1. 在线性空间$\mathbf{R}^3$ 中,$\boldsymbol{\alpha}=(x_1,x_2,x_3)^{\mathrm{T}}\in\mathbf{R}^3$. 判断下列变换是否为线性变换.

(1)$T_1(\boldsymbol{\alpha})=(2x_1-x_3,x_2+x_3,x_1+x_3)^{\mathrm{T}}$

(2)$T_2(\boldsymbol{\alpha})=(\sin(x_1),0,0)^{\mathrm{T}}$

(3)$T_3(\boldsymbol{\alpha})=(x_1^2,x_2^2,x_3^2)^{\mathrm{T}}$

(4)$T_4(\boldsymbol{\alpha})=(\sin(x_1),\cos(x_2),1)^{\mathrm{T}}$.

(1)$T_1(\boldsymbol{\alpha})=(2x_1-x_3,x_2+x_3,x_1+x_3)^{\mathrm{T}},T_1(\boldsymbol{\beta})=(2y_1-y_3,y_2+y_3,y_1+y_3)^{\mathrm{T}}$

$T_1(\boldsymbol{\alpha}+\boldsymbol{\beta})=(2(x_1+y_1)-(x_3+y_3),(x_2+y_2)+(x_3+y_3),(x_1+y_1)+(x_3+y_3))^{\mathrm{T}}=T_1(\boldsymbol{\alpha})+T_1(\boldsymbol{\beta}).$

$T_1(k\boldsymbol{\alpha})=(2kx_1-kx_3,kx_2+kx_3,kx_1+kx_3)^{\mathrm{T}}=kT_1(\boldsymbol{\alpha}).$

所以是线性变换

(2)$T_2(\boldsymbol{\alpha})=(\sin(x_1),0,0)^{\mathrm{T}},T_2(\boldsymbol{\beta})=(\sin(y_1),0,0)^{\mathrm{T}}$

$T_2(\boldsymbol{\alpha}+\boldsymbol{\beta})=(\sin(x_1+y_1),0,0)^{\mathrm{T}}\neq T_2(\boldsymbol{\alpha})+T_2(\boldsymbol{\beta}).$

所以不是线性变换

(3)$T_3(\boldsymbol{\alpha})=(x_1^2,x_2^2,x_3^2)^{\mathrm{T}},T_3(\boldsymbol{\beta})=(y_1^2,y_2^2,y_3^2)^{\mathrm{T}}$

$T_3(\boldsymbol{\alpha}+\boldsymbol{\beta})=((x_1+y_1)^2,(x_2+y_2)^2,(x_3+y_3)^2)^{\mathrm{T}}\neq T_3(\boldsymbol{\alpha})+T_3(\boldsymbol{\beta}).$

所以不是线性变换

(4)$T_4(\boldsymbol{\alpha})=(\sin(x_1),\cos(x_2),1)^{\mathrm{T}},T_4(\boldsymbol{\beta})=(\sin(y_1),\cos(y_2),1)^{\mathrm{T}}$

$T_4(\boldsymbol{\alpha}+\boldsymbol{\beta})=(\sin(x_1+y_1),\cos(x_2+y_2),1)^{\mathrm{T}}\neq T_4(\boldsymbol{\alpha})+T_4(\boldsymbol{\beta}).$

所以不是线性变换

2. 设$C[a,b]$为闭区间上的全体连续函数所组成的实数域上的线性空间

在$C[a,b]$内，定义变换如下：

$$\sigma: f(x) \to \int_a^x f(t)dt = F(x)$$

即让$C[a,b]$内每一个向量$f(x)$对应于它的变上限积分($f(x)$ 的一个原函数)

证明σ 是线性变换。

假设$f(x) \in C[a,b], g(x) \in C[a,b]$,根据积分的性质有

$$\sigma(f(x)+g(x)) = \int_a^x [f(t)+g(t)]dt = \int_a^x f(t)dt + \int_a^x g(t)dt = \sigma(f(x)) + \sigma(g(x)),$$

又对$\forall k \in \mathrm{R}$,有

$$\sigma(kf(x)) = \int_a^x kf(t)dt = k\int_a^x f(t)dt = k\sigma(f(x)).$$

故σ 是线性变换.

