CALCULUS & VECTORS
VECTORS CLASS 3: APPLICATIONS OF VECTORS

MERITUS ACADEMY

This handout is designed for instructional purposes. Students should **read the chapter notes** for a comprehensive understanding.

1. Velocity.
Velocity of an object is the rate of change of its displacement over time. It has both **direction** and **magnitude**. The magnitude is called **speed**.

Concept 1 (Relative Velocity). When an object A is moving with velocity $\vec{v}_A$ and an object B is moving with velocity $\vec{v}_B$, then the **relative velocity** of B relative to A is

$$\vec{v}_{BA} = \vec{v}_B - \vec{v}_A. \tag{1.1}$$

It can be interpreted as the velocity of B in the eyes of A.

Example 1. A river is 1000 m wide and flows westward at 8 m/s. Janice is driving a motorboat heading always **perpendicular** to the current. The speed of the motorboat in still water is 24 m/s.

(a) Draw a diagram to illustrate the situation.
(b) What is the speed of the boat **relative to the ground**?
(c) How long does it take to cross the river?
(d) How much downstream does Janice reach the opposite bank?

Example 2 (Example 1.2)**.** A boat is trying to cross a river that flows with a velocity of 3 m/s east. The boat's speed in still water is 5 m/s. The boat aims to reach a point directly north on the opposite bank.

(a) Draw a diagram to illustrate the situation.
(b) Determine the relative velocity of the boat to the ground.
(c) Find the angle the boat must steer relative to the north to reach the destination on the opposite bank.
(d) Calculate the time it takes for the boat to cross the river if the river is 100 m wide.

Key words and phrases. Velocity, Force, Newton's Laws of Motion, Equilibrium, Work, Torque.

2. Force.
Force is a physics concept that describes the interaction between objects. In classical mechanics, **Newton's Laws of Motion** describe the relationship between the **motion of an object** and the **forces** acting on it.

Concept 2 (Newton's Laws of Motion). The First Law is also called the Law of Inertia.

1. An object remains at rest, or in motion at a constant speed in a **straight line**, unless acted upon by a **force**.
2. The product of the **acceleration** $\vec{a}$ and the **mass** m of an object equals the amount of net force $\vec{F}_{\text{net}}$ applied; i.e. $\vec{F}_{\text{net}} = m\vec{a}$.
3. Whenever one object A exerts a force $\vec{F}_A$ on a second object B, object B also exerts an **equal and opposite** force $\vec{F}_B$ on object A; i.e. $\vec{F}_A = -\vec{F}_B$.

The magnitude of force is measured in N (Newtons), where N = kg $\times$ m/s^2.

- Object at rest:
- Object in motion:

Concept 3 (Net Force & Equilibrium). When several forces $\vec{F}_k$ act on an object, we may interpret them as **a single combined force** acting on the object. This resultant force is the **net force** $\vec{F}_{\text{net}}$ given by **vector addition**:

$$\vec{F}_{\text{net}} = \sum_k \vec{F}_k. \tag{2.1}$$

An object is at **equilibrium** if $\vec{F}_{\text{net}} = \vec{0}$. If $\vec{F}_{\text{net}} \neq \vec{0}$, then according to Newton's second law, the object will accelerate. In this case, the **equilibrant** is the force vector $\vec{F}_e = -\vec{F}_{\text{net}}$. It represents the force that must be applied to bring the object back to equilibrium.

Example 3. Find the net force and the equilibrant of the following forces on a single object:
50 N [N], 60 N [S], and 40 N [SE]

Concept 4 (Equilibrium with Three Forces). In the special case with only three forces $\vec{F_1}$, $\vec{F_2}$, $\vec{F_3}$, they can produce equilibrium when $\vec{F_1} + \vec{F_2} + \vec{F_3} = \vec{0}$.

- By triangle rule for vector subtraction, it is the case if and only if they can be arranged geometrically to **form a triangle**.
- **Triangle inequality**: $a + b > c$, where a, b, c are lengths of a triangle.

Example 4 (Example 2.3)**.** Which of the following sets of forces with magnitude given could produce a state of equilibrium? If so, find the angles between the largest force and the other two.

(a) 3N, 40N, 20N
(b) 7N, 21N, 15N

Example 5 (Free-body Diagram)**.** A car of 2500 kg is stopped on a road inclined at 12° to the horizontal.

