材料力学

第八章 组合变形

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§ 8–1　概述

§ 8–2　双对称轴梁非对称弯曲

§ 8–3　拉伸(压缩)与弯曲的组合

§ 8–4　偏心拉（压）· 截面核心

§ 8–5　弯曲与扭转的组合

§8–1　概 述

一、基本变形 ：

拉伸（压缩）、扭转、弯曲

二、组合变形 ：

两种或两种以上基本变形的组合。

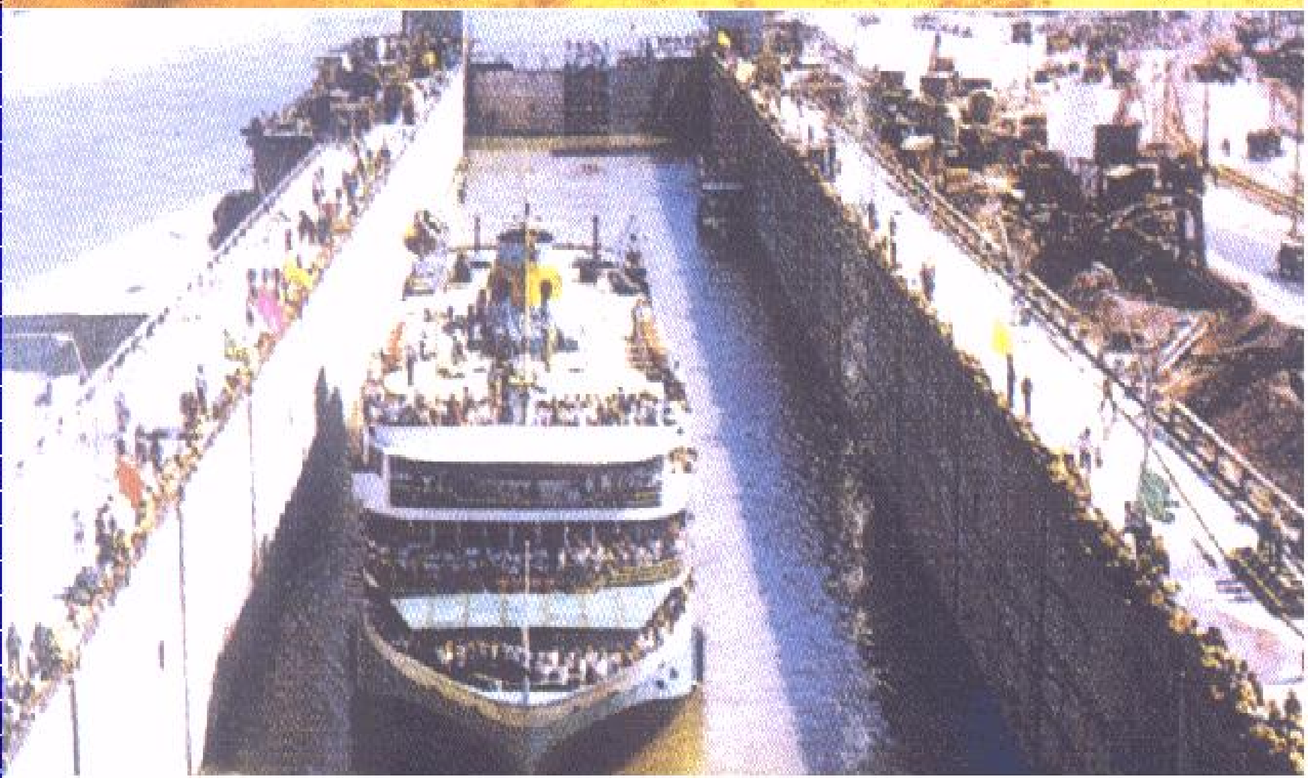
(1)拉伸（压缩）和弯曲的组合；

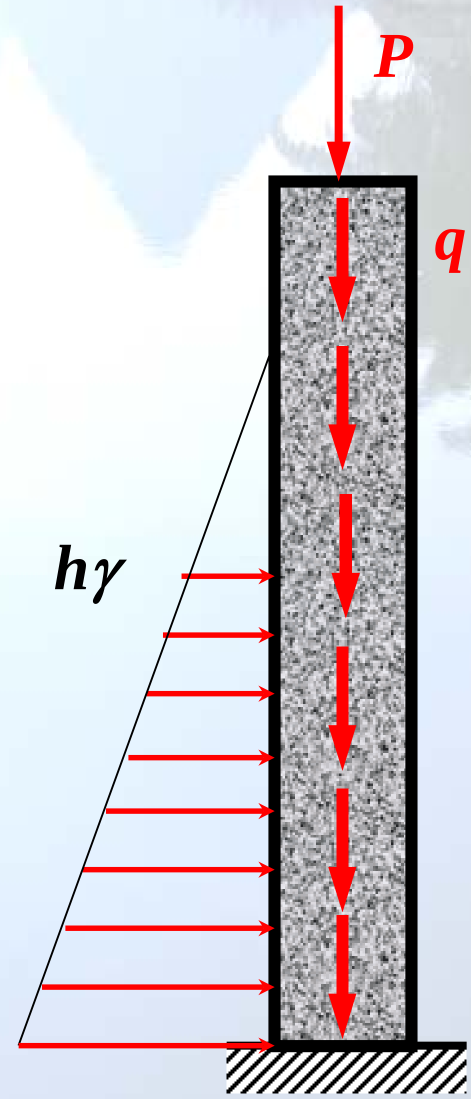
(2)拉伸（压缩）和扭转的组合；

(3)弯曲和扭转的组合；

(4)弯曲和弯曲的组合；

(5)拉、弯、扭组合。





水坝

三、组合变形的研究方法 —— 叠加原理

①外力分析：外力向形心简化并沿主惯性轴分解

②内力分析：求每个外力分量对应的内力方程和内力图，确定危险面。

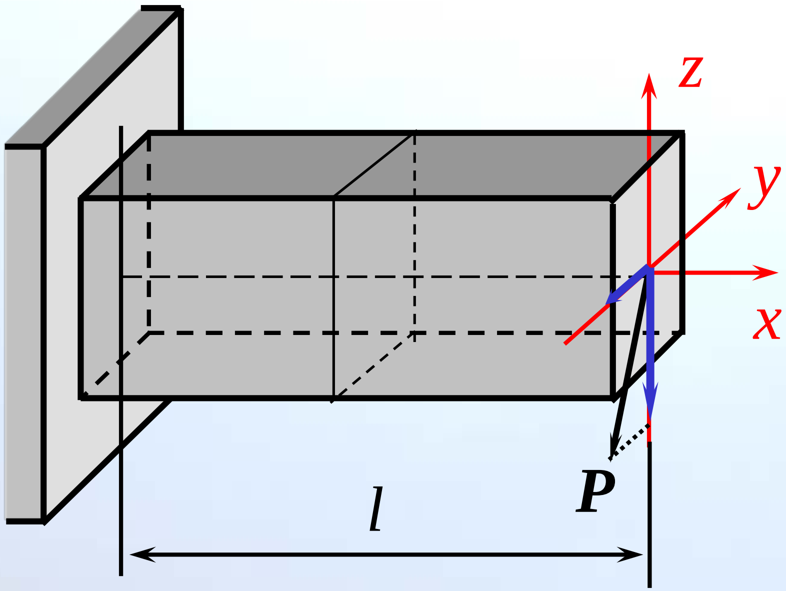
③应力分析：画危险面应力分布图，叠加，建立危险点的强度条件。

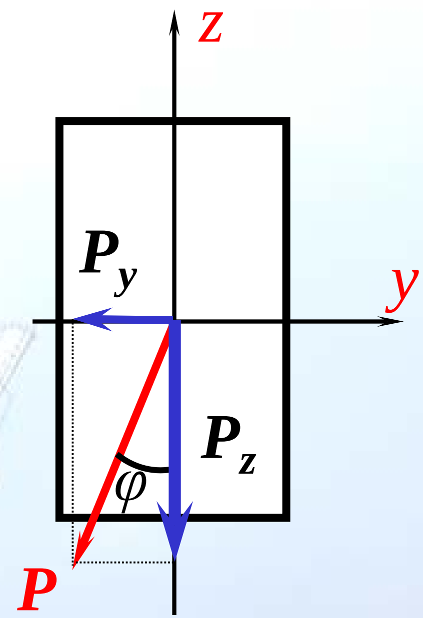
§ 8–2　双对称轴梁非对称弯曲

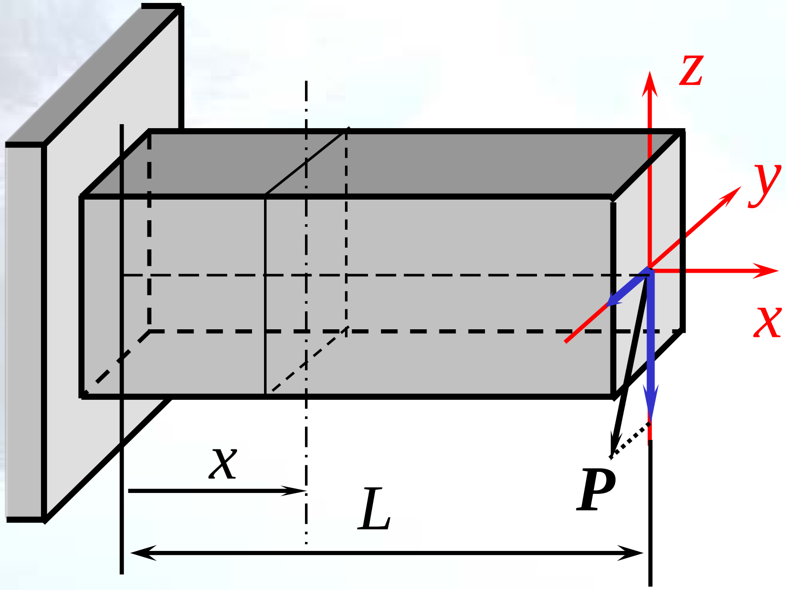
1. 分解：将外载沿横截面的两个形心主轴分解，于是得到两个正交的平面弯曲。

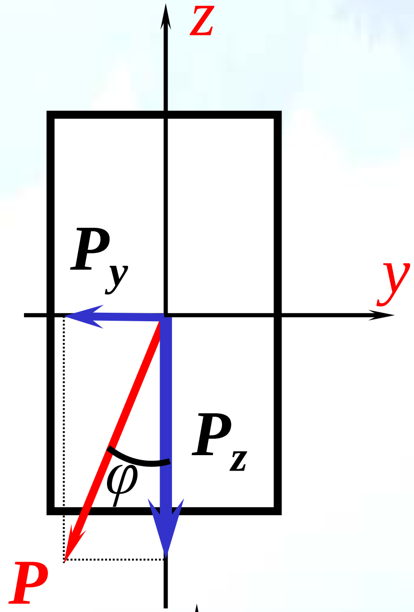
$P_y = P\sin\varphi$

$P_z = P\cos\varphi$









铅垂面内：

$$M_y = P_Z(l-x)$$

$$= P\cos\varphi\cdot(l-x)$$

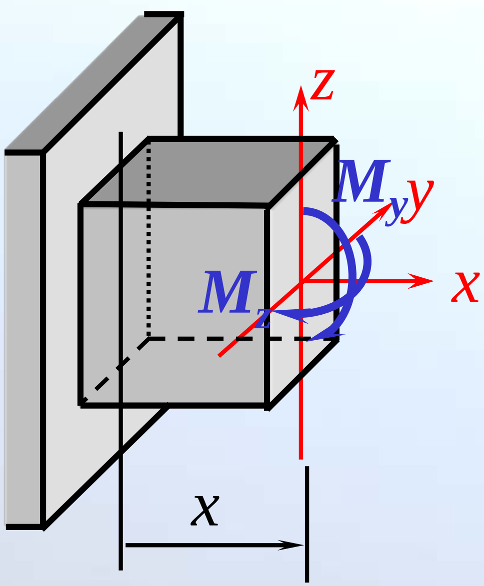
$$= M\cos\varphi$$

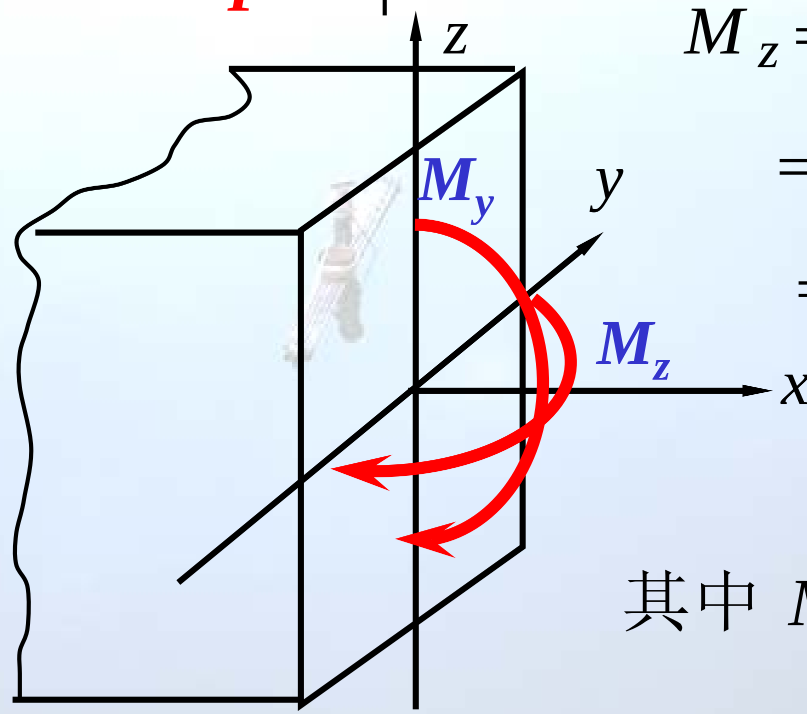
水平面内：

$$M_z = P_y(l-x)$$

$$= P\sin\varphi\cdot(l-x)$$

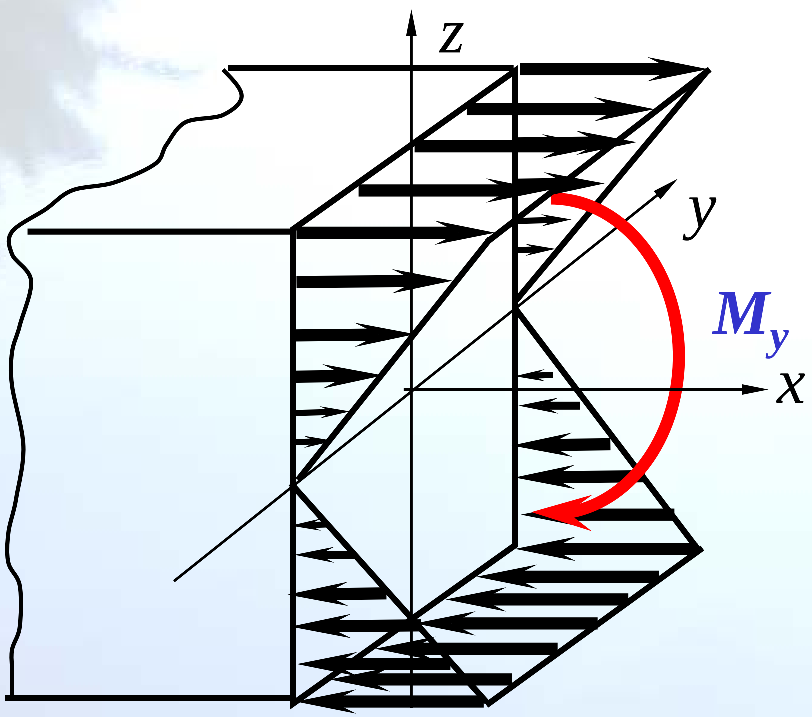
$$= M\sin\varphi$$

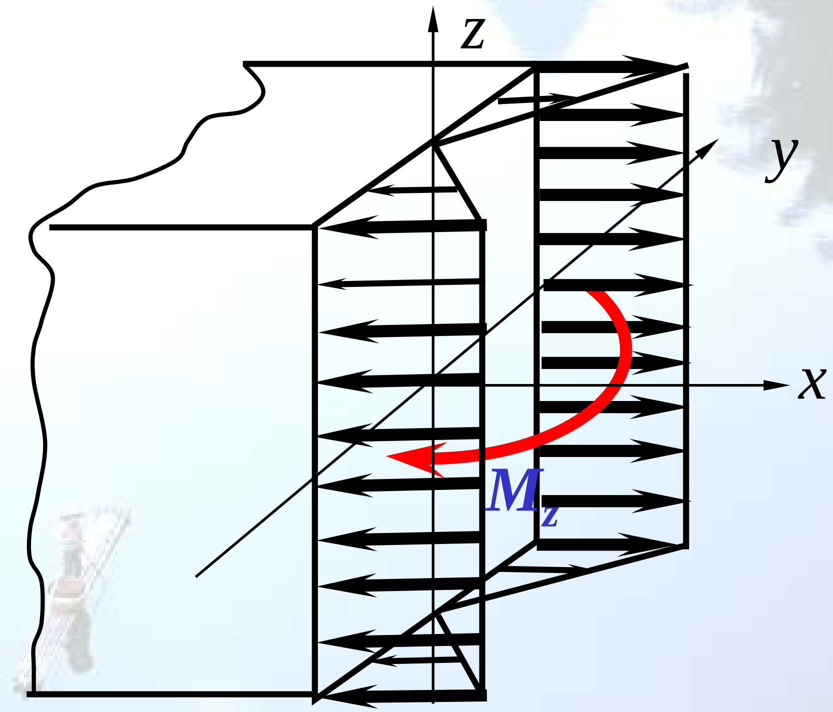




其中 $M = P(l-x)$

2. 叠加：对两个平面弯曲分别进行研究；然后将计算结果叠加起来。

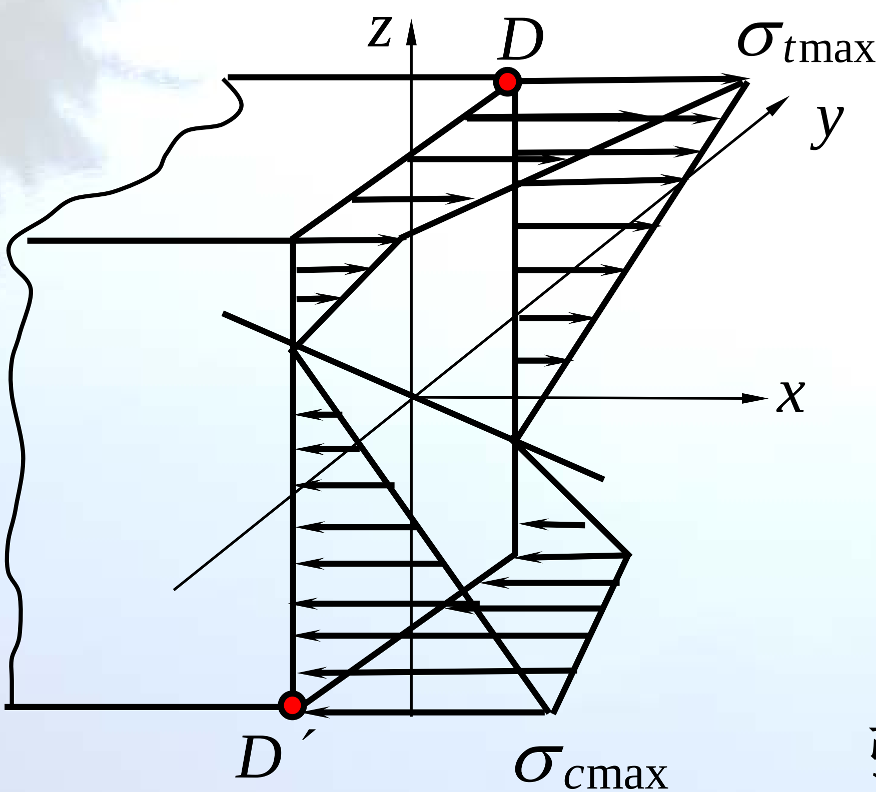




M_y引起的应力：$\sigma' = \dfrac{M_y z}{I_y}$

M_z引起的应力：$\sigma'' = \dfrac{M_z y}{I_z}$

合应力：$\sigma=\sigma'+\sigma''=\dfrac{M_y z}{I_y}+\dfrac{M_z y}{I_z}$



$$\sigma_{\max}=\sigma'_{\max}+\sigma''_{\max}$$

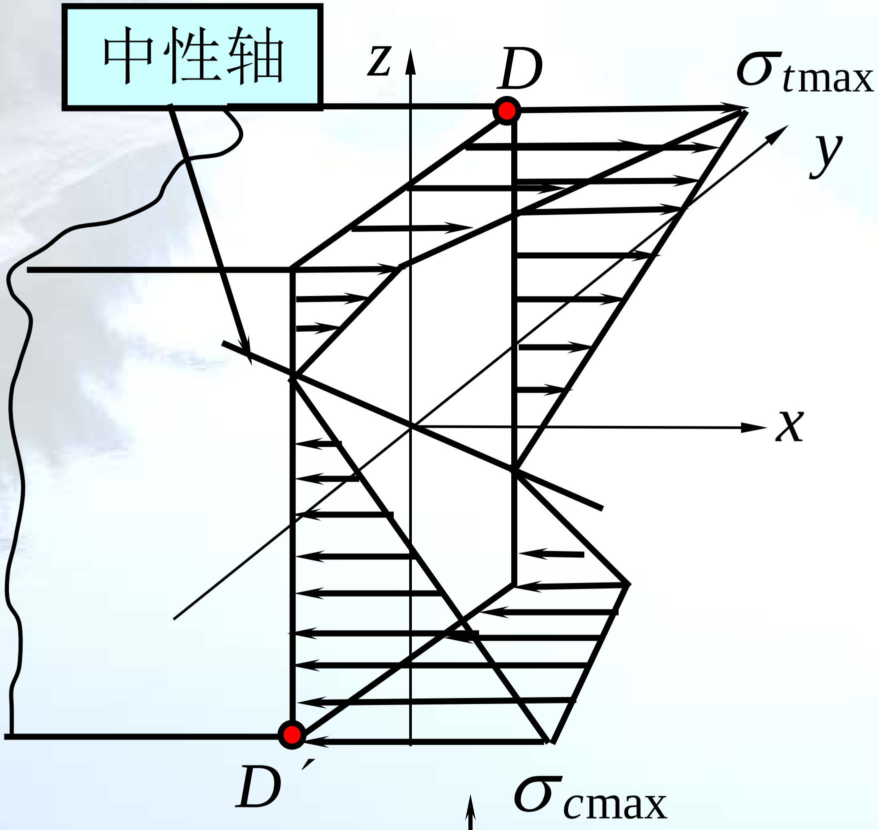
$$\therefore\sigma_{\max}=\frac{M_y}{W_y}+\frac{M_z}{W_z}$$