3. 设T是线性空间V上的线性变换,如果$T^k(\boldsymbol{\alpha}) \neq \mathbf{0}$ 且$T^n(\boldsymbol{\alpha}) = \mathbf{0}(n > k)$, 证明： $\boldsymbol{\alpha}, T(\boldsymbol{\alpha}), \cdots, T^k(\boldsymbol{\alpha})$ 线性无关。

设$\boldsymbol{\alpha}, T(\boldsymbol{\alpha}), \cdots, T^k(\boldsymbol{\alpha})$ 线性相关, 则存在不全为零的数$a_0, a_1, \cdots, a_k$

使$a_0\boldsymbol{\alpha} + a_1T(\boldsymbol{\alpha}) + \cdots + a_kT^k(\boldsymbol{\alpha}) = \mathbf{0}.$

设$a_0, a_1, \cdots, a_k$中第一个不为零的数为a_i

则$a_iT^i(\boldsymbol{\alpha}) + \cdots + a_kT^k(\boldsymbol{\alpha}) = \mathbf{0}.$

又m是$k+1, \cdots, n$ 中使得$T^m(\boldsymbol{\alpha}) = \mathbf{0}$ 最小的一个, 用T^{m-i-1} 作用于上式, 有$a_iT^{m-1}(\boldsymbol{\alpha}) = \mathbf{0}.$

因为$T^{m-1}(\boldsymbol{\alpha}) \neq \mathbf{0}$,从而$a_i = 0$,矛盾.

故$\boldsymbol{\alpha}, T(\boldsymbol{\alpha}), \cdots, T^k(\boldsymbol{\alpha})$ 线性无关.

设线性变换T在基$\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\cdots,\boldsymbol{\alpha}_n$下的矩阵为$A$, $|A|=5$, 且T 在基$\boldsymbol{\alpha}_n,\boldsymbol{\alpha}_{n-1},\cdots,\boldsymbol{\alpha}_1$ 下的矩阵为B. 求$|B|$.

$$设A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},则有\ B=\begin{bmatrix} a_{nn} & a_{n,n-1} & \cdots & a_{n1} \\ a_{n-1,n} & a_{n-1,n-1} & \cdots & a_{n-1,1} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{1,n-1} & \cdots & a_{11} \end{bmatrix}, |B|=|A|=5.$$

5. 已知$\mathbf{R}^3$中的一组基$\boldsymbol{\eta}_1=(-1,1,1)^{\mathrm{T}},\boldsymbol{\eta}_2=(1,0,-1)^{\mathrm{T}},\boldsymbol{\eta}_3=(0,1,1)^{\mathrm{T}}$,线性变换$T$ 在基 $\boldsymbol{\eta}_1,\boldsymbol{\eta}_2,\boldsymbol{\eta}_3$ 下的矩阵为

$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$，求$T$ 在基$\boldsymbol{\varepsilon}_1=(1,0,0)^{\mathrm{T}},\boldsymbol{\varepsilon}_2=(0,1,0)^{\mathrm{T}},\boldsymbol{\varepsilon}_3=(0,0,1)^{\mathrm{T}}$ 下的矩阵。

因为$(\boldsymbol{\eta}_1,\boldsymbol{\eta}_2,\boldsymbol{\eta}_3)=(\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2,\boldsymbol{\varepsilon}_3)\begin{bmatrix}-1&1&0\\1&0&1\\1&-1&1\end{bmatrix}$,从而由基$\boldsymbol{\eta}_1,\boldsymbol{\eta}_2,\boldsymbol{\eta}_3$ 到基$\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2,\boldsymbol{\varepsilon}_3$ 的过渡矩阵为$\begin{bmatrix}-1&1&0\\1&0&1\\1&-1&1\end{bmatrix}^{-1}$.

由于T在基$\boldsymbol{\eta}_1,\boldsymbol{\eta}_2,\boldsymbol{\eta}_3$下的矩阵为$\begin{bmatrix}1&0&1\\1&1&0\\-1&2&1\end{bmatrix}$,

所以T在基$\boldsymbol{\varepsilon}_1=(1,0,0),\boldsymbol{\varepsilon}_2=(0,1,0),\boldsymbol{\varepsilon}_3=(0,0,1)$下的矩阵为

$$\left[\begin{bmatrix}-1&1&0\\1&0&1\\1&-1&1\end{bmatrix}^{-1}\right]^{-1}\begin{bmatrix}1&0&1\\1&1&0\\-1&2&1\end{bmatrix}\begin{bmatrix}-1&1&0\\1&0&1\\1&-1&1\end{bmatrix}^{-1}=\begin{bmatrix}-1&1&0\\1&0&1\\1&-1&1\end{bmatrix}\begin{bmatrix}1&0&1\\1&1&0\\-1&2&1\end{bmatrix}\begin{bmatrix}-1&1&-1\\0&1&-1\\1&0&1\end{bmatrix}=\begin{bmatrix}-1&1&-2\\2&2&0\\3&0&2\end{bmatrix}.$$

6. 在$\mathbf{R}^3$中，已知$\boldsymbol{\alpha}_1=(-1,0,2),\boldsymbol{\alpha}_2=(0,1,1),\boldsymbol{\alpha}_3=(3,-1,0)$是$\mathbf{R}^3$ 的一组基，并且
$T\boldsymbol{\alpha}_1=(-5,0,3),T\boldsymbol{\alpha}_2=(0,-1,6),T\boldsymbol{\alpha}_3=(-5,-1,9)$,求线性变换$T$在基$\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3$下的矩阵，
T在基$\boldsymbol{\varepsilon}_1=(1,0,0),\boldsymbol{\varepsilon}_2=(0,1,0),\boldsymbol{\varepsilon}_3=(1,0,0)$下的矩阵。

