20220318课堂练习

答案

1.设$f(u,v)$具有二阶连续偏导数，且满足$f''_{uu}+f''_{vv}=1$，又$g(x,y)=f\left(xy,\frac{1}{2}(x^2-y^2)\right)$，求$g''_{xx}+g''_{yy}$

- 令$u=xy, v=\frac{1}{2}(x^2-y^2)$，则
- $g'_x=f'_u u'_x+f'_v v'_x=yf'_u+xf'_v$
- $g''_{xx}=(g'_x)'_x=(yf'_u+xf'_v)'_x$
- $=y(f'_u)'_x+x(f'_v)'_x+f'_v$ (偏导数的四则运算法则)
- $=y((f'_u)'_u u'_x+(f'_u)'_v v'_x)+x((f'_v)'_u u'_x+(f'_v)'_v v'_x)+f'_v$ (链式法则)
- $=y(yf''_{uu}+xf''_{uv})+x(yf''_{uv}+xf''_{vv})+f'_v$
- $=y^2f''_{uu}+2xyf''_{uv}+x^2f''_{vv}+f'_v$

1.设$f(u,v)$具有二阶连续偏导数，且满足$f''_{uu}+f''_{vv}=1$，又$g(x,y)=f\left(xy,\frac{1}{2}(x^2-y^2)\right)$，求$g''_{xx}+g''_{yy}$

- $g'_y=f'_u u'_y+f'_v v'_y=xf'_u-yf'_v$
- $g''_{yy}=\left(g'_y\right)'_y=(xf'_u-yf'_v)'_y$
- $=x(f'_u)'_y-y(f'_v)'_y-f'_v$ (偏导数的四则运算法则)
- $=x\left((f'_u)'_u u'_y+(f'_u)'_v v'_y\right)-y\left((f'_v)'_u u'_y+(f'_v)'_v v'_y\right)-f'_v$ (链式法则)
- $=x(xf''_{uu}-yf''_{uv})-y(xf''_{uv}-yf''_{vv})-f'_v$
- $=x^2f''_{uu}-2xyf''_{uv}+y^2f''_{vv}-f'_v$

1.设$f(u,v)$具有二阶连续偏导数，且满足$f''_{uu}+f''_{vv}=1$，又$g(x,y)=f\left(xy,\frac{1}{2}(x^2-y^2)\right)$，求$g''_{xx}+g''_{yy}$

- $g''_{xx}+g''_{yy}$
- $=y^2f''_{uu}+2xyf''_{uv}+x^2f''_{vv}+f'_v+x^2f''_{uu}-2xyf''_{uv}+y^2f''_{vv}-f'_v$
- $=(x^2+y^2)(f''_{uu}+f''_{vv})$
- $=x^2+y^2$

2.设$z=z(x,y)$具有连续的二阶偏导数，若用变换$\begin{cases}u=x-2y\\v=x+ay\end{cases}$可以把方程$6\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}-\frac{\partial^2 z}{\partial y^2}=0$化简为$\frac{\partial^2 z}{\partial u\partial v}=0$，求$a$的值

- $\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}$
- $\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=-2\frac{\partial z}{\partial u}+a\frac{\partial z}{\partial v}$
- $\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)$
- $=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)+\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$
- $=\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}+\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}$
- $=\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}$

2.设$z=z(x,y)$具有连续的二阶偏导数，若用变换$\begin{cases}u=x-2y\\v=x+ay\end{cases}$可以把方程$6\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}-\frac{\partial^2 z}{\partial y^2}=0$化简为$\frac{\partial^2 z}{\partial u\partial v}=0$，求$a$的值

- $\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)$
- $=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)+\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)$
- $=\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}+\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}$
- $=-2\frac{\partial^2 z}{\partial u^2}+(a-2)\frac{\partial^2 z}{\partial u\partial v}+a\frac{\partial^2 z}{\partial v^2}$

2.设$z=z(x,y)$具有连续的二阶偏导数，若用变换$\begin{cases}u=x-2y\\v=x+ay\end{cases}$可以把方程$6\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}-\frac{\partial^2 z}{\partial y^2}=0$化简为$\frac{\partial^2 z}{\partial u\partial v}=0$，求$a$的值

