Quantum Field Theory
Fall 2015 Seminar Notes

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Chapter 1

Klein-Gordon Field

1.1 Why Fields?

Volume 1 of Steven Weinberg's *Quantum Theory of Fields* is devoted to answering this question. A discussion of scattering experiments lead him to the S-matrix, and then to the local behaviour of experiments (which he calls the cluster decomposition principle), and then using Lorentz invariance, fields just practically fall out. Weinberg does a really good job of convincing us that QFT in some form or another really must exist if we assume Lorentz invariance and unitarity.

Peskin and Schroeder give a slightly different motivation, one that is closer to the historical reason of why fields were introduced:

- Single particle relativistic wave functions $\Longrightarrow$ inconsistencies in theory (negative energy eigenstates) **ELABORATE**
- $E = mc^2$ allows for particles to be created at high energies
- $\Delta E \cdot \Delta t = \hbar$ allows for virtual particles
- Causality violation. Set $H = \frac{\hat{p}^2}{2m}$

$$\begin{aligned} U(t) &= \langle \vec{x} | e^{-iHt} | \vec{x}_0 \rangle \\ &= \int \frac{d^3p}{(2\pi)^3} \langle \vec{x} | e^{-i(p^2/2m)t} | p \rangle \langle p | x \rangle \\ &= \int \frac{d^3p}{(2\pi)^3} e^{-i(p^2/2m)t} e^{i\vec{p}\cdot(\vec{x}-\vec{x}_0)} \\ &= \left(\frac{m}{2\pi i t} \right)^{3/2} e^{im(\vec{x}-\vec{x}_0)^2/2t} \end{aligned}$$

  This last quantity is non-zero, even for arbitariry x that may by space-like separated.

QFT seems to solve all of these mysteries. One very good feature of the theory is that it predicts a lot of experiments to very high accuracy. QED is something we will see very soon that has been very well tested and agrees very well with experiments.

1.2 Elements of Classical Field Theory

1.2.1 Lagrangian Field Theory

- Fundamental quantity in Lagrangian field theory is the action S. In high school, the Lagrangian is a function of time, positions, and velocities of a system: $L(t, x(t), \dot{x}(t))$. The action is given by

$S = \int dt\, L$. Fields can also be described in a Lagrangian formalism, for instance by considering every point in space-time as a "particle" that wiggles back and forth with the amplitude of wiggling characterizing the strength of the field.

Let $\varphi : M \to \mathbb{R}$, define a Lagrangian *density* $\mathcal{L}(t, \varphi, \partial_\mu \varphi)$, the honest Lagrangian $L = \int d^3x \mathcal{L}$, and finally define the action:

$$S = \int dt\, L = \int d^4x \mathcal{L}$$

Four-vector notation:

- Greek letters $\mu, \nu, \ldots \in \{0, 1, 2, 3\}$
- Roman letters $i, g, \ldots \in \{1, 2, 3\}$.
- $x^\mu = (x^0, x^1, x^2, x^3)$
- Signature $(+ - - -)$
- $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$
- $\partial_\mu f = \frac{\partial f}{\partial x^\mu} = (\partial_0 f, \partial_1 f, \partial_2 f, \partial_3 f)$.

- Extremize the action. Let $\delta f = f(\varphi + \xi) - f(\varphi)$.

$$\begin{aligned} 0 = \delta S &= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \underbrace{\delta(\partial_\mu}_{\text{commute}} \varphi) \right) \\ &= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi \right) - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \delta\varphi \right] \end{aligned}$$

By Stokes' theorem, we can break this integral up into two parts, one of which is called the boundary term. Taking a variation that is fixed along the boundary means $\delta\varphi \equiv 0$ on the boundary which means that the boundary term does not contribute to δS. Moreover, if we take $\delta S = 0$ for every variation, then we obtain the Euler Lagrange equations:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) - \frac{\partial \mathcal{L}}{\partial \varphi} = 0$$

Remark. The Lagragian formalism is useful for relativistic dynamics because all expressions are chosen to Lorentz invariant.

1.2.2 Hamiltonian Field Theory

- Introducing this makes the transition to the quantum theory easier.
- High school Hamiltonian formalism: $p = \frac{\partial L}{\partial \dot{q}}, H = \sum p\dot{q} - L$.
- Pretend that $\vec{x}$ enumerates points on the lattice of space-time:

$$\begin{aligned} p(\vec{x}) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}(\vec{x})} &= \frac{\partial}{\partial \dot{\varphi}(\vec{x})} \int d^3y\, \mathcal{L}(\varphi(y), \dot{\varphi}(y)) \\ &\sim \frac{\partial}{\partial \dot{\varphi}(\vec{x})} \sum \mathcal{L}(\varphi(y), \dot{\varphi}(y)) d^3y \\ &= \frac{\partial \mathcal{L}}{\partial \dot{\varphi}(\vec{x})} d^3x \\ &\equiv \pi(\vec{x}) d^3x \end{aligned}$$

since each point on the lattice represents a different variable, so the derivative just picks out the one at $\vec{x}$. We call $\pi(\vec{x})$ the momentum *density*. Therefore the Hamiltonian looks like:

$$H = \int d^3x \, [\pi(\vec{x})\dot{\varphi}(\vec{x}) - \mathcal{L}] \, .$$

(See the stress-energy tensor part for another derivation of the Hamiltonian which falls out of Noether's theorem for being the conserved quantity under time translations.)

