Project 1

Alessio Canclini, Filip von der Lippe

(Dated: September 9, 2022)

List a link to your github repository here!

PROBLEM 1

$$-\frac{d^2u}{dx^2} = f(x) \tag{1}$$

- source term: $f(x) = 100e^{-10x}$
- x range $x \in [0, 1]$
- boundary conditions: $u(0) = 0$ and $u(1) = 0$

$$u(x) = 1 - (1 - e^{-10})x - e^{-10x} \tag{2}$$

Checking analytically that an exact solution to Eq. 1 is given by Eq. 2:

$$\begin{aligned}
\frac{du}{dx} &= 1 - e^{-10} + 10e^{-10x} \\
\frac{d^2u}{dx^2} &= -100e^{-10x} \\
-\frac{d^2u}{dx^2} &= 100e^{-10x} \\
-\frac{d^2u}{dx^2} &= f(x)
\end{aligned}$$

PROBLEM 2

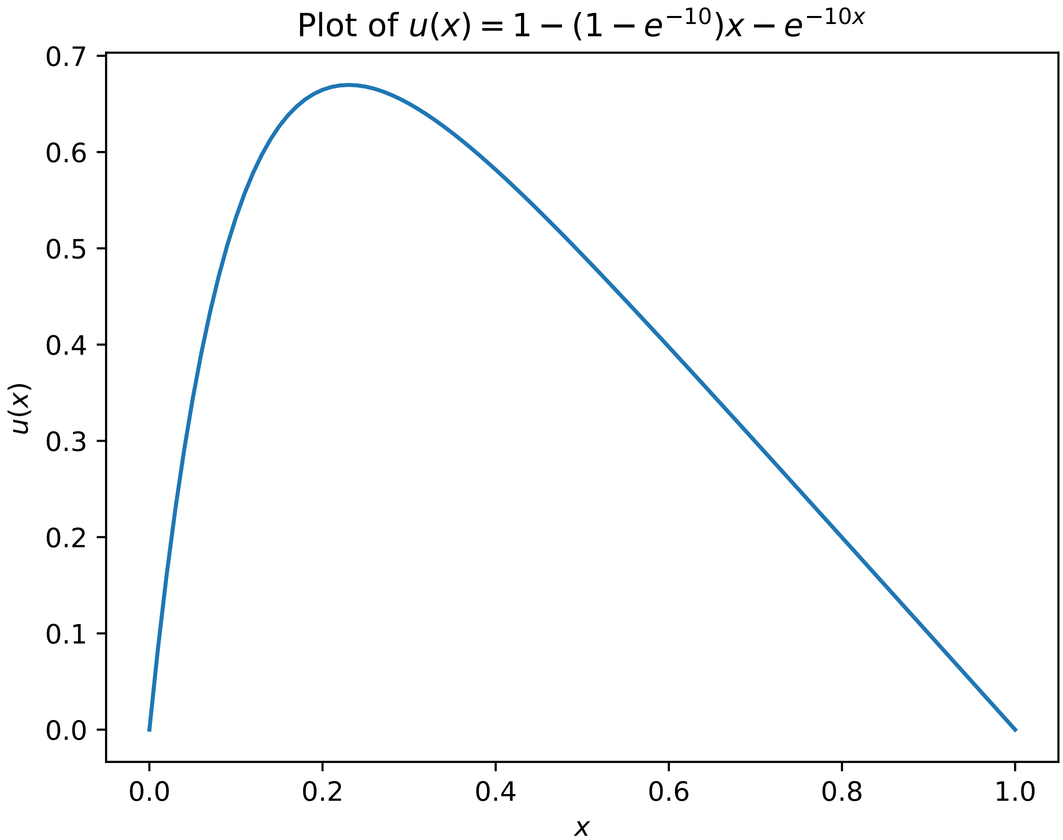


FIG. 1: Plot of $u(x)$.

