Calculus Class 6 Homework

First Name:____________________ Last Name: __________________ Student ID: _________

## Derivatives and their applications (1)

$c \in D_f \quad f'(c)=0$ or $f'(c)=DNE$

**1.** For each case, find the critical points. $\to (c, f(c))$

a. f(x) = $x^3 + 6x^2 + 9x + 2$ $\quad x \in \mathbb{R}$

$f'(x) = 3x^2 + 12x + 9$

$f'(x) = 3(x^2+4x+3) = 3(x+1)(x+3) = 0$

$x = -1, -3$

$f(-1) = -2 \quad f(-3) = 2$

$(-1,-2)$ and $(-3,2)$

b. f(x) = $x^3$ $\quad x \in \mathbb{R}$

$f'(x) = 3x^2$

$f'(x) = 0 \Rightarrow x = 0$

$\boxed{(0,0)}$

c. f(x) = $\sqrt[3]{x}$, $x \in \mathbb{R}$ $\quad x^{\frac{1}{3}} =$

$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$

$f'(x) = DNE$ when $x = 0$

$(0,0)$.

d. f(x) = $\frac{x^4+1}{x^2+1}$, $x \in \mathbb{R}$

$f'(x) = \frac{(4x^3)(x^2+1) - (x^4+1)(2x)}{(x^2+1)^2}$

$f'(x) = \frac{(2x)(x^4+2x^2-1)}{(x^2+1)^2}$

$f'(x) = 0$

$x=0$ or $x^4+2x^2-1=0$

$(0,1)$

$x^2 = \frac{-2 \pm \sqrt{4-4(1)(-1)}}{2(1)}$

$x^2 = \frac{-2\pm\sqrt{8}}{2} = -1 \pm \sqrt{2}$

$x = \sqrt{-1+\sqrt{2}}, -\sqrt{-1+\sqrt{2}}$

$f(0) = 1$

$f(\pm\sqrt{-1+\sqrt{2}})$

$= \frac{(\pm\sqrt{-1+\sqrt{2}})^4+1}{(\pm\sqrt{-1+\sqrt{2}})^2+1}$

$= \frac{(-1+\sqrt{2})^2+1}{-1+\sqrt{2}+1}$

$= \frac{1-2\sqrt{2}+2+1}{\sqrt{2}}$

$= 2\sqrt{2}-2$.

$\boxed{(\sqrt{-1+\sqrt{2}}, 2\sqrt{2}-2)}$

$\boxed{(-\sqrt{-1+\sqrt{2}}, 2\sqrt{2}-2)}$

$x^2-4>0$

$f'(2^+) = 4 \quad f'(2^-) = -4$

**2.** For each case, find any local extremum using the first derivative test.

a. f(x) = $|x^2 - 4|$ $= \begin{cases} x^2-4, & x \geq 2 \text{ or } x \leq -2 \\ 4-x^2, & -2 < x < 2 \end{cases}$

cont. at $x = 2, -2$

$f'(x) = \begin{cases} 2x, & x>2 \text{ or } x<-2 \\ -2x, & -2<x<2 \end{cases}$

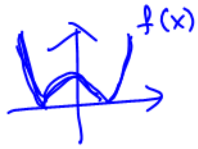
$f'(x)$ DNE at $x = \pm 2$

$f'(x) = 0$ when $x = 0$

| $x$ | | $-2$ | | $0$ | | $2$ | |
|---|---|---|---|---|---|---|---|
| $f'(x)$ | $-$ | DNE | $+$ | $0$ | $-$ | DNE | $+$ |
| $f(x)$ | $\searrow$ | | $\nearrow$ | | $\searrow$ | | $\nearrow$ |

local min at $x=-2$ $\boxed{(-2,0)}$ $\boxed{(2,0)}$

local max $\boxed{(0,4)}$




b. f(x) = $x^3 + 3x - 1$ $\quad x \in \mathbb{R}$

$f'(x) = 3x^2 + 3$.