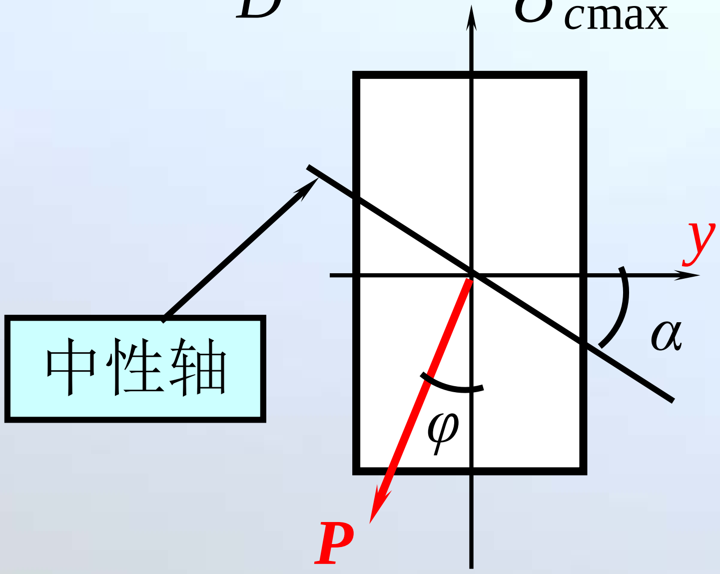
最大正应力在D和D'点

危险截面在固定端：

强度条件：

$$\sigma_{\max}=\frac{M_y}{W_y}+\frac{M_z}{W_z}\leq[\sigma]$$





中性轴上的正应力为零：

$$\sigma=0$$

令合应力等于零：

$$\sigma=\sigma'+\sigma''=\frac{M_y z}{I_y}+\frac{M_z y}{I_z}$$

$$=M\left(\frac{z}{I_y}\cos\varphi+\frac{y}{I_z}\sin\varphi\right)=0$$

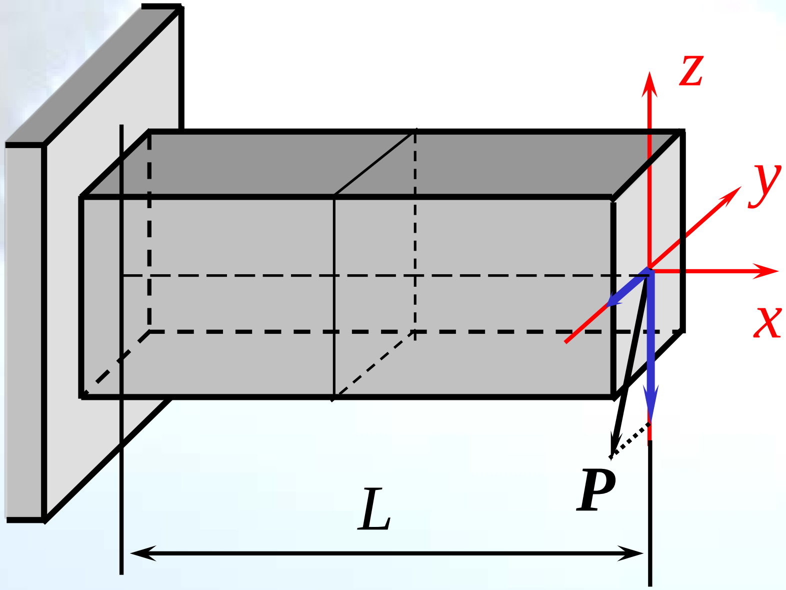
得中性轴方程：

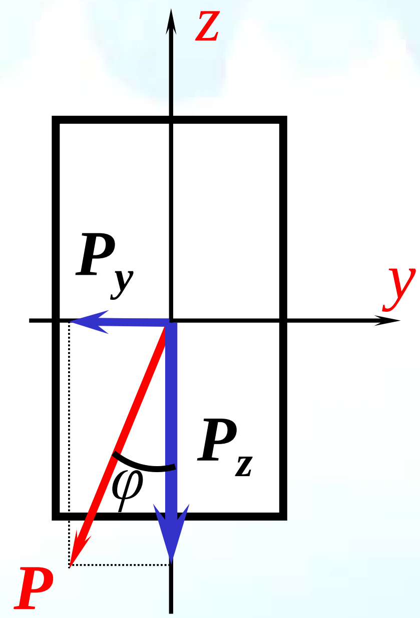
$$\frac{z}{I_y}\cos\varphi+\frac{y}{I_z}\sin\varphi=0$$

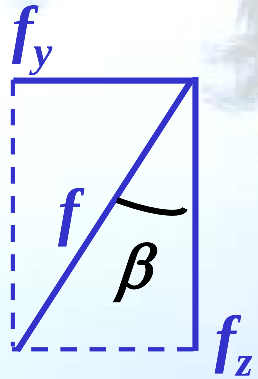
$$\frac{z}{y}=-\frac{I_y}{I_z}\tan\varphi$$

即：$\tan\alpha=-\dfrac{I_y}{I_z}\tan\varphi$

变形计算



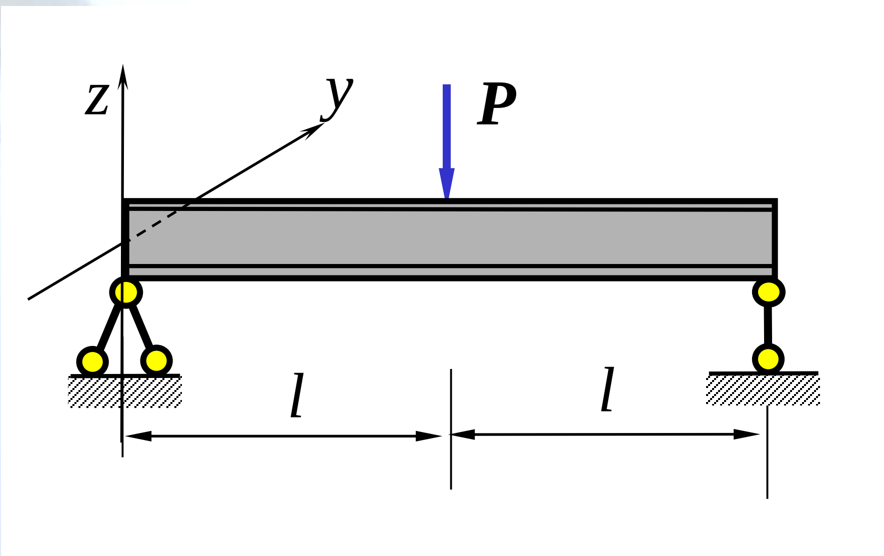


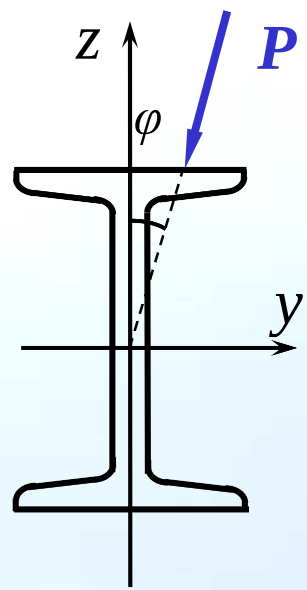


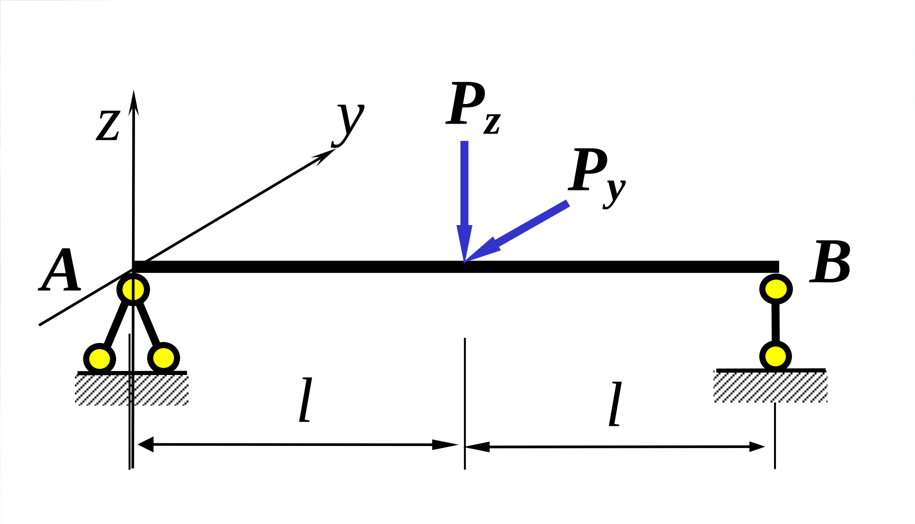
水平：$f_y = \dfrac{P_y L^3}{3EI_z}$　　合位移：$f = \sqrt{f_y^2 + f_z^2} = \sqrt{\left(\dfrac{P_y L^3}{3EI_z}\right)^2 + \left(\dfrac{P_z L^3}{3EI_y}\right)^2}$

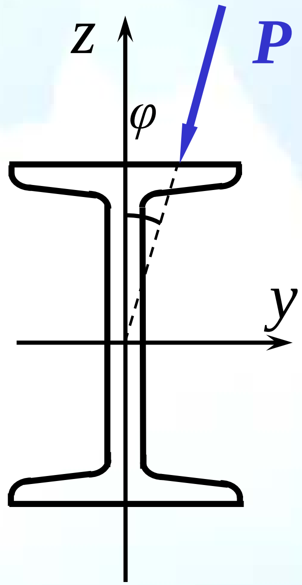
铅垂：$f_z = \dfrac{P_z L^3}{3EI_y}$　　$\tan\beta = \dfrac{f_y}{f_z} = \dfrac{I_y}{I_z}\tan\varphi$ ，当$I_y = I_z$时，$\tan\beta = \tan\varphi$

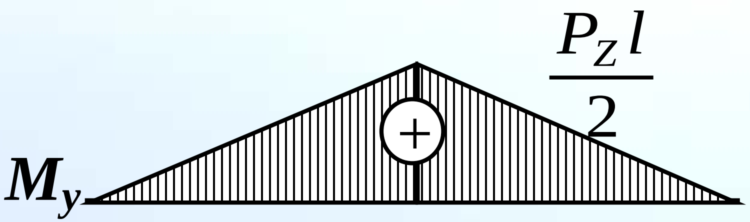
[例1] 已知：32a工字钢，l=2m，P=33kN，φ=15°，[σ]=170MPa，校核梁的强度。

