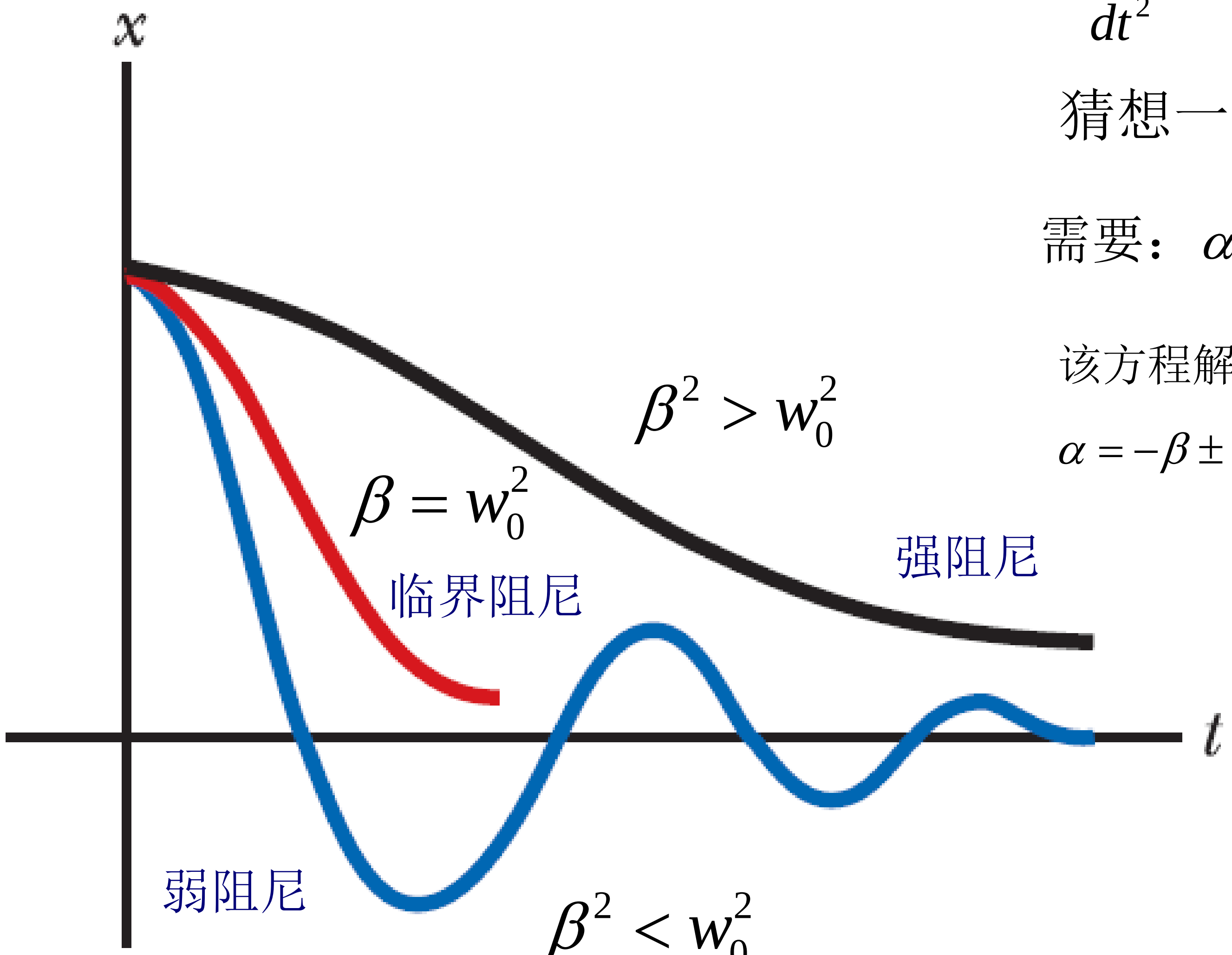
阻尼振动



$$\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + w_0^2 x = 0$$

猜想一个解$x = Ae^{\alpha t}$

需要： $\alpha^2 + 2\alpha\beta + w_0{}^2 = 0$

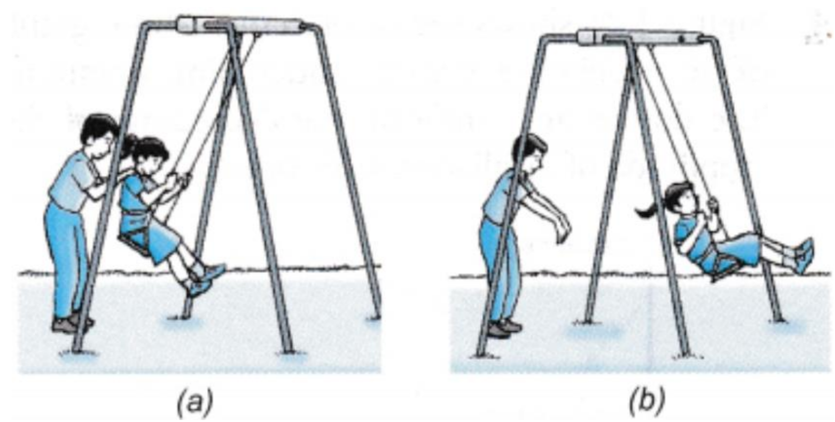
该方程解为：

$$\alpha = -\beta \pm \sqrt{\beta^2 - w_0^2}$$

受迫振动

周期性的驱动力 $f(t) = F\cos wt$

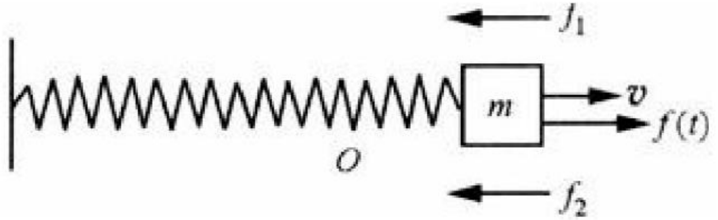
自由振动



受迫振动

弹性系统的受迫振动

周期性的驱动力 $f(t) = F\cos wt$



$$\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + w_0^2 x = f(t)$$

非其次线性二阶微分方程

$$\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + w_0^2 x = C\cos(wt)$$

$$= \frac{C}{2}(e^{iwt} + e^{-iwt})$$

其中$2\beta = \frac{\gamma}{m}, w_0^2 = \frac{k}{m}, C = F/m$