- **Important example:** Take $\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)^2 - \frac{1}{2}m^2\varphi^2$. Euler-Lagrange equations become $\partial^\mu(\partial_\mu\varphi) + m^2\varphi = 0$ which is the Klein Gordon equation. The Hamiltonian becomes:

$$H = \int d^3x\mathcal{H} = \int d^3x \left[\underbrace{\frac{\pi^2}{2}}_{\text{moving in time}} + \underbrace{\frac{(\nabla\varphi)^2}{2}}_{\text{shearing in space}} + \underbrace{\frac{m^2\varphi^2}{2}}_{\text{existing at all}} \right]$$

1.2.3 Noether's Theorem - How to Compute Conserved Quantities

To every continuous transformation of the field we can assign an infinitesmal transformation:

$$\varphi(x) \to \varphi'(x) = \varphi(x) + \alpha \underbrace{\Delta\varphi(x)}_{\text{deformation}}$$

Transformations might also change the Lagrangians. The interplay between how the infinitesmal transformation changes the Lagrangian and the field is what gives rise to conserved quantities, or sometimes known as Noether charges.

$$\begin{aligned}
\text{Symmetry} &\iff \text{Equations of motion – invariant} \\
&\iff \text{Action invariant (up to surface term)} \\
&\iff \mathcal{L}(x) \to \mathcal{L}(x) + \alpha\partial_m u\mathcal{J}^\mu(x)
\end{aligned}$$

Taylor expanding the perturbation:

$$\begin{aligned}
\Delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\varphi} \cdot \Delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\mu(\Delta\varphi) \\
&= \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\Delta\varphi \right) + \left[\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} \right) \right] \\
&= \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\Delta\varphi \right)
\end{aligned}$$

Since we claimed that under the symmetry $\Delta\mathcal{L} = \partial_\mu\mathcal{J}^\mu$ we have the following relations:

$$\begin{aligned}
j^\mu(x) &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\Delta\varphi - \mathcal{J}^\mu \\
\partial_\mu j^\mu &= 0 \\
\frac{\partial}{\partial t}j^0 &= \partial_i j^i
\end{aligned}$$

Define the charge $Q = \int d^3x \; j^0$. Then, if we assume that space does not have boundary, Stokes' theorem implies that $\partial Q/\partial t = 0$. Often, j^0 is called the charge density, and j^μ is called the current density.

Therefore to compute a conserved quantity we compare the deformation of the Lagrangian due to the φ changing with the deformation of the Lagrangian due to the symmetry transformation.

Examples:

1. $\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)^2$ has the following field symmetry, $\varphi \to \varphi + \alpha$, ie. $\Delta\varphi \equiv$ const. There is no change to the Lagrangian, so $j^\mu = \partial^\mu\varphi$.

2. Space-time transformation, $x^\mu \to x^\mu - a^\mu$, implies

$$\begin{aligned} \varphi(x) &\to \varphi(x+a) = \varphi(x) + a^\nu\partial_\nu\varphi(x) \\ \mathcal{L}(x) &\to \mathcal{L}(x+a) = \mathcal{L}(x) + a^\mu\partial_\mu\mathcal{L} \\ . &\qquad\qquad\quad = \mathcal{L}(x) + a^\nu\partial_\mu(\delta^\mu_\nu\mathcal{L}) \end{aligned}$$

Therefore we write

$$T^\mu_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\nu\varphi - \delta^\mu_\nu\mathcal{L}$$

we get four separately conserved quantities. **CHECK**

This is called the stress-energy tensor or the energy-momentum tensor in various contexts. The $T^{\bullet 0}$ quantity gives rise to the Hamiltonian:

$$\int d^3x T^{00} = \int d^3x \mathcal{H} \equiv H$$

1.2.4 Summary of Computing Noether Charges

Field or coordinate transformation $\rightsquigarrow \{\Delta\phi, \Delta\mathcal{L}\} \rightsquigarrow j^\mu(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\Delta\varphi - \mathcal{J}^\mu \rightsquigarrow Q = \int j^0 d^3x$ conserved charge.

1.3 Quantizing Klein Gordon Field

Quantization of field theories involves two steps:

1. Promote ϕ and π to operators,

2. Specify commutation relations:

$$\begin{aligned} [\phi(\vec{x}), \pi(\vec{y})] &= i\delta^{(3)}(\vec{x} - \vec{y}) \\ [\phi(\vec{x}), \phi(\vec{y})] &= [\pi(\vec{x}), \pi(\vec{y})] = 0 \end{aligned}$$

Note that here we are in the Schrodinger picture, so that ϕ, π are operators independent of time. Next, we are going to introduce a basis that will diagonalize the Hamiltonian, and we shall express ϕ and π in terms of these operators.

ELABORATE

The exact form of the following is a little tricky to motivate (but this is done in Weinberg, chapter 5). One way to motivate this is to look at the Klein-Gordon equation in the Fourier representation. This is exactly the harmonic oscillator equation and so it is reasonable to assume that the operators ϕ, π can be manipulated in a similar way to obtain the following ansatz:

$$\begin{aligned} \phi(\vec{x}) &= \\ \pi(\vec{x}) &= \end{aligned}$$