

# Numerical simulation of a Penning trap

Alessio Canclini, Filip von der Lippe  
(Dated: October 19, 2022)

## I. INTRODUCTION

The purpose of this report is to present the study of the effects of a Penning trap through numerical simulations. The Penning trap is a device used to store or "trap" charged particles using static electric and magnetic fields as shown in figure 1. These particles can then be used for a variety of experiments. Examples of this are the ALPHA, AEGIS and BASE experiments at CERN, these use Penning traps to control antimatter. The electric field is generated by two end caps (a), at the top and bottom, and a ring (b) (figure 1 only shows the ring cross-section). This electric field restricts the particles' movement in the  $z$  direction and the additional homogenous magnetic field hinders particles escaping in the  $xy$ -plane (radial direction) if it is strong enough. The magnetic field is set by a cylinder magnet (c) (figure 1 again only shows the ring cross-section).

Materials to construct a physical Penning trap are very costly, we will therefore be using a numerical approach to simulate a Penning trap. To implement such a simulation we will be working with a system of coupled non-linear differential equations. These are very difficult and often impossible to solve analytically. An example some readers might be familiar with are the famous Navier-Stokes equations, the solving of which would be rewarded with a million dollar prize. In addition to the material cost, the complexity of the equations therefore also leads us to the use of numerical methods.

Section II will describe the mathematical and physical background as well as concrete algorithms which in this case will be implemented in C++, but can be written in any programming language.

In section III we present...

A detailed discussion of the algorithms' and results is presented in section IV, followed by a summary and potential for further experiments in section V.

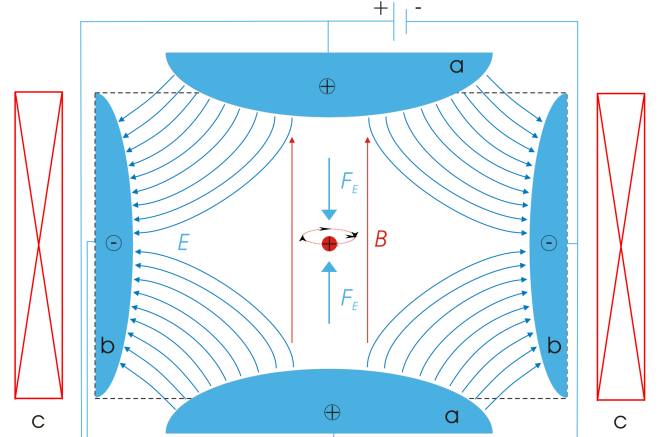


FIG. 1. This figure shows the idea of a Penning trap with a red positively charged particle in the center. Here blue lines represent the electric field generated by a quadrupole consisting of end caps (a) and a ring electrode (b). The red lines represent the magnetic field created by a surrounding cylinder magnet (c). Illustration by Arian Kriesch taken from Wikimedia Commons.

## II. METHODS

The physical laws used to implement the Penning trap simulation will be from electrodynamics and classical mechanics, we will not take quantum aspects into account. The following equations will be used:

$$\mathbf{E} = -\nabla V \quad (1)$$

$\mathbf{E}$  is the electric field and  $V$  the electric potential.

$$\mathbf{E}(\mathbf{r}) = k_e \sum_{j=1}^n q_j \frac{\mathbf{r} - \mathbf{r}_j}{|\mathbf{r} - \mathbf{r}_j|^3} \quad (2)$$

$\mathbf{E}(\mathbf{r})$  is the electric field at a point  $\mathbf{r}$ . This is set up by point charges  $q_1, \dots, q_n$  at points  $\mathbf{r}_1, \dots, \mathbf{r}_n$ .

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B} \quad (3)$$

This is the **Lorentz force**, the force  $\mathbf{F}$  on a particle with charge  $q$ , an electric field  $\mathbf{E}$ , magnetic field  $\mathbf{B}$  and velocity of the particle  $\mathbf{v}$ .

$$m\ddot{\mathbf{r}} = \sum_i \mathbf{F}_i \quad (4)$$

This is Newton's second law. Here  $m$  is the mass of the particle and  $\ddot{\mathbf{r}} \equiv \frac{d^2\mathbf{r}}{dt^2}$ . Famously expressing that the sum of forces equals mass times acceleration.

$$V(x, y, z) = \frac{V_0}{2d^2}(2z^2 - x^2 - y^2) \quad (5)$$

For this experiment we will be considering an ideal Penning trap for which the electric field  $\mathbf{E}$  is given by the electric potential  $V$ . Here  $V_0$  is the potential applied to the electrodes.  $d = \sqrt{z_0^2 + r_0^2/2}$  is the characteristic dimension representing the length scale for the region between electrodes. Here  $z_0$  is distance from the center to the end caps (a) and  $r_0$  is the distance from the center to the surrounding ring (b).

$$\mathbf{B} = B_0\hat{e}_z = (0, 0, B_0) \quad (6)$$

$\mathbf{B}$  is the homogenous magnetic field and is dictated by the field strength  $B_0$ . With  $B_0 > 0$ .

