1. 设$f(x)=\int_{x^2}^{x^3}\frac{dt}{\sqrt{1+t^4}}$，求$f'(1)$

$$f(x)=\int_{x^2}^{x^3}\frac{dt}{\sqrt{1+t^4}}=\int_{1}^{x^3}\frac{dt}{\sqrt{1+t^4}}-\int_{1}^{x^2}\frac{dt}{\sqrt{1+t^4}}$$

$$f'(x)=\frac{3x^2}{\sqrt{1+x^{12}}}-\frac{2x}{\sqrt{1+x^8}}$$

$$f'(1)=\frac{1}{\sqrt{2}}$$

2. (P176.12) 设函数f连续，且$\int_0^1 tf(2x-t)dt=\frac{1}{2}\arctan(x^2)$, $f(1)=1$，求$\int_1^2 f(x)dx$

令$u=2x-t$,则

$$\int_0^1 tf(2x-t)dt=\int_{2x}^{2x-1}(2x-u)f(u)d(2x-u)=2x\int_{2x-1}^{2x}f(u)du-\int_{2x-1}^{2x}uf(u)du=\frac{1}{2}\arctan(x^2)$$

上式两边求导

$$2\int_{2x-1}^{2x}f(u)du+2x(f(2x)\cdot 2-f(2x-1)\cdot 2)-(2xf(2x)\cdot 2-(2x-1)f(2x-1)\cdot 2)=2\int_{2x-1}^{2x}f(u)du-2f(2x-1)$$
$$=\frac{x}{1+x^4}$$

令$x=1$,得$2\int_1^2 f(u)du-2f(1)=\frac{1}{2}$,所以$\int_1^2 f(u)du=\frac{5}{4}$

3. 计算$\int\frac{xe^x}{\sqrt{e^x-1}}dx$

令$t=\sqrt{e^x-1}$,则$e^x=t^2+1, x=ln(t^2+1)$

$$\int\frac{xe^x}{\sqrt{e^x-1}}dx=\int\frac{ln(t^2+1)\cdot(t^2+1)}{t}\frac{2t}{(t^2+1)}dt=2\int ln(t^2+1)dt$$

$$=2tln(t^2+1)-2\int tdln(t^2+1)=2tln(t^2+1)-4\int\frac{t^2}{t^2+1}dt$$

$$=2tln(t^2+1)-4\int\frac{t^2+1-1}{t^2+1}dt$$

$$=2tln(t^2+1)-4t+4\arctan t+c$$
$$=2x\sqrt{e^x-1}-4\sqrt{e^x-1}+4\arctan\sqrt{e^x-1}+c$$

4. 计算$\int\frac{x}{1+\cos x}dx$

$$\int\frac{x}{1+\cos x}dx=\int\frac{x}{2\cos^2\frac{x}{2}}dx=\int x\,d\tan\frac{x}{2}=x\tan\frac{x}{2}-\int\tan\frac{x}{2}dx$$

$$=x\tan\frac{x}{2}+2\int\frac{d\cos\frac{x}{2}}{\cos\frac{x}{2}}=x\tan\frac{x}{2}+2\ln\left|\cos\frac{x}{2}\right|+c$$

5. 计算$\int\frac{\ln\left(x+\sqrt{1+x^2}\right)}{(1+x^2)^{\frac{3}{2}}}dx$

先求$\int\frac{dx}{(1+x^2)^{\frac{3}{2}}}$, 令$x=\tan t, t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right), \cos t=\frac{1}{\sqrt{1+x^2}}, \sin t=\tan t\cdot\cos t=\frac{x}{\sqrt{1+x^2}}$

$$\int\frac{dx}{(1+x^2)^{\frac{3}{2}}}=\int\frac{sec^2tdt}{sec^3t}=\int\cos t\,dt=\sin t+c=\frac{x}{\sqrt{1+x^2}}+c$$

$$\int\frac{ln\left(x+\sqrt{1+x^2}\right)}{(1+x^2)^{\frac{3}{2}}}dx=\int ln\left(x+\sqrt{1+x^2}\right)d\frac{x}{\sqrt{1+x^2}}$$

$$=\ln\left(x+\sqrt{1+x^2}\right)\frac{x}{\sqrt{1+x^2}}-\int\frac{x}{\sqrt{1+x^2}}dln\left(x+\sqrt{1+x^2}\right)$$

$$=\frac{x\ln\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}-\int\frac{x}{1+x^2}dx$$

$$=\frac{x\ln\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}-\frac{1}{2}\ln(1+x^2)+c$$