$f'(x) > 0$ for all $x \in \mathbb{R}$.

Function is always increasing

no local extrema

c. f(x) = $(\frac{1+x}{1-x})^2$ $\leftarrow x \in \mathbb{R}$, $x \neq 1$ $\to$ discont.

$f'(x) = 2\left(\frac{1+x}{1-x}\right)^1 \left(\frac{(1)(1-x)-(1+x)(-1)}{(1-x)^2}\right)$

$f'(x) = \frac{4(1+x)}{(1-x)^3}$

$f'(x) = 0$ when $4(1+x) = 0 \Rightarrow x = -1$

$f'(x) = DNE$ when $x = 1$ $\to$ not in $D_f$.

try $x=-2$ $\quad$ try $x=0$

| $x$ | | $-1$ | | $1$ |
|---|---|---|---|---|
| $f'(x)$ | $\ominus$ | $0$ | $\oplus$ | DNE |
| $f(x)$ | $\searrow$ | $f(-1)$ | $\nearrow$ | discont. |

$f(-1) = 0$

local min at $\boxed{(-1,0)}$

**3.** For each case, find the absolute extrema (maximum or minimum) points.

a. f(x) = -2x + 3, for x∈[−1,2]

$f'(x) = -2 \Rightarrow$ no critical #

$f(-1) = (-2)(-1)+3 = 5 \rightarrow$ abs. max $(-1,5)$

$f(2) = (-2)(2)+3 = -1 \Rightarrow$ abs. min $(2,-1)$.

b. f(x) = $\sqrt{x-2}$, for x∈[2,6]

$f'(x) = \frac{1}{2}(x-2)^{-\frac{1}{2}}(1) = \frac{1}{2\sqrt{x-2}}$

$f'(x)$ DNE at $x=2$ $\quad f'(x) > 0$ for $x \in (2,6]$

$f(2) = \sqrt{2-2} = 0 \rightarrow$ abs. min $(2,0)$

$f(6) = \sqrt{6-2} = 2 \Rightarrow$ abs. max $(6,2)$

**4.** Let f(x) = a$x^4$ + b$x^2$ + cx + d. Find f(x) such that it has a local maximum (LM) at (0,−6) and a local minimum (Lm) at (1,−8) .

LM: $(0,-6) \Rightarrow f(0) = -6 \Rightarrow d = -6$.

Lm: $(1,-8) \Rightarrow f(1) = -8 \Rightarrow a+b+c-6 = -8 \Rightarrow a+b+c = -2 \Rightarrow$

$f'(x) = 4ax^3 + 2bx + c$ $\quad \because$ Local extrema has to occur at critical points.

$\rightarrow x=0$, $x=1$ must be critical #s.

$f'(0) = 0 \quad 4a(0)^3 + 2b(0) + c = 0 \Rightarrow c = 0$.

$f'(1) = 0 \quad 4a + 2b + 0 = 0 \Rightarrow 2a+b = 0$

$a+b+c = -2 \Rightarrow a+b = -2$

$a = 2$

$b = -4$

$\boxed{f(x) = 2x^4 - 4x^2 - 6}$.

**5.** Find the LM (Local max) and Lm (local min) for f(x) = $x^n$, n is natural.

$f'(x) = nx^{n-1} \Rightarrow f'(x) = 0 \Rightarrow x = 0$

n is even, n−1 is odd, $f'(x) > 0 \quad x \in (0,\infty)$, $f'(x) < 0 \quad x \in (-\infty,0)$ $\Rightarrow$ Local min at $x = 0$.

n is odd, n−1 is even. $f'(x)$ is always ⊕ except $x=0$ $\Rightarrow$ no local extrema

**6.** For each case, use the first derivative sign to find the intervals of increase or decrease.

a. f(x) = $x^2$ – 2x

$f'(x) = 2x - 2$

$f'(x) = 0$ when $x = 1$

$f(x)$ is decreasing $x \in (-\infty, 1)$

$f(x)$ is increasing $x \in (1, \infty)$.

