材料力学

第七章 应力和应变分析 强度理论

第七章 应力和应变分析　　强度理论

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§ 7–1　应力状态概述

一、为什么要研究一点处的应力状态

分析拉(压)杆斜截面上的应力,任意一点所作各个截面上的应力都随着截面的方位而改变,一般说来，通过受力构件内任意一点所作的各个截面上，该点处的应力都随截面方位的不同而异。

构件的强度计算

轴向拉伸(压缩)和纯弯曲的构件，由于其材料处于单向拉伸或 压缩状态，横截面正应力与也是单向拉伸(压 缩)时材料的许用应力加以比较而建立强度条件。

自由扭转的构件，其材料处于纯剪切应力状态，横截面的剪应力与纯剪切时材料的许用应力相比较来建立强度条件。

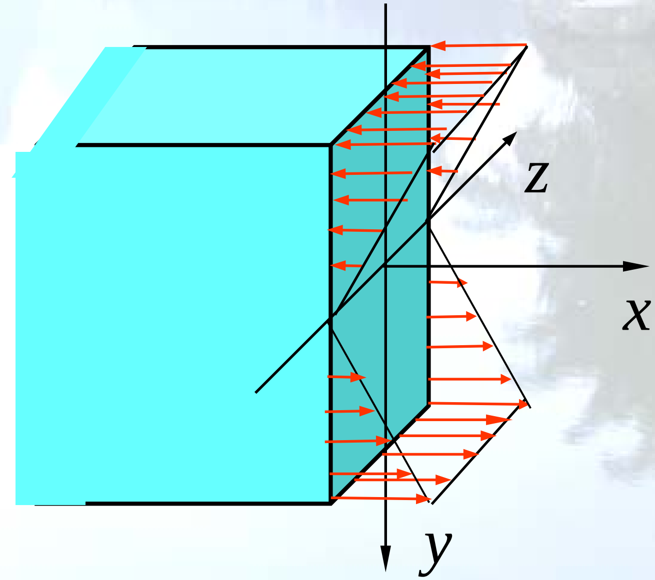
一般情况，梁在横力弯曲时，除去离中性轴最远的两边缘上和中性轴上的各点以外，在其它点处既有正应力又有剪应力，材料处于较复杂的应力状态。

对梁进行强度计算时，既不能认为材料处于单向拉伸或压缩状态，也不能认为材料处于纯剪切应力状态。

因此,不能只按单向拉 (压)，或只按剪应力来建立强度条件，必须考虑两种应力对材料强度的综合影响。

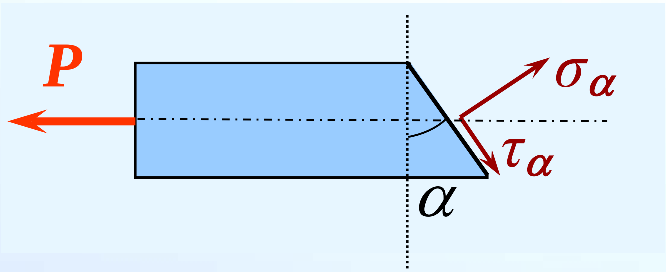
要解决在这类情况下的强度计算问题，研究一点处的应力状态。

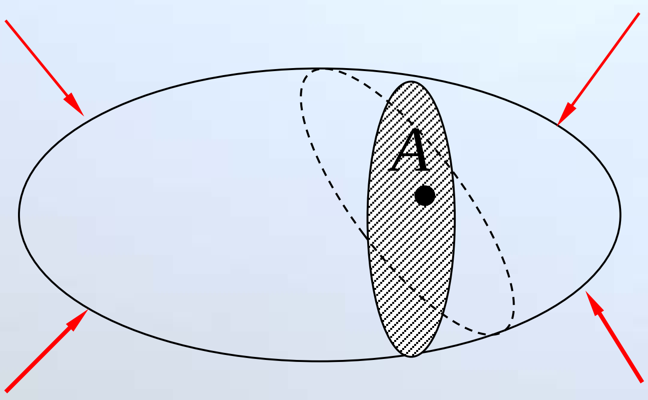
二、什么是一点处的应力状态？

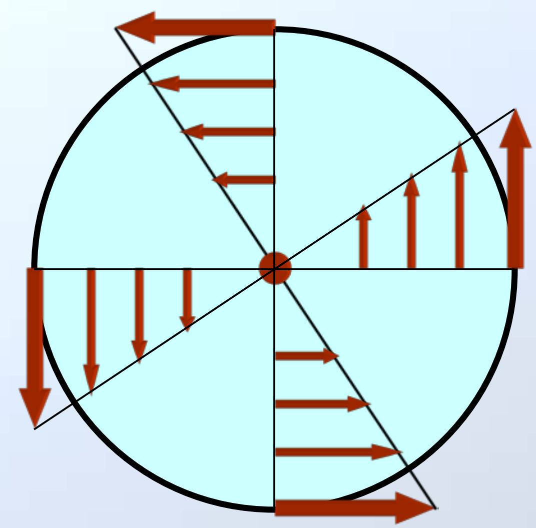


应力 → 与点的位置有关

应力 → 与作用面的方位有关



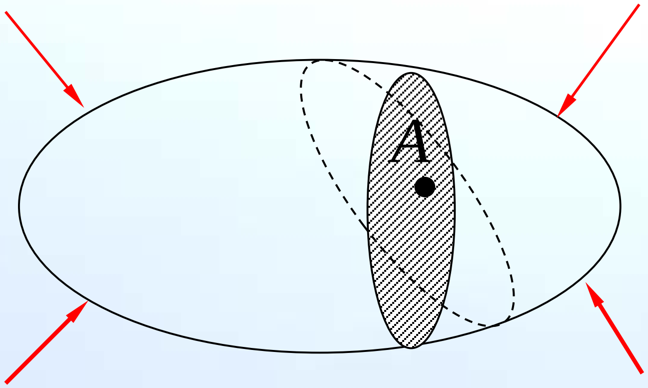




过一点有无数个不同方位的截面。

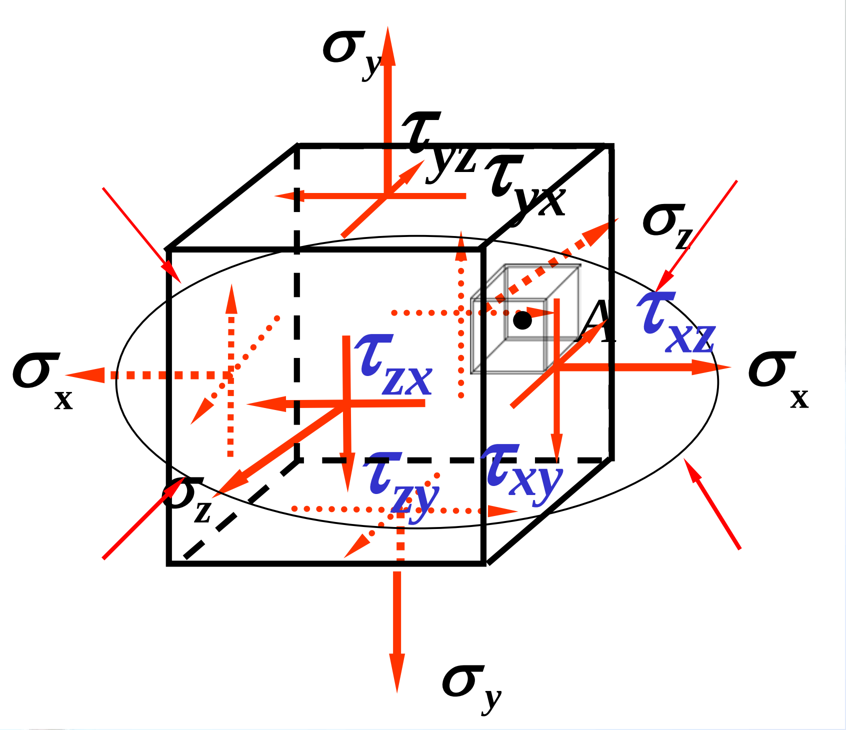
什么是一点处的应力状态？

一点处不同方位截面上应力的集合，称为这点的应力状态。



三、一点处应力状态的表示方法:

（1）单元体



单元体——构件内点的代表物，是包围被研究点的无限小的几何体，常用的是正六面体。

单元体各面上应力均布；相互平行的面上应力相等，面上的应力值即为该点所对应截面方位的应力大小。

应力单元体是一点受力状态的完整表示。

（2）应力分量

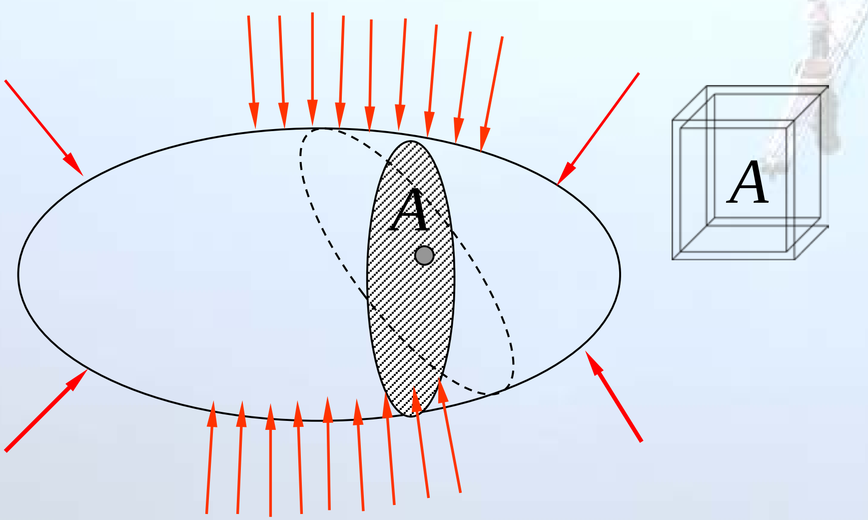
$$\sigma_x \quad \sigma_y \quad \sigma_z$$

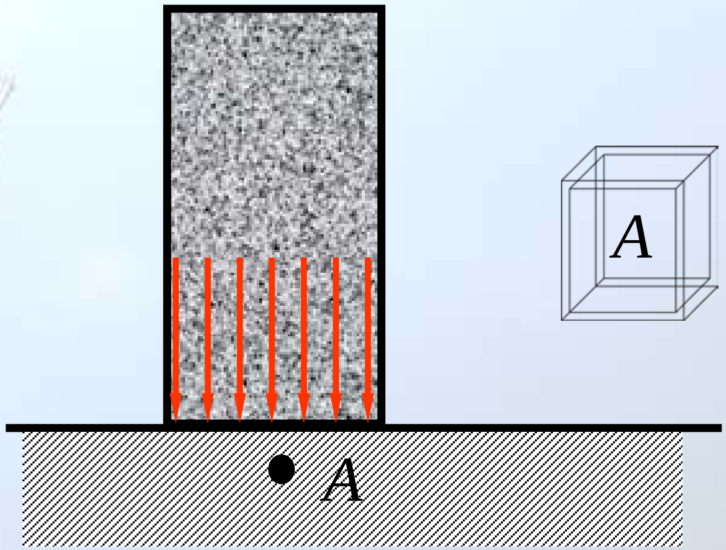
$$\tau_{xy} \quad \tau_{yz} \quad \tau_{zx}$$

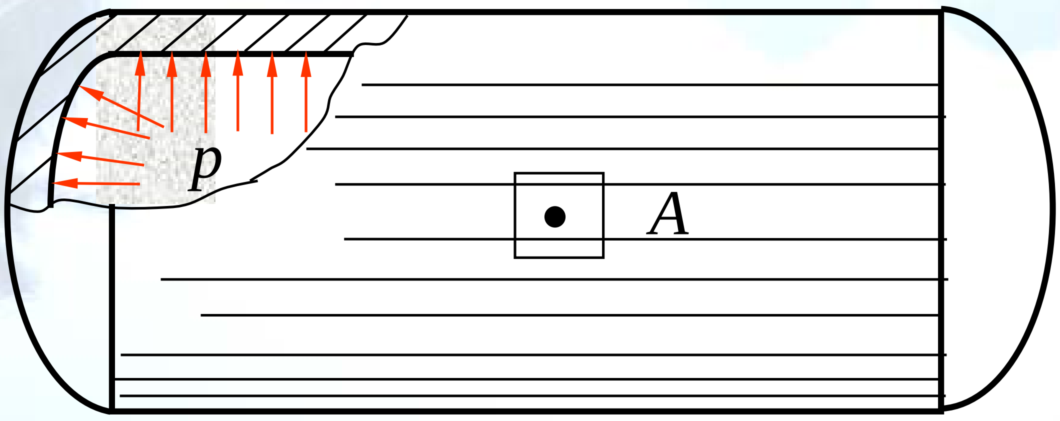
$$\cancel{\tau_{yx} \quad \tau_{zy} \quad \tau_{xz}}$$

一点有六个独立的应力分量

四、不同应力状态的实例



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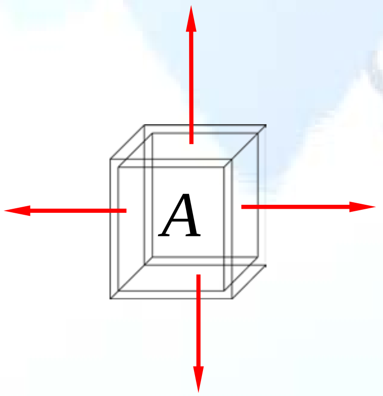
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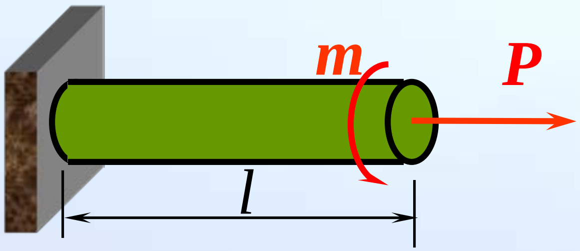
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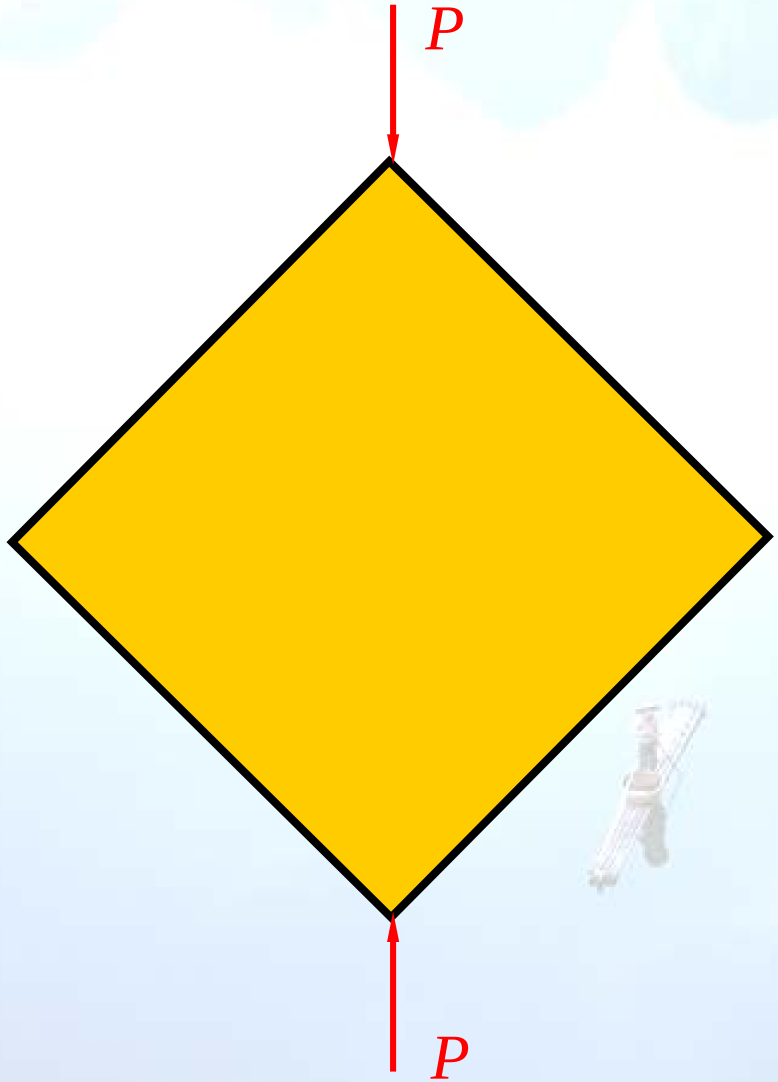
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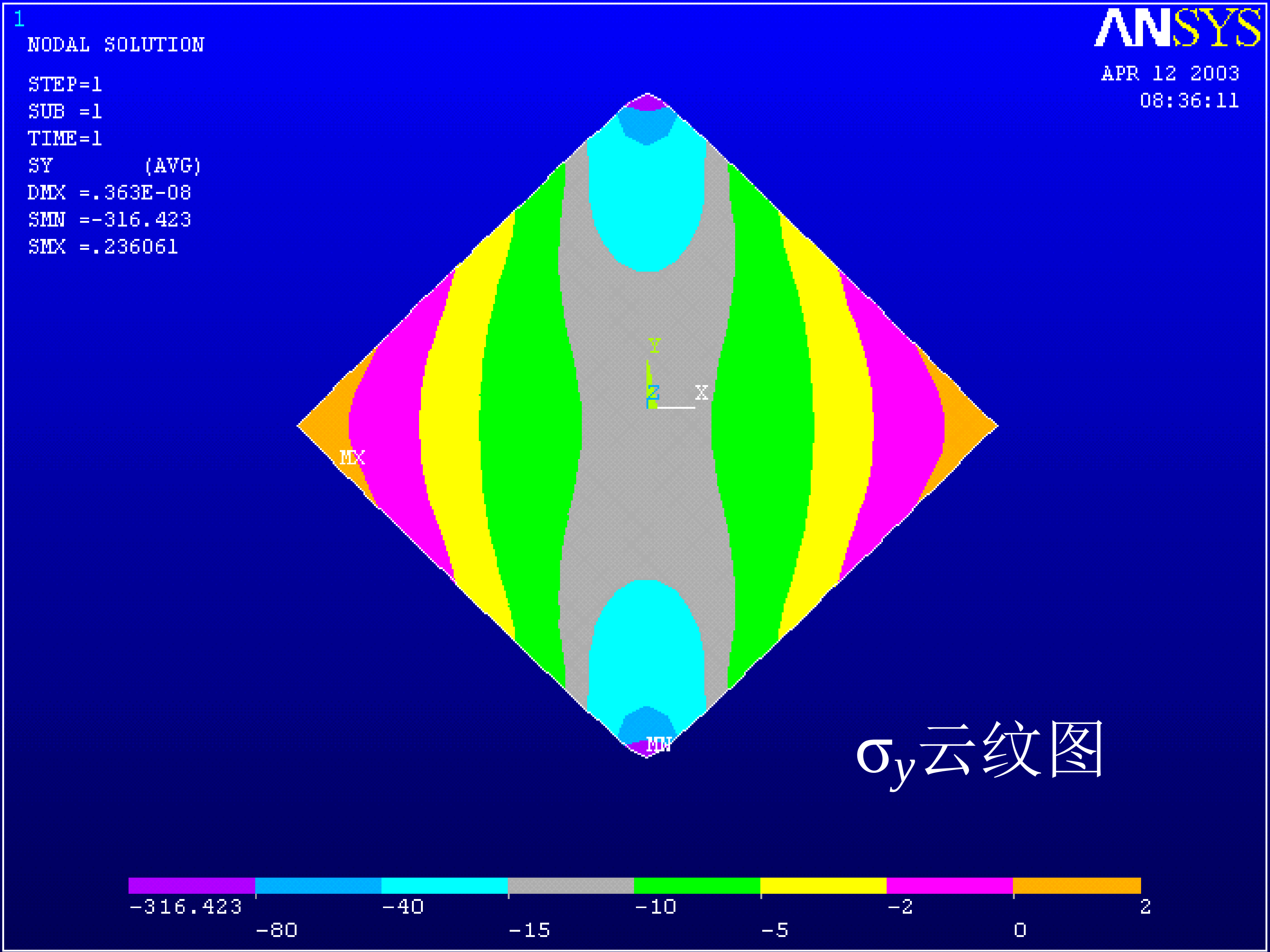
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08:36:11
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MN
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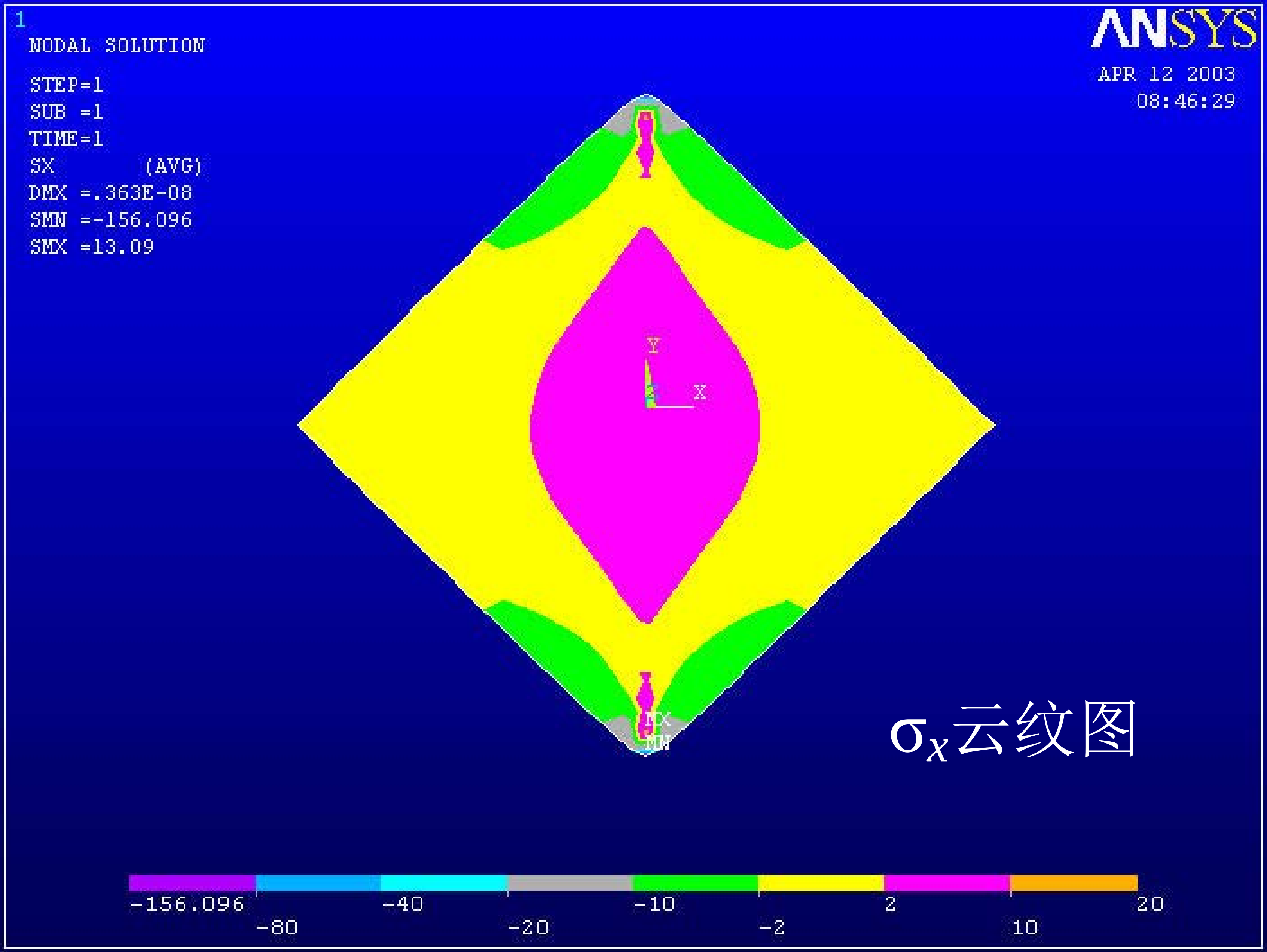
-40
-15
-10
-5
-2
0
2

σ_y云纹图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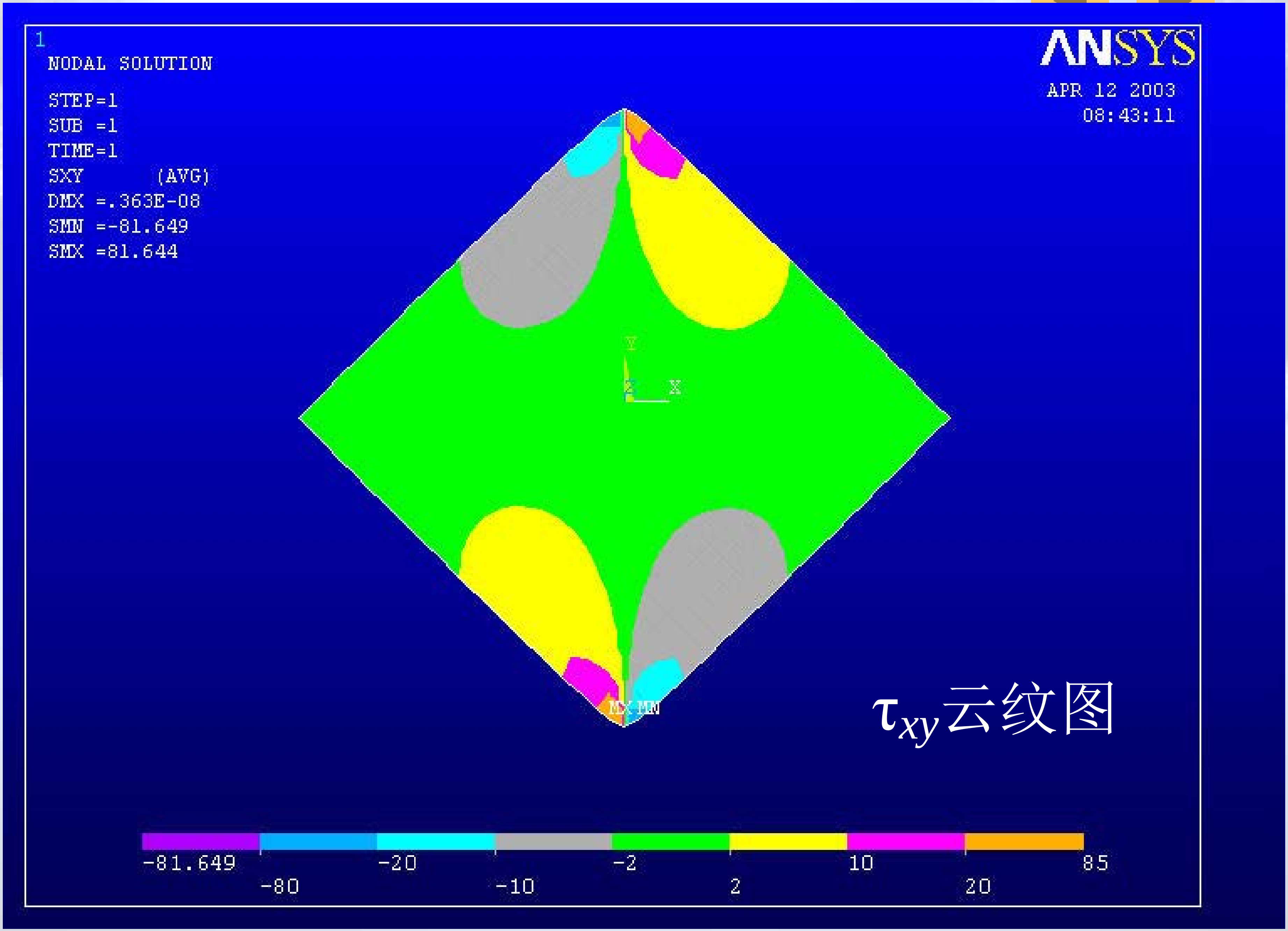
2
10
20

σ_x云纹图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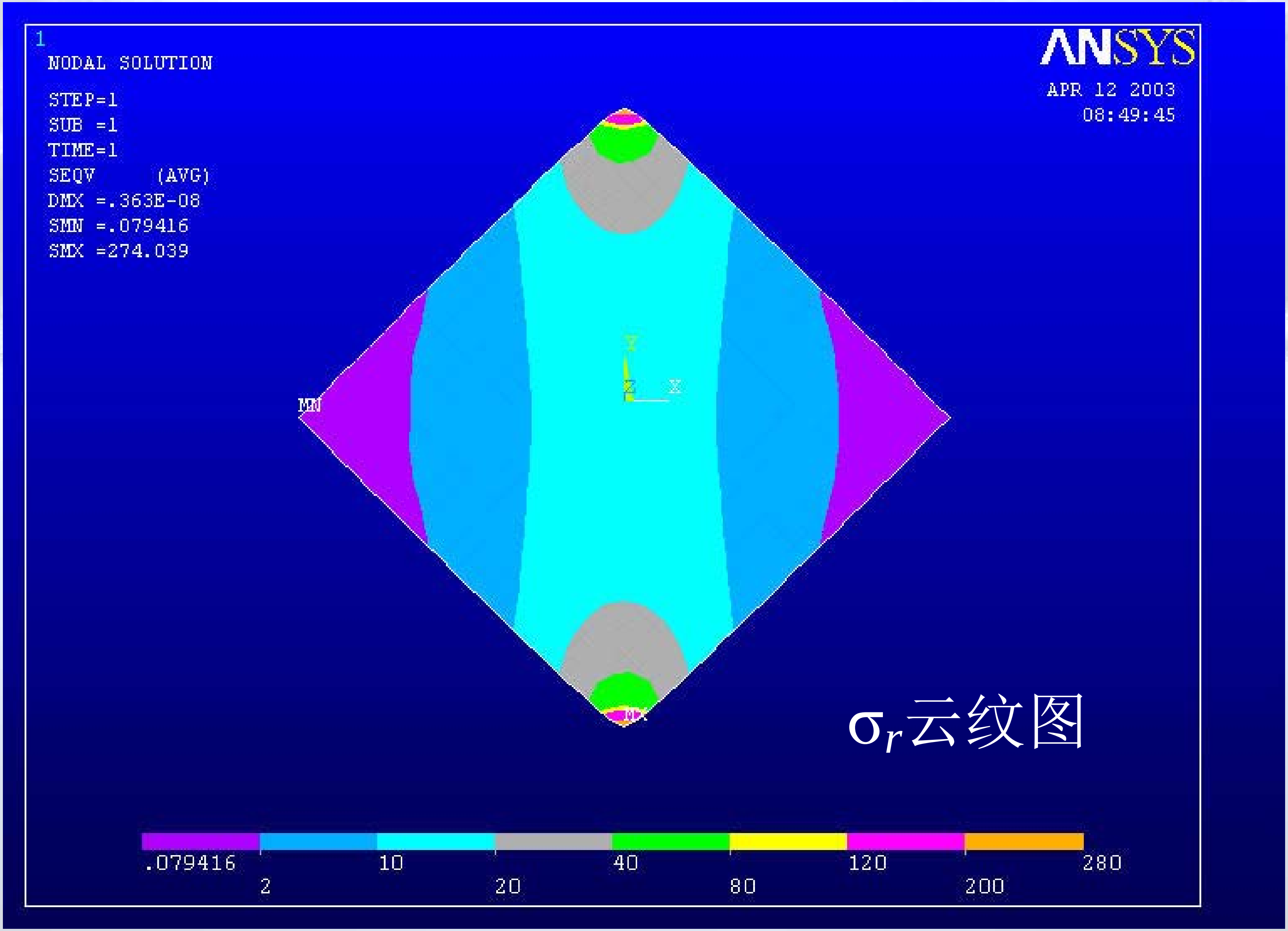
-10
-2
2
10
20
85

τ_{xy}云纹图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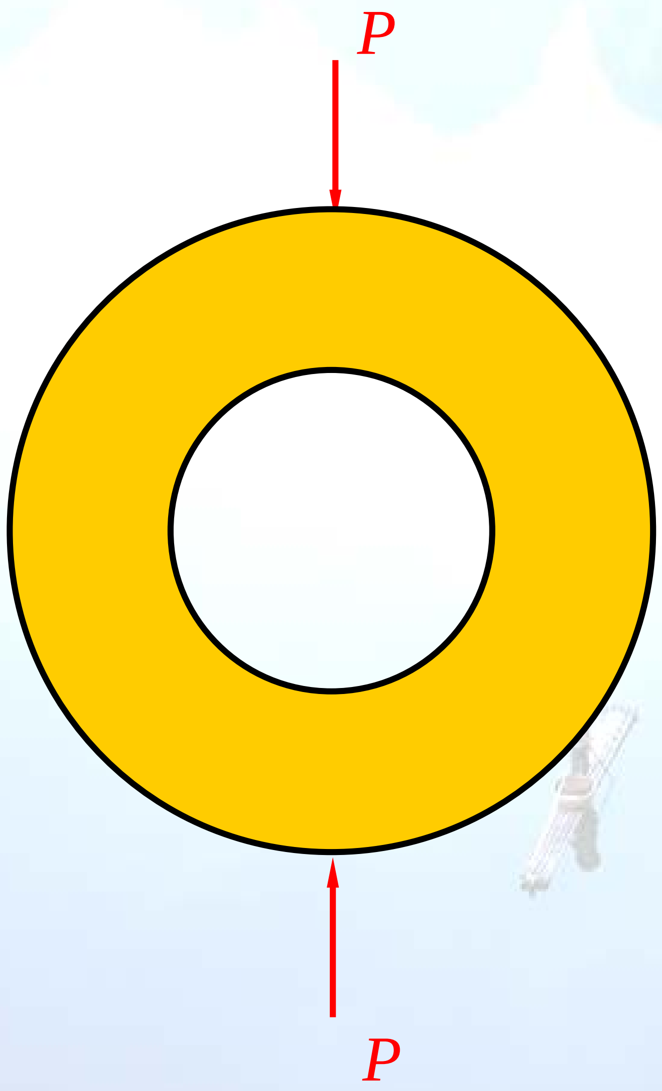
2
10
20
40
80
120
200
280