Now starting from Newton's second law and using the equations above we can express the time evolution of the particles motion. The sum of forces will be the Lonretz force, putting Eq. 3 into Eq. 4 leads to:

$$m\ddot{\mathbf{r}} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B} \quad (7)$$

Here  $\ddot{\mathbf{r}} = (\ddot{x}, \ddot{y}, \ddot{z})$  and  $\mathbf{v} = (\dot{x}, \dot{y}, \dot{z})$ . Putting Eq. 5 into Eq. 1 gives us:

$$\mathbf{E} = \left(x\frac{v_0}{d^2}, y\frac{v_0}{d^2}, -2z\frac{v_0}{d^2}\right) \quad (8)$$

Now looking at  $q\mathbf{v} \times \mathbf{B}$  we have:

$$(q\dot{x}, q\dot{y}, q\dot{z}) \times (0, 0, B_0) = (B_0q\dot{y}, -B_0q\dot{x}, 0) \quad (9)$$

Finally using results from Eq. 8 and 9 in Eq. 7 results in:

$$m \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = \begin{pmatrix} qx\frac{v_0}{d^2} \\ qy\frac{v_0}{d^2} \\ -2qz\frac{v_0}{d^2} \end{pmatrix} + \begin{pmatrix} B_0q\dot{y} \\ -B_0q\dot{x} \\ 0 \end{pmatrix}$$

Rewriting this as a set of equations leaves us with:

$$\ddot{x} - w_0\dot{y} - \frac{1}{2}w_z^2x = 0 \quad (10)$$

$$\ddot{y} + w_0\dot{x} - \frac{1}{2}w_z^2y = 0 \quad (11)$$

$$\ddot{z} + w_z^2z = 0 \quad (12)$$

Where  $w_0 = \frac{qB_0}{m}$  and  $w_z^2 = \frac{2qV_0}{md^2}$ . Taking a closer look at Eq. 12 we see that the general solution is:

$$z = A \cos(w_z^2t) + B \sin(w_z^2t) \quad (13)$$

Eq. 10 and 11 are coupled, thus introducing a challenge. This can be resolved by introducing a complex function  $f(t) = x(t) + iy(t)$  and rewriting them as a single differential equation.

## The algorithm

## III. RESULTS

## IV. DISCUSSION

*Note that you are free to merge the presentation and discussion of the results into a single section of your report. This can in many cases lead to a more fluid presentation. If you do this, we recommend you use "Results and discussion" or similar for the section title.*

From table ??, we note that our implementation reproduces the analytical results to four digits precision when the integration range is divided into  $n = 10^4$  subintervals. This indicates that that our implementation of the algorithm is correct.

From figure ??, we see that  $\log_{10}(\epsilon)$  decreases linearly with  $\log_2(n)$ . From this, it should be possible to extract the convergence rate of our implementation of the midpoint rule. From a theoretical point of view we know that the midpoint rule should have a convergence rate of  $\mathcal{O}(h^2)$ . To properly verify our implementation, we should have estimated the convergence rate from our results and compared it to this theoretical rate. Without doing so, we cannot know that the our implementation of the algorithm is correct, even though we have seen that the numerical approximation converges to the correct answer in ??.

*Although this is a somewhat silly example, please note the following: We are to-the-point in our discussion of the results, and we only make strong claims about what we are actually certain about. In the discussion it is important to try to be as concise as possible — long paragraphs that only make very general points are typically of limited interest. Note that we also highlight aspects of our analysis that could have been improved and that might form a topic for future work.*

## V. CONCLUSION

*In this section we state three things in a concise manner: what we have done, what we have found, and what should or could be done in the future.*

We have investigated an implementation of the midpoint rule for numerical integration. As a first validation test we have checked that our implementation of the method reproduces the analytical result for the definite integral of  $f(x) = x^3$  on  $x \in [0, 1]$ , achieving a four-digit precision when the integration range is divided into  $n = 10^4$  subintervals. Furthermore, we have presented results for how the relative error of the method varies with the number of subintervals. To use these results to extract a precise estimate for the convergence rate of the method remains a topic for future work. As such, while our implementation of the midpoint rule has passed the initial validation tests, more work is needed to fully assess

the validity of the implementation.