Pre-problem

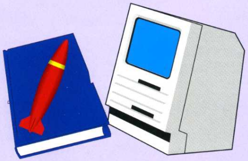
There's nothing wrong with this fictional story in itself. It uses a lot of acronyms (abbreviations using the first letters of words), but again that isn't the connection we're looking for.

What is the specific common link between all the acronyms used in this story?

I wanted to find out the ISBN number of a particular book I wanted, so I logged onto the Internet. My IBM PC computer is a little old – it uses programs written in BASIC code, running under the DOS operating system.

I was running short of cash, so I went to the nearest ATM machine. I entered my PIN number and the LCD display asked me how much money I wanted. I took out £50.

I found the book I wanted, which was about the SALT treaty. This concerns the decommissioning of ABM missiles that are placed all over the world.



Lecture #6: Filter design

Course 31606: Signals and Systems in Discrete Time

Bastian Epp

Hearing Systems group

latest change: Wednesday 10th October, 2018

What happened last time:

- Recall of Fourier transform
- Bi- and unilateral z-transform
- Connection between z-space and the frequency domain
- Frequency transfer function estimate from poles/zeros
- Difference equations in z
- Happy filter design by operating in the z-plane

What happened last time:

- Recall of Fourier transform
- Bi- and unilateral z-transform
- Connection between z-space and the frequency domain

What happened last time:

Bilateral z-transform

$$F(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \qquad x(n) = \frac{1}{2\pi j} \oint F(z) z^{n-1} dz$$

- Components are now exponentially growing/decaying harmonic functions
- With the special case being $\sigma = 0 \Rightarrow z = e^{j\omega}$

$\Rightarrow$ The frequency space lives on the unit circle within the z-plane (frequency transfer function!)

Unilateral z-transform

- For causal signals, the bilateral z-transform boils down to the unilateral z-transform

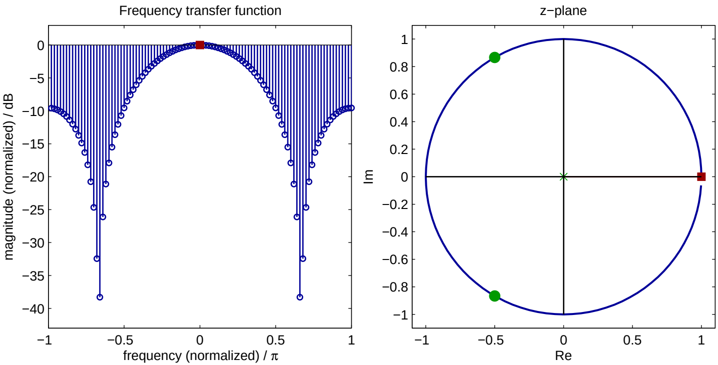
$$F(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

What happened last time:

- Frequency transfer function estimate from poles/zeros

What happened last time:

“Estimating” the frequency response from the z-transform



What happened last time:

- Difference equations in z
- Happy filter design by operating in the z-plane

What happened last time:

Back to schematic of system Application of the z-transform provides algebraic equation

$$
\begin{aligned}
Y(z) = \quad & -a_1\left(\tfrac{1}{z}Y(z) + y(-1)\right) + \dots \quad -a_0\left(\frac{1}{z^L}\sum_{n=1}^{L} y(-n)z^n + z^{-L}Y(z)\right) + \\
b_n X(z) \quad & +b_1\left(\tfrac{1}{z}F(z) + x(-1)\right) + \dots \quad +b_0\left(\frac{1}{z^M}\sum_{n=1}^{M} x(-n)z^n + z^{-M}X(z)\right)
\end{aligned}
$$

- A fourth order system is given by its transfer function

$$
\frac{Y(z)}{F(z)} = H(z) = \frac{b_3 z^3 + b_2 z^2 + b_1 z + b_0}{a_3 z^3 + a_2 z^2 + a_1 z + a_0}
$$

Cycling in the morning makes you happy

A digital 2-D signal

What is going to happen today:

- A closer look at the z-plane
- Controlling filter properties by poles and zeros
- Different filter types
- Examples of filters
- Design of recursive filters

Corresponding book chapter: 10.3

Quickie Nr 5

Does this ring a bell...?

Take some minutes to answer the following questions:

- Why is a signal synthesized from the Fourier domain always periodic?
- Where can the Fourier domain be found in the z-domain?
- Which are the “elements” used in the z-transform and how do they differ from the “elements” used in the Fourier transform?
- How can the poles and zeros that describe the behaviour of the system in the z-domain be obtained?

Quickie Nr 5

Does this ring a bell...?

Take some minutes to answer the following questions:

- Why is a signal synthesized from the Fourier domain always periodic?
  The elements in the Fourier domain are harmonic oscillations. The Fourier transform analyzes/synthesizes signals using these elements. Since each element is periodic, is the sum of such elements also periodic.
- Where can the Fourier domain be found in the z-domain?
- Which are the “elements” used in the z-transform and how do they differ from the “elements” used in the Fourier transform?
- How can the poles and zeros that describe the behaviour of the system in the z-domain be obtained?