弹性系统的受迫振动

$$\frac{d^2x}{dt^2}+2\beta\frac{dx}{dt}+w_0^2x=F\cos wt$$

受迫振动的频率和外力驱动频率相同

解：

$$x(t)=\frac{C}{R}\cos(wt+\phi_0)$$

受迫振动的振幅C/R和相位ϕ_0和初位移初速度无关，由驱动力F的大小，质量m, 驱动力频率ω,阻尼因数β和弹性系统本征频率w_0决定。

$$\frac{1}{R}=\frac{1}{\sqrt{(w_0^2-w^2)^2+(4\beta^2w^2)}}$$

受迫振动的振幅$C/R(w)$和相位$\phi_0(w)$和驱动力频率ω有关。

$$\varphi_0=\arctan\frac{-2\beta w}{w_0^2-w^2}$$

什么时候振幅最大（共振）？

受迫振动的共振

振动方程

$$x(t)=\frac{C}{R}\cos(wt+\phi_0)$$

$$\frac{1}{R}=\frac{1}{\sqrt{(w_0^2-w^2)^2+4\beta^2w^2}}$$

$$\varphi_0=\arctan\frac{-2\beta w}{w_0^2-w^2}$$

当$w_r=\sqrt{w_0^2-2\beta^2}$时,R最小（对R求导可得),

$1/R$最大，振幅最大

出现共振

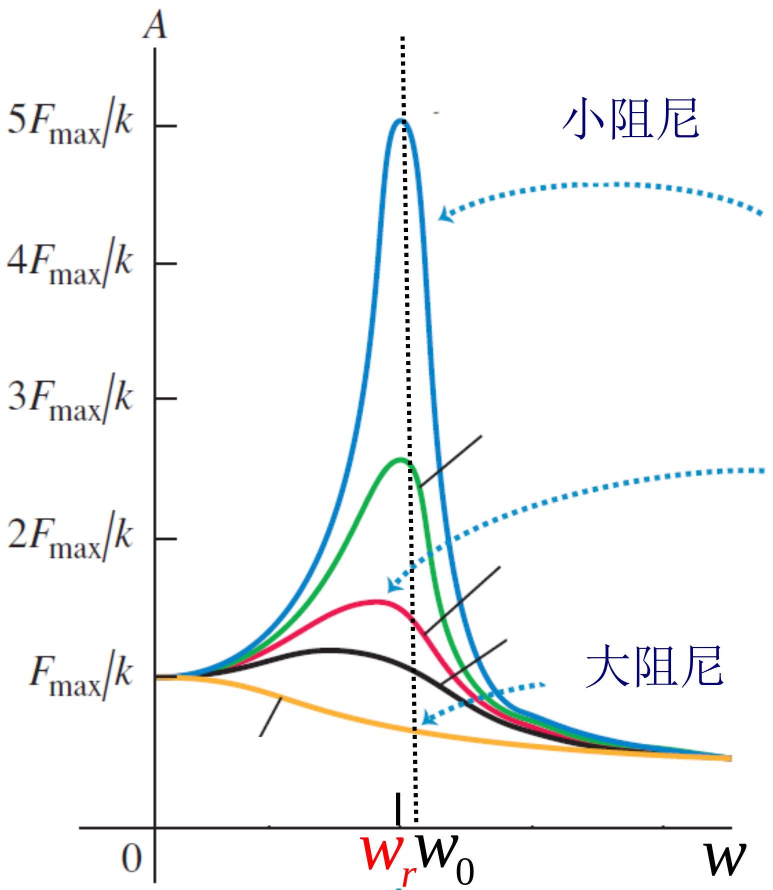
$$A_M=\frac{C}{2\beta\sqrt{w_0^2-\beta^2}}$$