- $\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(-2\frac{\partial z}{\partial u}+a\frac{\partial z}{\partial v}\right)$
- $=-2\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)+a\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)$
- $=-2\left(\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right)+a\left(\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right)$
- $=-2\left(-2\frac{\partial^2 z}{\partial u^2}+a\frac{\partial^2 z}{\partial u\partial v}\right)+a\left(-2\frac{\partial^2 z}{\partial u\partial v}+a\frac{\partial^2 z}{\partial v^2}\right)$
- $=4\frac{\partial^2 z}{\partial u^2}-4a\frac{\partial^2 z}{\partial u\partial v}+a^2\frac{\partial^2 z}{\partial v^2}$

2.设$z=z(x,y)$具有连续的二阶偏导数，若用变换$\begin{cases}u=x-2y\\v=x+ay\end{cases}$可以把方程$6\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}-\frac{\partial^2 z}{\partial y^2}=0$化简为$\frac{\partial^2 z}{\partial u\partial v}=0$，求$a$的值

- $6\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}-\frac{\partial^2 z}{\partial y^2}$
- $=6\left(\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}\right)+\left(-2\frac{\partial^2 z}{\partial u^2}+(a-2)\frac{\partial^2 z}{\partial u\partial v}+a\frac{\partial^2 z}{\partial v^2}\right)-\left(4\frac{\partial^2 z}{\partial u^2}-4a\frac{\partial^2 z}{\partial u\partial v}+a^2\frac{\partial^2 z}{\partial v^2}\right)$
- $=(10+5a)\frac{\partial^2 z}{\partial u\partial v}+(6+a-a^2)\frac{\partial^2 z}{\partial v^2}$

2.设$z=z(x,y)$具有连续的二阶偏导数，若用变换$\begin{cases}u=x-2y\\v=x+ay\end{cases}$可以把方程$6\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}-\frac{\partial^2 z}{\partial y^2}=0$化简为$\frac{\partial^2 z}{\partial u\partial v}=0$，求$a$的值

- 要将$6\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}-\frac{\partial^2 z}{\partial y^2}=0$化简为$\frac{\partial^2 z}{\partial u\partial v}=0$必须满足：
- $\begin{cases}10+5a\neq 0\\6+a-a^2=0\end{cases}$
- 所以：
- $a=3$

3.设$f(x,y)$二阶偏导数连续，且满足$f''_{xx}-f''_{yy}=0$，$f(x,2x)=x$，$f'_x(x,2x)=x^2$，求$f''_{xx}(x,2x)$, $f''_{xy}(x,2x)$

- 等式$f(x,2x)=x$两边关于自变量x求导，得:
- $\frac{d}{dx}\left(f(x,2x)\right)=1$
- 用链式法则将左边展开，得:
- $f'_x(x,2x)+2f'_y(x,2x)=1$
- 两边再次关于自变量x求导，并运用链式法则，得:
- $f''_{xx}(x,2x)+2f''_{xy}(x,2x)+2\left(f''_{xy}(x,2x)+2f''_{yy}(x,2x)\right)=0$
- 将$f''_{xx}-f''_{yy}=0$代入并化简
- $5f''_{xx}(x,2x)+4f''_{xy}(x,2x)=0$

3.设$f(x,y)$二阶偏导数连续，且满足$f''_{xx}-f''_{yy}=0$，$f(x,2x)=x$，$f'_x(x,2x)=x^2$，求$f''_{xx}(x,2x)$，$f''_{xy}(x,2x)$

- 等式$f'_x(x,2x)=x^2$两边关于自变量x求导，并运用链式法则，得：
- $f''_{xx}(x,2x)+2f''_{xy}(x,2x)=2x$
- 求解方程组：
- $\begin{cases}5f''_{xx}(x,2x)+4f''_{xy}(x,2x)=0\\ f''_{xx}(x,2x)+2f''_{xy}(x,2x)=2x\end{cases}$
- 得
- $\begin{cases}f''_{xx}(x,2x)=-\dfrac{4}{3}x\\ f''_{xy}(x,2x)=\dfrac{5}{3}x\end{cases}$

