String Theory and Supersymmetry
Winter 2016 Seminar Notes

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Chapter 1

Introduction to Strings

We asked "why fields?" when we started QFT; now we ask, why strings? Here are some potentially convincing reasons.

1. If we allow one more degree of freedom than particles, many IR/UV divergences disappear; we require less renormalization. If we allow more than one degree of freedom, new divergences arise from the increased internal degrees of freedom.
2. Every consistent string theory contains a massless spin-2 state, i.e. a graviton, whose interactions at low energies reduce to general relativity.
3. The Standard Model, based on QFT, has 25 adjustable constants. String theory has none, and leads to gauge groups big enough to include the Standard Model.
4. Consistent string theories force upon us supersymmetry and extra dimensions, which have arisen naturally from several different attempts to unify the Standard Model.

Regardless of whether they are convincing, we start in this chapter, as with any other physical model, by writing down an action. Specifically, we first write the action for a relativistic string by generalizing that of a relativistic point particle, and then we quantize the action. As with QFT there are different ways to quantize. We go through the analogue of canonical quantization in order to quickly compute the spectrum of a string, and then go through path integral quantization in preparation for studying string interactions.

As usual, we take $\hbar = c = 1$, and use **Einstein summation convention**: repeated indices are implicitly summed over.

1.1 Review of Relativity

We work in $\mathbb{R}^{D-1,1}$ where D is the **number of dimensions**. Recall that coordinates are written $x^\mu = (x^0, x^1, \dots, x^D) = (ct, x^1, \dots, x^D)$, and the metric is

$$-ds^2 \coloneqq \eta_{\mu\nu} dx^\mu dx^\nu, \quad \eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, \dots, 1).$$

Note that $\eta^\mu{}_\mu = D$. We use the dot product to stand for the **Lorentz inner product**, e.g. $-ds^2 = dx \cdot dx$.

Definition 1.1.1. Define the **proper time** of a system as the time elapsed measured by a clock traveling in the same Lorentz frame as the system itself. In such a Lorentz frame, $dx^i = 0$ and dt is the proper time elapsed, so $-ds^2 = -dt_p^2$; define

$$ds \coloneqq \sqrt{ds^2} = dt_p \qquad \text{whenever } ds^2 > 0,$$

i.e. for timelike intervals. Hence ds is the **proper time interval**. The **relativistic momentum** is $p^\mu := m(dx^\mu/ds)$. Conveniently,

$$p^\mu p_\mu = m^2 \frac{dx^\mu}{ds}\frac{dx_\mu}{ds} = -m^2 \frac{ds^2}{ds^2} = -m^2.$$

Definition 1.1.2. A **Lorentz transformation** $\Lambda^\mu{}_\nu$ is an element of the Lorentz group, the collection of all linear isometries of $\mathbb{R}^{D-1,1}$. We say a^μ is a **vector** if under Lorentz transformations, it changes as $a'^\mu = \Lambda^\mu{}_\nu a^\nu$. A **Poincaré transformation** is a Lorentz transformation possibly followed by a translation.

Definition 1.1.3. The **world line** of a point particle is the path in spacetime $\mathbb{R}^{D-1,1}$ traced out by the particle as it evolves in time.

The underlying principle of relativity says that physical laws are independent of Lorentz frame. In other words, any action we write down that we want to be compatible with relativity must have external symmetries: it must be invariant under Lorentz transformations. We call this **Lorentz invariance**. As long as superscripts and subscripts match up, we do not have to worry about Lorentz invariance.

The **action** for a free relativistic **point particle** is obtained by writing down the simplest Lorentz invariant action, and then making sure dimensions work out. If γ is the path taken by the particle, the action is therefore

$$S_{\text{pp}}[x] := -m\int_\gamma ds = -m\int_\gamma d\tau\,\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} = -m\int_\gamma d\tau\,\sqrt{-\dot{x}^\mu\cdot\dot{x}_\mu}$$

where a dot denotes a τ-derivative. Because ds is coordinate-independent, it does not what how we pick the parametrization τ. Physicists like to call this **reparametrization invariance**. This invariance is very important: without it, we have actually introduced a completely new parameter τ, thus increasing the number of degrees of freedom from $D-1$ to D.

Exercise 1.1.1. By computing $\delta(ds^2)$ in two different ways, show that

$$\delta S_{\text{pp}}[x] = m\int_\gamma \delta(dx^\mu)\frac{dx_\mu}{ds} = \int_\gamma d\tau\left(\frac{d}{d\tau}\delta x^\mu\right)p_\mu = \delta x^\mu p_\mu\Big|_{\tau_i}^{\tau_f} - \int d\tau\,\delta x^\mu\frac{dp_\mu}{d\tau}.$$

Argue that the first term vanishes if we specify **initial and final conditions**. Hence deduce the equation of motion $dp_\mu/d\tau = 0$.

The action S_{pp} seems simple in the $\int_\gamma ds$ form, but is messy when parametrized. Later when we quantize using path integrals, S_{pp} is difficult to work with because of the derivatives under the square root. There is a different, classically-equivalent action we can work with. Introduce an additional field $\gamma_{\tau\tau}(\tau)$ (sometimes called an **einbein** in general relativity), which we can view as a metric on the world line, and take the action

$$S'_{\text{pp}} := -\frac{1}{2}\int_\gamma d\tau\sqrt{-\gamma_{\tau\tau}}(\gamma^{\tau\tau}\dot{x}^\mu\dot{x}_\mu + m^2) = -\frac{1}{2}\int_\gamma d\tau\,(\eta^{-1}\dot{x}^\mu\dot{x}_\mu - \eta m^2), \quad \eta := \sqrt{-\gamma_{\tau\tau}(\tau)}.$$

It seems like we have arbitrarily added an extra degree of freedom, but in fact γ is completely specified by the equation of motion. The action S'_{pp} is much better to work with in a path integral, because it is **quadratic** in $\dot{x}^\mu$.

Exercise 1.1.2. Vary S'_{pp} with respect to $\gamma_{\tau\tau}$ to get the equation of motion $\gamma_{\tau\tau} = \dot{x}^\mu\dot{x}_\mu/m^2$. Substitute this expression back into S'_{pp} to obtain S_{pp}, and therefore conclude that the two actions are classically equivalent.

1.2 Nambu–Goto and Polyakov Actions

We graduate to **one-dimensional strings**; in this section we write down an action for them. There are two kinds of strings: those with two distinct endpoints, called **open strings**, and those which are loops, called **closed strings**. Because closed strings are just open strings with the extra constraint that the two endpoints match, we focus on open strings.

The action for the relativistic point particle is proportional to the proper time elapsed on the particle's world line. But the proper time, when multiplied by c, can be viewed as the "proper length" of the world line. The natural generalization, then, is to consider the surface in space-time traced out by the string as it evolves in time, called the **world sheet** M, and to define an action proportional to the "proper area" of the world sheet. The world sheet M is a two-dimensional surface, and therefore requires charts modeled on $\mathbb{R}^2$.

Definition 1.2.1. The **coordinates** we use on $\mathbb{R}^2$, the parameter space, are denoted (σ^0, σ^1), and so the **world sheet** M is locally a surface given by functions denoted $X^\mu(\sigma)$ (capitalized to disambiguate from the coordinates x^μ), called **string coordinates**. The lowercase Latin characters $a, b, \ldots$ are used to denote **indices** that run over values $0, 1$. Two notes:

1. The choice of parametrization (σ^0, σ^1) is, again, up to us, but usually we take the coordinate σ^0 to be the proper time, and σ^1 the position along the string.
2. For our purposes, $M = X^\mu$, i.e. the single chart X^μ describes the entire world sheet for the region of spacetime we care about.

Exercise 1.2.1. Show that the metric $\eta_{\mu\nu}$ on spacetime $\mathbb{R}^{D-1,1}$ induces a metric g on the world sheet via pullback along the inclusion $\iota\colon M \to \mathbb{R}^{D-1,1}$. Compute g and the area element:

$$g_{ab} = \partial_a X^\mu \partial_b X_\mu, \quad dA = d^2\sigma\, \sqrt{-\det g}.$$

A relativistic particle has a parameter we call mass. It turns out mass is not the appropriate physical interpretation of the corresponding parameter for strings. Instead, we interpret it as a **tension**, and denote it T_0. Old people write $T_0 = 1/2\pi\alpha'$ and call α' the **universal Regge slope**; we choose not to.

Definition 1.2.2. The **Nambu–Goto action** for a relativistic string is given by

$$S_{\mathrm{NG}}[X] \coloneqq -T_0 \int_M dA = -T_0 \int_M d^2\sigma\, \sqrt{-\det g}.$$

Again, note that it satisfies **reparametrization invariance**, literally by construction.

But again, we have a square root and derivatives inside it, and now we know how to get rid of it: introduce an independent world sheet metric $\gamma_{ab}(\sigma)$. This time the metric is on a surface, so we need to specify the signature. We take Lorentzian signature $(-, +)$.

