

Project 1

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List a link to your github repository here!

PROBLEM 1

$$-\frac{d^2u}{dx^2} = f(x) \tag{1}$$

- source term: $f(x) = 100e^{-10x}$
- x range $x \in [0, 1]$
- boundary conditions: $u(0) = 0$ and $u(1) = 0$

$$u(x) = 1 - (1 - e^{-10})x - e^{-10x} \tag{2}$$

Checking analytically that an exact solution to Eq. 1 is given by Eq. 2.

$$\begin{aligned} \frac{du}{dx} &= 1 - e^{-10} + 10e^{-10x} \\ \frac{d^2u}{dx^2} &= -100e^{-10x} \\ -\frac{d^2u}{dx^2} &= 100e^{-10x} \\ -\frac{d^2u}{dx^2} &= f(x) \end{aligned}$$

PROBLEM 2

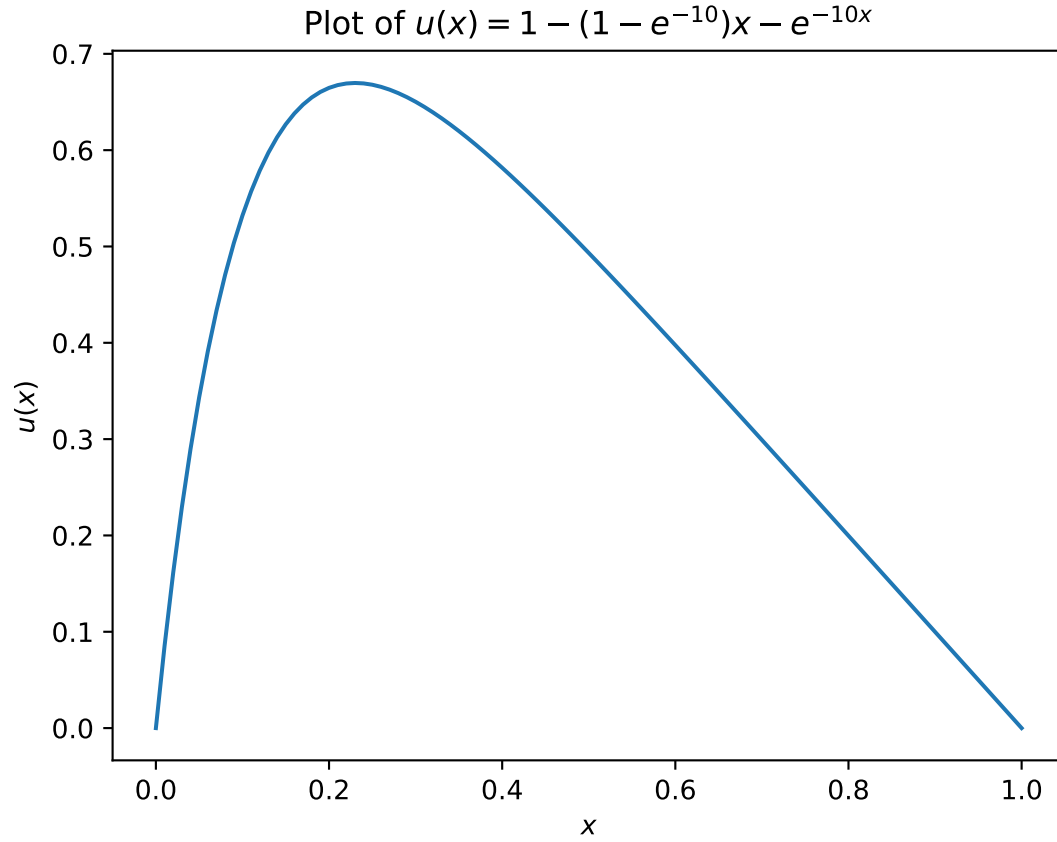


FIG. 1: Plot of $u(x)$.

