PROJECT: WOLF POPULATION MANAGEMENT

JASON HU AND QIAQIA JI

ABSTRACT. We modelled the change of wolf population in a hypothetical habitat under progressively more complicated assumptions, in order to apply our knowledge of ordinary differential equations.

1. Without harvesting penalty

1.1. Substitutions for K and M.

To find out the expression for K, the carrying capacity of the habitat, notice that wolf population decreases when its density exceeds 1 pack per 25 square miles, and vice versa. Knowing that a pack constitutes 10 wolves on average and the habitat occupies A square miles, it can be inferred that

$$K = \frac{10A}{25} = \frac{2A}{5}. \tag{1.1}$$

Similarly, to find out the expression for M, the critical minimum population, notice that the wolf population will decline when the density dips below 1 wolf per 25 square miles. It can be inferred that

$$M = \frac{A}{25}. \tag{1.2}$$

Given the original differential equation

$$\frac{dN}{dt} = rN(1 - \frac{N}{K})(\frac{N}{M} - 1), \tag{1.3}$$

we can substitute (1.3) with (1.1) and (1.2), so we have

$$\frac{dN}{dt} = rN(1 - \frac{5N}{2A})(\frac{25N}{A} - 1). \tag{1.4}$$

1.2. Classification of differential equation.

As we can see in Equation (1.4), on the left side is a first derivative of population over time, whereas on the right side, the dependent variable N is multiplied three times. Also, it is obvious that the independent variable t does not show up in (1.4). Thus we can conclude that the ODE is first-order, nonlinear, and autonomous.

In order to find the equilibrium solutions, let equation (1.4) be zero:

$$\frac{dN}{dt} = rN(1 - \frac{5N}{2A})(\frac{25N}{A} - 1) = 0 \tag{1.5}$$

The three roots we get are $N = 0$, $N = M = \frac{A}{25}$, and $N = K = \frac{2A}{5}$, for any non-specific values of r and A.

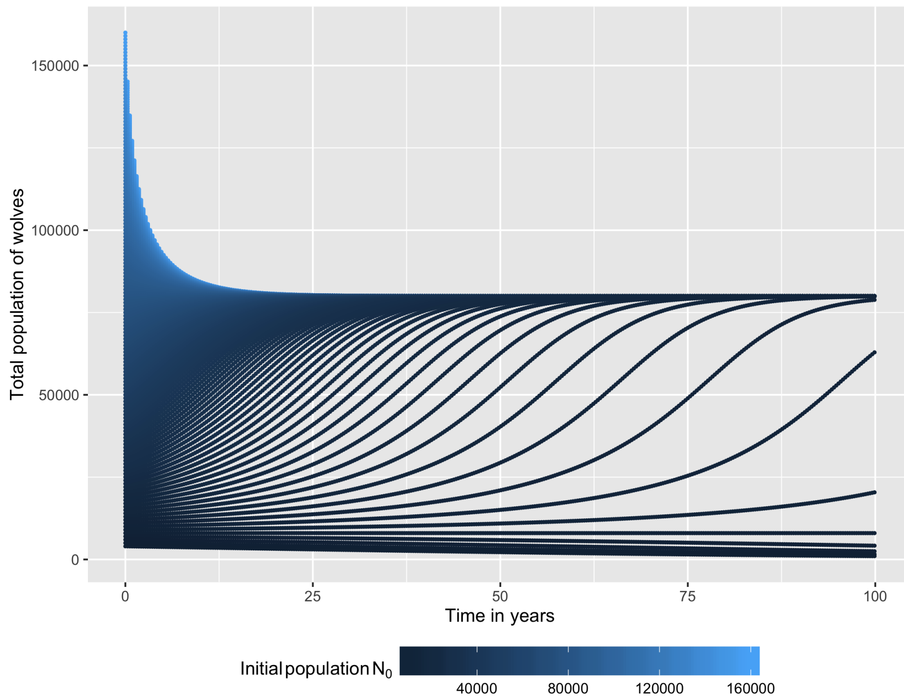

FIGURE 1. Plot for population N versus t, with a range of initial population

1.3. **Plot for N with different initial populations.** Closed form solution to differential equation (1.5) is beyond our knowledge, so we chose to use neumerical methods for approximation, specifically Runge-Kutta method (rk4).

Given parameter values $r = 0.02$ and $A = 200000$, we can plot N versus t for a wide range of initial population N_0, shown in Figure 1.

2. WITH CONSTANT HARVESTING PENALTY

2.1. **Qualitative analysis.** Given original differential equation

$$\frac{dN}{dt} = rN(1 - \frac{5N}{2A})(\frac{25N}{A} - 1) - h \tag{2.1}$$

To construct the bifurcation diagram, let equation (2.1) be zero and we have

$$h = rN(1 - \frac{5N}{2A})(\frac{25N}{A} - 1) \tag{2.2}$$

The solution set (h, N) of equation (2.2) is the bifurcation graph. Notice that parameter h is an affine term to the differential equation. It's safe to proclaim that the bifurcation graph should look like a x-y mirror of a third order polynomial.

