Gram-Schmidt 正交基的行列式表示法*

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摘 要 Gram-Schmidt 正交化方法是求正交基的一种算法. 基于行列式的性质和归纳法可以证明，其正交向量组的一般项可通过行列式表示出来.

关键词 Gram-Schmidt 正交基；行列式 **中图分类号** O151.2

Gram-Schmidt 正交化方法是求正交基的一种算法，其通项 β_n 的表示方法优美且规律性强，便于在计算机上进行递推运算. 本文受类似 Cramer 法则形式的启发，给出 Gram-Schmidt 正交基 β_n 的行列式表示法.

首先，设 $\alpha_1,\alpha_2,\cdots,\alpha_n$ 是 R^n 上的一组基. 由 Gram-Schmidt 正交化方法，可得

$$\beta_1=\alpha_1,\qquad \beta_2=\alpha_2-\frac{(\alpha_1,\alpha_2)}{(\alpha_1,\alpha_1)}\alpha_1,$$

$$\beta_3=\alpha_3-\frac{(\alpha_1,\alpha_1)(\alpha_2,\alpha_3)-(\alpha_1,\alpha_2)(\alpha_2,\alpha_3)}{(\alpha_1,\alpha_1)(\alpha_2,\alpha_2)-(\alpha_1,\alpha_2)(\alpha_1,\alpha_2)}\alpha_2+\frac{(\alpha_1,\alpha_2)(\alpha_2,\alpha_3)-(\alpha_2,\alpha_2)(\alpha_1,\alpha_3)}{(\alpha_1,\alpha_1)(\alpha_2,\alpha_2)-(\alpha_1,\alpha_2)(\alpha_1,\alpha_2)}\alpha_1.$$

上述 β_2、β_3 可用行列表示为如下形式：

$$\beta_2=\frac{\begin{vmatrix}(\alpha_1,\alpha_1) & (\alpha_1,\alpha_2)\\ \alpha_1 & \alpha_2\end{vmatrix}}{|(\alpha_1,\alpha_1)|},\qquad \beta_3=\frac{\begin{vmatrix}(\alpha_1,\alpha_1) & (\alpha_1,\alpha_2) & (\alpha_1,\alpha_3)\\ (\alpha_1,\alpha_2) & (\alpha_2,\alpha_2) & (\alpha_2,\alpha_3)\\ \alpha_1 & \alpha_2 & \alpha_3\end{vmatrix}}{\begin{vmatrix}(\alpha_1,\alpha_1) & (\alpha_1,\alpha_2)\\ (\alpha_2,\alpha_1) & (\alpha_2,\alpha_2)\end{vmatrix}}.$$

猜想 对于 R^n 中给定的一组基 $\alpha_1,\alpha_2,\cdots,\alpha_n$，则由 Gram-Schmidt 正交化所得到的正交基 $\beta_1,\beta_2,\cdots,\beta_n$，当 $2\leqslant m\leqslant n$ 时，β_m 的一般形式可以表示如下：

$$\beta_m=\frac{\begin{vmatrix}(\alpha_1,\alpha_1) & (\alpha_1,\alpha_2) & (\alpha_1,\alpha_3) & \cdots & (\alpha_1,\alpha_m)\\ (\alpha_2,\alpha_1) & (\alpha_2,\alpha_2) & (\alpha_2,\alpha_3) & \cdots & (\alpha_2,\alpha_m)\\ \vdots & \vdots & \vdots & & \vdots\\ (\alpha_{m-1},\alpha_1) & (\alpha_{m-1},\alpha_2) & (\alpha_{m-1},\alpha_3) & \cdots & (\alpha_{m-1},\alpha_m)\\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_m\end{vmatrix}}{\begin{vmatrix}(\alpha_1,\alpha_1) & (\alpha_1,\alpha_2) & \cdots & (\alpha_1,\alpha_{m-1})\\ (\alpha_2,\alpha_1) & (\alpha_2,\alpha_2) & \cdots & (\alpha_2,\alpha_{m-1})\\ \vdots & \vdots & & \vdots\\ (\alpha_{m-1},\alpha_1) & (\alpha_{m-1},\alpha_1) & \cdots & (\alpha_{m-1},\alpha_{m-1})\end{vmatrix}}.$$