Definition 1.2.3. The **Polyakov action** for a relativistic string is given by

$$S_{\mathrm{P}}[X, \gamma] \coloneqq -\frac{T_0}{2} \int_M d^2\sigma\, \sqrt{-\gamma}\, \gamma^{ab} \partial_a X^\mu \partial_b X_\mu,$$

where γ without indices stands for $\det(\gamma_{ab})$. From now on, we always refer to γ_{ab} as the **metric**, and g_{ab} as the **induced metric**. World sheet indices are raised/lowered using the metric γ_{ab}, not the induced metric g_{ab}. (In fact, from now on we basically forget about g_{ab}; we use it only to introduce the Nambu–Goto action, and the following exercise.)

Exercise 1.2.2. Show that $\delta\sqrt{-\gamma} = (1/2)\sqrt{-\gamma}\, \gamma^{ab}\delta\gamma_{ab}$, and therefore that

$$\delta_\gamma S_{\mathrm{P}}[X, \gamma] = -\frac{T_0}{2} \int_M d^2\sigma\, \sqrt{-\gamma}\, \delta\gamma^{ab} \left(g_{ab} - \frac{1}{2}\gamma_{ab}\gamma^{cd} g_{cd} \right).$$

Rearrange the obtained equation of motion and conclude that $g_{ab}\sqrt{-g} = \gamma_{ab}\sqrt{-\gamma}$. Hence replace γ in $S_{\mathrm{P}}[X, \gamma]$ with g, and obtain that $S_{\mathrm{P}}[X, \gamma] = S_{\mathrm{NG}}[X]$.

Definition 1.2.4. As in general relativity, define the **stress-energy tensor**

$$T_{ab}(\sigma) \coloneqq -\frac{4\pi}{\sqrt{-\gamma}}\delta_\gamma S_{\mathrm{P}}[X,\gamma] = -2\pi T_0 \left(\partial_a X^\mu \partial_b X_\mu - \frac{1}{2}\gamma_{ab}\partial_c X^\mu \partial^c X_\mu\right), \tag{1.1}$$

so that the equation of motion arising from varying γ says $T_{ab} = 0$. We call $T_{ab} = 0$ a **constraint** on the equation of motion for X^μ, which we derive soon.

Exercise 1.2.3. (Important!) Now vary $S_{\mathrm{P}}[X,\gamma]$ with respect to X^μ to obtain

$$\begin{aligned}\delta_X S_{\mathrm{P}}[X,\gamma] &= -T_0 \int_M d^2\sigma\, \sqrt{-\gamma}\, \gamma^{ab} \left(\partial_a(\delta X^\mu \partial_b X_\mu) - \partial_a\partial_b X_\mu \delta X^\mu\right) \\ &= -T_0 \int_0^\ell d\sigma^1\, \sqrt{-\gamma}\, \left[\delta X^\mu \partial^0 X_\mu\right]_{\sigma^0=\tau_i}^{\sigma^0=\tau_f} - T_0 \int_{\tau_i}^{\tau_f} d\sigma^0\, \sqrt{-\gamma}\, \left[\delta X^\mu \partial^1 X_\mu\right]_{\sigma^1=0}^{\sigma^1=\ell} \\ &\quad + T_0 \int_M d^2\sigma\, \sqrt{-\gamma}\, \delta X^\mu \nabla^2 X_\mu.\end{aligned}$$

A careful inspection of the terms in the variation $\delta_X S_{\mathrm{P}}[X,\gamma]$ yield interesting insights. For this variation to vanish, each of the terms must vanish independently, since they control different aspects of the string's behavior.

1. The last term is determined by the motion of the string in the domain $(0,\ell)\times(\tau_i,\tau_f)$, and therefore δX^μ is not constrained by any boundary conditions there. Hence we have the **equation of motion** $\sqrt{-\gamma}\,\nabla^2 X_\mu = 0$.
2. The first term is determined by the configuration of the string at times τ_i and τ_f. If we specify these configurations as **initial and final conditions**, then δX^μ is zero for the first term, so the term vanishes.
3. The second term is determined by the configuration of the endpoints of the string when $\sigma^0 \in (\tau_i,\tau_f)$. It does not vanish automatically. We have to impose **boundary conditions** in order to make it vanish.

Definition 1.2.5. There are two different kinds of boundary conditions.

- The **free (Neumann) boundary condition** is $\partial^1 X_\mu(\sigma^0,0) = \partial^1 X_\mu(\sigma^0,\ell) = 0$.
- The **Dirichlet boundary condition** is $\delta X^\mu(\sigma^0,0) = \delta X^\mu(\sigma^0,\ell) = 0$.

Alternatively, if the string is **closed**, i.e. we have the **periodicity** conditions

$$X^\mu(\sigma^0,0) = X^\mu(\sigma^0,\ell), \quad \partial^a X^\mu(\sigma^0,0) = \partial^a X^\mu(\sigma^0,\ell), \quad \gamma_{ab}(\sigma^0,0) = \gamma_{ab}(\sigma^0,\ell),$$

no additional boundary conditions are necessary.

For a long time, string theorists did not seriously consider the Dirichlet boundary condition. Why should the endpoints of an open string be fixed, and if they were, where would they be fixed onto? In particular, this fixing of endpoints would violate momentum conservation. Then Polchinski, in the 1990s, suggested that the endpoints are attached to **D-branes**, which should themselves be thought of as dynamical objects alongside strings. Conceptually, then,

1. a D0-brane is a particle, a D1-brane is a string, and so on, and they interact non-trivially;
2. the Dirichlet boundary condition says that a given D1-brane has fixed endpoints on a higher Dp-brane;
3. any momentum lost by the D1-brane is absorbed by the Dp-brane; and
4. the Neumann boundary condition is just saying there is a D-dimensional D-brane permeating all of space-time, i.e. the string endpoints are not fixed at all.

We return to this D-brane perspective much later on. It is hard enough to quantize strings without more dynamical objects floating around. We take **Neumann boundary conditions** for now.

1.3 Gauge Freedom and Gauge Fixing

There is another reason the Polyakov action is preferable over the Nambu–Goto action: it has more symmetries, and these symmetries make it easier to gauge fix (using Faddeev–Popov or otherwise) when we try to quantize. The Polyakov action is invariant under the following symmetries:

1. D-dimensional **Poincaré transformations**:
$$X^\mu(\sigma) \mapsto \Lambda^\mu{}_\nu X^\nu(\sigma) + a^\mu, \quad \gamma_{ab}(\sigma) \mapsto \gamma_{ab}(\sigma);$$
2. **Reparametrization** (i.e. diffeomorphisms): for new coordinates $\tilde{\sigma}^a(\sigma)$,
$$X^\mu(\sigma) \mapsto X^\mu(\tilde{\sigma}), \quad \gamma_{ab}(\sigma) \mapsto \frac{\partial \sigma^c}{\partial \tilde{\sigma}^a}\frac{\partial \sigma^d}{\partial \tilde{\sigma}^b}\gamma_{cd}(\sigma);$$
3. 2-dimensional **Weyl transformations**: for arbitrary $\omega(\sigma)$,
$$X^\mu(\sigma) \mapsto X^\mu(\sigma), \quad \gamma_{ab}(\sigma) \mapsto \exp(2\omega(\sigma))\gamma_{ab}(\sigma).$$

   The Nambu–Goto action is not invariant under Weyl transformations.

Exercise 1.3.1. Verify all these statements. (This should be quite straightforward.)

Definition 1.3.1. Let diff denote the group of diffeomorphisms acting on Σ, and Weyl the group of Weyl transformations acting on Σ; these are **internal symmetries**, while Poincaré transformations are **external symmetries**. The product diff $\times$ Weyl is the **gauge group**. The orbit, in the space of all possible fields and metrics, of a particular (X, γ) under the action of the gauge group is the **gauge orbit**.

A good exercise in working with the gauge and external symmetries is to make sure Polyakov action is as general as possible. This also reduces future work when we need the additional terms in the Polyakov action. Note that here, contrary to the case in QFT, the symmetries are very demanding. Weyl invariance in particular is very odd: it prevents us from adding terms such as

$$\int_M d^2\sigma \sqrt{-\gamma}\, V(X), \quad \mu \int_M d^2\sigma \sqrt{-\gamma}.$$

Exercise 1.3.2. Convince yourself that the action must contain one more γ^{ab} than γ_{ab} in order to satisfy Weyl invariance and counteract the change in $\sqrt{-\gamma}$. Since such a γ^{ab} can only pair up indices with derivatives, we need a second-order Lorentz-invariant term that is coordinate-independent. Convince yourself that other than $\partial_a X^\mu \partial_b X_\mu$, this term can only involve γ^{ab} and γ_{ab}, and that in fact it must be the **scalar curvature** R. Show that under a Weyl transformation,

$$\sqrt{-\gamma}\, R \mapsto \sqrt{-\gamma}(R - 2\nabla^2\omega).$$

Hence argue that we need another term integrated over ∂M to counteract $\nabla^2(\sqrt{-\gamma}\,\omega)$. Putting everything together, conclude that

$$\chi \coloneqq \frac{1}{4\pi}\int_M d^2\sigma \sqrt{-\gamma}\, R + \frac{1}{2\pi}\int_{\partial M} ds\, k$$

is Weyl invariant, and that it is essentially the only term we can add to the Polyakov action. Here ds is proper time along ∂M using the metric γ_{ab}, and $k \coloneqq \pm t^a n_b \nabla_a t^b$ is the **geodesic curvature** of the boundary, where t^a is a unit vector tangent to the boundary, and n_b an outward-pointing unit vector, and we choose $\pm$ depending on whether the boundary is timelike or spacelike.

