# String Theory and Supersymmetry
# Winter 2016 Seminar Notes

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# Chapter 1

# Introduction to Strings

We asked "why fields?" when we started QFT; now we ask, why strings? Here are some potentially convincing reasons.

1. If we allow one more degree of freedom than particles, many IR/UV divergences disappear; we require less renormalization. If we allow more than one degree of freedom, new divergences arise from the increased internal degrees of freedom.

2. Every consistent string theory contains a massless spin-2 state, i.e. a graviton, whose interactions at low energies reduce to general relativity.

3. The Standard Model, based on QFT, has 25 adjustable constants. String theory has none, and leads to gauge groups big enough to include the Standard Model.

4. Consistent string theories force upon us supersymmetry and extra dimensions, which have arisen naturally from several different attempts to unify the Standard Model.

Regardless of whether they are convincing, we start in this chapter, as with any other physical model, by writing down an action. Specifically, we first write the action for a relativistic string by generalizing that of a relativistic point particle, and then we quantize the action. As with QFT there are different ways to quantize. We go through the analogue of canonical quantization in order to quickly compute the spectrum of a string, and then go through path integral quantization in preparation for studying string interactions.

As usual, we take $\hbar = c = 1$, and use **Einstein summation convention**: repeated indices are implicitly summed over.

## 1.1 Review of Relativity

We work in $\mathbb{R}^{D-1,1}$ where $D$ is the **number of dimensions**. Recall that coordinates are written $x^\mu = (x^0, x^1, \ldots, x^D) = (ct, x^1, \ldots, x^D)$, and the metric is

$$-ds^2 := \eta_{\mu\nu} dx^\mu dx^\nu, \quad \eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, \ldots, 1).$$

Note that $\eta^\mu{}_\mu = D$. We use the dot product to stand for the **Lorentz inner product**, e.g. $-ds^2 = dx \cdot dx$.

**Definition 1.1.1.** Define the **proper time** of a system as the time elapsed measured by a clock traveling in the same Lorentz frame as the system itself. In such a Lorentz frame, $dx^i = 0$ and $dt$ is the proper time elapsed, so $-ds^2 = -dt_p^2$; define

$$ds := \sqrt{ds^2} = dt_p \qquad \text{whenever } ds^2 > 0,$$

i.e. for timelike intervals. Hence $ds$ is the **proper time interval**. The **relativistic momentum** is $p^\mu := m(dx^\mu/ds)$. Conveniently,

$$p^\mu p_\mu = m^2 \frac{dx^\mu}{ds}\frac{dx_\mu}{ds} = -m^2 \frac{ds^2}{ds^2} = -m^2.$$

**Definition 1.1.2.** A **Lorentz transformation** $\Lambda^\mu{}_\nu$ is an element of the Lorentz group, the collection of all linear isometries of $\mathbb{R}^{D-1,1}$. We say $a^\mu$ is a **vector** if under Lorentz transformations, it changes as $a'^\mu = \Lambda^\mu{}_\nu a^\nu$. A **Poincaré transformation** is a Lorentz transformation possibly followed by a translation.

**Definition 1.1.3.** The **world line** of a point particle is the path in spacetime $\mathbb{R}^{D-1,1}$ traced out by the particle as it evolves in time.

The underlying principle of relativity says that physical laws are independent of Lorentz frame. In other words, any action we write down that we want to be compatible with relativity must have external symmetries: it must be invariant under Lorentz transformations. We call this **Lorentz invariance**. As long as superscripts and subscripts match up, we do not have to worry about Lorentz invariance.

The **action** for a free relativistic **point particle** is obtained by writing down the simplest Lorentz invariant action, and then making sure dimensions work out. If $\gamma$ is the path taken by the particle, the action is therefore

$$S_{\mathrm{pp}}[x] := -m \int_\gamma ds = -m \int_\gamma d\tau \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} = -m \int_\gamma d\tau \sqrt{-\dot{x}^\mu \cdot \dot{x}_\mu}$$

where a dot denotes a $\tau$-derivative. Because $ds$ is coordinate-independent, it does not what how we pick the parametrization $\tau$. Physicists like to call this **reparametrization invariance**. This invariance is very important: without it, we have actually introduced a completely new parameter $\tau$, thus increasing the number of degrees of freedom from $D-1$ to $D$.

**Exercise 1.1.1.** By computing $\delta(ds^2)$ in two different ways, show that

$$\delta S_{\mathrm{pp}}[x] = m \int_\gamma \delta(dx^\mu)\frac{dx_\mu}{ds} = \int_\gamma d\tau \left(\frac{d}{d\tau}\delta x^\mu\right) p_\mu = \delta x^\mu p_\mu \Big|_{\tau_i}^{\tau_f} - \int d\tau \delta x^\mu \frac{dp_\mu}{d\tau}.$$

Argue that the first term vanishes if we specify **initial and final conditions**. Hence deduce the equation of motion $dp_\mu/d\tau = 0$.

The action $S_{\mathrm{pp}}$ seems simple in the $\int_\gamma ds$ form, but is messy when parametrized. Later when we quantize using path integrals, $S_{\mathrm{pp}}$ is difficult to work with because of the derivatives under the square root. There is a different, classically-equivalent action we can work with. Introduce an additional field $\gamma_{\tau\tau}(\tau)$ (sometimes called an **einbein** in general relativity), which we can view as a metric on the world line, and take the action

$$S'_{\mathrm{pp}} := -\frac{1}{2}\int_\gamma d\tau \sqrt{-\gamma_{\tau\tau}}(\gamma^{\tau\tau}\dot{x}^\mu\dot{x}_\mu + m^2) = -\frac{1}{2}\int_\gamma d\tau\,(\eta^{-1}\dot{x}^\mu\dot{x}_\mu - \eta m^2), \quad \eta := \sqrt{-\gamma_{\tau\tau}(\tau)}.$$

It seems like we have arbitrarily added an extra degree of freedom, but in fact $\gamma$ is completely specified by the equation of motion. The action $S'_{\mathrm{pp}}$ is much better to work with in a path integral, because it is **quadratic** in $\dot{x}^\mu$.

**Exercise 1.1.2.** Vary $S'_{\mathrm{pp}}$ with respect to $\gamma_{\tau\tau}$ to get the equation of motion $\gamma_{\tau\tau} = \dot{x}^\mu\dot{x}_\mu/m^2$. Substitute this expression back into $S'_{\mathrm{pp}}$ to obtain $S_{\mathrm{pp}}$, and therefore conclude that the two actions are classically equivalent.

## 1.2 Nambu–Goto and Polyakov Actions

We graduate to **one-dimensional strings**; in this section we write down an action for them. There are two kinds of strings: those with two distinct endpoints, called **open strings**, and those which are loops, called **closed strings**. Because closed strings are just open strings with the extra constraint that the two endpoints match, we focus on open strings.

The action for the relativistic point particle is proportional to the proper time elapsed on the particle's world line. But the proper time, when multiplied by $c$, can be viewed as the "proper length" of the world line. The natural generalization, then, is to consider the surface in space-time traced out by the string as it evolves in time, called the **world sheet** $\Sigma$, and to define an action proportional to the "proper area" of the world sheet. The world sheet $\Sigma$ is a two-dimensional surface, and therefore requires charts modeled on $\mathbb{R}^2$.

**Definition 1.2.1.** The **coordinates** we use on $\mathbb{R}^2$, the parameter space, are denoted $(\sigma^0, \sigma^1)$, and so the **world sheet** $\Sigma$ is locally a surface given by functions denoted $X^\mu(\sigma)$ (capitalized to disambiguate from the coordinates $x^\mu$), called **string coordinates**. The lowercase Latin characters $a, b, \ldots$ are used to denote **indices** that run over values $0, 1$. Two notes:

1. The choice of parametrization $(\sigma^0, \sigma^1)$ is, again, up to us, but usually we take the coordinate $\sigma^0$ to be the proper time, and $\sigma^1$ the position along the string.
2. For our purposes, $\Sigma = X^\mu$, i.e. the single chart $X^\mu$ describes the entire world sheet for the region of spacetime we care about.

**Exercise 1.2.1.** Show that the metric $\eta_{\mu\nu}$ on spacetime $\mathbb{R}^{D-1,1}$ induces a metric $g$ on the world sheet via pullback along the inclusion $\iota\colon \Sigma \to \mathbb{R}^{D-1,1}$. Compute $g$ and the area element:

$$g_{ab} = \partial_a X^\mu \partial_b X_\mu, \quad dA = d^2\sigma \sqrt{-\det g}.$$

A relativistic particle has a parameter we call mass. It turns out mass is not the appropriate physical interpretation of the corresponding parameter for strings. Instead, we interpret it as a **tension**, and denote it $T_0$. Old people write $T_0 = 1/2\pi\alpha'$ and call $\alpha'$ the **universal Regge slope**; we choose not to.

**Definition 1.2.2.** The **Nambu–Goto action** for a relativistic string is given by

$$S_{\mathrm{NG}}[X] \coloneqq -T_0 \int_\Sigma dA = -T_0 \int_\Sigma d^2\sigma \sqrt{-\det g}.$$

Again, note that it satisfies **reparametrization invariance**, literally by construction.

But again, we have a square root and derivatives inside it, and now we know how to get rid of it: introduce an independent world sheet metric $\gamma_{ab}(\sigma)$. This time the metric is on a surface, so we need to specify the signature. We take Lorentzian signature $(-, +)$.

**Definition 1.2.3.** The **Polyakov action** for a relativistic string is given by

$$S_{\mathrm{P}}[X, \gamma] \coloneqq -\frac{T_0}{2} \int_\Sigma d^2\sigma \sqrt{-\gamma}\, \gamma^{ab} \partial_a X^\mu \partial_b X_\mu,$$

where $\gamma$ without indices stands for $\det(\gamma_{ab})$. From now on, we always refer to $\gamma_{ab}$ as the **metric**, and $g_{ab}$ as the **induced metric**. World sheet indices are raised/lowered using the metric $\gamma_{ab}$, not the induced metric $g_{ab}$. (In fact, from now on we basically forget about $g_{ab}$; we use it only to introduce the Nambu–Goto action, and the following exercise.)

**Exercise 1.2.2.** Show that $\delta\sqrt{-\gamma} = (1/2)\sqrt{-\gamma}\, \gamma^{ab} \delta\gamma_{ab}$, and therefore that

$$\delta_\gamma S_{\mathrm{P}}[X, \gamma] = -\frac{T_0}{2} \int_\Sigma d^2\sigma \sqrt{-\gamma}\, \delta\gamma^{ab} \left( g_{ab} - \frac{1}{2} \gamma_{ab} \gamma^{cd} g_{cd} \right).$$

Rearrange the obtained equation of motion and conclude that $g_{ab}\sqrt{-g} = \gamma_{ab}\sqrt{-\gamma}$. Hence replace $\gamma$ in $S_{\mathrm{P}}[X, \gamma]$ with $g$, and obtain that $S_{\mathrm{P}}[X, \gamma] = S_{\mathrm{NG}}[X]$.

**Definition 1.2.4.** As in general relativity, define the **stress-energy tensor**

$$T_{ab}(\sigma) := -\frac{4\pi}{\sqrt{-\gamma}}\delta_\gamma S_{\mathrm{P}}[X,\gamma] = -2\pi T_0\left(\partial_a X^\mu \partial_b X_\mu - \frac{1}{2}\gamma_{ab}\partial_c X^\mu \partial^c X_\mu\right), \tag{1.1}$$

so that the equation of motion arising from varying $\gamma$ says $T_{ab} = 0$. We call $T_{ab} = 0$ a **constraint** on the equation of motion for $X^\mu$, which we derive soon.

**Exercise 1.2.3.** (Important!) Now vary $S_{\mathrm{P}}[X,\gamma]$ with respect to $X^\mu$ to obtain

$$\begin{aligned}\delta_X S_{\mathrm{P}}[X,\gamma] &= -T_0\int_\Sigma d^2\sigma\,\sqrt{-\gamma}\,\gamma^{ab}\left(\partial_a(\delta X^\mu \partial_b X_\mu) - \partial_a\partial_b X_\mu \delta X^\mu\right) \\ &= -T_0\int_0^\ell d\sigma^1\,\sqrt{-\gamma}\left[\delta X^\mu \partial^0 X_\mu\right]_{\sigma^0=\tau_i}^{\sigma^0=\tau_f} - T_0\int_{\tau_i}^{\tau_f} d\sigma^0\,\sqrt{-\gamma}\left[\delta X^\mu \partial^1 X_\mu\right]_{\sigma^1=0}^{\sigma^1=\ell} \\ &\quad + T_0\int_\Sigma d^2\sigma\,\sqrt{-\gamma}\,\delta X^\mu \nabla^2 X_\mu.\end{aligned}$$

A careful inspection of the terms in the variation $\delta_X S_{\mathrm{P}}[X,\gamma]$ yield interesting insights. For this variation to vanish, each of the terms must vanish independently, since they control different aspects of the string's behavior.

1. The last term is determined by the motion of the string in the domain $(0,\ell)\times(\tau_i,\tau_f)$, and therefore $\delta X^\mu$ is not constrained by any boundary conditions there. Hence we have the **equation of motion** $\sqrt{-\gamma}\,\nabla^2 X_\mu = 0$.
2. The first term is determined by the configuration of the string at times $\tau_i$ and $\tau_f$. If we specify these configurations as **initial and final conditions**, then $\delta X^\mu$ is zero for the first term, so the term vanishes.
3. The second term is determined by the configuration of the endpoints of the string when $\sigma^0 \in (\tau_i,\tau_f)$. It does not vanish automatically. We have to impose **boundary conditions** in order to make it vanish.

**Definition 1.2.5.** There are two different kinds of boundary conditions.

- The **free (Neumann) boundary condition** is $\partial^1 X_\mu(\sigma^0,0) = \partial^1 X_\mu(\sigma^0,\ell) = 0$.
- The **Dirichlet boundary condition** is $\delta X^\mu(\sigma^0,0) = \delta X^\mu(\sigma^0,\ell) = 0$.

Alternatively, if the string is **closed**, i.e. we have the **periodicity** conditions

$$X^\mu(\sigma^0,0) = X^\mu(\sigma^0,\ell), \quad \partial^a X^\mu(\sigma^0,0) = \partial^a X^\mu(\sigma^0,\ell), \quad \gamma_{ab}(\sigma^0,0) = \gamma_{ab}(\sigma^0,\ell),$$

no additional boundary conditions are necessary.

For a long time, string theorists did not seriously consider the Dirichlet boundary condition. Why should the endpoints of an open string be fixed, and if they were, where would they be fixed onto? In particular, this fixing of endpoints would violate momentum conservation. Then Polchinski, in the 1990s, suggested that the endpoints are attached to **D-branes**, which should themselves be thought of as dynamical objects alongside strings. Conceptually, then,

1. a D0-brane is a particle, a D1-brane is a string, and so on, and they interact non-trivially;
2. the Dirichlet boundary condition says that a given D1-brane has fixed endpoints on a higher D$p$-brane;
3. any momentum lost by the D1-brane is absorbed by the D$p$-brane; and
4. the Neumann boundary condition is just saying there is a D-dimensional D-brane permeating all of space-time, i.e. the string endpoints are not fixed at all.

We return to this D-brane perspective much later on. It is hard enough to quantize strings without more dynamical objects floating around. We take **Neumann boundary conditions** for now.

## 1.3 Gauge Freedom and Gauge Fixing

There is another reason the Polyakov action is preferable over the Nambu–Goto action: it has more symmetries, and these symmetries make it easier to gauge fix (using Faddeev–Popov or otherwise) when we try to quantize. The Polyakov action is invariant under the following symmetries:

1. $D$-dimensional **Poincaré transformations**:

$$X^\mu(\sigma) \mapsto \Lambda^\mu{}_\nu X^\nu(\sigma) + a^\mu, \quad \gamma_{ab}(\sigma) \mapsto \gamma_{ab}(\sigma);$$

2. **Reparametrization** (i.e. diffeomorphisms): for new coordinates $\tilde{\sigma}^a(\sigma)$,

$$X^\mu(\sigma) \mapsto X^\mu(\tilde{\sigma}), \quad \gamma_{ab}(\sigma) \mapsto \frac{\partial \sigma^c}{\partial \tilde{\sigma}^a}\frac{\partial \sigma^d}{\partial \tilde{\sigma}^b}\gamma_{cd}(\sigma);$$

3. 2-dimensional **Weyl transformations**: for arbitrary $\omega(\sigma)$,

$$X^\mu(\sigma) \mapsto X^\mu(\sigma), \quad \gamma_{ab}(\sigma) \mapsto \exp(2\omega(\sigma))\gamma_{ab}(\sigma).$$

The Nambu–Goto action is not invariant under Weyl transformations.

**Exercise 1.3.1.** Verify all these statements. (This should be quite straightforward.)

**Definition 1.3.1.** Let diff denote the group of diffeomorphisms acting on $\Sigma$, and Weyl the group of Weyl transformations acting on $\Sigma$; these are **internal symmetries**, while Poincaré transformations are **external symmetries**. The product diff $\times$ Weyl is the **gauge group**. The orbit, in the space of all possible fields and metrics, of a particular $(X, \gamma)$ under the action of the gauge group is the **gauge orbit**.

A good exercise in working with the gauge and external symmetries is to make sure Polyakov action is as general as possible. This also reduces future work when we need the additional terms in the Polyakov action. Note that here, contrary to the case in QFT, the symmetries are very demanding. Weyl invariance in particular is very odd: it prevents us from adding terms such as

$$\int_\Sigma d^2\sigma\, \sqrt{-\gamma}\, V(X), \quad \mu \int_\Sigma d^2\sigma\, \sqrt{-\gamma}.$$

**Exercise 1.3.2.** Convince yourself that the action must contain one more $\gamma^{ab}$ than $\gamma_{ab}$ in order to satisfy Weyl invariance and counteract the change in $\sqrt{-\gamma}$. Since such a $\gamma^{ab}$ can only pair up indices with derivatives, we need a second-order Lorentz-invariant term that is coordinate-independent. Convince yourself that other than $\partial_a X^\mu \partial_b X_\mu$, this term can only involve $\gamma^{ab}$ and $\gamma_{ab}$, and that in fact it must be the **scalar curvature** $R$. Show that under a Weyl transformation,

$$\sqrt{-\gamma}\, R \mapsto \sqrt{-\gamma}(R - 2\nabla^2\omega).$$

Hence argue that we need another term integrated over $\partial\Sigma$ to counteract $\nabla^2(\sqrt{-\gamma}\,\omega)$. Putting everything together, conclude that

$$\chi \coloneqq \frac{1}{4\pi}\int_\Sigma d^2\sigma\, \sqrt{-\gamma}\, R + \frac{1}{2\pi}\int_{\partial\Sigma} ds\, k$$

is Weyl invariant, and that it is essentially the only term we can add to the Polyakov action. Here $ds$ is proper time along $\partial\Sigma$ using the metric $\gamma_{ab}$, and $k \coloneqq \pm t^a n_b \nabla_a t^b$ is the **geodesic curvature** of the boundary, where $t^a$ is a unit vector tangent to the boundary, and $n_b$ an outward-pointing unit vector, and we choose $\pm$ depending on whether the boundary is timelike or spacelike.