b. f(x) = $\sqrt{x}$(x-1) $= (x^{\frac{1}{2}})(x-1) = x^{\frac{3}{2}} - x^{\frac{1}{2}}$

$x \geq 0$

$f'(x) = \frac{3}{2}x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2}x^{-\frac{1}{2}}(3x-1)$

$f'(x) = \frac{3x-1}{2\sqrt{x}}$ $\quad$ ⊖ ⊕ / ⊕ ⊕

$f'(x) = 0$ when $x = \frac{1}{3}$

$f'(x) =$ DNE when $x = 0$

| x | 0 | | $\frac{1}{3}$ | |
|---|---|---|---|---|
| f'(x) | DNE | ⊖ | 0 | ⊕ |
| f(x) | | ↘ | f(0) | ↗ |

Calculus Class 6 Homework

$f(x) \nearrow$ when $x \in (\frac{1}{3}, \infty)$

$f(x) \searrow$ when $x \in (0, \frac{1}{3})$

$x=1$ is a jump discont.

$f'(1^-) = \frac{1}{2}$

$f'(1^+) = 3$

c. f(x) = $x^3(x-1)^4$ $\quad x \in \mathbb{R}$.

$f'(x) = 3x^2(x-1)^4 + (x^3)(4(x-1)^3)$

$f'(x) = x^2(x-1)^3(3(x-1)+4x) = x^2(x-1)^3(7x-3)$.

$f'(x)=0$ when $x = 0, 1, \frac{3}{7}$

| $x$ | | 0 | | 3/7 | 0.5 | 1 | |
|---|---|---|---|---|---|---|---|
| $f'(x)$ | $+$ | 0 | $+$ | 0 | $-$ | 0 | $+$ |

$\oplus\ominus\ominus$, $\oplus\ominus\ominus$, $\oplus\ominus\oplus$, $\oplus\oplus\oplus$

$f(x) \nearrow$ when $x \in (-\infty, \frac{3}{7}) \cup (1, \infty)$

$f(x) \searrow$ when $x \in (\frac{3}{4}, 1)$

d. f(x) = $\begin{cases} \frac{x}{2} + 2, & x < 1 \\ x^3, & x \geq 1 \end{cases}$ $\quad x \in \mathbb{R}$,

$f'(x) = \begin{cases} \frac{1}{2}, & x < 1 \\ 3x^2, & x > 1 \end{cases}$

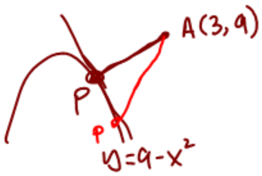
$f'(x) = $ DNE at $x = 1$.

$f'(x) = 0$ ~~when $3x^2 = 0$~~, $x = 0$, but $x=0$ not $x>1$

| $x$ | | 1 | |
|---|---|---|---|
| $f'(x)$ | $+$ | DNE | $+$ |
| $f(x)$ | $\nearrow$ | $f(1) = 1^3 = 1$ | $\nearrow$ |

$f(x) \nearrow$ when $x \in (-\infty, \infty)$.

**7.** Determine the coordinates of the point *P* on the parabola *y*=9−$x^2$ that is closest to the point *A*(3,9).




For shortest distance between AP.

$m_{AP} = m_{\text{Normal at } P}$. $\rightarrow \perp m_{\text{tangent at } P}$.

$P(x, y)$

$P(x, 9-x^2)$

$y' = -2x = m_{\text{tangent at } P} \Rightarrow m_{N_{\text{at } P}} = \frac{1}{2x}$

$m_{AP} = \frac{9-(9-x^2)}{3-x} = \frac{1}{2x} \Rightarrow \frac{x^2}{3-x} = \frac{1}{2x}$.

$2x^3 = 3 - x$.

$2x^3 + x - 3 = 0$. $\rightarrow (x-1)(2x^2+2x+3) = 0$.