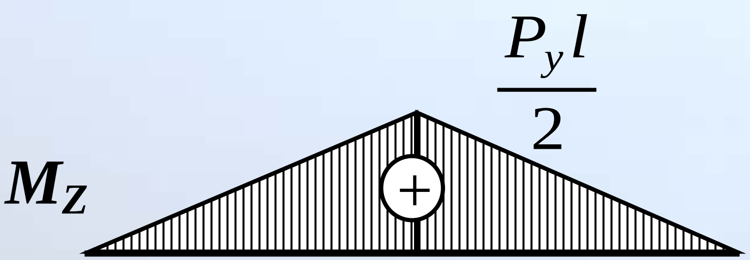


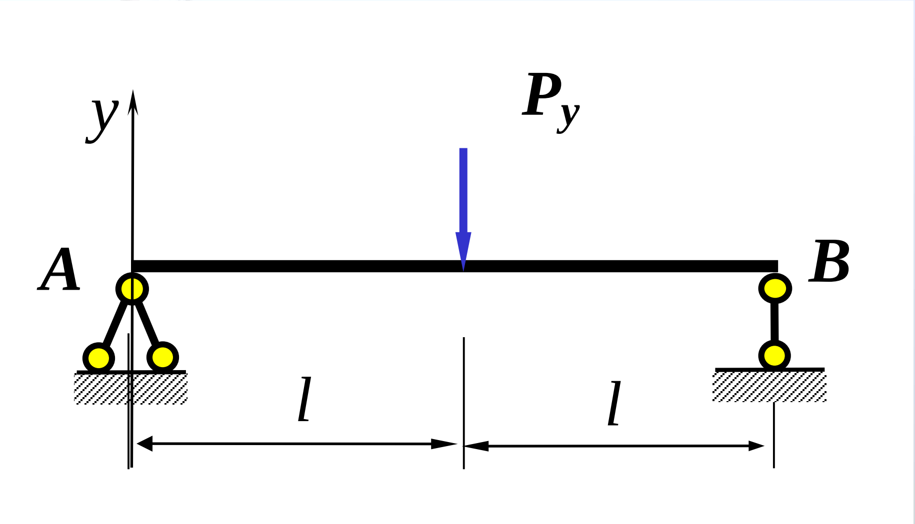


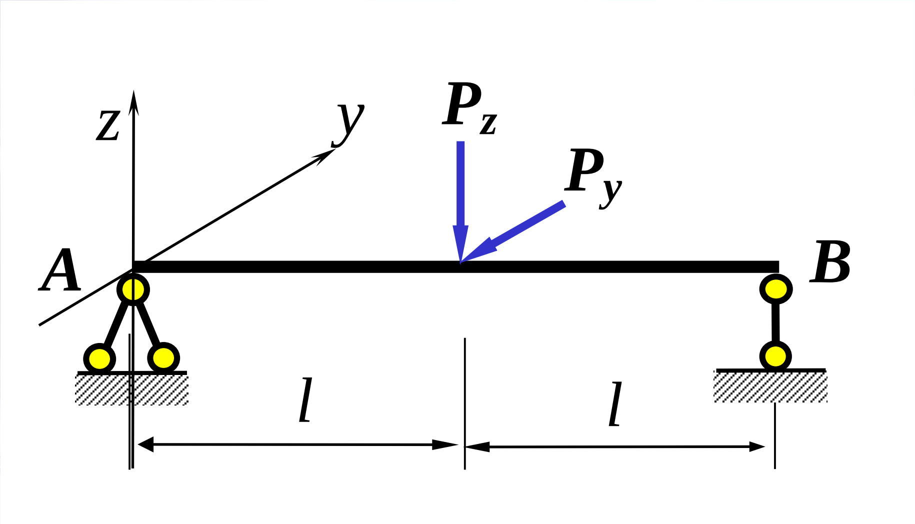

z
y
P_z
P_y
A
B
l
l


z
P
φ
y


$\frac{P_Z l}{2}$
+
M_y


$\frac{P_y l}{2}$
+
M_Z

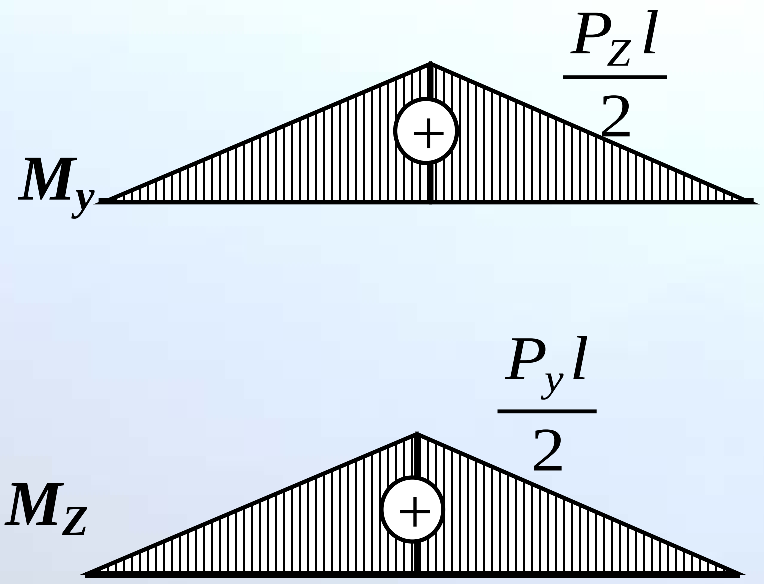

y
P_y
A
B
l
l

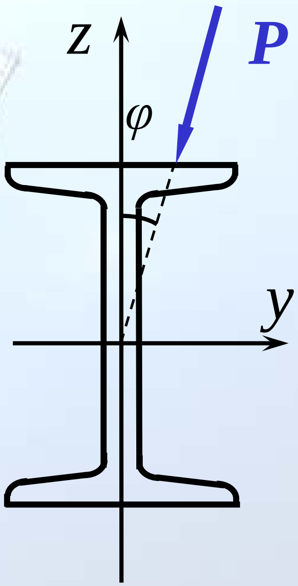


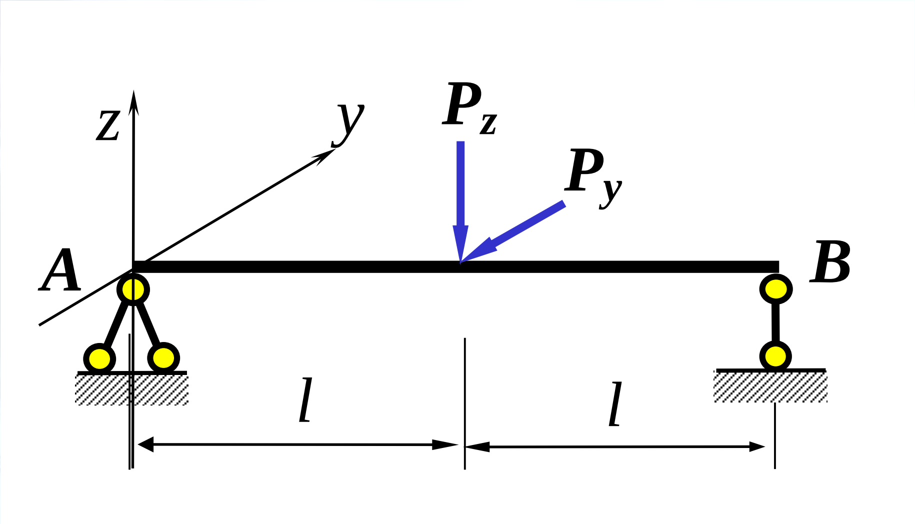
解： 危险截面在跨中

$$M_y=\frac{P_Z}{2}\times l=31.87(\mathbf{kN\cdot m})$$

$$M_z=\frac{P_y}{2}\times l=8.54(\mathbf{kN\cdot m})$$







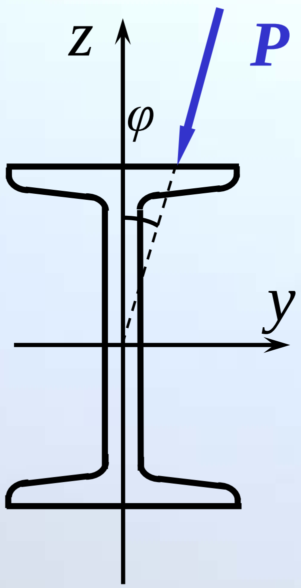
解：

$$M_y=\frac{P_z}{2}\times l=31.87(\mathbf{kN{\cdot}m})$$

$$M_z=\frac{P_y}{2}\times l=8.54(\mathbf{kN{\cdot}m})$$

查表得： $W_y=692(\mathbf{cm}^3)$

$$W_z=70.8(\mathbf{cm}^3)$$

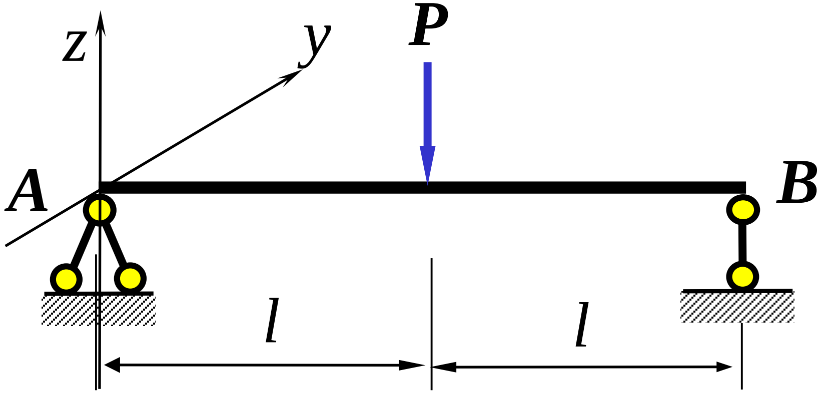


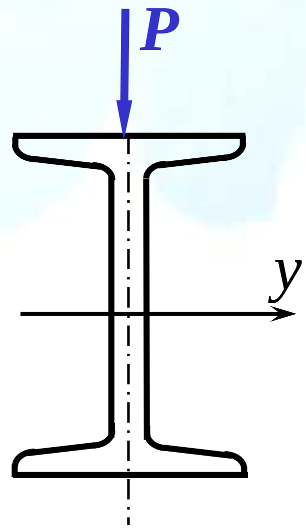
$$\therefore\sigma_{\max}=\frac{M_y}{W_y}+\frac{M_z}{W_z}$$

$$=\frac{31.87\times10^6}{692\times10^3}+\frac{8.54\times10^6}{70.8\times10^3}$$

$$=46.1+120.6$$

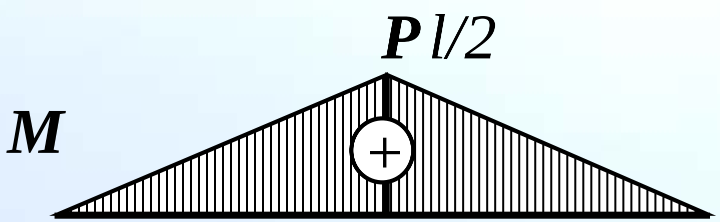
$=166.7(\mathbf{MPa})<[\sigma]$ ∴安全。





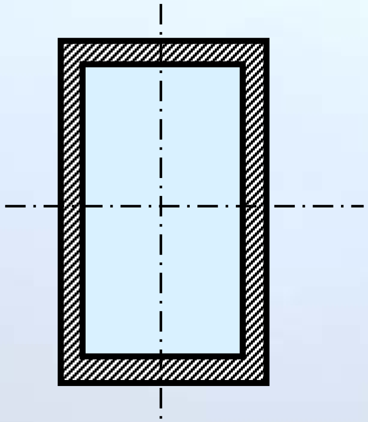
另：若$\varphi=0^\circ$，求梁内最大正应力。

解：$M_{max}=\dfrac{P}{2}\times l=33(\mathbf{kN\cdot m})$



查表得：$W_y=692(\mathbf{cm}^3)$

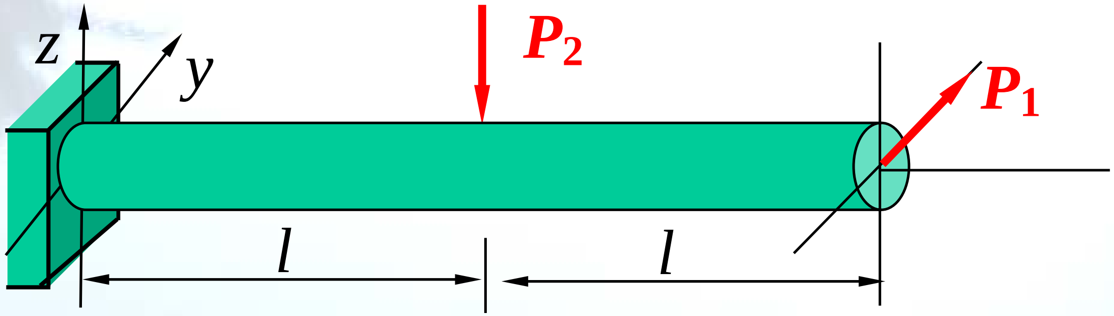
$$\therefore \sigma_{max}=\frac{M_{max}}{W_y}=\frac{33\times10^6}{692\times10^3}=47.7\ (\mathbf{MPa})$$

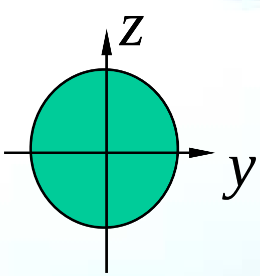


应力下降约3/4

[例2] 已知：P_1=1.7kN，P_2=1.6kN，l=1m，[σ]=160MPa，

试指出危险点的位置并设计圆截面杆的直径。

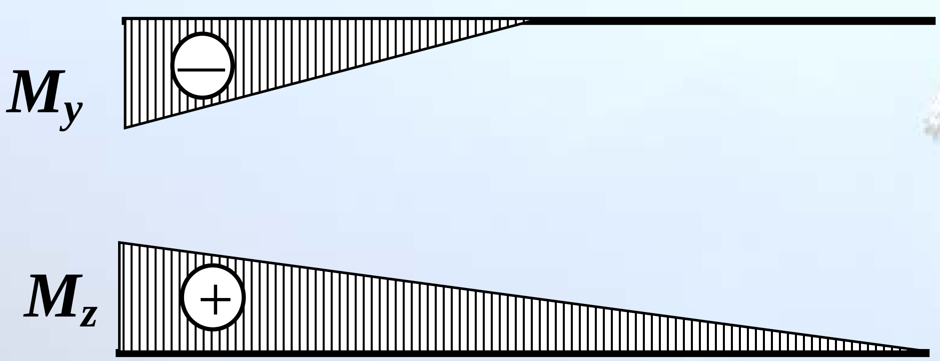


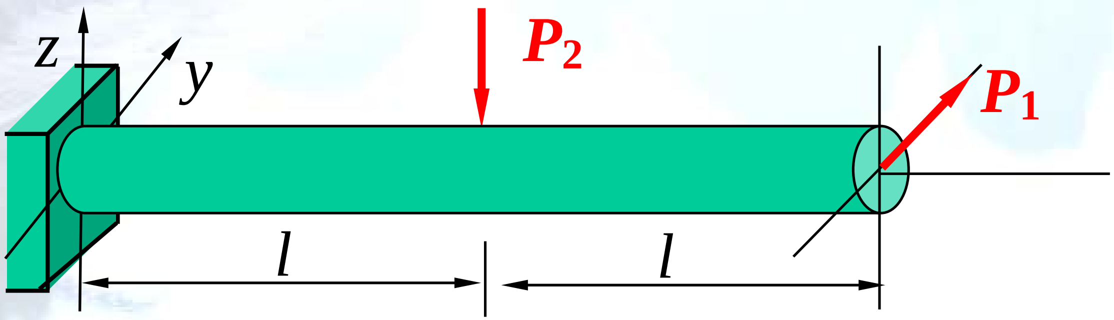


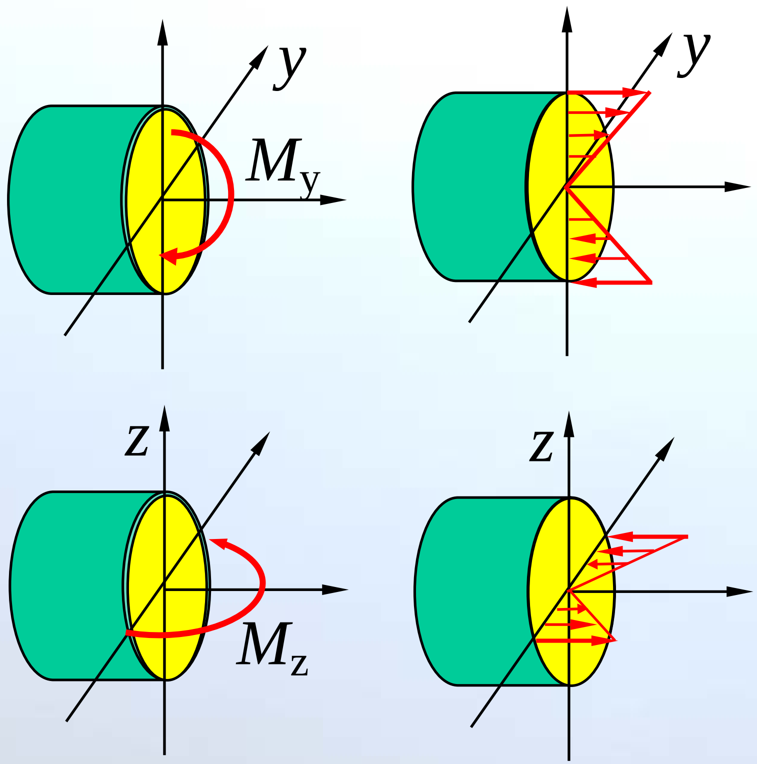
解：危险截面在固定端

$$M_y = P_2 \times l = 1.6(\mathbf{kN \cdot m})$$

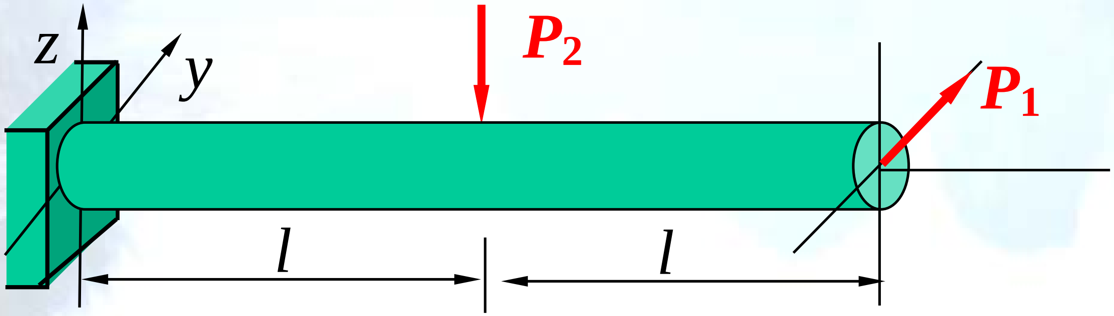
$$M_z = P_1 \times 2l = 3.4(\mathbf{kN \cdot m})$$



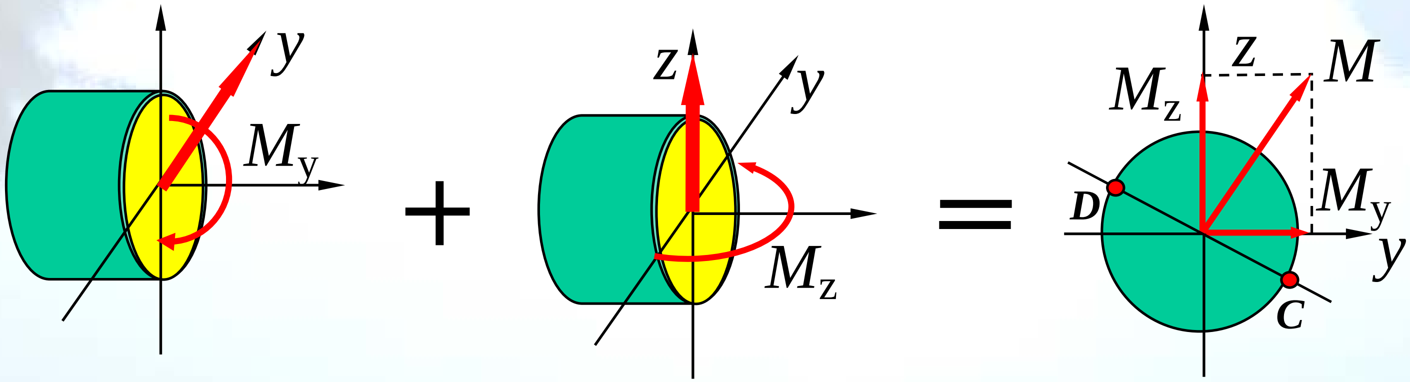




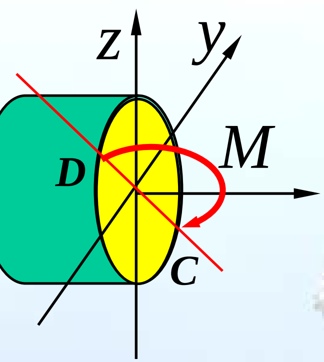
公式 $\sigma_{max} = \dfrac{M_y}{W_y} + \dfrac{M_z}{W_z}$ 不能使用！



危险点在C、D两点



$$M=\sqrt{M_y^2+M_z^2}=\sqrt{1.6^2+3.4^2}=3.76(\mathbf{kN\cdot m})$$

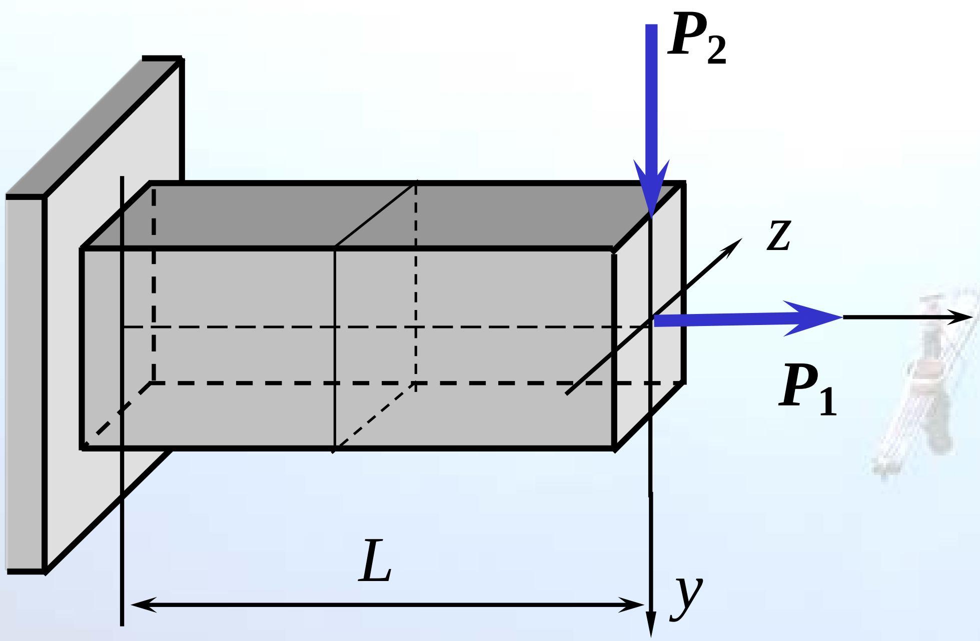


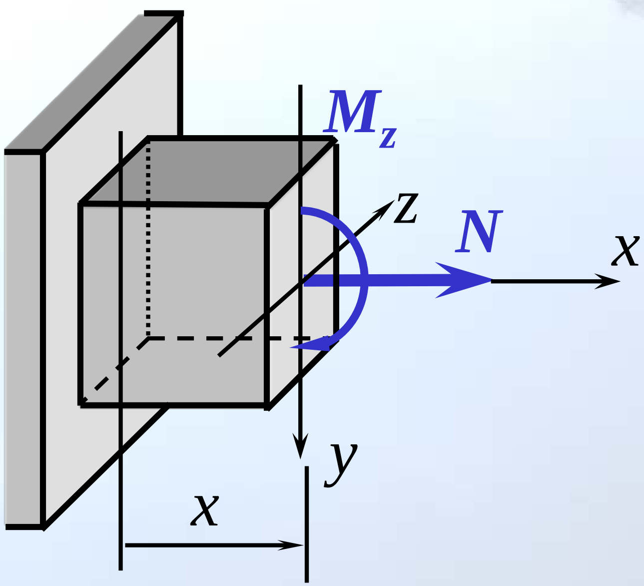
$$\sigma_{\max}=\frac{M}{W}=\frac{M}{\frac{\pi d^3}{32}}\le[\sigma]$$

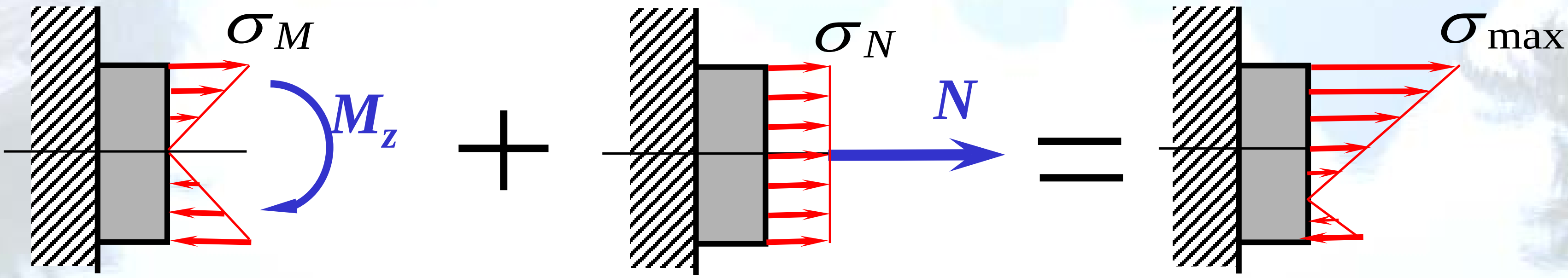
$$\therefore\quad d\ge\sqrt[3]{\frac{32M}{\pi[\sigma]}}=\sqrt[3]{\frac{32\times3.76\times10^6}{\pi\times160}}=62(\mathbf{mm})$$