Equilibrium solutions to original differential equation (2.1) is therefore all the points (h, N) on the bifurcation diagram.

To find bifurcation values of h^*, we can solve for all the local extrema (N^*, h^*) for polynomial (2.2), where h^* are all the bifurcation values. There are two solutions to polynomial (2.2), so there are two bifurcation values.

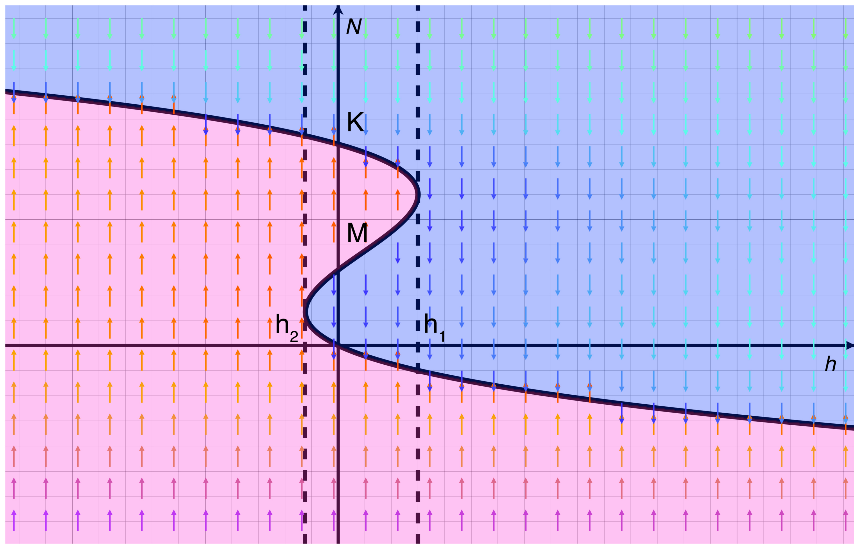

FIGURE 2. Sketch of the bifurcation diagram of the equation (2.1)
SINK? NODE?

A rough sketch of the bifurcation diagram is presented in Figure 2. $\frac{dN}{dt}$ is negative in the blue region with downward arrows and is positive in the red region with upward arrows. h_1 and h_2 are two bifurcation values. K and M are the carrying capacity and minimum population when $h = 0$.

2.2. **Search for the maximum h that preserves wolves in 100 years.** Closed form solution to differential equation 2.1 is again beyond our knowledge, so we used Runge-Kutta method to approximate $N(100)$ for different h. To find the maximum h so that $N(100) > 0$, note that $N(100)$ can be seen as a function of h, and we are looking for a root of function $N(100, h)$.

We can use some optimization scheme to find the root for function $N(100, h)$, but first we need to convince ourselves that there is only one root h for $N(100, h)$. Note that $\frac{dN}{dt}$ is continuous on N and h, and $\frac{dN}{dt}$ is negative for $N > 0, h > h_1$. For a given initial population N_0, as h continuously increases, $N(100, h)$ continuously decreases to the equilibrium point.

Given initial population $N(0) = 120000 > K$, we know that a root h for $N(100, h) = 0$ exists only when $h > h_1$, where $N(100, h)$ monotonically decreases as stated above. We conclude that there exists one and only one root h for $N(100, h) = 0$.

To search for maximal h such that $N(100, h) > 0$, it's possible to modify Runge-Kutta methods for a computationally efficient algorithm. But here we decided to use brute force instead. We relied on Runge-Kutta method to calculate $N(100, h)$ for every given h. We optimized h simply using binary search. Suppose we start with $h = 10000$ and aim for a one-decimal-place precision, given that $\log_2 10000 \approx 16.6$, the optimization is feasible. Results of the optimization is presented in Table 1.

Our optimization shows that the maximal harvest rate is $h \approx 2366.2$ wolves per year such that the wolf population does not deplete in 100 years.

TABLE 1

	h	$N(100, h)$
1	5000.00	−33721.89
2	2500.00	−22556.96
3	1250.00	71078.98
4	1875.00	62375.55
5	2187.50	47968.03
6	2343.75	9102.03
7	2421.88	−15772.75
8	2382.81	−5957.67
9	2363.28	1154.78
10	2373.05	−2553.67
11	2368.16	−732.39
12	2365.72	203.70
13	2366.94	−266.32
14	2366.33	−31.79
15	2366.03	85.83
16	2366.18	26.99
17	2366.26	−2.41

3. Environmental threat and harvesting scheme

3.1. Differential equation with human encroachment and climate fluctuations. We can model the proportional decrease of habitat with expression

$$A(t) = A_0 d^t. \tag{3.1}$$

Consequently, M and K are changed due to dependence on A:

$$M(t) = \frac{A}{25} = \frac{A_0 d^t}{25}, \tag{3.2}$$

$$K(t) = \frac{2A}{5} = \frac{2A_0 d^t}{5}. \tag{3.3}$$

Furthermore, we can model the impact of climate fluctuation with expression

$$K(t) = \frac{10A_0 d^t}{-\frac{75}{2}\cos(\frac{\pi}{11}t) + \frac{125}{2}} = \frac{4A_0 d^t}{-15\cos(\frac{\pi}{11}t) + 25}. \tag{3.4}$$