Quickie Nr 5

Does this ring a bell...?

Take some minutes to answer the following questions:

- Why is a signal synthesized from the Fourier domain always periodic?
  The elements in the Fourier domain are harmonic oscillations. The Fourier transform analyzes/synthesizes signals using these elements. Since each element is periodic, is the sum of such elements also periodic.
- Where can the Fourier domain be found in the z-domain?
  The Fourier domain is obtained from the z-domain by replacing $z \rightarrow e^{j\Omega}$ (unit circle).
- Which are the "elements" used in the z-transform and how do they differ from the "elements" used in the Fourier transform?
- How can the poles and zeros that describe the behaviour of the system in the z-domain be obtained?

Quickie Nr 5

Does this ring a bell...?

Take some minutes to answer the following questions:

- Why is a signal synthesized from the Fourier domain always periodic?
  The elements in the Fourier domain are harmonic oscillations. The Fourier transform analyzes/synthesizes signals using these elements. Since each element is periodic, is the sum of such elements also periodic.
- Where can the Fourier domain be found in the z-domain?
  The Fourier domain is obtained from the z-domain by replacing $z \to e^{j\Omega}$ (unit circle).
- Which are the "elements" used in the z-transform and how do they differ from the "elements" used in the Fourier transform?
  The elements in the z-transform are exponentially growing or decaying oscillations. The elements of the Fourier transform are a subset of these with an infinitely slow decay.
- How can the poles and zeros that describe the behaviour of the system in the z-domain be obtained?

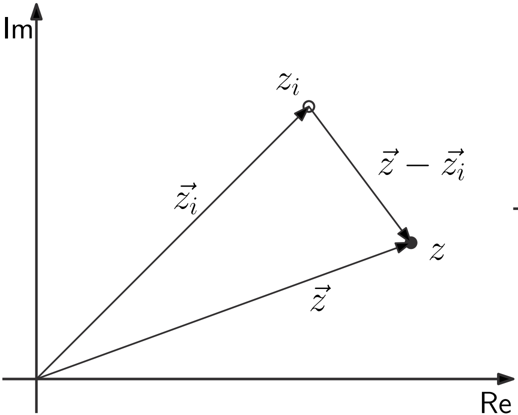
Quickie Nr 5

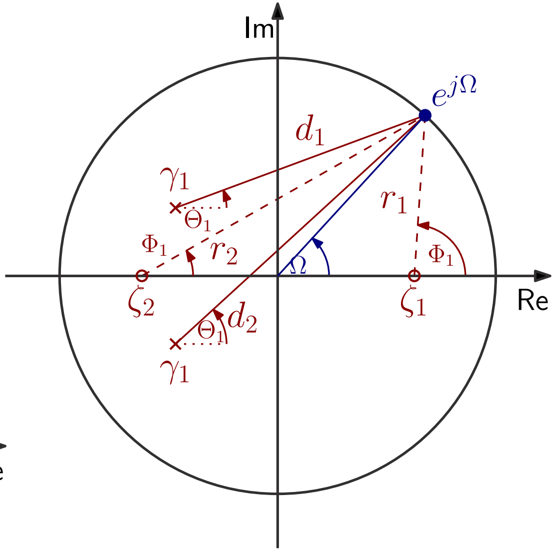
Does this ring a bell...?

Take some minutes to answer the following questions:

- Why is a signal synthesized from the Fourier domain always periodic?
  The elements in the Fourier domain are harmonic oscillations. The Fourier transform analyzes/synthesizes signals using these elements. Since each element is periodic, is the sum of such elements also periodic.
- Where can the Fourier domain be found in the z-domain?
  The Fourier domain is obtained from the z-domain by replacing $z \to e^{j\Omega}$ (unit circle).
- Which are the "elements" used in the z-transform and how do they differ from the "elements" used in the Fourier transform?
  The elements in the z-transform are exponentially growing or decaying oscillations. The elements of the Fourier transform are a subset of these with an infinitely slow decay.
- How can the poles and zeros that describe the behaviour of the system in the z-domain be obtained?
  Poles and zeros are obtained by computation of the roots of the polynomials in the numerator (zeros) and denominator (poles) of the transfer function in z.

Transfer function and z-plane





Transfer function and z-plane

Geometrical derivation of frequency transfer function

- Transfer function

$$H(z) = \frac{(z-\zeta_1)(z-\zeta_2)\ldots(z-\zeta_n)}{(z-\gamma_1)(z-\gamma_2)\ldots(z-\gamma_n)}$$

Transfer function and z-plane

Geometrical derivation of frequency transfer function

- Transfer function

$$H(z) = \frac{(z-\zeta_1)(z-\zeta_2)\ldots(z-\zeta_n)}{(z-\gamma_1)(z-\gamma_2)\ldots(z-\gamma_n)}$$