PROBLEM 3

By using the taylor approximation of the second derivative we can discretize the second derivative in the Poission equation:

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + O(h^2)$$

Here we have the stepsize h and the truncation error O. The one-dimensional Poisson equation can then be written for the approximated version of u as v like:

$$-\frac{v_{i+1} - 2v_i + v_{i-1}}{h^2} = f_i \tag{3}$$

PROBLEM 4

We can rewrite the discretized equation as a matrix equation for $n+1$ number of points and $n-1$ unknown points (v_0 and v_n are known) with the $n-1 \times n-1$ matrix $\boldsymbol{A}$. We rewrite the discretized Poisson funcion:

$$\begin{aligned}
2v_1 - v_2 &= f_1 h^2 \\
-v_1 + 2v_2 - v_3 &= f_2 h^2 \\
&\vdots \\
-v_{n-3} + 2v_{n-2} - v_{n-1} &= f_{n-2} h^2 \\
-v_{n-2} + 2v_{n-1} &= f_{n-1} h^2
\end{aligned}$$

This can be written in terms of the following matrix equation where we rewrite $f_i h^2$ for $i = 1, 2, \ldots n-1$ as g_i

$$\begin{bmatrix} 2 & -1 & 0 & 0 & \ldots & 0 \\ -1 & 2 & -1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & -1 & 2 & -1 \\ 0 & 0 & \ldots & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-2} \\ v_{n-1} \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_{n-2} \\ g_{n-1} \end{bmatrix}$$

PROBLEM 5

a) Since the vector $\vec{v*}$ of length m represents a complete solution of the descretized Poisson equation it contains all values in $\vec{v}$ in addition to the boundary conditions. The relation bethween n and m is therfore $m = n + 2$.

b) Solving $\boldsymbol{A}\vec{v} = \vec{g}$ for $\vec{v}$ gives us all but the first an last value in $\vec{v*}$. So all but the boundary values.

PROBLEM 6

a) We can use the Thomas algorithm to row reduce the matrix $\boldsymbol{A}$ to give us a solution of the equation $\boldsymbol{A}\vec{v} = \vec{g}$. This is done in two steps. We first define the three diagonals as three vectors. The sub diagonal $\vec{a}$ the diagonal $\vec{b}$ and the superdiagonal $\vec{c}$ where the i element in theese vectors corresponds to the i row of the matrix $\boldsymbol{A}$. We define n unknowns and the matrix $\boldsymbol{A}$ as $n \times n$

i) The first step is forwards substitution. We define a numer w for each step and overwrite the i element of both b and g starting at index 2. This means overwrite the values in the original vectors b and g, but at the same time we dont have to define new vectors to store new values. For large n this will reduce both the computation time and memory usage for the data machine

for $i = 2, .., n$:

$$\begin{aligned}
w &= \frac{a_i}{b_{i-1}} \\
b_i &= b_i - w c_{i-1} \\
g_i &= g_i - w g_{i-1}
\end{aligned}$$

ii) The second and last step is back substitution where we find an expression for $\vec{v}$. We start at our last element and work our way backwards:

$$\begin{aligned}
v_n &= \frac{g_n}{b_n} \\
v_i &= \frac{g_i - c_i v_{i+1}}{b_i} \quad \text{for } i = n-1, ..., 1
\end{aligned}$$

b) We find the number of FLOPs for this algorithm by counting the number of floating point operations the computer has to do. For the first step we have 3 FLOPs (1 subtraction 1 division and one multiplication) for defining b_i and g_i each. Since we loop this operation $n-1$ times we end up with a totalt number of $6(n-1)$ FLOPs.

For back substitution we have 1 FLOP calculating v_n, and three FLOPs calculating v_i $n-1$ times. This gives us $3(n-1)+1$ FLOPs

The total FLOPs of the general algorithm is $9n-8$

PROBLEM 7

PROBLEM 8

a) Figure 2 show the logarithm of the absolute error as a function of x_i. The different graphs show $\log_{10}(\Delta_i)$ for differen stepsizes.