设$T\boldsymbol{\alpha}_1=(x_1,x_2,x_3)\begin{pmatrix}\boldsymbol{\alpha}_1\\\boldsymbol{\alpha}_2\\\boldsymbol{\alpha}_3\end{pmatrix}$,有$\begin{cases}-x_1+3x_3=-5\\x_2-x_3=0\\2x_1+x_2=3\end{cases}$,解得$\begin{cases}x_1=2\\x_2=-1\\x_3=-1\end{cases}$,故$T\boldsymbol{\alpha}_1=2\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2-\boldsymbol{\alpha}_3$.同理$T\boldsymbol{\alpha}_2=3\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_3,T\boldsymbol{\alpha}_3=5\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2$.因此$T$在基$\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3$下的矩阵为$\begin{bmatrix}2&3&5\\-1&0&-1\\-1&1&0\end{bmatrix}$

设$\boldsymbol{\varepsilon}_1=(x_1,x_2,x_3)\begin{pmatrix}\boldsymbol{\alpha}_1\\\boldsymbol{\alpha}_2\\\boldsymbol{\alpha}_3\end{pmatrix}$,从而有$\begin{cases}-x_1+3x_3=1\\x_2-x_3=0\\2x_1+x_2=0\end{cases}$,解得$\begin{cases}x_1=-\dfrac{1}{7}\\x_2=\dfrac{2}{7}\\x_3=\dfrac{2}{7}\end{cases}$ 故$\boldsymbol{\varepsilon}_1=-\dfrac{1}{7}\boldsymbol{\alpha}_1+\dfrac{2}{7}\boldsymbol{\alpha}_2+\dfrac{2}{7}\boldsymbol{\alpha}_3$.同理$\boldsymbol{\varepsilon}_2=-\dfrac{3}{7}\boldsymbol{\alpha}_1+\dfrac{6}{7}\boldsymbol{\alpha}_2-\dfrac{1}{7}\boldsymbol{\alpha}_3,\boldsymbol{\varepsilon}_3=\dfrac{3}{7}\boldsymbol{\alpha}_1+\dfrac{1}{7}\boldsymbol{\alpha}_2+\dfrac{1}{7}\boldsymbol{\alpha}_3$.

又因为$\begin{cases}T\boldsymbol{\alpha}_1=(-5,0,3)=-5\boldsymbol{\varepsilon}_1+3\boldsymbol{\varepsilon}_3\\T\boldsymbol{\alpha}_2=(0,-1,6)=-\boldsymbol{\varepsilon}_2+6\boldsymbol{\varepsilon}_3\\T\boldsymbol{\alpha}_3=(-5,-1,9)=-5\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2+9\boldsymbol{\varepsilon}_3\end{cases}$,

代入得$T\boldsymbol{\varepsilon}_1=T(-\dfrac{1}{7}\boldsymbol{\alpha}_1+\dfrac{2}{7}\boldsymbol{\alpha}_2+\dfrac{2}{7}\boldsymbol{\alpha}_3)=-\dfrac{1}{7}T\boldsymbol{\alpha}_1+\dfrac{2}{7}T\boldsymbol{\alpha}_2+\dfrac{2}{7}T\boldsymbol{\alpha}_3=-\dfrac{1}{7}(-5\boldsymbol{\varepsilon}_1+3\boldsymbol{\varepsilon}_3)+\dfrac{2}{7}(-\boldsymbol{\varepsilon}_2+6\boldsymbol{\varepsilon}_3)+\dfrac{2}{7}(-5\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2+9\boldsymbol{\varepsilon}_3)=-\dfrac{5}{7}\boldsymbol{\varepsilon}_1-\dfrac{4}{7}\boldsymbol{\varepsilon}_2+\dfrac{27}{7}\boldsymbol{\varepsilon}_3$

同理$T\boldsymbol{\varepsilon}_2=\dfrac{20}{7}\boldsymbol{\varepsilon}_1-\dfrac{5}{7}\boldsymbol{\varepsilon}_2+\dfrac{18}{7}\boldsymbol{\varepsilon}_3,T\boldsymbol{\varepsilon}_3=-\dfrac{20}{7}\boldsymbol{\varepsilon}_1-\dfrac{2}{7}\boldsymbol{\varepsilon}_2+\dfrac{24}{7}\boldsymbol{\varepsilon}_3$.因此$T$在基$\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2,\boldsymbol{\varepsilon}_3$下的矩阵为$\begin{bmatrix}-\dfrac{5}{7}&\dfrac{20}{7}&-\dfrac{20}{7}\\-\dfrac{4}{7}&-\dfrac{5}{7}&-\dfrac{2}{7}\\\dfrac{27}{7}&\dfrac{18}{7}&\dfrac{24}{7}\end{bmatrix}$.