.1 Functional Quantization of Spinor Fields

Now that we have understood the essentials of the path integral formulation we would now like to apply this formalism to understand how fermions interact in an electromagnetic field. Before we can get anywhere let us revisit the Dirac field and attempt to quantize it using the path integral formalism.

Recall that in our discussion of the Dirac field we realized that fermionic quantization required anti-commutation relations. This suggests that the eigenfunctions of these operators also need to anticommute. These sorts of variables are called Grassmann numbers: $\theta\eta = -\eta\theta$. To proceed, we first need to understand how to integrate over such variables. It will turn out that integration over these variables is much easier than regular integration! Consider the Taylor expansion $f(\theta) = A + B\theta + C\theta^2 + D\theta^3 + \cdots$. If θ is a Grassmann variable then $\theta^2 = 0$ and so $f(\theta) = A + B\theta$. The two properties that we would like of integration is that it is linear and invariant under a translation of variables, $\theta \to \theta + \eta_0$. Thus:

$$\begin{aligned}\int d\theta\, f(\theta) = \int d\theta\, A + B\theta &= \int d(\theta + \eta_0)\, A + B(\theta + \eta_0) \\ &= \int d\theta\, (A + B\eta_0) + B\theta\end{aligned}$$

By linearity, we expect that this integral be a linear function of the constant term, A, and the linear term, B. The last equality implies that The integral does not depend on the constant term. Therefore, we may assume that the integral evaluates to:[1]

$$\int d\theta\, A + B\theta = B$$

If we integrate more than one Grassmann number we need the following convention that $\int d\eta \int d\theta\, \theta\eta = 1$. This definition makes some integrals very easy to evaluate. We compile a few that will be useful for us later on:

$$\int d\theta^* d\theta e^{-\theta^* b\theta} = b^{-1}, \quad \left(\prod_i \int d\theta_i^* d\theta_i\right) e^{-\theta_i^* B_{ij}\theta_j} = \prod_i b_i = \det B, \quad \left(\prod_i \int d\theta_i^* d\theta_i\right) \theta_k\theta_l^* e^{-\theta_i^* B_{ij}\theta_j} = (\det B)(B^{-1})_{kl} \tag{3.1}$$

[1] Integration by differentiation!

Recall that the Lagrangian for the Dirac field is given by $\mathcal{L}_{Dirac} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$ where $\partial\!\!\!/ = \gamma^\mu \partial_\mu$. Using the integrals given in (3.1) it is possible to immediately calculate the correlation functions for the free field theory directly. Instead, we will get some more practice with our generating functional method.

Take the generating functional,

$$Z[\bar{\eta}, \eta] = \int D\bar{\psi} D\psi \exp\left[i \int d^4x\, \bar{\psi}(i\partial\!\!\!/ - m)\psi + \bar{\eta}\psi + \bar{\psi}\eta\right]$$

So that if you make the substitution $\psi \to \psi + (i\partial\!\!\!/ - m)^{-1}\bar{\eta}$ then we can evaluate Z explicitly:

$$Z[\bar{\eta}, \eta] = Z_0 \cdot \exp\left[-\int d^4x d^4y \bar{\eta}(x) S_F(x-y) \eta(y)\right]$$

Therefore we are in position to evaluate arbitrary n-point correlation functions. This is where the generating functional method shines, on the one hand it is easy to compute explicitly, and on the other hand by differentiating it in a special way it gives rise to the correlation functions! In the free field case, here is an example of such a correlation function calculation:

$$\langle 0|T\psi(x_1)\psi(x_2)|0\rangle = Z_0^{-1} \cdot \left(\frac{-i\delta}{\delta\bar{\eta}(x_1)}\right)\left(\frac{+i\delta}{\delta\eta(x_2)}\right) Z[\bar{\eta}, \eta] = \left(\frac{-i\delta}{\delta\bar{\eta}(x_1)}\right)\left(\frac{+i\delta}{\delta\eta(x_2)}\right) \exp\left[-\int d^4x d^4y \bar{\eta}(x) S_F(x-y)\eta(y)\right]$$

What are possible interactions that we can add to the Dirac field? Since we've already discussed the scalar field, we may start by coupling the Dirac field to the scalar field and trying to compute the correlation functions.

To talk about QED we need the quantized version of the electromagnetic field. Here we go!

3.2 Functional Quantization of Electromagnetic Field

sketch

The electromagnetic field is described by a 4-vector, $A_\mu = (\phi, \vec{A})$, which packages the information about the electric field into ϕ and the magnetic field into $\vec{A}$. By taking the Lagrangian: $\mathcal{L}_{EM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, we recover Maxwell's equations (exercise!). We may write the action explicitly in terms of the A_μ:

$$\begin{aligned}
\int d^4x \mathcal{L}_{EM} &= \int d^4x\ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\
&= -\frac{1}{4}\int d^4x\ (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \\
&= \frac{1}{2}\int d^4x\, A_\mu(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu)A_\nu
\end{aligned}$$

Fourier transforming the variables, A_μ, would get:

$$\int d^4x \mathcal{L}_{EM} = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\, A_\mu(k^2 g^{\mu\nu} - k^\mu k^\nu)A_\nu$$

The 4×4 matrix $(k^2 g^{\mu\nu} - k^\mu k^\nu)$ is not invertible and so

Change of variables

$$\int_{\mathbb{R}^n} dx\, \delta(g(x))|g'(x)|f(g(x)) = \int_{g(\mathbb{R}^n)} du \delta(u) f(u)$$

which means that we can interpret $\int_{\mathbb{R}^n} dx\, \delta(g(x))|g'(x)|$ as taking a constant function and returning the constant function back. So

$$1 = \int_{\mathbb{R}^n} dx\, \delta(g(x))|g'(x)|$$
$$1 = \int \mathcal{D}\alpha(X)\delta(G(A^\alpha)) \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)$$

Note $\det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right) = \det(\partial^2/e)$ Then

$$\begin{aligned}\int \mathcal{D}A e^{iS[A]} &= \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)\int \mathcal{D}\alpha \int \mathcal{D}A e^{iS[A]}\delta(G(A^\alpha)) \\ &= \det(\partial^2/e)\left(\int \mathcal{D}\alpha\right)\int \mathcal{D}A^\alpha e^{iS[A^\alpha]}\delta(G(A^\alpha))\end{aligned}$$

Take a general gauge-fixing condition, $G(A) = \partial^\mu A_\mu(x) - \omega(x)$ for scalar function ω. The above holds for any linear combination of ω's so that we may take a very big linear combination, weigted by a Gaussian:

$$\begin{aligned}\int \mathcal{D}A e^{iS[A]} &= \underbrace{N(\xi)\int D\omega \exp\left[-i\int d^4x \frac{\omega^2}{2\xi}\right]}_{=1} \det(\partial^2/e)\left(\int \mathcal{D}\alpha\right)\int \mathcal{D}A e^{iS[A]}\delta(\partial^\mu A_\mu - \omega(x)) \\ &= N(\xi)\det(\partial^2/e)\left(\int \mathcal{D}\alpha\right)\int \mathcal{D}A e^{iS[A]} \exp\left[-i\int d^4x \frac{(\partial_\mu A^\mu)^2}{2\xi}\right] \\ &= N(\xi)\det(\partial^2/e)\left(\int \mathcal{D}\alpha\right)\int \mathcal{D}A \exp\left[\frac{i}{2}\int d^4x\, A_\mu(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu)A_\nu - \frac{(\partial_\mu A^\mu)^2}{\xi}\right] \\ &= N(\xi)\det(\partial^2/e)\left(\int \mathcal{D}\alpha\right)\int \mathcal{D}A \exp\left[\frac{i}{2}\int d^4x\, A_\mu(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu)A_\nu + \frac{A_\mu\partial^\mu\partial^\nu A_\nu}{\xi}\right] \\ &= N(\xi)\det(\partial^2/e)\left(\int \mathcal{D}\alpha\right)\int \mathcal{D}A \exp\left[\frac{i}{2}\int d^4x\, A_\mu(\partial^2 g^{\mu\nu} - \left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\nu)A_\nu\right] \\ &\propto \int \mathcal{D}A \exp\left[\frac{i}{2}\int \frac{d^4k}{(2\pi)^4} A_\mu(k^2 g^{\mu\nu} - \left(1-\frac{1}{\xi}\right)k^\mu k^\nu)A_\nu\right]\end{aligned}$$

Inverting,

$$D_F^\gamma(k) = \frac{-i}{k^2+i\epsilon}(g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}$$

The full free field QED Lagragian will then be of the form:

$$Z[J_\mu, \bar{\eta}, \eta] = \exp\left[\frac{-1}{2}\int d^4x d^4y\, \bar{\eta}S_F(x-y)\eta + J_\mu(D_F^\gamma)^{\mu\nu}J_\nu\right]$$

3.2.1 Interaction term: Fermions and Photons

Let's guess the interaction term:

- ψ is a column vector, $\bar{\psi}$ is a row vector.
- A_μ is a column vector.
- γ^ν are four matrices

After a bit of fiddling there are a few simple choices, up to prefactors,

$$\bar{\psi}\psi \not{A}, \bar{\psi}\psi A^2, \bar{\psi}\gamma^\mu\psi A_\mu.$$

But there are many higher order terms. How do we exclude these terms theoretically? I mean, we don't have to: we can guess one model, realize that it agrees well with experiment and be done. However, it's possible to theoretically justify why the simplest theory ought to be the correct one. This is the argument of **renormalizability**.