Let's explore a few choices of gauge, some which use up all the gauge freedom, and some which do not. We commonly use reparametrization invariance to simplify expressions, so let's explore some choices of gauge using reparametrization invariance first.

Definition 1.3.2. We can reparametrize (σ^0, σ^1) such that σ^0 corresponds to the time coordinate x^0, i.e. $X^0 = R\sigma^0$ for some dimensionful constant R. This is **static gauge**, named as such because then lines of constant σ^0 correspond to the string at fixed moments in time, i.e. the string is static. Another choice is **light cone gauge**, given by $X^+ = R\sigma^0$, where

$$X^\pm := \frac{1}{\sqrt{2}}(X^0 \pm X^1), \quad \sigma^\pm := \frac{1}{\sqrt{2}}(\sigma^0 \pm \sigma^1)$$

are **light cone coordinates** on Minkowski space and the world sheet respectively. When in light cone gauge, the indices $i, j, \ldots$ range over $\{2, \ldots, D\}$.

Clearly neither static gauge nor light cone gauge exhausts the gauge freedom: we haven't done anything with the metric! But it is hard to transform the metric in a useful way while staying in static or light cone gauge. Let's take a different approach and try to transform the metric first.

The transformation of the scalar curvature computed in the exercise above says we can use Weyl invariance to locally set the scalar curvature to zero, by solving $2\nabla^2\omega = R$ and then applying the Weyl transformation $\exp(2\omega)$. But we are in two dimensions, where the symmetries of the Riemann curvature tensor determine it from R:

$$R_{abcd} = R_{cdab},\ R_{abcd} = -R_{bacd} = -R_{abdc} \implies R_{abcd} = (1/2)(\gamma_{ac}\gamma_{bd} - \gamma_{ad}\gamma_{bc})R.$$

Hence we can always locally get a flat metric, which, possibly after applying a coordinate transformation, gives $\gamma_{ab} = \eta_{ab}$, the flat Minkowski metric.

Definition 1.3.3. If we consider only reparametrization and not Weyl transformations, the metric γ_{ab} can always be brought to the form $\exp(2\omega)\eta_{ab}$. Forcing the metric to be of that form is known as **conformal gauge**. Performing the additional Weyl transformation to obtain $\gamma_{ab} = \eta_{ab}$ is known as **unit gauge**. In general, the form of the metric we choose to put γ_{ab} in is called the **fiducial metric**.

Exercise 1.3.3. (Important!) Show that in unit gauge, the equation of motion and its constraints become

$$\partial_a\partial^a\vec{X} = 0, \quad \partial_0\vec{X}\cdot\partial_1\vec{X} = 0, \quad (\partial_0\vec{X})^2 + (\partial_1\vec{X})^2 = R^2.$$

In this form, the constraints are called **Virasoro conditions**. Argue that by tensoriality, the Virasoro conditions still hold in static gauge, where $X^\mu = (R\sigma^0, \vec{X})$. Hence show in static gauge that at the (free) endpoints an open string, i.e. endpoints satisfying the Neumann boundary condition, $|\partial_t\vec{X}| = 1$. (**Be careful**: ∂_t is not ∂_0. What is ∂_t?) Conclude that string endpoints always move at the speed of light.

How many internal degrees of freedom have we used up if we put the metric γ_{ab} in unit gauge? Well, diff has two degrees of freedom, one for each coordinate, and Weyl has one, for the scale of the metric. But the metric itself has three independent components, being symmetric. Hence we expect to be done with choosing a representative of each gauge orbit.

But, perhaps unexpectedly, there is more gauge freedom: there are non-trivial transformations in diff $\times$ Weyl that preserve unit gauge! The key to finding these transformations is to realize that Σ is actually a **Riemann surface**: let $z := \sigma^0 + i\sigma^1$, so that $ds^2 = dzd\bar{z}$. Now if $f(z)$ is a holomorphic change of coordinates, then

$$z \mapsto f(z), \quad ds^2 \mapsto |\partial_z f|^{-2} dzd\bar{z},$$

so now applying the Weyl transformation $\exp(2\ln|\partial_z f|)$ recovers ds^2. Clearly the composition of the two transformations is non-trivial.

What went wrong? Well, just because dimensions match up does not mean we have spanned the whole space of gauge transformations! The holomorphic diffeomorphisms above actually have **measure zero** in diff. When we stop working locally and work globally instead, these extra bits of freedom are removed by boundary conditions.

Definition 1.3.4. When we successfully pick a unique and continuously-varying choice of representative in each gauge orbit, our theory is **gauge-fixed**. When such a choice is impossible due to topological obstructions, our theory has **Gribov ambiguity**. (For us, there is no Gribov ambiguity; we are just failing to consider boundary conditions.)

1.4 Quantization via Canonical Commutation Relations

When we did QFT, we started by **canonically quantizing** the Klein-Gordon and Dirac fields, which allowed us to immediately investigate some aspects of the quantized free theories, such as that Klein-Gordon fields represent bosons and Dirac fields represent fermions, and to obtain the spectrum and Hilbert space of states. On the other hand, **path integral quantization** gave us an easy way to compute interactions in perturbative QFT, such as scattering amplitudes. We do the same for string theory: first, in this section, we canonically quantize in order to write down the spectrum and Hilbert space of states, and then, in the next section, we quantize using the path integral to work with interactions.

In string theory, canonical quantization is called **covariant quantization**. This is just a difference in terminology. The procedure is the same: take the classical object (e.g. Lagrangian, Hamiltonian, solutions) you want to quantize, and impose **canonical commutation relations** modeled on $[x, p] = i$ on dynamical variables, by promoting them all to operators.

1.4.1 Classical Solutions

We take classical solutions and quantize them in light cone gauge as well as two more gauge-fixing conditions for the metric: set

$$X^+ = \sigma^0, \quad \partial_1 \gamma_{11} = 0, \quad \det \gamma_{ab} = -1.$$

Note that we have dispensed with the dimensionful constant R; it can be reinserted via dimensional analysis. The first thing to do right after picking a gauge is to rewrite all the relevant objects in that gauge. To do so, we need some formulas.

Exercise 1.4.1. Show that in this gauge, $\gamma_{11}(\sigma^0)$ depends only on σ^0, and we have

$$\begin{pmatrix} \gamma^{00} & \gamma^{01} \\ \gamma^{10} & \gamma^{11} \end{pmatrix} = \begin{pmatrix} -\gamma_{11}(\sigma^0) & \gamma_{01}(\sigma) \\ \gamma_{01}(\sigma) & \gamma_{11}^{-1}(\sigma^0)(1 - \gamma_{01}^2(\sigma)) \end{pmatrix}.$$

Furthermore, show that $\partial_a X^\mu \partial_a X_\mu = 2\partial_a X^+ \partial_a X^- - \partial_a X^i \partial_a X^i$. (Recall that indices $i, j, \ldots$ range over $\{2, \ldots, D\}$).

Definition 1.4.1. Given a dynamical variable $V(\sigma)$, define its associated **center of mass** (conceptually at a fixed time) variables

$$v(\sigma^0) = \frac{1}{\ell} \int_0^\ell d\sigma^1 \, V(\sigma), \quad \tilde{V}(\sigma) = V(\sigma) - v(\sigma^0),$$

i.e. we split $V = v + \tilde{V}$ where v is the mean value of V, and $\tilde{V}$ has mean zero.

For example, using that $\partial_1 X^+ = 0$, we have

$$\partial_1 \tilde{X}^- = \partial_1 X^- = \frac{1}{\sqrt{2}}(\partial_1 X^0 - \partial_1 X^1) = \sqrt{2} \partial_1 X^0,$$

and using that $\partial_0 X^+ = 1$, we have

$$\partial_0 X^0 \partial_1 X^0 - \partial_0 X^1 \partial_1 X^1 = (\partial_0 X^0 + \partial_0 X^1)\partial_1 X^0 - \partial_0 X^1(\partial_1 X^0 + \partial_1 X^1) = \sqrt{2}\partial_1 X^0 - 0.$$

Exercise 1.4.2. Using all these calculations, show that the Polyakov Lagrangian in this gauge is

$$L = -\frac{T_0}{2}\int_0^\ell d\sigma^1 \left[\gamma_{11}(2\partial_0 X^- - \partial_0 X^i \partial_0 X^i) - 2\gamma_{01}(\partial_1 \tilde{X}^- - \partial_0 X^i \partial_1 X^i) + \gamma_{11}^{-1}(1-\gamma_{01}^2)\partial_1 X^i \partial_1 X^i\right].$$

Argue that because $\tilde{X}^-$ does not appear with time derivatives, it is not a dynamical variable, and therefore when we vary S_{P} with respect to γ, it constrains $\partial_1\gamma_{01}$ to be zero. Show that the Neumann boundary condition, in this gauge, gives $\gamma_{01} = 0$ at the endpoints $\sigma = 0, \ell$, and conclude that $\gamma_{01} = 0$ everywhere. Therefore write down the simplified **Lagrangian**:

$$L = -T_0 \ell \gamma_{11}\partial_0 x^- + \frac{T_0}{2}\int_0^\ell d\sigma^1 \, \left(\gamma_{11}\partial_0 X^i \partial_0 X^i - \gamma_{11}^{-1}\partial_1 X^i \partial_1 X^i\right),$$

