Math 452: Partial Differential Equations

Drying Time for Wood Logs with Different Geometries

Jason Hu and Qiaqia Ji

Introduction

- In this project, we use both analytical and numerical solutions of PDEs to explore the drying time for wood logs with different shapes and sizes.
- Usually, freshly cut wood contains approximately 50% of moisture content. Wood is considered ready to burn if the moisture content is below 20%, since wood with minimum moisture content is desirable to burn.
- Moisture in wood logs can diffuse longitudinally (along the length of the wood) and angularly (within each tree ring), but not radially (across tree ring).

General Diffusion PDE for Wood Logs

- Since there is no moisture diffusion radially, the heat equation in cylindrical coordinates reduces to

$$u_t = \frac{k}{r^2} u_{\phi\phi} + k u_{zz} \tag{1}$$

Diffusion in Unsplit Logs

- The wood log in our model has length of 0.5 meters and diameter of 0.25 meters. The diffusivity constant is $k = 0.013024368 m^2/s$.
- Unsplit wood logs are symmetric on angular dimension ϕ, so the moisture does not diffuse angularly, and the PDE only depends on longituinal dimension z and time t.
- Environment moisture is T(t), a sinusoidal function with period of 1 year, maximum at 30% and minimum at 2%.
- Moisture content at the ends of the logs match environment moisutre, which implies Dirichlet boundary conditions

$$\begin{cases} u_t = k u_{zz} & (2) \\ u(z,0) = 0.5 & (3) \\ u(0,t) = 0.14\sin(2\pi t) + 0.16 & (4) \\ u(0.5,t) = 0.14\sin(2\pi t) + 0.16 & (5) \end{cases}$$

Analytical Solution

- The final solution is

$$u(z,t) = \sum_{n=1}^{\infty} A_n(t) Z_n(z) T_n(t) \tag{6}$$

where

$$Z_n(z) = \sin(2n\pi z)$$
$$T_n(t) = e^{-4kn^2\pi^2 t}$$
$$\lambda_n = -4n^2\pi^2$$

$$A_n(t) = -0.28\pi N_n \cdot \left[\frac{e^{-k\lambda t}[-k\lambda\cos(2\pi t) + 2\pi\sin(2\pi t)]}{4\pi^2 + k^2\lambda^2}\right]_0^t + c_n$$

$$N_n = \frac{\int_0^{0.5}\sin(2\pi nz)dz}{\int_0^{0.5}\sin^2(2\pi nz)dz}$$

$$c_n = 0.34 N_n$$

- $Z_n(z)$ is the eigenfunction. λ_n is the eigenvalue.

Finite Sum Approximation

- The analytic solution consists of an infinite sum, which cannot be calculated exactly.
- We use the partial sum of the first 20 terms to approximate the solution.

Numerical Approximation

- We can use numerical methods to approximate the PDE. We used R language and reacTran package.
- We created grids of size $100 \times 100 \times 100$ in dimension r, ϕ, z with 200 time steps.
- The reacTran package automatically selects implicit method or explicit method depending on the stiffness of the PDE equation.
- The R code can be found on github.com/fuchai/PDE_project