§ 8–3　拉伸(压缩)与弯曲的组合

一、拉(压)弯组合变形：杆件同时受横向力和轴向力的作用而产生的变形。







$$\sigma_{\max} = \sigma_M + \sigma_N$$

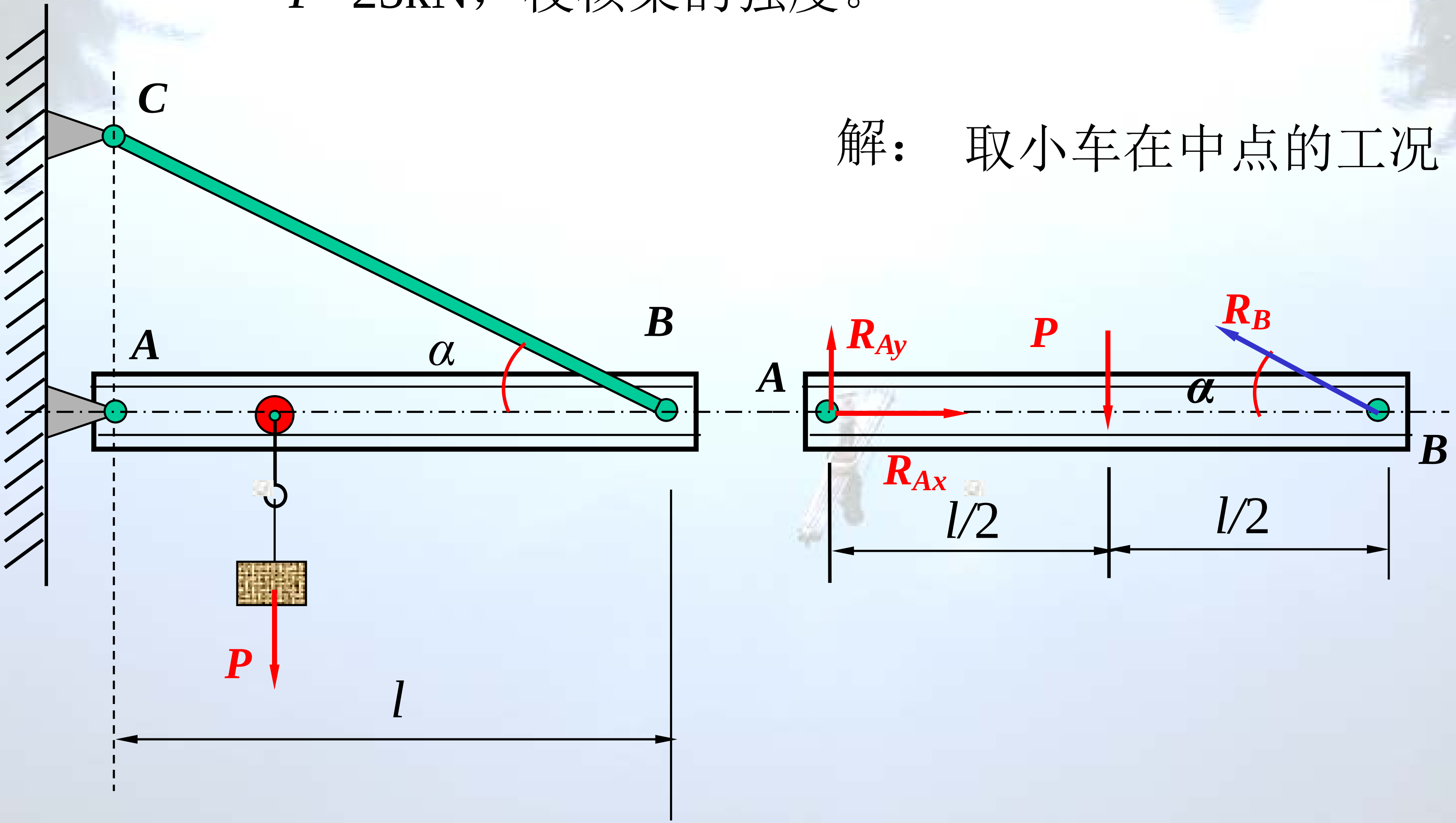
$$= \frac{M_z}{W_z} + \frac{P}{A}$$

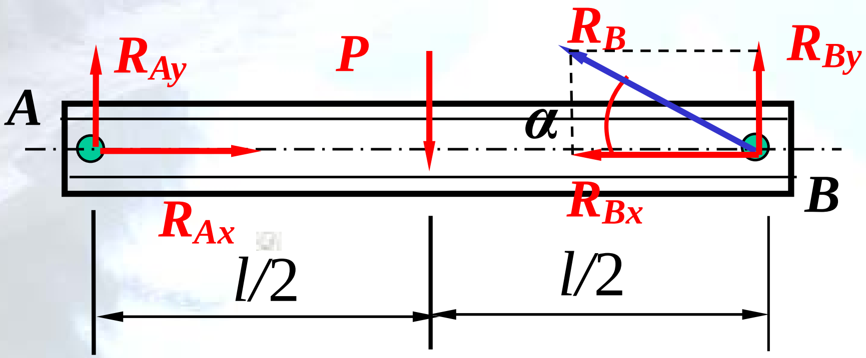
强度条件：

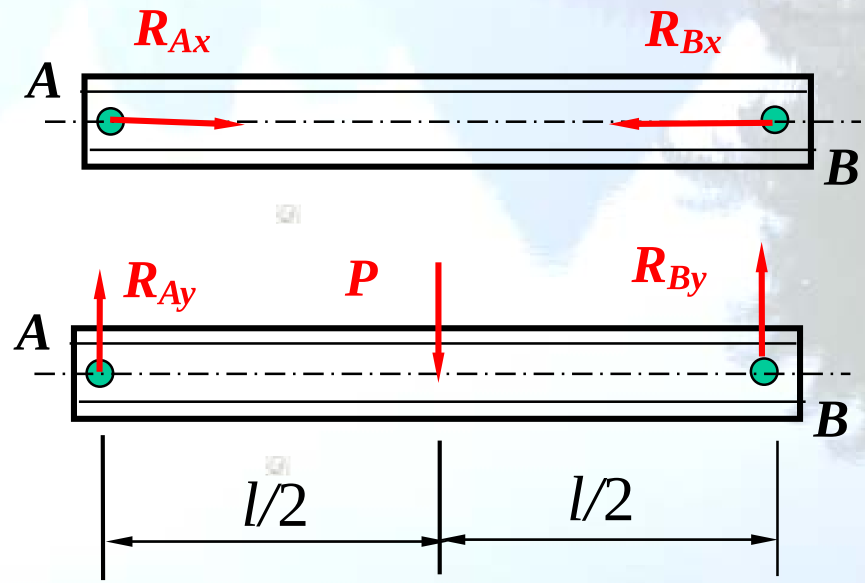
$$\sigma_{\max} = \frac{M_z}{W_z} + \frac{P}{A} \le [\sigma]$$

[例3] 简易吊车，AB梁为18号工字钢，W=185cm^3，A=30.6cm^2，梁长l=2.6m，α=30°，[σ]=120MPa，P=25kN，校核梁的强度。

解： 取小车在中点的工况







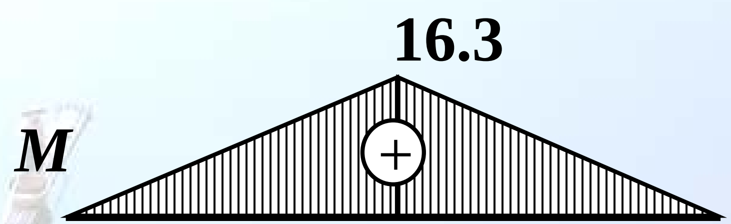
由理论力学得： $R_B=25(\text{kN})$

$$R_{Ax}=21.65(\text{kN})$$

$$R_{Ay}=\frac{P}{2}=12.5(\text{kN})$$

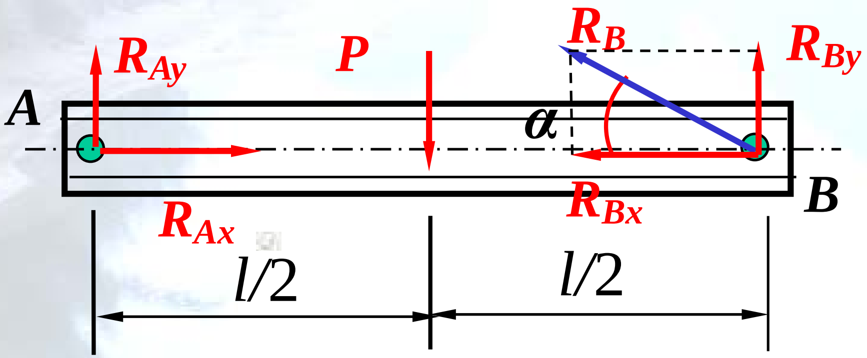
$$N=-R_{Ax}=-21.65(\text{kN})$$

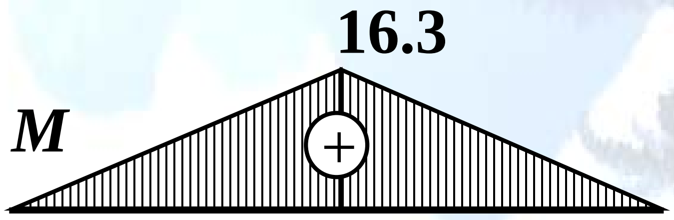
$$M=R_{Ay}\cdot\frac{l}{2}=16.3(\text{kN}\cdot\text{m})$$



AB梁受压弯组合，跨中为危险截面：

危险点在跨中上边缘，是压应力：





*AB*梁受压弯组合，跨中为危险截面：

$$N=-R_{Ax}=-21.65(\mathrm{kN})$$

$$M=R_{Ay}\cdot\frac{l}{2}=16.3(\mathrm{kN\cdot m})$$

危险点在跨中上边缘，是压应力：

$$\therefore\sigma_{\max}^{c}=\frac{|N|}{A}+\frac{M}{W}$$

$$=\frac{21.65\times10^{3}}{30.6\times10^{2}}+\frac{16.3\times10^{6}}{185\times10^{3}}$$

$$=7.09+88.1$$

$$=95.2(\mathrm{MPa})<[\sigma]$$

∴安全！

当小车在*B*点时：

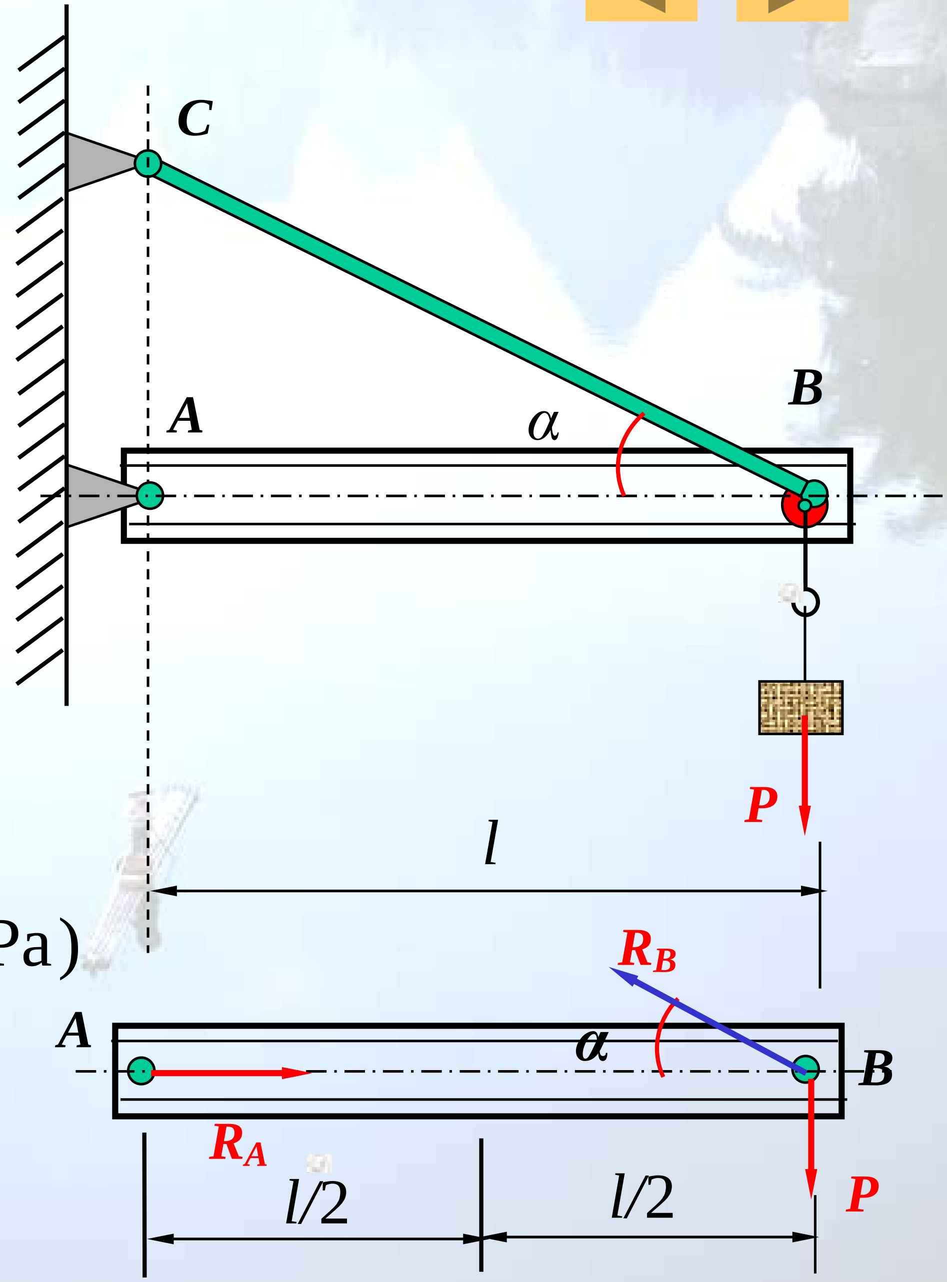
$R_B=50(\text{kN})$

$R_A=43.3(\text{kN})$

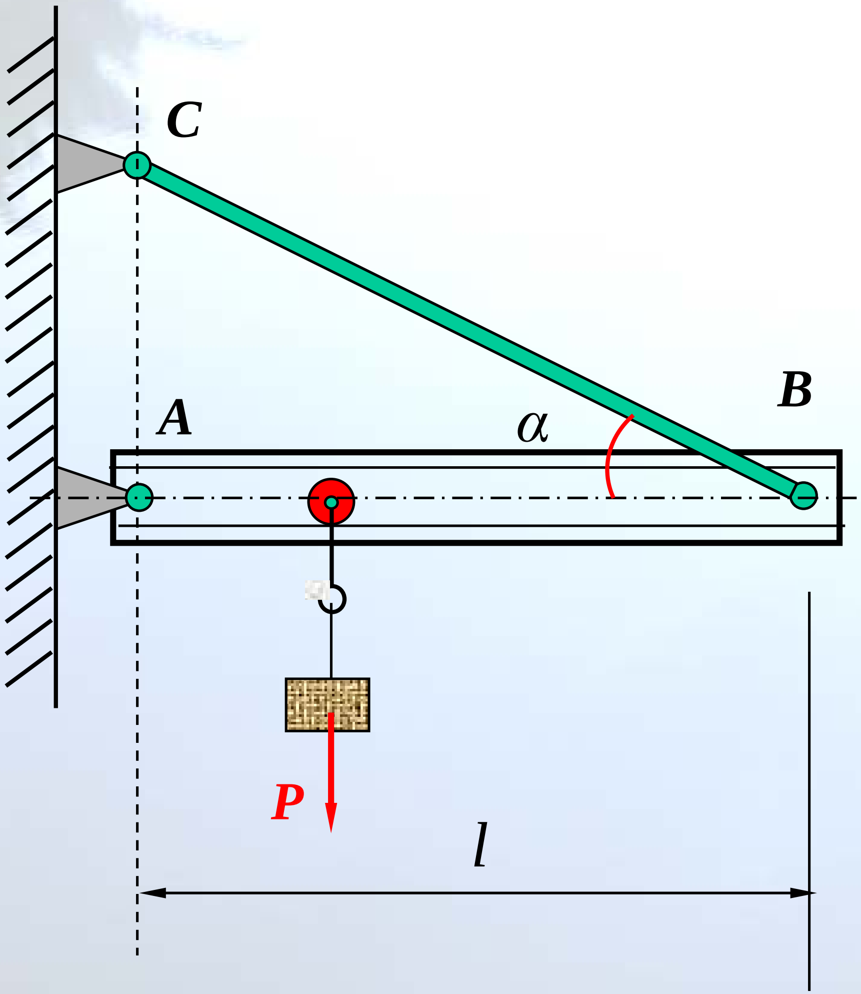
*AB*梁受轴向压缩：

$N=-R_A=-43.3(\text{kN})$

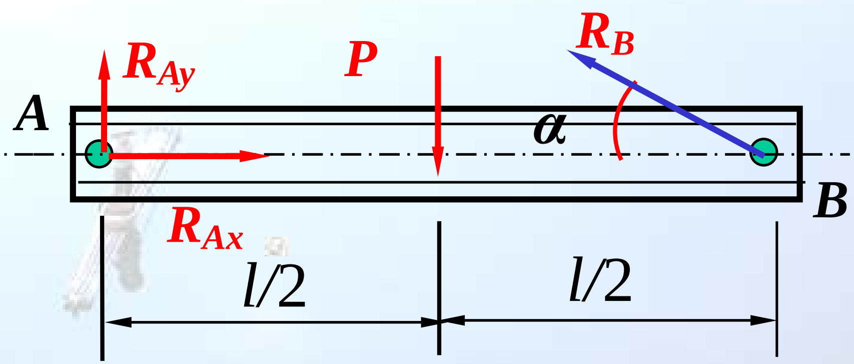
$$\therefore\sigma=\frac{N}{A}=\frac{43.3\times10^3}{30.6\times10^2}=14.1(\text{MPa})$$