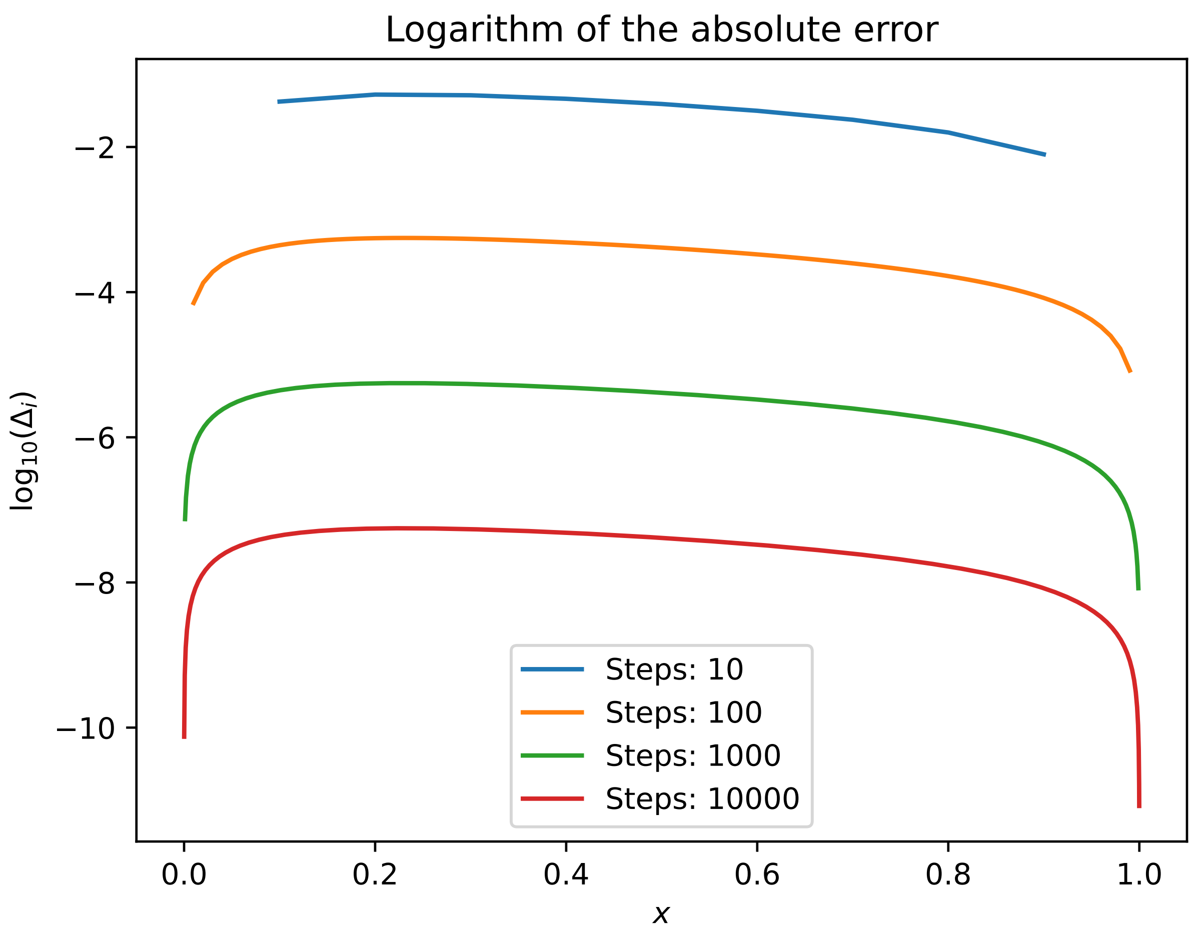


FIG. 2

b) Figure 3 below, shows the logarithm of the relative error for x_i. Again presented with a graph for each stepsize.