Introduction to Renormalizable Theories

Higher order terms in perturbation theory can be computed using integrals over 4-momenta. These integrals are seemingly divergent, so we

1. Introduce a large cutoff Λ.
2. Evaluate integral.
3. Take $\Lambda \to \infty$, hope answer is independent of Λ.

A theory in which this procedure works are called renormalizable. Whether a theory is or is not renormalizable can be heuristically checked by a dimensional analysis.

Using $E = 2\pi\hbar\nu = pc = p = mv = m$ and $L/T = c = 1$ we list two relations that are useful for us:

$$L = T$$
$$T^{-1} = [p] = M$$

The momentum cutoff has units, $[\Lambda] = [p] = M$. Moreover the Lagrangian has units of M^4 since

- Action dimensionless, $S = e^{i\int d^4x\mathcal{L}}$
- so $1 = [Length]^4 \cdot [\mathcal{L}] = [M]^{-4}[\mathcal{L}]$.

Examples of dimensions.

- $\mathcal{L} = \frac{1}{2}\partial^2\phi - \frac{1}{2}m^2\phi^2 \implies [\phi] = M$.
- $\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi \implies [\bar{\psi}m\psi] = M^4, [\psi] = M^{3/2}$.
- $\mathcal{L} = F^2 = A_\nu\partial^2 g^{\mu\nu}A_\mu + \dots \implies [A^2\partial^2] = [A^2m^2] = M^4 \implies [A_\mu] = M$

Scattering amplitudes.

$$[\text{Coupling const.}] \sim \frac{1}{M^k} \implies [Sc.Ampl] \sim \frac{1}{M^k} \cdot \Lambda^k \to \infty.$$

To combine ψ and A_μ we need:

$$[\text{Coupling const.}] \cdot [\psi]^k[A_\mu]^\ell = M^c M^{3k/2} M^\ell = 1$$

so

$$3k/2 + \ell \leq 4 \iff (k, \ell) \in \{(1,1),(2,1)\}.$$

3.3 Aside: Scattering Amplitudes

Let us briefly try to understand the big picture – or, in fact, I guess it's the small picture! A **cross section** is the fundamental physical quantity that an experimental particle physicist is interested in, here's roughly what it is. Suppose you want to understand the inner workings of protons. You will devise an experiment to smash some number of protons together and try to investigate the result. In fact, your theorist friends suggest to you some possible types of particles that can come out. Ideally, if we're not in one already, we will consider two protons colliding to produce other species: $pp \to abc\ldots$.

Your detectors cover only a portion of 4π, as the experimentalists call it, that is, the sphere surrounding the collision point. Therefore, we would like to compute the amount of scattering events in the direction of the detector. This is given the symbol:

$$\left(\frac{d\sigma}{d\Omega}\right)$$

This is a very technical object and it depends on a lot of parameters. Qualitatively, we may simplify this object by considering just the **scattering events** which are partly characterized by momenta (particle type, spin, etc. are also valid quantum numbers). The scattering events can be analyzed by looking at the **S-matrix**:

$$\begin{aligned}\langle \vec{p}_1 \cdots |S| k_A k_B\rangle &:= {}_{\text{out}}\langle \vec{p}_1\vec{p}_2 \cdots | \vec{k}_A \vec{k}_B\rangle_{\text{in}} \\ &= \lim_{T\to\infty} \langle \vec{p}_1 \cdots | e^{iH(2T)} | \vec{k}_A \vec{k}_B\rangle\end{aligned}$$

In words, this means that we assume that our particles are idealized momentum eigenstates and long after they interact they again will be idealized momentum eigenstates. We apply a trick, that sometimes particles don't interact at all, and we can take that out of the S matrix:

$$S = \mathbb{1} + iT$$

Assuming unitarity of S (which is a big assumption!) T is a Hermitian operator (think of $e^{i\theta}$) and it contains all of the interesting dynamics. By also considering energy conservation can be written as:

$$\langle p_1 \ldots |iT| k_A k_B\rangle = (2\pi)^4 \delta^{(4)}(k_A + k_B - \sum p_f)\mathcal{M}(k_A, k_B \to \{p_f\})$$

What this means is that the scattering amplitude is really a distribution and the density is given by the expression $\mathcal{M}$. To briefly connect this back to the cross section, in the case that there are four identical particles (two incoming and two outgoing) then the cross section will be (PS 4.85)

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{cm}^2}.$$

3.3.1 Scattering amplitudes via Feynman diagrams

Now we interpret the scattering amplitude as something very similar to a correlation function. Recall a correlation function for us was

$$\langle \Omega | T\{\phi(x_1)\phi(x_2)\}|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\int D\phi\, \phi(x_1)\phi(x_2) \exp[i\int_{-T}^{T}\mathcal{L}]}{\int D\phi\, \exp[i\int_{-T}^{T}\mathcal{L}]} \tag{3.2}$$

$$= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\{\phi(x_1)\phi(x_2)\exp[-i\int_{-T}^{T} dt\, H_I(t)]\}|0\rangle}{\langle 0|T\{\exp[-i\int_{-T}^{T} dt\, H_I(t)]\}|0\rangle} \tag{3.3}$$

Although we haven't explicitly mentioned this in our lectures, the formula in line (3.3) has appeared in the derivation of (3.2) (see (2.1) to compare).

Here's a quick review of how we derived a path integral representation for the correlation functions:

- Using a leap of intuition we define (or is it prove?)

$$\langle\phi(a)|e^{-iHt}|\phi(b)\rangle = \int \mathcal{D}\phi \exp\left[i\int d^4x\mathcal{L}\right]$$

- Investigate the expression

$$\int \mathcal{D}\phi\,\phi(x_1)\phi(x_2)\exp\left[i\int d^4x\mathcal{L}\right]$$

  by splitting up the path into three sections.

- Simplify this to get

$$\langle\phi_b(\vec{x})|\,e^{-i\hat{H}(t-x_2^0)}\phi(\vec{x}_2)e^{-i\hat{H}(x_2^0-x_1^0)}\phi(\vec{x}_1)e^{-i\hat{H}(x_1^0-(-t))}\,|\phi_a(\vec{x})\rangle \tag{3.4}$$

- The Canonical Quantizers prefer to rewrite this expression up to prefactors (which come from approximating $\phi_{a,b}(\vec{x})$ with $|\Omega\rangle$):

$$\langle 0|T\left\{\phi_I(x_1)\phi_I(x_2)\exp[-i\int_{-T}^{T} dt\,H_I(t)]\right\}|0\rangle$$

  The extra I's on the ϕ's absorb the free evolution into the original ϕ's.

Now we may interpret the scattering amplitude as the following:

$$\langle\vec{p}_1\ldots\vec{p}_n|iT|\vec{p}_A\vec{p}_B\rangle = \lim_{T\to\infty(1-i\epsilon)} {}_0\langle\vec{p}_1\ldots\vec{p}_n|e^{-iH(2T)}|\vec{p}_A\vec{p}_B\rangle_0 \tag{3.5}$$

$$\propto \lim_{T\to\infty(1-i\epsilon)}\left({}_0\langle\vec{p}_1\ldots\vec{p}_n|T\exp\left[-i\int_{-T}^{T}dt\,H_I(t)\right]|\vec{p}_A\vec{p}_B\rangle_0\right)_{\text{connected, amputated}} \tag{3.6}$$

The zeros on the end of the bracket refer to idealistic wavepackets.

3.3.2 Wick's Theorem

Our next goal will be to interpret this amplitude as a sum of Feynman diagrams. We will now introduce **Wick's theorem**, and thus **Wick contractions**, using which we'll be able to write down the Feynman diagrams very easily.