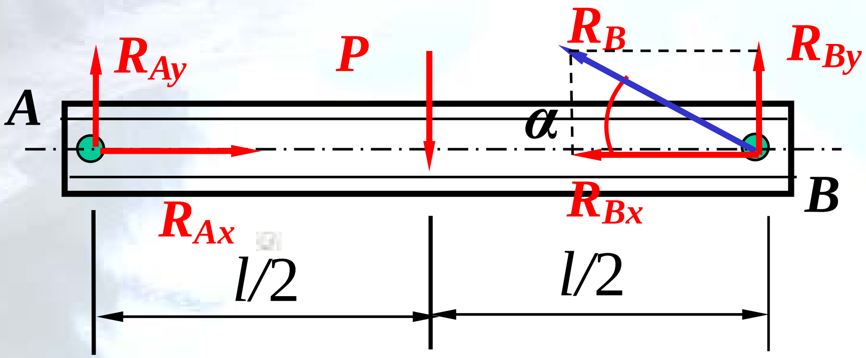


[例4] 简易吊车，梁长l=2.6m，α=30°，[σ]=120MPa，P=50kN，试选择工字钢型号。



解：取小车在中点的工况





$$R_B=50(\text{kN})$$

$$R_{Ax}=43.3(\text{kN})$$

$$R_{Ay}=\frac{P}{2}=25(\text{kN})$$

AB梁受压弯组合，跨中为危险截面：

$$N=-R_{Ax}=-43.3(\text{kN})$$

$$M=R_{Ay}\cdot\frac{l}{2}=32.5(\text{kN}\cdot\text{m})$$

由弯曲强度进行试算：

$$\sigma_{\max}=\frac{M}{W}\leq[\sigma]$$

$$W\geq\frac{M}{[\sigma]}=\frac{32.5\times10^6}{120}$$

$$=270\times10^3(\text{mm}^3)$$

∴选22a工字钢，W=309cm^3

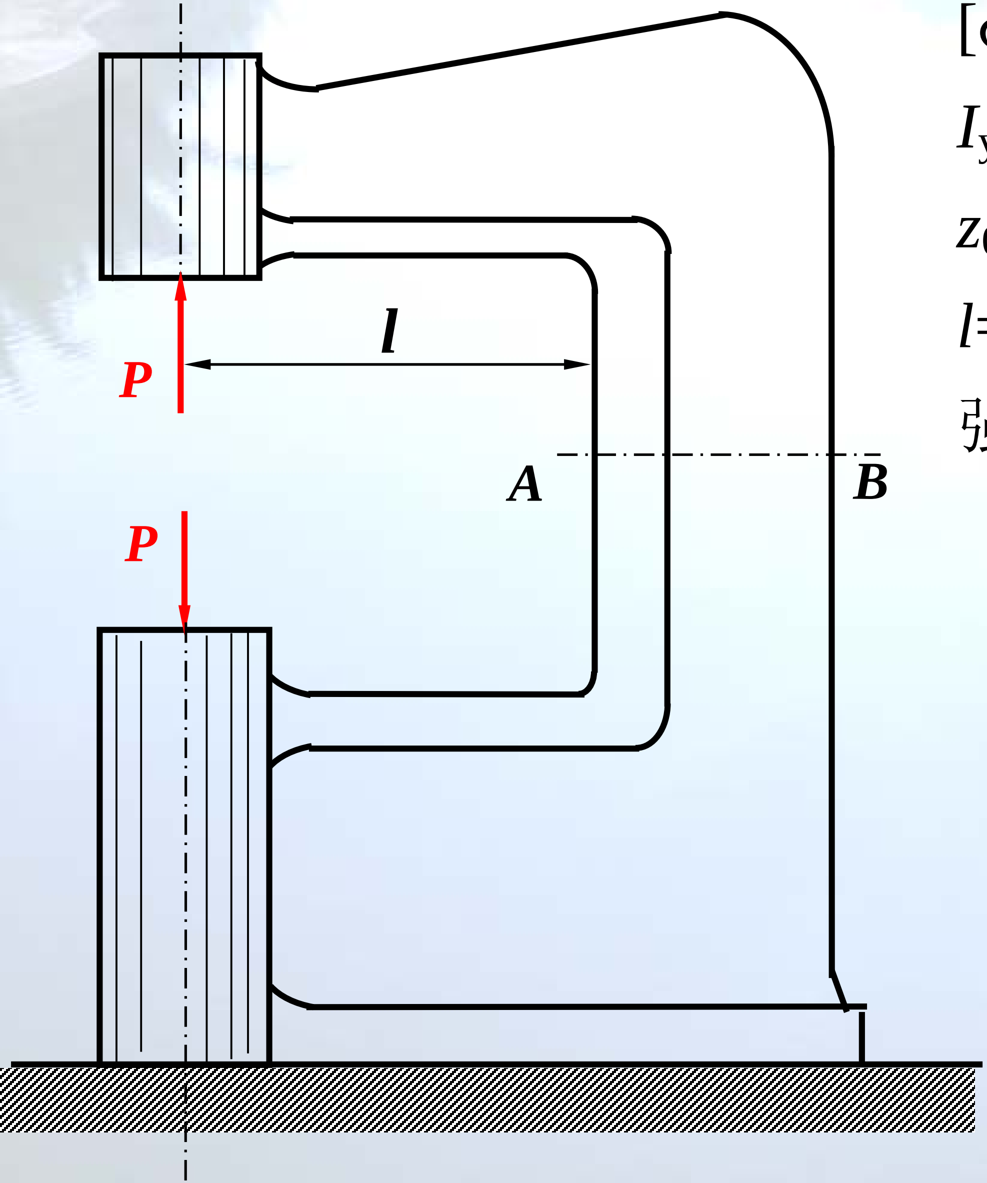
$$\therefore\sigma_{\max}^{c}=\frac{|N|}{A}+\frac{M}{W}$$

$$=\frac{43.3\times10^3}{42.128\times10^2}+\frac{32.5\times10^6}{309\times10^3}$$

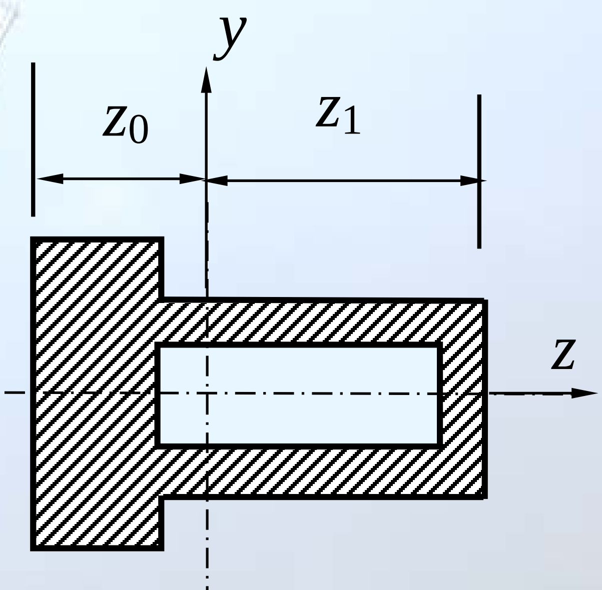
$$=115.5(\text{MPa})<[\sigma]$$

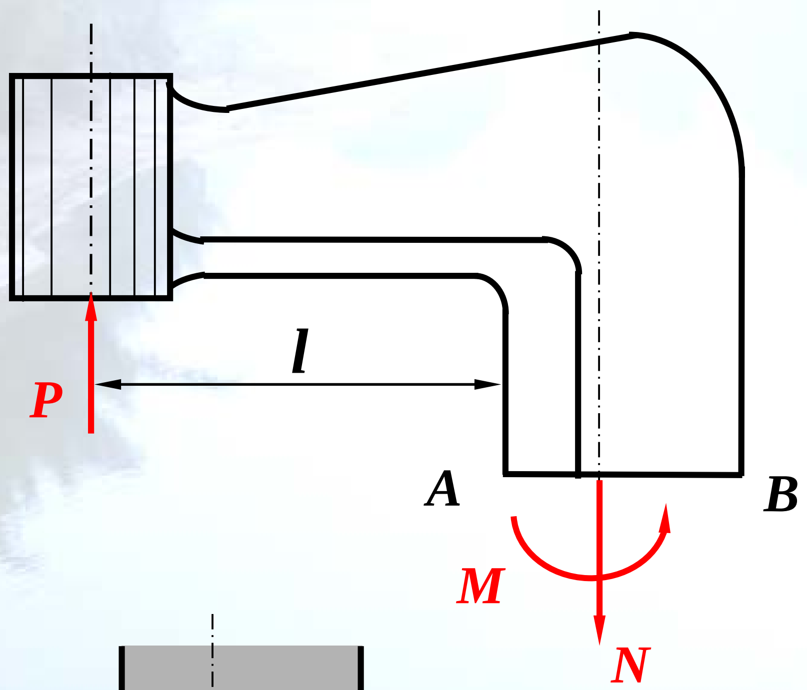
∴可以选22a工字钢！

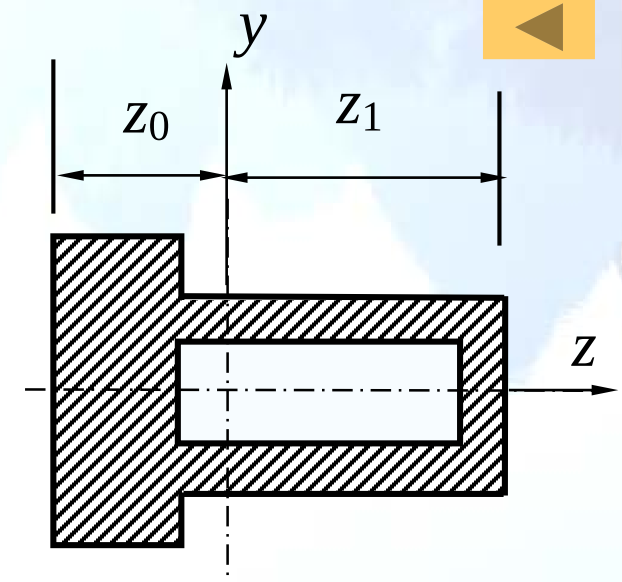
[例4]

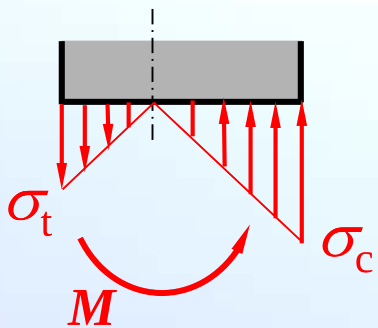


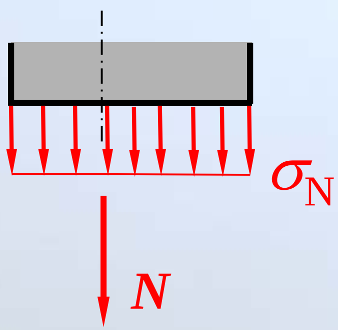
已知：冲压机，铸铁机身，$[\sigma_t]=30\text{MPa}$，$[\sigma_c]=160\text{MPa}$，$I_y=5310\text{cm}^4$，$A=150\text{cm}^2$，$z_0=7.5\text{cm}$，$z_1=12.5\text{cm}$，$l=35\text{cm}$，$P=40\text{kN}$，校核立柱强度。











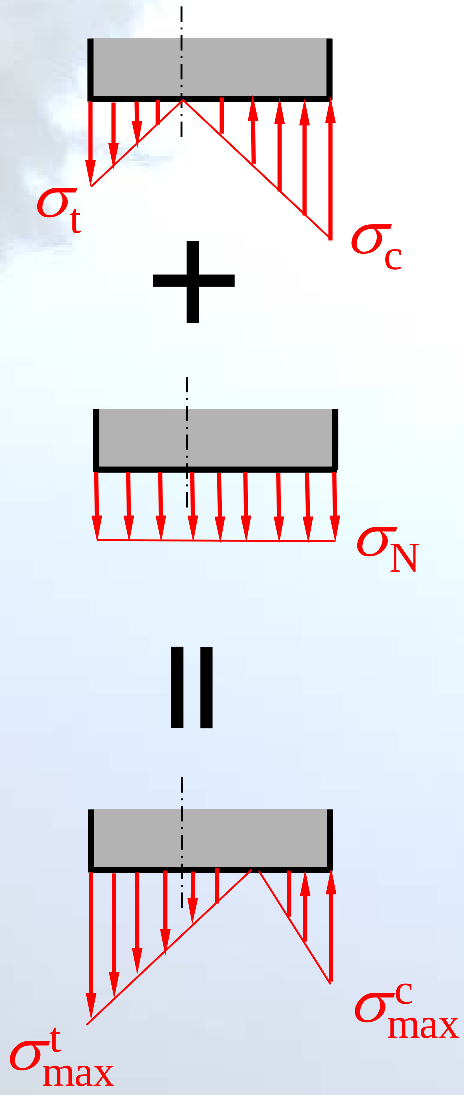
解： $N = P = 40(\mathrm{kN})$

$$M = P\cdot(l + z_0) = 17(\mathrm{kN}\cdot\mathrm{m})$$

$$\sigma_N = \frac{N}{A} = \frac{40\times10^3}{150\times10^2} = 2.67(\mathrm{MPa})$$

$$\sigma_t = \frac{M\cdot z_0}{I_y} = \frac{17\times10^6\times75}{5310\times10^4} = 24(\mathrm{MPa})$$

$$\sigma_c = \frac{M\cdot z_1}{I_y} = \frac{17\times10^6\times125}{5310\times10^4} = 40(\mathrm{MPa})$$



$$\sigma^t_{max} = \sigma_t + \sigma_N = 24 + 2.67$$

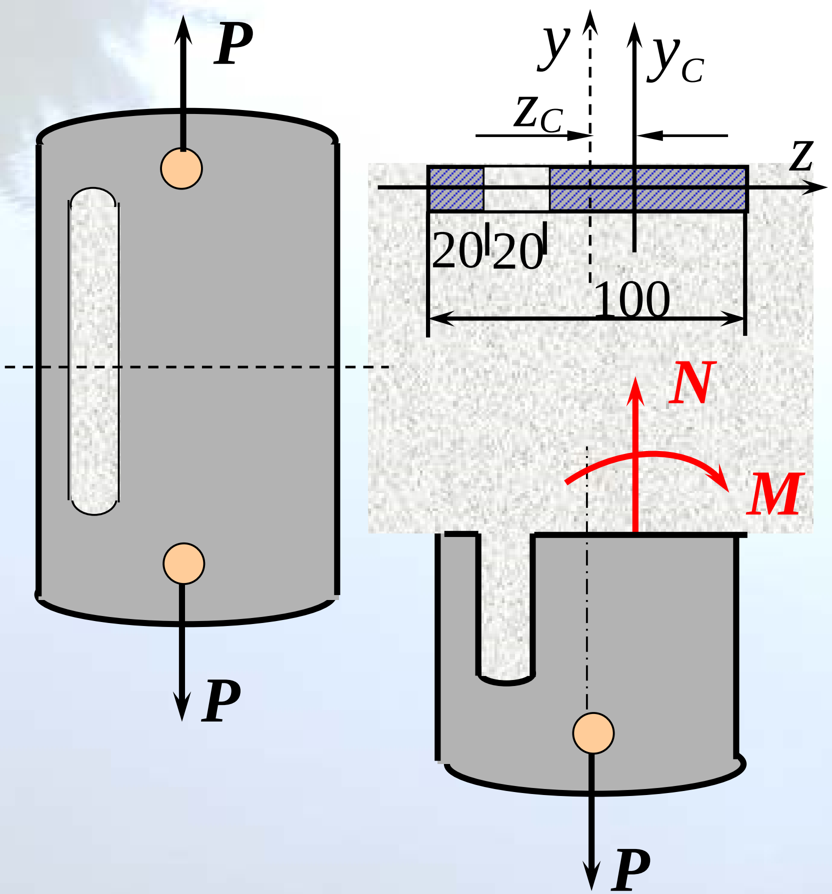
$$= 26.7(\text{MPa}) < [\sigma_t]$$

$$\sigma^c_{max} = \sigma_c - \sigma_N = 40 - 2.67$$

$$= 37.3(\text{MPa}) < [\sigma_c]$$

∴该立柱安全！

[例5] 图示钢板，厚度t=10mm，受力P=100kN，试求最大正应力；若将缺口移至板宽的中央，则最大正应力为多少？



解：内力分析如图

坐标如图，形心位置

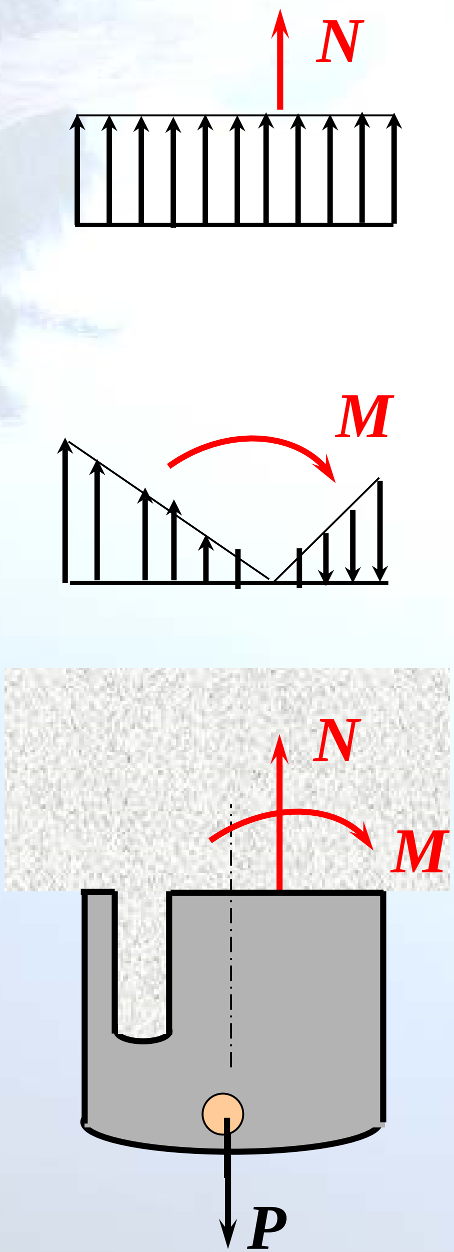
$$z_C=\frac{20\times10\times20}{100\times10-20\times10}=5\text{mm}$$

$$I_{y_C}=\frac{10\times100^3}{12}+10\times100\times5^2$$

$$-[\frac{10\times20^3}{12}+10\times20\times25^2]$$

$$=7.27\times10^5\ \text{mm}^4$$

$$M=P\cdot z_C=500\mathbf{N\cdot m}$$



应力分析如图

$$\sigma^{t}_{\max}=\frac{N}{A}+\frac{M|z|_{\max}}{I_{yc}}$$

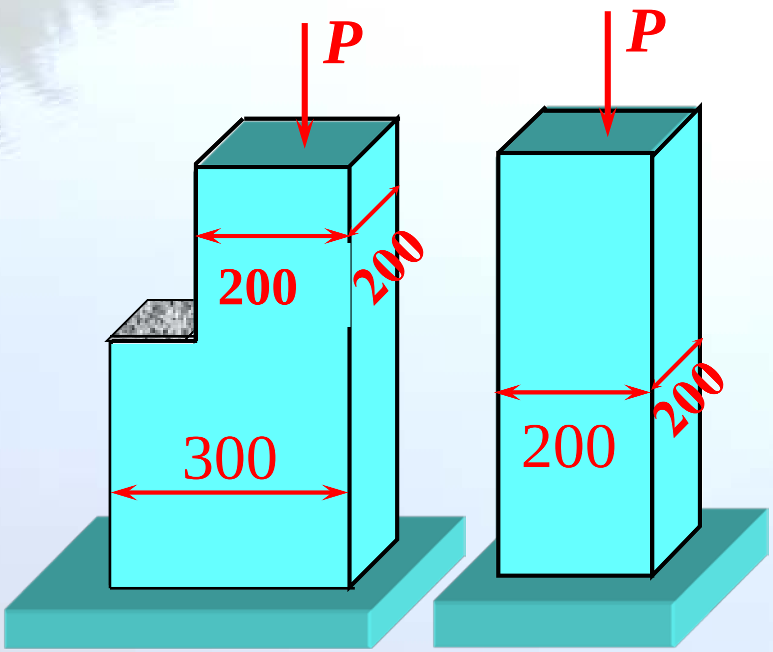
$$=\frac{100\times10^{3}}{800}+\frac{500\times10^{3}\times55}{7.27\times10^{5}}$$

$$=125+37.8=162.8\text{MPa}$$

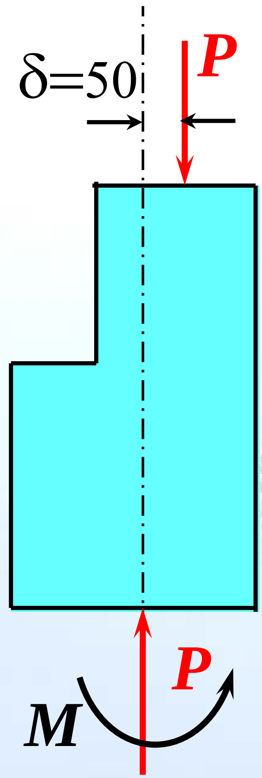
孔移至板中间时

$$\sigma_{\max}=\frac{N}{A}=\frac{100\times10^{3}}{800}=125(\mathbf{MPa})$$

[例6] 图示不等截面与等截面杆，受力P=350kN，试分别求出两柱内的最大正应力(绝对值)。



图（1）　　　　图（2）



解：图（1）

$$\sigma_{1\max}=\frac{P}{A_1}+\frac{M}{W_{z1}}$$

$$=\frac{350\times10^3}{0.2\times0.3}+\frac{350\times10^3\times0.05\times6}{0.2\times0.3^2}$$