Plot of Drying Time for Unsplit Logs

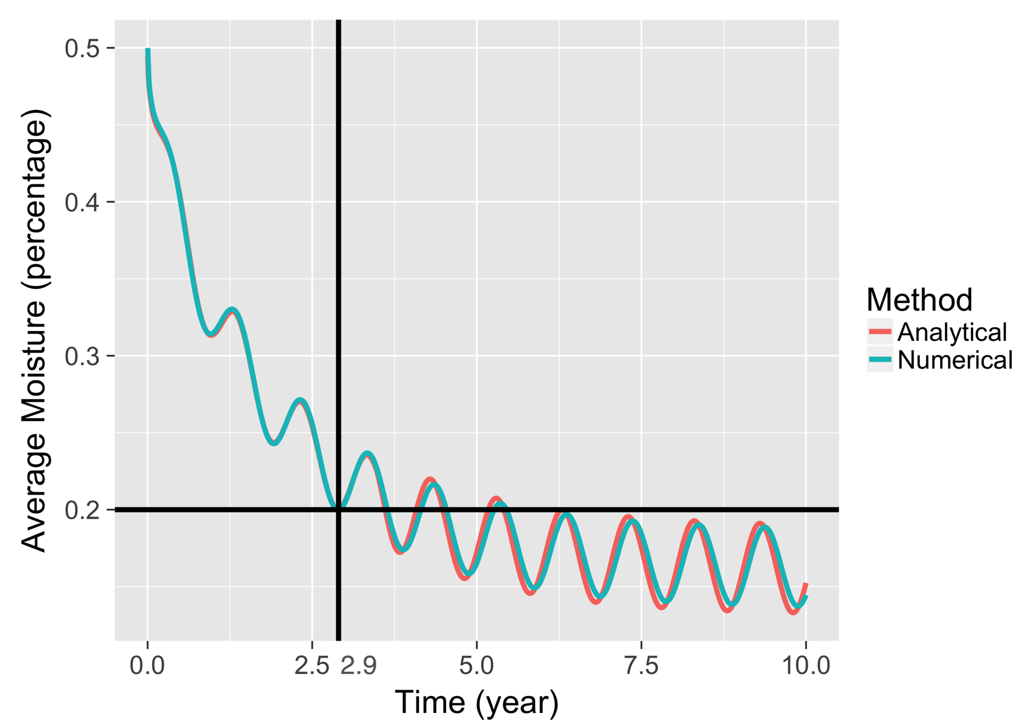

Figure 1: Finite sum approximation and numerical method approximation for unsplit wood logs.

- We plot the drying time needed for unsplit wood logs, and compare the results from finite sum approximation and numerical methods approximation, in Figure 1.
- Finite sum approximation and numerical approximation align well with each other.
- The first time unsplit wood logs reaches the 20% moisture is 2.9 year.

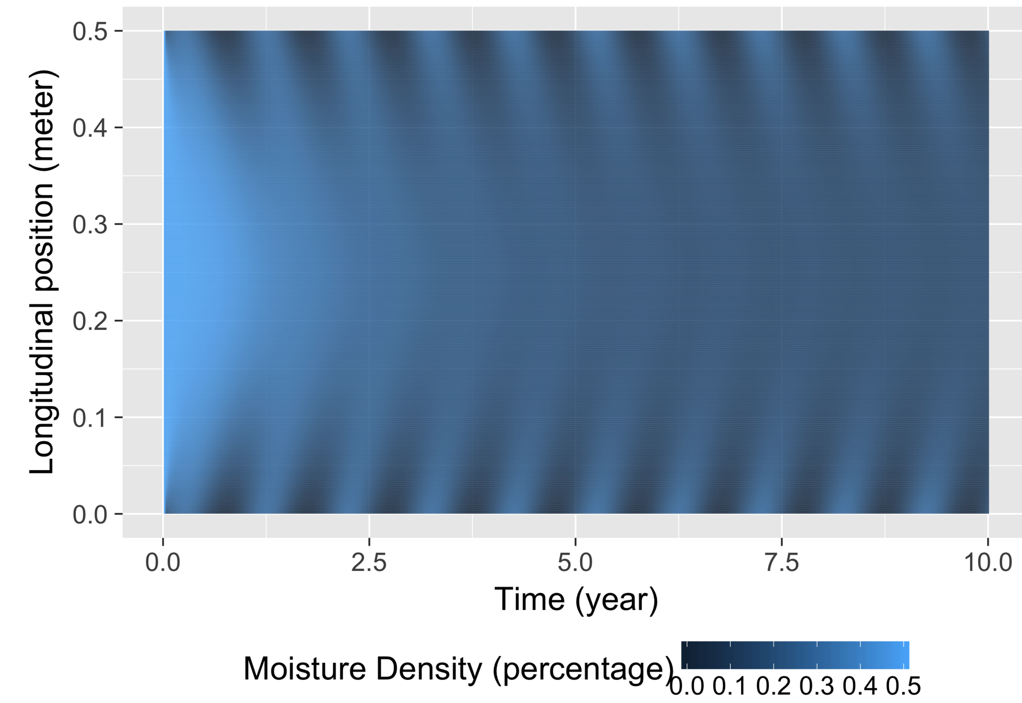

Figure 2: The changes of mositure density in the unsplit wood logs creates a ripple-like pattern in longitude and time dimension.