In canonical quantization there are raising and lowering operators. We introduce a particular order for expressions involving operators: raising on the left and lowering on the right – **Normal Order**. A Wick contraction is defined by:

$$\contraction{}{\phi}{(x)}{\phi}\phi(x)\phi(y) = D_F(x-y)$$

Finally Wick's theorem states the following:

Theorem 3.3.1.

$$T\{\phi(x_1)\phi(x_2)\cdots\phi(x_n)\} = N\{\phi(x_1)\phi(x_2)\cdots\phi(x_n) + \textit{all possible contractions}\}$$

This is a theorem with very major *computational* consequences for physics. In particular, this makes the Feynman diagrams immediate! Let's go through an example from Peskin and Schroeder:

$$\begin{aligned}
&\langle 0|T\left\{\phi(x)\phi(y)+\phi(x)\phi(y)\left[-i\int dt\int d^3z\frac{\lambda}{4!}\phi(z)^4\right]+\cdots\right\}|0\rangle \\
&=\langle 0|T\{\phi(x)\phi(y)\}|0\rangle+\frac{(-i\lambda}{4!}\langle 0|T\{\phi(x)\phi(y)\phi(z)\phi(z)\phi(z)\phi(z)|0\rangle
\end{aligned}$$

Apply Wick's Theorem:

$$\begin{aligned}
&\langle 0|N\{\phi(x)\phi(y)+\wick{\c1\phi(x)\c1\phi(y)}\}|0\rangle \\
&+\frac{(-i\lambda)}{4!}\langle 0|N\{\phi(x)\phi(y)\phi(z)\phi(z)\phi(z)\phi(z)\}|0\rangle \\
&+\frac{(-i\lambda)}{4!}\langle 0|N\{\wick{\c1\phi(x)\c1\phi(y)}\phi(z)\phi(z)\phi(z)\phi(z)\}|0\rangle \\
&+\frac{(-i\lambda)}{4!}\langle 0|N\{\wick{\c1\phi(x)\phi(y)\c1\phi(z)}\phi(z)\phi(z)\phi(z)\}|0\rangle+\cdots \\
&+\frac{(-i\lambda)}{4!}\langle 0|N\{\wick{\c1\phi(x)\c1\phi(y)\c2\phi(z)\c2\phi(z)}\phi(z)\phi(z)\}|0\rangle+\cdots \\
&+\frac{(-i\lambda)}{4!}\langle 0|N\{\wick{\c1\phi(x)\c2\phi(y)\c2\phi(z)\c1\phi(z)\c3\phi(z)\c3\phi(z)}\}|0\rangle+\cdots
\end{aligned}$$

Now you can probably see where the Feynman Diagrams will be appearing! They In fact, using this approach I would hazard that it's even easier to see where the Feynman diagrams are coming from! Because of the normal ordering most of these are zero, in fact only the Fuchsia coloured terms are non-zero.

Let's apply these rules to calculate the scattering amplitude (finally!). One last rule before we do this, since $|\vec{p}\rangle \propto a^\dagger_{\vec{p}}|0\rangle$, it makes sense to define:

$$\wick{\c1\phi(x)|\c1p\rangle}=e^{-ip\cdot x}\qquad\qquad \wick{\langle \c1p|\c1\phi(x)}=e^{+ip\cdot x}$$

Now, we compute a few examples. NOTE: the first amplitude does **not** have a time ordering so we compute it directly.

$$\langle p_1p_2|p_Ap_B\rangle=\sqrt{2E_12E_22E_A2E_B}\,\langle 0|a_1a_2a^\dagger_Aa^\dagger_B|0\rangle=(diagram)$$

$$\begin{aligned}
\langle p_1p_2|\mathcal{T}\left\{\frac{(-i\lambda)}{4!}\int d^4x\phi_x\phi_x\phi_x\phi_x\right\}|p_Ap_B\rangle=\frac{(-i\lambda)}{4!}\int d^4x\Big(&(\ )\times\langle p_1p_2|\wick{\c1\phi_x\c1\phi_x\c2\phi_x\c2\phi_x}|p_Ap_B\rangle \\
&+(\ )\times\langle p_1\wick{\c1p_2|\c1\phi_x\c2\phi_x\c2\phi_x\c3\phi_x|\c3p_A}p_B\rangle \\
&+(4!)\times\langle\wick{\c1p_1\c2p_2|\c2\phi_x\c1\phi_x\c3\phi_x\c4\phi_x|\c4p_A\c3p_B}\rangle
\end{aligned}$$

The blue terms are not interesting. The very first term is just zero because they contribute to the $\mathbb{1}$ in $S=\mathbb{1}+iT$; the second blue term is also not interesting because it also contributes to the $\mathbb{1}$. The fuchsia term is very interesting because by applying the rules that we've expressed above we get:

$$\langle p_1 p_2 | iT | p_A p_B \rangle \approx \frac{-i\lambda}{4!} \int d^4x \, (4!) \times e^{i(p_1+p_2-p_A-p_B)\cdot x} = (-i\lambda) \cdot (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_A - p_B)$$

But we also had,

$$\langle p_1 p_2 | iT | p_A p_B \rangle = i\mathcal{M}(p_1 p_2 \to p_A p_B) \cdot (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_A - p_B)$$

3.4 Amplitudes from QED

How do we handle any other theory? We have developed two procedures: first we wrote down the scattering amplitude; second we drew a Feynman diagram associated to it. Ideally, we'd like to go the other way because drawing diagrams is easier. But in order to do this we need to know the (Feynman) rules, of what we need to replace each arrow or vertex in the propogator by.

In the spin-0 theory, I defined the Wick contraction to be $D_F(x-y)$, but for other particles, we replace this with the associated propogator. The vertex terms can be read off the interaction Hamiltonian. All of the subtle points, always go back to the expression for the scattering amplitude and computed directly; the passage to the combinatorial Feynman diagrams is for now just a convenient representation. Without further ado, let's begin QED.

Let's quickly recall that all solutions to the Dirac equation can be spin up or down and the general Dirac field can be written as in (1.2) and (1.3), which we remind ourselves here:

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} a^s_{\vec{p}} u^s(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} + b^{s\dagger}_{\vec{p}} v^s(\vec{p}) e^{i\vec{p}\cdot\vec{x}}, \quad \bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} b^s_{\vec{p}} \bar{v}^s(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} + a^{s\dagger}_{\vec{p}} \bar{u}^s(\vec{p}) e^{i\vec{p}\cdot\vec{x}}, \quad \text{where } s = 1, 2$$

Feynman Rules for QED

1. Propogators:

$$\psi(x)\bar{\psi}(y) = \frac{i}{\not{p} - m + i\epsilon}$$

$$A_\mu(x) A_\nu(y) = \frac{-ig_\mu \nu}{q^2 + i\epsilon}$$

2. Vertex: $-ie$.

3. External legs contractions (for more details and Feynman diagrams see Peskin and Schroeder pg 118, 123):

$$\psi(x) \underbrace{|p, s\rangle}_{\text{fermion}} = u^s(p) |0\rangle \qquad \underbrace{\langle p, s|}_{\text{fermion}} \bar{\psi}(x) = \bar{u}^s(p) |0\rangle$$

$$\bar{\psi}(x) \underbrace{|k, s\rangle}_{\text{antifermion}} = v^s(k) |0\rangle \qquad \underbrace{\langle k, s|}_{\text{antifermion}} \psi(x) = v^s(p) |0\rangle$$

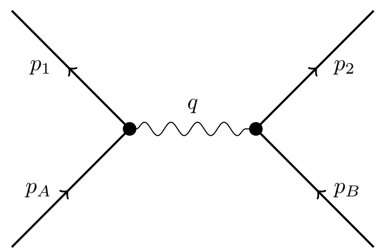
$$A_\mu |p\rangle = \epsilon_\mu(p) \qquad \langle p| A_\mu = \epsilon^*_\mu(p)$$

4. Impose momentum conservation at each vertex.
5. Integrate over each undetermined loop momentum.

Thus, a typical matrix element is given by

$$\begin{aligned}\langle p_1 p_2 | \int d^3x \bar{\psi}\gamma^\mu \psi A_\mu \cdot \bar{\psi}\gamma^\nu \psi A_\nu | p_A p_B\rangle &= \left(\int d^4q\right)\cdot(-ie)^2 \bar{u}(p_2)\gamma^\mu u(p_A)\left(\frac{-ig_{\mu\nu}}{q^2+i\epsilon}\right)\bar{u}(p_1)\gamma^\nu u(p_B) \\ . \qquad &\times (2\pi)^8 \delta^{(4)}(p_A+q-p_1)\delta^{(4)}(p_B+q-p_2) \\ &= i\mathcal{M}\times(2\pi)^4\delta^{(4)}(p_A+p_B-p_1-p_2)\end{aligned}$$

where the diagram corresponding to this choice of contractions is given by:



Born Approximation

In the nonrelativistic regime, one has the following identity:

$$\langle \vec{p}|iT|\vec{q}\rangle = -iV(\vec{p}-\vec{q})\delta(E_p - E_q)$$

In the case of the $e^-e^- \to e^-e^-$ vertex that we computed above we get

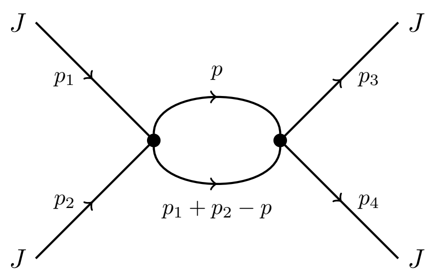
$$V(x) = -\frac{e^2}{4\pi r}$$

which is the Coulomb repulsion that we are familiar.