The next step is to write down the Hamiltonian, which is the **Legendre transform** of the Lagrangian. Recall that this means we write down momenta Π_μ corresponding to X^μ, and then define

$$H \coloneqq \int_0^\ell \Pi_\mu \partial_0 X^\mu - L = \int_0^\ell d\sigma^1 \, \left(\Pi_+ \partial_0 X^+ + \Pi_- \partial_0 X^- + \Pi_i \partial_0 X^i\right) - L = p_- \partial_0 x^- + \int_0^\ell \Pi_i \partial_0 X^i - L,$$

where p_- is the momentum conjugate to x^-, and Π^i is the momentum density conjugate to X^i:

$$p_- \coloneqq \frac{\partial L}{\partial \partial_0 x^-} = -T_0 \ell \gamma_{11}, \quad \Pi^i \coloneqq \frac{\delta L}{\delta \partial_0 X^i} = T_0 \gamma_{11} \partial_0 X^i = \frac{p^+}{\ell}\partial_0 X^i.$$

Note that $p_- = -p^+$. Simplifying, we get the Hamiltonian

$$H = \frac{\ell T_0}{2p^+}\int_0^\ell d\sigma^1 \, \left(\frac{1}{T_0}\Pi^i \Pi^i + T_0 \partial_1 X^i \partial_1 X^i\right),$$

which is precisely the **Hamiltonian** for $D-2$ free fields X^i, with $p^+ \propto \gamma_{11}$ a conserved quantity.

We can also directly write down **classical solutions**: the equation of motion in this gauge is $\partial_+\partial_- X^i = 0$, which has the general solution

$$X^i(\sigma) = X_L^i(\sigma^+) + X_R^i(\sigma^-)$$

for arbitrary functions X_L^i and X_R^i, describing **left-moving** and **right-moving** waves respectively, which we can expand as Fourier series: (**CHECK** Why are there $\sigma^\pm$ terms?)

$$X_L^i(\sigma^+) = \frac{1}{2}x^i(0) + \frac{1}{2T_0}p^i(0)\sigma^+ + i\frac{\ell}{\pi}\sqrt{\frac{1}{2T_0}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^i e^{-in\pi\sigma^+/\ell},$$
$$X_R^i(\sigma^-) = \frac{1}{2}x^i(0) + \frac{1}{2T_0}p^i(0)\sigma^- + i\frac{\ell}{\pi}\sqrt{\frac{1}{2T_0}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^i e^{-in\pi\sigma^-/\ell}.$$

(Here p^i are center of mass variables for Π^i with an extra factor of ℓ. We've also mucked around with the normalization factors for Fourier coefficients for later convenience.) Because the X^i are real fields, we have the **constraints** $\tilde{\alpha}_n^i = (\tilde{\alpha}_{-n}^i)^*$ and $\alpha_n^i = (\alpha_{-n}^i)^\dagger$ on the Fourier coefficients.

Exercise 1.4.3. Show that the Neumann boundary condition forces $\tilde{\alpha}_n^i = \alpha_n^i$, so that the general form of a **classical solution for an open string** is

$$X^i(\sigma) = x^i + \frac{1}{2\pi T_0}p^i\sigma^0 + i\frac{\ell}{\pi}\sqrt{\frac{1}{2T_0}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^i e^{-in\pi\sigma^0/\ell}\cos\frac{n\pi\sigma^1}{\ell}.$$

Finally, we must write down the **constraints**, i.e. the Virasoro conditions in this gauge. They become $(\partial_+ X)^2 = (\partial_- X)^2 = 0$, which give conditions on the momenta p^i and Fourier coefficients α_n^i. Both ∂_+ and ∂_- give the same result, so we compute

$$\partial_+ X^i = \partial_+ X_L^i = \frac{1}{2T_0} p^i(0) + \sqrt{\frac{1}{2T_0}} \sum_{n \neq 0} \alpha_n^i e^{-in\pi\sigma^+/\ell}.$$

Hence, writing $\alpha_0^i \coloneqq \sqrt{1/2T_0}\, p^i(0)$,

$$0 = (\partial_+ X)^2 = \frac{1}{T_0} \sum_n L_n e^{-i\pi n\sigma^+/\ell}, \quad L_n \coloneqq \frac{1}{2} \sum_{m,n} \alpha_m \cdot \alpha_{n-m}.$$

So the L_n are the Fourier coefficients of the constraints. By the linear independence of the Fourier basis, $L_n = 0$ for all $n \in \mathbb{Z}$. In particular, since $p_\mu p^\mu = -M^2$ is the effective mass and L_0 contains the momentum, $L_0 = 0$ implies that the **effective mass** of the string is

$$M^2 = -p \cdot p = 4T_0 \sum_{m>0} \alpha_m \cdot \alpha_{-m}.$$

1.4.2 Covariant Quantization

Quantization is now trivial: we impose the canonical **equal-time commutation relations**

$$[x^-, p^+] = i\eta^{-+} = -i, \quad [X^i(\sigma^0, \sigma^1), \Pi^j(\sigma^0, \sigma'^1)] = i\delta^{ij}\delta(\sigma - \sigma'),$$

with all other commutators vanishing. In terms of Fourier components,

$$[x^-, p^+] = -i, \quad [x^i, p^j] = i\delta^\mu{}_\nu, \quad [\alpha_m^i, \alpha_n^j] = m\delta^{ij}\delta_{m+n,0},$$

with all other commutators vanishing. So as in QFT, we can treat α_n^i as creation/annihilation operators (α is annihilation, $\alpha^\dagger$ is creation), and build up our state space using them. Note that instead of just a single creation/annihilation operator, we have an infinite tower of them!

Definition 1.4.2. The **creation/raising operators** are α_{-m}^i and the **annihilation/lowering operators** are α_n^i. The **ground state of a string with momentum** k is defined as the eigenstate $|0;k\rangle$ of p^i, the center of mass momenta, annihilated by the annihilation operators, i.e.

$$p^+ |0;k\rangle = k^+ |0;k\rangle, \quad p^i |0;k\rangle = k^i |0;k\rangle, \quad \alpha_m^i |0;k\rangle = 0 \quad \forall m > 0.$$

Note that the zero-momentum ground state $|0;0\rangle$ of a string is not the true **vacuum state**, which consists of no strings at all; we denote the true vacuum state $|\text{vacuum}\rangle$.

Unlike QFT, each raising operator α_{-m}^i (for varying m) creates a different mode. So the **independent states** are labeled using center of mass momenta $k = (k^+, k^i)$, and occupation numbers $N_{i,n}$ for $i = 2, \dots, D$ and $n = 1, 2, \dots$:

$$|N;k\rangle \coloneqq \left(\prod_{i=2}^{D} \prod_{n=1}^{\infty} \frac{(\alpha_{-n}^i)^{N_{i,n}}}{\sqrt{n^{N_{i,n}} N_{i,n}!}} \right) |0;k\rangle.$$

(The normalization is chosen for convenience.) Hence there are an infinite number of different first excitations of a single string. Let $\mathcal{H}_1$ denote the space of all possible single-string states:

$$\mathcal{H}_1 \coloneqq \text{span}\{|N;k\rangle : \text{all possible } N, k\}.$$

Definition 1.4.3. The **state space**, of any number of strings, is a **bosonic Fock space**

$$\mathrm{Sym}(\mathcal{H}_1) \coloneqq |\text{vacuum}\rangle \oplus \mathcal{H}_1 \oplus (\mathcal{H}_1 \odot \mathcal{H}_1) \oplus (\mathcal{H}_1 \odot \mathcal{H}_1 \odot \mathcal{H}_1) \oplus \cdots \oplus \cdots,$$

where $\odot$ is the symmetrized tensor product

$$v_1 \odot \cdots \odot v_n \coloneqq \frac{1}{n!} \sum_{\sigma \in S_k} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}.$$

The n-th term in the sum $\mathrm{Sym}(\mathcal{H}_1)$ is the state space of n strings. We symmetrize because it turns out the strings we are working are bosonic, i.e. they have integer spin, i.e. they commute, instead of anticommuting. ($\mathrm{Sym}(\mathcal{H}_1)$ is known by us mathematicians as a **symmetric algebra**; a fermionic Fock space, for objects with half-integer spins, is an exterior algebra.)

We still need to impose the constraints $L_n = 0$, coming from the Virasoro conditions. Naively one might just insist that as operators, $L_n = 0$, but this quickly runs into problems (cf. Gupta–Bleuler quantization of QED). Instead, we impose $L_n |\text{phys}\rangle = 0$ for any physical state $|\text{phys}\rangle$.