PROBLEM 3

By using the Taylor approximation of the second derivative we can discretize the second derivative in the Poisson equation:

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + O(h^2)$$

Here we have the stepsize h and the truncation error O . The one-dimensional Poisson equation can then be written for the approximated version of u as v like:

$$-\frac{v_{i+1} - 2v_i + v_{i-1}}{h^2} = f_i \quad (3)$$

PROBLEM 4

We can rewrite the discretized equation as a matrix equation for $n + 1$ number of points and $n - 1$ unknown points (v_0 and v_n are known) with the $n - 1 \times n - 1$ matrix \mathbf{A} . We rewrite the discretized Poisson function:

$$\begin{aligned} 2v_1 - v_2 &= f_1 h^2 \\ -v_1 + 2v_2 - v_3 &= f_2 h^2 \\ &\vdots \\ -v_{n-3} + 2v_{n-2} - v_{n-1} &= f_{n-2} h^2 \\ -v_{n-2} + 2v_{n-1} &= f_{n-1} h^2 \end{aligned}$$

This can be written in terms of the following matrix equation where we rewrite $f_i h^2$ for $i = 1, 2, \dots, n - 1$ as g_i

$$\begin{bmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & \dots & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-2} \\ v_{n-1} \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_{n-2} \\ g_{n-1} \end{bmatrix}$$

PROBLEM 5

- Since the vector \vec{v}^* of length m represents a complete solution of the discretized Poisson equation it contains all values in \vec{v} in addition to the boundary conditions. The relation between n and m is therefore $m = n + 2$.
- Solving $\mathbf{A}\vec{v} = \vec{g}$ for \vec{v} gives us all but the first and last value in \vec{v}^* . So all but the boundary values.

PROBLEM 6

- We can use the Thomas algorithm to row reduce the matrix \mathbf{A} to give us a solution of the equation $\mathbf{A}\vec{v} = \vec{g}$. This is done in two steps. We first define the three diagonals as three vectors. The sub diagonal \vec{a} the diagonal \vec{b} and the superdiagonal \vec{c} where the i element in these vectors corresponds to the i row of the matrix \mathbf{A} . We define n unknowns and the matrix \mathbf{A} as $n \times n$

- The first step is forwards substitution. We define a number w for each step and overwrite the i element of both b and g starting at index 2. This means overwrite the values in the original vectors b and g , but at the same time we don't have to define new vectors to store new values. For large n this will reduce both the computation time and memory usage for the data machine

for $i = 2, \dots, n$:

$$\begin{aligned} w &= \frac{a_i}{b_{i-1}} \\ b_i &= b_i - w c_{i-1} \\ g_i &= g_i - w g_{i-1} \end{aligned}$$

- The second and last step is back substitution where we find an expression for \vec{v} . We start at our last element and work our way backwards:

$$\begin{aligned} v_n &= \frac{g_n}{b_n} \\ v_i &= \frac{g_i - c_i v_{i+1}}{b_i} \quad \text{for } i = n - 1, \dots, 1 \end{aligned}$$

- b) We find the number of FLOPs for this algorithm by counting the number of floating point operations the computer has to do. For the first step we have 3 FLOPs (1 subtraction 1 division and one multiplication) for defining b_i and g_i each. Since we loop this operation $n - 1$ times we end up with a total number of $6(n - 1)$ FLOPs.

For back substitution we have 1 FLOP calculating v_n , and three FLOPs calculating v_i $n - 1$ times. This gives us $3(n - 1) + 1$ FLOPs

The total FLOPs of the general algorithm is $9n - 8$

PROBLEM 7

PROBLEM 8

PROBLEM 9

- a) For the special case we don't need to do new computations for every i element of the vectors \vec{a} and \vec{c} and thus don't need to assign and use these variables.

- i) The first step is forwards substitution. We have to define $b_1 = 2$ and then loop over the rest:

$$b_i = b_i + \frac{1}{b_{i-1}} \quad \text{for } i = 2, \dots, n$$

$$g_i = g_i + \frac{1}{b_{i-1}} g_{i-1} \quad \text{for } i = 2, \dots, n$$

- ii) The second and last step is back substitution where we find an expression for \vec{v} . We start at our last element and work our way backwards:

$$v_n = \frac{g_n}{b_n}$$

$$v_i = \frac{g_i + v_{i+1}}{b_i} \quad \text{for } i = n - 1, \dots, 1$$

- b) Since we know that $a, c = -1$ we are able to reduce the amount of FLOPs compared to the general algorithm. For the forward substitution we have $2(n - 1)$ FLOPs to compute b_i and the same for g_i . The back substitution requires 1 FLOP for v_n and $2(n - 1)$ FLOPs for v_i . This gives us a total of $6n - 5$ FLOPs for the special algorithm.