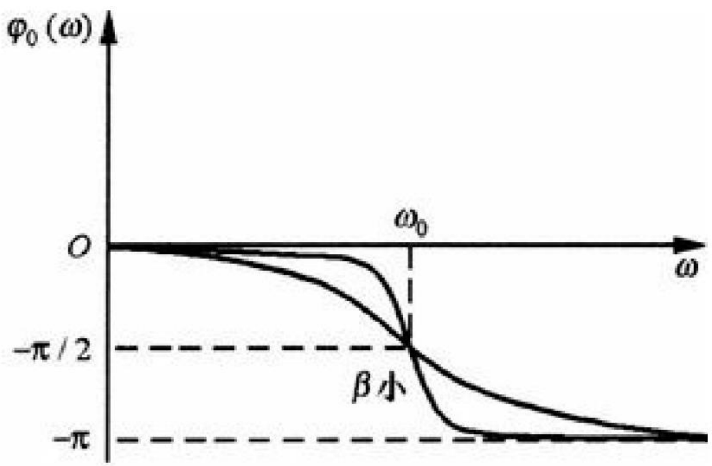
共振频率比本征频率小一点，和阻尼有关。

弱阻尼下，$\beta^2\ll w_0^2$,

$$w_r\approx w_0, A_M\approx\frac{C}{2\beta w_0}$$

受迫振动的共振





位移$x(t)$和驱动力$f(t)$之间的相位差

当$w_r = \sqrt{w_0^2 - 2\beta^2}$时,R最小（对R求导可得），

$1/R$最大，振幅最大

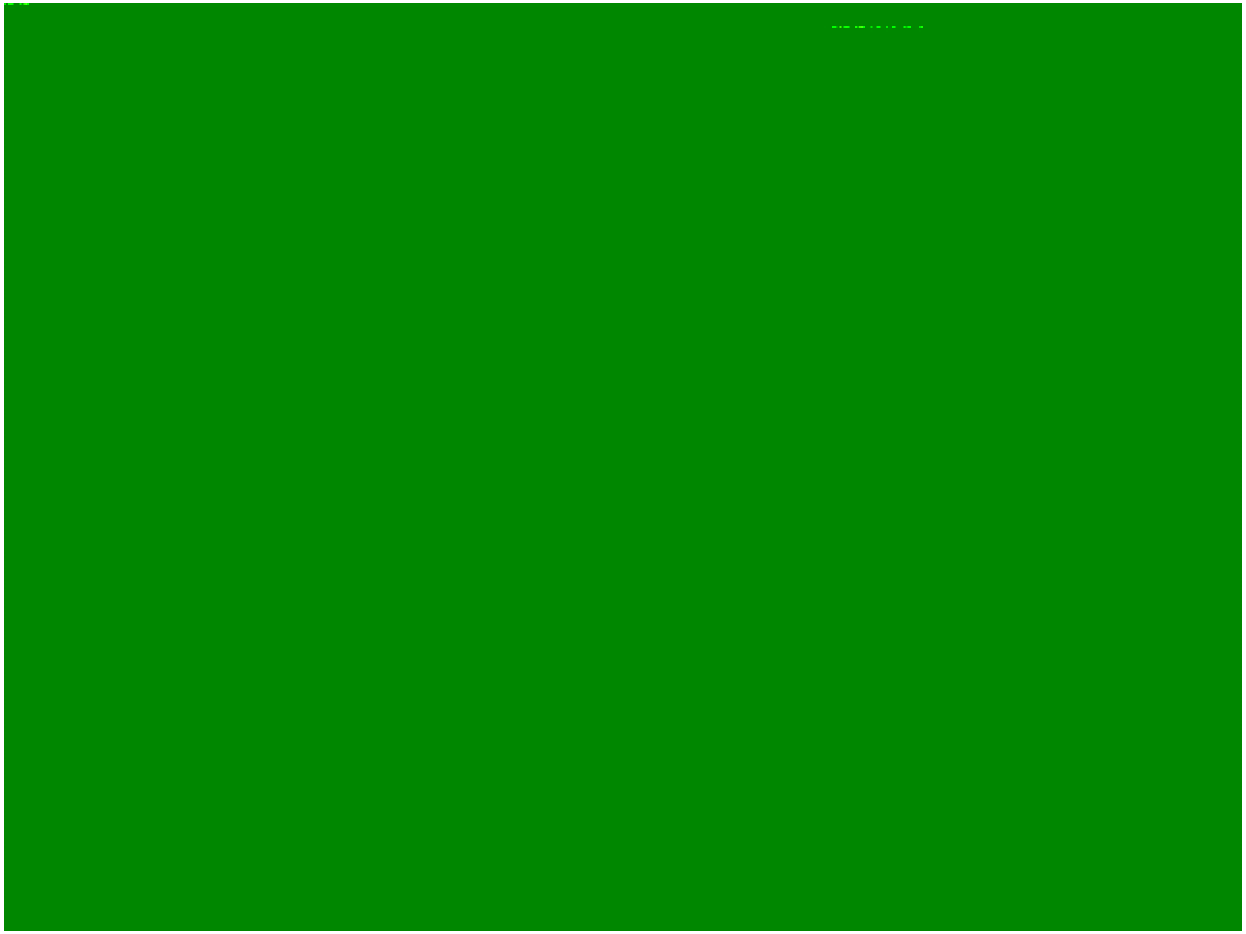
弱阻尼下，$\beta^2 \ll w_0^2$，

$$w_r \approx w_0, A_M \approx \frac{C}{2\beta w_0}$$

受迫振动的共振

Tacoma Narrows Bridge 1940

共振

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、微信、客户端。
为支持企业复工复产和助企纾困，5月6日召开的国务院常务会议推出和
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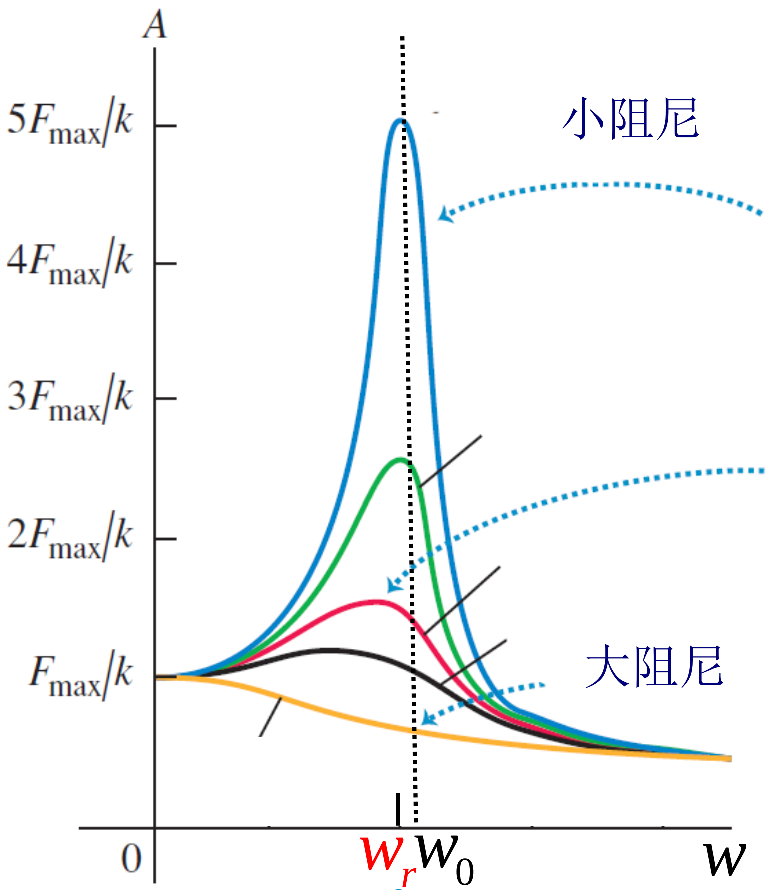
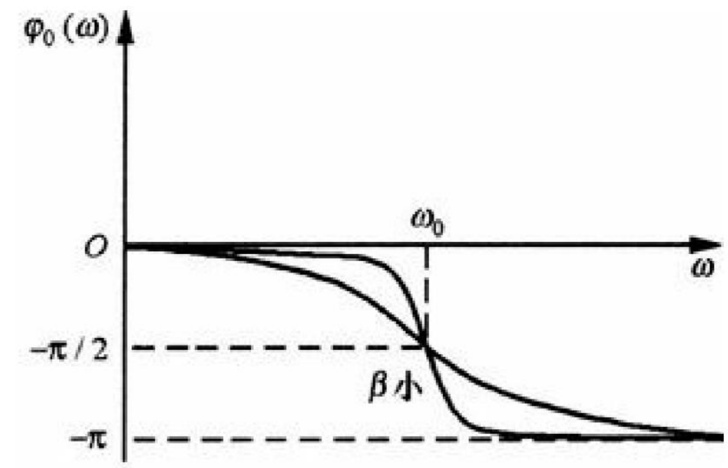
倍速
超清

受迫振动的共振





位移$x(t)$和驱动力$f(t)$之间的相位差

共振系统的品质因子Q

定义品质因子：$Q = 2\pi \dfrac{\Delta E_0}{\Delta E}$