Let's explore a few choices of gauge, some which use up all the gauge freedom, and some which do not. We commonly use reparametrization invariance to simplify expressions, so let's explore some choices of gauge using reparametrization invariance first.

**Definition 1.3.2.** We can reparametrize $(\sigma^0, \sigma^1)$ such that $\sigma^0$ corresponds to the time coordinate $x^0$, i.e. $X^0 = R\sigma^0$ for some dimensionful constant $R$. This is **static gauge**, named as such because then lines of constant $\sigma^0$ correspond to the string at fixed moments in time, i.e. the string is static. Another choice is **light cone gauge**, given by $X^+ = R\sigma^0$, where

$$X^{\pm} \coloneqq \frac{1}{\sqrt{2}}(X^0 \pm X^1), \quad \sigma^{\pm} \coloneqq \frac{1}{\sqrt{2}}(\sigma^0 \pm \sigma^1)$$

are **light cone coordinates** on Minkowski space and the world sheet respectively. When in light cone gauge, the indices $i, j, \ldots$ range over $\{2, \ldots, D\}$.

Clearly neither static gauge nor light cone gauge exhausts the gauge freedom: we haven't done anything with the metric! But it is hard to transform the metric in a useful way while staying in static or light cone gauge. Let's take a different approach and try to transform the metric first.

The transformation of the scalar curvature computed in the exercise above says we can use Weyl invariance to locally set the scalar curvature to zero, by solving $2\nabla^2\omega = R$ and then applying the Weyl transformation $\exp(2\omega)$. But we are in two dimensions, where the symmetries of the Riemann curvature tensor determine it from $R$:

$$R_{abcd} = R_{cdab},\ R_{abcd} = -R_{bacd} = -R_{abdc} \implies R_{abcd} = (1/2)(\gamma_{ac}\gamma_{bd} - \gamma_{ad}\gamma_{bc})R.$$

Hence we can always locally get a flat metric, which, possibly after applying a coordinate transformation, gives $\gamma_{ab} = \eta_{ab}$, the flat Minkowski metric.

**Definition 1.3.3.** If we consider only reparametrization and not Weyl transformations, the metric $\gamma_{ab}$ can always be brought to the form $\exp(2\omega)\eta_{ab}$. Forcing the metric to be of that form is known as **conformal gauge**. Performing the additional Weyl transformation to obtain $\gamma_{ab} = \eta_{ab}$ is known as **unit gauge**. In general, the form of the metric we choose to put $\gamma_{ab}$ in is called the **fiducial metric**.

**Exercise 1.3.3.** (Important!) Show that in unit gauge, the equation of motion and its constraints become

$$\partial_a\partial^a\vec{X} = 0, \quad \partial_0\vec{X}\cdot\partial_1\vec{X} = 0, \quad (\partial_0\vec{X})^2 + (\partial_1\vec{X})^2 = R^2.$$

In this form, the constraints are called **Virasoro conditions**. Argue that by tensoriality, the Virasoro conditions still hold in static gauge, where $X^\mu = (R\sigma^0, \vec{X})$. Hence show in static gauge that at the (free) endpoints an open string, i.e. endpoints satisfying the Neumann boundary condition, $|\partial_t\vec{X}| = 1$. (**Be careful**: $\partial_t$ is not $\partial_0$. What is $\partial_t$?) Conclude that string endpoints always move at the speed of light.

How many internal degrees of freedom have we used up if we put the metric $\gamma_{ab}$ in unit gauge? Well, diff has two degrees of freedom, one for each coordinate, and Weyl has one, for the scale of the metric. But the metric itself has three independent components, being symmetric. Hence we expect to be done with choosing a representative of each gauge orbit.

But, perhaps unexpectedly, there is more gauge freedom: there are non-trivial transformations in diff $\times$ Weyl that preserve unit gauge! The key to finding these transformations is to realize that $\Sigma$ is actually a **Riemann surface**: let $z \coloneqq \sigma^0 + i\sigma^1$, so that $ds^2 = dzd\bar{z}$. Now if $f(z)$ is a holomorphic change of coordinates, then

$$z \mapsto f(z), \quad ds^2 \mapsto |\partial_z f|^{-2} dzd\bar{z},$$

so now applying the Weyl transformation $\exp(2\ln|\partial_z f|)$ recovers $ds^2$. Clearly the composition of the two transformations is non-trivial.

What went wrong? Well, just because dimensions match up does not mean we have spanned the whole space of gauge transformations! The holomorphic diffeomorphisms above actually have **measure zero** in diff. When we stop working locally and work globally instead, these extra bits of freedom are removed by boundary conditions.

**Definition 1.3.4.** When we successfully pick a unique and continuously-varying choice of representative in each gauge orbit, our theory is **gauge-fixed**. When such a choice is impossible due to topological obstructions, our theory has **Gribov ambiguity**. (For us, there is no Gribov ambiguity; we are just failing to consider boundary conditions.)

## 1.4 Quantization via Canonical Commutation Relations

When we did QFT, we started by **canonically quantizing** the Klein-Gordon and Dirac fields, which allowed us to immediately investigate some aspects of the quantized free theories, such as that Klein-Gordon fields represent bosons and Dirac fields represent fermions, and to obtain the spectrum and Hilbert space of states. On the other hand, **path integral quantization** gave us an easy way to compute interactions in perturbative QFT, such as scattering amplitudes. We do the same for string theory: first, in this section, we canonically quantize in order to write down the spectrum and Hilbert space of states, and then, in the next section, we quantize using the path integral to work with interactions.

In string theory, canonical quantization no different from what we saw in QFT. The procedure is the same: take the classical object (e.g. Lagrangian, Hamiltonian, solutions) you want to quantize, and impose **canonical commutation relations** modeled on $[x,p] = i$ on dynamical variables, by promoting them all to operators.

### 1.4.1 Classical Solutions

We take classical solutions and quantize them in light cone gauge as well as two more gauge-fixing conditions for the metric: set

$$X^+ = \sigma^0, \quad \partial_1 \gamma_{11} = 0, \quad \det \gamma_{ab} = -1.$$

Note that we have dispensed with the dimensionful constant $R$; it can be reinserted via dimensional analysis. The first thing to do right after picking a gauge is to rewrite all the relevant objects in that gauge. To do so, we need some formulas.

**Exercise 1.4.1.** Show that in this gauge, $\gamma_{11}(\sigma^0)$ depends only on $\sigma^0$, and we have

$$\begin{pmatrix} \gamma^{00} & \gamma^{01} \\ \gamma^{10} & \gamma^{11} \end{pmatrix} = \begin{pmatrix} -\gamma_{11}(\sigma^0) & \gamma_{01}(\sigma) \\ \gamma_{01}(\sigma) & \gamma_{11}^{-1}(\sigma^0)(1-\gamma_{01}^2(\sigma)) \end{pmatrix}.$$

Furthermore, show that $\partial_a X^\mu \partial_a X_\mu = 2\partial_a X^+ \partial_a X^- - \partial_a X^i \partial_a X^i$. (Recall that indices $i, j, \ldots$ range over $\{2, \ldots, D\}$).

**Definition 1.4.1.** Given a dynamical variable $V(\sigma)$, define its associated **center of mass** (conceptually at a fixed time) variables

$$v(\sigma^0) = \frac{1}{\ell}\int_0^\ell d\sigma^1\, V(\sigma), \quad \tilde{V}(\sigma) = V(\sigma) - v(\sigma^0),$$

i.e. we split $V = v + \tilde{V}$ where $v$ is the mean value of $V$, and $\tilde{V}$ has mean zero.

For example, using that $\partial_1 X^+ = 0$, we have

$$\partial_1 \tilde{X}^- = \partial_1 X^- = \frac{1}{\sqrt{2}}(\partial_1 X^0 - \partial_1 X^1) = \sqrt{2}\partial_1 X^0,$$

and using that $\partial_0 X^+ = 1$, we have

$$\partial_0 X^0 \partial_1 X^0 - \partial_0 X^1 \partial_1 X^1 = (\partial_0 X^0 + \partial_0 X^1)\partial_1 X^0 - \partial_0 X^1(\partial_1 X^0 + \partial_1 X^1) = \sqrt{2}\partial_1 X^0 - 0.$$

**Exercise 1.4.2.** Using all these calculations, show that the Polyakov Lagrangian in this gauge is

$$L = -\frac{T_0}{2}\int_0^\ell d\sigma^1 \left[\gamma_{11}(2\partial_0 X^- - \partial_0 X^i \partial_0 X^i) - 2\gamma_{01}(\partial_1 \tilde{X}^- - \partial_0 X^i \partial_1 X^i) + \gamma_{11}^{-1}(1-\gamma_{01}^2)\partial_1 X^i \partial_1 X^i\right].$$

Argue that because $\tilde{X}^-$ does not appear with time derivatives, it is not a dynamical variable, and therefore when we vary $S_{\mathrm{P}}$ with respect to $\gamma$, it constrains $\partial_1\gamma_{01}$ to be zero. Show that the Neumann boundary condition, in this gauge, gives $\gamma_{01} = 0$ at the endpoints $\sigma = 0, \ell$, and conclude that $\gamma_{01} = 0$ everywhere. Therefore write down the simplified **Lagrangian**:

$$L = -T_0\ell\gamma_{11}\partial_0 x^- + \frac{T_0}{2}\int_0^\ell d\sigma^1 \, \left(\gamma_{11}\partial_0 X^i \partial_0 X^i - \gamma_{11}^{-1}\partial_1 X^i \partial_1 X^i\right),$$

The next step is to write down the Hamiltonian, which is the **Legendre transform** of the Lagrangian. Recall that this means we write down momenta $\Pi_\mu$ corresponding to $X^\mu$, and then define

$$H \coloneqq \int_0^\ell \Pi_\mu \partial_0 X^\mu - L = \int_0^\ell d\sigma^1 \, \left(\Pi_+\partial_0 X^+ + \Pi_-\partial_0 X^- + \Pi_i \partial_0 X^i\right) - L = p_-\partial_0 x^- + \int_0^\ell \Pi_i \partial_0 X^i - L,$$

where $p_-$ is the momentum conjugate to $x^-$, and $\Pi^i$ is the momentum density conjugate to $X^i$:

$$p_- \coloneqq \frac{\partial L}{\partial \partial_0 x^-} = -T_0\ell\gamma_{11}, \quad \Pi^i \coloneqq \frac{\delta L}{\delta \partial_0 X^i} = T_0\gamma_{11}\partial_0 X^i = \frac{p^+}{\ell}\partial_0 X^i.$$

Note that $p_- = -p^+$. Simplifying, we get the Hamiltonian

$$H = \frac{\ell T_0}{2p^+}\int_0^\ell d\sigma^1 \, \left(\frac{1}{T_0}\Pi^i\Pi^i + T_0\partial_1 X^i \partial_1 X^i\right),$$

which is precisely the **Hamiltonian** for $D-2$ free fields $X^i$, with $p^+ \propto \gamma_{11}$ a conserved quantity.

We can also directly write down **classical solutions**: the equation of motion in this gauge is $\partial_+\partial_- X^i = 0$, which has the general solution

$$X^i(\sigma) = X_L^i(\sigma^+) + X_R^i(\sigma^-)$$

for arbitrary functions $X_L^i$ and $X_R^i$, describing **left-moving** and **right-moving** waves respectively, which we can expand as Fourier series:

$$X_L^i(\sigma^+) = \frac{1}{2}x^i(0) + \frac{1}{2T_0}p^i(0)\sigma^+ + i\frac{\ell}{\pi}\sqrt{\frac{1}{2T_0}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^i e^{-in\pi\sigma^+/\ell},$$
$$X_R^i(\sigma^-) = \frac{1}{2}x^i(0) + \frac{1}{2T_0}p^i(0)\sigma^- + i\frac{\ell}{\pi}\sqrt{\frac{1}{2T_0}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^i e^{-in\pi\sigma^-/\ell}.$$

(Here $p^i$ are center of mass variables for $\Pi^i$ with an extra factor of $\ell$. We've also mucked around with the normalization factors for Fourier coefficients for later convenience.) Because the $X^i$ are real fields, we have the **constraints** $\tilde{\alpha}_n^i = (\tilde{\alpha}_{-n}^i)^*$ and $\alpha_n^i = (\alpha_{-n}^i)^\dagger$ on the Fourier coefficients.

**Exercise 1.4.3.** Show that the Neumann boundary condition forces $\tilde{\alpha}_n^i = \alpha_n^i$, so that the general form of a **classical solution for an open string** is

$$X^i(\sigma) = x^i(0) + \frac{1}{2T_0}p^i(0)\sigma^0 + i\frac{\ell}{\pi}\sqrt{\frac{1}{2T_0}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^i e^{-in\pi\sigma^0/\ell}\cos\frac{n\pi\sigma^1}{\ell}.$$

Finally, we must write down the **constraints**, i.e. the Virasoro conditions in this gauge. They become $(\partial_+ X)^2 = (\partial_- X)^2 = 0$, which give conditions on the momenta $p^i$ and Fourier coefficients $\alpha_n^i$. Both $\partial_+$ and $\partial_-$ give the same result, so we compute

$$\partial_+ X^i = \partial_+ X_L^i = \frac{1}{2T_0} p^i(0) + \sqrt{\frac{1}{2T_0}} \sum_{n \neq 0} \alpha_n^i e^{-in\pi\sigma^+/\ell}.$$

Hence, writing $\alpha_0^i \coloneqq \sqrt{1/2T_0}\, p^i(0)$,

$$0 = (\partial_+ X)^2 = \frac{1}{T_0} \sum_n L_n e^{-i\pi n \sigma^+/\ell}, \quad L_n \coloneqq \frac{1}{2} \sum_m \alpha_m \cdot \alpha_{n-m}.$$

So the $L_n$ are the Fourier coefficients of the constraints. By the linear independence of the Fourier basis, $L_n = 0$ for all $n \in \mathbb{Z}$. In particular, since $p_\mu p^\mu = -M^2$ is the effective mass and $L_0$ contains the momentum, $L_0 = 0$ implies that the **effective mass** of the string is

$$M^2 = -p \cdot p = 4T_0 \sum_{m>0} \alpha_m \cdot \alpha_{-m}.$$

### 1.4.2 Canonical Quantization

Quantization is now trivial: we impose the canonical **equal-time commutation relations**

$$[x^-, p^+] = i\eta^{-+} = -i, \quad [X^i(\sigma), \Pi^j(\sigma')] = i\delta^{ij}\delta(\sigma - \sigma'),$$

with all other commutators vanishing. In terms of Fourier components,

$$[x^-, p^+] = -i, \quad [x^i, p^j] = i\delta^\mu{}_\nu, \quad [\alpha_m^i, \alpha_n^j] = m\delta^{ij}\delta_{m+n,0},$$

with all other commutators vanishing. So as in QFT, we can treat $\alpha_n^i$ as creation/annihilation operators ($\alpha$ is annihilation, $\alpha^\dagger$ is creation), and build up our state space using them. Note that instead of just a single creation/annihilation operator, we have an infinite tower of them!

**Definition 1.4.2.** The **creation/raising operators** are $\alpha_{-m}^i$ and the **annihilation/lowering operators** are $\alpha_n^i$. The **ground state of a string with momentum** $k$ is defined as the eigenstate $|0; k\rangle$ of $p^i$, the center of mass momenta, annihilated by the annihilation operators, i.e.

$$p^+ |0;k\rangle = k^+ |0;k\rangle, \quad p^i |0;k\rangle = k^i |0;k\rangle, \quad \alpha_m^i |0;k\rangle = 0 \quad \forall m > 0.$$

Note that the zero-momentum ground state $|0;0\rangle$ of a string is not the true **vacuum state**, which consists of no strings at all; we denote the true vacuum state $|\text{vacuum}\rangle$.

Unlike QFT, each raising operator $\alpha_{-m}^i$ (for varying $m$) creates a different mode. So the **independent states** are labeled using center of mass momenta $k = (k^+, k^i)$, and occupation numbers $N_{i,n}$ for $i = 2, \ldots, D$ and $n = 1, 2, \ldots$:

$$|N;k\rangle \coloneqq \left( \prod_{i=2}^{D} \prod_{n=1}^{\infty} \frac{(\alpha_{-n}^i)^{N_{i,n}}}{\sqrt{n^{N_{i,n}} N_{i,n}!}} \right) |0;k\rangle.$$

(The normalization is chosen for convenience.) Hence there are an infinite number of different first excitations of a single string. Let $\mathcal{H}_1$ denote the space of all possible single-string states:

$$\mathcal{H}_1 \coloneqq \text{span}\{|N;k\rangle : \text{all possible } N, k\}.$$

**Definition 1.4.3.** The **state space**, of any number of strings, is a **bosonic Fock space**

$$\mathrm{Sym}(\mathcal{H}_1) \coloneqq |\text{vacuum}\rangle \oplus \mathcal{H}_1 \oplus (\mathcal{H}_1 \odot \mathcal{H}_1) \oplus (\mathcal{H}_1 \odot \mathcal{H}_1 \odot \mathcal{H}_1) \oplus \cdots \oplus \cdots,$$

where $\odot$ is the symmetrized tensor product

$$v_1 \odot \cdots \odot v_n \coloneqq \frac{1}{n!} \sum_{\sigma \in S_k} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}.$$

The $n$-th term in the sum $\mathrm{Sym}(\mathcal{H}_1)$ is the state space of $n$ strings. We symmetrize because it turns out the strings we are working are bosonic, i.e. they have integer spin, i.e. they commute, instead of anticommuting. ($\mathrm{Sym}(\mathcal{H}_1)$ is known by us mathematicians as a **symmetric algebra**; a fermionic Fock space, for objects with half-integer spins, is an exterior algebra.)

We still need to impose the constraints $L_n = 0$, coming from the Virasoro conditions. Naively one might just insist that as operators, $L_n = 0$, but this quickly runs into problems (cf. Gupta–Bleuler quantization of QED). Instead, we impose $L_n |\text{phys}\rangle = 0$ for any physical state $|\text{phys}\rangle$.

