Total Differentiation of a Vector in a Rotating Frame of Reference

- Before we can write Newton's second law of motion for a reference frame rotating with the earth, we need to develop a relationship between the total derivative of a vector in an inertial reference frame and the corresponding derivative in a rotating system.

Let $\vec{A}$ **be an arbitrary vector with Cartesian components**

$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ **in an inertial frame of reference, and**

$\vec{A} = A'_x\hat{i}' + A'_y\hat{j}' + A'_z\hat{k}'$ **in a rotating frame of reference.**

If $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ **in an inertial frame of reference, then**

$$\frac{d\vec{A}}{dt} = \left(A_x\frac{d\hat{i}}{dt} + \hat{i}\frac{dA_x}{dt}\right) + \left(A_y\frac{d\hat{j}}{dt} + \hat{j}\frac{dA_y}{dt}\right) + \left(A_z\frac{d\hat{k}}{dt} + \hat{k}\frac{dA_z}{dt}\right)$$

Since the coordinate axes are in an inertial frame of reference,

$$\frac{d\hat{i}}{dt} = \frac{d\hat{j}}{dt} = \frac{d\hat{k}}{dt} = 0$$

$$\frac{d\vec{A}}{dt} = \left(\cancel{A_x\frac{d\hat{i}}{dt}} + \hat{i}\frac{dA_x}{dt}\right) + \left(\cancel{A_y\frac{d\hat{j}}{dt}} + \hat{j}\frac{dA_y}{dt}\right) + \left(\cancel{A_z\frac{d\hat{k}}{dt}} + \hat{k}\frac{dA_z}{dt}\right)$$

$$\frac{d\vec{A}}{dt} = \frac{dA_x}{dt}\hat{i} + \frac{dA_y}{dt}\hat{j} + \frac{dA_z}{dt}\hat{k} \qquad \text{(Eq. 1)}$$

If $\vec{A} = A'_x\hat{i}' + A'_y\hat{j}' + A'_z\hat{k}'$ in a rotating frame of reference, then

$$\frac{d\vec{A}}{dt} = \left(A'_x\frac{d\hat{i}'}{dt} + \hat{i}'\frac{dA'_x}{dt}\right) + \left(A'_y\frac{d\hat{j}'}{dt} + \hat{j}'\frac{dA'_y}{dt}\right) + \left(A'_z\frac{d\hat{k}'}{dt} + \hat{k}'\frac{dA'_z}{dt}\right) \quad \text{(Eq. 2)}$$

Because the left hand sides of Eq. 1 and Eq. 2 are identical,

$$\frac{dA_x}{dt}\hat{i} + \frac{dA_y}{dt}\hat{j} + \frac{dA_z}{dt}\hat{k} = \left(A'_x\frac{d\hat{i}'}{dt} + \hat{i}'\frac{dA'_x}{dt}\right) + \left(A'_y\frac{d\hat{j}'}{dt} + \hat{j}'\frac{dA'_y}{dt}\right) + \left(A'_z\frac{d\hat{k}'}{dt} + \hat{k}'\frac{dA'_z}{dt}\right)$$

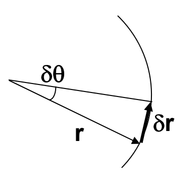
Regrouping the terms

$$\underbrace{\frac{dA_x}{dt}\hat{i} + \frac{dA_y}{dt}\hat{j} + \frac{dA_z}{dt}\hat{k}}_{\left(\frac{d\vec{A}}{dt}\right)_{inertial}} = \underbrace{\frac{dA'_x}{dt}\hat{i}' + \frac{dA'_y}{dt}\hat{j}' + \frac{dA'_z}{dt}\hat{k}'}_{\left(\frac{d\vec{A}}{dt}\right)_{rotating}} + \underbrace{A'_x\frac{d\hat{i}'}{dt} + A'_y\frac{d\hat{j}'}{dt} + A'_z\frac{d\hat{k}'}{dt}}_{\text{effects of rotation}}$$

$$\underbrace{\frac{dA_x}{dt}\hat{i} + \frac{dA_y}{dt}\hat{j} + \frac{dA_z}{dt}\hat{k}}_{\left(\frac{d\vec{A}}{dt}\right)_{inertial}} = \underbrace{\frac{dA'_x}{dt}\hat{i}' + \frac{dA'_y}{dt}\hat{j}' + \frac{dA'_z}{dt}\hat{k}'}_{\left(\frac{d\vec{A}}{dt}\right)_{rotating}} + \underbrace{A'_x\frac{d\hat{i}'}{dt} + A'_y\frac{d\hat{j}'}{dt} + A'_z\frac{d\hat{k}'}{dt}}_{\text{effects of rotation}}$$