$\Delta E_0 = \dfrac{1}{2} kA^2$，系统储能

ΔE为一周内系统能量损耗值

$\Delta E = \pi\gamma w A^2$

$$\mathrm{Q} = \frac{k}{\gamma w}$$

共振态附近$w \approx w_0, w_0 = \sqrt{k/m}$,最后得

$$Q = \frac{w_0 m}{\gamma}, 或者 Q = \frac{\sqrt{km}}{\gamma}$$

Q因子和阻尼有关。一般阻尼小，Q因子较高，容易共振，能量损失小，振荡的次数多。

思考： 一台电子显微镜需要减振，阻尼器Q因子高还是低好？一个门的阻尼器呢？

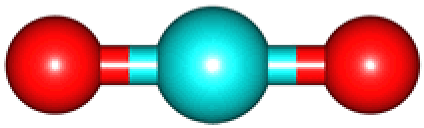
一台收音机，如果不希望串台，Q因子高还是低好？

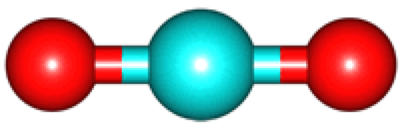
能量的共振转移与吸收

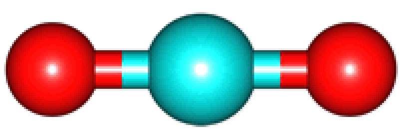
外部激励源*w*单一时，调控系统本征频率使得w=w_0，达到共振。称为调谐。

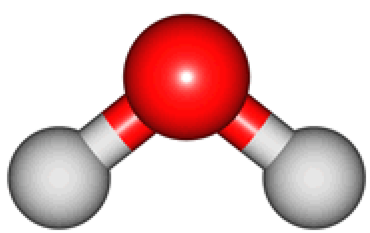
外部激励源*w*具有一定频带宽度，频率w=w0将会被强烈吸收。调频

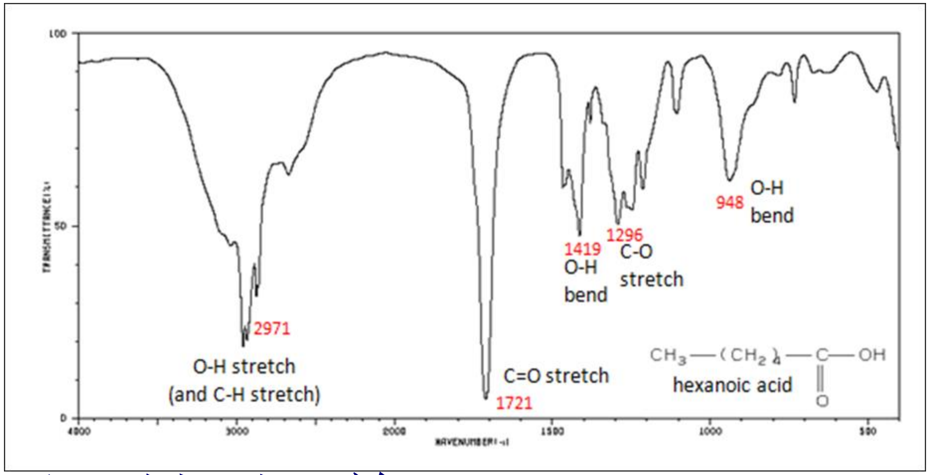
分子的振动










2971
O-H stretch
(and C-H stretch)
1721
C=O stretch
1419
O-H
bend
1296
C-O
stretch
948
O-H
bend
CH3—(CH2)4—C—OH
O
hexanoic acid