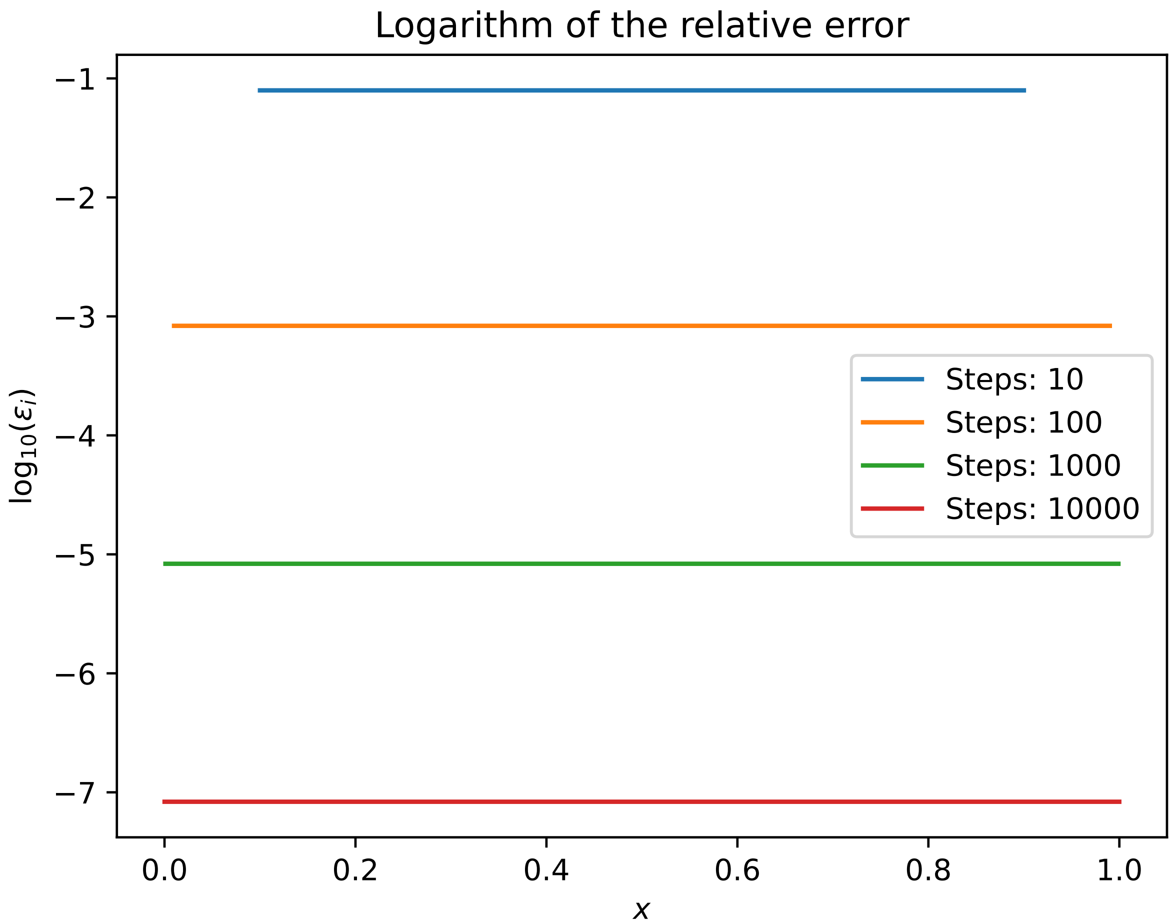


FIG. 3

c) Table ?? shows the maximum relative error for n_steps, up to $n_steps = 10^7$

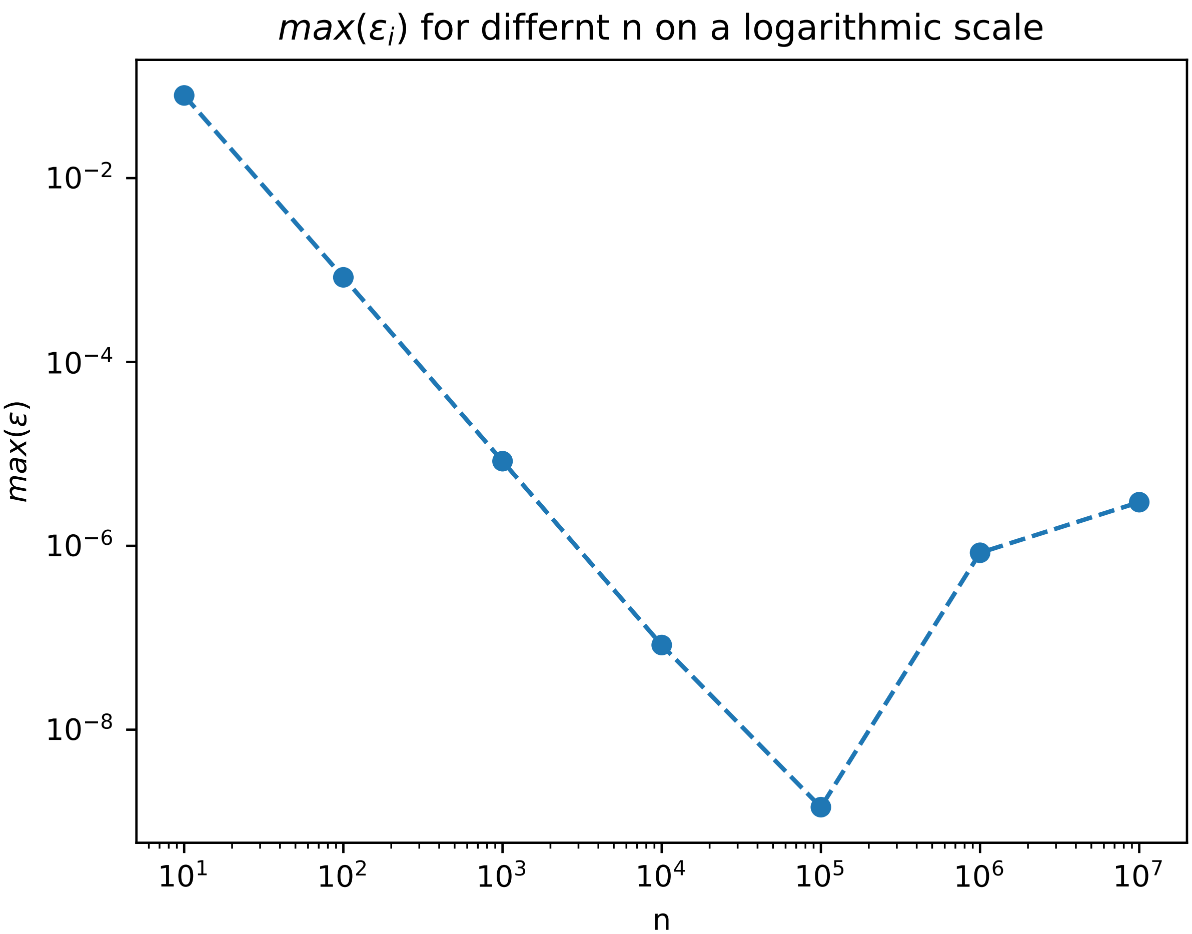


FIG. 4

As we can see from the plot in figure 4, our results show that the ideal stepsize is $n_steps = 10^5$ to minimize the maximum relative error. Hereby avoiding both major roundoff errors and major truncation errors.

PROBLEM 9

a) For the special case we dont need to do new computations for every i element of the vectors $\vec{a}$ and $\vec{c}$ and thus don't need to assign and use these variabeles.