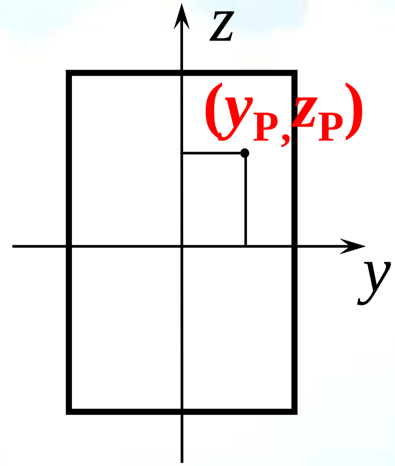
$$=11.7\textbf{MPa}$$

图（2）

$$\sigma_{2\max}=\frac{P}{A}=\frac{350000}{0.2\times0.2}=8.75\textbf{MPa}$$

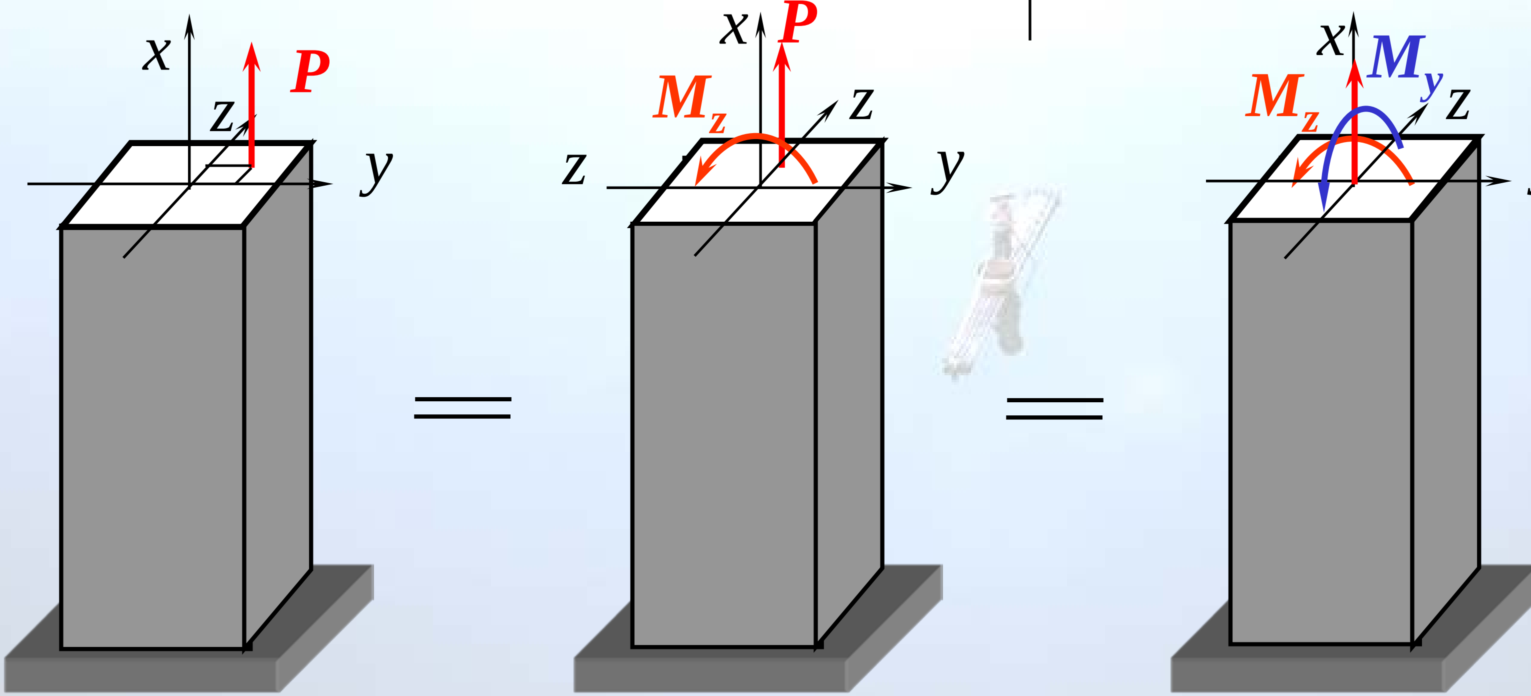
§8-4　偏心拉（压）· 截面核心

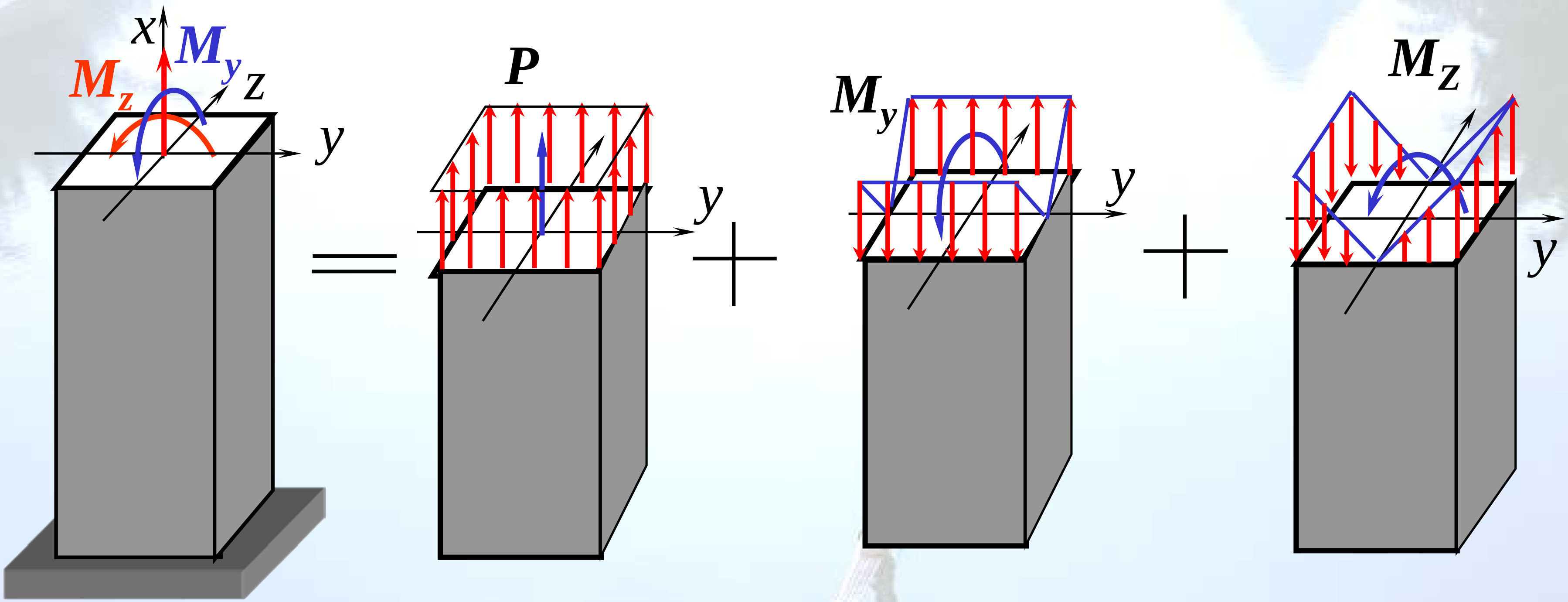
一、偏心拉（压）



$$M_z = P \cdot y_P$$

$$M_y = P \cdot z_P$$

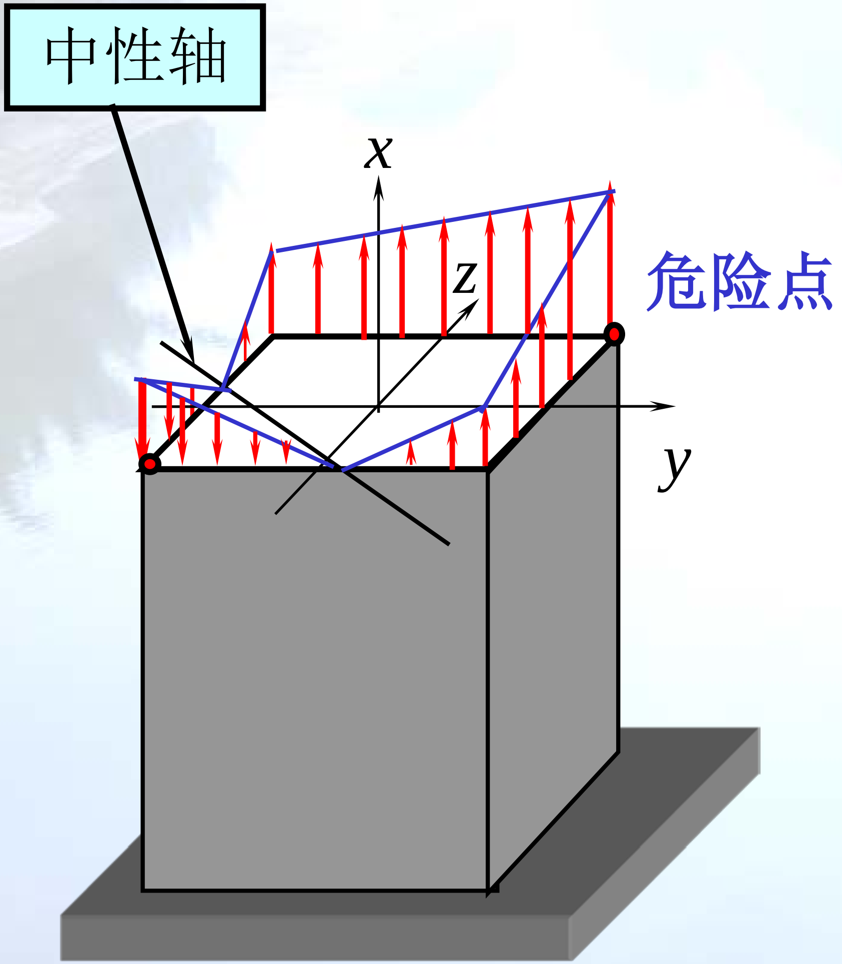




$$\sigma_N = \frac{P}{A}$$

$$\sigma_{M_y} = \frac{M_y z}{I_y}$$

$$\sigma_{M_z} = \frac{M_z y}{I_z}$$



$$\sigma=\frac{P}{A}+\frac{M_z y}{I_z}+\frac{M_y z}{I_y}$$

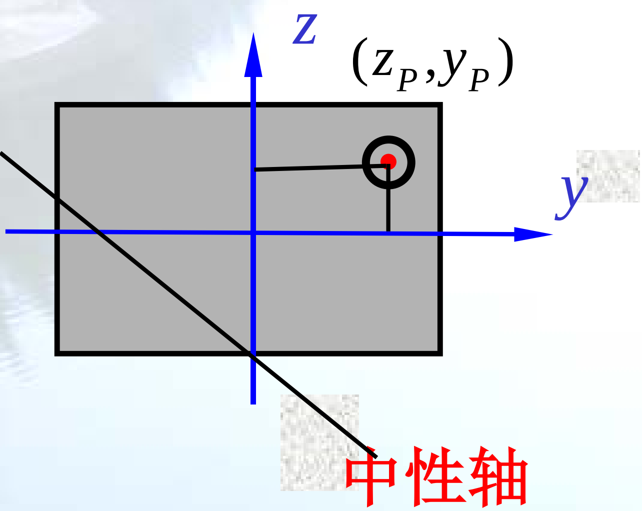
$$\sigma_{t\max}=\frac{P}{A}+\frac{M_z}{W_z}+\frac{M_y}{W_y}$$

$$\sigma_{c\max}=-\frac{P}{A}+\frac{M_z}{W_z}+\frac{M_y}{W_y}$$

强度条件：

$$\sigma_{\max}=\frac{P}{A}+\frac{M_z}{W_z}+\frac{M_y}{W_y}\le[\sigma]$$

二、中性轴方程



令： $\sigma_x = \dfrac{P}{A} + \dfrac{M_z y_0}{I_z} + \dfrac{M_y z_0}{I_y} = 0$

$$\frac{P}{A} + \frac{P y_P y_0}{A i_z^2} + \frac{P z_P z_0}{A i_y^2} = \frac{P}{A}\left(1 + \frac{y_P y_0}{i_z^2} + \frac{z_P z_0}{i_y^2}\right) = 0$$

$$1 + \frac{y_P y_0}{i_z^2} + \frac{z_P z_0}{i_y^2} = 0$$

中性轴在*y*和*z*轴上的截距a_y，a_z：

令$z_0 = 0$， $1 + \dfrac{y_P a_y}{i_z^2} = 0$，

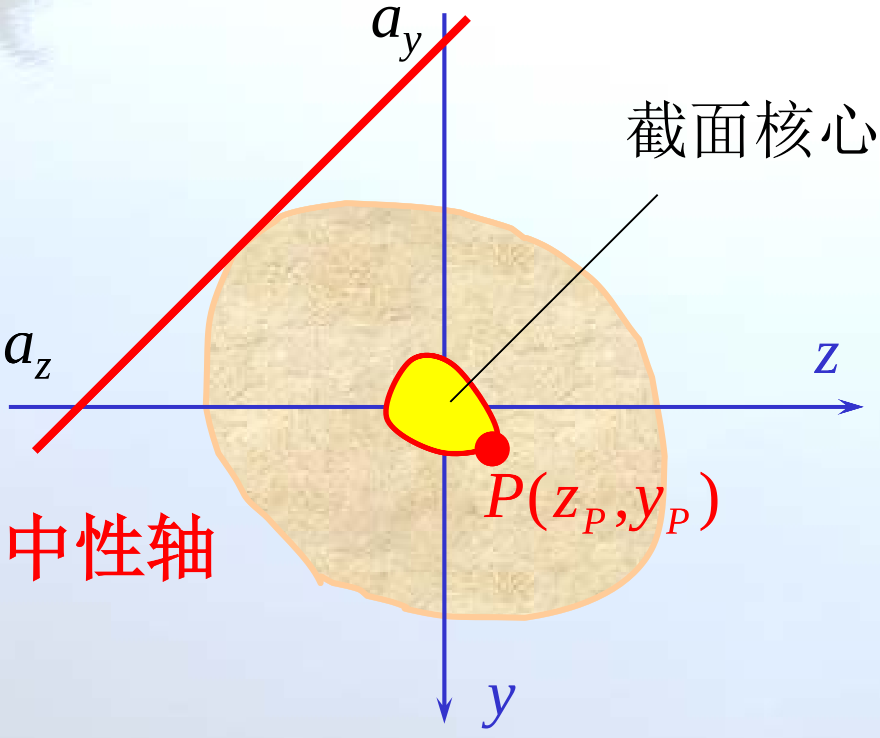
令$y_0 = 0$， $1 + \dfrac{z_P a_z}{i_y^2} = 0$

$$\therefore \begin{cases} a_y = -\dfrac{i_z^2}{y_P}, \\ a_z = -\dfrac{i_y^2}{z_P}, \end{cases}$$