σ_r云纹图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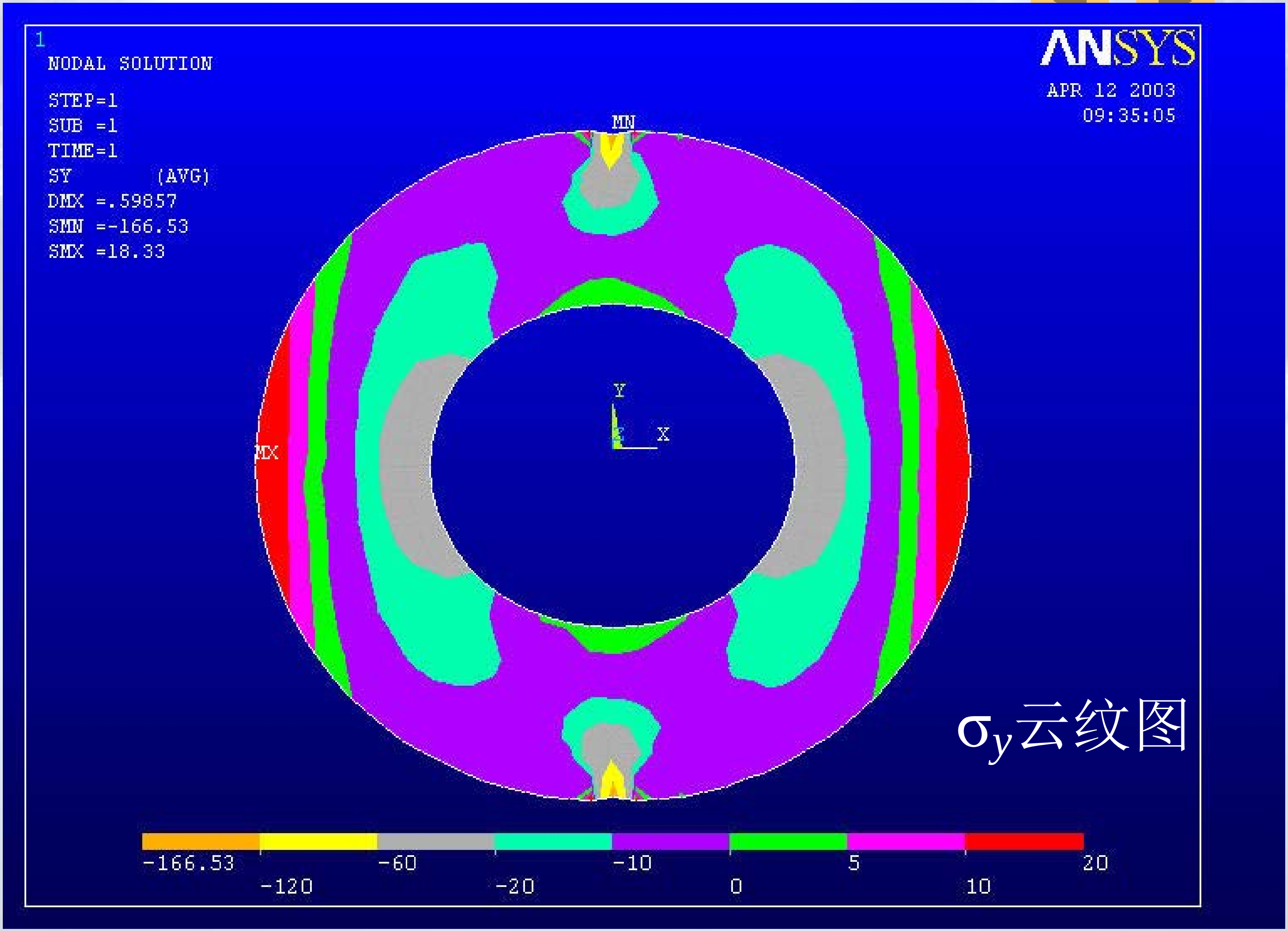
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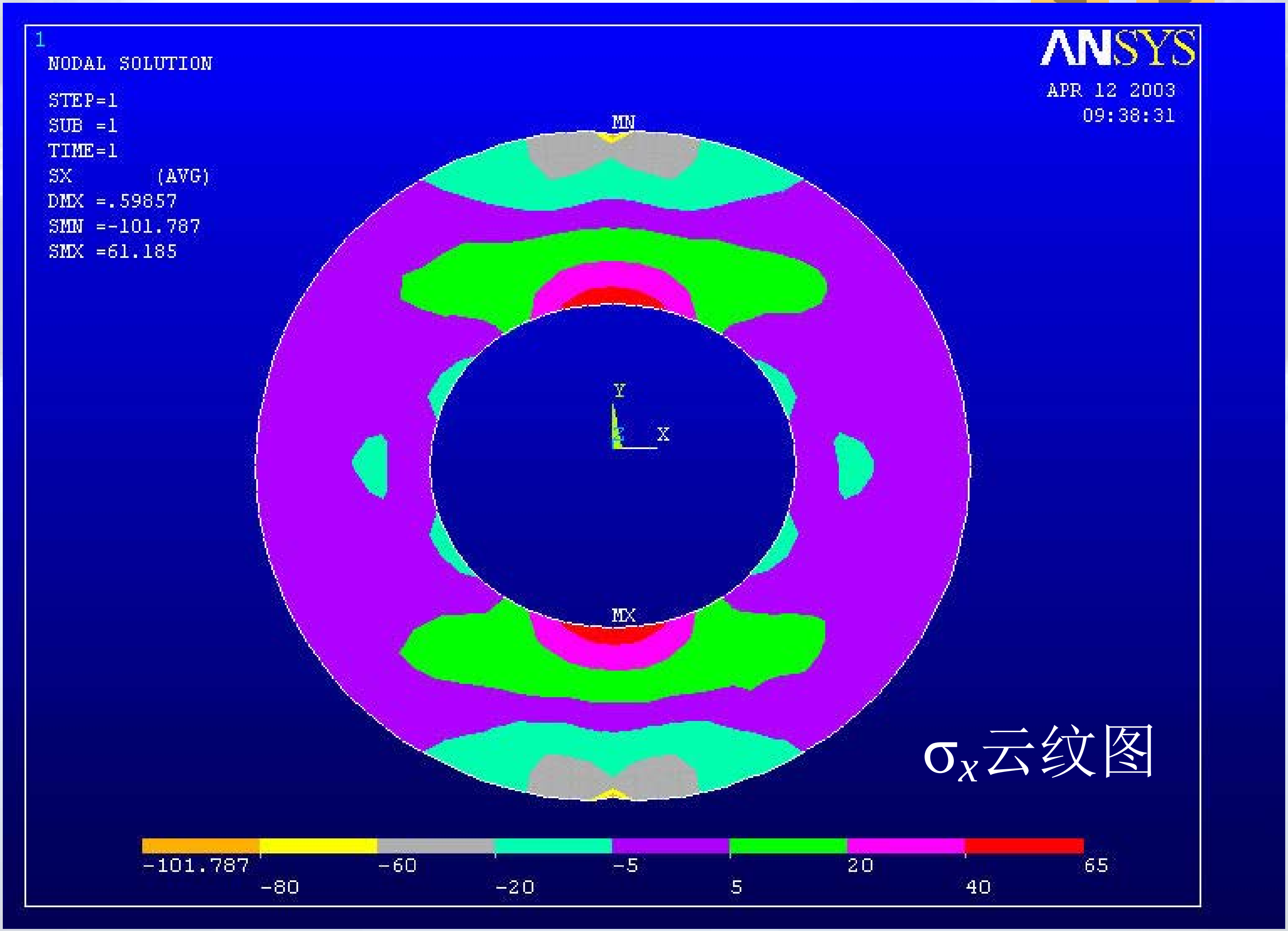
0
5
10
20

σ_y云纹图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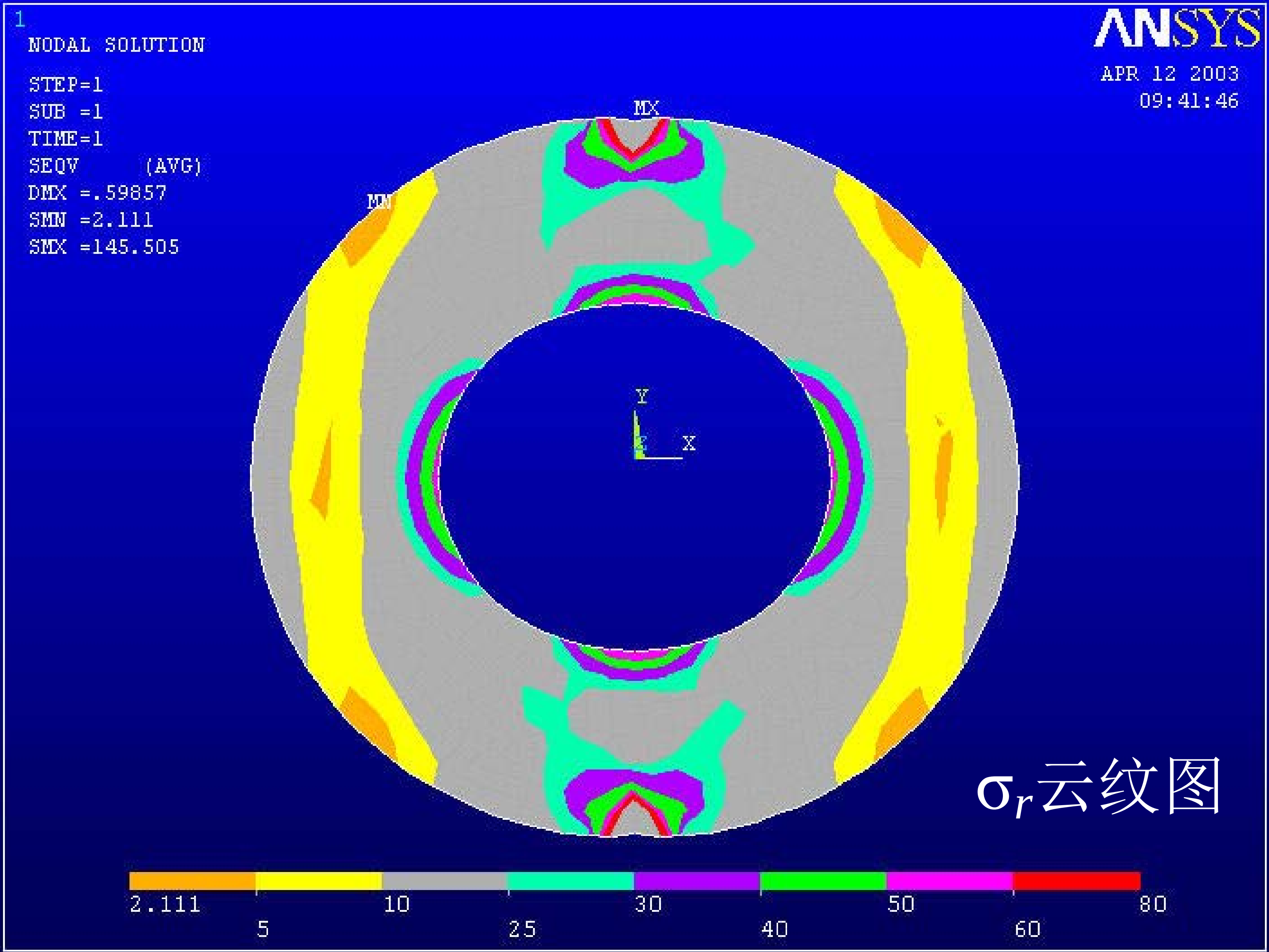
5
20
40
65

σ_x云纹图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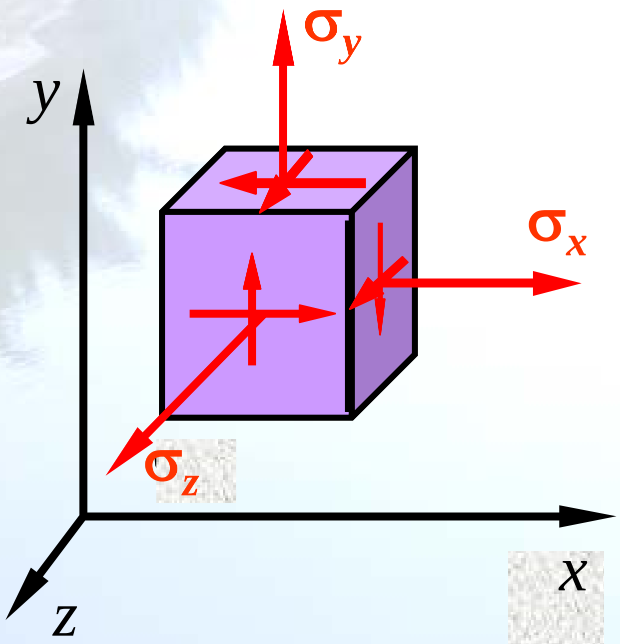
5
10
25
30
40
50
60
80

σ_r云纹图

五、主平面、主应力：

（1）主平面(Principal Plane)：

剪应力为零的截面。



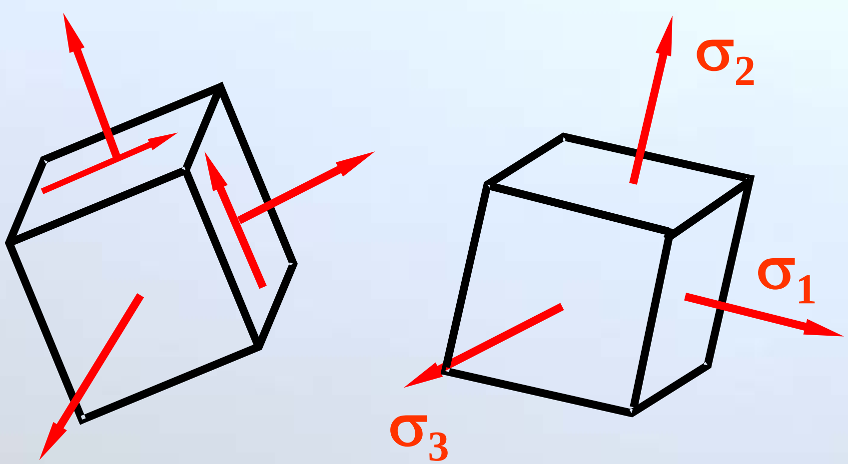
任意一点都可以找到三个相互垂直的主平面。

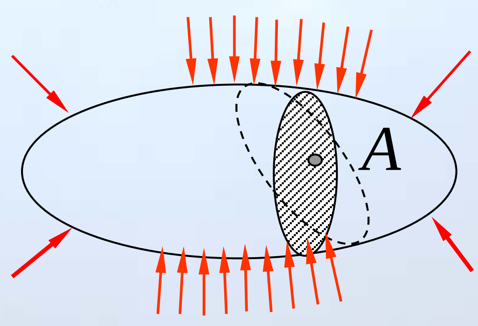
（2）主应力(Principal Stress)：

主平面上的正应力。

主应力排列规定：按代数值大小，

$$\sigma_1 \geq \sigma_2 \geq \sigma_3$$

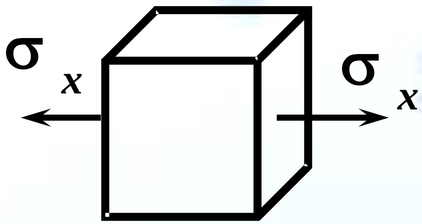




六、应力状态分类

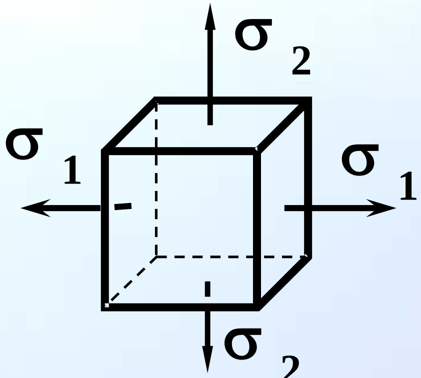
1、单向应力状态（Unidirectional State of Stress）：

一个主应力不为零的应力状态。



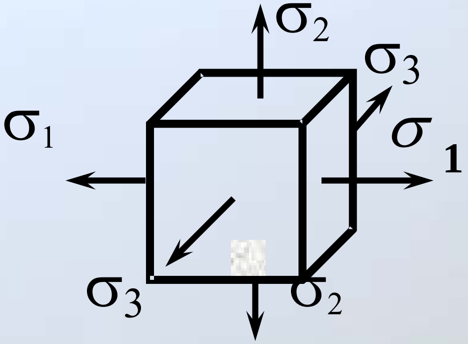
2、二向应力状态（Plane State of Stress）：

二个主应力不为零的应力状态。



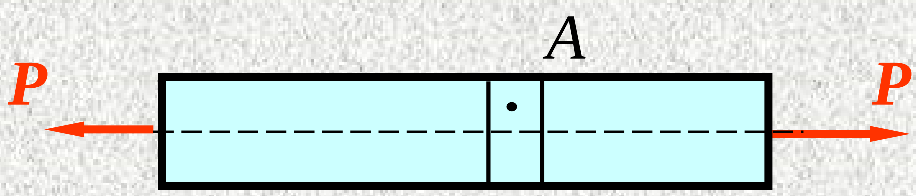
3、三向应力状态（ Three—Dimensional State of Stress）：