### 1.4.3 Spectrum and Critical Dimension

By mass-energy equivalence, to find the spectrum of our quantized string is equivalent to finding its effective mass, i.e. we must look at the quantized version of $M^2 = 4T_0 \sum_{m>0} |\alpha_m|^2$. But when we quantize, $\alpha_m$ and $\alpha_{-m}$ no longer commute, so there is an operator ordering ambiguity here. There are two choices: either we quantize $\alpha_m \cdot \alpha_{-m}$, or we quantize $\alpha_{-m} \cdot \alpha_m$. They both give

$$M^2 = 4T_0 \sum_{m>0} (N_m + a), \quad N_m \coloneqq \alpha_{-m} \cdot \alpha_m$$

(where by analogy with the harmonic oscillator, we've defined the **number operators** $N_m$), but the first with $a = m(D-2)/2$, using the commutation relation $[a_m^i, a_{-m}^i] = m$, and the second with $a = 0$. There are some physical arguments for why we pick the former during the quantization of the simple harmonic oscillator (Heisenberg uncertainty principle, etc.), but it boils down to the assertion that we want the ground state of the system to have non-zero energy. Hence we pick $a = m(D-2)/2$.

**Exercise 1.4.4.** Recall/review from QFT that $\sum_{m>0} m = -1/12$, and therefore conclude that the ground state and first excited states, i.e. $\alpha_{-m}^i |0; k\rangle$ for any $m$, have energies

$$M_0^2 = 4T_0 \frac{2-D}{24}, \quad M_1^2 = 4T_0 \frac{26-D}{24}.$$

Fix an $m$. The first excited state $\alpha_{-m}^i |0; k\rangle$ acts as a vector because it has a vector index $i$, so it better be Lorentz invariant. In particular, in the rest frame, the (spatial rotation subgroup of the) Lorentz group can act on a vector to make it point in any spatial direction, so vectors better have $D-1$ states. But $\alpha_{-m}^i$ lives in the standard representation of $\mathrm{SO}(D-2)$: it only has $D-2$ states contained in it. This is not good!

Here is the solution: we posit that $M_1^2 = 0$. Then there is no rest frame! Consequently, we are only free to rotate around the direction of motion, giving only $D-2$ states, exactly the number that we have. But this implies $D = 26$, known as the **critical dimension** of bosonic string theory. This entire argument is sketchy, and we (hopefully) give a more rigorous argument later that $D = 26$ is the only dimension that works, based on enforcing Weyl invariance.

There is another problem: $M_0^2 < 0$ for $D > 2$, especially for $D = 26$. We have **negative energy states**, known as **tachyons**! This is explained from a field-theoretic perspective: given a field $\phi$, its mass squared is just $\partial^2 V(\phi)/\partial\phi^2|_{\phi=0}$. We are actually expanding around a critical point of the potential that is a maximum, i.e. an **unstable** point, therefore resulting in a negative mass-squared. Currently it is unknown whether there are stable points in the purely bosonic theory. However, with the addition of fermions and **supersymmetry**, giving the **superstring**, the problem disappears. This is content for much later on.

## 1.5 Quantization via Path Integral

Now it is time to develop a different tool. Recall from QFT that we have a giant machine for quantizing classical theories and studying their interactive pictures: the path integral. However, before we begin plugging the Polyakov action into the machine, we need to make a modification. From now on, the world sheet is equipped with a **Euclidean metric** $g_{ab}$, instead of a Lorentzian one $\gamma_{ab}$. This is so that the path integral over metrics is better defined. The transition from Euclidean to Minkowski is, formally, done via **Wick rotation**: $x^0 \mapsto ix^0$ and similarly for the metric. The **Euclidean path integral**, and the Euclidean action (with the additional terms on top of the Wick-rotated Polyakov action), is therefore

$$Z \coloneqq \frac{1}{\text{Vol}} \int \mathcal{D}g\,\mathcal{D}X\,\exp(-S_{\text{P}}[X,g]),$$

$$S_{\text{P}}[X,g] = \frac{T_0}{2}\int_\Sigma d^2\sigma\,\sqrt{g}\,g^{ab}\partial_a X^\mu \partial_b X_\mu + \lambda\left(\frac{1}{4\pi}\int_\Sigma d^2\sigma\,\sqrt{g}\,R + \frac{1}{2\pi}\int_{\partial\Sigma} ds\,k\right)$$

where Vol is the volume of the gauge action on the **configuration space** consisting of all possible $X^\mu$ and $g$). More explicitly, we can imagine partitioning configuration space into gauge orbits; we actually want to integrate on a path through these gauge orbits. But now recall from QFT that we have another giant machine for doing so: the Faddeev-Popov method.

### 1.5.1 The Faddeev-Popov Method

Let's first recall that the idea behind Faddeev-Popov is very natural: we want to do a change of coordinates in configuration space so that instead of integrating over a mish-mash of $g$ and $X$, we integrate such that one variable goes along gauge orbits, and the other goes along the gauge-fixed path. Although this sounds technical, we perform procedures like this quite often without realizing it! For example, consider the calculation

$$\iint dx\,dy\,e^{-x^2-y^2} = \int d\theta \int dr\,re^{-r^2} = 2\pi\int dr\,re^{-r^2} = \pi.$$

What is really happening here is that we recognized the $U(1)$ symmetry of the original integrand, and changed variables in order to factor out that symmetry. Instead of integrating over $(x,y)$, we integrated over $(r,\theta)$, with $\theta$ parametrizing the gauge orbits. Furthermore, we picked out the $y=0$ representative of each gauge orbit for the remaining integral.

Armed with this motivation, we can proceed. Let $\hat{g}_{ab}$ be the fiducial metric; it represents our choice of gauge fixing, just like the choice $y=0$. Let $\zeta$ be shorthand for a combined coordinate and Weyl transformation:

$$\zeta\colon g_{ab} \mapsto g^\zeta_{ab} \coloneqq \exp(2\omega(\sigma))\frac{\partial\sigma^c}{\partial\sigma'^a}\frac{\partial\sigma^d}{\partial\sigma'^b}g_{cd}(\sigma).$$

**Definition 1.5.1.** Let $\mathcal{D}\zeta$ be a gauge invariant measure on diff $\times$ Weyl. (Whether such a measure exists is very relevant for us, but we disregard it for now.) Define the **Faddeev-Popov determinant** $\Delta_{\text{FP}}$ by

$$\Delta_{\text{FP}}^{-1}(g) \coloneqq \int \mathcal{D}\zeta\,\delta[\hat{g}^\zeta - g].$$

Here the $\delta$ is the **Dirac functional**, i.e. $\hat{g}^\zeta$ and $g$ must agree at every point $\sigma$.

**Exercise 1.5.1.** Show that $\Delta_{\text{FP}}(g)$ is gauge-invariant by computing that $\Delta_{\text{FP}}(g^\zeta)^{-1} = \Delta_{\text{FP}}(g)^{-1}$.

Now it is time to do the calculation to factor out the integral over the gauge orbits. The first step is to add a 1 to the integral:

$$Z = \int \frac{\mathcal{D}g\,\mathcal{D}X}{\text{Vol}}\exp(-S_{\text{P}}[X,g]) = \int \frac{\mathcal{D}g\,\mathcal{D}X\,\mathcal{D}\zeta}{\text{Vol}}\Delta_{\text{FP}}(g)\delta[\hat{g}^\zeta - g]\exp(-S_{\text{P}}[X,g]).$$

The second step is to do the integral over $g$, which, due to the $\delta[\hat{g}^\zeta - g]$, amounts to replacing $g$ with $\hat{g}^\zeta$:

$$Z = \int \frac{\mathcal{D}X\,\mathcal{D}\zeta}{\mathrm{Vol}} \Delta_{\mathrm{FP}}(\hat{g}^\zeta)\exp(-S_{\mathrm{P}}[X,\hat{g}^\zeta]).$$

Finally, since both $\Delta_{\mathrm{FP}}$ and $S_{\mathrm{P}}$ are gauge-invariant, we can replace $\hat{g}^\zeta$ with $\hat{g}$. Then nothing in the integrand depends on $\zeta$ anymore, so it factors out and cancels the volume normalization:

$$Z = \int \frac{\mathcal{D}\zeta}{\mathrm{Vol}} \int \mathcal{D}X\,\Delta_{\mathrm{FP}}(\hat{g})\exp(-S_{\mathrm{P}}[X,\hat{g}]) = \int \mathcal{D}X\,\Delta_{\mathrm{FP}}(\hat{g})\exp(-S_{\mathrm{P}}[X,\hat{g}]).$$

**Exercise 1.5.2.** Evaluate $\iint dx\,dy\,e^{-x^2-y^2}$ by applying the Faddeev-Popov method to its $U(1)$ symmetry and the gauge-fixing condition $y=0$. Conclude that the Faddeev-Popov method is completely rigorous in finite dimensions, and that $\Delta_{\mathrm{FP}}$ is actually a Jacobian (hence the name Faddeev-Popov determinant).

### 1.5.2 Computing the Faddeev-Popov Determinant

It remains to compute the Faddeev-Popov determinant $\Delta_{\mathrm{FP}}$ for the diff $\times$ Weyl action on world sheet metrics. To do so, we make the simplifying assumption that diff $\times$ Weyl actually acts freely on metrics $g$, i.e. for each $g$, there is exactly one $\zeta$ such that $\delta[\hat{g}^\zeta - g] = 0$. Obviously this assumption is false: we showed earlier that the action has fixed points (albeit a measure zero set of them). But it is true locally, so we deal with the global issues later. The reason we make this assumption is so that we can compute $\Delta_{\mathrm{FP}}(\hat{g})^{-1}$ by integrating only around a small neighborhood of $\zeta = 0$. In this neighborhood, we can take infinitesimal Weyl transformations $\omega(\sigma)$ and infinitesimal diffeomorphisms $\delta\sigma^\alpha = v^\alpha(\sigma)$, and write

$$\Delta_{\mathrm{FP}}^{-1}(\hat{g}) = \int \mathcal{D}\omega\,\mathcal{D}v\,\delta[2\omega\hat{g}_{ab} + \nabla_a v_b + \nabla_b v_a].$$

Note that now we are integrating over the Lie algebra of diff $\times$ Weyl. We want to get rid of the delta functional.

**Exercise 1.5.3.** For a function $\phi\colon \mathbb{R}^D \to \mathbb{R}$, derive the integral form

$$\delta[\phi] = \int_{j\colon \mathbb{R}^D \to \mathbb{R}} \mathcal{D}j(x)\,\exp\left(2\pi i \int d^D x\, j(x)\phi(x)\right)$$

by applying the one-dimensional identity $\delta(x) = \int dp\,\exp(2\pi i p x)$ to piecewise linear paths, and then taking the limit as the number of path segments goes to infinity.

In our case, the function inside the delta functional lives on the world sheet $\Sigma$, whose integration measure is $d^2\sigma\,\sqrt{\hat{g}}$ (remember we fixed the fiducial metric). Hence, if $\beta$ ranges over symmetric 2-tensors on $\Sigma$, then

$$\Delta_{\mathrm{FP}}^{-1}(\hat{g}) = \int \mathcal{D}\omega\,\mathcal{D}v\,\mathcal{D}\beta\,\exp\left(2\pi i \int d^2\sigma\,\sqrt{\hat{g}}\,\beta^{ab}(2\omega\hat{g}_{ab} + \nabla_a v_b + \nabla_b v_a)\right).$$

But we can directly do the integral over $\omega$. The one and only term containing an $\omega$ factors out to give a delta functional:

$$\int \mathcal{D}\omega\,\exp\left(2\pi i \int d^2\sigma\,\sqrt{\hat{g}}\,\beta^{ab}(2\omega\hat{g}_{ab})\right) = \delta[2\beta^{ab}\hat{g}_{ab}],$$

i.e. in the remaining integral, $\beta^{ab}$ is traceless:

$$\Delta_{\mathrm{FP}}^{-1}(\hat{g}) = \int \mathcal{D}v\,\mathcal{D}\beta\,\exp\left(2\pi i \int d^2\sigma\,\sqrt{\hat{g}}\,\beta^{ab}(\nabla_a v_b + \nabla_b v_a)\right).$$

**Recap**: we are integrating over vector fields $v$ and symmetric 2-tensors $\beta$ such that $\beta^{ab}$ is traceless, both living on $\Sigma$.

### 1.5.3 Faddeev-Popov Ghosts

We are not done: the path integral above is for $\Delta_{\mathrm{FP}}^{-1}$, but we want $\Delta_{\mathrm{FP}}$ itself. There is a general procedure for inverting $\Delta_{\mathrm{FP}}^{-1}$. To understand it, we must first clarify what $\Delta_{\mathrm{FP}}$ really is. Let $F$ is the gauge-fixing condition. (For us, $F$ is a function of $g$ and $\zeta$ and takes values in symmetric 2-tensors.) Note that via a change of variables from $\zeta$ to $F$,

$$\Delta_{\mathrm{FP}}^{-1} = \int D\zeta\, \delta(F) = \int DF\, \det\left[\frac{\delta\zeta}{\delta F}\right]\delta(F) = \det\left[\frac{\delta\zeta}{\delta F}\right]_{F=0}.$$

This change of variables is valid again because we assume $\zeta$ acts freely on gauge orbits, and $F$ is supposed to pick a unique representative from each gauge orbit, so $\zeta$ and $F$ "have the same number of degrees of freedom" as physicists like to say. Now all we have to do is invert the determinant. For this, we use a clever trick, which is developed in the following two exercises.

**Exercise 1.5.4.** Show by analogy from the finite dimensional case for two real fields $\phi^1$ and $\phi^2$ that

$$\int \mathcal{D}\phi^1\, \mathcal{D}\phi^2 \exp\left(i\int d^Dx\, \phi^1 A\phi^2\right) = (\det A)^{-1}.$$

**Exercise 1.5.5.** Recall from QFT that we defined **Grassmann numbers**: they are anti-commuting formal variables, i.e. $\theta\eta = -\eta\theta$, that form an algebra. We also worked out the **Berezin integral** for Grassmann-valued quantities, with the convention that $\int d\theta \int d\eta\, \eta\theta = 1$. If $\theta$ and $\eta$ are Grassmann variables, i.e. taking values in the Grassmann algebra, and $b \in \mathbb{R}$, review/show (in order) that

$$\theta^2 = 0, \quad \int d\theta\, f(\theta) = \frac{\partial f}{\partial\theta}, \quad \int d\theta\, d\eta\, \exp(-\theta b\eta) = b$$

Hence show by analogy with the finite dimensional case that for Grassmann-valued fields $\chi^1$ and $\chi^2$,

$$\int \mathcal{D}\chi^1\, \mathcal{D}\chi^2 \exp\left(-\int d^Dx\, \chi^1 A\chi^2\right) = \det A.$$

So here's the trick: if we have a path integral expression for $(\det A)^{-1}$, to get $\det A$ we simply replace ordinary variables with Grassmann variables! In particular, to get $\Delta_{\mathrm{FP}}(\hat{g})$ from $\Delta_{\mathrm{FP}}(\hat{g})^{-1}$, we replace $(\beta_{ab}, v^a)$ with Grassmann-valued fields $(b_{ab}, c^a)$, with $b^{ab}$, like $\beta^{ab}$, being traceless:

$$\Delta_{\mathrm{FP}}(\hat{g}) = \int \mathcal{D}b\, \mathcal{D}c\, \exp(S_{\mathrm{G}}), \quad S_{\mathrm{G}} \coloneqq \frac{1}{2\pi}\int d^2\sigma\, \sqrt{\hat{g}}\, b_{ab}\nabla^a c^b.$$

Note that we've implicitly made a few cosmetic changes:

1. Because $b$ is a symmetric 2-tensor (do **not** confuse the fact that $b$ is symmetric, i.e. $b_{ab} = b_{ba}$, with $b$ being anti-commutative, e.g. $b_{ab}\theta = -\theta b_{ab}$), we can rewrite
$$b^{ab}(\nabla_a c_b + \nabla_b c_a) = b^{ab}\nabla_a c_b + b^{ab}\nabla_a c_b = 2b^{ab}\nabla_a c_b = 2b_{ab}\nabla^a c^b.$$
2. We chose slightly different normalization factors to make later computations cleaner.

The quantity $S_{\mathrm{G}}$ is called the **ghost action**: when we plug $\Delta_{\mathrm{FP}}(\hat{g})$ back into the path integral, we get

$$Z = \int \mathcal{D}X\, \mathcal{D}b\, \mathcal{D}c\, \exp(-S_{\mathrm{P}}[X, \hat{g}] - S_{\mathrm{G}}[b, c]),$$

i.e. $S_{\mathrm{G}}$ becomes part of the action. The fields $b$ and $c$, which do not correspond physically to anything, are **Faddeev-Popov ghost fields**. The price of gauge fixing is the introduction of these unphysical ghosts.

**Exercise 1.5.6.** Repeat the computation of $\Delta_{\mathrm{FP}}$ for QED, and show that for QED, $\Delta_{\mathrm{FP}}$ is independent of any fields. Hence conclude that QED has no Faddeev-Popov ghosts. (That's why quantizing QED went a lot faster. QCD has ghosts, however.)

# Chapter 2

# Conformal Field Theory

String theory as we have defined it so far is a 2 dimensional theory where the fields are parameterized by two coordinates $(\sigma^1, \sigma^2)$. We shall now explore the conformal symmetry of the Polyakov action and deduce a number of important technical tools that will enable us to say a lot about the properties of this quantum field theory. This conformal symmetry is especially large in two dimensions and provides significant constraints. [1]

The technical tool that will drive this whole chapter is the **operator product expansion** (OPE). This is a canonical form for the product of two local operators:

$$\mathcal{A}_i(\sigma_1)\mathcal{A}_j(\sigma_2) = \sum_k c_{ij}^k(\sigma_1 - \sigma_2)\mathcal{A}_k(\sigma_2). \tag{2.1}$$

This will turn out to be much like a Laurent expansion and the form of $c_{ij}^k(\sigma_1 - \sigma_2)$ is severely restricted.

There are many reasons why it is useful for us to learn about CFT. Certain critical phase transitions can be described by a CFT and using the AdS/CFT correspondence we may be able to take a highly correlated system and rewrite it in terms of a weakly coupled theory of supergravity. Let's begin!

The plan for this chapter as of January 1, 2016 will be to showcase important details of chapter 2 from Polchinski's Volume 1 leaving out some technical details for as exercises. In the future it would be nice to include $d$-dimensional CFT and it's application to condensed matter systems.

## 2.1 Conformal Normal Order and Operator Product Expansions

In our QFT adventures we focused on computing correlation functions since every physical quantity could be expressed in terms of them. However, in our journey we focused a lot on operators of the form $\langle \phi_1 \phi_2 \cdots \phi_n \rangle$. We shall now generalize this ever so slightly.