4.证明：曲面$F(x-az,y-bz)=0$的切平面与某一给定的直线平行，其中a,b为常数

- 令$u=x-az,v=y-bz$
- 那么切平面的法向量为
- $\vec{n}=(F_u',F_v',-aF_u'-bF_v')$
- 设某一直线的方向向量为(l,m,n)，直线和平面平行，则
- $(F_u',F_v',-aF_u'-bF_v')\cdot(l,m,n)=0$
- $(l-na)F_u'+(m-nb)F_v'=0$
- 取$n=1,l=a,m=b$，则曲面上的所有切平面与以$(a,b,1)$为方向向量的直线平行

5.设椭球面$x^2+2y^2+3z^2=21$在某点处的切平面经过直线$\frac{x-6}{2}=\frac{y-3}{1}=\frac{2z-1}{-2}$，求此切平面方程

- 设椭球面上一点(x_0,y_0,z_0)的切平面经过直线$\frac{x-6}{2}=\frac{y-3}{1}=\frac{2z-1}{-2}$
- (x_0,y_0,z_0)处的切平面的法向量为$\vec{n}=(2x_0,4y_0,6z_0)$，提取出系数2，将法向量记为$\vec{n}=(x_0,2y_0,3z_0)$
- 将直线方程$\frac{x-6}{2}=\frac{y-3}{1}=\frac{2z-1}{-2}$标准化，得
- $\frac{x-6}{2}=\frac{y-3}{1}=\frac{z-\frac{1}{2}}{-1}$
- 因为直线$\frac{x-6}{2}=\frac{y-3}{1}=\frac{z-\frac{1}{2}}{-1}$在平面上，则
- $(x_0,2y_0,3z_0)\cdot(2,1,-1)=0$
- $(x_0,2y_0,3z_0)\cdot\left(x_0-6,y_0-3,z_0-\frac{1}{2}\right)=0$

5.设椭球面$x^2+2y^2+3z^2=21$在某点处的切平面经过直线$\frac{x-6}{2}=\frac{y-3}{1}=\frac{2z-1}{-2}$，求此切平面方程

- 求解方程组
- $\begin{cases} 2x_0+2y_0-3z_0=0 \\ 6x_0+6y_0+\frac{3}{2}z_0=21 \\ x_0^2+2y_0^2+3z_0^2=21 \end{cases}$
- 得到两个点
- $(3,0,2),(1,2,2)$
- 所以满足条件的切平面方程有两个:
- $x+2z-7=0$以及$x+4y+6z-21=0$

6.设$z = z(x,y)$是由方程$e^z - xy^3 + z^2 = 2$在点$(-1,1,0)$附近所定义的隐函数，问在点$(-1,1,0)$处自变量(x,y)沿着哪个方向变化时，函数值$z(x,y)$增加的速度最快，并求出该点处沿着此方向的方向导数

- 函数值增加最快的方向是梯度的方向
- 方程$e^z - xy^3 + z^2 = 2$两边关于自变量x求偏导，得:
- $e^z \cdot z_x' - y^3 + 2z \cdot z_x' = 0$
- $z_x' = \frac{y^3}{e^z+2z}$
- $z_x'(-1,1) = 1$
- 同理可得
- $z_y' = \frac{3xy^2}{e^z+2z}$
- $z_y'(-1,1) = -3$

6.设$z=z(x,y)$是由方程$e^z-xy^3+z^2=2$在点$(-1,1,0)$附近所定义的隐函数，问在点$(-1,1,0)$处自变量(x,y)沿着哪个方向变化时，函数值$z(x,y)$增加的速度最快，并求出该点处沿着此方向的方向导数

- 所以在$(-1,1)$处，函数$z(x,y)$的梯度为$(1,-3)$
- 在$(-1,1)$处，函数$z(x,y)$增加最快的方向为$(1,-3)$
- 沿着此方向的方向导数为$\sqrt{10}$

7.设函数$u=u(x)$由方程组$\begin{cases} u=f(x,y,z) \\ g(x,y,z)=0 \\ h(x,y,z)=0 \end{cases}$定义，其中$f,g,h$的偏导数都连续，求$\frac{du}{dx}$