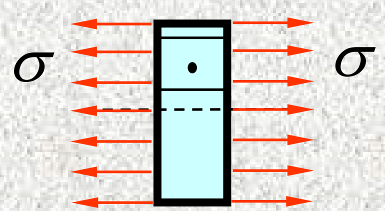
三个主应力都不为零的应力状态。

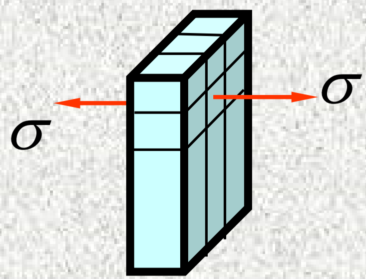


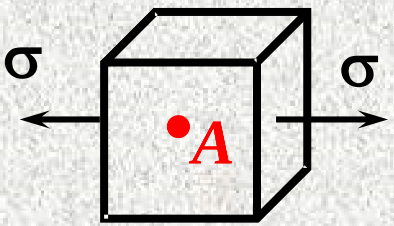
§ 7–2　　二向和三向应力状态的实例

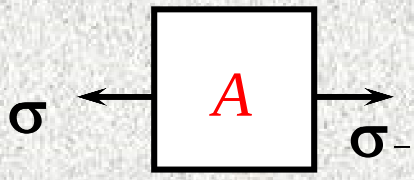
[例1]画出图中**A**点的应力单元体。



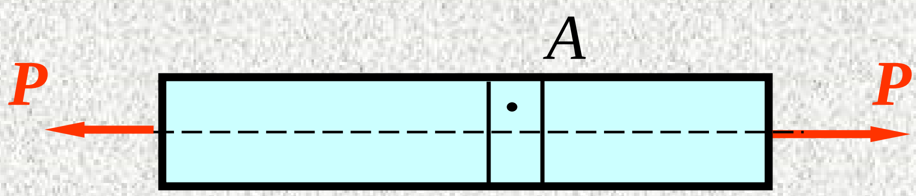


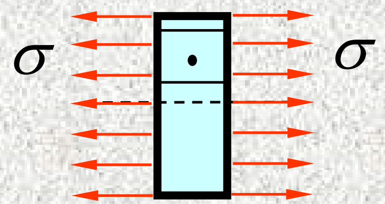


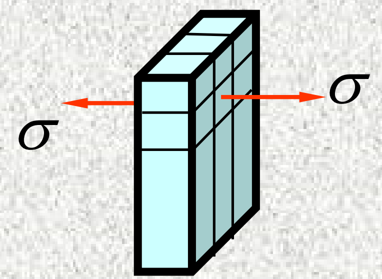


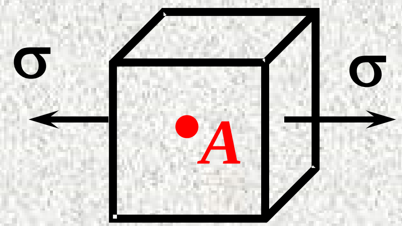


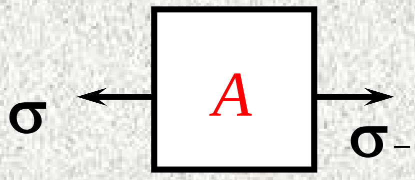
[例2] 画出图中**A**点的应力单元体。



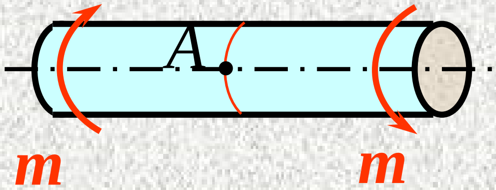


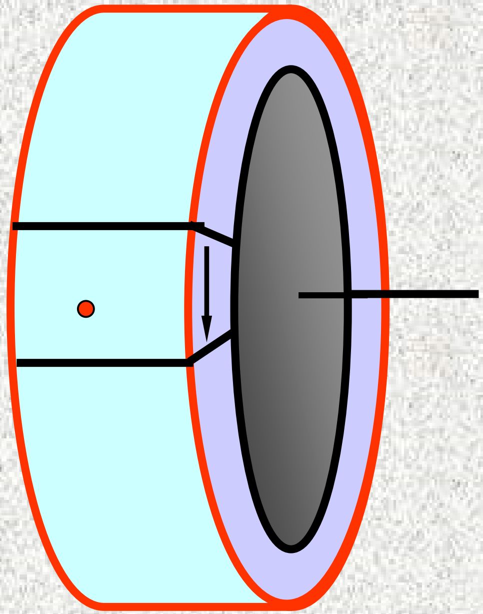


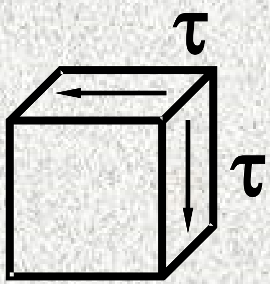


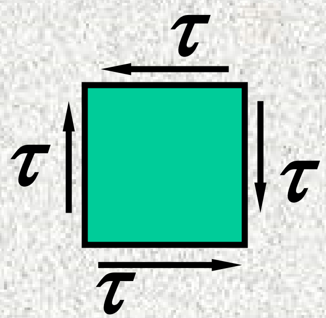


[例3]　画出图中A点的应力单元体。

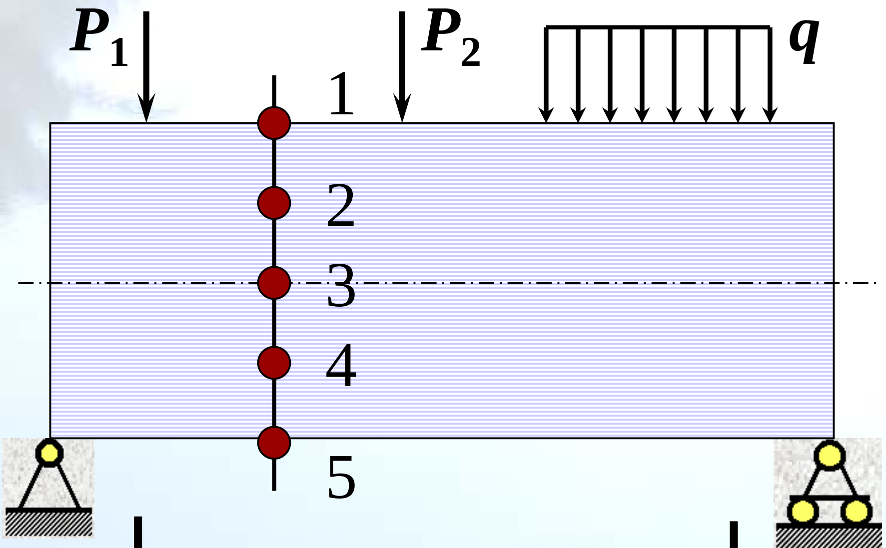


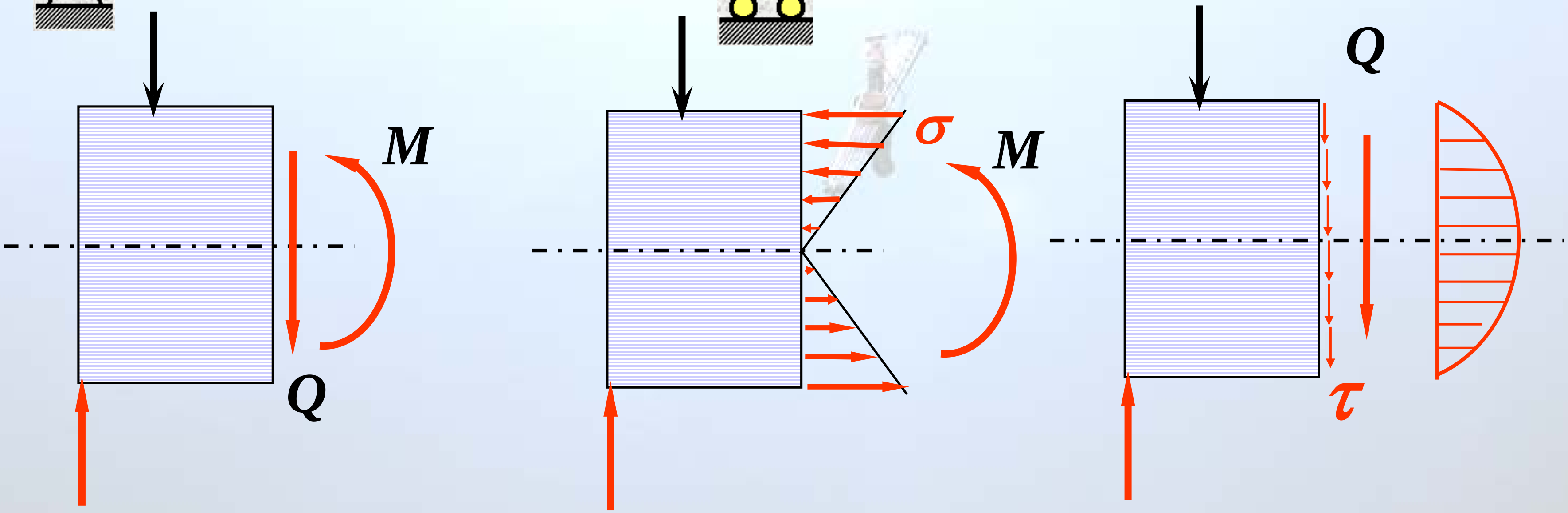


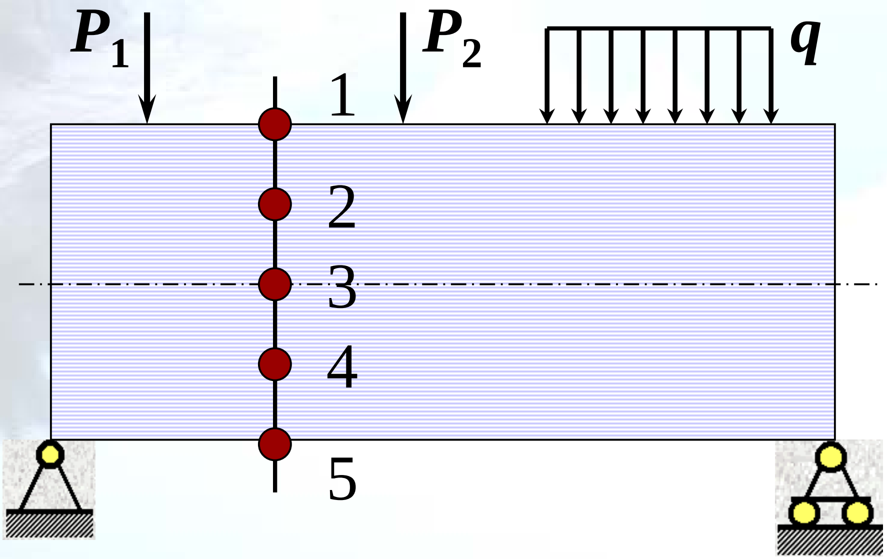




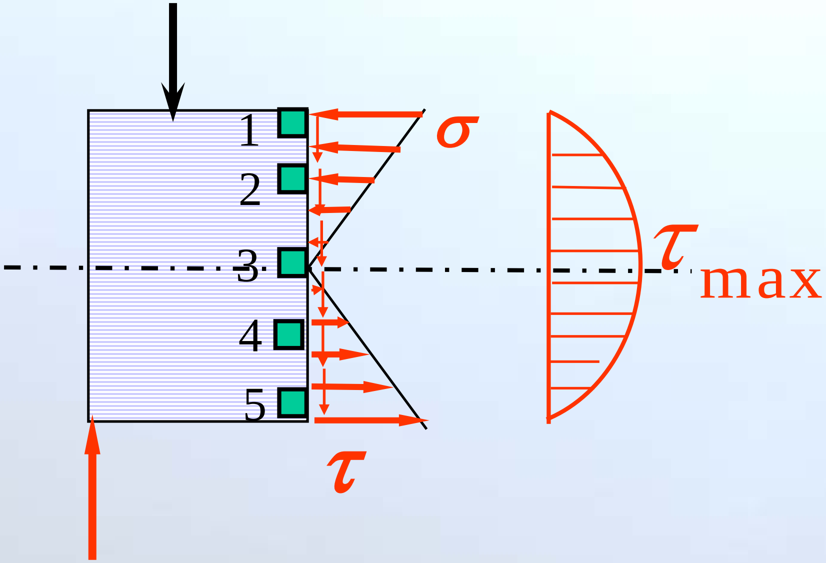
[例4] 画出图中各点的应力单元体。



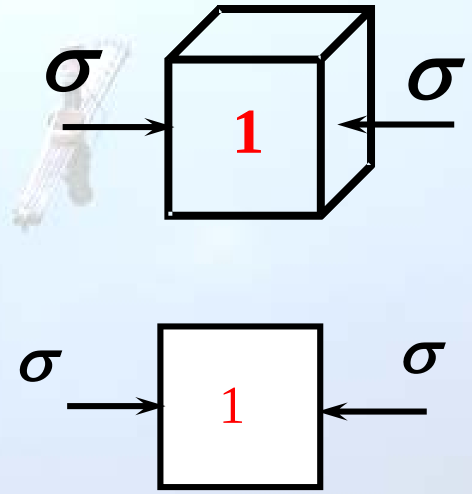




P_1
P_2
q
1
2
3
4
5



1
2
3
4
5
σ
τ_{max}
τ



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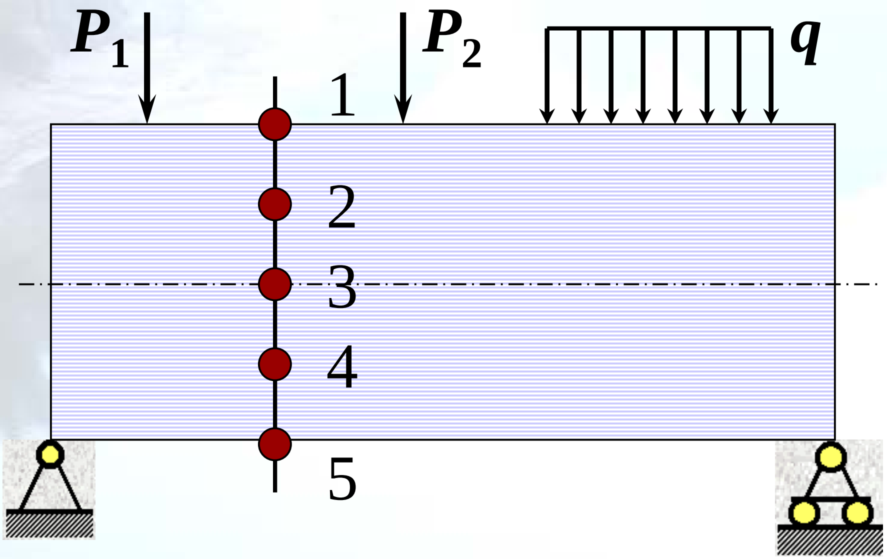
σ
σ
1
σ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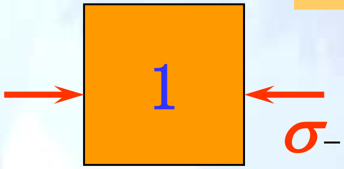
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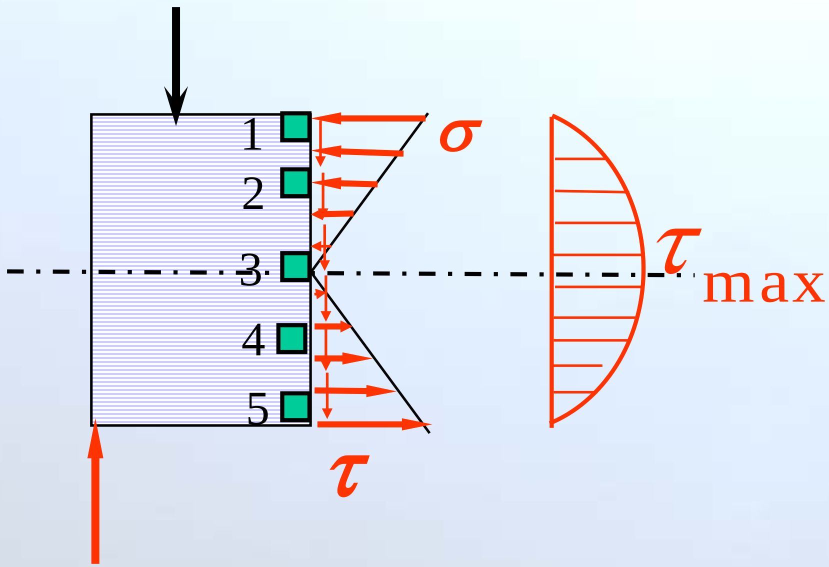
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σ
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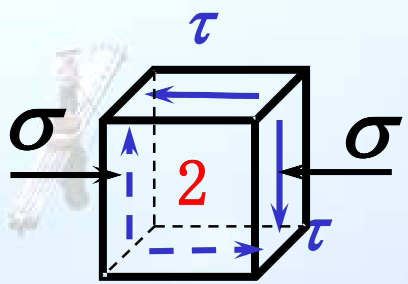
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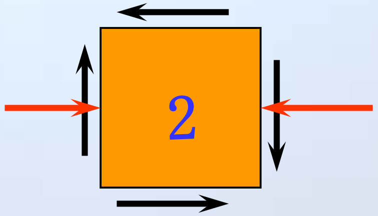
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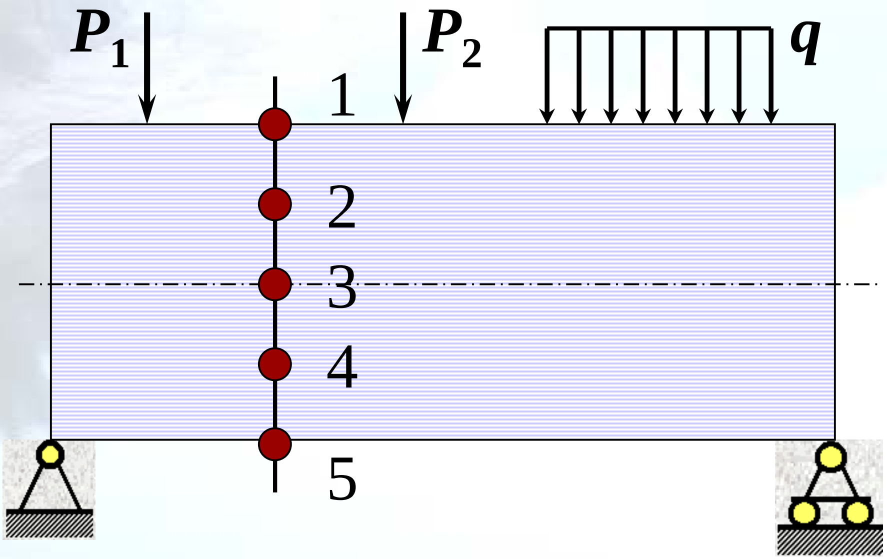
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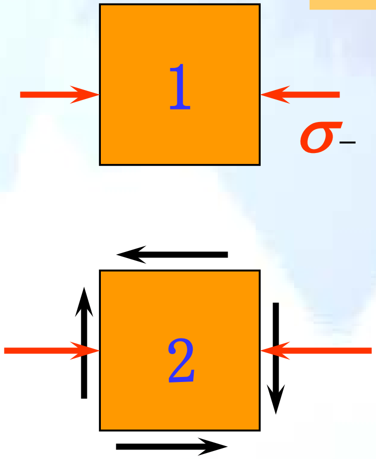
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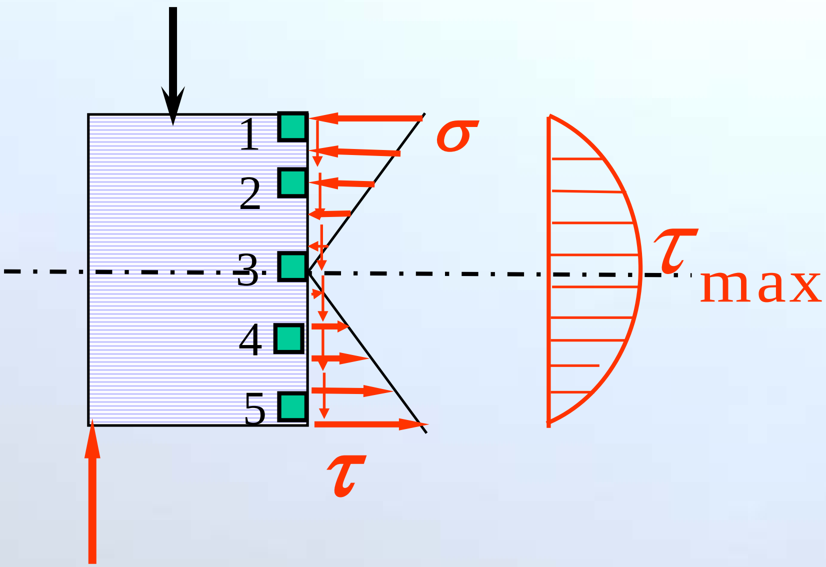
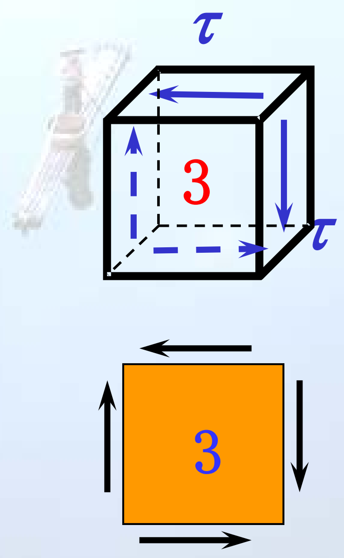
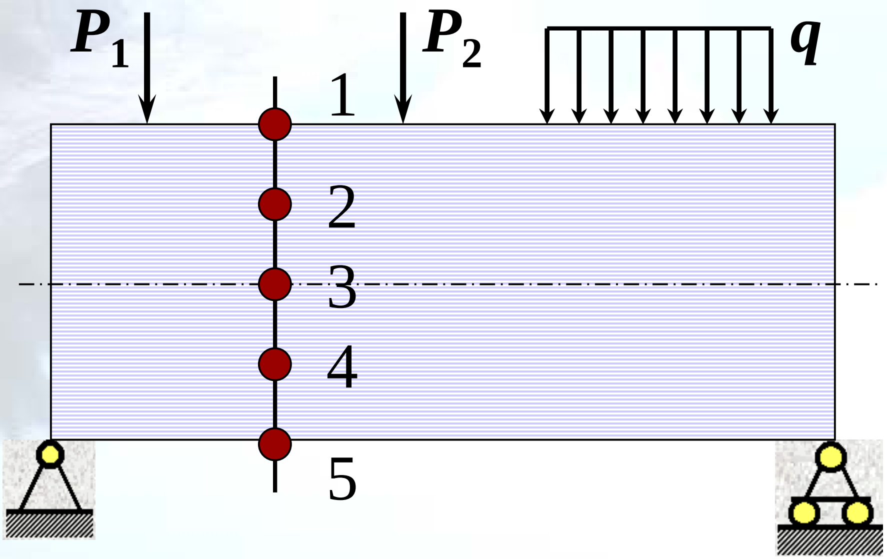
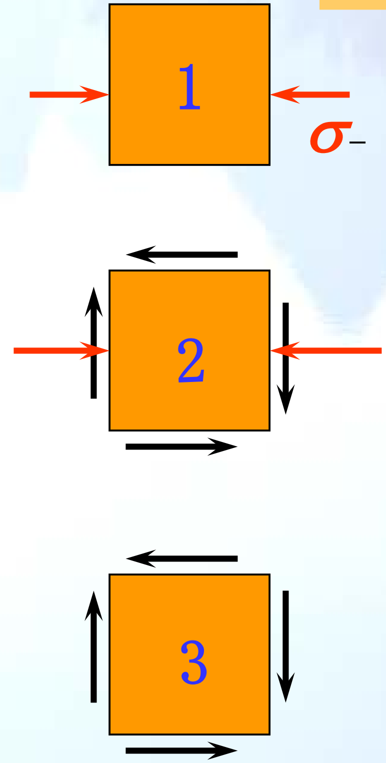
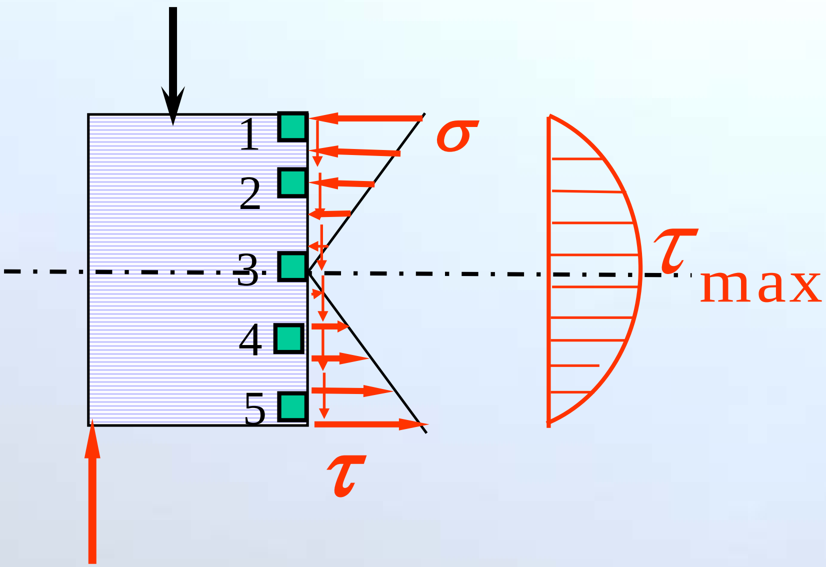
σ-
2

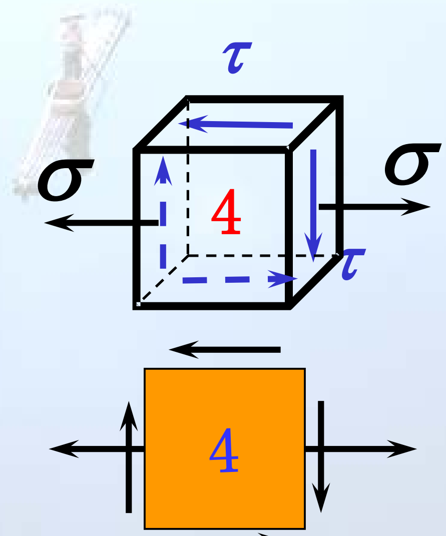

1
2
3
4
5
σ
τmax
τ

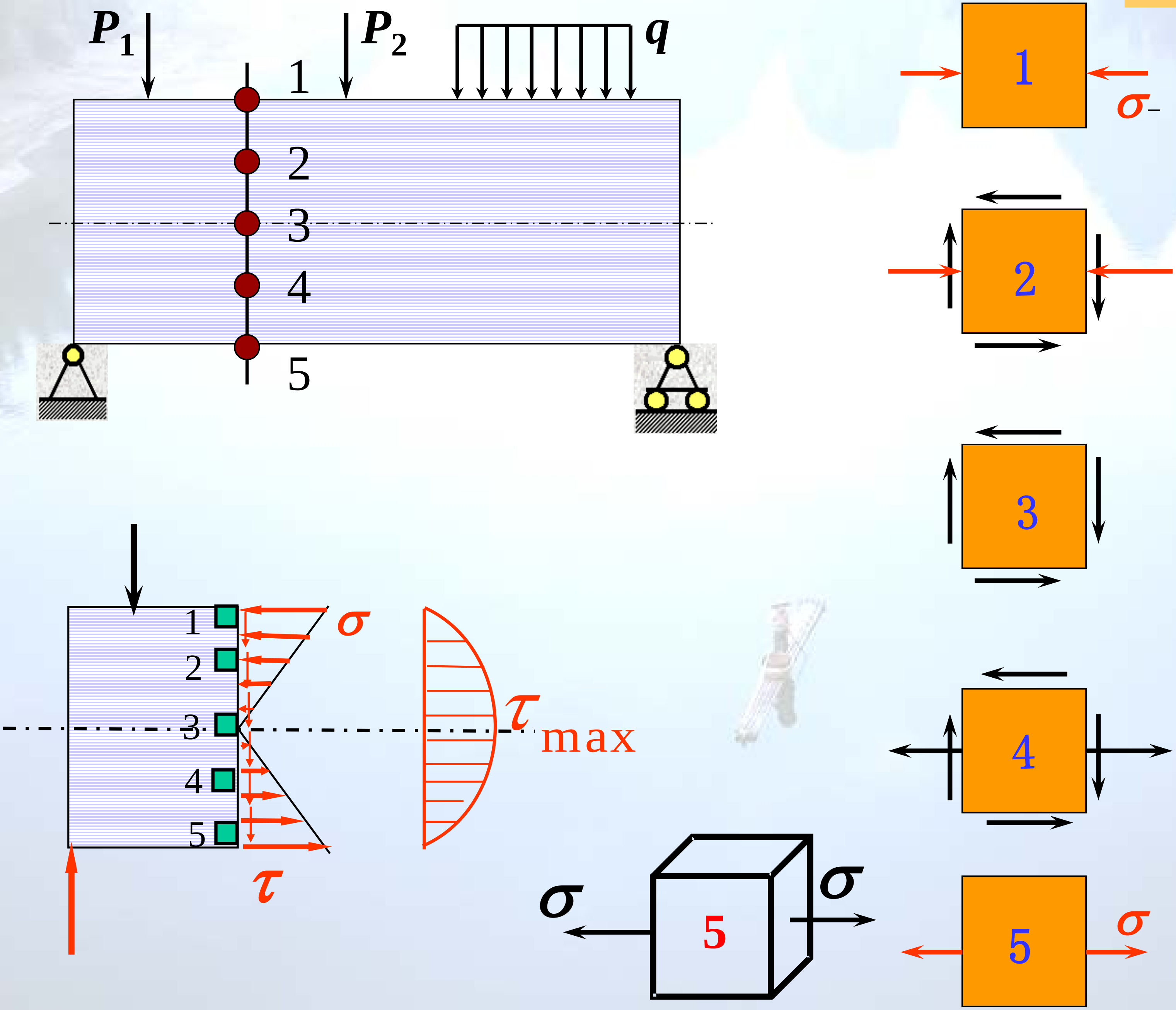

τ
3
τ
3


P1
1
P2
q
2
3
4
5


1
σ−
2
3


1
2
3
4
5
σ
τmax
τ

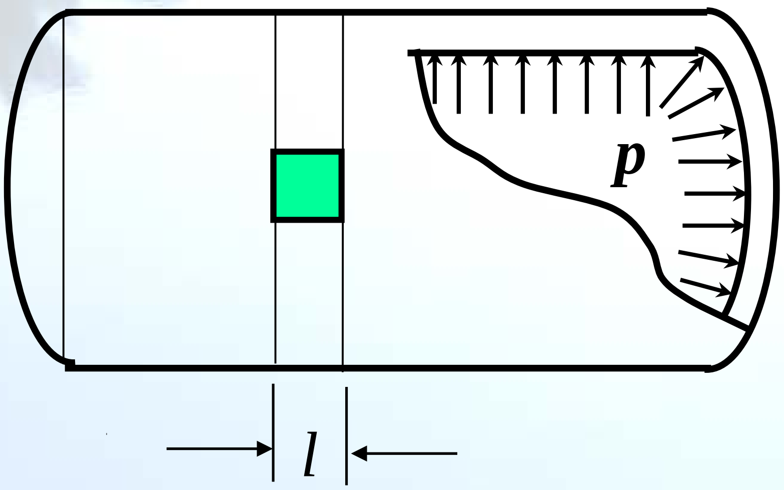

τ
σ
4
σ
τ
4

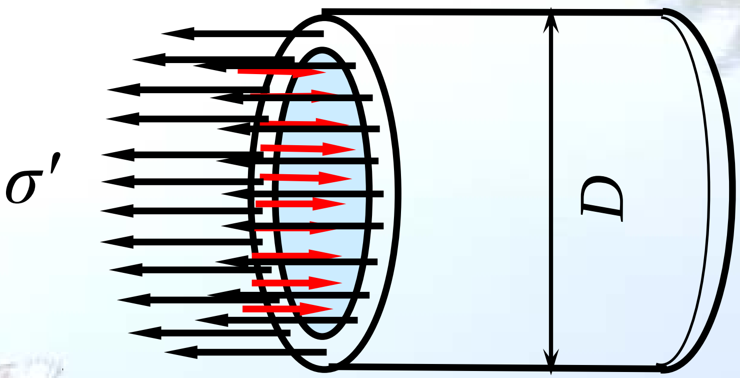


P1
P2
q
1
2
3
4
5
σ
τmax
τ
σ
5
σ
1
σ
2
3
4
5
σ

[例5] 如图所示为承受内压的薄壁容器。容器所承受的内压力为 p，容器直径D，壁厚δ。

$$\left(\frac{D}{\delta} \geq 20\right)$$

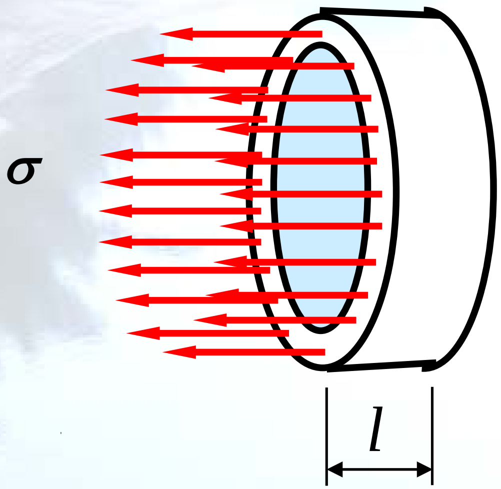


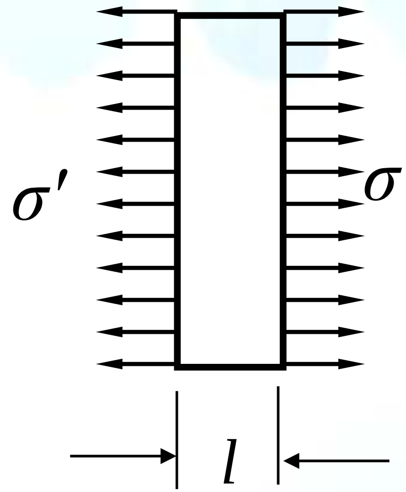


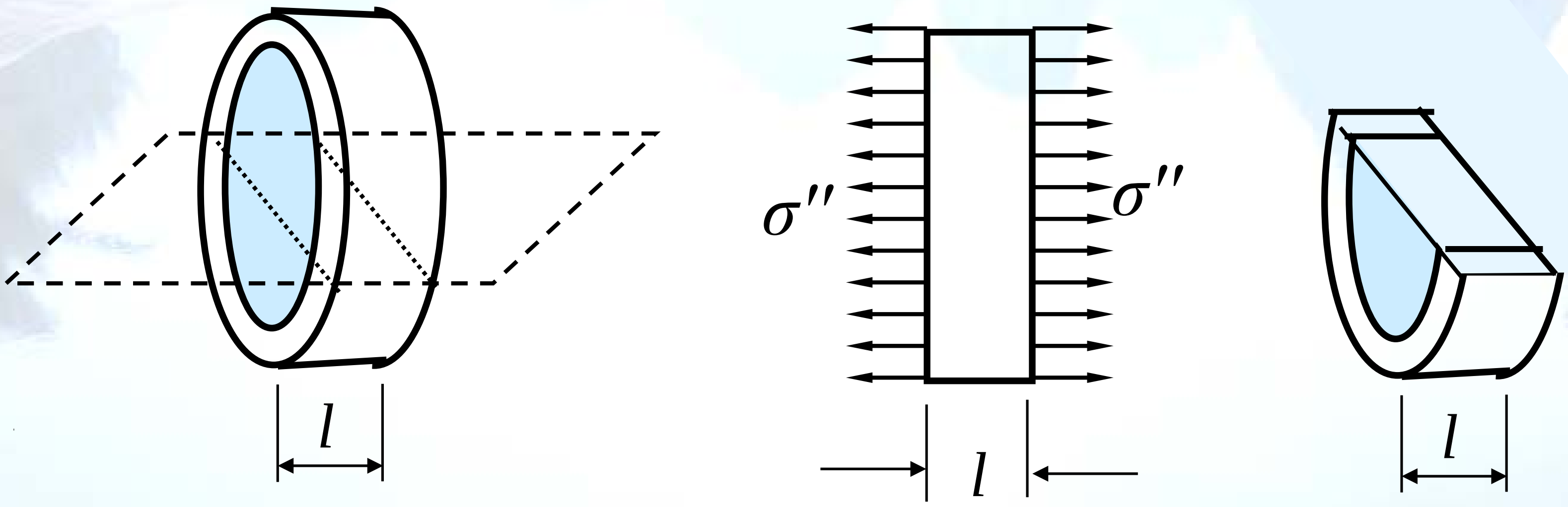
用横截面将容器截开，受力如图所示，根据平衡方程

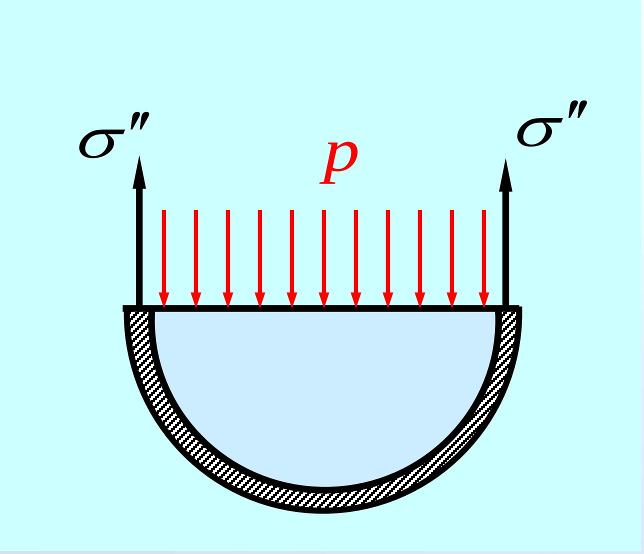
$$\sigma'(\pi D\delta) = p \times \frac{\pi D^2}{4}$$

$$\sigma' = \frac{pD}{4\delta}$$


σ
l

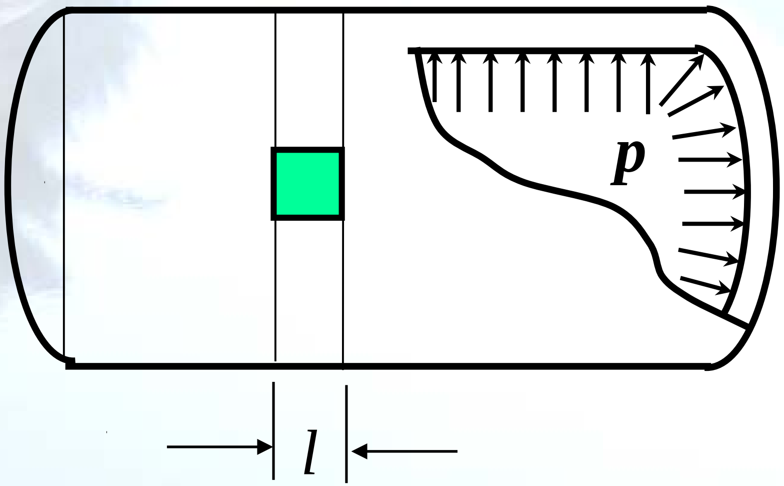

σ′
σ′
l





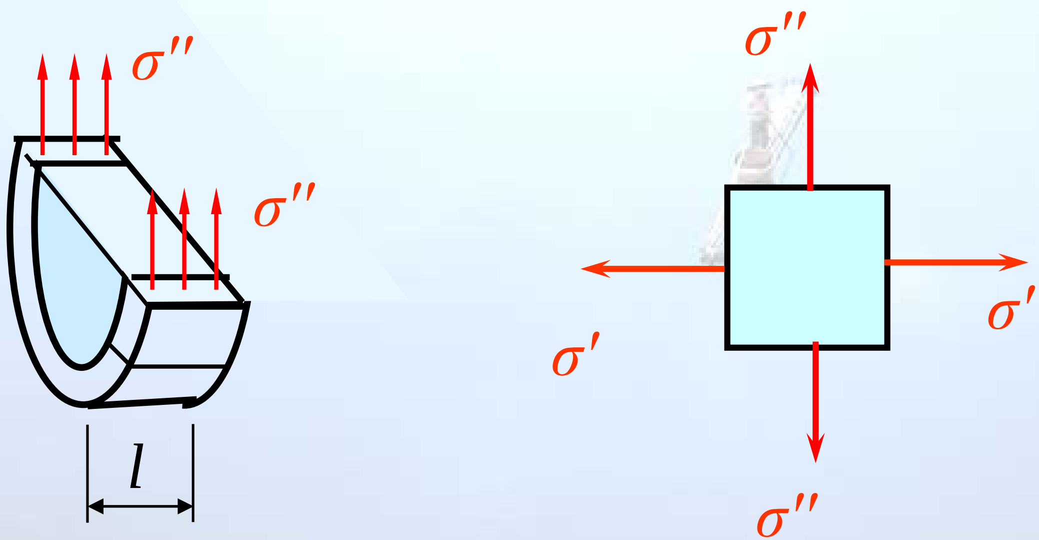
$$\sigma''\left(l\times\delta\right)\times 2 = p\times Dl$$

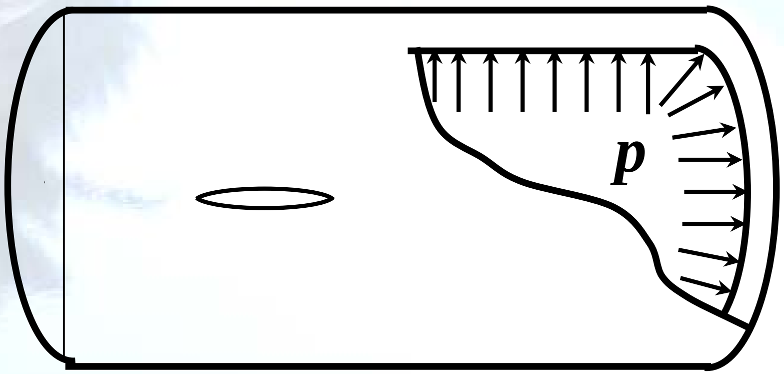
$$\sigma'' = \frac{pD}{2\delta}$$

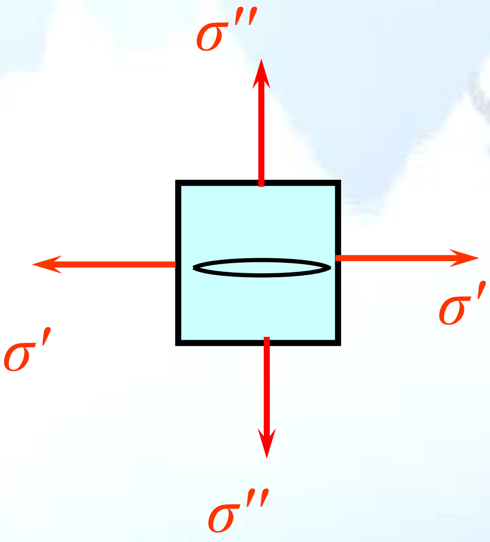


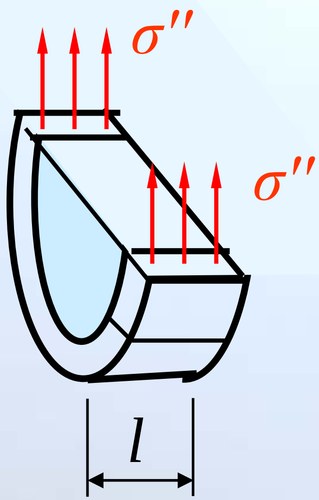
$$\sigma' = \frac{pD}{4\delta}$$

$$\sigma'' = \frac{pD}{2\delta}$$







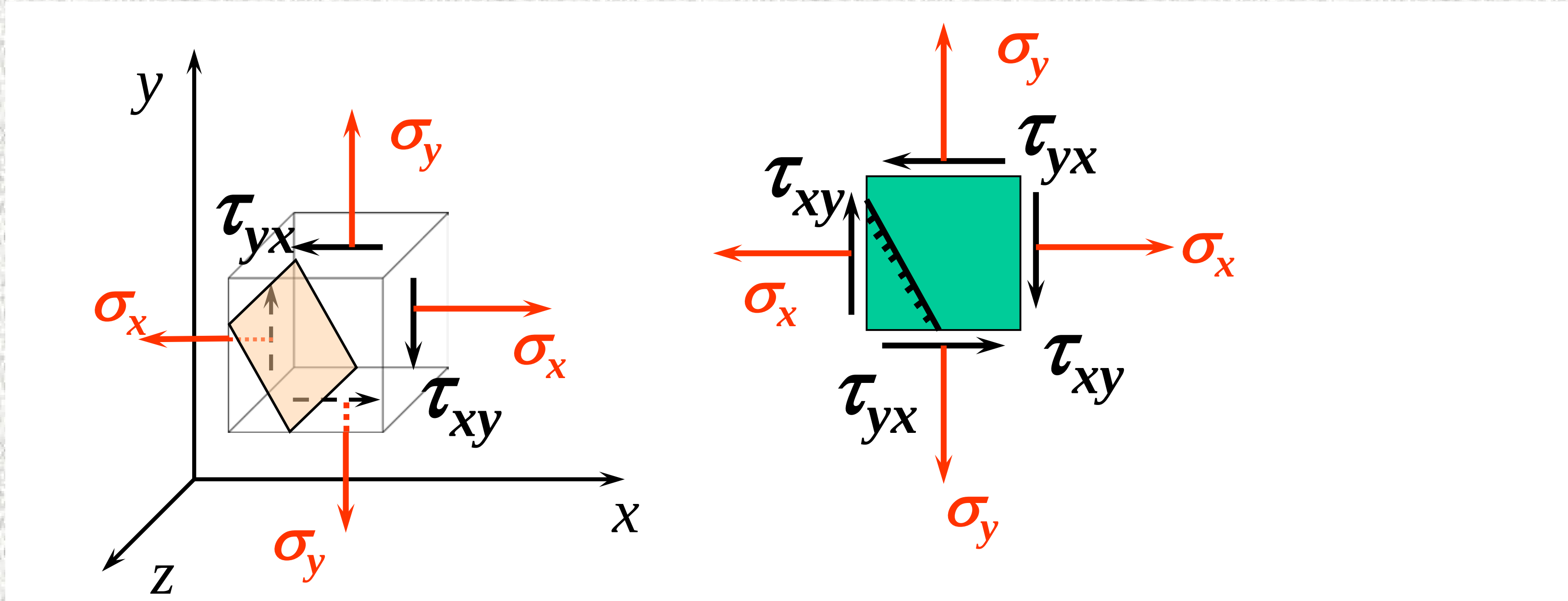


$$\sigma_1 = \sigma'' = \frac{pD}{2\delta}$$

$$\sigma_2 = \sigma' = \frac{pD}{4\delta}$$

§ 7–3　二向应力状态分析——解析法

平面应力状态： 单元体有一对平面上的应力为零。



一、任意斜截面上的应力

二、最大正应力和最小正应力

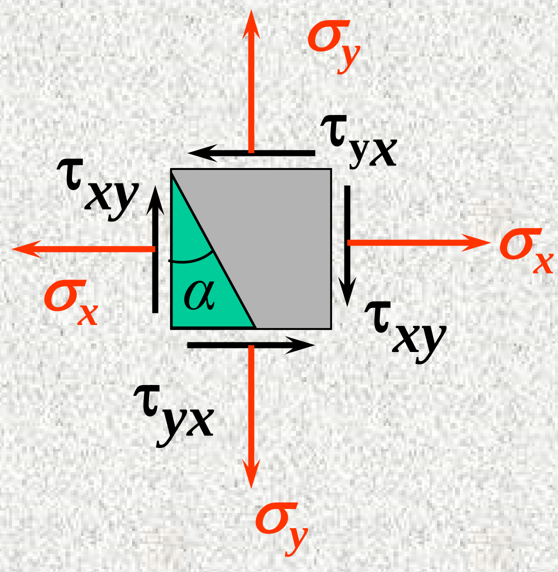
三、主平面和主应力

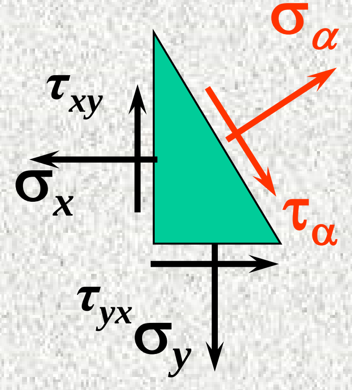
四、应力圆（莫尔圆）

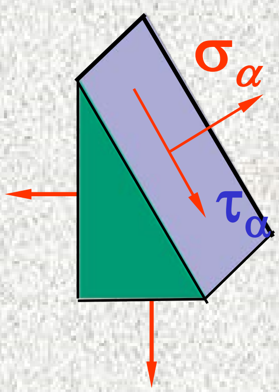
一、任意斜截面上的应力

已知：σ_x、σ_y、τ_{xy} 、α

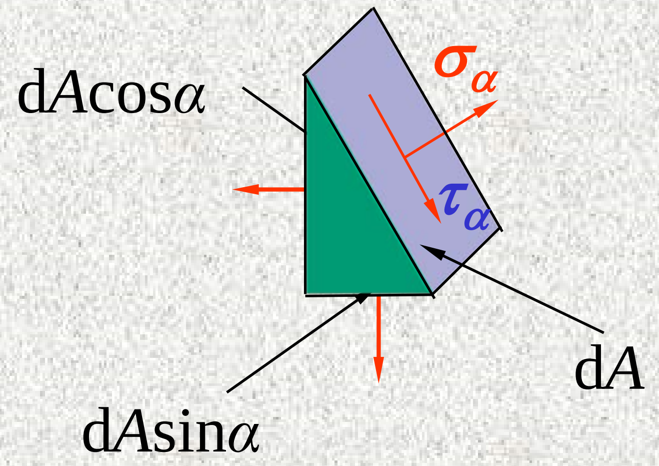
求：σ_α、τ_α

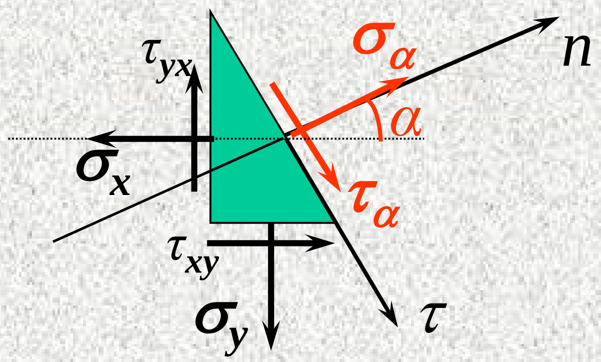






解：设斜截面面积为d*A*，





由平衡得： $\sum F_n = 0$ ， $\sigma_\alpha \cancel{dA} - \sigma_x \cancel{dA}\cos^2\alpha + \tau_{xy}\cancel{dA}\cos\alpha\sin\alpha$

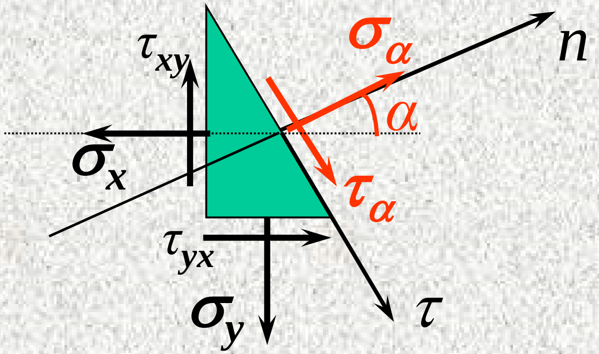
$$-\sigma_y \cancel{dA}\sin^2\alpha + \tau_{yx}\cancel{dA}\sin\alpha\cos\alpha = 0$$

$$\sigma_\alpha = \sigma_x \cos^2\alpha - \tau_{xy}\cos\alpha\sin\alpha + \sigma_y \sin^2\alpha - \tau_{yx}\sin\alpha\cos\alpha$$

由 $\tau_{yx}=\tau_{xy}$ 和三角变换，得：

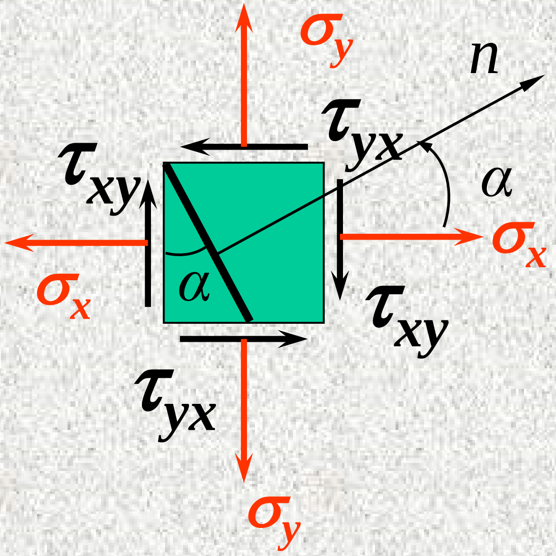
$$\sigma_\alpha = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha$$

同理：



$$\tau_\alpha = \frac{\sigma_x - \sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha$$

正负号规定：

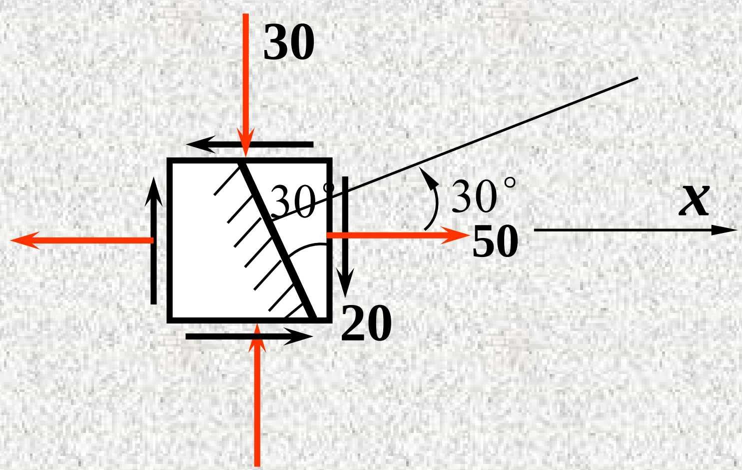


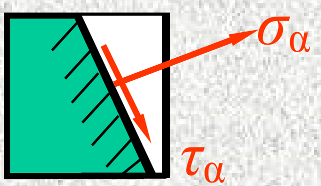
(1) 正应力拉为正；

(2)剪应力绕研究对象顺时针转为正；

(3)α逆时针为正。

[例6] 求斜截面上的应力，单位MPa





解：

$$\sigma_x=50\ ,\ \sigma_y=-30\ ,\ \tau_{xy}=20\ ,\ \alpha=30^\circ$$

$$\sigma_{30^\circ}=\frac{\sigma_x+\sigma_y}{2}+\frac{\sigma_x-\sigma_y}{2}\cos 2\alpha-\tau_{xy}\sin 2\alpha$$