1.4.3 Spectrum and Critical Dimension

By mass-energy equivalence, to find the spectrum of our quantized string is equivalent to finding its effective mass, i.e. we must look at the quantized version of $M^2 = 4T_0 \sum_{m>0} |\alpha_m|^2$. But when we quantize, α_m and α_{-m} no longer commute, so there is an operator ordering ambiguity here. There are two choices: either we quantize $\alpha_m \cdot \alpha_{-m}$, or we quantize $\alpha_{-m} \cdot \alpha_m$. They both give

$$M^2 = 4T_0 \sum_{m>0} (N_m + a), \quad N_m \coloneqq \alpha_{-m} \cdot \alpha_m$$

(where by analogy with the harmonic oscillator, we've defined the **number operators** N_m), but the first with $a = m(D-2)/2$, using the commutation relation $[a^i_m, a^i_{-m}] = m$, and the second with $a = 0$. There are some physical arguments for why we pick the former during the quantization of the simple harmonic oscillator (Heisenberg uncertainty principle, etc.), but it boils down to the assertion that we want the ground state of the system to have non-zero energy. Hence we pick $a = m(D-2)/2$.

Exercise 1.4.4. Recall/review from QFT that $\sum_{m>0} m = -1/12$, and therefore conclude that the ground state and first excited states, i.e. $\alpha^i_{-m} |0; k\rangle$ for any m, have energies

$$M_0^2 = 4T_0 \frac{2-D}{24}, \quad M_1^2 = 4T_0 \frac{26-D}{24}.$$

Fix an m. The first excited state $\alpha^i_{-m} |0; k\rangle$ acts as a vector because it has a vector index i, so it better be Lorentz invariant. In particular, in the rest frame, the (spatial rotation subgroup of the) Lorentz group can act on a vector to make it point in any spatial direction, so vectors better have $D-1$ states. But α^i_{-m} lives in the standard representation of $\mathrm{SO}(D-2)$: it only has $D-2$ states contained in it. This is not good!

Here is the solution: we posit that $M_1^2 = 0$. Then there is no rest frame! Consequently, we are only free to rotate around the direction of motion, giving only $D-2$ states, exactly the number that we have. But this implies $D = 26$, known as the **critical dimension** of bosonic string theory. This entire argument is sketchy, and we (hopefully) give a more rigorous argument later that $D = 26$ is the only dimension that works, based on enforcing Weyl invariance.

There is another problem: $M_0^2 < 0$ for $D > 2$, especially for $D = 26$. We have **negative energy states**, known as **tachyons**! This is explained from a field-theoretic perspective: given a field ϕ, its mass squared is just $\partial^2 V(\phi)/\partial\phi^2|_{\phi=0}$. We are actually expanding around a critical point of the potential that is a maximum, i.e. an **unstable** point, therefore resulting in a negative mass-squared. Currently it is unknown whether there are stable points in the purely bosonic theory. However, with the addition of fermions and **supersymmetry**, giving the **superstring**, the problem disappears. This is content for much later on.

1.5 Quantization via Path Integral

Now it is time to develop a different tool. Recall from QFT that we have a giant machine for quantizing classical theories and studying their interactive pictures: the path integral. However, before we begin plugging the Polyakov action into the machine, we need to make a modification. From now on, the world sheet is equipped with a **Euclidean metric** g_{ab}, instead of a Lorentzian one γ_{ab}. This is so that the path integral over metrics is better defined. The transition from Euclidean to Minkowski is, formally, done via **Wick rotation**: $x^0 \mapsto ix^0$ and similarly for the metric. The **Euclidean path integral**, and the Euclidean action (with the additional terms on top of the Wick-rotated Polyakov action), is therefore

$$Z \coloneqq \frac{1}{\mathrm{Vol}} \int \mathcal{D}g\, \mathcal{D}X\, \exp(-S_{\mathrm{P}}[X, g]),$$

$$S_{\mathrm{P}}[X, g] = \frac{T_0}{2} \int_M d^2\sigma\, \sqrt{g}\, g^{ab} \partial_a X^\mu \partial_b X_\mu + \lambda \left(\frac{1}{4\pi} \int_M d^2\sigma\, \sqrt{g}\, R + \frac{1}{2\pi} \int_{\partial M} ds\, k \right)$$

where Vol is the volume of the gauge action on the **configuration space** consisting of all possible X^μ and g). More explicitly, we can imagine partitioning configuration space into gauge orbits; we actually want to integrate on a path through these gauge orbits. But now recall from QFT that we have another giant machine for doing so: the Faddeev-Popov method.

1.5.1 The Faddeev-Popov Method

Let's first recall that the idea behind Faddeev-Popov is very natural: we want to do a change of coordinates in configuration space so that instead of integrating over a mish-mash of g and X, we integrate such that one variable goes along gauge orbits, and the other goes along the gauge-fixed path. Although this sounds technical, we perform procedures like this quite often without realizing it! For example, consider the calculation

$$\iint dx\, dy\, e^{-x^2-y^2} = \int d\theta \int dr\, re^{-r^2} = 2\pi \int dr\, re^{-r^2} = \pi.$$

What is really happening here is that we recognized the $U(1)$ symmetry of the original integrand, and changed variables in order to factor out that symmetry. Instead of integrating over (x, y), we integrated over (r, θ), with θ parametrizing the gauge orbits. Furthermore, we picked out the $y = 0$ representative of each gauge orbit for the remaining integral.

Armed with this motivation, we can proceed. Let $\hat{g}_{ab}$ be the fiducial metric; it represents our choice of gauge fixing, just like the choice $y = 0$. Let ζ be shorthand for a combined coordinate and Weyl transformation:

$$\zeta \colon g_{ab} \mapsto g^\zeta_{ab} \coloneqq \exp(2\omega(\sigma)) \frac{\partial\sigma^c}{\partial\sigma'^a} \frac{\partial\sigma^d}{\partial\sigma'^b} g_{cd}(\sigma).$$

Definition 1.5.1. Let $\mathcal{D}\zeta$ be a gauge invariant measure on diff $\times$ Weyl. (Whether such a measure exists is very relevant for us, but we disregard it for now.) Define the **Faddeev-Popov determinant** Δ_{FP} by

$$\Delta^{-1}_{\mathrm{FP}}(g) \coloneqq \int \mathcal{D}\zeta\, \delta[\hat{g}^\zeta - g].$$

Here the δ is the **Dirac functional**, i.e. $\hat{g}^\zeta$ and g must agree at every point σ.

Exercise 1.5.1. Show that $\Delta_{\mathrm{FP}}(g)$ is gauge-invariant by computing that $\Delta_{\mathrm{FP}}(g^\zeta)^{-1} = \Delta_{\mathrm{FP}}(g)^{-1}$.

Now it is time to do the calculation to factor out the integral over the gauge orbits. The first step is to add a 1 to the integral:

$$Z = \int \frac{\mathcal{D}g\, \mathcal{D}X}{\mathrm{Vol}} \exp(-S_{\mathrm{P}}[X, g]) = \int \frac{\mathcal{D}g\, \mathcal{D}X\, \mathcal{D}\zeta}{\mathrm{Vol}} \Delta_{\mathrm{FP}}(g) \delta[\hat{g}^\zeta - g] \exp(-S_{\mathrm{P}}[X, g]).$$

The second step is to do the integral over g, which, due to the $\delta[\hat{g}^\zeta - g]$, amounts to replacing g with $\hat{g}^\zeta$:

$$Z = \int \frac{\mathcal{D}X\,\mathcal{D}\zeta}{\mathrm{Vol}} \Delta_{\mathrm{FP}}(\hat{g}^\zeta) \exp(-S_{\mathrm{P}}[X, \hat{g}^\zeta]).$$

Finally, since both Δ_{FP} and S_{P} are gauge-invariant, we can replace $\hat{g}$ with $\hat{g}^\zeta$. Then nothing in the integrand depends on ζ anymore, so it factors out and cancels the volume normalization:

$$Z = \int \frac{\mathcal{D}\zeta}{\mathrm{Vol}} \int \mathcal{D}X\, \Delta_{\mathrm{FP}}(\hat{g}) \exp(-S_{\mathrm{P}}[X, \hat{g}]) = \int \mathcal{D}X\, \Delta_{\mathrm{FP}}(\hat{g}) \exp(-S_{\mathrm{P}}[X, \hat{g}]).$$

Exercise 1.5.2. Evaluate $\iint dx\, dy\, e^{-x^2-y^2}$ by applying the Faddeev-Popov method to its $U(1)$ symmetry and the gauge-fixing condition $y = 0$. Conclude that the Faddeev-Popov method is completely rigorous in finite dimensions, and that Δ_{FP} is actually a Jacobian (hence the name Faddeev-Popov determinant).

1.5.2 Computing the Faddeev-Popov Determinant

It remains to compute the Faddeev-Popov determinant Δ_{FP} for the diff $\times$ Weyl action on world sheet metrics. To do so, we make the simplifying assumption that diff $\times$ Weyl actually acts freely on metrics g, i.e. for each g, there is exactly one ζ such that $\delta[\hat{g}^\zeta - g] = 0$. Obviously this assumption is false: we showed earlier that the action has fixed points (albeit a measure zero set of them). But it is true locally, so we deal with the global issues later. The reason we make this assumption is so that we can compute $\Delta_{\mathrm{FP}}(\hat{g})^{-1}$ by integrating only around a small neighborhood of $\zeta = 0$. In this neighborhood, we can take infinitesimal Weyl transformations $\omega(\sigma)$ and infinitesimal diffeomorphisms $\delta\sigma^\alpha = v^\alpha(\sigma)$, and write

$$\Delta_{\mathrm{FP}}^{-1}(\hat{g}) = \int \mathcal{D}\omega\, \mathcal{D}v\, \delta[2\omega \hat{g}_{ab} + \nabla_a v_b + \nabla_b v_a].$$

Note that now we are integrating over the Lie algebra of diff $\times$ Weyl. We want to get rid of the delta functional.