**Definition 2.1.1.** Let $\sigma_0$ be a fixed point and consider a classical world-sheet field theory with fields $X_1(\sigma), \ldots, X_n(\sigma)$. A **local functional** is a function $\mathscr{F}[X]$ taking in, as arguments, $X_i(\sigma_0)$ and $\partial_a X_i(\sigma_0)$ which are taken at $\sigma_0$. A **local operator** is the quantized version of a local functional that has well defined expectation values. Often a local operator is given by the normal ordering of a local functional.

[1] References that were used for the preparation of this chapter include Polchinski's Vol 1, Polchinski's Little Book, and Ginsparg's Applied CFT arXiv:hep-th/9108028, IAS Vol 1 and 2, Gomis' PSI 14/15 Lectures on CFT, Green Schwarz Witten Vol 1.

Here are some examples of local operators $X^\mu(0,0), X^\mu(a,b)X_\mu(a,b), \partial_{\bar{z}}X^3(z_1,\bar{z}_1)$. However, $X^\mu(z_1,\bar{z}_1)+X^\mu(z_2,\bar{z}_2)$ is not a local operator. Before introducing the operator product expansion we need to introduce conformal normal ordering which will be used in the definition of OPE.

**Definition 2.1.2.** Write $z_{12} = z_1 - z_2$. Let $\mathscr{F}$ be an arbitrary function of $X$. Define the (free-field) **normal order** of $\mathscr{F}$ to be the functional:

$$:\mathscr{F}: = \mathscr{F} + \sum (\text{subtractions}) \tag{2.2}$$

$$= \exp\left(\frac{\alpha'}{2}\int d^2z_1 d^2z_2 \ln|z_{12}|^2 \frac{\delta}{\delta X^\mu(z_1,\bar{z}_1)}\frac{\delta}{\delta X_\mu(z_2,\bar{z}_2)}\right)\mathscr{F} \tag{2.3}$$

This is very analogous to the normal ordering that we saw in QFT where the coefficient $\eta^{\mu\nu}\ln|z_{12}|^2$ is replaced by the corresponding propogator $\Delta(z_1,z_2)$. The QFT version of normal ordering is useful for calculating matrix elements, while this version is useful for computing the OPE.

Here are two examples: $:X^\mu(z,\bar{z}): = X^\mu(z,\bar{z})$, and

$$:X^\mu(z_1,\bar{z}_2)X^\nu(z_2,\bar{z}_2): = X^\mu(z_1,\bar{z}_2)X^\nu(z_2,\bar{z}_2) + \frac{\alpha'}{2}\eta^{\mu\nu}\ln|z_{12}|^2. \tag{2.4}$$

The reason why this ordering is useful is because of the following:

**Conjecture 2.1.3** (Fundamental Property of Normal Ordering)**.** *Normal ordered expressions satisfy the classical equations of motion averaged over paths. In the case of classical bosonic field this amounts to saying:*

$$\langle \partial\bar{\partial} :\mathscr{F}: \rangle = 0.$$

This is true for instance in the case of $\langle \partial\bar{\partial} :X^\mu(z_1,\bar{z}_1)X^\nu(z_2,\bar{z}_2): \rangle = 0$ (Polchinski Vol 1 Page 36).

**Proposition 2.1.4.** *Let $\mathscr{F},\mathscr{G}$ be two local operators. Then,*

$$:\mathscr{F}::\mathscr{G}: =:\mathscr{F}\mathscr{G}: + \sum (\textit{cross-contractions}) \tag{2.5}$$

$$= \exp\left(-\frac{\alpha'}{2}\int d^2z_1 d^2z_2 \ln|z_{12}|^2 \frac{\delta}{\delta_F^\mu(z_1,\bar{z}_1)}\frac{\delta}{\delta_{\mu G(z_2,\bar{z}_2)}}\right) :\mathscr{F}G: \tag{2.6}$$

*Remark.* The operator product expansion, as given by (2.1), is our definition of the OPE. However, it turns out, just like in complex analysis, that the singular part of this expansion is the one that plays the most crucial role. What does the singular part of the OPE do for us? It turns out that it gives us a way to compute the variation of local operators under conformal transformations. Here's how. Using the Ward identity in $d = 2$ we relate $\delta\mathscr{A}(z_0,\bar{z}_0)$ with the residue of $j(z)\mathscr{A}(z_0,\bar{z}_0)$ at $z_0$. Rewrite $j$ in terms of the energy-momentum tensor and use the singular part of the OPE of $T\mathscr{A}$ to calculate the residue.

Now we describe how to compute the singular part of the OPE in free field theory. In particular, this means that the following derivation only works for the bosonic non-interacting string. We will revise our method later, if need be. The key observation is that harmonic functions can locally be written as a sum of a holomorphic and antiholomorphic part.

**Lemma 2.1.5.** *Let $f : \mathbb{C} \to \mathbb{C}$ be harmonic. Then $\partial\bar{\partial}f = 0$ and so $f = a(z) + b(\bar{z})$ where $a$ is holomorphic and $b$ is antiholomorphic.*

Using Prop. 2.1.4 we notice that, since $:\mathscr{F}:$ is non-singular, the singular part in the OPE of $:\mathscr{F}::\mathscr{G}:$ is given by the coefficient functions in the cross-contractions.

**Example 2.1.6.** Using the definition of normal ordering (2.2), we may write (cf. (2.4))

$$\begin{aligned}X^\mu(z_1,\bar z_1)X^\nu(z_2,\bar z_2) &= -\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z_{12}|^2 + \sum_{k=1}^{\infty}\frac{1}{k!}(z_{12})^k \;{:}X^\nu\partial^k X^\mu(z_2,\bar z_2){:}\; + (\bar z_{12})^k \;{:}X^\nu\bar\partial^k X^\mu(z_2,\bar z_2){:}\\ &\sim -\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z_{12}|^2\end{aligned}$$

The first equation is the full operator product expansion and the equivalence (up to singular terms) shows that $X^\mu X^\nu$ behaves like $\ln|z_{12}|^2$ for $z_1\to z_2$.

**Example 2.1.7.** Let's suppose we have a product of two composite operators. $\mathscr{F}(z)=\partial X^\mu(z)\partial X_\mu(z)$ and $\mathscr{G}(z')=\partial' X^\nu(z')\partial' X_\nu(z')$. Using the harmonicity of normal ordering we obtain:

$$\begin{aligned}{:}\mathscr{F}(z){:}{:}\mathscr{G}{:} &= {:}\mathscr{F}(z)\mathscr{G}(z'){:}\; -4\frac{\alpha'}{2}(\partial\partial'\ln|z-z'|^2)\;{:}\partial X^\mu\partial' X_\mu(z'){:}\; +2\eta^\mu_\mu\left(-\frac{\alpha'}{2}\partial\partial'\ln|z-z'|^2\right)^2\\ &\sim \frac{D\alpha'^2}{2(z-z')^4} - \frac{2\alpha'}{(z-z')^2}\;{:}\partial X^\mu\partial' X_\mu(z'){:}\; - \frac{2\alpha'}{z-z'}\;{:}\partial X^\mu\partial' X_\mu(z'){:}\end{aligned}$$

In general CFTs we require the basis in which we expand operator products to transform like a tensor under conformal transformations. Moreover, the conformal invariance then puts even more restrictions on the coefficient functions rendering them unique up to a constant.

### 2.1.1 Ward Identity

Although the idea of the OPE is what drives this chapter, the Ward identity is the oil that makes the engine turn. Suppose we are given a coordinate transformation $\sigma'=\sigma+\delta\sigma$, that is a symmetry of the theory, how do operators transform under this transformation? Denote the transformation of fields as follows: $X'_\mu(\sigma)=X_\mu(\sigma)+\delta X_\mu(\sigma)$. Now we consider a slightly more general transformation:

$$X'_\mu(\sigma)=X_\mu(\sigma)+\rho(\sigma)\delta X_\mu(\sigma).$$

Such a general transformation might not be a symmetry of the action. However, the path integral *is* invariant under change of coordinates, which means:

$$\begin{aligned}0=\delta\left(\int\mathcal{D}X\,e^{-S[X]}\mathscr{A}(\sigma_0)\right) &= \int\mathcal{D}X\,\delta(e^{-S[X]})\mathscr{A}(\sigma_0)+e^{-S[X]}\delta\mathscr{A}(\sigma_0)\\ &= \int\mathcal{D}X\,(d^d\sigma\sqrt{g})e^{-S[X]}j^a(\sigma)\partial_a\rho(\sigma)\mathscr{A}(\sigma_0)+e^{-S[X]}\delta\mathscr{A}(\sigma_0)\end{aligned}$$

Applying Stoke's theorem:

$$\begin{aligned}\langle\delta\mathscr{A}(\sigma_0)\rangle &= \frac{i\epsilon}{2\pi}\int d^d\sigma\sqrt{g}\,\langle\partial_a j^a(\sigma)\rangle\\ &= \frac{i\epsilon}{2\pi}\langle\oint_{\partial R}dA\,n^a j_a(\sigma)\mathscr{A}(\sigma_0)\rangle\end{aligned}$$

In operator form and in $d=2$ this looks like

$$\frac{2\pi}{\epsilon}\delta\mathscr{A}(\sigma_0)=\oint_{\partial R}(j_z dz - j_{\bar z}d\bar z)\mathscr{A}(z_0,\bar z_0)$$

In the case that $j_z$ and $j_{\bar z}$ are (anti)holomorphic then we have the following relation:

$$\mathrm{Res}_{z\to z_0}\,j(z)\mathscr{A}(z_0,\bar z_0)+\overline{\mathrm{Res}}_{\bar z\to\bar z_0}\tilde{j}(\bar z)\mathscr{A}(z_0,\bar z_0)=\frac{1}{i\epsilon}\delta\mathscr{A}(z_0,\bar z_0).$$

### 2.1.2 Applications of OPE

Let us show that the $X^\mu$-theory is conformally invariant. This amounts to showing if $z' = f(z)$, for some holomorphic $f$, then $X'^\mu(z', \bar{z}') = X(z, \bar{z})$. For our purposes it will be easier to check this infinitesimally. consider

$$z' = z + \epsilon v(z) \tag{2.7}$$

for holomorphic $v$ (and similarly for the $\bar{z}'$). We want to show that such a transformation gives rise to the following variation:

$$X'^\mu(z', \bar{z}') = X^\mu(z, \bar{z}) - \epsilon v^a(z)\partial_a X^\mu(z, \bar{z}) - \epsilon v^a(z)^* \bar{\partial} X^\mu$$

because this is the infinitesimal version of $X'^\mu(z', \bar{z}') = X(z, \bar{z})$. The idea will be to use the Ward identity,

$$\mathrm{Res}_{z \to z_0}\, j(z)\mathscr{A}(z_0, \bar{z}_0) + c.c. = \frac{1}{i\epsilon}\delta\mathscr{A}(z_0, \bar{z}_0),$$

to compute the variation of $\mathscr{A}$. Therefore, we must first compute the current $j^a(z)$ corresponding to $v^a(z)$, then compute the OPE $j(z)\mathscr{A}(z_0, \bar{z}_0)$ to understand the asymptotics around $(z_0, \bar{z}_0)$, and finally compute the residue to obtain the symmetry that we are interested in.

**Exercise 2.1.1.** Show that the Noether current, corresponding to the symmetry (2.7), is given by $j_a = iv^b T_{ab}$ where $T_{ab}$ is the normal ordered version of the stress-energy tensor

$$T_{ab} = -\frac{1}{\alpha'} :\!\left(\partial_a X^\mu \partial_b X_\mu - \frac{1}{2}\delta_{ab}\partial_c X^\mu \partial^c X_\mu\right)\!: \; .$$

Moreover, show that $T^a_{\ a} = 0$, that is the tensor is traceless. Rewriting this in complex coordinates, show this is equivalent to $T_{z\bar{z}} = 0$. Also, using $\partial^a T_{ab} = 0 = T^a_{\ a}$, we have $\bar{\partial} T_{zz} = \partial T_{\bar{z}\bar{z}} = 0$, showing that $T = T_{zz}, \tilde{T} = T_{\bar{z}\bar{z}}$ are holomorphic and anti-holomorphic. Next, show that the OPEs of $T\mathscr{A}$ and $\tilde{T}\mathscr{A}$ have the following asymptotics:

$$T(z)X^\mu(0) \sim \frac{1}{z}\partial_z X^\mu(0), \qquad \tilde{T}(\bar{z})X^\mu(0) \sim \frac{1}{\bar{z}}\bar{\partial} X^\mu(0).$$

### 2.1.3 Primary Fields

In a CFT, we would like to use a particular basis for the OPE (2.1). This is a set of local operators which transform under conformal transformations similar to a tensor:

$$\mathscr{O}'(z', \bar{z}') = (\partial z')^{-h}(\bar{\partial}\bar{z}')^{-\tilde{h}}\mathscr{O}(z, \bar{z}). \tag{2.8}$$

We call such a local operator a **primary field** or **conformal tensor** of weight $(h, \tilde{h})$. These quasi-primary fields, by definition, play nice with conformal transformations, thus we may expect that the OPE of $T\mathscr{O}$ will be particularly nice. In fact, this does turn out to be the case:

$$T(z)\mathscr{O}(0,0) = \frac{h}{z^2}\mathscr{O}(0,0) + \frac{1}{z}\partial\mathscr{O}(0,0) + \cdots \tag{2.9}$$

**Example 2.1.8.** The operator $:\!(\prod_i \partial^{m_i} X^{\mu_i})(\prod_j \partial^{n_j} X^{\nu_j})e^{ik\cdot X}\!:$ has weight $\left(\frac{\alpha' k^2}{4} + \sum_i m_i, \frac{\alpha' k^2}{4} + \sum_j n_j\right)$.

**Proposition 2.1.9** (Refined OPE)**.** *Using rigid translations, scaling and rotations to both sides of an OPE we can write, for any two primary operators $\mathscr{A}_i, \mathscr{A}_j$:*

$$\mathscr{A}_i(z_1, \bar{z}_1)\mathscr{A}_j(z_2, \bar{z}_2) = \sum_k z_{12}^{h_k - h_i - h_j}\bar{z}_{12}^{\tilde{h}_k - \tilde{h}_i - \tilde{h}_j}\mathscr{A}_k(z_2, \bar{z}_2) \tag{2.10}$$

**Theorem 2.1.10** (Conformal Bootstrap)**.** *We may express the OPEs (ie. the correlation functions) for a product of any two or more fields, using only the quasi-primary fields. (Cf. Chapter 15 – Vol 2 Polchinski)*

**Example 2.1.11.** *bc* CFT There are many different free conformal field theories. We have, in fact, already met with two in the first chapter. The first is the $X^\mu$ theory, which we have gotten to know quite well. The second comes from § 1.5.3: Faddeev-Popov ghosts $b_{ab}, c^a$ with action

$$S_G = \frac{1}{2\pi}\int d^2\sigma b_{ab}\partial^a c^b \tag{2.11}$$

is a free CFT where $b, c$ are primary fields (conformal tensors) with weights $(h_b, 0) = (\lambda, 0)$ and $(h_c, 0) = (1-\lambda, 0)$. For this theory we can compute the OPEs, and all of the other quantities in a similar manner to what we did above. All of these important facts are left as exercises with answers in Polchinski pg 50-51.

### 2.1.4 Virasoro Algebra

Let us now compute the spectrum of a CFT. First, we will be clever and apply a conformal transformation from $(w = \sigma^1 + i\sigma^2) \mapsto (z = e^{-iw} = e^{-i\sigma^1+\sigma^2})$.[2] Quantizing with parameter $z$ is usually referred to as **radial quantization**. Second, we will do a Laurent expansion of $T_{zz}(z)$ and $T_{\bar{z}\bar{z}}(\bar{z})$ to obtain the Virasoro generators:

$$T(z) = \sum_{m=-\infty}^{\infty}\frac{L_m}{z^{m+2}}, \qquad \tilde{T}(z) = \sum_{m=-\infty}^{\infty}\frac{\tilde{L}_m}{\bar{z}^{m+2}}, \qquad L_m = \oint_C \frac{dz}{2\pi i z} z^{m+2}T(z) \tag{2.12}$$

This definition of operators $L_m$ has a number of consequences. First, we notice that the shape of the contour is irrelevant and we may fix the contours $C$ to be circles centred at 0. This contours correspond to equal-time points of the world-sheet, since $|z| = e^{\sigma^2}$. Moreover, since these operators are invariant under the radius of the circle, it follows that $L_m$ are invariant under time translation! This means that $L_m$ are conserved charges with current $j_m(z) = z^{m+1}T(z)$.

Before we can talk about the Hamiltonian, we again need a technical lemma.

**Lemma 2.1.12** (Transformation of the Energy Momentum Tensor)**.** *In a general CFT, under a conformal transformation, $z \to z + \epsilon v(z)$, the energy momentum tensor transforms as:*

$$\delta T(z) = -\frac{c}{12}\partial^3 v(z) - 2\partial v(z)T(z) - v(z)\partial T(z) \tag{2.13}$$

*The quantity $c$ is called the central charge of the CFT.*

**Exercise 2.1.2.** Show that the Hamiltonian is the conserved quantity given by (in the $w = \sigma^1 + i\sigma^2$ frame)

$$H = \int_0^{2\pi}\frac{d\sigma^1}{2\pi}T_{22} = L_0 + \tilde{L}_0 - \frac{c+\tilde{c}}{24},$$

where $T_{ww} = (\partial_w z)^2 T_{zz} + \frac{c}{24}$.

The collection $\{L_m, \tilde{L}_m\}_{m\in\mathbb{Z}}$ are the generators for the Virasoro algebra. To compute their commutators, we use the Ward identity and a contour trick.

**Lemma 2.1.13.** *Let $Q_i = \oint \frac{dz}{2\pi i} j_i(z), i = 1, 2$ be conserved charges, with current $j_1, j_2$. Then*

$$\begin{aligned}[Q_1, Q_2]\{C_2\} &= \lim_{C_1, C_3 \to C_2} Q_1\{C_1\}Q_2\{C_2\} - Q_1\{C_3\}Q_2\{C_2\} \\ &= \oint_{C_2}\frac{dz_2}{2\pi i} = \mathrm{Res}_{z_1\to z_2}\, j_1(z_1)j_2(z_2)\end{aligned}$$

[2]In the previous sections what we meant by $z$ was actually the $w$ here.

*Proof.* For the first equality, imagine slicing the path integral into three chunks. For the second, draw the standard picture where $C_1, C_3$ are perturbed around a point $z_2 \in C_2$. $\square$

This lemma shows that knowing the singular terms means we understand the commutators between conserved charges. Now we may apply this lemma to the Virasoro generators.