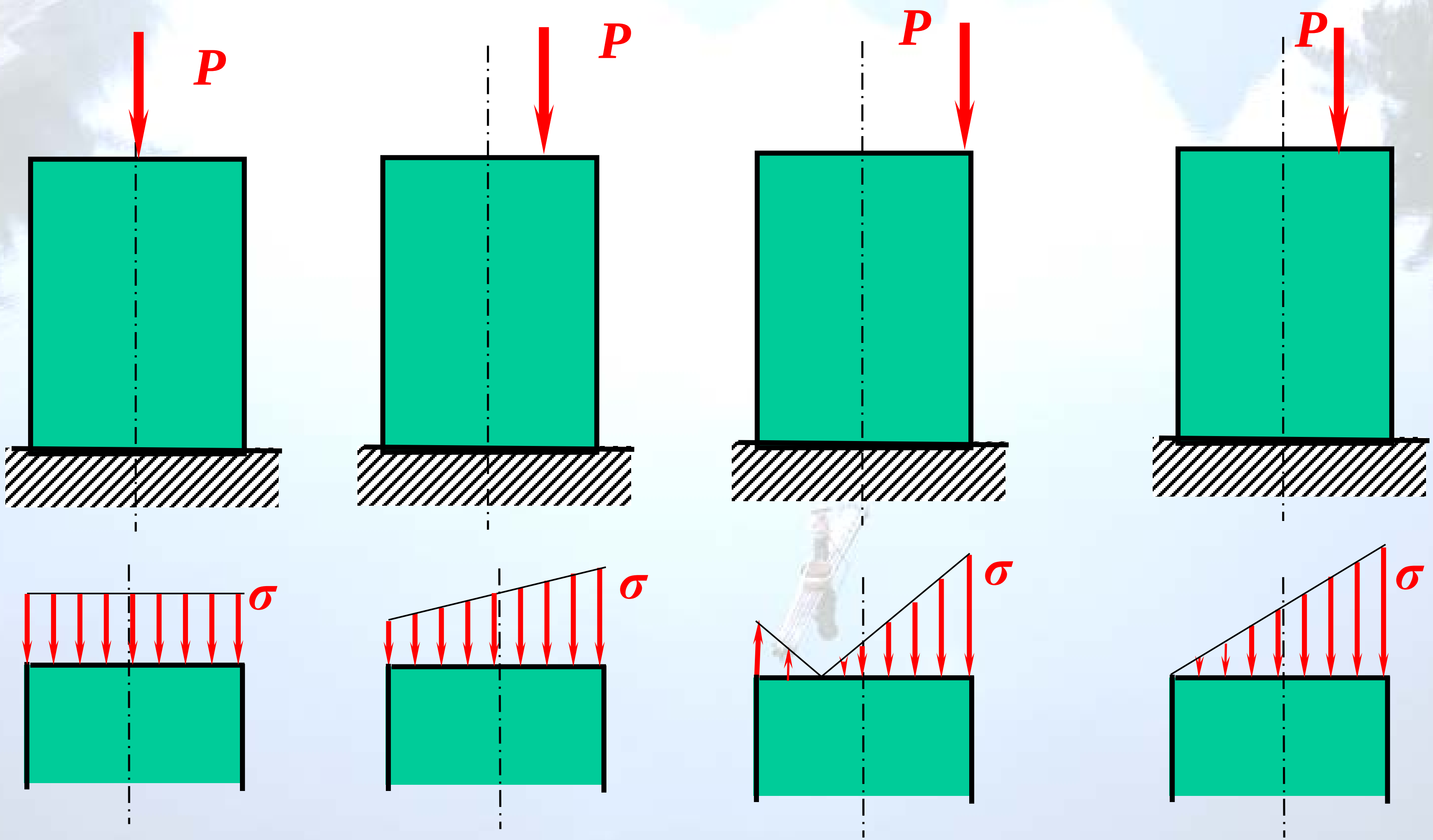
三、截面核心：

$$a_y=-\frac{i_z^2}{y_P}, \quad a_z=-\frac{i_y^2}{z_P},$$

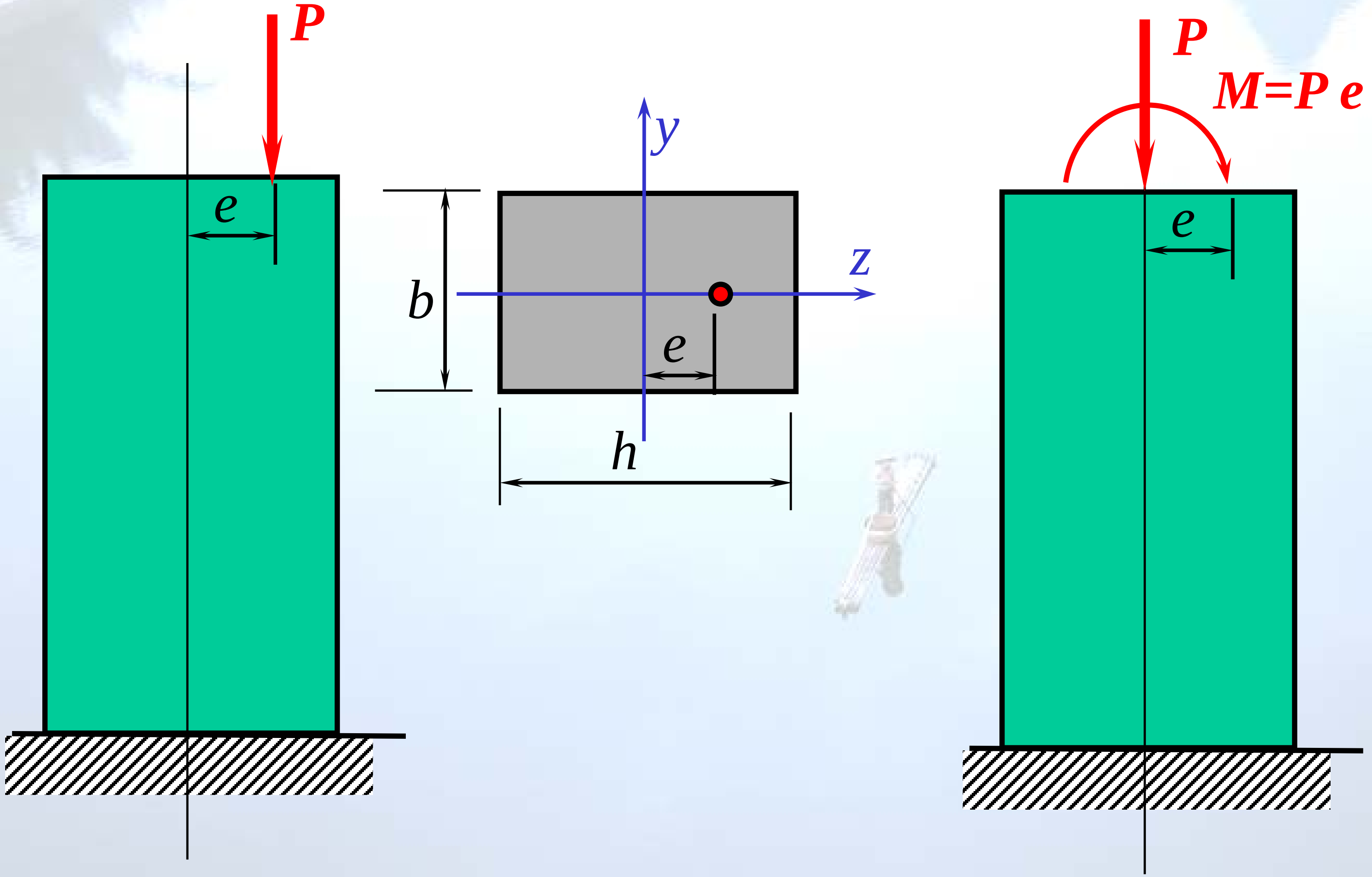
当y_P和z_P逐步减小时，中性轴将移出横截面，截面上只存在拉应力。

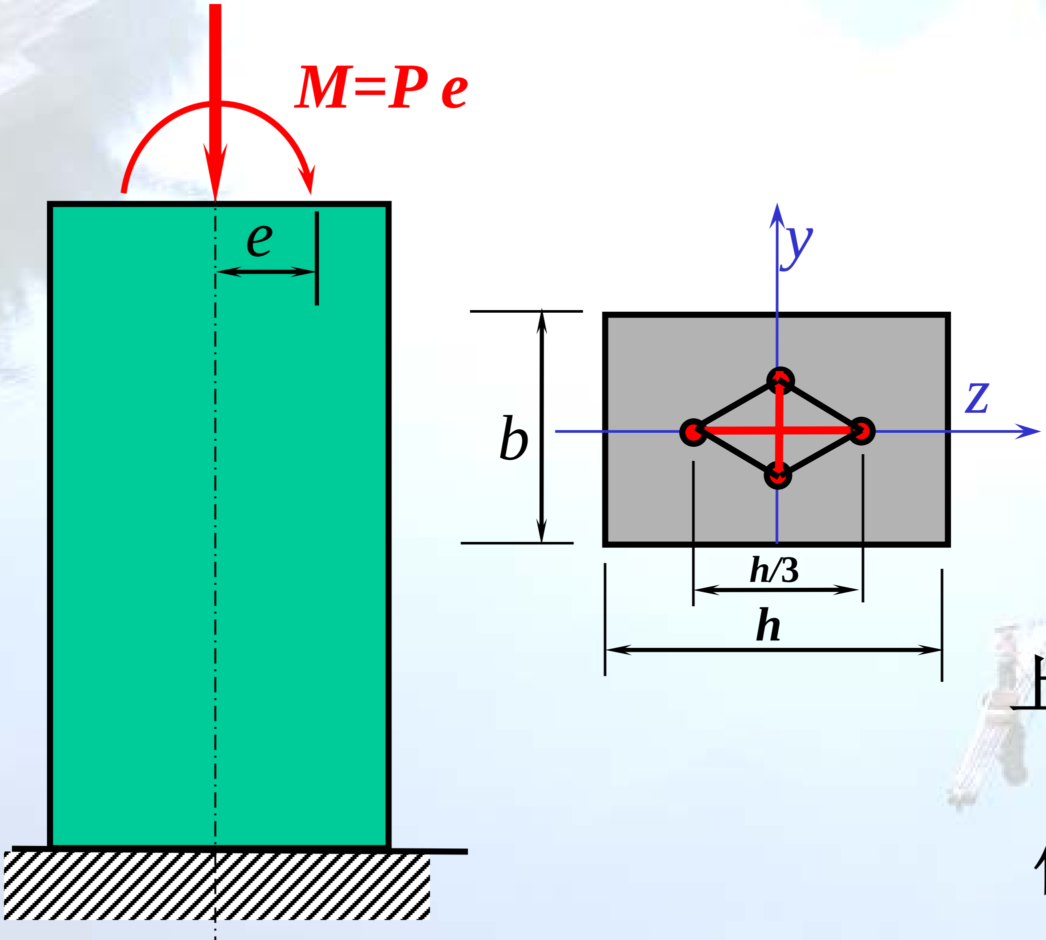


当外力作用点位于截面形心附近的一个区域内时，就可以保证中性轴不穿过横截面，横截面上无压应力（或拉应力），此区域称为截面核心。


P
P
P
P
σ
σ
σ
σ

[例]：矩形截面立柱，欲使柱内不出现拉应力，求*P*力的作用区域。





$$\sigma_{t\max}=\frac{M}{W_y}-\frac{P}{A}\le 0$$

$$\frac{Pe}{bh^2/6}-\frac{P}{bh}\le 0$$

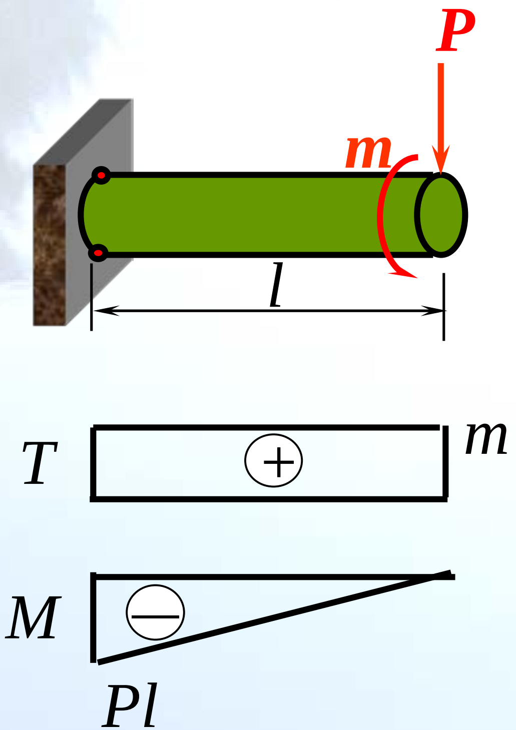
$$\therefore e\le\frac{h}{6}$$

由对称性可知，在z轴上的作用区域为 $h/3$

同理可知，在y轴上的作用区域为 $b/3$

可以证明，当P力作用在由此四点围成的菱形内时，横截面上无拉应力。该菱形区域称为截面核心

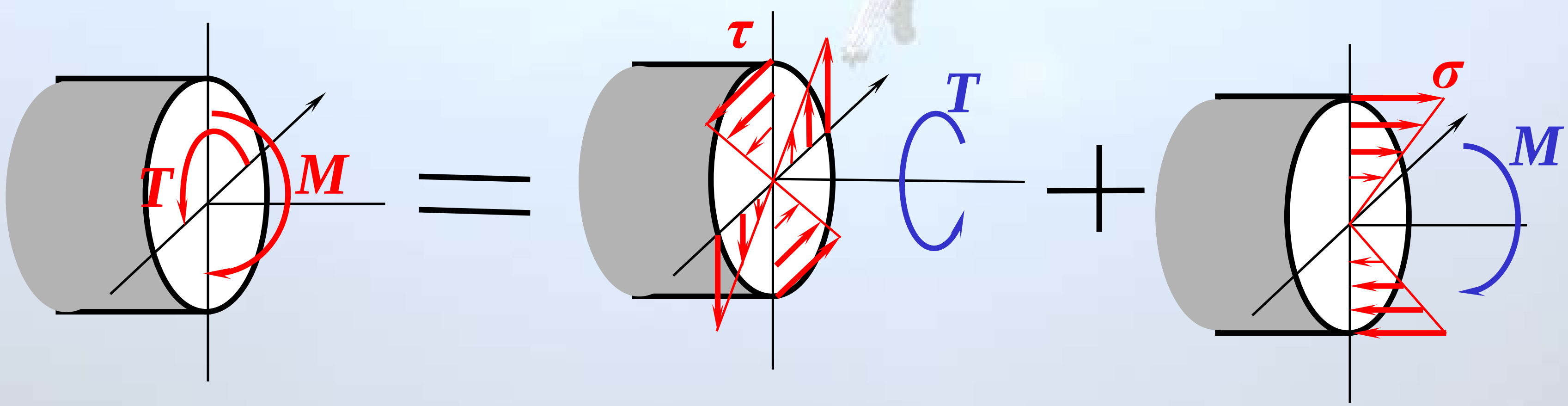
§ 8–5　弯曲与扭转的组合

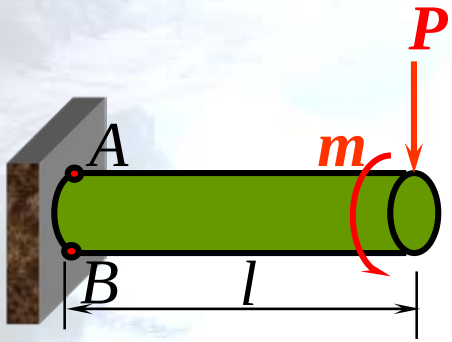


危险截面在固定端

危险点在固定端的上、下两点

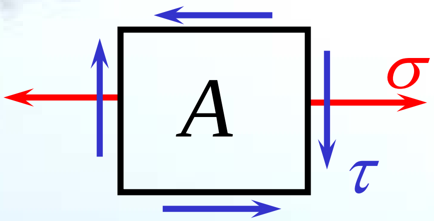
$$\tau=\frac{T}{W_t}\,,\qquad \sigma=\frac{M}{W}$$





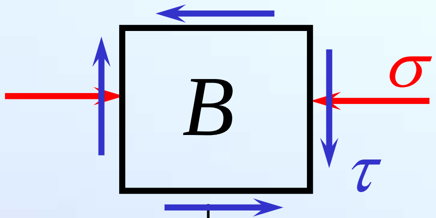
$$\sigma_{r3}=\sqrt{\sigma^2+4\tau^2}$$

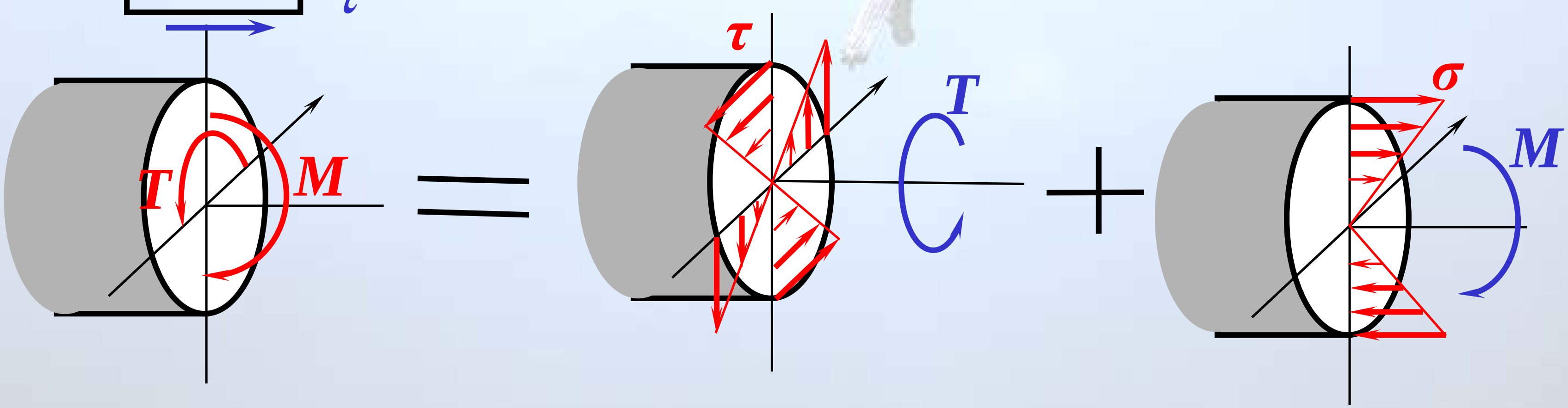
$$\sigma_{r4}=\sqrt{\sigma^2+3\tau^2}$$

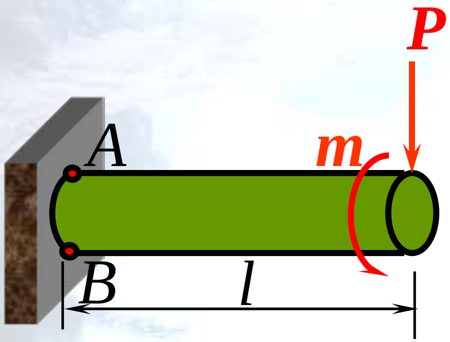


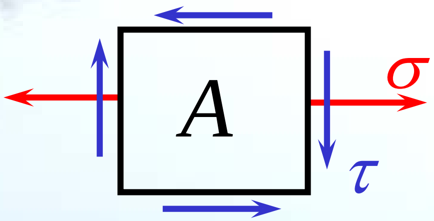
$$\sigma_{r3}=\sqrt{(\frac{M}{W})^2+4(\frac{T}{W_t})^2}$$

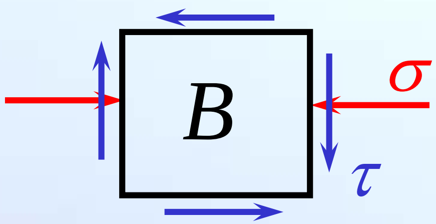
$$\sigma_{r4}=\sqrt{(\frac{M}{W})^2+3(\frac{T}{W_t})^2}$$











$$\because \quad W_t = 2W$$

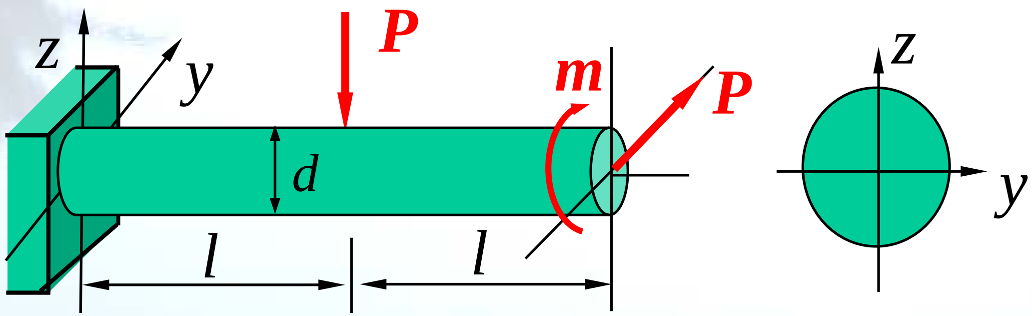
$$\therefore \quad \sigma_{r3} = \sqrt{\left(\frac{M}{W}\right)^2 + 4\left(\frac{T}{2W}\right)^2}$$

即：

$$\boxed{\sigma_{r3} = \frac{\sqrt{M^2 + T^2}}{W}}$$

$$\boxed{\sigma_{r4} = \frac{\sqrt{M^2 + 0.75T^2}}{W}}$$

[例7]　已知：P=4.2kN，m=1.5kN·m，l=0.5m，d=100mm，[σ]=80MPa，按第三强度理论校核杆的强度。

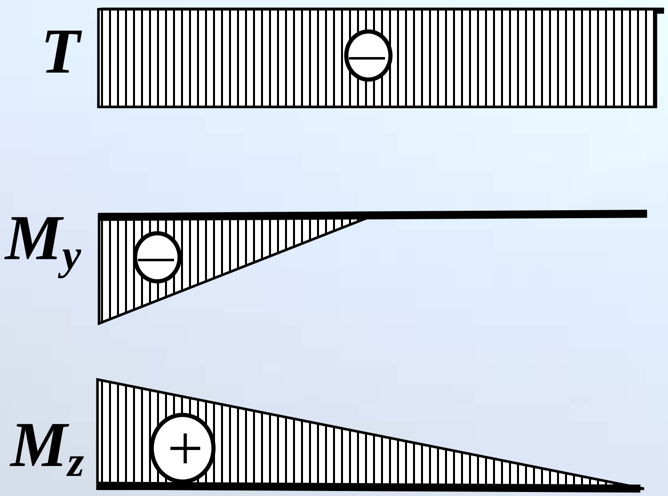


解：　危险截面在固定端

$$T=m=1.5(\mathbf{kN{\cdot}m})$$

$$M_y=P\times l=2.1(\mathbf{kN{\cdot}m})$$

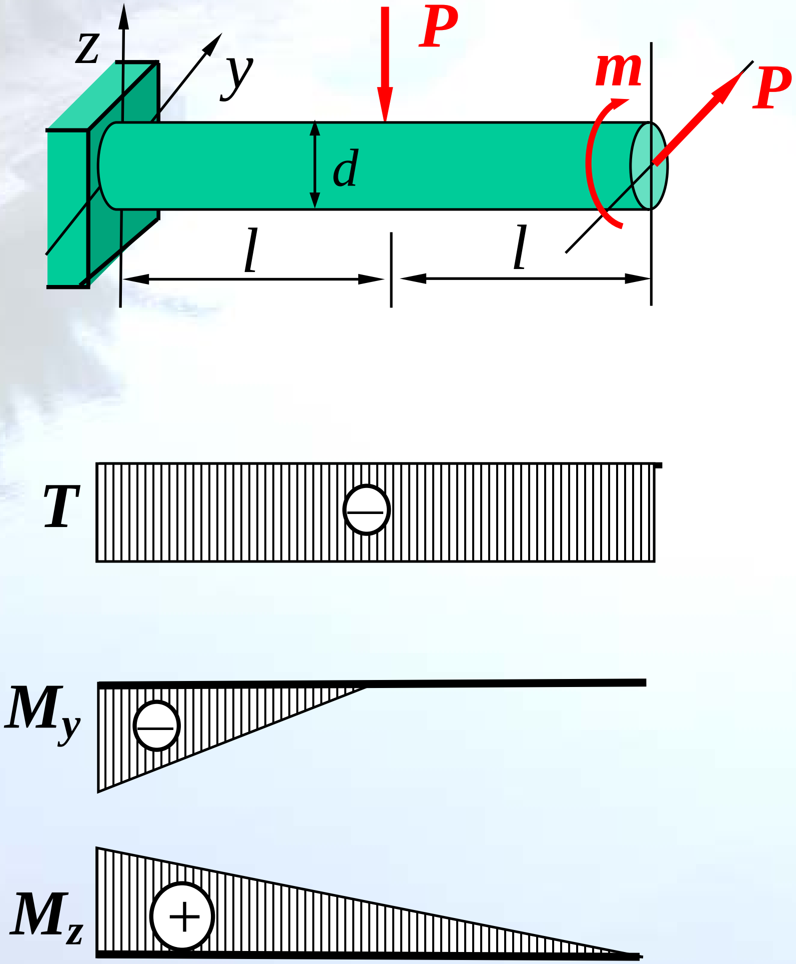
$$M_z=P\times 2l=4.2(\mathbf{kN{\cdot}m})$$