Exercise 1.5.3. For a function $\phi\colon \mathbb{R}^D \to \mathbb{R}$, derive the integral form

$$\delta[\phi] = \int_{j\colon \mathbb{R}^D \to \mathbb{R}} \mathcal{D}j(x)\, \exp\left(2\pi i \int d^D x\, j(x)\phi(x)\right)$$

by applying the one-dimensional identity $\delta(x) = \int dp\, \exp(2\pi i p x)$ to piecewise linear paths, and then taking the limit as the number of path segments goes to infinity.

In our case, the function inside the delta functional lives on the world sheet Σ, whose integration measure is $d^2\sigma\, \sqrt{\hat{g}}$ (remember we fixed the fiducial metric). Hence, if β ranges over symmetric 2-tensors on Σ, then

$$\Delta_{\mathrm{FP}}^{-1}(\hat{g}) = \int \mathcal{D}\omega\, \mathcal{D}v\, \mathcal{D}\beta\, \exp\left(2\pi i \int d^2\sigma\, \sqrt{\hat{g}}\, \beta^{ab}(2\omega \hat{g}_{ab} + \nabla_a v_b + \nabla_b v_a)\right).$$

But we can directly do the integral over ω. The one and only term containing an ω factors out to give a delta functional:

$$\int \mathcal{D}\omega\, \exp\left(2\pi i \int d^2\sigma\, \sqrt{\hat{g}}\, \beta^{ab}(2\omega \hat{g}_{ab})\right) = \delta[2\beta^{ab}\hat{g}_{ab}],$$

i.e. in the remaining integral, β^{ab} is traceless:

$$\Delta_{\mathrm{FP}}^{-1}(\hat{g}) = \int \mathcal{D}v\, \mathcal{D}\beta\, \exp\left(2\pi i \int d^2\sigma\, \sqrt{\hat{g}}\, \beta^{ab}(\nabla_a v_b + \nabla_b v_a)\right).$$

Recap: we are integrating over vector fields v and symmetric 2-tensors β such that β^{ab} is traceless, both living on Σ.

1.5.3 Faddeev-Popov Ghosts

We are not done: the path integral above is for $\Delta_{\mathrm{FP}}^{-1}$, but we want Δ_{FP} itself. There is a general procedure for inverting $\Delta_{\mathrm{FP}}^{-1}$. To understand it, we must first clarify what Δ_{FP} really is. Let F is the gauge-fixing condition. (For us, F is a function of g and ζ and takes values in symmetric 2-tensors.) Note that via a change of variables from ζ to F,

$$\Delta_{\mathrm{FP}}^{-1} = \int D\zeta\, \delta(F) = \int DF \,\det\left[\frac{\delta\zeta}{\delta F}\right] \delta(F) = \det\left[\frac{\delta\zeta}{\delta F}\right]_{F=0}.$$

This change of variables is valid again because we assume ζ acts freely on gauge orbits, and F is supposed to pick a unique representative from each gauge orbit, so ζ and F "have the same number of degrees of freedom" as physicists like to say. Now all we have to do is invert the determinant. For this, we use a clever trick, which is developed in the following two exercises.

Exercise 1.5.4. Show by analogy from the finite dimensional case for two real fields ϕ^1 and ϕ^2 that

$$\int \mathcal{D}\phi^1\, \mathcal{D}\phi^2 \exp\left(i \int d^dx\, \phi^1 A \phi^2\right) = (\det A)^{-1}.$$

Exercise 1.5.5. Recall from QFT that we defined **Grassmann numbers**: they are anti-commuting formal variables, i.e. $\theta\eta = -\eta\theta$, that form an algebra. We also worked out the **Berezin integral** for Grassmann-valued quantities, with the convention that $\int d\theta \int d\eta\, \eta\theta = 1$. If θ and η are Grassmann variables, i.e. taking values in the Grassmann algebra, and $b \in \mathbb{R}$, review/show (in order) that

$$\theta^2 = 0, \qquad \int d\theta\, f(\theta) = \frac{\partial f}{\partial\theta}, \qquad \int d\theta\, d\eta\, \exp(-\theta b\eta) = b$$

Hence show by analogy with the finite dimensional case that for Grassmann-valued fields χ^1 and χ^2,

$$\int \mathcal{D}\chi^1\, \mathcal{D}\chi^2 \exp\left(-\int d^dx\, \chi^1 A \chi^2\right) = \det A.$$

So here's the trick: if we have a path integral expression for $(\det A)^{-1}$, to get $\det A$ we simply replace ordinary variables with Grassmann variables! In particular, to get $\Delta_{\mathrm{FP}}(\hat{g})$ from $\Delta_{\mathrm{FP}}(\hat{g})^{-1}$, we replace (β_{ab}, v^a) with Grassmann-valued fields (b_{ab}, c^a), with b^{ab}, like β^{ab}, being traceless:

$$\Delta_{\mathrm{FP}}(\hat{g}) = \int \mathcal{D}b\, \mathcal{D}c\, \exp(S_{\mathrm{G}}), \quad S_{\mathrm{G}} \coloneqq \frac{1}{2\pi} \int d^2\sigma \sqrt{\hat{g}}\, b_{ab} \nabla^a c^b.$$

Note that we've implicitly made a few cosmetic changes:

1. Because b is a symmetric 2-tensor (do **not** confuse the fact that b is symmetric, i.e. $b_{ab} = b_{ba}$, with b being anti-commutative, e.g. $b_{ab}\theta = -\theta b_{ab}$), we can rewrite

$$b^{ab}(\nabla_a c_b + \nabla_b c_a) = b^{ab}\nabla_a c_b + b^{ab}\nabla_a c_b = 2b^{ab}\nabla_a c_b = 2b_{ab}\nabla^a c^b.$$

2. We chose slightly different normalization factors to make later computations cleaner.

The quantity S_{G} is called the **ghost action**: when we plug $\Delta_{\mathrm{FP}}(\hat{g})$ back into the path integral, we get

$$Z = \int \mathcal{D}X\, \mathcal{D}b\, \mathcal{D}c\, \exp(-S_{\mathrm{P}}[X, \hat{g}] - S_{\mathrm{G}}[b, c]),$$

i.e. S_{G} becomes part of the action. The fields b and c, which do not correspond physically to anything, are **Faddeev-Popov ghost fields**. The price of gauge fixing is the introduction of these unphysical ghosts.

Exercise 1.5.6. Repeat the computation of Δ_{FP} for QED, and show that for QED, Δ_{FP} is independent of any fields. Hence conclude that QED has no Faddeev-Popov ghosts. (That's why quantizing QED went a lot faster. QCD has ghosts, however.)

Chapter 2

Conformal Field Theory

String theory as we have defined it so far is a 2 dimensional theory where the fields are parameterized by two coordinates (σ^1, σ^2). We shall now explore the conformal symmetry of the Polyakov action and deduce a number of important technical tools that will enable us to say a lot about the properties of this quantum field theory. This conformal symmetry is especially large in two dimensions and provides significant constraints. [1]

The technical tool that will drive this whole chapter is the **operator product expansion** (OPE). This is a canonical form for the product of two local operators:

$$\mathcal{A}_i(\sigma_1)\mathcal{A}_j(\sigma_2) = \sum_k c_{ij}^k(\sigma_1 - \sigma_2)\mathcal{A}_k(\sigma_2). \tag{2.1}$$

This will turn out to be much like a Laurent expansion and the form of $c_{ij}^k(\sigma_1 - \sigma_2)$ is severely restricted.

There are many reasons why it is useful for us to learn about CFT. Certain critical phase transitions can be described by a CFT and using the AdS/CFT correspondence we may be able to take a highly correlated system and rewrite it in terms of a weakly coupled theory of supergravity. Let's begin!

The plan for this chapter as of January 1, 2016 will be to showcase important details of chapter 2 from Polchinski's Volume 1 leaving out some technical details for as exercises. In the future it would be nice to include d-dimensional CFT and it's application to condensed matter systems.

2.1 Conformal Normal Order and Operator Product Expansions

In our QFT adventures we focused on computing correlation functions since every physical quantity could be expressed in terms of them. However, in our journey we focused a lot on operators of the form $\langle\phi_1\phi_2\cdots\phi_n\rangle$. We shall now generalize this ever so slightly.

Definition 2.1.1. Let σ_0 be a fixed point and consider a classical world-sheet field theory with fields $X_1(\sigma), \ldots, X_n(\sigma)$. A **local functional** is a function $\mathscr{F}[X]$ taking in, as arguments, $X_i(\sigma_0)$ and $\partial_a X_i(\sigma_0)$ which are taken at σ_0. A **local operator** is the quantized version of a local functional that has well defined expectation values. Often a local operator is given by the normal ordering of a local functional.

[1]References that were used for the preparation of this chapter include Polchinski's Vol 1, Polchinski's Little Book, and Ginsparg's Applied CFT arXiv:hep-th/9108028, IAS Vol 1 and 2, Gomis' PSI 14/15 Lectures on CFT, Green Schwarz Witten Vol 1.