红外光谱


FDA

多自由度弹性系统

耦合方程

$$2\ddot{x}+w^2(5x-3y)=0$$

$$2\ddot{y}+w^2(5y-3x)=0$$

$$\ddot{x}=\frac{d^2x}{dt^2}$$

$$\dot{x}=\frac{dx}{dt}$$

如何求解？

方法1：在某种情况下（如上例）线性叠加

$(\ddot{x}+\ddot{y})+w^2(x+y)=0$　　可以看成$\ddot{z}+w^2z=0$

解为：

$$x+y=A_1\cos(wt+\phi_1)$$

A_1和ϕ_1由初始条件获得。

继续线性相减

$(\ddot{x}-\ddot{y})+4w^2(x-y)=0$

解得：$x-y=A_2\cos(2\omega t+\phi_2)$

由此可以解得：

$$x(t)=B_1\cos(wt+\phi_1)+B_2\cos(2wt+\phi_2)$$

$$y(t)=B_1\cos(wt+\phi_1)-B_2\cos(2wt+\phi_2)$$

B_i为A_i的一半

多自由度弹性系统

耦合方程

$$2\ddot{x}+w^2(5x-3y)=0$$

$$2\ddot{y}+w^2(5y-3x)=0$$

解法2：

尝试解$:x=Ae^{i\alpha t},y=Be^{i\alpha t},$写成

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}A\\B\end{pmatrix}e^{i\alpha t}$$

代入原方程，

$$2A(-\alpha^2)+5Aw^2-3Bw^2=0$$

$$2B(-\alpha^2)+5Bw^2-3Aw^2=0$$

写成矩阵形式

$$\begin{pmatrix}-2\alpha^2+5\omega^2 & -3\omega^2\\ -3\omega^2 & -2\alpha^2+5\omega^2\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

方程存在非零解，则矩阵行列式为0

$$0=\begin{vmatrix}-2\alpha^2+5\omega^2 & -3\omega^2\\ -3\omega^2 & -2\alpha^2+5\omega^2\end{vmatrix}$$

$$=4\alpha^4-20\alpha^2\omega^2+16\omega^4.$$

解得： $\alpha=\pm\omega$ 此时A=B

$\alpha=\pm2\omega.$ 此时A=-B

线性组合四个解，代回原方程组

$$\begin{pmatrix}x\\y\end{pmatrix}=A_1\begin{pmatrix}1\\1\end{pmatrix}e^{i\omega t}+A_2\begin{pmatrix}1\\1\end{pmatrix}e^{-i\omega t}$$

$$+A_3\begin{pmatrix}1\\-1\end{pmatrix}e^{2i\omega t}+A_4\begin{pmatrix}1\\-1\end{pmatrix}e^{-2i\omega t}.$$

多自由度弹性系统

定义：

$$A_2^* = A_1 \equiv (B_1/2)e^{i\phi_1} \text{ 及 } A_4^* = A_3 \equiv (B_2/2)e^{i\phi_2}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = B_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega t + \phi_1) + B_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(2\omega t + \phi_2),$$

等价于上一种解法：

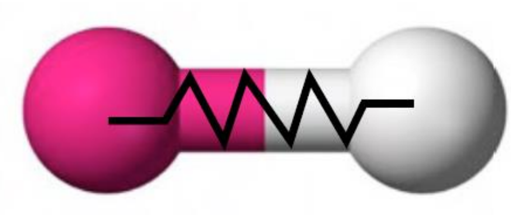
$$x(t) = B_1 \cos(wt + \phi_1) + B_2 \cos(2wt + \phi_2)$$