将弯矩合成：

$$M=\sqrt{M_y^2+M_z^2}=\sqrt{2.1^2+4.2^2}$$

$$=4.7(\mathbf{kN{\cdot}m})$$



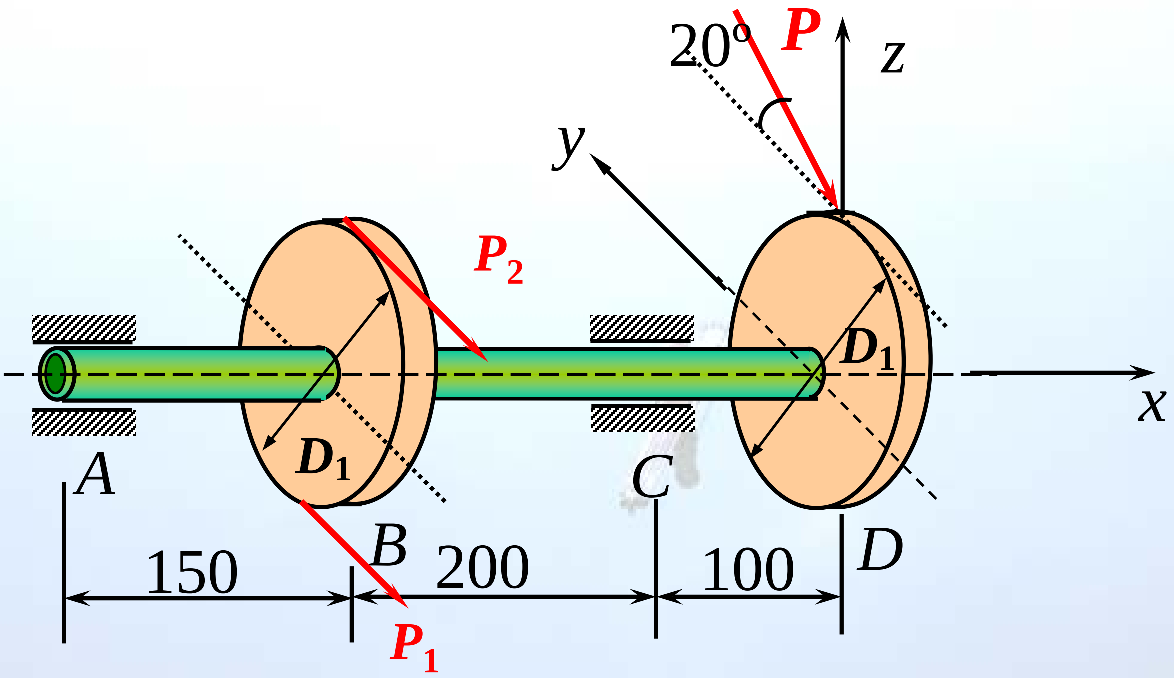
$$\sigma_{r3}=\frac{\sqrt{M^2+T^2}}{W}$$

$$=\frac{\sqrt{4.7^2+1.5^2}\times10^6}{\frac{3.14\times100^3}{32}}$$

$$=50.3\textbf{MPa}<[\sigma]$$

安全!

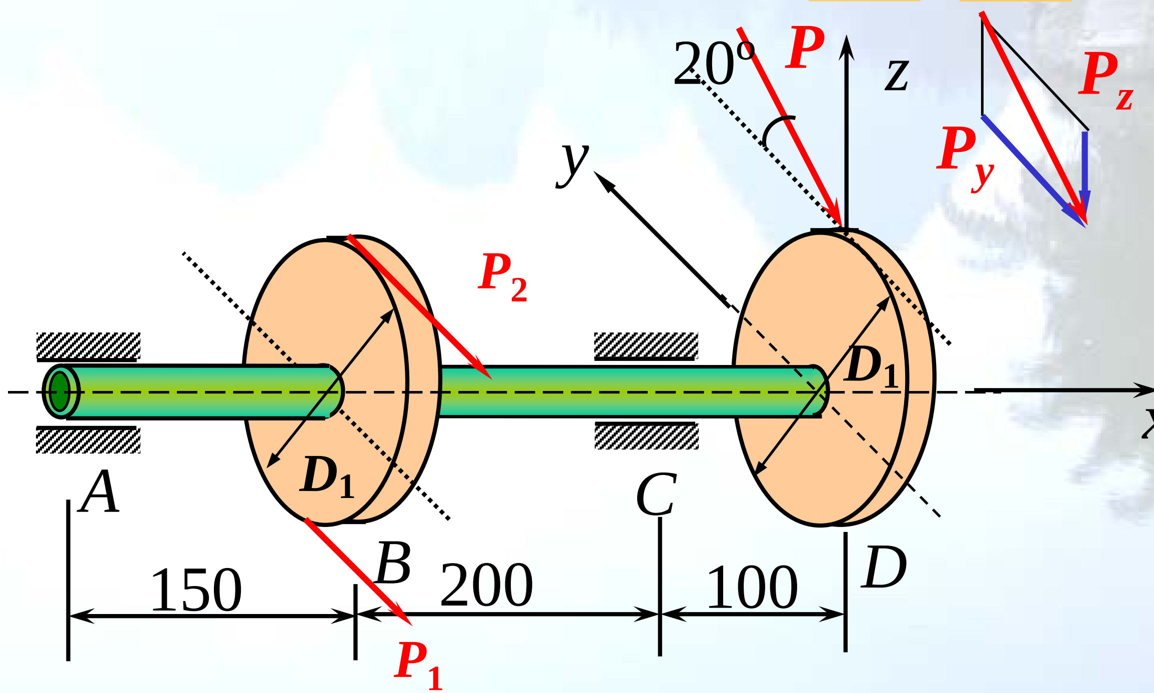
[例8] 图示空心圆轴，内径d=24mm，外径D=30mm，轮子直径D_1=400mm，P_1=1.2kN，$P_1=2P_2$，[σ]=120MPa，试用第三强度理论校核此轴的强度。



解：外力分析：

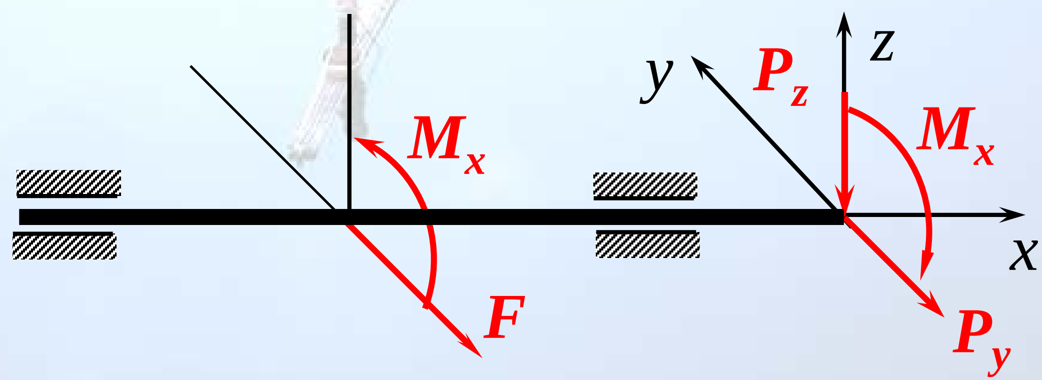
$$M_x = P_1 \cdot \frac{D_1}{2} - P_2 \cdot \frac{D_1}{2}$$

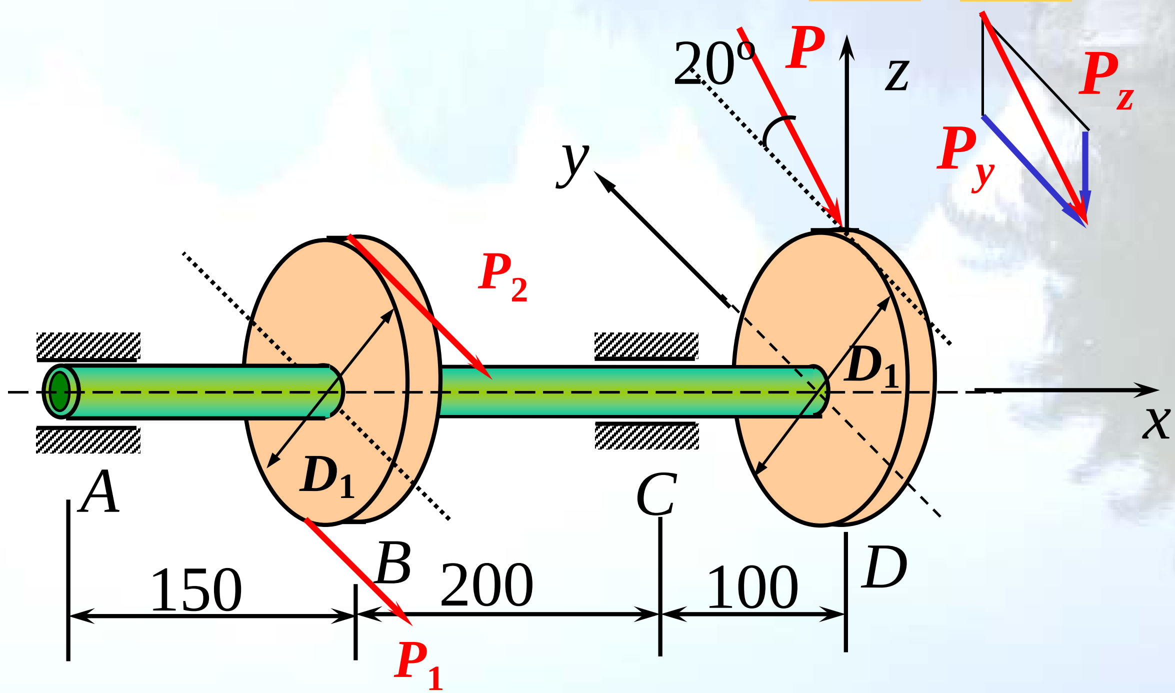
$$= 120(\mathrm{N \cdot m})$$



$$F = P_1 + P_2$$

$$= 1.8(\mathrm{kN})$$



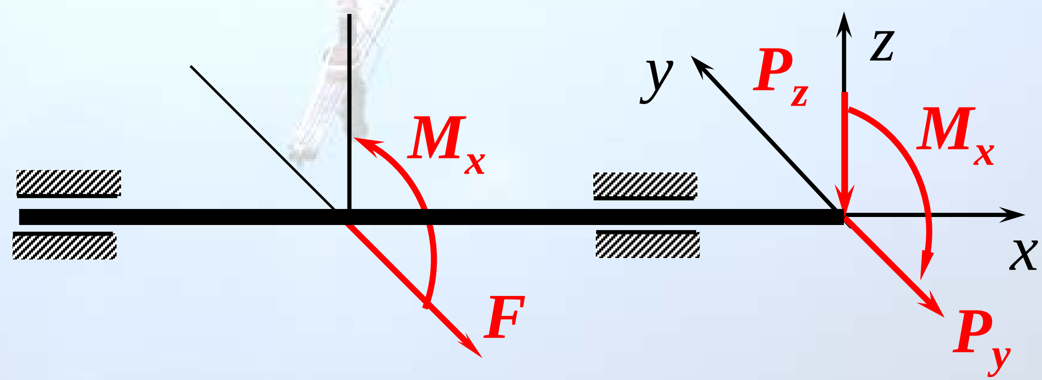


由 $M_x = P_y \cdot \dfrac{D_1}{2}$

得：$P_y = 0.6(\mathbf{kN})$

$P_z = P_y \cdot \tan 20°$

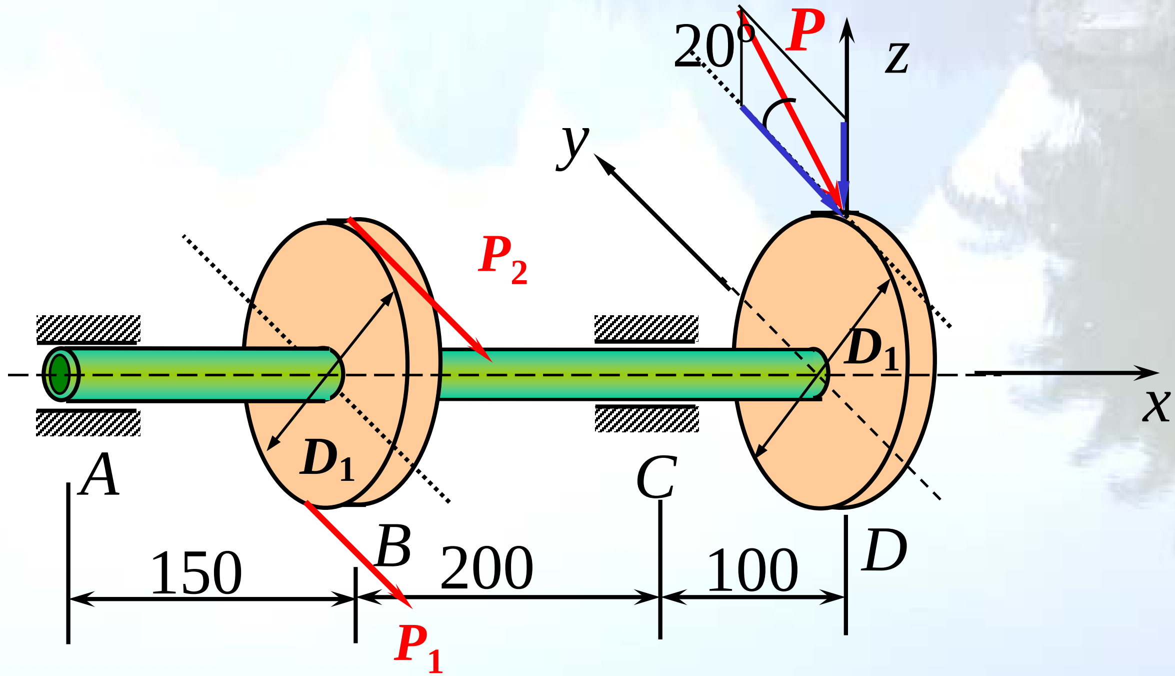
$= 0.218(\text{kN})$



$$M_x = 120(\mathrm{N\cdot m})$$

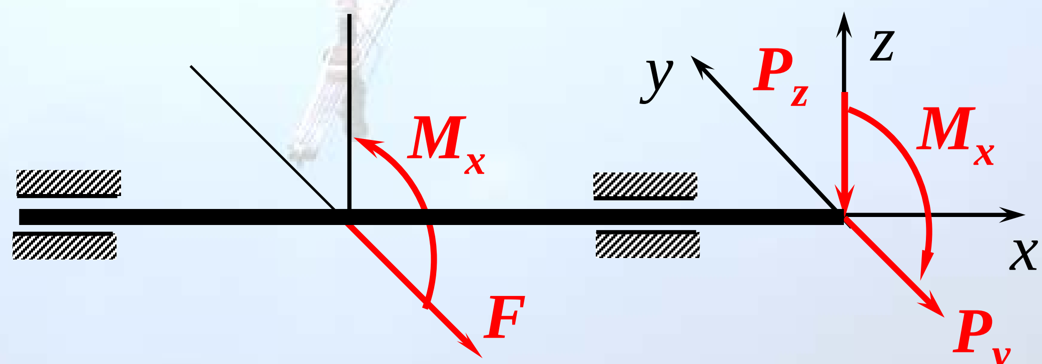
$$F = P_1 + P_2$$

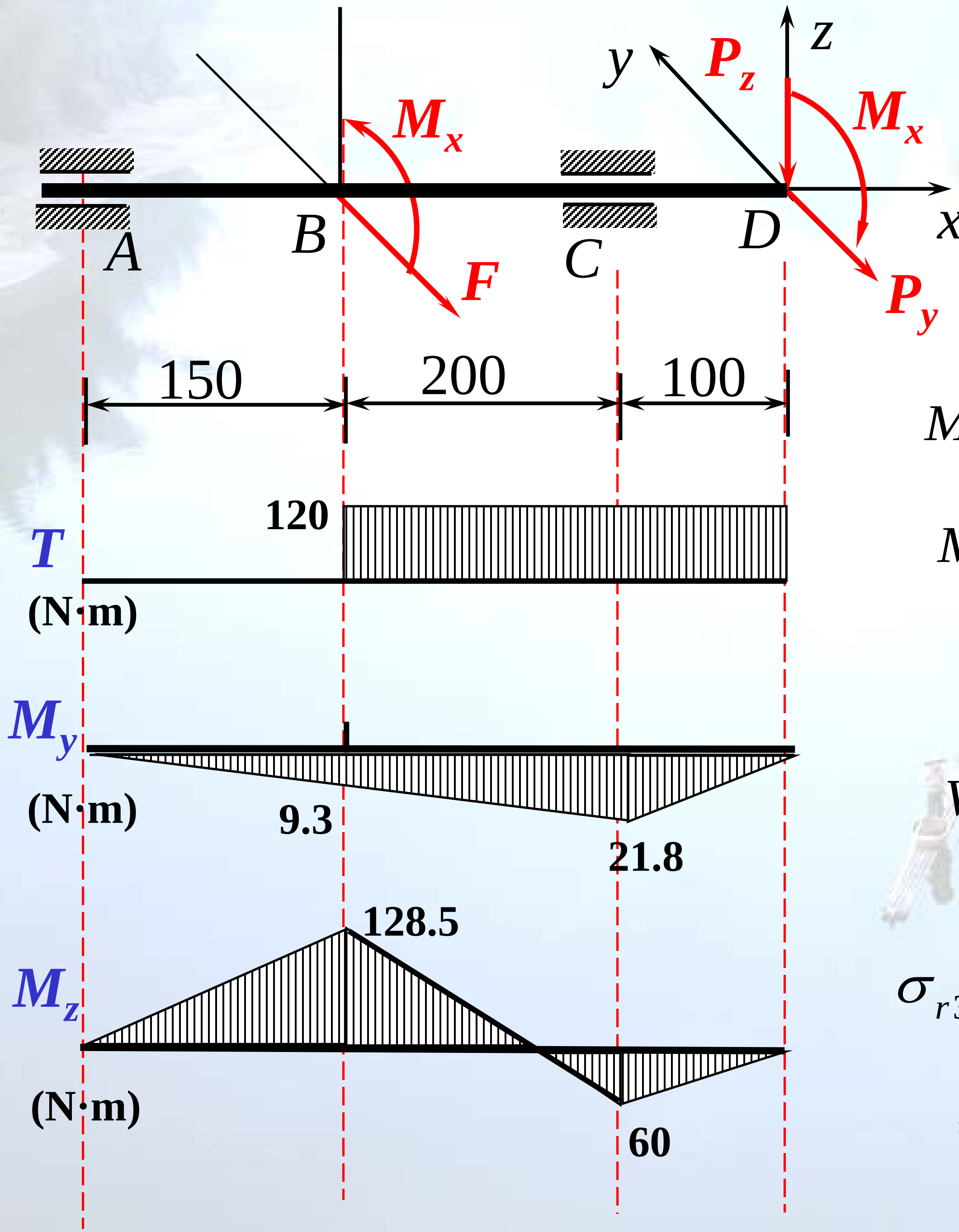
$$= 1.8(\mathrm{kN})$$



$$P_y = 0.6(\mathbf{kN})$$

$$P_z = 0.218(\mathbf{kN})$$





内力分析：弯扭组合变形

危险面内力为：

$$M_B=\sqrt{9.3^2+128.5^2}=128.8(\mathbf{N{\cdot}m})$$

$$M_C=\sqrt{21.8^2+60^2}=63.8(\mathbf{N{\cdot}m})$$

∴ B截面是危险面。

$$W=\frac{\pi D^3}{32}\left[1-\left(\frac{d}{D}\right)^4\right]=1564(\mathbf{mm^3})$$

$$\sigma_{r3}=\frac{\sqrt{M_B^2+T^2}}{W}=\frac{\sqrt{128.8^2+120^2}\times10^3}{1564}$$

$$=112.5(\mathbf{MPa})<[\sigma]$$

∴ 安 全


本章结束