To interpret $\frac{d\hat{i}'}{dt}, \frac{d\hat{j}'}{dt}, \frac{d\hat{k}'}{dt}$ **think of each unit vector as a position vector.**

linear velocity = angular velocity x position vector → $\vec{V} = \vec{\Omega} \times \vec{r}$



Because $\vec{V} = \frac{d\vec{r}}{dt}$, $\frac{d\vec{r}}{dt} = \vec{\Omega} \times \vec{r}$

Thus $\frac{d\hat{i}'}{dt} = \vec{\Omega} \times \hat{i}'$, $\frac{d\hat{j}'}{dt} = \vec{\Omega} \times \hat{j}'$, $\frac{d\hat{k}'}{dt} = \vec{\Omega} \times \hat{k}'$

$$\underbrace{\frac{dA_x}{dt}\hat{i}+\frac{dA_y}{dt}\hat{j}+\frac{dA_z}{dt}\hat{k}}_{\left(\frac{d\vec{A}}{dt}\right)_{inertial}} = \underbrace{\frac{dA'_x}{dt}\hat{i}'+\frac{dA'_y}{dt}\hat{j}'+\frac{dA'_z}{dt}\hat{k}'}_{\left(\frac{d\vec{A}}{dt}\right)_{rotating}} + \underbrace{A'_x\left(\vec{\Omega}\times\hat{i}'\right)+A'_y\left(\vec{\Omega}\times\hat{j}'\right)+A'_z\left(\vec{\Omega}\times\hat{k}'\right)}_{\vec{\Omega}\times\vec{A}\ \text{(effects of rotation)}}$$

$$\boxed{\left(\frac{d\vec{A}}{dt}\right)_{inertial} = \left(\frac{d\vec{A}}{dt}\right)_{rotating} + \vec{\Omega}\times\vec{A}}$$

This equation provides us with a formal way of expressing the balance of forces on a fluid parcel in a rotating coordinate system.

Newton's second law in an inertial reference frame:

$$\left(\frac{d\vec{V}}{dt}\right)_{inertial} = \frac{\sum\vec{F}}{m}$$

To transform to rotating coordinates:

$$\left(\frac{d\vec{r}}{dt}\right)_{inertial} = \left(\frac{d\vec{r}}{dt}\right)_{rotating} + \vec{\Omega}\times\vec{r}$$

$\vec{r}$ is the position vector for an air parcel on the rotating earth.

$$\vec{V}_{inertial} = \vec{V} + \vec{\Omega}\times\vec{r}$$

Velocity is the rate of change of the position vector with time.

$$\left(\frac{d\vec{V}_{inertial}}{dt}\right)_{inertial} = \frac{d\vec{V}_{inertial}}{dt} + \vec{\Omega}\times\vec{V}_{inertial}$$

Using the transformation of the total derivative.

$$\left(\frac{d\vec{V}_{inertial}}{dt}\right)_{inertial} = \frac{d\vec{V}_{inertial}}{dt} + \vec{\Omega}\times\vec{V}_{inertial}$$

$$\left(\frac{d\vec{V}}{dt}\right)_{inertial} = \frac{d}{dt}\left(\vec{V}+\vec{\Omega}\times\vec{r}\right)+\vec{\Omega}\times\left(\vec{V}+\vec{\Omega}\times\vec{r}\right)$$

Substituting for $\vec{V}_{inertial}$

Using some vector identities and defining $\vec{R}$ as a vector perpendicular to the axis of rotation with magnitude equal to the distance to the axis of rotation.

$$\left(\frac{d\vec{V}}{dt}\right)_{inertial} = \frac{d\vec{V}}{dt} + 2\vec{\Omega}\times\vec{V} - \Omega^2\vec{R}$$

Acceleration following the motion in an inertial system

Rate of change of relative velocity following the relative motion in a rotating reference frame.

Coriolis acceleration

Centrifugal acceleration

Substituting into Newton's second law:

$$\left(\frac{d\vec{V}}{dt}\right)_{inertial} = \frac{\sum\vec{F}}{m}$$

$$\frac{d\vec{V}}{dt} + 2\vec{\Omega}\times\vec{V} - \Omega^2\vec{R} = \frac{\sum\vec{F}}{m}$$

If the real forces acting on a fluid parcel are the pressure gradient force, gravitation and friction, then

$$\frac{d\vec{V}}{dt} = -2\vec{\Omega}\times\vec{V} - \frac{1}{\rho}\nabla p + \vec{g} + \vec{F}_r$$

Rate of change of relative velocity following the relative motion in a rotating reference frame.

Coriolis acceleration

Pressure gradient force (per unit mass)

Gravity term (gravitation + centrifugal)

Friction

Vector momentum equation in rotating coordinates