$$y(t) = B_1 \cos(wt + \phi_1) - B_2 \cos(2wt + \phi_2)$$

双振子和耦合双振子

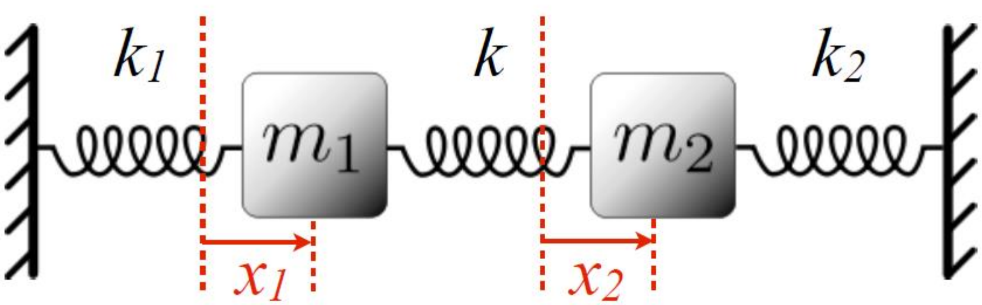
双振子：

H_2、O_2、N_2、CO

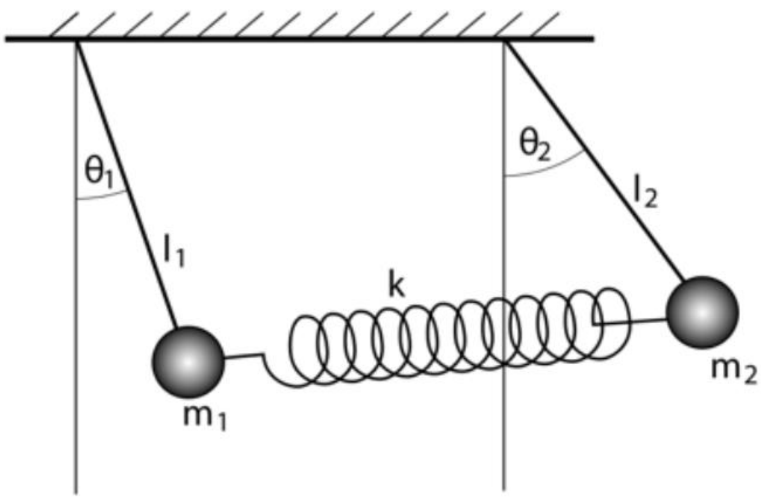


振动自由度：1

耦合双振子：

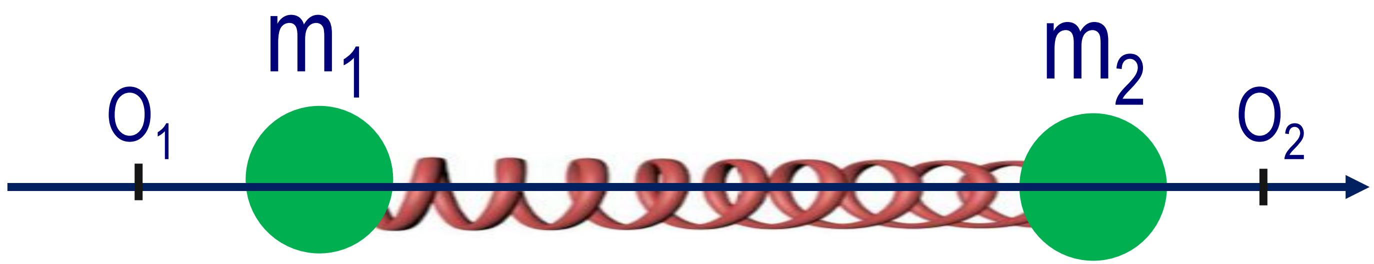


振动自由度：2



振动自由度：2

双振子-双原子分子振动



m_1距平衡点O_1 位移x_1
m_2距平衡点O_2位移x_2

$$m_1\frac{d^2x_1}{dt^2}=-k(x_1-x_2),m_2\frac{d^2x_2}{dt^2}=-k(x_2-x_1)$$

$$\text{尝试解}:x_1=Ae^{i\alpha t},x_2=Be^{i\alpha t}$$

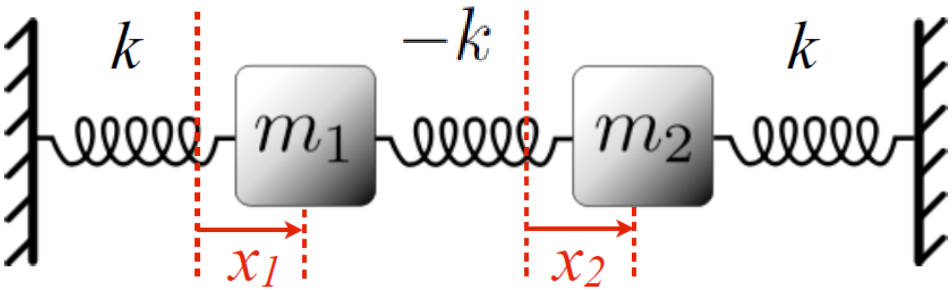
带入原方程：

$$A(k-m_1a^2)-Bk=0$$
$$A(-k)+B(k-m_2a^2)=0$$

$$\begin{vmatrix} k-m_1a^2 & -k \\ -k & k-m_2a^2 \end{vmatrix}=0$$

$$a_1=0,a_2=\sqrt{\frac{(m_1+m_2)k}{m_1m_2}}$$

耦合双振子



振动自由度：2

$$x_{1,2} = A_{1,2}e^{i\omega t}$$

$$m_1\ddot{x}_1 = -k_1x_1 - k(x_1 - x_2)$$
$$m_2\ddot{x}_2 = -k_2x_2 - k(x_2 - x_1)$$

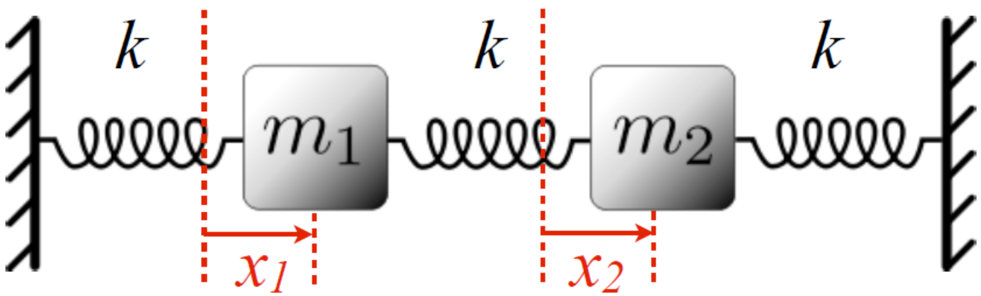
$$\begin{pmatrix} k + k_1 - m_1\omega^2 & -k \\ -k & k + k_2 - m_2\omega^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

$$(k + k_1 - m_1\omega^2)(k + k_2 - m_2\omega^2) - k^2 = 0$$

设:

$$k_1 = k_2 = k \quad \text{and} \quad m_1 = m_2 \quad \Rightarrow \quad w_1^2 = \frac{k}{m} \quad \text{and} \quad w_2^2 = \frac{3k}{m}$$