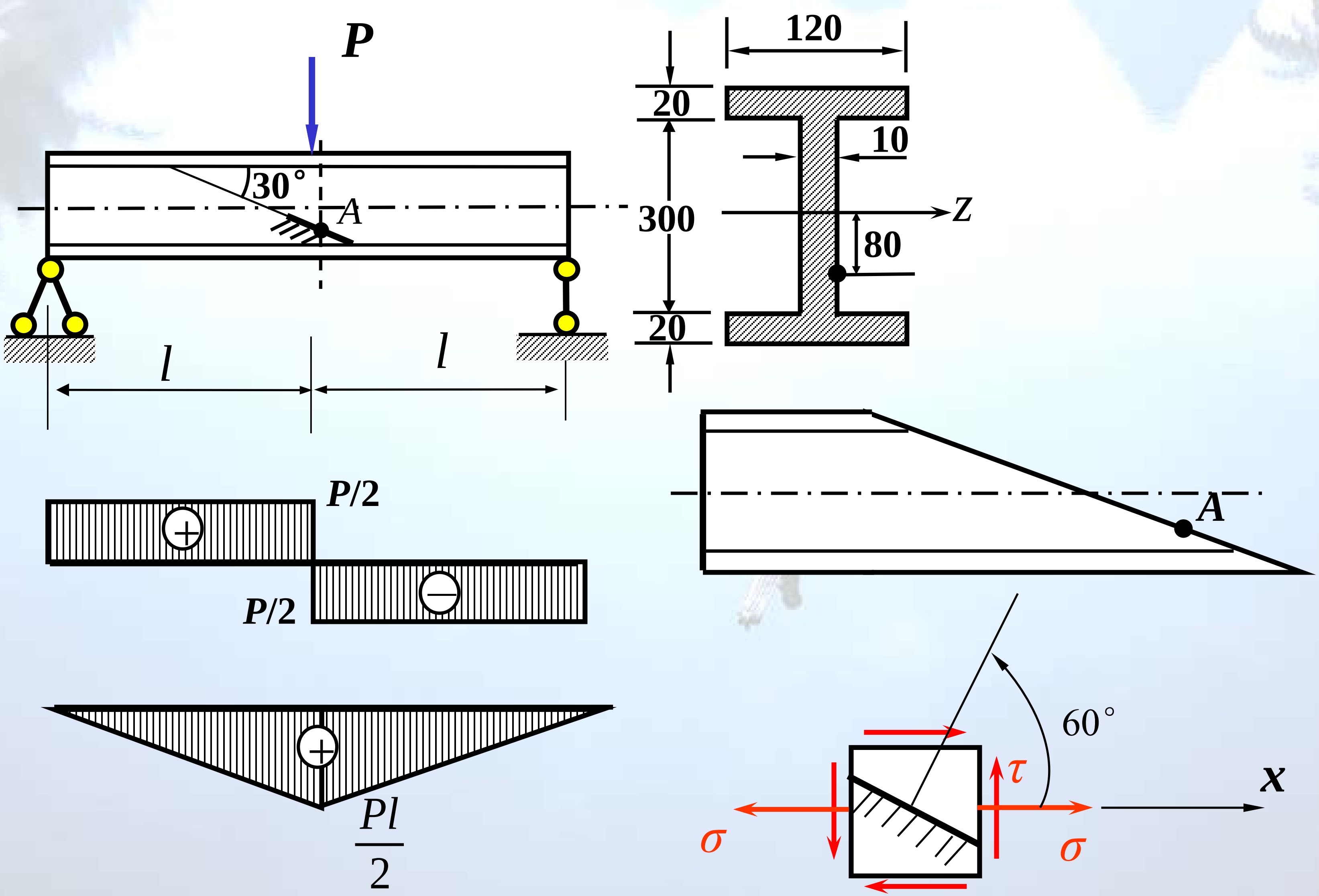
$$=\frac{50-30}{2}+\frac{50+30}{2}\cos 60^\circ-20\sin 60^\circ$$

$$=12.7(\text{MPa})$$

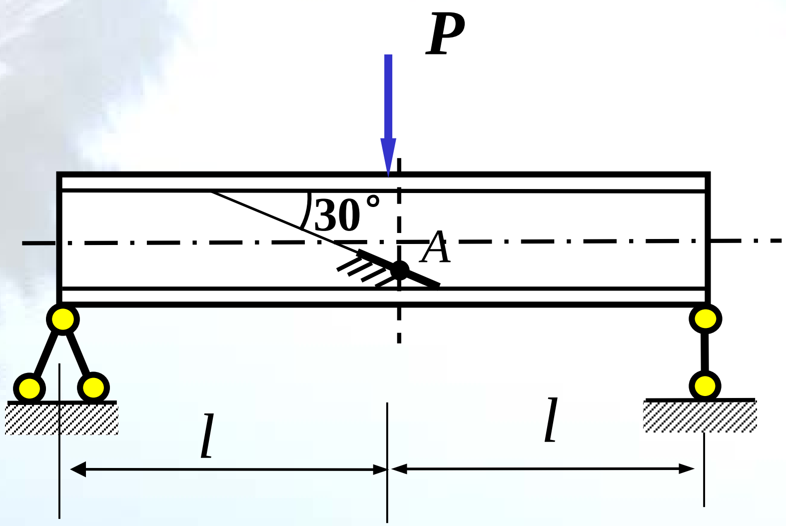
$$\tau_{30^\circ}=\frac{\sigma_x-\sigma_y}{2}\sin 2\alpha+\tau_{xy}\cos 2\alpha=\frac{50+30}{2}\sin 60^\circ+20\cos 60^\circ$$

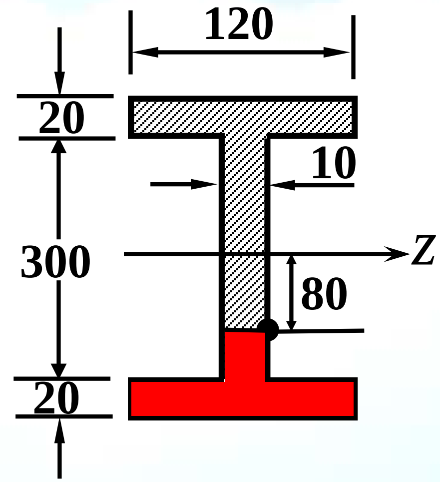
$$=44.6(\text{MPa})$$

[例7] 已知：P=180kN，l=1.5m，求A点斜截面上的应力。



[例7] 已知：P=180kN，l=1.5m，求A点斜截面上的应力。

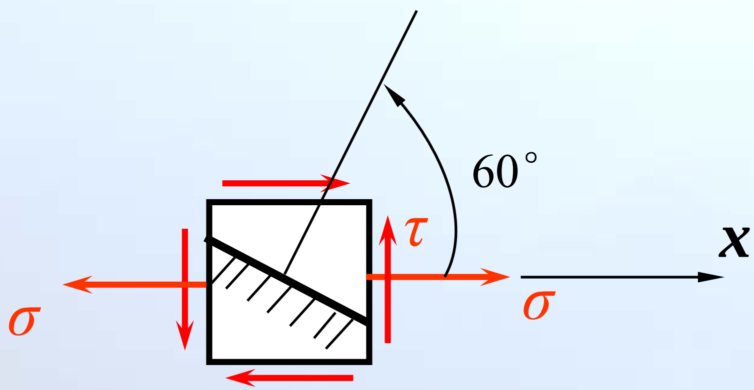




解：

$$I_z = 1.46\times10^{8}\,(\mathrm{mm}^4)$$

$$S_z^* = 4.65\times10^{5}\,(\mathrm{mm}^3)$$



$$\sigma = \frac{M\cdot y}{I_z} = 74(\mathrm{MPa})$$

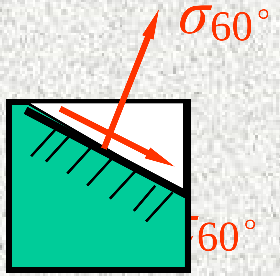
$$\tau = \frac{QS_z^*}{I_z b} = 29(\mathrm{MPa})$$

$$\therefore\ \sigma_x = 74\ ,\ \sigma_y = 0\ ,\ \tau_{xy} = -29\ ,\ \alpha = 60^\circ$$

$$\sigma_{60^\circ}=\frac{\sigma_x+\sigma_y}{2}+\frac{\sigma_x-\sigma_y}{2}\cos 2\alpha-\tau_{xy}\sin 2\alpha$$

$$=\frac{74}{2}+\frac{74}{2}\cos 120^\circ-(-29)\sin 120^\circ$$

$$=43.6(\text{MPa})$$



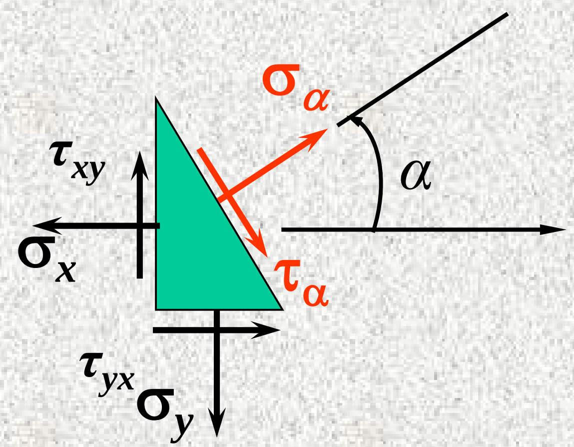
$$\tau_{60^\circ}=\frac{\sigma_x-\sigma_y}{2}\sin 2\alpha+\tau_{xy}\cos 2\alpha$$

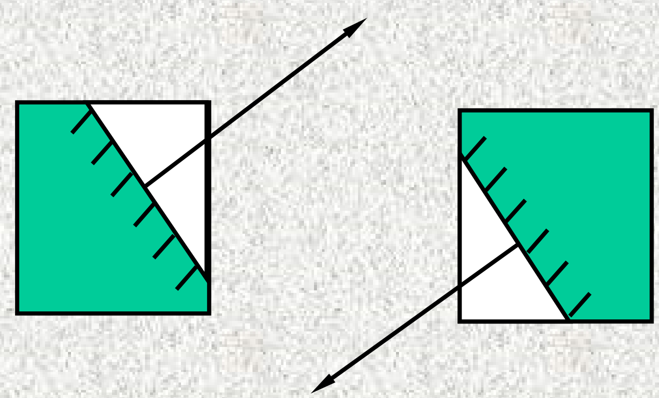
$$=\frac{74}{2}\sin 120^\circ+(-29)\cos 120^\circ$$

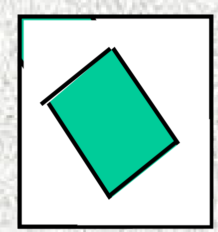
$$=46.5(\text{MPa})$$

二、最大正应力和最小正应力

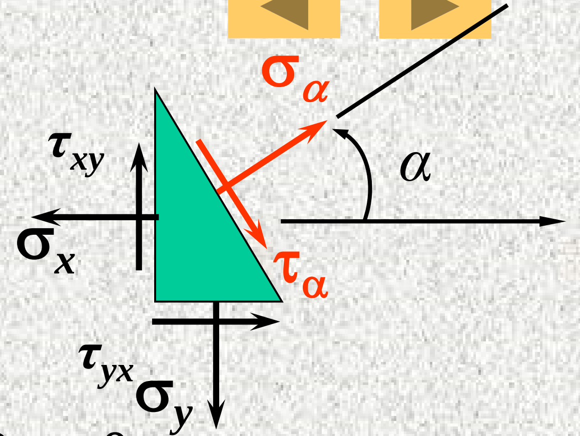
$$\sigma_\alpha = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha$$







二、最大正应力和最小正应力



$$\sigma_\alpha = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha$$

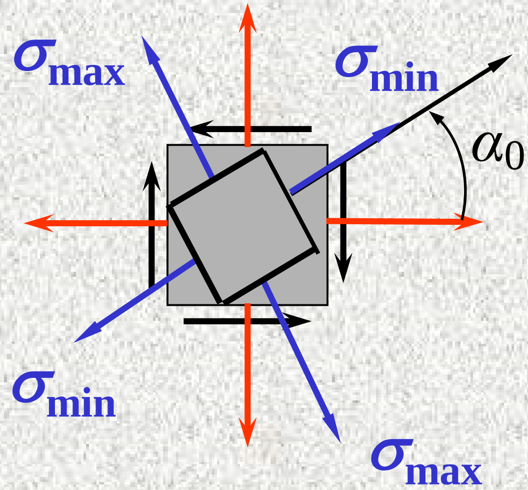
令：$\dfrac{d\sigma_\alpha}{d\alpha} = 0$ ，得：$-(\sigma_x - \sigma_y)\sin 2\alpha - 2\tau_{xy}\cos 2\alpha = 0$

$$\mathrm{tg}2\alpha_0 = -\frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

剪应力**箭头**所在象限就是最大正应力所在象限。

由此得两个驻点:α_0和α_0'

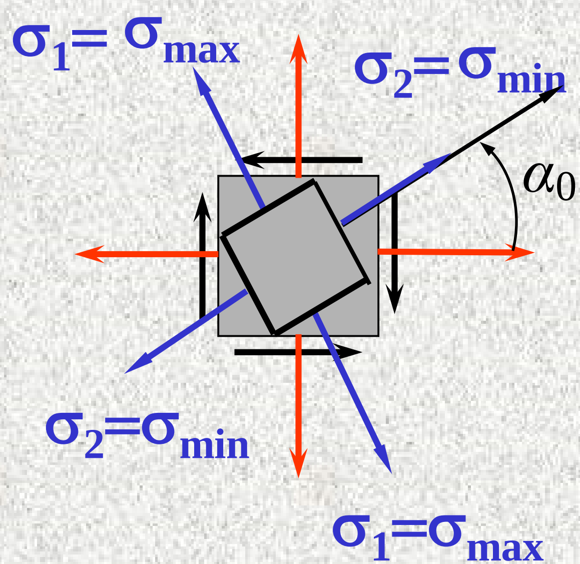
$$(\alpha_0' = \alpha_0 + \frac{\pi}{2})$$



$$\left.\begin{matrix}\sigma_{max}\\ \sigma_{min}\end{matrix}\right\} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

三、主平面和主应力

令 $\tau_\alpha=0$ ， 可得主平面的方位：



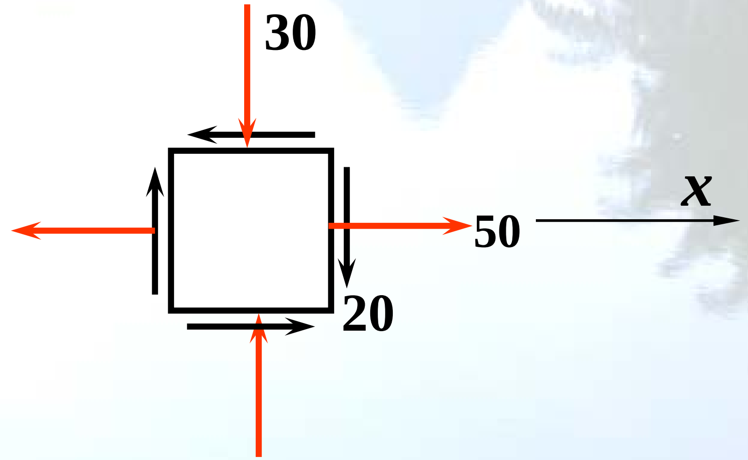
由 $$\tau_\alpha=\frac{\sigma_x-\sigma_y}{2}\sin 2\alpha+\tau_{xy}\cos 2\alpha$$

得 $$\frac{\sigma_x-\sigma_y}{2}\sin 2\alpha+\tau_{xy}\cos 2\alpha=0$$

$$\therefore \operatorname{tg}2\alpha_0=-\frac{2\tau_{xy}}{\sigma_x-\sigma_y}$$

即：主应力就是最大或最小的正应力。

[例8] 求主应力大小和主平面方位，并在单元体上画出主平面和主应力。单位MPa



解： $\sigma_x=50$，$\sigma_y=-30$，$\tau_{xy}=20$

$$\left\{\begin{matrix}\sigma_{max}\\ \sigma_{min}\end{matrix}\right.=\frac{\sigma_x+\sigma_y}{2}\pm\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}$$

$$=\frac{50-30}{2}\pm\sqrt{\left(\frac{50+30}{2}\right)^2+20^2}$$

$$=10\pm44.7=\begin{matrix}54.7\\ -34.7\end{matrix}$$

$\therefore\ \sigma_1=54.7\text{MPa}$

$\sigma_2=0$

$\sigma_3=-34.7\text{MPa}$

$$\mathrm{tg}2\alpha_0=-\frac{2\tau_{xy}}{\sigma_x-\sigma_y}$$

$$=-\frac{2\times20}{50+30}$$

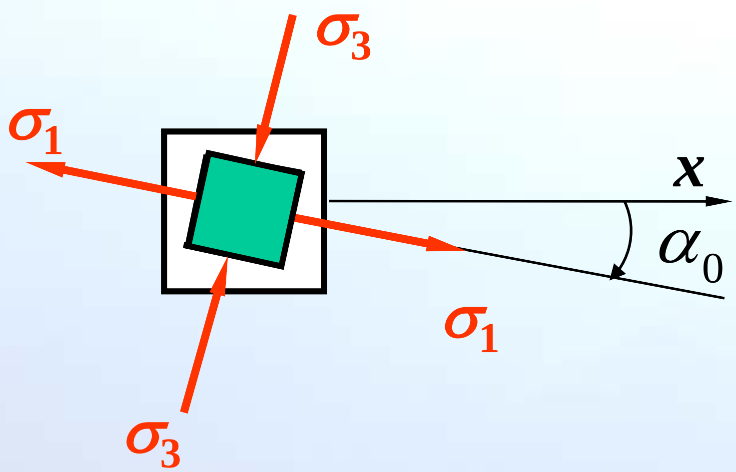
$$\therefore\ \mathrm{tg}2\alpha_0=-0.5$$

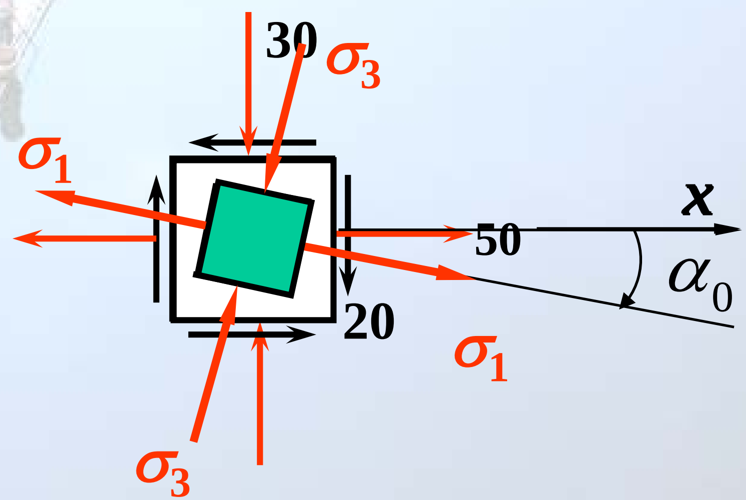
$$\therefore\ 2\alpha_0 = -26.56^\circ$$

$$\therefore \alpha_0 = -13.28^\circ$$

$$\alpha_0' = 76.72^\circ$$

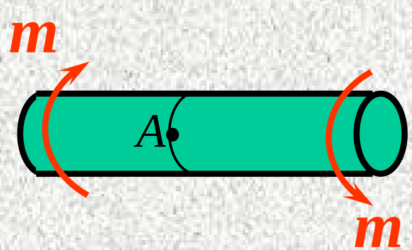
在单元体上画出主平面和主应力

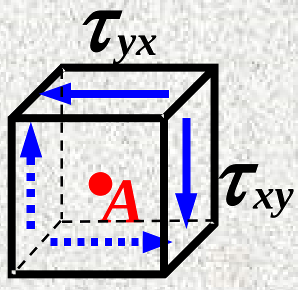


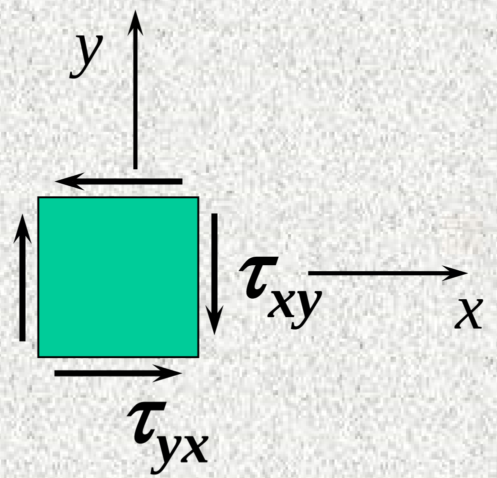


剪应力**箭头**所在象限就
是最大正应力所在象限。

[例9] 分析受扭构件的应力状态。







解：(1) 单元体如图所示

$$\sigma_x = \sigma_y = 0$$

$$\tau_{xy} = \tau = \frac{T}{W_P}$$

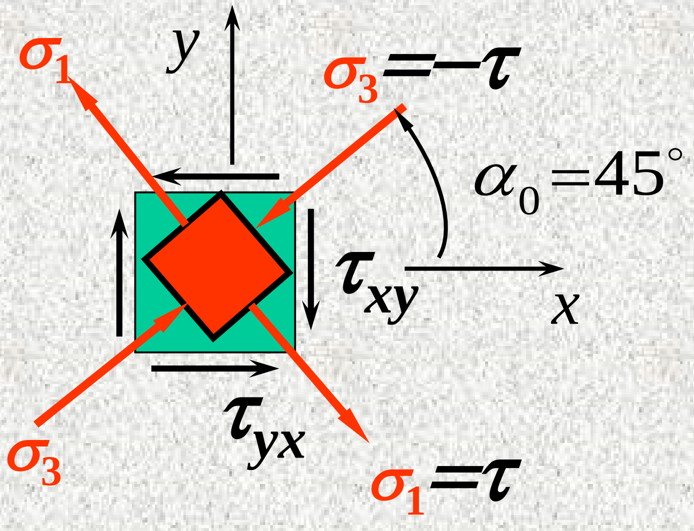
(2) 主应力

$$\left.\begin{cases}\sigma_{max} \\ \sigma_{min}\end{cases}\right. = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \pm\sqrt{\tau_{xy}^2} = \pm\tau$$

$$\therefore\ \sigma_1 = \tau,\quad \sigma_2 = 0,\quad \sigma_3 = -\tau$$

(2)主平面所在方位



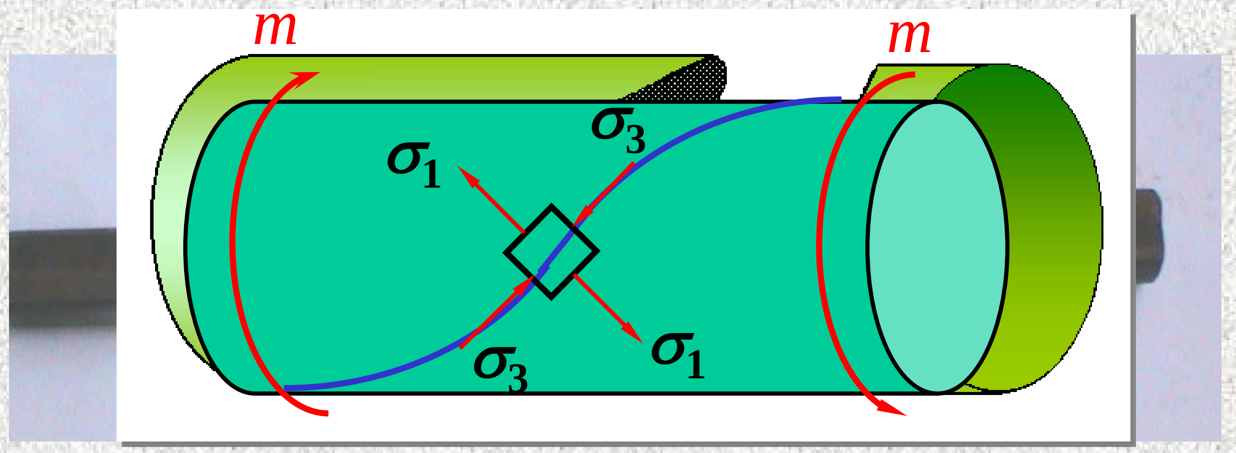
$$\mathrm{tg}2\alpha_0=\frac{2\tau_{xy}}{\sigma_x-\sigma_y}=-\infty$$

$$\therefore 2\alpha_0=-90^\circ$$

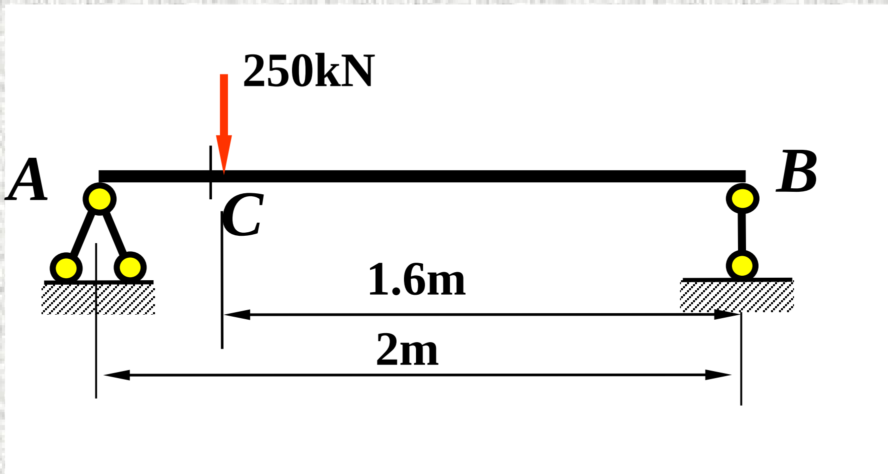
$$\therefore \alpha_0=-45^\circ$$

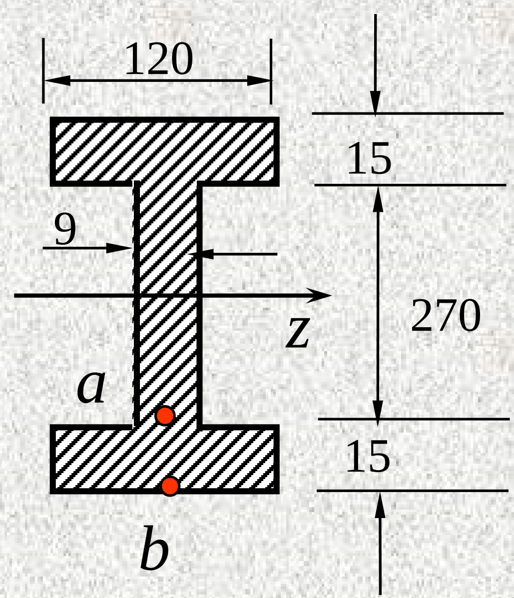
铸铁扭转破坏

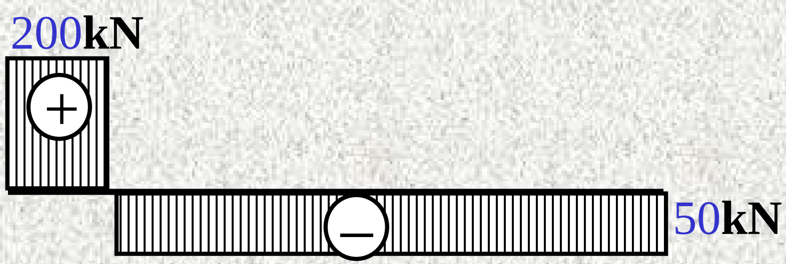
断口分析

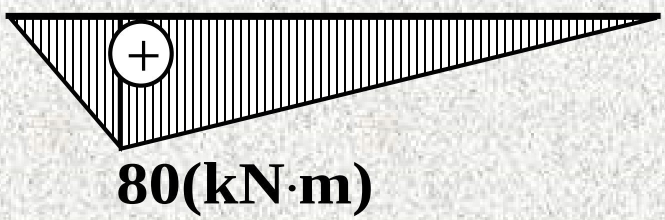


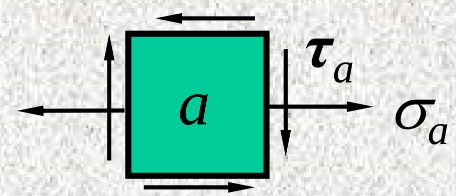
[例10] 求C截面左侧 a、b两点的主应力及主平面。

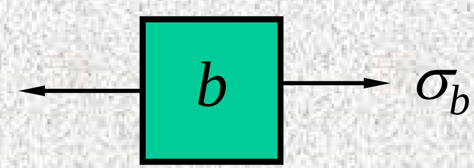












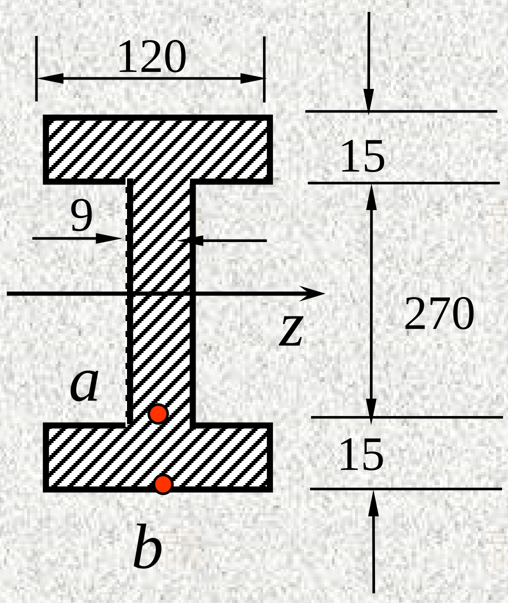
解：$I_z = \dfrac{120\times300^3}{12} - \dfrac{110\times270^3}{12}$

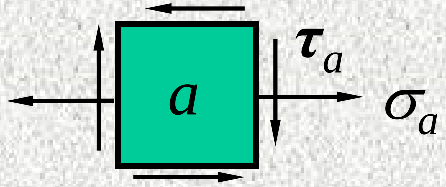
$= 88\times10^6\,(\mathbf{mm}^4)$

$\sigma_a = \dfrac{My_a}{I_z} = 122.5(\mathbf{MPa})$

$\tau_a = \dfrac{Q_{\max}S_z^*}{I_z b} = \dfrac{200\times10^3\times[120\times15\times(150-7.5)]}{88\times10^6\times9}$

$= 64.6(\mathbf{MPa})$





$$\left\{\begin{matrix}\sigma_{\max}\\ \sigma_{\min}\end{matrix}\right. = \frac{\sigma_x+\sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2} = \left\{\begin{matrix}150\mathbf{MPa}\\ -27\mathbf{MPa}\end{matrix}\right.$$

$\sigma_1 = 150\mathbf{MPa},\quad \sigma_2 = 0,\quad \sigma_3 = -27\mathbf{MPa}$

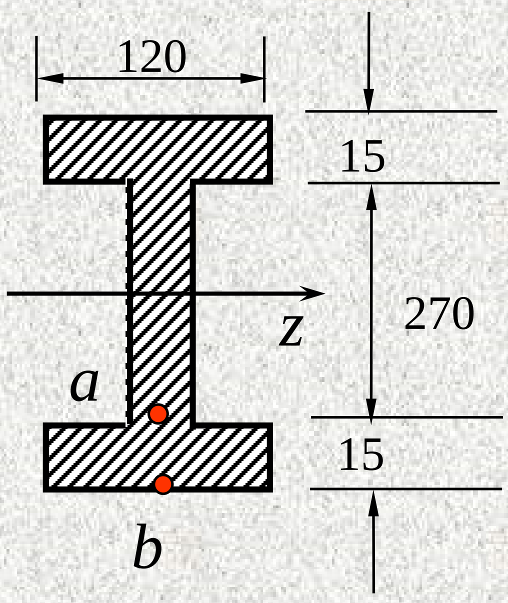
$$\mathrm{tg}2\alpha_0 = -\frac{2\tau_{xy}}{\sigma_x - \sigma_y} = -\frac{2\times 64.6}{122.5}$$