**Theorem 2.1.14** (Virasoro Algebra Relations)**.** *Let $L_m, m \in \mathbb{Z}$ be the generators of the Virasoro algebra. Then,*

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n} \tag{2.14}$$

*Proof.* Let us calculate the right-hand side of the equation of the lemma.

$$\mathrm{Res}_{z_1 \to z_2}(z_1^{m+1}T(z_1))\,(z_2^{n+1}T(z_2)) = \mathrm{Res}_{z_1 \to z_2}\, z_1^{m+1} z_2^{n+1} \cdot \mathrm{OPE}\{T(z_1)T(z_2)\}.$$

Expanding the OPE, and then doing the contour integral, we obtain the result. $\square$

## 2.2 Vertex Operators, Scattering Amplitudes, Anomalies

# Chapter 3

# BRST Quantization

We now have enough background in conformal field theory to continue our exploration into string theory. However, for completeness, there are a few loose ends we must tie up first, and a few more tools we must develop.

The first problem we tackle is that our canonical and path integral quantizations of the Polyakov action were somewhat unsatisfactory.

1. We were vague about how canonical quantization imposes the quantized Virasoro conditions: how can we tell which states are physical using our theory?
2. We did not derive the spectrum or state space using the path integral. With the introduction of ghost fields, not every configuration that is integrated over is a valid state.

**Becchi-Rouet-Stora-Tyutin (BRST) quantization** cures these problems simultaneously. It is a much more advanced method for quantizing a field theory with gauge symmetries and constraints. Instead of applying it directly to the open bosonic string, we first develop it in general.

The setting is any $D$-dimensional field theory with action $S$, and fields $\phi_r$, infinitesimal gauge symmetries $K_\alpha$, and gauge-fixing conditions $F^A[\phi]$. The infinitesimal gauge symmetries form a Lie algebra $G$, i.e. $[K_\alpha, K_\beta] = f_{\alpha\beta}{}^\gamma K_\gamma$ where $f_{\alpha\beta}{}^\gamma$ are the **structure constants** of $G$. Importantly, the $f_{\alpha\beta}{}^\gamma$ must be independent of the fields $\phi_r$. This condition does not hold for all gauge theories: when it does not, we must rely on some even more sophisticated machinery known as the **Batalin-Vilkovisky (BV) formalism**. Fortunately, we do not need to for the bosonic string.

**Note**: indices become messy in this section. We use $A$ (and not $B$, which appears as a non-indexing subscript later) to index the gauge-fixing conditions, i.e. the degrees of gauge freedom, $\alpha, \beta, \ldots$ to index the gauge symmetries, and $r, s, \ldots$ to index the fields. This avoids conflict with $a, b, \ldots$, which index worldsheet coordinates, and $i, j, \ldots$ and $\mu, \nu, \ldots$, which index space and spacetime coordinates.

**More important note**: throughout this section we use **deWitt notation**, where indices on fields not only index field components, but also spacetime. For example, $\phi_r$ is really $\phi_i(x)$ where $r$ ranges over all possible values of $i$ and $x$. Consequently, when contracting two deWitt indices, we must also integrate over the spacetime variable(s), with the appropriate measure.

**Exercise 3.0.1.** Apply the Faddeev-Popov method to this more general setting to obtain the gauge-fixed path integral

$$\int \mathcal{D}\phi_r \, \mathcal{D}B_A \, \mathcal{D}b_A \, \mathcal{D}c^\alpha \, \exp(-S - S_{\text{gf}} - S_{\text{G}})$$

Here, as for the bosonic string,

- $S$ is the original gauge-invariant action,
- $S_{\text{gf}} := -iB_A F^A[\phi]$ is the **gauge fixing action**, and
- $S_{\text{G}} := b_A c^\alpha K_\alpha F^A[\phi]$ is the **Faddeev-Popov ghost action**.

(We did not explicitly see $S_{\text{gf}}$ for the bosonic string because we immediately integrated it away: there, $\int \mathcal{D}B_A \exp(-S_{\text{gf}}) = \delta[\hat{g}^\zeta - g]$. If we had written the $\delta$ functional as a path integral, its variable of integration would have been $B_A$.) The auxiliary fields $B_A$ are sometimes called **Nakanishi-Lautrup fields**.

The resulting action $S + S_{\text{gf}} + S_{\text{G}}$ is not gauge-invariant anymore, but it has a very important symmetry called **BRST symmetry**.

## 3.1 BRST Symmetry

**Definition 3.1.1.** The infinitesimal **BRST transformation** $\delta_B$ is given by

$$\begin{aligned}
\delta_B \phi_r &:= -i\theta c^\alpha K_\alpha \phi_r, \\
\delta_B B_A &:= 0, \\
\delta_B b_A &:= \theta B_A, \\
\delta_B c^\alpha &:= \frac{i\theta}{2} f_{\beta\gamma}{}^\alpha c^\beta c^\gamma.
\end{aligned}$$

(This is not the cleanest way of writing the BRST transformation; be assured that this definition can actually be very well-motivated, as we see soon.)

**Proposition 3.1.2.** *The BRST transformation $\delta_B$ is a symmetry of the action $S + S_{gf} + S_G$, i.e.*

$$\delta_B(S + S_{gf} + S_G) = 0.$$

*Proof/Exercise.* First we establish two small identities, both of which are fairly straightforward:

$$\delta_B(F^A[\phi]) = -i\theta c^\alpha K_\alpha F^A[\phi], \qquad c^\alpha c^\beta K_\alpha K_\beta = \frac{1}{2} c^\alpha c^\beta f_{\alpha\beta}{}^\gamma K_\gamma.$$

Using these two identities and that $c^\alpha$ and $c^\beta$ anti-commute,

$$\begin{aligned}
\delta_B(c^\alpha K_\alpha F^A[\phi]) &= \left(\frac{i}{2}\theta f_{\beta\gamma}{}^\alpha c^\beta c^\gamma\right) K_\alpha F^A[\phi] + c^\alpha K_\alpha(-i\theta c^\beta K_\beta F^A[\phi]) \\
&= \frac{i}{2}\theta f_{\beta\gamma}{}^\alpha c^\beta c^\gamma K_\alpha F^A[\phi] - \frac{i\theta}{2} c^\alpha c^\beta f_{\alpha\beta}{}^\gamma K_\gamma F^A[\phi] = 0.
\end{aligned}$$

Hence when we compute $\delta_B(S_{\text{G}})$, the only non-vanishing term comes from $\delta_B b_\lambda$. Similarly, since $\delta_B(B_A) = 0$, the only non-vanishing term in $\delta_B(S_{\text{gf}})$ comes from $\delta_B(F^A[\phi])$. But then again using the first of the two identities,

$$\delta_B(S_{\text{gf}} + S_{\text{G}}) = -iB_A \delta_B(F^A[\phi]) + \theta B_A c^\alpha K_\alpha F^A[\phi] = 0.$$

Finally, $\delta_B(S) = 0$ since $S$ is the original gauge-invariant action and is only a function of $\phi_r$, but the transformation $\delta_B \phi_r$ is no more than an infinitesimal gauge transformation. □

Note that $\delta_B$ mixes commuting (e.g. $\phi_r$ and $B_A$) and anti-commuting (e.g. $b_A$ and $c^\alpha$) objects. For example, $b_A$ is supposed to anti-commute, but $\delta_B b_A = \theta B_A$, and $B_A$ commutes. Hence $\theta$ must be anti-commuting, i.e. a Grassmann variable. Because of this mixing, physicists say that the BRST symmetry is a **supersymmetry**. The field $c^\alpha$ has ghost number $+1$, the field $b_A$ and parameter $\theta$ have ghost number $-1$, and all other fields have ghost number 0.

**Exercise 3.1.1.** Let $\delta$ be an infinitesimal symmetry of the fields $\phi_r$. Review/show that by Noether's theorem, associated to the symmetry $\delta$ is a conserved charge

$$Q := \int d^{D-1}\vec{x} \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} \delta\phi_r - \delta\mathcal{L} \right) = \int d^{D-1}\vec{x} \, (\Pi^r \delta\phi_r - \delta\mathcal{L}) \, .$$

Suppose that $\delta\mathcal{L} = 0$. When quantized, $Q = :\!\int d^{D-1}\vec{x}\,\Pi^r \delta\phi_r\!:$ is a **generator** of the symmetry $\delta$: show that

$$\delta\phi_r = [Q, \phi_r],$$

and more generally, $\delta G = [Q, G]$ for any function $G$ depending only on the fields $\phi_r$ and not $x$. Finally, argue that $\delta\theta_r = \{Q, \theta_r\}$ for fermionic, i.e. Grassmann-valued, fields $\theta_r$. Let

$$[\cdot,\cdot]_- := [\cdot,\cdot], \quad [\cdot,\cdot]_+ := \{\cdot,\cdot\},$$

so that for a general field $G$, we can write $\delta G = [Q, G]_\pm$.

Hence there is a conserved charge $Q_B$ associated to the BRST symmetry $\delta_B$. There are two important properties of $Q_B$. To establish both of them, we first need to calculate that

$$\delta_B(b_A F^A) = (\theta B_A) F^A - b_A(i\theta c^\alpha K_\alpha F^A) = i\theta(S_{\text{gf}} + S_{\text{G}}).$$

**Proposition 3.1.3.** *Physical states satisfy* $Q_B \left|\text{phys}\right\rangle = 0$.

*Proof/Exercise.* Let $\delta$ be an infinitesimal transformation of the gauge-fixing functionals $F^A$, i.e. $F^A \mapsto F^A + \delta F^A$. Physical amplitudes $\langle f|i\rangle$ should be independent of our choice of gauge, so we require

$$0 = \delta \langle f|i\rangle = i\theta \langle f| \, \delta_B(b_A \delta F^A) \, |i\rangle = -\langle f| \left\{ Q_B, b_A \delta F^A \right\} |i\rangle \, .$$

(The second equality follows from writing $\langle f|i\rangle$ as a path integral.) For this equality to hold for all variations $\delta F^A$, we must have $Q_B^\dagger \left|f\right\rangle = Q_B \left|i\right\rangle = 0$. (It must be that $Q_B^\dagger = Q_B$, otherwise there would be another symmetry associated with $Q_B^\dagger$.) Since $|f\rangle$ and $|i\rangle$ are arbitrary physical states, we are done. ☐

**Proposition 3.1.4.** $Q_B$ *is nilpotent, i.e.* $Q_B^2 = 0$.

*Proof/Exercise.* A few calculations show that $\delta_B^2 = 0$. A few more calculations show that $[Q_B, [Q_B, G]_\pm]_\mp = [Q_B^2, G]$ for any formal variable $G$. Hence for any function $G$ of the fields,

$$0 = \delta_B \delta_B G = i\theta'[Q_B, i\theta[Q_B, G]_\pm]_\pm = \theta'\theta[Q_B, [Q_B, G]_\pm]_\mp = \theta'\theta[Q_B^2, G].$$

Since $G$ is arbitrary, $Q_B^2$ must be a scalar multiple of the identity. But $Q_B$ increases ghost number by 1, as one can verify from the definition of $\delta_B$, and the identity operator does not do this. Hence $Q_B^2 = 0$. ☐

These two properties allow us to directly construct the **physical (BRST) state space** $\mathcal{H}_{\text{BRST}}$. First, note that states of the form $Q_B \left|\chi\right\rangle$, called **null states**, are automatically physical states, since $Q_B^2 = 0$. But they are orthogonal to all physical states, since $\langle\psi| \left(Q_B \left|\chi\right\rangle\right) = \left(\langle\psi| Q_B\right) \left|\chi\right\rangle = 0$. So their presence is never measurable: two physical states differing by a null state are physically equivalent. Hence

$$\mathcal{H}_{\text{BRST}} := \frac{\{|\psi\rangle : Q_B \left|\psi\right\rangle = 0\}}{\{|\psi\rangle : |\psi\rangle = Q_B \left|\chi\right\rangle\}},$$

i.e. the **cohomology** of the operator $Q_B$.

## 3.2 Mathematical Formalism

**Note**: this subsection is **very optional**: it can be completely skipped. But if you are curious about the mathematical underpinnings of BRST, and what we actually calculated in the previous section, read on! This subsection contains a very rough outline of BRST quantization as Lie algebra cohomology, in the context of symplectic reduction. For more detail, see José Figueroa-O'Farrill's PhD thesis.[1]

What is BRST quantization really doing? Let's first restate the underlying problem of quantizing a constrained system: we have a phase space $M$ and a Hamiltonian action of a Lie group $G$ on $M$. For us, $M$ is the space of field configurations $\{\phi_r, B_A, b_A, c^\alpha\}$ and $G$ is the gauge group diff $\times$ Weyl. But for now, let's imagine $M$ is finite-dimensional.

There is a correspondence, which can be made into a diffeomorphism, between $M$ and the maximal closed ideals of $C^\infty(M)$, so it suffices to study $C^\infty(M)$. But we actually want to understand $M$ after modding out by the action of $G$. Since the action of $G$ on $M$ is Hamiltonian, we get via **symplectic reduction** the "quotient" symplectic manifold $\tilde{M}$, sometimes denoted $M//G$. The symplectic reduction is done via the **moment map** $\Phi\colon M \to \mathfrak{g}^*$, where $\mathfrak{g}$ is the Lie algebra of $G$:

$$M_0 \coloneqq \Phi^{-1}(0), \quad \tilde{M} \coloneqq M_0/G.$$

The Marsden-Weinstein theorem says that if $G$ acts freely on $\Phi^{-1}(0)$, then $\tilde{M}$ is a symplectic manifold, inheriting the symplectic form from $M$. Note that this reduction is often, more precisely, called **coisotropic reduction**. [2] We want to understand $C^\infty(\tilde{M})$.

The Hamiltonian action of $G$ on $M$ induces a Hamiltonian action of $\mathfrak{g}$ on $C^\infty(M)$, and therefore on $C^\infty(M_0)$. Quite unsurprisingly, it turns out that

$$C^\infty(\tilde{M}) = C^\infty(M_0)^{\mathfrak{g}} = H^0(\mathfrak{g}; C^\infty(M_0)),$$

where the second equality is the definition of **Lie algebra cohomology**: it is the derived functor associated with the **invariants functor** $(-)^{\mathfrak{g}}$. But even $M_0$ is hard to understand; we really want $C^\infty(\tilde{M})$ in terms of $C^\infty(M)$. A standard tool called the **Koszul complex** provides a resolution of $C^\infty(M_0)$ in terms $C^\infty(M)$-modules:

$$\cdots \xrightarrow{\delta} \Lambda^2\mathfrak{g} \otimes C^\infty(M) \xrightarrow{\delta} \mathfrak{g} \otimes C^\infty(M) \xrightarrow{\delta} C^\infty(M) \to C^\infty(M_0) \to 0.$$

Here $\delta$ extends the action $\mathfrak{g} \otimes C^\infty(M) \to C^\infty(M)$ as an odd derivation, i.e.

$$\delta(X_1 \wedge \cdots \wedge X_n \otimes f) \coloneqq \sum_{k=1}^n X_1 \wedge \cdots \wedge \hat{X}_k \wedge \cdots \wedge X_n \otimes \delta(X_k \otimes f).$$

Given this Koszul complex, denoted $K^\bullet$, we can now construct its **Chevalley-Eilenberg resolution**

$$K^{\bullet,\bullet} \coloneqq \mathrm{Hom}(\Lambda^\bullet\mathfrak{g}, \Lambda^\bullet\mathfrak{g} \otimes C^\infty(M)) = \Lambda^\bullet\mathfrak{g}^* \otimes \Lambda^\bullet\mathfrak{g} \otimes C^\infty(M)$$

in order to compute Lie algebra cohomology. The horizontal sequences come from the Koszul complex, while the vertical differentials $d$ come from Lie algebra cohomology. [3] Form the **total complex** $\mathrm{Tot}(K)^\bullet \coloneqq \bigoplus_{p-q=n} K^{p,q}$ with differential $D = d + \delta$. (Note that this is a little different than usual, since the horizontal complex is homological while the vertical complex is cohomological.)

[1] http://www.maths.ed.ac.uk/~jmf/Research/PVBLICATIONS/Thesis.pdf.

[2] The terminology describes how the zero locus $M_0 \coloneqq \Phi^{-1}(0)$ arising in symplectic reduction sits inside $M$: if $T_pM_0^\perp \subset T_pM_0$, then $M_0$ is **coisotropic**. This is in contrast with when $T_0M_0^\perp \cap T_pM_0 = \{0\}$, in which case we call $M_0$ **symplectic**. It turns out that for an equivariant moment map $\Phi\colon M \to \mathfrak{g}^*$, the zero locus $M_0$ is always coisotropic.

[3] If $\rho\colon \mathfrak{g} \to \mathrm{End}(V)$ is a representation, then $C^p(\mathfrak{g}; V) \coloneqq \mathrm{Hom}(\Lambda^p\mathfrak{g}, V)$ are the **$p$-cochains**, and the differential $d$ is given by extending

$$d\colon C^0(\mathfrak{g}; V) = V \to C^1(\mathfrak{g}; V), \quad (dv)(X) = \rho(X)v$$

to $\Lambda^\bullet\mathfrak{g}^*$ as an anti-derivation, and to $\Lambda^\bullet\mathfrak{g}^* \otimes V$ by $d(\alpha \otimes v) = d\alpha \otimes v + (-1)^{|\alpha|}\alpha \otimes dv$. In this case, $V = \Lambda^\bullet\mathfrak{g} \otimes C^\infty(M)$.

**Proposition 3.2.1.** *$H^p(\mathrm{Tot}(K)) = C^\infty(\tilde{M})$ for $p = 0$ and is zero otherwise.*

*Proof.* Let $E_n^{i,j}$ be the (cohomological, vertical) spectral sequence associated to $K^{\bullet,\bullet}$, and recall that it converges to $H^\bullet(\mathrm{Tot}(K))$.

1. The first page is the cohomology of the horizontal differentials $\delta$, so $E_1^{p,q} = H^q(K^{p,\bullet}) = \Lambda^p\mathfrak{g}^* \otimes C^\infty(M_0)$ if $q = 0$, and is zero otherwise.
2. The second page is the cohomology of the vertical differentials $d$ on $E_1^{\bullet,\bullet}$, so $E_2^{p,q} = H^p(E_1^{\bullet,q}) = C^\infty(M_0)^{\mathfrak{g}} = C^\infty(\tilde{M})$ if $p = q = 0$, and is zero otherwise.

Hence the spectral sequence collapses at the second page, and we conclude that $H^0(\mathrm{Tot}(K)) = C^\infty(\tilde{M})$. $\square$

So now we have a direct way to compute $C^\infty(\tilde{M})$: just form the total complex $\mathrm{Tot}(K)$ and compute its cohomology. The total complex $\mathrm{Tot}(K)$ has another name: it is usually called the **BRST complex**, and its differential $D$ is the **BRST symmetry**.