耦合双振子



振动自由度：2

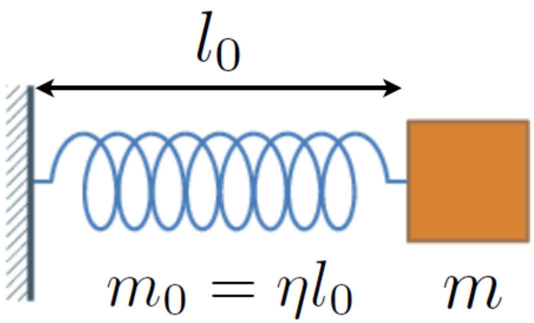
$$x_{1,2} = A_{1,2} e^{i\omega t}$$

$$\begin{pmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

$$\omega_1 = \sqrt{\frac{k}{m}} \quad \Rightarrow \quad \frac{A_1}{A_2} = 1 \quad \Rightarrow \quad \begin{aligned} x_1(t) &= A\cos(w_1 t + \varphi_a) \\ x_2(t) &= A\cos(w_1 t + \varphi_a) \end{aligned}$$

$$\omega_2 = \sqrt{\frac{3k}{m}} \quad \Rightarrow \quad \frac{A_1}{A_2} = -1 \quad \Rightarrow \quad \begin{aligned} x_1(t) &= B\cos(w_2 t + \varphi_b) \\ x_2(t) &= -B\cos(w_2 t + \varphi_b) \end{aligned}$$

弹簧的有效质量



如果不计弹簧质量

$$x(t) = x_0 \cos \omega_0 t$$

$$\frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \frac{1}{2}kx_0^2 \quad \Rightarrow \quad \omega_0 = \sqrt{\frac{k}{m}}$$

考虑弹簧质量

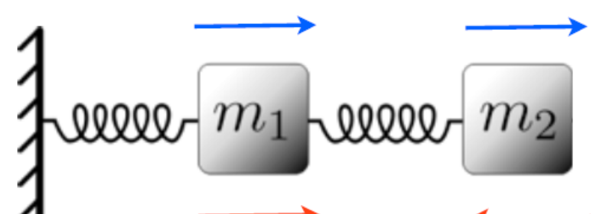
假设弹簧上各点同步伸长或缩短： $dm = \eta du \quad \text{and} \quad v(u) = \frac{u}{l_0} v$

$$E_{ks} = \int_0^{l_0} \frac{1}{2} v(u)^2 dm = \frac{1}{2}\frac{m_0 v^2}{l_0^3}\int_0^{l_0} u^2 du = \frac{1}{2}\frac{m_0}{3}v^2$$

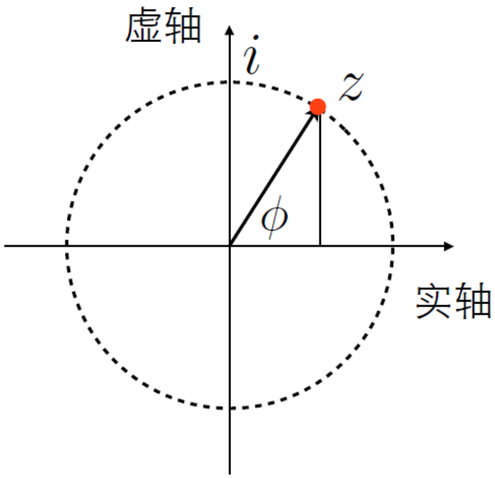
$$\omega_0 \ll \frac{V_s}{l_0}$$

只需要替换质量

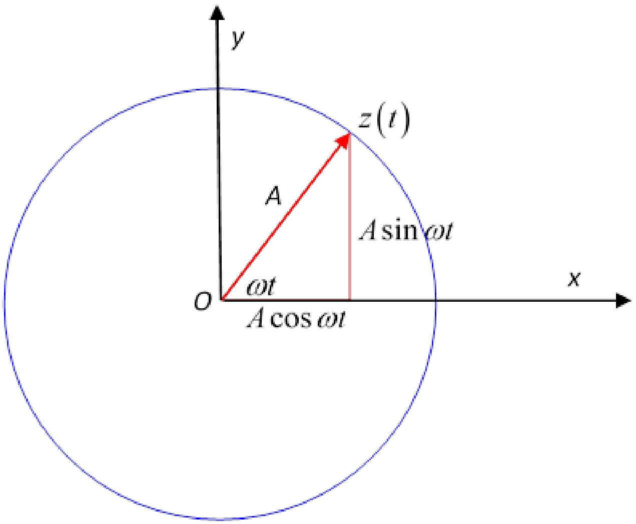
$$m \to m + \frac{m_0}{3} \quad \Rightarrow \quad \omega_0 = \sqrt{\frac{k}{m + m_0/3}}$$