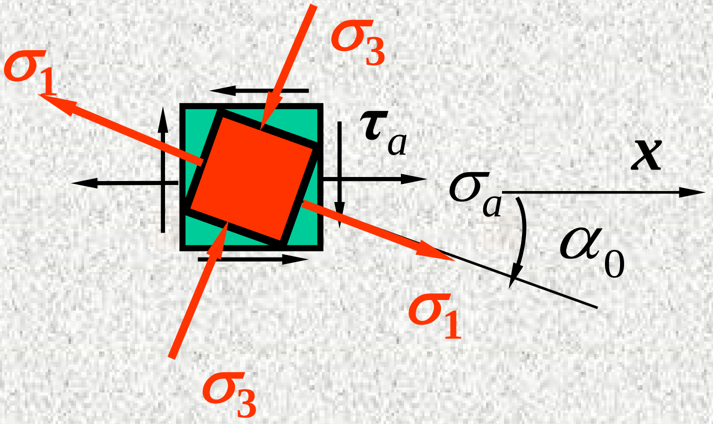
$$= -1.055$$

$$\therefore 2\alpha_0 = -46.5^\circ$$

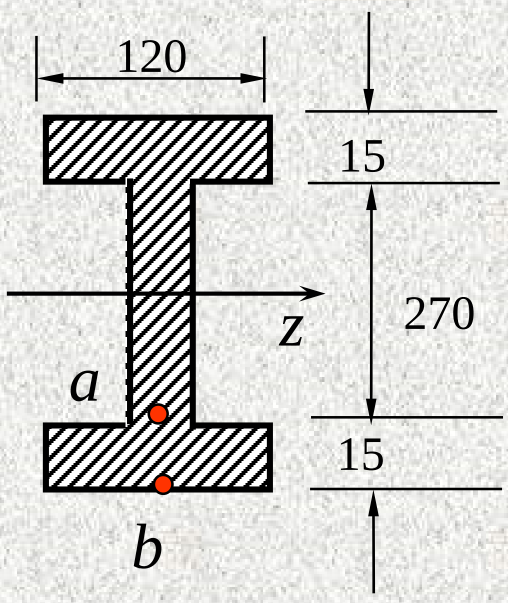
$$\alpha_0 = -23.26^\circ$$

$$\alpha_0' = 66.7^\circ$$

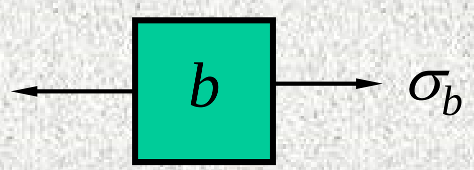




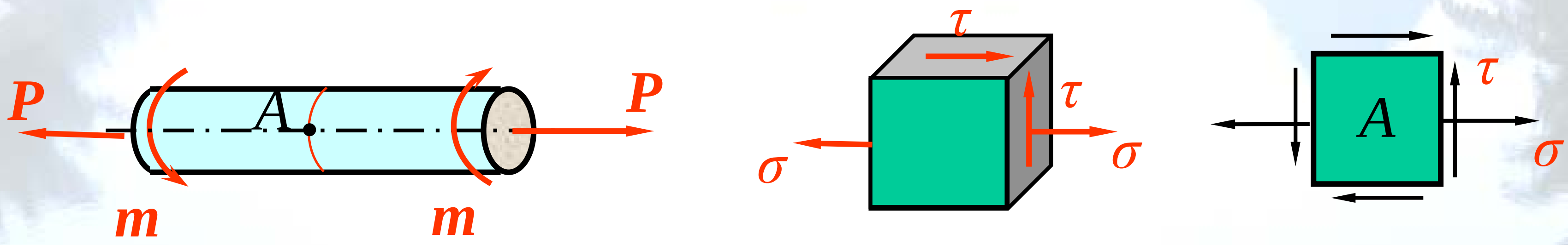
$$\sigma_b = \frac{My_{\max}}{I_z} = 136.5(\mathbf{MPa})$$



$$\sigma_1 = 136.5\mathbf{MPa}, \quad \sigma_2 = \sigma_3 = 0$$



[例11] 求圆杆表面 A点的主应力及主平面。已知：P=6.28kN，m=47.1N·m，d=20mm。



解：$\sigma = \dfrac{P}{A} = 20(\mathbf{MPa})$

$$\tau = \frac{T}{W_t} = \frac{m}{\dfrac{\pi d^3}{16}} = 30(\mathrm{MPa})$$

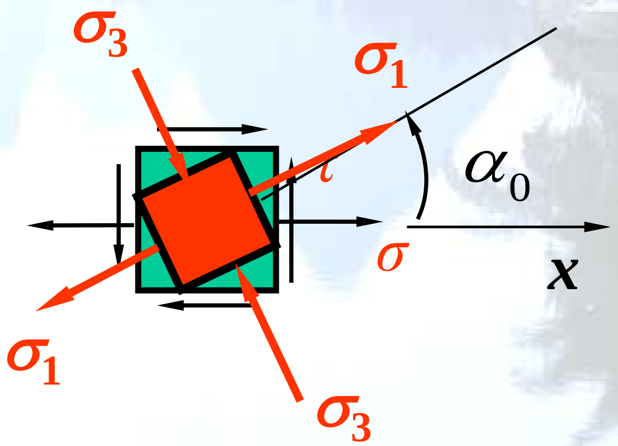
$$\left.\begin{cases}\sigma_{\max} \\ \sigma_{\min}\end{cases}\right\} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \begin{cases}\mathbf{41.6MPa} \\ -21.6\mathbf{MPa}\end{cases}$$

$\sigma_1 = 41.6\mathbf{MPa}$,　$\sigma_2 = 0$,　$\sigma_3 = -21.6\mathbf{MPa}$

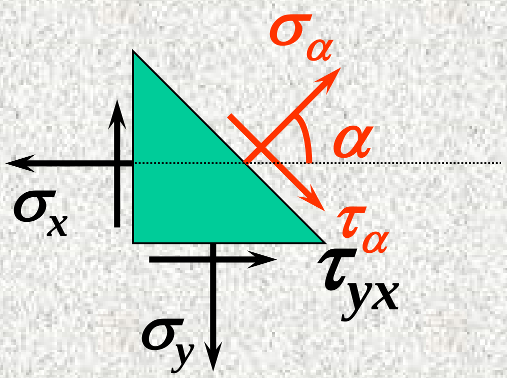
$$\operatorname{tg}2\alpha_0 = -\frac{2\tau_{xy}}{\sigma_x - \sigma_y} = -\frac{2\times(-30)}{20} = 3$$

$$\therefore 2\alpha_0 = 71.6^\circ$$

$$\alpha_0 = 35.8^\circ$$



§ 7–4　二向应力状态分析——图解法



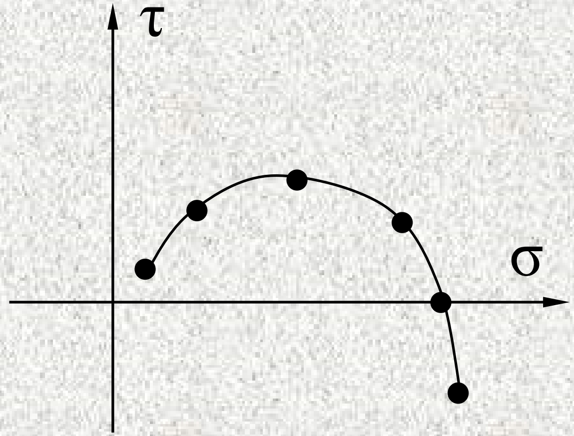
（1）应力圆（ Stress Circle）

$$\begin{cases}\sigma_\alpha = \dfrac{\sigma_x+\sigma_y}{2}+\dfrac{\sigma_x-\sigma_y}{2}\cos 2\alpha-\tau_{xy}\sin 2\alpha \\ \tau_\alpha = \dfrac{\sigma_x-\sigma_y}{2}\sin 2\alpha+\tau_{xy}\cos 2\alpha\end{cases}$$

对上述方程消去参数（2α），得到曲线的表达式：

$$(\sigma_\alpha-\frac{\sigma_x+\sigma_y}{2})^2=(\frac{\sigma_x-\sigma_y}{2}\cos 2\alpha-\tau_{xy}\sin 2\alpha)^2$$

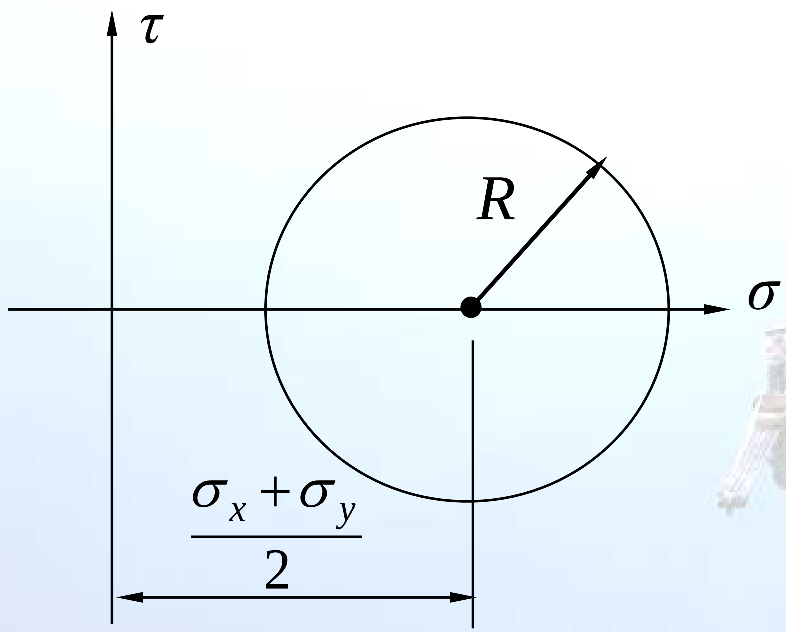
$$\tau^2{}_\alpha=(\frac{\sigma_x-\sigma_y}{2}\sin 2\alpha+\tau_{xy}\cos 2\alpha)^2$$



两边相加得：

$$\left(\sigma_\alpha - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_\alpha^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$

与圆方程相比较： $(x-a)^2 + y^2 = R^2$

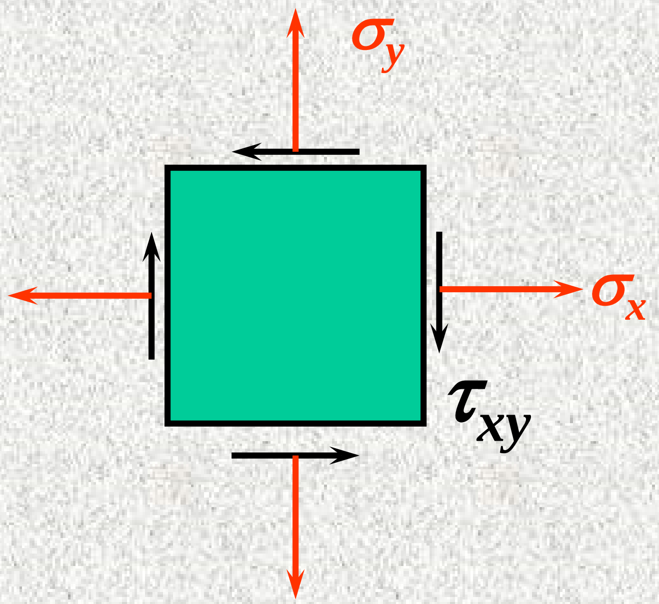


$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

此圆称为应力圆（或莫尔圆，由德国工程师：Otto Mohr引入）

（2）应力圆的画法

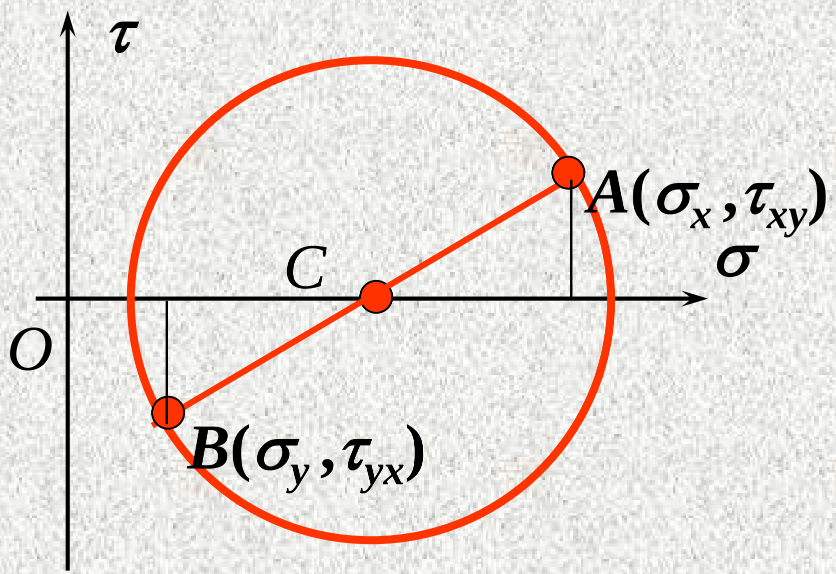
①建立应力坐标系，如下图所示，（注意选好比例尺）

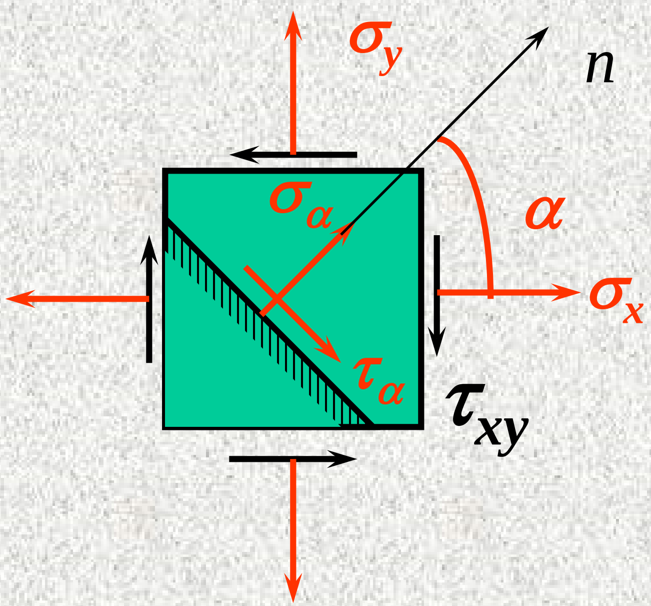


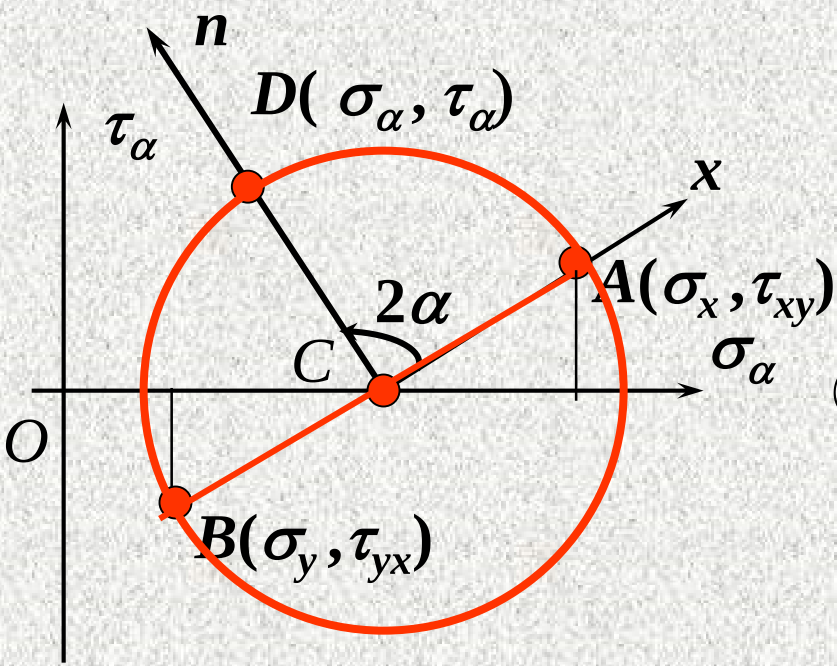
②在坐标系内画出点$A(\sigma_x，\tau_{xy})$和 $B(\sigma_y，\tau_{yx})$

③AB与σ轴的交点C便是圆心。

④以C为圆心，以AC为半径画圆——应力圆；







（3）单元体与应力圆的对应关系

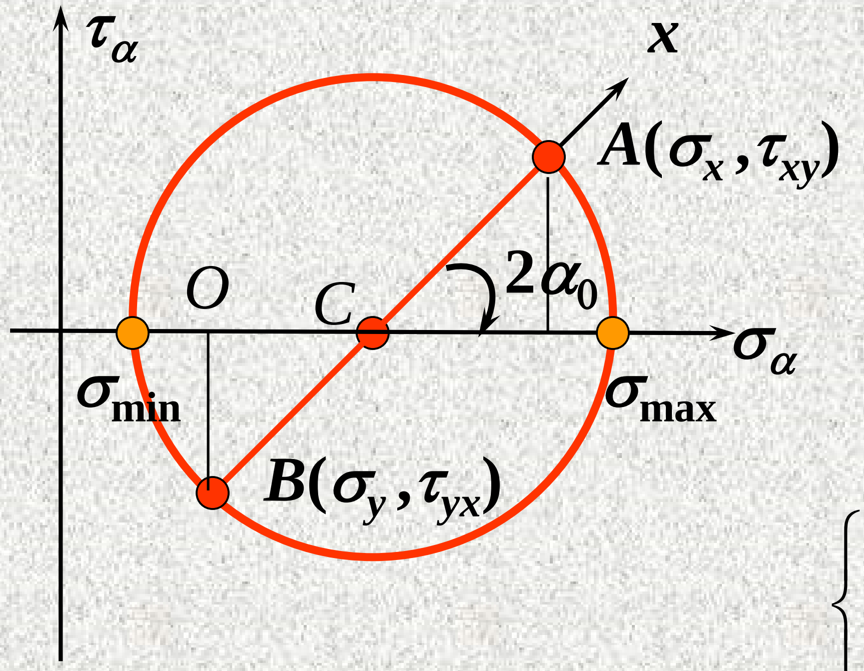
点面对应，转向相同，转角两倍。

①斜截面面上的应力(σ_α，τ_α) ⟷ 应力圆上一点(σ_α，τ_α)

②斜截面的法线 ⟷ 应力圆的半径

③x轴与斜截面的夹角为α ⟷ 两半径夹角2α；且转向一致。

（4）在应力圆上标出主应力

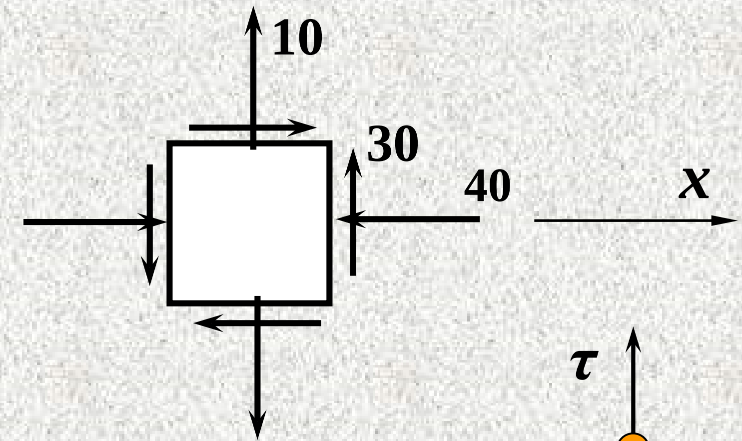


$$\left.\begin{matrix}\sigma_{max}\\ \sigma_{min}\end{matrix}\right\} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\mathrm{tg}2\alpha_0 = -\frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

[例12]求图示单元体的主应力及主平面的位置。(单位：MPa)

解：应力坐标系如图

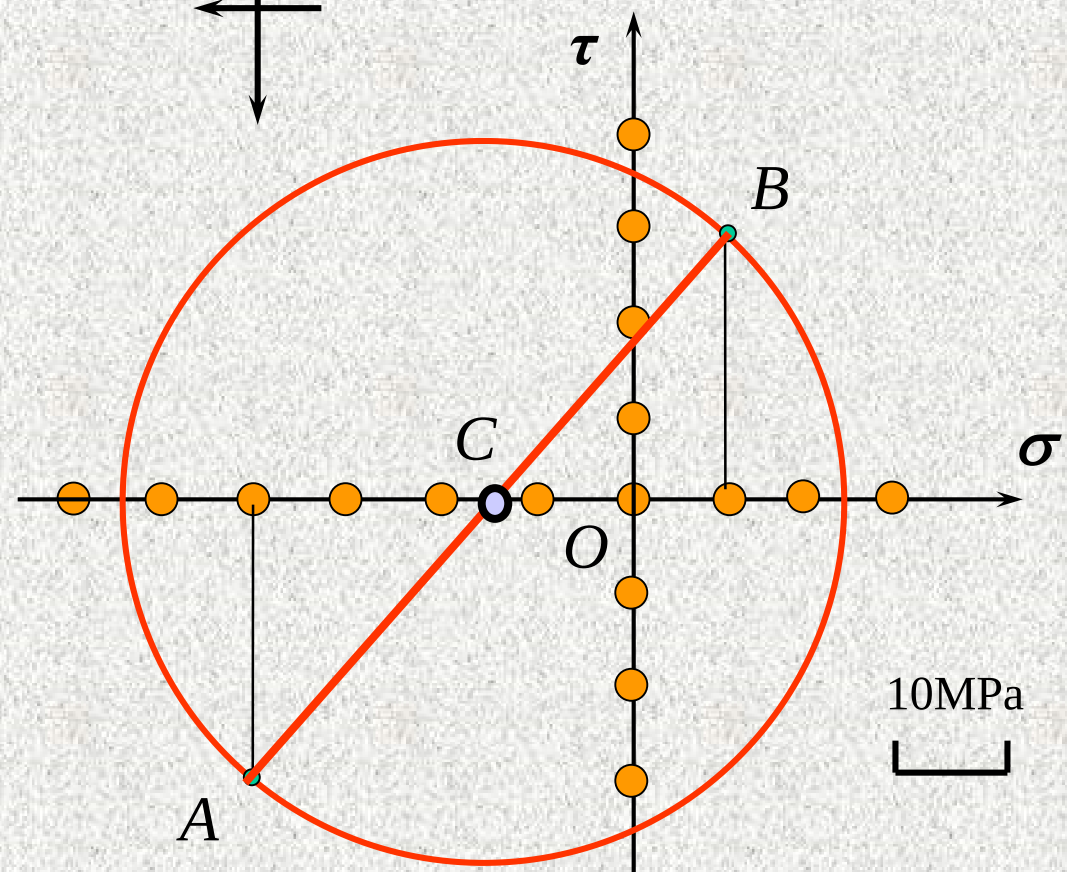


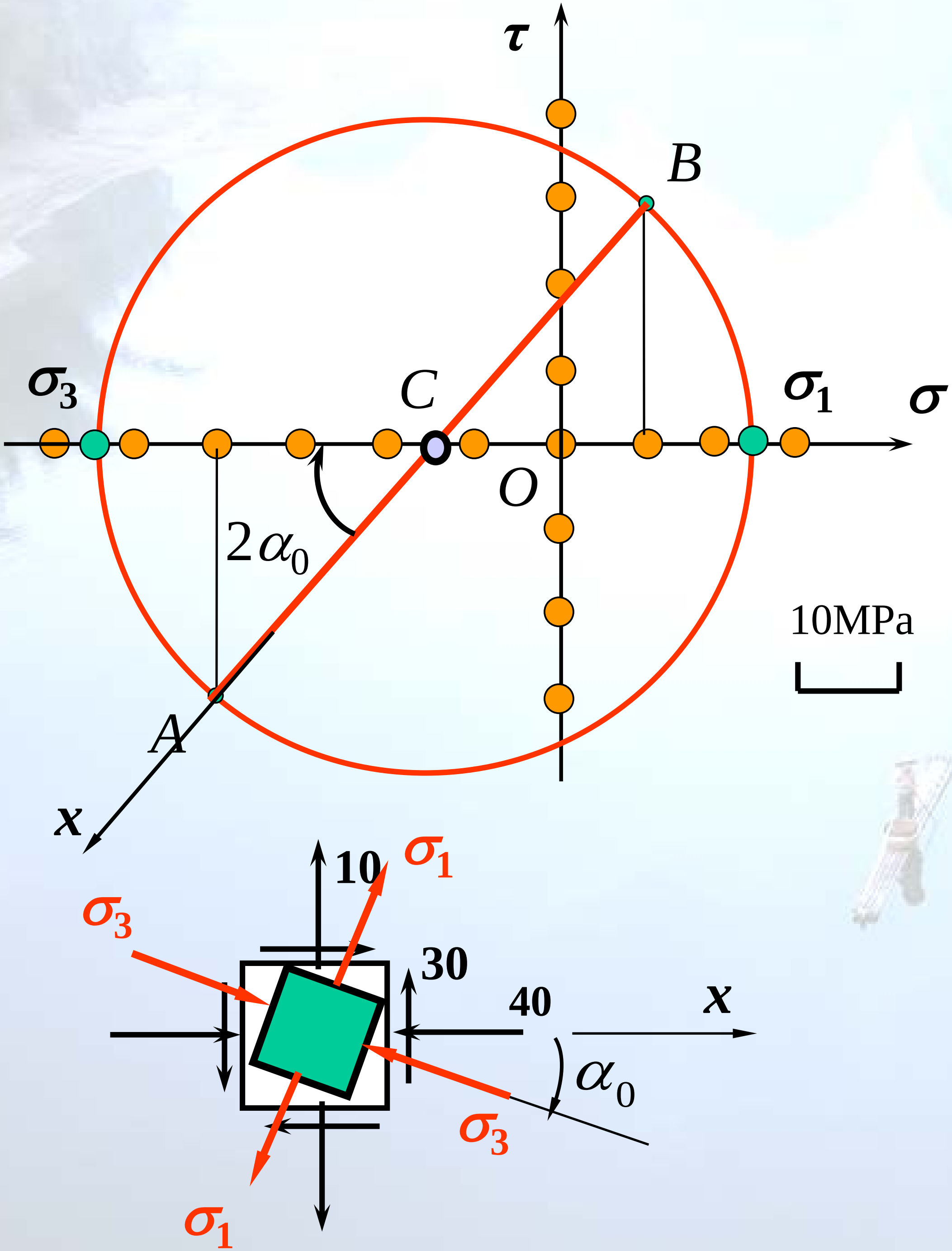
在坐标系内画出点

$A(-40,-30)$

$B(10,30)$

连接A、B两点，与σ轴的交点C便是圆心，以C为圆心，以AC为半径画圆得应力圆。





应力圆与σ轴的交点便是主应力，根据比例量得主应力的大小：

$$\sigma_1=24$$

$$\sigma_2=0$$

$$\sigma_3=-54$$

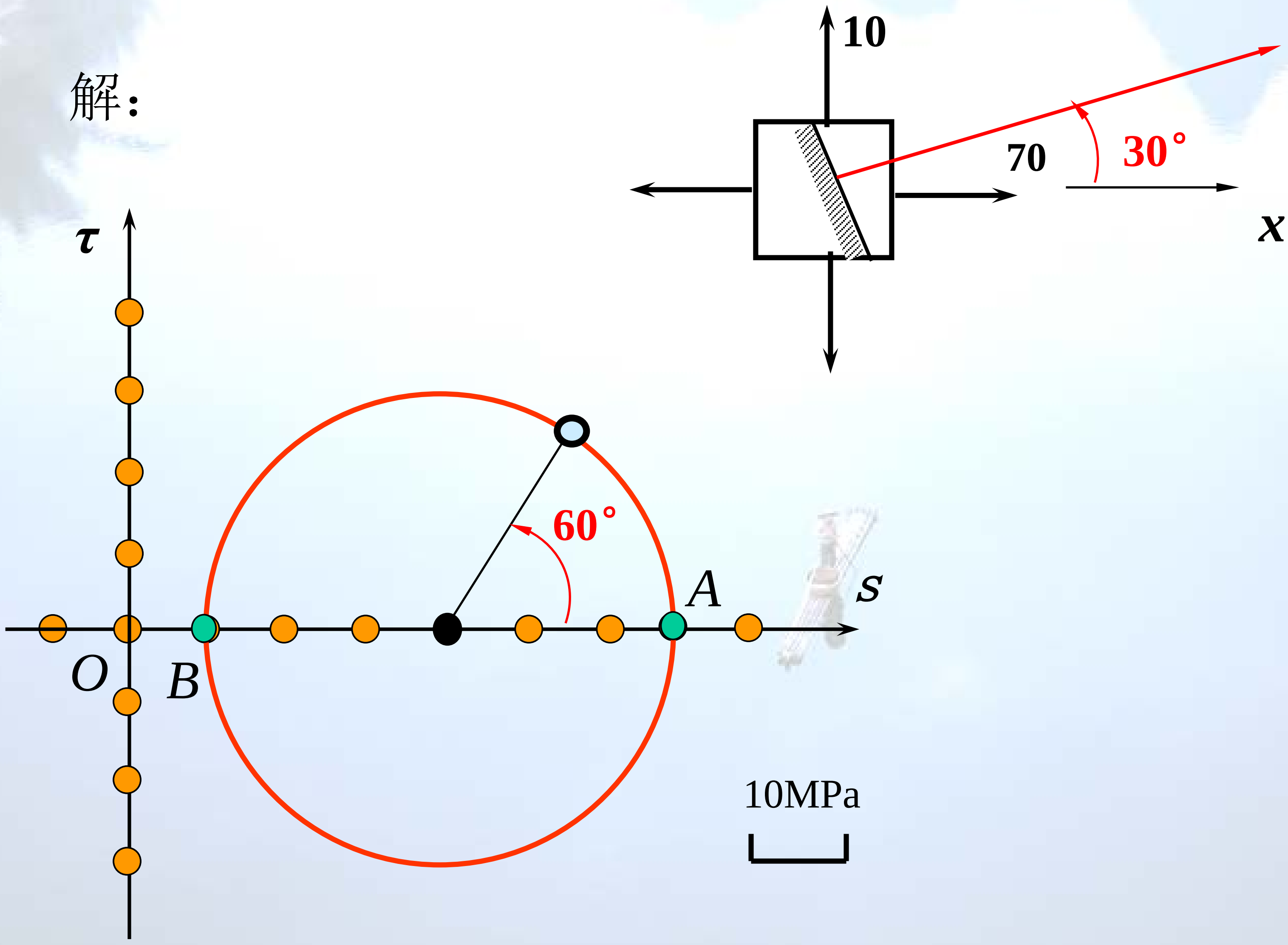
从应力圆上可得σ_3与x轴的夹角为：

$$2\alpha_0=-50^\circ$$

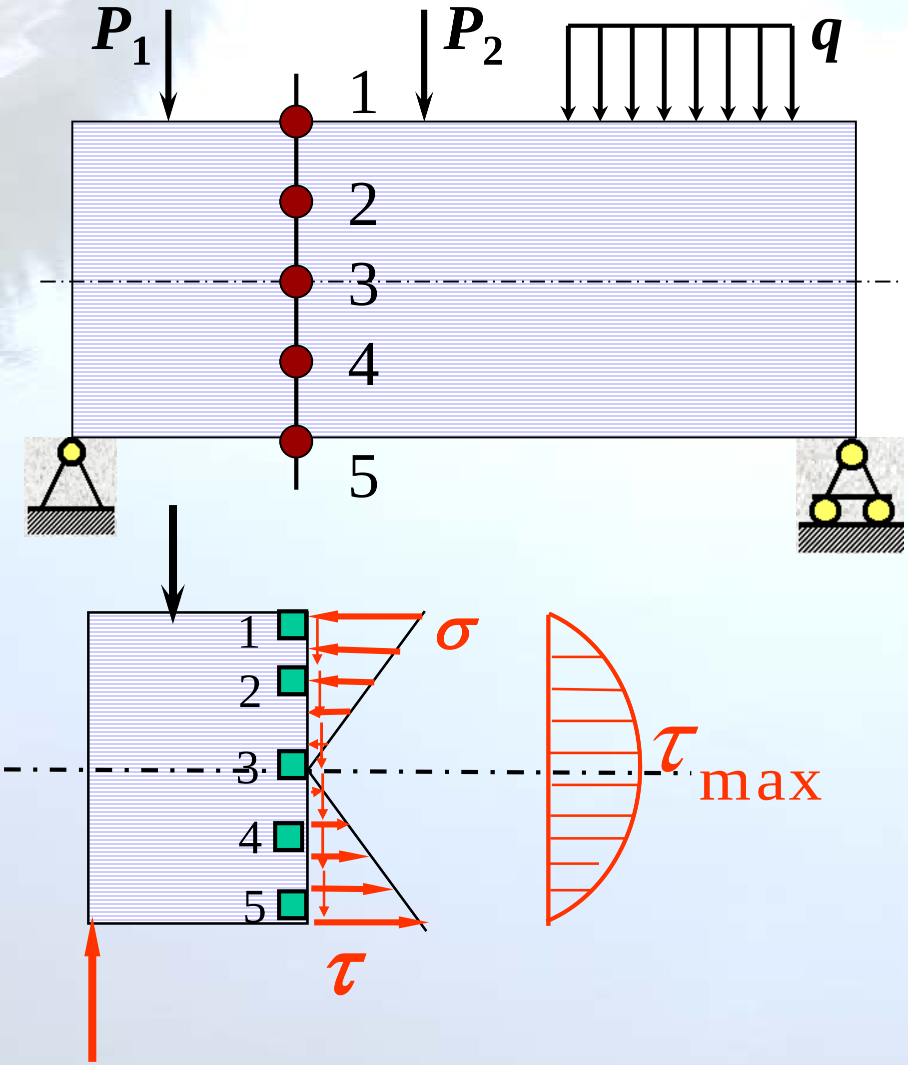
$$\therefore \alpha_0=-25^\circ$$

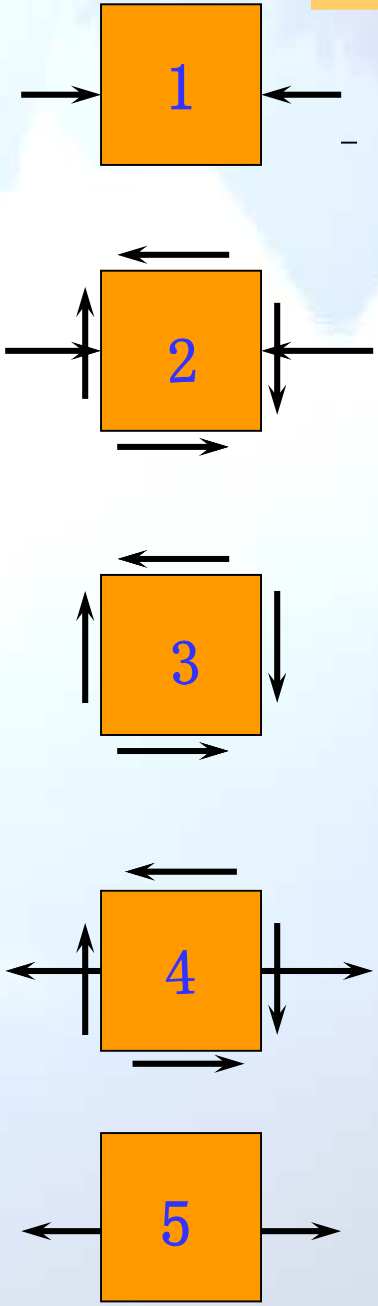
[例13]用图解法求图示单元体斜截面上的应力。(单位：MPa)