- In terms of distances of zeros/poles to some frequency

$$H(e^{j\omega T}) = \frac{(r_1 e^{j\Phi_1})(r_2 e^{j\Phi_2})\ldots(r_n e^{j\Phi_n})}{(d_1 e^{j\Theta_1})(d_2 e^{j\Theta_2})\ldots(d_n e^{j\Theta_n})}$$

Transfer function and z-plane

Geometrical derivation of frequency transfer function

- Transfer function

$$H(z) = \frac{(z-\zeta_1)(z-\zeta_2)\dots(z-\zeta_n)}{(z-\gamma_1)(z-\gamma_2)\dots(z-\gamma_n)}$$

- In terms of distances of zeros/poles to some frequency

$$H(e^{j\omega T}) = \frac{(r_1 e^{j\Phi_1})(r_2 e^{j\Phi_2})\dots(r_n e^{j\Phi_n})}{(d_1 e^{j\Theta_1})(d_2 e^{j\Theta_2})\dots(d_n e^{j\Theta_n})}$$

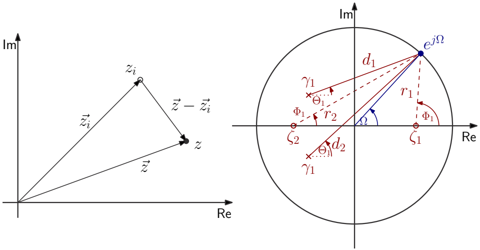
- And even further simplified

$$H(e^{j\omega T}) = \frac{r_1 r_2 \cdots r_n}{d_1 d_2 \cdots d_n} \cdot e^{j(\Phi_1+\Phi_2+\dots+\Phi_n-\Theta_1-\Theta_2-\dots-\Theta_n)}$$

$$= \frac{\prod_k r_k}{\prod_l d_l} \cdot e^{j\left(\sum_k \Phi_k - \sum_l \Theta_l\right)}$$

Transfer function and z-plane

Geometrical derivation of frequency transfer function

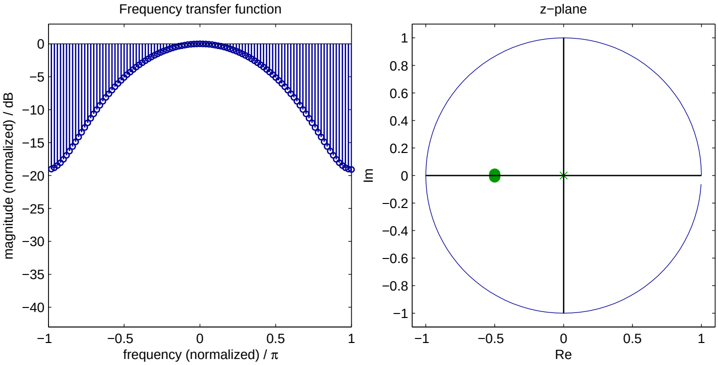


$$H(e^{j\omega T}) = \frac{r_1 r_2 \cdots r_n}{d_1 d_2 \cdots d_n} \cdot e^{j(\Phi_1 + \Phi_2 + \cdots + \Phi_n - \Theta_1 - \Theta_2 - \cdots - \Theta_n)}$$

$$= \frac{\prod_k r_k}{\prod_l d_l} \cdot e^{j\left(\sum_k \Phi_k - \sum_l \Theta_l\right)}$$

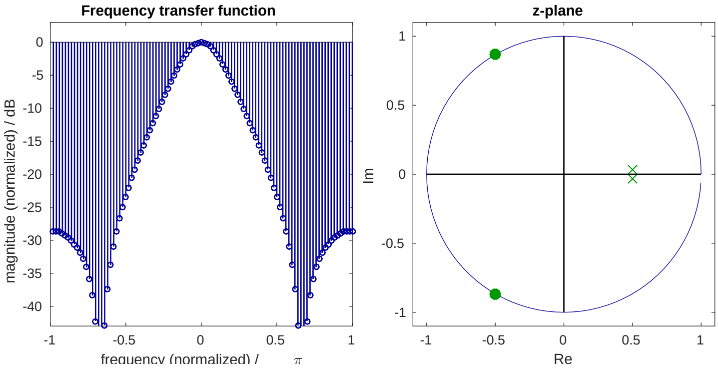
Influence of poles and zeros

What do zeros do to the frequency transfer function?



Influence of poles and zeros

What do poles do to the frequency transfer function?



Different filter types

The name tells it all...:

Different filter types

The name tells it all...: There exist 5 standard classes of filters

Different filter types

The name tells it all...: There exist 5 standard classes of filters

- Low-pass (LP)
  - Passes low frequencies and attenuates high frequencies

Different filter types

The name tells it all...: There exist 5 standard classes of filters

- ▶ Low-pass (LP)
- Passes low frequencies and attenuates high frequencies
- ▶ High-pass (HP)
- Passes high frequencies and attenuates low frequencies

Different filter types

The name tells it all...: There exist 5 standard classes of filters

- ▶ Low-pass (LP)
- \- Passes low frequencies and attenuates high frequencies
- ▶ High-pass (HP)
- \- Passes high frequencies and attenuates low frequencies
- ▶ Band-pass (BP)
- \- Passes frequencies around a certain center frequency