简谐振动的复数表示



$$z = \cos\phi + i\sin\phi = e^{i\phi}$$



$$x(t) = A\cos(wt + \phi)$$

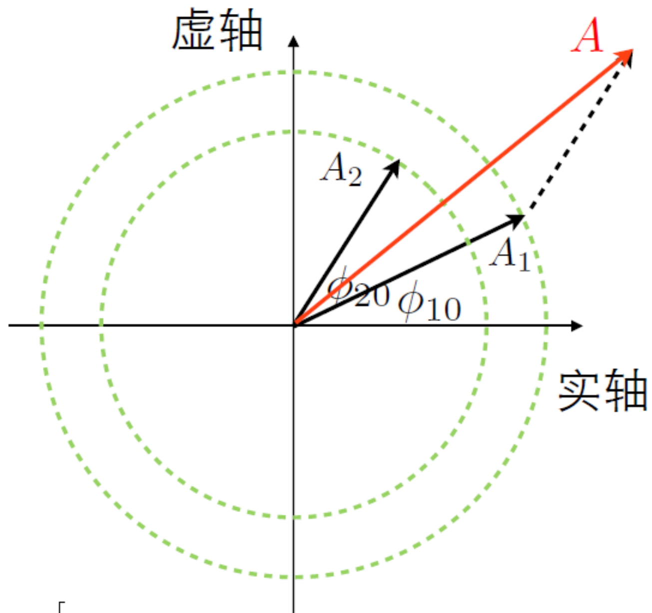
$$x(t) = \mathrm{Re}[Ae^{i(wt+\phi)}]$$

$$= \mathrm{Re}[z(t)]$$

$$\dot{z} = i\omega z \quad \text{and} \quad \ddot{z} = -\omega^2 z$$

$$\ddot{z} + \frac{k}{m} z = 0$$

一维同频振动的合成

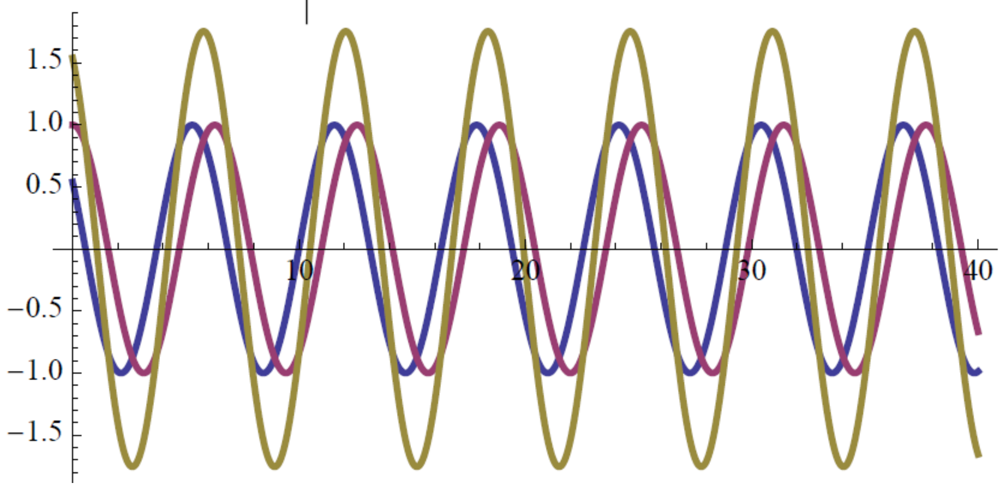


$$x_1(t) = A_1\cos(\omega t + \phi_{10})$$

$$x_2(t) = A_2\cos(\omega t + \phi_{20})$$

$$x(t) = x_1(t) + x_2(t) = A\cos(\omega t + \phi_0)$$

$$A^2 = A_1^2 + A_2^2 + 2A_1A_2\cos(\phi_{20} - \phi_{10})$$

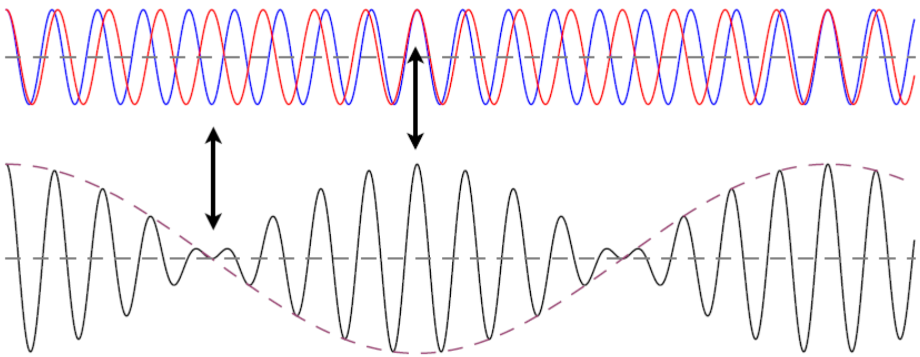


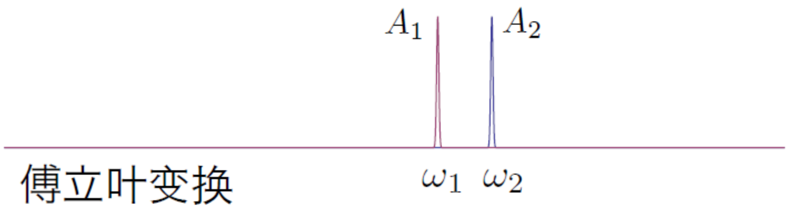
$$\tan\phi_0 = \frac{A_1\sin\phi_{10} + A_2\sin\phi_{20}}{A_1\cos\phi_{10} + A_2\cos\phi_{20}}$$

一维差频振动的合成-拍

$$x_1(t) = A_1\cos(\omega_1 t + \phi_{10}) \quad \text{and} \quad x_2(t) = A_2\cos(\omega_2 t + \phi_{20}), \quad x(t) = x_1(t) + x_2(t)$$

$$A_1 = A_2 \quad \text{and} \quad \phi_{10} = \phi_{20}$$





$$A_1 \neq A_2 \quad \text{and} \quad \phi_{10} = \phi_{20}$$