**Proposition 3.2.2.** *The complex* $\mathrm{Tot}(K)$ *has the structure of a* **graded Poisson superalgebra**, *which for us just means that it has a* **Poisson bracket** $\{\cdot,\cdot\}$ *that respects the grading. Let $b_i$ be a basis for $\mathfrak{g}$, and $c^i$ be a dual basis for $\mathfrak{g}^*$. If $\phi_i \coloneqq \Phi(b_i)$ are the components of the moment map, then the differential $D$ on* $\mathrm{Tot}(K)$ *arises as $D = \{Q, -\}$, with*

$$Q = c^i\phi_i - \frac{1}{2}f^i_{jk}c^jc^kb_i \in \mathrm{Tot}(K)^1,$$

*where $f^i_{jk}$ are the structure constants of $\mathfrak{g}$.*

*Proof.* Check that $\{Q, -\}$ acts as $D$ on the generators, i.e. on $f \in C^\infty(M)$, $X \in \mathfrak{g}$, and $\alpha \in \mathfrak{g}^*$. $\square$

Physicists call $c^i$ and $b_i$ **ghosts** and **anti-ghosts** respectively. The total degree in $\mathrm{Tot}(K)$ is called the **ghost number** (so note that the action of $Q$ increases the ghost number by 1). In fact, this is how BRST quantization works in general for the physicists: given a system with symmetries $K_i$ forming a Lie algebra $\mathfrak{g}$, we

1. introduce a basis $\{b_i\}$ for $\mathfrak{g}$ and $\{c^i\}$ for $\mathfrak{g}^*$,
2. write down the BRST operator $Q = c^iK_i - (1/2)f^i_{jk}c^jc^kb_i$, and
3. compute cohomology, treating everything as operators, to find physical states.

The underlying hypothesis is, of course, that the following diagram commutes:

$$\begin{tikzcd}
C^\infty(M) \arrow[r, "\text{quantization}"] \arrow[d, "\text{classical BRST}"'] & \mathcal{H} \arrow[d, "\text{quantum BRST}"] \\
C^\infty(\tilde{M}) \arrow[r, "\text{quantization}"] & \mathcal{H}_{\text{gauge-fixed}},
\end{tikzcd}$$

so that instead of having to quantize $C^\infty(\tilde{M})$ directly, which we have no idea how to do in general, we quantize $C^\infty(M)$ and then apply the machinery of BRST, which is easy, quantized or not. In fact, the quantization of $C^\infty(\tilde{M})$ is usually defined using this diagram.

## 3.3 BRST Quantization of the Bosonic String

We can now apply the general machinery of BRST quantization and our knowledge of CFT to the open bosonic string. (We need CFT to apply Noether's theorem, to get the BRST charge $Q_B$.) This involves quite a lot of calculation that can be safely skipped. We record them here in grisly detail for reference.

### 3.3.1 BRST Symmetry and Critical Dimension

First, in order to apply CFT, we need to rewrite the action in complex coordinates $(z, \bar{z})$. Pick **unit gauge**, i.e. $g_{ab} = \delta_{ab}$, so that $\nabla = \partial$, and

$$S_{\mathrm{X}} = T_0 \int_\Sigma d^2z\, \partial X \cdot \bar{\partial} X, \quad S_{\mathrm{G}} = \frac{1}{2\pi} \int_m d^2z\, \left( b_{zz} \partial_{\bar{z}} c^z + b_{\bar{z}\bar{z}} \partial_z c^{\bar{z}} \right).$$

**Notation**: we write $S_{\mathrm{X}}$ instead of $S_{\mathrm{P}}$ for the Polyakov action from now on. Similarly, $T_{\mathrm{X}}$ is the holomorphic part of the energy-momentum tensor for $S_{\mathrm{X}}$, and $T_{\mathrm{G}}$ for $S_{\mathrm{G}}$. In general, X as a subscript refers to quantities involving the **matter fields** $X^\mu$, and G as a subscript refers to quantities involving the **ghost fields** $b_{ab}$ and $c^a$. Remember that tildes indicate the anti-holomorphic part.

The equations of motion for the ghosts give $\partial_{\bar{z}} b_{zz} = \partial_z b_{\bar{z}\bar{z}} = 0$ and $\partial_{\bar{z}} c^z = \partial_z c^{\bar{z}} = 0$, so define

$$b(z) \coloneqq b_{zz}(z, \bar{z}), \quad \tilde{b}(\bar{z}) \coloneqq b_{\bar{z}\bar{z}}(z, \bar{z}), \quad c(z) \coloneqq c^z(z, \bar{z}), \quad \tilde{c}(\bar{z}) \coloneqq c^{\bar{z}}(z, \bar{z}).$$

Using this new notation, the ghost action is $S_{\mathrm{G}} = (1/2\pi) \int d^2z\, (b\bar{\partial}c + \tilde{b}\partial\tilde{c})$. It is straightforward now to write down the BRST symmetry, since we only need to consider diffeomorphisms and not Weyl transformations: nothing depends on the metric anymore.

**Proposition 3.3.1.** *The BRST symmetry for the bosonic string is*

$$\begin{aligned} \delta_B X^\mu &= i\theta(c\partial X^\mu + \tilde{c}\bar{\partial} X^\mu), \\ \delta_B b &= i\theta(T_X + T_G), \qquad & \delta_B \tilde{b} &= i\theta(\tilde{T}_X + \tilde{T}_G), \\ \delta_B c &= i\theta c\partial c, & \delta_B \tilde{c} &= i\theta \tilde{c}\bar{\partial}\tilde{c}, \end{aligned}$$

*where* $T_X = -(2\pi T_0)(\partial X)^2$ *and* $T_G = -2 :b\partial c: + :c\partial b:$.

*Proof.* The gauge transformations here are diffeomorphisms and Weyl transformations, but Weyl transformations leave $X^\mu$ invariant, so the expression for $\delta_B X^\mu$ comes directly from the definition.

To calculate $\delta_B b = \theta B$, we need to find the equation of motion for $B$, which we integrated out, so that we can rewrite it in terms of the other fields. Well it is easy to find the equation of motion: simply write down the total non-gauge-fixed action and vary it with respect to the metric to get

$$S = S_{\mathrm{X}} - iB^{ab}(\delta_{ab} - g_{ab}) + S_{\mathrm{G}} \implies \delta_g S = \frac{\sqrt{g}}{4\pi}\, \delta g_{ab}(T_{\mathrm{X}} - iB^{ab} + T_{\mathrm{G}}) \implies B^{ab} = -i(T_{\mathrm{X}} + T_{\mathrm{G}}).$$

Finally, to calculate $\delta_B c^a = (i\theta/2) f_{bc}{}^a c^b c^c$, we need the structure constants $f_{bc}{}^a$. Since $[K_\alpha, K_\beta]X^\mu = 0$ when either of $K_\alpha$ or $K_\beta$ is a Weyl transformation, again we only need to consider diffeomorphisms. Since infinitesimal diffeomorphisms have as a basis $\delta_w X^\mu(z) = \delta(z - w)\partial_z X^\mu(z)$, we can compute

$$f_{bc}{}^a = -\delta^a_c \delta(z_1 - z_2)\partial_b \delta(z_1 - z_3) + \delta^a_b \delta(z_1 - z_3)\partial_a \delta(z_1 - z_2),$$

where $z_1, z_2, z_3$ are the coordinates corresponding to $a, b, c$ respectively. Plugging this into the definition of $\delta_B c^a$ and rewriting in complex coordinates gives the desired result. □

Note that we disregard the Nakanishi-Lautrup fields $B_A$ in the BRST construction here, because we could and did integrate them away when we gauge-fixed the action.

**Proposition 3.3.2.** *The holomorphic part* $j_B(z)$ *of the conserved BRST current is*

$$j_B(z) \coloneqq cT_X + \frac{1}{2} :cT_G: + \frac{3}{2}\partial^2 c = cT_X + :bc\partial c: + \frac{3}{2}\partial^2 c,$$

*where the* $\partial^2 c$ *term is added manually to make* $j_B(z)$ *a tensor.*

*Proof.* Let $\mathcal{J}^x$ denote the variation in the action, in standard coordinates, i.e.

$$\frac{\theta}{2}\int d^2z\,\partial_x\mathcal{J}^x := \delta_B(S_{\mathrm{X}}+S_{\mathrm{G}}) = iT_0\theta\int d^2z\,\partial_x(c^x\partial X\cdot\bar{\partial}X).$$

Hence, by the standard Noether procedure, the conserved current is

$$(j_B)^x(z,\bar{z}) = :\frac{\delta(\mathcal{L}_{\mathrm{X}}+\mathcal{L}_{\mathrm{G}})}{\delta(\partial_x\Phi_i(z,\bar{z}))}\delta_B\Phi_i(z,\bar{z}) - \mathcal{J}^x:$$

where $\Phi_i$ ranges over all dynamical fields, i.e. $X, b, c$. Switching to complex coordinates,

$$\begin{aligned} j_B(z) = 2\pi i\left(\frac{1}{2}(j_B)^{\bar{z}}\right) &= \pi i:\left(\frac{T_0}{2}\partial X\cdot\delta_B X - \frac{1}{\pi}b\delta_B c - \frac{iT_0}{2}\tilde{c}\partial X\cdot\bar{\partial}X\right): \\ &= -2\pi T_0 :c\partial X\cdot\partial X: + :bc\partial c: = :cT_{\mathrm{X}}: + :bc\partial c: . \end{aligned}$$

□

The analogous formula clearly holds for $\tilde{j}_B$. Now recall that the BRST charge, by definition, is the conserved charge associated with this conserved current: $Q_B = (1/2\pi i)\int(dz\,j_B - d\bar{z}\,\tilde{j}_B)$.

**Proposition 3.3.3.** *$Q_B^2 = 0$ if and only if $D = 26$.*

*Proof.* We directly compute the OPE, ignoring the total derivative term:

$$\begin{aligned} 2Q_B^2 = \{Q_B, Q_B\} &= \int dz\,dw :c(z)\left(T_{\mathrm{X}}(z)+\frac{1}{2}T_{\mathrm{G}}(z)\right)::c(w)\left(T_{\mathrm{X}}(w)+\frac{1}{2}T_{\mathrm{G}}(w)\right): \\ &= -\frac{1}{12}\int dw\,\partial_w^3 c(w)c(w)(D-26), \end{aligned}$$

which vanishes if and only if $D = 26$. □

This calculation is the rigorous derivation of the **critical dimension**. The failure of $Q_B^2$ to be zero automatically for any $D$ is known as the **Weyl anomaly**. Alternatively, we could compute this result from the mode expansion of $Q_B$, which we record here for use in the next subsection:

$$Q_B = 2\sum_{n=-\infty}^{\infty} c_n L_{-n}^X + \sum_{m,n=-\infty}^{\infty}(m-n)\,\mathring{\circ}\, c_m c_n b_{-m-n}\,\mathring{\circ} - 2c_0$$

where $L_{-n}^X$ are the modes of $T_X$, and $\mathring{\circ}\ \mathring{\circ}$ denotes **creation-annihilation normal ordering**.

### 3.3.2 Physical State Space and the No-Ghost Theorem

We can now find the **physical state space** by calculating the BRST cohomology of $Q_B$ acting on the entire state space. First, we need to write down the state space. This is entirely analogous to when we canonically quantized naively, using the mode expansion coefficients $\alpha_m^i$ as **raising/lowering operators**. But now we are not in lightcone gauge anymore, and we have two additional ghost fields. Hence we now have three sets $\{\alpha_m^\mu\}$, $\{b_m\}$, $\{c_m\}$ of raising/lowering operators, arising from the mode expansions

$$\partial X^\mu(z) = -i\sqrt{\frac{1}{2\pi T_0}}\sum_{m=-\infty}^{\infty}\frac{\alpha_m^\mu}{z^{m+1}}, \quad b = \sum_{m=-\infty}^{\infty}\frac{b_m}{z^{m+2}}, \quad c = \sum_{m=-\infty}^{\infty}\frac{c_m}{z^{m-1}}.$$

We also define, as usual, the **number operators** for each set of operators, and the total number operator:

$$N_n^\mu := \alpha_{-n}^\mu\alpha_n^\mu, \quad N_n^b := nb_{-n}c_n, \quad N_n^c := nc_{-n}b_n, \quad N := \sum_{n=1}^{\infty}\left(\sum_{\mu=0}^{25}N_n^\mu + N_n^b + N_n^c\right).$$

The associated eigenvalue of $N$ is called the **level** of the string. The most general level-$N$ open bosonic string is, as expected,

$$|N;k\rangle = \sum_{N_n^\mu, N_n^b, N_n^c} C_{N_n^\mu, N_n^b, N_n^c} \left( \prod_{\mu=0}^{25} \prod_{n=1}^{\infty} \frac{(\alpha_{-n}^\mu)^{N_n^\mu}}{\sqrt{n^{N_n^\mu} N_n^\mu!}} \right) \left( \prod_{n=1}^{\infty} (b_{-n})^{N_n^b} \right) \left( \prod_{n=0}^{\infty} (c_{-n})^{N_n^c} \right) |0;k\rangle \,,$$

where the sum is of course over the quantum numbers $\{N_n^\mu, N_n^b, N_n^c\}$ giving a level-$N$ string, and $C_{N_n^\mu, N_n^b, N_n^c}$ are normalization constants. What are the ground states $|0;k\rangle$? From canonical quantization, we have a definition of the ground states $|0;k\rangle_{\mathrm{X}}$ for the matter fields, but not for the ghost fields. Well, ground states should always be annihilated by lowering operators, so now we define the **ground state** $|0;k\rangle$ such that

$$p^\mu |0;k\rangle = k^\mu |0;k\rangle \,, \quad b_0 |0;k\rangle = 0, \quad \alpha_n^\mu |0;k\rangle = b_n |0;k\rangle = c_n |0;k\rangle_{\mathrm{G}} = 0 \quad \forall n > 0.$$

**Exercise 3.3.1.** Note that $c_0$ is a creation operator while $b_0$ is not: this exercise explains why. Show that if $b_0$ were also a creation operator, the system generated by $b_0$ and $c_0$ has a two-fold degeneracy at every state $|0;k\rangle$, which splits into states $|0;k,\uparrow\rangle$ and $|0;k,\downarrow\rangle$ satisfying

$$\begin{aligned} b_0 |0;k,\uparrow\rangle &= |0;k,\downarrow\rangle \,, \quad b_0 |0;k,\downarrow\rangle = 0 \\ c_0 |0;k,\downarrow\rangle &= |0;k,\uparrow\rangle \,, \quad c_0 |0;k,\uparrow\rangle = 0. \end{aligned}$$

Using the mode expansion of $Q_B$ from the previous subsection, show that $Q_B$ acts on either of these states as $2c_0((L_X)_0 - 1)$, and therefore that

$$Q_B |0;k,\downarrow\rangle = 2\left(\frac{k^2}{2\pi T_0} - 1\right) |0;k,\uparrow\rangle \,, \quad Q_B |0;k,\uparrow\rangle = 0.$$

Hence if $k^2 \neq 2\pi T_0$, the states $|0;k,\uparrow\rangle$ are BRST-exact. But not all the states $|0;k,\downarrow\rangle$ are BRST-closed, hence $\langle \mathrm{phys}|0;k,\downarrow\rangle \propto \delta(k^2 - 2\pi T_0)$. Since amplitudes in QFTs cannot have delta functions, conclude that $|0;k,\uparrow\rangle$ for $k^2 = 2\pi T_0$ are the real physical states, which therefore satisfy $b_0 |0;k,\uparrow\rangle = 0$.

This also explains why we require the ground state $|0;k\rangle$ to be annihilated by $b_0$: when we write $|0;k\rangle$, we implicitly mean $|0;k,\uparrow\rangle$.

**Definition 3.3.4.** The **physicality condition** for states $|N;k\rangle$ to be physical is the criteria $Q_B |N;k\rangle = b_0 |N;k\rangle = 0$. Note that $L_0 = \{Q_B, b_0\}$, so $L_0 = (1/2\pi T_0)(k^2 + m^2)$ annihilates all physical states, i.e. $k^2 = -m^2$ for all physical states. Hence we index using $\vec{k}$ instead of $k$, since given $\vec{k}$ we can solve for $k^0$.

For the ground state, then, $m^2 = -k^2 = -2\pi T_0$, which is, of course, the same result that we got via canonical quantization. We essentially worked out in the exercise that $|0;\vec{k}\rangle$ for $k^2 = 2\pi T_0$ is the unique ground state. (There are no BRST-exact states here to remove, since $Q_B |0;\vec{k}\rangle = 0$.)

For $N = 1$, the most general state is of the form

$$|1;\vec{k}\rangle = (e \cdot \alpha_{-1} + \beta b_{-1} + \gamma c_{-1}) |0;\vec{k}\rangle \,, \quad k^2 = 0$$

where $e^\mu$ is a 26-vector and $\beta, \gamma \in \mathbb{C}$ are scalars. The physicality condition requires

$$\begin{aligned} 0 = Q_B |1;\vec{k}\rangle &= 2(c_1 (L_X)_{-1} + c_0 (L_X)_0 + c_{-1} (L_X)_1 + c_0 c_{-1} b_1 + c_1 c_0 b_{-1} - c_0) |1;\vec{k}\rangle \\ &= 2\sqrt{\frac{1}{\pi T_0}} (\beta k \cdot \alpha_{-1} + k \cdot e c_{-1}) |0;\vec{k}\rangle \,, \end{aligned}$$

i.e. $k \cdot e = \beta = 0$. (Note that $b_0 |1;\vec{k}\rangle$ is automatically satisfied.) Also, note that this calculation also gives us the BRST-exact states, $c_{-1} |0;\vec{k}\rangle$ and $k \cdot \alpha_{-1} |0;\vec{k}\rangle$. Hence, after removing these BRST-exact states, the remaining **physical states** are

$$|1;\vec{k}\rangle = e \cdot \alpha_{-1} |0;\vec{k}\rangle \,, \quad k^2 = k \cdot e = 0, \quad e^\mu \sim e^\mu + \zeta k^\mu.$$

A basis for these remaining physical states is $\{\alpha^i_{-1}\left|1;\vec{k}\right\rangle : i = 2,\dots,25\}$. Note that all these results for $N=1$ matches up exactly with our heuristic derivation a long time ago when we applied canonical quantization. But now that we have obtained these results rigorously, it is time to reveal that an open string at $N=1$ excitation is usually identified with a **photon** with **polarization** $e^\mu$.