Chapter 4

Renormalization

There is a problem with the machinery we have been developing that we have been sweeping under the rug. This problem is best illustrated by explicitly computing the following Feynman diagram for ϕ^4 theory (it is the only type of second-order interaction we are interested in when computing 4-particle scattering amplitudes):



Here we've labeled each source and propagator with its momentum, where since p is an internal momentum, we must integrate over it. The value of the diagram is

$$\frac{(-i\lambda)^2}{2} \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} \frac{i}{(p_1 + p_2 - p)^2 - m^2 + i\epsilon}.$$

As $|p| \to \infty$, the integrand is $O(1/p^4)$, which diverges logarithmically when integrated over! This divergence cannot simply be ignored, since it does not get absorbed into the $Z[0,0]$ constant that we can ignore; it is part of a J^4 term. In fact, there are many such divergences. An easy observation is that any diagram with this diagram as a sub-diagram will have the same divergence. So we must develop a way to remove these divergences, or, rather, explain why they come about in our naive theory, and how to avoid them in a more sophisticated theory. In this chapter, we will develop such a theory and systematically remove these divergences using **renormalization theory**.

4.1 Counting Divergences

The first step to getting rid of divergences is to understand what kind of divergences occur, and this we will do by analyzing the divergent Feynman diagrams. For this section, we will work in d dimensions (instead of 4) to make the general structure of the results clearer.

Why did the above diagram diverge? The obvious answer is that there weren't enough powers of p in the denominator.

Definition 4.1.1. The **superficial degree of divergence** D is the difference

$$D = (\text{power of } p \text{ in numerator}) - (\text{power of } p \text{ in denominator}).$$

Every integration dp counts as a power of p in the numerator. If $D \geq 0$, we say the integral is **superficially divergent**.

If $D \geq 0$ then surely the integral is divergent. However, this is not an if and only if. For example, take $\iint dx\,dy\, 1/(1+x^2)^2$ in $\mathbb{R}^2$, for which $D = -2$, yet the integral still diverges because it does not decay in y. The good news is that this is the only sort of pathological divergence we will get.

Theorem 4.1.2 (Weinberg-Dyson)**.** *A connected Feynman diagram is convergent if and only if it and all of its connected subdiagrams are not superficially divergent.*

Suppose now that we are given a Feynman diagram for some process in ϕ^4 theory, with V internal vertices, E external propagators (from sources J to internal vertices), and I internal propagators. Using the momentum-space Feynman rules, each internal propagator decreases D by 2, and every momentum being integrated over increases D by 4. How many such momenta are there? Well, each internal propagator is associated to one, but every internal vertex adds a single linear constraint on them. So there are $I - V + 1$ independent momenta (since global momentum conservation frees one constraint), and

$$D = -2I + d(I - V + 1) = (d-2)I - dV + d.$$

Now note that every external propagator connects to one internal vertex, every internal propagator to two, and every vertex connects four propagators, so $2I + E = 4V$. Then

$$D = \frac{1}{2}(d-2)(4V - E) - dV + d = (d-4)V - \frac{1}{2}(d-2)E + d.$$

The behavior of D depends on the dimensionality d.

- If $d = 2$ or $d = 3$, as the number of vertices increases, D decreases. Hence there can only be a finite number of superficially divergent diagrams. QFTs with this property are **super-renormalizable**.
- If $d = 4$ (the case we care about), $D = 4 - E$. Hence the only superficially divergent diagrams are those with at most four sources, but there are infinitely many of them, and $D \leq d$. QFTs with this sort of property are **renormalizable**.
- If $d \geq 5$, there are diagrams of all orders that are superficially divergent (due to the differing signs of V and E). QFTs with this property are **non-renormalizable**.

Exercise 4.1.1. Consider QED. Let V be the number of internal vertices, E_e, E_γ be the number of external electron and photon propagators, respectively, and I_e, I_γ the number of internal ones, respectively. Compute that

$$D = \frac{d-4}{2}V - \frac{d-2}{2}E_\gamma - \frac{d-1}{2}E_e,$$

and thus conclude that QED is renormalizable in $d = 4$.

Exercise 4.1.2. In general, show that a QFT in d dimensions with several different fields f and several different interactions i has

$$D = d - \sum_f a_f E_f - \sum_i b_i V_i,$$

where

- E_f is the number of external propagators of field type f,

- V_i is the number of internal vertices of interaction type i,
- $-d + 2a_f$ is the degree of the propagator for field type f,
- $[m^{b_i}]$ is the dimension (in natural units, where $[l] = [t] = [m^{-1}]$) on the coupling constant for interaction type i.

Exercise 4.1.3 (Optional, for those who know general relativity or Riemannian geometry)**.** The **Einstein-Hilbert action** in general relativity is

$$S = \int d^4x \left(\frac{R}{16\pi} + \frac{\Lambda}{8\pi} \right) \sqrt{-g}$$

where R is the Ricci curvature, Λ is the cosmological constant, and g the metric. Compute the dimension of R and therefore conclude that gravity is non-renormalizable.

4.2 Regularization

Obviously, we want to call a QFT renormalizable only if we can really make the divergences vanish in a systematic way. Of course, it turns out we can. There are two distinct steps in doing so: **regularization**, and **renormalization**. We first tackle regularization.

The idea behind regularization is simple: if we have a physical theory, there are limitations when it is valid. For example, classical mechanics applied to particles is a good approximation to quantum mechanics in low energy regimes, where we can pretend $\hbar = 0$, but it is by no means valid at, say, energies where mass-energy equivalence allows for the creation of new particles. (In fact, we need QFT for those regimes.) At those energies, we must incorporate more subtle corrections to the dynamics that come from QM, or QFT. Here's an example of regularization in action, alongside some cool physics.

Example 4.2.1 (Casimir effect)**.** Take a large $L \times L \times L$ vacuum cavity and insert another $L \times L$ plate parallel to one of the walls at a distance $d \ll L$. The plate disturbs the electromagnetic field in the vacuum cavity, and create some energy E relative to the ground state energy 0. Then we expect to detect a force $F = -\partial E/\partial d$ on the plate. Since this is just an example of regularization, let's simplify a bit. Instead of the EM field, which is a spinor field, we work with a massless scalar field, and in $1+1$ dimensions. From EM/QM, we know the EM modes in the cavity are quantized, with wave vector $k = n\pi/d$, i.e. modes $\sin(n\pi x/d)$. Naively, then, $E = f(d) + f(L-d)$ where

$$f(d) = \frac{1}{2} \sum_{n=1}^{\infty} \omega_n = \frac{\pi}{2d} \sum_{n=1}^{\infty} n \to \infty.$$

But we are neglecting the physics: high-frequency waves are not "seen" by the plates, whose electrons move at a finite speed. So introduce a parameter s and a factor $e^{-sn/d}$ to dampen the energy contribution of higher-frequency modes:

$$f(d) = \frac{1}{2d} \sum_{n=1}^{\infty} n e^{-sn/d} = -\frac{\pi}{2} \frac{\partial}{\partial s} \sum_{n=1}^{\infty} e^{-sn/d} = -\frac{\pi}{2} \frac{\partial}{\partial s} \frac{1}{1 - e^{-s/d}} = \frac{\pi}{2d} \frac{e^{s/d}}{(e^{s/d} - 1)^2}.$$

As a series,

$$f(d) = \frac{\pi d}{2s^2} - \frac{\pi}{24d} + \frac{\pi s^2}{480 d^3} + O(s^4).$$

It remains to compute

$$F = -\frac{\partial E}{\partial d} = -(f'(d) - f'(L-d)) \xrightarrow{s \to 0} -\frac{\pi}{24} \left(\frac{1}{d^2} - \frac{1}{(L-d)^2} \right).$$

For $d \ll L$, this is approximately $F = -\pi/24d^2$, the attractive **Casimir force**. In particular, it is finite. In addition, there are other things to note here.

- The general technique used here is known as **zeta function regularization**.
- One may object to this calculation, because it depended on a choice of **regularization** term $e^{-sn/d}$. However, it can be shown that F is independent of the choice of regularization.
- This calculation, which is often repeated in a different context in string theory, is the source of all the silliness surrounding $\sum_{n=1}^{\infty} n = -1/12$: the "equality" arises from matching the $\pi/2d$ terms in the unregularized and regularized $f(d)$ (since other terms cancel or vanish as $s \to 0$ in F).

Let's look at the divergent diagram at the beginning of this chapter. The integral is over the momentum p. As $p \to \infty$, the energy of the internal propagator also goes to infinity. But we don't expect QFT to be valid up to arbitrarily high energies! The solution, as with the Casimir effect, is to impose a cutoff: instead of integrating to infinity, we integrate up to some value Λ. To do the resulting integral, we introduce **Mandelstam variables** $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$, and $u = (p_1 - p_4)^2$. Then it turns out the amplitude is

$$\frac{i\lambda^2}{32\pi^2}\left[\log\frac{\Lambda^2}{s} + \log\frac{\Lambda^2}{t} + \log\frac{\Lambda^2}{u}\right].$$

This method is essentially **Pauli-Villars regularization**. But we are not going to regularize this way, because it is not gauge covariant: later on, when we have a non-abelian gauge group, this kind of regularization is going to upset our choice of gauge. Instead, we are going to use **dimensional regularization**.