   i) The first step is forwards substitution. We have to define $b_1 = 2$ and then loop over the rest:

$$b_i = b_i + \frac{1}{b_{i-1}} \quad \text{for } i = 2, .., n$$
$$g_i = g_i + \frac{1}{b_{i-1}} g_{i-1} \quad \text{for } i = 2, .., n$$

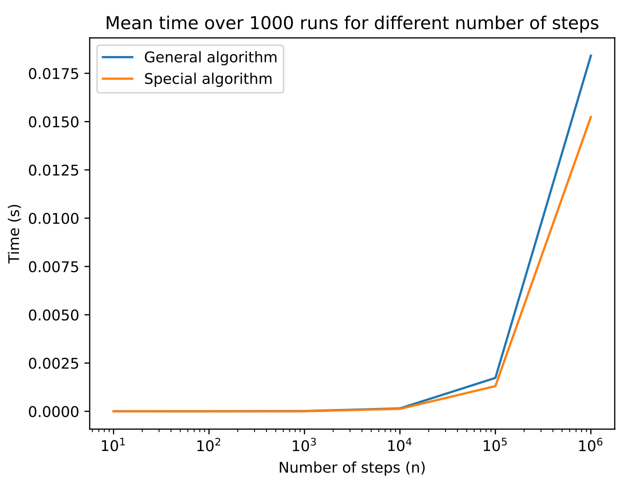
   ii) The second and last step is back substitution where we find an expression for $\vec{v}$. We start at our last element and work our way backwards:

$$v_n = \frac{g_n}{b_n}$$
$$v_i = \frac{g_i + v_{i+1}}{b_i} \quad \text{for } i = n-1, ..., 1$$