Diffusion in Split Wood Logs

Moisture Equations in Half-Split Logs

$$\begin{cases} u_t = \frac{k}{r^2}u_{\phi\phi} + k u_{zz} & (7) \\ u(\phi,z,0) = 0.5 & (8) \\ u(0,z,t) = 0.14\sin(2\pi t) + 0.16 & (9) \\ u(\pi,z,t) = 0.14\sin(2\pi t) + 0.16 & (10) \\ u(\phi,0,t) = 0.14\sin(2\pi t) + 0.16 & (11) \\ u(\phi,0.5,t) = 0.14\sin(2\pi t) + 0.16 & (12) \end{cases}$$

- Using similar methods, we derive the PDE equations and boundary conditions for half-split wood logs.
- Quarter-split wood logs have similar conditions. The only difference is that the boundary condition (9) is $\phi = \pi/2$, so we have omitted the analytic solution for quarter-split wood logs.

Analytic Solution for Half-Split Logs

- Let $\lambda = \alpha + \beta$, the eigenvalue is $\lambda(n,m) = -4n^2 - 4m^2\pi^2$, where $\alpha = -4n^2$, $\beta = -4m^2\pi^2$.
- Let $x = \phi r$, the eigenfunctions are $X_n(x) = \sin(\sqrt{-\alpha})$, $Z_m(z) = \sin(\sqrt{-\beta})$, $T_{n,m}(t) = e^{k\lambda t}$.
- The solution for moisture in half-split wood logs is

$$u(x,z,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_n(t) X_n(x) Z_m(z) T_{n,m}(t) \tag{13}$$

where

$$A_{n,m}(t) = -0.28\pi N_n \cdot \left[\frac{e^{-k\lambda t}[-k\lambda\cos(2\pi t) + 2\pi\sin(2\pi t)]}{4\pi^2 + k^2\lambda^2}\right]_0^t + c_n$$

$$N_{n,m} = \frac{\int_0^{0.5}\int_0^{\pi}\sin(\sqrt{-\alpha x})\sin(\sqrt{-\beta z})dxdz}{\int_0^{0.5}\int_0^{\pi}\sin^2(\sqrt{-\alpha x})\sin^2(\sqrt{-\beta z})dxdz}$$

$$c_{n,m} = 0.34 N_{n,m}$$

Comparison of Drying Time for Wood Logs with Different Splits

- We use numerical methods to calculate the average moisture for different splits, plotted in Figure 3.
- Quarter-split wood logs dry the fastest, followed by half-split logs, and unsplit logs dries the slowest.
- Environment moisture has bigger impact than split shapes for half-split and quarter-split logs.