**Theorem 3.3.5.** *The BRST state space, i.e. the cohomology of $Q_B$, is isomorphic to the state space obtained from the subspace (of the Hilbert space arising from canonical quantization) annihilated by the Virasoro algebra.*

*Proof.* See Polchinski, section 4.4, pg. 141. □

**Exercise 3.3.2.** By repeating the BRST calculation for $N=1$, conclude that at $N=2$ there are a total of 324 proper physical states, 300 of which come from $\alpha^i_{-1}\alpha^j_{-1}\left|0;\vec{k}\right\rangle$ and 24 of which come from $\alpha^i_{-2}\left|0;\vec{k}\right\rangle$.

It turns out, quite fortunately, that we do not need to go through the tedious BRST procedure to actually find what the proper physical states are; for the bosonic string, at least, BRST quantization is just a tool used to establish the following very important and practical theorem.

**Theorem 3.3.6** (No-ghost theorem)**.** *The BRST state space is isomorphic to the subspace (of the Hilbert space arising from canonical quantization) with no $X^0$, $X^1$, $b$, or $c$ excitations.*

*Proof.* See Polchinski, section 4.4, pg. 137. □

**Exercise 3.3.3.** Show, as a corollary of the no-ghost theorem, that the generating function for the number of states at excitation $N$ is

$$\prod_{n=1}^{\infty} \frac{1}{(1-x^n)^{24}} = 1 + 24x + 324x^2 + 3200x^3 + 25650x^4 + \cdots = \eta(x)^{-24} \propto \Delta(x),$$

where $\eta(q)$ is the **Dedekind eta function** and $\Delta(x)$ is the **modular discriminant**. Conclude that there is something magical about the number 24, and that string theory is worth studying for its mathematics (even if the physicists laugh at it.)

# Chapter 4

# Amplitudes

In QFT, after we studied the spectrum of a free particle, we proceeded to add an interaction term in the form of $(\lambda/4!)\phi^4$. Then we studied how to compute $n$-point functions in $\phi^4$ theory. We shall do the same now for the bosonic string. However two things are different.

1. We do not need to add an interaction term to the Polyakov action: interactions are implicitly encoded.
2. Computing amplitudes is a lot harder unless the string sources are taken to infinity, which corresponds to computing S-matrix elements.

For simplicity, this chapter works out the theory for **closed strings** only. The simplicity arises from the worldsheet of interacting closed strings being a **closed surface**, i.e. we do not need to deal with boundary components. Henceforth when we say surface, we mean closed surface.

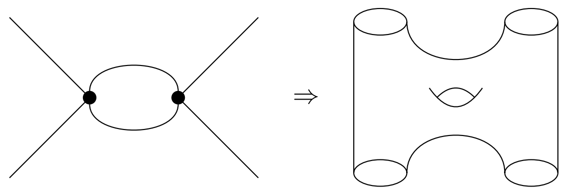
**Theorem 4.0.7** (Classification of closed surfaces)**.** *Any closed surface $\Sigma$ is homeomorphic to the connected sum of a 2-sphere with $p \geq 0$ tori, and has Euler characteristic $\chi(\Sigma) = 2 - 2p$.*

## 4.1 Computing Scattering Amplitudes

As a first example, let's look at the scattering amplitude for four closed strings, with sources approaching $X^0 = \pm\infty$. Diagrammatically, this looks like four infinitely long cylinders attached to a sphere in the middle. On these cylinders, we have the complex coordinate $w = x + iy$ where $x \cong x + 2\pi$ goes around the string, and $-\infty < y \leq 0$ is identified with the time coordinate $X^0$.

Consider the new coordinate $z = \exp(-iw)$. In this coordinate, the cylinders taper off conically as $y \to -\infty$, and therefore as the cylinders become infinitely long, we end up with a sphere with 4 **marked points**. This is essentially the **state-operator correspondence** from CFT: at each of these 4 marked points is now a local **vertex operator** $\mathcal{V}_j(k)$. (Incoming and outgoing states are distinguished by the sign of $k^0$.) Generally, we look at ground state, i.e. **tachyon**, vertex operators $:\!\exp(ik \cdot X)\!:$.

The same argument can be repeated for open strings to obtain the closed disk with marked points on it. In both cases, the resulting (conformally-equivalent) worldsheet is compact, with some number of marked points on it. Note that the **genus** of the worldsheet encodes different interactions! **Tree-level amplitudes** in QFT correspond to a sphere with marked points; **one-loop amplitudes** correspond to a torus with marked points; and so on. For example,



$\cong$ $T^2$ with 4 marked points.

In QFT, to obtain the 4-point correlation function, we summed over all tree-level, i.e. zero-loop, Feynman diagrams, followed by one-loop diagrams, etc. Similarly, to obtain the 4-point scattering amplitude here, we must sum over all genus zero surfaces with four marked points and their metrics, then genus one surfaces with four marked points and their metrics, etc.

Do surfaces of different genus contribute differently to the path integral? Recall that when we first discussed the gauge symmetries of the Polyakov action, we found that aside from the already-existing term, there could only be one other term satisfying all the gauge symmetries, given by

$$\chi := \frac{1}{4\pi}\int_\Sigma d^2\sigma\,\sqrt{-\gamma}\,R + \frac{1}{2\pi}\int_{\partial\Sigma} ds\,k$$

Of course, by Gauss-Bonnet, $\chi$ is nothing more than the **Euler characteristic** of the worldsheet. It acts as the **coupling constant** for worldsheets of different genus.

**Definition 4.1.1.** An $n$**-string scattering amplitude** involving $n$ strings with momenta $(k_i)^\mu$ and states $j_i$ is given by

$$S_{j_1\cdots j_n}(k_1,\ldots,k_n) := \sum_{\substack{\text{closed}\\ \text{surfaces } \Sigma}} \int \frac{\mathcal{D}X\,\mathcal{D}g}{\text{diff}\times\text{Weyl}} \exp(-S_{\mathrm{X}} - \lambda\chi) \prod_{i=1}^n \int d^2\sigma_i\,\sqrt{g(\sigma_i)}\,\mathcal{V}_{j_i}(k_i,\sigma_i)$$

where the $\mathcal{V}_{j_i}$ are the local operators arising from the state-operator correspondence. Note that we integrate them over the worldsheet to preserve diffeomorphism invariance. Also, since $\exp(-\lambda\chi)$ is a topological invariant, it does not depend on the **embedding** $X^\mu$ of $\Sigma$ into spacetime, nor the metric $g$ on $\Sigma$, so we define $g_s := \exp(-\lambda)$, the **string coupling constant**, and write

$$S_{j_1\cdots j_n}(k_1,\ldots,k_n) = \sum_{\text{genus } p=0}^{\infty} g_s^{-(2-2p)} \int \frac{\mathcal{D}X\,\mathcal{D}g}{\text{diff}\times\text{Weyl}} \exp(-S_{\mathrm{X}}) \prod_{i=1}^n \int d^2\sigma_i\,\sqrt{g(\sigma_i)}\,\mathcal{V}_{j_i}(k_i,\sigma_i)$$

**Definition 4.1.2.** For a fixed genus $p$, let $\mathrm{Met}(\Sigma_p)$ be the space of all metrics on any surface $\Sigma_p$ of genus $p$. Then the configuration space we are integrating over is

$$\frac{\mathrm{Met}(\Sigma_p)\times C^\infty(\Sigma_p,\mathbb{R}^D)\times(\Sigma_p)^n}{\text{diff}\times\text{Weyl}} \cong \mathcal{M}(\Sigma_p)\times C^\infty(\Sigma_p,\mathbb{R}^D)\times(\Sigma_p)^n$$

where $\mathcal{M}(\Sigma_p)$ is the **moduli space of genus-$p$ Riemann surfaces**. Here, the action of Weyl on $\mathrm{Met}(\Sigma_p)$ simply defines the **conformal classes** $\mathrm{Conf}(\Sigma_p)$ of metrics on $\Sigma_p$, i.e. $\mathrm{Met}(\Sigma_p)/\text{Weyl} \cong \mathrm{Conf}(\Sigma_p)$.

Recall that $\mathcal{M}(\Sigma_p)$ is actually finite-dimensional, so these is some hope of ending up with a well-defined integral! There are a few things we must do in order to have this happen: we must

1. completely understand the action of diff $\times$ Weyl on $\mathrm{Met}(\Sigma_p)$ (including the remnant gauge symmetry we encountered, in the form of holomorphic diffeomorphisms, even after gauge-fixing),
2. use Faddeev-Popov to write $S_{j_1\cdots j_n}(k_1,\ldots,k_n)$ as a finite-dimensional integral over the moduli space $\mathcal{M}(\Sigma_p)$, and
3. define a measure on $\mathcal{M}(\Sigma_p)$, hopefully descending from $\mathrm{Met}(\Sigma_p)$, so that this integral is well-defined.

Physicists call elements of the moduli space **metric moduli**, or just **moduli** for short.

### 4.1.1 Integration Measure on the Moduli Space

We shall begin with items 1 and 3; they are related. But the former is surprisingly non-trivial for genus $p < 2$. (Usually, people avoid such difficulties by defining $\mathcal{M}(\Sigma_p)$ differently for $p < 2$. But we cannot.)

1. For genus $p < 2$, there exist **conformal Killing vectors** (CKVs) on $\Sigma_p$, i.e. non-trivial transformations in diff $\times$ Weyl that act as the identity on the metric. We examined some, given by holomorphic diffeomorphisms, when we introduced gauge-fixing.
2. The CKVs form a group, called the **conformal Killing group** $\mathrm{CKG}(\Sigma_p)$. If $\mathrm{CKG}(\Sigma_p)$ has real dimension $k$, then we can remove these extra degrees of gauge freedom by fixing $k$ (real) coordinates of marked points on $\Sigma_p$.

The CKG causes us problems; we must understand how to identify which infinitesimal diff $\times$ Weyl transformations are in the CKG. So consider an infinitesimal diff $\times$ Weyl transformation

$$\delta g_{ab} = 2\delta\omega g_{ab} - \nabla_a \delta\sigma_b - \nabla_b \delta\sigma_a = (2\delta\omega - \nabla_c \delta\sigma^c) g_{ab} - 2(P_1 \delta\sigma)_{ab}$$

where we define a differential operator $P_1$ taking vectors into traceless symmetric 2-tensors:

$$(P_1 \delta\sigma)_{ab} = \frac{1}{2}(\nabla_a \delta\sigma_b + \nabla_b \delta\sigma_a - g_{ab} \nabla_c \delta\sigma^c).$$

By definition, $\delta$ is a CKV iff $\delta g_{ab} = 0$, which, from the variation $\delta g_{ab}$, happens iff $(P_1\delta\sigma)_{ab} = 0$, called the **conformal Killing equation**. (To get this, just take the trace of $\delta g_{ab} = 0$, which enforces $2\delta\omega = \nabla \cdot \delta\sigma$.)

In much the same way, we can identify infinitesimal transformations of $\mathrm{Met}(\Sigma_p)$ that correspond to moduli: they are variations $\delta' g_{ab}$ that are **orthogonal** (hang on, we'll define the metric soon) to all diff$\times$Weyl variations, i.e.

$$\begin{aligned} 0 &= \int d^2\sigma \sqrt{g}\, \delta' g_{ab} \left( (2\delta\omega - \nabla \cdot \delta\sigma) g^{ab} - 2(P_1 \delta\sigma)^{ab} \right) \\ &= \int d^2\sigma \sqrt{g} \left( \delta' g_{ab} g^{ab} (2\delta\omega - \nabla \cdot \delta\sigma) - 2(P_1^\dagger \delta' g)_a \delta\sigma^a \right), \end{aligned}$$

where we define $(P_1^\dagger u)_a = -\nabla^b u_{ab}$. We see from this calculation that $\delta'$ is a modulus iff

$$g^{ab} \delta' g_{ab} = 0 \text{ and } (P_1^\dagger \delta' g)_a = 0.$$

**Exercise 4.1.1.** Show that in conformal gauge, the conformal Killing equation and the condition that $\delta'$ is a modulus become, respectively,

$$\partial_{\bar{z}} \delta z = \partial_z \delta \bar{z} = 0, \quad \partial_{\bar{z}} \delta' g_{zz} = \partial_z \delta' g_{\bar{z}\bar{z}} = 0,$$

i.e. CKVs are **holomorphic vector fields** and moduli are **holomorphic quadratic differentials**.

**Definition 4.1.3.** Define an inner product on symmetric tensors (of any rank) on a surface with metric $g$ by

$$(A, B)_g := \int d^2\sigma \sqrt{g}\, A \cdot B,$$

where the dot denotes contraction on all indices. Using this inner product, define a **metric** $\langle \cdot, \cdot \rangle_{\mathrm{Met}}$ on $\mathrm{Met}(\Sigma)$ as follows. Fix $g \in \mathrm{Met}(\Sigma)$, and let $(g^I)$ be coordinates in an open neighborhood of $g$. Given $X, Y \in T_g \mathrm{Met}(\Sigma)$, we can expand them in coordinates as

$$X = X^I \frac{\partial}{\partial g^I} = X^I_{a_1 b_1} \frac{\partial}{\partial g^I_{a_1 b_1}} \in T_g \mathrm{Met}(\Sigma), \quad Y = Y^J \frac{\partial}{\partial g^J} = Y^J_{a_2 b_2} \frac{\partial}{\partial g^J_{a_2 b_2}} \in T_g \mathrm{Met}(\Sigma).$$

Then define

$$\langle X, Y\rangle_{\text{Met}} \coloneqq (X^I, Y^I)_g = \int_\Sigma d^2\sigma\,\sqrt{g}\,g^{a_1a_2}(\sigma)g^{b_1b_2}(\sigma)X^I_{a_1b_1}(\sigma)Y^I_{a_2b_2}(\sigma).$$

Consequently, we get a well-defined measure on $\text{Met}(\Sigma)$.

**Exercise 4.1.2.** Check that, under this metric, moduli are indeed orthogonal to the diff $\times$ Weyl action. Indeed, check that we have an **orthogonal decomposition** of metric variations

$$\delta g = \{\text{Weyl}\} \oplus \{\text{diff}\} \oplus \{\text{moduli}\} = \{\text{Weyl}\} \oplus \operatorname{Im} P_1 \oplus \ker P_1^\dagger$$

by showing that $P_1^\dagger$ is the adjoint of $P_1$ under $(\cdot,\cdot)_{g(x)}$, and recalling that $\ker A^\dagger = (\operatorname{Im} A)^\perp$ for an operator $A$. Also, verify that $\ker P_1$ is the CKG. $P_1$ is an important operator!

It is fairly easy to check that $\langle\cdot,\cdot\rangle_{\text{Met}}$ is invariant under the diff $\times$ Weyl action, so it descends from $\text{Met}(\Sigma_p)$ to a well-defined metric on $\mathcal{M}(\Sigma_p)$, known as the **Weil-Petersson metric**. Hence we also obtain a well-defined measure on $\mathcal{M}(\Sigma_p)$.

### 4.1.2 Calculating the Measure Using Faddeev-Popov

Now we shall apply Faddeev-Popov in order to compute what this measure on $\mathcal{M}(\Sigma_p)$ is. Essentially, we shall perform a change of variables, from integrating over metrics and vertex positions to integrating over variables corresponding to the orthogonal decomposition, i.e. integrating over the gauge group, moduli (of dimension $\tau$), and unfixed vertex positions. In other words, we wish to find the Jacobian for the transformation

$$\frac{1}{\text{diff}\times\text{Weyl}}\int_{\text{Met}(\Sigma_p)}\mathcal{D}g\,d^{2n}\sigma \to \frac{1}{\text{diff}\times\text{Weyl}}\int_{\text{diff}\times\text{Weyl}}\mathcal{D}\zeta\int_{\mathcal{M}(\Sigma_p)}d^\tau m\int_{(\Sigma_p)^{n-k/2}}d^{2n-k}\sigma.$$

Note that because of the additional gauge freedom given by the CKG, we can choose the gauge-fixing conditions now to also fix $\kappa = \dim\text{CKG}(\Sigma_p)$ of the vertex operator coordinates, so that $\sigma_i^a \to \hat\sigma_i^a$ for some set, denoted Fixed, of fixed coordinates $(a,i)$.

To find the appropriate Jacobian for this change of variables, we use Faddeev-Popov. We can directly write down the **Faddeev-Popov determinant**:

$$\Delta_{\text{FP}}(g,\sigma)^{-1} = \int_{\mathcal{M}(\Sigma)}d^\tau m\int_{\text{diff}\times\text{Weyl}}\mathcal{D}\zeta\,\delta(\hat g(m)^\zeta - g)\prod_{(a,i)\in\text{Fixed}}\delta((\hat\sigma_i^\zeta)^a - \sigma_i^a).$$

Previously, we rewrote the right hand side as an integration over infinitesimal transformations under the false assumption that the gauge group acted freely on the configuration space of embeddings and metrics. This was OK because we did not explicitly compute anything that depended on getting factors correct in numerical results. However, now we care. Note that:

1. an integral over infinitesimal transformations, i.e. the Lie algebra, only captures behavior in the connected component of the identity in diff $\times$ Weyl;
2. by homogeneity, the value of the integral in the connected component of the identity is the same as the value in any other connected component;
3. after removing the CKG, the action of the connected component of the identity $(\text{diff}\times\text{Weyl})_0 = \text{diff}_0\times\text{Weyl}$ is free.

Hence we can, again, write the integral as an integral over the Lie algebra, but now we have a constant, finite factor $n_R$ in front:

$$\Delta_{\text{FP}}(\hat g,\sigma)^{-1} = n_R\int d^\tau(\delta m)\,\mathcal{D}(\delta\omega)\,\mathcal{D}(\delta\sigma)\,\delta(\delta g_{ab})\prod_{(a,i)\in\text{Fixed}}\delta(\delta\sigma^a(\hat\sigma_i)),$$

where $n_R$ is the size of the **mapping class group** $\text{Mod}(\Sigma) = |\text{diff}/\text{diff}_0|$. (We don't care about Weyl, since it is trivially connected.)