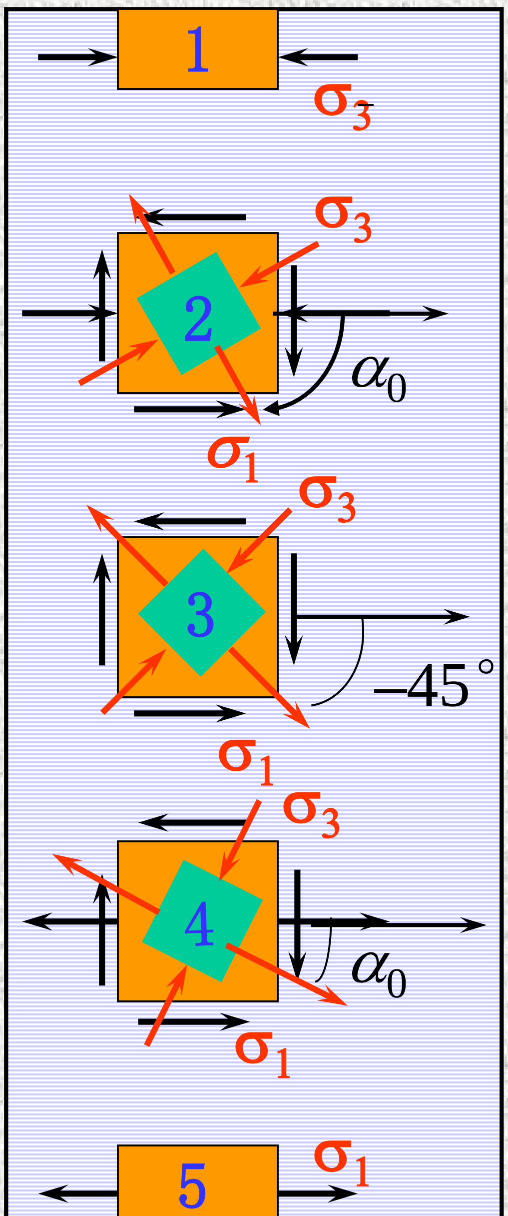
解：



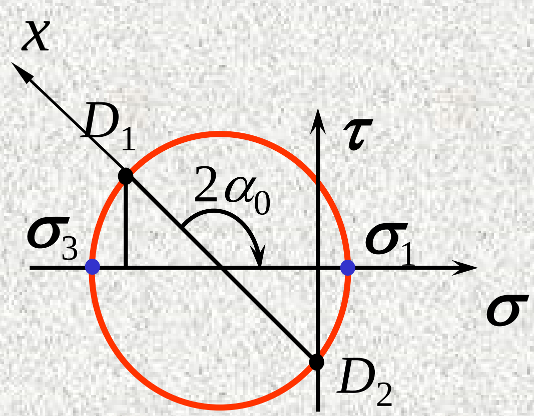
梁的主应力及其主应力迹线



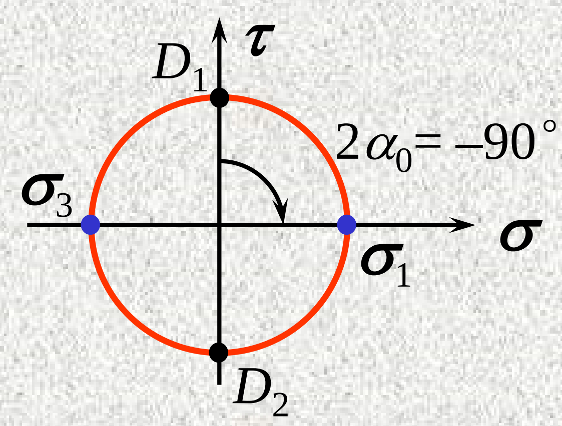




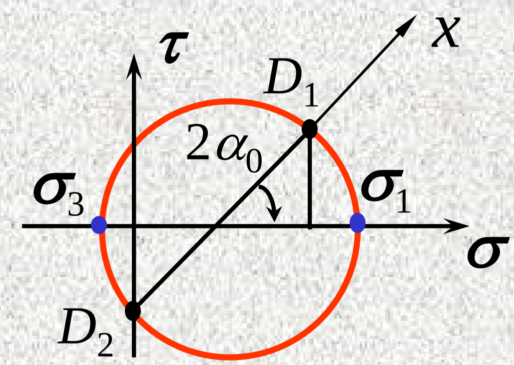
1
σ3
2
σ3
α0
σ1
3
σ3
–45°
σ1
4
σ3
α0
σ1
5
σ1



x
D1
τ
2α0
σ3
σ1
σ
D2



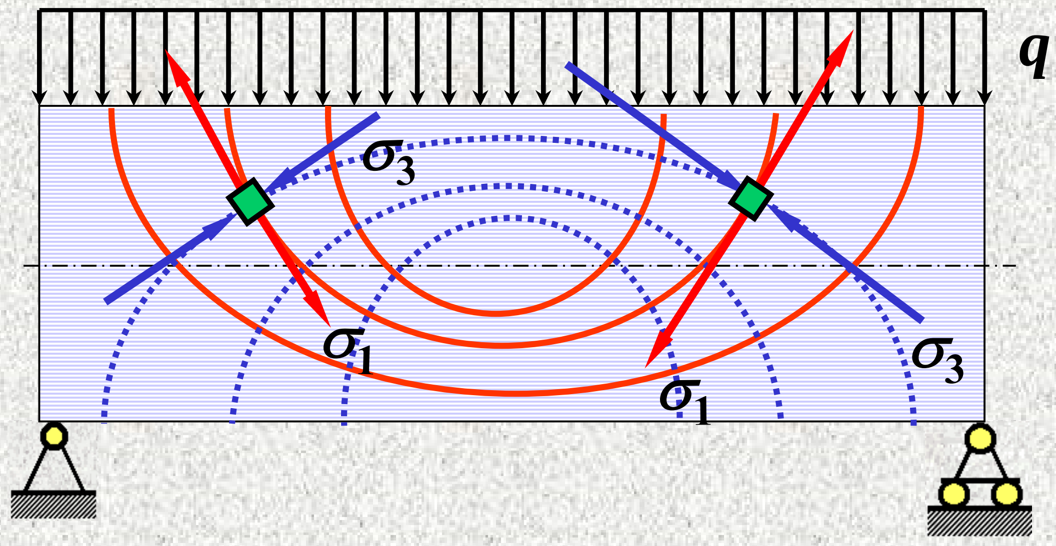
D1
τ
2α0= –90°
σ3
σ
σ1
D2



τ
x
D1
2α0
σ3
σ1
σ
D2

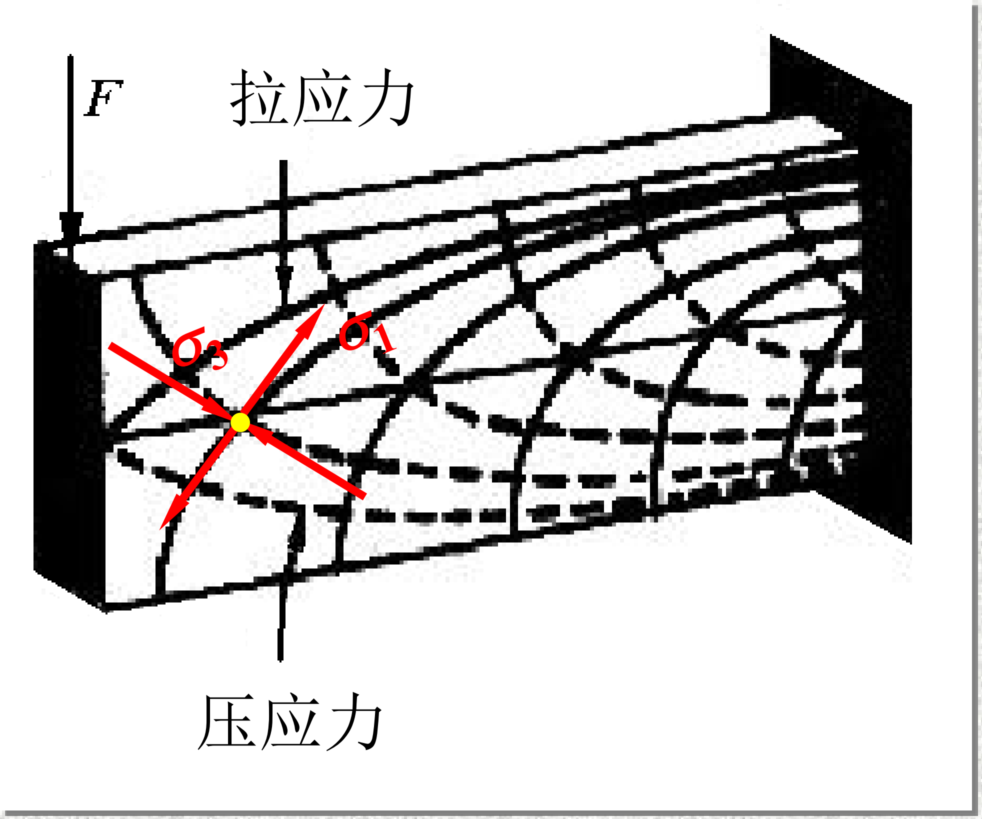
主应力迹线（Stress Trajectories）：

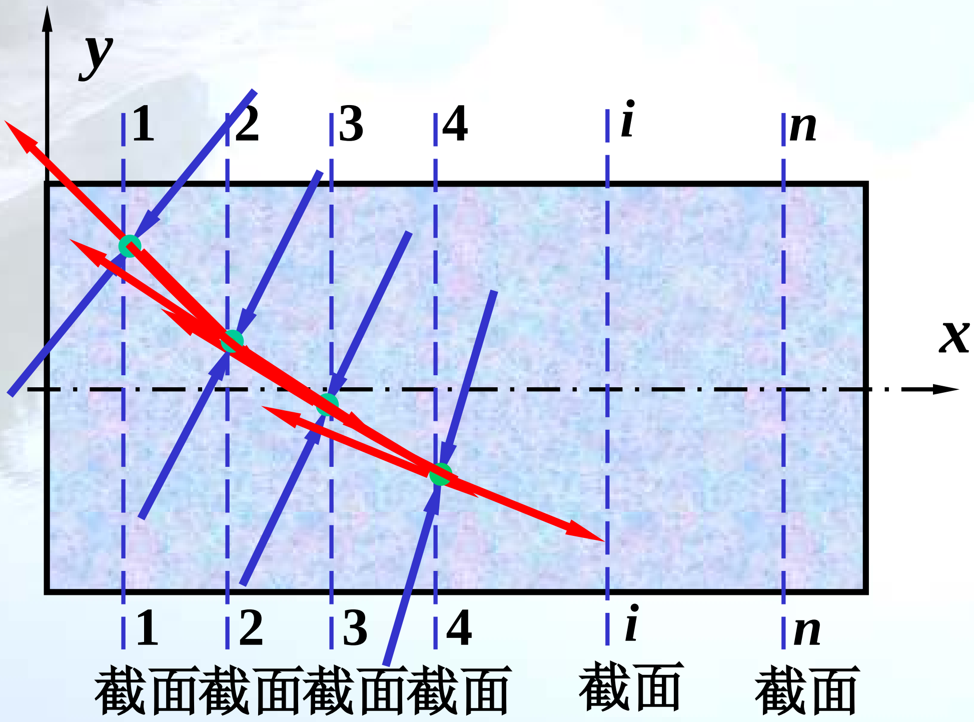
主应力方向线的包络线——曲线上每一点的切线都指示着该点的拉主应力方位（或压主应力方位）。



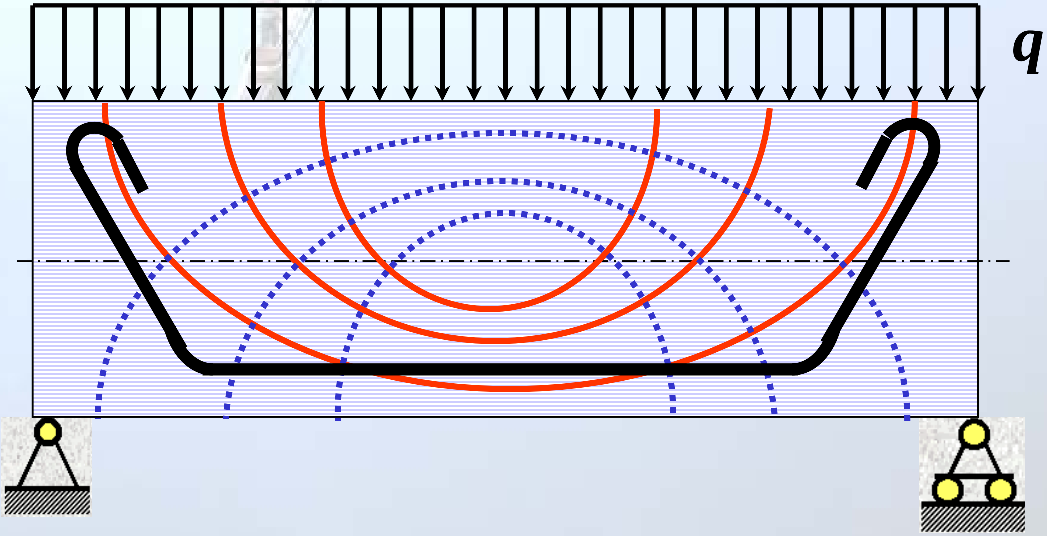
红实线表示拉主应力迹线；

蓝虚线表示压主应力迹线。


F
拉应力
σ3
σ1
压应力

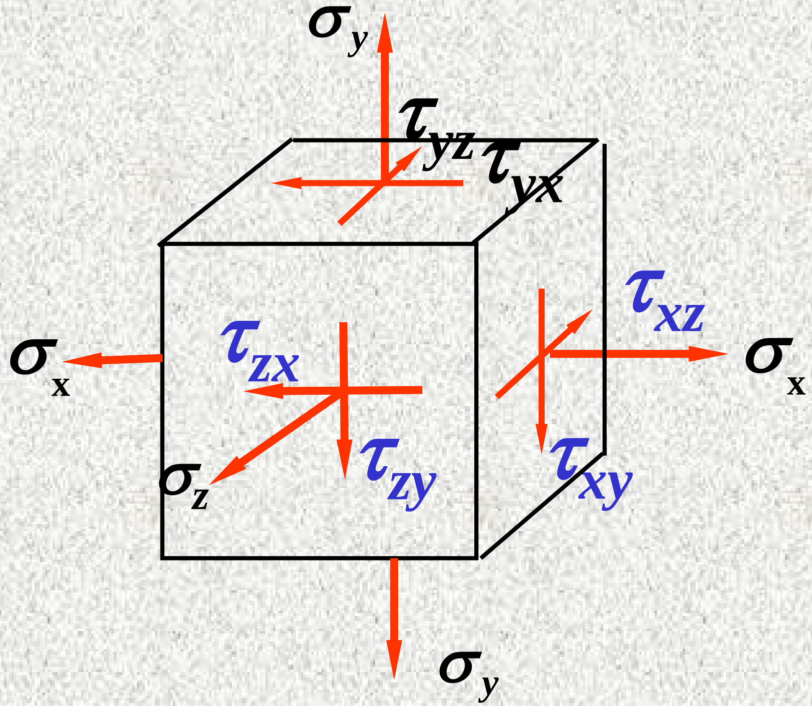


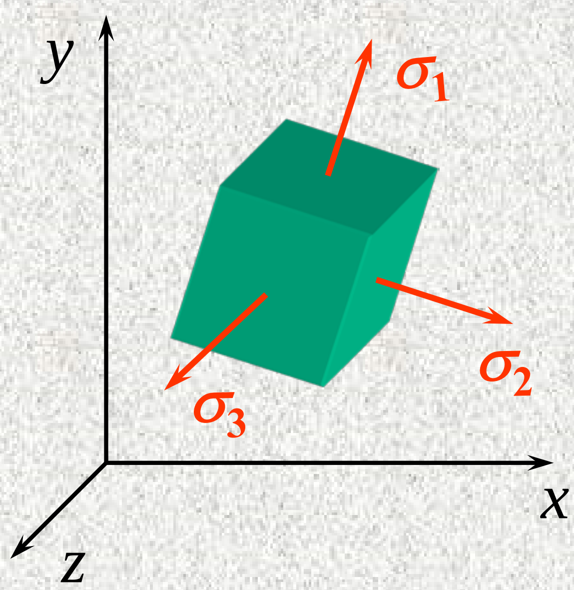
主应力迹线的画法：



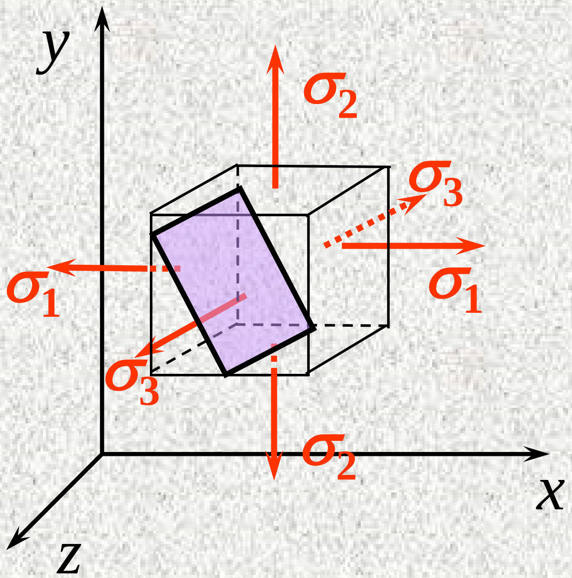
§ 7–5　三向应力状态

1、空间应力状态

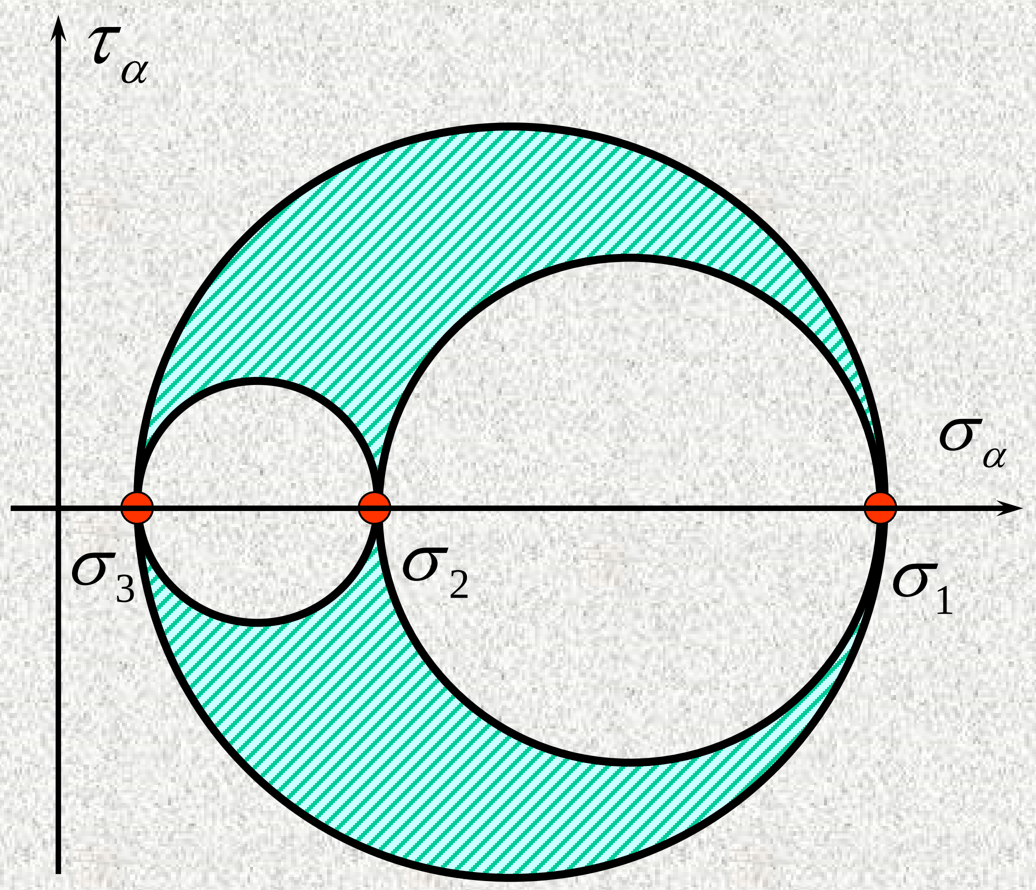




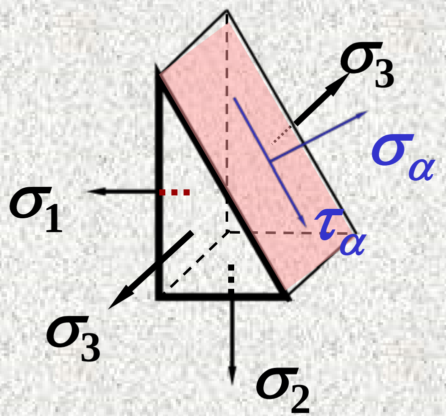
2、三向应力分析



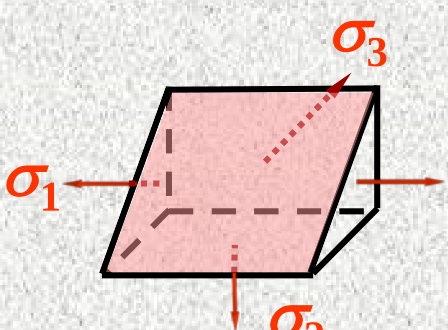
y
x
z
σ2
σ3
σ1
σ1
σ3
σ2



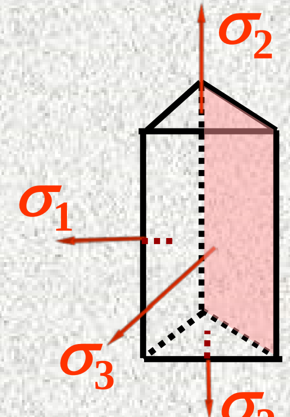
τα
σα
σ3
σ2
σ1



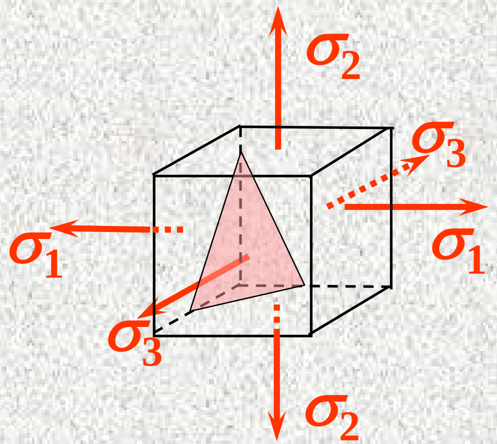
σ3
σα
σ1
τα
σ3
σ2



σ3
σ1
σ1
σ2

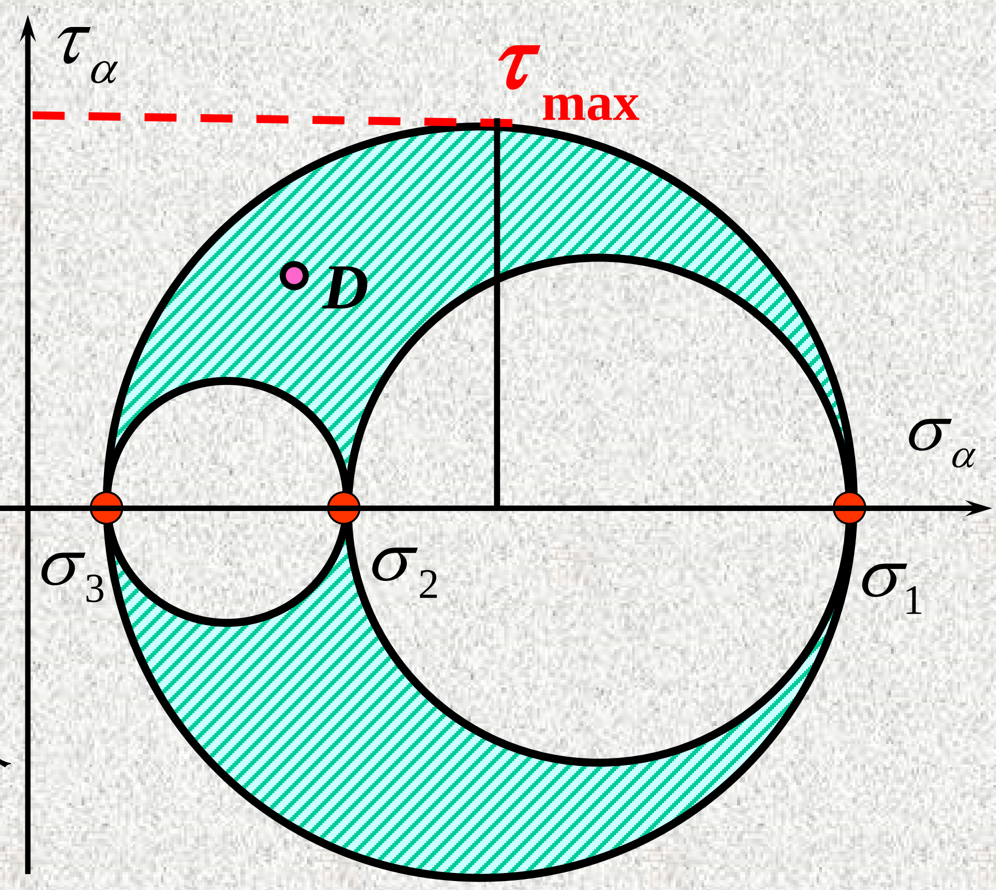


σ2
σ1
σ3
σ2



斜面上的应力

在三向应力圆的阴影内

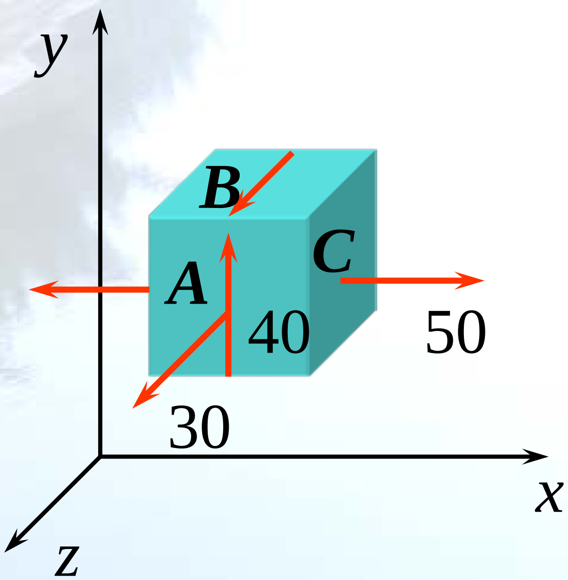


三向应力圆是一点处所有各个不同方位截面上应力的集合。

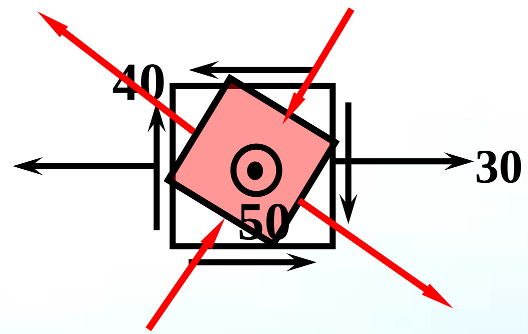
一点的最大正应力为： $\sigma_{\max}=\sigma_1$

一点的最大剪应力为： $\tau_{\max}=\dfrac{\sigma_1-\sigma_3}{2}$

[例14] 求图示单元体的主应力和最大剪应力。（MPa）



解：　由单元体图知：$y\,z$面为主面



$$\left\{\begin{matrix}\sigma_{\max}\\ \sigma_{\min}\end{matrix}\right. = \frac{30}{2} \pm \sqrt{\left(\frac{30}{2}\right)^2 + 40^2} = \left\{\begin{matrix}57.7\text{MPa}\\ -27\text{MPa}\end{matrix}\right.$$

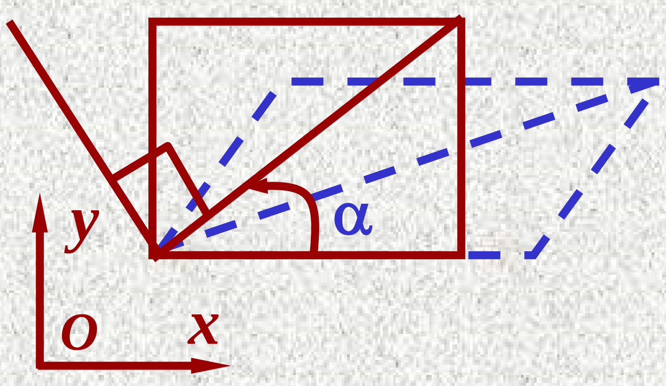
$$\therefore\ \sigma_1 = 57.7\text{MPa},\ \sigma_2 = 50\text{MPa},\quad \sigma_3 = -27\text{MPa}$$

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = 42.3\text{MPa}$$

§ 7–6　平面应力状态下的应变分析

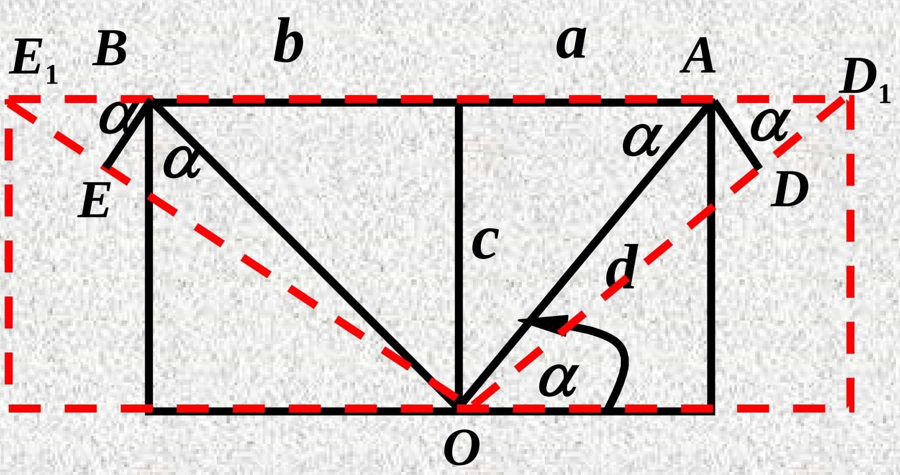
一、叠加法求应变分析公式

剪应变：　直角的增大量！（只有这样，前后才对应）



平面应力问题：薄板（ s_z=0 、τ_{zx}=0 、τ_{zy}=0 ）

平面应变问题：长柱形（w=0 、τ_{zx}=0 、τ_{zy}=0、 s_z#0 ）



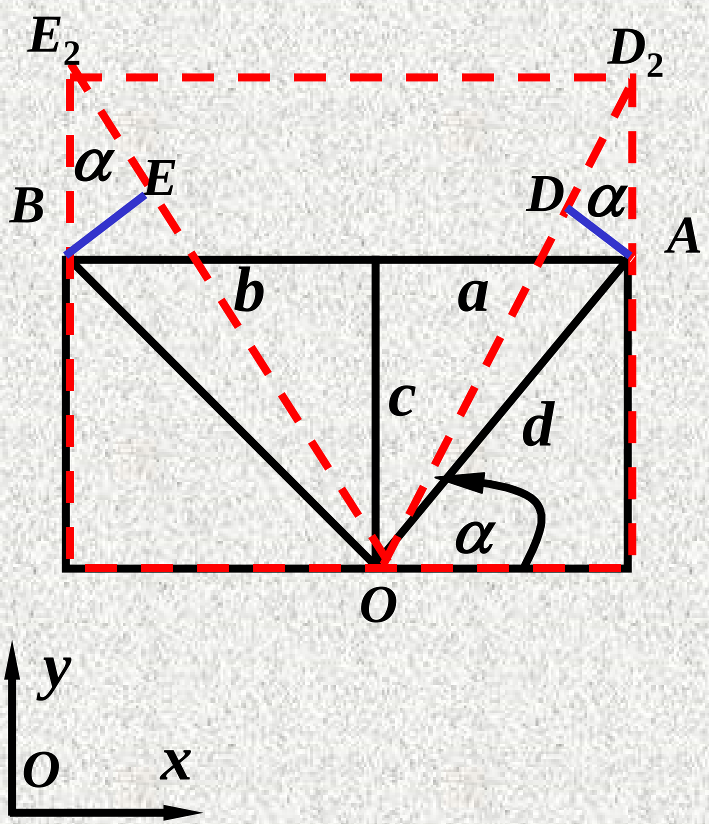
$$DD_1 = \mathrm{d}\varepsilon_{\alpha 1} = a\varepsilon_x \cos\alpha$$

$$\varepsilon_{\alpha 1} = \varepsilon_x \cos^2\alpha$$

$$\gamma_{\alpha 1} = +\angle AOD + \angle BOE$$

$$= +\frac{b\varepsilon_x \cos\alpha}{b/\sin\alpha} + \frac{a\varepsilon_x \sin\alpha}{a/\cos\alpha}$$

$$= +\varepsilon_x \sin 2\alpha$$



$$DD_2 = d\varepsilon_{\alpha 2} = c\varepsilon_y \sin\alpha$$

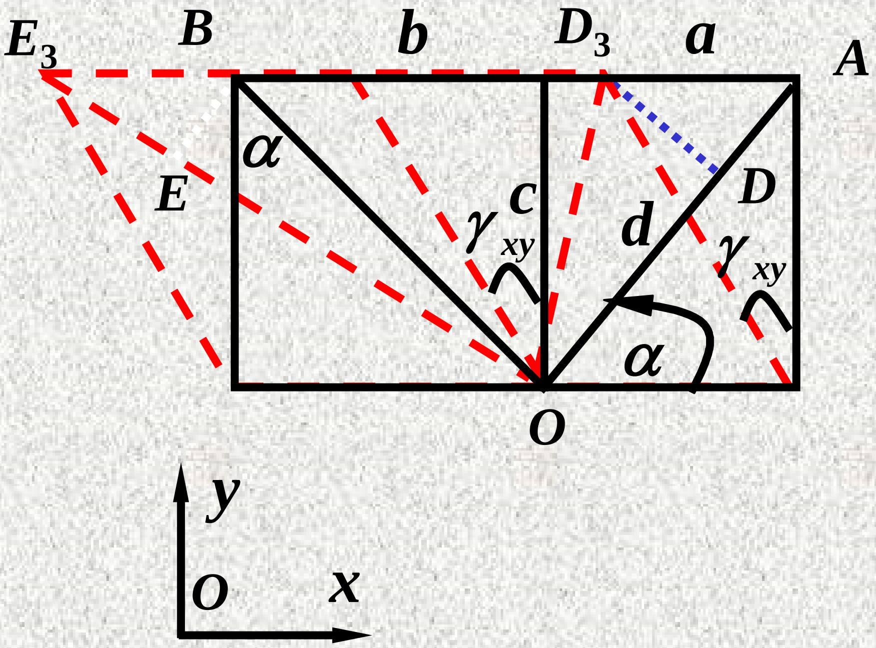
$$\varepsilon_{\alpha 2} = \varepsilon_y \sin^2\alpha$$

$$\gamma_{\alpha 2} = -\angle AOD - \angle BOE$$

$$= -\frac{c\varepsilon_y \sin\alpha}{c/\cos\alpha} - \frac{c\varepsilon_y \sin\alpha}{c/\cos\alpha}$$

$$= -\varepsilon_y \sin 2\alpha$$

$$\Delta d_3 = -AD = -\mathrm{d}\varepsilon_{\alpha 3} = -c\gamma_{xy}\cos\alpha$$



$$\varepsilon_{\alpha 3} = -\gamma_{xy}\sin\alpha soc\,\alpha$$

$$\gamma_{\alpha 3} = -\angle AOD_3 + \angle BOE$$

$$= -\frac{c\gamma_{xy}\sin\alpha}{c/\sin\alpha} + \frac{c\gamma_{xy}\cos\alpha}{c/\cos\alpha}$$

$$= \gamma_{xy}\left(\cos^2\alpha - \sin^2\alpha\right)$$

$$\varepsilon_\alpha=\sum_{i=1}^{3}\varepsilon_{\alpha i}=\varepsilon_x\cos^2\alpha+\varepsilon_y\sin^2\alpha-\gamma_{xy}\sin\alpha\cos\alpha$$

$$\gamma_\alpha=\sum_{i=1}^{3}\gamma_{\alpha i}=-\varepsilon_x\sin2\alpha+\varepsilon_y\sin2\alpha+\gamma_{xy}\left(\cos^2\alpha-\sin^2\alpha\right)$$

$$\begin{cases}\varepsilon_\alpha=\dfrac{\varepsilon_x+\varepsilon_y}{2}+\dfrac{\varepsilon_x-\varepsilon_y}{2}\cos2\alpha-\dfrac{1}{2}\gamma_{xy}\sin2\alpha\\[2ex]\dfrac{\gamma_\alpha}{2}=\dfrac{\varepsilon_x-\varepsilon_y}{2}\sin2\alpha+\dfrac{1}{2}\gamma_{xy}\cos2\alpha\end{cases}$$