(a) Find the magnitude of the **force of friction** $\vec{F_f}$ that prevents it from sliding down the hill.
(b) Find the magnitude of the **normal force** $\vec{F_N}$ exerted on the car perpendicular to the road surface.

Example 6 (Tension)**.** An object weighing 30 kg is suspended by two wires of equal length 50 cm. How far apart must they be attached to the surface above so that the tension in each is 150 N.

3. Work. **Work** is done when a force $\vec{F}$ on an object causes the object to **move in displacement** $\vec{d}$.

Concept 5 (Work as Dot Product). Work W is a **scalar** defined to be the **dot product** of **force vector** $\vec{F}$ and a **displacement vector** $\vec{d}$,

$$W = \vec{F} \cdot \vec{d} = |\vec{F}||\vec{d}| \cos\theta, \tag{3.1}$$

where $\vec{F}$ is the force acting on an object, $\vec{d}$ is the displacement of the object in meter m, and $\theta = \angle(\vec{F}, \vec{d})$. The unit for work is J (Joules) and J = N·m.

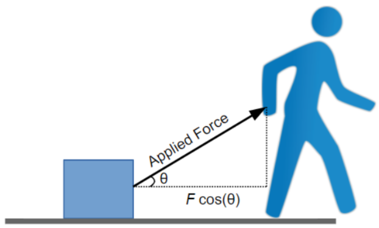


FIGURE 1. Work done

Example 7. A box is pulled a horizontal distance of 100 m by a force of 500 N applied at an angle of 30° to the horizontal line. Calculate the amount of work done.

Concept 6 (Work & Energy). Work is the measure of **energy transfer** when a force moves an object through a distance. When the work done is **negative**, it then **removes** energy from the system. Similarly, when the work done is **positive**, it **adds** energy to the system.

Example 8 (Example 3.2)**.** A wagon comes to a stop after moving a horizontal distance of 10 m by a force of 250 N applied downward at an angle of 20° to the horizontal line. Calculate the amount of work done. What is the interpretation of this answer?

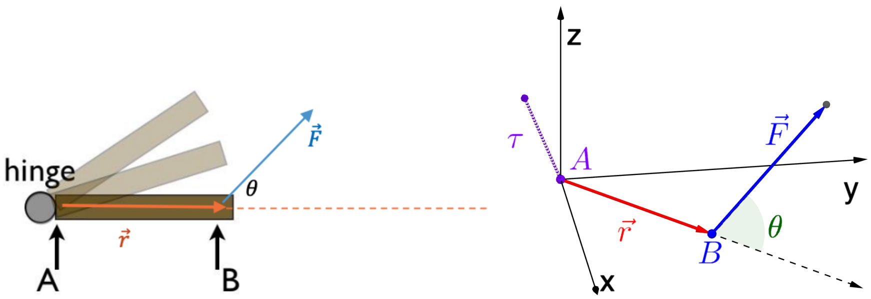
4. Torque. **Torque** is the **rotational** analogue of linear force.

- When a force $\vec{F}$ is applied and causes an object to **rotate** about a **pivot point**, resulting in **angular displacement** (in radians), then **torque** is generated.
- Torque is also called **moment of force** or **turning effect**, for it measures the force's **tendency to cause an object to rotate** about the pivot point.
- Since rotation can be clockwise and counter-clockwise, torque is a **vector**.

Concept 7 (Torque as Cross Product). In 3D space, **torque** τ is a **vector** defined as

$$\tau = \vec{r} \times \vec{F} = |\vec{r}||\vec{F}| \sin\theta, \tag{4.1}$$

where $\vec{r}$ is the position vector from the pivot point A where torque is measured to the point B, where the force $\vec{F}$ is applied to cause rotation of the object, and $\theta = \angle(\vec{r}, \vec{F})$. The unit of torque is m·N.



(a) Rotating a hinge (b) Vector perspective

FIGURE 2. Torque

Example 9. You are to apply a force of 200 N to the arm of a wrench to loosen a bolt. Determine the torque if you apply the force in the following ways:
(a) 10 cm from the bolt perpendicularly to the wrench
(b) 15 cm from the bolt at an angle of 60° to the wrench
Which method is more effective?

Example 10. Daniel applies a force $\vec{F_1}$ of 20 N perpendicularly to rotate a merry-go-round of radius 2 m in counterclockwise direction. To keep it in equilibrium, you apply a force $\vec{F_2}$ at an angle of 120° to the arm that is 1 m from the center. Calculate the magnitude of $\vec{F_2}$.