Now we follow the exact same procedure as we did for inverting the Faddeev-Popov determinant for the Polyakov action: compute $\delta g_{ab}$, introduce auxiliary variables of integration $x$ and $\beta_{ab}$ to write the delta functionals as exponentials, integrate out $\delta\omega$ to obtain the constraint that $\beta_{ab}$ is traceless, and replace the bosonic variables $(\delta\sigma^a, \beta_{ab}, x_{ai}, \delta m^t)$ with fermionic variables $(c^a, b_{ab}, \eta_{ai}, \zeta^t)$, to get

$$\Delta_{\text{FP}}(\hat{g}, \sigma) = \frac{1}{n_R} \int \mathcal{D}b\, \mathcal{D}c\, d^\tau\zeta\, d^\kappa\eta\, \exp\left( -\frac{1}{4\pi}(b, 2P_1 c - \zeta^j \partial_j \hat{g})_{\hat{g}} + \sum_{(a,i)\in\text{Fixed}} \eta_{ai} c^a(\hat{\sigma}_i) \right).$$

(We've added in some convenient normalization factors.) Finally, we do the integratation over the Grassmann parameters $\eta_{ai}$ and $\zeta^i$. The (somewhat elegant) result is

$$\Delta_{\text{FP}}(\hat{g}, \sigma) = \frac{1}{n_R} \int \mathcal{D}b\, \mathcal{D}c\, \exp(-S_{\text{G}}) \prod_{j=1}^{\tau} \frac{(b, \partial_j \hat{g})_{\hat{g}}}{4\pi} \prod_{(a,i)\in\text{Fixed}} c^a(\hat{\sigma}_i)$$

But there is a nicer way to write $\Delta_{\text{FP}}$. As a Jacobian, it is a determinant, and as a Jacobian for a change of variables into an orthogonal decomposition, we expect it to decompose as a product of determinants. Indeed, it does. First, obtain real eigenbases $\{\mathcal{C}_J\}$ and $\{\mathcal{B}_K\}$ for the operators $P_1^\dagger P_1$ and $P_1 P_1^\dagger$, i.e.

$$P_1^\dagger P_1 (\mathcal{C}_J)^a = v_J^2 (\mathcal{C}_J)^a, \quad P_1 P_1^\dagger (\mathcal{B}_K)_{ab} = w_K^2 (\mathcal{B}_K)_{ab},$$

normalized such that $(\mathcal{C}_J, \mathcal{C}_{J'}) = \delta_{JJ'}$ and $(\mathcal{B}_K, \mathcal{B}_{K'}) = \delta_{KK'}$. These eigenbases are related. Note that $P_1\mathcal{C}_J$ is an eigenfunction of $P_1 P_1^\dagger$, and similarly, $P_1\mathcal{B}_K$ is an eigenfunction of $P_1^\dagger P_1$. Hence there is a one-to-one correspondence between the eigenbases $\{\mathcal{C}_J\}$ and $\{\mathcal{B}_K\}$ except for when $P_1\mathcal{C}_J = 0$ or $P_1\mathcal{B}_K = 0$, i.e. when the eigenfunction has a zero eigenvalue. The $\mathcal{C}_J$ with zero eigenvalue correspond to $\kappa$ CKVs, and the $\mathcal{B}_K$ with zero eigenvalue correspond to $\tau$ moduli. Denote these eigenfunctions of zero eigenvalue $\{(\mathcal{C}_{0j})^a\}_{j=1}^\kappa$ and $\{(\mathcal{B}_{0k})_{ab}\}_{k=1}^\tau$. The rest (of non-zero eigenvalue) are indexed as normal with $J, K = 1, \ldots$, and satisfy $(\mathcal{B}_J)_{ab} = (1/w_J)(P_1\mathcal{C}_J)_{ab}$.

If we rewrite the integral for $\Delta_{\text{FP}}$ in terms of these eigenbases, we get

$$\begin{aligned} \Delta_{\text{FP}} = \int \prod_{k=1}^{\tau} db_{0k} \prod_{j=1}^{\kappa} dc_{0j} \prod_J db_J\, dc_J\, \exp\left( -\frac{w_J b_J c_J}{2\pi} \right) \\ \times \left( \prod_{k'=1}^{\tau} \sum_{k''=1}^{\tau} \frac{b_{0k''}}{4\pi} (\mathcal{B}_{0k''}, \partial_{k'} \hat{g}) \right) \left( \prod_{(a,i)\in\text{Fixed}} \sum_{j'=1}^{\kappa} c_{0j'} (\mathcal{C}_{0j'})^a(\sigma_i) \right), \end{aligned}$$

which is just a product of three Gaussian integrals. Recalling how to do Gaussian integrals with insertions, i.e. of the form $\int dx\, \exp(-x^2) x^k$, we get

$$\Delta_{\text{FP}} = \left[ \det \left( \frac{(\mathcal{B}_{0k}, \partial_{k'} \hat{g})}{4\pi} \right)_{k,k'=1}^{\tau} \right] \left[ \det\left( (\mathcal{C}_{0j})^a(\sigma_i) \right)_{j=1,(a,i)\in\text{Fixed}}^{\kappa} \right] \left[ \det{}' \left( \frac{P_1^\dagger P_1}{2\pi} \right)^{1/2} \right],$$

where $\det'$ indicates that we omit zero eigenvalues. Otherwise the whole expression is trivially zero.

For tree-level and one-loop amplitudes, evaluating this expression for $\Delta_{\text{FP}}$ is not too bad. For higher-loop amplitudes, there is more work involved. But in that case we must put in more work anyway: to even integrate on the moduli space requires us to put **Fenchel-Nielsen** coordinates on it, which is not an easy task.

### 4.1.3 The $X^\mu$ Integration

It remains to handle the $X^\mu$ integration

$$\left\langle \prod_{i=1}^n \int d^2\sigma_i \sqrt{g(\sigma_i)}\, \mathcal{V}_{j_i}(k_i,\sigma_i) \right\rangle := \int \mathcal{D}X \, \exp(-S_X) \prod_{i=1}^n \int d^2\sigma_i \sqrt{g(\sigma_i)}\, \mathcal{V}_{j_i}(k_i,\sigma_i).$$

To evaluate this integral, we use the same trick we used in QFT: introduce a formal variable $J$, write down the **generating functional**

$$Z[J] = \left\langle \exp\left( i \int d^2\sigma \, J_\mu X^\mu \right) \right\rangle$$

and do tricks with $J^\mu(\sigma)$ in order to get the vertex operator insertions we want. We can directly compute $Z[J]$ by writing the Polyakov action as

$$S_X = \frac{T_0}{2} \int d^2\sigma \, X_\mu \Delta_g X^\mu, \quad \Delta_g := \nabla^2 = \frac{1}{\sqrt{g}} \partial_a \sqrt{g}\, g^{ab} \partial_b,$$

which is a Gaussian. Expand $X$ and $J$ in terms of an eigenbasis of the Laplacian:

$$\Delta_g \mathcal{X}_I = -\omega_I^2 \mathcal{X}_I, \quad X^\mu = \sum_I x_I^\mu \mathcal{X}_I, \quad J^\mu = \sum_I J_I^\mu \mathcal{X}_I,$$

where the eigenfunctions $\psi_I$ are chosen to be orthogonal with respect to $(\cdot,\cdot)_g$. Then

$$Z[J] = \prod_{I,\mu} \int \mathcal{D}X^\mu \, \exp\left( -\frac{T_0}{2} \omega_I^2 x_I \cdot x_I + i J_I \cdot x_I \right).$$

Before doing the Gaussian integrals, note that the zero mode $\mathcal{X}_0$ is special: since $\nabla^2 \mathcal{X}_0 = 0$, it has an extremum in the interior of the Riemann surface $\Sigma$. By the maximum modulus principle, $\mathcal{X}_0$ must be constant. Hence its corresponding Gaussian integral becomes $\int \mathcal{D}X^\mu \exp(iJ_0 \cdot x_0) = \delta^D(J_0)$.

**Exercise 4.1.3.** Review how to do Gaussian integrals, in particular the $Z[J]$ integral from QFT, and calculate that

$$Z[J] = i(2\pi)^D \delta^D(J_0) \det{}' \left( \frac{T_0}{\pi} \Delta_g \right)^{-d/2} \exp\left( -\frac{1}{2} \int d^2\sigma_1 \, d^2\sigma_2 \, J(\sigma_1) G'(\sigma_1,\sigma_2) J(\sigma_2) \right),$$

where the **Green's function** (excluding zero modes) is

$$G'(\sigma_1,\sigma_2) := \sum_{I \neq 0} \frac{1}{T_0} \frac{1}{\omega_I^2} \mathcal{X}_I(\sigma_1) \mathcal{X}_I(\sigma_2).$$

Verify that $G'(\sigma_1,\sigma_2)$ satisfies

$$-T_0 \Delta_g G'(\sigma_1,\sigma_2) = \sum_{I \neq 0} \mathcal{X}_I(\sigma_1) \mathcal{X}_I(\sigma_2) = \frac{1}{\sqrt{g}} \delta^2(\sigma_1 - \sigma_2) - \mathcal{X}_0^2$$

using the completeness relation for the eigenbasis $\{\mathcal{X}_I\}$.

We shall use this differential equation for the Green's function in order find out what it is. Note that it depends on $\Delta_g$, which changes with different moduli, and, in particular, different genus. Hence for every $n$, to compute the $n$-loop corrections to the amplitude, we need to find $G'$ for every element in the moduli space.

## 4.2 Tree-Level Amplitudes

In this section, we compute tree-level amplitudes for $n$**-tachyon scattering**, which, for closed strings, corresponds to $\Sigma = S^2$ with $n$ marked points with **tachyon vertex operators** $\mathcal{V}_{j_i}(k_i, \sigma_i) = :e^{ik\cdot X}:$. Hence we must first understand $\mathcal{M}(S^2)$ and $\mathrm{CKG}(S^2)$.

**Proposition 4.2.1.** *$\mathcal{M}(S^2)$ is a single point, and $\mathrm{CKG}(S^2) \cong \mathrm{PSL}(2,\mathbb{C})$, with real dimension 6.*

*Proof.* Take the standard atlas for $S^2$ given by stereographic projection: one coordinate $z$ on $S^2 \setminus \{N\}$, i.e. everywhere except the north pole, and another coordinate $u$ on $S^2 \setminus \{S\}$, i.e. everywhere except the south pole. In conformal gauge, we know from a previous exercise that moduli are holomorphic quadratic differentials $\delta g_{zz}(z)$ and CKVs are holomorphic vector fields $\delta z(z)$. But these objects must be well-defined globally, so we need to consider them on the $u$ patch:

$$\delta u = \frac{\partial u}{\partial z}\delta z = -z^{-2}\delta z, \quad \delta g_{uu} = \left(\frac{\partial u}{\partial z}\right)^{-2} \delta g_{zz} = z^4 \delta g_{zz}.$$

If $\delta g_{uu}$ is well-defined at $u = 0$, i.e. the north pole, then $\delta g_{zz} \propto z^{-4}$ as $|z| \to \infty$. Hence $\delta g_{zz}$ is a bounded entire function, which, by Liouville's theorem, is constant. In the moduli space, we have already modded out by Weyl transformations, so all constant functions are equivalent. (In fact, by the uniformization theorem, every metric is equivalent to the constant-curvature metric induced by the inclusion $S^2 \to \mathbb{R}^3$.) However, for CKVs, we only need $\delta z \propto z^2$ as $|z| \to \infty$, so in general,

$$\delta z = a_0 + a_1 z + a_2 z^2, \quad \delta\bar{z} = a_0^* + a_1^* \bar{z} + a_2^* \bar{z}^2.$$

Hence $\dim_{\mathbb{R}} \mathrm{CKG}(S^2) = 6$. It is well-known that these are precisely the infinitesimal transformations of the Möbius group

$$\mathrm{PSL}(2,\mathbb{C}) = \left\{ \frac{az+b}{cz+d} : ad - bc = 1 \right\} / ((a,b,c,d) \sim (-a,-b,-c,-d)). \qquad \square$$

We first compute the $X^\mu$ integral using the results of the previous section. Using the fact that $\partial\bar{\partial} \ln|z| = 2\pi\delta(z)$, by solving the DE, we get the Green's function

$$G'(\sigma_1, \sigma_2) = -\frac{1}{4\pi T_0} \ln|z_1 - z_2|^2 + f(z_1, \bar{z}_1) + f(z_2, \bar{z}_2), \quad f(z, \bar{z}) := \frac{\mathcal{X}_0^2}{8\pi T_0} \int d^2 w\, e^{2\omega(z,\bar{z})} \ln|z - w|^2 + C$$

where the function $f$ acts as a **regulator** that cancels out in the final amplitudes, and $C$ is just a constant to make sure $G'$ is orthogonal to $\mathcal{X}_0$, a constant. The tachyon vertex operators $:e^{ik_i\cdot X}:$ correspond to $J(\sigma) = \sum_i \delta(\sigma - \sigma_i)$. Consequently,

$$Z[J] = C^{\mathrm{X}}_{S^2} \delta^D(\sum_i k_i) \exp\left( -\sum_{i<j} k_i \cdot k_j G'(\sigma_i, \sigma_j) - \frac{1}{2}\sum_i k_i^2 G'_r(\sigma_i, \sigma_i) \right), \; C^{\mathrm{X}}_{S^2} := \frac{i(2\pi)^D}{\mathcal{X}_0^D} \det{}' \left( \frac{T_0}{\pi}\Delta_g \right)^{-\frac{D}{2}},$$

where the $G'_r \neq G'$ because of the normal ordering. In fact, for normal-ordered vertex operators, we get $G'_r(\sigma, \sigma) = 2f(z, \bar{z})$, so

$$Z[J] = C^{\mathrm{X}}_{S^2} \delta^D\left(\sum_i k_i\right) \prod_{i<j} |z_i - z_j|^{k_i\cdot k_j/2\pi T_0}.$$

Note that the regulator $f$ has canceled.

Since the sphere has no moduli, we are free to ignore the integral over moduli space. Most of the Faddeev-Popov measure becomes constant, except the Jacobian for the CKG. From our computation of the CKG, we know a basis for CKVs is given by $\mathcal{C}^z = 1, z, z^2$ and $\mathcal{C}^{\bar{z}} = 1, \bar{z}, \bar{z}^2$, so

$$\Delta_{\mathrm{FP}} = C^{\mathrm{G}}_{S^2} \det(\mathcal{C}^a_{0j}(\sigma_i)) = C^{\mathrm{G}}_{S^2} \det((z_i)^{j-1})^3_{i,j=1} \det((\bar{z}_i)^{j-1})^3_{i,j=1} = C^{\mathrm{G}}_{S^2} |z_1 - z_2|^2 |z_1 - z_3|^2 |z_2 - z_3|^2,$$

absorbing the two other (now constant) determinants in $C^{\mathrm{G}}_{S^2}$. Assume that we are scattering $n > 2$ tachyons, so that the $\dim_{\mathbb{C}} \mathrm{CKG}(S^2) = 3$ degrees of freedom can fix the positions $\hat{z}_1, \hat{z}_2, \hat{z}_3$ of the first three local operators: it is well-known that the Möbius transformations $\mathrm{PSL}(2,\mathbb{C})$ act transitively on triplets of points, i.e. is 3-transitive.

**Proposition 4.2.2.** *The closed string, tree-level, $n$-tachyon scattering amplitude is given by*

$$S_{j_1\cdots j_n}(k_1,\ldots,k_n) = g_s^{-2+n} C^X_{S^2} C^G_{S^2} \delta^D\left(\sum_{i=1}^n k_i\right) |\hat{z}_1 - \hat{z}_2|^2 |\hat{z}_1 - \hat{z}_3|^2 |\hat{z}_2 - \hat{z}_3|^2 \int \prod_{i=4}^n d^2 z_i \prod_{i<j} |z_i - z_j|^{k_i \cdot k_j / 2\pi T_0}$$

*for any choice of $\hat{z}_1, \hat{z}_2, \hat{z}_3 \in \mathbb{C}^2$.*

The closed string, four-tachyon scattering amplitude is the first computed non-trivial string scattering amplitude, and therefore has a name: the **Virasoro-Shapiro amplitude**. We shall compute it explicitly.

**Example 4.2.3** (Virasoro-Shapiro amplitude)**.** Write $\alpha' = 1/2\pi T_0$. Pick $\hat{z}_1 = 1$, $\hat{z}_2 = 0$, and $\hat{z}_3 = \infty$. Then any term $|\hat{z}_3 - z_i|$ can be treated as $|\hat{z}_3|$, so

$$|\hat{z}_1 - \hat{z}_3|^{2+\alpha' k_1\cdot k_3} |\hat{z}_2 - \hat{z}_3|^{2+\alpha' k_2 \cdot k_3} |\hat{z}_3 - z_4|^{\alpha' k_3 \cdot k_4} = |\hat{z}_3|^{4+\alpha' k_3 \cdot (k_1 + k_2 + k_4)} = |\hat{z}_3|^0 = 1,$$

by recalling that on-shell tachyons satisfy $k^2 = 4/\alpha'$, and therefore, using momentum conservation,

$$4 + \alpha' k_3 (k_1 + k_2 + k_4) = 4 - \alpha' k_3^2 = 4 - 4 = 0.$$

Introduce **Mandelstam variables** like we did for 4-point scattering in QFT:

$$s \coloneqq -(k_1 + k_2)^2, \quad t \coloneqq -(k_1 + k_3)^2, \quad u \coloneqq -(k_1 + k_4)^2,$$

which satisfy $s + t + u = -16/\alpha'$. Again using the on-shell condition and momentum conservation, we can show that the remaining part of the integral is

$$\int d^2 z_4\, |1 - z_4|^{\alpha' k_1 \cdot k_4} |z_4|^{\alpha' k_2 \cdot k_4} = \int d^2 z_4 |1 - z_4|^{-(\alpha' u)/2 - 4} |z_4|^{-(\alpha' t)/2 - 4}.$$

Now recall that using the **Euler beta function**,

$$\int d^2 z\, |z|^{2a-2} |1 - z|^{2b-2} = 2\pi \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b)\Gamma(a+c)\Gamma(b+c)}, \quad a + b + c = 1,$$

so that our final expression is

$$S_{j_1\cdots j_4}(k_1,\ldots,k_4) = g_s^2 C^{\mathrm{X}}_{S^2} C^{\mathrm{G}}_{S^2} \delta^D(k_1 + \cdots + k_4) 2\pi \frac{\Gamma(-1 - \frac{\alpha'}{4}s)\Gamma(-1 - \frac{\alpha'}{4}t)\Gamma(-1 - \frac{\alpha'}{4}u)}{\Gamma(2 + \frac{\alpha'}{4}s)\Gamma(2 + \frac{\alpha'}{4}t)\Gamma(2 + \frac{\alpha'}{4}u)}.$$

The analogous amplitude for open strings, called the **Veneziano amplitude**, led to the birth of string theory in the 1970s.

## 4.3 One-Loop Amplitudes

In this section, we compute one-loop corrections to amplitudes for $n$-tachyon scattering, which, for closed strings, corresponds to $\Sigma = T^2$, the torus, with $n$ marked points. Hence we must understand $\mathcal{M}(T^2)$ and $\mathrm{CKG}(T^2)$. Recall that $\mathbb{H}^2$ is the **hyperbolic plane**, which can be identified with the **upper half plane**.

**Proposition 4.3.1.** $\mathcal{M}(T^2) = \mathbb{H}^2 / \mathrm{PSL}(2,\mathbb{Z})$, *and* $\mathrm{CKG}(T^2) = U(1) \times U(1)$.

## 4.4 Higher-Order Amplitudes