证明 用归纳法证明. 当 $m=2$ 时，

$$(\beta_1,\beta_2)=\left(\alpha_1,\alpha_2-\frac{(\alpha_1,\alpha_2)}{(\alpha_1,\alpha_1)}\alpha_1\right)=(\alpha_1,\alpha_2)-(\alpha_1,\alpha_1)\cdot\frac{(\alpha_1,\alpha_2)}{(\alpha_1,\alpha_1)}=0,$$

即 β_1 和 β_2 正交. 当 $m=k(2\leqslant k\leqslant n-1)$ 时，假设 $\beta_1,\beta_2,\cdots,\beta_k$ 两两正交，现证 β_{k+1} 和 $\beta_1,\beta_2,\cdots,\beta_k$ 均正

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交.

对任一 $\beta_j(1 \leqslant j \leqslant k)$，由猜想表达式，$\beta_j = A_{j,1}\alpha_1 + A_{j,2}\alpha_2 + \cdots + A_{j,j}\alpha_j$. 其中当 $j = 1$ 时，$A_{1,1} = 1$. 当 $2 \leqslant j \leqslant k$ 时，

$$A_{j,s} = (-1)^{j+s}\frac{\begin{vmatrix} (\alpha_1,\alpha_1) & (\alpha_1,\alpha_2) & \cdots & (\alpha_1,\alpha_{s-1}) & (\alpha_1,\alpha_{s+1}) & \cdots & (\alpha_1,\alpha_j) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ (\alpha_{j-1},\alpha_1) & (\alpha_{j-1},\alpha_2) & \cdots & (\alpha_{j-1},\alpha_{s-1}) & (\alpha_{j-1},\alpha_{s+1} & \cdots & (\alpha_{j-1},\alpha_j) \end{vmatrix}}{\begin{vmatrix} (\alpha_1,\alpha_1) & (\alpha_1,\alpha_2) & \cdots & (\alpha_1,\alpha_{s-1}) & (\alpha_1,\alpha_s) & \cdots & (\alpha_1,\alpha_{j-1}) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ (\alpha_{j-1},\alpha_1) & (\alpha_{j-1},\alpha_2) & \cdots & (\alpha_{j-1},\alpha_{s-1}) & (\alpha_{j-1},\alpha_s) & \cdots & (a_{j-1},\alpha_{j-1}) \end{vmatrix}},(1 \leqslant s \leqslant j).$$

$$(\beta_{k+1},\beta_j) = (\beta_{k+1},\sum_{s=1}^{j}A_{j,s}\alpha_s) = \sum_{s=1}^{j}A_{j,s}(\alpha_s,\beta_{k+1}).$$

又因为
$$\beta_{k+1} = C_{k+1,1}\alpha_1 + C_{k+1,2}\alpha_2 + \cdots + C_{k+1,k+1}\alpha_{k+1},$$

$$(\beta_{k+1},\beta_j) = \sum_{s=1}^{j}A_{j,s}(\alpha_s,\beta_{k+1}) = \sum_{s=1}^{j}A_{j,s}(\alpha_s,\sum_{i=1}^{k+1}C_{k+1,i}\alpha_i) = \sum_{i=1}^{k+1}C_{k+1,i}\sum_{s=1}^{j}A_{j,s}(\alpha_s,\alpha_i) =$$
$$C_{k+1,1}\sum_{s=1}^{j}A_{j,s}(\alpha_s,\alpha_1) + C_{k+1,2}\sum_{s=1}^{j}A_{j,s}(\alpha_s,\alpha_2) + \ldots + C_{k+1,k+1}\sum_{s=1}^{j}A_{j,s}(\alpha_s,\alpha_{k+1}).$$