So we can modify the original differential equation (2.1) to be

$$\frac{dN}{dt} = rN\left(1 + \frac{15N\cos(\frac{\pi}{11}t) - 25N}{4A_0d^t}\right)\left(\frac{25N}{A_0d^t} - 1\right) - h \tag{3.5}$$

3.2. **Search for maximal harvesting rate h under harsher conditions.** Again we will use Runge-Kutta method combined with binary search, to a precision of one decimal place. We have assumed the parameter for encroachment proportion per year $d = 0.99$. The optimization results are presented in Table 2.

Table 2

	h	$N(100, h)$
1	5000.00	-14516.73
2	2500.00	-11063.22
3	1250.00	-8382.14
4	625.00	-6208.68
5	312.50	6982.19
6	468.75	-3927.95
7	390.62	1534.07
8	429.69	-1706.57
9	410.16	-159.06
10	400.39	680.42
11	405.27	257.43
12	407.71	48.19
13	408.94	-55.70
14	408.33	-3.82

As a result, the maximum rate of harvest under harsh environmental constraints is smaller than that without, which is about $h \approx 408.2$ wolves per year.

3.3. **Design of variable harvesting scheme.** We are asked to maintain the wolf population to be $N_{target} \approx 40000$, that is, when $N = N_{target}$, it's ideal that $\frac{dn}{dt} = 0$. Substitute this condition into expression (3.5) and eliminate parameters, we have

$$\frac{dN}{dt} = 0.02 \cdot 40000\left(1 + \frac{15 \cdot 40000\cos(\frac{\pi}{11}t) - 25 \cdot 40000}{4 \cdot 200000d^t}\right)\left(\frac{25 \cdot 40000}{200000d^t} - 1\right) - h. \tag{3.6}$$

Assume that $d = 0.99$, let expression (3.6) be zero, we have:

$$h(t) = 800\left(1 + \frac{3\cos(\frac{\pi}{11}t) - 5}{4 \cdot 0.99^t}\right)\left(\frac{5}{0.99^t} - 1\right) \tag{3.7}$$

Expression (3.7) represents the harvest scheme that will eventually maintain wolf population to be 40000, if it ever reaches there. We can use Runge-Kutta methods to verify this solution. The approximation process is charted in Table 3, and the plot of population against time is provided in Figure 3.

We can see that indeed the population was indeed maintained at 40000, and the population was less variable compared to that under no harvesting scheme.

TABLE 3

Time Step	Time	$N(t)$
0	0	120000.00
20	4	72721.74
40	8	47459.88
60	12	41137.40
80	16	40231.04
100	20	40129.27
120	24	40131.29
140	28	40059.65
160	32	40005.41

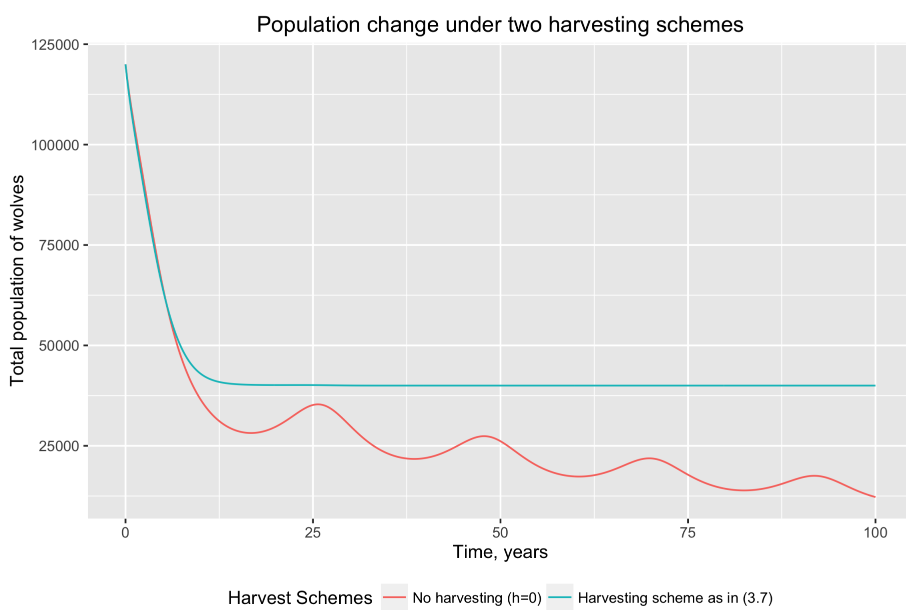

FIGURE 3. Plot for population change under two harvesting schemes