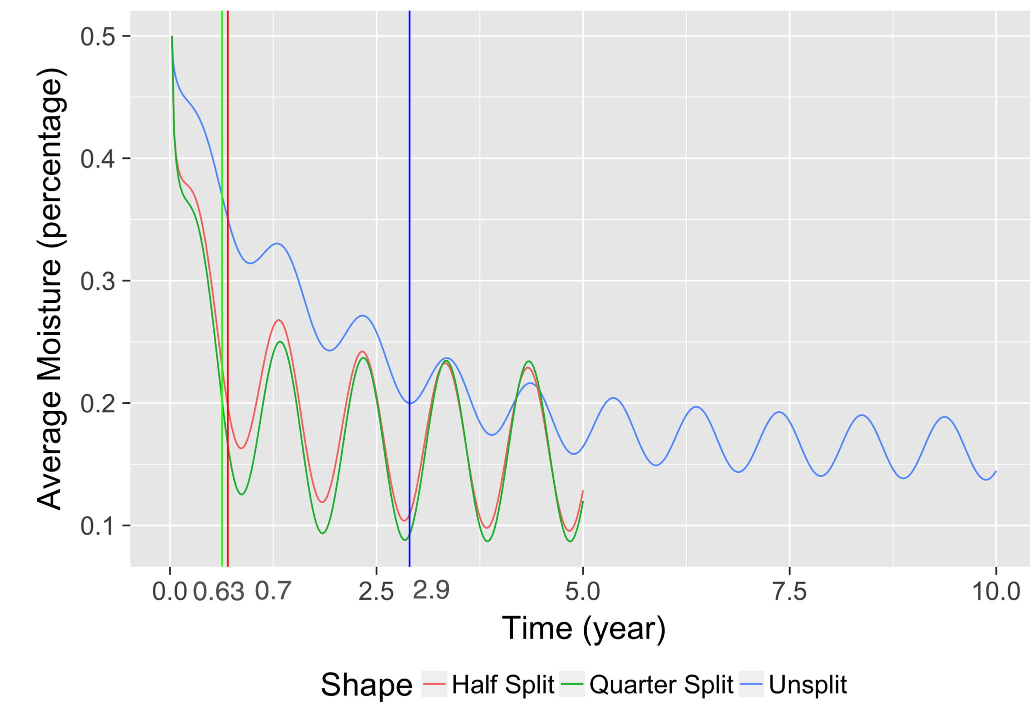

Figure 3: Comparison of the change in average moisture for wood logs with different splits.

Shape	Unsplit	Half-Split	Quarter-Split
Year	2.9	0.7	0.63

Table 1: Comparison of the time needed to reach 20% for the first time for wood logs with different splits.

Variation in Wood Log Sizes

- We vary the dimensions of the wood logs and compare their drying time. The parameters are $\frac{r_{big}}{r_{normal}} = \frac{z_{big}}{z_{normal}} = \frac{r_{normal}}{r_{small}} = \frac{z_{normal}}{z_{small}} = \frac{r_{wide}}{r_{normal}} = \frac{z_{long}}{z_{normal}} = 2$.
- Small logs in general dry faster.
- Change in radius and longitude has similar effects.
- The oscillation amplitude of average moisture in bigger logs is smaller.

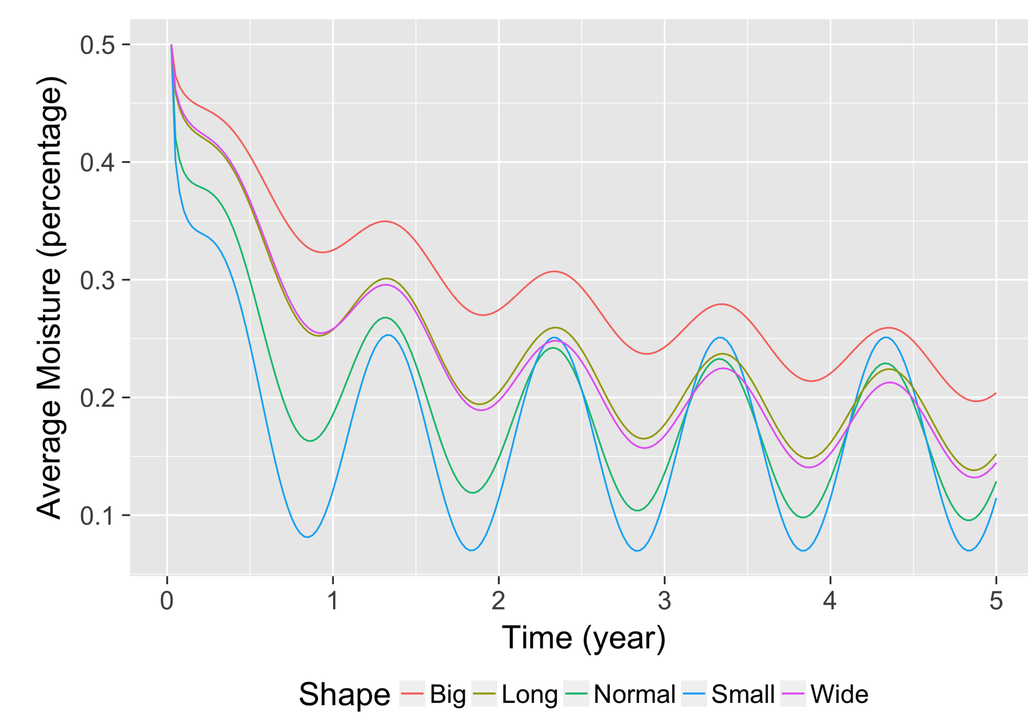

Figure 4: Change of average moisture for wood logs of different sizes.

Advisor: Kyle Fey