$2(1)^3 + 1 - 3 = 0$

$x = 1$ or not real

$x - 1 \overline{)\,2x^3 + 0x^2 + x - 3}$

$$\begin{array}{c|cccc} 1 & 2 & 0 & 1 & -3 \\ & \downarrow & 2 & 2 & 3 \\ \hline & 2 & 2 & 3 & 0 \end{array}$$

$2x^2 + 2x + 3$

$\boxed{P(1, 8)}$

**8.** Determine the maximum and minimum values of the function f(x) = $sin^2x$ on the interval 0 ≤ x ≤ 2**π**.

abs.

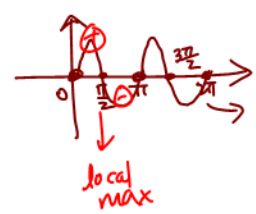
$\begin{cases} \square^2 \\ \sin x \end{cases}$

$f'(x) = 2\sin x \cos x = \sin(2x)$

period of $\sin(2x)$ is $\pi$

when $f'(x) = 0$ $\quad \sin x = 0$ or $\cos x = 0$.

$x = 0, \pi, 2\pi, \frac{\pi}{2}, \frac{3\pi}{2}$.




$f(0) = 0$

$f(\pi) = 0$

$f(2\pi) = 0$

abs. min at $(0,0)$ $(\pi, 0)$, $(2\pi, 0)$

$f(\frac{\pi}{2}) = 1$

$f(\frac{3\pi}{2}) = 1$

abs. max $(\frac{\pi}{2}, 1)$, $(\frac{3\pi}{2}, 1)$.

**9.** Find all the critical points of the function f(x) = cos($x + \frac{1}{x}$).

$f'(x) = 0$ when $x = \pm 1$

$f'(x) = -\sin(x + \frac{1}{x})(1 - x^{-2})$

or $x = \frac{n\pi \pm \sqrt{n^2\pi^2 - 4}}{2}$ $\quad n \in \mathbb{Z} \setminus \{0\}$.

from Last HW Q12

$f(1) = \cos(2) \rightarrow (1, 2)$.

$f(-1) = \cos(-2) = \cos(2) \rightarrow (-1, 2)$

$\sin(x + \frac{1}{x}) = 0$ or $1 - x^{-2} = 0$

$x + \frac{1}{x} = n\pi$.

$x^2 + 1 = n\pi x$

$x^2 - n\pi x + 1 = 0$.

$x + \frac{1}{x} = \frac{n\pi \pm \sqrt{n^2\pi^2 - 4}}{2} + \frac{2}{n\pi \pm \sqrt{n^2\pi^2 - 4}}$

$(a \pm b)^2 \leftarrow$

$= \frac{(n\pi \pm \sqrt{n^2\pi^2-4})^2 + (2)(2)}{2(n\pi \pm \sqrt{n^2\pi^2-4})} = \frac{n^2\pi^2 \pm 2n\pi\sqrt{n^2\pi^2-4} + n^2\pi^2 \cancel{-4} \cancel{+4}}{2(n\pi \pm \sqrt{n^2\pi^2-4})} = \frac{2n^2\pi^2 + 2n\pi\sqrt{n^2\pi^2-4}}{2(n\pi \pm \sqrt{n^2\pi^2-4})} = \frac{2n\pi\cancel{(n\pi + \sqrt{n^2\pi^2-4})}}{2\cancel{(n\pi \pm \sqrt{n^2\pi^2-4})}}$

$= n\pi$.

$\cos(x + \frac{1}{x}) = \cos(n\pi) \Rightarrow$ n is odd $\Rightarrow -1$

n is even $\Rightarrow 1$

$\boxed{\begin{array}{ll} (\frac{n\pi \pm \sqrt{n^2\pi^2-4}}{2}, -1) & n \text{ is odd} \\ (\frac{n\pi \pm \sqrt{n^2\pi^2-4}}{2}, 1) & n \text{ is even} \end{array}}$