Different filter types

The name tells it all...: There exist 5 standard classes of filters

- Low-pass (LP)
  - Passes low frequencies and attenuates high frequencies
- High-pass (HP)
  - Passes high frequencies and attenuates low frequencies
- Band-pass (BP)
  - Passes frequencies around a certain center frequency
- Band-stop (BS)
  - Passes all frequencies but around a certain center frequency

Different filter types

The name tells it all...: There exist 5 standard classes of filters

- Low-pass (LP)
- Passes low frequencies and attenuates high frequencies
- High-pass (HP)
- Passes high frequencies and attenuates low frequencies
- Band-pass (BP)
- Passes frequencies around a certain center frequency
- Band-stop (BS)
- Passes all frequencies but around a certain center frequency
- All-pass (AP)
- Passes all frequencies

What do we already know about filters?

What is a filter, what does it do and how is it done?

What do we already know about filters?

What is a filter, what does it do and how is it done?

- The filter is the process (or the system) that does the filtering

What do we already know about filters?

What is a filter, what does it do and how is it done?

- The filter is the process (or the system) that does the filtering
- The filtered signal is obtained by convolution of the input signal with the impulse response in the time domain

What do we already know about filters?

What is a filter, what does it do and how is it done?

- The filter is the process (or the system) that does the filtering
- The filtered signal is obtained by convolution of the input signal with the impulse response in the time domain
- Filtering can also be done in the frequency domain by (complex) multiplication with the frequency response (transfer function)

What do we already know about filters?

What is a filter, what does it do and how is it done?

- The filter is the process (or the system) that does the filtering
- The filtered signal is obtained by convolution of the input signal with the impulse response in the time domain
- Filtering can also be done in the frequency domain by (complex) multiplication with the frequency response (transfer function)
- Task is to selectively attenuate/amplify certain frequencies (for time signals)

What do we already know about filters?

What is a filter, what does it do and how is it done?

- The filter is the process (or the system) that does the filtering
- The filtered signal is obtained by convolution of the input signal with the impulse response in the time domain
- Filtering can also be done in the frequency domain by (complex) multiplication with the frequency response (transfer function)
- Task is to selectively attenuate/amplify certain frequencies (for time signals)

Two different types

- Finite impulse response (FIR)
- Infinite impulse response (IIR)

What do we already know about filters?

Back to lecture 2: The moving average order N difference equation:

$$y(n) = \frac{1}{N}(x(n) + x(n-1) + x(n-2) + \dots x(n-N)) \tag{1}$$

What do we already know about filters?

Back to lecture 2: The moving average order N difference equation:

$$y(n) = \frac{1}{N}(x(n) + x(n-1) + x(n-2) + \ldots x(n-N)) \tag{1}$$

The general system difference equation:

$$\begin{aligned} y(n) = & & -a_1 y(n-1) + \ldots & \quad -a_L y(n-L) + \\ & b_0 x(n) & +b_1 x(n-1) + \ldots & \quad +b_M x(n-M) \end{aligned}$$

What do we already know about filters?

Back to lecture 2: The moving average order N difference equation:

$$y(n) = \frac{1}{N}(x(n) + x(n-1) + x(n-2) + \dots x(n-N)) \tag{1}$$

The general system difference equation:

$$\begin{aligned} y(n) = & & -a_1 y(n-1) + \dots & \quad -a_L y(n-L) + \\ & b_0 x(n) & +b_1 x(n-1) + \dots & \quad +b_M x(n-M) \end{aligned}$$

- The running sum has only delayed input
- We see that $a_0 = 1$
- $\Rightarrow$ The b_n are the coefficients for delayed input ('feed-forward') $\Rightarrow$ "Finite impulse response (FIR)"
- $\Rightarrow$ The a_n are the coefficients for delayed output ('feed-back') $\Rightarrow$ "Infinite impulse response (IIR)"

A brief review of the Laplace transform

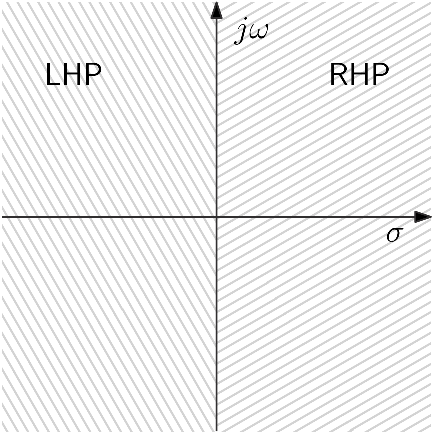
S- (or Laplace) transform

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{st}dt \qquad s = \sigma + j\omega$$

Space of:

- exponentially growing (RHP)
- exponentially decaying (LHP)
- harmonic (y-axis)

signals



Let's do the math: How do we define a filter?

Time domain equivalence ("impulse response invariance method"):

"The impulse response of the digital is a sampled version of the impulse response of the filter in continuous time"

$$h(n) = \lim_{T \to 0} T h_a(nT)$$

Let's do the math: How do we define a filter?