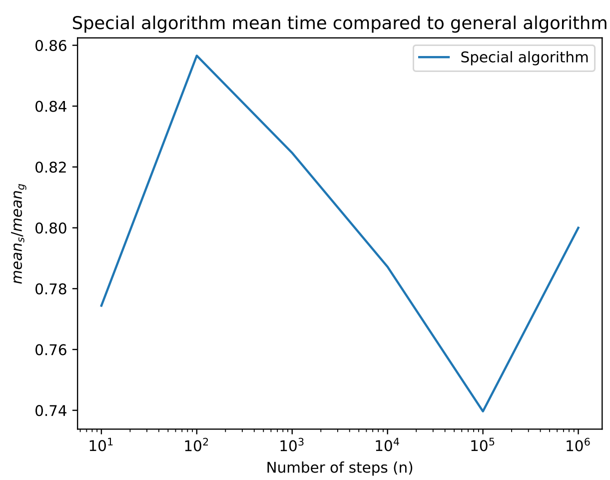
b) Since we know that $a, c = -1$ we are abel to reduce the amount of FLOPs compared to the general algorithm. For the forward substitution we have $2(n-1)$ FLOPs to compute b_i and the same for g_i. The back substitution requires 1 FLOP for v_n and $2(n-1)$ FLOPs for v_i. This gives us a total of $6n-5$ FLOPs for the special algorithm.

PROBLEM 10

The plots presented below show results from run timing tests done for the general and special algorithm. The test has been run 1000 times for each stepsize $n = 10^i$ for $i = 1, ..., 6$. Figure 6 shows the standard deviation for these same tests.



(a)



(b)

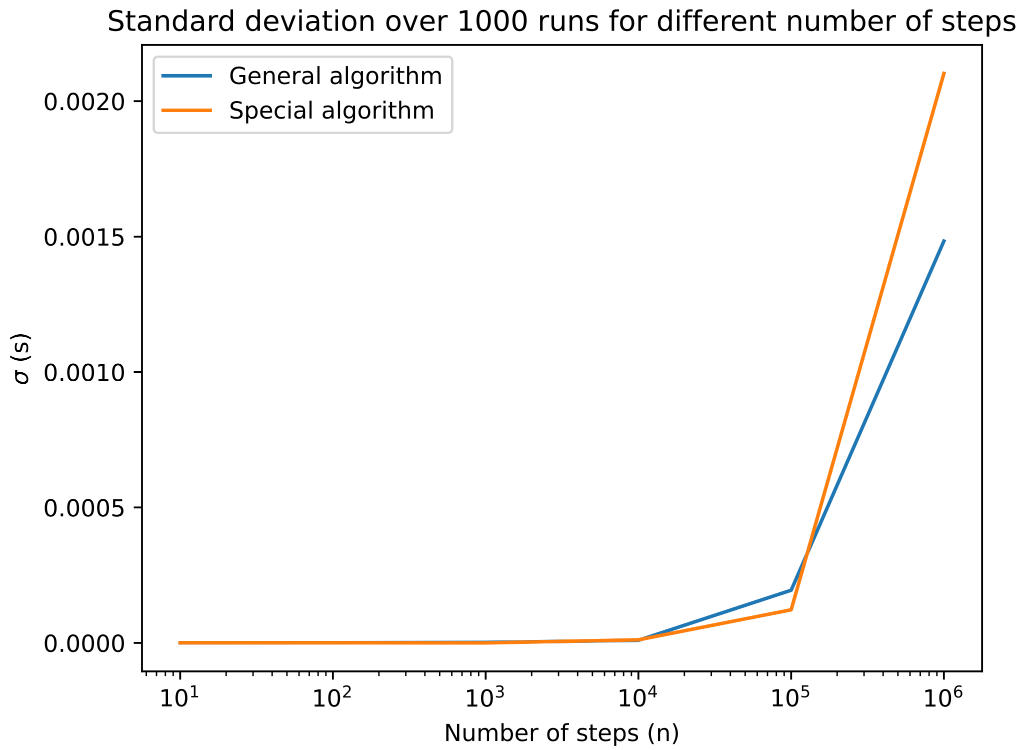

Standard deviation over 1000 runs for different number of steps
General algorithm
Special algorithm
0.0020
0.0015
0.0010
0.0005
0.0000
σ (s)
10^1
10^2
10^3
10^4
10^5
10^6
Number of steps (n)

FIG. 6