Dimensional regularization is exactly what it sounds like. We notice that some integrals are divergent at $d = 4$, but not necessarily divergent at lower dimensions. So we perform analytic continuation in the number of dimensions! This is really not as stupid as it sounds. To save some time, we'll do an example, and along the way, introduce the necessary machinery.

4.2.1 Basic One-Loop Diagram in ϕ^4

At the beginning of this chapter, we considered the basic one-loop diagram in ϕ^4 theory, whose amplitude (rewritten a little) is given by terms of the form

$$I(q) = \frac{(-i\lambda)^2}{2}\int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - m^2 + i\epsilon}\frac{i}{(p+q)^2 - m^2 + i\epsilon}.$$

Here q is some combination of external momenta. This integral is has superficial degree of divergence 0; we are going to regularize it and get the same result as we would have gotten using Pauli-Villars regularization.

We begin by doing a **Wick rotation**. This is where we change from integrating in Minkowski space to Euclidean space, by substituting q^0 with iq^0, and p^0 with ip^0:

$$I(q) = \frac{(-i\lambda)^2}{2}\int \frac{d^4p}{(2\pi)^4}\frac{-i}{(|p|^2 + m^2)(|p+q|^2 + m^2)},$$

where here $|p|^2$ denotes the Euclidean (as opposed to Minkowski) norm. Note that we can remove the $i\epsilon$ now that there are no poles.

The next step is to apply the following trick, called **Feynman's formula** to turn the product of quadratics in the denominator into a power of a single quadratic.

Proposition 4.2.2 (Feynman's formula)**.** *Suppose $c_1, \ldots, c_n \in \mathbb{C}$ such that their convex hull does not contain the origin. Then*

$$\frac{1}{c_1 \cdots c_n} = (n-1)!\int_{[0,1]^n} d^n x\,\frac{\delta(1 - \sum x_j)}{\left(\sum c_j x_j\right)^n}.$$

Exercise 4.2.1. Prove Feynman's formula by induction on n using the following steps. First prove the base case $n = 2$:

$$\int_0^1 \frac{dx}{(c_1 x + c_2(1-x))^2} = \frac{1}{c_1 c_2}.$$

Then differentiate both sides $n-1$ times with respect to c_1 to get a formula for $1/c_1^n c_2$. Finally, do the inductive step using this formula.

To apply Feynman's formula to our Wick-rotated integral, we let c_j be the j-th quadratic term in the denominator, to get

$$I(q) = \frac{(-i\lambda)^2}{2} \int \frac{d^4 p}{(2\pi)^4} \int_0^1 dx \, \frac{i}{(|p|^2 + 2xp \cdot q + x|q|^2 + m^2)^2}.$$

The third step is to do a **linear change of variables** so that the denominator looks like $(|p|^2 + c(q,x))^{I+1}$ for some positive quadratic function c. In our case, the substitution $k = p + xq$ suffices:

$$I(q) = \frac{(-i\lambda)^2}{2} \int_0^1 dx \int \frac{d^4 p}{(2\pi)^4} \frac{i}{(|k|^2 + x(1-x)|q|^2 + m^2)^2}.$$

The final step is to **evaluate the inner integral over d dimensions**, instead of 4, and Wick-rotate back to Minkowski space. This is not hard to do, since now the inner integral is guaranteed to be a radial function, so we can work in polar coordinates. The key to evaluating the polar integral is the following formula.

Proposition 4.2.3. *Let $k \in \mathbb{Z}$ and $0 < d < 2n - k$. Then*

$$\int_0^\infty dr \, \frac{r^{2k+d-1}}{(r^2 + c^2)^n} = \frac{c^{2k+d-2n}}{2} \frac{\Gamma(k + d/2)\Gamma(n - k - d/2)}{\Gamma(n)}.$$

Proof. Substitute $t = (r/c)^2$ and evaluate the integral in terms of the beta function $B(x,y)$. Then recall that $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$. $\square$

Let's carry through with the calculation:

$$\begin{aligned}
\int \frac{d^d p}{(2\pi)^d} \frac{1}{(|k|^2 + x(1-x)|q|^2 + m^2)^2} &= \frac{2\pi^{d/2}}{\Gamma(d/2)(2\pi)^d} \int_0^\infty \frac{r^{d-1}\, dr}{(r^2 + x(1-x)|q|^2 + m^2)^2} \\
&= \frac{2}{\Gamma(d/2)(4\pi)^{d/2}} \frac{1}{2(x(1-x)|q|^2 + m^2)^{(4-d)/2}} \frac{\Gamma(d/2)\Gamma(2 - d/2)}{\Gamma(2)} \\
&= \frac{\Gamma(2 - d/2)}{(4\pi)^{d/2}(x(1-x)|q|^2 + m^2)^{(4-d)/2}}.
\end{aligned}$$

Wick-rotating back and plugging this result back into the outer integral for $I(q)$, which we now write as $I_d(q)$ to indicate that it depends on the dimensionality d,

$$I_d(q) = -i \frac{(-i\lambda)^2}{2} \frac{\Gamma(2 - d/2)}{(4\pi)^{d/2}} \int_0^1 \frac{dx}{(m^2 - x(1-x)q^2)^{(4-d)/2}}.$$

If we carefully keep track of convergence issues, we will discover that as long as $d \in (3,4)$, we are okay in this case. For later convenience, define a new function $J_d(s)$ such that $I_d(q) = (-i\lambda)^2 J_d(q^2)$.

Exercise 4.2.2 (Very optional)**.** By expanding $x^\epsilon = 1 + \epsilon \log x + O(\epsilon^2)$ for $\epsilon = 4 - d$ near zero, compute that

$$J_d(s) = \frac{i}{32\pi^2} \int_0^1 \log(m^2 - sx(1-x))\, dx - \frac{i}{32\pi^2} \left(\frac{2}{4-d} - \gamma + \log(4\pi) \right) + O(4-d)$$

where γ is the Euler-Mascheroni constant.

Now since q depends on which momenta are entering and leaving the internal vertices, there are actually three different ways we can draw the one-loop diagram. Introducing the Mandelstam variables $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$, and $u = (p_1 - p_4)^2$ again, we get that the amplitude is

$$\lim_{d \to 4} (-i\lambda)^2 (J_d(s) + J_d(t) + J_d(u)).$$

This agrees with the result we "got" from Pauli-Villars.

4.3 Renormalization

Now we seem to be stuck again: as $d \to 4$, the pole at $d = 4$ in J_d will cause divergences. For completeness, let's write out the terms we know for the 4-particle scattering amplitude $\mathcal{M}$:

$$i\,\mathcal{M} = \lim_{d \to 4} \left(-i\lambda + (-i\lambda)^2 (J_d(s) + J_d(t) + J_d(u)) + O(\lambda^3) \right).$$

The idea behind renormalization is as follows. So far, λ has represented a **coupling constant**, dictating how strongly the ϕ^4 term affects the free theory. But what does it mean physically? Physically it means nothing until an experimentalist goes into a laboratory, scatters four particles off each other, and measures the scattering amplitude $\mathcal{M}$. Then we can relate $\mathcal{M}$ to λ by saying that, perhaps, the actual value of λ, which we call λ_{phys}, should be $\mathcal{M}$ in the limit where the particles are stationary, i.e.

$$-i\lambda_{phys} = i\,\mathcal{M}\,|_{p_1 = p_2 = p_3 = p_4 = (m, \vec{0})}.$$

This is called the **renormalization condition**.

Definition 4.3.1. The **bare coupling constant** is the parameter λ we have been using all along. The λ_{phys} arising from physically measuring $\mathcal{M}$ is called the **physical coupling constant**.

So now we have two "unphysical" quantities. Instead of writing the scattering amplitude $\mathcal{M}$ in terms of λ, then, let's write it in terms of λ_{phys}. We first do this up to second order in $\hbar$, both for simplicity and because we only computed the expression for the scattering amplitude up to the second order diagrams. In principle it can be done up to any order.

Note that we are expanding in $\hbar$ now, not λ. This generalizes everything we have been doing up until now, because λ secretly contains a factor of $\hbar$ (do the dimensional analysis to figure this out). The reason for doing this will become clear later on, when we introduce renormalized perturbation theory. For now, check that the $\hbar$-dimensions of the various quantities are $[\phi] = [\lambda] = [m] = \hbar$.