6. 求$\int_{-2}^{2}\frac{\sin x+|x|}{2+x^2}dx$

用奇函数、偶函数的性质

$$\int_{-2}^{2}\frac{\sin x+|x|}{2+x^2}dx=\int_{-2}^{2}\frac{\sin x}{2+x^2}dx+\int_{-2}^{2}\frac{|x|}{2+x^2}dx=2\int_{0}^{2}\frac{x}{2+x^2}dx=\ln(2+x^2)|_0^2=\ln 3$$

7. 求$\int_{0}^{\frac{\pi}{2}}\frac{\sin(2n-1)x}{\sin x}dx$

当 n=1 时，$\int_{0}^{\frac{\pi}{2}}\frac{\sin x}{\sin x}dx=\frac{\pi}{2}$

当$n=2$时，

$$\int_{0}^{\frac{\pi}{2}}\frac{\sin 3x}{\sin x}dx=\int_{0}^{\frac{\pi}{2}}\frac{sin2x\ cosx+cos2x\ sinx}{sinx}dx=\int_{0}^{\frac{\pi}{2}}(2cos^2x+cos2x)\,dx=\int_{0}^{\frac{\pi}{2}}(1+2cos2x)\,dx=\frac{\pi}{2}$$

当$n>2$时，$\int_{0}^{\frac{\pi}{2}}cos(2n-2)xdx=0$

$$\int_{0}^{\frac{\pi}{2}}\frac{sin(2n-1)x}{sinx}dx=\int_{0}^{\frac{\pi}{2}}\frac{sin(2n-2)x\ cosx+cos(2n-2)x\ sinx}{sinx}dx=\int_{0}^{\frac{\pi}{2}}\frac{\sin(2n-2)x\cos x}{\sin x}dx$$

$$=\int_{0}^{\frac{\pi}{2}}\frac{(\sin(2n-3)x\cos x+\cos(2n-3)x\sin x)\cos x}{sinx}dx$$

$$=\int_{0}^{\frac{\pi}{2}}\left(\frac{\sin(2n-3)x\cos^2 x}{sinx}+\cos(2n-3)x\cos x\right)dx$$

$$=\int_{0}^{\frac{\pi}{2}}\left(\frac{\sin(2n-3)x}{sinx}-\sin(2n-3)x\sin x+\cos(2n-3)x\cos x\right)dx$$

$$=\int_{0}^{\frac{\pi}{2}}\left(\frac{\sin(2n-3)x}{sinx}+\cos(2n-2)x\right)dx=\int_{0}^{\frac{\pi}{2}}\frac{\sin(2n-3)x}{sinx}dx$$

所以$\forall n\in Z^+, \int_{0}^{\frac{\pi}{2}}\frac{sin(2n-1)x}{sinx}dx=\frac{\pi}{2}$

8. 求$\lim\limits_{n\to\infty}\int_{0}^{1}\frac{x^n}{2+sinnx}dx$

本题不能算出被积函数的原函数

当$x\in[0,1]$时，

$$0\le\frac{x^n}{2+sinnx}\le x^n$$

所以

$$0\le\int_{0}^{1}\frac{x^n}{2+sinnx}dx\le\int_{0}^{1}x^n dx=\frac{1}{n+1}x^{n+1}\Big|_0^1=\frac{1}{n+1}$$