Here are some examples of local operators $X^\mu(0,0), X^\mu(a,b)X_\mu(a,b), \partial_{\bar z}X^3(z_1,\bar z_1)$. However, $X^\mu(z_1,\bar z_1)+X^\mu(z_2,\bar z_2)$ is not a local operator. Before introducing the operator product expansion we need to introduce conformal normal ordering which will be used in the definition of OPE.

Definition 2.1.2. Write $z_{12}=z_1-z_2$. Let $\mathscr{F}$ be an arbitrary function of X. Define the (free-field) **normal order** of $\mathscr{F}$ to be the functional:

$$:\mathscr{F}: = \mathscr{F} + \sum(\text{subtractions}) \tag{2.2}$$

$$= \exp\left(\frac{\alpha'}{2}\int d^2z_1d^2z_2 \ln|z_{12}|^2 \frac{\delta}{\delta X^\mu(z_1,\bar z_1)}\frac{\delta}{\delta X_\mu(z_2,\bar z_2)}\right)\mathscr{F} \tag{2.3}$$

This is very analogous to the normal ordering that we saw in QFT where the coefficient $\eta^{\mu\nu}\ln|z_{12}|^2$ is replaced by the corresponding propogator $\Delta(z_1,z_2)$. The QFT version of normal ordering is useful for calculating matrix elements, while this version is useful for computing the OPE.

Here are two examples: $:X^\mu(z,\bar z): = X^\mu(z,\bar z)$, and

$$:X^\mu(z_1,\bar z_2)X^\nu(z_2,\bar z_2): = X^\mu(z_1,\bar z_2)X^\nu(z_2,\bar z_2) + \frac{\alpha'}{2}\eta^{\mu\nu}\ln|z_{12}|^2. \tag{2.4}$$

The reason why this ordering is useful is because of the following:

Conjecture 2.1.3 (Fundamental Property of Normal Ordering)**.** *Normal ordered expressions satisfy the classical equations of motion averaged over paths. In the case of classical bosonic field this amounts to saying:*

$$\langle\partial\bar\partial :\mathscr{F}:\rangle = 0.$$

This is true for instance in the case of $\langle\partial\bar\partial :X^\mu(z_1,\bar z_1)X^\nu(z_2,\bar z_2):\rangle = 0$ (Polchinski Vol 1 Page 36).

Proposition 2.1.4. *Let $\mathscr{F},\mathscr{G}$ be two local operators. Then,*

$$:\mathscr{F}::\mathscr{G}: = :\mathscr{F}\mathscr{G}: + \sum(\textit{cross-contractions}) \tag{2.5}$$

$$= \exp\left(-\frac{\alpha'}{2}\int d^2z_1d^2z_2\ln|z_{12}|^2\frac{\delta}{\delta^\mu_F(z_1,\bar z_1)}\frac{\delta}{\delta_{\mu G(z_2,\bar z_2)}}\right):\mathscr{F}G: \tag{2.6}$$

Remark. The operator product expansion, as given by (2.1), is our definition of the OPE. However, it turns out, just like in complex analysis, that the singular part of this expansion is the one that plays the most crucial role. What does the singular part of the OPE do for us? It turns out that it gives us a way to compute the variation of local operators under conformal transformations. Here's how. Using the Ward identity in $d=2$ we relate $\delta\mathscr{A}(z_0,\bar z_0)$ with the residue of $j(z)\mathscr{A}(z_0,\bar z_0)$ at z_0. Rewrite j in terms of the energy-momentum tensor and use the singular part of the OPE of $T\mathscr{A}$ to calculate the residue.

Now we describe how to compute the singular part of the OPE in free field theory. In particular, this means that the following derivation only works for the bosonic non-interacting string. We will revise our method later, if need be. The key observation is that harmonic functions can locally be written as a sum of a holomorphic and antiholomorphic part.

Lemma 2.1.5. *Let $f:\mathbb{C}\to\mathbb{C}$ be harmonic. Then $\partial\bar\partial f=0$ and so $f=a(z)+b(\bar z)$ where a is holomorphic and b is antiholomorphic.*

Using Prop. 2.1.4 we notice that, since $:\mathscr{F}:$ is non-singular, the singular part in the OPE of $:\mathscr{F}::\mathscr{G}:$ is given by the coefficient functions in the cross-contractions.

Example 2.1.6. Using the definition of normal ordering (2.2), we may write (cf. (2.4))

$$
\begin{aligned}
X^\mu(z_1,\bar z_1)X^\nu(z_2,\bar z_2) &= -\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z_{12}|^2 + \sum_{k=1}^{\infty}\frac{1}{k!}(z_{12})^k \, {:}X^\nu\partial^k X^\mu(z_2,\bar z_2){:} \, + (\bar z_{12})^k \, {:}X^\nu\bar\partial^k X^\mu(z_2,\bar z_2){:} \\
&\sim -\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z_{12}|^2
\end{aligned}
$$

The first equation is the full operator product expansion and the equivalence (up to singular terms) shows that $X^\mu X^\nu$ behaves like $\ln|z_{12}|^2$ for $z_1 \to z_2$.

Example 2.1.7. Let's suppose we have a product of two composite operators. $\mathscr{F}(z) = \partial X^\mu(z)\partial X_\mu(z)$ and $\mathscr{G}(z') = \partial' X^\nu(z')\partial' X_\nu(z')$. Using the harmonicity of normal ordering we obtain:

$$
\begin{aligned}
{:}\mathscr{F}(z){:}{:}\mathscr{G}{:} &= {:}\mathscr{F}(z)\mathscr{G}(z'){:} \, -4\frac{\alpha'}{2}(\partial\partial'\ln|z-z'|^2) \, {:}\partial X^\mu \partial' X_\mu(z'){:} \, +2\eta^\mu_\mu\left(-\frac{\alpha'}{2}\partial\partial'\ln|z-z'|^2\right)^2 \\
&\sim \frac{D\alpha'^2}{2(z-z')^4} - \frac{2\alpha'}{(z-z')^2} \, {:}\partial X^\mu\partial' X_\mu(z'){:} \, - \frac{2\alpha'}{z-z'} \, {:}\partial X^\mu \partial' X_\mu(z'){:}
\end{aligned}
$$

In general CFTs we require the basis in which we expand operator products to transform like a tensor under conformal transformations. Moreover, the conformal invariance then puts even more restrictions on the coefficient functions rendering them unique up to a constant.

2.1.1 Ward Identity

Although the idea of the OPE is what drives this chapter, the Ward identity is the oil that makes the engine turn. Suppose we are given a coordinate transformation $\sigma' = \sigma + \delta\sigma$, that is a symmetry of the theory, how do operators transform under this transformation? Denote the transformation of fields as follows: $X'_\mu(\sigma) = X_\mu(\sigma) + \delta X_\mu(\sigma)$. Now we consider a slightly more general transformation:

$$X'_\mu(\sigma) = X_\mu(\sigma) + \rho(\sigma)\delta X_\mu(\sigma).$$

Such a general transformation might not be a symmetry of the action. However, the path integral *is* invariant under change of coordinates, which means:

$$
\begin{aligned}
0 = \delta\left(\int \mathcal{D}X\, e^{-S[X]}\mathscr{A}(\sigma_0)\right) &= \int \mathcal{D}X \delta(e^{-S[X]})\mathscr{A}(\sigma_0) + e^{-S[X]}\delta\mathscr{A}(\sigma_0) \\
&= \int \mathcal{D}X\, (d^d\sigma\sqrt{g})e^{-S[X]}j^a(\sigma)\partial_a\rho(\sigma)\mathscr{A}(\sigma_0) + e^{-S[X]}\delta\mathscr{A}(\sigma_0)
\end{aligned}
$$

Applying Stoke's theorem:

$$
\begin{aligned}
\langle\delta\mathscr{A}(\sigma_0)\rangle &= \frac{i\epsilon}{2\pi}\int d^d\sigma\sqrt{g}\,\langle\partial_a j^a(\sigma)\rangle \\
&= \frac{i\epsilon}{2\pi}\langle\oint_{\partial R} dA\, n^a j_a(\sigma)\mathscr{A}(\sigma_0)\rangle
\end{aligned}
$$

In operator form and in $d = 2$ this looks like

$$\frac{2\pi}{\epsilon}\delta\mathscr{A}(\sigma_0) = \oint_{\partial R}(j_z dz - j_{\bar z} d\bar z)\mathscr{A}(z_0,\bar z_0)$$

In the case that j_z and $j_{\bar z}$ are (anti)holomorphic then we have the following relation:

$$\mathrm{Res}_{z\to z_0}\, j(z)\mathscr{A}(z_0,\bar z_0) + \overline{\mathrm{Res}}_{\bar z\to\bar z_0}\tilde{j}(\bar z)\mathscr{A}(z_0,\bar z_0) = \frac{1}{i\epsilon}\delta\mathscr{A}(z_0,\bar z_0).$$

2.1.2 Applications of OPE

Let us show that the X^μ-theory is conformally invariant. This amounts to showing if $z' = f(z)$, for some holomorphic f, then $X'^\mu(z', \bar{z}') = X(z, \bar{z})$. For our purposes it will be easier to check this infinitesimally. consider

$$z' = z + \epsilon v(z) \tag{2.7}$$

for holomorphic v (and similarly for the $\bar{z}'$). We want to show that such a transformation gives rise to the following variation:

$$X'^\mu(z', \bar{z}') = X^\mu(z, \bar{z}) - \epsilon v^a(z)\partial_a X^\mu(z, \bar{z}) - \epsilon v^a(z)^* \bar{\partial} X^\mu$$

because this is the infinitesimal version of $X'^\mu(z', \bar{z}') = X(z, \bar{z})$. The idea will be to use the Ward identity,

$$\mathrm{Res}_{z \to z_0}\, j(z)\mathscr{A}(z_0, \bar{z}_0) + c.c. = \frac{1}{i\epsilon}\delta\mathscr{A}(z_0, \bar{z}_0),$$

to compute the variation of $\mathscr{A}$. Therefore, we must first compute the current $j^a(z)$ corresponding to $v^a(z)$, then compute the OPE $j(z)\mathscr{A}(z_0, \bar{z}_0)$ to understand the asymptotics around $(z_0, \bar{z}_0)$, and finally compute the residue to obtain the symmetry that we are interested in.