Time domain equivalence ("impulse response invariance method"):

"The impulse response of the digital is a sampled version of the impulse response of the filter in continuous time"

$$h(n) = \lim_{T \to 0} T h_a(nT)$$

Cooking recipe

1. "Ignore" the limit $T \to 0$ (keep it in mind though)
2. Consider the desired transfer function in the s (Laplace) domain
3. "Sample" the corresponding impulse response (continuous time)
4. Transfer the sampled impulse response into the z-domain
5. Chose your sampling interval to match the desired frequency range
6. Done!

A practical example

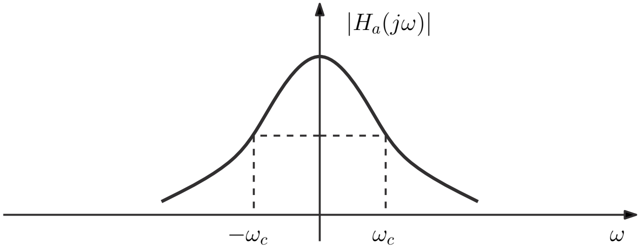
Define a filter using the impulse response invariance method such that it matches the analogue filter (1st order Butterworth LP)

- Transfer function

$$H_a(s) = \frac{\omega_c}{s + \omega_c}$$

- Cut-off frequency

$$\omega_c = 10^5$$



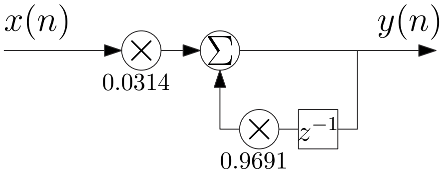
A practical example

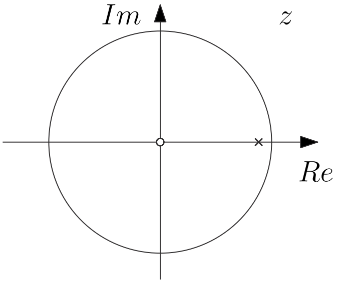
1. “Ignore“ the limit $T \to 0$ (keep it in mind though)
2. Consider the desired transfer function in the s (Laplace) domain
3. ”Sample“ the corresponding impulse response (continuous time)
4. Transfer the sampled impulse response into the z-domain
5. Chose your sampling interval to match the desired frequency range
6. Done!

A practical example

Transfer function with $T = 10^{-7}\pi$:

$$H(z) = \frac{0.0314 \cdot z}{z - 0.9691} = \frac{0.0314}{1 - 0.9691 z^{-1}}$$





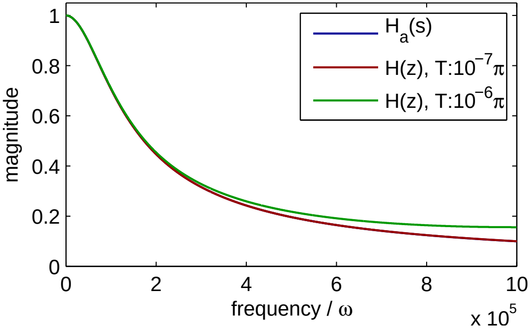

```
>> zeros = [0]; poles = [.9691];
>> zplane(zeros, poles); figure(); freqz(poly(zeros), poly(poles))
```

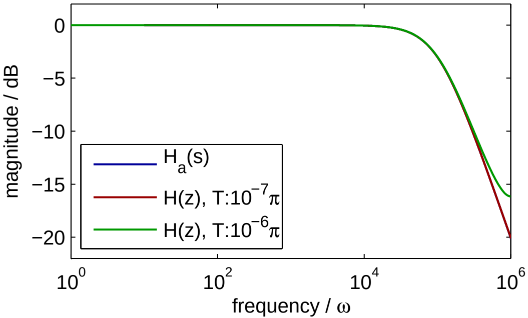
A practical example

Transfer function with $T = 10^{-7}\pi$:

$$H(z) = \frac{0.0314 \cdot z}{z - 0.9691} = \frac{0.0314}{1 - 0.9691 z^{-1}}$$

useful command: `impinvar`





Let’s do the math: How do we define a filter?

Frequency domain equivalence:

“The transfer function of the digital filter is a sampled version of the transfer function of the filter in continuous time”

$$\lim_{T \to 0} H(e^{sT}) = H_a(s)$$

Let's do the math: How do we define a filter?