由夹逼性知

$$\lim_{n\to\infty}\int_{0}^{1}\frac{x^n}{2+sinnx}dx=0$$

9. 设$x_n=\frac{1}{n^4}\prod_{k=1}^{2n}(n^2+k^2)^{\frac{1}{n}}$,求$\lim\limits_{n\to\infty}x_n$

$$\ln x_n = \ln\left(\frac{1}{n^4}\prod_{k=1}^{2n}(n^2+k^2)^{\frac{1}{n}}\right) = \sum_{k=1}^{2n}\frac{1}{n}\ln(n^2+k^2) - 4\ln n = \sum_{k=1}^{2n}\frac{1}{n}\ln n^2 + \sum_{n=1}^{2n}\frac{1}{n}\ln\left(1+\left(\frac{k}{n}\right)^2\right) - 4\ln n = \sum_{n=1}^{2n}\frac{1}{n}\ln\left(1+\left(\frac{k}{n}\right)^2\right)$$

$$\lim_{n\to\infty}\ln x_n = \lim_{n\to\infty}\left(\sum_{n=1}^{2n}\frac{1}{n}\ln\left(1+\left(\frac{k}{n}\right)^2\right)\right) = \int_0^2 \ln(1+x^2)\,dx = (x\ln(x^2+1) - 2x + 2\arctan x)|_0^2 = 2\ln 5 - 4 + 2\arctan 2$$

$$\lim_{n\to\infty} x_n = e^{2\ln 5 - 4 + 2\arctan 2} = 25e^{-4+2\arctan 2}$$

10. 设$f(x) = \int_1^{x^2} e^{-t^2}dt$，求$\int_0^1 xf(x)dx$

$$\int_0^1 xf(x)dx = \frac{1}{2}\int_0^1 f(x)dx^2 = \frac{1}{2}x^2 f(x)\Big|_0^1 - \frac{1}{2}\int_0^1 x^2 df(x) = -\frac{1}{2}\int_0^1 x^2 e^{-x^4}dx^2 = -\frac{1}{2}\int_0^1 ue^{-u^2}du$$

$$= -\frac{1}{4}\int_0^1 e^{-u^2}du^2 = \frac{1}{4}e^{-u^2}\Big|_0^1 = \frac{1}{4}\left(\frac{1}{e} - 1\right)$$

11. (P191. 6) 求$x^4 + y^4 = a^2(x^2+y^2)$围成的图形的面积

令$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, $r \geq 0, \theta \in [0,2\pi]$，则$x^4 + y^4 = a^2(x^2+y^2)$变为$r^2 = \frac{a^2}{\cos^4\theta + \sin^4\theta}$

$$S = \frac{1}{2}\int_0^{2\pi}\frac{a^2}{\cos^4\theta + \sin^4\theta}d\theta = \frac{4}{2}\int_0^{\frac{\pi}{2}}\frac{a^2}{\cos^4\theta + \sin^4\theta}d\theta = 4\int_0^{\frac{\pi}{4}}\frac{a^2}{\cos^4\theta + \sin^4\theta}d\theta = 4a^2\int_0^{\frac{\pi}{4}}\frac{\sec^2\theta}{1+\tan^4\theta}d\tan\theta$$

$$= 4a^2\int_0^1\frac{1+t^2}{1+t^4}dt = 4a^2\int_0^1\frac{\frac{1}{t^2}+1}{\frac{1}{t^2}+t^2}dt = 4a^2\int_0^1\frac{d\left(t-\frac{1}{t}\right)}{\left(t-\frac{1}{t}\right)^2+2}dt = \frac{4a^2}{\sqrt{2}}\arctan\frac{t-\frac{1}{t}}{\sqrt{2}}\Bigg|_0^1 = \sqrt{2}\pi a^2$$

12. 求$x^3 + y^3 = 3axy\ (a > 0)$所围成的封闭区域的面积

令$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, $r \geq 0, \theta \in [0,2\pi]$,则$x^3 + y^3 = 3axy$变为$r = \frac{3a\cos\theta\sin\theta}{\cos^3\theta + \sin^3\theta}$

确定θ的范围：

1) $\begin{cases} \cos\theta\sin\theta \geq 0 \\ \cos^3\theta + \sin^3\theta > 0 \end{cases} \Rightarrow \begin{cases} \theta \in \left[0,\frac{\pi}{2}\right] \cup \left[\pi,\frac{3\pi}{2}\right] \\ \theta \in \left[0,\frac{3\pi}{4}\right) \cup \left(\frac{7\pi}{4},2\pi\right] \end{cases} \Rightarrow \theta \in \left[0,\frac{\pi}{2}\right]$