Exercise 2.1.1. Show that the Noether current, corresponding to the symmetry (2.7), is given by $j_a = iv^b T_{ab}$ where T_{ab} is the normal ordered version of the stress-energy tensor

$$T_{ab} = -\frac{1}{\alpha'} :\!\left(\partial_a X^\mu \partial_b X_\mu - \frac{1}{2}\delta_{ab}\partial_c X^\mu \partial^c X_\mu\right)\!: .$$

Moreover, show that $T^a_a = 0$, that is the tensor is traceless. Rewriting this in complex coordinates, show this is equivalent to $T_{z\bar{z}} = 0$. Also, using $\partial^a T_{ab} = 0 = T^a_a$, we have $\bar{\partial} T_{zz} = \partial T_{\bar{z}\bar{z}} = 0$, showing that $T = T_{zz}, \tilde{T} = T_{\bar{z}\bar{z}}$ are holomorphic and anti-holomorphic. Next, show that the OPEs of $T\mathscr{A}$ and $\tilde{T}\mathscr{A}$ have the following asymptotics:

$$T(z)X^\mu(0) \sim \frac{1}{z}\partial_z X^\mu(0), \qquad \tilde{T}(\bar{z})X^\mu(0) \sim \frac{1}{\bar{z}}\bar{\partial} X^\mu(0).$$

2.1.3 Primary Fields

In a CFT, we would like to use a particular basis for the OPE (2.1). This is a set of local operators which transform under conformal transformations similar to a tensor:

$$\mathscr{O}'(z', \bar{z}') = (\partial z')^{-h}(\bar{\partial}\bar{z}')^{-\tilde{h}}\mathscr{O}(z, \bar{z}). \tag{2.8}$$

We call such a local operator a **primary field** or **conformal tensor** of weight $(h, \tilde{h})$. These quasi-primary fields, by definition, play nice with conformal transformations, thus we may expect that the OPE of $T\mathscr{O}$ will be particularly nice. In fact, this does turn out to be the case:

$$T(z)\mathscr{O}(0,0) = \frac{h}{z^2}\mathscr{O}(0,0) + \frac{1}{z}\partial\mathscr{O}(0,0) + \cdots \tag{2.9}$$

Example 2.1.8. The operator $:\!(\prod_i \partial^{m_i} X^{\mu_i})(\prod_j \partial^{n_j} X^{\nu_j})e^{ik\cdot X}\!:$ has weight $\left(\frac{\alpha' k^2}{4} + \sum_i m_i, \frac{\alpha' k^2}{4} + \sum_j n_j\right)$.

Proposition 2.1.9 (Refined OPE). *Using rigid translations, scaling and rotations to both sides of an OPE we can write, for any two primary operators $\mathscr{A}_i, \mathscr{A}_j$:*

$$\mathscr{A}_i(z_1, \bar{z}_1)\mathscr{A}_j(z_2, \bar{z}_2) = \sum_k z_{12}^{h_k - h_i - h_j}\bar{z}_{12}^{\tilde{h}_k - \tilde{h}_i - \tilde{h}_j}\mathscr{A}_k(z_2, \bar{z}_2) \tag{2.10}$$

Theorem 2.1.10 (Conformal Bootstrap)**.** *We may express the OPEs (ie. the correlation functions) for a product of any two or more fields, using only the quasi-primary fields. (Cf. Chapter 15 – Vol 2 Polchinski)*

Example 2.1.11. *bc* CFT There are many different free conformal field theories. We have, in fact, already met with two in the first chapter. The first is the X^μ theory, which we have gotten to know quite well. The second comes from § 1.5.3: Faddeev-Popov ghosts b_{ab}, c^a with action

$$S_G = \frac{1}{2\pi}\int d^2\sigma b_{ab}\partial^a c^b \tag{2.11}$$

is a free CFT where b, c are primary fields (conformal tensors) with weights $(h_b, 0) = (\lambda, 0)$ and $(h_c, 0) = (1-\lambda, 0)$. For this theory we can compute the OPEs, and all of the other quantities in a similar manner to what we did above. All of these important facts are left as exercises with answers in Polchinski pg 50-51.

2.1.4 Virasoro Algebra

Let us now compute the spectrum of a CFT. First, we will be clever and apply a conformal transformation from $(w = \sigma^1 + i\sigma^2) \mapsto (z = e^{-iw} = e^{-i\sigma^1+\sigma^2})$.[2] Quantizing with parameter z is usually referred to as **radial quantization**. Second, we will do a Laurent expansion of $T_{zz}(z)$ and $T_{\bar{z}\bar{z}}(\bar{z})$ to obtain the Virasoro generators:

$$T(z) = \sum_{m=-\infty}^{\infty}\frac{L_m}{z^{m+2}}, \qquad \tilde{T}(z) = \sum_{m=-\infty}^{\infty}\frac{\tilde{L}_m}{\bar{z}^{m+2}}, \qquad L_m = \oint_C \frac{dz}{2\pi i z} z^{m+2}T(z) \tag{2.12}$$

This definition of operators L_m has a number of consequences. First, we notice that the shape of the contour is irrelevant and we may fix the contours C to be circles centred at 0. This contours correspond to equal-time points of the world-sheet, since $|z| = e^{\sigma^2}$. Moreover, since these operators are invariant under the radius of the circle, it follows that L_m are invariant under time translation! This means that L_m are conserved charges with current $j_m(z) = z^{m+1}T(z)$.

Before we can talk about the Hamiltonian, we again need a technical lemma.

Lemma 2.1.12 (Transformation of the Energy Momentum Tensor)**.** *In a general CFT, under a conformal transformation, $z \to z + \epsilon v(z)$, the energy momentum tensor transforms as:*

$$\delta T(z) = -\frac{c}{12}\partial^3 v(z) - 2\partial v(z)T(z) - v(z)\partial T(z) \tag{2.13}$$

The quantity c is called the central charge of the CFT.

Exercise 2.1.2. Show that the Hamiltonian is the conserved quantity given by (in the $w = \sigma^1 + i\sigma^2$ frame)

$$H = \int_0^{2\pi}\frac{d\sigma^1}{2\pi}T_{22} = L_0 + \tilde{L}_0 - \frac{c+\tilde{c}}{24},$$

where $T_{ww} = (\partial_w z)^2 T_{zz} + \frac{c}{24}$.

The collection $\{L_m, \tilde{L}_m\}_{m\in\mathbb{Z}}$ are the generators for the Virasoro algebra. To compute their commutators, we use the Ward identity and a contour trick.

Lemma 2.1.13. *Let $Q_i = \oint \frac{dz}{2\pi i}j_i(z), i = 1, 2$ be conserved charges, with current j_1, j_2. Then*

$$\begin{aligned}[Q_1, Q_2]\{C_2\} &= \lim_{C_1, C_3\to C_2} Q_1\{C_1\}Q_2\{C_2\} - Q_1\{C_3\}Q_2\{C_2\} \\ &= \oint_{C_2}\frac{dz_2}{2\pi i} = \mathrm{Res}_{z_1\to z_2}\, j_1(z_1)j_2(z_2)\end{aligned}$$

[2] In the previous sections what we meant by z was actually the w here.

Proof. For the first equality, imagine slicing the path integral into three chunks. For the second, draw the standard picture where C_1, C_3 are perturbed around a point $z_2 \in C_2$. $\square$

This lemma shows that knowing the singular terms means we understand the commutators between conserved charges. Now we may apply this lemma to the Virasoro generators.

Theorem 2.1.14 (Virasoro Algebra Relations)**.** *Let $L_m, m \in \mathbb{Z}$ be the generators of the Virasoro algebra. Then,*

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n} \tag{2.14}$$

Proof. Let us calculate the right-hand side of the equation of the lemma. $\mathrm{Res}_{z_1 \to z_2}(z_1^{m+1}T(z_1))\,(z_2^{n+1}T(z_2)) = \mathrm{Res}_{z_1 \to z_2}\, z_1^{m+1} z_2^{n+1} \cdot \mathrm{OPE}\{T(z_1)T(z_2)\}$. Expanding the OPE, and then doing the contour integral, we obtain the result. $\square$

2.2 Vertex Operators, Scattering Amplitudes, Anomolies

Chapter 3

String Theory Revisited

3.1 BRST Quantization

3.2 S-matrix