$$C_{k+1.q} = (-1)^{k+1+q}\frac{\begin{vmatrix} (\alpha_1,\alpha_1) & (\alpha_1,\alpha_2) & \cdots & (\alpha_1,\alpha_{q-1}) & (\alpha_1,\alpha_{q+1}) & \cdots & (\alpha_1,\alpha_{k+1}) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ (\alpha_k,\alpha_1) & (\alpha_k,\alpha_2) & \cdots & (\alpha_k,\alpha_{q-1}) & (\alpha_k,\alpha_{q+1}) & \cdots & (\alpha_k,\alpha_{k+1}) \end{vmatrix}}{\begin{vmatrix} (\alpha_1,\alpha_1) & (\alpha_1,\alpha_2) & \cdots & (\alpha_1,\alpha_{q-1}) & (\alpha_1,\alpha_q) & \cdots & (\alpha_1,\alpha_k) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ (\alpha_k,\alpha_1) & (\alpha_k,\alpha_2) & \cdots & (\alpha_k,\alpha_{q-1}) & (\alpha_k,\alpha_q) & \cdots & (\alpha_k,\alpha_k) \end{vmatrix}},(1 \leqslant q \leqslant k+1).$$

所以，

$$(\beta_{k+1},\beta_j) = \frac{\begin{vmatrix} (\alpha_1,\alpha_1) & (\alpha_1,\alpha_2) & \cdots & (\alpha_1,\alpha_{k+1}) \\ (\alpha_2,\alpha_1) & (\alpha_2,\alpha_2) & \cdots & (\alpha_2,\alpha_{k+1}) \\ \cdots & \cdots & \cdots & \cdots \\ (\alpha_k,\alpha_1) & (\alpha_k,\alpha_2) & \cdots & (\alpha_k,\alpha_{k+1}) \\ \sum_{s=1}^{j}A_{j,s}(\alpha_s,\alpha_1) & \sum_{s=1}^{j}A_{j,s}(\alpha_s,\alpha_2) & \cdots & \sum_{s=1}^{j}A_{j,s}(\alpha_s,\alpha_{k+1}) \end{vmatrix}}{\begin{vmatrix} (\alpha_1,\alpha_1) & (\alpha_1,\alpha_2) & \cdots & (\alpha_1,\alpha_k) \\ (\alpha_2,\alpha_1) & (\alpha_2,\alpha_2) & \cdots & (\alpha_2,\alpha_k) \\ \cdots & \cdots & \cdots & \cdots \\ (\alpha_k,\alpha_1) & (\alpha_k,\alpha_2) & \cdots & (\alpha_k,\alpha_k) \end{vmatrix}}.$$

记分子各行对应的 $k+1$ 维向量是 $\gamma_1,\gamma_2,\cdots,\gamma_k,\gamma_{k+1}$. 则 $\gamma_{k+1} = \sum_{s=1}^{i}A_{j,s}\gamma_s$. 即 γ_{k+1} 可以用 $\gamma_1,\gamma_2,\cdots,\gamma_j$ 线性表示. 由行列式性质，即知 $(\beta_{k+1},\beta_j) = 0,(j = 1,2,\cdots,k)$. 故得到一组正交向量 $\beta_1,\beta_2,\cdots,\beta_{k+1}$. 由归纳法，当 $k = n-1$ 时，得 $\beta_1,\beta_2,\cdots,\beta_n$ 是一组正交向量. 又正交向量必线性无关，且 R^n 中任 n 个线性无关的向量是一组基. 故 $\beta_1,\beta_2,\cdots,\beta_n$ 是 R^n 中一组正交基. 证毕.

至此，上文给出 Gram-Schmidt 正交基的行列式表示. 它是类似于 Gramer 法则的优美的表示形式. 但是，必须指出：从计算方法角度考虑仍没有任何实质性的改进.

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