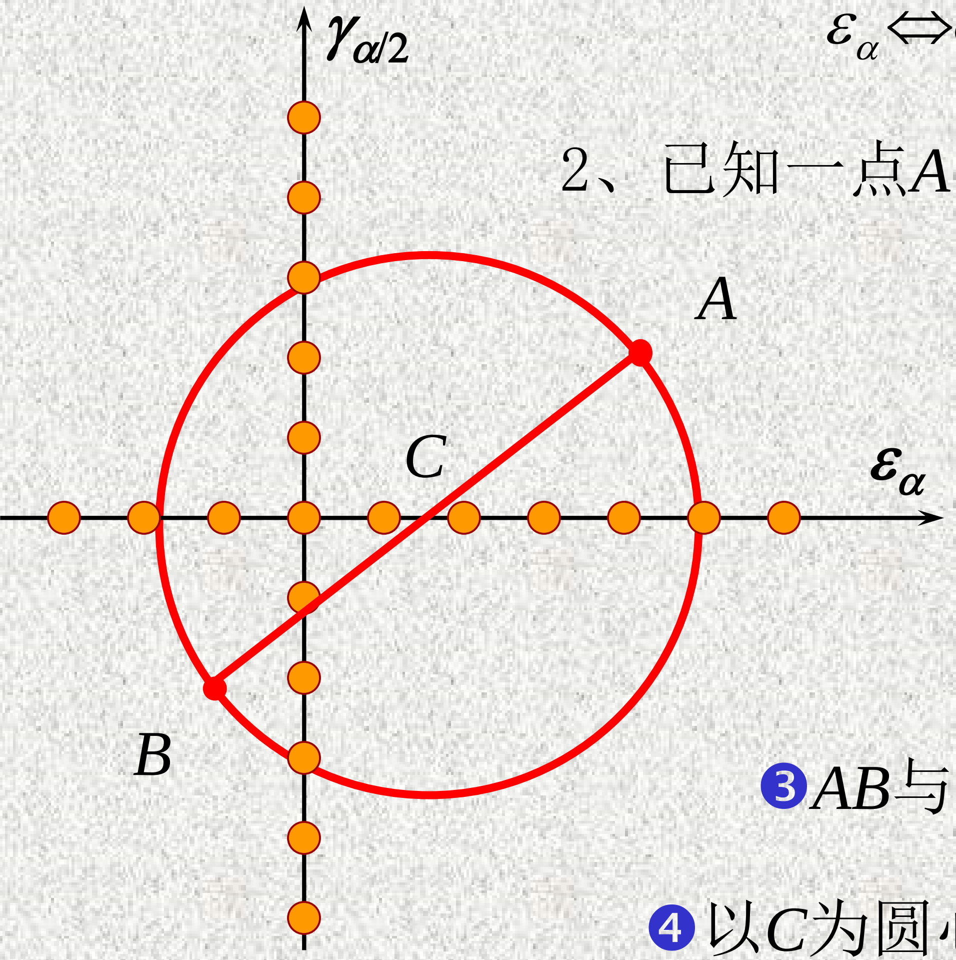
$$\begin{cases}\sigma_\alpha=\dfrac{\sigma_x+\sigma_y}{2}+\dfrac{\sigma_x-\sigma_y}{2}\cos2\alpha-\tau_{xy}\sin2\alpha\\[2ex]\tau_\alpha=\dfrac{\sigma_x-\sigma_y}{2}\sin2\alpha+\tau_{xy}\cos2\alpha\end{cases}$$

二、应变分析图解法——应变圆（ Strain Circle）

1、应变圆与应力圆的类比关系

$$\varepsilon_\alpha \Leftrightarrow \sigma_\alpha \;;\; \frac{\gamma_\alpha}{2} \Leftrightarrow \tau_\alpha \;;\; 2\alpha \Leftrightarrow 2\alpha$$

2、已知一点A的应变（$\varepsilon_x, \varepsilon_y, \gamma_{xy}$ ），画应变圆



❶建立应变坐标系如图

❷在坐标系内画出点

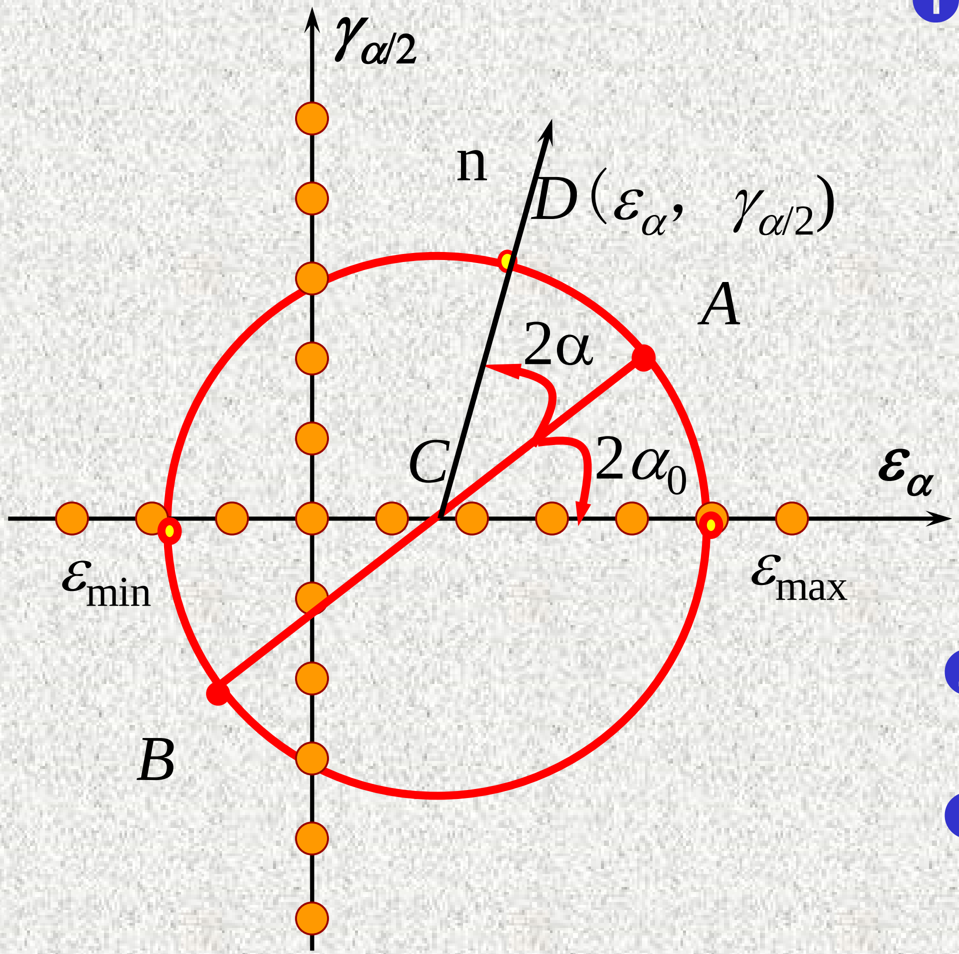
$A(\varepsilon_x,\ \gamma_{xy/2})$

$B(\varepsilon_y,\ -\gamma_{yx/2})$

❸AB与ε_α轴的交点C便是圆心

❹以C为圆心，以AC为半径画圆——应变圆。

三、α方向上的应变与应变圆的对应关系



❶α方向上的应变$(\varepsilon_\alpha, \gamma_{\alpha/2})$ ⟶ 应变圆上一点$(\varepsilon_\alpha, \gamma_{\alpha/2})$

❷α 方向线⟶应变圆的半径

❸两方向间夹角α ⟶ 两半径夹角2α；且转向一致。

四、主应力数值及其方位

$$\left\{\begin{matrix}\sigma_{\max}\\ \sigma_{\min}\end{matrix}\right.=\frac{\sigma_x+\sigma_y}{2}\pm\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}$$

$$tg\,2\alpha_0=-\frac{2\tau_{xy}}{\sigma_x-\sigma_y}$$

$$\varepsilon_\alpha\Leftrightarrow\sigma_\alpha\ ;\ \frac{\gamma_\alpha}{2}\Leftrightarrow\tau_\alpha\ ;\ 2\alpha\Leftrightarrow 2\alpha$$

$$\left\{\begin{matrix}\varepsilon_{\max}\\ \varepsilon_{\min}\end{matrix}\right.=\frac{1}{2}\left[\left(\varepsilon_x+\varepsilon_y\right)\pm\sqrt{(\varepsilon_x-\varepsilon_y)^2+\gamma_{xy}^2}\right]$$

$$\mathrm{tg}\,2\alpha_0=\frac{-\gamma_{xy}}{\varepsilon_x-\varepsilon_y}$$

例5 已知一点在某一平面内的 α_1、 α_2、 α_3、方向上的应变 $\varepsilon_{\alpha 1}$、 $\varepsilon_{\alpha 2}$、 $\varepsilon_{\alpha 3}$，三个线应变，求该面内的主应变。

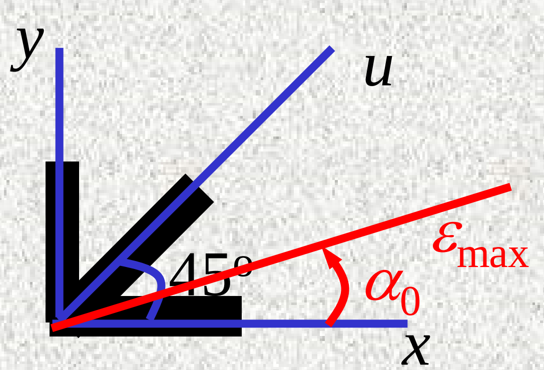
解：由

$$\varepsilon_{\alpha_i}=\varepsilon_x\cos^2\alpha_i+\varepsilon_y\sin^2\alpha_i-\gamma_{xy}\sin\alpha_i\cos\alpha_i$$

i =1,2,3这三个方程求出 ε_x， ε_y， γ_{xy}；然后在求主应变。

$$\left.\begin{cases}\varepsilon_{\max}\\ \varepsilon_{\min}\end{cases}\right.=\frac{1}{2}\left[\left(\varepsilon_x+\varepsilon_y\right)\pm\sqrt{\left(\varepsilon_x-\varepsilon_y\right)^2+\gamma_{xy}^2}\right]$$

例6 用45°应变花测得一点的三个线应变后，求该点的主应变。



$$\varepsilon_{max}=\frac{1}{2}\left[(\varepsilon_x+\varepsilon_y)+\sqrt{2[(\varepsilon_x-\varepsilon_u)^2+(\varepsilon_u-\varepsilon_y)^2]}\right]$$

$$\varepsilon_{min}=\frac{1}{2}\left[(\varepsilon_x+\varepsilon_y)-\sqrt{2[(\varepsilon_x-\varepsilon_u)^2+(\varepsilon_u-\varepsilon_y)^2]}\right]$$

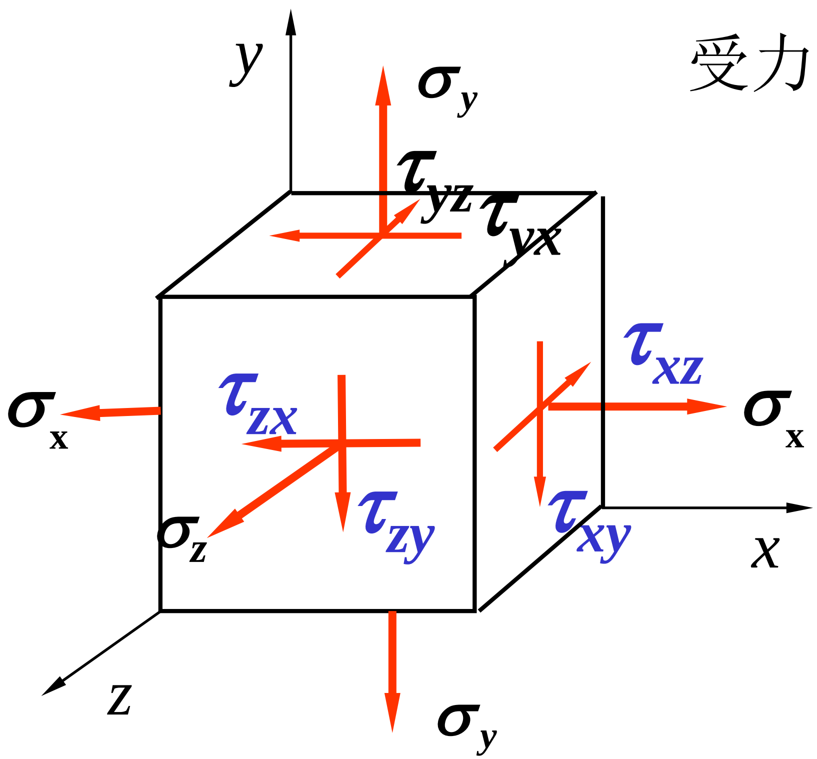
$$\mathrm{tg}\,2\alpha_0=\frac{2\varepsilon_u-\varepsilon_x-\varepsilon_y}{\varepsilon_x-\varepsilon_y}$$

§ 7–8广义胡克定律

一、一点的变形

(线应变和角应变)

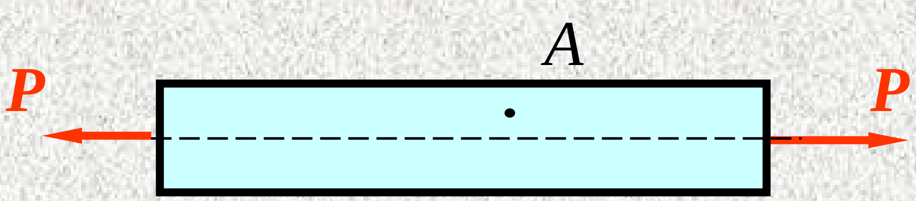
设单元体的三个边长分别为l_x、l_y、l_z



受力后三个边长分别伸长Δx、Δy、Δz

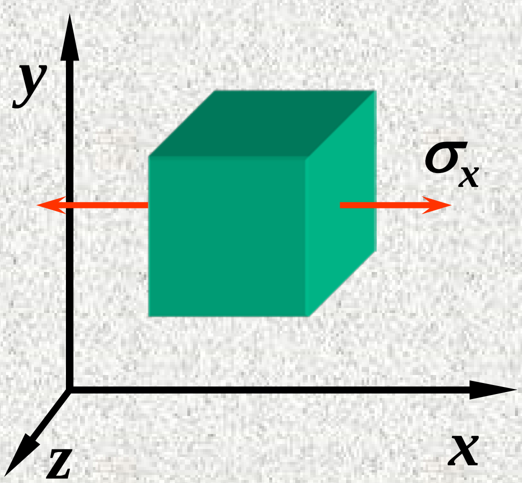
线应变	角应变
$\varepsilon_x=\dfrac{\Delta x}{l_x}$	γ_{xy}
$\varepsilon_y=\dfrac{\Delta y}{l_y}$	γ_{yz}
$\varepsilon_z=\dfrac{\Delta z}{l_z}$	γ_{zx}

二、单向拉（压）时的胡克定律



$$\sigma=E\cdot\varepsilon \qquad \left|\frac{\varepsilon'}{\varepsilon}\right|=\nu$$

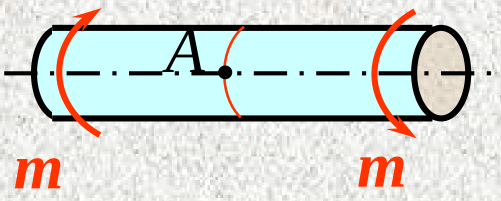
$$\varepsilon'=-\nu\varepsilon$$

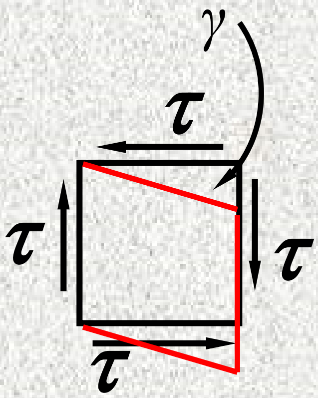


$$\varepsilon_x=\frac{\sigma_x}{E} \qquad \varepsilon_y=-\nu\frac{\sigma_x}{E} \qquad \varepsilon_z=-\nu\frac{\sigma_x}{E}{}_x$$

$$\gamma_{ij}=0\ \ (i,j=x,y,z)$$

二、纯剪的应力--应变关系

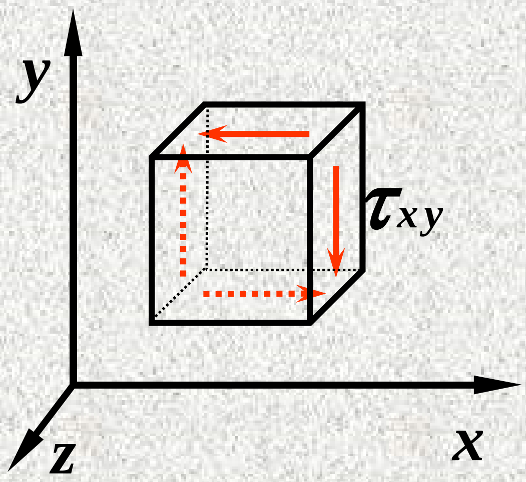




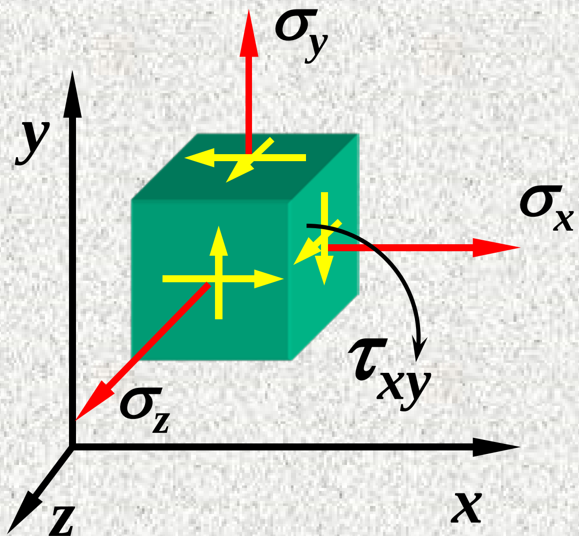
$$\tau = G \cdot \gamma$$

$$\varepsilon_i = 0 \ (i = x, y, z)$$

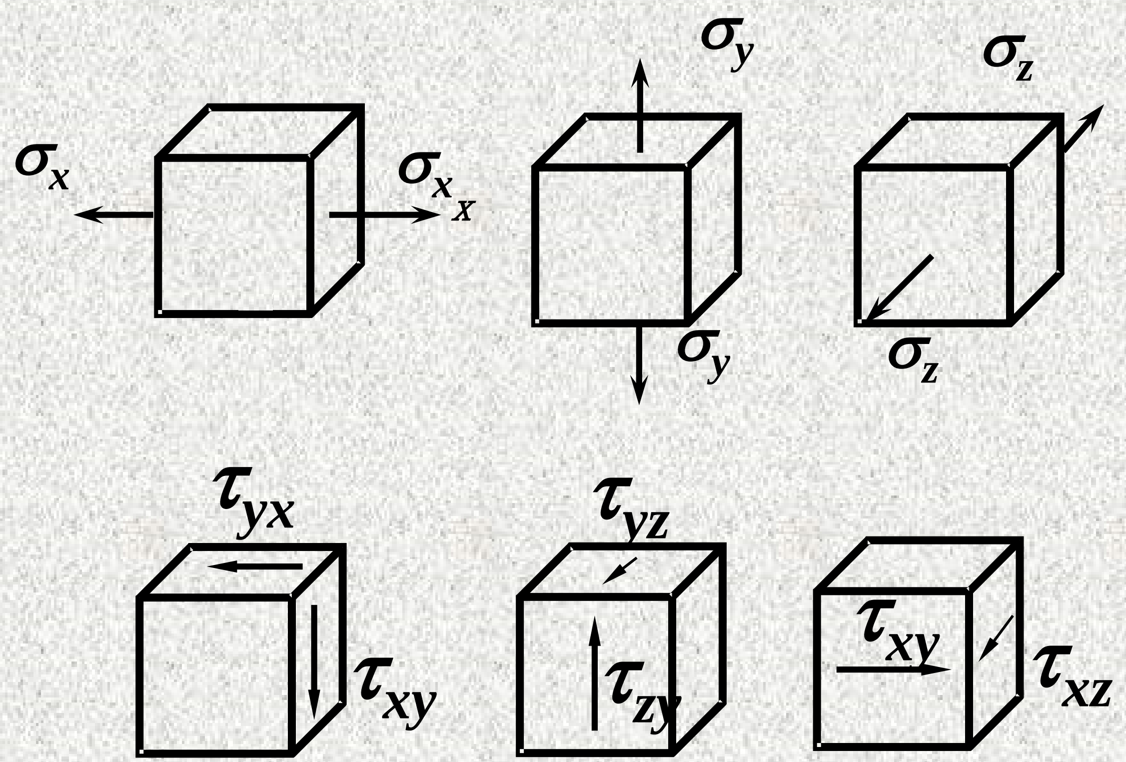
$$\gamma_{xy} = \frac{\tau_{xy}}{G} \qquad \gamma_{yz} = \gamma_{zx} = 0$$



三、复杂应力状态下的应力 — 应变关系



依叠加原理, 得:

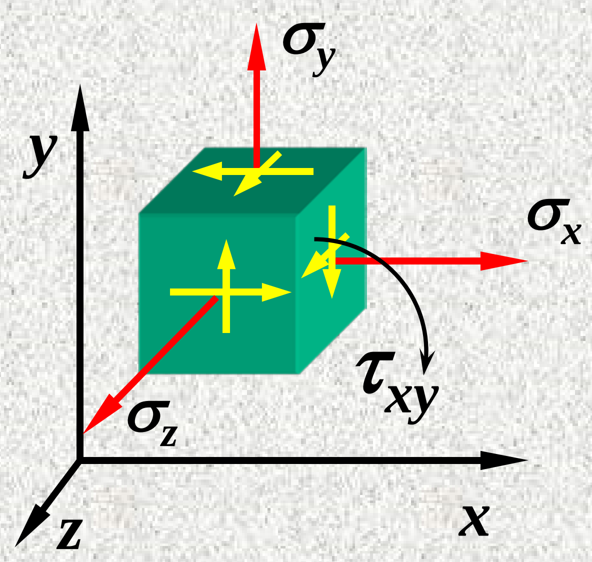


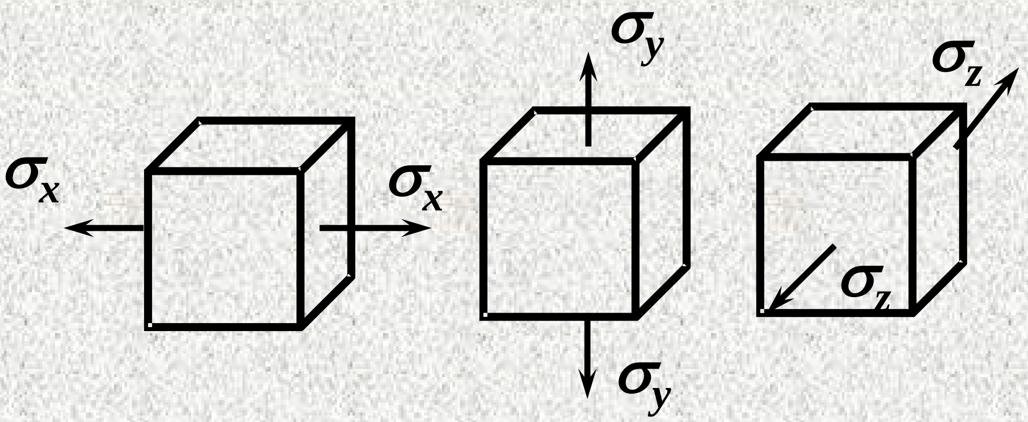
$$\varepsilon_x = \frac{\Delta x}{l_x}$$

$$= \frac{\Delta x' + \Delta x'' + \Delta x'''}{l_x}$$

$$= \frac{\Delta x'}{l_x} + \frac{\Delta x''}{l_x} + \frac{\Delta x'''}{l_x}$$

$$= \varepsilon'_x + \varepsilon''_x + \varepsilon'''_x$$





$$\varepsilon'_x = \frac{\sigma_x}{E} \qquad \varepsilon''_y = \frac{\sigma_y}{E} \qquad \varepsilon'''_x = -\nu\frac{\sigma_z}{E}$$

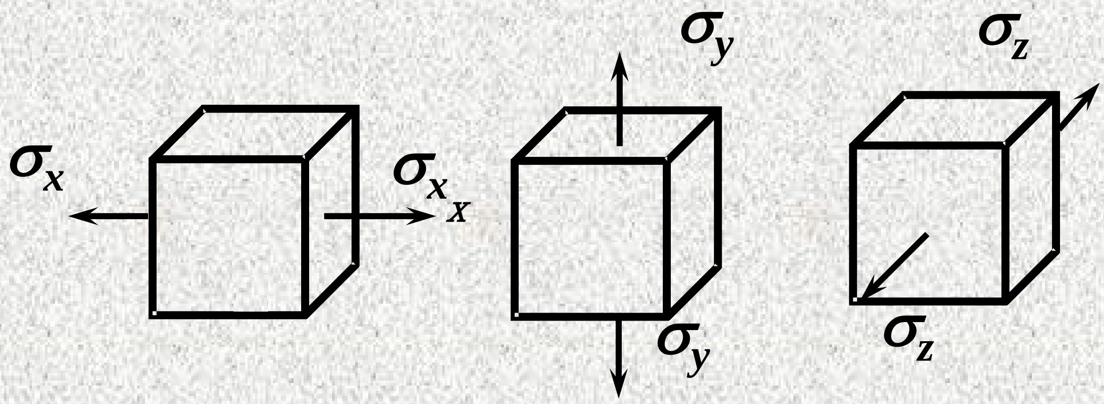
$$\varepsilon''_x = -\nu\varepsilon''_y$$

$$= -\nu\frac{\sigma_y}{E}$$

$$\varepsilon_x = \varepsilon'_x + \varepsilon''_x + \varepsilon'''_x$$

$$= \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$

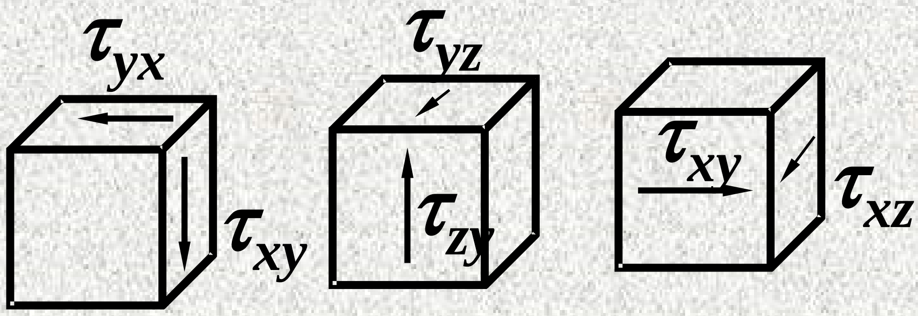
$$\boxed{\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right]}$$



$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right]$$

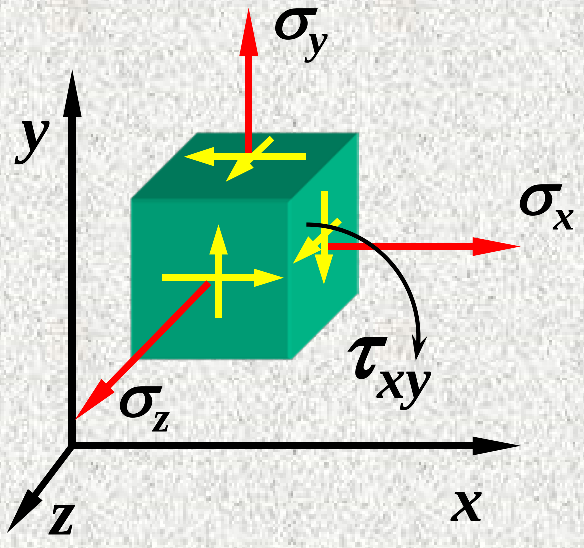
$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu\left(\sigma_z + \sigma_x\right)\right]$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right]$$



$$\gamma_{xy} = \frac{\tau_{xy}}{G} \qquad \gamma_{yz} = \frac{\tau_{yz}}{G} \qquad \gamma_{zx} = \frac{\tau_{zx}}{G}$$

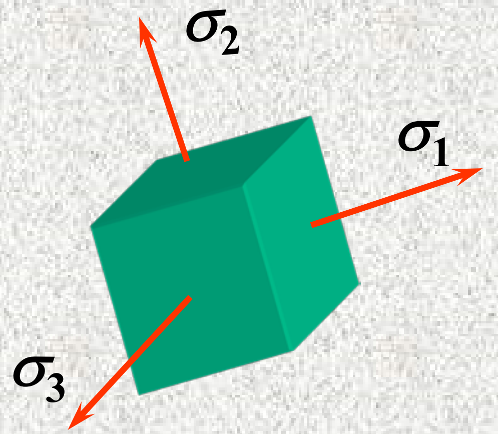
上式称为广义胡克定律



$$
\begin{cases}
\varepsilon_x = \dfrac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] \\
\varepsilon_y = \dfrac{1}{E}\left[\sigma_y - \nu\left(\sigma_z + \sigma_x\right)\right] \\
\varepsilon_z = \dfrac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right] \\
\gamma_{xy} = \dfrac{\tau_{xy}}{G} \\
\gamma_{yz} = \dfrac{\tau_{yz}}{G} \\
\gamma_{zx} = \dfrac{\tau_{zx}}{G}
\end{cases}
$$

上式称为广义胡克定律

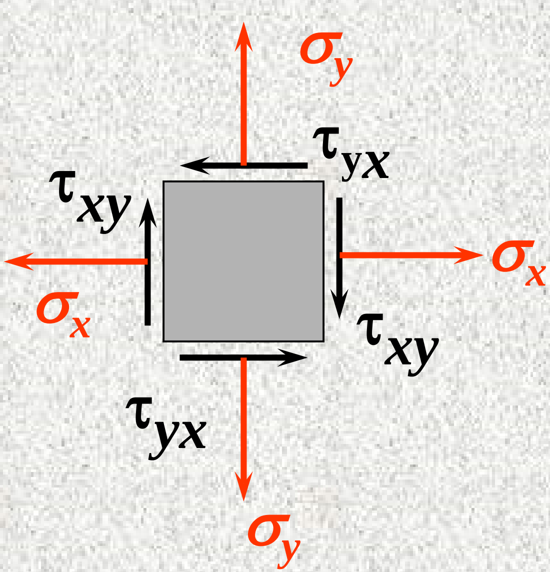
主应力 --- 主应变关系



$$\begin{cases} \varepsilon_1 = \dfrac{1}{E}[\sigma_1 - \nu(\sigma_2 + \sigma_3)] \\ \varepsilon_2 = \dfrac{1}{E}[\sigma_2 - \nu(\sigma_3 + \sigma_1)] \\ \varepsilon_3 = \dfrac{1}{E}[\sigma_3 - \nu(\sigma_2 + \sigma_1)] \end{cases}$$

对于平面应力状态问题：

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right]$$

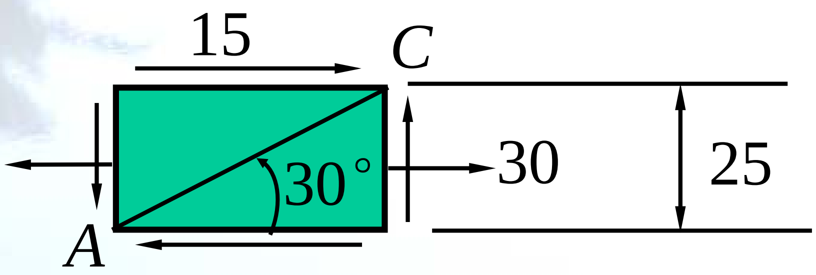


$$\because \quad \sigma_z = 0$$

$$\therefore \quad \varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\sigma_y\right]$$

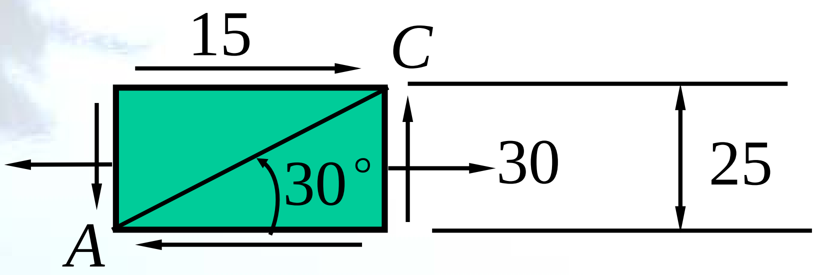
$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu\sigma_x\right]$$

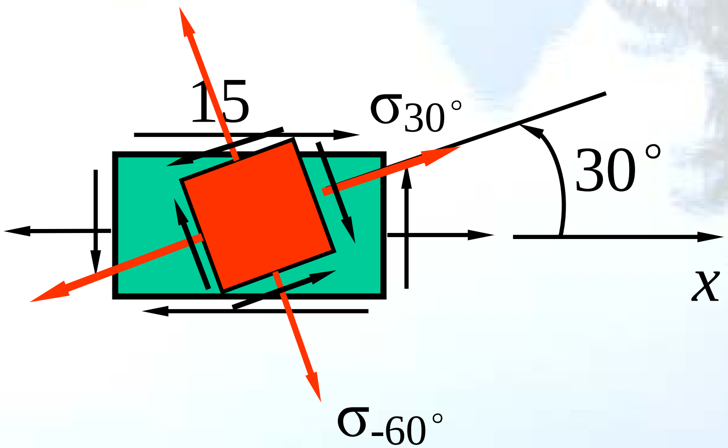
[例15] 已知：E=200GPa，μ=0.3，应力单位为MPa，求对角线AC的改变量Δl_{AC}



$$\varepsilon_{AC}=\frac{\Delta l_{AC}}{\overline{AC}}$$

[例15] 已知：E=200GPa，μ=0.3，应力单位为MPa，求对角线AC的改变量Δl_{AC}





$$\Delta l_{AC} = \overline{AC} \cdot \varepsilon_{AC} = \overline{AC} \cdot \varepsilon_{30^\circ}$$

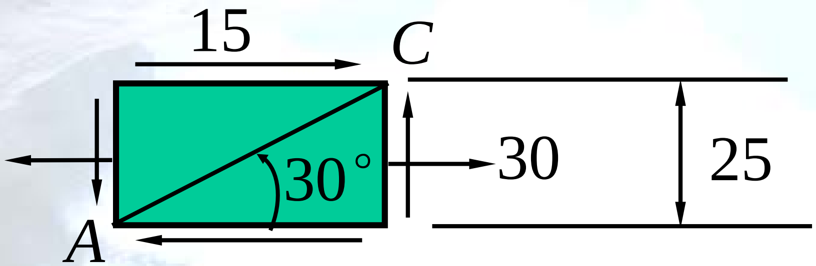
$$\varepsilon_{30^\circ} = \frac{1}{E}\left[\sigma_{30^\circ} - \mu(\sigma_{-60^\circ} + \sigma_z)\right]$$