Frequency domain equivalence:

"The transfer function of the digital filter is a sampled version of the transfer function of the filter in continuous time"

$$\lim_{T \to 0} H(e^{sT}) = H_a(s)$$

Cooking recipe

1. Design your filter in the s-domain
2. Map your s-plane into something than can be handled better - ∞ is a problem (using the bilinear transform)

$$H(z) = H_a(s)\big|_{\frac{2}{T}\frac{z-1}{z+1}}$$

3. Chose the suitable sampling period T
4. ~~Done!~~

also called "Tustin approximation" in control theory

A practical example

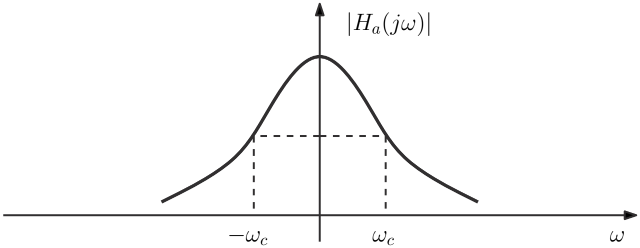
Define a filter using the bilinear transformation such that it matches the analogue filter (1st order Butterworth LP)

- Transfer function

$$H_a(s) = \frac{\omega_c}{s + \omega_c}$$

- Cut-off frequency

$$\omega_c = 10^5$$



A practical example

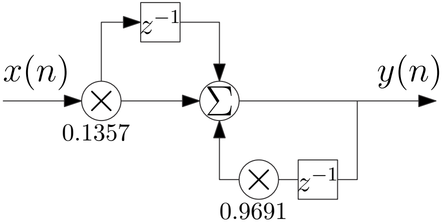
1. Design your filter in the s-domain
2. Map your s-plane into something than can be handled better - ∞ is a problem (using the bilinear transform)
$$H(z) = H_a(s)|_{\frac{2}{T}\frac{z-1}{z+1}}$$
3. Chose the suitable sampling period T
4. Done!

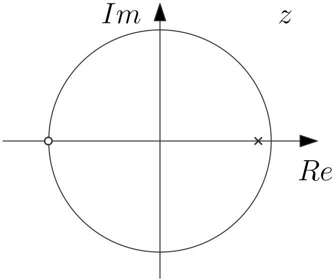
also called "Tustin approximation" in control theory

A practical example

Transfer function with $T = 10^{-7}\pi$:

$$H(z) = 0.1357 \cdot \frac{z+1}{z-0.9691} = 0.1357 \cdot \frac{1+z^{-1}}{1-0.9691z^{-1}}$$



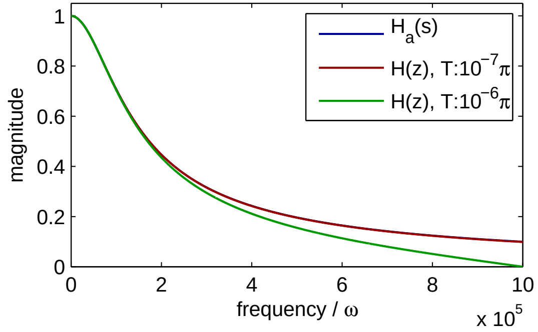


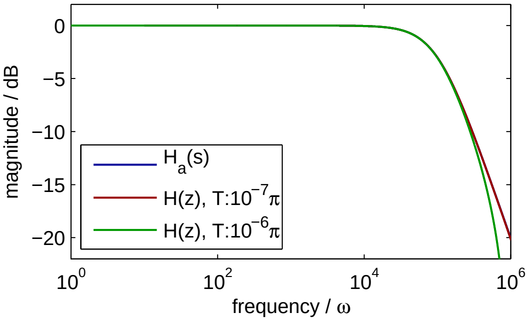

```
>> zeros = [-1]; poles = [.9691];
>> zplane(zeros, poles); figure(); freqz(poly(zeros), poly(poles))
```

A practical example

Transfer function with $T = 10^{-7}\pi$:

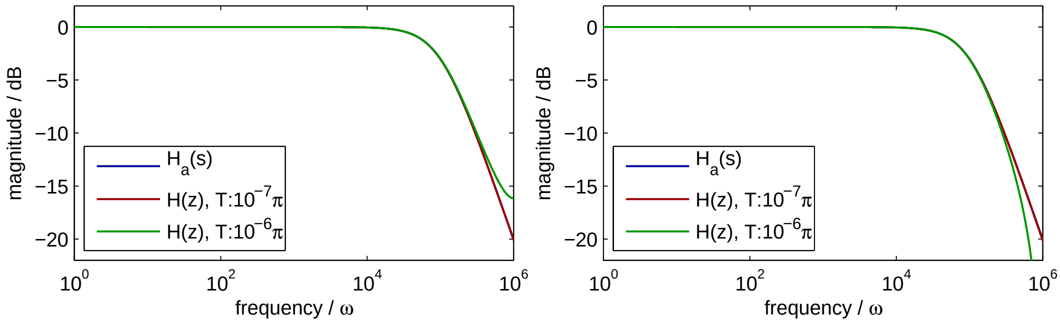
$$0.1357 \cdot \frac{1 + z^{-1}}{1 - 0.9691 z^{-1}} \qquad \texttt{useful command: bilinear}$$