Up to second order in $\hbar$, then,

$$\lambda_{phys} = -\,\mathcal{M}\,|_{p_1 = p_2 = p_3 = p_4 = (m, \vec{0})} = \lambda \left(1 - i\lambda (J_d(4m^2) + J_d(0) + J_d(0)) \right),$$

where we substituted $s = (p_1 + p_2)^2 = 4m^2$ and so on. Using this formula, we can write λ in terms of λ_{phys}, up to second order. Since λ_{phys} and λ agree at first order, $\lambda_{phys} = \lambda + O(\hbar^2)$, so

$$\lambda_{phys} = \lambda \left(1 - i\lambda_{phys} (J_d(4m^2) + 2J_d(0)) \right) + O(\hbar^3).$$

Solving for λ in this equation, we get

$$\lambda = \lambda_{phys} + i\lambda_{phys}^2 (J_d(4m^2) + 2J_d(0)) + O(\hbar^3).$$

Substituting this into our expression for $\mathcal{M}$, we get

$$i\,\mathcal{M} = \lim_{d \to 4} \left(-i\lambda_{phys} + (-i\lambda_{phys})^2 (J_d(s) + J_d(t) + J_d(u) - J_d(4m^2) - 2J_d(0)) + O(\hbar^3) \right).$$

Aha! The term responsible for the divergences, $2/(4-d)$, is cancelled by the additional J_d's. This process to remove the divergences by introducing the physical coupling constant is called **renormalization**.

Exercise 4.3.1. We began this derivation by defining $-i\lambda_{phys} = i\,\mathcal{M}\,|_{p_1=p_2=p_3=p_4=(m,\vec{0})}$, which may seem rather arbitrary. Convince yourself that we could have picked any values at all for p_1, p_2, p_3, p_4 to define λ_{phys}, and the divergences would still have been removed. Note that upon re-measuring λ_{phys} with these new momenta, we must change not only λ_{phys}, but also the values of s, t, and u in the formula for the scattering amplitude. Convince yourself that the scattering amplitude is unchanged.

Let's summarize what has happened so far in our exploration of QFT. First we write down a Lagrangian whose parameters are bare masses and bare coupling constants, and we compute Feynman diagrams using it. Then we relate the bare parameters to the physical ones, and rewrite the value of the diagrams in terms of the physical parameters, thus canceling the divergences. This entire procedure is known as **bare perturbation theory**.

But this seems rather roundabout. Why not start at the beginning with the physical parameters, and avoid all the fussing around? The answer is yes, that is exactly what we should do, and this alternative process is simply known as **renormalized perturbation theory**. Let $\mathcal{L} = \mathcal{L}_{free} + \mathcal{L}_{int}$ be the bare Lagrangian, and $\mathcal{L}' = \mathcal{L}'_{free} + \mathcal{L}'_{int}$ be the physical Lagrangian. Then

$$\mathcal{L} = \mathcal{L}' + \mathcal{L}_{ct} = \mathcal{L}'_{free} + \mathcal{L}'_{int} + \mathcal{L}_{ct},$$

where $\mathcal{L}_{ct}$ contains terms of the same form as the ones in $\mathcal{L}'$, but with undetermined coefficients that are determined iteratively as we go to higher and higher order in perturbation theory. These terms are known as **counterterms**. The idea now is to plug-and-chug through the usual Feynman diagram machinery, except using $\mathcal{L}'_{int} + \mathcal{L}_{ct}$ as the interaction term.

Exercise 4.3.2. Convince yourself, by looking at how we derived the Feynman rules, that the diagrams arising from the interaction term $\mathcal{L}'_{int} + \mathcal{L}_{ct}$ are precisely those arising from $\mathcal{L}'_{int}$ and $\mathcal{L}_{ct}$ individually. In particular, the ones arising from $\mathcal{L}'_{int}$ are precisely the "bare" Feynman diagrams we already know how to compute, except with bare masses and coupling constants replaced with their physical equivalents.

Let's use ϕ^4 theory (as usual) as an example. Write

$$\mathcal{L} = \frac{1}{2}(\partial\phi_B)^2 - \frac{1}{2}m_B^2\phi_B^2 - \frac{\lambda_B}{4!}\phi_B^4,$$

where m_B, λ_B, and ϕ_B are the bare mass, coupling constant, and field. Let m and λ be the physical mass and coupling constant, and $\phi = Z^{-1/2}\phi_B$ be the renormalized field. (We do this to the field in order to get rid of the infinite vacuum energy, so that we don't even have to worry about vacuum bubbles after renormalization.) We therefore have that

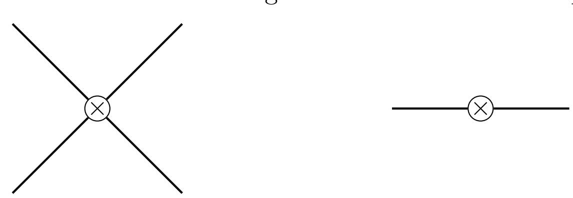
$$\mathcal{L}_{ct} = \frac{1}{2}(Z-1)(\partial\phi)^2 - \frac{1}{2}(Zm_B^2 - m^2)\phi^2 - \frac{1}{4!}(Z^2\lambda_B - \lambda)\phi^4.$$

For convenience, write

$$\delta Z = Z - 1, \quad \delta m^2 = Zm_B^2 - m^2, \quad \delta\lambda = Z^2\lambda_B - \lambda,$$

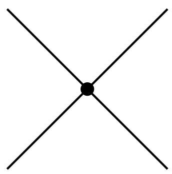
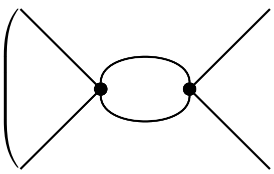
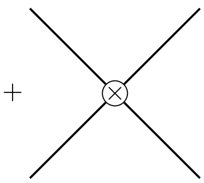
so that $\mathcal{L}_{ct} = (\delta Z/2)(\partial\phi)^2 - (\delta m^2/2)\phi^2 - (\delta\lambda/4!)\phi^4$. Note that $[\delta Z] = \hbar^0$, and $[\delta m^2] = \hbar^2$, and $[\delta\lambda] = \hbar^2$, not $\hbar$. The reason for this is that $\delta\lambda = Z^2\lambda_B - \lambda$ will turn out to be of order λ^2 because the first-order λ terms cancel out.

The counterterms in $\mathcal{L}_{ct}$ contribute two new diagrams to the momentum-space Feynman rules:

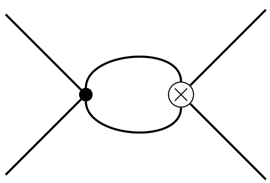
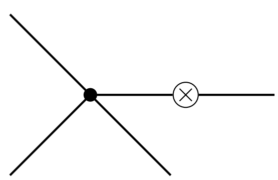
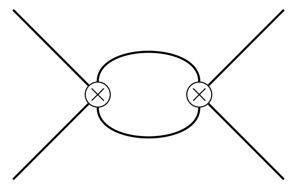


We denote the **counterterm vertices** by a crossed circle to distinguish them from regular vertices arising from $\mathcal{L}'$. The first diagram corresponds to the $(\delta\lambda/4!)\phi^4$ term, and contributes a factor of $-i\delta\lambda$, just as the $(\lambda/4!)\phi^4$ term contributes a factor of $-i\lambda$. The second diagram corresponds to the sum $(1/2)((\delta Z)(\partial\phi)^2 - (\delta m^2)\phi^2)$ considered as a single unit, and contributes a factor of $i((\delta Z)p^2 - \delta m^2)$, where p is the momentum of the incoming/outgoing line. This factor is not trivial to calculate, and is the content of the upcoming subsection. The second diagram is actually very special: it is responsible for counteracting the divergences arising from any loops occuring along a propagator.

Ignoring the second diagram for now, let's repeat the calculation of the 4-particle scattering amplitude up to first order using renormalized perturbation theory. We have

$i\,\mathcal{M} =$  $+ \Bigg($  $+$ permutations $\Bigg) +$  $+ O(\hbar^3).$

Why does this suffice for $O(\hbar^3)$? Aren't there more diagrams with two counterterm vertices, such as the following?

The answer is no. Remember that a counterterm vertex turns out to contribute a factor of $\hbar^2$, instead of just $\hbar$. So the first and second diagrams above are $\hbar^3$ terms (the second should not even be considered; we will tackle this in the upcoming subsection), and the third is a $\hbar^4$ term.

This might be more obvious upon doing a calculation. Using the sum above,

$$i\,\mathcal{M} = \lim_{d\to 4}(-i\lambda + (-i\lambda)^2[J_d(s) + J_d(t) + J_d(u)] - i\delta\lambda + O(\hbar^3)).$$

On the other hand, recall that earlier, we posited the **renormalization condition**

$$-i\lambda = i\,\mathcal{M}_{p_1=p_2=p_3=p_4=(m,\vec{0})} = \lim_{d\to 4}(-i\lambda + (-i\lambda)^2[J_d(4m^2) + 2J_d(0)] - i\delta\lambda + O(\hbar^3)),$$

so it follows that up to order $\hbar^2$,

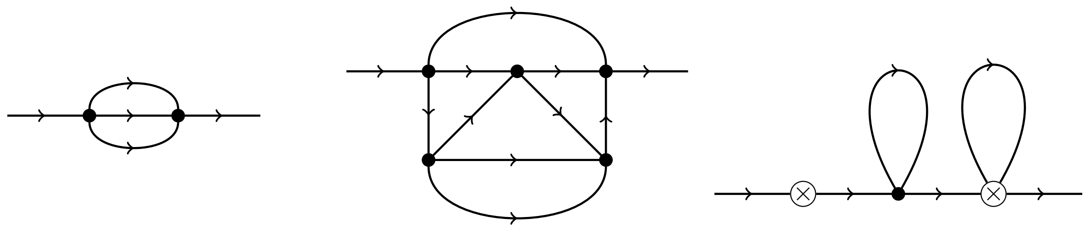
$$\delta\lambda = i\lambda^2[J_d(4m^2) + J_d(0)].$$

Hence the four-point counterterm vertex is indeed a $\hbar^2$ term. In general, to compute the scattering amplitude $\mathcal{M}$ up to a certain order, one writes down all the Feynman diagrams (including both vertices and counterterms) up to that order, computes them, and then adjusts δZ, δm, and $\delta\lambda$ until the renormalization condition is satisfied. So, really, δZ, δm^2, and $\delta\lambda$ are variables whose values are iteratively computed.