- 由题意知，方程组
- $\begin{cases} g(x,y,z)=0 \\ h(x,y,z)=0 \end{cases}$确定了隐函数$y=y(x),z=z(x)$，则
- $\begin{cases} g'_x+g'_y\dfrac{dy}{dx}+g'_z\dfrac{dz}{dx}=0 \\ h'_x+h'_y\dfrac{dy}{dx}+h'_z\dfrac{dz}{dx}=0 \end{cases}$
- 解此方程组，得
- $\frac{dy}{dx}=\frac{\frac{D(g,h)}{D(z,x)}}{\frac{D(g,h)}{D(y,z)}},\frac{dz}{dx}=\frac{\frac{D(g,h)}{D(x,y)}}{\frac{D(g,h)}{D(y,z)}}$

7. 设函数$u=u(x)$由方程组$\begin{cases} u=f(x,y,z) \\ g(x,y,z)=0 \\ h(x,y,z)=0 \end{cases}$定义，其中$f,g,h$的偏导数都连续，求$\frac{du}{dx}$

- $\frac{du}{dx}=f_x'+f_y'\frac{dy}{dx}+f_z'\frac{dz}{dx}$
- $=\frac{\frac{D(g,h)}{D(y,z)}f_x'+\frac{D(g,h)}{D(z,x)}f_y'+\frac{D(g,h)}{D(x,y)}f_z'}{\frac{D(g,h)}{D(y,z)}}$

8.1 问函数$f(x,y)=\begin{cases}\frac{2xy^2}{x^2+y^2}, x^2+y^2\neq 0\\ 0, \quad x^2+y^2=0\end{cases}$在(0,0)处是否可微分？在(0,0)处沿着方向$\vec{l}=(\cos\alpha,\sin\alpha)$的方向导数是否能用定理7.5.1的结论计算

- 首先计算出$f(x,y)$在(0,0)处的两个偏导数：
  - $\frac{\partial f}{\partial x}(0,0)=\frac{\partial f}{\partial y}(0,0)=0$
- 判断在(0,0)处是否可微分
  - 令$\Delta=f(x,y)-f(0,0)-\left(\frac{\partial f}{\partial x}(0,0)x+\frac{\partial f}{\partial y}(0,0)y\right)=\frac{2xy^2}{x^2+y^2}$
  - $\lim\limits_{(x,y)\to(0,0)}\frac{\Delta}{\sqrt{x^2+y^2}}=\lim\limits_{(x,y)\to(0,0)}\frac{2xy^2}{(x^2+y^2)^{\frac{3}{2}}}\neq 0$
  - 所以$f(x,y)$在(0,0)处不可微分
- 如果不看条件直接用定理7.5.1的结论在(0,0)处沿着方向$\vec{l}=(\cos\alpha,\sin\alpha)$的方向导数
  - $\frac{\partial f}{\partial \vec{l}}(0,0)=\frac{\partial f}{\partial x}(0,0)\cos\alpha+\frac{\partial f}{\partial y}(0,0)\sin\alpha=0$
- 直接用方向导数的定义计算在(0,0)处沿着方向$\vec{l}=(\cos\alpha,\sin\alpha)$的方向导数
  - $\frac{\partial f}{\partial \vec{l}}(0,0)=\lim\limits_{t\to 0+}\frac{f(t\cos\alpha,t\sin\alpha)-f(0,0)}{t}=\lim\limits_{t\to 0+}\frac{2t^3\cos\alpha\sin^2\alpha}{t^3}=2\cos\alpha\sin^2\alpha$
  - 当$\cos\alpha\cdot\sin\alpha\neq 0$时，$\frac{\partial f}{\partial \vec{l}}(0,0)\neq 0$
- 所以不能用定理7.5.1的结论计算

8.2求函数$f(x,y)=|x^2-y^2|^{1/2}$在$(0,0)$处沿着方向$\vec{l}=(\cos\alpha,\sin\alpha)$的方向导数，问此函数在$(0,0)$处是否可偏导