PROBLEM 10

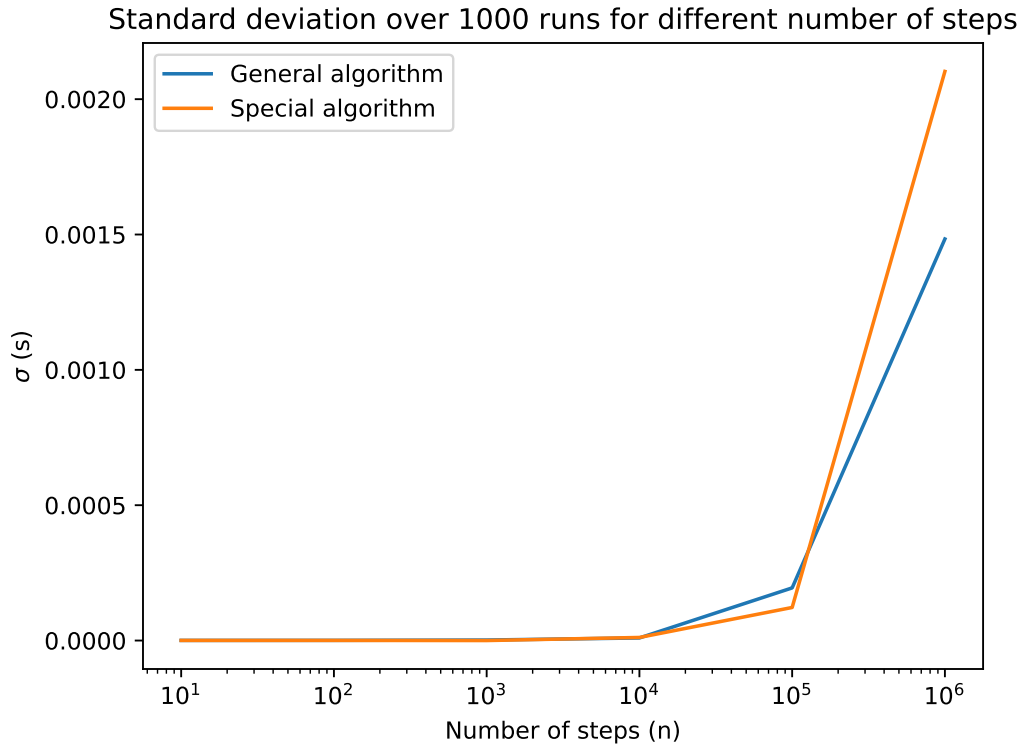
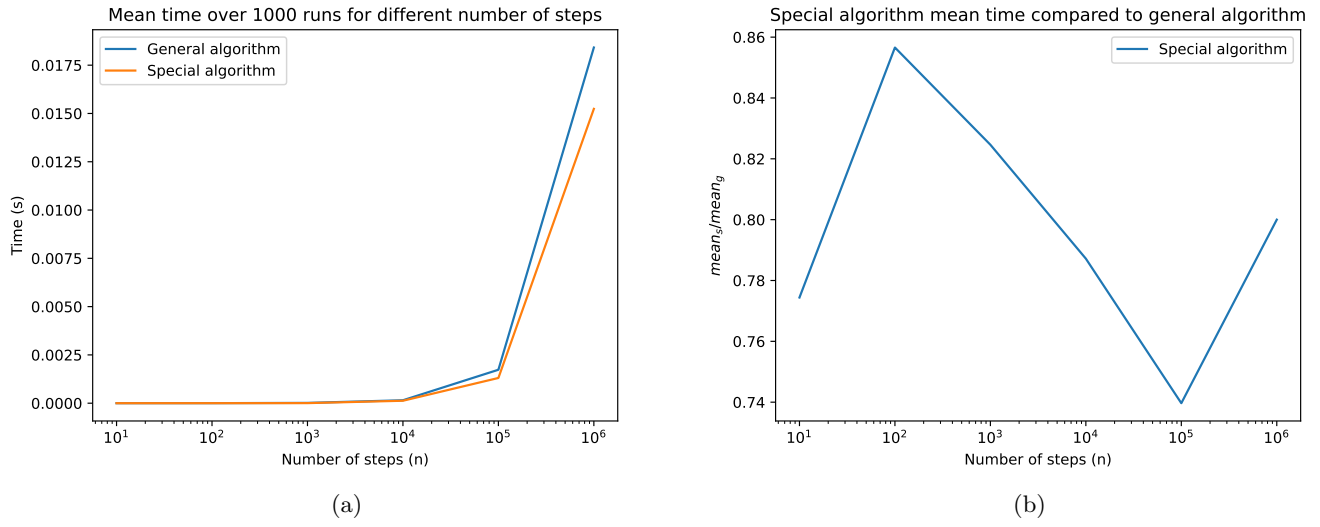


FIG. 3