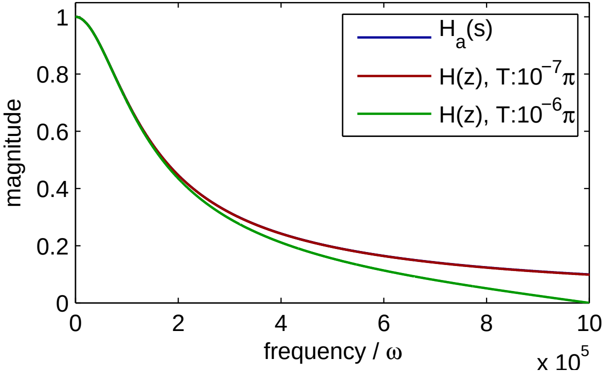
A practical example

Impulse invariance vs. bilinear transformation



Something happens to the frequencies

Why the deviation at higher frequencies?: Frequency (pre-)warping



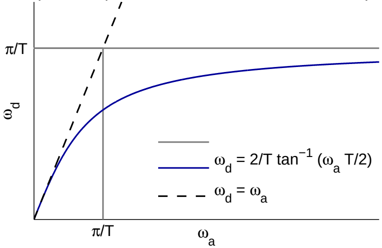
Something happens to the frequencies

Why the deviation at higher frequencies?: Frequency (pre-)warping

$$H(e^{j\omega T}) = H_a\left(j\frac{2}{T}\tan(\frac{\omega_d T}{2})\right)$$

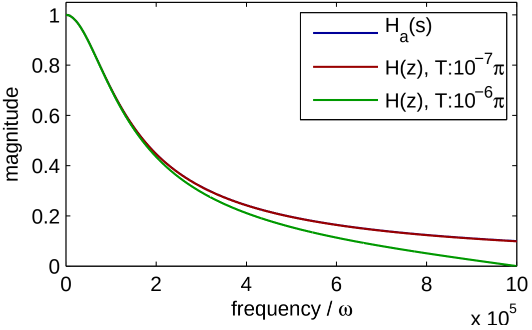
"Analogue" frequencies are being mapped on a "digital" frequency (and vice versa):

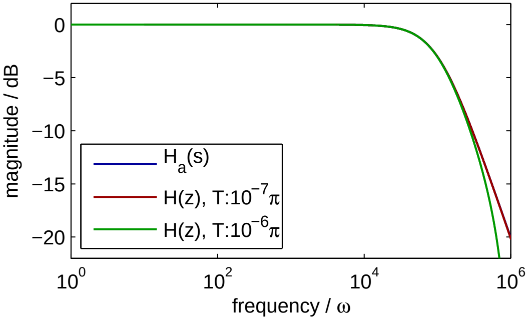
$$\omega_a \to \frac{2}{T}\tan\left(\frac{\omega_d T}{2}\right) \qquad \omega_d \to \frac{2}{T}\tan^{-1}\left(\frac{\omega_a T}{2}\right)$$



Something happens to the frequencies

Why the deviation at higher frequencies?: Frequency (pre-)warping





Something happens to the frequencies

Why the deviation at higher frequencies?: Frequency (pre-)warping Pre-warp your frequencies to have an exact match after bilinear transformation

$$\omega' = \frac{2}{T}\tan\left(\frac{\omega T}{2}\right) \longrightarrow \frac{2}{T}\tan^{-1}\left(\frac{\omega' T}{T}\right) = \frac{2}{T}\tan^{-1}\left(\frac{\frac{2}{T}\tan\left(\frac{\omega T}{2}\right)T}{T}\right) = \omega$$

The new recipe:

1. Pre-warp you frequencies
2. Design your filter in the s-domain
3. Map your s-plane into something than can be handled better - ∞ is a problem (using the bilinear transform)
4. Chose the suitable sampling period T
5. Done!

also called "Tustin approximation" in control theory

A practical example

Define a filter using the bilinear transformation meeting the following specs:

- Gain of unity at $\omega = 0$
- Gain G_p not less than -2dB (0.794) for the band $0 \leq \omega \leq 8$ (pass band)
- Gain G_s not greater than -11dB (0.282) for the band $\omega \geq 15$ (stop band)
- Highest frequency to process $\omega_h = 35$ rad/s

1. Pre-warp you frequencies
2. Transform your s-plane into something than can be handled better (∞ is a problem)
3. Map on z-plane
4. Chose the suitable sampling period T
5. Done!

A practical example

What was again....a Butterworth filter:

- Butterworth approximation to have maximally flat gain in pass band
- Poles arranged in semicircle with n (oder of filter poles) uniformaly distributed
- Slow roll-off compared to other filters (Chebyshev, Elliptic)
- Amplitude response:

$$|H(j\omega)| = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_c}\right)^{2n}}} \qquad |H(j\omega_c)| = \frac{1}{\sqrt{2}}$$

- Order determination and cut-off

$$n = \frac{\log\left[\left(10^{-G_s/10} - 1\right)/\left(10^{-G_p/10} - 1\right)\right]}{2\log(\omega_s/\omega_p)} \qquad w_c = \frac{w_p}{\left[10^{-G_p/10} - 1\right]^{1/2n}}$$

An intermediate summary on filters

How to get a digital filter from an analogue one: **Impulse invariance**

- Match the digital and analogue impulse responses

$$h(n) = \lim_{T \to 0} Th_a(kT)$$

- Suitable for LP filters only due to aliasing at higher frequencies (or high sampling rates required)