4.3.1 Two-Point Counterterm Vertex

There are two remaining issues. First, we must compute the value of the two-point counterterm vertex (we only stated it earlier, without justification). We must also specify more renormalization conditions: we only have one renormalization condition so far, and three free parameters δZ, δm^2, and $\delta\lambda$, so we are missing two more renormalization conditions. Let's talk about some physics to justify how we get the remaining two renormalization conditions, and along the way, compute the two-point counterterm vertex.

We wish for the propagator to express the amplitude of propagation between two points x and y. But the problem is that as a particle propagates between two points, it can go on all sorts of crazy adventures. Here are some examples.



The idea is that we should sum all of these possible adventures to form a **renormalized propagator** Δ. So let's compute the renormalized propagator. First we simplify the sum a little.

Definition 4.3.2. A **one-particle irreducible** (1P1) diagram is one that cannot be disconnected by removing just one propagator. For example, the first two diagrams above are 1P1, but the last one is not.

Hence if $\Pi(p^2)$ denotes the sum of all possible non-trivial 1P1 diagrams with initial and final momentum p and external lines amputated (we write p^2 because Π must be a function of p^2 by Lorentz invariance),

$$\widehat{\Delta}(p) = \widehat{D}_F(p) + \widehat{D}_F(p)\Pi(p^2)\widehat{D}_F(p) + \cdots = \widehat{D}_F(p)\sum_{n=0}^{\infty}(\Pi(p^2)\widehat{D}_F(p))^n = [\widehat{D}_F(p)^{-1} - \Pi(p^2)]^{-1}.$$

Perturbatively, i.e. up to a finite order, this is valid; non-perturbatively, there are convergence issues, which we conveniently ignore. Here $\widehat{\Delta}(p)$ is the free propagator with physical mass and coupling constant. Using this idea, we can compute the value of the two-point counterterm vertex $\Pi_0(p^2)$.

Proposition 4.3.3. *The two-point counterterm vertex contributes a factor $i((\delta Z)p^2 - \delta m^2)$.*

Proof. Suppose in the expression for $\widehat{\Delta}(p)$ above, we only inserted the mass renormalization, i.e. the two-point counterterm vertices, and not the normal 1P1 terms, giving the **counterterm propagator**. In other words, if $\Pi_0(p^2)$ is the factor given by the counterterm vertex, then

$$\widehat{\Delta}_{ct}(p) = \frac{i}{p^2 - m^2 + i\epsilon}\sum_{n=0}^{\infty}\left(\frac{i\Pi_0(p^2)}{p^2 - m^2 + i\epsilon}\right)^n.$$

But we can compute this in another way: start with $\widehat{D}_F(p) = i/(p^2 - m_B^2 + i\epsilon)$, and renormalize the field and its mass. The counterterm propagator thus picks up a factor of Z^{-1}, coming from $\phi(x)\phi(y) = Z^{-1}\phi_B(x)\phi_B(y)$. Hence, ignoring the $i\epsilon$ term for simplicity, $\widehat{\Delta}_{ct}(p) = Z^{-1}\widehat{D}_F(p)$. Write $Z = 1 + \delta Z$ and $Zm_B^2 = m^2 + \delta m^2$ to get

$$\begin{aligned}\widehat{\Delta}_{ct}(p) &= \frac{i}{Zp^2 - Zm_B^2} = \frac{i}{(1+\delta Z)p^2 - m^2 - \delta m^2} \\ &= \frac{i}{p^2 - m^2}\sum_{n=0}^{\infty}\left(-\frac{(\delta Z)p^2 - \delta m^2}{p^2 - m^2}\right)^n.\end{aligned}$$

Matching this with our previous expression for $\widehat{\Delta}_{ct}(p)$, it follows that the two-point counterterm vertex has value $\Pi_0(p^2) = i((\delta Z)p^2 - \delta m^2)$. ◻

Now it's time to get the remaining renormalization conditions. Recall that we got a renormalization condition for $\delta\lambda$ by considering the physical meaning of what we wrote down. Now we will get a renormalization condition for δm^2 and δZ by thinking about the physical meaning of the renormalized propagator. Suppose

$p^2 \to m^2$ in Δ. Then Δ should represent a real particle. In particular, it's not going on any adventures with its virtual buddies, so we expect the singularity structure of $\widehat{\Delta}(p)$ and $\widehat{D}_F(p)$ to be the same at $p^2 = m^2$:

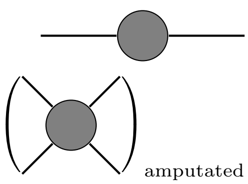
$$\widehat{\Delta}(p) = \widehat{D}_F(p)|_m + (\text{terms regular at } p^2 = m^2).$$

We insist that this indeed does hold; it is another **renormalization condition**. Rephrased, we insist that $\widehat{\Delta}(p)$ has a simple pole of residue i at $p^2 = m^2$. Note that really, this specifies two conditions: that there is a simple pole at $p^2 = m^2$, and that the residue is i. Hence this suffices to determine δm^2 and δZ.

4.3.2 The Renormalization Algorithm

In short, we now have a systematic algorithm for renormalizing ϕ^4 theory. Underlying the entire algorithm are the following two **renormalization conditions**, which arose from imposing physical considerations on our perturbative calculations.

Definition 4.3.4 (Renormalization conditions for ϕ^4 theory)**.** The two-point scattering amplitude, i.e. the renormalized propagator, and the amputated four-point correlation function, i.e. the four-point scattering amplitude, satisfy, respectively,



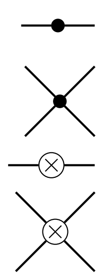
$$= \frac{i}{p^2 - m^2} + (\text{terms regular at } p^2 = m^2).$$

$$= -i\lambda \quad \text{at } s = 4m^2,\ t = u = 0.$$

The big gray circles represent the sum over valid interactions.

Definition 4.3.5 (Algorithm for renormalized perturbation theory)**.** The following algorithm renormalizes ϕ^4 theory (and can be suitably generalized to renormalize any renormalizable theory). Let m be the physical mass and λ be the physical coupling constant.

- Pick an order $O(\hbar^n)$ to which everything is to be computed.
- Use the Feynman rules



$$= \frac{i}{p^2 - m^2 + i\epsilon}$$

$$= -i\lambda$$

$$= i((\delta Z)p^2 - \delta m^2)$$

$$= -i\delta\lambda.$$

  to compute the value of the desired diagram(s), up to $O(\hbar^n)$. Write down the resulting value as a function of d, the dimensionality.
- Compute the two-point and four-point scattering amplitudes up to $O(\hbar^n)$, and adjust δZ, $\delta\lambda$, and δm^2 such that these scattering amplitudes satisfy the two renormalization conditions up to $O(\hbar^n)$.
- The resulting value of the desired diagram(s), up to $O(\hbar^n)$, should now be finite and independent of the dimension d. Take the limit $d \to 4$ to get the final result.

4.4 Renormalization Group

So far, we have focused on the practicalities of removing divergences, and how to compute the actual (renormalized) values of Feynman diagrams. There is a slightly more abstract way of viewing renormalization that is less practical in terms of calculations, but more revealing in terms of what renormalization is really doing. To demonstrate this new perspective, for this section, we drop the technique of dimensional regularization and stick with imposing a momentum cutoff, purely for simplicity. But whereas previously we imposed a cutoff when evaluating Feynman diagrams, now we impose a cutoff in the path integral itself.

To impose such a cutoff, we look at the momentum representation of the path integral:

$$Z[J] = \int D\phi\, e^{i\int(\mathcal{L}+J\phi)} = \left(\prod_k \int d\phi(k)\right) e^{i\int(\mathcal{L}+J\phi)}.$$

It is tempting to integrate over $k^2 \leq \Lambda^2$ for a momentum cutoff Λ, but this choice is inappropriate in Minkowski space. Instead, we need to impose $|k|^2 \leq \Lambda^2$ to completely rule out large momenta, which means we must also Wick rotate the path integral. The result is (taking $J = 0$ for simplicity)

$$Z = \left(\prod_{|k|<\Lambda} \int d\phi(k)\right) \exp\left(-\int d^dx\, \left(\frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\right)\right).$$

For convenience, write the integration measure $\prod_{|k|<\Lambda} d\phi(k)$ as $[D\phi]_\Lambda$.