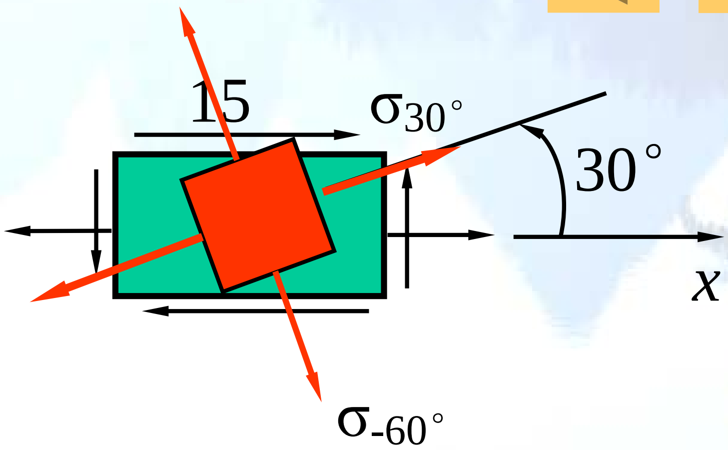
$$\sigma_{30^\circ} = \frac{\sigma_x}{2} + \frac{\sigma_x}{2}\cos 60^\circ - \tau_{xy}\sin 60^\circ$$

$$\sigma_{-60^\circ} = \frac{\sigma_x}{2} + \frac{\sigma_x}{2}\cos(-120^\circ) - \tau_{xy}\sin(-120^\circ)$$

$$\sigma_x = 30\text{MPa}$$

$$\tau_{xy} = -15\text{MPa}$$





解：$\sigma_{30^\circ}=\frac{\sigma_x}{2}+\frac{\sigma_x}{2}\cos 60^\circ-\tau_{xy}\sin 60^\circ$

$=\frac{30}{2}+\frac{30}{2}\cos 60^\circ-(-15)\sin 60^\circ$

$=35.5(\mathbf{MPa})$

$$\sigma_{-60^\circ}=\frac{\sigma_x}{2}+\frac{\sigma_x}{2}\cos 120^\circ-\tau_{xy}\sin(-120^\circ)$$

$$=\frac{30_x}{2}+\frac{30}{2}\cos 120^\circ-(-15)\sin(-120^\circ)$$

$$=-5.5(\mathbf{MPa})$$

$$\varepsilon_{30^\circ}=\frac{1}{E}\left[\sigma_{30^\circ}-\mu\sigma_{-60^\circ}\right]$$

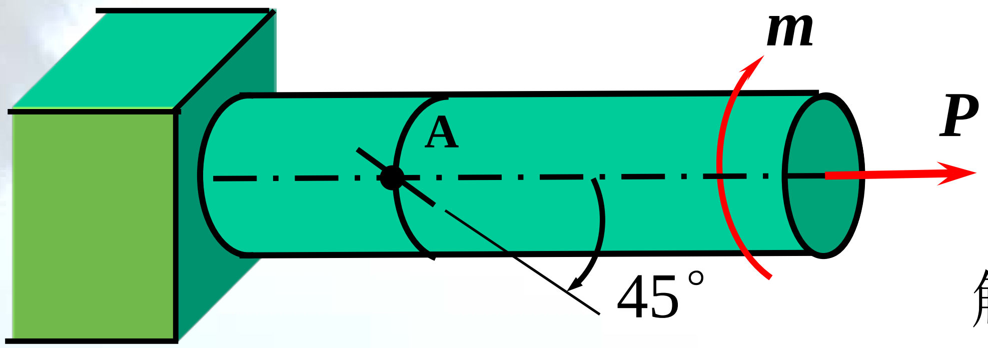
$$=186\times 10^{-6}$$

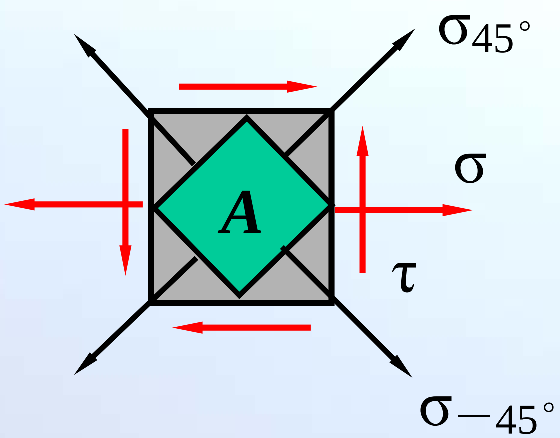
$$\Delta l_{AC}=\overline{AC}\cdot\varepsilon_{30^\circ}$$

$$=50\times 186\times 10^{-6}$$

$$=9.3\times 10^{-3}(\mathbf{mm})$$

[例16] 已知：E=200GPa， μ=0.3，P=3kN，m=12N·m，d=10mm，求A点图示方向的线应变。



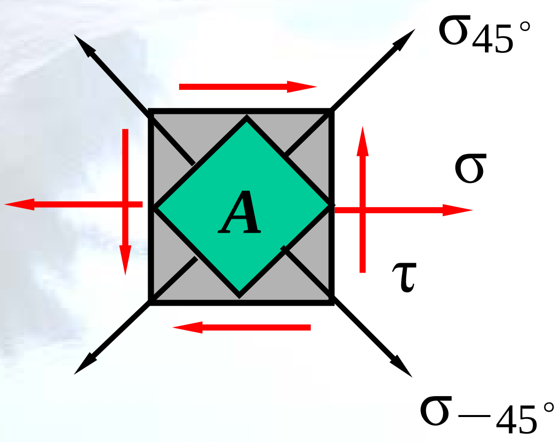


解：

$$\sigma=\frac{P}{A}=\frac{4P}{\pi d^2}=38\text{MPa}$$

$$\tau=\frac{m}{W_t}=\frac{16m}{\pi d^3}=61\text{MPa}$$

$$\therefore\ \sigma_x=38\text{MPa},\quad \tau_{xy}=-61\text{MPa}$$



$$\sigma_{45^\circ}=\frac{\sigma_x}{2}+\frac{\sigma_x}{2}\cos 90^\circ-\tau_{xy}\sin 90^\circ$$

$$=\frac{\sigma_x}{2}-\tau_{xy}=\frac{38}{2}-(-61)=80(\mathrm{MPa})$$

$$\sigma_{-45^\circ}=\frac{\sigma_x}{2}+\frac{\sigma_x}{2}\cos 90^\circ+\tau_{xy}\sin 90^\circ$$

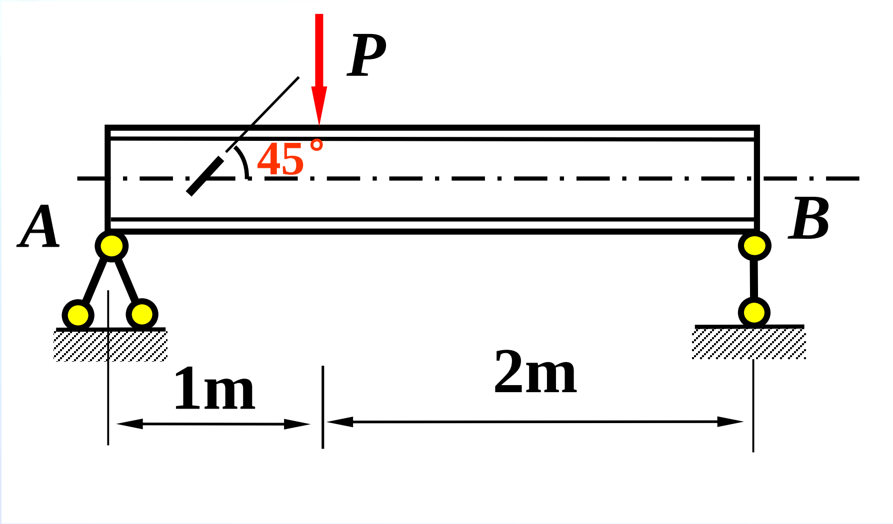
$$=\frac{\sigma_x}{2}+\tau_{xy}=\frac{38}{2}+(-61)=-42(\mathrm{MPa})$$

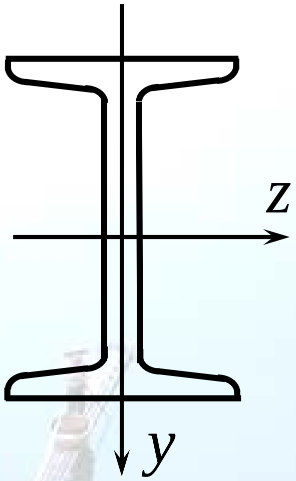
$$\varepsilon_{-45^\circ}=\frac{1}{E}\left[\sigma_{-45^\circ}-\mu\sigma_{45^\circ}\right]$$

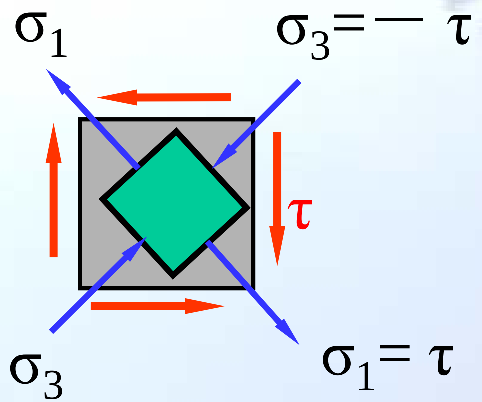
$$=\frac{1}{200\times 10^{3}}\left[(-42)-0.3\times 80\right]$$

$$=-330\times 10^{-6}$$

[例17] 图示28a工字钢梁，查表知，I_Z/S_Z=24.62cm,腹板厚 d=8.5mm，材料的E=200GPa， μ=0.3，在梁中性层处粘贴应变片，测得与轴线成45°方向的线应变为$\varepsilon=-2.6\times10^{-4}$，求载荷$P$的大小。

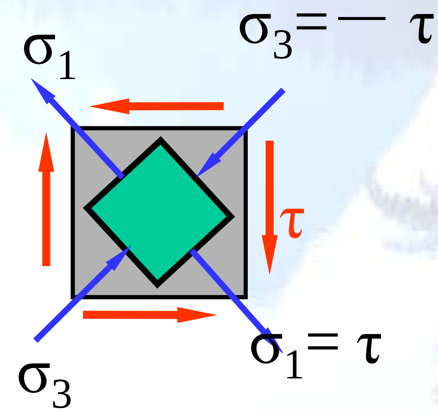






$$\tau=\frac{QS_z^*}{I_z d}=\frac{Q}{\left(\dfrac{I_z}{S_z^*}\right)d}$$

$$Q=\frac{2P}{3}$$



解： $\because \quad \varepsilon_3=\frac{1}{E}\left[\sigma_3-\mu\sigma_1\right]$

$$=\frac{1}{E}\left[-\tau-\mu\tau\right]$$

$$=-\frac{(1+\mu)}{E}\tau$$

$$\therefore \tau=-\frac{E\varepsilon_3}{(1+\mu)}$$

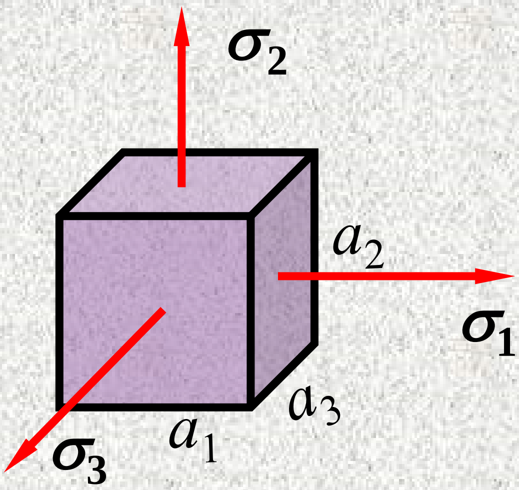
由 $\varepsilon_3=-2.6\times10^{-4}$

$\therefore$解得 $\tau=40$（**MPa**）

$$\because \quad \tau=\frac{Q}{\left(\frac{I_z}{S_z^*}\right)d}=\frac{\frac{2P}{3}}{\left(\frac{I_z}{S_z^*}\right)d}$$

$$\therefore P=\frac{3}{2}\cdot\tau\cdot\frac{I_z}{S_z^*}\cdot d=125.6(\mathrm{kN})$$

五、体积应变



变形前：a_1、a_2、a_3

变形后：

$$a_1' = a_1 + \Delta a_1 = a_1(1+\varepsilon_1)$$

$$a_2' = a_2 + \Delta a_2 = a_2(1+\varepsilon_2)$$

$$a_3' = a_3 + \Delta a_3 = a_3(1+\varepsilon_3)$$

变形前：$V = a_1 a_2 a_3$

变形后：$V' = a_1' a_2' a_3'$

$$= a_1(1+\varepsilon_1)a_2(1+\varepsilon_2)a_3(1+\varepsilon_3)$$

体积应变：

$$\Theta = \frac{V'-V}{V} = \frac{\cancel{a_1 a_2 a_3}(1+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3) - \overset{1}{\cancel{a_1 a_2 a_3}}}{\cancel{a_1 a_2 a_3}}$$

展开并

忽略高阶微量 $\varepsilon_1\varepsilon_2$、$\varepsilon_2\varepsilon_3$、$\varepsilon_3\varepsilon_1$、$\varepsilon_1\varepsilon_2\varepsilon_3$项

$$\therefore \Theta = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$$

$$\Theta = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (1)$$

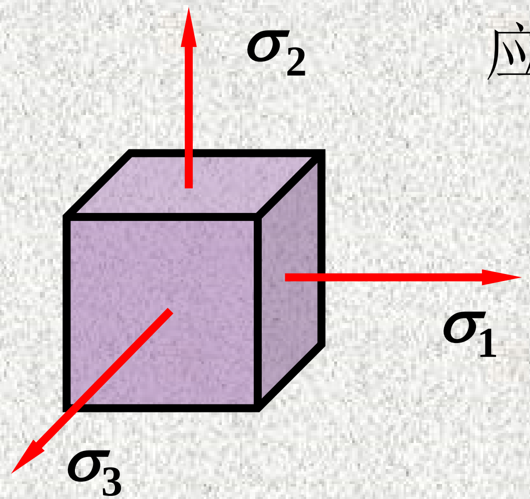
体积应变与应力分量间的关系:

$$\text{由:}\begin{cases} \varepsilon_1 = \dfrac{1}{E}[\sigma_1 - \nu(\sigma_2 + \sigma_3)] \\ \varepsilon_2 = \dfrac{1}{E}[\sigma_2 - \nu(\sigma_3 + \sigma_1)] \\ \varepsilon_3 = \dfrac{1}{E}[\sigma_3 - \nu(\sigma_2 + \sigma_1)] \end{cases}$$

代入（1）式，得

$$\Theta = \frac{1-2\nu}{E}(\sigma_1 + \sigma_2 + \sigma_3)$$

$$\Theta=\frac{1-2\nu}{E}(\sigma_1+\sigma_2+\sigma_3)$$



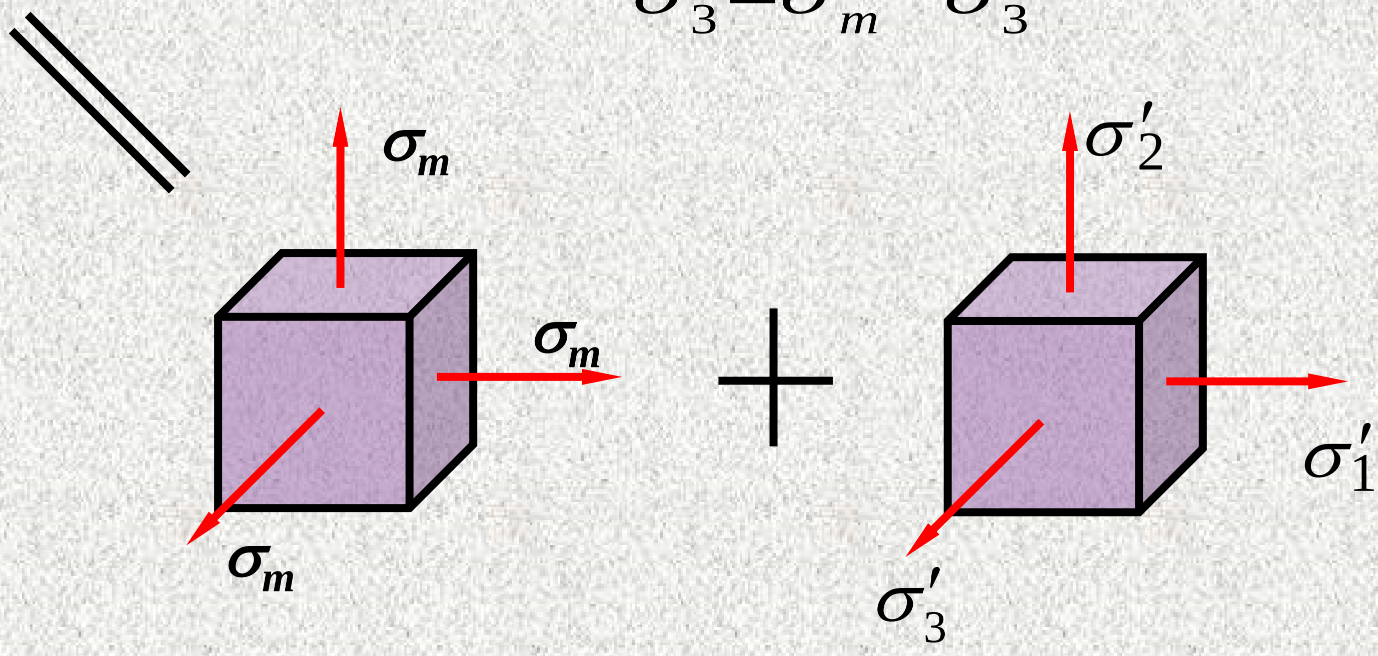
应力状态分解：

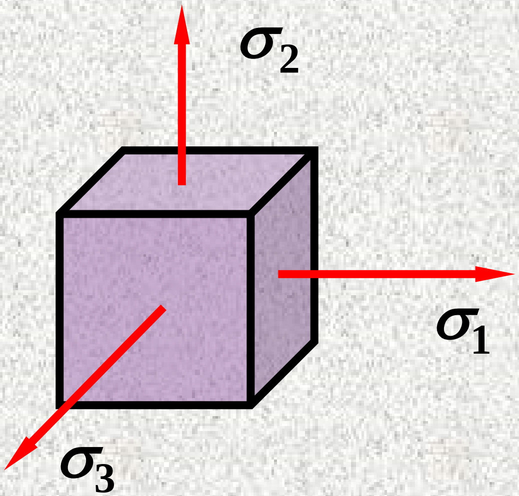
$$\sigma_m=\frac{1}{3}(\sigma_1+\sigma_2+\sigma_3)$$

$$\sigma_1'=\sigma_m-\sigma_1$$

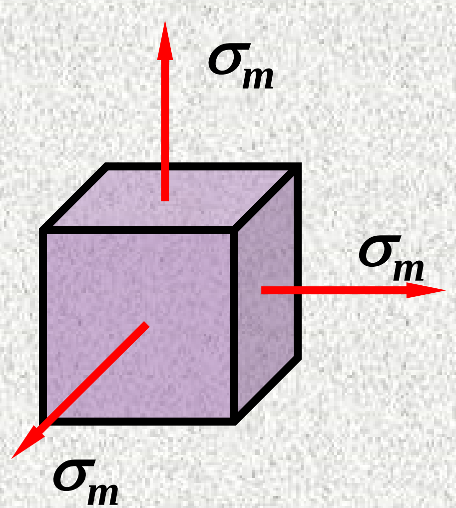
$$\sigma_2'=\sigma_m-\sigma_2$$

$$\sigma_3'=\sigma_m-\sigma_3$$





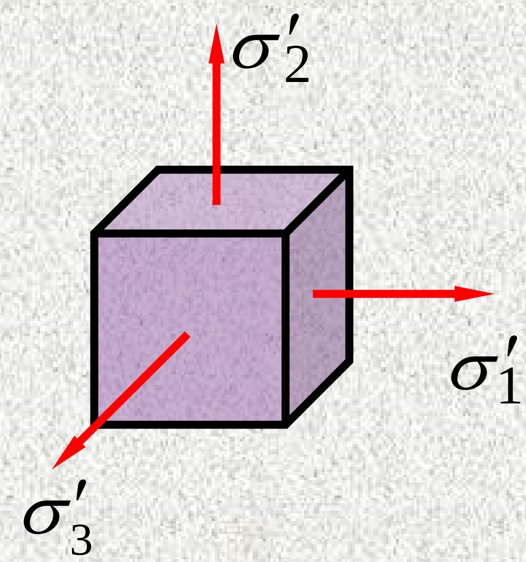
$$\Theta=\frac{1-2\nu}{E}(\sigma_1+\sigma_2+\sigma_3)$$



$$\Theta=\frac{1-2\nu}{E}(\sigma_m+\sigma_m+\sigma_m)$$

$$=\frac{1-2\nu}{E}(\sigma_1+\sigma_2+\sigma_3)$$

体积改变，
形状不变

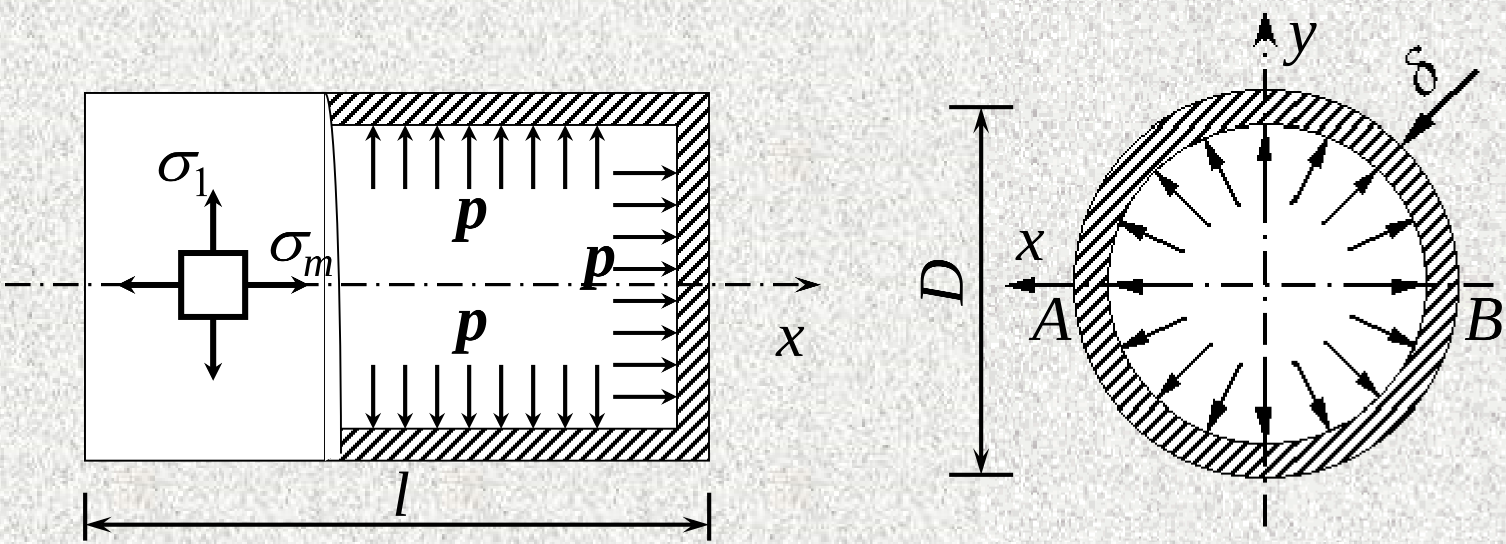


$$\Theta=\frac{1-2\nu}{E}(\sigma_1'+\sigma_2'+\sigma_3')$$

$$=0$$

体积不变，
形状改变

例8 图a所示为承受内压的薄壁容器。为测量容器所承受的内压力值，在容器表面用电阻应变片测得环向应变$\varepsilon_t=350\times10^6$，若已知容器平均直径$D$=500 mm，壁厚$\delta$=10 mm，容器材料的$E$=210GPa，$\mu$=0.25，试求:1. 导出容器横截面和纵截面上的正应力表达式；2. 计算容器所受的内压力。



图a

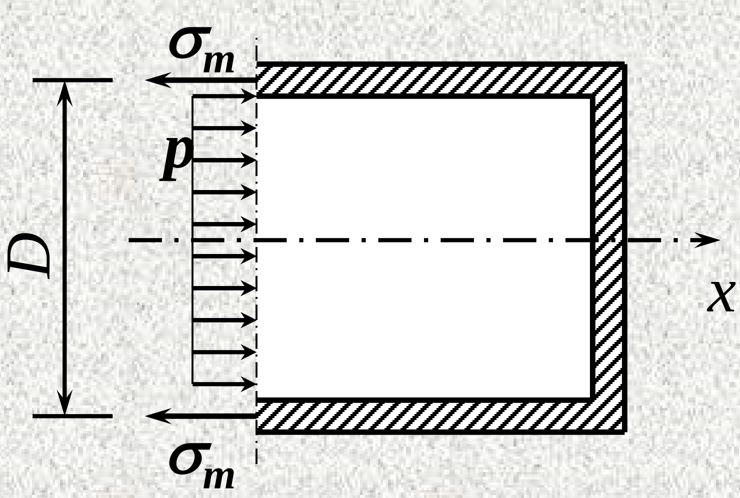
解：容器的环向和纵向应力表达式

1、轴向应力:（longitudinal stress）

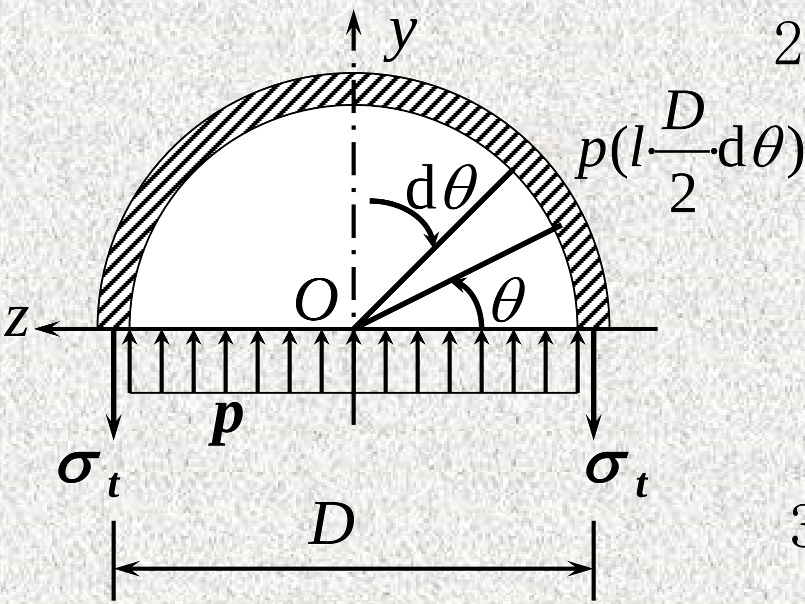
用横截面将容器截开，受力如图b所示, 根据平衡方程

$$\sigma_m(\pi D\delta)=p\times\pi D^2/4$$

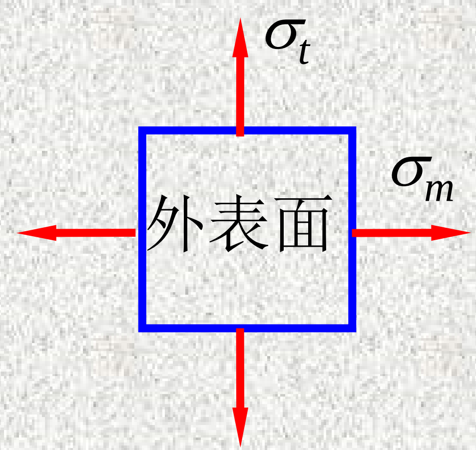
$$\sigma_m=\frac{pD}{4\delta}$$



图b



图c



2、环向应力：（hoop stress）

用纵截面将容器截开，受力如图c所示

$$\sigma_t(l\times 2\delta)=p\times Dl \qquad \sigma_t=\frac{pD}{2\delta}$$

3、求内压（以应力应变关系求之）

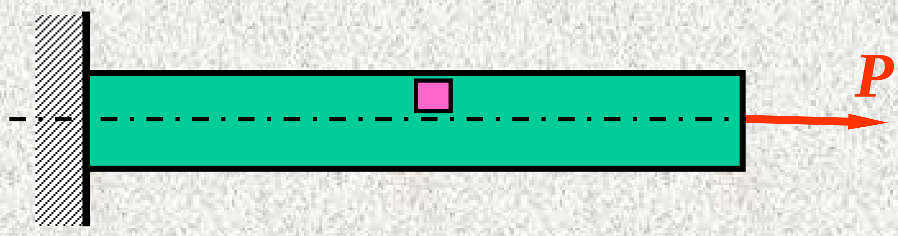
$$\varepsilon_t=\frac{1}{E}[\sigma_t-\mu\sigma_m]=\frac{pD}{4\delta E}[2-\mu]$$

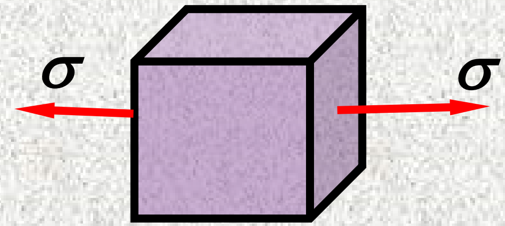
$$p=\frac{4\delta E\varepsilon_t}{D(2-\mu)}$$

$$=\frac{4\times 210\times 10^9\times 0.01\times 350\times 10^{-6}}{0.5\times(2-0.25)}=3.36\text{MPa}$$

§7—9 复杂应力状态下的比能

一、单向应力状态下的应变能



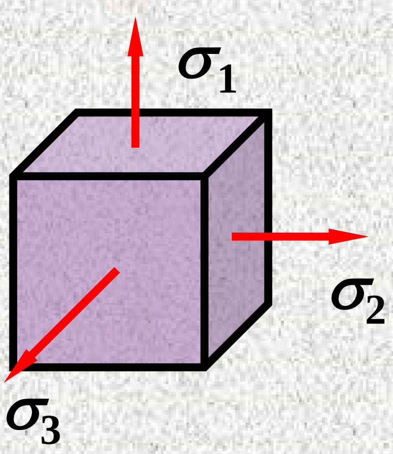


杆件的变形能：
$$U=\frac{1}{2}P\cdot\Delta l$$

比能：
$$u=\frac{U}{V}=\frac{\frac{1}{2}P\cdot\Delta l}{Al}=\frac{1}{2}\cdot\frac{P}{A}\cdot\frac{\Delta l}{l}=\frac{1}{2}\cdot\sigma\cdot\varepsilon$$

$$\therefore\ u=\frac{1}{2}\sigma\cdot\varepsilon$$

二、复杂应力状态下的比能



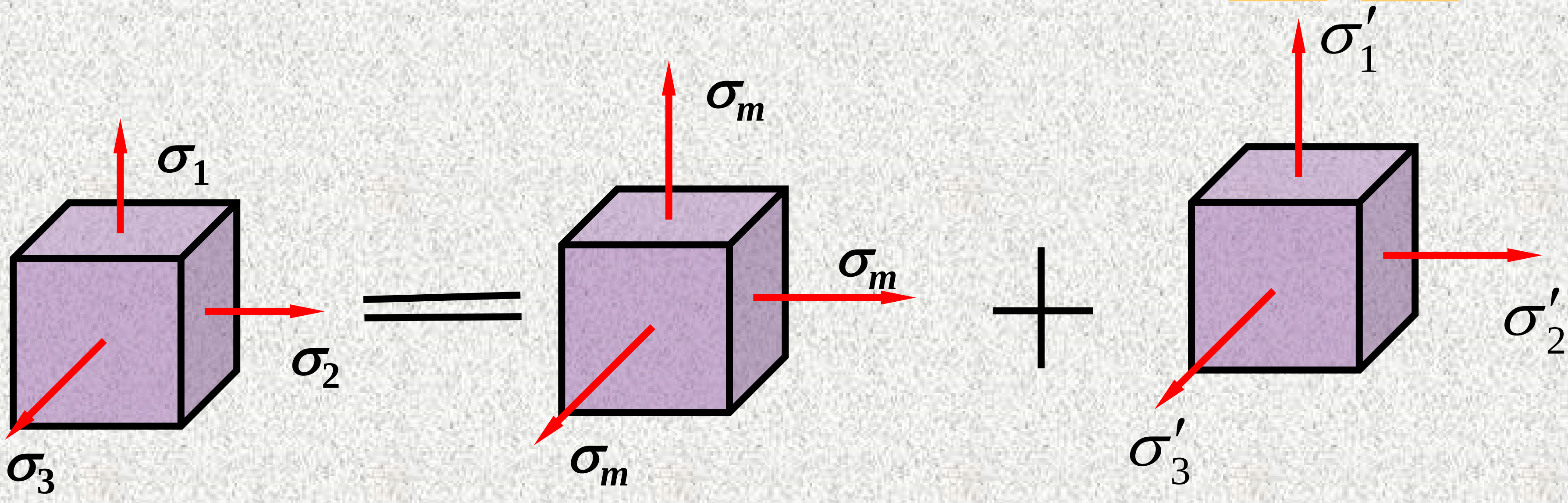
$$u=\frac{1}{2}\sigma_1\varepsilon_1+\frac{1}{2}\sigma_2\varepsilon_2+\frac{1}{2}\sigma_3\varepsilon_3$$

由广义胡克定律

$$\begin{cases}\varepsilon_1=\dfrac{1}{E}[\sigma_1-\nu(\sigma_2+\sigma_3)]\\ \varepsilon_2=\dfrac{1}{E}[\sigma_2-\nu(\sigma_3+\sigma_1)]\\ \varepsilon_3=\dfrac{1}{E}[\sigma_3-\nu(\sigma_2+\sigma_1)]\end{cases}$$

代入上式得：

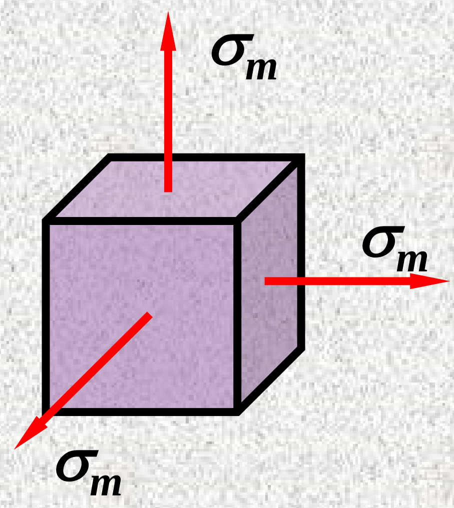
$$u=\frac{1}{2E}\left[\sigma_1^2+\sigma_2^2+\sigma_3^2-2\nu(\sigma_1\sigma_2+\sigma_3\sigma_2+\sigma_1\sigma_3)\right]$$



体积改变比能

形状改变比能

体积改变比能



$$u_v = \frac{1}{2}\sigma_m\varepsilon_m + \frac{1}{2}\sigma_m\varepsilon_m + \frac{1}{2}\sigma_m\varepsilon_m$$

$$= \frac{3}{2}\sigma_m\varepsilon_m$$

$$\because\ \varepsilon_m = \frac{1}{E}[\sigma_m - \nu(\sigma_m + \sigma_m)] = \frac{1-2\nu}{E}\sigma_m$$

$$\therefore\ u_v = \frac{3(1-2\nu)}{2E}\sigma_m^2 = \frac{1-2\nu}{6E}(\sigma_1+\sigma_2+\sigma_3)^2$$

形状改变比能

$$u_x = u - u_v$$

$$= \frac{1+\nu}{6E}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right]$$