2) $\begin{cases} \cos\theta\sin\theta \leq 0 \\ \cos^3\theta + \sin^3\theta < 0 \end{cases} \Rightarrow \begin{cases} \theta \in \left[\frac{\pi}{2},\pi\right] \cup \left[\frac{3\pi}{2},2\pi\right] \\ \theta \in \left(\frac{3\pi}{4},\frac{7\pi}{4}\right) \end{cases} \Rightarrow \theta \in \left(\frac{3\pi}{4},\pi\right] \cup \left[\frac{3\pi}{2},\frac{7\pi}{4}\right)$

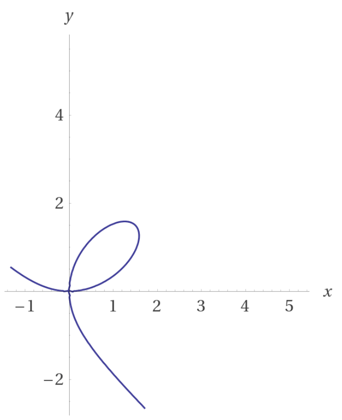
当$\theta \in \left[0,\frac{\pi}{2}\right]$时，$r(0) = r\left(\frac{\pi}{2}\right) = 0$，围成一个封闭区域

当$\theta \in \left(\frac{3\pi}{4},\pi\right]$时，当且仅当$\theta = \pi$，$r(\theta) = 0$所以不构成封闭区域

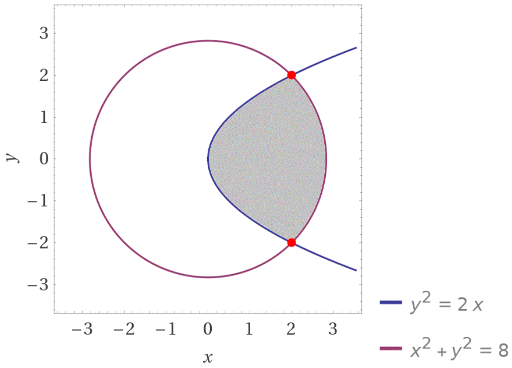
当$\theta \in \left[\frac{3\pi}{2},\frac{7\pi}{4}\right)$时，当且仅当$\theta = \frac{3\pi}{2}$，$r(\theta) = 0$，所以不构成封闭区域

$$S = \frac{9a^2}{2}\int_0^{\frac{\pi}{2}}\left(\frac{\cos\theta\sin\theta}{\cos^3\theta + \sin^3\theta}\right)^2 d\theta = 9a^2\int_0^{\frac{\pi}{4}}\left(\frac{\cos\theta\sin\theta}{\cos^3\theta + \sin^3\theta}\right)^2 d\theta = 9a^2\int_0^{\frac{\pi}{4}}\frac{\cos^2\theta\sin^2\theta}{\cos^6\theta\,(1+\tan^3\theta)^2}d\theta$$

$$= 9a^2\int_0^{\frac{\pi}{4}}\frac{\tan^2\theta}{(1+\tan^3\theta)^2}d\tan\theta = 3a^2\int_0^{\frac{\pi}{4}}\frac{1}{(1+\tan^3\theta)^2}d\tan^3\theta = -3a^2\frac{1}{1+\tan^3\theta}\Bigg|_0^{\frac{\pi}{4}} = \frac{3a^2}{2}$$



13. 求由抛物线$y^2 = 2x$与圆$x^2 + y^2 = 8$所围成的平面有界区域（取$y^2 \le 2x$的部分），绕x轴旋转一周生成的旋转体的体积



$y^2 = 2x$与$x^2 + y^2 = 8$的交点

$x^2 + 2x - 8 = 0$ 得 $x = 2, x = -4$(舍)

$$V = \pi\int_0^2 2x dx + \pi\int_2^{2\sqrt{2}}(8 - x^2)dx = \pi\left(\frac{32\sqrt{2}}{3} - \frac{28}{3}\right)$$