The insight that becomes relevant now is that if we are imposing a momentum cutoff Λ, we are free to choose Λ. In particular, if we pick $\Lambda_1 < \Lambda_2$, the theory we obtain by using Λ_1 had better be fully described by the theory we obtain by using the higher Λ_2 cutoff. In fact, we expect that imposing the Λ_2 cutoff and then imposing the Λ_1 cutoff gives the exact same result as directly imposing the Λ_1 cutoff.

Pick some real number $b < 1$, and split ϕ into two pieces $\phi_s + \phi_b$:

$$\phi_s(k) = \begin{cases} \phi(k) & |k| \in [0, b\Lambda) \\ 0 & \text{otherwise,} \end{cases} \qquad \phi_s(k) = \begin{cases} \phi(k) & |k| \in [b\Lambda, \Lambda) \\ 0 & \text{otherwise.} \end{cases}$$

(Here the subscript s stands for "small" (momentum), and b for "big"). The idea is that by plugging $\phi = \phi_s + \phi_b$ into the path integral Z and then integrating over ϕ_b, we are left the **effective path integral**

$$Z = \int [D\phi]_{b\Lambda} e^{-\int d^dx\ \mathcal{L}_{eff}}$$

such that $\mathcal{L}_{eff}$ involves only field components $\phi(k)$ with $k \in [0, b\Lambda)$.

Exercise 4.4.1. Plug $\phi = \phi_s + \phi_b$ into Z and expand everything, noting that quadratic terms of the form $\phi_s\phi_b$ vanish since Fourier components of different wavelengths are orthogonal. You should obtain

$$Z = \int D\phi_s e^{-\int \mathcal{L}(\phi_s)} \int D\phi_b e^{-\int L'(\phi_s,\phi_b)},$$

$$L'(\phi_s, \phi_b) = \frac{1}{2}(\partial_\mu\phi_b)^2 + \frac{1}{2}m^2\phi_b^2 + \lambda\left(\frac{1}{6}\phi_s^3\phi_b + \frac{1}{4}\phi_s^2\phi_b^2 + \frac{1}{6}\phi_s\phi_b^3 + \frac{1}{4!}\phi_b^4\right).$$

We treat the $(1/2)m^2\phi_b^2$ as a perturbation as well, since we are mainly interested in the regime $m^2 \ll \Lambda^2$. Hence compute the propgator $(1/p^2)(2\pi)^d\Theta(|p|)$ where Θ is the indicator function for $[b\Lambda, \Lambda)$. Now convince yourself that in principle, one can

- derive the Feynman rules for the "Lagrangian" $L'(\phi_s, \phi_b)$, where each vertex term now contains factors of ϕ_s in addition to numerical factors and λ,
- note that the total amplitude, i.e. the path integral, is the sum of all possible such Feynman diagrams, and
- note that that the sum factors as an exponential of the sum of connected diagrams.

Conclude that

$$\mathcal{L}_{eff}(\phi) = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 + (\text{sum of connected diagrams}).$$

But there is a much easier way to express the relationship between the two integrals

$$\int [D\phi]_\Lambda \exp(i\int \mathcal{L}(\phi)) \qquad \text{and} \qquad \int [D\phi]_{b\Lambda} \exp(i\int \mathcal{L}_{eff}(\phi)).$$

Namely, in the effective path integral, let's write the variables $k' = k/b$ and $x' = xb$, so that now the integration limit is $|k'| < \Lambda$. Note that the effective path integral must be of the form

$$\int d^dx\ \mathcal{L}_{eff} = \int d^dx\ \left(\frac{1}{2}(1+\Delta Z)(\partial_\mu\phi)^2 + \frac{1}{2}(m^2+\Delta m^2)\phi^2 + \frac{1}{4}(\lambda + \Delta\lambda)\phi^4 + \Delta C(\partial_\mu \phi)^4 + \Delta D\phi^6 + \cdots\right),$$

Exercise 4.4.2. Perform the substitution $k' = k/b$ and $x' = xb$ to get a new Lagrangian

$$\int d^dx\ \mathcal{L}_{eff} = \int d^dx' b^{-d} \left(\frac{1}{2}(1+\Delta Z)b^2(\partial'_\mu\phi)^2 + \frac{1}{2}(m^2+\Delta m^2)\phi^2 + \frac{1}{4}(\lambda + \Delta\lambda)\phi^4 + \Delta C b^4(\partial'_\mu \phi)^4 + \cdots\right).$$

Show that this new Lagrangian will give the same propagator as the original Lagrangian $\mathcal{L}$ (with cutoff Λ) when $\phi' = (b^{2-d}(1+\Delta Z))^{1/2}\phi$, and that we can write

$$\int d^dx\ \mathcal{L}_{eff} = \int d^dx' \left(\frac{1}{2}(\partial'_\mu\phi')^2 + \frac{1}{2}m'^2\phi'^2 + \frac{1}{4}\lambda'\phi'^4 + C'(\partial'_\mu\phi')^4 + D'\phi'^6 + \cdots\right).$$

where

$$m'^2 = (m^2+\Delta m^2)(1+\Delta Z)^{-1}b^{-2}, \quad \lambda' = (\lambda+\Delta\lambda)(1+\Delta Z)^{-2}b^{d-4}, \quad C' = (C+\Delta C)(1+\Delta Z)^{-2}b^d, \quad \ldots.$$

So given a momentum cutoff $b\Lambda$, we can integrate out the higher momentums, and then apply this rescaling procedure. The end result is that we obtain a path integral over the same momentum range $[0, \Lambda]$, but with a transformed Lagrangian, whose coefficients transformed in accordance to the rules for m' and λ' (and, in principle, C', D', and so on) above. We can iterate this procedure, taking b very close to 1 each time. The limit, as b is infinitesimally close to 1, is a continuous transformation of the Lagrangian, i.e. a flow in the space of all possible Lagrangians.

Definition 4.4.1. The **renormalization group** is $\mathbb{R}_{\geq 0}$, which acts on the original Lagrangian to produce the Lagrangians in this flow, which is more appropriately called the **renormalization group flow**. Note that the renormalization group **is not a group**, since the action of integrating out higher momentums is not invertible.

Now let's do a thought experiment. Suppose that we have found a Lagrangian for a QFT that is valid up to extremely high energies. But all the phenomena we can currently measure are at a much smaller energy scale, so we act on the original Lagrangian with the renormalization group until the effective energy cutoff is small enough. The resulting effective Lagrangian has finite parameters, obviously, but as we act on the original Lagrangian, its coefficients blow up. It turns out that the parameters of the original Lagrangian are the bare parameters, and those of the effective Lagrangian are the physical parameters. If we computed amplitudes directly using the original Lagrangian, we get mysterious divergences popping up when we integrate over momentum, hence invalidating the use of perturbation theory. But if we computed amplitudes from the effective Lagrangian, these high-momentum contributions have already been absorbed into the parameters of the effective Lagrangian, so when we compute amplitudes order-by-order, perturbative techniques are still valid.

Hence the renormalization group gives us a very intuitive and conceptually straightforward picture of renormalization, and we can use it to gain some insight into renormalizability. For example, the free-field Lagrangian $\mathcal{L}_0 = (1/2)(\partial_\mu \phi)^2$ is a fixed point of the renormalization group, and around $\mathcal{L}_0$ we can ignore Δm^2, $\Delta\lambda$, etc. and keep only the linear perturbations:

$$m'^2 = m^2 b^{-2}, \quad \lambda' = \lambda b^{d-4}, \quad C' = C b^d, \quad D' = D b^{2d-6}, \quad \ldots.$$

Definition 4.4.2. Parameters which grow during the action of the renormalization group are **relevant**, those which die away are **irrelevant**, and those which are unaffected by b are **marginal** (we must look at higher-order corrections to determine the effect of marginal parameters).

In general, this kind of analysis shows that a term in the Lagrangian involving N powers of ϕ and M derivatives has a coefficient that transforms as $C'_{N,M} = b^{N(d/2-1)+M-d} C_{N,M}$.

Exercise 4.4.3. Using the definition of renormalizability and the exercises immediately after the definition, show that $N(d/2-1)+M-d$ is precisely the mass dimension of the coefficient, and therefore superrenormalizable terms correspond to relevant terms, and renormalizable terms correspond to marginal terms.

The terms "relevant", "irrelevant", and "marginal" are misnomers: they came from condensed matter physicists, who are interested in the low-energy (and therefore long-distance) limit instead of the high-energy (and therefore short-distance) limit. So to them, irrelevant terms are irrelevant, but to us, the irrelevant terms are the most relevant ones, since as energy goes up, they become dominant. And now we see why terms with negative mass dimension are bad: as we move from lower energies to higher energies, more and more terms of higher and higher order become non-negligible, and the coefficients of all of these terms must be determined by experiment. In effect, a non-renormalizable theory has infinite degrees of freedom, requiring infinite experiments to specify the theory, and therefore has no predictive power at all.