- $\frac{\partial f}{\partial \vec{l}}(0,0)=\lim\limits_{t\to 0+}\frac{f(t\cos\alpha,t\sin\alpha)-f(0,0)}{t}=\lim\limits_{t\to 0+}\frac{\left|\mathrm{t}^2\cos^2\alpha-t^2\sin^2\alpha\right|^{1/2}}{t}=\sqrt{|\cos 2\alpha|}$
- 因为$\lim\limits_{x\to 0}\frac{f(x,0)-f(0,0)}{x}=\lim\limits_{x\to 0}\frac{|x|}{x}$不存在，所以$f(x,y)$在$(0,0)$处关于$x$的偏导数不存在
- 同理$f(x,y)$在$(0,0)$处关于y的偏导数也不存在

8.3求函数$f(x,y)=\begin{cases}\frac{xy}{x^2+y^2}, x^2+y^2\neq 0\\ 0, \quad x^2+y^2=0\end{cases}$在(0,0)处的两个偏导数，问此函数在(0,0)处沿着方向$\vec{l}=(\cos\alpha,\sin\alpha)$的方向导数是否存在

- 偏导数
  - $\frac{\partial f}{\partial x}(0,0)=\frac{\partial f}{\partial y}(0,0)=0$
- $\lim_{t\to 0+}\frac{f(t\cos\alpha,t\sin\alpha)-f(0,0)}{t}=\lim_{t\to 0+}\frac{t^2\cos\alpha\sin\alpha}{t^3}$
- 当$\cos\alpha\sin\alpha\neq 0$时，极限不存在
- 当$\cos\alpha\sin\alpha=0$时，极限为0
- 所以当$\vec{l}=(\pm 1,0),(0,\pm 1)$时，$\frac{\partial f}{\partial \vec{l}}(0,0)=0$；而在其他方向上，方向导数不存在

8.4设函数$f(x,y)$在(x_0,y_0)处沿着任意方向$\vec{l}=(\cos\alpha,\sin\alpha)$的方向导数都存在，问方向导数满足什么条件时$f(x,y)$在$(x_0,y_0)$处可偏导

- $f(x,y)$在(x_0,y_0)处关于x的偏导数存在，当且仅当
  - $\frac{\partial f}{\partial x}(x_0,y_0)=\lim\limits_{t\to 0}\frac{f(x_0+t,y_0)-f(x_0,y_0)}{t}=\lim\limits_{t\to 0+}\frac{f(x_0+t,y_0)-f(x_0,y_0)}{t}=\lim\limits_{t\to 0-}\frac{f(x_0+t,y_0)-f(x_0,y_0)}{t}$
- 而
  - $\lim\limits_{t\to 0+}\frac{f(x_0+t,y_0)-f(x_0,y_0)}{t}=\frac{\partial f}{\partial \vec{l}_1}(0,0)$，其中$\vec{l}_1=(1,0)$
  - $\lim\limits_{t\to 0-}\frac{f(x_0+t,y_0)-f(x_0,y_0)}{t}=-\lim\limits_{t\to 0+}\frac{f(x_0-t,y_0)-f(x_0,y_0)}{t}=-\frac{\partial f}{\partial \vec{l}_2}(0,0)$，其中$\vec{l}_2=(-1,0)$
- 所以当$f(x,y)$在(x_0,y_0)处沿着方向$(1,0)$和$(-1,0)$的两个方向导数互为相反数时，$f(x,y)$在(x_0,y_0)处关于x的偏导数存在
- 同理当$f(x,y)$在(x_0,y_0)处沿着方向$(0,1)$和$(0,-1)$的两个方向导数互为相反数时，$f(x,y)$在(x_0,y_0)处关于y的偏导数存在

8.5 根据你的理解，写出方向导数、偏导数、全微分之间的关系

- (此处只写出全微分和方向导数、偏导数和方向导数之间的关系)
- 在某一点处可微分：该点处任意方向上的方向导数都存在(定理7.5.1)
- 在某一点处可偏导：在该点处沿着与同一坐标轴平行的两个方向上的方向导数存在且互为相反数，但是在其他方向上的方向导数可能不存在
- 在某一点处沿着任意方向的方向导数都存在：该点处可能不可偏导，但是如果该点处沿着与同一坐标轴平行的两个方向上的方向导数互为相反数，则该点处关于此自变量的偏导数存在