An intermediate summary on filters

How to get a digital filter from an analogue one: **Impulse invariance**

- Match the digital and analogue impulse responses

$$h(n) = \lim_{T \to 0} T h_a(kT)$$

- Suitable for LP filters only due to aliasing at higher frequencies (or high sampling rates required)

Bilinear transformation

- Match the digital and analogue transfer function

$$\lim_{T \to 0} H(e^{sT}) = H_a(s)$$

- Frequency prewarping compensates for frequency shift
- Suitable for all types of filters due to lack of aliasing (lower sampling rates possible)

Hands-on for today (2nd assignment)

- Design of a IIR Butterworth filter
- Solving practical issues with a filter
- Hands-on 5 - 3

Please remember:

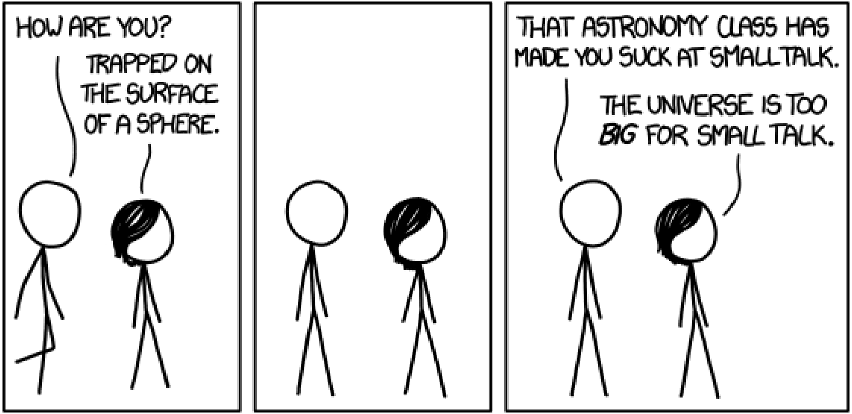
- Work in groups up to three
- Share knowledge (team work)
- Make code readable (add comments, use functions...)
- Address all points in the guide (using logical arguments)
- Hand in before start of next lecture (LEARN + hardcopy)

Take-home messages of today

What just happened

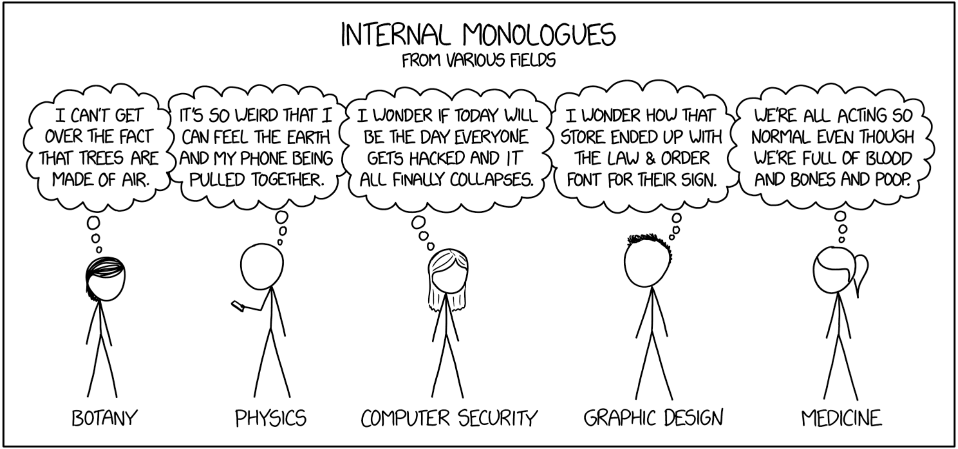
- Geometrical derivation of the transfer function
- Different filter types
- Filter properties in time and frequency
- Design of IIR filters
- Time domain equivalence
- Frequency domain equivalence

...need a break?



HOW ARE YOU?
TRAPPED ON THE SURFACE OF A SPHERE.
THAT ASTRONOMY CLASS HAS MADE YOU SUCK AT SMALLTALK.
THE UNIVERSE IS TOO *BIG* FOR SMALL TALK.

...need a break?


INTERNAL MONOLOGUES
FROM VARIOUS FIELDS
I CAN'T GET OVER THE FACT THAT TREES ARE MADE OF AIR.
IT'S SO WEIRD THAT I CAN FEEL THE EARTH AND MY PHONE BEING PULLED TOGETHER.
I WONDER IF TODAY WILL BE THE DAY EVERYONE GETS HACKED AND IT ALL FINALLY COLLAPSES.
I WONDER HOW THAT STORE ENDED UP WITH THE LAW & ORDER FONT FOR THEIR SIGN.
WE'RE ALL ACTING SO NORMAL EVEN THOUGH WE'RE FULL OF BLOOD AND BONES AND POOP.
BOTANY
PHYSICS
COMPUTER SECURITY
GRAPHIC DESIGN
MEDICINE